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Abstract

In this note we show that the characterizations of the Kalai-Smorodinsky solution given in Kalai and Smorodinsky (1975), and of the Egalitarian solution given in Kalai (1977) for the domain of convex bargaining problems can be extended to a domain of comprehensive (free disposal) bargaining problems. We also discuss the literature in this area.

1. Introduction

An *n-person bargaining problem* consists of a pair (S, d) where S is a non-empty subset of $\Re^n$, and $d \in S$. The set S is interpreted as the set of utility allocations that are attainable through joint action on the part of all n agents. If the agents fail to reach an agreement, then the problem is settled at the point d, which is called the *disagreement point*. A *bargaining solution* F, defined on a class of problems Σ^n, is a map that associates with each problem $(S, d) \in \Sigma^n$ a unique point in S. In the axiomatic approach to bargaining we start by specifying a list of properties (Pareto-optimality, for example) that we would like a solution to have. If it can be shown that there is a unique solution that satisfies a given list of axioms, then the solution is said to be *characterized* this list.

It is common to restrict the domain to problems with convex feasible sets. However, bargaining problems can arise from a variety of political, social and economic situations. The requirement that S be convex seems to remove many important cases from consideration. For example, the image in utility space of a finite set of resource allocations will be a finite set of points, not a convex set. The standard justification for restricting attention to convex problems is an assumption that agents' preferences can be represented by von Neumann–Morgenstern utility functions. The feasible set may then be convexified by using lotteries. We find this approach unappealing for two reasons. First, the von Neumann-Morgenstern hypothesis is often rejected in empirical studies, and there is no shortage of alternatives in the literature. See Fishburn (1989) for a systematic exposition. Second, allowing problems to be settled at lotteries gives rise to serious questions in the interpretation of the axioms. We discuss this a length in Conley and Wilkie (1989).

In this paper we require only that the feasible set be comprehensive. This is equivalent to assuming free disposal in the underlying economic problem. Our results may be stated succinctly: (1) on our domain, there does not exit a solution that satisfies strong Pareto optimality and symmetry; (2) if we replace strong Pareto-optimality with weak Pareto-optimality, then Kalai and Smorodinsky's characterization of their solution on the domain of convex problems may be carried over to the domain of comprehensive problems; and (3) Kalai's characterization of the egalitarian solution on the domain of convex and comprehensive problems may be extended directly to the domain of comprehensive problems.

2. Definitions and Axioms

We start with some definitions and formal statements of the axioms used in the characterizations. Given a point $d \in \Re^n$, and a set $S \subset \Re^n$, we say S is *d-comprehensive* if $d \leq x \leq y$ and $y \in S$ implies $x \in S$.[1]

The *comprehensive hull of a set* $S \subset \Re^n$, *with respect to a point* $d \in \Re^n$ is the smallest d-comprehensive set containing S:

$$comp(S; d) \equiv \{x \in \Re^n \mid x \in S \text{ or } \exists\, y \in S \text{ such that } d \leq x \leq y\}. \tag{1}$$

The *convex hull of a set* $S \subset \Re^n$ is the smallest convex set containing the set S:

$$con(s) \equiv \left\{ x \in \Re^n \mid x = \sum_{i=1}^{n+1} \lambda_i y_i \text{ where } \sum_{i=1}^{n+1} \lambda_i = 1, \lambda_i \geq 0\ \forall\, i, \text{ and } y_i \in S\ \forall\, i \right\}. \tag{2}$$

Define the *weak Pareto set of* S as:

$$WP(S) \equiv \{x \in S \mid y \gg x \text{ implies } y \notin S\}. \tag{3}$$

Define the *strong Pareto set of* S as:

$$P(S) \equiv \{x \in S \mid y \geq x \text{ implies } y \notin S\}. \tag{4}$$

The domain of bargaining problems considered in this paper is Σ^n_d. This is defined as the class of pairs (S, d) where $S \subset \Re^n$ and $d \in \Re^n$ such that:

A1) S is compact.

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[1] The vector inequalities are represented by $\geq, >, and \gg$.

A2) S is d-comprehensive.

A3) There exists $x \in S$ and $x \gg d$.

We now present the axioms used in this paper.

Weak Pareto-Optimality (W.P.O.): $F(S,d) \in WP(S)$.

A *permutation operator*, π, is a bijection from $\{1,2,\ldots,n\}$ to $\{1,2,\ldots,n\}$. Π^n is the class of all such operators. Let $\pi(x) = (x^{\pi(1)}, x^{\pi(2)}, \ldots, x^{\pi(n)})$.[2] and $\pi(S) = \{y \in \Re^n \mid y = \pi(x)x \in S\}$.

Symmetry (SYM): If for all permutation operators $\pi \in \Pi^n$, $\pi(S) = S$ and $\pi(d) = d$, then $F^i(S,d) = F^j(S,d)\ \forall\, i,j$.

An *affine transformation on* $\Re^n$ is a map, $\lambda : \Re^n \rightarrow \Re^n$, where $\lambda(x) = a + bx$ for some $a \in \Re^n, b \in \Re^n_{++}$. Λ^n is the class of all such transformations. Let $\lambda(S) = \{y \in \Re^n \mid y = \lambda(x), x \in S\}$.

Scale Invariance (S.INV): $\forall\ \lambda \in \Lambda^n$, $F(\lambda(S), \lambda(d)) = \lambda(F(S,d))$.

Translation Invariance (T.INV): $\forall\ x \in \Re^n$, $F(S + \{x\}, d + x) = F(S,d) + x$.

Strong Monotonicity (S.MON): If $S \subset S'$ and $d = d'$, then $F(S', d') \geq F(S,d)$.

The *Ideal Point* of a problem (S,d) is defined as:

$$a(S,d) \equiv (\max_{\substack{x \in S \\ x \geq d}} x^1, \max_{\substack{x \in S \\ x \geq d}} x^2, \ldots, \max_{\substack{x \in S \\ x \geq d}} x^n). \tag{5}$$

Restricted Monotonicity (R.MON): If $S \subset S'$, $d = d'$, and $a(S,d) = a(S',d')$, then $F(S',d') \geq F(S,d)$.

[2] Superscripts stand for the components of a vector

3. The Results

First we show the impossibility result.

Theorem 1. *$\nexists\, f : \Sigma^n_c \rightarrow \Re^n$ such that f satisfies SYM and PO.*

<u>Proof/</u>

Consider the problem (S, d) where $S \equiv comp(\{(1,2) \bigcup (2,1)\}; (0,0))$ and $d \equiv (0,0)$. By PO, $f(S,d) = (2,1)$ or $f(S,d) = (1,2)$. But this contradicts SYM.

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Now we consider the Kalai-Smorodinsky solution, K:

$$K(S,d) \equiv \max\left[x \in S \mid x \in con(a(S,d),d)\right]. \tag{6}$$

The axioms used are those employed by Kalai and Smorodinsky(1975) to characterize K on the convex domain with two agents, except that only <u>weak</u> Pareto-optimality is used. The generalization to more agents is not immediate since K does not even satisfy WPO on Σ^n_{con} for $n > 2$. No such difficulty arises on the comprehensive domain. For further discussion see Kalai and Smorodinsky(1975) and Thomson(1986).

Theorem 2. *A solution F on Σ^n_c satisfies SYM, S.INV, W.P.O. and R.MON if and only if it is the Kalai-Smorodinsky solution.*

<u>Proof/</u>

The proof that K satisfies the axioms is elementary and is omitted.

Conversely let F be a solution satisfying the four axioms. Given any $(S,d) \in \Sigma^n_c$, assume by S.INV that the problem has been normalized such that $d = 0$ and $a(S,d) = (\beta, \ldots, \beta) \equiv y$. Then $K(S,d) = (\alpha, \ldots, \alpha) \equiv x$ for some $\alpha > 0$. Let T be defined as:

$$T \equiv comp(y; 0) \setminus \{x + \Re^n_{++}\} \tag{7}$$

and consider the problem $(T, 0)$. We distinguish two cases:

Case 1) $S \subset \Re^n_+$. Since S is comprehensive and $x \in WP(S)$, we have $S \subseteq T$. Also, since T is symmetric, $d = 0$, and x is the only symmetric element $WP(T)$, by W.P.O. and SYM, $F(T, 0) = x$. However, since $S \subset T$, and $a(S, 0) = a(T, 0) = y$, by R.MON $F(S, 0) \leq F(T, 0) = x$

Now let T' be defined by,

$$T' \equiv comp((\beta, 0, \ldots, 0), (0, \beta, \ldots, 0), \ldots, (0, \ldots, \beta), x; 0). \tag{8}$$

Consider the problem $(T', 0)$. Since T is symmetric, $d = 0$, and x is the only symmetric element in $WP(T')$, then by W.P.O. and SYM, $F(T', 0) = x$. Also, since $T' \subset S$ and $a(S, d) = a(T', 0) = y$, by R.MON, $F(S, d) \geq F(T', d) = x$. Thus $F(S, d) = x = K(S, d)$.

Case 2) $S \not\subset \Re^n_+$. Let V be defined as follows,

$$V \equiv T \bigcup \left\{ \bigcup_{\pi \in \Pi} \pi(S) \right\}. \tag{9}$$

Note that V is symmetric and $S \subset V$. If we replace $(T, 0)$ with $(V, 0)$ then the argument of case 1 goes through as before.

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Last we turn to the egalitarian solution. We show that Kalai's (1977) characterization is true on the comprehensive domain.

$$E(S, d) \equiv \{\max [x \in S \mid x_i - d_i = x_j - d_j \; \forall \, i, j \in (1, \ldots n)]\}. \tag{10}$$

Theorem 3. *A solution F on Σ^n_c satisfies SYM, T.INV, W.P.O. and S.MON if and only if it is the egalitarian solution.*

Proof/

The proof that E satisfies the four axioms is elementary and is omitted. Conversely let F be a solution satisfying the four axioms. Given any $(S, d) \in \Sigma_c^n$, we can assume by T.INV that the problem has been normalized such that $d = 0$. Thus $E(S, d) = (\alpha, \ldots, \alpha) \equiv x$ for some $\alpha > 0$. Now let T be defined by:

$$T \equiv comp(x; 0), \tag{11}$$

and consider the problem $(T, 0)$. Since T is symmetric, $d = 0$, and x is the only symmetric element of $WP(T)$, by W.P.O. and SYM, $F(T, d) = x$. Also, since S is comprehensive $T \subseteq S$. Hence, by S.MON, $F(S, d) \geq x$.

By assumption, S is compact. Thus, there exists $\beta \in \Re$ such that $x \in S$ implies $(-\beta, -\beta, \ldots, -\beta) \leq (x^1, x^2, \ldots, x^n) \leq (\beta, \beta, \ldots, \beta)$. Let Z be the symmetric closed hypercube defined by:

$$Z \equiv \{y \in \Re^n \mid |y| \leq (\beta, \beta, \ldots, \beta)\}. \tag{12}$$

Also define T' as:

$$T' \equiv Z \setminus \{x + \Re^n_{++}\}. \tag{13}$$

Consider the problem $(T'; 0)$. Since T' is symmetric, $d = 0$ and x is the only symmetric element of $WP(T')$, by W.P.O. and SYM, $F(T', d) = x$. But since $S \subseteq T'$, by S.MON, $F(S, d) \leq x$. Thus, $F(S, d) = x = E(S, d)$.

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4. Conclusion

In a recent paper, Anant *et al* [1990] show that the Kalai-Smorodinsky theorem can be extended directly on the domain of "NE-Regular" problems. Our first theorem shows this characterization is not true on the domain of comprehensive problems. However, since the set of comprehensive problems includes this class of NE-Regular problems, and the Kalai-Smorodinsky solution is always strongly Pareto-optimal on this class, our axioms imply strong Pareto-optimality on the domain of NE-Regular problems. Thus our second theorem implies the Anat *et al* [1990] theorem. In addition, the comprehensive domain arises naturally from an assumption of free disposal on the underlying economic problem. It is not clear what class of economic problems would give rise to NE-Regular feasible sets.

In general, work suggests that the assumption of a convex feasible set is not essential for any Monotone Path Solution. Since any Monotone Path Solution is well-defined on the domain of comprehensive problems any characterization found on the domain of convex problems should be easy to adapt. This class of solutions is discussed and axiomatized Thomson (1986), pp 52-57. The solution proposed by Nash (1950) is not well defined on our domain. We examine an approach to extending the Nash solution in a companion paper, Conley-Wilkie (1989).

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