BACKBONE OF PERSPECTIVE

TAYLOR

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PREFACE

These notes have been given in the form of lectures and drawing-board exercises for many years. They are here reduced to print to save time in note-taking on the part of the student. A reader that finds no errors in these pages should read again. I am indebted to E. E. Howard, a former instructor, for much valuable assistance. Prof. E. C. H. Bantel, and Instructor S. P. Finch have rendered substantial aid. The chapter on Axometric Projections is a modification of notes taken under Dr. W. M. Thornton of the University of Virginia, when I was a student there.

T. U. Taylor.

Austin, Texas.
June, 1910.

BACKBONE OF PERSPECTIVE

CHAPTER ONE.

Primary Methods.

1. **Plan and Projection.**—When a perpendicular is dropped from any point P to a plane, the point of intersection p of the perpendicular and the plane is called the foot of the perpendicular and p is called the *projection* of P. If from two points, A and B, perpendiculars are dropped on any plane, and their feet a and b be joined, the line ab is called the projection of AB on the given plane. The projection of a point on a horizontal plane is called the *plan* of

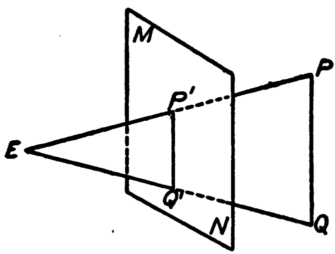

Figure 1.

the point, and its projection on a vertical plane is called its *elevation*. We shall denote the horizontal plane by H in these pages.

2. **Perspective.**—The perspective of a point P

(Fig. 1) with reference to any other point E and a plane MN is the point of intersection of the line PE and the plane. Thus if P be any point, the perspective of P with reference to the point E and the plane MN is the point P′ where EP cuts the plane MN. Similarly the perspective of Q with reference to the point E and plane MN is the point Q′ where EQ cuts MN.

The point E is the eye of the observer. The perspective of any point P with reference to any plane MN is the intersection of the line of sight PE with the plane. The plane MN is called the perspective plane.

3. **Perspective of a Line.**—The perspective of a line will be found by joining all points of the line to the eye and finding their intersections with the perspective plane. The perspective of a straight line will be a straight line. Since by joining P and Q with E we have a triangle EPQ (Fig. 1), and the plane of the triangle will cut the perspective plane MN in a straight line P′Q′, it is evident that a line joining any point in PQ with E will cut the plane MN somewhere in P′Q′, as such a line lies in the plane EPQ, and as the plane cuts the perspective plane MN in P′Q′. It is sufficient in determining the perspective of a definite part PQ of a straight line, to find the perspective of the two points P and Q and join these points by a straight line.

4. **Point of View.** — A perspective is defined with reference to the *point of view,* which is the eye of the observer. If we stand in a room and look through a window glass at a point A, the intersection of the line of sight with the plane of the win-

dow glass will be the perspective of the point with reference to that particular location of the eye. The point A will have a perspective for every position of the eye in the room.

5. **Perspective of a Point.**—Let P (Fig. 2) be any point and E the eye and MN the perspective plane which is taken as vertical. Let the plane MN cut the horizontal plane in GL. Project P and E on the horizontal plane in p and e. Join pe, cutting GL in D. The plane PpeE will cut plane MN in a vertical line, as the plane PpeE is itself vertical. The line of intersection of the two vertical planes PpeE and MN will be parallel to Pp and Ee. The perspective of P will lie somewhere on the vertical line Dt, and as it must lie on EP it will be the point where EP intersects this vertical.

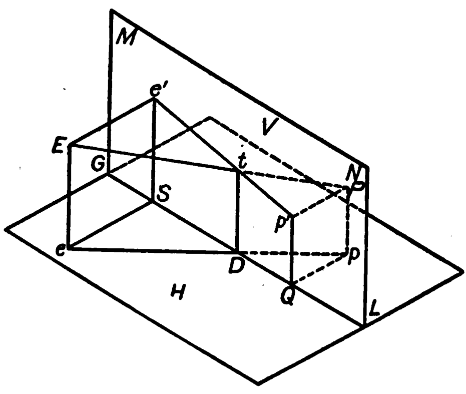

Figure 2.

It is our object to develop some method of finding this point of intersection.

Draw the perpendiculars from P and E to MN, and let the feet of these perpendiculars be p′ and e′ respectively. The projection of PE on MN will be p′e′. The perspective t of the point P will be on EP and it will lie on e′p′, because PE will intersect the plane MN at some point on p′e′. Hence the perspective of the point P will lie on p′e′, and on Dt, and therefore at their intersection.

6. Perspective Plane in H.—If in Fig. 3 we revolve the plane MN about GL as an axis, each point in the plane will describe a circle whose center lies on and whose plane is perpendicular to GL. If the plane is folded to the horizontal position, the points p′ and e′ will fall as far from GL as the points P and E are above H. Thus p′ will describe a circle whose radius is Qp′ and whose center

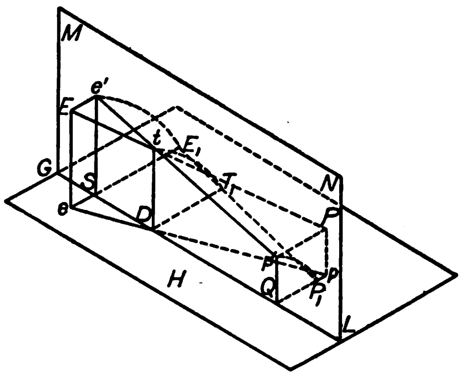

Figure 3.

is Q. When MN is folded into the horizontal position p′ will fall at P_1. Similarly e′ will fall at E_1 and t will fall at T_1.

7. Example.—In Fig. 3 let SQ=4″, Qp=2″,

$Pp = p'Q = QP_1 = 1\frac{1}{2}''$, $eS = 3''$, $Ee = Se' = SE_1 = 2\frac{1}{2}''$

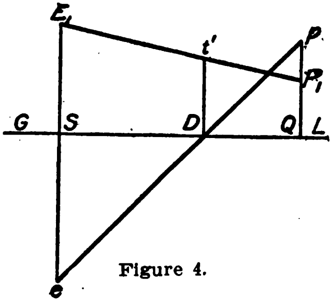

Figure 4.

Find the perspective of the point P. Draw as in Fig. 4 a line GL and make $SQ = 4''$, and make $Se = 3''$, $pQ = 2''$. Join pe, cutting GL at D. When MN is folded into the horizontal position p′ and e′ (Fig. 3) will fall at P_1 and at E_1, respectively. The folded position of the perspective will lie on P_1E_1 and the folded position of the vertical line Dt of Figs. 2 and 3 will lie on Dt′.

PROBLEM 1. If $SQ = 2\frac{1}{2}''$, $Qp = 2$, $QP_1 = 0$, $eS = 2\frac{1}{2}''$, and $SE_1 = 1''$, find the perspective of P.

PROBLEM 2. If $SQ = 2''$, $Qp = 2''$, $QP_1 = 1''$, $eS = 3''$, and $SE_1 = 2''$, find the perspective of P.

8. General Method.— The method just outlined is perfectly general, and can be used to find the perspective of any structure, however complicated. The plane of the paper upon which the construction is made represents H, and the perspective plane (which in all practical cases is vertical) is folded into H. The horizontal projection p of the point P is located on the paper, and the perspective is found to be at *t*, which (after folding) is in the same plane as *p*. The points p′ and e′ are the projections of P and E on the perspective plane, and P_1 and E_1 are the folded positions of these points.

9. Translation of the Perspective Plane.—The horizontal projection of points and the folded position of their projections on the perspective plane are generally above GL, as the perspective plane is taken between the eye of the observer and the object. Where there are many points the figure may become complicated. To prevent confusion and to keep the plan and perspective of the points separate, the perspective plane before folding is brought from position GL in Fig. 5 to position G′L′, while the point e and the plan p of the point remain fixed. It is clear that E_1 and P_1 will occupy exactly the same positions with respect to G′L′ that they did with respect to GL; *that is,* $S'E''=SE_1$, $Q'P''=QP_1$. The perspective T_1 of the point P lies on the perpendicular at D to GL and on P_1E_1; and it is clear that in bringing GL to G′L′ we have simply brought all the points except the horizontal projection p into the position $P''T_1E''$, by a motion of translation which has not altered their relative positions.

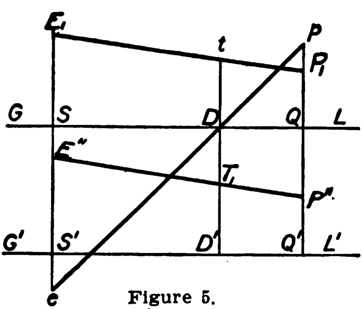

Figure 5.

10. True Height Line of a Point.—From Fig. 5 we see that the line pP_1 is perpendicular to GL, and that $Q'P''$ is equal to the *true height* of the point above H, and that the point P″ is joined to E″. This can be expressed in the following constructive rule:

(1) *Join the plan of the point to the plan of the*

eye, cutting GL in D, and at D draw DD′ perpendicular to the ground line.

(2) *Drop a perpendicular from p, the plan of the point, on the new ground line (G′L′) and from G′L′ lay off on this perpendicular the true height of the point=Q′P″. Join the point thus located to E″, and where the line E″P″ cuts DD′ is the perspective of the point P.*

In the following articles the perspective of the point P will be marked P′, that of K will be marked K′, etc.

11. Perspective Triangle.—Two points P and K whose heights above H are Q′P″ and Q′K″ respectively lie on a common vertical line. Let the height of E=S′E″ and e and p be located with respect to GL and G′L′ as in Fig. 6. Join pe, cutting

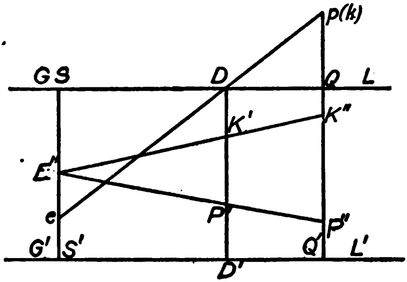

Figure 6.

GL at D and on the true height line Qp of P and K lay off Q′K″ equal to the height of K and Q′P″ equal to the height of P. Take S′E″ equal to height of E and join E″P″ and E″K″, cutting DD′

in K′ and P′. The triangle E″P′K′ is called the perspective triangle, and it is the perspective of the triangle EPK.

PROBLEM 3. Given SQ=3″, Q′P″=1″, Q′K″=3″, S′E″=2″, find the perspective of PK.

12. Special Case.—When the plan of a point lies on or near eE″, the foregoing method of article 10 is indeterminate or lacks exactness. In the first case the former method can not, and in the latter case should not, be used. In the first case the perspective triangle resolves into the straight line eE″, and in the latter case its sides cut DD′ at such sharp angles that the solution is not definite.

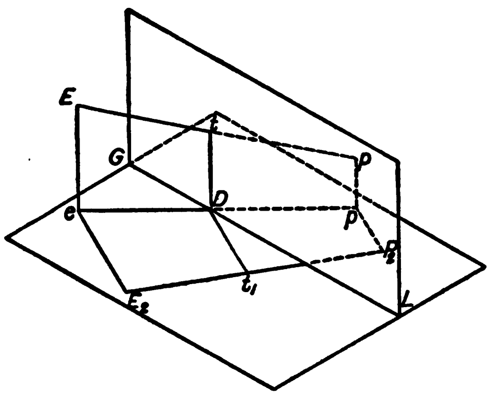

Figure 7.

In Fig. 7 let p and e be the plans of the point P and of the eye E. If the trapezoid PEep be folded around pe into H, PE will fall at P_2E_2, and E_2e, Dt_1 and P_2p will be perpendicular to pe, and Dt_1 is equal to the height of the perspective of P above GL.

In Fig. 8 draw lines at e, D, and p perpendicular to pe and make eE_2 equal to the height of the eye, and P_2p equal to the height of the point. Join P_2E_2, cutting Dt_1 in t_1. Lay off D′P′ equal to Dt_1. The point P′ is the perspective of P.

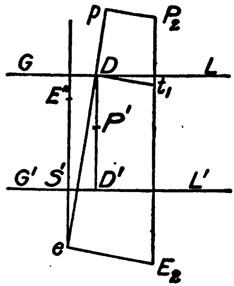

Figure 8.

PROBLEM 4. If Fig. 5 SQ=o, Qp=2″, Q′P″=1″ eS=2″, and $S'E_2$=3″, find the perspective of P.

PROBLEM 5. If SQ=o, Qp=1″, Q′P″=o, eS=2″. and $S'E_2$=1″, find the perspective of P (Fig. 5).

PROBLEM 6. The plan of a point whose height above H is 3″ is 2″ behind GL, and the plan of the eye whose height is 1 inch is 5 inches from GL. If pe is at right angles to GL, find the perspective of P.

13. Perspective of a Block.—Let ABCD (Fig. 9) be the base of a rectangular block, whose side and end views are shown by ABFT and ADHT. The block rests on the horizontal plane and has its edge in the perspective plane. We have AB=15′, AD=9′, height of block=10′, baL=30°. The eye is taken 25′ in front of the perspective plane and at a height of 6′.

To find the perspective of AT join ea, cutting GL at a and from this point drop a perpendicular to G′L′, cutting it at A′. As AT lies in the perspective plane, it will be its own perspective, and all we have to do is to lay off A′T′=10′, thus determining the perspective of AT. Then drop a perpendicular from b on G′L′, cutting it at Q′. Now as Q′b is the "true height" line for all points that lie on

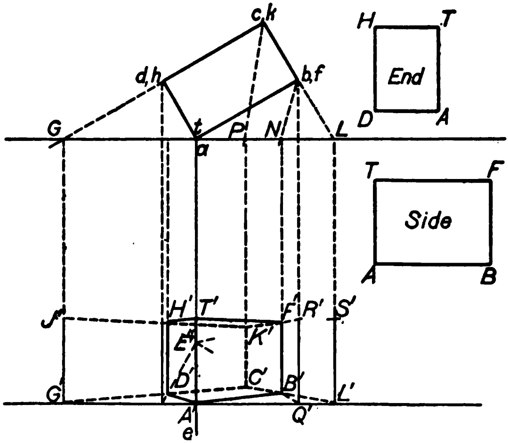

Figure 9.

Q′b lay off the height of B and F on Q′b. The height of B is zero and that of F=10′=Q′R′. Join eb, cutting GL at N. The points B′ and F′ where E″Q′ and E″R′ cut the perpendicular to GL at N are the perspectives of B and F, respectively. The perspectives of DH and CK are found as that of BF. The whole perspective is easily completed, and appears as A′B′F′T′H′D′C′K′.

PROBLEM 7. Given (in Fig. 9) AB=12′, AD=8′, ea=16′, height of block=8′, height of eye=4′. The perspective plane GL passes 1′ below a and AB makes 30° with GL. Construct the perspective of the block.

PROBLEM 8. Find the perspective of the block of Fig. 9 when AB makes 45° with GL.

14. True Height Line of a Horizontal Line.—If the line HK (Fig. 9) be extended to the perspective plane it will intersect this plane directly above G (or G′) at a distance equal to the height of HK above the horizontal plane. The true height of HK above the horizontal plane can be laid off on GG′ equal to G′J′. The point J′ is the perspective of the point of the line HK that lies in the perspective plane. Now, if we join J′H′ we have the perspective of HK extended to the perspective plane. The perspective of K lies somewhere on this line and can be found in two ways:

First. Join ek, and from the point where it cuts

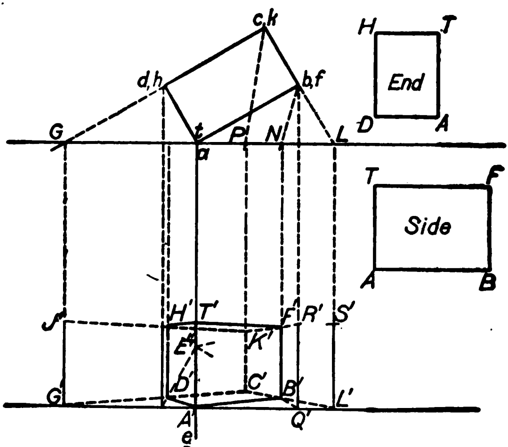

Figure 9.

GL drop a perpendicular, cutting J′H′ at K′. This determines the perspective of K. In the same way we can find the perspective of any point not in eE″.

Second. Produce kf to cut GL at L. Then LL' is the "true height" line of all horizontal lines that lie in the face of BFKC. Lay off L'S' equal to the height (10') of line KF. Join S'F' and produce it to cut J'H' at K'.

One of these methods should always be used for points near eE", for the construction lines of the usual method intersect at such sharp angles that their point of intersection is not sufficiently definite.

15. **Horizontal Squares.**—A square abcd, 6'x6', lies on H and its side dc, Fig. 10, is parallel to and 1' from GL. The eye (whose height is 6') is 10' in front of the perspective plane and lies in a perpendicular to GL 2' to right of c. Find the perspective of abcd and of the nine squares into which it is divided.

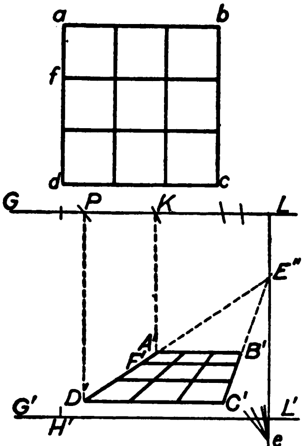

Figure 10.

SOLUTION.—The true height line of ad is H'd, and as D and A lie on the horizontal plane, their true heights are zero. We therefore join ea and ed cutting GL in K and P, respectively. The perspective of D will lie on E"H' vertically below P, and that of a will lie on E"H' vertically below K. The perspectives of a and d are thus found at A' and D'. In the same way we find the perspectives of b and c at B' and C'. Join A'B', B'C', C'D' and D'A', forming A′B′C′D′, the perspective of abcd. The perspective of any point f can be found by the usual method.

16. Perpendiculars and Parallels.—It will be observed from the preceding example that the perspective of all lines perpendicular to the perspective plane pass through a common point E″. When the perspective of a series of parallels passes through a common point as E″, they are said to *vanish* at E″, and E″ is called the *vanishing point* of this system of parallel lines. It will be shown later that all systems of parallel lines have a vanishing point.

The point E″ (the vanishing point of perpendiculars) is called the *center* of the *picture.* The perspectives of all horizontal lines parallel to the perspective plane are parallel to GL.

PROBLEM 9. Find the perspective of the square in Fig. 10 when eE″ lies on BC produced, other data remaining the same.

PROBLEM 10. Find the perspective of the same when eE″ bisects AB.

PROBLEM 11. Find the perspective of the squares when the eye is 2′ to the left of AD.

PROBLEM 12. Find the perspective of a regular hexagon of 2′ side that lies on H when one of its sides is parallel to and its center is 3′ from GL, when the eye (height 4′) lies 6′ in front of the perspective plan and in a perpendicular from the center of hexagon.

PROBLEM 13. Find the perspective of hexagon in problem 12 when one side is perpendicular to GL.

PROBLEM 14. A circle of diameter 4′ lies on H and has its center 3′ from the perspective plane. Find the perspective of the circle when the eye (height 5′) lies in a perpendicular through center to GL and 5′ in front of GL.

PROBLEM 15. Find the perspective of a circle whose plane is parallel to the perspective plane and at a distance of 3′ from it, the height of its center above H being 5′, when the eye (height 5′) lies in

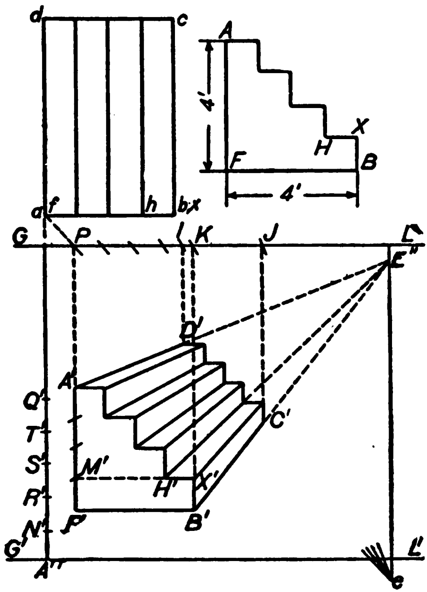

Figure 11.

the perpendicular to the perspective plane from the center of the circle and 6′ in front of the perspective plane. Diameter of circle=4′.

17. **Perspective of Steps.** — A series of four steps (Fig. 11) six feet long, each having a tread and rise of 1′, has one end parallel to the perspective plane at a distance 1′ from it. Find the perspective of the steps when the eye (height 8′) is 6′ to the right of lowest step and 9′ in front of the perspective plane. Height of base fbc=1′.

On aA″ the true height line of ad lay off the heights of the different steps from G′L′ equal to 1, 2, 3, and 4 feet at N′, R′; S′, T′ and Q′. Join ea and ed, cutting GL at P and I. Drop perpendiculars from P and I, cutting E″Q′ in A′ and D′, the perspectives of a and d. The perspective of f lies on E″N′ and vertically below P. Join R′, S′, and T′ to E″, cutting A′F′ at M′, etc., through which draw lines parallel to G′L′. To find the perspective B′X′ of an edge BX, join eb, and from the point where it cuts through GL drop a perpendicular to cut the horizontals through F′ and M′ at B′ and X′. Join B′ and X′ to E″ and drop a perpendicular from point of intersection of ec and GL, cutting E″B′ at C′, the perspective of c. In the same way we can find the perspective of all other edges.

PROBLEMS.

Problem 16. Find the perspective of the steps in Fig. 11 when height of eye is 9′ and lies 4′ to the right of the lowest step, other dimensions remaining the same.

17. Find the perspective of a hollow box 4′x6′, height 5′, that rests on H with the long face perpendicular to the perspective plane. The open end next the perspective plane is 2′ from it, and the eye

(height 3′) lies on a line bisecting the plan of the box and at a distance of 10′ from the perspective plane.

18. Find the perspective in problem 17 when the height of the eye is 1′, other dimensions remaining the same.

19. One corner of the box in problem 17 lies in the perspective plane and the short end makes an angle of 30° with it. If the eye (height 2½′) lies in a perpendicular to GL at the nearest corner, find the perspective of the box.

20. The base of a monument is 4′x6′ by 1′ high. Upon the base rests a rectangular block 3′x5′ by 3′ high. The block is capped by a pyramid whose base is the upper base of the block and whose height is 2′. The longest face of the base makes 30° with the perspective plane and the nearest corner of base is 1′ from the perspective plane. The eye (height 2′) lies in a perpendicular from the nearest corner of base and is 8′ in front of the perspective plane. Find the perspective.

21. Find the perspective in problem 20 when the height of the eye is 5′, other dimensions remaining fixed.

22. Find the perspective in problem 20 when the long face of base makes 45° with the perspective plane, other dimensions remaining as in problem 20.

23. Find the perspective in problem 20 if the height of the eye is 1′, other dimensions remaining the same.

24. Eight cubes are the corners of a larger cube of 12′ edge. One face of the cubes is parallel to the perspective plane and 2′ from it. The eye (height

6′) lies in a perpendicular to GL from the central line of the plan and is 12′ from the perspective plane. Draw the perspective of the cubes if edge =2′.

25. A framework whose outer dimensions form a cube 12′ edge is composed of pieces 1′x1′ along each edge of the cube. One corner of the cube lies in the perspective plane and the plane of one face makes 30° with it. If the eye (height 6′) lies in a perpendicular to GL through the corner in the perspective plane and is 15′ from it, draw the perspective of the framework.

26. Draw the perspective of the framework in problem 25 when the eye is moved 2′ to the right, other dimensions remaining the same.

27. Draw the perspective of the framework in problem 25 when the height of the eye is 4′, other dimensions remaining the same.

28. Draw the perspective of the framework in problem 25 when the front face of the cube lies in the perspective plane and the eye (height 8′) lies in the central line of plan that is perpendicular to GL. Other dimensions are the same as in problem 25.

CHAPTER TWO.

Vanishing Point Method.

18. Vanishing Points.—To find the perspective of a line AB in Fig. 12 whose height above H is h, we can find the perspective of A and B as before. But if one of the points is on eE″ or near it the solution by the former method lacks exactness. It is

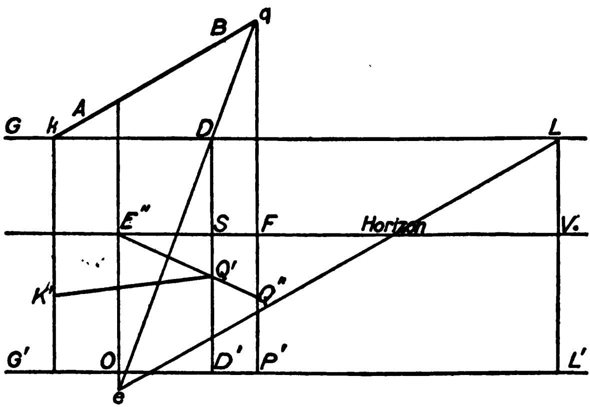

Figure 12.

advisable to find the perspective of points on AB favorably located. One of these favorable points is K where AB cuts the perspective plane at a height h above GL. The second favorable point should be taken as far from eE″ as the limits of the paper will permit. Let Q be such a point. The perspective

Q′ of Q is found as follows: Drop a perpendicular from q on G′L′, cutting it at P′; lay off P′Q″ equal to the true height of point Q. Join Q″ and E″; join eq cutting GL at D and draw DD′ perpendicular to GL, cutting E″Q″ at Q′, the perspective of Q. K′Q′ is the perspective of the line desired. The perspectives of A and B are found by joining A and B to e and by dropping perpendiculars from the points where these lines cut GL to cut the line K′Q′.

The line E″F is drawn parallel to GL and G′L′.

In the similar triangles E″SQ′ and E″FQ″,

$$\frac{SQ'}{FQ''} = \frac{E''S}{E''F}.$$

But E″S=OD′, E″F=OP′.

$$\frac{SQ'}{FQ''} = \frac{OD'}{OP'}.$$

Let t = height of eye above H = OE″.

h = height of AB above H = P′Q″.

x = OP′ = distance from eye to the true height line of Q.

z = OD′ = distance from eye line to the vertical through the perspective of the point.

y = SQ′ = height of perspective of point from horizontal through E″, called the *horizon*.

$$\therefore\ y = (t-h)\frac{z}{x}$$

Now, let Q move along AB to an infinite distance from K. The line eq that is drawn from q to e will then be parallel to AB and will cut GL at L, and D′ will move to L′. But x becomes infinity,

$$\therefore\ y = (t-h)\frac{OL'}{\infty} = 0$$

Therefore the perspective of the point on AB that is at infinity is zero distance from E″V in LL′. The line E″V is called the *horizon*.

Corollary: *The perspectives of all lines parallel to AB pass through* V, and are said to *vanish* at V.

From these results we derive the following rule for obtaining the vanishing point of a system of parallel lines:

Rule.—*Through the plan* (e) *of the eye draw a line parallel to the lines whose vanishing point is desired. From the point where this line cuts the perspective plane, GL, drop a perpendicular to cut the horizon at* V.

Thus the vanishing point is absolutely independent of the height of the line above H and of the position of its plan. The only controlling factors that locate the vanishing point are the angle the system of lines makes with GL and the position of the eye.

19. **Perspective of Cross.**—Given the side and end views of a cross and base as shown in Fig. 13. Let abcd be the plan of the base and let the perspective plane pass through the corner a. Through e draw lines parallel to the sides of the base, cutting GL at G and L. From these points draw perpendiculars to GL, cutting the horizon line at V and V′. As the corner of the base is in the perspective plane, lay off A′F′ equal to the

height of base and join A′ and F′ to V and V′. Join b and d to e, cutting GL at 1 and 4, and from these points draw perpendiculars to GL, cutting F′V and F′V′ at B′ and D′. The base is thus defined.

To draw the perspective of the arms of the cross produce one side to cut the perspective plane at M.

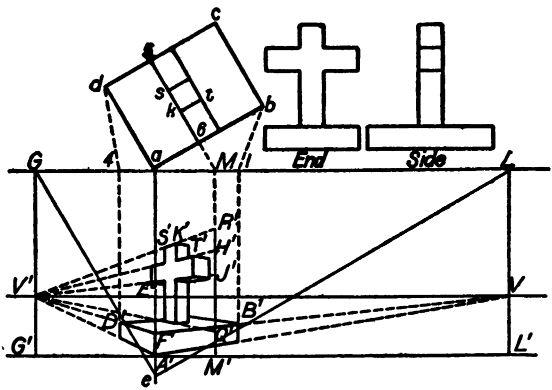

Figure 13.

Then MM′ is the true height line for all lines in this face of the arm. Lay off M′J′ and M′H′ equal to the true heights of the upper and lower surfaces of the arm. Join J′V′ and H′V′. Join 5 and 6 to e and from the points where these lines cut GL draw perpendiculars to GL. The intersection of these perpendiculars with J′V′ and H′V′ will define the front face of the arm.

To draw the upright, lay off the height of the base equal to M′Q′ and height of top=M′R′ on M′M from M′. Join these to V′ and join s, k and t to e, and from the points of intersection of these lines with GL draw perpendiculars to GL, cutting R′V′ in K′ and S′. Draw K′ V and join t to e and from point where et cuts GL drop perpendicular to cut K′ V at T^1. The rest can be drawn in the same way.

PROBLEM 29. Construct the perspective of the cross in Fig. 13 when side of base = 7′, end of base = 5′, length of crossarm = 6′, other dimensions as in Fig. 20.

20. **Perspective of a House.**—Let one corner a of the plan abcd (Fig. 14) be in the perspective plane. Through e draw lines parallel to ab and ad, cutting GL in L and G. From these points drop perpendiculars, cutting the horizon in V and V′, the vanishing points for the systems of lines parallel to ab and ad, respectively. Draw a line from a to e. The point A′ where it cuts G′L′ will be the perspective of a. Join A′V and A′V′ and from the points where eb and ed cut GL drop perpendiculars cutting A′V and A′V′ in B′ and D′, the perspective of the points b and d. Lay off the true heights of all points in the visible sides of the house on vertical A′a from G′L′, and the perspective of these points will lie on the line joining their proper height to their vanishing point. To draw the window pq, lay off the height of the top and bottom of window from A′ on aA′ at K′ and T′. Join T′ and K′ to V, and from the points where ep and eq cut GL drop perpendiculars to cut K′V and T′V in P′, M′, Q′, and O′. To locate the roof, extend the comb to cut the GL at U, and drop a perpendicular from U on

G′L′, cutting it at U′. Lay off on UU′ from U′ the true height of the comb, equal to U′x, and join x to V. Join the ends of the comb to e, and where these lines cut GL, drop perpendiculars, cutting the line xV in Z′ and Y′, the perspective of the ends of the ridge. To find the perspective of the eaves, extend the plan of the eave line to cut GL say at N, and

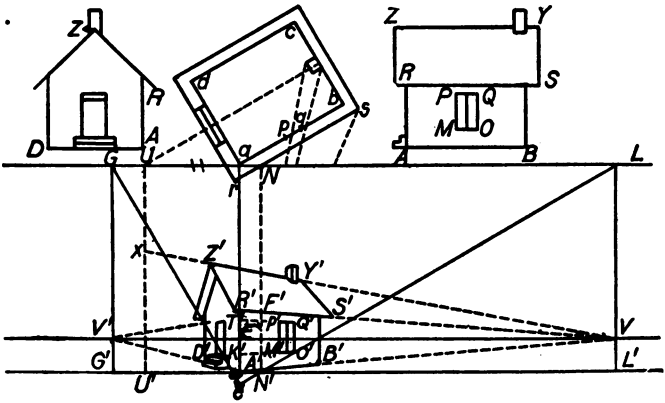

Figure 14.

from N′ lay off true height of eave line equal to N′F′, and join F′ to V. From the points where er and es cut GL, drop perpendiculars, cuting F′V in R′ and S′ the perspectives of r and s. In the same way the perspectives of the other eave-lines can be found.

PROBLEM 30. Construct the complete perspective of the house in Fig. 14 when AB = 6′, AD = 4′, height of eaves = 4′, height of comb = 6½′, height of eye = 2′, distance from eye to perspective plane = 12′, other dimensions remaining as in Fig. 14.

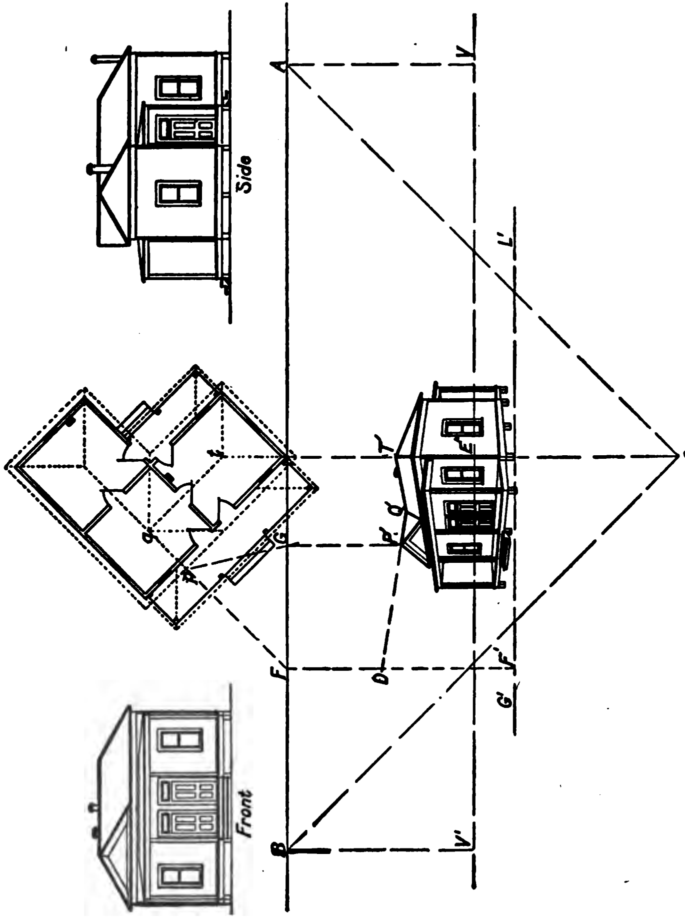

Figure 15.

21. **Architectural Perspective.**—In finding the perspective of a house, it is convenient to let one corner of the house pass through or lie in the perspective plane. Thus, if is it desired to construct the perspective of the house whose side and front elevations are marked "front" and "side" (Fig. 15), it will shorten the work if we let a corner common to the two views given lie in the perspective plane GL. It is always best to take e as low as the size of the drawing sheet will permit. After e is located draw lines through e parallel to the sides of the house, cutting GL in A and B, and then drop perpendiculars from A and B to the horizon line through E″, cutting the horizon in V and V′. The corner that lies in the perspective plane will be the true height line for all points lying in the planes of the two faces or sides seen in the two views. These heights are laid off from G′L′ on verticals through E″ and joined to V or V′. To find the perspective of any horizontal line like the ridge or comb pq, we produce pq to cut the perspective plane at F and from F drop a perpendicular cutting G′L′ at F′. On line F′F lay off the true height of the ridge equal to F′D. Join D and V, and join e and p, cutting GL at G, and from G drop a perpendicular, cutting DV at P′, which is the perspective of p, or the left end of ridge. In the same way we can find T′, the perspective of t. Join T′ to V′ and where it cuts DV will be the perspective of q. Other horizontal lines can be found in the same way. If a line is not horizontal in the building it is best to find the perspective of each end separately by the projective method.

CHAPTER THREE.

Axometric Projections.

22. In order to show the different parts, connections and relations of a framework, it is often desirable to take its projection on some plane not parallel to any of the plane faces. The basal planes of most frameworks are composed of a series of surfaces each of which is at right angles to the other two. Three axes each at right angles to the other and parallel to the edges of the framework can be drawn, and all lines can then be located with references to these axes.

23. Axes.—Let PA, PB, and PC in Fig. 16 be the axes in space each at right angles to the other two, and let these axes be projected normally on any plane ABC in KA, KB, and KC. Let PAK = a, PBK = b, PCK = c. Let AK, BK, and CK be produced to intersect BC, AC, and AB in D, E, and F. The lines KA, KB and KC are the axes of projection.

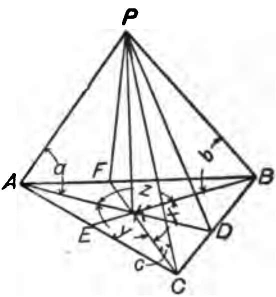

Figure 16.

Then as PA and PK are at right angles respectively to the planes BPC and ABC, the plane of these two lines is at right angles to the intersection of the two planes BPC and ABC. That is, APD is at right angles to BC. Hence AD and PD are the altitudes of the triangles ABC and BPC. Similar relations are true of BE and CF.

The right angles BPC, APC, and APB are projected in the angles BKC, AKC, and AKB, which we represent by x, y, and z.

24. Reduction Cosines.—The axis PA has been projected in KA and its length has been reduced from PA to KA. But in the right triangle PAK we have

$$KA = PA \cos a.$$

Similarly

$$KB = PB \cos b.$$

$$KC = PC \cos c.$$

Thus each axis is reduced in the ratio of the cosine of the angle it makes with the plane of projection. All lines that are parallel to PA, PB, and PC will, when projected on the plane ABC be reduced in the ratio of the cosines of a, b, and c, and these are therefore called the reduction cosines.

25. Fundamental Formula.—In the right triangles PBK and PCK of Fig. 16 we have

$$\sin b = \frac{PK}{PB} = \frac{h}{PB}$$

$$\sin c = \frac{PK}{PC} = \frac{h}{PC}$$

where $PK = h$

Squaring and adding,

$$\sin^2 b + \sin^2 c = \frac{h^2}{PB^2} + \frac{h^2}{PC^2} = h^2 \frac{PB^2 + PC^2}{PB^2 \times PC^2} \quad \ldots\ldots \text{(A)}$$

In the right triangle PBC we have

$$PB^2 + PC^2 = BC^2$$

Also as expressions for the area M of PBC we have the following:

$$M = \frac{PB \times PC}{2} = \frac{PD \times BC}{2}$$

$$\text{or } \overline{PB}^2\overline{PC}^2 = \overline{PD}^2\overline{BC}^2$$

Substituting in (A) gives

$$\sin^2 b + \sin^2 c = h^2 \frac{BC^2}{PD^2 \times BC^2} = \frac{h^2}{PD^2} \; \ldots\ldots \text{(B)}$$

But in the right triangle APD, DPK = PAK = a, and $\cos DPK = \frac{h}{PD} = \cos a$. Putting this value of $\frac{h}{PD}$ in (B) we get

$$\sin^2 b + \sin^2 c = \cos^2 a$$

$$\therefore \sin^2 a + \sin^2 b + \sin^2 c = 1$$

$$\text{or, } \cos^2 a + \cos^2 b + \cos^2 c = 2 \qquad (1)$$

26. **Angles between Axes.**— In the triangle CKD of Fig. 16 we have

$$\cos DKC = \frac{DK}{CK} \; \ldots\ldots\ldots\ldots \text{(C)}$$

Now DKC = 180° — y, and in the right triangles PKD and PKC,

$$DK = PK \tan DPK = h \tan a$$

$$CK = \frac{PK}{\tan PCK} = \frac{h}{\tan c}$$

Substituting these values of PK and CK in (C) we get

$$\cos(180° - y) = \tan a \tan c$$

$$\therefore \cos y = -\tan a \tan c$$

Similarly

$$\left.\begin{array}{l}\cos x = -\tan b \tan c \\ \cos z = -\tan a \tan b\end{array}\right\} \ldots\ldots\ldots, \quad (2)$$

PROBLEM 31. In the right triangle DKC, $\sin^2 y = \frac{DC^2}{KC^2} = \frac{PC^2 - PD^2}{KC^2}$ also $PC = \frac{h}{\sin c}$, $PD = \frac{h}{\cos a}$ and $KC = h \cot c$, then show that $\cos y = \frac{\sin b}{\cos a \cos c}$

PROBLEM 32. Given $\cos a = 2/3$, $\cos b = 1/2$, find the cosines of x, y, and z.

27. Relation of Reduction Angles.—It is assumed that the axes make some definite angles with the plane of projection. These angles must lie between 0° and 90°. As neither of the cosines can be 0 or unity, it follows that the square of either cosine must be less than unity; and therefore that the sum of the squares of the other two cosines must be greater than unity. If $a = 32°$, then $90 - a = 58°$. Now if we assume that $b = 58°$, we have

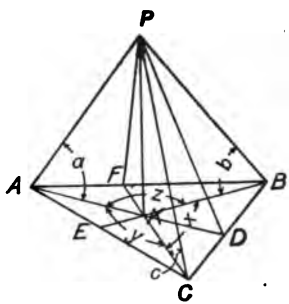

Figure 17.

$$\cos^2 a + \cos^2 b = \cos^2 32° + \cos^2 58° =$$
$$\cos^2 32° + \sin^2 32° = 1.$$

But as $\cos^2 a + \cos^2 b + \cos^2 c = 2$,

then $\cos^2 c = 1, \therefore \cos c = 1, \therefore c = 0$

This is contrary to hypothesis. Now as the cosine of an angle decreases as the angle increases, we see that for possible values b must be less than 58°. It can have any value between 0 and 58°. Thus none of the angles a, b, or c can be equal to or greater than the complement of either of the others.

Case I: If $a = 90° - b$, then $\cos^2 a + \cos^2 b = 1$

$\therefore \cos^2 c = 1 \therefore c = 0°$.

One axis is parallel to the plane of projection and the plane of the other two perpendicular to the plane of projection. This is the case of ordinary plans and elevations, and is not treated here.

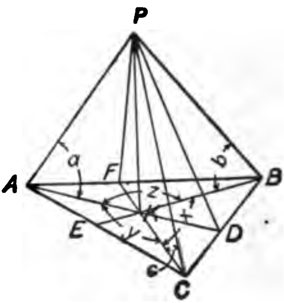

Figure 17.

Case II: $b > 90° - a$. Then $\cos^2 a + \cos^2 b < 1$
$\therefore \cos^2 c > 1$. c is impossible.

Case III: If $b < 90° - a$, then $\cos^2 a + \cos^2 b > 1$
$\therefore \cos^2 c < 1$, c is possible.

28. Application.—If we assume the angles a, b, and c, then each dimension of the structure parallel to PA, PB, and PC will be reduced in the ratio of the cosine of a, b, and c.

Example.—Let $\cos a = \frac{1}{2}\sqrt{3}$, $\cos b = \sqrt{\frac{3}{8}}$, then

$$\cos c = \sqrt{\tfrac{7}{8}}.$$

$$\sin a = \tfrac{1}{2},\ \sin b = \sqrt{\tfrac{5}{8}},\ \sin c = \sqrt{\tfrac{1}{8}}.$$

$$\tan a = \frac{1}{\sqrt{3}},\ \tan b = \sqrt{\tfrac{5}{3}},\ \tan c = \sqrt{\tfrac{1}{7}}.$$

$$\cos x = -\tan b \tan c = -\sqrt{\tfrac{5}{21}}.$$

$$\cos y = -\tan a \tan c = -\sqrt{\tfrac{1}{21}}.$$

$$\cos z = -\tan a \tan b = -\sqrt{\tfrac{5}{9}}.$$

$\log \cos x = \frac{1}{2} [\log 5 - \log 21]$
$= \frac{1}{2} [0.698970 - 1.322219]$
$= \frac{1}{2} [19.376751 - 20] = 9.6883755$
$\therefore x = 119° \; 12' \; 21''$
$\log \cos y = \frac{1}{2} [\log 1 - \log 21]$
$= \frac{1}{2} [0 - 1.322219]$
$= \frac{1}{2} [18.677781 - 20] = 9.3388905$
$\therefore y = 102° \; 36' \; 16''$
$\log \cos z = \frac{1}{2} [\log 5 - \log 9]$
$= \frac{1}{2} [0.698970 - .954243]$
$= \frac{1}{2} [19.744727 - 20] = 9.8723635$
$\therefore z = 138° \; 11' \; 23''$
Check : $x + y + z = 360°$

Now, all dimensions of the structure will be reduced in the ratios of cos a (= .8660), cos b (= .6124), and cos c (= .9354), and a separate scale will have to be made for each axis.

29. Reduction Ratios.— Instead of assuming the angles, we can assume their ratios. Thus

Let $\cos a = lp$, $\cos b = mp$, $\cos c = np$.

$\cos a : \cos b : \cos c :: l : m : n$.

Then from (1)

$$l^2p^2 + m^2p^2 + n^2p^2 = 2$$

$$\therefore \; p^2 = \frac{2}{l^2 + m^2 + n^2} \; \ldots\ldots\ldots\ldots \; (3)$$

$$\cos^2 a = \frac{2l^2}{l^2+m^2+n^2}, \; \cos^2 b = \frac{2m^2}{l^2+m^2+n^2},$$

$$\cos^2 c = \frac{2n^2}{l^2+m^2+n^2}$$

From these we can find tan a, tan b, and tan c, and then x, y, and z.

30. Practical Application.—We can, instead of using the incommensurable fractions represented by the cosines of a, b, and c, use their ratios, l, m, and n, and thus get the relative dimensions. The object of axometric projections is to show the connections and relations more fully and completely than ordinary projections or perspectives would do. It is therefore allowable to vary these ratios proportionally. It will be far more expeditious and far more satisfactory to assume the ratios l, m, and n, and find the values of x, y, and z from these.

31. Systems.— There are three general systems:

Isometric: $l : m : n$, where $l = m = n$; all equal.
Dimetric: $l = m$ or n, or $m = n$; two equal.
Trimetric: $l : m : n$; all different.

In any system the ratios must fulfill formulas (1) and (2).

32. Isometric System.—The ratios are $l : m : n = 1 : 1 : 1$.

$$p^2 = \frac{2}{l^2 + m^2 + n^2} = \tfrac{2}{3}.$$

$$\cos^2 a = \tfrac{2}{3}, \cos^2 b = \tfrac{2}{3}, \cos^2 c = \tfrac{2}{3}.$$
$$\sin^2 a = \tfrac{1}{3}, \sin^2 b = \tfrac{1}{3}, \sin^2 c = \tfrac{1}{3}.$$
$$\tan^2 a = \tfrac{1}{2}, \tan^2 b = \tfrac{1}{2}, \tan^2 c = \tfrac{1}{2}.$$
$$\cos x = -\tan b \tan c = -\tfrac{1}{2}, \therefore x = 120^\circ.$$
$$\cos y = -\tan a \tan c = -\tfrac{1}{2}, \therefore y = 120^\circ.$$
$$\cos z = -\tan a \tan b = -\tfrac{1}{2}, \therefore z = 120^\circ.$$

33. Isometric of Stone Cap.—In the isometric system, the line OX is drawn vertically, and then the angles XOA and XOB are made equal to 120°, thus making AOB 120°. Let OH equal the total length of a stone cap, OB the total width and HG the total height. Make OB = its true width, and OC equal to its true length, and draw the parallelogram OBDC. Then draw CF and DE parallel to axis OA and make CF equal to its true length, and draw FE parallel to axis OB. To locate the top GMLK, through G draw lines parallel to the axis OA and OB, and make GM and GK equal to their true lengths in the stone cap, and draw lines GK, ML, and KL. Then join KF and EL, forming the parallelogram FKLE. To draw the back face AGMR, make OA equal to its true length and join AG. Draw BR parallel to OA and AR parallel to OB, intersecting at R. Join R with M, thus completing the figure.

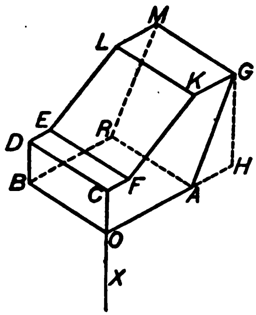

Figure 18.

PROBLEM 33. Draw an isometric of a cube 4′ edge that has a square hole (1′x1′) connecting each pair of parallel faces.

34. One-Half Dimetric System.—In the dimetric system two of the ratios are equal, but they can bear a variety of relations to the third.

The one-half dimetric: (l : m : n) = (1 : ½ : 1)

$$p^2 = \frac{2}{l^2+m^2+n^2} = \frac{8}{9}$$

$$\cos^2 a = \tfrac{8}{9},\ \cos^2 b = \tfrac{2}{9},\ \cos^2 c = \tfrac{8}{9}:$$
$$\sin^2 a = \tfrac{1}{9},\ \sin^2 b = \tfrac{7}{9},\ \sin^2 c = \tfrac{1}{9}.$$
$$\tan^2 a = \tfrac{1}{8},\ \tan^2 b = \tfrac{7}{2},\ \tan^2 c = \tfrac{1}{8}.$$
$$\therefore \cos x = -\tan b \tan c = -\tfrac{1}{4}\sqrt{7}$$
$$\cos y = -\tan a \tan c = -\tfrac{1}{8}$$
$$\cos z = -\tan a \tan b = -\tfrac{1}{4}\sqrt{7}$$

As an example of the application of this system,

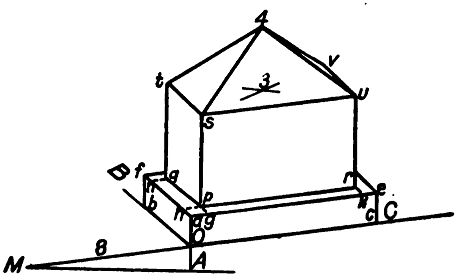

Figure 19.

draw the dimetric of the monument in problem 20.

Draw in Fig. 19 a vertical line and make OA = 1, and draw AM at right angles to OA. With O as a center and 8 as a radius, cut AM at M. Join OM and produce in OC.

$$\text{Now} \cos MOA = \frac{OA}{OM} = \tfrac{1}{8}$$

$$\therefore \cos COA = -\tfrac{1}{8} \ \therefore COA = y.$$

As x and z are equal, bisect the angle COA and produce in OB, then COB = x, and BOA = z. Full dimensions are laid off on OA and OC and one-half of the real dimensions are laid off on OB. Thus Oa and Oc are made equal to 1′ and 6′ respectively—the height and length of the side of the base of the monument, while Ob is laid off equal to one-half of the real length (4′) of the end of the base. Through a, b, and c draw lines parallel to the axes intersecting in e and f. Lay off ag and ke equal to 1′ to the full scale, while ah and fn are laid off to the half scale. Through g and k draw lines parallel to the axis OB, and through h and n lines parallel to OC, intersecting the former lines in p, q, and r. Through p draw a vertical (parallel to OA) and make ps = 3′ (the height of the block). Then through s draw lines parallel to the axes and make su = 5′, and st = 3′ on the half scale. Complete the parallelogram tsuv, the top of the block. Draw the diagonals and from the point of intersection 3 lay off 34 = 2′ and join 4 to s, t, u, and v.

35. Three-Fourths Dimetric.—In this system the same scale is used on two of the axes, while a scale equal to three-fourths of the first is used on the other one.

In the three-fourth system we have,

$$(l : m : n) = (1 : \tfrac{3}{4} : 1)$$

$$p^2 = \frac{2}{l^2 + m^2 + n^2} = \frac{2}{1 + \frac{9}{16} + 1} = \frac{32}{41}.$$

$$\cos^2 a = \tfrac{32}{41},\ \sin^2 a = \tfrac{9}{41},\ \tan^2 a = \tfrac{9}{32};$$
$$\cos^2 b = \tfrac{18}{41},\ \sin^2 b = \tfrac{23}{41},\ \tan^2 b = \tfrac{23}{18};$$
$$\cos^2 c = \tfrac{32}{41},\ \sin^2 c = \tfrac{9}{41},\ \tan^2 c = \tfrac{9}{32}.$$
$$\cos y = -\tan a \tan c = -\tfrac{9}{32}.$$

In Fig. 20 draw OA = 9 parts on some scale, draw AM perpendicular to OA, and with O as a center and a radius equal to 32, cut AM at M. Join OM and produce in OC. Then bisect angle AOC and produce the bisector in the line OB. Now, full dimensions are laid off on the axes OA and OC and three-fourth dimensions on OB. On axis OC make O2 = 7″, the full length of the side O2. Now, the side O3 is equal in the original figure to eight inches, but we lay off only three-fourths of this on

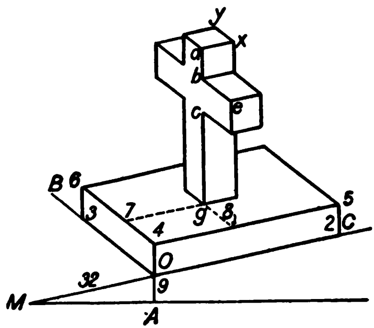

Figure 20.

O3; that is, we make O3 = 6″. Draw lines O4, 25, and 36 parallel to axis OA each equal to one inch, and through 4, 5, and 6 draw lines parallel to the axes, marking out the top of the base. The shaft of the cross is 1″ by 1″. Make distance 48=3″, and draw 8g = three-fourths of 3.5″. Then through g draw lines parallel to the axes and lay off dimensions equal to one inch on axis parallel to OC and to three-fourths of one inch on axis parallel to axis OB. Make gc = 4″, bc = 1″, ba = 1″, and through these points draw lines parallel to the axes. Make ax = 1″, xy = ¾″, be = 2⅝″, etc.

PROBLEM 34. Draw the one-half dimetric of a box without top, whose outside dimensions are 2′x3′x2½′, the thickness of the material being 2″.

36. The One-Third Dimetric.—(1 :1/3 :1). We have

$$p^2 = \frac{2}{l^2+m^2+n^2} = \tfrac{18}{19}$$

$$\cos^2 a = \tfrac{18}{19},\ \sin^2 a = \tfrac{1}{19},\ \tan^2 a = \tfrac{1}{18}$$
$$\cos^2 b = \tfrac{2}{19},\ \sin^2 b = \tfrac{17}{19},\ \tan^2 b = \tfrac{17}{2}$$
$$\cos^2 c = \tfrac{18}{19},\ \sin^2 c = \tfrac{1}{19},\ \tan^2 c = \tfrac{1}{18}$$
$$\cos x = -\tan b \tan c = -\sqrt{\tfrac{17}{36}}$$
$$\cos y = -\tan a \tan c = -\tfrac{1}{18}$$
$$\cos z = -\tan a \tan b = -\sqrt{\tfrac{17}{36}}$$

The axes can be laid off as in Fig. 19, except that, while OA equals 1, OM is 18. The drawing of a structure in the one-third dimetric is made in all respects like the one-half dimetric, except that one-third is measured along the OB axis instead of one-half.

PROBLEM 35. Show in the (m : 1 : m) dimetric that $\cos y = -\frac{1}{2m^2}$.

PROBLEM 36. Show in the (1 : m : 1) dimetric that $\cos y = -\frac{m^2}{2}$.

PROBLEM 37. In the system (1 : ¾ : 1) find the cosines of x, y, and z, and draw the cross in Fig. 13 in this system.

PROBLEM 38. In the system (1 : ⅔ : 1) show that $\cos y = -2/9$, and draw the monument in

Problem 20.

37. **Trimetric System.**—(l : m : n). In this system the ratios are all different, but must fulfill the conditions of formula (3).

Find the angles between the axes for the (⅓ : ⅔ : 1) system.

$$p^2 = \tfrac{18}{14} = \tfrac{9}{7}$$

$$\cos^2 a = \tfrac{1}{7},\ \cos^2 b = \tfrac{4}{7},\ \cos^2 c = \tfrac{9}{7}.$$

As no cosine can be greater than unity, the system is impossible.

For the system (9 : 10 : 11) we have

$$p^2 = \frac{2}{81+100+121} = \frac{2}{302}$$

$$\cos^2 a = \tfrac{162}{302},\ \cos^2 b = \tfrac{200}{302},\ \cos^2 c = \tfrac{242}{302}.$$

$$\sin^2 a = \tfrac{140}{302},\ \sin^2 b = \tfrac{102}{302},\ \sin^2 c = \tfrac{60}{302}.$$

$$\tan^2 a = \tfrac{140}{162},\ \tan^2 b = \tfrac{102}{200},\ \tan^2 c = \tfrac{60}{242}.$$

$$\cos x = -\tan b \quad \tan c = -\sqrt{\tfrac{102 \times 60}{200 \times 242}}$$

$$\cos y = -\tan a \quad \tan c = -\sqrt{\tfrac{140 \times 60}{162 \times 242}}$$

$$\cos z = -\tan a \quad \tan b = -\sqrt{\tfrac{140 \times 102}{162 \times 200}}$$

Log cos x = ½ [log 102 + log 60 — log 200 — log 242]

log 102 = 2.008600	log 200 = 2.301030
log 60 = 1.778151	log 242 = 2.383815
3.786751	4.684845

∴ Log cos x = ½ [3.786751 — 4.684845]
= ½ [19.101906 — 20] = 9.550953

∴ x = 110° 49′ 47″

Log cos y = ½ [log 140 + log 60 — log 162 — log 242]

log 140 = 2.146128	log 162 = 2.209515
log 60 = 1.778151	log 242 = 2.383815
3.924279	4.593330

$$\text{Log} \cos y = \tfrac{1}{2}\,[3.924279 - 4.593330]$$
$$= \tfrac{1}{2}\,[19.330949 - 20] = 9.6654745$$
$$\therefore y = 117^\circ\ 34'\ 25''$$

$$\text{Log} \cos z = \tfrac{1}{2}[\log 140 + \log 102 - \log 162 - \log 200]$$

$\log 140 = 2.146128$	$\log 162 = 2.209515$
$\log 102 = 2.008600$	$\log 200 = 2.301030$
4.154728	4.510545

$$\text{Log} \cos z = \tfrac{1}{2}\,[4.154728 - 4.510545]$$
$$= \tfrac{1}{2}\,[19.644183 - 20] = 9.8220915$$
$$\therefore z = 131^\circ\ 35'\ 48''$$

Check: $x + y + z = 360^\circ$.

38. Example.—Draw the perspective of a 4″ cube that has a hole 1″ square connecting each set of opposite faces, the axis of the hole coinciding with the axis of the cube, in the (4:5:6) system.

$$\text{Now } p^2 = \frac{2}{16+25+36} = \tfrac{2}{77}.$$

$$\therefore \cos^2 a = \tfrac{32}{77},\ \sin^2 a = \tfrac{45}{77},\ \tan^2 a = \tfrac{45}{82}.$$
$$\cos^2 b = \tfrac{50}{77},\ \sin^2 b = \tfrac{27}{77},\ \tan^2 b = \tfrac{27}{50}.$$
$$\cos^2 c = \tfrac{72}{77},\ \sin^2 c = \tfrac{5}{77},\ \tan^2 c = \tfrac{5}{72}.$$

$$\cos x = -\tan b \tan c = -\sqrt{\tfrac{27 \times 5}{50 \times 72}} \quad \therefore x = 101^\circ\ 9'\ 57''$$
$$\cos y = -\tan a \tan c = -\sqrt{\tfrac{45 \times 5}{82 \times 72}} \quad \therefore y = 108^\circ\ 12'\ 36''$$
$$\cos z = -\tan a \tan b = -\sqrt{\tfrac{45 \times 27}{82 \times 50}} \quad \therefore z = 150^\circ\ 37'\ 27''$$

Check: $x + y + z = 360^\circ\ 0'\ 0''$.

This system is equivalent to (4/6:5/6:1). Draw a vertical line for the axis OA (Fig. 20) and draw OB making angle AOB $= 108^\circ\ 12'\ 30''$, and angle AOC $= 150^\circ\ 37'\ 27''$. The angle BOC is then $= 101^\circ\ 9'\ 57''$. Now as full sizes are laid off on OC and the dimension on OA must be reduced by multiplying by 4/6, or 2/3, we can use the 4 0 scale on OC and the 6 0 scale on OA, as this makes the reduction for us. Similarly we could use the 5 0 scale on OC and the 6 0 on OB. The cube is then easily drawn.

PROBLEM 39. Draw a cube in the (3 : 4 : 5) system.

PROBLEM 40. Find the angles x, y, and z in the (5 : 6 : 7) system.

PROBLEM 41. Find the angles x, y, and z in the (10 : 11 : 12) system.

PROBLEM 42. Find the angles x, y, and z in the (8 : 15 : 17) system.

CHAPTER FOUR.

Shades and Shadows.

39. **Shade.**—An opaque body exposed to light will be illuminated on the side next the light and dark on the opposite side. Thus if an opaque

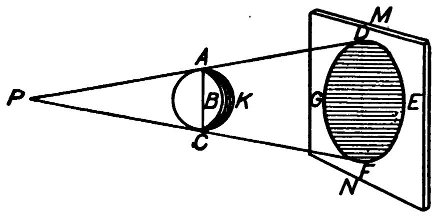

Figure 21.

sphere be exposed to a source of light P, a set of rays of light from the source P will be tangent to the sphere. If the source of light is a point P the rays of light that are tangent to the sphere will form the surface of a cone whose vertex is the source of light P and which touches the sphere in the circle of contact, ABC. The part ABCK of the body, that does not receive any light, is in *shade* and the line of the contact ABC is the curve of shade. That part of space from which the source of light cannot be seen is called the shadow in space.

40. **Shadow.**—If light is excluded from a second body MN, that part of the surface of the second body from which the source of light cannot be seen is said to be in shadow, and the dark part is called shadow. The line of contact of the second body

with the cone of rays tangent to the first is called the curve of shadow.

Thus in Fig. 21, P is the source of light, ABCK is an opaque sphere and MN is another body that intersects the shadow cone in the curve of shadow DEFG. The area DEFG is called the shadow area of the sphere on the body MN.

41. **Drawings in Projection.**— The ordinary drawings of structures consist of a side and an end view called *elevations,* and a top view called the *plan.* If the structure is a house, we ordinarily have the end elevation, side elevation, and a rear elevation.

In addition to this we have the ground plan and roof plan and these elevations and plans define the structure completely, from which we get a conception of the structure as a whole.

Projective drawings are referred to a horizontal plane (called H) and a vertical plane (called V), while their intersection is called the ground line (G. L.). The phrases "horizontal projection" and "vertical projection" will hereafter be abbreviated into hp and vp respectively.

42. **Shadow of Points.**—The shadow of a point on a plane is where the ray of light through the point cuts the plane. If the point is a material point it is supposed to intercept the light and its shadow will appear as a dark spot on the plane.

43. **Directions of the Rays of Light.**—The conventional direction of any ray of light is such that the elevation and plan of the ray make 45° with the ground line. The ray of light is supposed to come over the left shoulder as we face the drawing

in such a way that its projections make 45° with the ground line (G. L.).

44. Angle with H or V.—In Fig. 22, let M be a plane perpendicular to H and V cutting H in LA and V in LB.

PQ is a ray of light whose hp is pQ, vp is p′Q and whose projection on M is LR. The lines pQ

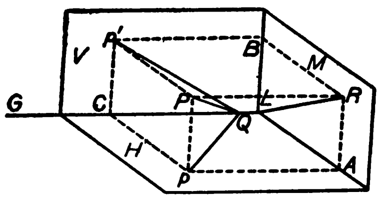

Figure 22.

and p′Q make 45° with GL. ∴ CQp′=45°= CQp. But if angles CQp and CQp′=45°, then Cp = CQ, CQ = Cp′ ∴ Cp = Cp′.

$$\text{Let } CQ = a, \text{ then } Cp = a$$

$$\therefore pQ = a\sqrt{2}$$

$$\text{Tangent } PQp = \frac{Pp}{pQ} = \frac{a}{a\sqrt{2}} = \frac{\sqrt{2}}{2} = .707$$

$$\therefore PQp = 35° \ 16'$$

Now R is the projection of P on M, and PR is parallel to GL and perpendicular to M. Hence LR is the projection of the ray PQ on M. Draw RA and RB ⊥ to H and V. As GL is ⊥ to M, pA and p′B will be parallel to GL.

$$\therefore Cp = LA$$

$$Cp' = LB$$

$$\text{But } CQ = Cp = Cp'$$

$$\therefore LA = LB \text{ or } LA = RA$$

$$\therefore \text{The angle } RLA \text{ is } 45°$$

Hence the projection of the ray of light on a plane M ⊥ to H and V makes an angle of 45° with lines ⊥ to H and V, or with new ground line LA.

45. Shadow of Vertical Rod.—a. Shadow all on H. Let cd-c′d′ Fig. 23 (a) be the projections of a vertical rod. To find the shadow of a point C on H, draw rays through c and c′, making 45° with GL. Find the horizontal trace at C′.

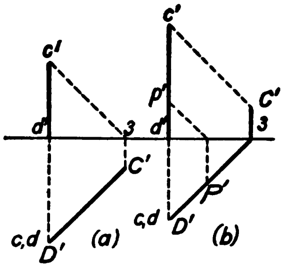

Figure 23.

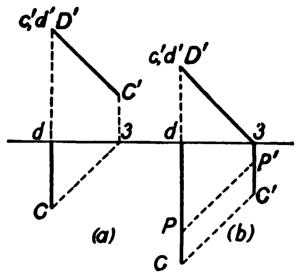

Figure 24.

The point D is in H and is its own shadow. The shadow of the rod will be the heavy line D′C′.

b. Shadow on H and V.—Through cc′ draw projections of a ray making 45° with GL and find the vertical trace at C′. Find the shadow P′ of some point P on the rod. Join P′c and produce to cut GL at 3; then join C′3. The broken line C′3c is the shadow. The part C′3 on V makes 90° with GL, while the part c3 on H makes 45°.

46. Horizontal Lines.—a. Shadow on V. Let cd-c′d′ be the projection of a line perpendicular to V. Fig. 24 (a). Draw rays through c and c′ and find shadow of C at C′. In the same way find the shadow of D at D′. Connect C′ and D′, giving the shadow C′D′.

b. Shadow on H and V. Fig. 24 (b). Find the shadow of point C at C′ and D at D′, also find the shadow of P at P′. Join C′P′ to cut GL at 3 and then join 3D′. The broken line D′3C′ is the shadow required.

47. Shadow of Any Line.—Given a line ab-a′b′.

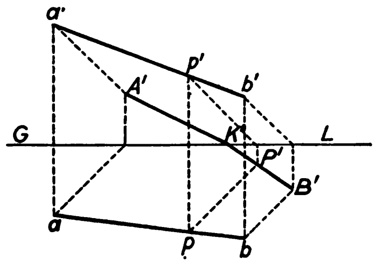

Figure 25.

1. Find the shadow of A at A′, and shadow of B at B′. 2. Take any intermediate point P on the line AB and find its shadow at P′. Produce B′P′ to cut GL at K′. Join A′ and K′. The broken line A′K′B′ is the shadow of the line AB.

48. Shadow of a Shelf.—Let a′b′c′f′ be the vertical projection of the shelf and aebc the horizontal projection.

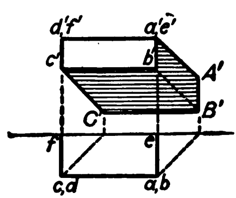

Figure 26.

1. Find shadow of the line cb-c′b′ at C′B′;

2. Find shadow of line ab-a′b′ at A′B′;

3. Find shadow of a line cf-c′f′ in c′C′.

∴ Area e′A′B′C′c′b′e′ is the shadow of the shelf on V as shaded in Fig. 26.

49. Shadow of Block on H.—Let abcd be the plan of the top and a′b′c′f′ be the elevation of a rectangular block resting on H.

1. Find shadow of line bc-b′c′ at B′C′;

2. Find shadow of line cd-c′d′ at C′D′;

3. Find shadow of line df-d′f′ at D′d;

4. Find shadow of line be-b′e′ at B′e.

The area bB′C′D′dcb (shaded) is the shadow of block on H.

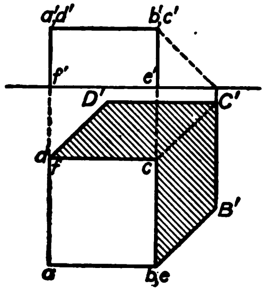

Figure 27.

50. Shadow of a Circular Disc on H.—Given the projections ef-e′f′ of the circular disc whose plane is parallel to H and perpendicular to V. The rays of light that touch the circle form a cylindrical surface of rays which is cut by H parallel to plane of circle. Hence the section cut by H from cylinder of rays is similar to that cut by the plane of circle. Thus the shadow is a circle. To find this shadow, find the shadow of the center of the circle at C′and with C′ as a center and radius equal to that of the circle draw the circle A′B′D′, which is the required shadow.

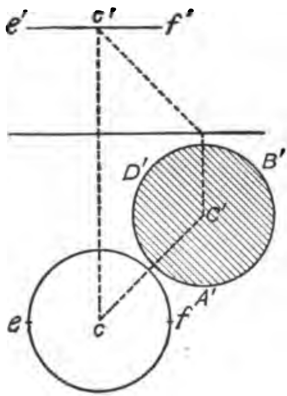

Figure 28.

51. Shadow of a Horizontal Circular Disc.—

Given a circle abcd-a′b′c′d′ whose plane is horizontal and perpendicular to V. Find the shadow of a series of points as a-a′, b-b′, c-c′, d-d,′ at A′, B′, C′, D′, etc. Through these points draw a curve. In this case the shadow falls on V and forms an ellipse.

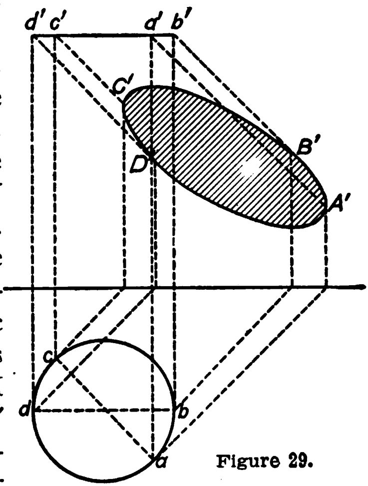

Figure 29.

52. Construct the shadow of a horizontal circular disc where its center is equidistant from H and V. Let abcd-a′b′ c′d′ be the projections of the circular disc. Find the shadow of the two semi-circular halves abc and adc. The part of shadow that falls on H is a circular curve, while that on V is an elliptical curve. First find the shadow of center o-o′ at O′ and with center O′ and radius equal to oa draw semicircle B′A′

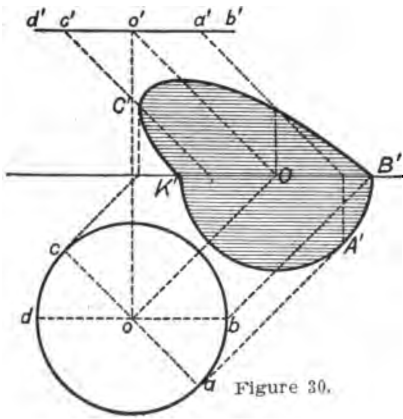

Figure 30.

K′. Then construct elliptical part B′C′K′ by points.

53. A circular disc 2″ in diameter touches H at a point 3″ in front of V and its plane is perpendicular to V. Construct its shadow. Let abcd-

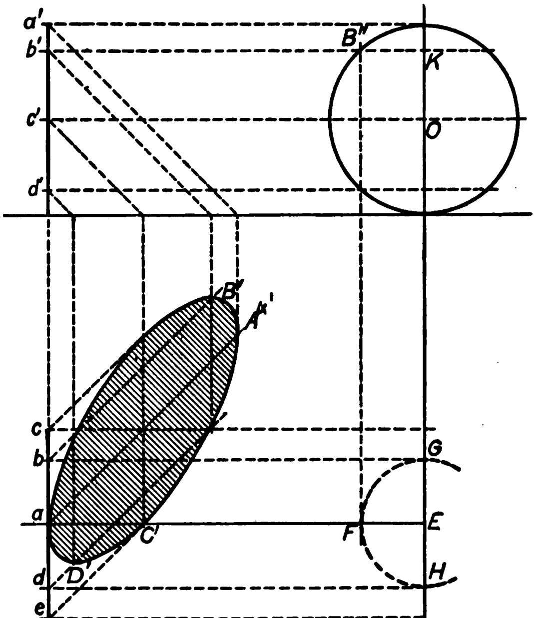

Figure 31.

a′b′c′d′ be points on the circumference of the circular disc. Find the shadows of these points at A′B′C′D′, etc., and sketch a curve through these points. The shadow will be an ellipse. If we have vp (b′) of a point B, the hp can be found by drawing the circle O. Draw b′B″ parallel to GL to cut circle and diameter at B″ and K. Lay off KB″

from a on ac making ab = KB″. In the same way the projections of the other points can be found.

54. The plane of a circular disc 2″ in diameter is parallel to V and perpendicular to H. Its center is 3″ from V and 1″ from H. Construct its shadow.

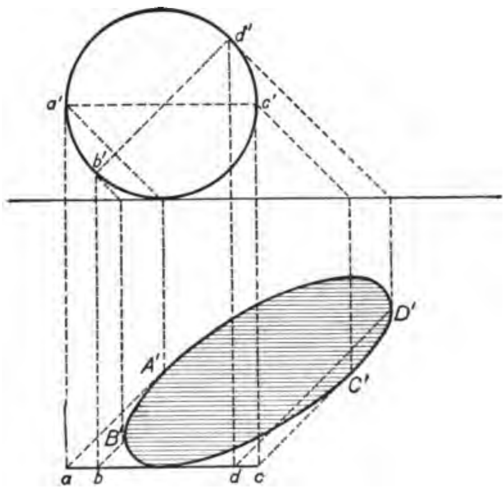

Figure 32.

(See Fig. 32.) Locate points abcd-a′b′c′d′ and find their shadow at A′, B′, C′, D′, etc. Draw the curve through them, defining the ellipse of shadow.

55. **Shadow of a Chimney.**— In Fig. 33 let epqf be the plan of a hip roof and p′e′f′s′ the elevation, while abc and a′b′c′ represent the plan and elevation of a chimney. The line ab-a′b′ is perpendicular to V and its shadow on H will be in the line ef, which is found by drawing a′e′ through a′ at

45° with GL. Produce e′e to cut the eaves of roof at e and f. If the line AB were indefinite in extent, its shadow on the roof would pass through e and f. The shadow of AB will cut the comb of the roof at o-o′ where a′e′ cuts the comb s′t′. Then the shadow

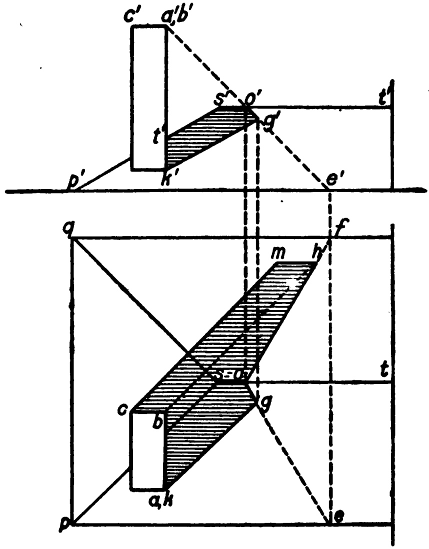

Figure 33.

of AB will lie in the lines oe and fo. Through a and b draw rays, ag and bh, making 45° with GL cutting oe at g and fo at h. The broken line goh will be the shadow of ab-a′b′ on the roof. The shadow of the line bc-b′c′ will be parallel to GL and hence will be mh; while the shadow of the vertical corner ak-a′k′ will be ag; and that at C will be cm. The shaded area agohmcba will be the plan of the shadow, and k′g′o′s′t′ will be the elevation of the shadow.

56. Shadow of Cap on Cylinder.—Given a vertical cylinder 2″ diameter and a cap 3½″, required

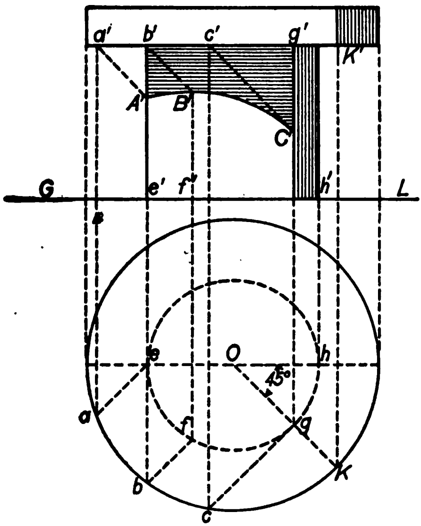

Figure 34.

to find shadow of cap on the cylinder. See Fig. 34. Let abc-a′b′c′ be the projections of the cap and efg-e′f′g′ the projections of the cylinder. Draw diameter eh parallel to GL, and from e draw ray ea at 45° with GL. Mark a′ vertically above a. Draw a′A′ at 45° with GL, cutting a vertical through e at A′. Draw any intermediate ray through bb′, cutting surface of cylinder at f. Locate b′ vertically above b, and draw b′B′ at 45° with GL to cut vertical through f at B′. The extreme point of shadow curve will be above g where Og makes 45° with GL. Draw gc perpendicular to Og, cutting rim of cap at c. Find c′ vertically above c and draw c′C′ at 45° with GL to cut vertical through g at C′. The line A′B′C′ is the shadow line. The rest of the visible part of the cylinder to right of vertical through g is in shade as indicated by area h′g′. In the same way the part of cap to right of vertical through k is in shade.

57. **Shadow of a Straight Line with the Projections Perpendicular to GL.**—Let ab-a′b′ be

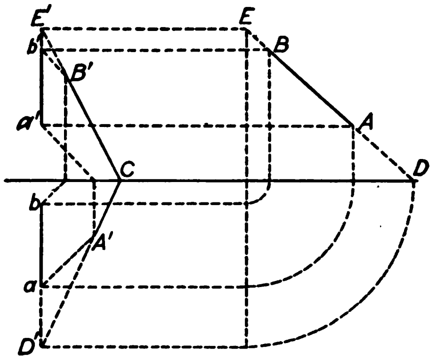

Figure 35.

the projections of a straight line where ab and a′b′ are perpendicular to GL, in Fig. 35. Find the shadow of A at A′ and B at B′. If the line AB is revolved around its vertical projection into the vertical plane, it will appear at AB and its horizontal trace and vertical traces will be D and E respectively. If these points D and E are taken back to their true positions they will appear at D′ and E′ and each point will be its own shadow. Join D′A′ and produce to cut GL at C and then join CB′. The broken line A′CB′ will be the required shadow. The shadows of all lines parallel to AB will be parallel to A′C and B′C.

58. **Shadow of Wall on Steps.**—Let 1, 2, 3, 4, 5, 6, 7 be the end view of a series of steps and ABC

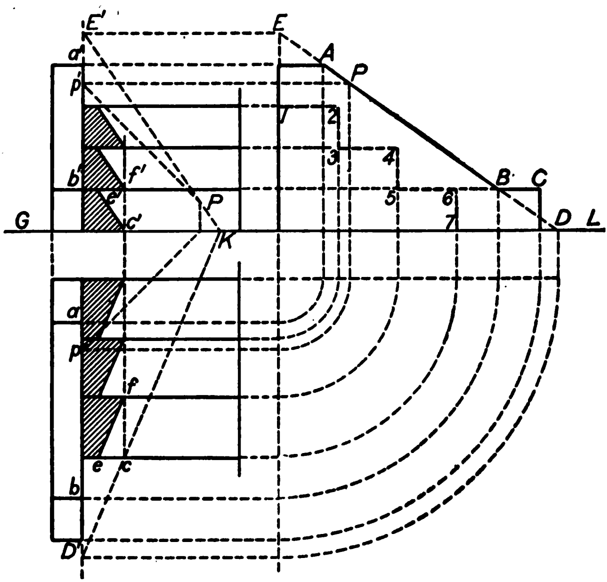

Figure 36.

that of a wing-wall, in Fig. 36. The vertical and horizontal projections are shown above and below GL. Now the shadow of a line AB is found as in Art. 57 in the broken line E′KD′. That part D′c is on H and at point c the shadow strikes the vertical face of the first step. The shadow on the vertical face of the first step will pass through c′ and be parallel to KE′; i. e., c′e′ will be this shadow. Through e, the horizontal projection of e′ draw ef parallel to D′K. The shadows on the other steps will be parallel to c′e′ and ef respectively. The shaded areas represent the shadows on the vertical and horizontal planes of the steps.

59. Shadow on a Semicylinder.—Let abc-a′b′c′ represent the horizontal and vertical projections of a semicylindrical surface. (See Fig. 37.) Required to find the shadow of the rim on the cylindrical surface. Through a draw ad at 45° with GL to cut the cylindrical surface at d and through a′ draw a′A′ at 45° with GL to cut vertical through d at A′. To find the shadow of any intermediate point C, through c draw c3 at 45° with GL to cut surface of cylinder at 3. Through c′ draw c′C′ at 45° with GL to cut vertical through 3 at C′. If Ob makes 45° with GL, b′ is a point in shadow line. Thus the shadow of rim is A′C′b′. The area a′b′A′d′G is the required shadow.

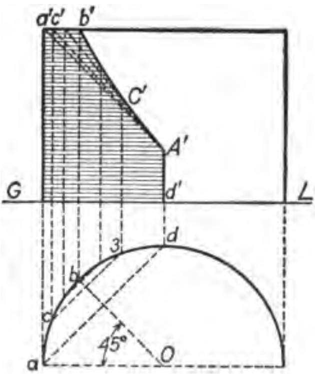

Figure 37.

60. Shadow of Half-Cylinder on H. — Given a b c d and a′ c′, in Fig. 38, the horizontal and vertical projections of a half-cylinder whose axis is parallel to GL. It is required to construct the shadow cast by the cylinder on its curved surface and on H. The shadow of the semi-circle ab-a′b′ will be a semi-ellipse A′OB′ which is found as in Art. 53. That of cd-c′d′ is similarly found, but

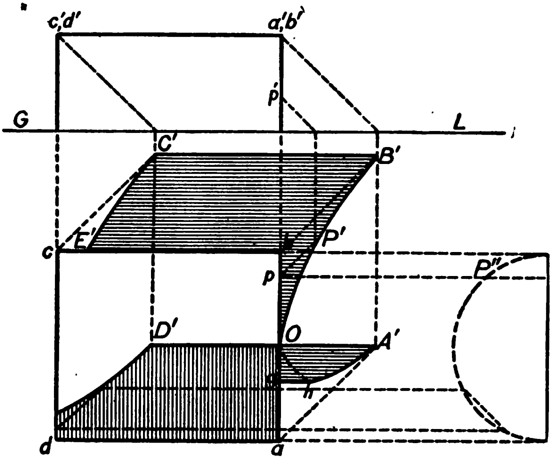

Figure 38.

only that part C′E′ will be seen. The shadow of the left end on the interior surface of the cylinder will be found as in Art. 59. The shaded area D′ Oad is on the cylinder while A′gOB′C′E′b is on H.

61. Shadow of Cone.—Let Fig. 39 represent a right cone whose axis is vertical with base resting on the horizontal plane. Find the shadow of the vertex at A′ and from A′ draw A′b and A′c tangent to the circle bec. The area A′cebA′ will be the shadow that the cone casts on H. The area cebac is the horizontal projection of the shade cast by the cone on its own surface, while a′b′f′ is the vertical projection of that part of the shade that is visible. Fig. 40 shows how the shadow of the cone falls when the shadow of the vertex falls on the vertical plane. Draw the rays of light through the vertex at 45° with GL. The horizontal trace of the ray will be at A″ and its vertical trace at A′. From

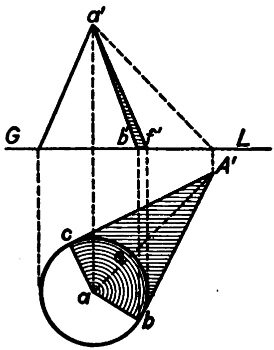

Figure 39.

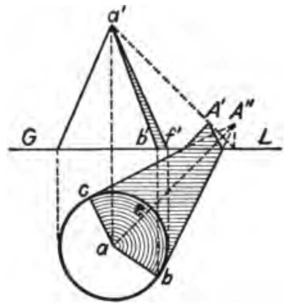

Figure 40.

A″ draw lines A″b and A″c tangent to the circle. Join the points where A″b and A″c cut GL to the vertical trace A′. The shaded area A′ceb will be the shadow of the cone on the horizontal and vertical planes. The areas ceba and a′b′f′ will be the horizontal and vertical projections of the shade the cone casts on its own surface.