BEECROFT'S

GENERAL METHOD

OF

FINDING ALL THE ROOTS,

BOTH REAL AND IMAGINARY,

OF

ALGEBRAICAL EQUATIONS,

WITHOUT

THE AID OF AUXILIARY EQUATIONS

OF HIGHER DEGREES.

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Beecroft, Philip C.

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PREFACE.

THE publication of the following researches, comprehending a new and general method of solving numerical equations, &c., has been hastened by the surprise, not unmixed with the appearance of doubt, expressed by any gentleman to whom the author happened to mention that he possessed a method of simultaneously obtaining all the coefficients of a quadratic factor of any equation, in the same form; as all the writers on this subject, and to no branch of mathematics has more attention been paid, have only succeeded, up to the present time, in giving methods for obtaining them as functions of each other, eliciting, by elimination, an equation of the $\frac{n(n+1)}{2}$th order for their separate determination when the proposed equation is of the nth. This uncertainty is however easily removed by publishing a small work containing a portion of my investigations on equations, reserving the rest to appear in a larger work, which, if I meet with sufficient encouragement, I hope to produce at a future period; and it was for this reason that I was led to issue a few circulars amongst my friends and those gentlemen with whose correspondence on mathematical subjects I had been previously favoured, and afterwards to such of their friends and other gentlemen distinguished for scientific attainments as they thought likely to take an interest in the subject, in order to ascertain nearly the number of copies likely to be required, so as not to incur the risk of any pecuniary loss by publishing my first work at the price announced in the circular.

The establishment of a universal method for finding the roots of any equation independent of their depression, as this method may be made, will fully disprove the opinion that seems to have been generally held by analysts that depression and continuous approximation are one and the same; the idea of which must have originated Horner's method of solution, which indeed is nothing more than an elegant mode of depressing the roots of an equation leading to their determination, when real, to any degree of exactness. It was, however, a most valuable improvement in the theory of equations. The following method also distinguishes itself from every other yet given, in its process being always the same for any equation, and in the values of the coefficients of the quadratic factors, &c., being directly approximated to, without a previous knowledge of the nature of the roots, whether real or imaginary.

The other method following the general one in this work, though not of so great efficacy for finding the quadratic factors of an equation as the general method, is yet better adapted than either it, or than any other known method for finding the value of any particular root to a great degree of exactness. It may also be interesting as enabling us to express the roots of an equation in a form of series similar to the common trigonometrical expressions for the signs and cosines of an arc in terms of its length.

The short chapter on the solution of equations by general algebraical formulas, contains only a few of the least extensive of the formulæ which I have obtained for the reduction of the higher equations, as the discussion of the more extensive ones would have interfered too much with the design of the present work, which is intended to establish, chiefly, a new method of approximation, that will be of much greater practical utility, and may probably prove more interesting, than any formulæ that might possibly be obtained for the solution of equations by combinations of algebraic symbols. The method of solving equations of the fifth degree, given at the conclusion, appears, to me, to controvert the opinion generally entertained, of the impossibility of solving the general equation of the fifth degree by the help of an auxiliary equation below the sixth. Professor Young, after remarking, in page 469 of his work on the higher equations, that no such solution has yet been devised, adds, that "It has indeed been the object of analysts of late to demonstrate the utter impossibility of such a solution by any combination of algebraic symbols; on which subject the student is referred to the profound paper of Sir William Hamilton in the *Transactions of the Royal Irish Academy, Vol.* XVI." The method, however, which I have added, solves completely the general equations of the fifth degree, without using any auxiliary equation of a higher order, and that by a process equally as definite as the general method of solving a cubic equation complete in all its terms, though of course more laborious, the expressions for p, q, &c., in the solution, being merely assumed for greater convenience in forming the cubic by help of which four of the roots are determined, after first determining one real root, as in Dr. Rutherford's method of solving an equation of the fifth degree deprived of its second term by help of an auxiliary equation of the tenth order.—"*Ruth. Eq., pages* 20, 21." This method of solving equations of the fifth order is not likely, however, to be considered very interesting, after a method depending on so neat an algebraic form, and so general in its application as that contained in this work for determining the quadratic factors of any equation, but it may aid in directing attention to new methods of algebraical reduction.

PHILIP BEECROFT.

Hyde,
August 15th, 1854.

NEW METHOD

OF

RESOLVING NUMERICAL EQUATIONS.

1. The following general method is founded upon the principle that imaginary roots greater than their reciprocals and real roots greater than unity have identical properties, and are capable of being determined by the same common process, and, it is therefore evident, that, it will always be necessary first to determine the number of roots in an equation exceeding their reciprocals in magnitude; for this purpose we may make use of the following simple rule.

Substitute x_1+p, and $\dfrac{1}{x_2+p}$, respectively for the root (x) of the proposed equation, then, if the equation resulting for x_1 contains m variations of sign from $+$ to $-$ and $-$ to $+$ amongst the signs of its coefficients, and the equation for x_2 contains n variations amongst the signs of its coefficients, the equation proposed will contain $m-n$ roots greater than their reciprocals; and, by decreasing the value of p, and taking the equation for both positive and negative values of x, we can thus determine all the roots greater than their reciprocals, and also the limits between which they are placed.

This method depends upon an extension of the well-known rule of signs, and will be at once comprehended by the practised analyst, but, as other methods will be hereafter given, it may be omitted by the reader until he has perused the examples commencing in the third page of this work, the operations in which are so simple that, with a previous knowledge of the first four rules of common algebra only, he will find no difficulty in understanding them.

Every algebraical equation may be made to take the form

$$x^n + ax^{n-1} + bx^{n-2} + cx^{n-3} + \&c. \ldots\ldots\ldots\ldots = 0 \ldots\ldots\ldots\ldots (1)$$

and if this equation be multiplied by the polynomial

$$x^m + p_1 x^{m-1} + p_2 x^{m-2} + \ldots\ldots\ldots\ldots p_{m-1}x + p_m,$$

we get, putting $m+n=r$,

$$x^r + (p_1+a)x^{r-1} + (p_2+ap_1+b)x^{r-2} + \&c. \ldots\ldots\ldots = 0 \ldots\ldots\ldots (2)$$

Multiplying (1) by x^m, and substracting the product from (2), there remains

$$p_1x^{r-1} + (p_2+ap_1)x^{r-2} + (p_3+ap_2+bp_1)x^{r-3} + \&c. \ldots\ldots = 0 \ldots\ldots (3)$$

and, if we take the coefficients

$$p_2+ap_1=0,\ p_3+ap_2+bp_1=0, \ldots \text{to } p_m+ap_{m-1}+bp_{m-2}+\&c.=0, \ldots (a)$$

it then becomes

$$p_1x^{r-1} + (ap_m+bp_{m-1}+cp_{m-2}+\ldots\ldots)x^{n-1} + (bp_m+cp_{m-1}+\ldots\ldots)x^{n-2} + \&c. = 0, \ldots\ldots (4)$$

or, dividing by the coefficient (p_1) of the first term, and then denoting the coefficients of the other terms by a_m, b_m, c_m, &c..., it will be

$$x^{r-1} + a_m x^{n-1} + b_m x^{n-2} + c_m x^{n-3} + \&c. \ldots\ldots\ldots = 0 \ldots\ldots\ldots (5)$$

Now, if we form three equations from this, by substituting $m-1$, $m-2$, $m-3$, successively for m, and then eliminate the terms containing x^{n-1}, x^{n-2}, from these three equations, we get, after a little reducing, an equation of the form

$$(Rx^2+Sx+T)x^{r-1} + u_3x^{n-3} + u_4x^{n-4} + \&c \ldots\ldots\ldots = 0 \ldots\ldots\ldots (6)$$

in which x^{r-1} will always be the highest of the powers of x, since $r=m+n$, and, by dividing by this power, we immediately perceive that when x is greater than unity or than its reciprocal, the farther we extend the calculation, or the higher we take m, the nearer will the part within the parenthesis approximate in value to zero; and hence we may take for an approximation the quadratic

$$Rx^2+Sx+T=0, \ldots\ldots\ldots\ldots\ldots (7)$$

in which the coefficients obtained by the above mentioned operation, will be found to have the values,

$$R = a_{m-3}b_{m-2} - a_{m-2}b_{m-3},\quad S = a_{m-1}b_{m-3} - a_{m-3}b_{m-1},$$
$$T = a_{m-2}b_{m-1} - a_{m-1}b_{m-2}$$

Putting $m-1$, for m, in (5), and then multiplying by x, we get

$$x^{r-1}+a_{m-1}x^{n}+b_{m-1}x^{n-1}+c_{m-1}x^{n-2}+\&c.\dots\dots=0\dots\dots(8)$$

and, eliminating x^n from this equation, by means of (1), we get

$$x^{r-1}+(b_{m-1}-aa_{m-1})x^{n-1}+(c_{m-1}-ba_{m-1})x^{n-2}+\&c.\dots=0\dots(9)$$

Making the last equation identical with (5) we shall have

$$a_m=b_{m-1}-aa_{m-1},\text{ or }b_{m-1}=a_m+aa_{m-1}\dots\dots\dots(10)$$

similarly $b_{m-2}=a_{m-1}+aa_{m-2}$, $b_{m-3}=a_{m-2}+aa_{m-3}$.

Substituting the three last values in the above expressions for R, S, T, we readily get

$$R=a_{m-1}a_{m-3}-a_{m-2}^2,\ S=a_{m-1}a_{m-2}-a_ma_{m-3},\ T=a_ma_{m-2}-a_{m-1}^2\,..(b)$$

Put $p_m=a_{m-1}p_1$, for any value of m, then equations (a) become

$$a_1+a=0,\ a_2+aa_1+b=0,\ a_3+aa_2+ba_1+c=0,\ \&c.\dots\dots\dots(c)$$

and since these relations would result by taking

$$(1+ax+bx^2+cx^3+\&c.\dots)(1+a_1x+a_2x^2+a_3x^3+\&c.\dots)=1,$$

we shall have from (7) and the last values of R, S, T, the following simple method of finding the quadratic factors of an equation.

R_2. Let the proposed equation be transformed into one containing two roots greater than their reciprocals, and let the transformed equation be denoted by

$$x^n+ax^{n-1}+bx^{n-2}+cx^{n-3}+\&c.\dots\dots=0.$$

Then, if we put $f(x)=1+ax+bx^2+cx^3+\&c.\dots\dots\dots\dots$,

$$\text{and }\frac{1}{f(x)}=1+a_1x+a_2x^2+a_3x^3+\&c.\dots\dots x_m a^m,$$

the quadratic factor containing the two roots exceeding their reciprocals, may be found to any degree of exactness, by calculating the values of a_m, by common algebraical division, to any extent, and substituting them in the equation

$$(a_{m-1}a_{m-3}-a_{m-2}^2)x^2+(a_{m-1}a_{m-2}-a_ma_{m-3})x+a_ma_{m-2}-a_{m-1}^2=0\dots,\quad(11)$$

EXAMPLE I.

To find a quadratic factor of $x^5-3x^4+6x^3+1=0$.

This equation needs no transformation, as it already contains two roots exceeding their reciprocals, and dividing unity by

$1-3x+6x^2+x^5=f(x)$, we easily get $\frac{1}{f(x)}=1+3x+3x^2-9x^3-45x^4-82x^5+21x^6+552x^7+1539x^8+$&c., and taking the coefficients of x^5, x^6, x^7, x^8, we shall have, $m=8$,

$$a_5=-82,\ a_6=21,\ a_7=552,\ a_8=1539,$$

$$a_5a_7-a_6^2=-45705,\ a_6a_7-a_5a_8=137790,\ a_6a_8-a_7^2=-272385,$$

Hence "(11)" we have the approximate quadratic factor

$$45705x^2-137790x+272385=0,$$

or $$x^2-3{\cdot}014768x+5{\cdot}95963=0.$$

Similarly, by advancing the division another step, and taking $m=9$, we get

$$30265x^2-91242x+180369=0,$$

or $$x^2-3{\cdot}0147695x+5{\cdot}959656=0.$$

Which is the same as the preceding quadratic for five or six places of figures, showing the former quadratic to be correct to about that extent, and the latter being obtained at a step more in advance of the operation is probably correct to the last figure.

EXAMPLE II.

Solve the equation $x^4-18x^3+80x^2+1=0$.

This equation has two roots greater than their reciprocals, and we have

$$\frac{1}{1-18x+80x^2+x^4}=1+18x+244x^2+2952x^3+33615x^4+368892x^5+\text{\&c.}$$

Here $a_2=244$, $a_3=2952$, $a_4=33615$, $a_5=368892$;

$$a_3^2-a_2a_4=512244,\ a_3a_4-a_2a_5=9221832,\ a_4^2-a_3a_5=40999041,$$

and $\therefore$ $512244x^2-9221832x+40999041=0$, nearly,

or $$56916x^2-1024648x+4555449=0.$$

Solving this quadratic we get the two real roots

$$8{\cdot}007829\ldots,\ \text{and}\ 9{\cdot}9949823\ldots,$$

the greater of which is correct to the last figure inclusive, and the lesser value is nearly as exact, the true root being 8·007828......

The two roots less than their reciprocals may be found in the same manner by taking the reciprocal equation, these two roots being, in it, greater than their reciprocals, and we shall then have

$$\frac{1}{1+80x^2-18x^3+x^4}=1-80x^2+18x^3+6399x^4-2880x^5-\&c\ldots,$$

Here $a_2=-80,\ a_3=18,\ a_4=6399,\ a_5=-2880,$

or, dividing these by 9 we may take

$$a_2=-8\tfrac{8}{9},\ a_3=2,\ a_4=711,\ a_5=-320;$$

then $a_2a_4-a_3^2=-6324,\ a_3a_4-a_2a_5=-1422\tfrac{4}{9},\ a_3a_5-a_4^2=-506161.$ Hence, reversing the coefficients, since we have been finding the roots exceeding their reciprocals for the reciprocal equation, we have

$$506161x^2+1422\tfrac{4}{9}x+6324=0,$$

Which contains two roots of the equation proposed correct to six decimals.

Obs. In the above solution it will be seen that the approximation advances the most rapidly for the greater of the two roots exceeding their reciprocals, which will always be the case whether the roots sought be real or imaginary; in the latter case, when the equation contains more than one pair of imaginary roots, the approximation will be most rapid for that pair the greatest in proportion to their reciprocals.

EXAMPLE III.

To determine the quadratic factors of the equation

$$x^5+5x^3-10x^2+5x-1=0.$$ "*Young's Math. Dis., p.* 154."

Since the positive coefficients have the same aggregate as the negative ones, one root is unity, and dividing by $x-1$, we get

$$x^4+x^3+6x^2-4x+1=o\ldots\ldots\ldots\ldots(12)$$

This equation has two roots greater than their reciprocals, and here we shall have $f(x)=1+x+6x^2-4x^3+x^4$, and

$$\frac{1}{f(x)}=1-x-5x^2+15x^3+10x^4-119x^5+124x^6+615x^7-1845x^8-\&c.$$

Here $a_7=615,\ a_8=-1845,\ a_9=-1230,\ a_{10}=14636,$

or, we may put $a_7=u,\ a_8=-3u,\ a_9=-2u,\ a_{10}=14636$, and substituting these values in the expressions (b) we get

$$R=-11u^2,\ S=(6u-14636)u,\ T=-(43908+4u)u,$$

$$\therefore\quad 11ux^2+(14636-6u)x+43908+4u=0, \text{ nearly,}$$

or, restoring the value of $u=615$, this becomes

$$6765x^2+10946x+46368=0,$$

or $$x^2+1{\cdot}61803398x+6{\cdot}85410199=0.$$

The reciprocal equation of (12) has also two roots exceeding their reciprocals, and putting $f(x)=1-4x+6x^2+x^3+x^4$, and proceeding as before, we get $a_7=-1230$, $a_8=-1845$, $a_9=615$, $a_{10}=15251$, and substituting these in the equation (11) when $m=10$, and dividing by 615, we get the approximate quadratic

$$46368x^2-28657x+6765=0,$$

or $$x^2-\cdot618033989x+\cdot145898033=0.$$

Advancing the process a few more steps we obtain the quadratic

$$832040x^2-514229x+121393=0.$$

or $$x^2-\cdot6180339887505x+\cdot145898033748=0.$$

This may be proved by means of the following finite factor of the equation proposed,

$$x^2-\frac{\sqrt{5}-1}{2}x+\frac{7-3\sqrt{5}}{2}=0.$$

EXAMPLE IV.

Solve the equation $x^4-4x^3+8x^2-16x+20=0$. "*L. & G. Diary for* 1846, *p.* 96."

Substituting $x=y+2$. we get

$$y^4+4y^3+8y^2+4=0\ldots\ldots\ldots\ldots(13)$$

which has two roots greater and two less than their reciprocals, and to find the lesser roots, we may take the reciprocal equation, since the corresponding roots in it will be greater than their reciprocals, and putting $f(y)=4+8y^2+4y^3+y^4$, we get, by division

$$\frac{4}{f(y)}=1-2y^2-y^3+\frac{15}{4}y^4+4y^5-6y^6-\frac{46}{4}y^7+\frac{113}{16}y^8+28y^9-\frac{9}{8}y^{10}-\&c.\ldots\ldots$$

Here $a_7=-\frac{23}{2}$, $a_8=\frac{113}{16}$, $a_9=28$, $a_{10}=-\frac{9}{8}$, and substituting these in the equation

$$a_7a_9-a_8^2+(a_8a_9-a_7a_{10})y+(a_8a_{10}-a_9^2)y^2=0,\ \ldots\ldots\ldots(14)$$

we get $$95201-47312y+202738y^2=0,$$

or dividing by the coefficient of y^2, we get

$$y^2-\cdot23336523y+\cdot469576=0.$$

By proceeding another step, we get

$$431748y^2-100755y+202738=0,$$

or, $$y^2-\cdot23336529y+\cdot46957485=0.$$

Substituting $y=x-2$, in the last quadratic, we get

$$x^2-4{\cdot}23336529x+4{\cdot}93630543=0.$$

Which, I believe, will give two roots of the proposed equation correctly to the eighth decimal.

When we make use of the reciprocal equation, to find the roots of the proposed one, less than their reciprocals, as in the last examples, we must then reverse the coefficients in equation (11) and take for the approximate quadratic factors containing two roots of the proposed equation less than their reciprocals, the equation

$$(a_{m-1}^2-a_m a_{m-2})x^2+(a_m a_{m-3}-a_{m-1}a_{m-2})x+a_{m-2}^2-a_{m-1}a_{m-3}=0, \quad (15)$$

Thus, taking $m=10$, we have

$$(a_9^2-a_8 a_{10})x^2+(a_7 a_{10}-a_8 a_9)x+a_8^2-a_7 a_9=0,$$

the same as (14), when x is substituted for y.

The two roots greater than their reciprocals in the last example may be approximated to with great ease and rapidity by taking $f(y)=1+4y+8y^2+4y^4$, from (13), and proceeding as before.

EXAMPLE V.

To find a quadratic factor of the equation $x^6-6x+6=0$.

Putting $x=x_1+1$, this equation becomes

$$x_1^6+6x_1^5+15x_1^4+20x_1^3+15x_1^2+1=o.$$

The coefficient of x is ± 0, and, taking the lower sign, it is immediately seen that the equation has two roots less than their reciprocals, and putting

$$f(x_1)=1+15x_1^2+20x_1^3+15x_1^4+6x_1^5+x_1^6$$

and dividing unity by this, as before, we find the coefficients of x_1^7, x_1^8, x_1^9, x_1^{10}, to be

$$a_7=-12720,\ a_8=22995,\ a_9=231170,\ a_{10}=-56409.$$

Taking $m=10$, in (15), and substituting these values, we get

$$10947338771x_1^2-919646334x_1+693850485=0,$$

or $$x_1^2-{\cdot}08400638x_1+{\cdot}06388074=0,$$

which I have proved to be exact to the seventh place of decimals at least, and by restoring the value of x, we immediately obtain a quadratic factor of the equation proposed.

EXAMPLE VI.

To find two roots of the equation

$$x^6-6x^5+40x^3+60x^2-x-1=0. \quad \text{“*Young Eq.*, p. 200.”}$$

This equation has two roots less than their reciprocals, and we have

$$\frac{1}{1+x-60x^2-40x^3+6x^5-x^6}=1-x+61x^2-81x^3+3701x^4$$
$$-6127x^5+224594x^6-444901x^7+\&c.$$

Here $a_4=3701$, $a_5=-6127$, $a_6=224594$, $a_7=-444901$, and substituting those values in (15) for $m=7$, we get

$$47878393689x^2-268285443x-795014625=0,$$

or $$x^2-\cdot005603476x-\cdot016604872=0 \ldots\ldots\ldots\ldots(16)$$

By extending the above division another term, we find the same as the last equation to the extent here given, showing its correctness to the ninth place of decimals; and if the division had been carried only to the seventh term, we should have got the equation

$$795014625x^2-4454753x-13201114=0,$$

or $$x^2-\cdot00560336x-\cdot01660487=0,$$

which quadratic is the next in degree of correctness to (16). Solving the quadratic (16) we obtain the two roots

$$x=\cdot131692087, \quad x=-\cdot126088611.$$

EXAMPLE VII.

Solve the equation $x^6+x^5-x^4-x^3+x^2-x+1=0$.

Decreasing the roots by unity we get the equation

$$x_1^6+7x_1^5+19x_1^4+25x_1^3+17x_1^2+5x_1+1=0 \ldots\ldots\ldots(17)$$

where $x_1=x-1$.

The last equation has two roots less than $\frac{1}{3}$, and since its reciprocal equation contains none so small, the equation (17) contains two roots less than their reciprocals, and putting

$$f(x)=1+5x+17x^2+25x^3+19x^4+7x^5+x^6,$$

$$\frac{1}{f(x)}=1-5x+8x^2+20x^3-130x^4+198x^5+602x^6-\&c\ldots$$

$$\therefore \quad a_3=20, \ a_4=-130, \ a_5=198, \ a_6=602,$$

or dividing by 2, since they all contain this common factor, we may take $\quad a_3=10, \ a_4=-65, \ a_5=99, \ a_6=301.$

$$a_5^2 - a_4a_6 = 29366,\ a_3a_6 - a_4a_5 = 9445,\ a_4^2 - a_3a_5 = 3235,$$

and $\therefore\ 29366x_1^2 + 9445x_1 + 3235 = 0$, nearly.

And restoring the value of x by substituting $x_1 = x - 1$, we get

$$29366x^2 - 49287x + 23156 = 0,$$

or $$x^2 - 1{\cdot}67837x + {\cdot}78855 = 0.$$

Carrying the process as far as a_8, we obtain the quadratic

$$9688603x_1^2 + 3116068x_1 + 1066690 = 0,$$

and restoring the value of x, this gives

$$9688603x^2 - 16261138x + 7639225 = 0,$$

or $$x^2 - 1{\cdot}6783779x + {\cdot}788475 = 0 \ldots\ldots\ldots(18)$$

The two last quadratic factors for x, show the former of them to be correct to about the fifth decimal, and the latter, being obtained at a more advanced stage of the same process, is probably exact to the sixth decimal or more. It will however be otherwise tested hereafter.

For two other roots take the reciprocal equation and x negative, then

$$x^6 + x^5 + x^4 + x^3 - x^2 - x + 1 = 0 \ldots\ldots\ldots(19)$$

and putting $x = x_1 + 1$, this becomes

$$x_1^6 + 7x_1^5 + 21x_1^4 + 35x_1^3 + 33x_1^2 + 15x_1 + 3 = 0 \ldots\ldots(20)$$

This equation has two negative roots less than their reciprocals, and putting

$$f(x_1) = 3 + 15x_1 + 33x_1^2 + 35x_1^3 + 21x_1^4 + 7x_1^5 + x_1^6,\ \text{then}$$

$$\frac{3}{f(x_1)} = 1 - 5x_1 + 14x_1^2 - \frac{80}{3}x_1^3 + \frac{92}{3}x_1^4 + \frac{28}{3}x_1^5 - \frac{1436}{9}x_1^6 - \&c.,$$

and multiplying these coefficients by $\frac{9}{4}$ to clear them of fractions, the coefficients of the four last terms exhibited, will then be

$$a_3 = -60,\ a_4 = 69,\ a_5 = 21,\ a_6 = -359,$$

and $a_4^2 - a_3a_5 = 6021,\ a_3a_6 - a_4a_5 = 20091,\ a_5^2 - a_4a_6 = 25212,$

$\therefore\ 25212x_1^2 + 20091x_1 + 6021 = 0$, nearly "(15)."

Substituting $x_1 = x - 1$, in this equation, we get

$$25212x^2 - 30333x + 11142 = 0,$$

which is an approximate quadratic of equation (19), and therefore, for an approximate quadratic factor of the proposed equation, we

shall have $$11142x^2+30333x+25212=0,$$
or $$x^2+2{\cdot}723x+2{\cdot}262=0.$$

By proceeding a step farther in the last division for finding the values of a_m, we get, for the next approximation,

$$186886x^2+508385x+422347=0,$$
or $$x^2+2{\cdot}7203x+2{\cdot}2599=0\ldots\ldots\ldots(21)$$

Extending the last division by $f(x_1)$, we may easily obtain the quadratic factor to a much greater degree of correctness, but all the factors of the equation proposed will be given more exactly hereafter, in illustrating a subsequent method of transformation.

2. In the preceding examples the operation for the division of unity by $f(x)$ has not been exhibited, as the labour would not be very great, for these examples, by the common method of algebraical division, but, it will generally be found most convenient to develop the coefficients a_m successively, in the following manner:

$$\begin{array}{cccccc}
-a & -b & -c & -d & -e & -\&c\ldots\ldots\ldots \\
 & -a_1a & -a_1b & -a_1c & -a_1d & -\ldots\ldots\ldots\ldots \\
\hline
 & a_2 & -a_2a & -a_2b & -a_2c & -\ldots\ldots\ldots\ldots \\
\hline
 & & a_3 & -a_3a & -a_3b & -\ldots\ldots\ldots\ldots \\
\hline
 & & & a_4 & -a_4a & -\ldots\ldots\ldots\ldots \\
\hline
 & & & & a_5 & \\
 & & & & & \&c\ldots\ldots\ldots\ldots(b)
\end{array}$$

in which the first line contains the coefficients, with their signs changed, of the divisor, and the second line is the product of the first by $a_1(=-a)$; then a_2 is the sum of the quantities above it in the same column, and the third line is formed by multiplying the first by a_2; then a_3 is the sum of the quantities above it in the same column, and the fourth line is formed by multiplying the first by a_3; a_4 is the sum of the quantities above it in the same column, and the fifth line is obtained by multiplying the first by a_4; and so on, we may continue determining the coefficients a_5, a_6, a_7, &c... , to any extent.

The above process is at once suggested by the equation (c), but, another method of obtaining the coefficients a_m, when the coefficient of the first term of the equation is not unity, or when some of the others *a*, *b*, *c*, *d*, &c. contain several places of figures, will be subsequently given in the present work.

EXAMPLE VIII.

To solve the equation $3x^4-4x^3+10x^2+x+1=0$.

This equation has two roots less and two roots greater than their reciprocals, and to find the two lesser roots, we must take the reciprocal equation

$$x^4+x^3+10x^2-4x+3=0.$$

and the operation, according to the above method of division, will be

-1	-10	$+4$	-3					
	1	$+10$	-4	$+3$				
	-9	9	$+90$	-36	$+27$			
		23	-23	-230	$+92$	-69		
			60	-60	-600	$+240$	-180	
			$a_5=$	-323	323	$+3230$	-1292	$+$&c.
					-158	158	$+1580$	$-\ldots$
						3559	-3559	$-\ldots$
							-3451	

which gives $a_5=-323$, $a_6=-158$, $a_7=3559$, $a_8=-3451$, and substituting these values in the equation

$$(a_7^2-a_6a_8)x^2+(a_5a_8-a_6a_7)x+a_6^2-a_5a_7=0. \quad \text{“(15)”}$$

we get $$12121223x^2+1676995x+1174521=0,$$

or $$x^2+\cdot13835196x+\cdot09689789=0\ldots\ldots\ldots(22)$$

By extending the process another step, we get $a_9=31802$, and substituting as before, we have

$$125092719x^2+17306825x+12121223=0,$$

or $$x^2+\cdot138351978x+\cdot096897901=0\ldots\ldots\ldots(23)$$

Comparing the equations (22), (23), the former is evidently correct to the seventh place of decimals, and the latter will be found to be so to the last figure.

EXAMPLE IX.

Take the equation $x^4-12x^2+12x-3=0$. "*Bonnycastle's Alg. Biq. Eq. Ex. 7.*"

This equation contains two roots greater than their reciprocals, and applying the process (b) we have

$$\begin{array}{rrrrrrrrl}
0+12 & -12 & +3 & & & & & & \\
 & 0 & +144 & -144 & +36 & & & & \\
\hline
 & -12 & 0 & -144 & +144 & -36 & & & \\
 & & 147 & 0 & +1764 & -1764 & +441 & & \\
 & & & -288 & 0 & -3456 & +3456 & -864 & \\
 & & & & a_6=1944 & 0 & +23328 & -23328 & -\&c. \\
 & & & & & a_7=-5256 & 0 & -63072 & +\ldots. \\
 & & & & & & a_8=27225 & 0 & \\
 & & & & & & & a_9=-87264 &
\end{array}$$

and, dividing the last four values of a_m by 9, we may take

$$a_6=216,\ a_7=-584,\ a_8=3025,\ a_9=-9696,$$

and, substituting these in equation (11), when $m=9$, we get

$$312344x^2+327736x-3488161=0,$$

which contains a positive and a negative root of the proposed equation correct to about six places of figures.

The degree of approximation is not so rapid in this example as in some of the preceding ones, from the roots not being so much in excess of their reciprocals, but the operation may be extended with little trouble, or we may transform the equation into others better adapted for rapidly approximating to the roots by this method, which can be easily done by applying some of the following methods of transformation.

ON TRANSFORMATION.

All the methods of approximation apprehended by other writers on equations have been made, I believe, to depend upon the restricted principle, that *the rapidity of approximation is increased in proportion as the root is decreased;* but this is only a simple consequence and part of a general principle, which may be thus enunciated:—*The rapidity of approximation is increased in proportion as the quotients of the roots divided by their reciprocals are made to decrease or increase according as they are less or greater than their reciprocals.*

3. To increase the greatest roots of an equation.

Find m such a quantity that when $\frac{x_1+1}{m}$, $\frac{x_2+1}{m-1}$, are respectively substituted for x in the proposed equation, the equations for x_1 and x_2, shall each contain the same number of variations of sign from $+$ to $-$ and from $-$ to $+$ amongst their respective sets of coefficients, then the greatest roots will be increased in the equation for x_1.

For, by these assumptions $mx=x_1+1$, $(m-1)x=x_2+1$,
and, by subtraction, $x=x_1-x_2$............(24)
Therefore the greatest roots of the equation for x_1, exceed those of the proposed one, by the corresponding greatest roots of the equation for x_2.

4. To decrease the least roots of an equation.

Find m such a quantity that when the above substitutions are made for x, the equation for x_1 shall contain p more variations amongst the signs of its coefficients than shall subsist amongst the signs of the coefficients of the equation for x_2, then the p least roots of the equation for x shall be decreased in the equations for x_1 and x_2.

For, in this case the least roots will be positive quantities in the equation for x_1, and negative ones in that for x_2, and when x_2 is negative, (24) becomes $x = x_1 + x_2$............(25)

Hence, the least roots of the proposed equation will then be respectively equal to the sum of their corresponding roots in the equations for x_1 and x_2, and consequently these roots will be diminished in both these equations.

Obs. The equation for x_1 may be easily obtained by multiplying the coefficients of x^{n-1}, x^{n-2}, x^{n-3}, &c..., by m, m^2, m^3, &c..., respectively, and then decreasing the roots of the equation, with enlarged coefficients, by unity.

Cor. If we take m the least integer that will cause no change in the number of variations of sign amongst the coefficients of the equation for x_1, from the number of variations amongst the coefficients of the proposed equation, the separation of the least from the greatest roots will be widened in the equations for x_1 and x_2, and these equations will be better adapted for rapid approximation by this method than the proposed one.

EXAMPLE X.

To find a quadratic factor of the equation

$$x^4 + x^3 + 4x^2 - 4x + 1 = 0. \quad \text{"*Ruth. Eq. p.* 16."}$$

This equation has two roots greater than their reciprocals, and we find that by taking $m=3$, two variations of sign are lost in the equation for x_2, and none in that for x_1, and putting $x = \frac{x_1 + 1}{3}$,

we get $x_1^4 + 7x_1^3 + 51x_1^2 - 23x_1 + 13 = 0$.........(26)

And, applying the preceding method of division, we get

$$\frac{1}{1 + 7x + 51x^2 - 23x^3 + 13x^4}$$

$$= 1 - 7x - 2x^2 + 394x^3 - 2830x^4 - 239x^5 + 155091x^6 - 1143660x^7 - \&c.$$

Here $a_4 = -2830$, $a_5 = -239$, $a_6 = 155091$, $a_7 = -1143660$.

and substituting these values in the equation

$$(a_5^2 - a_4 a_6)x_1^2 + (a_4 a_7 - a_5 a_6)x_1 + a_6^2 - a_5 a_7 = 0, \quad \text{"(11)."}$$

we get the approximate quadratic factor

$$438964651x_1^2 + 3273624549x_1 + 23779883541 = 0,$$

and, substituting $x_1=3x-1$, this gives

$$3950681859x^2+7187085741x+20945223643=0,$$

or $$x^2+1{\cdot}81920134x+5{\cdot}30167307=0 \ldots\ldots\ldots (27)$$

which will give the values of two roots of the equation proposed to eight places of decimals, and, by extending the above process, we may easily obtain these roots to a much greater degree of exactness.

If we had substituted $x=\frac{1+x_2}{2}$, we should then have had

$$x_2^4+6x_2^3+28x_2^2+10x_2+3=0, \ldots\ldots\ldots\ldots (28)$$

and, proceeding with this equation in the same manner as we have above made use of in finding the quadratic factor for the equation x_1, we easily get, at the same step

$$3125506x_2^2+17622861x_2+80778996=0,$$

and substituting $x_2=2x-1$, in this equation, we get

$$12502024x^2+22743698x+66281641=0,$$

or $$x^2+1{\cdot}81920127x+5{\cdot}30167283=0 \ldots\ldots\ldots (29)$$

Comparing (27) with (29), they are seen to agree to nearly the last figures of the coefficients.

In the last example, the other biquadratic factor may be directly found by operating at once upon the reciprocal equation of that proposed, that is, by taking

$$f(x)=1-4x+4x^2+x^3+x^4,$$

and proceeding as before, which will give the approximate factors with tolerable rapidity of approximation; but the degree of approximation will be much more rapid if we take the reciprocal equations of (26) or (28), of which the last will involve less trouble than the other in the division part of the operation.

EXAMPLE XI.

To find a quadratic factor of the equation

$4x^7-6x^6-7x^5+8x^4+7x^3-23x^2-22x-5=0$. "*Young's Eq. p.* 201."

This equation contains two positive roots less than their reciprocals, and decreasing the negative roots by putting $-x=\frac{x_1+1}{2}$, we get the equation $x_1^7+10x_1^6+32x_1^5+29x_1^4-11x_1^3+168x_1^2+26x_1+1=0$...(30), and dividing unity by this function of x, we find the coefficients of x^3, x^4, x^5, x^6, in the result to be,

$$a_3=-8829,\quad a_4=143895,\quad a_5=-2251688,\quad a_6=34258499,$$

and substituting these in (15), when $m=6$, we get the equation

$$140472135739x_1^2+21538357089x_1+825617673=0,$$

and substituting $x_1=2x-1$, we get

$$561888542956x^2-518811828778x+119759396323=0,$$

Hence we get $x = \cdot4616679190935 \pm \sqrt{-000000013573...}$

$$= \cdot4616679190935 \pm \cdot000116503\sqrt{-1}.$$

5. Another method of forming an equation better adapted for rapidly approximating to the roots than the proposed one, consists in obtaining the equation for the squares of its roots, which can always be obtained by the multiplication of the equation itself by the form it takes when the alternate signs of its coefficients are changed.

Thus, if we multiply the general form of equations

$$ax^n+bx^{n-1}+cx^{n-2}+dx^{n-3}+\&c.........=0$$

by........... $ax^n-bx^{n-1}+cx^{n-2}-dx^{n-3}+\&c.........=0,$

we get an equation of the form

$$Ax^{2n}-Bx^{2n-2}+Cx^{2n-4}-Dx^{2n-6}+\&c.........=0,$$

where $A=a^2$, $B=b^2-2ac$, $C=c^2-2bd+2ae$, &c....., the same values for the coefficients of the equation for the squares of the roots as are commonly given in the elementary works on equations, and the process for obtaining these values may be conveniently arranged in the following manner:

$$\begin{array}{llllll}
a & +b & +c & +d & +e & +\&c...... \\ \hline
a^2 & +b^2 & +c^2 & +d^2 & +e^2 & +......... \\
 & -2ac & -2bd & -2ce & -2df & -......... \\
 & & & 2ae & +2bf & +2cg+......... \\
 & & & & -2ag & -2bh-......... \\
 & & & & & 2ak+......... \\
 & & & & & \&c......... \\ \hline
A & +B & +C & +D & +E & +\&c...............(\iota),
\end{array}$$

in which A, B, C, &c..., are the sums of the quantities above them in the same columns, excepting those in the first line.

The following expressions, derived from the preceding values, will be useful in finding the quadratic factors of an equation from those of the equation for the squares of its roots.

Taking $d=e=f=\&c.=0$, then we shall have

$$A=a^2,\quad B=b^2-2ac,\quad C=c^2,$$

and $$a=\sqrt{A},\quad c=\sqrt{C},\quad b=\sqrt{B+2ac}\ldots\ldots\ldots\ldots(d)$$

EXAMPLE XII.

Take the equation $x^4+x^3+4x^2-4x+1=0$. "*Ruth. Eq. p.* 16."

Operating according to the process (c) we shall have

$$\begin{array}{l} 1+1\ +4\ -4+1 \\ \hline 1+1+16+16+1 \\ \quad -8\ +8\ -8 \\ \qquad\quad 2 \\ \hline 1-7+26\ +8+1 \end{array}$$

Therefore, the equation for the squares of the roots of the proposed one will be $$y^4+7y^3+26y^2-8y+1=0\ldots\ldots\ldots(31)$$

This equation has evidently two roots less than their reciprocals, and to find the quadratic factor containing these two roots, put $f(y)=0$, for the equation, then

$$\frac{1}{f(y)}=1+8y+38y^2+89y^3-333y^4-5252y^5-34019y^6-133358y^7-\&c.$$

Here $a_4=-333$, $a_5=-5252$, $a_6=-34019$, $a_7=-133358$, and substituting these in the equation (15), when $m=7$, we get

$$456896145y^2-134259574y+16255177=0,$$

or $$y^2-{\cdot}29385140468y+{\cdot}0355774002=0,$$

Here $A=1$, $B={\cdot}29385140468$, $C={\cdot}0355774002$,

and "(d)" $a=1$, $c=\sqrt{C}={\cdot}1886197238$, $b=\pm{\cdot}8192013502$,

and we have, for an approximate quadratic true to the tenth decimal,

$$x^2-{\cdot}8192013502x+{\cdot}1886197238=0.$$

The other two roots may also be rapidly found by taking the equation for two roots greater than their reciprocals and proceeding as before.

EXAMPLE XIII.

Take $x^5-x^4+x^3+2x^2+2x+1=0$. "*Ruth. Eq. p.* 21."

Applying the process (𝔠) to the coefficients of this equation, we shall have

```
 1−1+1+2+2+1
─────────────
 1+1+1+4+4+1
  −2+4−4−4
      4−2
─────────────
 1−1+9−2+0+1
```

Therefore the equation for the squares of the roots is

$$y^5+y^4+9y^3+2y^2-1=0\text{............}(32)$$

This equation contains two roots exceeding their reciprocals, and performing the division of unity by $f(y)=1+y+9y^2+2y^3-y^5$, according to the process (𝔟), we have

```
−1−9−2 −0  +1
  1+9 +2  +0 −1
 −8  8+72 +16  +0   −8
     15−15−135 −30   −0  +15
         59 −59−531 −118   −0   +59
            −177  177+1593 +345   +0   −177
                 −385  385+3465 +770    −0 −
                  a₇=1852−1852−16668 −3704 −
                            1982 −1982−17838 −&c..
                                −17821  17821 −
                                        −3898
```

Here $a_7=1852$, $a_8=1982$, $a_9=-17821$, $a_{10}=-3898$, and substituting these values in eq. (11) when $m=10$, we get

$$36932816y^2+28102126y+325313877=0.$$

Similarly, by advancing the operation another step (to $a_{11}=159938$), we get $325313878y^2+247530858y+2865449502=0$.

Dividing the last equations by their first coefficients, we get the two equations $y^2+\cdot 76089854y+8\cdot 80826088=0$,

$$y^2+\cdot 760898552y+8\cdot 808260897=0,$$

which will give the same roots for eight places of figures. Hence the former of these two quadratics gives the roots of eq. (32), correctly to that extent, and the latter being obtained at a more advanced stage of the operation, will give the same two roots still more correctly.

Putting A=1, B=·760898552, C=8·808260897, we get from equations (*d*),

$$a=1,\ b=2\cdot 2748284178,\ c=2\cdot 9678714421.$$

$$\text{Therefore } x^2-2\cdot 2748284178x+2\cdot 9678714421=0.$$

It will be seen that the two roots given by this quadratic coincide with the values of the same roots obtained by Dr. Rutherford to seven places of decimals only, and this is owing to the roots he obtains for the equation of the tenth degree, used as an auxiliary in his solution, being correct for only seven places of figures, as may be seen by examining and proceeding with the determination of the root alluded to of his equation of the tenth order.

EXAMPLE XIV.

To solve the equation $x^6+x^5-x^4-x^3+x^2-x+1=0$. "*Young's Eq. p.* 235."

Operating in the manner of process (c) we have

$$\begin{array}{c} 1+1-1-1+1-1+1 \\ \hline 1+1+1+1+1+1+1 \\ 2+2+2-2-2 \\ 2-2-2 \\ -2 \\ \hline 1+3+5-1-3-1+1 \end{array}$$

Therefore the equation for the squares of the roots of the proposed equation is

$$y^6-3y^5+5y^4+y^3-3y^2+y+1=0 \ldots\ldots\ldots\ldots(33).$$

Which contains two roots exceeding their reciprocals, and performing the division of unity by $f(y)=1-3y+5y^2+y^3-3y^4+y^5+y^6$, according to the process (b), we shall have

3−5 −1 +3 −1 −1
9−15 −3 +9 −3 −3
4 12−20 −9 +12 −4 −4
−4−12+20 +4 −12 +4 +4
−32−96+160 +32 −96 +32 +32
−72−216+360 +72 −216 +72
−44−132+220 +44 −132
$a_7=241$ 723−1205 −241
919 2757−4595
1416 4248
−616

Here $a_7=241$, $a_8=919$, $a_9=1416$, $a_{10}=-616$, and substituting these in eq. (11), when $m=10$, we get

$$100661y^2-289952y+514232=0\text{.............}(34)$$

Extending the operation to a_{12}, and reducing by 8, we get

$$a_9=177,\ a_{10}=-77,\ a_{11}=-1126,\ a_{12}=-2850.$$

and substituting the last four values in (11) we get

$$205231y^2-591152y+1048426=0.$$

Or, dividing by the first coefficient

$$y^2-2{\cdot}8804225y+5{\cdot}1085167=0\text{............}(35)$$

If we divide the preceding quadratic by the coefficient of its first term, we get the same equation as the last for five places of figures in the coefficients, and since (35) was obtained by advancing the operation two more steps, it will evidently be correct to a still greater extent, probably to the last figure expressed.

Take y negative then (33) becomes

$$y^6+3y^5+5y^4-y^3-3y^2-y+1=0\text{............}(36)$$

This equation contains two roots less than their reciprocals, and decreasing these roots, according to the first method of transformation, by substituting $y=\dfrac{y_1+1}{2}$, we get the equation

$$y_1^6+12y_1^5+65y_1^4+152y_1^3+123y_1^2-36y_1+3=0,$$

and denoting this equation by $f(y_1)=0$, we get

$$\frac{3}{f(y_1)} = 1 + 12y_1 + 103y_1^2 + 693\tfrac{1}{3}y_1^3 + 3467\tfrac{1}{3}y_1^4 + 7698\tfrac{2}{3}y_1^5$$
$$-87185\tfrac{5}{9}y_1^6 - \&c.$$

Multiplying these coefficients by 3, we may take

$$a_2 = 309, \quad a_3 = 2080, \quad a_4 = 10402, \quad a_5 = 23096,$$

and substituting these values in the general equation (15) for $m=5$, we get $\quad 60161924y_1^2 - 14499496y_1 + 1112182 = 0.$

Similarly by proceeding another step we get

$$9762412988y_1^2 - 2352847376y_1 + 180485772 = 0,$$

or, $\qquad y_1^2 - \cdot 2410108y_1 + \cdot 0184878 = 0 \ldots\ldots\ldots\ldots (37)$

The preceding equation gives, when divided by its first coefficient,

$$y_1^2 - \cdot 2410078y_1 + \cdot 0184864 = 0,$$

which will give the same roots as (37) to five places of figures, and I believe (37) to be exact to the last figure; it can, however be easily tested by extending the above division another term.

The value of y in the proposed equation may be obtained by substituting $-\dfrac{y_1+1}{2}$ for y, or $y_1 = -1 - 2y$ in (37) and then we get

$$y^2 + 1\cdot 1205054y + \cdot 3148746 = 0 \ldots\ldots\ldots\ldots (38).$$

By substituting $y = x^2$, and applying the expressions (d) for finding the quadratic factors for x, the equations (35), (38), will give, respectively $\qquad x^2 + 2\cdot 7204463x + 2\cdot 2602028 = 0.$

$$x^2 - \cdot 04205211x + \cdot 56113689 = 0.$$

By another method of transformation we get, for an approximate quadratic for the other two roots, the equation

$$x^2 - 1\cdot 6783942x + \cdot 78846745 = 0.$$

The correctness of the three last factors can easily be tested by equating the product of all three with the proposed equation for x.

R_1. When an algebraical equation contains only one root greater than its reciprocal, this root may be found to any degree of exactness by taking

$$x = \frac{a_m}{a_{m-1}} \text{..................}(40)$$

For, if x be less than unity or less than its reciprocal, the first term of eq. (5) may, by taking m sufficiently large, be always made so small in comparison with the other terms that for an approximation it may be omitted, and we may take

$$a_m x^{n-1} + b_m x^{n-2} + c_m x^{n-3} + \&c...... = 0 \text{.........}(41)$$

Hence, if the equation contains $n-1$ roots less than their reciprocals they will be contained approximately in the equation (41), and since the sum of the roots of this equation is, by the theory of equations, $= -\frac{b_m}{a_m}$, and the sum of the roots of the proposed equation (1), is $= -a$, the other root of the proposed equation, or that exceeding its reciprocal, will be equal to the difference between these two sums, and will therefore be

$$= -a + \frac{b_m}{a_m} = \frac{b_m - aa_m}{a_m} = \frac{a_{m+1}}{a_m} \quad \text{"(10.)"}$$

This theorem, though neat, is not so valuable as the preceding theorem for finding the quadratic factors of an equation, nor is it, generally, so convenient for finding the real or single roots of an equation to a great extent of figures, as Horner's or another method that I shall hereafter give; but it possesses one advantage over Horner's, in not being tentative but always certain in its operation for obtaining fractions successively approximating more nearly in value to a root of the equation, previously transformed so as to contain either one root less or one root greater than its reciprocal.

Ex. 1. Take the equation $x^3 - 6x + 7 = 0$. "*Ruth. Eq. Ex.* I."

Putting $x = y + 1$, we get the equation

$$y^3 + 3y^2 - 3y + 1 = 0 \text{...........}(42.)$$

This equation has only one root greater than its reciprocal, and by division which will, in this case, be most easily performed according to the process (b) we get

$$\frac{1}{1+3y-3y^2+y^3}=1-3y+12y^2-46y^3+177y^4-681y^5+2620y^6$$
$$-10080y^7+38781y^8-149203y^9+\&c.......$$

Taking the coefficients of y^8, y^9, we have $a_8=38781$, $a_9=-149203$.

and "(40)" $\quad y=\frac{a_9}{a_8}=\frac{-149203}{38781}=-3{\cdot}8473221......$

which, by proceeding another step, may be shewn to be correct to the seventh decimal inclusive.

Hence $\quad x=y+1=-2{\cdot}8473221......$

To obtain more convenient equations for rapid approximation, we may apply the method of transformation given in the preceding part of this work, thus—

Substitute $y=\frac{x_1+1}{2}$, and $y=\frac{x_2+1}{3}$, respectively, in (42), then

we get $\quad x^3+9x^2+3x+3=0,\ x^3+12x^2-6x+10=0,$

either of which will be found to give the value of the root greater than its reciprocal much more rapidly by this method than the equation above used.

Ex. 2. Take the equation $x^3-17x^2+54x-350=0$. "*Ruth. Eq. Ex.* II."

Putting $x=y+15$, we get from the proposed equation

$$y^3+28y^2+219y+10=0.$$

This equation has one negative root less than its reciprocal, and for this root we shall have

$$\frac{10}{10+219y+28y^2+y^3}=1-21{\cdot}9y+476{\cdot}81y^2-10380{\cdot}919y^3$$
$$+226028{\cdot}9581y^4-\&c.$$

And taking the coefficients of y^3, y^4, we shall have, inverting the fractional expression (40), since the root is less than its reciprocal

$$y=\frac{a_3}{a_4}=\frac{-10380{\cdot}919}{226028{\cdot}9581}=-{\cdot}04593139037$$

$$\therefore\quad x=15+y=14{\cdot}95406860963.$$

Ex. 3. Take $x^3-7x-7=0$. "*Ruth. Eq. Ex.* III."

Putting $x=y+3$, we get the equation

$$y^3+9y^2+20y-1=0.$$

This equation contains one root less than unity, and the determination of the coefficient a_m according to the process of division (𝔟) will be

$$\begin{array}{llllll}
20 & +9 & +1 & & & \\
400 & +180 & & +20 & & \\
409 & 8180 & & +3681 & +409 & \\
a_3=8361 & & 167220 & & +75249 & \\
 & & a_4=170921 & & 3418420+\&c\ldots\ldots & \\
 & & & a_5=3494078 & & \&c\ldots\ldots
\end{array}$$

And here $y=\dfrac{a_4}{a_5}=\dfrac{170921}{3494078}=\cdot 04891733954$

Hence $x=y+3=3\cdot 04891733954$.

The Theorem R_2 is expressed in a form best adapted for general application, but it will be most easily committed to memory in the following more elegant form, which will also be more convenient for obtaining the quadratic factors with reduced coefficients when several of the coefficients a_m have a common factor.

$R_{1,2}$. Let the proposed equation be transformed into one containing two roots greater than their reciprocals, and let the transformed equation be denoted by

$$x^n+ax^{n-1}+bx^{n-2}+cx^{n-3}+\&c\ldots=0\ldots\ldots\ldots(1)$$

Then, if we put

$$\frac{1}{1+ax+bx^2+cx^3+\&c.}=1+a_1x+a_2x^2+a_3x^3+\&c\ldots a_mx^m,$$

the quadratic factor containing these two roots may be found to any degree of exactness from the equation

$$(a_{m-1}x-a_m)(a_{m-3}x-a_{m-2})=(a_{m-2}x-a_{m-1})^2\ldots\ldots\ldots(\mathrm{m})$$

Ex. 1. Take the equation $x^4-5x^3+10x^2+5=0$.

By the common method, or the more convenient process (𝔟) of division, we get

$$\frac{1}{1-5x+10x^2-5x^4}=1+5x+15x^2+25x^3-30x^4-425x^5-1900x^6-5375x^7-7725x^8+17250x^9-\&c...$$

Hence, taking the coefficients of x^6, x^7, x^8, x^9, we have "(m)"

$$(1900x-5375)\,(7725x+17250)=(5375x-7725)^2,$$

or, dividing by 5^4, this becomes

$$(76x-215)\,(309x+690)=(215x-690)^2,$$

which, by multiplying out and transposing, becomes

$$22741x^2-118875x+243831-0,$$

or $$x^2-5{\cdot}227342x+10{\cdot}72208=0.$$

The correctness of this quadratic factor may be proved by carrying the above division another step further, but the degree of approximation is not very rapid for this example since the reciprocals of the roots considered are not very small; other equations may however be obtained by transforming the proposed one, as directed in the first method of transformation, for which this method will approximate to the values of the quadratic factors with great rapidity.

Ex. 2. Take $x^5+5x^3-10x^2+5x-1=0$. "*Young's Dis. p.* 154."

Dividing this equation by $x-1$, since one root is unity, we get

$$x^4+x^3+6x^2-4x+1=0,$$

and dividing unity by this equation, we get as in Ex. III to Theorem R_2, which is here repeated to show the advantage gained in elimination by using the expression (m),

$$a_7=615,\ a_8=-1845,\ a_9=-1230,\ a_{10}=14636.$$

Hence "(m)" $(615x+1845)\,(-1230x-14636)=(1845x-1230)^2$,

or, dividing by 615^2, this becomes

$$(x+3)\,(-2x-23\tfrac{491}{615})=(3x-2)^2,$$

or $$11x^2+17\tfrac{491}{615}x+75\tfrac{243}{615}=0,$$

which contains two roots of the proposed equation correct to eight places of decimals, as shown in Ex. III to Theorem R_2.

The equation (m) will apply, generally, whether the equation contains one or two roots exceeding their reciprocals. For, when the equation contains only one root greater than its reciprocal, we shall have, from (40), as approximations,

$$a_{m-1}x=a_m,\ a_{m-2}x=a_{m-1},\ a_{m-3}x=a_{m-2},$$

by which values the equation (m) will be fulfilled.

ANOTHER METHOD

OF FINDING THE

REAL ROOTS OF ANY NUMERICAL EQUATION.

Let r be any quantity approximating to a root of the proposed equation for x, and substitute $r+rx_1$ for x; then find r_1 any quantity approximating to a root of the equation for x_1, and substitute $r_1+r_1x_2$, for x_1 in this equation, then find r_2 an approximation to a root of the equation for x_2, and substitute in this equation $x_2=r_2+r_2x_3$, and continue in this manner to find the values, r_3, r_4, r_5, &c., and substituting $x_3=r_3+r_3x_4$, $x_4=r_4+r_4x_5$, &c...; then, we shall have

$$x=r+rx_1=r+rr_1+rr_1x_2$$
$$=r+rr_1+rr_1r_2+rr_1r_2x_3$$
$$=\&c.......................$$
$$=r+rr_1+rr_1r_2+rr_1r_2r_3+rr_1r_2r_3r_4+\&c......(\iota)$$

The equation for x_1 may be found from the equation for x by multiplying the coefficients for x^n, x^{n-1}, x^{n-2}, &c..., by r^s, r^{s-1}, r^{s-2}, &c..., respectively, and then decreasing the roots of the equation, with its coefficients thus enlarged, by unity; and in the same manner the equation for x_{m+1}, may be found from that for x_m, by multiplying the first, second, third, &c..., coefficients of that equation by the descending powers of r_m, and then decreasing the roots of the equation, with its coefficients thus enlarged or diminished, by unity. In the following examples, the common process (Budan's) is employed in decreasing the roots by unity, but in Simple Equations a slight modification is made in the arrangement, which, though at once obvious to those accustomed to the common process, is fully explained in the first example.

Ex. Take the equation $3{\cdot}14159265x=3$, to express x in the form (ι).

The operation will stand thus :

$$\begin{array}{rl}
3{\cdot}14159265-3 & (1=r \\
3{\cdot}14159265 & \\
\hline
+{\cdot}14159265 & ({\cdot}04=r_1 \\
-{\cdot}12566371 & \\
\hline
+1592894 & ({\cdot}12=r_2 \\
-1507964 & \\
\hline
+84930 & ({\cdot}05=r_3 \\
-75398 & \\
\hline
+9532 & ({\cdot}12=r_4 \\
-9047 & \\
\hline
+485 & ({\cdot}05=r_5 \\
-452 & \\
\hline
-33 & ({\cdot}07=r_6
\end{array}$$

Of which the first three lines are like common division, and by dividing, mentally, the third line by the second, the fifth by the fourth, the seventh by the sixth, and so on, we find the leading figures of the quotients to be ·04, ·12, ·05, ·12, ·05, and ·07; also the fourth line is got by multiplying the second by ·04, the sixth by multiplying the fourth by ·12, the eighth by multiplying the sixth by ·05, and so on. The values of r_m being in this case negative with respect to r, we have

$$x=1-{\cdot}04-{\cdot}04\times{\cdot}12-{\cdot}04\times{\cdot}12\times{\cdot}05-\&c\ldots\ldots,$$

which series is very easily summed, for six terms thus:

$$\begin{array}{ll}
 & 1 \\
\hline
- & {\cdot}04 \\
- & \quad 048 \\
- & \qquad 240 \\
- & \qquad\; 2880 \\
- & \qquad\quad 14400 \\
- & \qquad\quad\; 1008 \\
\hline
\end{array}$$

$$x=1-{\cdot}0450703408={\cdot}9549296592.$$

In which if we denote the first, second, third, fourth, &c... , lines by (1), (2), (3), (4), &c....., then (2)=(1)×·04, (3)=(2)×·12, (4)=(3)×·05, (5)=(4)×·12, &c... , the ciphers being omitted at the commencement of each line after the second.

The above process may be materially shortened by occasionally expressing the values of the convergent factors r_m as reciprocals of integral numbers, that is of the form $r_m = \frac{1}{u_m}$, each value of u_m being then obtained by mental division of the $2m$ th line of the process by the following one.

Ex. 2. Take $3{\cdot}14159265358979323846x - 3 = 0$.

3·14159265358789323846 − 3	$(1 = r$
	3·14159265358979323846
	14159265358979323846 $(\cdot 04 = r_1$
	−12566370614359172954
	1592894744620150892 $(\frac{1}{8} = r_2$
	−1570796326794896619
	22098417825254273 $(\frac{1}{80} = r_3$
	−19634954084936207
	2463463740318066 $(\frac{1}{8} = r_4$
	−2454369260617026
	9094479701040 $(\frac{1}{390} = r_5$
1	−8181230868723
·04	913248832317 $(\frac{1}{9} = r_6$
05	−909025652080
00625	4223180237 $(\cdot 004 = r_7$
78125	−3636102608
26041666666666	587077629 $(\frac{1}{7} = r_8$
2893518518518	−519443229
11574074074	67634400 $(\frac{1}{8} = r_9$
1653439153	−64930404
206679894	2703996 $(\cdot 04 = r_{10}$
8267196	−2597216
330688	106780 $(\cdot 04 = r_{11}$
8267	−103888
918	2892 $(\frac{1}{40} = r_{12}$
22	−2597
1 − ·04507034144862798540	295 $(\frac{1}{9} = r_{13}$
∴ $x = \cdot 95492965855137201460$	−288
	7 $(\frac{1}{40} = r_{14}$

In the above operation the values 8, 80, 8, 300, &c..., of u_2, u_3, u_4, &c..., or the reciprocals of r_2, r_3, r_4, &c..., are found mentally, by the division of the leading digits in the fourth, sixth, eighth, &c..., lines, by the first digits in the fifth, seventh, ninth, &c..., lines, respectively; these values being taken the next integers, suitable for short division, above the quotients thus mentally obtained; and the sixth, eighth, tenth, &c..., lines are obtained by dividing the fourth, sixth, eighth, &c..., lines as they are successively obtained by the quantities 8, 80, 8, &c..., respectively. And the m th line in the summation of the series (ε), is obtained by multiplication of the preceding line the same as in the preceding example when r_m occurs, and by division of the preceding line by u_{m-1}, or $\frac{1}{r_{m-1}}$, when u_m takes place.

The last value of x gives the diameter of a circle whose circumference is 3, true to twenty places of figures, and will serve to show the average amount of labour for division by this method when the divisor shall contain that number of figures, as I have not met with another instance in which the approximation was not more rapid. The rapidity with which the left hand digits vanish at each step of subtraction by this method of division, will more than compensate for the additional labour involved in the summation of the series (ε), and each convergent factor is obtained with just the same ease as a digit of the quotient by the common method of division; but a greater advantage in the application of this method to simple equations exists in the use which may be made of it in forming rules for ready computations, as in the following instance.

To find the diameter of a circle of a given circumference.

Rule.—Divide the given circumference by ·4, the result by 8, the last result by 60, then multiply the last result by ·11, and divide the result thus produced by 20. The sum of the last four results will give the diameter correctly to eight places of figures.

Ex. 3. Find the diameter of a circle whose circumference is 876495820.

The operation will be

$$\begin{array}{rr} 4) & \underline{876495820} \\ 8) & \underline{219123955} \\ 60) & 27390494{\cdot}3 \\ \times{\cdot}11 & 456508{\cdot}2 \\ 20) & 50215{\cdot}9 \\ & \underline{2510{\cdot}7} \\ & 27899729{\cdot}1 \ \textit{Ans.} \end{array}$$

Here the first line is divided by 4, the second by 8, the third by 60, the fourth is multiplied by ·11, and the fifth is divided by 20. The answer in the last line is the sum of the last four lines above it.

This rule was immediately found by the above method applied to the equation $3{\cdot}14159265x = {\cdot}4$,
and there seems no doubt that others still better may be found by this method with a little more trouble.

An equally convenient rule may be given for finding the circumference from the diameter, and an exceedingly convenient one for finding common from hyperbolic logarithms will be given with a method for finding these logarithms without the aid of any tables, which I hope to publish, with some other researches, next after another small work on equations intended to follow the present one, as an appendix to it.

Ex. 4. To find a root of the equation

$$87315x^3 - 174544x^2 + 354632x - 698336 = 0.$$

Dividing by x^2, $\quad 87315x - 174544 + \dfrac{354632}{x} - \dfrac{698336}{x^2} = 0,$

from which we immediately see that $x=2$ nearly, and taking $r=2$, the operation for finding the equation for x_1 will be

$$\begin{array}{l} 349260 - 349088 + 354632 - 349168 \\ \qquad\qquad 172 + 354804 + \quad 5636 \\ \qquad 349432 + 704236 \\ \qquad 698692 \end{array}$$

the first line containing the products of the coefficients of x^3, x^2, x, x^0, by r^2, r, r^0, r^{-1}, respectively, or the products of the coefficients of the proposed equation by 4, 2, 1, ½, respectively; and each of the other lines is obtained, from that preceding it, by simple addition, each coefficient in any of them being equal to that preceding it in the same line added to the one placed immediately above it, the first coefficient being understood in every line.

Hence, $349260x_1^3 + 698692x_1^2 + 704236x_1 + 5636 = 0$,

the coefficients of the equation for x_1 being the last in each column of the preceding operation.

Since the last coefficient but one of this equation is the greatest, we may obtain an approximate value of x_1 by omitting the two first terms, and then we get, by division, $x_1 = -{\cdot}008$ nearly, and taking $r_1 = -{\cdot}008$, and proceeding as before, we shall have

$$\begin{array}{llll} 22{\cdot}35264 & -5589{\cdot}536 & +704236 & -704500 \\ & -5567{\cdot}18336 & +698668{\cdot}81664 & -\ \ 5831{\cdot}18336 \\ & -5544{\cdot}83072 & +693123{\cdot}98592 & \\ & -5522{\cdot}47808 & & \end{array}$$

the first line of which is obtained by multiplying the coefficients of the preceding equation for x_1, by $(-{\cdot}008)^2$, $-{\cdot}008$, 1, $\frac{1}{-{\cdot}008}$, respectively, the same as in the preceding operation, and the coefficients in the succeeding lines are obtained by addition as before. The bottom coefficients being those of the equation for x_2, we have

$$22{\cdot}35264x_2^3 - 5522{\cdot}47808x_2^2 + 693123{\cdot}98592x_2 - 5831{\cdot}18336 = 0.$$

Dividing the last coefficient of this equation by the last but one, we find that $x_2 = {\cdot}008$, nearly, and putting $r_2 = {\cdot}008$, and proceeding as before, we have, carrying the operations to five places of decimals only,

$$\begin{array}{llll} {\cdot}00143 & -44{\cdot}17982 & +693123{\cdot}98592 & -728897{\cdot}92000 \\ & -44{\cdot}17839 & +693079{\cdot}80753 & -\ \ 35818{\cdot}11247 \\ & -44{\cdot}17696 & +693035{\cdot}63057 & \\ & -44{\cdot}17553 & & \end{array}$$

the first line being obtained as before, by multiplying the coefficients of the equation for x_2, by $(.008)^2$, ${\cdot}008$, 1, $\frac{1}{{\cdot}008}$, respectively, and the rest by addition as before. The bottom coefficients give the equation for x_3, that is,

$$\cdot 00143x_3^3 - 44\cdot 17553x_3^2 + 693035\cdot 63057x_3 - 35818\cdot 11247 = 0.$$

Dividing the last coefficient by the last but one, we get $x_3 = \cdot 05$ nearly, and taking $r_3 = \cdot 05$, and proceeding as before, the first term will vanish as its first five decimal digits will be ciphers, and the operation is

$$\begin{array}{r} -2\cdot 208 + 693035\cdot 630 - 716362\cdot 249 \\ 693033\cdot 422 - 23328\cdot 827 \\ 693031\cdot 214 \end{array}$$

the first line of which is obtained by multiplying the coefficients of x_3^2, x_3, x_3^0, by $\cdot 05$, 1, and $\frac{1}{\cdot 05}$ respectively, and the others are obtained by addition as before. The last results give the equation for x_4,

$$-2\cdot 208x_4^2 + 693031\cdot 214x_4 - 23328\cdot 827 = 0.$$

Proceeding as before we have $x_4 = \cdot 03$ nearly, and, if we multiply the coefficients of the last equation by the quantities $\cdot 03$, 1, $\frac{1}{\cdot 03}$, respectively, the last result will only be correct to one place of decimals, and the first becomes $\cdot 06624$, which may therefore be omitted in the succeeding part of the operation, and we may then proceed as in Ex. 1, thus

```
693031·214 – 23328·827 (.03
             20790937
             --------
              2537890 (·12
              2494912
              -------
                42978 (·01
                24949
                -----
                18029 (·7
                17464
                -----
                  565 (·03
                  524
                  ---
                   41 (·08
```

Collecting the preceding processes, the entire work may be arranged as follows:

87315	−174544	+354632	−698336(2 = r
349260	−349088	+354632	−349168
	+ 172	+354804	+ 5636 (−·008 = r_1
	+349432	+704236	−704500
	+698692	+698668·81664	−5831·18336(·008 = r_2
22·35264	−5589·536	+693123·98592	−728897·92
	−5567·18336	+693079·80753	−35818·11247 (·05
	−5544·83072	+693035·63057	−716362·249
	−5522·47808	+693033·422	−23328·827(·03
·00143	−44·17982	+693031·214	20790936
	−44·17839		2537891(·12
	−44·17696		2494912
	−44·17553		42979(·01
0	−2·208		24949
			18039(·7
			17464
			566(·03
			524
			42(·08

And summing the series (ε) with the last values of r_m, we shall have

$r =$	2
$rr_1 =$	− ·016
$rr_1r_2 =$	− 0128
$rr_1r_2r_3 =$	− 64
&c. $=$	− 192
:	2304
:	2304
:	16128
:	484
:	39

$x = 2 - {\cdot}016134615436 91 = 1{\cdot}983865384563 08.$

In which the first line contains r; the second its product by r_1; the third the preceding line multiplied by r_2; the fourth the third multiplied by r_3; and so on to the tenth, which is produced by multiplying the ninth by $r_9 = {\cdot}08$. The last line is the sum of all the others and gives the numerical value of x.

F

Ex. 5. Take the equation $x^6-2x^5+4x^4-8x^3-1=0$.

By proceeding as in the last solution, we immediately see that $x=2$ nearly, and the operation will stand

1–2+	4–	8 – 0	– 0		– 1 (2 = r
8–8+	8–	8 – 0	– 0		–·125
0+	8–	0 – 0	– 0		–·125 (–·008 = r_1
8+	16+	16 +16	+16		–244140·625
16+	32+	48 +64		250000	–2037·329(·008 = r_2
24+	56+104		–8000	242103·296	–2037·329
32+	88	104	–7896·704+	234309·184	–163·348(·08 = r_3
40–·704		103·296	–7794·112	+1874·473	149·879
0		102·592	–7692·224	1873·981	13·469(·08 = r_4
		101·888	–·492	+1873·489	11·990
		+101·184	0	149·879	1479(·12 = r_5
		0 + 0			1439
					40(·02 = r_6
					29
					11(·4 = r_7

In this example we first multiply the coefficients of the given equation by 2^3, 2^2, 2, 2^0, 2^{-1}, 2^{-2}, 2^{-3}, or 8, 4, 2, 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, respectively, and then decreasing by unity as before, we obtain the coefficients 8, 40, 88, 104, 64, 16, –·125, of the equation for x_1; from the two last of which we get by division $r_1 = -·008$. The equation for x_2 is then found in the same manner, by multiplying the last set of coefficients by the descending powers of r_1, and applying the common process for decreasing the roots of an equation by unity, neglecting in the multiplication the two first coefficients, since the operation is here only carried to three places of decimals, and the values of $8r_1^3$, $40r_1^2$, will be less than $\frac{1}{10^3}$: the coefficients of the equation for x_2, thus obtained, carried only to three places of decimals being –7692·224, 234309·184, –2037·329, we get from the two last, as before by division, $r_2 = ·008$. Proceeding in the same manner with r_2, we find that two other coefficients will vanish for three places of decimals, and we get for x_3 a quadratic whose coefficients are –·492, 1873·489, –163·348, and dividing the last by the preceding one, we get $r_3 = \cdot 08$. Proceeding in the same manner with r_3 we get the simple equation

$$149\cdot 879 x_4 - 13\cdot 469 = 0,$$

which gives $r_4 = \cdot 08$, and the rest of the operation is performed in the same manner as Ex. 1.

The summation of (ϵ) with these values of r_m will stand thus

$$\begin{array}{l} 2 \\ \hline -\cdot 016 \\ \quad\quad 0128 \\ \quad\quad\quad 1024 \\ \quad\quad\quad\quad 8192 \\ \quad\quad\quad\quad\quad 98304 \\ \quad\quad\quad\quad\quad\quad 1966 \\ \quad\quad\quad\quad\quad\quad\quad 786 \\ \hline \end{array}$$

$$x = 2 - \cdot 016139160256 = 1\cdot 938860839743.$$

This root agrees with that found in the preceding example for six places of figures, and the roots of the cubic equation forming the preceding example will be found to agree to about the same extent with three of the roots of the last equation of the sixth order.

10. In the preceding solutions the convergent factors r_m have been taken any values approximating to a root of the equation for x_m, but if this equation be represented by

$$A_n x^{n-1} + A_{n-1} x^{n-2} + A_{n-2} x^{n-2} + \&c. \ldots\ldots A_1 x + A = 0, \ldots\ldots (g)$$

and we take u_m any value that will nearly fulfil the condition

$$A + \frac{A_1}{u_m} + \frac{A_2}{u_m^2} + \frac{A_3}{u_m^3} + \&c. \ldots = 0, \ldots\ldots (45),$$

that is take u_m any value approximating to a root of the reciprocal equation of (g), the root may, by operating in every other respect as before, be developed in the form

$$x = r + \frac{r}{u_1} + \frac{r}{u_1 u_2} + \frac{r}{u_1 u_2 u_3} + \frac{r}{u_1 u_2 u_3 u_4} + \ldots\ldots\ldots (\mathrm{f})$$

When A_1 is greater than the other coefficients, which will generally be the case when the roots are decreased below unity, it will be sufficient to take the two first terms of (45) only, that is

$$A+\frac{A_1}{u_m}=0, \text{ or } u_m=-\frac{A_1}{A}, \text{ nearly,} \ldots\ldots.(h).$$

This process will immediately be seen to be identical with my method of decreasing the roots of an equation, given in the preceding general method of finding both the real and unreal roots; and the equation for x_{m+1} may be formed from that for x_m, by multiplying its first, second, third, &c..., coefficients by 1, r_m, r_m^2, r_m^3, &c..., respectively, and then decreasing the roots of the equation with enlarged coefficients by unity, as before, or we may divide them by the descending powers of r_m, commencing with A_n, or the ascending powers commencing with A_1, as in the expression (45), and then apply the process for decreasing the roots of an equation by unity.

In assuming the values of r_m, or u_m, it is not necessary that they shall always be positive quantities, and in some cases the integral fulfilling most nearly the condition (45) or (h), will exceed the root of the equation for x_m, and then the value of the next convergent factor u_{m+1} must be taken a negative quantity. This assumption of both positive and negative values in (f) will add very little to the trouble of its summation, while it will materially shorten the process for obtaining a sufficient number of convergent factors, for obtaining the root correctly to a great extent.

11. Substitute ru_1 for u_1 in (f) then we get

$$x=r+\frac{1}{u_1}+\frac{1}{u_1 u_2}+\frac{1}{u_1 u_2 u_3}+\frac{1}{u_1 u_2 u_3 u_4}+\&c.\ldots\ldots(g)$$

Hence if we decrease the root by r and then operate as in the preceding examples, we may express it in the form (g).

Ex. Take the equation $x^3-2x=5$.

One root is nearly 2, and substituting x_1+2, for x, this equation becomes

$$x_1^3+6x_1^2+10x_1-1=0.$$

and proceeding as in the preceding example, the operation will be

1	+ 6	+ 10	−1	(11
1	+66	+1210	−1331	
	67	1277	− 54	(25
	68	1345	−843750	
	69	840625	−1399	(600
	1725	842351	−302184000000	
	1726	844078	1685116801	(−200
	1727	303868080000	165766058·905075(10	
	1728	303869116801	13830987·546791(11	
	1036800	+303870153603	18708·358107(800	
	1036801	−1519350768·015000	1443·009124(12	
	1036802	−1519350742·094925	−1438·779082	
	+1036803	−1519350716·174850	4·230042(400	
−·000000	+25·920075	−151935071·617485	3596947	
	·259201	−151935071·358284	633095(6	
		−151935071·099083	599491	
	·002142	−13812279·190826	33604(20	
		−13812279·188684	29974	
		−13812279·186542	3630(9	
	·000000	− 17265·348983	3330	
			300(12	
			278	
			22(13	

And summing (g) for these values we have

$r=2$

·09090909090909090909090909
36363636363636363636
606060606060606060606
−30303030303030303
−3030303030303030
−275482093663911
−344352617079
−28696051423
−71740128
−11956688
−597834
−66426
−5535
−425

$x=2·09455148154232659148236 9$

This value of the root agrees to the twenty-third decimal with that given by Professor de Morgan, "*Math. p.* 290, *Vol.* III," and it may be easily obtained to a much greater extent, by proceeding a few more steps in the equation u_m, before commencing with the abbreviation; which, in the above solution, commences with u_4, by carrying the division of the preceding set of coefficients, $1+1036803+303870153603+1685116801$, of the equation for x_4 by $(-200)^3$, $(-200)^2$, -200, 1, respectively, to only six places of decimals. The subsequent factors may be obtained much larger, and therefore the approximation be made more rapidly, by taking $u_5=11$, but the above, in which it will be seen that many figures are unnecessarily repeated, is sufficiently rapid in approximation.

12. The last method of expressing the real roots of an equation may be very conveniently employed in determining, by elimination, whether any two equations of the same degree contain a common factor. My method is first to eliminate the first terms and then the last from the proposed equations, thus obtaining two equations one degree lower than those proposed; then, by treating the resulting equations in the same manner, to obtain two other equations another degree lower; then, by proceeding in the same manner with the two last equations and continuing this process of elimination until we arrive at two equations of which the coefficients of any two terms in one equation, shall have the same ratio to each other as the coefficients of the corresponding terms in the other, the two equations consequently containing the same roots.

Ex. To determine the common factor, if any, of the two following equations:

$$17821x^4-21719x^3+4349x^2+1852x-1982=0,$$
$$3898x^4+156040x^3-37494x^2+1982x+17821=0.$$

Taking $17821y=3898$, and proceeding as in the preceding methods we find that

$$y=\frac{1}{5}+\frac{1}{5.11}+\frac{1}{5.11.40}+\frac{1}{5.11.40.5}+\frac{1}{5.11.40.5.30}+\&c...,$$

and dividing all the coefficients in the first of the two above equations omitting the leading one, by 5, all the results by 11, the last set of results by 110, then the last set by 5, and so on to any extent, and then adding the results in each column, the process will stand

5)	−21719	+4349	+1852	−1982
11)	−4343·8	+869·8	+370·4	−396·4
40)	394·8909	79·07272	33·67272	36·03636
5)	9·8722	1·97681	·84181	·90090
30)	1·9744	·39536	·16836	·18018
8)	658	1317	561	606
12)	82	164	70	75
	6	13	5	6
	−4750.6121	+951·25983	+405·08925	−433·52433

and since the first coefficients of each equation have the same signs, we must subtract from these the coefficients of all the terms but the first in the second equation, that is the coefficients

156040	−37494	+1982	+1782

and we get the coefficients of the cubic

$$-160790{\cdot}6121x^3+38445{\cdot}25983x^2-1576{\cdot}91075x-18254{\cdot}52433 = 0, \ldots\ldots(46)$$

resulting from the elimination of the first terms of the proposed equations.

In the same manner, by eliminating the two last terms of the same equations, we obtain another cubic equation, and it will be found that the coefficients of the two cubics have the same ratios to one another in each equation, for nine places of figures. Hence the three roots of the cubic (46), will also be the roots of the two proposed equations for nine places of figures correctly, or it is a common factor of both equations to that extent.

The two equations forming the last example, respectively contain the same three roots of the equation

$$x^5-x^4+9x^3-2x^2+1=0,$$

to about ten or more places of figures correctly, and the factor containing the two other roots may be found by the same method of elimination, repeated on each pair of resulting equations successively diminishing in degree, from the equations

$$2366x^4-3505x^3+22966x^2-15646x+7298=0,$$
$$7298x^4-9664x^3+69817x^2-37562x+15646=0,$$

which will be found to contain two of the imaginary roots of the preceding equation of the fifth order to a greater degree of exactness than the three nearly common roots of the two equations in the preceding example, are found to coincide.

13. The same method of elimination may also be applied in determining the values of the coefficients a_m, b_m, &c..., in my general method for finding the quadratic factors of an equation in the following manner.

Ex. Take the equation $x^{10}+2x^3-5=0$. "*Math. Vol.* III. *p.* 286." Putting x_1+1, for x this becomes

$$x_1^{10}+10x_1^9+45x_1^8+120x_1^7+210x_1^6+252x_1^5+210x_1^4+122x_1^3$$
$$+51x_1^2+16x_1-2=0\ldots\ldots(47)$$

And putting $x^2=y$, we get by the process (ϵ)

$$y^{10}+10y^9+45y^8+120y^7+182y^6+318y^5+12y^4+2368y^3-2143y^2$$
$$+460y+4=0.$$

We find, by my first method of transformation, that two of these roots may be decreased by putting $y=\frac{y_1+1}{3}$, and then we get

$$y_1^{10}+40y_1^9+720y_1^8+7680y_1^7+51492y_1^6+260478y_1^5+761988y_1^4$$
$$+6420108y_1^3+2603088y_1^2-2993804y_1+513409=0,$$

and decreasing the roots of this equation in the same manner by putting $y_1=\frac{y_2+1}{4}$, which I perform, in this case, by dividing the coefficients of the last equation by 4^{10}, 4^9, 4^8, &c..., in their order respectively, and then decreasing the roots by unity as in the previous examples, first omitting the four leading terms, since they become less than unity by this division, and the determination of the quadratic factor containing the two roots to about six or seven places of figures, will be sufficient to show the method of elimination; we then get the equation

$$13y_2^6+332y_2^5+4441y_2^4+115018y_2^3+484226y_2^2-108871y_2+31208=0\ldots\ldots(48)$$

And the operation for finding the two roots less than their reciprocals will be

$$\begin{array}{rrrr}
-108871+ & 484226+ & 115018+ & 4441+\&c\ldots \\
 & -326613+ & 1452678+ & 345054+\ldots\ldots \\
 & -\ 54435+ & 242113+ & 57509 \\
 & +\ 1088- & 4842- & 1150 \\
 & +\ 155- & 691- & 164 \\
 & +\ 3- & 13- & 3 \\
\hline
 & 104424+ & 1804263+ & 405687 \\
 & & 326613- & 1452678 \\
 & & 36290- & 161408 \\
 & & 1209- & 5382 \\
 & & 172- & 768 \\
 & & 10- & 48 \\
\hline
 & & 2168557- & 1214597 \\
 & & & \&c\ldots
\end{array}$$

Here, we find, by the preceding method of obtaining the real roots of an equation, that $\frac{108871}{31208}=3+\frac{3}{6}-\frac{3}{6.50}-\frac{3}{6.50.7}-\frac{3}{6.50.7.50}$ $-$&c..., and, arranging the coefficients of the equation for y_2, omitting the last, in their reverse order, the second line consists of the coefficients in the first multiplied by 3; the third line of those in the second divided by 6; the fourth of the third divided by -50; the fifth of the fourth divided by 7; the sixth of the fifth divided by 50; and the next line contains the sums of the coefficients of the preceding lines in each column. Then we find that $\frac{104424}{31208}=3+\frac{3}{9}+\frac{3}{9.30}+\frac{3}{9.30.7}+\frac{3}{9.30.7.16}+$&c..., and the eighth line in the above operation is obtained by multiplying the first by -3, the ninth by dividing the eighth by 9, the tenth by dividing the ninth by 30, the the eleventh by dividing the tenth by 7, the next by dividing the preceding one by 16, and the next by adding the columns in the last six lines. In this manner we might continue the process by taking the value of $\frac{2168557}{31208}$, and proceeding as before, but it is already sufficiently advanced, for obtaining the two roots in consideration, to the extent of about six or seven places of decimals. The first lines of coefficients may be taken to represent the values of a_1, b_1, c_1, d_1, &c..., in eq. (5) at the commencement of this work, for the equation (48), the seventh the values a_2, b_2, c_2, &c..., the thirteenth the values a_3, b_3, c_3, &c..., and we may therefore take

$$a_1 = -\cdot 108871,\quad b_1 = \cdot 484226,$$

$$a_2 = \cdot 104424,\quad b_2 = 1\cdot 804263,\quad a_3 = 2\cdot 168557,\quad b_3 = -1\cdot 214597.$$

And substituting these values in the equation

$$(a_1 b_2 - a_2 b_1)\frac{1}{y_2^2} + (a_3 b_1 - a_1 b_3)\frac{1}{y_2} + a_2 b_3 - a_3 b_2 = 0, \text{ “}\textit{From } (7)\ p.\ 2.\text{”}$$

we get $\quad 4\cdot 039480 y_2^2 - \cdot 9178374 y_2 + \cdot 2469967 = 0 \ldots\ldots(49)$

And restoring y_1, from $y_2 = 4y_1 - 1$, we get

$$4\cdot 039480 y_1^2 - 2\cdot 249199 y_1 + \cdot 325269 = 0,$$

And substituting $y_1 = 3y - 1$, we get

$$4\cdot 03948 y^2 - 3\cdot 442719 y + \cdot 734883 = 0.$$

or

$$y^2 - \cdot 8522679 y + \cdot 1819255 = 0.$$

And substituting $y = x_1^2$, applying eqs. (d) we get

$$x_1^2 - \cdot 0280314 x_1 + \cdot 4265272 = 0.$$

Restoring the value of x from $x_1 = x - 1$, this gives

$$x^2 - 2\cdot 0280314 x + 1\cdot 4545586 = 0 \ldots\ldots\ldots(50).$$

Which agrees very nearly with one of the approximating factors given by the late Mr. Weddle, by means of his ingenious method of approximating, trionometrically, to the imaginary roots of a trinomial equation. “*See Math. Vol.* III. *p.* 288.”

Several of the other factors may be obtained by means of my first method of transformation and the above method of elimination, without taking the equation for the squares of the roots of any of the transformed equations, and some of them with more rapidity than (50) was obtained; but the approximation to the roots considered in eq. (48), is very rapid, and they may be found to ten or more places of decimals with but little more trouble, by the above method, and retaining the whole of its terms as derived from the substitution of $y_1 = \frac{1+y_2}{4}$, in the equation for y_1

One of the real roots may be easily obtained at once from eq. (47), by means of eq. (40), but I intend to reconsider this, and some of the other examples in this work, in a small work that I am preparing to publish, to be considered, though separate, in the light of an Appendix to the present one.

The determinaticn of the above factor by any other algebraical method yet published, would have involved the finding of one root of an equation of the *forty-fifth* order.

ON THE

SOLUTION OF EQUATIONS

BY

GENERAL FORMULÆ.

1. The following Theorem is connected with the general principle of division, partially considered in the introductory investigations at the commencement of my general method for finding the approximate quadratic factors of any equation, upon which that method depends; and it is not entirely unconnected with the "*General Theory of Reciprocity for points referred to co-ordinate axes*," commenced by me in the *Lady's and Gentleman's Diary for* 1849. It admits of different enunciations, but the following one seems to be sufficiently explicit.

Put $$\frac{1+ax+bx^2+cx^3+dx^4+ex^5+\&c\ldots\ldots}{1+a_1x+a_2x^2+a_3x^3+a_4x^4+a_5x^5+\&c\ldots\ldots}$$

$$=1+(b_1+a)x+(b_2+b_1a+b)x^2+(b_3+ba_2+b_1b+c)x^3+\&c\ldots\ldots(\mathfrak{h}),$$

then, this equation will be fulfilled by taking for every value of m,

$$a_m=-b_m-b_{m-1}a_1-b_{m-2}a_2-b_{m-3}a_3-\&c\ldots\ldots(h_1)$$

or $$b_m=-a_m-a_{m-1}b_1-a_{m-2}b_2-a_{m-3}b_3-\&c\ldots\ldots$$

the latter of which will always subsist with the former, that is the quantities a_m, b_m, are interchangeable in (h) for any value of m.

Taking $m=1, =2, =3, =4, =$&c......, successively, and substituting, we readily get the values

$$a_1=-b_1,\ a_2=-b_2+b_1^2,\ a_3=-b_3+2b_1b_2-b_1^3,\ \&c......$$

or $$b_1=-a_1,\ b_2=-a_2+a_1^2,\ b_3=-a_3+2a_1a_2-a_1^3,\ \&c......(k)$$

In biquadratic equations, we take "(𝔥)"

$$1+ax+bx^2+cx^3+dx^4=(1+a_1x+a_2x^2)\left\{1+(b_1+a)x+(b_2+b_1a+b)x^2\right\}$$

$$="(h)"\left\{1-b_1x+(b_1^2-b_2)x^2\right\}\left\{1+(b_1+a)x+(b_2+b_1a+b)x^2\right\}$$

Multiplying out these factors and equating the coefficients of the same powers of x on both sides of the last equation, we get

$$b^2(a+2b_1)=b_1^3-bb_1-c...(\text{k}'),\quad (b_1^2-b_2)(b_2+ab_1+b)=d...(\text{k}'').$$

If we eliminate b_2, by substituting in (k$''$), its value from (k$'$), we shall obtain an equation whose roots are double those of the equation (7), in page 15, of Dr. Rutherford's small tract on the complete solution of numerical equations, but, I think, the equation for a_2 affords a more convenient equation for the general solution of biquadratics, and it may be easily found from the last two equations by substituting $b_1^2-a_2$, for b_2, and then eliminating b_1. The resulting equation, putting y for a_2 will be

$$y^6-by^5+(ac-d)y^4+(2bd-a^2d-c^2)y^3+(ac-d)dy^2-bd^2y+d^3=0..(51)$$

which is the equation for the products of each pair of the roots of the equation

$$x^4+ax^3+bx^2+cx+d=0,$$

and can easily be resolved into three factors of the form

$$(y^2+p_1y+d)(y^2+p_2y+d)(y^2+p_3y+d)=0......(52),$$

Of which the coefficients p_1, p_2, p_3, will be the roots of the equation

$$p^3+bp^2+(ac-4d)p+a^2d+c^2-4bd=0......(l).$$

In the same manner we may proceed with the general equation of the sixth order. Resolving it by means of (𝔥) into two cubic factors involving only three unknown quantities b_1, b_2, b_3, or a_1, a_2, a_3, for the determination of which equations may be formed by multiplying the two factors, involving these quantities, together, and equating, as above, the coefficients of the same powers of x. One of the auxiliary equations resulting from the elimination of two of these quantities from the equations involving all three will admit of reduction, but the low price of the present work will hardly permit its greater extension, and I also feel anxious for its completion, as it has already been unwillingly delayed beyond the time advertised for its appearance.

2. By another method, too extensive to be discussed in this small work after the investigations it now contains, I have succeeded in obtaining greater improvements in the solution of equations by general formulæ than by means of equation (ḫ), from which I take the following as the first instance of the general equation of the fifth order being solved, algebraically, without the help of an auxiliary equation of a higher order.

To solve the general equation of the fifth order

$$x^5-ax^4+bx^3-cx^2+dx-e=0.$$

Every equation of the fifth order contains one real root at least, that may be determined without the help of any auxiliary equation, which denote by r, and put

$$q=\frac{e-dr}{(r-a)r^2},\quad p=\frac{e}{r}-q^2,\quad s_1=a-r,$$

$$s_2=\frac{c-aq}{r}-q,\quad h=s_1^2-4s_2 \ldots\ldots\ldots\ldots (r)$$

Then if we solve the cubic equation

$$z^3+\left(\frac{16p}{hs_1}+\frac{4q+s_2}{s_1}\right)z^2+\left(\frac{8p}{h}+q\right)z+\frac{ps_1}{h}=0\ldots\ldots\ldots(53)$$

and moreover put $\text{k}=q-\dfrac{hz}{8z+2s_1}$,

$$\text{M}=2z+\frac{s_1}{2}+\sqrt{\left(\frac{h}{4}+2\text{k}-q\right)},\quad \text{N}=z^2+\frac{1}{2}s_1z+\text{k},$$

we shall have $x=\dfrac{\text{M}}{2}-z\pm\sqrt{\left(\dfrac{\text{M}^2}{4}-\text{N}\right)}\ldots\ldots\ldots\ldots(54)$

Ex. Solve the equation $x^5+3x^3-8x^2+5x-1=0$.

This equation contains four imaginary roots and one real one $=1$, and, here, we have

$$a=0,\ b=3,\ c=8,\ d=5,\ e=1,\ \text{and}\ r=1,$$

and substituting these values in equations (r), we get

$$q=-4,\ p=-15,\ s_1=-1,\ s_2=12,\ h=-47,$$

and substituting these values in the cubic (53), we get

$$47z^3-52z^2-68z-15=0.$$

The positive root of this equation is very easily found by the preceding method for finding the real root of an equation, to be $z=1{\cdot}93793493$...... and substituting this and the preceding values in the above expressions for k, M, N, we get

$k=2{\cdot}74514601$, $M=3{\cdot}37586986+\sqrt{1{\cdot}740292202}$, which gives for M the two values $M=4{\cdot}69507121$, or $M=2{\cdot}05666851$, according as we take the surd quantity, positive or negative, also

$$N=z^2+\frac{1}{2}zs_1+k=3{\cdot}75559179-\frac{1}{2}z+k=5{\cdot}53177043...$$

Substituting this value of N, and the first of the above values of M in (54) we get

$$x={\cdot}409600675\pm\sqrt{-{\cdot}02084701}.$$

And if we take the last value of M and the rest as before, then, the same equation (54) gives

$$x=-{\cdot}909600675\pm\sqrt{-4{\cdot}47429909}.$$

Ex. 2. Take the equation $x^5-5x^4+x^3+16x^2-20x+16=0.$
"*Young's Eqs. p.* 327."

We easily find 2 to be an integral root of this equation, and to find the others by the above method, we shall have

$$a=5,\ b=1,\ c=-16,\ d=-20,\ e=-16,\ r=2.$$

And substituting these values in the equations (r), we get

$$q=-2,\ p=-12,\ s_1=3,\ s_2=-1,\ h=13,$$

and substituting these values in the cubic (53), it becomes

$$39z^3-309z^2-326z-108=0,$$

one root of which we find to be 9, and substituting $z=9$, and the preceding values in the above expressions for M and N, we easily get

$$M=19\ \text{or}\ M=20,\ \text{and}\ N=91.$$

Hence, taking the first value of M, we get "(54)" $x=\frac{1}{2}\pm\sqrt{-\frac{3}{4}}$, and taking $M=20$, then (54) gives $x=1\pm 3=4$ or -2.

The investigation of this method of solving equations of the fifth degree will be given, with many others, in a small work intended to appear, in the course of the present year, as an appendix to this.

By applying the above method to the solution of the quintic, solved in pages 21, 22, of Dr. Rutherford's tract, it will be seen that the advantage gained by this over the method employed in that work, consists in the same being here effected by means of the cubic (53), as is effected by means of an equation of the tenth order in the tract referred to; one root being previously determined in both methods. In the appendix, to appear hereafter, will also be added several other methods of increasing the rapidity of approximation in applying my general theorems, inclnding an elegant mode of finding the coefficients of the different powers of x in the result of unity divided by $f(x+p)$, from the coefficients (a_m), resulting from the division of unity by $f(x)$, as in the first part of this work. The theorem R_2, in this work, will also be generalized in the appendix, for any number of roots exceeding their reciprocals, and it will also contain some important applications of the general theorem to the summation of series, rationalising of surd equations, &c.

NOTE TO THEOREM R_2.

As cases will frequently occur of equations containing one root nearly equal to unity, it may be remarked, before concluding, that, in place of $\frac{1}{f(x)}$, we must then take

$$\frac{1-x}{f(x)}=1+a_1x+a_2x^2+a_3x^3+a_4x^4+\&c.\ldots\ldots,$$

and proceed as before for finding the values of the roots exceeding their reciprocals.

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