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A BAYESIAN APPROACH TO ESTIMATE THE TIME VARYING SECURITY BETA

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Summary:

Based upon the random coefficient model and the maximum principle, Vasicek's static Bayesian beta coefficient adjustment model is extended to a dynamic model. It is shown that the time varying security beta model can be used to investigate the existence of non-stationarity and the regression tendency associated with beta coefficient over time.

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I. Introduction

Vasicek's (1973) Bayesian approach for estimating security beta has been extensively accepted in empirical finance research. However, this method cannot be used to take care of the possible non-stationary nature of security beta. In addition, Fisher and Kamin (1978) have argued that Vasicek's Bayesian estimator of security beta is a static estimator and is generally not unbiased. The main purpose of this paper is to generalize Vasicek's Bayesian beta coefficient adjustment model so it can be used to investigate the existence of the non-stationarity and the regression tendency associated with beta coefficients over time.

In the second section the model is defined. In the third section the estimators of related parameters are derived. In the fourth section the Bayesian estimation of time-varying beta coefficient is derived in accordance with the maximum likelihood principle. In the fifth section, the relationship between the Bayesian beta estimator derived in this study and Vasicek's beta estimator are explored. In the sixth section possible implications associated with the time-varying security beta are indicated. Finally, results of this paper are summarized.

II. The Model

Following Sharpe (1964), Lintner (1965) and Mossin (1966), the capital asset pricing model (CAPM) is defined as

$$E(y_j) = \beta_j E(x) \tag{1}$$

where $y_j = R_j - R_f$ = the excess rates of return on security j.

$x = R_m - R_f$ = the excess market rates of return.

If we allow the security beta β_j to be a random coefficient, then the empirical version of random coefficient CAPM can be defined as[1] (j subscript is omitted to simplify the notations)

$$\begin{aligned} &(a)\ y_t = \beta_t x_t + \varepsilon_t \\ &(b)\ \beta_t = \beta_0 + u_t \end{aligned} \qquad (2)$$

where $E(\varepsilon_t) = 0$, $E(\varepsilon_t^2) = \sigma_\varepsilon^2$,

$E(\beta_t \varepsilon_t) = 0$,

$E(u_t) = 0$, $E(u_t^2) = \sigma_u^2$,

$E(u_t u_{t'}) = 0$, $t \neq t'$.

Then (2) can be written as

$$y_t = \beta_0 x_t + \varepsilon_t^* \qquad (3)$$

where

$$\varepsilon_t^* = (\beta_t - \beta_0)x_t + \varepsilon_t.$$

Then,

$$\mathrm{Var}(\varepsilon_t^*) = x_t^2 \sigma_u^2 + \sigma_u^2 \qquad (4)$$

To find $\mathrm{Cov}(\varepsilon_t^*, \varepsilon_{t'}^*)$:

$$\begin{aligned} \mathrm{Cov}(\varepsilon_t^*, \varepsilon_{t'}^*) &= \mathrm{Cov}[(\beta_t - \beta_0)x_t + \varepsilon_t, (\beta_{t'} - \beta_0)x_{t'} + \varepsilon_{t'}] \\ &= x_t x_{t'} \mathrm{Cov}[(\beta_t - \beta_0), (\beta_{t'} - \beta_0)] + \mathrm{Cov}(\varepsilon_t, \varepsilon_{t'}) \\ &= 0 + 0 = 0, \quad t \neq t' \end{aligned}$$

Thus,

$$\mathrm{Cov}(\varepsilon_t^*, \varepsilon_{t'}^*) = 0, \quad t \neq t'.$$

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[1] Chen and Lee (1977) have first developed random coefficient CAPM; Fabozzi and Francis (1978) have estimated the random coefficient market model.

Hence, equation (4) can be written in terms of vector notation as

$$\mathrm{Var}(\underline{\varepsilon}^*) = \sigma_\varepsilon^2 I + \sigma_u^2 D \tag{5}$$

where I = a (nxn) identity matrix

D = a diagonal matrix with diagonal elements x_t^2

Rewrite (5) as follows:

$$\begin{aligned}\mathrm{Var}(\underline{\varepsilon}^*) &= \sigma_\varepsilon^2 (I + \gamma D) \\ &= \sigma_\varepsilon^2 D^*(\gamma)\end{aligned} \tag{6}$$

where $\gamma = \sigma_u^2/\sigma_\varepsilon^2$

$D^*(\gamma)$ = a diagonal matrix with diagonal elements $1 + \gamma x_t^2$.

Thus, the covariance matrix of $\underline{\varepsilon}^*$ has two parameters, σ_ε^2 and γ to be estimated. In the following section, the method of estimating β_0, σ_ε^2 and σ_u^2 will be derived.

III. The Estimation of the Best Prior for β_0, σ_ε^2 and σ_u^2

If the disturbances are normally distributed, then the logarithm of the likelihood function of $\underline{y}$ can be written as

$$L(\underline{y};\ \underline{x},\ \beta_0,\ \sigma_\varepsilon^2,\ \gamma) = -\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln\sigma_\varepsilon^2 - \frac{1}{2}\ln\left[\sum_{t=1}^{n}(1+\gamma x_t^2)\right] - \frac{1}{2\sigma_\varepsilon^2}(\underline{y} - \beta_0\underline{x})' D^*(\gamma)^{-1}(\underline{y} - \beta_0\underline{x}), \tag{7}$$

where

$$\underline{y}' = (y_1, y_2, \ldots, y_n),\ \underline{x}' = (x_1, x_2, \ldots, x_n).$$

It can be shown that the maximum likelihood estimators (MLE's) of β_0 and σ_ε^2 of (7) can be written as

$$\begin{cases} \hat{\beta}_0(\gamma) = [\underline{x}'D^*(\gamma)^{-1}\underline{x}]^{-1}\cdot\underline{x}'D^*(\gamma)^{-1}\underline{y} = \left(\sum_{t=1}^{n} \frac{x_t y_t}{1+\gamma x_t^2}\right)\left(\sum_{t=1}^{n} \frac{x_t^2}{1+\gamma x_t^2}\right) & (8) \\ \hat{\sigma}_\varepsilon^2(\gamma) = \frac{1}{n}[\underline{y} - \underline{x}\hat{\beta}_0(\gamma)]'D^*(\gamma)^{-1}[\underline{y} - \underline{x}\hat{\beta}_0(\gamma)] & (9) \end{cases}$$

$$= \frac{1}{n} \sum_{t=1}^{n} \left\{\frac{[y_t - x_t\hat{\beta}_0(\gamma)]^2}{1+\gamma x_t^2}\right\}$$

The concentrated likelihood function of γ is obtained by substituting (8) and (9) into (7)[2]

$$L(\underline{y};\underline{x},\gamma) = -\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln\frac{1}{n}\{\underline{y}'D^{*-1}[I - \underline{x}(\underline{x}'D^{*-1}\underline{x})^{-1}\underline{x}'D^{*-1}]\underline{y}\} \quad (10)$$

$$- \frac{1}{2}\ln[\sum_{t=1}^{n} (1+\gamma x_t^2)] - \frac{n}{2}$$

The MLE of γ can be obtained by maximizing numerically the concentrated likelihood function for γ. Equation (8), (9) and (10) can be used in the iterative algorithms to obtain the MLE's of β_0, σ_ε^2 and γ.

The MLE estimator of β_0 defined as in (8) is unbiased and is identical to the generalized least squares (GLS) estimator. The $\hat{\sigma}_\varepsilon^2(\gamma)$ defined as in (9) is biased since

$$E(\hat{\sigma}_\varepsilon^2(\gamma)) = \frac{n-1}{n}\sigma_\varepsilon^2.$$

Thus, the unbiased estimator of $\hat{\sigma}_\varepsilon^2(\gamma)$ is

$$\hat{\sigma}_\varepsilon^{*2} = \frac{n}{n-1}\hat{\sigma}_\varepsilon^2(\gamma). \quad (11)$$

IV. Bayesian Estimation of Time Varying Security Beta

Assume that β_0, σ_ε^2 and $\gamma = (\sigma_u^2/\sigma_\varepsilon^2)$ are known. Consider the model of (2.a) in vector notation

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[2]The explicit form of this equation can be found in the appendix (A).

$$\underline{y} = \overline{\underline{x}}\,\underline{\beta} + \underline{\varepsilon}$$

where $\underline{y}' = (y_1, y_2, \ldots, y_n)$,

$\underline{\beta}' = (\beta_1, \beta_2, \ldots, \beta_n)$,

$\underline{\varepsilon}' = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$,

and

$$\underset{(n\times n)}{\overline{\underline{x}}} = \begin{bmatrix} x_1 & & & 0 \\ & x_2 & & \\ & & \ddots & \\ 0 & & & x_n \end{bmatrix}$$

Assume that $\underline{\beta}$ has a multivariate normal prior density with mean

$$\underline{\mu} = E(\underline{\beta}) = E(\underline{j}\beta_0 + \underline{u}) = \underline{j}\beta_0 \qquad (12)$$

where $\underline{j}$ = a column vector of ones = $\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$,

and covariance matrix

$$\Sigma = \mathrm{Var}(\underline{\beta}) = \mathrm{Var}(\underline{j}\beta_0 + \underline{u}) = \mathrm{Var}(\underline{u}) = \sigma_u^2 I. \qquad (13)$$

The likelihood of the data $\underline{y}$ given $\underline{\beta}$ is multivariate normal with mean $\overline{\underline{x}}\,\underline{\beta}$ and variance $\sigma_\varepsilon^2 I$. Thus, the joint density of $\underline{y}$ and $\underline{\beta}$ is

$$g(\underline{y},\underline{\beta}) = f(\underline{\beta}) \cdot \ell(\underline{y};\overline{\underline{x}},\underline{\beta})$$

$$= \frac{1}{(2\pi)^{n/2}\,|\Sigma|^{1/2}} \exp\{-\tfrac{1}{2}(\underline{\beta}-\underline{\mu})'\Sigma^{-1}(\underline{\beta}-\underline{\mu})\}$$

$$\cdot \frac{1}{(2\pi)^{n/2}\,\sigma_\varepsilon^{\,n}} \exp\{-\frac{1}{2\sigma_\varepsilon^2}(\underline{y}-\overline{\underline{x}}\,\underline{\beta})'(\underline{y}-\overline{\underline{x}}\,\underline{\beta})\}$$

$$= \frac{1}{(2\pi)^{n/2}\,\sigma_u^{\,n}} \exp\{-\frac{1}{2\sigma_u^2}(\underline{\beta}-\underline{\mu})'(\underline{\beta}-\underline{\mu})\}$$

$$\frac{1}{(2\pi)^{n/2}\,\sigma_\varepsilon^{\,n}} \exp\{-\frac{1}{2\sigma_\varepsilon^2}(\underline{y}-\overline{\underline{x}}\,\underline{\beta})'(\underline{y}-\overline{\underline{x}}\,\underline{\beta})\}. \qquad (14)$$

The posterior density of $\underline{\beta}$ can be obtained by dividing the marginal density of $\underline{y}$ into the joint density defined in (14). Thus,

$$g(\underline{\beta}|\underline{y}) = \frac{g(\underline{y},\underline{\beta})}{g_1(\underline{y})}$$

$$= \frac{\left|(\frac{1}{\sigma_\varepsilon^2})\bar{\underline{x}}'\bar{\underline{x}} + (\frac{1}{\sigma_u^2}I)\right|}{(2\pi)^{n/2}} \exp\{\tfrac{-1}{2}[\underline{\beta} - \underline{\beta}^*]'\Omega^{-1}[\underline{\beta} - \underline{\beta}^*]\}, \quad (15)$$

where $\underline{\beta}^* = E(\underline{\beta}|\underline{y}) = (\frac{1}{\sigma_\varepsilon^2}\bar{\underline{x}}'\bar{\underline{x}} + \frac{1}{\sigma_u^2}I)^{-1}(\frac{1}{\sigma_u^2}\underline{\mu} + \frac{1}{\sigma_\varepsilon^2}\bar{\underline{x}}'\underline{y})$ (16)

and $\quad \Omega = Var(\underline{\beta}|\underline{y}) = (\frac{1}{\sigma_\varepsilon^2}\bar{\underline{x}}'\bar{\underline{x}} + \frac{1}{\sigma_u^2}I)^{-1}$ (17)

If the expected quadratic loss, $E(\underline{\beta}-\underline{\beta}^*)'(\underline{\beta}-\underline{\beta}^*)$, is used to determine the best estimator of $\underline{\beta}$, it is well known that the mean of the posterior density of $\underline{\beta}$ is the best estimator. Thus, the Bayesian estimator of $\underline{\beta}$ is $\underline{\beta}^*$ defined as in (16).

$\underline{\beta}^*$ can be simplified as follows:

Since $\bar{\underline{x}}'\bar{\underline{x}} = \begin{bmatrix} x_1^2 & & & 0 \\ & x_2^2 & & \\ & & \ddots & \\ 0 & & & x_n^2 \end{bmatrix}$

$$\bar{\underline{x}}'\underline{y} = \begin{bmatrix} x_1 & & & 0 \\ & x_2 & & \\ & & \ddots & \\ 0 & & & x_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1y_1 \\ x_2y_2 \\ \vdots \\ x_ny_n \end{bmatrix}$$

$$\underline{\mu} = \beta_0\underline{i} = \begin{bmatrix} \beta_0 \\ \beta_0 \\ \vdots \\ \beta_0 \end{bmatrix}$$

Thus, the matrix $(\frac{1}{\sigma_\varepsilon^2}\underline{\bar{x}}'\underline{\bar{x}} + \frac{1}{\sigma_u^2} I)$ is a diagonal matrix with diagonal elements, $\frac{1}{\sigma_\varepsilon^2} x_t^2 + \frac{1}{\sigma_u^2}$. And $\frac{1}{\sigma_u^2}\underline{\mu} + \frac{1}{\sigma_\varepsilon^2}\underline{\bar{x}}'\underline{y} = \frac{\beta_0}{\sigma_u^2}\underline{i} + \frac{1}{\sigma_\varepsilon^2}\begin{bmatrix} x_1y_1 \\ x_2y_2 \\ \vdots \\ x_ny_n \end{bmatrix}$

$$= \begin{bmatrix} \frac{\beta_0}{\sigma_u^2} + \frac{x_1y_1}{\sigma_\varepsilon^2} \\ \frac{\beta_0}{\sigma_u^2} + \frac{x_2y_2}{\sigma_\varepsilon^2} \\ \vdots \\ \frac{\beta_0}{\sigma_u^2} + \frac{x_ny_n}{\sigma_\varepsilon^2} \end{bmatrix}$$

, a column vector with the typical element $(\frac{\beta_0}{\sigma_u^2} + \frac{x_ty_t}{\sigma_\varepsilon^2})$.

Thus, the optimal estimator of β_t can be written explicitly as

$$\hat{\beta}_t = (\frac{\beta_0}{\sigma_u^2} + \frac{x_ty_t}{\sigma_\varepsilon^2}) \Big/ (\frac{x_t^2}{\sigma_\varepsilon^2} + \frac{1}{\sigma_u^2}) = \frac{\beta_0 + \gamma x_t y_t}{1 + \gamma x_t^2}, \; t = 1, 2, \ldots n \qquad (18)$$

and

$$\mathrm{Var}(\hat{\beta}_t) = (\frac{x_t^2}{\sigma_\varepsilon^2} + \frac{1}{\sigma_u^2})^{-1} \quad . \qquad (19)$$

It should be noted that β_0, σ_u^2, and σ_ε^2 are prior estimators. These estimators can be obtained by using either

(i) the maximum likelihood estimators as discussed in the previous section or

(ii) the random coefficient method as discussed by Chen and Lee (1977) or Fabozzi and Francis (1978).

It is easy to show that the Bayesian estimator, $\hat{\beta}_t$, is unbiased.

$$E(\hat{\beta}_t) = (\frac{\beta_0}{\sigma_u^2} + \frac{\beta_0 x_t^2}{\sigma_\varepsilon^2}) \Big/ (\frac{x_t^2}{\sigma_\varepsilon^2} + \frac{1}{\sigma_u^2}) = \beta_0.$$

Thus, $\hat{\beta}_t$ is a linear unbiased estimator of β_t. Following (18), $\hat{\beta}_t$ is linear in y_t. $\hat{\beta}_t$ can be shown to have minimum variance among the class of linear unbiased estimators of β_t.[3] Thus, the $\hat{\beta}_t$ defined as in (18) is the unbiased and minimum-variance estimator if the priors β_0, σ_u^2 and σ_ε^2 are known with certainty.

If the priors β_0, σ_u^2 and σ_ε^2 are consistent and efficient, Cooley (1971) has proven in a similar context that the results of the previous two sections are certainly true for large samples.

V. The Relationship Between the Time-Varying Bayesian Beta and Vasicek Bayesian Beta.

The Bayesian estimator and its variance defined as in equations (18) and (19), respectively, are the generalized cases of Vasicek's equations (15) and (16), respectively. The reasons are analyzed as follows. To obtain a cross-sectional Bayesian estimator for a security beta,[4] Vasicek (1973) has employed the following reparametrized regression model

$$y_t = \eta + \beta_c(x_t - \bar{x}) + \varepsilon_t \ , \ t = 1, 2, \ldots n \qquad (20)$$

where β_c indicates the beta coefficient to be estimated by using cross-sectional information, $\eta = \alpha + \beta_c\bar{x}$ (α is the constant term of the market model and $\bar{x} = \sum_t x_{t/n}$. Equation (20) can easily be rewritten in an equivalent form

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[3]See Appendix B for proof.

[4]A cross-sectional Bayesian estimator refers to the use of cross-sectional information in estimating a security beta.

$$y_t^* = \beta_c x_t^* + \varepsilon_t, \quad t = 1, 2, \ldots n \tag{21}$$

where $x_t^* = x_t - \bar{x}$, $y_t^* = y_t - \bar{y}$ and $\bar{y} = \sum_{t=1}^{n} y_t/n$.

The β_c defined as in (21) is not assumed to be a time varying beta coefficient. The cross-sectional beta coefficient, β_c, can easily be obtained by using equation (21) in the above-mentioned analysis. Under the assumption of stationarity of β_c, equations (12) and (13) reduce to the following equations, respectively,[5]

$$E(\beta_c) = b' \tag{22}$$

and

$$\mathrm{Var}(\beta_c) = \sigma_u^2 \quad , \tag{23}$$

where $\beta_c = \beta_t$ and $b' = \beta_0$ (the vector $\underline{\beta}$ in (12) and (13) reduces to β_c). Then, the joint density of $\underline{y}^*$ and β_c defined as in (14) will reduce to the following equation

$$g(\underline{y}^*, \beta_c) = \frac{1}{\sqrt{2\pi}\sigma_u} \exp\{-\frac{1}{2\sigma_u^2} (\beta_c - b')^2\}.$$

$$\frac{1}{(2\pi)^{n/2}\sigma_\varepsilon^n} \exp\{-\frac{1}{2\sigma_\varepsilon^2} (\underline{y}^* - \underline{x}^*\beta_c)'(\underline{y}^* - \underline{x}^*\beta_c)\}, \tag{24}$$

where $\underline{x}^{*\prime} = (x_1^*, x_2^*, \ldots, x_n^*)$ and $\underline{y}^{*\prime} = (y_1^*, y_2^*, \ldots, y_n^*)$.

The posterior of β_c defined as in (14) becomes

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[5]Vasicek (1973) has assumed that the cross-sectional distribution of betas is approximately normal with mean b' and variance σ_u^2.

$$g(\beta_c|\underline{y}^*) = \frac{[(\frac{1}{\sigma_\varepsilon^2})\underline{x}^{*\prime}\underline{x}' + \frac{1}{\sigma_u^2}]}{(2\pi)^{n/2}} \quad \exp\{-\frac{(\beta_c - b^*)^2}{2\gamma^2}\} ,$$

where $b^* = E(\beta_c|\underline{y}^*)$

$$= (\frac{1}{\sigma_\varepsilon^2}\underline{x}^{*\prime}\underline{x}^* + \frac{1}{\sigma_u^2})^{-1} (\frac{b'}{\sigma_u^2} + \frac{1}{\sigma_\varepsilon^2}\underline{x}^{*\prime}\underline{y}^*) \tag{25}$$

and

$$Var(b^*) = Var(\beta_c|\underline{y}^*) = (\frac{1}{\sigma_\varepsilon^2}\underline{x}^{*\prime}\underline{x}^* + \frac{1}{\sigma_u^2})^{-1} \tag{26}$$

Equations (25) and (26) are the reduced forms of the Bayesian estimator, $\hat{\beta}_t$, of the time varying beta coefficient and its variance, $Var(\hat{\beta}_t)$, defined as in equations (16) and (17), respectively. Under the criterion of minimizing the expected quadratic loss function, $E(\beta_c - b')^2$, the optimum estimator of β_c is defined as in (25) and its variance is defined as in (26). Explicitly, the Bayesian estimator of β_c can be written as

$$b^* = (\frac{b'}{\sigma_u^2} + \frac{\sum_t x_t^* y_t^*}{\sigma_\varepsilon^{*2}}) \Big/ (\frac{\sum_t x_t^{*2}}{\sigma_\varepsilon^2} + \frac{1}{\sigma_u^2})$$

$$= [\frac{b'}{\sigma_u^2} + \frac{b}{\sigma_\varepsilon^2/\sum_t (x_t - \bar{x})^2}] \Big/ [\frac{1}{\sigma_u^2} + \frac{1}{\sigma_\varepsilon^2/\sum_t (x_t - \bar{x})^2}] \tag{27}$$

and its variance is

$$Var(b^*) = [\frac{1}{\sigma_u^2} + \frac{1}{\sigma_\varepsilon^2/\sum_t (x_t - \bar{x})^2}]^{-1}, \tag{28}$$

where b = the estimated β_c obtained by the ordinary least squares method

$$= \frac{\sum_t x_t^* y_t^*}{\sum_t x_t^{*2}} .$$

Equations (27) and (28) are exactly equal to Vasicek's equations (16) and (17), respectively.[6] Therefore, the Bayesian estimator and its variance defined as in equations (16) and (17), respectively, are the generalization of the Vasicek's estimator, b*, and its variance, Var(b*), respectively.

In the following section, possible implications associated with time-varying Bayesian security beta are discussed.

VI. Some Implications

Possible implications associated with the time-varying Bayesian security beta are now explored.

(i) The $\mathrm{Var}(\hat{\beta}_t)$ approaches zero if the variance, σ_u^2, associated with the beta coefficient approaches zero. This is simply because

$$\lim_{\sigma_u^2 \to 0} \mathrm{Var}(\hat{\beta}_t) = \lim_{\sigma_u^2 \to 0} \frac{1}{\frac{x_t^2}{\sigma_\varepsilon^2} + \frac{1}{\sigma_u^2}} = 0 ,$$

where x_t^2 and σ_ε^2 are known to be finite. Hence, the precision of the Bayesian estimator, $\hat{\beta}_t$, depdends on the magnitude of the variation of the beta coefficient. The smaller the variance associated with the beta coefficient, the smaller the variance of the Bayesian estimator. Furthermore, the Bayesian estimator also implies that $\hat{\beta}_t$ is equal to the prior, β_0, if σ_u^2 approaches to zero.
This can be shown as follows

[6]The prior σ_ε^2 in Vasicek's analysis is estimated by

$$s^2 = \frac{1}{n-2} \sum_t (y_t - a - bx_t)^2$$

where a and b are the ordinary least squares estimators of the regression parameters α and β_c $(y_t = \alpha + \beta_c x_t + \varepsilon_t)$.

$$\lim_{\sigma_u^2 \to 0} \hat{\beta}_t = \lim_{\sigma_u^2 \to 0} \left[\left(\frac{\beta_0}{\sigma_u^2} + \frac{x_t y_t}{\sigma_\varepsilon^2} \right) \Big/ \left(\frac{x_t^2}{\sigma_\varepsilon^2} + \frac{1}{\sigma_u^2} \right) \right]$$

$$= \lim_{\sigma_u^2 \to 0} [\beta_0 \sigma_\varepsilon^2 + \sigma_u^2 x_t y_t / \sigma_u^2 x_t^2 + \sigma_\varepsilon^2]$$

$$= \lim_{\sigma_u^2 \to 0} \beta_0 = \beta_0.$$

Thus, the random coefficient CAPM defined in (2) becomes the fixed coefficient CAPM as the variance of the beta coefficient reduces to zero. And, the Bayesian estimator, $\hat{\beta}_t$, defined as in (18) is a generalized case of the estimator of the beta coefficient associated with the fixed coefficient CAPM.

(ii) The estimated time-varying beta coefficient, $\hat{\beta}_t$, is a function of y_t, x_t, σ_u^2 and σ_ε^2. Thus, the $\hat{\beta}_t$ is jointly determined by the current market conditions, firm's structure changes in response to economic events and micro-economic changes of a firm. Rosenberg and Guy (1976) have shown that the beta for any security is the weighted average of the relative variances of particular economic events to the variance of the market returns, where each weight is the relative response coefficient that represents the proportion of total variance in market return due to a particular economic event. Thus, the Bayesian estimator defined as in (18) not only is the optimal estimator but possesses the practical significance of the beta coefficient as discussed by Rosenberg and Guy.

(iii) The estimated beta coefficients of a security over a sampling period can be employed to identify the behavior of the beta

coefficient in response to prominent changes in certain economic events such as rate of inflation, interest rates, growth rate of real GNP, etc. For example, when uncertain inflation is considered, Chen and Boness (1975) have shown that the systematic risk of a stock contains inflation risk represented by the covariance between the security's return and the rate of inflation. A positive covariance between a security's return and the rate of inflation indicates that an inflation preferred stock is likely to have higher return when inflation exists. On the other hand, the market return, x_t, will also possibly be affected by changes in the rate of inflation. Hence, the Bayesian estimator, $\hat{\beta}_t$, defined in (18) implies that an inflation-preferred security would experience a positive change in magnitude of the beta coefficient if inflation is anticipated. Likewise, an inflation-averse stock tends to have a negative change in magnitude of the beta coefficient.[7]

(iv) The $\hat{\beta}_t$ defined in (18) can be used to forecast beta coefficients once the priors β_0, σ_u^2 and σ_ε^2 are properly determined. To forecast beta coefficient, both expected excess security return and expected excess market return must be estimated. If the related excess rates of return can be estimated precisely, then the estimated beta coefficient will be a useful predictor for future systematic risk.

(v) The $\hat{\beta}_t$ can also be used to measure the risk level of a mutual fund. Kon and Jen (1978) have argued that the magnitude of β_t for a

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[7]See Chen and Boness (1975) for detail discussion of "inflation-preferred" or "inflation-averse" stocks.

mutual fund depends on expectations at the beginning of the time interval of future security and market movements over that interval conditional on the manager's information set. The Bayesian estimators defined as in (18) indicates that the numerator, $[\frac{\beta_0}{\sigma_u^2} + (\frac{y_t}{\sigma_\varepsilon^2})x_t]$, represents over-all expectation at the beginning of the time interval. In other words, the future security movements expressed by β_0/σ_u^2, and market movements over that interval, signified by $(y_t/\sigma_\varepsilon^2)x_t$.[8] The denominator, $(\frac{x_t^2}{\sigma_\varepsilon^2} + \frac{1}{\sigma_u^2})$, contains information about market conditions, unsystematic risk of a firm and variation of firm's structure changes in response to the market. This implies that a manager's information set can be represented by $(x_t^2/\sigma_\varepsilon^2 + 1/\sigma_u^2)$. Thus, the Bayesian estimator $\hat{\beta}_t$, may well be an appropriate risk proxy for time-varying systematic risk of a managed fund.

(vi) Finally, it should be noted that the time-varying beta $\hat{\beta}_t$ as indicated in equation (18) can also be used to estimate the time path of historical beta coefficient. The time path of historical beta can be used to test whether individual firm's beta is non-stationary or not. Furthermore, the time path of historical beta coefficient can also be used to analyze the sources of regression tendency discovered by Blume (1971, 1975). Following equation (18), the change of beta coefficient over time can be defined as

$$\hat{\beta}_t - \beta_0 = \frac{\gamma(\beta_0 x_t^2 - x_t y_t)}{1 + \gamma x_t^2} \qquad (29)$$

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[8] β_0 $[=E(\beta_t)]$ is the expected value of β_t during the period t.

If estimated γ is constant over time, then the change of $\hat{\beta}_t$ over time is due to the change of $(\beta_0 x_t^2 - x_t y_t)$ and γx_t^2. If the ratio between y_t and x_t (y_t/x_t) is equal to β_0, then the estimated systematic risk is constant over time. If y_t/x_t is larger (or smaller) than β_0, the the $\hat{\beta}_t$ will decrease (or increase) over time. This analysis has given the existence of regression tendency a formal interpretation.

VII. Summary

In this study, Vasicek's (1973) static Bayesian security beta estimator is generalized to obtain a time-varying Bayesian security beta estimator. It is shown the prior estimators associated with the dynamic Bayesian security beta can either be obtained by the maximum likelihood estimator or the generalized least squares estimator associated with random coefficient CAPM.

The implications related to the dynamic Bayesian security beta estimator in obtaining historical beta and predicting future beta are discussed. The relationship between the dynamic Bayesian beta with the traditional beta estimator is also explored. Furthermore, the possible existence of regression tendency has given a formal interpretation.

Appendix A

The explicit form of equation (10) can be derived as follows:

Since

(a) $(\underline{x}'D^{*-1}\underline{x})^{-1} = \left(\sum_{t=1}^{n} \frac{x_t^2}{1+\gamma x_t^2}\right)^{-1}$

(b) $\underline{x}'D^{*-1}\underline{y} = \sum_{t=1}^{n} \frac{x_t y_t}{1+\gamma x_t^2}$

(c) $(\underline{x}'D^{*-1}\underline{x})^{-1}\underline{x}'D^{*-1}\underline{y} = \left(\sum_{t=1}^{n} \frac{x_t y_t}{1+\gamma x_t^2}\right) \Big/ \left(\sum_{t=1}^{n} \frac{x_t^2}{1+\gamma x_t^2}\right)$

(d) $\underline{y}'D^{*-1}\underline{x} \cdot (\underline{x}'D^{*-1}\underline{x})^{-1} \cdot \underline{x}'D^{*-1}\underline{y} = \left(\sum_{t=1}^{n} \frac{x_t y_t}{1+\gamma x_t^2}\right)^2 \Big/ \left(\sum_{t=1}^{n} \frac{x_t^2}{1+\gamma x_t^2}\right)$

(e) $\underline{y}'D^{*-1}\underline{y} = \sum_{t=1}^{n} \frac{y_t^2}{1+\gamma x_t^2}$

(f) Thus,

$$\underline{y}'D^{*-1}[I - \underline{x}(\underline{x}'D^{*-1}\underline{x})^{-1}\underline{x}'D^{*-1}]\underline{y}$$

$$= \sum_{t=1}^{n} \frac{y_t^2}{1+\gamma x_t^2} - \left[\left(\sum_{t=1}^{n} \frac{x_t y_t}{1+\gamma x_t^2}\right)^2 \quad \left(\sum_{t=1}^{n} \frac{x_t^2}{1+\gamma x_t^2}\right)\right] > 0$$

since the likelihood function, $L(\underline{y};x,\alpha)$ is a ln function and it is well-defined. Thus, equation (10) can be written as

$$L(\underline{y};\underline{x},\gamma) = -\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln\frac{1}{n}\left\{\sum_{t=1}^{n} \frac{y_t^2}{1+\gamma x_t^2} - \left(\sum_{t=1}^{n} \frac{x_t y_t}{1+\gamma x_t^2}\right)^2 \Big/ \left(\sum_{t=1}^{n} \frac{x_t^2}{1+\gamma x_t^2}\right)\right\}$$

$$- \frac{1}{2}\ln\left[\sum_{t=1}^{n}(1+\gamma x_t^2)\right] - \frac{n}{2} .$$

Appendix B

To show that $\hat{\beta}_t$ has minimum variance among the class of linear unbiased estimators of β_t, the following theorem is first established.

<u>Theorem 1</u>: $E[(\hat{\beta}_t - \beta_t)y_t] = 0$

Proof: $E[(\hat{\beta}_t - \beta_t)y_t] = E[(\hat{\beta}_t - \beta_t)(\beta_t x_t + \varepsilon_t)]$

$$= x_t E[(\hat{\beta}_t - \beta_t)\beta_t] + E[(\hat{\beta}_t - \beta_t)\varepsilon_t]$$

But, $E[(\hat{\beta}_t - \beta_t)\beta_t] = E(\hat{\beta}_t\beta_t) - E(\beta_t^2)$

$$= E\,\frac{\beta_0\beta_t + \gamma x_t y_t \beta_t}{1 + \gamma x_t^2} - (\sigma_u^2 + \beta_0^2)$$

$$= \frac{\beta_0^2 + \gamma x_t E(y_t\beta_t)}{1 + \gamma x_t^2} - (\sigma_u^2 + \beta_0^2)$$

$$= \frac{\beta_0^2 + \gamma x_t^2(\sigma_u^2 + \beta_0^2)}{1 + \gamma x_t^2} - (\sigma_u^2 + \beta_0^2), \quad E(\beta_t\varepsilon_t) = 0$$

$$= \frac{\gamma x_t^2 \sigma_u^2}{1 + \gamma x_t^2} - \sigma_u^2 = \frac{-\sigma_u^2}{1 + \gamma x_t}$$

And

$$E[(\hat{\beta}_t - \beta_t)\varepsilon_t] = E(\hat{\beta}_t\varepsilon_t) = \frac{E(\beta_0\varepsilon_t + \gamma x_t y_t \varepsilon_t)}{1 + \gamma x_t^2}$$

$$= \frac{\gamma x_t E(y_t\varepsilon_t)}{1 + \gamma x_t^2} = \frac{\gamma x_t E(x_t\beta_t\varepsilon_t + \varepsilon_t^2)}{1 + \gamma x_t^2}$$

$$= \frac{\gamma x_t \sigma_\varepsilon^2}{1 + \gamma x_t^2} = \frac{x_t\sigma_u^2}{1 + \gamma x_t^2}, \text{ since } \gamma = \frac{\sigma_u^2}{\sigma_\varepsilon^2}.$$

Hence, $E[(\hat{\beta}_t - \beta_t)y_t] = \frac{-x_t\sigma_u^2}{1 + \gamma x_t^2} + \frac{x_t\sigma_u^2}{1 + \gamma x_t^2} = 0.$

Q.E.D.

<u>Theorem 2</u>: Let $\tilde{\beta}_t$ be any estimator of β_t that is linear in y_t and unbiased. Then,

$$E(\tilde{\beta}_t - \beta_t)^2 - E(\tilde{\beta}_t - \beta_t)^2 \geq 0.$$

Proof: Since $\tilde{\beta}_t$ and $\hat{\beta}_t$ are linear in y_t, we may write

$$\tilde{\beta}_t = \hat{\beta}_t + ay_t + b,$$

where a and be are real numbers.

Then,

$$E(\tilde{\beta}_t - \beta_t)^2 = E[(\hat{\beta}_t - \beta_t) + (ay_t + b)]^2$$

$$= E(\tilde{\beta}_t - \beta_t)^2 + E(ay_t + b)^2 + 2E[(\tilde{\beta}_t - \beta_t)(ay_t + b)]$$

$$= E(\tilde{\beta}_t - \beta_t)^2 + E(ay_t + b)^2,$$

where the cross-product term equals to zero by Theorem 1.

Thus, $E(\tilde{\beta}_t - \beta_t)^2 \geq E(\tilde{\beta}_t - \beta_t)^2$,

since $E(ay_t + b)^2 \geq 0$.

Q.E.D.

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