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BANKRUPTCY, LIMITED LIABILITY AND FINANCIAL INTERMEDIATION: A GENERAL EQUILIBRIUM APPROACH

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1. Introduction

In Arrow-Debrew formulation of general equilibrium under uncertainty, two rather unrealistic assumptions are made. First, they assume that all uncertainties are of the exogeneous nature and are objective, while, more important uncertainties may be endogeneous ones. Second, partially as a result of the first assumption, they assumed that there are complete markets for each contingency.

When we relax the second assumptions because of transaction costs, incomplete information, moral hazard or some other reasons, we obtain incomplete market structure. It is one of the important results in incomplete market theory that efficiency of equilibria depends upon underlying market structure, especially upon availability of different financial securities. For example, Diamond [1] showed that, in a restricted economy, with incomplete markets and stock markets, an equilibrium with firm's value maximization behavior will achieve "constrained" efficiency. Namely, an equilibrium is Pareto efficient among all the possible allocations which are feasible with stock markets. Since, in general, such an equilibrium is not "full" Pareto efficient, there is a good reason to expect other types of financial institutions should emerge.

In this paper, we would like to analyze the property of an economy with a banking sector, which provides a riskless asset by pooling individual risks inherent to each individual firm. Following Modigliani-Jaffee [4], we assume that banks provide loans to a firm whose return is contingent upon the solvency of the firm.

In order to avoid possiblities of inefficiency caused by reasons

other than the existence of banking sector (e.g., see Hart [3]), we assume the same technology as in Diamond. Our main interest lies in the question if our economy will achieve constrained efficiency. Because of the well-known problem of value maximization criterion due to the possibility of bankruptcy, we shall focus upon the stockholder's (investor's) preference on firm's action.

Many works on stockmarkets assume the existence of a riskless asset. However, unless we analyze the mechanism of a financial intermediary which provides such an asset, the conclusion may be misleading. In fact, we shall show in this paper that even in Diamond economy with bank (and therefore a certain rule of bankruptcy and a certain rule of limited liability), a constrained Pareto efficient allocation is unstable in the sense that all the stockholders may want firms to change their actions, leading to inefficient state of the economy.

In section 2, we will present the model. In section 3, we will show that for each choice of firm's action, there is a corresponding price and allocation. In section 4, our main result will be presented. Section 5 concludes the paper. A mathematical proof is relegated to appendix.

2. Model

The economy consists of three different types of agents; consumers, firms, and banks. There is only one good and there are two periods; period 0 and 1. In period 0, consumer can invest either in firms which will yield a risky return in period 1 but with limited liability, or in bank deposit which will yield a safe return (with possibly, positive interest rate). Firms invest a certain amount of good in period 0 which can be financed either directly by consumers or by banks in the form of bank loans. We assume that firms have individual risks (Malinvand [5]). That is, given a certain amount of investment, a firm may find itself in period 1 in one of several possible states with respect to output. The probability of the occurrence of a state for a firm is independent of the probability for any other firm. Given the amount of investment and loan in period 0, if a firm is in a state such that its output falls short of its obligation to banks, it goes bankrupt and all the output will be confiscated by creditors (banks). On the other hand, if a firm finds itself in a state where it can meet its loan obligation, the firm will distribute the difference between the output and the loan as dividends to investors proportionally to the amount of investment. Finally banks provide safe asset (deposit) to consumers and provides loan to firms. Banks can pool the risk inherent to loans, for we assume that there are an infinite number of identical firms of each type with independent but identical individual risks. By the law of large numbers, the proportion of a certain type of firms in a certain state is equal to the objective probability of the state to occur.

Formally, there are m types of consumers indexed by $i = 1,\ldots,m$, and n firms indexed by $j = 1,\ldots,n$. There are infinitely many consumers and firms of each type. Let (i, k) denote the k-th consumer of i-th type and (j, k) denote the k-th consumer of type j. Define E_k to be a set of

consumers (i, k) for all i and firms (j, k) for all j. We assume;

(A.1) For all i and k, a consumer (i, k) can invest only in firms in E_k.

Without such an assumption, a consumer can invest in infinitely many firms and completely pool the risk. In the reality, a consumer can invest only in a finite number of firms because of indivisibility of stocks and transaction costs. In this sense, our formulation may be considered as an approximation of the reality. For the firms in which a consumer invests should be determined endogeneously rather than exogeneously as in our formulation.

Let us start with a firm (j, k). For the sake of simplicity, we drop k from subscripts. Let $e_{ij} \in R_+$ be the units of good invested in j-th firm by i-th consumer (in E_k) in period 0 and let $\ell_j \in R_+$ be the units of loan j-th firm borrows from banks. A unit of loan is a contract such that the firm will pay one unit of good in period 1 if it does not go bankrupt and will surrender all the outputs to banks if it goes bankrupt. Let $p_j \in R_+$ be the price (in terms of good in period 0) of a unit of loan to a firm of type j. $1/p_j - 1$ is the rate of interest on loan for firm of type j.

Let S_j be the set of all individual states for j-th firm which is a finite set. Let $a_{js_j} \in R_+$ for all $s_j \in S_j$ be the output-input ratio for j-th firm if state s_j occurs. We assume a_{js_j} is constant and therefore firms have constant returns to scale (both in the sense of stochastic and in the usual sense for each state). Let $\pi_{js_j} \in R_+$ be the probability that j-th firm finds itself in state $s_j \in S_j$ in period 1. Hence in period 1, exactly π_{js_j} portion of j-th firm is in the state s_j. By definition $\sum_{s_j \in S_j} \pi_{s_j} = 1$.

Let $x_j = \ell_j / \sum_{i=1}^{m} e_{ij}$, the debt-equity ratio j-th firm chooses. Since we assume that all firms and consumers of the same type behave in the same manner, the ratio x_j is common for all j-th type firms. Given the

choice of x_j and the price of loan for j-th type firm, p_j, the total dividend payment by j-th firm in state $s_j \in S_j$ is

$$\text{Max}[a_{js_j}(\sum_{i=1}^{m} e_{ij} + p_j \ell_j) - \ell_j,\ 0] = \text{Max}[\{a_{js_j}(1 + p_j x_j) - x_j\} \sum_{i=1}^{m} e_{ij},\ 0].$$

Since dividend is distributed proportionally to the amount of investment, the return for a unit of investment if state s_j occurs, ρ_{js_j}, is

$$\rho_{js_j}(x_j, p_j) = \text{Max}[a_{js_j}(1 + p_j x_j) - x_j,\ 0]. \qquad [1]$$

Given the price p_j, firm will choose the optimal return profile by choosing a suitable debt-equity ratio $x_j \in R_+$. We shall come back to this point later.

For the sake of simplicity, we assume that consumers will face no uncertainty except risks involved in returns from investment in firms. Let $w_i \equiv (w_{i0}, w_{i1}) \in R_+^2$ be the pair of endowments for each period for a consumer of type i. In period 1, consumer's income in E_k depends upon what state has been realized in each firm in E_k. We call an array of individual states $\sigma \equiv (s_j)_{j=1}^n \in \overset{n}{\underset{j=1}{X}} S_j \equiv S$ as a social state of E_k. A consumer's plan is a plan of consumptions $c_i \equiv \left(c_{i0}, (c_{i\sigma})_{\sigma \in S}\right) \in R_+^{S+1}$, i.e., consumption plan for period 0 and contingent plans for each social state, investments $e_i \equiv (e_{ij})_{j=1}^n \in R_+^n$ and purchase of bank deposit $m_i \in R_+$. A unit of bank deposit is a contract such that a bank will deliver one unit of good in period 1 regardless of the social state in exchange for investment in bank in period 0. We write $p_0 \in R_+$ for the price for deposit in peirod 0 in terms of good in period 0. $1/p_0 - 1$ is the interest rate for deposit. Under prices $p \equiv \left(p_0, (p_j)_{j=1}^n\right) \in R_+^{n+1}$ and firms decisions $x \equiv (x_j)_{j=1}^n \in R_+^n$, a consumer's plan (c_i, e_i, m_i) is <u>budget feasible</u> if it satisfies

$$c_{i0} \leq w_{i0} - \sum_{j=1}^{n} e_{ij} - p_0 m_i \tag{2}$$

$$c_{i\sigma} \leq w_{i1} + \sum_{j=1}^{n} a_{js_j}(x_j, p_j) e_{ij} + m_i$$

$$\text{for all } \sigma = (s_j)_{j=1}^{n} \quad S \; . \tag{3}$$

Each consumer of type i is endowed with the same utility function, $u_i = R_+^{S+1} \to R$. Consumer's plan (c_i, e_i, m_i) is optimal if it is budget feasible and if there is no other budget feasible plan which would yield higher utility.

Number of banks is arbitrary. However, for the convenience, we treat this sector as though it consists of one bank in the whole economy. This treatment is justified, for competitive banking implies constant returns to scale for banks as will be shown in the following. Let $m_0 \in R_+$ be the units of deposit banks offer to consumers in each subeconomy, E_k. Let $\ell_{0j} \in R_+$ be the units of loan they finance to each of j-th type firm. Since we are interested in some kind of equilibrium, we assume theat banks have correct expectations about firms' return. Otherwise, banks will have profit or loss which would result in new entry or exit. Let f_j be the expected return for banks from a unit loan to a firm of type j. If j-th firm chooses the debt-equity ratio x_j and banks lend ℓ_{0j} units of loan to the firm, total investment of j-th firm must be $p_j \ell_{0j} + \ell_{0j}/x_j$, provided $x_j > 0$. In period 1, banks expect to receive average return from each of j-th type firms of the amount,

$$\sum_{s_j \in S_j} \pi_{js_j} [\ell_{0j}, \; a_{js_j}(p_j \ell_{0j} + \ell_{0j}/x_j)] .$$

If $x_j = 0$, firm of type j will never go bankrupt, for $a_{js_j} \geq 0$ for all $s_j \in S_j$.

Hence expected return should be ℓ_{0j}, although firm would never demand loans. Therefore,

$$f_j(x_j, p_j) = \begin{cases} \sum_{s_j \in S_j} \pi_{js_j} \min[1, a_{js_j}(p_j + 1/x_j)] & \text{if } x > 0 \\ 1 & \text{if } x = 0 \end{cases} \tag{4}$$

Banks' plan is a pair $(m_0, \ell_0) \equiv \left(m_0, (\ell_{0j})_{j=1}^{n}\right) \in R_+^{n+1}$. Banks' plan is <u>feasible</u> if they can meet the amount of loan by the amount of deposit,

$$\sum_{j=1}^{n} p_j \ell_{0j} \leq p_0 m_0. \tag{5}$$

We consider two different types of banks.

(A.2) Banks are perfectly competitive and

(A.2)' Banks are regulated so that they cannot make any positive profit (namely, banks must choose a feasible plan which would yield zero profit.)

Under competitive assumption (A.2), banks' plan is <u>optimal</u> if, given prices p, it is feasible and it maximizes profit.

$$\sum_{j=1}^{n} f_j(x_j, p_j)\ell_{0j} - m_0$$

among all feasible plans.

Under regulatory assumption (A.2)', banks' plan is <u>optimal</u> if, given prices p, it is feasible and

$$\sum_{j=1}^{n} f_j(x_j, p_j)\ell_{0j} - m_0 = 0$$

3. Equilibrium

Throughout this paper, we assume the following in additon to (A.1) and (A.2) (or (A.2)').

(A.3) For all i, u_i is strictly quasi-concave, differentiable and monotone in consumption in peirod 1.

(A.4) $w_{i0} > 0$ and $w_{i1} \geq 0$ for all i.

(A.5) $a_{js_j} > 0$ for all j and all $s_j \in S_j$.

Assume that, by whatever reason, each type of firm has chosen a debt-equity ratio, x_j, and $x = (x_j)_{j=1}^n$ is temporarily fixed. Then the return from investment, ρ_{js_j}, and return from loan, f_j, depend only upon prices. Under assumptions (A.1), (A.2), (A.3) - (A.5), we can prove:

Proposition 1 For each $x \in R_+^n$, there exists prices $p \in R_+^{n+1}$, consumer's plan $(c_i, e_i, m_i)_{i=1}^m$, banks' plan (ℓ_0, m_0) such that

(a) (c_i, e_i, m_i) is optimal under p for all i

(b) (ℓ_0, m_0) is optimal under p

(c) $\sum_{i=1}^{m} m_i \leq m_0$

(d) $\ell_{0j} \leq x_j \sum_{i=1}^{m} e_{ij}$ for all j.

Proof See Appendix.

Note that the same proposition holds under (A.1), (A.2)', (A.3) - (A.5), for banks technology is that of constant returns to scale given the prices. We call a pair $(x, p) \in R_+^n \times R_+^{n+1}$, a pseudo-equilibrium of the economy if p with $(c_i, e_j, m_i)_{i=1}^m$ and (ℓ_0, m_0) satisfies (a) - (d) under x. In other words, if (x, p) is a pseudo-equilibrium, then p is a true equilibrium price when firms choose x as their decisions under p.

One may note that if (x, p) is a pseudo-equilibrium of the economy, $p_j \leq p_0$ for all j. Namely, interest on loan is always higher than interest on money. It is so because under the condition of zero profit (recall that banking technology is that of constant returns to scale), bank must charge higher interest on loan in order to compensate a possibility of bankruptcy.

In what way do firms make their decisions? Since incomplete markets and a possibility of bankruptcy are our major concern, neither profit maximization nor value maximization can be an accepted doctrine. Therefore, we will take a rather naive approach that firm's action is determined by stockholders (investors). Since there are an infinite number of firms and consumers, prices, p, must be viewed as given by them. Under a pseudo-equilibri (x, p), a consumer <u>prefers</u> a decision x_j to x_j' for a firm of type j if the optimal plan under (x, p) yields higher utility to him than the optimal plan under $(x(x_j'), p)$ where $x(x_j') = (x_1, \ldots, x_{j-1}, x_j', x_{j+1}, \ldots, x_n)$. Similarly, a consumer prefers x_j' to x_j under (x, p) if the optimal plan under $(x(x_j'), p)$ is preferred to the optimal plan under (x, p).

Under assumptions (A.1), (A.2)', (A.3) - (A.5);

<u>Proposition 2</u> There exists a pseudo-equilibrium (x, p) where no consumer prefers any other debt equity ratio to x_j for any firm j.

The pseudo-equilibrium in <u>Proposition 2</u> is very stable, for there is no consumer who wants to defect from the situation. However surprising, the proposition is rather trivial.

Let $x_j = 0$ for all j. Whatever the prices are, $\rho_{js_j} > 0$ for all j and s_j by (A.5). Hence there is a price of deposit p_0 (which is large and corresponding interest rate is negative) such that there is no demand for deposit by consumers. On the other hand, sales of loan by firms are zero, for $x_j = 0$ for all j. Therefore, if $m_0 = 0$ and $\ell_{0j} = 0$ for all j, (ℓ_0, m_0)

is optimal for any p_j and clears both loan and money markets under (A.2)'. Hence we can ignore banking sector and markets of deposit and loan. Then, whatever p_j's are, consumer's plans are determined and they will clear the markets. In particular, if $p_j = 0$ for all j, firm's returns $\rho_{js_j}(\cdot, p)$ is maximized at $x_j = 0$ for all j and s_j and the proposition holds.

The question of the existence of a true equilibrium, both in the sense of non-trivial one under regulation assumption (A.2)' and in the sense of one under competitive assumption (A.2), will not be pursued here. One of the reasons is that our bankruptcy rule will inevitably introduce non-convexity and we can only hope some approximate equilibrium or local equilibrium. A simple example in the next section is an example where we cannot have a true equilibrium but only a local one. However, our main concern being that of efficiency, we are content with an economy which possesses only a local equilibrium.

4. Efficiency

Since we are concerned with incomplete markets, there is no great hope that we can obtain a Pareto efficient equilibrium. Therefore, we consider much weaker notion of efficiency introduced by Diamond [1]. A pseudo-equilibrium of the economy (x, p) is called constrained efficient if there is no other pseudo-equilibrium, (x', p'), which will make at least one consumer better off without making anybody worse off.

Proposition 3 There may exist a constrained efficient pseudo-equilibrium, (p, x), and x_j' such that for all $\hat{x}_j \in (x_j, x_j')$ all the investors prefer $\hat{x}_j$ to x_j'.

In other words, we may find a constrained efficient pseudo-equilibrium so that not only all the investors want to change the debt-equity ratio even locally but also they agree on the direction. Such a pseudo-equilibrium can never be stable, and any equilibrium which reflects stockholders preference may be inefficient. Different from the result on stockmarkets with value maximization (Stiglitz [7], Ekern-Wilson [2], Radner [6]), our inefficiency is the result of the existence of bankruptcy and limited liability.

The proposition above is a consequence of the following example. Let $n = 1$ and let us drop subscript j. (Therefore, p denotes the price of loan.) Let $S_j = S = \{1, 2\}$ and $a_1 > a_2 > 0$. Let $w_{i0} > 0$ and $w_{i1} = 0$ for all i. Assume that utility functions u_i's are identical for all i and are homothetic so that we can aggregate indifference curves. Finally assume that u_i's depends only upon consumption in period 1. Namely, $u_i = (c_0, c_1, c_2) = u_i(c_0', c_1, c_2)$ for all c_0 and c_0'.

Consider a state of the economy (x, p) so that amount of loan demand by firms is $\ell(x)$. By (A.3) and definition of equilibrium,

$$(1 + px) \sum_{i=1}^{m} e_i = \sum_{i=1}^{m} e_i + p\ell(x) = \sum_{i=1}^{m} e_i + p_0 \sum_{i=1}^{m} m_i = \sum_{i=1}^{m} w_{i0}$$

Therefore, for $S = 1, 2$,

$$\sum_{i=1}^{m} c_{is} = \mathrm{Max}[a_s \sum_{i=1}^{m} w_{i0} - \ell(x), 0] + \sum_{s=1}^{2} \pi_s \min[a_s \sum_{i=1}^{m} w_{i0}, \ell(x)].$$

Hence the following holds.

Case 1:

$$\ell(x) \leq a_2 \sum_{i=1}^{m} w_{i0} < a_1 \sum_{i=1}^{m} w_{i0} .$$

$$\sum_{i=1}^{m} c_{i1} = a_1 \sum_{i=1}^{m} w_{i0}$$

$$\sum_{i=1}^{m} c_{i2} = a_2 \sum_{i=1}^{m} w_{i0}$$

Case 2:

$$a_2 \sum_{i=1}^{m} w_{i0} < \ell(x) < a_1 \sum_{i=1}^{m} w_{i0}$$

$$\sum_{i=1}^{m} c_{i1} = a_1 \sum_{i=1}^{m} w_{i0} - \pi_2[\ell(x) - a_2 \sum_{i=1}^{m} w_{i0}]$$

$$\sum_{i=1}^{m} c_{i2} = \ell(x) - \pi_2[\ell(x) - a_2 \sum_{i=1}^{m} w_{i0}].$$

Case 3:

$$a_2 \sum_{i=1}^{m} w_{i0} < a_1 \sum_{i=1}^{m} w_{i0} \leq \ell(x)$$

$$\sum_{i=1}^{m} c_{i1} = \sum_{i=1}^{m} c_{i2} = [\pi_1 a_1 + \pi_2 a_2] \sum_{i=1}^{m} w_{i0}$$

Therefore, depending upon the amount of loan to a firm, $\ell(x)$, the economy (E_k)'s aggregate consumption lies on the segment AB in Figure 1. AB has a slope $-\pi_1/\pi_2$ for $\pi_1 \sum_{i=1}^{m} c_{i1} + \pi_2 \sum_{i=1}^{m} c_{i2} = \pi_1 a_1 \sum_{i=1}^{m} w_{i0} + \pi_2 a_2 \sum_{i=1}^{m} w_{i0}$.

If we obtain some pseudo-equilibrium (x, p) and its associated amount of loan $\ell(x)$ s that the aggregate consumption vector $(\sum_{i=1}^{m} c_{i1}, \sum_{i=1}^{m} c_{i2})$ is on an open segment AB (i.e., Case 2), say C in Figure 1, then the prices p are shown indirectly in the Figure. Let I_1 be the aggregate indifference curve through C. Then C being the aggregate optimal consumption, the aggregate budget line must be tangent to I_1 which is depicted as DE in Figure 1. Since E represents the aggregate consumption plan consumers as a whole would expect to obtain by investing all their endowments in firms (note that $\rho_2(x, p) = 0$ in this case), $E = (\rho_1(x, p) \sum_{i=1}^{m} w_{i0}, 0)$. Similarly, D represents the aggregate consumption plan when they invest all their endowments in bank deposit, $D = (\sum_{i=1}^{m} w_{i0}/p_0, \sum_{i=1}^{m} w_{i0}/p_0)$. In Figure 1, E lies in the east of B, and $\rho_1(x, p) > a_1$ or $a_1 p - 1 > 0$. Therefore, investors unanimously prefer higher debt-equity ratio which would increase return for investor in state 1, ρ_1. If E lies in the west of B, the same argument shows that investors unanimously prefer lower debt-equity ratio. Only at point F in Figure 2, investors are locally satisfied with the situation, i.e., investors prefer to stay with the ongoing debt-equity ratio if their alternatives are restricted to small changes only. In other words, in the open segment AB, F is the only possible (local) equilibrium. Of course, F cannot be an efficient state in our sense, for point G in Figure 2 makes the all consumers better off. On the other hand, at efficient state G, investors unanimously prefer a higher debt-equity ratio. Finally, a point F cannot be a true global equilibrium, since at F, $\rho_1(x, p) = a_1$ and $\rho_2(x, p) = 0$ which is preferred to by $x = 0$ which would guarantee $\rho_1 = a_1$ and $\rho_2 = a_2$.

5. Conclusions

In our economy, the rule of limited liability (equation (1)) severs the process of information transmission by prices. Because of the rule, it is irrelevant for stockholder's preference how much output a firm will produce in a state where the firm goes bankrupt. However, this level of output is at least as important as the level of output in non-bankrupt states from banks viewpoints, and therefore crucial for prices. In other words, the marginal rate of product transformation between two states (a state where a firm goes bankrupt and a state where bank does not go bankrupt) would be equal to the marginal rate of substitution when investors consider the change of output in bankruptcy state when the firm changes its action. Namely, when the rate of return from a firm could be written as

$$\rho_{js_j}(x_j, p_j) = a_{js_j}(1 + p_j x_j) - x_j,$$

as a simple computation would show, MRS and MRPT would coincide.

However, with this modified rate of return, the rule of limited liability would no longer exist and banks would be unable to pool risks. In this sense, although one cannot hope that price mechanism is likely to lead to an efficient equilibrium in an economy with limited liability and financial intermediaries, it is an inevitable price one must pay in order to introduce the mechanism of financial intermediation.

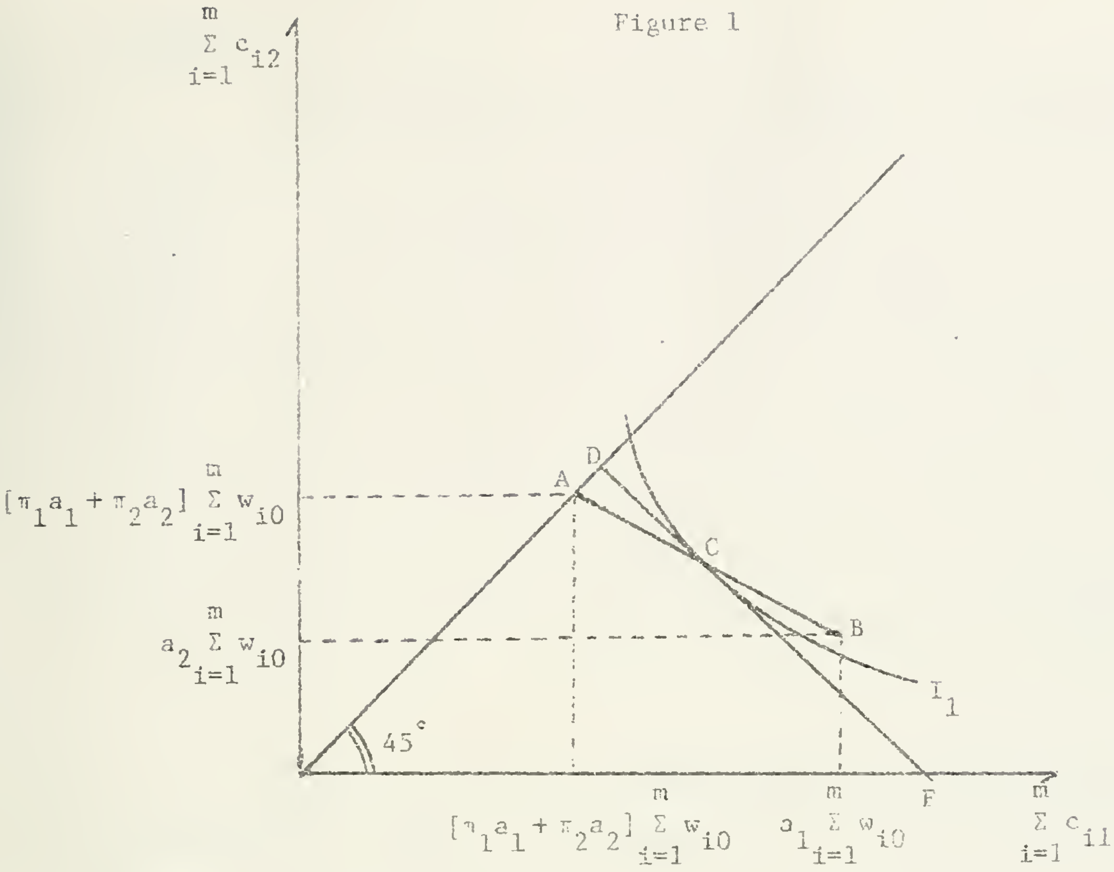

Figure 1

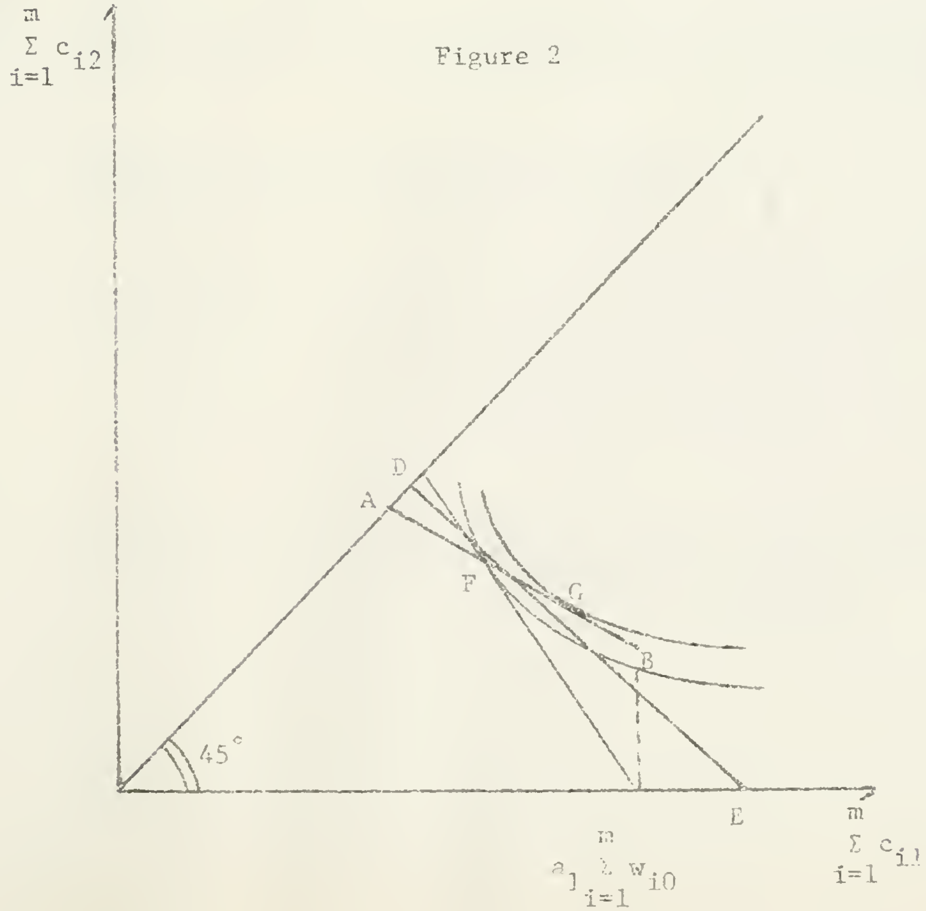

Figure 2

Appendix 1

We shall prove Proposition 1 and therefore $x \in R^n_+$ is fixed throughout this section. Let $q_j(p_j, p_0) = p_0 f(x_j, p_j)$. Then

Lemma 1 $q_j(\cdot, p_0)$ has a unique fixed point for all $p_0 \in R_+$.

Proof $q_j(\cdot, p_0)$ is continuous and non-decreasing for each $p_0 \in R_+$, for $f_j(x_j, p_j)$ has the same property. The range of $q_j(\cdot, p_0)$ is contained in a compact set, $[0, p]$. Unless $p_0 = 0$, $p_j = 0$ cannot be a fixed point. Hence, if $q_j(\cdot, p_0)$ is concave for given $p_0 \in R_+$, there is a unique fixed point. To see this, observe that $\min[1, a_{js_j}(p_j + 1/x_j)]$ is concave in p_j. $f_j(x_j, p_j)$ as a sum of concave functions is also concave and hence $q_j(p_j)$ is concave.

Q.E.D.

Because of the above lemma, $\phi_j(p_0) = \{p_j \mid q_j(p_j, p_0) = p_j\}$ is a well-defined, single-valued function.

Lemma 2 $\phi_j(p_0)$ is continuous.

Proof It follows from the continuity of $q_j(p_j, \cdot)$ for each p_j and the concavity of $q_j(\cdot, 0)$.

Q.E.D.

Let $B_i : R_+ \to R^{S+1}_+ \times R^n_+ \times R_+$ such that

$$B_i(p_0) = \left\{ (c_i, e_i, m_i) \geq 0 \;\middle|\; \begin{array}{l} (c_i, e_i, m_i) \text{ satisfies (2) and (3)} \\ \text{with } p_j = \phi_j(p_0) \end{array} \right\},$$

and

$$\xi_i(p_0) = \left\{ (c_i, e_i, m_i) \in B_i(p_0) \;\middle|\; \begin{array}{l} u_i(c_i) \geq u_i(c) \text{ for all} \\ (c, e, m) \in B_i(p_0) \end{array} \right\}.$$

<u>Lemma 3</u> $\xi_i(p_0)$ is a continuous function for all $p_0 > 0$.

<u>Proof</u> $\xi_i(p_0)$ is a well-defined function for all $p_0 > 0$, for $B^i(p_0)$ is non-empty, compact valued for all $p_0 > 0$. Continuity follows from the continuity of u_i and $B^j(p_0)$. Continuity of $B^i(p_0)$ is guaranteed by A.4 by usual reasons.

<u>Q.E.D.</u>

$$\text{Let } \psi(p_0) = \{p_0 \sum_{i=1}^{m} m_i - \sum_{j=1}^{n} \phi_j(p_0) \sum_{i=1}^{m} x_j e_{ij} \mid (c_i, e_i, m_i) \in \xi_i(p_0)\}$$

$$\text{and } I_k = [\tfrac{1}{k}, k] \text{ for each } k = 2, 3, \ldots$$

Define a mapping $\gamma_k : I_k \to I_k$ such that

$$\gamma_k(p_0) = \min[k, \max(\tfrac{1}{k}, p_0 + \psi(p_0))]$$

Since $\psi(p_0)$ is a continuous single-valued function,

$\gamma_k(p_0)$ is continuous and using Brower's fixed point theorem,

<u>Lemma 4</u> For each $k = 2, 3, \ldots$, there exists a fixed point, p^*_{0k} such that $p^*_{0k} = \gamma_k(p^*_{0k})$.

<u>Proof of Proposition 1</u>

Consider a sequence $\{p^*_{0k}\}_{k=2}^{\infty}$. Let us show that $0 < \liminf_{k\to\infty} p^*_{0k} \leqq \limsup_{k\to\infty} p^*_{0k} < \infty$. First, let $\varepsilon > 0$ be small enough so that $a_{js_j}(1 + p_0 x_j) - x_j < \frac{1}{p_0}$ for all p_0 in an open interval, $(0, \varepsilon)$. Since $\phi_j(p_0) \leqq p_0$ for all p_0, for any $p_0 \in (0, \varepsilon)$, $\rho_{js_j}(\phi_j(p_0), x_j) < \frac{1}{p_0}$ for all j and s_j. For such $p_0 \in (0, \varepsilon)$, therefore, $m_i > 0$ and $e_{ij} = 0$ for all i and j whenever $(c_i, e_i, m_i) \in \xi_i(p_0)$. In view of the definition of γ_k, $p_0 \in (0, \varepsilon)$ cannot be a fixed point for $\gamma_k(p_0)$ for any k. Hence $\liminf_{k\to\infty} p^*_{0k} \geqq \varepsilon > 0$.

Next, let $N > 0$ be large enough so that $a_{js_j}(p_0 + 1/x_j) > 1 + 1/x_j N$ for all j and s_j when $p_0 > N$. Then, for $p_0 > N$, $p_j = p_0$ is a fixed point of $q_j(\cdot, p_0)$ for all j and since the fixed point must be unique, $\phi_j(p_0) = p_0$ for all j. But then, $p_{js_j}(x_j, \phi_j(p_0)) > 1/N > 1/p_0$ for all j and s_j for $p_0 > N$. Namely, $\sum_{j=1}^{n} e_{ij} > 0$ and $m_i = 0$ whenever $(c_i, e_i, m_i) \in \xi_i(p_0)$ for all i. But because of definition, such $p_0 > N$ cannot be a fixed point for γ_k. Hence, $\limsup_{k \to \infty} p_{0k}^* \leqq N < \infty$. Therefore, there is a subsequence of $\{p_{0k}^*\}_{k=2}^{\infty}$ which converges to $p_0^* \in [\varepsilon, N]$.

Let $(c_i^*, e_i^*, m_i^*) \in \xi_i(p_0^*)$ for all i. Then $p_0^* \sum_{i=1}^{m} m_i^* - \sum_{j=1}^{n} \phi_j(p_0^*) \sum_{i=1}^{m} x_j e_{ij}^* = 0$ must hold. Let $m_{0i}^* = \sum_{i=1}^{m} m_i^*$, $\ell_{0j}^* = \sum_{i=1}^{m} x_j e_{ij}^*$, and $p_j^* = \phi_j(p_0^*)$ for all i and j. Then $(p_0^*, (p_j^*)_{j=1}^{n}, (c_i^*, e_i^*, m_i^*)_{i=1}^{m}, (m_0^*, (\ell_{0j}^*)_{j=1}^{n})$ satisfies (a) - (d) of the proposition.

<u>Q.E.D.</u>

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