BEGINNERS' ALGEBRA

SYKES
COMSTOCK

BEGINNERS' ALGEBRA

BEGINNERS' ALGEBRA

By

MABEL SYKES

Instructor in Mathematics, Bowen High School, Chicago
Author of "A Source Book of Problems for Geometry"
and (with Clarence E. Comstock) of "Plane
and Solid Geometry"

and

CLARENCE E. COMSTOCK

Professor of Mathematics, Bradley Polytechnic Institute,
Peoria, Illinois

RAND McNALLY & COMPANY

CHICAGO NEW YORK

Made in U. S. A.

B-23

THE CONTENTS

THE PREFACE

This book is the first of a two-book series. It has been the intention of the authors to include in *Beginners' Algebra* no more work than can be completed easily in one year. It is believed that after this course has been finished the pupil will have a clear understanding of the principles presented and the ability to apply them to algebraic expression of less complicated forms.

The order in which some of the topics occur may be a little unusual; yet it is felt that the pupil will find his progress is easy and invigorating. The subject is presented in a manner to stimulate the power of the pupil to think in mathematical terms. It is hoped that the language is so simple and direct that the pupil will find little or no difficulty in reading the book with understanding and pleasure. Technical language has been reduced to a minimum.

In view of the fact that many high-school pupils have difficulty in using fractions and decimals with speed and accuracy, an introductory chapter is devoted to such matters. In this chapter attention is also called to the proper number of figures to be retained when calculating with approximate numbers.

The close connection between algebra and arithmetic is stressed throughout the book. The idea that letters stand for numbers is brought to the surface of the pupil's mind by frequent exercises in the evaluation of algebraic expressions.

The equation as the means of solving problems has been taken for central consideration. The methods of solving equations are developed very gradually in connection with problems giving rise to them. These, as well as other principles and methods, receive a gradual development at first from a more or less common-sense point of view, followed by a more formal presentation.

Several features of the book should receive special mention.

A large number of simple exercises are given. More complicated and difficult expressions are left to the later course. This is notably true in the case of fractions, factoring, and radicals. Certain types often given in a first-year course are omitted, including the sum and difference of cubes and forms factored after grouping terms. These have been placed in the Appendix for the convenience of teachers who wish to include them in the one-year course.

It is in the stating of problems that most pupils find their greatest difficulty. The problems in this book have been chosen from topics with which the average pupil has considerable familiarity. The approach to problems of the various types is made easy. General rules of procedure are given which it is hoped will be found to be very helpful.

The graph is considered of so much importance that it is made an essential part of the course. Many ways in which graphs, both statistical and algebraic, may be used are indicated. Place is given to the solution of problems by graphic methods.

Although the word "function" does not appear in the book, the idea is frequently present, notably in connection with the graph which is called for when various algebraic forms are considered.

The important distinction between equations of the first and second degrees and fractional equations is taken into account, and a separate treatment is given to the solution of fractional equations.

It is believed that the book is in close accord with the country-wide movement for the improvement of the teaching of algebra to those beginning the study of the subject.

C. E. C.

January, 1922 M. S.

BEGINNERS' ALGEBRA

CHAPTER I

Introductory: Fractions and Decimals of Arithmetic

1. Equal fractions. Certain fractions that look very unlike have the same value. $\frac{26}{65}$, $\frac{14}{35}$, and $\frac{2}{5}$ are equal in value though they differ in form. We may reduce $\frac{14}{35}$ to the form $\frac{2}{5}$ by dividing both terms by the number 7.

To reduce a fraction to lower terms means to divide both terms of the fraction by 2 or 3 or some larger number.

A fraction is said to be in its lowest terms when the numerator and denominator cannot be divided by 2 or 3 or some larger whole number without a remainder.

We may reverse this operation and reduce a fraction to higher terms if we so desire. We may multiply both terms of $\frac{2}{5}$ by 13 and get $\frac{26}{65}$. To reduce $\frac{2}{3}$ to a fraction whose denominator is 12, we multiply both terms by 4, a number which multiplied by 3 gives 12; the resulting fraction is $\frac{8}{12}$.

For some purposes it is desirable to reduce fractions to their lowest terms, and for other purposes it will be found necessary to reduce them to higher terms. One very important fact must be thoroughly fixed in mind: **If both terms of a fraction are divided (or multiplied) by the same number, the value of the fraction is unchanged.** This may be called the *Fraction Law.*

EXERCISES

Reduce to lower terms:

1. $\frac{12}{18}, \frac{6}{9}, \frac{15}{30}, \frac{45}{81}, \frac{24}{48}$

2. $\frac{15}{25}, \frac{21}{63}, \frac{8}{9}, \frac{34}{51}, \frac{24}{30}$

3. $\frac{27}{51}$, $\frac{36}{54}$, $\frac{30}{75}$, $\frac{121}{33}$, $\frac{126}{168}$

4. Reduce to fractions having denominators indicated:

Fraction	$\frac{3}{7}$, $\frac{7}{8}$, $\frac{13}{15}$, $\frac{3}{4}$, $\frac{5}{9}$
Denominator	21, 40, 45, 32, 63
Fraction	$\frac{2}{3}$, $\frac{3}{4}$, $\frac{3}{5}$, $\frac{2}{7}$, $\frac{1}{2}$
Denominator	12, 12, 70, 70, 70

5. Find all the fractions with denominators less than 100 which are equal to $\frac{5}{13}$.

2. To multiply together an integer and a fraction.

$$\frac{2}{3}\times 7=\frac{2\times 7}{3}=\frac{14}{3}$$

RULE. **Multiply the numerator of the fraction by the integer.**

Notice the very important special case in which the integer and the denominator are the same:

$$\frac{2}{3}\times 3=2$$

In this case the product is the numerator. The rule would call for work like this:

$$\frac{2}{3}\times 3=\frac{6}{3}=2$$

But the work is done more quickly thus:

$$\frac{2}{3}\times 3=\frac{2\times \cancel{3}}{\cancel{3}}=2$$

or, better still:

$$\frac{2}{\cancel{3}}\times \cancel{3}=2$$

It is foolish to do unnecessary multiplications such as:

$$\frac{13}{36}\times 48=\frac{624}{36}=17\tfrac{1}{3}$$

It is much better to work as follows:

$$\frac{13}{\underset{3}{\cancel{36}}} \times \overset{4}{\cancel{48}} = \frac{52}{3}$$

Whether the result should be left in the form of $\frac{52}{3}$ or reduced to the form $17\frac{1}{3}$ or to the form $17.33+$ depends upon what use is to be made of it. As an answer $17\frac{1}{3}$ lb. is preferable to $\frac{52}{3}$ lb. \$17.33 is better than $\frac{52}{3}$ dollars, but in computation it is often more convenient to use the form $\frac{52}{3}$.

EXERCISES

Find products:

1. $7\times\frac{2}{3}$, $9\times\frac{7}{5}$, $3\times\frac{5}{3}$
2. $7\times\frac{2}{7}$, $8\times\frac{7}{24}$, $9\times\frac{5}{6}$
3. $15\times\frac{3}{5}$, $25\times\frac{3}{75}$, $\frac{5}{7}\times 49$
4. $46\times\frac{5}{8}$, $32\times\frac{5}{64}$, $\frac{7}{9}\times 56$
5. $24\times\frac{3}{16}$, $45\times\frac{2}{75}$, $\frac{13}{9}\times 54$

3. To multiply two fractions together.

$$\frac{2}{3}\times\frac{5}{7}=\frac{2\times 5}{3\times 7}=\frac{10}{21}$$

Rule. **The product of two fractions is a new fraction whose numerator is the product of the numerators of the two fractions and whose denominator is the product of the denominators of the two fractions.**

In practice do not actually carry out the multiplications until you have made as much use as possible of the principle of dividing both terms by the same number. Thus:

$$\frac{36}{35}\times\frac{10}{27}$$

$$\frac{4\cdot\cancel{9}}{7\cdot\cancel{5}}\times\frac{2\cdot\cancel{5}}{3\cdot\cancel{9}}=\frac{8}{21}$$

EXERCISES

Multiply:

1. $\frac{2}{3}\times\frac{3}{7}, \frac{2}{3}\times\frac{3}{4}, \frac{5}{7}\times\frac{3}{8}$

2. $\frac{12}{5}\times\frac{15}{4}, \frac{9}{8}\times\frac{12}{27}, \frac{45}{32}\times\frac{18}{25}$

3. $\frac{8}{15}\times\frac{30}{64}, \frac{50}{17}\times\frac{34}{50}, \frac{8}{19}\times\frac{19}{7}$

4. $\frac{9}{5}\times\frac{3}{7}\times\frac{15}{2}\times\frac{14}{81}, \frac{7}{6}\times\frac{6}{7}$

5. $\frac{9}{5}\times\frac{20}{36}, \frac{21}{35}\times\frac{10}{6}$

6. $\frac{3}{6}\times\frac{10}{15}\times\frac{21}{35}\times\frac{10}{14}, \frac{5}{7}\times\frac{2}{3}\times\frac{21}{10}$

4. Reciprocal of a number.

$\frac{1}{2}$ is the reciprocal of 2.
$\frac{1}{3}$ is the reciprocal of 3.
2 is the reciprocal of $\frac{1}{2}$.
3 is the reciprocal of $\frac{1}{3}$.
5 is the reciprocal of $\frac{1}{5}$.
$\frac{3}{2}$ is the reciprocal of $\frac{2}{3}$.
What is the reciprocal of $\frac{3}{4}$?
What is the reciprocal of $\frac{5}{3}$?
What is the reciprocal of 5?
What is the reciprocal of $\frac{1}{7}$?
What is the reciprocal of 4?

5. To divide a fraction.

$$\frac{2}{3}\div 7=\frac{2}{3}\times\frac{1}{7}=\frac{2}{21}$$

$$\frac{2}{3}\div\frac{5}{7}=\frac{2}{3}\times\frac{7}{5}=\frac{14}{15}$$

To divide by a number is the same as to multiply by the reciprocal of that number.

RULE. **To divide by a fraction is the same as to multiply by the reciprocal of the fraction.**

EXERCISES

Divide:

1. $\frac{7}{5}\div 2, \frac{6}{7}\div 3, \frac{15}{16}\div 3$

2. $\frac{7}{5}\div\frac{2}{3}, \frac{1}{5}\div\frac{2}{5}, \frac{6}{5}\div\frac{3}{7}$

3. $\frac{3}{4}\div\frac{7}{9}$, $\frac{21}{16}\div\frac{9}{8}$, $\frac{63}{25}\div\frac{7}{15}$

4. $\frac{15}{16}\div 4$, $\frac{3}{8}\div 5$, $\frac{25}{30}\div 15$

5. $\frac{7}{3}\div\frac{2}{3}$, $\frac{6}{5}\div\frac{6}{3}$, $\frac{8}{9}\div\frac{9}{8}$

6. $\frac{7}{12}\div\frac{4}{7}$, $\frac{49}{75}\div 35$, $\frac{7}{16}\div 3$

6. To add two fractions.

(*a*) If their denominators are the same:

$$\frac{2}{3}+\frac{5}{3}=\frac{7}{3}$$

RULE. **Add the numerators for the numerator of the sum and use the given denominator.**

(*b*) If the denominators are not the same:

$$\frac{2}{3}+\frac{5}{7}=\frac{14}{21}+\frac{15}{21}=\frac{29}{21}$$

RULE. **Reduce to fractions having the same denominator and then add as in (*a*).**

EXERCISES

Add:

1. $\frac{1}{2}+\frac{1}{3}$, $\frac{2}{3}+\frac{3}{5}$, $\frac{7}{9}+\frac{2}{5}$

2. $\frac{2}{3}+\frac{5}{6}$, $\frac{3}{2}+\frac{2}{3}$, $\frac{3}{4}+\frac{5}{6}$

3. $\frac{9}{10}+\frac{7}{15}$, $\frac{8}{7}+\frac{9}{14}$, $\frac{3}{16}+\frac{3}{4}$

4. $\frac{1}{8}+\frac{3}{32}+\frac{1}{16}$, $\frac{1}{5}+\frac{1}{6}+\frac{1}{3}$

5. $\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\frac{4}{5}$, $\frac{1}{2}+\frac{2}{3}+\frac{8}{9}$

6. $\frac{3}{7}+\frac{4}{5}+\frac{9}{20}+\frac{5}{14}$, $\frac{4}{5}+\frac{5}{6}+\frac{1}{12}$

7. To subtract one fraction from another. Follow the rule for adding fractions, but subtract when the rule says add.

EXERCISES

Subtract:

1. $\frac{1}{2}-\frac{1}{3}$, $\frac{1}{7}-\frac{1}{9}$, $\frac{2}{3}-\frac{1}{5}$

2. $\frac{9}{7}-\frac{3}{5}$, $\frac{3}{4}-\frac{3}{5}$, $\frac{1}{9}-\frac{1}{12}$

3. $\frac{7}{12}-\frac{2}{8}$, $\frac{9}{16}-\frac{2}{5}$, $\frac{11}{5}-\frac{3}{2}$

4. $\frac{4}{5}-\frac{3}{4}$, $\frac{1}{2}-\frac{2}{5}$, $\frac{7}{6}-\frac{3}{4}$

8. Operations with mixed numbers. The following typical illustrations offer suggestions:

(*a*) Reduce a fraction to a mixed number:

$$\frac{33}{7}=4+\frac{5}{7} \text{ or } 4\tfrac{5}{7}$$

(*b*) Reduce a mixed number to a fraction:

$$7\tfrac{3}{5}=\frac{35}{5}+\frac{3}{5}=\frac{35+3}{5}=\frac{38}{5}$$

(*c*) Add mixed numbers:

$$2\tfrac{2}{5}+3\tfrac{4}{5}=5\tfrac{6}{5}=6\tfrac{1}{5}$$

$$4\tfrac{1}{2}+3\tfrac{1}{3}=7\tfrac{5}{6}$$

(*d*) Multiply mixed numbers:

$$\begin{array}{rl} 32 & \\ 3\tfrac{1}{2} & \\ \hline 96 & 32\times3 \\ 16 & 32\times\tfrac{1}{2} \\ \hline 102 & \end{array} \qquad\qquad \begin{array}{r} 23 \\ 3\tfrac{1}{3} \\ \hline 69 \\ 7\tfrac{2}{3} \\ \hline 76\tfrac{2}{3} \end{array}$$

In multiplying, one might, if he chose, reduce the mixed numbers to fractions and then multiply:

$$23\times3\tfrac{1}{3}=23\times\frac{10}{3}=\frac{230}{3}=76\tfrac{2}{3}$$

The following is generally the more convenient method in case both fractions are mixed numbers:

$$7\tfrac{1}{2}\times2\tfrac{1}{3}=\frac{\overset{5}{\cancel{15}}}{2}\times\frac{7}{\cancel{3}}=\frac{35}{2}$$

EXERCISES

1. Reduce to mixed numbers: $\frac{9}{5}, \frac{81}{7}, \frac{43}{12}, \frac{75}{32}, \frac{17}{8}$.

2. Reduce to pure fractions: $3\frac{1}{2}, 7\frac{2}{9}, 8\frac{3}{7}, 9\frac{2}{3}, 23\frac{2}{3}$.

3. Copy the table at the top of the following page and fill in by adding each number in the horizontal row to each number in the vertical column.

	$2\frac{1}{3}$	$2\frac{1}{2}$	$\frac{2}{3}$	$2\frac{1}{7}$	$4\frac{4}{5}$	5
$2\frac{1}{3}$						
$2\frac{1}{2}$						
$5\frac{2}{3}$						
$6\frac{1}{5}$						
$1\frac{3}{5}$						
3						

4. Copy and fill in the blanks of Exercise 3 by subtracting each number in the horizontal row from each number in the vertical column whenever possible.

5. Copy and fill in the blanks of Exercise 3 by multiplication.

6. Copy and fill in the blanks of Exercise 3 by dividing the numbers in the vertical column by those in the horizontal rows.

7. Multiply: $349 \times 2\frac{1}{3}$, $743 \times 23\frac{2}{5}$, $936 \times 14\frac{3}{7}$, $349 \times 3\frac{1}{7}$.

9. Ratio. When one quantity is twice another quantity, the two quantities are said to be in the ratio 2 to 1. If one quantity is $1\frac{1}{2}$ times another, they are said to be in the ratio 3 to 2.

What is the ratio of 8 to 4?
What is the ratio of 15 to 3?
What is the ratio of 6 to 2?
How many times larger is 9 than 6?
What is the ratio of 9 to 6?
What is the ratio of 3 to 4?

The **ratio** of two numbers is their quotient. To find the ratio of two quantities such as 3 pints and 2 quarts, it is necessary that these quantities be expressed in the same unit.

EXERCISES

1. Find the ratio of 36 to 12.

2. Find the ratio of 12 to 15.

3. Find the ratio of 3 feet to 2 feet.

4. Find the ratio of 3 inches to 1 foot.

5. Is the ratio of 14 feet to 2 feet, 7 feet?

6. What is the ratio of 4 to 12?

7. What is the ratio of $\frac{4}{3}$ to $\frac{2}{3}$?

8. What is the ratio of $\frac{2}{3}$ to $\frac{3}{4}$?

9. Express the ratio 3 to 12 in a decimal form.

10. Find the ratio of 2 to 7 correct to three decimal places.

11. What is the ratio of a mile to a kilometer? 1 mile = 1.6093 kilometers.

12. The population of the United States was 91,972,266 in 1910 and 105,610,720 in 1920. Find the ratio of the population in 1920 to that in 1910 correct to two decimal places.

13. Two rectangles have the dimensions 6 by 15 and 10 by 18. What is the ratio of their areas?

14. The dimensions of one box are 4 by 6 by 10; the dimensions of another are 6 by 8 by 12. What is the ratio of the volume of the first to the volume of the second?

15. If the ratio of two numbers is 3, the first is how many times the second?

16. If the ratio of two numbers is $\frac{5}{3}$, the first is how many times the second?

10. Decimals. Fractions with 10, 100, 1000, etc., for denominators are more conveniently written and used in the decimal form. Thus:

$$\frac{75}{100} = .75, \quad \frac{34}{10} = 3.4$$

11. Addition and subtraction. To **add** decimal numbers, place the numbers with the units digits in the same column and add as in the case of integers. The decimal point of the result will fall directly under the decimal point of the numbers to be added.

To **subtract** decimal numbers, follow the same form as in adding, subtracting instead of adding.

Add	Subtract
3.45	13.45
14.3	7.63
7.6	
25.35	5.82

EXERCISES

Add:

1. 34.26, 32.5, 1.72
2. .432, 7.64, 32.55
3. .036, 72.5, 89.2
4. 96.3, 493.67, 32.4

Subtract:

5. 96.37 from 239.42
6. 3.75 from 432.7
7. 2.346 from 32.49
8. 362.975 from 762.37
9. 40.325 from 40.990
10. 937.8 from 1000

12. Multiplication and division by 10, 100, 1000, etc. To multiply by 10, 100, 1000, simply move the decimal point the proper number of places to the right, as follows:

$2.356 \times 10 = 23.56$, one place to right
$2.356 \times 100 = 235.6$, two places to right
$2.356 \times 1000 =$? , ? places to right

To multiply by 200, move the decimal point two places to the right and multiply by 2:

$$2.314 \times 200 = 231.4 \times 2 = 462.8$$

To divide by 10, 100, 1000, etc., move the decimal point the proper number of places to the left:

$247.3 \div 10 = 24.73$, one place to left
$247.3 \div 100 = 2.473$, two places to left
$2345.6 \div 10{,}000 =$?, ? places to left

To divide by 200, move the decimal point two places to the left and divide by 2.

To multiply by .01 is the same as to multiply by $\frac{1}{100}$, is the same as dividing by 100; therefore move the decimal point two places to the left.

EXERCISES

1. Copy the following table and multiply each number in the horizontal row by each number in the vertical column and place the results in the proper places:

	32	24.2	3.456	.237
10				
100				
3000				
.1				
.02				

2. Copy the following table and divide each number in the horizontal row by each number in the vertical column and place the results in the proper places:

	3.42	47.34	.354	906
10				
1000				
200				
300				

13. Multiplication. RULE. **Multiply as if with integers and point off as many decimal places in the product as there are in both numbers that are to be multiplied together.**

```
  32.7
   2.43
   ----
    981
 1308
 654
 ------
 79.461
```

The following rule gives a better method of multiplying decimal numbers:

RULE. **Write down the numbers as in addition with the units digit in the same column—that is, with the decimal points under each other—and then begin to multiply at the left of the multiplier.**

```
(a)   32.7               (b)   32.7
       2.43                    25.3
      ------                  ------
      65.4                    654
      13.08                   163 5
        .981                    9 81
      ------                  ------
      79.461                  827.31
```

It will be noticed that the decimal point of the product falls directly under the decimal point of the number to be multiplied. The decimal points of the partial products are shown in (*a*), but are omitted in the other illustrations. It is important to determine where to place the first partial product. In (*a*) we find the first partial product by multiplying by 2 .The position of the decimal point is unchanged, and the last figure of the partial product comes under the last figure of the upper factor. The other partial products follow on just as in ordinary multiplication.

In (*b*) we first multiply by 20, that is, we move the decimal point one place to the right and multiply by 2, getting 654; consequently 4 must come in the units column, that is, one place to the left of the last figure of the multiplicand.

(*c*)

```
 32.7
  .32
-----
 9 81
  654
------
10.464
```

(*d*)

```
 32.7
 .032
------
  981
   654
------
1.0464
```

The student can now readily see how to find the position of the first partial product in any case. When that is placed, the others follow as in ordinarv multiplication.

EXERCISES

Multiply:

1. 2.3×7.2	2. 14×3.5	3. $14 \times .72$
4. 4.2×6.71	5. $2\frac{1}{2} \times 7.42$	6. $5\frac{1}{3} \times 83.7$
7. 24.7×39.2	8. 3.52×69.3	9. $7\frac{1}{2} \times 36.26$
10. $.36 \times .0362$	11. $6.34 \times .052$	12. $83.25 \times .6321$

14. Division. Probably the best way of dividing by a decimal is to reduce the divisor to an integer by multiplying by its decimal denominator. Of course the dividend must then be multiplied by the same number. This amounts to moving the decimal point the same number of places to the right in both divisor and dividend. The quotient is to be placed above the dividend; the first figure of the quotient above the right-hand figure of the first partial dividend. Thus:

(*a*) $43.27 \div 3.2$

```
     13.5
  32|432.7
     32
     ---
     112
      96
     ---
      167
      160
      ---
        7
```

(*b*) $3.764 \div 12.37$

```
        .304
  1237|376.4
       371 1
       -----
         5 300
```

The decimal point of the quotient falls above the decimal point of the dividend.

EXERCISES

Divide:

1. $.73 \div 9$	2. $36.2 \div 9.3$	3. $5.73 \div .6$
4. $423 \div .73$	5. $.025 \div .73$	6. $45.2 \div 423$
7. $.73 \div 423$	8. $45.2 \div .73$	9. $9.3 \div .025$
10. $3.52 \div 69.3$	11. $.0036 \div .092$	12. $.035 \div .0023$

15. Use of approximate numbers. Find the circumference of a circle of diameter 27 inches. You have been taught to compute the circumference by multiplying the diameter by $\frac{22}{7} = 3\frac{1}{7}$ or by 3.1416.

Using each of these values, we find:

$$\text{circumference} = 27 \times 3\tfrac{1}{7} = 84\tfrac{6}{7} = 84.857$$

$$\text{circumference} = 27 \times 3.1416 = 84.8232$$

These results agree in the first three figures on the left, but differ in the other figures. Both are correct as far as they agree; neither is correct beyond that point. The reason for this is that the number used as a multiplier of the diameter is not accurately known. The values $3\frac{1}{7}$ and 3.1416 are merely approximations to the number that should be used. The number that should be used is denoted by the Greek letter π (pronounced *pī*). It has been shown that π cannot be exactly expressed by a fraction or a decimal. Out to ten places the number is 3.1415926536.

3.1416 is a five-figure approximation.

3.142 is a four-figure approximation.

$\frac{22}{7}$ or $3\frac{1}{7}$ is a fractional approximation that has been known for two thousand years or more. It is not as accurate as either of the other two, as is shown when it is reduced to the decimal form:

$$3\tfrac{1}{7} = 3.14287+$$

This value $3\frac{1}{7}$ is somewhat easier to use and gives results close enough in ordinary cases where only three figures are used.

Comparing results, we have

$$27\times3\tfrac{1}{7}=84.857$$
$$27\times3.142=84.834$$
$$27\times3.1416=84.8232$$

The last is the most accurate. A still more accurate result would be obtained by using a more accurate approximation for π, as, 3.14159:

$$27\times3.14159=84.82293$$

You will notice, in the comparison above, that each result has figures at the right that are shown to be incorrect when closer approximations are made. Such figures are useless. This fact leads to the very important rule that should always be observed when calculations with approximate numbers are being made:

Rule. **A result of a calculation with approximate numbers should not show more figures than do the approximate numbers used.**

For instance, in the calculations given above, the product 27×3.1416 should not show more than five figures; the product 27×3.142 should not show more than four figures; the result of the multiplication should be cut back to the proper number of figures.

For discarding the meaningless figures at the right of such calculated results the common practice is indicated in the following rule:

Rule. **Determine how many figures you wish to retain, then, if the next figure to the right is less than 5, throw away all figures not to be retained; if the next figure to the right is 5 or greater than 5, add one to the last digit retained and throw away all figures not to be retained.**

For instance;

$$27\times3.14=84.7|8$$

Retain three figures, result 84.8

$$27 \times 3.142 = 84.83|4$$

Retain four figures, result 84.83

$$27 \times 3.1416 = 84.823|2$$

Retain five figures, result 84.823

If $3\frac{1}{7}$ is used for π, only three figures should be retained because $3\frac{1}{7} = 3.1429$ is itself accurate to only three places; hence:

$$27 \times 3\tfrac{1}{7} = 84.857+$$

Retain three figures, result 84.9

Remember whenever $3\frac{1}{7}$ is used for π never to give a result showing more than three figures. For instance, find the circumference of a circle 526 feet in diameter:

$$526 \times 3\tfrac{1}{7} = 1653.1$$

Retain three figures, result 1650

When the 3 is thrown away, its place must be filled by 0 so that the decimal point shall come in the right place. It must be remembered that in all such cut-backs the last figure retained is doubtful, being either too large or too small.

16. Numbers obtained by measuring. There is still another matter to be taken into consideration. The machinist uses calipers (Fig. 1) for measuring directly the diameter of a pipe, shaft, or circular rod.

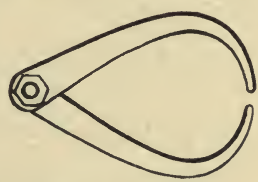

FIG. 1. *Calipers*

Suppose the diameter of a pipe has been measured to the nearest tenth of an inch and found to be 2.7 inches. Using $\pi = 3.1416$,

$$\text{circumference} = 2.7 \times 3.1416 = 8.4823$$

When the diameter was measured more accurately, say to hundredths of an inch, it was found to be 2.73 inches. Now,

$$\text{circumference} = 2.73 \times 3.1416 = 8.5766$$

The last result is the more accurate because a more accurate measurement of the diameter is used.

The measure of the diameter is itself an approximate number. This leads to a slight amendment to the rule, as follows:

RULE. **Do not retain more figures in the result of a multiplication of approximate numbers than there are in that one of the approximate numbers used which has the least number of figures.**

In the multiplication

$$2.73 \times 3.1416 = 8.5766$$

since 2.73 has but three figures, the result should be cut back to three figures, namely 8.58. Had $3\frac{1}{7}$ been used instead of 3.1416, the same result would have been reached, 8.58.

To give another illustration: The edges of a rectangular card when measured to the nearest tenth of an inch are found to be 6.7 and 3.8. In the calculation the doubtful figures are in black type. Only two figures are to be retained in the result.

$$\begin{array}{r} 6.7 \\ 3.8 \\ \hline 20\,1 \\ 5\,36 \\ \hline 25.46 \end{array} \quad \text{Result, } 25$$

When more accurate measurements were made, the edges were found to be 6.76 and 3.82. The calculation is as follows:

$$\begin{array}{r} 6.76 \\ 3.82 \\ \hline 20\,28 \\ 5\,408 \\ 1352 \\ \hline 25.8232 \end{array} \quad \text{Result, } 25.8$$

17. Division of approximate numbers. The rule just given for determining the proper number of figures to show in the result of a multiplication applies also to division.

EXERCISES

The problems given below are supposed to be practical; the numbers given are the results of measurement and are therefore approximate, the right-hand figure being in doubt.

1. Find the circumference of a circle of diameter 1.86 inches; use $3\frac{1}{7}$, 3.142, and 3.1416 and compare results.

2. Find the circumference of a circle of diameter 5.7; use $3\frac{1}{7}$ and 3.142 and compare results.

3. A tree measures $25\frac{1}{2}$ feet around its trunk. What is its diameter?

4. The area of a circle is found by multiplying the square of the radius by π. Find the area of a circle of radius 4.27 inches.

5. A 5-foot concrete walk is laid outside a circular grass plot 60 feet in diameter. What will it cost at 9 cents a square foot?

6. Find the diameter of a circle whose circumference is 43.62 inches.

7. Find the diameter of a stove pipe, steam pipe, water pipe, or any convenient circular pipe or tank by measuring the circumference.

8. Find the diameter of a number of trees in your vicinity. Is there any other way of finding the diameter of a tree?

NOTE. The forester uses an instrument called tree-calipers for finding the diameter of a tree.

9. Measure the diameter of a pipe with calipers; also measure the circumference of the same pipe with a tape. To test the accuracy of the measurements compute the circumference and compare with the direct measure. Explain the resulting situation.

10. A certain rectangular lot was found to be 34.62 feet by 74.08 feet. What is its area?

11. The inside dimensions of a box are carefully measured and found to be 3.55, 6.24, and 5.17 inches. What is its capacity?

12. If the dimensions of the bottom of a box are 2 feet 6 inches and 4 feet 3 inches, how deep should the box be that it may contain 1 cubic yard?

13. If 1 cubic foot of water weighs 63.255 pounds, what does a quart of water weigh? 57.75 cubic inches is a liquid quart.

14. A dry quart is equivalent to 67.20 cubic inches. Does the ordinary square strawberry box contain a quart when level full?

18. Order of operations. As far as the result is concerned, it does not matter in what order several additions are made:

$$3+7+5=10+5=15$$

or $$=3+12=15$$

or $$=8+7=15$$

Sometimes the result is obtained more quickly if the additions are grouped in tens:

$$5+9+4+7+1+3+6=35$$

$$5+10+10+10=35$$

So also in a sum of additions and subtractions, the order in which the work is done does not matter.

$$\begin{aligned}7-2+5&=7+5-2\\&=12-2\\&=10\end{aligned}$$

We usually proceed from left to right when possible, thus:

$$\begin{aligned}23-4+3-7&=19+3-7\\&=22-7\\&=15\end{aligned}$$

But in $$7+2-12+8=9-12+8,$$

as we cannot subtract 12 from 9, we must add $9+8$ first and get

$$17-12=5$$

In a series of additions, subtractions, multiplications, and divisions the multiplications and divisions are to be done before the other operations, thus:

$$\begin{aligned}2+3\times5-7+6\div3&=2+15-7+2\\&=17-7+2\\&=10+2\\&=12\end{aligned}$$

The order in which the operations are to be done may be changed by the use of parentheses, thus:

$$\begin{aligned}(2+3)\times5-7+6\div3&=5\times5-7+6\div3\\&=25-7+2\\&=18+2\\&=20\end{aligned}$$

The operations within the parentheses must be done first

EXERCISES

1. $25+13+15$
2. $7+19+3$
3. $8+7+2+3$
4. $4+9+1+6+8$
5. $18-7+3$
6. $18-(7+3)$
7. $9\times6+3$
8. $9+6\times3$
9. $9\times(6+3)$
10. $18-9+3-7$
11. $9-13+15$
12. $\frac{2}{3}\times6+2$
13. $\frac{2}{3}\times(6+2)$
14. $3\times6+5-7+3\times2$
15. $2\times(7-3)+2-7$
16. $5\times\frac{7+2}{6}$
17. $\frac{5+2}{8}-\frac{7-3}{16}$

CHAPTER II

FORMULAS AND EQUATIONS

19. The formula. How many square feet are there in the floor of a room 14 feet by 18 feet? How many square yards are there in a tennis court 36 feet by 78 feet? What is the area of a city lot 40 feet wide and 150 feet deep? How would you find the area of a card having square corners?

In words the answer is: The area of a rectangle is found by multiplying its length by its width.

This rule may be stated more briefly by the omission of unnecessary words and by the use of certain symbols:

$$\text{area} = \text{length} \times \text{width}$$

This may be still further shortened by the use of the first letter of each word:

$$a = l \times w$$

We may even omit the times sign and agree that the two letters lw standing together shall mean "length times width" and write the rule in the form

$$a = lw$$

The multiplication sign cannot be omitted when two arithmetical numbers are used such as:

2×5, for 25 already has another meaning

$2 \times \frac{2}{3}$, for $2\frac{2}{3}$ already means $2 + \frac{2}{3}$

It is convenient to have a special name for such a shorthand way of writing a rule. The Latin word *formula* is an appropriate word to use, for it means "short form."

It has become a very common practice to write rules in this shorthand fashion. Such a formula is easy to remember, its meaning is quickly grasped, and it is easily used in any particular example. The use of formulas leads to orderly ways of setting down work.

Illustration. How many square feet are there in a 5-foot concrete walk in front of a 40-foot lot?

Rule or formula to be used:

$$a=lw$$

Here l is 40 and w is 5

Hence we have

$$a=40\times 5$$
$$=200$$

EXERCISES

Work the following examples in the same way:

1. Find the area of a window glass 13 inches by 15 inches.
2. Find the area of a wall 9 feet by 22 feet.

Express the following rules as formulas, using the first letters of the quantities:

3. Rule for finding the total cost of a number of articles when the price of one article is given.
4. Rule for finding the number of inches in a given number of feet.
5. Rule for changing pounds into ounces.
6. Rule for computing the circumference of a circle if the diameter is given.
7. Any other rules that you can call to mind in which one number is found by multiplying together two other numbers.
8. Rule for finding the volume of a room.
9. Rule for finding the area of a triangle.
10. Make numerical examples to fit each of the formulas of Exercises 3–9.
11. What must be the length of a board 9 inches wide if its area is to be 144 square inches?

The rule for finding the length of a rectangle when the area and width are given is

$$\text{length}=\text{area divided by width}$$

or, better,

$$\text{length}=\frac{\text{area}}{\text{width}}$$

What is the shorthand form? Find the answer to the exercise

12. Write in shorthand the rules for finding:

   a) The price of one article when the cost of a number is given.

   b) The diameter of a circle when the circumference is given.

   c) The number of dozens in a given number of eggs.

   d) The number of yards in a given number of feet.

13. Make numerical exercises to fit each of the formulas in Exercise 12.

14. Find the number of inches in 3 feet 5 inches.

15. Write the formula for:

   a) The number of inches in a measurement given in feet and inches.

   b) The number of pints in a measurement given in quarts and pints.

   c) Computing the year in which you were born from your age.

   d) Rule for making coffee, one teaspoonful for each person and one for the pot.

16. Make numerical examples for each of the formulas in Exercise 15.

17. Write a formula for finding the perimeter of a rectangle.

20. The formula states a relation. The formula

$$a = lw$$

is a short way of writing the rule for finding the area of a rectangle when the length and width are given.

The formula $$l = \frac{a}{w}$$

is a short way of writing the rule for finding the length when the area and width are given.

The important thing for us to notice is that the formula

$$a = lw$$

really includes the formula $l = \frac{a}{w}$

One involves multiplication and the other division.

One asks the question $? = 4 \times 5$

the other asks the question $? = \frac{20}{5}$

or, in another form, $5 \times ? = 20$

What question does $a = 7 \times 8$ ask? How answered?

What question does $l \times 7 = 28$ ask? How answered?

What does $9w = 26$ ask?

The formula $a = lw$ is something more than a rule: it is a statement of the fact that the length, width, and area of a rectangle are so related to one another that the area equals the product of the length and the width. This may be called the **area law of the rectangle.** If any two of these three numbers are known, the other one can be determined. We then really have need of only the one formula $a = lw$ for this class of problems, or $lw = a$, for it matters not which way the equality is written.

Illustration. How wide is a room if its length is 22 feet and its area is 396 square feet?

Formula $lw = a$ where $l = 22$, $a = 396$

$$22w = 396$$

$$w = \frac{396}{22}$$

$$= 18$$

PROBLEMS

In solving the following nine problems write a formula for each and then solve the problem:

1. A farm of 60 acres sold for $4,500. What was the price per acre?

2. A city lot sells for $5000, the rate being $125 a front foot. Find the number of front feet in the lot.

3. A man who eats his dinners at restaurants finds that for a certain week his dinners cost him $4.50. What was the average cost of a dinner?

4. A certain grade of pencils sells at wholesale for $12.96 a gross. What is the cost per pencil?

5. How much water flows through a pipe in one minute if 450 gallons are delivered in $2\frac{1}{2}$ hours?

6. The dimensions of a box are $9\frac{1}{2}$, $3\frac{1}{4}$, and 16 inches. How many cubic inches does the box contain?

7. What is the height of a box 8 inches wide and $18\frac{1}{2}$ inches long if it contains 988 cubic inches?

8. What must be the depth of a box 6 inches in length and $3\frac{1}{2}$ inches in width to contain a pint (231 cu. in. = 1 gal.)?

9. A printers' rule for computing the charges to be made on a certain kind of circular is 75 cents for setting up and 25 cents a hundred for printing. What was the bill for 700 circulars?

Solve the following six problems by use of but one formula:

10. What is 5 per cent of 75?

11. 21 is what per cent of 350?

12. 24 is 8 per cent of what number?

13. 35 is what per cent of 210?

14. 36 is 5 per cent of what number?

15. 42 is 6 per cent of what number?

16. Write formulas for the following rule: To find gain or loss multiply the cost by the rate per cent of gain or loss. Use c for cost, r for rate, g for gain, l for loss.

17. What does a man gain on goods that cost $720 if he sells them $12\frac{1}{2}$ per cent above cost?

18. If goods cost $480 and are sold at a gain of $80, what is the gain per cent?

19. What is the cost of goods if $96 is gained when they are sold $8\frac{1}{3}$ per cent above cost?

20. What does a man lose on goods that cost $360 if he sells them at 6 per cent below cost?

21. What is the cost of goods if $20 is lost when they are sold at 5 per cent below cost?

22. What connection is there between Problems 16–21 and Problems 10–15?

23. From the formula $\pi D = 15$, find D. What is the meaning of the formula?

24. If a train travels at the average rate of 40 miles an hour, how far does it go in 2 hours? In 3 hours? In $\frac{3}{4}$ hour? In $\frac{1}{4}$ hour? In 10 minutes? In t hours?

State as a formula the rule just used, putting d for distance, r for rate, and t for time. Use this formula in working the following five problems.

25. How far can an airplane fly in 3 hours at the rate of 120 miles an hour?

26. A train travels from Chicago to St. Louis, a distance of 294 miles, in 8 hours. What is the average speed per hour, not counting stops?

27. How long will it take an automobile going at the rate of 15 miles an hour to travel 35 miles?

28. How long will it take the automobile mentioned in Problem 27 to go the length of a city block of 300 feet?

29. It was noticed that an automobile traveled a block 300 feet long in 10 seconds. Was it exceeding the speed limit of 15 miles an hour?

Three times a number added to 2 times the same number is 30, may be written in the short form, n being used for the number:

$$3n + 2n = 30$$

The directions for finding the number are not given. Can you find out what to do in order to find the number? Ask yourself the question: How many times a number is 3 times the number and 2 times the number?

Work out the following four problems in the same way:

30. Nine times a number is 35. What is the number?

31. Seven times a number less 4 times the number is 123. What is the number?

32. Seven times a number increased by 4 times the number is 132. What is the number?

33. What number added to 5 times itself equals 102?

21. Directions for work not given. A formula states a rule for finding some required number. For instance, when the area and width of a rectangle are given, the length is found by use of the formula

$$l=\frac{a}{w}$$

The directions for the work are all indicated. We could use the formula

$$lw=a$$

for solving the same problem. In this case not all the directions for finding l are given, and we must think a little to find out just what to do. We must answer the question

$$?\times w=a$$

The name **equation** is used for both types of formulas.

22. Any letter used for the unknown. In the formulas thus far we have been using the first letters of the quantities to stand for the numerical values of those quantities. This is sometimes very helpful and convenient, but it is not necessary to confine ourselves to first letters. We may use any letter we choose to represent the unknown in a formula or an equation.

Five times an unknown number plus 2 times the same number equals 14.

$$5?+2\,?=14$$

$$5n+2n=14$$

$$5x+2x=14$$

The three expressions all have the same meaning. The letter x has been used so often to represent the unknown number that when Roentgen discovered a new ray which was almost entirely strange to him he called it the X-ray, the unknown ray.

The use of a letter to stand for a number whose value is unknown is one of the important ideas of algebra.

23. Operations on equations. Let us examine carefully the way in which such an equation as is given in the last article is worked out to find the unknown number.

The problem stated in algebraic language is

$$5n+2n=14$$

We are to regard this equation as asking the question: What value can be given to n that will make the statement true? One sees at once that 5 times a number added to 2 times the same number is 7 times that number, and we may write

$$7n=14$$

The 7 was found by adding together the 5 and the 2. This operation may be expressed on paper thus:

$$5n+2n=(5+2)n$$
$$=7n$$

The **parenthesis** ( ) is used to denote the fact that 5 and 2 are to be added together and the result used as a multiplier of n.

Now taking the equation

$$7n=14$$

we say that if 7 times a number equals 14, the number will be $\frac{1}{7}$ of 14.

This operation may be written

$$n=\frac{1}{7} \text{ of } 14$$

or, better,

$$n=\frac{14}{7}$$
$$=2$$

That is, we may divide the 14 by 7.

It is to be noticed that all the things that were done to find the unknown number may be written down in algebraic shorthand and arranged in a clear and orderly form:

$$5n+2n=14 \quad (1)$$

$$(5+2)n=14 \quad (2)$$

$$7n=14 \quad (3)$$

$$n=\frac{14}{7} \quad (4)$$

$$n=2 \quad (5)$$

The important thing to notice now is the way in which we passed from one equation to the next. We did something to each equation in order to get the next equation. To get (2) from (1), we added 5 and 2; (3) is merely a condensed form of (2). In getting (4) from (3) we divided 14 by 7.

We may, however, look at this last step in a slightly different way. If we divide 3×5 by 3, the result is 5. If we divide 7×6 by 7, the result is 6. If we divide the product of any two factors by one of the factors, the result is the other factor. If we divide $7n$ by 7, the result is n. We may, then, consider that we obtain equation (4) from (3) by dividing both sides by 7. Equation (5) shows (4) in a more condensed form.

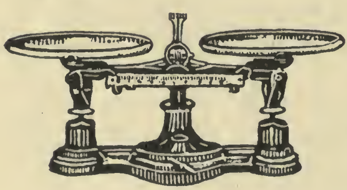

FIG. 2

This way of looking at the matter opens up a very important point of view. The equation states the fact that two numbers are equal to each other. It is to be regarded as a sort of balance, like a druggist's pan balance (Fig. 2).

If any object be placed in one pan, that pan will go down unless an equal weight is put in the other pan. The same weight must be in each pan to keep the instrument in balance.

The two sides of an equation are like the two arms of the balance (Fig. 3). Nothing must ever be done to the

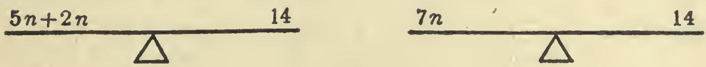

FIG. 3

equation that will disturb this balance. If the value of one side be changed in any way, the other side must be changed in exactly the same way. The problem at issue is always to find the value of the unknown. The unknown number is more or less tangled up with the known numbers in the equation. We must do something to the equation to untangle the unknown. We have just discovered two operations that can be used in untangling the unknown without disturbing the balance of the equation, namely:

OPERATION 1. **Any indicated operation within a side of the equation can be carried out.**

OPERATION 2. **Both sides of an equation can be divided by the same number.**

Illustration.

$$9n-5n=12$$
$$4n=12 \quad \text{(operation 1)}$$
$$n=3 \quad \text{(operation 2)}$$

EXERCISES

Find the value of the unknowns in the following equations, indicating in each step the operation used:

1. $4x+7x=69$
2. $5a+9a=98$
3. $12x-2x+5x=75$
4. $8n-5n+6n-4n=30$
5. $11t-5t+9t-3t=72$
6. $5y+7y-6y-y=16+6-7$
7. $8t-3t+4t-2t=30-5+3$
8. $7x+5x-9x+2x=30-12+2$

Find value of:

9. $3x+4x-2x$ if $x=5$
10. $7a+2-3a-1$ if $a=7$
11. $9b-7-2b+12$ if $b=5$
12. $8c+6-5c-3$ if $c=4$

24. Definitions. It is convenient to have names for things to which we constantly refer. We have already spoken of the **sides** of an equation, the parts on either side of the equality sign. The parts on one side that are separated by addition or subtraction signs are called **terms.** In $7n+4n$, $7n$ is a term; so also is $4n$.

A term may have only one number in it or it may be made up of several factors. The part of the term that is represented by an Arabic numeral is called the **numerical coefficient** of the term or, more simply, the **coefficient** of the term. In the term $7n$, 7 is the numerical coefficient. In the term $3hw$, 3 is the numerical coefficient. What is the coefficient of the term $17t$? Of $.06pt$?

No coefficient appears in the term n; in such cases it is to be understood that the coefficient is 1.

The letter used to stand for a number the actual value of which is not known is called the **unknown.**

To **solve** an equation is to find the value of the unknown.

The actual value of the unknown is called the **root** of the equation.

In the solving of an equation certain operations are carried out which result in the finding of a certain value for the unknown. Mistakes in the work may have been made. In order to discover if the value found is really the true value, we must **check** the work.

25. Checking. The pupil checks the solution of an equation by putting the value found in place of the letter that stands for it in the equation, and then working down each side by itself, arithmetically. If the two sides come out the same, it is generally safe to say that the work is correct and that the value found is the root of the equation. If they do not come out the same, there is trouble somewhere and a search should be made for the mistake. Mistakes may occur either in finding the root or in checking

Illustration. Take the example solved in Art. **23**, page 29,

$$9n-5n=12$$

where we found $n=3$

Substitute 3 for n, $9\cdot 3-5\cdot 3=12$

$$27-15=12$$

$$12=12$$

The number 3 satisfies the equation; the work is correct.

EXERCISES

Solve and check:

1. $6n+3n=72$
2. $5x+8x-9x=48$
3. $7t-2t+5t=95$
4. $8x+7x-9x=70$
5. $9a+8a-3a=70$
6. $6x-2x+9x-3x=72$
7. $3b+5b-2b-b=17-9+2$
8. $9c-3c+2c-c=14+10-3$

Find the value of:

9. $11c-3-2c+7$ if $c=4$
10. $5a+a-8+12-3a$ if $a=5$

26. The subtraction operation. Consider the problem: What number added to 7 equals 15? This can be worked very quickly by arithmetic by the simple device of subtracting 7 from 15. Translating the problem into algebraic language, we have the equation

$$7+n=15$$

which asks the question $7+?=15$

By arithmetic we have at once

$$n=15-7$$

$$=8$$

Supply the missing number in each of the following:

$$?+4=7$$

$$n+2=9$$

$$3+x=11$$

$$12=3+?$$

$$15=n+7$$

$$17+n=17$$

Now let us consider the matter from the algebraic point of view. The statement

$$7+n=15$$

is an equation. We must do nothing to it that will destroy its balance. Since we obtained the answer by subtracting 7 from the right-hand side, we must also subtract (or take away) 7 from the left side of the equation. We have thus found another operation that can be applied to an equation, namely:

OPERATION 3. **The same number can be subtracted from both sides of an equation.**

The solution of the equation should be written down in the following orderly form:

$$7+n=15$$

$$n=15-7$$

$$=8$$

Solve the equation, $3n+8=20$

The unknown number $3n$ is tangled up with 8 by addition. We may untangle it by using the subtraction operation and

$$3n=12$$

Here n is tangled up with 3 by multiplication. We may untangle it by the use of the division operation and

$$n=4$$

Notice that the end in view was to get the unknown number n all alone on one side of the equation and at the same time have nothing on the other side except known numbers. To do this it was necessary to get rid of the term that had no unknown in it and also to get rid of the multiplier 3. We chose the operations that would do these things:

Subtraction of a term to get rid of a term

Division by a multiplier to get rid of a multiplier

The work should be set down as follows:

$$3n+8=20$$

Subtract 8 from both sides,

$$3n=20-8$$

$$3n=12$$

Divide both sides by 3, $n=\frac{12}{3}$

$$=4$$

EXERCISES

Solve and check:

1. $x+18=23$	2. $x+3=15$	3. $19+n=25$
4. $15+x=40$	5. $27=n+17$	6. $23=3x+5$
7. $3a+4=19$	8. $100=85+t$	9. $5a=3a+14$
10. $3x+12=54$	11. $9x=30+4x$	12. $7a=45+2a$
13. $6a+a=36+3a$	14. $7y-y+2=44$	15. $4x+x+3=23$

Find value of:

16. $13x-4x-7$ if $x=2$

17. $9x+3x-2-6x$ if $x=3$

18. $12a-6-2a+8$ if $a=10$

19. $13n-3+2n-5$ if $n=4$

27. Subtraction operation repeated. Consider the equation

$$8+6n=23+n$$

To solve, take one needed step at a time. On the left side $6n$ and 8 are tangled by addition, hence to get rid of 8,

Subtract 8 from both sides,

$$6n=15+n$$

On the right side 15 and n are tangled by addition, hence to get rid of n,

Subtract n from both sides,

$$5n=15$$

n is tangled with 5 by multiplication, hence to get rid of 5,

Divide both sides by 5, $n=3$

The solution should be put down as follows:

$$8+6n=23+n$$

Subtract 8 from both sides,

$$6n=15+n$$

Subtract n from both sides,

$$5n=15$$

Divide both sides by 5, $n=3$

*Check:**

$8+6\cdot 3$	$23+3$
$8+18$	$23+3$
26	26

EXERCISES

Solve and check equations 1–10:

1. $2x+5=x+15$
2. $5n+2=3n+4$
3. $5n+2=3n+11$
4. $7x+5=x+29$
5. $12a+3=5a+17$
6. $8x+4=5x+20$
7. $9x+7=2x+35$
8. $7a+2=3a+22$
9. $10x+8=3x+25$
10. $14x+6=6x+24$
11. $7x-2x+8-3x=?x+?$
12. $3x+8+5x-2=?x+?$
13. $2x+9+6x-4=?$
14. $7x+3-2x+8=?$

28. More complicated equations. Consider the equation

$$5+6n+3=28+2n$$

This equation may be solved by three applications of the subtraction operation. We may shorten the work, however, by adding terms within each side whenever possible before using the subtraction operation. We cannot add 5 and 6, because $5+6n+3$ means $5+6\times n+3$ and we must obey the laws of order (see Art. **18**). We may change the order of addition, however, and add 5 and 3. The work is as follows:

*We use the vertical line to separate the two sides to be checked.

$$5+6n+3=28+2n$$

Add 5+3, $8+6n=28+2n$

Subtract 8 from both sides, $6n=20+2n$

Subtract $2n$ from both sides, $4n=20$

Divide both sides by 4, $n=5$

EXERCISES

Solve and check:

1. $x+6x+2=14+3x$
2. $9n-2n+6=27+4n$
3. $7a+a+5=21+4a$
4. $7a+8-6=26+3a$
5. $19c+9-5=37+8c$
6. $8x+5-3=5x+17$
7. $7x-2x+8=23$
8. $12+9x-3x=48$
9. $2+11p+3=7p+25$
10. $21x-7x+5=18+3x$

Find value of:

11. $3a-2b$ if $a=6$, $b=2$
12. $5x+2y$ if $x=4$, $y=5$
13. $6a-2b+12$ if $a=5$, $b=4$
14. $9a-2+4b$ if $a=3$, $b=5$

29. The addition operation. Five times a number less 17 equals 8. What is the number? In algebraic language this is

$$5n-17=8$$

Each side of the equation is 17 less than 5 times the number. If 17 be added to both sides, the deficiency will be made up.

$$5n=25$$

that is, $5n-17+17=8+17$

Notice that on the left side the same number 17 is subtracted and added; these operations destroy each other. The solution is then

$$5n-17=8$$

Add 17 to both sides, $5n=25$

Divide both sides by 5, $n=5$

Check:

$5 \cdot 5-17$	8
$25-17$	8
8	8

We have thus brought to light another operation that can be performed on both sides of an equation. Stated in words we have

OPERATION 4. **The same number can be added to both sides of an equation.**

EXERCISES

Solve and check:

1. $x-15=30$
2. $x-9=21$
3. $a-16=46$
4. $3t-5=15$
5. $7t-3=8$
6. $3y-10=20$
7. $n-7-6=3$
8. $3x-10+2x=40$
9. $8n-1=5n+41$
10. $3+5x=11-2x$
11. $6x-4=24-x$
12. $3x-21=2x-11$
13. $13n-6=8n+14$
14. $7x+15=2x+25$

Although we usually read an equation from left to right, it is sometimes more convenient to read it from right to left and keep the unknown on the right instead of on the left. For instance,

$$8+n=3n$$
$$8=2n$$
$$4=n$$

15. $3a+13=23a-7$
16. $2x+24=8x+6$
17. $2x+21=9x-3$
18. $4x+37-x=9x+9-2x$
19. $3x+32-x+2x=12x+12-3x$
20. $3n+29+4n-2-2n=10n+9$
21. $2h=5h-5-2h$
22. $4-x=2$

30. The multiplication operation. The problem: One-third of a number is 4, what is the number? can be answered very quickly by arithmetic. Simply multiply 4 by 3. Translated into algebraic language, the problem is

$$\frac{1}{3}n=4$$

or

$$\frac{n}{3}=4$$

for just as $\frac{1}{3}$ of 5 may be written either as

$$\frac{1}{3}\times 5 \text{ or } \frac{5}{3}$$

so also $\frac{1}{3}$ of n may be written

$$\frac{1}{3}n \text{ or } \frac{n}{3}$$

so also $\frac{2}{3}$ of n may be written either as

$$\frac{2}{3}n \text{ or } \frac{2n}{3}$$

To get the value of n from $\frac{n}{3}=4$, we multiplied 4 by 3. To preserve the balance of the equation, we must multiply the left side also by 3. That is,

$$3 \cdot \frac{n}{3}=3 \cdot 4$$

$$n=12 \quad \text{(See Art. 2)}$$

This discloses another operation that can be used in handling equations:

OPERATION 5. **Both sides of an equation may be multiplied by the same number.**

Apply to the equation

$$\frac{1}{3}n+8=15$$

n is tangled up with 3 by division, hence to untangle n, multiply by 3. That means both sides must be multiplied by 3. Why?

The work proceeds as follows:

$$\frac{1}{3}n+8=15$$

Multiply both sides by 3, $\quad n+24=45$

Subtract 24 from both sides, $\quad n=21$

Check:

$$\begin{array}{r|l} \frac{1}{3}\cdot 21+8 & 15 \\ 7+8 & 15 \\ 15 & 15 \end{array}$$

A more complicated case:

$$\frac{1}{2}x-2=\frac{1}{3}x+\frac{1}{3}$$

To untangle the x on left side, multiply both sides by 2:

$$x-4=\frac{2}{3}x+\frac{2}{3}$$

To get rid of the fractions on the right side, multiply both sides by 3:

$$3x-12=2x+2$$

Add 12 to both sides, $3x=2x+14$

Subtract $2x$ from both sides, $x=14$

The first two steps may be taken at the same time by multiplying both sides by $2\times 3=6$.

Thus $\frac{1}{2}x-2=\frac{1}{3}x+\frac{1}{3}$

Multiply both sides by 6, $3x-12=2x+2$

Add 12 to both sides, $3x=2x+14$

Subtract $2x$ from both sides, $x=14$

Check:

$$\begin{array}{r|l} \frac{1}{2}\cdot 14-2 & \frac{1}{3}\cdot 14+\frac{1}{3} \\ 7-2 & \frac{14}{3}+\frac{1}{3} \\ 5 & \frac{15}{3} \\ 5 & 5 \end{array}$$

EXERCISE I

Find value of:

1. $4\left(\frac{1}{4}x+3\right)$ if $x=2$, $x=3$, $x=10$

2. $3\left(\frac{x}{3}+5\right)$ if $x=2$, $x=5$, $x=7$

3. $6\left(\frac{a}{3}+8\right)$ if $a=2$, $a=5$, $a=7$

4. Show that $6\left(\frac{a}{3}+5\right)=2a+30$. Find value of both sides if $a=2$; if $a=7$.

5. Show that $10\left(\frac{2x}{5}+7\right)=4x+70$. Find value if $x=3$; if $x=5$.

6. Are the statements in Exercises 4 and 5 true for any values of a and x that you may choose?

EXERCISE II

Solve and check:

1. $\frac{1}{3}x=5$
2. $\frac{3}{2}n=12$
3. $\frac{a}{7}=3$
4. $\frac{5x}{11}=15$
5. $\frac{4x}{5}=16$
6. $\frac{1}{2}n+n=9$
7. $\frac{n}{3}+n=12$
8. $t-\frac{t}{3}=6$
9. $\frac{x}{3}=\frac{2}{3}+5$
10. $\frac{1}{5}a-2=\frac{3}{5}$
11. $m-\frac{3m}{4}=8$
12. $\frac{t}{2}+\frac{t}{3}=5$
13. $\frac{x}{12}+\frac{x}{6}=5$
14. $\frac{k}{4}+\frac{k}{2}-\frac{k}{5}=11$
15. $\frac{4}{7}=\frac{a}{3}$
16. $\frac{1}{2}y-\frac{3}{4}=\frac{1}{3}y-\frac{1}{3}$
17. $\frac{4x}{3}-2=\frac{x}{5}-\frac{2}{5}x$
18. $7n-5n+\frac{1}{2}n=\frac{5}{6}$

19. One-third of a number increased by one-half the same number equals 35. What is the number?

31. Summary of allowable operations on equations. In the preceding articles we have been learning how to solve equations. Our effort has been to untangle the unknown number from all the known numbers and thus determine its actual value. In doing this we have found five operations that can be used upon an equation without disturbing the balance of the equation. These operations do not in any way change the value of the unknown number; they merely get rid of certain known numbers that are in positions where they are not desired. We may use any of the five operations that seem necessary with the certainty that the number sought will be found if there is such a number.

Operation 1. Either side of an equation may be altered in form by the performance of any of the indicated operations that are possible.

Operation 2. Both sides may be divided by the same known number, excepting the number zero.*

Operation 3. The same number may be subtracted from both sides.

Operation 4. The same number may be added to both sides.

Operation 5. Both sides may be multiplied by the same known number.

We shall find later that there are other operations that can be used in solving equations, but we do not need to consider them here.

The operations to be used in solving any given equation are the ones that will do what is needed to be done. There is no particular order in which these operations should be used. Simply decide what needs to be done and then choose the operation that accomplishes your purpose. Often a thoughtful choice of one operation rather than another will greatly reduce the amount of work necessary.

Illustration. $5x-7-3x=21$

Here it is best first to make any alteration in the left side that may be possible; that is, $3x$ can be subtracted from $5x$, and we have

$$2x-7=21$$

We then need to get rid of the term 7, and we can do this by adding 7 to both sides:

$$2x=28$$

*Division by zero is not permitted in mathematics.

To get rid of the coefficient 2, use division:

$$x=14$$

Check:

$5\times14-7-3\times14$	21
$70-7-42$	21
$63-42$	21
21	21

MISCELLANEOUS EXERCISES

Solve and check. In each case indicate the operation used.

1. $6n+5=47$
2. $7+n=31$
3. $3n+2n+16=31$
4. $5n+3n+10=90$
5. $\frac{n}{3}+\frac{n}{5}=24$
6. $46-3n=7$
7. $5+\frac{1}{4}n=n$
8. $32=\frac{1}{5}x$
9. $48=\frac{5}{100}x$
10. $n+5=11$
11. $15+b=40$
12. $9n-3n=66$
13. $4n+n=4$
14. $8x-2=6x+6$
15. $9n-5=6n-2$
16. $2n-5-n-3=0$
17. $18-n=n-18$
18. $5+4x=2x+10$
19. $4n+5=5n+5$
20. $5-3y=y-1$
21. $6x-4=3x-2$
22. $15-7t=9t+3$
23. $2x-8+3x=x+12$
24. $\frac{2}{3}x+5=\frac{1}{3}x+11$
25. $\frac{1}{2}x-2=\frac{1}{3}x+1$
26. $7h+5+2h-2=4h+3$

Suggestion. It is often better to add a term to both sides or to subtract a term from both sides before any attempt is made to combine terms on the same side. In some cases this must be done.

$$7+3x-3=x-3+15$$

Add 3 to both sides, $7+3x=x+15$

Then $3x=x+8$

$$2x=8$$

$$x=4$$

27. $15x+42=3+10x+42$
28. $11x+13-3x=29-3x$
29. $15a-42-7a=93-7a-50$
30. $3+2x-6=1$
31. $4a+7-5a=6a-42$

CHAPTER III

THE SOLUTION OF PROBLEMS

32. Introductory. One of the most important uses of algebra is the solution of problems. In fact, the algebraic language has been developed in an effort to simplify the working out of problems. When a problem is presented, the first thing to be done is to translate its ideas into algebraic shorthand.

33. Translations. In translating from English words into algebraic symbols notice:

(1) An Arabic figure is used to stand for a known number.

(2) A letter is used to stand for an unknown number.

(3) The plus sign, $+$, is used to express a sum.

(4) The minus sign, $-$, is used to express a difference.

(5) In the writing of products the factors are written close together without a sign between them: $7a$ instead of $7 \times a$. If, however, we have the product of two known numbers, a sign must be used between them: 7×6 or $7 \cdot 6$.

(6) The quotient of two numbers is written as a fraction, $\frac{4}{5}, \frac{a}{5}$.

34. Exercises in translation. State the following ideas in algebraic language:

1. Three added to an unknown number.
2. An unknown number minus 5.
3. Eleven times a number.
4. One-fifth of a number.
5. Nine divided by an unknown number.
6. The quotient of 7 and an unknown number.
7. One-sixth of 5 times a number.

8. The sum of an unknown number and 7.
9. The sum of a number and 7, divided by 9.
10. A number plus 7, divided by 9.
11. The difference between a number and 5.
12. The difference between a number and 7, divided by 7.
13. Three times a number, plus 7.
14. Three times the sum of a number and 8.
15. The ratio between two numbers.
16. Three times the difference between a number and 4.

When it is necessary to multiply or to divide a sum or a difference by any number, inclose the sum or the difference in parentheses; for example, $4(x+1)$ represents 4 times the sum of a number and one.

17. Translate the following into English, using the words "sum," "difference," "product," and "quotient" wherever they are called for:

(*a*) $4+a$	(*b*) $5x$	(*c*) $\frac{5}{x}$
(*d*) $\frac{a}{5}$	(*e*) $a-1$	(*f*) $6b+2$
(*g*) $8-2x$	(*h*) $a+\frac{4}{5}$	(*i*) $\frac{a+4}{5}$
(*j*) $5(a+4)$	(*k*) $5a+4$	(*l*) $5a+2b$
(*m*) $5a-4$	(*n*) $5(a-4)$	(*o*) $\frac{2b}{3}$
(*p*) $\frac{2}{3}b$	(*q*) $(1-x)5$	(*r*) $\frac{7}{x-2}$

18. Evaluate each of the expressions given in Exercise 17, using $a=10$, $b=6$, $x=2$.

Translate the following into algebraic language:

19. A man is y years old. How old will he be in 3 years?
20. A boy is x years old. How old was he 5 years ago?
21. If a boy receives x cents a day, how much will he receive in a month?

22. How many feet in 6 yards? How many feet in b yards?

23. A boy earns x dollars a week. How much does he earn in 10 weeks? How much does he receive in 10 weeks if he receives a raise of one dollar a week?

24. What is the average of 3, 10, 17?

25. What is the average of a, b, c?

26. If sweaters sell at wholesale at $75 a dozen, what is the cost price of one sweater? Of b sweaters?

27. If the wholesale price of the sweaters in Exercise 26 is increased x dollars a dozen, what will be the cost of one sweater at the increased price?

28. What is the reciprocal of 2? Of 7? Of $\frac{1}{3}$? Of $\frac{2}{3}$?

29. What is the reciprocal of a?

30. If the difference between two numbers is 5 and the larger is represented by a, what will represent the other? What will represent their sum? Their product? Their quotient?

31. The weight of a cup containing 250 shot is W; the weight of the empty cup is w. What will express the weight of one shot?

32. The dimensions of one wall of a room are x and y feet. What will express the area of the wall? If the wall has 3 windows each a by b feet, what will express the area of each window? Of all 3 windows? Of the wall space?

Translate into words the following algebraic statements:

33. $3a$	34. $5a+7$	35. $2x-3$
36. $\frac{x-2}{3}$	37. $5-n$	38. $2t+5t-3$
39. $3(x-2)$	40. $\frac{5}{n}$	41. $\frac{7}{n}-3$
42. $10n+3(n-1)$	43. $100-\frac{n}{3}$	44. $\frac{n+2}{n-2}$

35. Problems relating to numbers. In trying to solve the following problems: (a) translate the sentence into an equation, using some letter for the unknown; (b) solve the equation; (c) check the result.

Problems:

1. The sum of an unknown number and 8 is 23. Find the number.

2. The difference between an unknown number and 10 is 2. Find the number. Can there be more than one answer.

3. The sum of 3 times a number and 2 is 17. Find the number.

4. Ten times a number minus 2 is 23. Find the number.

5. The sum of three-fourths of a number and 7 is 16. Find the number.

6. Three-fourths of a number minus 6 is 9. Find the number.

7. If 5 is added to 6 times a number, the result is 47. What is the number?

8. The sum of 7 and a certain number is 31. What is the unknown number?

9. If 3 times a number, twice the number, and 16 are added, the result is 31. Find the number.

10. The sum of 5 times a number and 3 times the number and 10 is 90. What is the number?

11. The sum of a third of a number and a fifth of a number is 24. What is the number?

12. There is a number such that 3 times the number subtracted from 46 is 7. What is the number?

13. If a fourth of a certain number is added to 5, the result will be the number itself. What is the number?

14. Thirty-two is one-fifth of what number?

15. Forty-eight is 5 per cent of what number?

16. A number increased by 5 equals 11. Find the number.

17. A ball and a bat together cost 40 cents; the bat cost 15 cents. What did the ball cost?

18. The difference between 3 times a certain number and 9 times the same number is 66. What is the number?

19. The sum of 4 times a number and the number is 4. What is the number?

Problems are not always so easily stated as the preceding, which are almost word-for-word translations.

The sum of two numbers is 78; one of the numbers is 5 times the other. What are the numbers?

The equality that must be stated is obvious:

$$\text{one number} + \text{other number} = 78$$

Both numbers are unknown; we will choose one of them as the unknown of the equation and represent it by n. The problem states that one is 5 times the other, consequently the second number is $5n$, and the equation becomes

$$n + 5n = 78$$
$$6n = 78$$
$$n = 13, \text{ one number}$$
$$5n = 65, \text{ the other number}$$

20. The sum of two numbers is 72. One of them is 6 less than the other. What are the numbers?

21. The larger of two numbers is 27. Their difference is 13. What are the numbers?

22. Find the value of the following if $x = 5$:

(*a*) $x + 34 + x$ (*b*) $3x + 8 - x$
(*c*) $4x - 12 - x$ (*d*) $6x - 3 + 2x$
(*e*) $x + 4 + 2x + 8$ (*f*) $x + x + 3x + 5$

23. State each of the expressions in Exercise 22 in a shorter form.

24. In a class of 48 pupils there are twice as many girls as boys. How many are there of each?

25. Separate 100 into two parts one of which shall be 4 times the other.

26. A horse and a wagon cost $540. What was the cost of each if the wagon cost twice as much as the horse?

27. Tom, Sam, and John have 63 cents. Sam has twice as many as Tom, and John has twice as many as Sam. How many does each have?

28. Two hundred and seventy dollars is divided among 3 children. The second receives twice as much as the first, while the third receives twice as much as the other two. What was the share of each?

36. Solving problems. In each problem to be solved certain steps are to be taken:

(1) Find an equality.

(2) Choose an unknown of the problem to be the unknown of the equation and represent it by a letter.

(3) Restate, or translate, the equality in algebraic symbols, using the letter for the unknown. In the problems of this chapter it will be found that after one of the unknowns of the problem is expressed by a letter the other unknown can be expressed by that letter and the other known number of the problem.

(4) Solve the equation.

(5) Check, so as to be sure that the working out of the solution has been done without mistake.

(6) Verify by applying the numbers to the problem itself to see if they actually work.

Illustration. Three times a certain even number plus 7 is 22. Find the number.

(1) Simply translate into algebraic language.

(2) Let x represent the unknown.

(3) The equation is $3x+7=22$

(4) The solution is

$$3x=15$$
$$x=5$$

(5) *Check:*

$3\times5+7$	22
$15+7$	22
22	22

The equation has been worked out without mistake.

(6) *Verification.* The answer 5 will not work in the problem, for it is not an even number.

37. Rectangle problems. Put the following into algebraic form:

Translations:

1. If one side of a rectangle is 5 and the other is unknown, what will represent the perimeter? What will represent the area? (Fig. 4.)

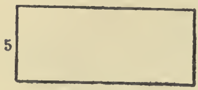

Fig. 4

2. If one side of a rectangle is 3 inches longer than the other, what will represent the perimeter? What will represent the area?

3. If one side of a rectangle is 2 inches shorter than the other, what will represent the perimeter? What will represent the area?

4. If two sides of a rectangle are in the ratio 2 to 1, write a number to express each side.

5. If two sides of a rectangle are in the ratio 2 to 3, write a number to express each side.

Problems:

1. It takes 410 feet of wire fencing to inclose a lot that is 172 feet deep. How wide is the lot?

The equality is,	the perimeter $=410$
One side is known,	use x for the unknown side
The equation is then	$x+172+x+172=410$

Let the pupil solve the equation. Check and verify. Can the perimeter be expressed in a more condensed form?

2. It takes 420 feet of fence to inclose a square lot. Find one side of the lot.

3. The length of a rectangular field is 3 times the width. It requires 840 feet of fence to inclose it. What are the dimensions of the field?

4. If the perimeter of a rectangle is 122 feet and the width is 21 feet, find the length. What will represent the perimeter? The area?

5. The length of a rectangle is 3 feet more than its width. If the perimeter is 42 feet, find the dimensions.

6. The length of a rectangle is $2\frac{1}{2}$ times the width. What are the dimensions if the perimeter is 49?

7. The width of a certain rectangle is $\frac{2}{3}$ of its length. If the perimeter is 110 inches, find the dimensions.

8. The perimeter of a rectangle is 130 feet. If the length is 5 feet more than 3 times the width, find the dimensions.

9. The perimeter of a rectangle is 130 feet. If the length is 5 feet less than its width, find the dimensions.

10. The perimeter of a certain rectangle is 70 feet. If the length is 5 feet less than 3 times the width, find the dimensions.

11. Two sides of a rectangle are in the ratio 2 to 1; the perimeter is 36. What are the dimensions of the rectangle?

12. Two sides of a rectangle are in the ratio 2 to 3. If the perimeter is 60, what are the dimensions of the rectangle?

13. Express the following in shorter form:

(*a*) $x+3x+5+2x+10$ (*b*) $3x-x+5+2x-2$
(*c*) $6x+10-x-5+2x$ (*d*) $3x+2+4x+5$

14. Find the value of the expressions in Exercise 13 if $x=3$.

38. Triangle problems. The triangle receives its name from the fact that it has 3 corners or angles. A rectangle has 4 angles and receives its name from the fact that its angles are all right angles. That the sum of the angles of a rectangle is 4 right angles is easily seen from Fig. 5. It is a rather remarkable fact that the sum of the angles of a triangle is 2 right angles. As a right angle is too large an angle to be a convenient unit for measuring angles, it is divided into 90 equal parts, each of which is called a degree. Hence a right angle contains 90 degrees. We may say, then, that the sum of the angles of a triangle is 180 degrees. If we use a, b, and c for the

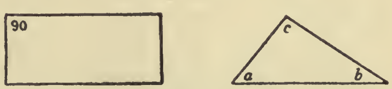

FIG. 5

number of degrees in each of the angles of a triangle, we may state this very important fact in the algebraic form

$$a+b+c=180$$

You may verify this equality by tearing off the corners and rearranging.

Translations:

1. If the sum of two angles of a triangle is s, what will represent the other angle?

2. If one angle of a triangle is n, what will represent the sum of the other two?

3. If a triangle has three equal sides, what will represent its perimeter?

Problems:

1. The perimeter of a triangle is 105 inches. What is the third side if two of the sides are 30 inches and 40 inches?

2. One side of a triangle is 2 feet more than a second, the third is 3 feet more than the second, and the perimeter is 50 feet. Find the length of each side.

3. One side of a triangle is twice a second, and the third is $1\frac{1}{2}$ times the second. If the perimeter is 18 feet, find the length of each side.

4. Two sides of a triangle are equal, the third side is 4, and the perimeter is 16. Find the sides of the triangle.

5. Two sides of a triangle are equal, the third side is $\frac{1}{2}$ of one of the equal sides, and the perimeter is 30. Find the sides of the triangle.

6. The sides of a certain triangle are all equal. If the sides in succession are increased by 1, 2, 3, the perimeter of the resulting triangle will be 12. Find the sides of the triangles.

7. One angle of a triangle is twice a second, and the third is 3 times the second. Find the angles of the triangle.

8. One angle of a triangle is 20 degrees more than a second, and the third is twice the second. Find the angles.

9. If one angle of a triangle is 42 degrees more than a second and 10 degrees more than the third, find the number of degrees in each angle.

10. If one angle of a triangle is 20 degrees less than a second and 20 degrees more than the third, find the number of degrees in each angle.

11. One angle of a triangle is 25 degrees more than a second, and the third is 15 degrees more than twice the second. Find the number of degrees in each angle.

12. One angle of a triangle is $\frac{2}{3}$ of a second, and the third is 4 degrees less than the second. Find the number of degrees in each angle.

13. One angle of a triangle is 30 degrees, and the other two angles are equal. Find the number of degrees in each angle.

14. One of the two equal angles of a certain triangle is $\frac{1}{4}$ of the third angle. Find all the angles.

39. Digit problems. In a number of two figures or digits, what does the right-hand figure mean? What does the left-hand figure mean? If the left-hand digit is 2 and the right-hand digit is 6, how is the value of the number obtained?

Translations:

1. The left-hand digit is 2 and the right-hand digit is a. Represent the number.

2. The left-hand figure is a and the right-hand figure is 5. Represent the number.

3. The left-hand figure is a and the right-hand figure is b. Represent the number.

4. Write a number of three digits in which the digits beginning at the left are 2, 3, and a.

5. Write a number of three digits in which the digits beginning at the right are (1) 5, 7, a; (2) b, 7, a; (3) 7, c, 5.

6. Write a number whose digits from left to right are x, y, z.

7. Write a number of three digits in which the digits from left to right are a, 0, and 3.

8. If the units digit of a number is 5 less than the tens digit, how may the number be represented?

9. If the tens digit of a number is 3 more than the units digit, how will you represent the number?

Problems:

1. In a number of 2 digits, the tens digit is 3 more than the units digit. If the sum of the digits is 15, find the number.

2. In a number of 2 digits, the units digit is 4 more than the tens digit. If the sum of the digits is 14, what is the number?

3. The units digit of a certain number is $\frac{3}{4}$ of the tens digit. The difference between the two digits is 2. What is the number?

4. The right-hand digit of a certain number is 4 less than the left-hand digit. The number is 6 more than 9 times the left-hand digit. Find the number.

5. The right-hand digit of a certain number is one more than the left-hand digit. The number is equal to 12 times the left-hand digit minus 5. Find the number.

6. The units digit of a number is twice the tens digit. The number is equal to 4 more than 11 times the tens digit. Find the number.

7. The units digit of a certain number is 3 times the tens digit. The number is equal to 5 times the units digit minus 6. Find the number.

8. The units digit of a certain number is 3 times the tens digit. The number is 5 more than 11 times the tens digit. What is the number?

40. Consecutive integer problems. Give illustrations of two consecutive integers; of two consecutive fractions.

Translations:

1. If n is an integer, what is the next higher integer? The next lower integer?

2. If n is an integer, write an expression for an even number; for an odd number.

3. Write expressions for three consecutive integers.

Problems:

1. The sum of two consecutive integers is 99. What are the integers?

2. Find three consecutive integers whose sum is 63.

3. Find four consecutive integers whose sum is 126.

41. Coin problems. In solving coin problems keep in mind the relation of the values of the coins to one another.

Translations:

1. How many cents in d dimes? In q quarters? In n nickels?

2. How many nickels in d dimes? In q quarters? In c cents?

3. How many quarters in c cents? In n nickels?

4. Write an expression to represent the number of cents in n nickels and 2 cents; in d dimes and 7 cents; in d dimes and n nickels and c cents.

Problems:

1. I have 72 cents in nickels and one-cent pieces, and have the same number of each. How many nickels have I?

Suggestion. The value of one-cent pieces plus the value of the nickels equals 72 cents.

2. I have the same number of nickels and dimes, amounting in all to $1.05. What is the number of dimes?

3. In a pocket full of change amounting in all to $4.05 there are twice as many dimes as quarters. How many are there of each?

4. I bought some 3-cent stamps and twice as many 2-cent stamps. If I paid for them all 70 cents, how many of each did I buy?

5. I have 81 cents in quarters and one-cent pieces. If I have 3 more one-cent pieces than I have quarters, how many have I of each?

6. I have 98 cents in nickels and one-cent pieces. If I have 2 more one-cent pieces than I have nickels, how many have I of each?

7. I have 95 cents in nickels and dimes. If I have one more nickel than I have dimes, how many have I of each?

42. Speed problems. If a cyclist rides 18 miles in 2 hours, we say that his average rate of speed is $\frac{18}{2}$ miles per hour. In other words,

$$\frac{\text{distance}}{\text{time}} = \text{speed} \qquad d = st$$

Translations:

1. A man walks at the rate of 4 miles an hour. How far can he walk in n hours?

2. How long will it take the man mentioned in Exercise 1 to walk m miles?

3. How long will it take a man to cycle x miles at the rate of 15 miles an hour?

4. If 10 gallons of water flow from a pipe in one hour, how much will flow from it in 7 hours? In b hours? In one day? In x days?

5. If a train moves at the rate of r miles an hour, how far will it move in 3 hours? In t hours?

6. If a train moves at the average rate of r miles an hour, how long will it take it to travel 30 miles? To travel m miles?

7. What is the average rate of a train that travels 36 miles in 2 hours? That travels m miles in 3 hours? That travels 60 miles in t hours? That travels m miles in t hours?

8. A man can row in still water at the rate of 6 miles an hour. A river is flowing at the rate of x miles an hour. How far can he row downstream in one hour? How far can he row upstream in one hour?

9. If a man can row at the rate of r miles an hour in still water, what will be his rate rowing downstream if the current is 2 miles an hour? What is his rate upstream?

Problems:

1. A man makes a journey of 147 miles. He cycles 5 times as many miles as he walks, and rides on a train 15 miles more than he cycles. How far does he ride on the train?

2. A certain man can cycle 5 times as fast as he can walk. How many miles does he walk in one hour if he makes a journey of 69 miles, cycling 4 hours and walking 3 hours?

3. A man started for a place 28 miles away. He traveled $1\frac{1}{2}$ hours by cycle when an accident occurred, and he walked for the rest of the way. He finished the trip in 4 hours. At what rate did he walk if he cycled at three times his walking rate?

4. Two pipes supply a cistern which can hold 150 gallons of water. One supplies 2 gallons per minute, the other 3 gallons per minute. How soon will the cistern be full?

5. Two pipes supply a cistern which can hold 180 gallons of water. One pipe supplies 2 gallons per minute, the other 5 gallons per minute. At the same time water is being pumped from the cistern at the rate of 4 gallons per minute. How long will it take to fill the cistern?

6. A certain tank holds 30 gallons of water. Two pipes flow into it. One pipe supplies 3 gallons more per minute than the other. If the tank is filled when the larger pipe is open 2 minutes and the smaller pipe 4 minutes, how much water flows through each pipe per minute?

43. Generalized statements. Number symbols, special, general. When we wish to represent on paper the number five, we usually write the figure 5, though sometimes for special purposes we use the forms V, :·: It is interesting to notice that men have not always written numbers as we do now. A very ancient and a very natural way of writing five was |||||. Other forms are given in Fig. 6.

Early Egyptian — Ancient Hindu — Greek — Roman — West Arabian

FIG. 6

Most of these systems of writing numbers were so inconvenient for making arithmetical calculations that the ancients for the most part did their figuring on their fingers, on the abacus, or on the dustboard. The multiplication of XIX by XIII is very difficult when the Roman notation is used, but very simple with our present notation 19 by 13. Some time during the Middle Ages the Arabs introduced into Europe the notation we now use: the ten symbols, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, with the very important idea of place value, by which we write 23 for twenty-three. They learned it from the Hindus. The story of the origin and development of this way of writing numbers is of great interest. It is known that the Hindus used nine of these figures, all except 0, as early as 631 A.D. It may be that the origin of some of the symbols can be traced back to the Babylonians, who lived thousands of years before that time.

As long as we wish to write down only definite fixed numbers, the Hindu notation serves our purpose admirably. But if we wish to write down a number whose particular value is unknown, or to write down the idea "any number" without having in mind any particular number, the Hindu notation fails us. We have seen how we may use a letter of the alphabet to represent some unknown number whose value we wish to find. In some of the formulas of chapter ii a letter was used to represent "any number." For instance, in the formula for the area of a rectangle

$$a=lw$$

l stands for any number. So also w stands for any number. Two different letters are used to indicate that the two dimensions need not be the same.

A letter, then, such as a, can be used to stand for any number whatsoever.

$a+b$ means the sum of any two numbers.

$\frac{a}{b}$ means the quotient of any two numbers.

What does ac mean?

What does $a-c$ mean?

EXERCISES

State in words the meaning of:

1. $\frac{a+b}{a-b}$ 2. $a+b+c$ 3. $\frac{a+b}{2}$

4. $\frac{a+b}{2}+\frac{a-b}{2}$ 5. $x-y$ 6. $xy+\frac{x}{y}$

7. $a(x-y)$ 8. $ax+bx$ 9. abc

10. $\frac{abc}{3}$ 11. $(x-y)b$ 12. $abcx$

Translate into algebraic symbols:

13. The sum of three numbers.

14. Twice the difference between two numbers.

15. The product of a number and itself.

16. The product of two numbers divided by a third number.

17. The product of a number and the sum of two other numbers.

18. A number divided by the difference between two other numbers.

19. Any number that is divisible by 2.

20. The area of any rectangle.

21. The volume of a brick.

22. The area of any triangle.

23. The average of any three numbers.

24. The surface of a brick.

25. The surface of a cube.

26. The volume of a cube.

27–38. Find the value of Exercises 1–12 when $a=17$, $b=9$, $x=12$, $y=4$, $c=2$.

39. Evaluate $\frac{5a+3b}{c}$ when $a=3$, $b=4$, $c=6$.

40. Evaluate $\frac{2a-6b}{3c}$ when $a=7$, $b=2$, $c=1$.

41. Evaluate $\frac{a-2b+3c}{2a-b}$ when $a=13$, $b=4$, $c=13$.

44. General equations. Let us consider a number of special problems.

(1) What number added to 3 equals 5?
The algebraic statement is $3+x=5$.

(2) What number added to 2 equals 6?
The algebraic statement is $2+x=6$.

(3) What number added to 7 equals 13?
The algebraic statement is $7+x=13$.

The three problems are of the same type. They differ only in the particular known numbers used. We may make the statement of the problem so general that it will include not only these three, but all problems of this type that can be thought of.

What number added to a equals b?
The algebraic statement is $a+x=b$.

We call this a **general equation.** Our interest is centered on one unknown which we represent by the letter x so that it will stand out clearly from the other numbers of the equations. It is quite common to use one of the last letters of the alphabet for the unknown of an equation, though this is not at all necessary.

Any special case may be obtained from the general equation by the insertion, in place of a and b, of particular numbers, such as 49 and 151:

$$49+x=151$$

A question: Can a and b be any numbers whatsoever?

Now let us attend to the solution of the following equations:

(1) $3+x=5$
$x=5-3$
$=2$

(2) $2+x=6$
$x=6-2$
$=4$

(3) $7+x=13$
$x=13-7$
$=6$

(4) $a+x=b$
$x=b-a$

All four are solved in exactly the same way. In each case subtraction is used to get the known number out of the place where it is not wanted. It will be noticed that we can obtain the answers to the first three from the solution of the fourth or general case by simply putting the proper numbers in place of a and b.

For instance, in Exercise 2, $a=2$, $b=6$.

The general solution is

$$x=b-a$$

The solution of Exercise 2 is

$$x=6-2=4$$

What is the solution of a case in which $a=30$ and $b=17$?

In the solution of the general case

$$a+x=b$$

every example of the kind is solved once for all. The answer to the general equation may be used as a formula for finding the answer for any particular case. We may call the a and the b general numbers. Such a general equation is sometimes called a **literal** equation.

Equations in which all the numbers except the unknown are figures are called **numerical** equations.

The numerical equations $4x=12$, $6x=15$, $7x=28$ are special cases of the general equation

$$ax=b$$

The solution of the general equation is

$$x=\frac{b}{a}$$

Using this result, write down the solution for each of the other equations. Notice that as 4 is called the numerical coefficient of x in $4x$, a is called the general coefficient of x in ax. Ordinarily we simply say that a is the coefficient of x.

In working with general numbers we use them just as we use special numbers except that we cannot combine them into one number and express them in a single symbol:

$$2+3=5$$

but $$a+b$$

has to be left in that form.

EXERCISES

Solve the following equations for x:

1. $4x+5=9$
2. $2x-7=5$
3. $2x+11=21$
4. $ax+3b=8b$
5. $ax-2b=9b$
6. $bx-3c=12c$
7. $cx+4a=15a$
8. $ax-b=c$
9. $bx+c=2b$
10. $cx-3a=5b$
11. $a+cx=3$
12. $a+b=2x$
13. $a=cx+b$
14. $a-b+cx=0$

45. Generalized problems. Consider the problem: The sum of 9 times a number and 3 times the number is 72. What is the number?

Let n be the number and the statement is

$$9n+3n=72$$
$$12n=72$$
$$n=6$$

or, indicating each step, $$9n+3n=72$$
$$(9+3)n=72$$
$$n=\frac{72}{9+3}$$
$$=6$$

Let us state a problem that will include all problems of this kind as special cases. The sum of a times a number and b times the number is c. What is the number?

$$an+bn=c$$

It will be solved in exactly the same way as the special case above:

$$(a+b)n=c$$

$$n=\frac{c}{a+b}$$

The checking of this problem may give you some trouble. If you can think it out, well and good; if not, let it go to a later time. You can easily check your result by taking a special case. For instance, take

$$a=2, b=3, c=12, \text{ or } a=1, b=2, c=5$$

Substitute these numbers in the general answer. Then substitute the result and the other special numbers in the original equation. Is there any choice between the two sets of numbers suggested for this particular example? Does it make any difference what values are assigned to the general numbers for the purpose of checking results?

It is to be noticed that such expressions as $ax+bx$ can be added into one term in the same way as $2x+3x$, by the simple addition of the coefficients. The unknown number in each term is the same.

Special case, $2x+3x=(2+3)x=5x$
General case, $ax+bx=(a+b)x$

In the special case the coefficients can be actually added together into one number. In the general case the addition can be expressed only. That is, this addition of terms cannot be worked out with a case like $ax+bx$.

MISCELLANEOUS PROBLEMS

State as equations and solve:

1. Three times a number is two less than 17. What is the number?

2. Three times a number is b less than 17. What is the number?

3. Write the equation called for in Exercise 2 for the special cases $b=4, 7, 3, 5, 9$. What is the required number in each case?

4. Three times a number is b less than c. What is the number?

5. State the special cases of Exercise 4 if

(1) $b=5, c=12$ (2) $b=3, c=5$
(3) $b=6, c=21$ (4) $b=7, c=19$

What is the required number in each case?

6. a times a number is b less than c. What is the number?

7. Write the equation called for in Exercise 6 for the special cases

(1) $a=12,\ b=10,\ c=46$
(2) $a=5,\ b=0,\ c=17$
(3) $a=7,\ b=3,\ c=17$
(4) $a=9,\ b=5,\ c=32$
(5) $a=15,\ b=5,\ c=50$

Find the required number in each case by using the solution of the general equation found in Exercise 6 as a formula.

8. Find two consecutive integers whose sum is 77; whose sum is 99.

9. Find two consecutive integers whose sum is s.

10. Use the answer found in Exercise 9 as a formula and find two consecutive integers whose sum is 311, 401, 2395, 151, 28.

11. What kind of a number must s be in Exercise 9 in order that the problem can have an answer?

12. State a rule for finding two consecutive integers if their sum is given.

13. The perimeter of a rectangular field is p feet; one side is a feet longer than the other side. What is the width of the field? The length?

14. One side of a triangle is 3 feet longer than the base; the other side is 5 feet longer than the base. What is the length of the base if the perimeter is 32 feet?

15. Generalize Problem 14.

16. The angles of a certain triangle are such that the second is a more than the first and the third is b less than the first. What are the angles? Apply to a special case.

17. In a certain triangle the second angle is a times the first angle and the third angle is b times the first. What are the angles?

18. Divide any number into two parts such that the difference between them shall be n. Apply to the special case when the number is 241 and $n=5$. Can you use any numbers you may choose for the number and for n?

19. Divide a number into two parts such that one part shall be n times the other. What can you say about the special values that can be given to the number and to n?

20. The tens figure of a number of two figures is twice the units figure. What are the figures of the number if their sum is a? What sort of a number must a be? Why?

21. The tens figure of a number of two figures is n times the units figure. What are the figures if their sum is a? Apply to the special case $n=3$, $a=12$.

46. Finding values of formulas. The results obtained by the solution of generalized problems can be used as formulas for finding the desired answers in special cases without its being necessary to go through all the work of solving the problem again. All one needs to do with such a formula is to put in the special numbers and perform the indicated arithmetical work. Such formulas are of great use to engineers, mechanics, manufacturers, and many others. Important formulas have been worked out by men well acquainted with the kind of work in which they are to be used and then preserved in handbooks for the use of those who may need them. The ability to use such formulas and get the desired results from them is of great importance. Such formulas are shorthand rules for calculating certain desired numbers.

EXERCISES

Substitute in the following formulas and make the necessary arithmetical calculations.

NOTE. All of these formulas are of use in practical work. They are taken from various trades and professions.

1. $A=\frac{a+b}{2}$, $a=12$, $b=26$

2. $x=\frac{M+N}{2P}$, $M=30$, $N=24$, $P=14$

3. $W=\frac{Ps}{d}$, $P=45$, $s=5$, $d=3$

4. $W=\frac{Ps}{d}$, $P=36$, $s=2\frac{1}{2}$, $d=12$

5. $C=\frac{.5P}{10}$, $P=\frac{1}{2}$

6. $W=7.6D-1.5$, $D=\frac{3}{4}$

7. $F=\frac{D}{r+1}$, $D=30$, $r=4$

8. $h=\frac{1}{2n}+.01$, $n=5$

9. $R=D-\frac{1.73}{T}$, $D=2$, $T=12$

10. $S=\frac{rl-a}{r-1}$, $a=5$, $r=3$, $l=125$

11. $T=212-\frac{h}{520}$, $h=7000$

12. $H=20v\frac{t}{n}$, $t=3$, $n=15$, $v=1320$

13. $y=\frac{a}{2+b}$, $a=1.175$, $b=2.578$

14. $W=P\frac{2R}{a-b}$, $P=223$, $R=19$, $a=12$, $b=4$

15. $F=\frac{9C}{5}+32$, $C=30$

16. $l=a+(n-1)d$, $a=3$, $n=10$, $d=4$

17. $S=\frac{n}{2}(a+l)$, $n=100$, $a=1$, $l=100$

18. $L=\frac{Wh}{d(W+P)}$, $W=6$, $h=4$, $P=15$, $d=1\frac{1}{2}$

CHAPTER IV

NEGATIVE NUMBERS

47. A difficulty. Consider the general equation

$$x-b=a$$

Its solution is $x=a+b$

No matter what values are given to a and b, their sum can always be found. Every equation of this kind can be solved.

But consider the general equation

$$x+b=a$$

The solution is $x=a-b$

For the special case $a=9,\ b=5$

this is $x=9-5$

$=4$

But for the special case $a=9,\ b=12$

we have $x=9-12$

Here we meet a difficulty, for we have learned in arithmetic that we cannot subtract a larger number from a smaller one, and we are led to say that the equation $x+b=a$ cannot be solved in all cases, but only in those special cases when b is not greater than a. And so man thought for many years, until the Hindus of the fifth or sixth century, who were not satisfied to let the matter rest there, invented a way of overcoming the difficulty.

48. Some practical illustrations. It is worth while to examine a little more carefully the statement made in the last article that "we cannot subtract a larger number from a smaller." It is true that it is impossible to take 5 apples from a plate on which there are but 3. But one can easily determine the resulting temperature when 15° have been taken from the temperature of a room in which the temperature is registered as 10°. There has been a drop of 15°; 15° have been taken away. We say the result is 5° below zero. Five degrees above zero and 5° below zero have very definite meanings.

Or again, it is easy to determine the result of buying a coat costing $25 when one has only $15. A person cannot take $25 from his pocket when there is but $15 in it, but he can buy the coat by going in debt for $10.

A man with only $3000 can buy a house for $5000 by paying the $3000 and going in debt for the remaining $2000. He takes $5000 from his assets, which leaves him a debt of $2000.

Assets and debts, temperature above zero and temperature below zero are certainly different kinds of numbers.

49. Usefulness of negative numbers. It is convenient to have some way to distinguish clearly the one kind of number from the other. To meet this need a new kind of number has been invented. We will use the numbers already invented, such as 3, 7, $\frac{1}{2}$, for one kind of quantity, such as assets, above zero, etc., and call them **positive numbers,** and denote them by using the sign $+$ in front of them: $+25$. We will invent a new kind of number to represent the other quantities, such as debt, below zero, and call them **negative numbers,** using the same figures but placing the sign $-$ in front of them: -25. Thus $+75°$ means 75° above zero, and $-75°$ means 75° below zero.

NOTE. The origin of the signs $+$ and $-$ is not quite certain, but it is probable that some mathematician saw them on the chests of goods in a German warehouse about five hundred years ago. They were used to indicate whether the weight of a given chest was over or under a given standard weight. They appear for the first time in a mathematics book in one written by Widmann in 1489.

The plus sign ($+$) may be omitted in the case of positive numbers.

EXERCISES

1. If +\$25 represents a credit, how would you represent a debt of \$10?

2. If 53 represents a gain, what does −47 denote?

3. In a certain dictionary you will find the dates of the births of George Washington and Julius Caesar put down thus: 1732, −100. What do the numbers mean?

4. In the same dictionary you will find the elevations, Mt. Vesuvius 3948 feet, Dead Sea 1312 feet below Mediterranean Sea. Express these elevations by using positive and negative numbers. Elevation means the vertical distance from sea level.

5. How would you express in numbers the latitudes of New York and Rio de Janeiro?

6. Express in algebraic numbers 10 miles west and 15 miles east; the freezing point of water and the lowest temperature reached in the place where you live; also the boiling point of water.

7. Express in algebraic symbols the speed of a train going ahead and the speed of a train backing; the weight of a wagon and the weight of a balloon which is ready to be cut loose; underweight and overweight.

50. Number system. With the addition of negative numbers we have now three kinds of numbers that can be used when needed—integers, fractions, including both common and decimal, and negative numbers. Integers and fractions have been used a very long time. We do not know when they were first invented. The early Egyptians used fractions with unit numerators, $\frac{1}{2}$, $\frac{1}{7}$, $\frac{1}{21}$, etc. These are found in the most ancient book on arithmetic, written by an Egyptian named Ahmes. The fractions used by the Babylonians, as shown on tablets that have been unearthed, had the same denominator, 60 being used for that purpose. For instance, $\frac{1}{2}$ was represented by the symbol for 30, the word "sixtieths" being left for the reader to supply. The Greeks used fractions of a more general type, such as $\frac{17}{21}$. None of these ancient peoples used decimal fractions or negative numbers. Strange as it may seem, negative numbers were invented before decimal fractions. It is probable that negative numbers are due to the Hindus, who were very skillful in arithmetic and algebra. Aryabhatta, who was born in 476 A.D., distinguished between positive and negative numbers, but without the signs plus and minus.

The decimal fraction is a comparatively recent innovation. The idea was of slow growth, but was used effectively toward the close of the sixteenth century by Simon Stevin of Bruges, Belgium, and others in making certain calculations. It was not until the beginning of the nineteenth century that decimals were brought into ordinary arithmetic. Various notations have been used for fractions. The Hindus wrote a fraction in the form of $\genfrac{}{}{0pt}{}{2}{3}$; the bar we use between the two parts of a fraction $\frac{2}{3}$ was probably introduced by the Arabs. One of the forms Stevin used for a decimal was 5 (0) 9 (1) 1 (2) 2 (3); our form is 5.912.

51. Number scale. You know how the numbers are arranged on a thermometer. Each mark indicates a temperature lower than the mark above it. (Fig. 7.) Usually on the thermometers we use at our houses the change from one mark to another is the change of one degree.

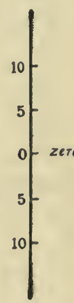

FIG. 7

Such a scale furnishes a very convenient way of arranging positive and negative numbers, the integers being placed at equal distances along a straight line in such a way as to show their distance from a point chosen as the starting, or zero point. It is customary to put the positive numbers on the right. Each number represents not only a numerical value, but also that numerical

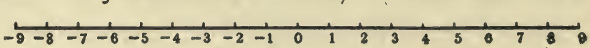

FIG. 8

value in a given direction from the zero point. Each integer is one less than the next integer to its right; each is less than any integer to its right: 2 is less than 5, −4 is less than −2.

EXERCISES

1. Where on the scale would you place $+12$, -15, -11, $-\frac{1}{2}$, $-\frac{1}{3}$, $-2\frac{1}{2}$, $+\frac{2}{3}$, $-\frac{7}{2}$, $.5$, -2.2, -3.5?

2. What is the number that is halfway between 5 and 6? Halfway between -5 and -6?

52. Adding a positive number. We may think of passing from one point on the scale to the next point on the right as counting on one or adding one positive unit.

To add 3 to 7, we may start at 7 on the scale and count on 3 units to the right and reach 10. To add 3 to -7, start at -7 and count on 3 units to the right and reach -4.

EXERCISES

By means of the scale add:

1. 5 to 9, 6 to 3, 3 to 6, 2 to -3
2. 5 to -7, 9 to -3, 5 to -5, 7 to -2
3. 10 to -7, 7 to -9, 3 to 0, 3 to 5

53. Subtracting a positive number. In a similar way we may think of passing from right to left as counting off or subtracting a unit at each step. Subtraction is the undoing of addition. Therefore in subtraction we reverse the direction and pass to the left.

To subtract 3 from 7, we may start at 7 and count off 3 to the left and reach 4. To subtract 3 from -5, start at -5 and count off 3 to the left and reach -8.

EXERCISE I

By the use of the scale subtract:

1. 2 from 7, 7 from 7, 9 from 7
2. 5 from 2, 3 from 5, 8 from 1
3. 5 from 5, 5 from 0, 2 from -3
4. 3 from -2, 4 from -1, 2 from -6

We may write such examples in mathematical symbols, using the distinguishing marks for positive and negative numbers.

Add 2 to -3, $(-3)+(+2)$

Subtract 3 from (-5), $(-5)-(+3)$

or, omitting the sign for the positive number,

$$(-3)+2$$
$$(-5)-3$$

since 3, $+3$, and $(+3)$ all mean the same thing.

EXERCISE II

Find values of the following from the number scale:

1. $(-3)+(+5)$
2. $(6)+(+3)$
3. $(-7)+(+2)$
4. $(-7)+(+9)$
5. $9-8$
6. $9-12$
7. $(-9)+3$
8. $(-9)-2$
9. $(5)-(+6)+(+2)$
10. $9-5+3$
11. $7-6+8$
12. $3-(+7)+(+1)$
13. $2-5+6$
14. $2-7+4$
15. $3-9+10$
16. $7-(+2)-(+3)$
17. $7+2-3$
18. $(-2)+7-3$
19. $(-2)+(3)-(+7)$
20. $(-2)+5-6$
21. $(-9)+6-1$
22. $(-9)+9-2$
23. $8-5+3$
24. $8-12+2$

25. If the temperature is now $+10°$, what will represent the temperature after a fall of 8°? After a fall of 15°?

26. If the temperature is now $-5°$, what will represent the temperature after a rise of 8°? After a fall of 8°?

27. A boy has $3 and buys some tools worth $5. What will represent the condition of his finances?

28. While playing a certain game a man finds himself "3 in the hole." What will represent his score if he gains 8 points? If he loses 2 points?

29. Jerusalem is 2500 feet above sea level. To reach the Dead Sea, one descends 3800 feet. What number represents the level of the Dead Sea?

30. A boy is 15 pounds overweight. He loses 20 pounds. What number represents his condition?

54. Solution of equations. In the last article it was shown that the invention of negative numbers enables us to subtract a larger number from a smaller. It therefore enables us to solve the equation that started this discussion:

$$x+9=5$$
$$x=5-9$$
$$=-4$$

Check: $(-4)+9=5$

Use the number scale in finding value of the left side.

EXERCISES

Solve and check:

1. $x+9=8$ 2. $x+12=4$ 3. $x+13=4$
4. $x+15=12$ 5. $x+n=p$ 6. $y+7=1$

7. Derive equations from Exercise 5 in which

(1) $n=12,\ p=6$ (2) $n=9,\ p=2$
(3) $n=13,\ p=3$ (4) $n=15,\ p=8$

8. Find the answers to the equations obtained in Exercise 7 by substituting the numbers given in the answer to Exercise 5.

55. Adding on the scale. Adding a positive number means counting to the right. Adding a negative number means counting to the left.

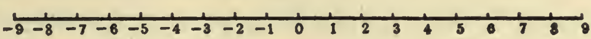

FIG. 9

Add 3 to 5, start at 5, move 3 places to right, reach 8.
Add −3 to 5, start at 5, move 3 places to left, reach 2.
Add 7 to −2, start at −2, move 7 places to right, reach 5.
Add −5 to −2, start at −5, move 2 places to left, reach −7.
Add 2 to −5, start at ?, move ? places to ?, reach ?

EXERCISES

Find the value of the following from the number scale:

1. $(6)+(-8)$
2. $(8)+(-12)$
3. $(-2)+(-9)$
4. $(-6)+(+3)$
5. $(7)+-10$
6. $7+(-3)$
7. $(-7)+(-10)$
8. $(-9)+(-3)$
9. $6+(-4)+2$
10. $7+(-10)+3$
11. $7+(-10)+(-3)$
12. $(-3)+6+(-8)$
13. $(-3)+(-6)+8$
14. $(-3)+(-6)+(-8)$
15. $(+3)+(-6)+8$
16. $3+(-6)+(-8)$

56. Addition. The way to proceed is made clear by the special problem of combining debts and credits.

The result of adding a credit to a credit is clearly a credit, equal to the sum of the two amounts of money:

$$\begin{array}{r} +25 \\ \underline{+30} \\ +55 \end{array}$$

The result of adding a debt to another debt is clearly a debt, equal in value to the sum of the two amounts of money:

$$\begin{array}{r} -25 \\ \underline{-30} \\ -55 \end{array}$$

There is no difficulty in finding the result of combining a debt and a credit. It will be a credit if the amount of the given credit is greater than the amount of the given debt; and a debt if the amount of the given debt is greater than the amount of the given credit:

$$\begin{array}{r} +25 \\ \underline{-15} \\ +10 \end{array} \qquad \begin{array}{r} +25 \\ \underline{-30} \\ -\ 5 \end{array}$$

In each case the amount of the result is the difference between the given amounts.

The illustration suggests an extension of the idea of adding. In each case the term "adding" has been used. When the two numbers were both positive or both negative, the adding implied the finding of an arithmetical sum. When one number was positive and the other negative, the adding implied the finding of an arithmetical difference. Both operations are included in what we call **addition.**

In arithmetic the sum of two numbers is always greater than either number. This is not always true in algebra. The sum of two numbers may be less than either or less than both the numbers. Show that this last statement is true.

57. Definitions. These illustrations suggest certain definitions. The numerical value of a number (disregarding its sign) is called its **absolute value.**

The **sum** of **two positive numbers** is a positive number whose absolute value is the sum of the absolute values of the numbers.

The **sum** of **two negative numbers** is a negative number whose absolute value is the sum of the absolute values of the two numbers.

The **sum** of a **positive** and a **negative number,** or of a negative number and a positive number, is a positive or a negative number according as the positive or the negative number is in excess. Its absolute value is the difference between the absolute values of the two numbers.

58. Rule for addition. We may write a practical rule for addition in this simple form:

(*a*) If the numbers have the **same** sign, find the **sum** of their absolute values and prefix the common sign.

(*b*) If the numbers have **different** signs, find the **difference** between the absolute values and prefix the sign of the one having the larger absolute value.

EXERCISES

Add:

1.

30	5	-9	-3	-9	-9	25	-6	19	3.8
-10	8	4	-7	$+3$	$+15$	-25	-2	23	-2.9

2.

2	$3a$	$9a$	$-23x$	$-25n$	$-6n$	-2
3	$-2a$	$-11a$	$-2x$	$+30n$	$6n$	$+3$

3. Find the sum of the following credits and debits:
$+25, +10, -15, +3, -10, -7$

We may find the sum of several numbers by adding each one in succession. A preferable method is to find the sum of the positive numbers and the sum of the negative numbers, and then find the sum of these two results. There is less chance of making a mistake when the latter plan is followed.

4. Find the sum of:
$15, -7, +6, 3, -5, +2, -8$

5. Add:
$-7, 9, -3, 2, 11, -12, +3, -10$

6. Find the sum of:
$3a, -5a, -2a, +8a, -10a$

7. Add together:
$6x, -2x, 25x, -15x, 28x, -4x$

8. Find the sum of:
$7n, 2n, -3n, -4n, +5n$

9. Find the sum of:
$2y, -5y, +3y, -2y, +y, -7y$

10. Add together:
$15x, -7x, -2x, +4x, -10x$

11. Find the sum of:
$-3a, -2a, 6a, 7a, -4a, 5a$

12. Find the average of:
6, 7, 6, 9, 7, 10, 6, 5

13. Find the average of:
65, 80, 0, 75

14. Find the average of:
6, -2, 8, -5, 3

15. What is the average temperature when the readings taken each hour are 28, 12, 8, 4, 0, -2, -5, -7, -6, 0, 5, 5?

16. Let the members of the class guess the length of a line drawn on the blackboard. The teacher will measure the line and announce the length. Let each student subtract the true length from his guess and announce his result to be written on the blackboard. Then let each member of the class compute the average of all these errors.

59. Double meaning of signs +, −. We have used the expression $(+7)+(-3)$ to mean the sum of positive 7 and negative 3, the signs within the parentheses marking the distinction between positive and negative. And we have also used a simpler expression for positive numbers, simply 7 without any sign: $7+(-3)$.

In adding -3 to 7 on the number scale we move three places to the left and reach 4, which is the same operation as subtracting 3 from 7, and the same result is obtained. We may say then that adding a negative number gives the same result as subtracting a positive number of the same absolute value.

We may then replace

$$7+(-3) \text{ by } 7-3$$

so that $7-3$ may be thought of in two ways: as meaning 7 minus 3; or as meaning the sum of positive 7 and negative 3, or, as we often speak of it, the sum of $+7$ and -3.

It is the usual practice to regard $7x-3x$ as the sum of the two terms or numbers:

$$7x \text{ and } -3x$$

Thus

$$3+7a-5$$

is thought of as the sum of

$$3,\ 7a, \text{ and } -5$$

EXERCISES

Write each of the following expressions in another way, explain the meaning of each, and find results:

1. $6+(-3)$	2. $6-3$	3. $2+(-5)$
4. $2-5$	5. $7+(-5)$	6. $8-2$
7. $9-11$	8. $7a+(-8a)$	9. $5x-7x$
10. $3a-12a$	11. $2a+(-9a)$	12. $7a+(-3a)$

60. To find the sum of several positive and negative numbers. It is a simple matter to express the sum of several positive and negative numbers. To find the sum of x, $-a$, $+b$, -3, write down each number in order with its sign in front:

$$+x-a+b-3$$

The sum of -2, $4x$, -5, $-2x$, is $-2+4x-5-2x$ or $2x-7$. If the first number in the series is a positive number, the sign need not be written. The sum of x, -2, $-3a$, 7, is $x-2-3a+7$. We usually regard the sign in front of a term as belonging to the term and speak of positive and negative terms. In $7x-3+4x-10x$ the terms are $+7x$, -3, $+4x$, $-10x$.

EXERCISES

Find the sum of the following:

1. $3, -2, 5, -7, -9, +3$
2. $3a, -7a, +2a, +5a, -a, -9a$
3. $9x, -x, +7x, -5x, +10x, -3x$
4. $3A, A, 25A, -3A, -41A, +31A, 6A$
5. $4\pi, -6\pi, +3\pi, -2\pi, -9\pi$
6. $5\pi-3\pi+2\pi-9\pi+7\pi$
7. $5+5n-7+2n-8n+9+7n-2$
8. $7, -3x, -9, -a, +11, -10a, 15a, -10x$
9. $7, x, -a, -b, +3a, -2x, +7x, +3b$

10. $2x,\ -\frac{2}{3},\ +\frac{3}{2}x,\ -4,\ +\frac{3}{5},\ -\frac{5}{2}x,\ \frac{1}{3},\ -\frac{1}{5}x,\ -\frac{1}{2}x$
11. $3t-5+2t-7t+9+11t$
12. $3x-2,\ 2x+5,\ 5-3x,\ 2x-5$
13. $3-y,\ 4+2y,\ 7+5y,\ -3-4y$

61. Adding terms of different kinds. When a number of expressions with several kinds of terms are to be added together, it is often convenient to arrange them in columns with like terms under each other just as we do in arithmetic. This makes the addition somewhat easier.

$$\left.\begin{array}{r} 345 \\ 26 \\ \underline{605} \\ 976 \end{array}\right\} \quad \text{which means} \quad \left\{\begin{array}{l} 300+40+5 \\ \qquad\ \ 20+6 \\ \underline{600 \qquad +5} \\ 900+60+16 \end{array}\right.$$

$$\begin{array}{r} 7x+3a-7y+5b \\ -a\ +2y-3b \\ \underline{x-5a-3y\qquad} \\ 8x-3a-8y+2b \end{array}$$

EXERCISES

Find the sums of the following:

1. 4.2, 32.97
2. 4.2, −3.46
3. 4.31, −5.37
4. 4, −6.23
5. 3.24, −7.41, +3.46
6. 3, 6.84, 7.6, −6.7, −.98
7. $2a-4b,\ 6a+5b,\ 2b-3a,\ 3b-a$
8. $3a-2b,\ 5b-3c,\ 2c-3a$
9. $ax-3y,\ 2ax-5y,\ 7y-6ax$
10. $3x+2a+3b,\ 2x-5a+2b,\ -5x-2a-9b,\ 3x+a+b$
11. $a+b-c,\ a-b+c,\ -a+b+c,\ a+b+c$

12. $\frac{1}{2}a-\frac{1}{3}x,\ \frac{1}{3}a-\frac{1}{5}x,\ \frac{1}{3}a+\frac{1}{2}x$

13. $3ax-2by,\ -7ax+9by,\ -2ax-3by$

14. $15x-9,\ -10x+3,\ -47x-31,\ +19x+3,\ -7x-21,\ -98x+35$

15. $7x+41a-3,\ 35a-23x+91,\ 72-8x-71a,\ -13+16a-25x$

16. $7x-3b-2a, 7y+3a-10b, 13b-5a-2y-3x, 14a-10y-6b-x$

17. $3x-2a-b,\ \frac{1}{2}a-\frac{1}{3}b-\frac{1}{4}x,\ \frac{1}{2}b-\frac{1}{3}a-\frac{1}{3}x$

18. $5x-6y-7z,\ 5y+9z-4x,\ 2x-3y-z,\ y+4z-6x$

19. $3a-2b+4c,\ 7a+3b-5c,\ 2c-2a-4b,\ a+b-6c$

20. $1.6x-1.4y-z,\ 1.2x+1.3y+.5z,\ x-2y+1.6z$

21. $9x-10y-9z,\ 11y+3z-2x,\ 4z-5x+6y,\ x-7y-8z$

22. $\frac{2}{3}x-3y-z,\ 2x-\frac{1}{2}y-\frac{1}{3}z,\ \frac{1}{4}x-y+2z$

23. $7x-8y+10a,\ 9y-5a-3x,\ 4a-2y-6x,\ 3x-9y+2a,\ 7y-6x-9a$

24. $.7x-.5y-.6z,\ .7y-.5z-.6x,\ .5x+.7y+1.5z$

25. $2x-3y,\ 4y-5x,\ 6x-7y,\ 5y-6x,\ 3x+9y,\ -2y-11x$

26. $a-b,\ 2b-4c,\ c-3a,\ 3a-5b-2c,\ 6b-4a,\ 5a+c$

27. $\frac{1}{2}x-\frac{1}{3}y+\frac{1}{3}x+\frac{1}{4}y-\frac{1}{6}x-\frac{5}{6}y$

28. $2n+3h,\ 5n-7h+2,\ -3n-h-5$

62. Subtraction. We have seen that the adding of a negative number and the subtracting of a positive number of the same absolute value had the same result. That is, the subtraction of a positive number may be replaced by the addition of a negative number.

The question then arises, how shall we subtract a negative number? Subtracting is the reversal of adding. If adding a negative number means counting to the left on the number scale, then subtracting a negative number must mean counting to the right, which is the same operation as adding a positive number.

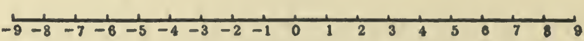

FIG. 10

Subtract 3 from 5

Subtract 3 from -5

Subtract -3 from 5

Subtract -3 from -5

Subtract -2 from 7

Subtract -7 from 2

Subtract -8 from 3

We will assume, then, that in every case the operation of subtraction can be replaced by addition. There is nothing new or strange in this idea of replacing subtraction by addition. It is used constantly in making change. A five-dollar bill is offered in payment of a bill of $2.87. How much money is to be paid back? The man in the store does not find the amount to be returned by subtraction, $5.00-2.87=2.13$, but by addition, $2.87+.03+.10+2.00$.

The operation of subtracting a number may then be expressed in the form of a rule:

RULE. **To subtract a positive number, add a negative number of the same absolute value.**

To subtract a negative number, add a positive number of the same absolute value.

To subtract 4 from 7, add -4 to 7, $7-4=3$.

To subtract 4 from 3, add -4 to 3, $3-4=-1$.

To subtract -5 from 3, add 5 to 3, $3+5=8$.

To subtract -5 from -7, add 5 to -7, $-7+5=-2$.

Consider a practical case. If the temperature at 6:00 A.M. is 30° and at noon is 70°, we find the rise in temperature by subtracting 30° from 70°.

$$70-30=40$$

If the temperature at 6:00 A.M. is $-10°$ and at noon is 40°, we find the rise in temperature by subtracting -10

from 40, that is, by adding +10 to 40, which gives 50. These illustrations may be expressed in algebraic symbols:

$$(7)-(4)=7-4=3$$
$$(3)-(4)=3-4=-1$$
$$(3)-(-5)=3+5=8$$
$$(-7)-(-5)=-7+5=-2$$
$$(40)-(-10)=40+10=50$$

EXERCISE I

1. Subtract 17 from 23, 42 from 23, 38 from 9, −12 from 30, −20 from 16, 3 from −9, −6 from −10, −20 from 9, −9 from 20, −10 from −6.

2. From $5x$ subtract $2x$, from $7x$ subtract $-9x$, from $-2x$ subtract $-3x$, from $-12x$ subtract $-4x$.

3. Copy and fill in the blanks with the values of $a-b$, where a represents the numbers in the column at the left and b represents the numbers in the row at the top.

	7	−13	3	−5	−9	4	1	12	−6	10
9										
−6										
7										
−10										
1										
−3										

4. Copy and fill in the blanks in the table at the top of page 81 with the values of $a-b$, where a represents the numbers in the column at the left and b represents the numbers in the row at the top.

	$3x$	$-2x$	$-5x$	$-7x$	$4n$	$-3n$	$9t$	$-12t$
$7x$								
$-5x$								
$6x$								
x								
$3n$								
$-7n$								
$-5t$								
$+7t$								

EXERCISE II

Find the value of the following:

1. $3-(-6)$
2. $4+(-6)$
3. $7+(-2)$
4. $7-(-2)$
5. $7-(-9)$
6. $7+(-9)$
7. $4x-(-3x)$
8. $5x+(-3x)$
9. $(-2)-(4)$
10. $(-2)-(-4)$
11. $(-4)+(-5)$
12. $(-3x)+(-2x)$
13. $6-(2)+(-3)$
14. $3-(-2)+(-3)$
15. $7-(5)-(-9)$
16. $(-5)-(-7)+(10)$
17. $(-8)+(-6)-(-4)$
18. $(10)+(-6)-(-3)$
19. $(-3)-(-5)+(-5)$
20. $(-3)+(-6)-(-7)$
21. $(2)-(4)-(-6)$
22. $(-2)-(-4)+(-6)$
23. $(-2)+(-4)-(+6)$
24. $(+2)-(-4)+(-6)$
25. If $a=5$ and $b=6$, evaluate $a-b$, $2a-3b$.

26. If $a=-2$ and $b=7$, evaluate $a+b$, $a-b$, $a-3b$, $a+3b$.

27. If $a=-2$, $b=-6$, $c=4$, evaluate $a+b$, $a-b$, $a+b-c$, $a-b+c$, $b-c-a$, $b+c-a$.

28. If $x=-3$, $y=2$, evaluate $x-5$, $x+y+5$, $y-x+7$, $3y-x-5$.

63. Positive and negative numbers. Answers must be given in algebraic language.

EXERCISES

1. Add 7, subtract 10, add −3, subtract −5, add 9, subtract 5, add 6, add −10, subtract −14, subtract 16, add 25, subtract −8.

2. Take $5a$, add $7a$, subtract $-9a$, subtract $11a$, add $-6a$, add $9a$, subtract a, subtract $-3a$, add $5a$, add $-10a$.

3. François Vieta, the first man to write ab for $a \times b$, was born in 1540 and died in 1603. How old was he when he died?

4. Augustus Caesar was born in 63 B. C. and died in 14 A. D. How old was he when he died?

5. Pythagoras was born in 580 B.C. and died in 501 B. C. How old was he when he died?

6. At 3:00 P.M. the temperature was 93°; at 6:00 P.M. it was 75°. How much of a drop was there in temperature?

7. What was the drop in temperature from 10° at noon to 15° below zero at 6:00 P.M.?

8. What is the difference in latitude between New York, 40° 40′, and Baltimore, 39° 18′?

9. What is the difference in latitude between New York, 40° 40′, and Rio de Janeiro, 23° south?

10. What is the difference in latitude between the North and South Poles?

11. Two trains, one behind the other, are moving in the same direction with speeds of 45 and 32 miles an hour. What is the difference in their speeds? In answering the next question does it make any difference which train is ahead? How fast is the one approaching the other?

12. The same two trains are moving toward each other. How fast are they approaching each other?

13. An airplane is maintaining a speed of 80 miles an hour. How fast is it moving across the country if flying with a 35-mile an hour wind? If flying against a head wind of 35 miles an hour?

14. If the speed of the airplane is a miles an hour and the rate of the wind is b miles an hour, state the rate at which the airplane travels with and against the wind.

15. A man in a boat is rowing upstream. He can row at the rate of 5 miles an hour. What is his rate upstream if there is a current of 2 miles an hour? Of 5 miles an hour? Of 6 miles an hour?

64. Subtraction—expressions of several terms. If the number to be subtracted is composed of two or more terms, the same procedure is followed (see page 79): each term is subtracted in turn. To subtract $2x-4$ from $5x-3$, subtract each term of $2x-4$ in turn. $2x$ is positive, so add $-2x$. -4 is negative, so add $+4$ and obtain

$$5x-3-2x+4$$

Now add terms of the same kind and we have

$$3x+1$$

The work may be written in the form

$$\begin{aligned}(5x-3)-(2x-4)&=5x-3-2x+4\\&=3x+1\end{aligned}$$

Another illustration of this follows:

Subtract $\quad -3x+7$ from $4x-2$

It may be written

$$(4x+2)-(-3x+7)$$

$-3x$ is negative, so add $3x$

$+7$ is positive, so add -7

$$4x+2+3x-7$$

Add like terms, $\quad 7x-5$

EXERCISES

1. From $7x+3$ subtract $2x+1$.
2. From $9x+7$ subtract $5x+9$.
3. From $13x+2$ subtract $4x-3$.
4. From $13x-5$ subtract $5x-2$.
5. From $19x-8$ subtract $12x+7$.
6. From $8x+3$ subtract $12x-4$.
7. From $3a-5b$ subtract $a+7b$.
8. From $5x-2y+6$ subtract $2x-9y-7$.
9. From $7n-3t+a$ subtract $2n+7t-a$.
10. $(7x-3y+2)-(4x+2y-5)=$?
11. $(13n+10p-5)-(7n-5p-9)=$?
12. $(40A-60B-7C)-(50A-9C+3B)=$?

65. Arrangement of work. It may be desirable, sometimes, to arrange the work of subtraction as in addition with like terms under each other.

$$\begin{array}{ll} \text{From} & 7x-2y-6 \\ \text{subtract} & \underline{5x+3y-2} \\ & 2x-5y-4 \end{array}$$

In subtracting a term you must be very careful to add a term of opposite sign. The change of sign is carried in the mind and not actually put down on paper as in the other way of doing the work:

$$(7x-2y-6)-(5x+3y-2)=7x-2y-6-5x-3y+2= 2x-5y-4$$

66. Double use of word negative. We call $2x$ a positive term and $-2x$ a negative term. The two terms have opposite signs. We often say that $-2x$ is the negative of $2x$. We may also say that $2x$ is the negative of the term $-2x$. That is, we may use the word **negative** to mean **opposite in sign.** This is a very reasonable use of the word, for we do not know whether $2x$ stands for a positive number or a negative number, for that depends upon whether x is a positive or negative number. So $a-b$ and $-a+b$, or $b-a$, are to be regarded as the negatives of each other. $4-2x$ is the negative of $2x-4$. The negative of $-3x-2$ is $3x+2$. The rule for subtraction may then be put in the more concise form:

RULE. **To subtract a number add its negative.**

EXERCISE I

1. Subtract $7x-2y+3xy-5$ from $35x+11y-12xy+5$.
2. From $9x-2xy+5-5y$ subtract $7x-2y+3xy-9$.
3. Subtract $17y+3x+5-2b+a$ from $19y-6x+2a-7b-6$.
4. From $3ab-2bc+6ca-9$ subtract $5ab-2ca+7bc+6$.
5. Subtract $7x-21a-3b-2c$ from $9x-10a+7b+c$.
6. From $3aa-7a+3$ subtract $2aa-2a+7$.
7. From $2a-5b+d$ subtract $7a-9c-15d+f$.
8. Subtract $21x-35xy+9$ from $4x+2xy-13$.
9. Subtract $\frac{1}{3}a+\frac{1}{2}b$ from $\frac{1}{2}a-\frac{1}{3}b$.
10. Subtract $3x-2y-.5z$ from $2x-y-.2z$.
11. From $6x-7y-3z+4a$ subtract $9x+6y-4z+5a$.
12. From $8x-3y+9z-5a$ subtract $6x-4y+10z-3a$.
13. From $3x-.5y+.7z$ subtract $4x+8y-.5z$.

Add the following expressions and from the sum subtract each expression in turn until nothing is left:

14.
$$\begin{array}{r} 3x-4y+2z \\ -8x+5y-7z \\ +2x-6y+3z \\ -5x+5y-6z \\ -x-3y+4z \end{array}$$

15.
$$\begin{array}{r} 2x-\ y-\ z \\ x+2y-2z \\ -3x-3y+3z \\ x-4y+5z \\ -2x+5y-4z \end{array}$$

In an exercise when there are many numbers, some to be added and some to be subtracted, all the subtractions may be replaced by additions and the exercise treated as suggested in the article on addition.

Add $3x$, subtract $2x$, subtract -3, subtract $-7x$, add $5x$, subtract x, subtract -4, write thus:

$$3x-2x+3+7x+5x-x+4=12x+7$$

So also, to $13x-2$ add $4x-3$, subtract $5x+2$, subtract $2x-7$, write thus:

$$13x-2+4x-3-5x-2-2x+7=?$$

16. Add $7x-3$, subtract $2x-5$, subtract $5x-2$.
17. Add $13n+5$, subtract $9n-6$, subtract $7n+7$.
18. Subtract $4x-2$, subtract $14x-3$, add $9x+7$, subtract $10x+8$.
19. Add $x-b+a$, subtract $2x-3b+2a$, subtract $-3x+2b-5a$.

EXERCISE II

Solve and check:

1. $3x-5=7x-10$
2. $9x+8=4x+23$
3. $5x+17=10x-3$
4. $8x+1-6x=9x-19+3x$
5. $3x+8-5x-2=6x+4-x-19$
6. $3y-12+y+6=24-8y-2+5y$
7. $6x-9+x+4=3x+15-4x$
8. $5a+10-7a+2=25-4a-3$
9. $7a-4-2a-3=8a+4-a-19$
10. $x+16-5x-2=18x-4-10x-6$
11. $4-n+7-3n=15-8n+4+2n$
12. $x-8+2x-2=6-3x-2-8x$
13. $2x-3-(x+5)=3x-12$
14. $4y-9=12-(y-3)+1$
15. $14-(x-2)=27-(2+4x)$

16. $3n-(n-5)=16-(3+2n)$

17. $27-(x-3)=15-(5-4x)$

18. $4p-3-(9-3p)=9$

19. $3a+9=15-(a-8)+2$

20. $5x+6-20x=3-5x-(5x-2)$

21. $6x-(4x-3)=8x$

22. $10x-(x-2)=2-(6x-5)$

23. $3x-(x-a)=b$

24. Using the answer to Exercise 23, find answers for the special equations when:

(1) $a=2, \quad b=6$
(2) $a=-5, b=11$
(3) $a=3, \quad b=-8$

25. $5x=b-(a-2x)$

26. $4x-(x-b)=2b-(x+a)$

PROBLEMS

1. I have three numbers; the second is 2 more than the first, and the third is 4 less than twice the first. If their sum is 34, find the numbers.

2. Of the three sides of a triangle, the second is 3 inches more than the first, and the third 2 inches less than the first. If the perimeter is 22 inches, find the sides.

3. Of the angles of a triangle, the first is 20° less than the second, and the third is 40° less than twice the second. Find the number of degrees in each.

4. The length of a certain rectangle is 6 feet less than 4 times its width. If the perimeter is 108 feet, find the dimensions.

5. If 4 more than a certain number is subtracted from 4 times the number, the result is 26. Find the number.

6. If 3 times a certain number is subtracted from the sum of the number and 20, the result is 4. Find the number.

7. If 3 is subtracted from a certain number and this remainder is subtracted from 4 times the number, the result is 18. Find the number.

8. If 2 is added to 4 times a number and the sum is subtracted from 6 times the number, the result is 20. Find the number.

9. If 3 is subtracted from 7 times a number, and the difference is subtracted from 10 times the number, the result is 33. Find the number.

10. Of the three angles of a triangle, the second is 20° less than the first, and the third is 12° more than the sum of the other two. Find the number of degrees in each angle.

11. There are two consecutive integers such that if the larger is subtracted from twice the smaller the result will be 14. Find the numbers.

12. There are two consecutive integers such that if 4 times the smaller is subtracted from the larger the result is -26. Find the numbers.

13. There are three consecutive integers such that if the third is subtracted from the sum of the first two the result is 9. Find the numbers.

14. There are three consecutive integers such that if the second is subtracted from the sum of twice the first and the third the result is 31. Find the numbers.

15. There are three consecutive integers such that if 3 times the first is subtracted from the sum of the second and the third the result is -16. Find the numbers.

16. The length of a certain rectangle is 2 feet more than its width. The side of a certain square is twice the width of the rectangle. The perimeter of the square is 16 feet more than the perimeter of the rectangle. Find the dimensions of each.

67. Multiplication. We know that in arithmetic the product of 5 times 3 is the same as the product of 3 times 5; that is,

$$5\times3=3\times5$$

The order in which the multiplying is done makes no difference in the result. The idea is just as true in algebra. Whatever may be the product of 3 times -5, -5 times 3 must give the same result. With this understanding three cases arise in multiplying positive and negative numbers:

A positive number times a positive number, $(+3)\times(+5)$
A positive number times a negative number, $(+3)\times(-5)$
A negative number times a negative number, $(-3)\times(-5)$

How to handle the first case is known to us in arithmetic:

$$(+3)\times(+5)=+15$$

We may think of it as adding 5 three times. On the number scale this would mean moving to the right from zero 5 unit places 3 times, and give $+15$. The product of a positive 3 and a positive 5 is a positive 15.

In like manner, on the number scale $(3)\times(-5)$ would mean adding -5 three times, or, what is the same thing, subtracting $+5$ three times, that is, moving to the left 5 places three times, and give -15. Thus the product of positive 3 and negative 5 is negative 15.

So also $(-3)\times(-5)$ might be thought of as subtracting -5 three times or, what is the same thing, adding $+5$ three times, which would give $+15$. Thus the product of negative 3 and negative 5 would be positive 15.

These illustrations suggest the following definitions: The **product** of **two positive numbers** is a positive number. The **product** of **two negative numbers** is a positive number. The **product** of a **positive number** and a **negative number** is a negative number.

Thus, to multiply two numbers together, find the product of their absolute values and write in front of this product the proper sign.

In symbols $(+3)\times(+5)=+15, \quad (+3)\times(-5)=-15$
$(-3)\times(-5)=+15, \quad (-3)\times(+5)=-15$

EXERCISES

1. $(3)\times(+7)=$? 2. $(-3)\times(7)=$? 3. $(-5)\times(-11)=$?
4. $9\times(-3)=$? 5. $(-8)\times(-9)=$? 6. $(-7)\cdot(a)=$?
7. $(-a)\cdot(-5)=$? 8. $(-a)\cdot(-b)=$? 9. $a\cdot(-b)=$?
10. $(-a)\cdot(-6)=$? 11. $n\cdot(-7)=$? 12. $-x\cdot(-y)=$?

13. Copy and fill in the blanks in the following table, placing the products of a number from the horizontal row with a number from the vertical column at the intersection of its row and column.

0	2	-5	$\frac{2}{3}$	-6	$2\frac{1}{2}$	n
3						
-2						
$\frac{1}{2}$						
.7						
-8						
$-a$						

14. $(-3)(-5)(6) =$?
15. $(-10)(\frac{1}{2})(-\frac{2}{3})(-\frac{9}{7}) =$?
16. $(-4)(-3)(-a) =$?
17. $(-5a)(3)(-2b) =$?
18. $(-6)(-3)(+a)(-b) =$?
19. $(-7)(-a)(-a)(-2) =$?
20. $(-5)(-9)(+3) =$?
21. $(-2)(-3a)(-4b)(-7) =$?
22. $(+x)(-9)(-3x)(-7) =$?
23. $(+5a)(-3x)(-7) =$?
24. $5(-x)(-2)(a) =$?
25. $7(-n)(+3t)(-2) =$?

68. Multiplying a sum. In arithmetic we multiply $7+3$ by 5 in two ways. We may add the 7 and 3, getting 10, and then multiply by 5, getting 50, or we may multiply 7 by 5, getting 35, and multiply 3 by 5, getting 15, and then add 35 and 15, getting 50. If we wish to express the example in symbols, we must inclose the $7+3$ in a parenthesis, thus:

$$5(7+3)=5\cdot 10=50$$
$$5(7+3)=35+15=50$$

The second way is exactly what you do when you multiply 32 by 3:

$$32 \text{ is } 30+2$$
$$3(30+2)=90+6$$
$$=96$$

though you usually write it

$$\begin{array}{r} 32 \\ \underline{3} \\ 96 \end{array}$$

RULE. **To multiply a sum by any number, multiply each term of the sum by the number.**

EXERCISES

Work both ways when possible:

1. $5(7+4)$
2. $5(7-4)$
3. $3(8+2-5)$
4. $6(\pi-3)$ (Use $\pi=3.142$)
5. $\pi(9+4)$
6. $\pi(9-4)$
7. $(-2)(4+3)$
8. $(-2)(6-3)$
9. $(-3)(7-2+3)$
10. $5(8-10+2)$
11. $3(a-4)$
12. $-a(x-4)$
13. $5(x+y-7)$
14. $20(x-y+3)$
15. $3x(a-7)$
16. $\pi(a-b)$
17. $-4(x-y)$
18. $4x(x-a+2)$
19. $5a(a-x-y)$
20. $-7x(a-2x+4)$
21. $7y(a-2b+3y)$
22. $-12x(a-4b+3x)$
23. $-2(3-5x)+2$
24. $-1(-7+x-3b)$

It is to be noticed that when we multiply a sum by a number we introduce the number as a factor in each term of the sum:

$$2(3+7)=2\times3+2\times7$$
$$2(a-b)=2a-2b$$

But when we multiply a product by a number we simply introduce it as a factor but once:

$$2a(xy)=2axy$$
$$2(3\cdot7)=2\cdot3\cdot7$$

It should also be noticed that the result is the same in whatever order we combine the factors:

$$2\times3\times7=6\times7=42$$
$$3\times7\times2=21\times2=42$$
$$7\times2\times3=14\times3=42$$

Find the values when $a=2$, $b=-2$, $c=3$, $x=-4$, $y=-5$, $z=6$:

25. $a(b-c)$	26. $b(a-c-x)$	27. $b(c-x+y)$
28. $x(a+5y-4z)$	29. $x(ab-2ay+bc)$	30. $bx(a+3y-2)$
31. $ax(b-5y-4c)$	32. $bc(3a+2y-2b)$	33. $a(bcx)$
34. $ax(bayz)$	35. $b(a+b+c)$	36. $(abc)(x+y)$

69. Division. Multiplication is the operation of finding the product when the factors are given. Division is the operation of finding one of the factors when the product and the other factor are given.

$$7\times3=?$$
$$7\times?=35$$

Because of this relation **division** is called the inverse of multiplication. Show that this fact enables us to write down the following rule for dividing positive and negative numbers:

RULE. **The quotient of two positive numbers is a positive number.**

The quotient of two negative numbers is a positive number.

The quotient of a positive number and a negative number is a negative number.

$(+7)\times(\quad)=+28$	$(\quad)\times(9)=45$
$(+7)\times(\quad)=-28$	$(\quad)\times(9)=-45$
$(-7)\times(\quad)=+28$	$(\quad)\times(-9)=+45$
$(-7)\times(\quad)=-28$	$(\quad)\times(-9)=-45$

EXERCISES

Divide:

1. 45 by 5
2. 72 by -12
3. -96 by 3
4. -25 by -50
5. 91 by -91
6. $\frac{2}{3}$ by $-\frac{1}{2}$
7. $-\frac{7}{12}$ by $-\frac{14}{3}$
8. -2.3 by 7
9. 3.92 by -1.33
10. a by $-a$
11. $-a$ by $-a$
12. $2a$ by 2
13. $2a$ by -2
14. $3a$ by $-a$
15. $6a$ by $-2a$

70. Mastery of the laws of signs. A thorough mastery of the laws of signs for the four fundamental operations is of great importance. There should be no hesitation in your mind as to the proper sign to be used in any case that comes up. Practice these operations until they become automatic. Such mastery will save you much trouble in the future.

EXERCISES

1. Copy and fill in the blanks of this table using the values of a and b given in the first two columns.

a	b	$a+b$	$a-b$	ab	$\frac{a}{b}$
12	4				
9	15				
8	-4				
-9	$+3$				
-7	-4				

Evaluate, that is, find the value of, the following if $a=15$, $b=8$, $c=2$, $x=3$, $y=5$, working the first four exercises in more than one way:

2. $\frac{ab}{2}$ 3. $\frac{abc}{y}$ 4. $\frac{acy}{x}$ 5. $\frac{abcx}{2}$ 6. $\frac{bc}{2c}$

7. $\frac{a}{y}+\frac{b}{c}$ 8. $\frac{a}{x}-\frac{b}{c}$ 9. $\frac{ab}{xy}+\frac{by}{2c}$ 10. $\frac{a}{2}+\frac{b}{3}+\frac{c}{4}$

11. Work the same exercises when $a=15$, $b=-8$, $c=2$, $x=-3$, $y=5$.

Divide:

12. 12π by 4
13. $15\pi a$ by $3a$
14. $33x$ by -3
15. $-24x$ by 8
16. $-18ax$ by $9a$
17. $+15bx$ by $-3b$

71. Dividing a sum. In arithmetic we may divide $27-15$ by 3 in two ways.

Thus, $(27-15)\div 3=12\div 3=4$

or $(27-15)\div 3=(27\div 3)-(15\div 3)$
$=9-5=4$

A better manner of expressing it is

$$\frac{27-15}{3}=\frac{12}{3}=4$$

or

$$\frac{27-15}{3}=\frac{27}{3}-\frac{15}{3}=9-5=4$$

RULE. **To divide a sum by a number, divide each term of the sum by the number.**

EXERCISE I

Divide:

1. $(81-27)$ by 9
2. $9\pi-6\pi$ by 3
3. $9\pi-6\pi$ by 3π
4. $27a-15a$ by 3
5. $45x-27x$ by 9
6. $9x+15$ by 3
7. $4x-12$ by -4
8. $-3x+21y$ by -3
9. $3ax-2ay+5a$ by a
10. $36ax-9x$ by $9x$

11. $-36ax+9x$ by $-3x$
12. $3x+8$ by 3
13. $5x-7$ by 2
14. $7x-8$ by -3
15. $-2a-3b+5c$ by -3

EXERCISE II

Evaluate when $x=4$, $y=-3$, $z=8$, $a=2$, $b=-3$:

1. $\frac{x+2y+3z}{a}$
2. $\frac{3x-12y+6z}{b}$
3. $\frac{ax-4a}{z}$
4. $\frac{ax-4by+z}{ab}$
5. $\frac{ay-bx+z}{a-b}$
6. $\frac{2a+x+2y}{ab}$

If $a=2$, $b=-3$, $c=5$, $x=-4$, find the values of the expressions:

7. $2(a-b)$
8. $5(a-c)$
9. $a(b-c)$
10. $\frac{a+b}{c}$
11. $\frac{a-b}{c}x$
12. $\frac{ab-c}{a}$
13. $\frac{a+b}{2}+\frac{a-c}{2}$
14. $\frac{2a-3b+2c}{4a-2b}$
15. $\frac{ab+bc+ca}{x}$
16. $\frac{abcx}{ab+ca}$
17. $(a-b)+(c-x)-(a-x+c)$

72. Equations. It may save you some trouble in changing the forms of such expressions as

$$(1)\ x+2(2x-5)$$
$$(2)\ x-2(5x-2)$$

if you take care to read out in words the meaning of the expressions. (1) means that 2 times each term in the parenthesis is to be added to x, while (2) means that 2 times each term in the parenthesis is to be subtracted from x. Remember also what is to be done when one number is to be subtracted from another in algebra.

EXERCISES

Solve and check:

1. $x+9=2$
2. $x+11=4$
3. $x+13=7$
4. $3x+14=2x+10$
5. $2x+10=x+8$
6. $x+15=10$
7. $x+9=2x+15$
8. $x+5=2x+10$

9. $4x+15=3x+12$

10. $5x+16=4x+13$

11. $x-a=b$

12. Derive equations from Exercise 11 in which:

(1) $a=12,\quad b=6$ (2) $a=-9,\ b=2$

(3) $a=-13,\ b=-3$ (4) $a=10,\quad b=8$

13. Find the answers to the equations obtained in Exercise 12 by adding the numbers as indicated in the answer to Exercise 11.

14. If a is subtracted from a certain number and this remainder is subtracted from twice the number, the result is $4a$. Find the number.

15. Solve Exercise 14 when a is 5, 20, -6, -3, 9, -18, 15.

16. $6x+4(x-2)=22$

17. $8x+2(x-5)=90$

18. $6x-3(x-4)=36$

19. $15n-2(3n-8)=34$

20. $8x-2(2x-5)=2$

21. $3n-4(2n-9)=6$

22. $7n+2(3n+14)=2$

23. $3y-5(3y+2)=38$

24. $5x-2(5x+20)=10$

25. $7t+3(5t+1)=-8$

26. $9x-3(x-12)=-6$

27. $4(3n-5)-5(5n-7)=2$

28. $12x=56-4(x-6)$

29. $9p=12+3(p-8)$

30. $2x-7(25-x)=2(25-3x)$

31. $8(x-3)-(6-2x)=2(x+2)-5(5-x)$

32. $2(y-3)-5(y-1)=7+(2-y)$

73. The fraction as a parenthesis. In the solving of the equation

$$\frac{x-2}{5}=\frac{x}{6}$$

the numerator $x-2$ must be regarded as one number. Thus when we multiply both sides of the equation by 30 we write

$$6(x-2)=5x$$

The fractional sign acts the same as a parenthesis:

$$6x-12=5x$$
$$x=12$$

In the same way we have

$$x-\frac{3(x-2)}{5}=4$$
$$5x-3(x-2)=20$$
$$5x-3x+6=20$$
$$2x=14$$
$$x=7$$

and also

$$x-\frac{x-2}{3}=4$$
$$3x-(x-2)=12$$
$$3x-x+2=12$$
$$2x=10$$
$$x=5$$

EXERCISES

Solve and check:

1. $\frac{x+32}{9}=\frac{x}{5}$
2. $\frac{3a}{4}-11=\frac{2a}{2}$
3. $\frac{7-3(x-5)}{4}=1$
4. $4-\frac{x+3}{4}=1$
5. $x-\frac{x-1}{5}=5$
6. $\frac{n}{2}-\frac{n-3}{5}=9$
7. $\frac{n}{3}-\frac{n+3}{6}=2$
8. $7-\frac{x+13}{5}=5$
9. $11-\frac{x+10}{2}=-2x$
10. $9-\frac{y-4}{3}=12$
11. $x-\frac{x-29}{5}=1$
12. $\frac{x}{5}-\frac{x+3}{6}=-1$
13. $\frac{x}{5}-\frac{x-14}{4}=4$
14. $\frac{x+3}{2}-\frac{x-3}{5}=9$
15. $\frac{x-6}{4}-\frac{x-7}{3}=1$
16. $\frac{x+6}{8}-\frac{x+2}{6}=1$
17. $\frac{n-5}{4}=1-\frac{3n+17}{10}$
18. $\frac{x-7}{6}=10-\frac{4(x-1)}{9}$
19. $\frac{c-2}{2}-\frac{c+2}{5}-\frac{c-1}{7}=0$
20. $\frac{x-4}{3}-\frac{x-3}{5}=\frac{x-6}{7}$

74. Problems with impossible answers. A problem to be solved is stated in the form of an equation. The equation is solved; that is, a satisfactory value for the unknown is found. It does not always follow that the answer of the equation will serve as an answer to the problem. For instance, consider the problem: The sum of two consecutive integers is 10, what are the integers?

$$x+(x+1)=10$$
$$2x=9$$
$$x=\frac{9}{2}$$

$\frac{9}{2}$ satisfies the equation, but is not a correct answer to the problem, for $\frac{9}{2}$ is not an integer. The answer to this equation when used as an answer to the problem is nonsensical. If there were integers of the kind sought in the problem, the solution of the equation would have given them. The solution of the equation shows there are no such integers. The problem is an impossible problem; that is, it has no solution.

This illustration shows the great importance of trying the answer obtained from the equation in the problem itself to see if it actually works. It is often quite as important to learn that a given problem cannot be solved as to be able to find the answer to one that can be solved. The roots of equations arising from problems should always be interpreted by being applied to the problem itself.

To illustrate this in another way: Two sides of a certain triangle were found to be respectively 1 and 2 inches longer than the third side. The sum of the two shorter sides is 3 inches shorter than the longest side. What are the sides of the triangle?

The equation is

$$\text{shortest side}+\text{next larger}=\text{largest}-3$$
$$x+(x+1)=(x+2)-3$$

whence $$x=-2$$

But -2 has no meaning when used as the side of the triangle. The problem is impossible. Clearly the man who made the measurement made some mistake in furnishing the data for the problem.

PROBLEMS

Interpret carefully the results obtained in solving the following problems and determine which problems really have answers:

1. Is there a number such that the result of subtracting 3 times itself from itself will be 15 more than itself?

2. Three times an integer minus 5 times the next larger integer is 37. Find the number.

3. Find an integer such that 5 times it plus 2 is 49.

4. There are three consecutive integers; if 5 times the middle one is subtracted from 3 times the sum of the other two, the result will be 4. Find the numbers.

5. There are four consecutive integers; if twice the sum of the first and third is subtracted from 3 times the sum of the second and fourth, the result will be 24. Find the numbers.

6. There are three consecutive integers; if the first is subtracted from twice the third, the result is 3 times the second. Find the numbers.

7. The sum of three consecutive integers equals the middle one. What are the numbers?

8. I have \$4 in dimes and quarters. If I have 25 pieces of money altogether, how many of each have I

9. I have \$5 in dimes and quarters, having 30 pieces in all. How many of each have I?

10. Twenty-five dollars were to be divided between two boys in the ratio of 1 to 3. What was the share of each?

11. Twenty-five electric light bulbs were to be divided between two boys in the ratio of 1 to 3. What was the share of each?

12. Two boys were talking about their ages; the older boy said, "I am 3 times as old as you, but in 5 years I shall be only twice as old as you. How old am I?"

13. The younger, not to be outdone, said, "All right, but I have a brother who is 3 times as old as I, and in 5 years he will be 5 times as old as I. How old is he?"

14. One side of a triangle is 2 inches longer than a second side, the second side is 4 inches longer than the third side, the perimeter is 7 inches. What are the sides of the triangle?

15. The sides of a triangle have the same relation to one another as in the last problem, but the perimeter is 22. What are the sides of the triangle?

16. Find the dimensions of a square such that when one side is increased by 21 inches and the adjacent side decreased by 3 inches the perimeter of the resulting rectangle is 34.

17. A certain mixture of two substances containing 5 times as many pounds of one substance as of the other weighs 45 pounds. What is the quantity of each substance?

18. In a certain lot of eggs there were twice as many bad eggs as good ones. There were 40 in all. How many good eggs were there?

19. There are three consecutive even numbers; the sum of the first two is 3 times the third. What are the numbers?

20. There are two consecutive even numbers whose sum is 3 times the number midway between them. What are the numbers?

21. In a lot of 7 dozen eggs the ratio of the good to the bad was 2 to 3. How many good eggs were there in the lot?

CHAPTER V

GRAPHICS

75. Comparison of numbers. In our everyday life we are frequently called upon to compare numbers, such as the population of cities, the size of crops, records in athletics, prices of goods. It has become a very common practice to represent numbers by lines or bars drawn to some convenient scale. The eye can more readily compare a lot of lines than the numbers in a table.

Compare the two ways of presenting the number of acres planted in the following crops in the United States for the year 1919:

	Acres
Corn	102,075,000
Oats	42,400,000
Wheat	73,243,000
Barley	7,420,000
Hay	72,034,000
Cotton	33,344,000

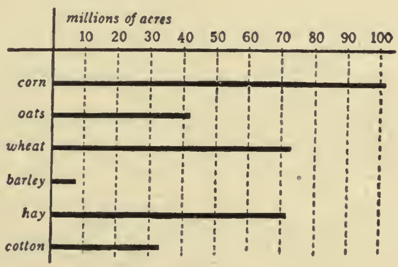

FIG. 11

To present a table of numbers graphically in this way, some convenient unit is chosen, and lines are drawn to scale. The lines or bars may be drawn in either a vertical or a horizontal position. It will help to make the comparison clearer if the bars are arranged in the order of length if there is no other order of arrangement suggested. The scale should be marked on paper as in the figure given. Where the numbers are large, approximate values should be used. Round the numbers off to convenient size for plotting. In the illustration the numbers are used to the nearest million acres.

EXERCISES

1. Compare the lengths of the following important rivers by means of lines. Use $\frac{1}{10}$ or $\frac{1}{8}$ of an inch to the 100 miles.

	Miles		Miles
Mississippi	2800	Amazon	3600
Rio Grande	1800	Danube	1800
Yukon	1600	Nile	3900
Columbia	1200	Congo	1600

2. Show by a bar diagram the comparative heights of the following structures:

	Feet		Feet
Eiffel Tower	984	Woolworth Building	729
Washington Monument	555	St. Peter's, Rome	435
		Pyramid of Cheops	451

3. Use a bar diagram in comparing the number of telephones per hundred people in the following countries:

Argentina	1.1	Great Britain	1.9
Australia	4.0	Italy	.3
Canada	8.1	Norway	4.5
Denmark	7.3	Sweden	6.4
France	.9	Switzerland	3.0
Germany	2.3	United States	11.4

76. Use of cross-ruled paper. The making of such diagrams is much easier when cross-ruled paper is used. A scale is chosen and marked along one edge. The length of the bar is estimated as nearly as possible.

The following table and diagram show the density of population of a number of states—that is, the average number of people to the square mile. Let the student complete the figure.

Massachusetts	479	Illinois	116
Rhode Island	566	Michigan	64
Connecticut	286	Kansas	22
New York	218	Texas	18
Pennsylvania	195	Colorado	9
Ohio	141		

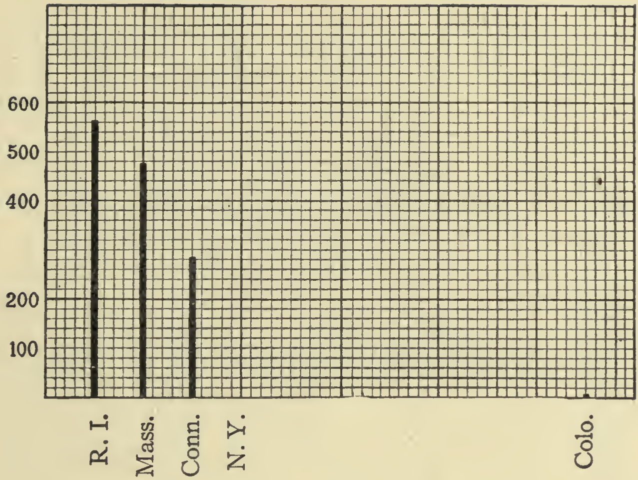

FIG. 12

EXERCISES

1. Display the comparative heights of mountains listed below. Use scale 1 inch to 4000 feet. Taking numbers from the table to the nearest hundred feet, arrange in order.

	Feet		Feet
St. Elias	18024	Blanc	15780
Ranier	14408	Vesuvius	4267
Pike's Peak	14109	Fujiyama	12395
Popocatepetl	17844	Kilauea	4040
Chimborazo	20517	Everest	29141

2. Compare on cross-ruled paper the population of the ten largest cities in the United States, census of 1920. Use numbers to the nearest 10,000.

New York City	5,621,151	St. Louis	772,897
Chicago	2,701,705	Boston	748,060
Philadelphia	1,823,158	Baltimore	733,826
Detroit	993,739	Pittsburgh	588,193
Cleveland	796,841	Los Angeles	568,886

77. Diagrams involving time. In the exercises we have been considering there was no particular order other than magnitude in which the bars should be arranged. With statistics taken at stated intervals of time there is, however, a natural order. The bars should be arranged in the order of time and at proper intervals. Two scales should be marked off, preferably at the left and lower edges of the diagram, as in Fig. 13, which shows the percentage of the total population of the United States living in cities of over 8000 inhabitants.

Year	Per Cent
1790	3.4
1800	4.0
1810	4.9
1820	4.9
1830	6.7
1840	8.5
1850	12.5
1860	16.1
1870	20.9
1880	22.6
1890	29.2
1900	33.1
1910	39.0
1920	43.8

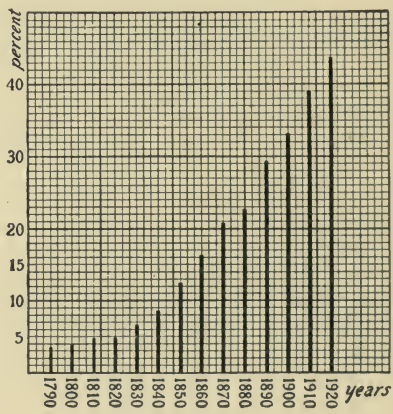

FIG. 13

EXERCISES

1. Display by means of a bar diagram the growth of population of the United States. Use the numbers to the nearest hundred thousand. Draw the bars vertically.

1790	3,929,214	1860	31,443,321
1800	5,308,483	1870	38,558,371
1810	7,239,881	1880	50,155,783
1820	9,633,453	1890	62,947,714
1830	12,866,020	1900	75,994,575
1840	17,069,453	1910	91,972,266
1850	23,191,876	1920	105,710,620

A glance at the diagram gives one a pretty clear idea regarding the increase in numbers.

2. Show also the percentage of increase in each decade as given in the table below:

Year	Per Cent	Year	Per Cent	Year	Per Cent
1800	35.1	1850	35.9	1900	20.7
1810	36.4	1860	35.6	1910	21.0
1820	33.1	1870	26.6	1920	14.9
1830	33.5	1880	26.0		
1840	32.7	1890	24.9		

78. Line diagrams. As we are especially interested in the way quantities change from time to time, it is the ends of the bars that hold our attention. Such changes are more easily seen if we plot merely the ends of the bars and then connect these ends by a series of straight lines or a smooth curve as in Fig. 14 (obtained from table of Fig. 13) and Fig. 15 (number of pupils enrolled in a certain school).

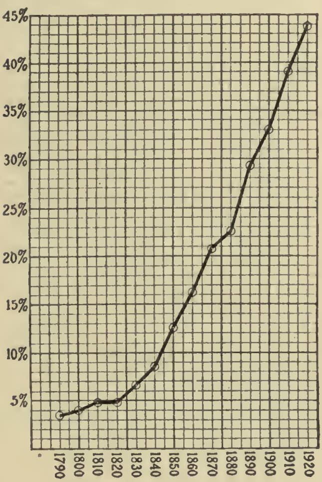

FIG. 14

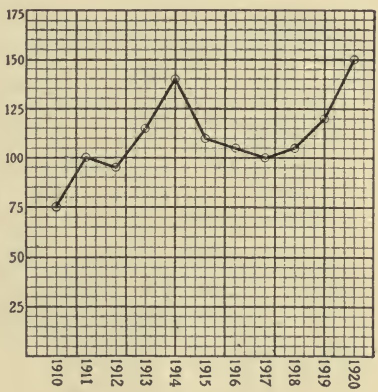

FIG. 15

Such figures are called **graphs.** A point is plotted to correspond to each pair of numbers in the table, and these points connected by a straight line or a smooth curve.

EXERCISES

1. Draw the graph for the following table showing the average weight of boys compared with their height:

Height in In.	Weight in Lbs.	Height in In.	Weight in Lbs.	Height in In.	Weight in Lbs.	Height in In.	Weight in Lbs.
35	32.0	43	43.5	50	59.5	57	83.5
36	33.5	44	45.5	51	63.0	58	87.5
37	34.5	45	47.5	52	66.0	59	91.5
38	36	46	49.5	53	69.0	60	95.0
39	37.5	47	51.5	54	72.5	61	99.5
40	39	48	53.5	55	75.5	62	105.0
41	40.5	49	55.5	56	79.5	63	109.5
42	42.0						

2. Draw the temperature graph from the reading for a certain day in June. Lay off the time scale horizontally to the right from the lower left-hand corner of the ruled paper, using $\frac{1}{4}$ of an inch to each hour. Lay off the temperature scale vertically on the left, using $\frac{1}{10}$ of an inch to a degree.

Hourly Temperature Reading

MIDNIGHT	77°	8:00 A.M.	81°	3:00 P.M.	83°
1:00 A.M.	75°	9:00 A.M.	84°	4:00 P.M.	68°
2:00 A.M.	74°	10:00 A.M.	81°	5:00 P.M.	70°
3:00 A.M.	73°	11:00 A.M.	88°	6:00 P.M.	76°
4:00 A.M.	73°	NOON	91°	7:00 P.M.	76°
5:00 A.M.	72°	1:00 P.M.	92°	8:00 P.M.	76°
6:00 A.M.	75°	2:00 P.M.	92°	9:00 P.M.	76°
7:00 A.M.	76°				

Note the sudden drop in temperature between 2:00 and 5:00 o'clock. There was a severe storm at that time.

3. Contrast the graphs just drawn with that of the coldest day the next winter:

Temperature Readings for December 25, 26, 1903

Dec. 25:

6:00 A.M.	18°	NOON	28°	6:00 P.M.	4°
8:00 A.M.	18°	2:00 P.M.	12°	8:00 P.M.	0°
10:00 A.M.	21°	4:00 P.M.	8°	10:00 P.M.	−2°

Dec. 26:

MIDNIGHT . . . −5°	10:00 A.M. . . . . 0°	4:00 P.M. . . . 14°
6:00 A.M. . . . −7°	NOON . . . . 5°	6:00 P.M. . . . 16°
8:00 A.M. . . . −6°	2:00 P.M. . . . .12°	8:00 P.M. . . . 18°

4. Plot a temperature graph for your own locality. Get the data for several days from the newspaper weather reports.

5. The following table gives the records for height in the pole vault for the United States and the years when made. Display these on a diagram.

Year	Height	Year	Height	Year	Height
1892	11′ $5\frac{3}{8}$″	1906	12′ $4\frac{7}{8}$″	1910	12′ $10\frac{7}{8}$″
1898	11′ $10\frac{1}{2}$″	1907	12′ $5\frac{1}{2}$″	1912	13′ $2\frac{1}{4}$″
1904	12′ $1\frac{1}{3}$″	1908	12′ $9\frac{1}{2}$″	1919	13′ $3\frac{1}{16}$″

6. Plot the following table showing the height above sea level of points on the Rock Island Railroad between Chicago and Rock Island:

Miles	Height in Feet	Station	Miles	Height in Feet	Station
0	603	Chicago	114	466	Bureau
30	716	Mokena	137	670	Sheffield
40	538	Joliet	159	641	Geneseo
51	612	Minooka	181	570	Rock Island
85	484	Ottawa			

7. Draw a graph from the following table showing the retail price of eggs per dozen:

	Cents		Cents		Cents		Cents
1913	33	1915	31	1917	46	1919	60
1914	33	1916	36	1918	54	1920	64

8. Plot profile of the bottom of a river from the following table of soundings made at points a hundred feet apart straight across.

0.0	10.1	9.4	7.4	2.8
8.1	10.3	9.5	6.5	1.4
10.6	10.5	9.2	5.3	0.8
10.3	10.1	8.2	3.9	0.0

A sounding is the depth of the water found by dropping into the water a chunk of lead fastened to a string.

Where is the channel of the river? Answer from the graph,

79. Comparison of graphs. It is sometimes of interest to compare two graphs. This may be done more easily if the graphs are drawn on the same diagram. To prevent confusion, the graphs may be distinguished by the use of different colors or different kinds of lines, such as:

The figure below shows the growth of the population of the cities of New York, Chicago, and Philadelphia:

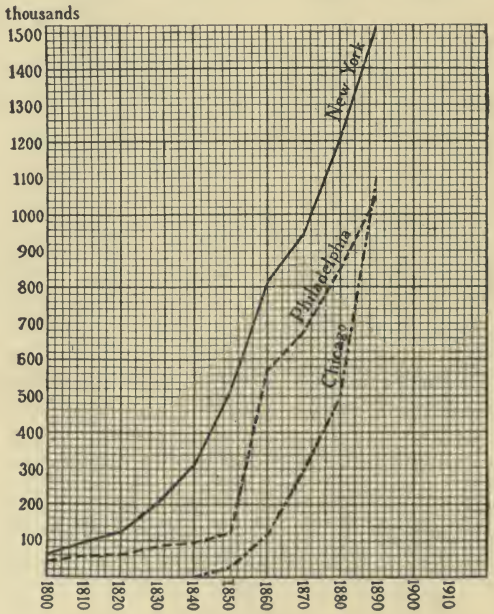

FIG. 16

What does the steepness of the graph indicate?

Which of the three cities was growing the most rapidly from 1880 to 1890?

Which had the slowest growth?

EXERCISES

1. Compare the enrollment in two different departments of a certain school for successive years.

A	A	A	B	B	B
210	191	54	192	142	180
207	164	77	193	125	202
216	168	55	184	109	243
186	27	72	173	103	285
178	38	84	184	138	318
161	37	104	147	157	337

2. Plot on one diagram graphs of the average prices given in table:

Years	Wheat	Oats	Corn	Potatoes
1899	.58	.24	.30	.39
1904	.93	.31	.44	.45
1909	.99	.40	.60	.54
1912	.76	.32	.49	.50
1913	.80	.39	.69	.69
1914	.89	.44	.64	.49

3. Draw on the same diagram graphs showing the imports and exports of the United States for the following years:

Year	Imports Millions of Dollars	Exports Millions of Dollars	Year	Imports Millions of Dollars	Exports Millions of Dollars
1900	850	1371	1908	1194	1835
1901	823	1460	1909	1312	1638
1902	903	1355	1910	1557	1710
1903	1026	1392	1911	1527	2014
1904	991	1435	1912	1653	2170
1905	1118	1492	1913	1813	2429
1906	1227	1718	1914	1893	2330
1907	1434	1854	....	....	....

80. Graphs mechanically drawn. The temperature readings given in several exercises of Art. **78** were obtained by readings of the thermometer every hour. If readings had been taken more frequently — say every fifteen minutes — the plotted points would have been closer and the graph would indicate more accurately the actual changes in temperature.

An instrument, called the thermograph, has been invented to draw a temperature graph automatically. A pen attached to a metal thermometer draws a continuous line on a sheet of paper rolled on a cylinder which is kept revolving at a constant rate. The thermograph and the line drawn by it are shown in Figs. 17 and 18. The rise and fall of the graph denote the rise and fall of the temperature. We

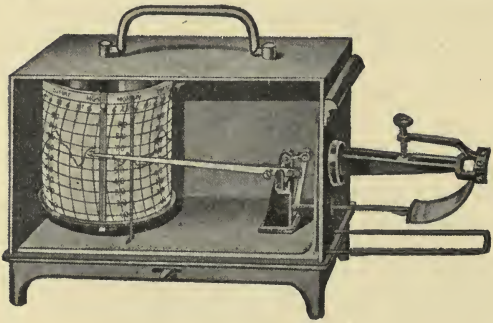

FIG. 17

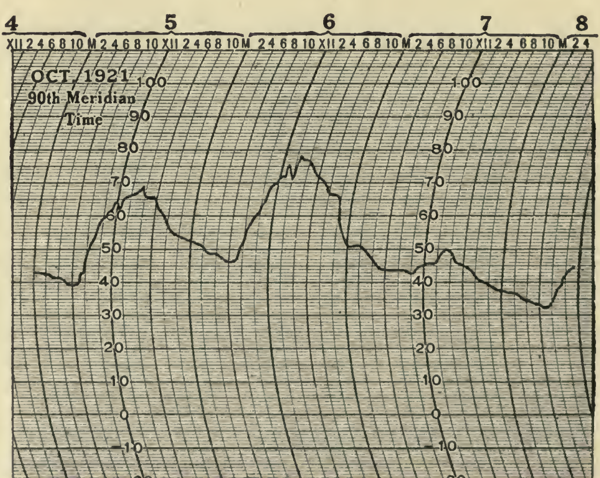

FIG. 18

can read the temperature at any moment directly from the graph by finding the time on the time scale and reading off the height of the graph at that point. For instance, at 4:00 P.M. October 6 the temperature was 76°.

EXERCISE

1. From the temperature graph in Fig. 18 read the temperature at noon October 5, at noon October 6, at 6:00 P.M. October 7, at 6:00 A.M. October 8.

When was the temperature 78°? 32°?
What hours during the day was it warmest?
What hours during the day was it coldest?

81. Graphs used for reckoning. Make a table of the prices of different lengths of gingham—say from 1 to 10 yards at 15 cents a yard—and plot the graph (Fig. 19). Lay off the scale of yards along a horizontal line or axis and the scale of costs along a vertical linear axis.

Yards	Cents
1	15
2	30
3	45
4	60
5	75
6	90
7	105
8	120
9	
10	

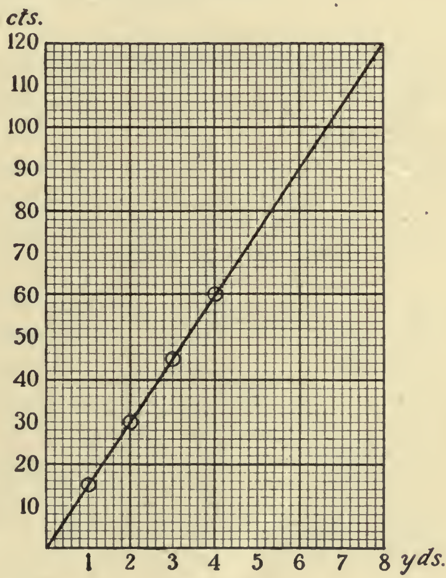

FIG. 19

What is peculiar about these points? Plot the price of $5\frac{1}{3}$ yards; of $3\frac{1}{2}$ yards.

Taking it for granted that the graph is a straight line, read directly from the graph the price of 8 yards; the price of $7\frac{1}{2}$ yards; the price of $2\frac{2}{3}$ yards.

How many yards can be bought for 50 cents? Read the number of yards from the graph and then test your result by computing the amount.

This graph differs very materially from those we have been considering (excepting the automatically drawn graphs). In the graphs hitherto drawn only those points that were actually plotted had a definite value. The lines joining them were merely to aid the eye. In this price graph every point on the graph has a meaning. It corresponds to a certain number of yards at a certain price. This graph shows the relation between the price and the number of yards of goods in the form of a picture. The same relation may be shown algebraically by means of a formula:

$$\text{Price} = 15 \times \text{number of yards}$$
$$p = 15y$$

Results may be read off from such graphs very readily. Graphs of this kind are used very extensively as ready reckoners. A few illustrations are given in the exercises below.

EXERCISES

Draw graphs and give formula for each.

1. Garden hose sells at 17 cents a foot. Draw a sales graph and read from the graph the price of 25 feet, 33 feet, and $12\frac{1}{2}$ feet. How many feet can be purchased for $1.50? Write down an algebraic formula and test your results by computation.

2. Tenpenny nails sell at 7 cents a pound. Draw a sales graph. How many pounds will 25 cents buy?

3. Draw a graph that can be used in filling out an order for any number of a certain kind of bolt by weighing, allowing 18 bolts to the pound.

4. Cold-storage eggs sell at 50 cents a dozen, and fresh eggs at 60 cents a dozen. Make a sales graph for each, putting the two graphs on the same paper and on the same axes. Find the

answers to the following questions on the graphs: How many more cold-storage eggs can you get for $4.50 than fresh eggs? What is the difference in the price of 3 dozen eggs of each kind?

5. Water is flowing into a tank at the rate of 2 gallons a second. Draw a graph showing the number of gallons of water in the tank at any time after the water is turned on.

6. Water is flowing into a tank from two pipes—from one at the rate of 2 gallons a second, from the other at the rate of 3 gallons a second. Draw graphs showing the number of gallons of water in the tank at any time after the water is turned on. Answer the following questions from the graphs: How much water flows in from each pipe in 10 seconds? From both pipes? How much more flows in from the larger than from the smaller in 15 seconds? How long will it take 10 gallons to flow in from each pipe? If the tank holds 25 gallons, how much longer will it take the smaller pipe to fill it than the larger?

7. An automobile is moving at the rate of 20 miles an hour. Draw a graph showing the distance traveled in any given time less than 3 hours.

8. One automobile is moving at the rate of 20 miles an hour, another at the rate of 25 miles an hour. Draw graphs showing the distances traveled at any given time. How far apart will the machines be in 8 hours? How long will it take each machine to go 65 miles?

9. Draw a graph that can be used to change inches into centimeters. 1 inch = 2.5 cm. (about).

10. Draw a graph that can be used to change kilometers to miles. Show by means of this graph the speed in miles per hour of a French train that is reported as having a speed of 90 kilometers per hour. 1 mile = 1.609 kilometers.

11. Draw a graph to be used in changing cubic inches into gallons, a gallon being 231 cubic inches.

12. Draw an interest graph for the interest on one dollar at 5 per cent for a number of years.

13. A saves \$5 a week. B has \$10 given to him and then starts saving at the rate of \$4 a week. Draw graphs showing how much money each one has at the end of a given number of weeks. When will they have the same amount?

14. The cost of setting up the type for printing a certain circular is two dollars. The printer charges fifty cents per hundred for copies. Draw a graph which could be used for finding the total cost of any number of circulars.

15. If the value of a house costing \$5,000 decreases \$200 each year, draw a graph showing what the house is worth at any time.

16. A saves at the rate of \$3 a week. After 3 weeks B begins to save at the rate of \$4 a week. When will they have saved the same amount? Answer from graphs.

17. The schedule of a train running from town A to town E is given in tabular form on the left below, and in graphical form in the figure.

Miles	Station	Daily	
0	A		12 Noon
20	B	arr.	12:40
		lve.	12:50
24	C		1:10
44	D	arr.	1:30
		lve.	1:35
50	E		1:40

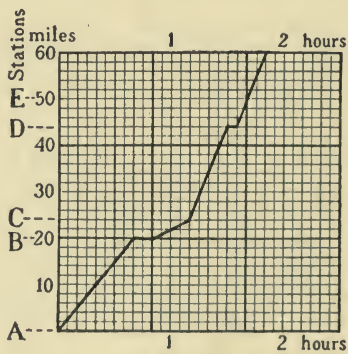

FIG. 20

The graph is worth careful study. What do the horizontal lines mean? Between what points was the train traveling at the fastest rate? At the slowest? What was the average rate between A and E, including stops?

It is a common practice in railroad offices to chart the movement of trains in this way. Many trains are charted on the same paper.

The schedule of a train leaving E at 1:00 P.M. and running to A in $1\frac{1}{2}$ hours without stop would be indicated by a line drawn from near the top of the diagram down to the right. Draw the graph.

18. Chart the following train schedules on one chart.

Miles	Station	1 Daily	2 Daily
0	A	lve. 1:00 P.M.	1:30 P.M.
6	B		
16	C	arr. 1:50	1:55
		lve. 2:00	
30	D	arr. 2:40	
		lve. 2:55	
40	E	3:20	2:30 arr.
			2:35 lve.
46	F		2:50

82. Problems solved graphically. Many problems can be solved by graphs without any use of equations. Especially is this true of problems concerning bodies moving at a constant speed, such as an automobile moving at the rate of 15 miles an hour or train at the rate of 50 miles an hour. In such cases we think of the object as moving through equal distances in equal times. The last two exercises in the last article deal with this kind of motion.

EXERCISES

1. A and B start at the same place at the same time and travel in the same direction, B at the rate of 10 miles an hour and A at the rate of 12 miles an hour. How far apart are they in 5 hours? When were they 8 miles apart?

Let the pupil answer the question from the graph at the right (Fig. 21).

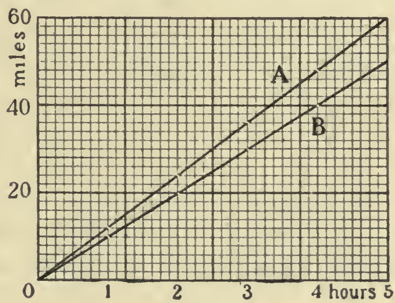

FIG. 21

2. A and B start from the same place and travel in the same direction. If A travels at the rate of 6 miles an hour and B at the rate of 9 miles an hour, after how many hours will they be 6 miles apart? Solve by graph.

3. A travels at the rate of 3 miles an hour. After 4 hours B, who started at the same time and place, is 15 miles ahead of A. Find by graph the rate at which B travels. An approximate answer is sufficient.

Solve the three preceding examples by arithmetic.

4. A and B start from the same place at the same time and travel in opposite directions. If A travels at the rate of 5 miles an hour and B travels at the rate of 7 miles an hour, how far apart will they be in 6 hours? At what time will they be 18 miles apart?

5. A and B start from the same place and travel in the same direction. A travels at the rate of 8 miles an hour. Two hours after A has left, B starts after him at the rate of 12 miles an hour. When will B overtake A? (See Fig. 22.)

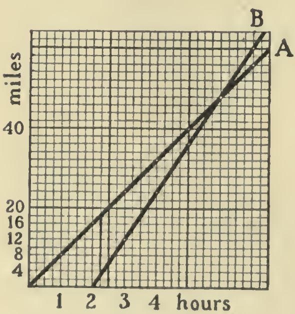

Fig. 22

6. An express train leaves a station two hours after a freight traveling in the same direction. The freight is running at the rate of 20 miles an hour, the express at the rate of 45 miles an hour. In how many hours will the express overtake the freight?

7. A is walking at the rate of $3\frac{1}{2}$ miles an hour. After 4 hours B starts after him on his bicycle at the rate of 15 miles an hour. When will B overtake A?

8. A rode away from town on his wheel. Two hours after he left, B started after him in his automobile at the rate of 40 miles an hour and overtook him in half an hour. How fast was A riding? What distance had he covered?

9. A and B start at the same time and travel east. A starts from a place 12 miles west of the place from which B starts and travels at the rate of 20 miles an hour. If B is going at the rate of 12 miles an hour, when will A overtake B?

10. A and B start at the same time from places 70 miles apart and travel toward each other, A at the rate of 20 miles an hour and B at the rate of 15 miles an hour. When and where will they meet? (See Fig. 23.)

Let the pupil read the answers from the graphs.

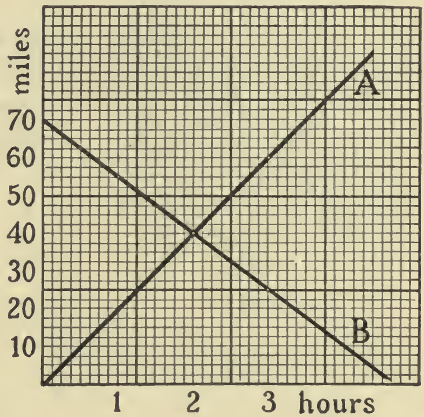

FIG. 23

83. Graphs of algebraic expressions. Any algebraic expression in one unknown can be represented graphically. As an illustration take the expression

$$x-2$$

Now x may be any number. If x is given any particular value, say 5, then $x-2$ will become $5-2$, or 3. Now, by giving x a number of values in succession and finding the corresponding value of $x-2$ in each case, we can build up a table of pairs of values for x and $x-2$.

If $x=2$ then $x-2=0$
$x=3$ $x-2=1$
$x=4$ $x-2=2$
$x=6$ $x-2=4$
$x=9$ $x-2=7$
$x=12$ $x-2=10$

Because the expressions x and $x-2$ can have various values, they are called **variables**.

These pairs of values may be plotted, each pair giving one point on the graph. Use the horizontal scale for the x values and the vertical scale for the $x-2$ values.

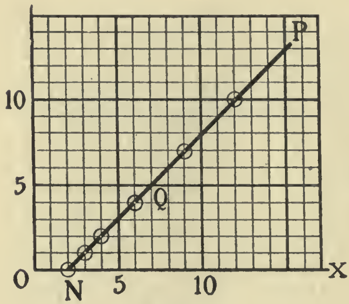

FIG. 24

The plotted points all seem to lie on a straight line NP (Fig. 24). Compute the value of $x-2$ for some value of x between the values plotted, say $x=3\frac{1}{2}$ and $x=5$, and plot.

Two important facts are to be noticed here:

(1) Any set of values of the two variables x and $x-2$ determine a point on the line NP.

(2) For any point on the line NP there are two numbers, one from each scale, that fit the variables x and $x-2$.

For instance, the scale readings for the point Q are:

$$\left.\begin{array}{ll}\text{horizontal,} & 7\\ \text{vertical,} & 5\end{array}\right\} \text{ that is, } \left\{\begin{array}{l}7=x\\ 5=x-2\end{array}\right.$$

The scale readings for a point are called the **coördinates of the point.** They are often written in the form (7, 5); the horizontal coördinate is written first.

From the graph find:

(1) values of $x-2$ when $x=8$, 10, 11
(2) values of x when $x-2=5$, 3, 9, 13

The pairs of values may be extended by taking either smaller or larger values of x.

$$\text{If } x=1 \text{ then } x-2=-1$$
$$\text{If } x=-3 \text{ then } x-2=-5$$

We may plot the pairs $(1, -1)$ and $(-3, -5)$ by extending the scales, the x scale to the left and the $x-2$ scale downward. The point for the pair (1, −1) then falls one unit below the +1 on the x scale. The point for (−3, −5) falls five units below the −3 on the x scale. (Fig. 25.)

Positive numbers are to be laid off to the right of the vertical axis or above the horizontal axis.

Negative numbers are to be laid off to the left or downward.

The straight line L is the graph of the expression $x-2$.

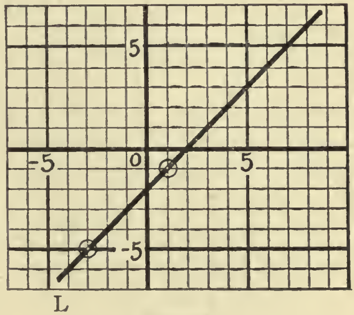

FIG. 25

EXERCISES

1. Read the coördinates for the points marked in Fig. 26.

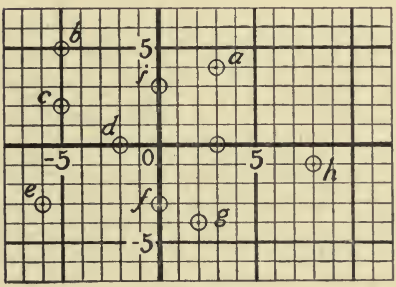

FIG. 26

2. Plot the following points. The horizontal coördinate is given first. (3, 4), (−2, 3), (2, −3), (−3, −5), (0, 0), (3, 0), (0, 5), (−2, 1), (−5, −4), (0, −5), (−7, 0).

3. Find the values of $x-5$ for $x=8$, 7, 6, 5, 4, 3, 2, 1, 0, −1, −2, −3, −4, −5, −6, −7, −8, and plot on cross-section paper.

4. Find the values of $3-x$ for integral values of x between $x=-6$ and $x=+6$ and plot.

5. Plot $2x$.

6. Plot $2x-3$.

7. Plot $4-3x$.

8. Plot $-2x+5$.

9. Plot $-2x-3$.

10. Plot $x+3$.

11. What kind of line do you think the graph of an expression of the form $ax+b$ is where a and b are supposed to be special known numbers?

84. Graphs of equations. The exercises of the last article may be put in a slightly different form.

For convenience we may denote the second of the two variables x and $x+3$ by a single letter, thus:

$$y=x+3$$

giving a letter for each variable. It is to be understood that the y stands for $x+3$ in this case. The graph, of course, will be exactly the same as before. We assign values to x and find the corresponding value of y and plot as before, with the horizontal line as the ***x*-axis** and the vertical line as the ***y*-axis.**

The equation $y=x+3$ may be put in the form $y-x=3$, but the graph will not be changed.

The fact that the difference between two numbers equals 3 may be stated in two ways:

Algebraic Statement

$$y-x=3$$

Graphic Statement

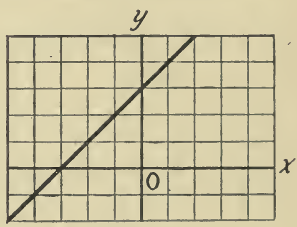

FIG. 27

In general, all equations in two variables may be represented graphically. There are many different kinds of equations. Each kind has its own peculiar graph. Some of these graphs are very simple, and others are rather complicated. But each displays the characteristics of its equation in very striking manner. It was a great step in mathematics when this relation between equations and lines was found. We owe the idea to a Frenchman named Descartes who lived from 1596 to 1650.

Illustration. Draw the graph of the equation $2x+y=10$.

Values are assigned to x and the corresponding values of y found. (Fig. 28.)

If $x=1 \quad 2+y=10 \quad \therefore \quad y=8$
$x=3 \quad 6+y=10 \quad y=4$
$x=5 \quad 10+y=10 \quad y=0$

Plot the pairs
(1, 8)
(3, 4)
(5, 0)

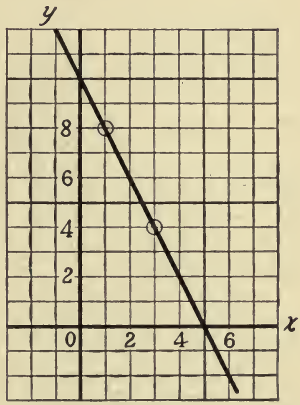

FIG. 28

EXERCISES

Draw a graph for the following equations:

1. $y=2x+4$
2. $y+4x=4$
3. $y+4x+4=0$
4. $2y=3x$

85. Solving equations for one variable. The work of graphing an equation is usually made simple if the equation is first **solved for one of the unknowns** in terms of the other.

Consider the equation

$$2x+y=9$$

Untangle the y and get

$$y=9-2x$$

and then find the value of $9-2x$ for various values of x. Why is this any better than the way given in the last article?

In assigning values to x it is better to select numbers in regular order rather than to choose them hit or miss.

PROBLEMS

Solve these equations for y in terms of x and then plot the graph.

1. $2x+3y=6$
2. $2x-3y=6$
3. $5y+2x=0$
4. $3x-4y=12$

GENERAL EXERCISES

1. The difference between two numbers is 12. Write the equation and draw the graph.

2. The quotient of two numbers is 4. Write the equation and draw the graph.

3. Twice a number plus a second number plus 4 is always zero. Write the equation and plot the graph.

4. If one side of a rectangle is always 3 less than twice the other side, write the equation and draw the graph showing the relation between the sides of all rectangles filling the requirement.

5. The value of a number of nickels is always 3 times the value of a number of one-cent pieces. Draw the graph showing the relation between the number of nickels and the number of one-cent pieces.

6. The value of a number of nickels is always 2 less than 3 times the value of a number of one-cent pieces. Draw the graph showing the relation between the number of nickels and the number of one-cent pieces.

7. In five years A will be twice as old as B. Draw the graph showing the relation between their ages.

8. The sum of the two digits of a number is 12. Draw a graph showing the relation between the digits.

9. A man can row one mile up a stream against a current in 25 minutes. Draw a graph showing the relation between the rate of the current and the rate the man can row in still water.

10. On the same diagram draw the graphs of $x+y=2$, $3x-y=6$, $3y-x=6$.

11. Draw a graph showing the position of all points that are twice as far from the y-axis as they are from the x-axis.

CHAPTER VI

LINEAR EQUATIONS IN TWO UNKNOWNS

86. Graphic solutions. Let us consider the following problem. What are the numbers whose sum is 4? The question may be stated algebraically or graphically:

Algebraic Statement

(1) $x+y=4$

Graphic Statement

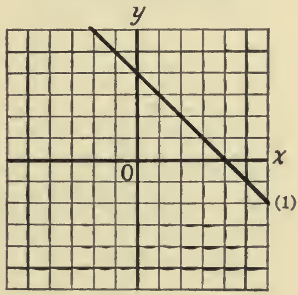

FIG. 29

There are two ways of getting an answer to the question:

(1) Any pair of values of x and y that satisfy the equation may serve as an answer to the problem, for instance, 1 and 3, or -6 and 10. Find four other sets of answers from the equation.

(2) The coördinates of any point on line (1) may serve as answers, for instance, the points that give 2 and 2, or -2 and 6. Find from the graph four other sets of answers to the question.

There is an unlimited number of answers to the question. Any pair of values that satisfies the equation is called a **solution** of the equation.

But suppose another fact is known about these two numbers. Suppose the question reads: What are the numbers whose sum is 4 and whose difference is 1? (Fig. 30.)

Algebraic Statement

(1) $x+y=4$

(2) $x-y=1$

Graphic Statement

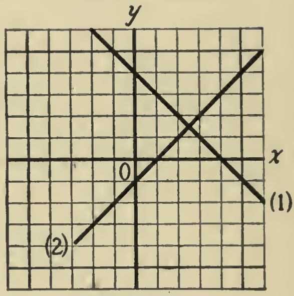

FIG. 30

The coördinates of any point on line (1) will satisfy the sum condition; the coördinates of any point on line (2) will satisfy the difference condition. As there is but one point common to the two lines, there is but one pair of values that will satisfy both conditions of the problem. If the graphs have been carefully drawn, the coördinates of this common point may be read from the scales on the axes with some degree of accuracy. To check the result substitute the numbers in the two equations.

Reading the coördinates of the common point of the case in hand, we get

$$x=2\tfrac{1}{2} \qquad y=1\tfrac{1}{2}$$

These check when substituted in the equations

$$x+y=4 \qquad x-y=1$$

$$2\tfrac{1}{2}+1\tfrac{1}{2}=4 \qquad 2\tfrac{1}{2}-1\tfrac{1}{2}=1$$

We have thus solved the problem graphically. The algebraic solution will be given later.

87. Short method of drawing graph of a linear equation. Any equation of the form

$$ax+by=c$$

such as

$$3x+5y=18$$
$$2x-3y=5$$
$$x-7y=-3$$

is called a **linear** equation because its graph is a straight line. The x and y are the two unknowns, while a, b, and c are any known numbers whatsoever.

Only one straight line can be drawn through two points. The graph of a linear equation is a straight line. Consequently only two points are needed for drawing the graph.

To find the coördinates of two points, assign to one of the letters, say x, any two values and find the corresponding values of y. (Fig. 31.)

For instance, $3x+2y=9$

Put $x=1$, find $y=3$

Put $x=5$, find $y=-3$

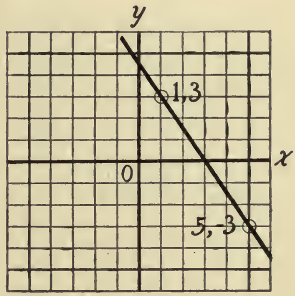

FIG. 31

It is well to choose points that are rather far apart. Why?

The points where the line crosses the axes are often very satisfactory points to choose. The x of the point where the line crosses the y-axis must be zero. Why? To find the coördinates of the y-axis crossing, put $x=0$ and find the corresponding value of y. (Fig. 32.)

Thus, $2x-3y=12$

For $x=0$ $y=-4$

What must be the value of y for the point where the line crosses the x-axis? Find the corresponding value of x.

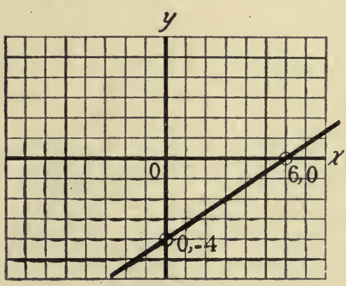

FIG. 32

EXERCISES

Solve by the graphic method:

1. $x+2y=1$
   $x-2y=5$

2. $x+y=1$
   $x-y=1$

3. $x-2y=2$
   $2y-6x=3$

4. $2x+3y=19$
   $3x+2y=16$

5. $12x-5y=3$
   $3x+4y=3$

6. $3x-2y=3$
   $4x+2y=5$

88. Algebraic solution by addition. The results obtained from the last examples indicate that the graphic method, depending as it does upon the actual measurement of lines, is not as accurate as might be desired, for the answers do not always satisfy the conditions exactly. A method is needed that will give better results.

Consider the two equations

$$x+2y=1 \qquad (1)$$
$$x-2y=5 \qquad (2)$$

the solution of which was found to be

$$x=3 \qquad y=-1$$

If the corresponding sides of the equations be added together, the y disappears and we have

$$x+2y=1 \qquad (1)$$
$$x-2y=5 \qquad (2)$$

Adding (1) and (2), $2x=6$ (3)

$$x=3$$

We have cleared equation (3) of y. We say we have **eliminated** y. The word "eliminate" is derived from the Latin verb *elimino*, which means "to turn out of doors."

The corresponding value of y can then be found by the substitution of $x=3$ in one of the given equations.

Substitute $x=3$ in (1), $3+2y=1$

$$2y=-2$$

$$y=-1$$

The work is checked by the substitution of both values in the other equation:

Check by putting $x=3$, $y=-1$ in (2)

$$\begin{array}{r|l} 3-(2\cdot -1) & 5 \\ 3+2 & 5 \\ 5 & 5 \end{array}$$

Exercise 6 of the last article was found difficult by the graphic method. Let us solve it by the algebraic method:

$$3x-2y=3 \qquad (1)$$

$$4x+2y=5 \qquad (2)$$

Eliminate y by adding, $7x=8$ (3)

$$x=\frac{8}{7}$$

Find value of y by substituting in (1),

$$3\cdot\frac{8}{7}-2y=3$$

$$\frac{24}{7}-\frac{21}{7}=2y$$

$$\frac{3}{14}=y \qquad (4)$$

Solution is $x=\frac{8}{7}$, $y=\frac{3}{14}$

Check by substituting in (2):

$$\begin{array}{r|l} 4\cdot\frac{8}{7}+2\cdot\frac{3}{14} & 5 \\ \frac{32}{7}+\frac{3}{7} & 5 \\ 5 & 5 \end{array}$$

Why is this method better for this problem than the graphic method?

EXERCISES

Solve by algebraic method:

1. What change in the method is needed to eliminate the y from the equations?

$$4x+2y=5$$
$$3x+2y=3$$

NOTE. Use either subtraction or addition, according to which operation is needed to eliminate one of the unknowns.

2. Which letter would you eliminate from

$$2x+y=7$$
$$3y-2x=13$$

3. $11n+8m=76$
 $11n+7m=72$

4. $7x-3y=15$
 $2x-3y=5$

5. $3x+2y=11$
 $x+2y=5$

6. $7a-5b=52$
 $2a+5b=47$

89. More complicated sets. The equations that arise in actual practice are seldom so simple as those of the last article, which have one term the same in both equations.

The set
$$5t+s=6 \quad (1)$$
$$2t-3s=16 \quad (2)$$

is not of this common term type. But equation (1) can be made to have a term just like a term in (2), all except the sign, if both sides of (1) are multiplied by 3.

Multiply (1) by 3,
$$15t+3s=18$$
$$2t-3s=16$$

and this set can be solved as before.

Consider a still more general case:

$$2x-5y=1 \quad (1)$$
$$7x+3y=24 \quad (2)$$

There is no particular choice of the letter to eliminate. Let us choose to eliminate y. To do this, we must make the y terms the same.

Multiply both sides of (1) by 3,

$$6x-15y=3$$

Multiply both sides of (2) by 5,

$$35x+15y=120$$

Add,

$$41x=123$$

$$x=3$$

Substitute $x=3$ in (1),

$$6-5y=1$$

$$-5y=-5$$

$$y=1$$

Check.

EXERCISES

Solve by the addition method and draw the graphs for the first six exercises.

1. $3x+7y=27$ $5x+2y=16$	2. $7x+2y=47$ $5x-4y=1$	3. $3v+4w=3$ $12v-5w=3$
4. $17a-18b=15$ $5a+12b=39$	5. $2x+7y=38$ $3x+4y=31$	6. $2n-3p=12$ $3n+5p=-1$
7. $5+p+2y=0$ $7+5p+y=0$	8. $17x-y=31$ $15x+3y=-27$	9. $5x+3y+2=0$ $3x+2y+1=0$
10. $9x-2y=41$ $4x+3y=-9$	11. $6x-5y=13$ $5x+2y=-20$	12. $5t+2s=6$ $4t-3s=37$

90. Standard form. Before attempting to eliminate one of the unknowns the equations should be put into the standard form, $ax+by=c$

For example, $4x-4=5-6y$

should be changed to the form,

$$4x+6y=9$$

EXERCISE I

Arrange the following equations in the standard form:

1. $5+2x-3=7y+8$
2. $2x-2y-3=7-x+4y$
3. $2(x-3)+5=3(y-2)$
4. $7-2(x+3)=5-(2y-7)$
5. $3(y-x)+2x=7-x-2(y+7)$
6. $\frac{x}{3}-8=\frac{2y}{5}-10$

EXERCISE II

Solve:

1. $13x-11y=5+5x$
   $4x+3y+2=1-2y$
2. $10x+8=2y$
   $5y-44=-15x$
3. $3(x-y)=35+3y$
   $3y=15-5x$
4. $\frac{4x}{3}+\frac{y}{6}=\frac{2}{3}$
   $5x-\frac{y}{8}=1$
5. $11n-y=3$
   $9n-\frac{y}{11}=1$
6. $27a-36b=1$
   $\frac{3}{2}a-\frac{6}{7}b=\frac{1}{2}$
7. $p-2r=8r+1$
   $2p-4r=p+r-9$
8. $\frac{x}{3}-4=y$
   $\frac{y}{2}+5=\frac{x}{5}$
9. $x+y=2y-7$
   $3y=5(x+1)$

91. Algebraic solution by substitution. In many examples the work of eliminating one of the unknowns may be shortened if we substitute from one equation into the other.

For instance,

$$a=3b \qquad (1)$$

$$4a-7b=20 \qquad (2)$$

In (1) a is given in terms of b. Substitute this value $3b$ for a in equation (2):

$$4\cdot 3b-7b=20$$
$$12b-7b=20$$
$$5b=20$$
$$b=4$$

Find the numerical value of a by substituting $b=4$ in (1).

$$a=3\cdot 4$$
$$=12$$

Check by substituting both values in (2).

EXERCISES

Solve by substitution, drawing graphs of the first 5:

1. $5x-y=16$
   $x=y$
2. $3x=-2y$
   $x=35+11y$
3. $n-5=-p$
   $p-n-1=3$
4. $x=2+6y$
   $3y-8x=29$

5. $5x-3y-72=5y$
$x-1=15y$

6. $5x-2y=20$
$x=y-2$

7. $5x-8y=7$
$x=1+y$

8. $7x+20=5-8y$
$x+y=-3$

9. $6x=7\frac{1}{2}-3y$
$x=-y$

10. $7x-6=5y+28$
$y+2=3x$

11. $8x-7y=17\frac{1}{3}$
$x=3y-\frac{2}{3}$

12. $5x+3y=105-3x+10y$
$x+y=0$

13. $7x-y=21-2x-5y$
$x+y=-1$

14. $4x-3y=1\frac{1}{2}$
$y-x=1$

15. $4x-5y=3-4x-2y$
$2x=y-1$

16. $3y-6x=\frac{3}{4}$
$3x=y+1$

17. $6x-\frac{1}{3}y=\frac{5}{11}$
$3x=y$

18. $14x+15y=5$
$7x=5y$

19. $\frac{x}{5}+y=8$
$x=5y$

20. $\frac{9}{2}y-\frac{2}{7}x=\frac{5}{14}$
$2x=9y$

21. $8x-\frac{3y}{5}=1$
$5x=2y$

22. $3y-\frac{3x}{10}=1$
$x=2y$

23. $\frac{1+y}{4}-\frac{x+3}{12}=\frac{3}{4}$
$x=1+y$

24. $a+2b=0$
$\frac{4a}{3}-b=11$

92. Choice of methods. Either of these two algebraic methods of solving two linear equations in two unknowns is applicable in any given case, but there is often an advantage in using one method rather than the other. Use your judgment in determining which method you will use for any one set of equations. Select the method that seems most suitable to the case in hand. Substitution is the better method for

$$5x-y=18$$
$$x=2y$$

Why?

But for

$$3x-5y=-11$$
$$7x-4y=5$$

the addition method is much to be preferred. Why? Convince yourself of the truth of the suggestion by solving both examples by both methods.

EXERCISES

In solving the following use the method that seems to you the most appropriate:

1. $4x+3y=6000$
   $x+2y=2000$

2. $3A=6$
   $5A+B=5$

3. $24=u+\frac{21f}{2}$
   $32=u+\frac{29f}{2}$

4. $4x-7y=19$
   $4x+9y=67$

5. $x-y=18$
   $x=4y$

6. $y=3x-19$
   $x=3y-7$

7. $8x+3y=41$
   $7x-5y=13$

8. $x+2y=5$
   $2x+y=1$

9. $a=5b$
   $5=2b-3a$

10. Solve for x and y.
    $x+y=a$
    $x-y=b$

11. $p-2r=8r+1$
    $2p-4r=p-2r+9$

12. $a+b=7$
    $3a=b$

13. $2n-3p=12$
    $3n-5p=17$

14. $\frac{x-2}{2}-y=5$
    $x-y=7$

15. $2x-y=1$
    $7x+2y=5$

16. $3n-2m=21$
    $\frac{n}{4}-\frac{m}{5}=\frac{5}{4}$

17. $7x+\frac{1}{2}y=3$
    $x+y=1$

18. $.1x-.01y=.296$
    $x+y=10$

93. Problems solved by means of two unknowns. The sum of two numbers is 33, the difference is 17. What are the numbers? This problem may be solved in two ways.

(*a*) With one unknown:

The equality is

$$\text{the larger} - \text{the smaller} = 17$$

Let the smaller number $= x$, then the larger $= 33 - x$

Hence the equation $33 - x - x = 17$

$$33 - 2x = 17$$

$$16 = 2x$$

$$8 = x, \text{ the smaller}$$

Hence $33 - 8 = 25$, the larger

(*b*) With two unknowns:

The problem furnishes two equalities:

$$\text{the larger} + \text{the smaller} = 33$$

$$\text{the larger} - \text{the smaller} = 17$$

Using x for the larger number and y for the smaller number, we have

$$x + y = 33 \quad (1)$$

$$x - y = 17 \quad (2)$$

Eliminate y by addition, $2x = 50$

$$x = 25, \text{ the larger}$$

Substitute $x = 25$ in equation (1),

$$25 + y = 33$$

$$y = 8, \text{ the smaller}$$

Which method do you regard as the better in this case? Why?

Many problems can be solved in both ways, and often there is little choice between the two methods. In many cases, however, the method of two unknowns is to be preferred; in some cases it alone can be used successfully. To be solved by two unknowns the problem must contain two separate statements concerning the two unknowns.

PROBLEMS

Number problems:

1. The sum of two numbers is 33 and their difference is 7. What are the numbers? Draw graphs.

2. The sum of two numbers is 32 and their difference is 47. What are the numbers?

3. The sum of two numbers is 35 and their difference is 20. What are the numbers?

4. The sum of two numbers is m and their difference is n. What are the numbers?

5. Apply the formula found in the last problem to the special cases: the sum is 18, difference is 12; sum is 16, difference is 20; sum is 97, difference is 63; sum is 3675, difference is 2691. The formula is a rule for finding two numbers when their sum and their difference are given. State the rule in words.

6. The sum of two integers is 87 and their difference is 32. What are the integers?

7. The sum of two numbers is 27; twice the first added to 3 times the second is 39. What are the numbers?

8. The difference between two numbers is 28; 5 times the first less the second is 197. What are the numbers?

9. The sum of two numbers is twice their difference, and twice the larger one is 7 more than 5 times the smaller one. What are the numbers? Draw graph.

Digit problems:

The **digits** or **figures** of 93 are the same as those of 39, but they are **reversed** in order. If x represents the tens digit and y represents the units digit of a number, what represents the number itself? What will represent the number with the digits reversed?

10. The sum of the digits of a number is 10. If 36 is subtracted from the number, the digits will be reversed. Find the number.

Solution:

Each of the first two sentences of the problem states an equality.

(1) One digit + other digit = 10

(2) The number − 36 = a number with the same digits but reversed

The two digits are the unknowns.

Let the tens digit $=x$ and the units digit $=y$.

Translate into the equations

$$(1)\quad x+y=10$$
$$(2)\quad 10x+y-36=10y+x$$

and solve.

11. The tens digit of a certain number is 2 less than twice the units digit. If 27 is subtracted from the number, the digits will be reversed. Find the number.

12. The tens digit of a certain number is 1 more than the units digit. The number itself is 6 times the sum of the digits. Find the number.

13. The tens digit of a certain number is twice the units digit. The number itself is 6 less than 12 times the tens digit. Find the number.

14. The tens digit of a certain number is 1 less than twice the units digit. If 18 is subtracted from the number, the digits are reversed. Find the number.

15. The units digit of a certain number is 2 less than the tens digit. The number formed by reversing the digits is 7 times the tens digit. Find the number.

16. The tens digit of a certain number is 2 more than the units digit. The number formed by reversing the digits is 3 less than 12 times the units digit. Find the number.

17. The tens digit of a certain number is 4 less than twice the units digit. The number itself is 2 more than 6 times the sum of the digits. Find the number.

18. The sum of the digits of a number is s. When n is added to the number, the digits are reversed. Find the following formulas for computing the digits of the number, x being the tens digit and y being the units digit.

$$x=\frac{9s-n}{18} \qquad y=\frac{9s+n}{18}$$

Mixture problems:

19. How many pounds each of nuts at 20 cents a pound and nuts

at 45 cents a pound should be mixed to make up 10 pounds worth 35 cents a pound?

Suggestion. The two equalities are not clearly evident, but they can be found by a little careful thinking; one will be between numbers of pounds and the other between values. Evidently,

no. of lbs. of first kind + no. of lbs. of second kind = 10
cost of first kind + cost of second kind = cost of mixture

Choose as the unknowns for the equations the number of pounds of each kind used. Translate equalities and solve.

20. How many pounds each of 40-cent coffee and 60-cent coffee must be mixed to make 12 pounds of 45-cent coffee?

21. How much candy worth 45 cents a pound and candy worth 85 cents a pound must be mixed to make 20 pounds worth 60 cents a pound?

22. If a and b are the costs per pound of two articles that are to be mixed to make a mixture worth c a pound, find formulas for calculating the quantity of each to make a mixture of n pounds.

CHAPTER VII

Special Products; Factoring; Equations Solved by Factoring

94. A problem. In the foregoing chapters we have seen that certain problems can be solved by means of one equation in one unknown, while others may be solved by two linear equations in two unknowns. There are many problems that lead to other kinds of equations more or less complicated. As an illustration consider the following problem:

A man wishes to double the area of an 8 by 12 rod field by adding to one end and one side strips of equal width. How wide must these strips be?

If we let s represent the width of strip to be added, the plan of addition is given in Fig. 33. The area of the added part equals the area of the original field.

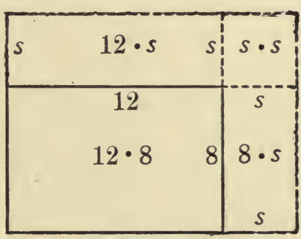

Fig. 33

That is $$12s+s\cdot s+8s=12\cdot 8$$
or $$ss+20s=96$$

This equation is very different from any we have considered thus far, for the unknown appears twice in one term. Any attempt you may make to solve it will soon lead you to see that none of the methods of solving equations you have yet learned will apply here. We need an entirely new method for the solution of this equation. Before we can attack the problem of finding a new method with any hope of success, it will be necessary for us to develop a little more algebraic machinerv involving both new ideas and new ways of working.

95. Powers. Exponents. In the product $2 \times 3 = 6$, 2 and 3 are factors of 6. Each of two or more numbers whose product is a given number is called a **factor** of that number.

A number that is the product of two or more equal factors is called a **power** of one of the equal factors.

$25 = 5 \cdot 5$ is the second power of 5.
$8 = 2 \cdot 2 \cdot 2$ is the third power of 2.
$243 = 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3$ is the fifth power of 3.
bb is the second power of b.
$aaaa$ is the fourth power of a.

In the latter part of the sixteenth century Simon Stevin introduced a better way of writing a power. In place of a string of letters or numbers, one factor is used with a number written to the right and above the factor to indicate how many equal factors there are.

$$5 \cdot 5 = 5^2 \qquad 2 \cdot 2 \cdot 2 = 2^3$$
$$bb = b^2 \qquad aaaa = a^4$$

The number indicating the number of factors is called an **exponent.** b^2 is read "the second power of b" or "b second power" or more commonly "the square of "b" or "b square." a^3 is read "the third power of a" or "a third power" or more commonly "a cube." a^4 is read "the fourth power of a" or "a fourth power."

Those who are of an inquiring turn of mind might ask why the special names "a square," "a cube" are used for a^2 and a^3. Can you answer the question?

EXERCISES

1. Find values of 2^3, 5^4, 7^2, 2^5, 25^2.
2. Write 3^2 in two other ways.
3. Write 2^5 in two other ways.
4. Write $4 \times 4 \times 4$ in two other ways.
5. Write a^2 in two other ways.
6. Write $nnnnnnn$ in another way.

7. If a is 7, what does a^2 equal?

8. If n is 3, what does n^3 equal?

9. If a is $\frac{2}{3}$, what does a^2 equal?

10. If $x=\frac{1}{2}$, what does x^3 equal?

11. If $x=-2$, what does x^2 equal?

12. If $x=-2$, what does x^3 equal?

13. If $x=2.5$, what does x^2 equal?

14. If $a=.05$, what does a^3 equal?

15. If $a=3$ and $b=2$, find value of a^2+b^2, a^3-b^3, $a-b^2+a^2$, a^2b^3, a^2-ab.

16. Find value of 3×5^2, $3\cdot2^3\cdot5^2$.

17. Find value of $5a^3$ when $a=2$.

NOTE. It must be carefully noted that in such an expression as $5a^3$ the exponent affects only the letter over which it is placed.

$5a^3$ means $5\cdot a\cdot a\cdot a$

If $a=2$, $5a^3$ means $5\cdot2\cdot2\cdot2$

18. Find value of $2a^3+3a$ when $a=5$.

19. Find value of $12x^2y$ when $x=-2$, $y=\frac{1}{3}$.

20. Find value of $2ax^2-3x+5$ when $x=5$, $a=1$.

21. Draw the graphs of x^3, x^2, $x+2$, x.

96. Names of algebraic expressions. An algebraic expression having but one term is called a **monomial.** An algebraic expression having two terms is called a **binomial.** One with three terms is called a **trinomial.**

$3x^2$ is a monomial.
$2x+3$ is a binomial.
$2x^2-3x+2$ is a trinomial.

97. Products of monomials. The product of $2a$ and $3x$ is $2a\cdot3x$ or, better, $6ax$. In like manner the product of $2ax^2$ and $3ax$ is

$$2ax^2\cdot3ax=2\cdot a\cdot x\cdot x\cdot3\cdot a\cdot x$$

or, rearranging, $=2\cdot3\cdot a\cdot a\cdot x\cdot x\cdot x$

or, using better notation, $=6a^2x^3$

It is unnecessary to write out all the factors as is done above. All that is necessary is to count up the number of times each letter occurs as a factor. This can be done in the simplest way by the addition of the exponents of that letter, thus:

$$2ax^3 \cdot 5a^2x^2 = 2 \cdot 5a^{1+2}x^{3+2}$$
$$= 10a^3x^5$$

This way of finding the product of two monomials may be stated as a rule:

RULE. **The product of two monomials is a monomial in which the numerical coefficient is the product of the numerical coefficients of the factors and in which the exponent of each letter is the sum of the exponents of that letter in the factors.**

If a letter has no exponent placed over it, it is to be understood that the exponent is one. This is evident from the purpose for which an exponent is used.

EXERCISES

Find the products:

1. $2x \cdot 5x$	2. $3a \cdot 7ax.$	3. $2x \cdot 3x^2$
4. $-3n \cdot 8n^3$	5. $x^2 \cdot ax$	6. $-x \cdot -2x^3$
7. $7an \cdot 2a^2n$	8. $-3ay \cdot 7a^2y$	9. $5at \cdot 3a^2t$
10. $2ax \cdot 3a$	11. $8x^2 \cdot -3x$	12. $6ax \cdot 5a^2$
13. $4a^2x \cdot 2x$	14. $3x^2 \cdot -2x^2$	15. $-5an^2 \cdot 2a^2n$
16. $-4a^2 \cdot -5n^2$	17. $3a^2 \cdot -6an^2$	18. $5an \cdot -2a^3$
19. $3n^2 \cdot 3n^2$	20. $2ax^3 \cdot 2ax^3$	21. $(3xy^2)^2$

Evaluate the following when $a=3$, $n=-3$, $x=-2$:

22. $3a^2n$, $3an^2$, $3anx$, $2a^3n$, $2a^2n^3$, $6n^2x$, $6nx^2$, $5an^3$, $5a^3n$

23. $2x^2 \cdot 3a$, $3a^2 \cdot 2x$, $4x \cdot 2n^2$, $2nx \cdot x^2$, $5ax \cdot n$, $5a^2n \cdot n^2$

98. Quotient of monomials. If one factor of a product is given, the other may be found by the division of the product by the given factor.

For example, suppose we have,

$$7x^2 \cdot (?) = 35x^3$$

or

$$35x^3 \div 7x^2 = ?$$

As division is the reverse of multiplication, we have simply to reverse the rule for multiplication, dividing coefficients instead of multiplying and subtracting exponents instead of adding:

$$35x^3 \div 7x^2 = 5x$$

so also, $$12x^5 \div 3x^2 = \frac{12}{3}x^{5-2} = 4x^3$$

This may be stated as a rule:

RULE. **To divide one monomial by another, divide the numerical coefficient of the dividend by the numerical coefficient of the divisor and subtract the exponent of each letter in the divisor from the exponent of that letter in the dividend.**

As we have thus far used only positive numbers for exponents, this rule applies, at present, only to cases when the exponent of the letter in the divisor is less than that of the same letter in the dividend. If the exponents are the same and the resulting exponents become zero, the meaning is simply that that letter does not appear in the result.

EXERCISES

1. If a product is 35 and one factor is 7, what is the other factor?
2. If $3x$ is one factor of $6x^2$, what is the other factor?

Divide:

3. $10x^3$ by $5x^2$
4. $24x^3$ by $4x$
5. $35x^4$ by $7x^3$
6. $3ax^3$ by ax
7. $27ab^2x$ by $9ab$
8. $45nx^4$ by $9nx$
9. $-28ny^3$ by $7y^2$
10. $-9nt^5$ by $-3t$
11. $51ay^5$ by $3y^3$
12. $63h^2t^3$ by $-9ht$
13. $10x^2y$ by $-2x$
14. $10x^2y$ by $-2xy$
15. $10x^2y$ by $5x^2$
16. $25a^2b^2$ by $-5ab$
17. $25a^2b^2$ by $-5a^2b$
18. $16ax^4$ by $8ax^2$
19. $18a^2x^4$ by $9ax^2$
20. $32a^2b^3$ by $4ab$
21. $-48x^3$ by $-8x^2$
22. $-51nh^3$ by $-3h$
23. $-72anx$ by $36an$
24. Evaluate the following if $x = -2$, $a = 3$, $b = -3$:

$$\frac{15ax^2}{ax}, \quad \frac{18a^2x}{ax}, \quad \frac{16a^2b^2}{4ab}, \quad \frac{14a^2bx}{7ab}, \quad \frac{12ab^2}{3b}, \quad \frac{20ab^3}{4b^2}$$

99. Factoring numbers. When one of the factors of a product is given, the other factor is found by division. The process of finding the factors of a product when only the product is given is called **factoring.**

You are more or less familiar with factoring in arithmetic. You know that the factors of 35 are 5 and 7 because you remember from the multiplication table that $5 \times 7 = 35$.

EXERCISES

1. Find two factors of 56, 72, 81, 121, 32, 54, 84, 96.

It is a little more difficult to factor numbers that do not appear in the multiplication tables we have learned. In such cases one has to experiment with different numbers as possible factors.

2. What are the factors of 39, 91, 51, 65, 221, 323?

There are simple tests for determining whether a number is divisible by some of the smaller integers which often help in factoring numbers. An even number is divisible by 2. Why? A number ending with 0 or 5 is divisible by 5. Why? If the sum of the digits of a number is divisible by 3, the number is divisible by 3. If the sum of the digits of a number is divisible by 9, the number is divisible by 9.

3. Factor 62, 115, 111, 171, 207, 57, 306, 855, 269, 441, 385, 495, 234, 104, 408, 195.

100. Prime factors. A **prime number** is an integer that is exactly divisible by no integer except itself and one.

A number is said to be factored into its **prime factors** when all the factors whose product is the given number are prime. Thus:

$$28 = 7 \cdot 2 \cdot 2$$

Exponents may be used for repeated factors:

$$28 = 7 \cdot 2^2$$

Factor into prime factors 18, 36, 98, 75, 250, 240, 512, 288. 350, 360, 448, 108, 196, 225, 175, 484, 728, 357, 372, 450.

101. Factors of monomials. $6ab^2$ is a **monomial.** Such an expression is already in a factored form; it is the product of several factors. These factors can be separated in various ways:

Into two factors such as $6a \cdot b^2$ or $a \cdot 6b^2$
Into three factors such as $6 \cdot a \cdot b^2$
Into prime factors such as $2 \cdot 3 \cdot a \cdot b \cdot b$

EXERCISES

1. Separate into prime factors $10ax$, $15bx^2$, $32ax^2$, $12a^2x^3$, $25abx^3$.
2. Separate into two factors 42, $7xy$, $15a^2$, 90, $18ax^2$, $36a^2x$.
3. Separate $24ax^2$ into two factors one of which is $3a$.
4. Separate $24ax^2$ into two factors, one being $6ax$.
5. Separate $35x^3$ into two factors one of which is $7x^2$.
6. Separate $51a^2n^2$ into two factors one of which is $3an$.
7. Find a number that is a factor of both $6x$ and $3a$.
8. Find a number that is a factor of both ax and bx.
9. Find a factor that is common to both $4x$ and $6x^2$. How many can you find?

102. Product of a binomial and a monomial. We have seen that

$$3(x-5)=3x-15$$
$$a(x+b)=ax+ab$$
$$x(x+2)=x^2+2x$$

In each case there are two forms for the same number. To find the second form, multiply each term of the binomial by the monomial.

The same method may be extended to the multiplication of an expression of any number of terms by a monomial:

$$3(x^2+x-2)=3x^2+3x-6$$

EXERCISES

Multiply:

1. $3(x-2)$ 2. $x(x-3)$ 3. $2x(x-5)$
4. $-3x(2x-3)$ 5. $n(n-a)$ 6. $n(n^2+2)$

7. $-a(3-b)$	8. $2a(a+b)$	9. $a(a^3-5)$
10. $a^2(a-b)$	11. $x(x^2-x+2)$	12. $x^2(2x+5)$
13. $2x^2(5x-2)$	14. $3n(2+n^3)$	15. $-5a(2a-3b)$
16. $m(2m-n)$	17. $a(b-c)$	18. $a^2(a+b)$
19. $-x^2(2x^3-x+5)$	20. $3a^2x(x^2-5x+3)$	21. $-3x(x^2-1)$

103. Equivalent forms. The expressions

$$a(x+b), \qquad ax+ab$$

are two forms for the same number:

$a(x+b)$, a product form showing two factors
$ax+ab$, a sum form showing two terms.

We have already found it useful to change a product form such as $3(x+4)$ into the form $3x+12$ in order to solve the equation in which the expression appeared. We shall find the reverse process of undoing the multiplication just as useful. This reverse process is called **factoring.** It consists in changing a sum of several terms into the product of several factors; that is, a sum form into a product form.

It should be noticed that the factoring of Art. **99** is also changing a sum form into a product form.

Factor 35:

35 means $30+5$, a sum
$5 \cdot 7$ is a product form

If one of the factors is known, the other factor is easily found by division as is shown in the next article.

104. Division by a monomial. If one factor of $7x-21$ is 7, the other is found by division of each term of $7x-21$ by 7. It is $x-3$.

Hence $$7x-21=7(x-3)$$

EXERCISES

Divide:

1. $27a-9$ by 9	2. $8ax-a$ by a
3. $7a-14b$ by -7	4. a^2+a by a

5. $8a^2-4a$ by $4a$

6. $5a+3a^2$ by $-a$

7. $15a^2-10ax$ by $5a$

8. $7a+14ay$ by $7a$

9. $15x^2-25x+10$ by -5

10. $3x^2a+4x^2b+x^2$ by x^2

11. $14x^3a-21x^2a^2+28xa^3$ by $7ax$

12. $ax^2-bx^3-cx^4$ by $-x^2$

13. If one factor of $12a^2x-4bx$ is $4x$, what is the other factor?

14. One factor of $9ax+12a^2x$ is $3ax$. Find the other factor.

15. One factor of $9a^4-27a^3-18a$ is $9a$. Find the other.

105. Factoring. The question of changing a sum form into a factored form when no one of the factors is given is a much more difficult matter. All expressions cannot be so changed. Only certain kinds can be changed in this way, or, as we commonly say, can be factored. We shall consider in this book only a few of the simpler kinds or types. We discover such types when we compare the result of a multiplication with the factors from which it came.

106. Common factor type. $a(b+3)=ab+3a$

The factor a is in every term of $ab+3a$. It may then be chosen as one factor and the other factor found by the division of $ab+3a$ by a. So in x^2-4x, x is a number common to the two terms.

Hence $$x^2-4x=x(x-4)$$

So $$3ax+12a=3a(x+4)$$

Any expression in which the terms do not have a common letter or number cannot be factored in this way.

EXERCISE I

Factor and then check the results by multiplying:

1. $5n+35$	2. $5a+15b$	3. a^2+3a
4. $5a^2-18a$	5. $3t+t^2$	6. $5\pi-25$
7. $7x-14$	8. $2x-3$	9. $\pi a^2-\pi b^2$
10. $3n^2+21$	11. $7n^2-14n$	12. $2a-4b$
13. $a^2\pi-a$	14. $12x-8x^2$	15. $3n^2-2n$

16. $25+65$
17. $72+27$
18. $25\pi+20$
19. $7n-6$
20. $4n^3-12n^2$
21. $2a+4ac$
22. $ax+bx$
23. $7x^2+51x$
24. $3n^2-9n-6$
25. $3n^3-15n^2+6n$
26. $2n^2-4n-3$
27. $3m^2-6mn$
28. ax^2+bxy
29. $5m^2-25m$
30. $\pi ab-\pi a^2$
31. $25\cdot 37+25\cdot 19$
32. $36\cdot 7+18\cdot 5$
33. $12\cdot 21-28\cdot 5$
34. ax^2+ax
35. $abc+ab$
36. $ax+bx-cx$
37. ax^3-bx^2
38. $4\pi a^2-\pi b^2$
39. $7+35A+161A^2$
40. $10s^2n+5mn^2$
41. $8x^3-4x^2-x$
42. $10x^3y-5xy$

EXERCISE II

1. Draw the graph of $2x-3$.
2. Draw the graph of $x-4$.
3. Draw the graph of x^2-4x. (See Fig. 34.)

If $x=$	6	5	4	3	2	1	0	-1	-2
Then $x^2-4x=$	12	5	0	-3	-4	-3	0	5	12

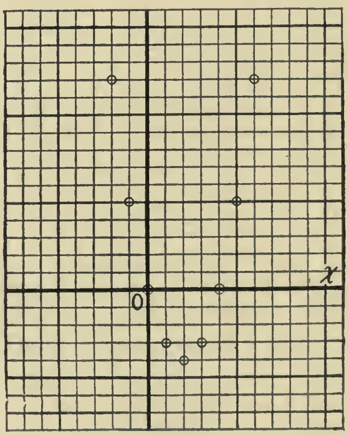

FIG. 34

The points on the figure are rather far apart. Let us find other points in between them. (See Fig. 35.)

Take $x=$	$5\frac{1}{2}$	$4\frac{1}{2}$	$3\frac{1}{2}$	$2\frac{1}{2}$	$1\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-1\frac{1}{2}$
Then $x^2-4x=$	$8\frac{1}{4}$	$2\frac{1}{4}$	$-1\frac{3}{4}$	$-3\frac{3}{4}$	$-3\frac{3}{4}$	?	?	?

You may insert more points if you desire, but these are sufficient to show the form of the smooth curve that can be drawn through these points. (Fig. 36.)

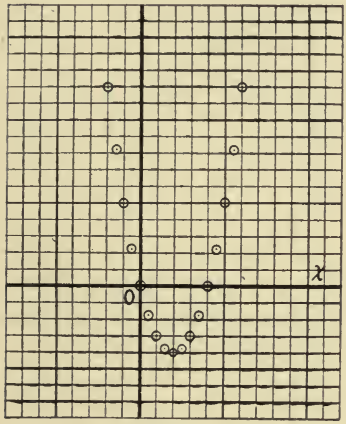

FIG. 35

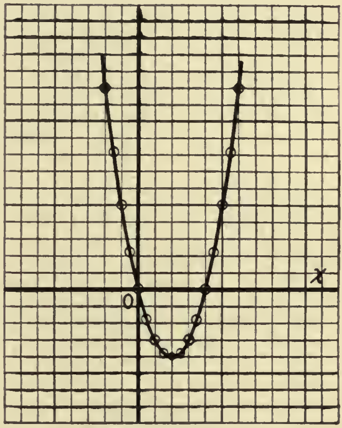

FIG. 36

You will notice that the graphs of Exercises 2 and 3 are quite different in form.

Expressions of the kind $x-4$ are said to be of the **first degree.** Why?

Expressions of the kind x^2-4x are said to be of the **second degree.** Why? How do you pick out an expression of the second degree?

What is the degree of x^3-3x? Why?

Expressions of the first degree are often called linear expressions. An expression of the second degree is commonly called a quadratic expression.

After you have drawn the graphs of the following expressions what conclusion do you think you could safely make?

4. x^2-6x	5. x^2+4x	6. $2x^2-5x$
7. $\frac{1}{2}x^2+6x$	8. $x(x-3)$	9. $x(x+5)$

107. Checking work by substitution in identities. The statement

$$n(n-5)=n^2-5n$$

is an **identity.** The right side is but another form for the number represented by the left side.

If we let $n=7$

we have

$$\begin{array}{r|l} 7(7-5) & 7^2-5\cdot 7 \\ 7\cdot 2 & 49-35 \\ 14 & 14 \end{array}$$

If $n=10$, what is the result?

If $n=2$, what is the result?

What conclusion do you think it is safe to make? It should be noted that the numerical calculation of the two sides must be worked out by different methods. Why?

$$3x+6ax^2=3x(1+2ax)$$

To check, put $x=3,\ a=2$

$$\begin{array}{r|l} 3\cdot 3+6\cdot 2\cdot 3^2 & 3\cdot 3(1+2\cdot 2\cdot 3) \\ 9+108 & 9\cdot 13 \\ 117 & 117 \end{array}$$

Try checking with some other values for x and a.

Thus we may check the truth of any identity by substituting some number for each letter and working out the two sides independently. If the two sides come out to be the same number, it is safe to say that the identity is true, provided, of course, that you have made no mistakes in the numerical calculations.

EXERCISES

In the following exercises multiply or factor as the case may be and check your results:

1. $3(x-2)$
2. $2a(3x+a)$
3. $2x^2(x-2)$
4. $6x+3y$
5. $6x-3x^2$
6. $2a(3a-2b)$
7. $x(2x^2-x+3)$
8. $3x^3-6x^2$
9. $3x^2-9x$

If in the following exercises any are untrue, write in correct form and check:

10. $x(x^2-3x)=x^3-3x$
11. $2a(a-b)=2a^2-2ab$
12. $2a(x-a)=2ax-2a$
13. $3x^2+6a=3x(x+2a)$
14. $15x^2-12x=5x(3x-4)$
15. $7x+21a=7(x+3a)$
16. $8x-4x^2=4x(2+x)$
17. $6x-2=2(2x-1)$

108. A factor equal to zero. In drawing the graph of such products as $x(x-3)$ it will be noticed that when one of the factors is zero the product will be zero.

This is an instance of a general principle of very great importance, namely, a product is zero when and only when one of its factors is zero.

If a factor is zero, the product must be zero; if a product is zero, one or more of its factors must be zero.

EXERCISES

1. If x is 7, what is the value of $x(x-7)$; $3(x-7)$?

2. If x is 0, what is the value of $x(x-3)$; $7(x-3)$?

3. For what values of x are the following products zero?

$3(x-2)$, $4(x+3)$, $x(x-1)$, $x(x-7)$, $x(x+2)$, $(x-8)x$

109. A problem solved. We are now prepared to solve a problem leading to an equation of the kind suggested at the beginning of this chapter. Consider a problem like this: The difference between two numbers is 7; twice the square of the smaller equals their product. What are the numbers?

The equality is

twice (smaller)^2 = smaller × larger

Let x = smaller number

then $x+7$ = larger

and the equality becomes

$$2x^2=x(x+7)$$

Expanding, $2x^2=x^2+7x$

Collecting terms on one side,

$$2x^2-x^2-7x=0$$

or $x^2-7x=0$

This equation, $x^2-7x=0$, asks the question: For what values of x is x^2-7x equal to zero? The answer is to be found by the method of putting x^2-7x in the factored or product form and applying the principle given in the last article:

Thus, $\qquad x^2-7x=0$

Factoring, $\qquad x(x-7)=0$

The product is zero if either factor is zero; that is,

$$\text{if } x=0$$
$$\text{or if } x-7=0$$
$$\text{that is, } x=7$$

Both $x=0$ and $x=7$ satisfy the equation, as shown below:

If $x=0$

$$\begin{array}{r|l} 2\cdot 0^2 & 0(0+7) \\ 0 & 0 \end{array}$$

If $x=7$

$$\begin{array}{r|l} 2\cdot 7^2 & 7(7+7) \\ 2\cdot 49 & 7\cdot 14 \\ 98 & 98 \end{array}$$

There are then two different answers to the problem itself. There are two sets of numbers that fulfil the requirements of the problem, namely, 0 and 7, or 7 and 14.

110. Root of an equation. Any number that will reduce an equation to an identity when that number is put in place of the unknown is called a **root** of the equation. An equation of the first degree in one unknown has but **one** root. An equation of the second degree in one unknown has **two** roots.

In the solution of an equation of the second degree both roots should be found, as one is just as important as the other.

EXERCISES

Solve and check:

1. $x(x-2)=0$
2. $x(x+3)=0$
3. $2x(x-5)=0$
4. $2x(3x-7)=0$
5. $n^2+9n=0$
6. $3x^2+9x=0$
7. $9s^2=4s$
8. $10n^2=7n$
9. $0=3r-6r^2$
10. $7x-3=21x^2-3$

11. $3-2x^2=5+5x-2$

12. $8x^2-9=2x-3(6x+3)$

13. $12-11x^2=8-2(11x-2)$

14. $\frac{n^2}{4}+1=2-\frac{n+2}{2}$

15. $3x^2+x-7=8x+x^2-7$

16. The product of a certain number and 3 less than its double is 7 times the number. What is the number?

17. The product of a certain number and itself increased by 5 is twice the square of the number. What is the number?

18. Divide 10 into 2 parts so that 4 times the product of the 2 parts shall equal the square of the first part.

19. Find the side of a square such that the area of the square shall be numerically equal to twice its perimeter.

20. The area of a certain square is numerically equal to $\frac{2}{3}$ of its perimeter. Find the side of the square.

21. The width of a certain rectangle is $\frac{2}{3}$ of its length. The area is numerically equal to 6 times its perimeter. Find its dimensions.

22. If 4 is subtracted from 9 times a number and this difference is divided by 4, the result is one less than 9 times the square of the number. Find the number.

23. The volume of a certain cube is numerically equal to $\frac{2}{3}$ of its surface. Find an edge of the cube.

24. The volume of a certain cube is numerically equal to a times its surface. Find the edge of the cube.

25. What is the answer to Exercise 24 if a is 8, 5, 3?

26. The units digit of a certain number is twice the tens digit. The number itself is 3 times the square of the tens digit. Find the number.

27. $x^2-cx=0$

28. $x^2=ax$

29. $bx^2=x$

30. $ax^2=bx$

31. $x^2-ax=bx$

32. $x^2+ax=-bx$

33. $x(x-6)=8-4(x+2)$

34. $7y-4(y-3)=3(4-y^2)$

35. $8n-n(n-3)=n(n-9)$

36. $6(n-2)=4(n^2-3)$

111. Product of two binomials. The floor space of the four rooms represented in the diagram (Fig. 37) may be computed in several ways. We may find the floor space of each room in the following way:

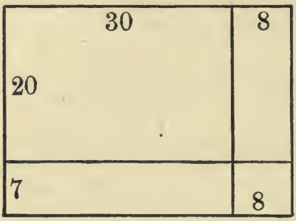

FIG. 37

$$20 \cdot 30 + 20 \cdot 8 + 7 \cdot 30 + 7 \cdot 8$$
$$= 600 + 160 + 210 + 56$$
$$= 1026$$

or we may compute the whole floor space at once:

$$(20+7)(30+8) = 27 \cdot 38 = 1026$$

Notice how the multiplication is done:

$$\begin{array}{r} 38 \\ \underline{27} \\ 266 \\ \underline{76} \\ 1026 \end{array}$$

or in a longer form

$$\begin{array}{r} 30+8 \\ \underline{20+7} \\ 210+56 \\ \underline{600+160} \\ 600+370+56 = 1026 \end{array}$$

Evidently $(20+7)(30+8) = 20 \cdot 30 + 20 \cdot 8 + 7 \cdot 30 + 7 \cdot 8$.

So also for the floor space in Fig. 38.

The area room by room is

$$x^2 + 5x + 4x + 5 \cdot 4 = x^2 + 9x + 20$$

The area of the whole is

$$(x+5)(x+4)$$

The multiplication is to be carried on as before:

$$\begin{array}{r} x+4 \\ \underline{x+5} \\ 5x+20 \\ \underline{x^2+4x} \\ x^2+9x+20 \end{array}$$

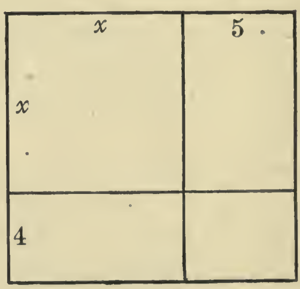

FIG. 38

Since we write from left to right, it is more convenient in algebra to begin multiplying at the left and put down the work from left to right:

$$\begin{array}{l} x+4 \\ \underline{x+5} \\ x^2+4x \\ \quad \underline{+5x+20} \\ x^2+9x+20 \end{array} \qquad \begin{array}{l} x-3 \\ \underline{x+6} \\ x^2-3x \\ \quad \underline{+6x-18} \\ x^2+3x-18 \end{array}$$

The product of the two binomial factors is made up of the sum of the products of every term of one factor with every term of the other factor.

It is not necessary to write the multiplication down in the arithmetical form given above. The work can be written just as well thus:

$$\begin{aligned}(x+5)(x+4) &= x(x+4)+5(x+4) \\ &= x^2+4x+5x+20 \\ &= x^2+9x+20\end{aligned}$$

$$\begin{aligned}(x-3)(x+6) &= x(x+6)-3(x+6) \\ &= x^2+6x-3x-18 \\ &= x^2+3x-18\end{aligned}$$

EXERCISES

Find the following products:

1. $(x+7)(x+9)$ 2. $(x+6)(x+2)$ 3. $(x-7)(x-6)$
4. $(x-8)(x-3)$ 5. $(x+9)(x-5)$ 6. $(x+11)(x-7)$
7. $(n+1)(n-6)$ 8. $(n-11)(n-12)$ 9. $(t+15)(t+7)$
10. $(t+15)(t-7)$ 11. $(t-19)(t+3)$ 12. $(a+8)(a-5)$
13. $(a-8)(a+3)$ 14. $(x+a)(x+b)$ 15. $(n+t)(n+s)$

Can you discover a short way of writing the product down at once without writing the intermediate steps?

$$(x+5)(x+6)=x^2+11x+30$$

Apply where possible to the following:

16. $(x+7)(x+2)$ 17. $(x-5)(x-3)$ 18. $(n-5)(n+7)$
19. $(n-8)(n+4)$ 20. $(a-15)(a-3)$ 21. $(a-15)(a+4)$

22. $(a-1)(a+7)$
23. $(a+9)(a+3)$
24. $(x-8)(x-9)$
25. $(x+3)(x-9)$
26. $(a+10)(a+8)$
27. $(a-10)(a+8)$
28. $(y-6)(y+8)$
29. $(x-\frac{3}{4})(x+\frac{1}{4})$
30. $(x+7)(x-7)$
31. $(x-3)(x+4)$
32. $(b-8)(b+10)$
33. $(x-\frac{1}{4})(x+\frac{1}{2})$
34. $(r-8)(r-12)$
35. $(r-8)(r+12)$
36. $(r+8)(r-12)$
37. $(x-\frac{1}{6})(x+\frac{5}{6})$
38. $(x-\frac{1}{4})(x-\frac{1}{2})$
39. $(s-6)(s-11)$
40. $(x-3)(x+13)$
41. $(x+\frac{1}{4})(x-\frac{1}{2})$
42. $(n-1)(n-3)$
43. $(a-6)(a+9)$
44. $(a+6)(a-9)$
45. $(x-11)(x+12)$
46. $(x+3)(x+13)$
47. $(40+3)(40-2)$
48. $(30+3)(30+5)$
49. $81 \cdot 83$
50. $69 \cdot 73$
51. $33 \cdot 27$
52. $(x^2-x-3)(x-1)$
53. $(x^2-2x+1)(x+3)$
54. $(x^2-3x+2)(x-3)$
55. $(x^2-x+4)(x+2)$
56. $(x^2+x-2)(x-2)$
57. We have seen that

$$(x+a)(x+b)=x^2+ax+bx+ab$$
$$=x^2+(a+b)x+ab$$

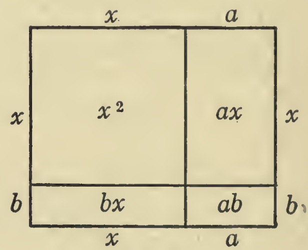

Fig. 39

58. Show that, in Fig. 40, $C=S-A-B-D$.

Translate into algebraic symbols, using data on figure.

59. Draw a figure similar to Fig. 39 to illustrate the product $(x-a)(x-b)$.

60. Draw a similar figure to represent $(x-a)(x+b)$.

61. Draw a similar figure to represent $(x-a)(x+a)$.

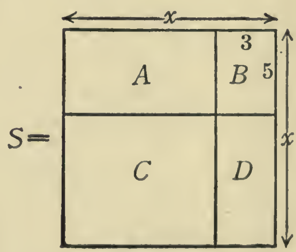

Fig. 40

112. Factoring a trinomial. We have seen that the product of two expressions of the first degree is an expression of the second degree. We may reverse the point of view and ask the question: What are the two factors whose product is a given quadratic expression?

What are the factors of $x^2+12x+35$? We have noticed that in the multiplication

$$(x+3)(x+5)=x^2+(3+5)x+3\cdot 5$$
$$=x^2+8x+15$$

(1) The coefficient of the x^2 term is 1.

(2) The coefficient of the x term is the sum of 3 and 5.

(3) The term that contains no x, called the **absolute** term, is the product of 3 and 5.

These facts may be used to undo the multiplication that gave $x^2+8x+15$: 15 is the product of the absolute terms of the two factors, and 8 is the sum of these same absolute terms. Now apply this to $x^2+12x+35$.

Thirty-five is to be regarded as the product of two unknown numbers whose sum is 12.

Separate 35 into two factors whose sum is 12:

$$35=5\cdot 7 \text{ and } 5+7=12$$

Hence 5 and 7 will do, and $x+5$ and $x+7$ are the desired factors; that is,

$$x^2+12x+35=(x+5)(x+7)$$

In most cases the absolute term of the trinomial has several pairs of factors. The various pairs must be tried until a pair is found that works, and then the factors will fall apart as if by magic.

$$x^2+9x+20$$

The pairs of factors of 20 are $2\cdot 10$, $1\cdot 20$, and $4\cdot 5$. The $2\cdot 10$ pair does not work. The $4\cdot 5$ pair does work.

Hence $\quad x^2+9x+20=(x+5)(x+4)$

Consider x^2-5x+6

The pairs of factors of 6 are $2\cdot 3$, $1\cdot 6$; neither pair works, but the pair $-2\cdot -3$ works.

Hence $x^2-5x+6=(x-2)(x-3)$

Consider x^2-x-6

A pair of factors of -6 whose sum is -1 is to be found.

-3 and $+2$ is such a pair

Hence $x^2-x-6=(x-3)(x+2)$

In selecting the proper pair of factors of the absolute term it will be of service to note that:

(1) If the absolute term is negative, one of the pair must be negative and one positive. How can you determine which is to be negative?

(2) If the absolute term is positive, the numbers of the pair are of the same sign. How can you determine which sign? If no satisfactory pair can be found, you will have to say that you cannot factor the expression. This does not mean that it cannot be factored, but only that, as far as you know with your present knowledge, it cannot be factored.

Practice will improve your skill in selecting the proper pairs of numbers.

EXERCISES

Factor if possible:

1. $x^2+7x+12$	2. $n^2+15n+50$	3. $n^2+5n+12$
4. x^2-5x+6	5. x^2-7x+5	6. $a^2+8x+15$
7. t^2+3x+1	8. a^2+a+5	9. x^2-6x+5
10. a^2+2a-8	11. x^2+5x-6	12. n^2-7n-6
13. $x^2-3x-28$	14. t^2+6t-7	15. n^2+7n+5
16. n^2+7n+6	17. t^2-6t+7	18. $x^2-7x-18$
19. $p^2+p-132$	20. $t^2+23t+102$	21. s^2-4s+3
22. $r^2-11r-60$	23. $x^2-5x-84$	24. $x^2+10x+24$
25. $s^2+7s-60$	26. s^2-5s+4	27. $t^2+13t-300$

28. $x^2+11x+24$
29. $n^2-8n+12$
30. $y^2-19y+48$
31. $z^2-9z-36$
32. $z^2-12z+36$
33. $x^2-x+\frac{1}{4}$
34. $x^2+\frac{2}{8}x-\frac{15}{64}$
35. $x^2+14x+24$
36. $z^2-5z-36$
37. $x^2-2x-35$
38. $x^2-11x+30$
39. $x^2-17x+30$
40. x^2-x-30
41. $a^2+13a+12$
42. $a^2+9a+18$
43. $n^2-10n-56$
44. $a^2+55a-56$
45. $b^2+11b+30$
46. a^2-a-12
47. $x^2-15x+56$
48. x^2-x-56
49. $x^2+10x-56$
50. x^2+4x+3
51. $x^2+26x-56$
52. x^2-x-2
53. $x^2-\frac{2}{3}x+\frac{1}{9}$
54. $x^2+\frac{1}{3}x-\frac{2}{9}$

Multiply:

55. $(5-x)(6+x)$
56. $(7-x)(8-x)$
57. $(7-x)(5+x)$
58. $(9+x)(3-x)$
59. $(\frac{1}{2}-x)(\frac{1}{2}+x)$
60. $(3-x)(5-x)$
61. $(3-x)(5+x)$
62. $(\frac{1}{3}-x)(\frac{2}{3}+x)$
63. $(8-x)(6+x)$
64. $(8+x)(5-x)$
65. $(2-x)(13+x)$
66. $(3+x)(11+x)$
67. $(\frac{1}{6}-x)(\frac{1}{3}+x)$
68. $(\frac{1}{2}-x)(\frac{1}{2}-x)$
69. $(9-x)(11+x)$
70. $(9+x)(10-x)$
71. $(\frac{1}{3}-x)(\frac{5}{6}+x)$
72. $(12-x)(4+x)$
73. $(10-x)(12+x)$
74. $(5-x)(20+x)$
75. $(7-x)(1+x)$

Factor:

76. $7-6t-t^2$
77. $18-7x-x^2$
78. $132+p-p^2$
79. $60-11x-x^2$
80. $60+17x-x^2$
81. $80-16x-x^2$
82. $200-10x-x^2$
83. $\frac{1}{4}-x+x^2$
84. $90+13x-x^2$
85. $90-23x+x^2$
86. $80-21x+x^2$
87. $\frac{1}{9}-\frac{2}{3}x+x^2$
88. $80-11x-x^2$
89. $75-20x+x^2$
90. $75+10x-x^2$
91. $75-28x+x^2$
92. $100-25x+x^2$
93. $100-15x-x^2$
94. $100-29x+x^2$
95. $48-13x-x^2$
96. $2-x+x^2$

Supply numbers that will make the following expressions factorable. If more than one such number is possible, supply as many as you can:

97. $x^2+?x+21$
98. $x^2-3x-?$
99. $n^2+?n+9$
100. $n^2-4n+?$
101. $x^2+?x+7$
102. $28+?x-x^2$
103. $?+4x-x^2$
104. $n^2+6n-(\ )$
105. $x^2+8x+(\ )$
106. $(\ )-9x+x^2$
107. $30+?x-x^2$
108. $x^2-?x+15$

109. $15-?x-x^2$	110. $x^2-(\ )x+14$	111. $(\ )+5x-x^2$
112. $x^2+7x+(\ )$	113. $x^2-(\ )x+18$	114. $x^2-(\ )x-18$
115. $x^2-(\ )x-36$	116. $x^2-(\ )x+36$	117. $?+11x-x^2$
118. $(\ )+x^2-3x$	119. $x^2-(\ )x+21$	120. $x^2-3x+?$

Draw the graphs of the following expressions of the second degree. Let the unit of the vertical scale be one-half the unit of the horizontal scale. Note the points where the graph crosses the x-axis.

121. x^2+2x-8	122. x^2+2x-3	123. x^2+2x+1
124. x^2+2x+2	125. x^2-4x-5	126. $(x-2)(x+1)$
127. $(x-3)(x-2)$	128. $8-2x-x^2$	129. x^2+x+1

113. Solution of equations of the form $x^2+mx+n=0$. We are now ready to solve the problem proposed at the beginning of this chapter (Art. **94**). A man wishes to double the area of an 8 by 12 rod field by adding to one end and one side strips of equal width. How wide must each strip be? The equation to be solved is

$$s^2+20s=96$$

An equation of this type may be solved by factoring as follows:

Put in form $\quad s^2+20s-96=0$

Factor, $\quad (s+24)(s-4)=0$

Either factor put equal to zero will give a root of the equation:

$$s-4=0 \text{ gives } s=4$$
$$s+24=0 \text{ gives } s=-24$$

Check the results by substituting them in the original equation.

Both 4 and -24 are roots of the equation. Can both be used as answers to the problem that gave rise to the equation? Why?

It will be noticed that this method of solving equations of this form applies when the coefficients m and n are such that n has two factors whose sum is m. Cases in which this is not true are considered in Art. **178**.

EXERCISES AND PROBLEMS

Solve:

1. $(x-4)(x-2)=0$	2. $(x-2)(x+3)=0$	3. $(a+5)(a+8)=0$
4. $x^2+5x+6=0$	5. $a^2+a-6=0$	6. $x^2+18=11x$
7. $n^2+24=11n$	8. $x^2=x+56$	9. $0=x^2+4x-21$
10. $n^2=6n+18$	11. $n^2+36=15n$	12. $x^2+5x=6$
13. $y^2+12y=45$	14. $y^2=5y+14$	15. $y^2-y=72$

16. The length of a rectangle is 2 feet more than its width. If its area is 63 square feet, find its dimensions.

17. If 5 feet be subtracted from two opposite sides of a square, the area of the resulting rectangle will be 25 square feet. Find the side of the square.

18. The length of a rectangle is 3 feet more than its width. The area is 40 square feet. Find length and width of the field.

19. To find the area of a triangle multiply the base by the altitude and divide the result by 2. Is this the same as multiplying the base by half the altitude? As multiplying the altitude by half the base? Why? Express the rule for the area of a triangle as a formula.

20. Find the area of the following triangles:

(1) Base $=12$, altitude $=6$
(2) Base $=15$, altitude $=7$
(3) Base $=16$, altitude $=8$
(4) Base $=a$, altitude $=5$
(5) Base $=a$, altitude $=a-3$

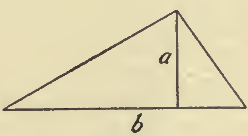

FIG. 41

21. The base of a triangle is 4 feet more than the altitude. If the area is 30 square feet, find the dimensions.

22. The base of a triangle is 3 feet less than the height. What are the dimensions if the area is 90 square feet?

23. The altitude of a triangle is 3 feet less than the length of the base. If the area is 27 square feet, find the dimensions.

24. The altitude of a triangle is 10 feet more than the length of the base. If the area is 72 square feet, find the dimensions.

25. One leg of a right triangle is 2 feet less than the other. Find the dimensions if the area is 40 square feet.

26. One leg of a right triangle is 7 feet more than the other. Find the other leg if the area is 72 square feet.

27. The sum of the base and altitude of a rectangle is 20 feet, the area is 91 square feet. Find the dimensions.

28. The sum of the base and altitude of a triangle is 21 inches. Find both if the area is 45 square inches. Interpret the two answers.

29. The difference between the sides of a rectangle is 2 feet; the area is 99 square feet. Find the dimensions.

30. The difference between the base and the altitude of a triangle is 3 feet; the area is 77 square feet. Find the base and the altitude.

31. The sum of the squares of two consecutive integers is 841. Find the numbers.

32. Find two consecutive integers such that if the larger is subtracted from the square of the smaller the result is 19.

33. A rectangle is 3 times as long as it is wide. If 2 feet are added to the length and 2 feet added to the width, the area is increased by 36 square feet. Find the dimensions of the rectangle.

34. A rectangle is 3 times as long as it is wide. If its length is decreased by 4 feet and its width increased by 2 feet, the area will be unchanged. Find its dimensions.

35. $(x-5)(x-6)-(x-5)(x-4)=2$

36. $(x-3)(x+2)-(x-4)(x+6)=27$

37. $(x+3)(x-4)+(x-3)(x-2)=2$

38. $(x+5)(x-1)+(x-2)(x+6)+25=0$

39. $(n-3)(n+8)-(n-6)(n+2)=24$

40. $3x^2-(x-4)(x-2)=12$

41. $4y^2-(3y-1)(y+3)=68$

42. $(x-5)(x+1)=7-(x-2)(x-6)$

43. $50-(x+2)(x-3)=(x-8)(x-3)$

114. The general quadratic trinomial. We have been factoring expressions of the form

$$x^2-3x+2$$

in which the coefficient of the square term is 1. A question that might naturally arise is: Would it be possible to factor an expression in which the coefficient of the square is some other number than 1—for instance, $5x^2+7x-6$? To guide us to the best way of untangling the factors, if there be any, we must first consider certain multiplications.

115. Products of binomials.

Multiply $3x+5$ by $2x+7$

$$\begin{array}{r} 3x+5 \\ 2x+7 \\ \hline 6x^2+10x \\ +21x+35 \\ \hline 6x^2+31x+35 \end{array}$$

Multiply $3x+5$ by $2x-7$

$$\begin{array}{r} 3x+5 \\ 2x-7 \\ \hline 6x^2+10x \\ -21x-35 \\ \hline 6x^2-11x-35 \end{array}$$

EXERCISES

Find the following products:

1. $(3x+2)(x+1)$
2. $(3x-1)(2x-3)$
3. $(4x+3)(x-2)$
4. $(4x-3)(x+2)$
5. $(x-3)(5x-8)$
6. $(3x+1)(2x+3)$
7. $(3x+2)(x-1)$
8. $(3x-1)(5x+2)$
9. $(2x-3)(7x+2)$

Can you find a short way of writing the product at once without putting down the intermediate steps? Apply to following:

10. $(3x-5)(5x-3)$
11. $(2x-7)(9x+5)$
12. $(4x-5)(5x-6)$
13. $(2+5x)(3-2x)$
14. $(3-5x)(1+2x)$
15. $(3x+7)(5x+9)$
16. $(6n-1)(3n-2)$
17. $(n-7)(7n-1)$
18. $(8a-3)(5a+4)$
19. $(2y-5)(3y+7)$
20. $(1-9y)(5-y)$
21. $(x+2)(5x-7)$
22. $(8+9x)(5-2x)$
23. $(2n+7)(3n-2)$
24. $(3-n)(7+n)$
25. $(8t-3)(t+1)$
26. $(s+7)(s-8)$
27. $(2s-3)(7s+5)$
28. $(y-10)(2y-3)$
29. $(y-3)(y-7)$
30. $(2y+7)(y+9)$
31. $(2n-3)(3n-2)$
32. $(n+1)(3+n)$
33. $(2n+3)(2+n)$

116. Factors of quadratic trinomial. We wish to factor, if possible, expressions like $21x^2+29x-10$. The way to untangle the factors will be made clear if we look over carefully the multiplications of the last article; for instance, $3x+5$ by $2x-7$, given at the beginning of the article.

$$\begin{array}{l} 3x+5 \\ 2x-7 \end{array}$$

It is seen that $6x^2$ in the result is the product of $3x$ and $2x$,

-35 in the result is the product of 5 and -7,

$-11x$ is the algebraic sum of the cross products, $2x$ and 5, and $3x$ and -7, that is, of $10x$ and $-21x$.

$$\begin{array}{rrr} 3x & +5 & \\ 2x & -7 & \\ \hline 6x^2 & +\mathbf{10x} & \\ & -\mathbf{21x} & -35 \\ \hline & -\mathbf{11x} & \end{array}$$

Use these facts in factoring $21x^2+29x-10$.

Try pairs of factors of 21 and -10. For instance, try 3 and 7 for 21, -5 and 2 for -10. Find whether the sum of the cross products is 29.

$$\begin{array}{r} 3x-5 \\ 7x+2 \\ \hline -\mathbf{35x} \\ +\ \mathbf{6x} \\ \hline -\mathbf{29x} \end{array}$$

This does not work, as the coefficient should be $+29$ instead of -29.

Again try

$$\begin{array}{r} 3x+5 \\ 7x-2 \\ \hline +\mathbf{35} \\ -\ \mathbf{6} \\ \hline +\mathbf{29} \end{array}$$

This works. Hence the factors are $3x+5$ and $7x-2$. Therefore

$$21x^2+29x-10=(3x+5)(7x-2)$$

EXERCISE I

Factor:

1. $10x^2+19x+6$	2. $10x^2-x-21$	3. $10x^2+16x+6$
4. $10x^2+4x-6$	5. $15x^2+16x+4$	6. $15x^2-4x-4$
7. $2x^2+11x+5$	8. $2x^2+11x-8$	9. $4c^2+23c+15$
10. $6+5a-6a^2$	11. $3x^2+7x-6$	12. $4-5x-6x^2$
13. $2h^2+h-28$	14. $10-19t-15t^2$	15. $2f^2+3f-2$
16. $2-5h-3h^2$	17. $4t^2+t-14$	18. $5x^2+23x-10$
19. $3+7x-20x^2$	20. $18x^2-9x-5$	21. $5-33x+18x^2$
22. $3-19x-14x^2$	23. $7x^2-x-8$	24. $8n^2+n-9$
25. $5+14n-3n^2$	26. $9y^2-44y-5$	27. $5-8a-4a^2$
28. $24y^2-29y-4$	29. $6y^2+7y-3$	30. $3+19b-14b^2$
31. $12x^2-23x+10$	32. $7+10n+3n^2$	33. $x^2+43x+390$
34. $3x^2+10x+3$	35. $9t^2+16-24t$	36. $7n^2-15n-18$

EXERCISE II

Insert numbers where indicated so that the following expressions can be factored:

1. $2x^2-5x+?$	2. $3+?x-2x^2$	3. $5x^2-4x-?$
4. $5x^2+?x+12$	5. $?-4x+3x^2$	6. $10+?x-3x^2$
7. $7+?x+3x^2$	8. $7x^2-4x-?$	9. $2x^2-?x+35$
10. $?-11x+7x^2$	11. $9x^2-25x+?$	12. $5x^2-19x+?$
13. $18+?x-5x^2$	14. $?-x-15x^2$	15. $15x^2-?x+2$

EQUATIONS

Solve the following equations and problems by the method of factoring:

1. $2x^2+x-21=0$	2. $3x^2-16x-35=0$
3. $2x^2+3x=35$	4. $7x^2=15-32x$
5. $5x^2+14=37x$	6. $3x^2=25x-28$
7. $2x^2-6x=7x-15$	8. $3x^2+8x=3x+28$
9. $9x^2=(3x+7)(x+2)-19$	10. $6x^2-(2x+3)(x+3)=4$
11. $\frac{x(x-4)}{6}+\frac{1}{6}=\frac{x-1}{4}$	12. $\frac{1-2x}{2}-\frac{x(x-2)}{3}=\frac{x}{2}$

13. $(2x-5)(3x-5)-(x-4)(3x-1)=9$

14. $(3x-1)(3x+4)-(x+4)(2x-5)=17$

15. $(3x-1)(x-5)-(2x+5)(x-1)=30$

16. $(5x-4)(x-1)-(3x+2)(x+2)=30$

17. The length of a rectangle is 5 feet less than 3 times its width. If its area is 78 square feet, find its dimensions.

18. The length of a rectangle is 1 foot more than 4 times the width. Find the dimensions if the area is 39 square feet.

19. The base of a triangle is 1 foot more than twice the altitude. If the area is $52\frac{1}{2}$ square feet, find the dimensions.

20. The altitude of a triangle is 1 foot less than 3 times the base. Find the dimensions if the area is 22 square feet.

21. The area of a certain rectangle is 59 square feet more than the area of a certain square. If the length of the rectangle is 7 feet less than twice the side of the square and its width is 1 foot more than the side of the square, find the dimensions of each.

22. The sum of the areas of a square and a triangle is 58 square feet. The base of the triangle is 1 foot longer than the side of the square and its altitude 1 foot more than twice a side of the square. Find the dimensions of each.

23. The units digit of a certain number is 2 more than the tens digit. The number itself is 1 less than 4 times the square of the tens digit. Find the number.

24. Find three consecutive integers such that the square of the first added to the product of the other two is 67.

25. If the area of a certain rectangle is subtracted from 3 times the area of a certain square, the result is 28 square feet. The base of the rectangle is 7 feet more, and its altitude is 6 feet more than a side of the square. Find the dimensions.

26. Divide 17 into two parts so that the product of the two parts is 21 more than the square of the smaller part.

27. The tens digit of a certain number is 2 more than the units digit. The number itself is 5 more than twice the product of the digits. Find the number.

117. Special forms. Square of a binomial. If the two binomial factors are equal, the product takes on a special form of considerable importance.

For instance, $(x+3)(x+3)$

$$\begin{array}{r}
x+3\\
\underline{x+3}\\
x^2+3x\\
\underline{+3x+9}\\
x^2+6x+9
\end{array}$$

The special characteristics of this form stand out more clearly in a more general case:

$$\begin{array}{r}
a+b\\
\underline{a+b}\\
a^2+ab\\
\underline{+ab+b^2}\\
a^2+2ab+b^2
\end{array}$$

that is, $(a+b)(a+b)=(a+b)^2=a^2+2ab+b^2$

Let the pupil verify by using various numbers for a and b.

Before reading any further, discover, if you can, the special features of the identity given above. Do the same characteristics appear in $(a-b)^2=$? What difference do you see in the two illustrations $(a+b)^2=$? and $(a-b)^2=$? It is well to express the identity in words.

The square of the sum of two numbers equals the sum of the squares of the numbers plus twice their product. Make a similar statement for the case $(a-b)^2$.

This statement is to be regarded as a formula or blank form which is to be filled in:

$$(\square+\bigcirc)^2=\square^2+2\square\bigcirc+\bigcirc^2$$

FIG. 42

$$(2x+3)^2=(2x)^2 + 2\cdot 2x\cdot 3 + 3^2=4x^2+12x+9$$

EXERCISES

Write out the products by use of the formula:

1. $(x+5)^2$	2. $(y+1)^2$	3. $(t+11)^2$
4. $(n-6)^2$	5. $(x-9)^2$	6. $(4-n)^2$
7. $(2a+3)^2$	8. $(3n-2)^2$	9. $(5x+1)^2$
10. $(3-7t)^2$	11. $(r-t)^2$	12. $(2r-3t)^2$
13. $(5r-2x)^2$	14. $(2a+9b)^2$	15. $(20+1)^2$
16. $(30-1)^2$	17. $41^2=(40+1)^2$	18. 33^2
19. 61^2	20. 52^2	21. 48^2
22. 99^2	23. 88^2	24. 69^2
25. 18^2	26. $\left(\frac{x}{2}+3\right)^2$	27. $\left(\frac{x}{3}-4\right)^2$
28. $\left(x-\frac{1}{2}\right)^2$	29. $\left(5-\frac{x}{4}\right)^2$	30. $\left(x+\frac{1}{2}y\right)^2$
31. $\left(4-\frac{1}{2}x\right)^2$	32. $\left(a+\frac{2}{3}b\right)^2$	33. $\left(\frac{1}{3}a-\frac{1}{2}b\right)^2$

34. Plot $(x-2)^2$ for values of x from $x=-5$ to $x=+5$.

35. Plot $(2-x)^2$ for values of x from $x=-5$ to $x=+5$.

36. Plot $(x+1)^2$ for same values.

37. Plot $(-3)^2$ for same values.

38. Plot x^2-2x+1 for same values.

39. Plot x^2+4x+4 for same values.

40. What is the relation of these graphs to the x-axis?

41. Plot $(x-2)^2$ for same values.

42. What is the difference between $-(x-2)^2$, $(x-2)^2$, $(2-x)^2$? What is the difference between their graphs?

118. Special forms. Factoring of a trinomial square. Factoring $4x^2+12x+9$ by the crisscross method, we get the factors $(2x+3)(2x+3)$. The factoring might be done more quickly if one noticed that the expression was a trinomial square, that is, the form obtained as the product of two equal binomials. It is rather easy to recognize such a form.

$$a^2+2ab+b^2=(a+b)^2$$

(1) The expression on the left has three terms.

(2) Two terms are positive and squares.

(3) The third term is twice the product of the square roots of the square terms.

a^2 is the square of a. We also say that a is the square root of a^2.

9 is the square of 3
3 is the square root of 9

One of the two equal factors of a number is the **square root** of that number.

So in
$$4x^2+12x+9$$
the two end terms are squares, their square roots being
$$2x \text{ and } 3$$

The third term is twice the product of these two roots,
$$2 \cdot 2x \cdot 3 = 12x$$
therefore
$$4x^2+12x+9=(2x+3)^2$$

The trinomial square is of so much importance in mathematics that it is well to become very familiar with it.

EXERCISES

Factor when possible:

1. x^2+6x+9
2. x^2+2x+1
3. x^2-2x+1
4. $x^2+8x+16$
5. $x^2-16x+64$
6. $x^2+12x-36$
7. $n^2+12n+36$
8. n^2-4n+4
9. $9-6n+n^2$
10. t^2-3t+9
11. $A^2-14A+49$
12. $x^2+x+\frac{1}{4}$
13. t^2+4t+6
14. $x^2+\frac{1}{2}x+\frac{1}{16}$
15. $4x^2-13x+9$
16. $4y^2+20y+25$
17. $t^2+121-22t$
18. $x^2-\frac{2}{3}x+\frac{1}{9}$
19. $x^2-\frac{2}{5}x+\frac{1}{25}$
20. $9n^2+25-30n$

21. $a^2-13a+36$
22. x^2+y^2+2xy
23. $4s^2+1+4s$
24. $4x^2+12xy+9y^2$
25. $25n^2+16-20n$
26. $81a^2+25b^2+90ab$
27. $9x^2-2x+\frac{1}{9}$
28. $\frac{x^2}{25}+\frac{3}{5}x+\frac{9}{4}$
29. $x^2-\frac{7}{12}x+\frac{1}{12}$
30. $x^2-\frac{1}{3}x-\frac{2}{9}$

Supply the terms that will make the following expressions trinomial squares:

31. $n^2+2n+(\quad)$
32. $m^2+18m+(\quad)$
33. $c^2+(\quad)+9$
34. $a^2+2ab+(\quad)$
35. $1+(\quad)+4x^2$
36. $x^2-16x+(\quad)$
37. $x^2-2x+(\quad)$
38. $n^2+(\quad)+49$
39. $x^2-10x+(\quad)$
40. $4n^2-12n+(\quad)$
41. $9x^2+6xy+(\quad)$
42. $64c^2-16bc+(\quad)$
43. $25a^2+(\quad)-30a$
44. $4c^2+4cd-(\quad)$
45. $16n^2+25m^2-(\quad)$
46. $9x^2+12x+(\quad)$
47. $49x^2+(\quad)+25y^2$
48. $x^2+3x+(\quad)$
49. $x^2-5x+(\quad)$
50. $4x^2+6x+(\quad)$
51. Plot x^2-4x+4
52. Plot $9-6x+x^2$
53. Plot x^2+6x+9
54. Plot $1+2x+x^2$

119. The product of the sum and difference of two numbers. In the multiplication

$$\begin{array}{l} x-3 \\ \underline{x+3} \\ x^2-3x \\ \underline{\quad\;\, +3x-9} \\ x^2 \qquad\;\; -9 \end{array}$$

the term of first degree drops out, as the sum of the cross products is zero. The general formula for this type of multiplication is

$$(a+b)(a-b)=a^2-b^2$$

Let the pupil verify by putting numbers in place of a and b.

The verbal statement is: **The product of the sum and difference of two numbers equals the square of the first number minus the square of the second.**

It is a very important identity, and is used frequently in mathematics. Keep in mind that it is true for any values that may be assigned to the letters.

$(7+5)(7-5)$	7^2-5^2
$12\cdot 2$	$49-25$
24	24

EXERCISES

Apply the identity given above to the following exercises:

1. $(x-2)(x+2)$
2. $(x-1)(x+1)$
3. $(20-1)(20+1)$
4. $(40-3)(40+3)$
5. $57\cdot 63$
6. $48\cdot 52$
7. $(x-7)(x+7)$
8. $(x-6)(x+6)$
9. $(5-x)(5+x)$
10. $(\frac{1}{2}x-y)(\frac{1}{2}x+y)$
11. $(\frac{1}{4}-x)(\frac{1}{4}+x)$
12. $(11-x)(11+x)$
13. $(t-16)(t+16)$
14. $98\cdot 102$
15. $23\cdot 17$
16. $(a-9)(a+9)$
17. $(s-10)(s+10)$
18. $(6-y)(6+y)$
19. $(9-n)(9+n)$
20. $(x-\frac{1}{3})(x+\frac{1}{3})$
21. $(3+2n)(3-2n)$
22. $(3x+5)(3x-5)$
23. $(2a+3b)(2a-3b)$
24. $199\cdot 201$
25. $(x-\frac{2}{3})(x+\frac{2}{3})$
26. $(2x-5)(2x+5)$
27. $(3x+7a)(3x-7a)$
28. $(x-\frac{1}{5}a)(x+\frac{1}{5}a)$
29. $(x+3a)(x-3a)$
30. $(n-5t)(n+5t)$

31. Evaluate x^2-4 for all integral values of x from $x=-4$ to $x=+4$ and plot on cross-section paper.

32. Plot $9-x^2$.

33. Evaluate x^2+4 for all values of x from -3 to $+3$, and plot on cross-section paper.

34. Note the points where these graphs cross the x-axis.

120. Factoring the difference between two squares. Reading the identity of the last article backward,

$$a^2-b^2=(a+b)(a-b)$$

we have an easily recognized form that can always be factored; namely, **the difference between two squares.** Why is it so called?

$9n^2-49$ is of this type

hence, $9n^2-49=(3n+7)(3n-7)$

The identity stated in words is:

The difference between the squares of two numbers equals the product of the sum of the two numbers and the difference between the two numbers.

EXERCISES

Factor:

1. x^2-16
2. s^2-36
3. n^2-9
4. y^2-25
5. $49-x^2$
6. $16-n^2$
7. y^2-1
8. $x^2-\frac{1}{4}$
9. $1-x^2$
10. $1-9m^2$
11. $64n^2-25$
12. $x^2-\frac{1}{25}$
13. $4x^2-81$
14. $a^2-\frac{1}{9}$
15. n^2+36
16. n^2-81
17. $121x^2-144$
18. $x^2-\frac{9}{16}$
19. n^2-t^2
20. x^2-4y^2
21. 5^2-4^2
22. 41^2-39^2
23. $\left(\frac{a}{2}\right)^2-\left(\frac{b}{3}\right)^2$
24. $\frac{x^2}{16}-\frac{y^2}{25}$
25. 52^2-48^2
26. 63^2-47^2
27. 27^2-23^2

EQUATIONS AND PROBLEMS

Solve the following equations:

1. $(n-3)(n+3)=0$
2. $(2n-7)(2n+7)=0$
3. $x^2-25=0$
4. $4x^2-49=0$
5. $n^2=144$
6. $9a^2=49$
7. $25x^2-169=0$
8. $196=16x^2$

9. What must be the edge of a cube so that the surface of the cube shall be 24 square inches? Is there more than one answer?

10. The width of a certain rectangle is $\frac{5}{6}$ of the length. If the area is 120 square feet, find the dimensions.

11. The altitude of a certain triangle is $\frac{3}{4}$ of its base. Find the dimensions if the area is 150 square inches.

12. The length of a certain rectangle is 6 feet more than its width. If its length is diminished by 4 feet and its width is multiplied by 3, its area will be increased by 8 square feet. Find its dimensions.

13. The square of a number increased by 15 equals 184. What is the number? Is there more than one answer to this question?

14. Three-fourths of the square of a certain number is 108. Find the number.

15. If 25 is added to $\frac{1}{5}$ of the square of a certain number, the result is 150. Find the number.

16. If 1 is subtracted from the square of a certain number and the remainder divided by 4, the answer will be the same as if 21 had been subtracted from the square of the number and the result divided by 3. Find the number.

Many equations that do not appear at first sight to be of the kind considered here may be reduced to this standard form:

$$x^2 - a^2 = 0$$

17. $5x^2 - 20 = x^2 - 16$

18. $x^2 - 4(2 - x) = 24 - x(x - 4)$

19. $x(x+4) - 75 = 6x - 2x(x+1)$

20. $(4x-3)(x-2) - (x-8)(x-3) = 57$

21. $2x - x(x-4) = 18 - x(3x-6)$

22. $(3x-2)(2x-5) - (x-20)(x+1) = 75$

23. $4x^2 - x(x+15) = 32 + x(5x-15)$

24. $\frac{x^2}{2} - 1 = \frac{x^2}{4} + 8$

25. $\frac{x^2}{3} + 6 = \frac{2x^2+3}{5}$

26. $3s^2 - 41 = 39 - 2s^2$

27. $98 - 8n^2 = 0$

28. $75t^2 = 27$

29. $x^2 = 25a^2$

30. $7x^2 - 252a^2 = 0$

121. Summary of important identities. In the preceding articles we have considered several important identities:

(1) $$a(b+c)=ab+ac$$
$$x(x+2)=x^2+2x$$

(2) $$(a+b)(a-b)=a^2-b^2$$
$$(x+3)(x-3)=x^2-9$$

(3) $$(a+b)^2=a^2+2ab+b^2$$
$$(x+5)^2=x^2+10x+25$$
$$(a-b)^2=a^2-2ab+b^2$$
$$(x-5)^2=x^2-10x+25$$

(4) $$(x+a)(x+b)=x^2+(a+b)x+ab$$
$$(x+2)(x-5)=x^2-3x-10$$

(5) $$(mx+a)(nx+b)=mnx^2+(an+bm)x+ab$$
$$(2x-3)(3x+5)=6x^2+x-15$$

In each identity there are two forms of the same number, a product or factored form and a sum or expanded form. Tell which is the product form and which the sum form in each case. Both forms have important uses. One form is to be used for one purpose. Another situation may require the other form. It is, therefore, important for you to be able to change an expression from one form into the other quickly and correctly; that is, to expand an expression or factor an expression. A product form can always be expanded. The reverse process, factoring a sum form, can be done in certain cases only.

122. Uses of the identities. These identities may be used for three different purposes: (1) to change complicated expressions into less complicated forms; (2) to change an expression into a form that can be used for a given purpose; (3) to reduce the amount of work in making numerical calculations.

(1) *To illustrate the first use*, take the expression,

$$(x-2)^2+(x+2)(x-3)$$

A knowledge of the identities enables us to write out the **expansion,** or multiplications indicated, at sight,

$$x^2-4x+4+x^2-x-6$$

which, upon collecting terms, becomes

$$2x^2-5x-2$$

A similar use of these identities will come to light in our study of fractions in a later chapter.

(2) *To illustrate the second use*, notice that the factored form is used in solving equations.

In solving $\quad 2x^2-5x+2=0$

we factor $\quad 2x^2-5x+2=0$

getting $\quad (2x-1)(x-2)=0$

(3) *To illustrate the third use*, notice that the labor of numerical calculation can often be greatly reduced by factoring.

For instance, $\quad 46\cdot 37+46\cdot 15=46(37+15)$

$$=46\cdot 52$$

Compute both ways and compare the number of figures used.

So also in a case like

$$48^2-35^2=(48+35)(48-35)$$
$$=83\cdot 13$$

Compare both ways of making the computation and determine whether the factoring method does really lessen the amount of work.

It is possible to use expanding in the same way:

$$93^2=(90+3)^2$$
$$=8100+540+9$$
$$=8649$$
$$19\cdot 21=(20-1)(20+1)$$
$$=400-1$$
$$=399$$

Such use is rather limited and is hardly worth while, for in most cases the ordinary way of computation is simpler and shorter.

123. Products of several factors. When three or more factors are to be multiplied together, they may be multiplied in any order:

$$3 \cdot 5 \cdot 7 = 15 \cdot 7 = 105$$

or

$$21 \cdot 5 = 105$$

or

$$3 \cdot 35 = 105$$

EXERCISES

Find the following products in more than one way. Have you any choice in the order used?

1. $9 \cdot 6 \cdot 5$
2. $25 \cdot 14 \cdot 8$
3. $x(x-4)(x-11)$
4. $8x(x-5)(x+5)$
5. $x^2(x-1)(x+2)$
6. $3x^2(x+7)(x-3)$
7. $7x(3-x)(3+x)$
8. $4x^2(x+7)(x-2)$
9. $9x(5-x)(x+6)$
10. $(x+10)(x-3)x$
11. $(x-11)x(x+5)$
12. $(x+2)x^2(x-2)$
13. $3(x+7)x^2(x+7)$
14. $5x(2x-1)(2x+7)$
15. $3x(3x-1)(3x+1)$
16. $2x(x-1)(3x+4)$
17. $2x^2(2x-1)(3x+2)$
18. $3x(2x-3)(2x-3)$
19. $5x^2(1-x)(3+2x)$
20. $4x^2(2x-1)(x-3)$
21. $6x(8+x)(1-7x)$
22. $3x(1-x)(7+2x)$

124. Possibility of factoring. As has been said, an unexpanded expression can always be expanded. But it is not always possible to factor a given expression. And it is just as important to know that a given expression cannot be factored as it is to know that it can be factored. You must, then, become so thoroughly acquainted with factorable types that you can recognize them almost at a glance. The factorable types we have discovered so far in this book are:

(1) the common factor form $ab+ac$
(2) the difference of two squares a^2-b^2
(3) the trinomial square $a^2+2ab+b^2$
(4) the trinomial quadratic ax^2+bx+c

Types 1, 2, and 3 are always factorable. Type 4 is factorable only in certain special cases, depending upon the values of a, b, and c.

There are many other types of expressions that can be factored, but these are sufficient for our present needs.

To factor a given expression:

1st. Find out whether the expression belongs to one of the types with which you are acquainted.

2d. If it does, it is readily factored.

3d. If it does not, it is not factorable by the methods with which you are acquainted.

The work may be checked by multiplying the factors together. This will answer in most cases.

Another method of checking is the substitution of some convenient number for the letter:

$$x^2+3x-70=(x+10)(x-7)$$

Check by substituting 10 for x:

$$\begin{array}{r|l} 10^2+3\cdot 10-70 & (10+10)(10-7) \\ 100+30-70 & 20\cdot 3 \\ 60 & 60 \end{array}$$

125. Several factors. Many expressions have more than two factors.

Illustration:

Factor $\qquad x^3-x^2-20x$

x is a factor, as it is in every term.

$$x^3-x^2-20x=x(x^2-x-20)$$

Then factor $\qquad x^2-x-20$

$$x^2-x-20=(x-5)(x+4)$$

Therefore $\qquad x^3-x^2-20x=x(x^2-x-20)$

$$=x(x-5)(x+4)$$

The expression has been factored into its prime factors.

EXERCISE I

Factor:

1. x^3+x^2-12x
2. $2x^3-6x^2-56x$
3. $a^4-12a^3+32a^2$
4. $24a-2a^2-a^3$
5. $2x^3+18x^2-72x$
6. $8x^4-2x^3-15x^2$
7. $8y^3+14y^2-15y$
8. $16a^3+4a^2-6a$
9. $3t^2-2t^3-8t^4$
10. $3x^3+27x^2-66x$
11. $24x^3-42x^2+9x$
12. $3y^2+10y^3-8y^4$
13. $10x^3-6x^2-14x$
14. $5y^4-33y^3-14y^2$
15. $7x^4-21x^3+28x^2$
16. $24x+3x^2-21x^3$
17. $8x^4-17x^3+9x^2$
18. $15n-48n^2+9n^3$

EXERCISE II

Expand and check:

1. $(x-3)(x+4)$
2. $(x-3)(x-4)$
3. $(x+3)(x-4)$
4. $(x-3)(x-3)$
5. $(x+3)(x+3)$
6. $(x+4)^2$
7. $(x-5)(x+6)$
8. $(x-5)(x-9)$
9. $(x+5)^2$
10. $(x-5)(x-5)$
11. $(x-8)(x+4)$
12. $(2x-1)(2x+1)$
13. $(2x-1)(2x-3)$
14. $(5-x)(7+x)$
15. $(x-7)^2$
16. $(5-x)^2$
17. $(5-x)(5+x)$
18. $(2x-5)(3x-4)$
19. $(x-4)(2x-1)$
20. $(2x-1)^2$
21. $(3x-1)(3x+1)$
22. $(3x+4)^2$
23. $(3x+4)(3x-5)$
24. $(3x-5)(3x+5)$
25. $(3x-2)(2x-3)$
26. $(6-2x)(7+2x)$
27. $(6-2x)^2$
28. $(3-x)(5-2x)$
29. $(1-3x)(1-5x)$
30. $(1-3a)^2$
31. $(1-5n)(1+5n)$
32. $23 \cdot 17$
33. $38 \cdot 42$
34. $56 \cdot 64$
35. 73^2
36. 93^2

EXERCISE III

Factor and check:

1. x^2-25
2. n^2-n
3. $a^2+2a-80$
4. p^2-2p+1
5. $4x^3-8x^2$
6. $9n^2-36t^2$
7. b^2+16
8. $p^2+2p-32$
9. $x^2-14x+49$
10. n^2+n+1
11. $3ax^2+7bx^2$
12. t^2-t-90
13. $64-9m^2$
14. $3x^4-3x^3-36x^2$
15. $2ax-6ay$
16. $2ab^2-4a^2b$
17. $6x^2-13x+6$
18. $6x^2-5x+6$
19. $5x^2-2x-3$
20. $12ax+20ax^2-32ax^3$
21. $10x^2-160$
22. $2x^2+9x-56$
23. $2x^2-23x+56$
24. $3x^2-27$
25. $4x^2+6x-70$
26. $5x^2-20x$
27. $5x^3-20x$

28. $2x^2-17x+35$
29. $81-121x^2$
30. $3x^2-7x-20$
31. $56x^3-9x^2-2x$
32. $1-144x^2$
33. $7x^2-21x$
34. $7x^3-63x$
35. $2x^3-24x^2+90x$
36. $2x^3+3x^2-27x$
37. $6x^3-14x^2+8x$
38. $9x^3-39x^2-30x$
39. $12x^3-21x^2-45x$
40. $6x^3+28x^2-10x$

Use factoring in the following exercises:

41. Find the value of 75^7-25^2.

42. Find the value of 323^2-277^2.

43. Find the value of $35(43^2-37^2)$.

44. Find the value of $17\cdot 81^2-17\cdot 69^2$.

45. Factor $\pi a^2-\pi r^2$. Evaluate the result when $a=10$, $r=4$; when $a=12$, $r=3$.

46. Draw two circles with the same center, and with radii $2\frac{1}{2}$ inches and 2 inches. What is the area between them? (Fig. 43.)

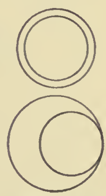

FIG. 43

47. Draw two circles with the radii 3 inches and 2 inches but so that the circles touch each other on the inside of the larger one. Find the area of the crescent-shaped figure. (Fig. 43.)

48. What will be the cost of laying a 4-foot concrete walk around a circular tower 20 feet in diameter, at 90 cents a square yard?

The area of the curved surface of a cylinder (Fig. 44) is the product of the altitude and the circumference of the base. If r is the radius of the base and h is the height of the cylinder, the lateral area $=2\pi rh$. The area of each end is πr^2. The total area, then, is

$$2\pi rh+2\pi r^2$$

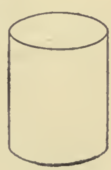

FIG. 44

49. What is the total area of a cylinder if $h=3$ and $r=2$? $h=7$ and $r=5$?

50. What is the area of the total surface of a cylindrical box 14 inches in diameter and 8 inches high?

51. Allowing nothing for overlapping, how much tin is used in making a can, the diameter of the base being 6.2 inches and the height being 7.5 inches?

52. What will it cost to cement the walls and floor of a cylindrical silo 30 feet high and 15 feet in diameter at 75 cents a square yard?

53. Plot x^2-4.

54. Plot x^2-9.

55. Plot $4x^2-25$.

56. Plot $4-x^2$.

126. Miscellaneous equations and problems. Solve and check:

1. $x^2=121$
2. x^2-7x+6
3. $5x^2=70x$
4. $3x^2=5x$
5. $ax^2=bx$
6. $x^2+2x=63$
7. $t^2-t-30=0$
8. $3n^2+7n=6$
9. $n^2-4n=45$
10. $m^2-10m=0$
11. $x^2+32=12x$
12. $(x-3)(x+4)=0$
13. $(x-2)(x+3)x=0$
14. $x(x-7)(x-1)=0$

NOTE. Observe that when there are three factors containing the unknown there are three roots to the equation. Why?

15. $3(x+5)(x-3)=0$
16. $x^3+5x^2-6x=0$
17. $2n^3-10n^2=28n$

18. Plot the expression x^3-3x^2+2x. Note the points where the curve cuts the x-axis. Solve the equation $x^3-3x^2+2x=0$.

19. Solve the equation $x^2+x-20=0$. Plot the expression x^2+x-20.

20. Plot x^3-9x; solve $x^3-9x=0$.

21. Solve $x^3-4x^2=0$; plot x^3-4x^2.

Solve:

22. $(t+10)^2=16t^2$

23. $144=n[48-4(n-1)]$

24. $11(11+d)(11-d)=792$

25. $(y+4)(y-5)+(y-2)(y+7)=-28$

26. $x^2(x-10)=x^2-18x$

27. $n(n^2-11)-7n(n+1)=0$

28. $\dfrac{x^3+5x^2}{3}=8x$

29. If 2 rods be added to one side of a square field and 4 added to the other side, the area of the resulting rectangular field will be 48 square rods. What was the area of the original field? Will both roots of the equation serve as answers to the problem? Why?

30. Find two numbers one of which is 4 times the other and whose product is 196.

31. Find a number such that 5 times its square increased by 10 times itself equals 495.

32. Find three consecutive numbers such that their sum is equal to three-sevenths of the product of the last 2.

33. The difference between the squares of two consecutive numbers is 49. What are the numbers?

34. The sum of the squares of two consecutive integers is 265. Find the numbers.

35. The sum of the squares of three consecutive integers is 245. Find the numbers.

36. The length of a rectangle is 1 foot more than 3 times its width. If the area is 200 square feet, find the dimensions.

37. The base of a triangle is 2 feet more than twice the height. If the area is 110 square feet, find the dimensions.

38. If one side of a square is increased by 3 and the other side by 1, the area of the rectangle formed is 12 less than the area of the square formed by increasing each side by 3. Find the area of the original square.

39. The length of a rectangle is 7 more than the width. The perimeter of a square of the same area is 48. Find the dimensions of the rectangle.

40. One side of a certain rectangle is 16 feet. Its area is 18 square feet less than twice the square on the other side. Find the dimensions of the rectangle.

41. A side of one square is 6 feet more than the side of another square. The difference between their areas is 96 square feet. Find the side of each.

42. An open box 3 inches high and containing 108 cubic inches is to be made from a square piece of tin by cutting out the corners and turning up the sides. What must be the dimensions of the square piece of tin?

In Fig. 45 a trapezoid is shown; a is its height or altitude, b is its lower base, and c is its upper base. It is shown in geometry that its area may be found by the formula

$$\text{area} = \frac{a(b+c)}{2}$$

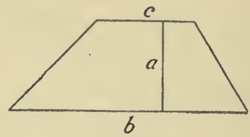

FIG. 45

43. Find the area of a trapezoid if:

(1) Its altitude is 6 feet and its bases are 10 feet and 15 feet.

(2) Its altitude is 9 feet and its bases are 7 feet and 12 feet.

(3) Its altitude is 4 feet and its bases are 20 feet and 30 feet.

(4) Its altitude is a feet and its bases are $\frac{2}{3}a$ and $\frac{3}{5}a$ feet.

(5) Its altitude is a feet and its bases are $a-2$ feet and $a+9$ feet.

44. Find the dimensions of a trapezoid if the longer base is twice the shorter base and the altitude is 5 feet less than the shorter base, the area being 75 square feet.

45. Find the dimensions of a trapezoid if one base is 1 foot more than twice the other, and the altitude is 4 feet less than the shorter base, the area being 148 square feet.

46. The bases of a trapezoid are respectively 6 feet and 8 feet longer than the altitude. Find the dimensions if the area is 60 square feet.

47. The length of a certain rectangle is 5 feet more than its width. Its area is $2\frac{1}{4}$ times the area of the square whose side is the width of the rectangle. Find the dimensions of the rectangle.

CHAPTER VIII

Review and Extension of Fundamental Operations

127. General view. Let us stop to take account of what we have learned in the preceding chapters. The most important thing that we have learned is that we may solve problems by using letters for the unknown numbers, stating the problems as equations, and solving the equations. We found a number of different kinds of equations and developed methods for solving them.

For solving these equations it was found that we needed to know how to add and subtract, multiply and divide, and to factor algebraic expressions. The examples we have worked were comparatively simple, though they seemed somewhat difficult because of their newness to us. In the present chapter we wish to gather together some of these algebraic ideas, review them, and extend them to more difficult examples and problems.

DEFINITIONS

128. Terms, coefficients, exponents. An algebraic expression may be made up of parts that are separated from one another by plus or minus signs. These parts with the signs in front of them are called **terms.** The expression $3x^2-5x+2$ has three terms: $3x^2$, $-5x$, $+2$.

An expression of one term is called a **monomial**; one of more than one term is called a **polynomial.** A two-term expression is called a **binomial**; a three-term expression is called a **trinomial.** Special names are not often used for polynomials of a larger number of terms.

$3x^2$ is a monomial.
x^2-9 is a binomial.
x^2-3x-2 is a trinomial.

A term may be made up of two parts, a numerical part and a literal part. The numerical part is called the **numerical coefficient** or simply the coefficient.

In $3x^2$, 3 is the coefficient of x^2. The coefficient of a term like x^2 is understood to be one. The coefficient of $-2x$ is -2.

The name coefficient is used in a broader sense also. In a product either of two factors may be called the **coefficient** of the other.

In ax^2, a is the coefficient of x^2. In ax, a is the coefficient of x; also x may be called the coefficient of a.

The choice of what shall be taken as the coefficient depends upon the choice of the letters which are the center of our interest.

129. Like or similar terms. Terms are said to be **like** or **similar** when they differ only in the numerical or literal coefficients.

$3x^2$ and $5x^2$ are like terms.
$5x$ and $3y$ are not.
ax and bx may be regarded as like terms.

130. Degree. The **degree** of a term is the sum of the exponents of the unknowns that it contains.

$3x^2$ is of the second degree in x.
ax^2 is of the second degree in x.
xy is of the second degree in x and y.
x^2y is of the third degree in x and y.

EXERCISES

Evaluate the following when $x=2$, $y=3$, $a=5$, $b=4$:

1. x^3y, a^2x^3, by^4
2. $8xy^4$, $\frac{1}{2}ax^5$, $5a^2b$
3. axy^3, ax^3y, a^3xy
4. a^4x, b^4y, a^4xy
5. a^5x, x^5a^2, x^4y^3

The **degree** of an **integral** expression is given by the term of highest degree.

$3x^2-4x+2$ is of the second degree in x.
$4x^3-5x-3$ is of the third degree in x.
$xy-3x$ is of the second degree in x and y.

THE FUNDAMENTAL OPERATIONS WITH INTEGRAL EXPRESSIONS

ADDITION

131. Addition of terms. RULE. **To add like terms, multiply the sum of the coefficients by the common factor:**

$$3x^2+7x^2=10x^2$$
$$9xy-5xy=4xy$$
$$ax+bx=(a+b)x$$

The addition of unlike terms can be indicated only:

$$ax+by$$

EXERCISES

Find the sum of:

1. x^3, $2x$, $3x^3$, $5x^2$, $-3x$, -5, 4
2. xy, $3xy$, $-5x^2$, $-7y^2$, $+x^2$, $-5xy$, $-7x^2$, $3y^2$, $3xy$

Put the following polynomials into more condensed forms by adding like terms:

3. $4m+3n-2n+5m-7n-3m$
4. $3x+4y-2z-7x+9y+11z-13y-10z$
5. $3x^2+5x-2x^2+2-3x-5x^2-5$
6. $3ab+6a^2b+7ab^2-2ab-3ab^2+3a^2b$
7. $3x^2y^2+5x^2y-3xy^2-3x^2y+2xy^2-6x^2y^2$
8. $7a^2+\frac{7}{3}a+\frac{9}{5}a^2+3-\frac{3}{7}a+2a^2$
9. $k^3-4k^2n+6kn^2+2k^2n-3kn^2-2n^3+n^3+3k^2n+4kn^2$
10. $6\pi-8\pi+9\pi-3\pi$
11. $2\pi R-\pi R^2+8\pi R-3\pi R+7\pi R^2$

132. Addition of polynomials. How are polynomials added? (See Art. **61.**)

EXERCISE I

Add the polynomials;

1. $3x+2$, $7x-5$, $2x-4$, $-3x+7$
2. x^2+3x+2, $2x^2-7x+3$, $2x^2+5x-8$, x^2-10, $3x^2-2x$, $-3x+2$
3. $x^2+x^3+x^4$, $2x^2-2x-3x^3$, $3x-3x^2+3$

4. $9x+3y+3,\ 7x-2y-7,\ 3x-5y+1,\ 3y-2x,\ 7x-4$

5. $\frac{1}{2}n^3-\frac{1}{3}n+\frac{1}{15},\ -\frac{1}{3}+n^3-\frac{1}{3}n,\ \frac{2}{15}+\frac{2}{3}n-\frac{3}{2}n^3$

6. $ax^2+bx,\ bx^2+a,\ ax+b,\ ax^2+bx+c,\ bx^2+cx+a,\ cx^2+ax+b$

7. $x^4+3x^2-2x+3,\ x^3-2x^2+3x-7,\ x^4-6x+4,\ x^3-5x+2$

Expressions in parentheses may be treated as a single term.

Thus, $$4(a-b)+5(a-b)=9(a-b)$$
for that is exactly what is often intended when parentheses are used.

EXERCISE II

In this way add:

1. $6(x+y)-5(x+y)+8(x+y)$

2. $4(x+y)-7(a-b)+8(x+y)-9(a-b)$

3. $6x^2-(a-b),\ 5x^2-4(a-b),\ 11(a-b)-9x^2,\ +7x^2+2(a-b)$

4. $a(x-y)+b(x-y)$

5. $7x^2y-(x-y),\ -9x^2y+9(x-y),\ 3x^2y-5(x-y),\ -6x^2y+4(x-y)$

6. $2x^2y^3-(x^2-y^2),\quad -12x^2y^3-7(x^2-y^2),\quad +15x^2y^3+5(x^2-y^2),\ -4x^2y^3+6(x^2-y^2)$

7. $(x-y)-9xy^3,\ -6(x-y)+7xy^3,\ 4(x-y)-2xy^3,\ -3(x-y)-5xy^3$

8. Add $4a+2b+3c,\ 6a+3b+2c,\ 5a+7b+9c$

What does each expression and their sum become when $a=100$, $b=10$, $c=1$?

9. Add $5x-2y+z,\ 8x+3y-4z,\ 6x-2y-3z$

Evaluate each expression and the sum when $x=100$, $y=10$, $z=1$.

10. Add $8x-2y,\ 15x-7y,\ 9x+3y,\ 12x-y,\ 8x+9y$

If $x=12y$, find the value of each and the value of the sum in terms of y.

11. Add $3x+2y-z,\ 4x-y+9z,\ 6x+y-10z,\ 5x-2y+11z$

Find the value of each expression and the sum in terms of z when $x=3z$ and $y=2z$.

MULTIPLICATION

133. Products of monomials. RULE. **Multiply the numerical coefficients and affix to each letter an exponent which is the sum of the exponents of that letter in the monomials to be multiplied.**

State the rule of signs. (See Art. **67.**)

EXERCISES

1. $3x \cdot 5x$
2. $5x \cdot 2x^2$
3. $3x^3 \cdot 5ax$
4. $-2x^2 \cdot 3ax$
5. $a \cdot 3ab$
6. $3x^2 \cdot -7xy$
7. $ab^2c \cdot a^2bc$
8. $21a^2b \cdot -3a^3b$
9. $3n^2 \cdot 2p^2 \cdot 4n^2$
10. $3t^2 \cdot 4t^3 \cdot ts^2 \cdot 2ts^4$
11. $-9x^4y \cdot -8x^3y^2$
12. $-3a^3y^3 \cdot -4x^2y \cdot 2xy^3$
13. $\frac{1}{2}x^3y \cdot 4xy^3 \cdot -\frac{2}{3}x^2y^2$
14. $\frac{1}{2}r^2st \cdot -\frac{3}{4}rs^2t^2$
15. $3x \cdot \frac{1}{3}xy \cdot -5y$
16. $\frac{2}{3}ax \cdot -\frac{3}{4}by \cdot -2xy$

134. Multiplication of polynomials. The product of two polynomials is the sum of the products of each term of the one with each term of the other.

As was indicated in Art. **111,** it is generally most convenient to arrange the multiplication as in arithmetic with similar terms under similar terms in the partial products:

$$\begin{array}{l} x^2+3x-5 \\ 2x-3 \\ \hline 2x^3+6x^2-10x \\ \quad\;\; -3x^2-9x+15 \\ \hline 2x^3+3x^2-19x+15 \end{array}$$

Check by substituting a value for x, say 3, in the expressions to be multiplied and in the product, then:

$$\begin{array}{lr} 9+9-5 & 13 \\ 6-3 & 3 \\ \hline 54+27-57+15 & 39 \end{array}$$

EXERCISE I

Simpler exercises. Find the products:

1. $x(x-1)$
2. $-x(x-2)$
3. $\pi(a-4)$
4. $x(x^2+3)$
5. $2\pi r(r-h)$
6. $(n+2)(n-5)$
7. $(3n+1)(2n-3)$
8. $(x^2-7x+2)(x-3)$
9. $(x^2-5x+3)(x^2+2x-3)$
10. $(3n^2-2n+4)(4n^2+n-3)$
11. $(2a^2-3a+1)(2a^2+3a-1)$
12. $(x^2+3x-5)(x^2-x+2)$
13. $(2t^2-3t+5)(2t-3)$
14. $(x^3-x^2-x+1)(x^2-2x+2)$

Arrange the following in the order of the descending powers of x, the highest power first and so on:

15. x^2+x^3-1+2x
16. $4+x^3-2x-x^2$
17. $3(x^2+3x+1)+2(x-3)-5$
18. $3(x-1)(x+2)-2(x+2)x$
19. $(x^2-x)(x-2)-(x-3)(x-2)$
20. $(x+3)^3-2(x+3)^2+(x+3)-5$

In multiplying two polynomials it is desirable, though not absolutely necessary, to arrange them in some regular order; for instance, according to decreasing or increasing exponents. Why?

21. $(3t-2t^2+5)(2-3t)$
22. $(h^2-2h+h^3+2)(2+3h^2-h)$
23. $(2+x^3-x-2x^2)(x^2-1-x)$

When terms of the regular order are missing, it is often well to leave room for them when setting down the work.

24. $(h^3-3h+2)(2h^2-3)$
25. $(4x^3-3x-1)(2x^3+2x^2-2)$
26. $(x+x^3-2)(3x^2+1)$

Find the shortest way of multiplying Exercises 27 to 33:

27. $n(n+2)(n+3)$
28. $(n-1)(n+2)(n-3)$
29. $(p-1)(p+1)(p^2+1)$
30. $(x-2)(x-2)(x+2)(x+2)$
31. $(x-y)(x^2-xy+y^2)(x+y)$

32. $(a-2)(a^2+4)(a+2)$

33. $(3x-1)(3x-1)(3x-1)$

EXERCISE II

More difficult exercises:

1. $(y^3+2y^2+y-1)(y^2-y+1)$
2. $(n^3+3n^2-2n+1)(n^2-2n+1)$
3. $(x^2-2xy+y^2)(x^2+2xy+y^2)$
4. $(4n-\frac{1}{2})(5n+\frac{3}{4})$
5. $(3x^4+6x^3y-9xy^3+12y^4)(2x^2-3xy+y^2)$
6. $(1+2n+3n^2+4n^3+5n^4)(2-3n+4n^2-5n^3)$
7. $(3x-2x^3+2x^4-7x^2+5)(3+3x^2+2x)$
8. $(x^2-4xy-2y^2)(x^2+4xy-2y^2)$
9. $(x^2-3xy-5y^2)(x^2-3xy+5y^2)$
10. $(x^3-2x^2y+xy^2)(x^2+2xy-y^2)$
11. $(x^4+6x^2y-y^2)(x^4+6x^2y-4y^2)$
12. $(a+x^2)(a^2-2ax^2+x^4)$
13. $(ax^2+4x^3)(a^2-ax+x^2)$
14. $(3x^2-5)(3x^2+5)(3x^2+5)(3x^2-5)$
15. $(x-1)(x^2+1)(x+1)(x^4+1)$
16. $(2x-1)(4x^2+1)(2x+1)$
17. $(a^2+ab+b^2)(a^2-ab+b^2)$
18. $(3x^3y-2x^4+5x^2y^2+y^4)(x^2-xy+2y^2)$
19. $(5x^4-7x^3y-2xy^3-y^4)(x^2-2xy-y^2)$
20. $(x^3-3x^2y+3xy^2-y^3)(x^2-y^2)$
21. $(x^2-4xy-5y^2)(x^2-y^2-xy)$
22. $(a-b)(b-a)(c-a)$
23. $(a^2+b^2+c^2-bc-ca-ab)(a+b+c)$
24. $(b+c)(b-c)+(c+a)(c-a)+(a+b)(a-b)$
25. $(x+y)(x^4-x^3y+x^2y^2-xy^3+y^4)$
26. $(x-y)(x+2)(x-3)(x+4)$
27. $(x+1)(x^2-2)+(x^2-1)(x+3)$
28. $(a-b)(b-c)(c-a)$

SUBTRACTION

135. Subtraction of polynomials. How is one polynomial subtracted from another? See Art. **64.**

EXERCISES

1. Subtract $3n-2$ from $5n+7$.
2. Subtract $4n^2-2n+3$ from $7n^2-5n+1$.
3. From $-7x^2y^2+13x^3y+15x^4y$ take $4x^2y^2+7x^3y-8x^4y$.
4. From $5k^4-3k^3-2k^2+5k+2$ take $2k^4-5k^3+2k^2-3k-5$.
5. From $3x^2y-2xy^2-x^3$ take $9x^2y-6xy^2+3x^3$.
6. From $2a^3b-6a^2b^2+7ab^3-b^4$ take $7a^3b+8a^2b^2-3ab^3-2b^4$.
7. Take ax from bx.
8. Subtract $ax-by$ from $bx+ay$.
9. Subtract $3a^3x-8a^2x^2+9ax^3-2x^4$ from $7a^3x-9a^2x^2-10ax^3-3x^4$
10. Subtract $3(x+y)$ from $-7(x+y)$.
11. Subtract $2x^4+6(x-y)$ from $3x^4-(x-y)$.
12. Subtract $3(x-y)-7(a+b)$ from $2(x-y)+3(a+b)$.
13. From $x^2-(a-b)+(x-y)$ take $3x^2-2(a-b)+4(x-y)$.
14. From $ax-(y-z)$ take $-3(y-z)$.
15. Subtract $bx-2(a-b)$ from $cx+2(a-b)$.
16. Subtract $3x^2-7x+1$ from $2x^2-5x-3$, subtract that difference from zero, and add this result to $2x^2-5x-4$.
17. $(x^3-x^2+x+1)-(x^3+x^2-x+1)=?$

Reduce to simpler form:

18. $(-3a^3+5b^3-c^3)-(a^3+b^3-c^3-3abc)$
19. $(2uv-v^2-3u^2)-(u^2+v^2-2uv)$
20. $(3xy+2x^2y)-(4x^3-2xy^2+3yx)$
21. $3x-2(2x^2-3y)-3(x+2x^2-y)$
22. $x^2-(x+5)(x-1)$
23. $h^3-(h+1)^2+2(h-2)-5$
24. $(n-2)-2(n-2)-(3n-6)+5$

136. Parentheses. Parentheses are used for grouping terms that are to be treated alike. They are removed from the work when the indicated operations are worked out. (See Art. **18.**)

EXERCISES

Perform the operations indicated:

1. $(x-3)(x-4)$
2. $x-3(x-4)$
3. $(x-3)x-4$
4. $x(y-2)-y(x-3)$
5. $x(y-z)-a(b-c)$

137. Parentheses within parentheses. Brackets [ ], braces { }, and the bar, $\overline{a-b}$, are often used for the same purpose as parentheses, especially to avoid any confusion when one set of parentheses incloses another.

$5(r-(n-1)3)$ means that $n-1$ is to be multiplied by 3, the result subtracted from r, and that result multiplied by 5. When such expressions are written with brackets, there is less danger of confusing the pairs of signs:

$$5[r-(n-1)3]$$

In such cases we may best simplify the expression by performing the indicated operations, beginning with the inner parenthesis:

$$\begin{aligned} 5[r-(n-1)3] &= 5[r-(3n-3)] \\ &= 5[r-3n+3] \\ &= 5r-15n+15 \end{aligned}$$

EXERCISES

Perform operations indicated:

1. $a-(+n-[3t+5])$
2. $3+[(x-2)+(x+2)]$
3. $3-[(x-2)-(x+2)]$
4. $x[(x-3)^2-4x]$
5. $x^2-(x-y)-[x-(3x+4y)]$
6. $x-(x^2-2)+[(x-3)-(2x+5)]$
7. $3a^2-[a(a-b)-b(a-3b)+ab]$

138. Insertion of parentheses. It is often important to put certain terms of an expression into parentheses. It will be noticed that when parentheses with a plus sign in front are removed the signs of the terms taken out remain unchanged. So in the inverse operation of putting the term back in parentheses the signs of the terms are not to be changed:

$$a+n-3=a+(n-3)$$

On the other hand, if the sign before the parentheses to be removed was minus, the sign of every term taken out was changed. So also in putting the terms back into parentheses with a minus sign in front.

Since $a-(n+3)=a-n-3$

$$a-n-3=a-(n+3)$$

EXERCISES

Put the last two terms into parentheses in Exercises 1 to 8:

1. $a+b+c$ 2. $a+b-c$ 3. $a-b+c$
4. $a-b-c$ 5. $3x-2a-4$ 6. $(a+b)^2-a-b$
7. $(a-b)^2-a+b$ 8. x^2-a^2-x+a 9. x^3-x^2-x-1

In Exercises 10 to 18 put first two terms in one parenthesis and last two in another:

10. $a+b-c-d$ 11. $a+b+c+d$ 12. $a-b+c-d$
13. $a-b-c+d$ 14. $3x^2-2+6x^2-4$ 15. x^2-2-2x^2+4
16. $x^2-2x+a-b$ 17. $x^2+4x-c-d$ 18. x^3-x^2-x-1

Group in pairs:

19. $x^3-1+2x^2-4-3x+3$ 20. $a-b-c-d+e-f$

Group in threes:

21. $x^2-3x+2-2y^2+6y-4$ 22. $x^2+5x-3-2y^2-7y+5$

DIVISION

139. Exact division by a monomial. To multiply one number by another we introduce the second as a factor.

To multiply $3a$ by x, introduce the factor x; the result is $3ax$.

The inverse operation of dividing a product by one of its factors is accomplished by rejecting that factor.

Divide $3ax$ by a; the result is $3\not{a}x$ or $3x$

Divide 63 by 7; the result is $9 \cdot \not{7}$ or 9.

$$3x \div 3 = \not{3}x = x$$

So also $$a \div a = 1 \cdot \not{a} \div \not{a} = 1$$

In all of these cases the quotient is the rest of the term with the division factor omitted.

EXERCISES

Divide:

1. ab by a
2. xy by y
3. abc by ab
4. abc by abc
5. $12a$ by 3
6. $17xy$ by x
7. $36st$ by $12t$
8. $10x^2$ by x
9. $ax+bx$ by x
10. $ab-ac+ad$ by a
11. $(2x^3-4x^2+6x)$ by $2x$

140. Use of exponents. In multiplying and dividing expressions what is done with the exponents? (See Arts. **97, 98.**)

$$x^5 \cdot x^3 = ?$$

$$x^5 \div x^3 = ?$$

141. A peculiar form: x^0. In applying the exponent rule to such a case as

$$x^3 \div x^3$$

we have $$x^3 \div x^3 = x^{3-3} = x^0$$

What does x^0 mean? We will give it a meaning consistent with some of our other work.

If we divide x^3 by x^3

by discarding the factors we have

$$x^3 \div x^3 = 1 \cdot \not{x^3} \div \not{x^3} = 1$$

To make the rules consistent, we must say that x^0 means 1. Note that x is x^1, not x^0.

142. Division of monomials by monomials. Make your own rule. (See Art. **98.**)

EXERCISES

1. Divide a^7 by a^3
2. Divide $5n^9$ by n^5
3. Divide $\frac{3}{4}\pi R^3$ by $\frac{1}{2}R$
4. Divide $35x^3y^5$ by $7x^2y^3$
5. Divide 2^5 by 2^3
6. $25^3 \div 25^2$
7. $36ax^3 \div 12x^2$
8. $6x^4 \div -2x$
9. $-8x^2y \div xy$
10. $12x^2y^3 \div x^2y$
11. $10x^2y^3 \div -5xy^2$
12. $-12a^3x^2y \div -6a^2xy$
13. $15a^4bx^2y^3 \div 5a^3x^2$
14. $25a^7b^3c^2 \div 5a^3bc$
15. Divide $39m^7n^3$ by $-3m^5n$
16. Divide $-49x^2y$ by $+7xy$
17. Divide $-6a^2bc$ by $-2ab$
18. $28a^2x^2y \div 7x^2y$

143. Division of a polynomial by a monomial. RULE. **Divide each term of the polynomial by the monomial.**

EXERCISES

1. $(6x^3-4x) \div 2x$
2. $(24x^5-36x^4-20x^3) \div 4x^2$
3. $(36x^2y^4-42x^6y^7) \div 6xy^3$
4. $(9x^4-12x^3) \div -3x^2$
5. $(10x^3y^2-15x^3y^3+5x^3y^4) \div 5x^2y$
6. $(32k^2n^3-16k^3n^5) \div 8kn^2$
7. $(a^2x^3-a^3x^2) \div a^2x$
8. $(a^3x^4-a^4x^3) \div ax^3$
9. $(a^3x^2-a^2x^3-ax^4) \div ax^2$
10. $(4a^2b-6a^3b^2+12a^4b^3) \div 2ab$
11. $(10a^4x^3-15a^3x^4+20a^2x^5) \div -5a^2x^2$
12. $(6n^3m^2+12n^2m^3-18nm^4) \div 6nm^2$
13. $4a(x-5) \div (x-5)$
14. $(1-x) \div (x-1)$
15. $3a(x-1) \div (x-1)$
16. $(x-2)^2 \div (x-2)$
17. $36a(x-y) \div 9(y-x)$
18. $36a(x-y) \div 4a$
19. $36a(x-y) \div 6(x-y)$
20. $5(x+1)(x-3) \div 5(x-3)$
21. $[(x-y)^2-2(x-y)] \div (x-y)$
22. $[4x(a-b)-2(a-b)^2] \div (a-b)$
23. $[7x(a-b)-8a(a-b)] \div (a-b)$
24. $(x^2-1) \div (x-1)$
25. $(x^2-2x+1) \div (x-1)$
26. $(x^2-3x-4) \div (x+1)$
27. $(x^2-4) \div (x+2)$

28. $(x^2-3x+2)\div(x-2)$ 29. $3(x^2-6x+9)\div(x-3)$
30. $(x^2+10x+25)\div(x+5)$

Find exact divisor for:

31. x^2-25 32. $x^2-8x+16$ 33. $x^2-2x-15$
34. a^2-100 35. $y^2-10y+21$ 36. x^4-49
37. a^2-81 38. $2a^2+19a+9$ 39. $4x^2-25$

144. Division of a polynomial by a polynomial. The following example from arithmetic will show how to proceed when it is desired to divide one polynomial by another.

Divide 8673 by 21:

$$\begin{array}{r|l} & 413, \text{ quotient} \\ 21 & 8673 \\ & \underline{84} \\ & \ \ 27 \\ & \ \ \underline{21} \\ & \ \ \ \ 63 \\ & \ \ \ \ \underline{63} \end{array}$$

This may be put in a fuller form:

$$\begin{array}{r|l} & 400+10+3, \text{ quotient} \\ 20+1 & 8000+600+70+3 \\ & \underline{8000+400} \\ & \qquad 200+70 \\ & \qquad \underline{200+10} \\ & \qquad\qquad 60+3 \\ & \qquad\qquad \underline{60+3} \end{array}$$

If we represent 10 by t, the work takes this form:

$$\begin{array}{r|l} & 4t^2+\ t+3, \text{ quotient} \\ 2t+1 & 8t^3+6t^2+7t+3 \\ & \underline{8t^3+4t^2} \\ & \qquad 2t^2+7t \\ & \qquad \underline{2t^2+t} \\ & \qquad\qquad 6t+3 \\ & \qquad\qquad \underline{6t+3} \end{array}$$

Apply to $(x^2-3x+2)\div(x-2)$

$$\begin{array}{r|l} & x-1, \text{ quotient} \\ \hline x-2 & x^2-3x+2 \\ & \underline{x^2-2x} \\ & -x+2 \\ & \underline{-x+2} \end{array}$$

EXERCISES

Divide:

1. $x^3+9x^2+26x+24$ by $x+2$
2. x^3-5x^2+x+10 by $x-2$
3. $x^3-6x^2+5x+12$ by $x-4$
4. $x^3-3x^2-4x+12$ by $x+2$
5. $2n^4-n^3+4n^2+4n-3$ by n^2-n+3
6. $4+12a+13a^2+6a^3+a^4$ by $2+3a+a^2$

In the multiplication of polynomials the arrangement of terms in ascending or descending powers of some letter is merely a matter of convenience. But in division such an arrangement is necessary. It is desirable to keep like terms under each other. This may be accomplished by means of a vacant place when there is a missing term. In arithmetic such vacant places are filled by means of a zero.

Divide:

7. $x^3-40+23x-8x^2$ by $x-5$
8. $4m^3-14m+2m^2+3$ by $4m+2m^2-1$
9. $x^4-10x^2+15x-6$ by x^2+3x-3
10. $x^4-x^3+2x^2-x+1$ by x^2-x+1
11. $x^4-5x^3+15x^2-23x+20$ by x^2-3x+4
12. $5k^3-3-5k+6k^5-3k^4$ by $3k^3-2k-1$

SUPPLEMENTARY EXERCISES

1. $(6a^2-10+a-4a^3+a^4)\div(5-3a+a^2)$
2. $(3a-8a^2+9+a^4-a^3)\div(a^2-3-2a)$
3. $(b^4-4a^3b+a^2b^2+a^4+4ab^3)\div(a^2-b^2-ab)$
4. $(6h^5-9h^4+10h^3-18h^2-5)\div(3h^2+5)$

5. $(x^4+x^2y^2+y^4) \div (y^2-xy+x^2)$
6. $(a^4-1) \div (a+1)$
7. $(x^4-12x^3+36x^2-25) \div (x^2-6x+5)$
8. $(4x^4-13x^2y^2+9y^4) \div (2x^2+xy-3y^2)$
9. $(x^9-a^9) \div (x^6+x^3a^3+a^6)$
10. $(x^{12}-b^{12}) \div (x^4-b^4)$
11. $(9x^4-49x^2y^2+16y^4) \div (3x^2-5xy-4y^2)$
12. $(8a^6+1) \div (2a^2+1)$
13. $(a^5+2b^5-2a^4b-5ab^4+a^3b^2+a^2b^3) \div (a^2+2b^2-ab)$
14. $(a^2-4ab+4b^2-c^2) \div (a-2b+c)$

145. Inexact division. If the division is inexact, that is, if a remainder is left after all the dividend is used, proceed in the same way. The division should be carried on until the highest exponent of the remainder is less than the highest exponent of the divisor.

Illustration 1.

$$
\begin{array}{r|l}
 & x-5+, \text{ quotient} \\
x+2 & x^2-3x+2 \\
 & \underline{x^2+2x} \\
 & \quad -5x+2 \\
 & \quad \underline{-5x-10} \\
 & \qquad +12, \text{ remainder}
\end{array}
$$

Illustration 2.

$$
\begin{array}{r|l}
 & x-1+, \text{ quotient} \\
x+3 & x^2+2x-5 \\
 & \underline{x^2+3x} \\
 & \quad -x-5 \\
 & \quad \underline{-x-3} \\
 & \qquad -2, \text{ remainder}
\end{array}
$$

EXERCISES

Divide:

1. x^3-x^2-3 by x^2
2. $2y^3-2y^2-3y+1$ by y
3. $3y^4-5y^3-2y^2-y+1$ by y^2

4. $x+1$ by $x-1$
5. x^2+1 by $x+1$
6. x^2+x-25 by $x-3$
7. h^3-11h^2+21 by $h-5$
8. $2x^2-5x+7$ by $x+3$
9. x^3+5x^2+7x-3 by x^2+3x+1
10. $(4x^3+18-9x^2-15x)\div(x^2-4x+3)$
11. $(20x^3-5x^2-4x+7)\div(5x^2-1)$
12. $(6x^3-13x^2+9x-2)\div(2x^2-3x+1)$

146. Division as a fraction. Instead of carrying out the division of one polynomial by another as in the preceding articles, it is the usual practice in mathematics to express the division in the fractional form and to treat the problem from that point of view. This method will be considered in a later chapter. It is, however, sometimes very important that the long division be actually performed.

SOLUTION OF EQUATIONS

147. Linear equations in one unknown. An equation of the first degree in one unknown is called a linear equation. State the rule for solving a linear equation. Why should the solution be checked?

EXERCISES

Solve and check:

1. $3x-2+7x=3x-5$
2. $50-10x-3+2x=3$
3. $2n+3=16-(2n-3)$
4. $8(t-3)-(6-2t)=2(t+2)-5(5-t)$
5. $7(25-x)=2x+2(3x-25)$
6. $15(x-1)=2(7+x)-4(x+3)$
7. $ax=-3x+b$
8. $at=c(t+h)$. Solve for t.
9. $.5x+2.75=3.25x-1$

10. $2(x-a)+3(x-2a)=2a$

11. $3(x+a+b)+2(x+a-b)=6b$

12. $(m+n)t+(n-m)t=n^2$. Solve for t.

13. $35n(5-2)-7(n-3)=17$

14. $8[t-(3-5t)+7]=5t-11$

15. $\frac{x}{2}-\frac{x-3}{4}=15$

16. $\frac{x}{3}-\frac{x-1}{5}=\frac{7}{2}-\frac{x-1}{2}$

17. $10-\frac{x+1}{3}=16+\frac{x-9}{2}$

18. $\frac{x-3}{2}-\frac{x+2}{7}-\frac{2}{3}+\frac{x-7}{3}$

19. $\frac{6(x-4)}{5}-\frac{3(x-8)}{4}=12$

20. $x(a-b)=a^2-b^2$

21. $x(a+b)=a^2-b^2$

22. $ax-bx=a^2+2a-ab-2b$

23. $ax+a=a^3-x+a^2-1$

24. $\frac{a(x-a)}{4}=\frac{x-2}{2}$

25. $\frac{3x-a}{4}=\frac{2x-a}{3}$

26. $\frac{5x-a}{5}=\frac{2x-a}{3}$

148. Factorable equations in one unknown. If an equation of a higher degree than the first can be put in the factored form, it can be easily solved. What is meant by the factored form? State the methods of solving such equations.

An equation of the second degree is called a **quadratic** equation. With your present knowledge can you solve any quadratic equation given you?

How many roots does an equation of the first degree have? How many roots does an equation of the second degree have?

EXERCISES

Solve and check:

1. $x^2-7x=0$

2. $6x+91=x^2$

3. $n^2+12=7n$

4. $3-2x-x^2=0$

5. $225=x^2$

6. $3x^2+8x+4=0$

7. $2x^2-5x=3$

8. $x^2-2ax+a^2=0$

9. $(a+b)x^2=a^3+a^2b$

10. $3x^2-2ax-bx=0$

11. $9x^2-16(1-x^2)=0$

12. $9x^2=13-4x$

13. $(x-7)(x-5)=0$

14. $(x-2)(x-3)=(x-2)3$

15. $(x-3)(x-4)-(2x-3)(x-7)=12$

16. $\frac{x-3}{2}-\frac{x^2-4}{5}=\frac{x-5}{6}$

17. $(x-3)(2x+1)-1=4x-(3x-1)(2x+5)+3$

18. $\frac{x^2-1}{2}-\frac{x-7}{7}=\frac{x+5}{5}$

19. $(2x+a)(x+a)-2a(x+2a)=0$

20. $\frac{x(x+a)}{2}-\frac{3a^2}{5}=\frac{x^2-2a^2}{5}$

21. $(x-1)^2+(x-1)=0$

22. $(x-2)^2+3(x-2)=0$

23. $x(x-3)-5(x-3)=0$

24. $7(x-3)=(x-3)2x$

149. A set of linear equations in two unknowns. State rules for solving a set of linear equations in two unknowns and show under what circumstances each method is the most desirable. How many solutions has such a set? What kind of a figure is the graph of a linear equation?

EXERCISES

Solve and check. Draw graphs for the first three.

1. $5-p+2q=0$
   $7-5p+q=0$

2. $2x+7y=38$
   $3x+4y=31$

3. $y=17x-31$
   $15x+3y=39$

4. $3x=15+5y$
   $2x+y=205$

5. Solve for v and u:
   $v-u=a$
   $v+u=e$

6. Find a and b in terms of x and y:
   $x=3a+2b$
   $y=a-5b$

7. $P+Q=112$
   $5P=9Q$

8. $\frac{x}{4}-5=\frac{y}{5}$
   $x-15=y$

9. $\frac{2}{3}x+y=1$
   $x-\frac{1}{2}y=\frac{1}{6}$

10. $7x+4=4y-3x+3$
    $\frac{5x}{3}+2y=1\frac{5}{6}$

11. $\frac{x}{3}+\frac{y}{2}=1$

$\frac{2y}{5}-x=16$

12. $\frac{3x}{7}-\frac{y}{2}=2$

$2x-\frac{y}{5}=-12$

13. $x+\frac{y-3}{2}=8$

$\frac{x+4}{2}-y=13$

14. $5a-11b=\frac{1}{6}$

$6a-\frac{10b-1}{2}=2$

15. $\frac{7p}{6}+7n=2$

$2n+\frac{7p+8}{5}=\frac{12}{7}$

16. $x-\frac{y-1}{4}=-3$

$y-\frac{x+5}{3}=4$

17. $\frac{t-1}{2}-\frac{s-2}{3}=1$

$s=2t$

18. $\frac{x+6}{5}-\frac{y+7}{6}=\frac{2}{5}$

$x+y=0$

150. The stating of problems. The problem to be solved must be stated in algebraic language. Definite rules that are sure to lead to the correct statement cannot be laid down. But certain ways of attacking problems may be suggested as affording some help. Two things are absolutely essential to success in stating problems:

First, a clear understanding of the relations between the quantities involved in the problem.

For instance, in rate problems such as involve walking, rowing, etc., it is necessary to know the fundamental relation

$$\text{distance} = \text{rate} \times \text{time}$$
$$d = rt$$

In digit problems the law underlying the Arabic notation must be known:

$$273=2\times 100 + 7\times 10 + 3$$

Second, familiarity with the ways in which ideas are expressed in algebraic symbols.

For instance, the square of the sum of two numbers.

$$(x+y)^2$$

The area of a rectangle when one side is 3 more than the other:

$$x(x+3) \text{ or } x(x-3)$$

Furthermore, in some cases quite a little ingenuity is required to detect the relations that will be of use in the given situation.

With this equipment one is ready to attack a given problem. The following four steps form a satisfactory plan of attack. (See Art. **36.**)

(*a*) Determine the nature of the problem. Is it a rate problem, a digit problem, a triangle problem? Recall the fundamental relations involved in such problems.

(*b*) Look for some equality that is implied in the problem, and state this equality in words. Several equalities may be suggested by the problem, but choose one that seems satisfactory.

(*c*) Select some one of the unknown quantities of the problem to be the unknown of the equation.

(*d*) Translate this equality in terms of the unknown and the known quantities of the problem.

Show how this method has been followed in the illustrations on pages 46, 47, and 48.

Let us apply the plan to the following problem:

An automobile running at the rate of 25 miles an hour made the distance between two towns in 2 hours less than a second automobile running 15 miles an hour. How long did it take the first automobile to make the trip?

a) This is a speed problem (see Art. **42**). It has to do with what we call **uniform motion.** If a man travels at the rate of r miles an hour for t hours, he travels a distance of rt miles.

$$\text{distance} = \text{rate} \times \text{time}$$

b) The distances in the problem are equal.

$$\left.\begin{array}{l}\text{Distance first}\\ \text{automobile runs}\end{array}\right\} = \left\{\begin{array}{l}\text{Distance second}\\ \text{automobile runs}\end{array}\right.$$

c) The time is unknown. It is convenient to arrange the various quantities of the problem in a table:

	Rate	Time	Distance
First automobile......	25	t	$25t$
Second automobile....	15	$t+2$	$15(t+2)$

d) The equality becomes

$$25t=15(t+2)$$

REVIEW PROBLEMS

1. Divide \$270 among A and B and C so that B may have \$25 more than A, and C may have \$10 more than B.

2. Three men entered into a contract to divide the profits of a certain business deal in the following way: A was to receive twice as much as B and a bonus of \$10; C was to receive 3 times as much as B and a bonus of \$20. If the profits were \$1041.66, what was the share of each?

3. A sum of money s is to be distributed among 3 men. B is to have twice as much as A, and C is to have $\frac{1}{3}$ as much as B. How much did each receive?

4. The sum of 2 numbers is 35. Their difference is 27. What are the numbers?

5. The sum of 2 numbers equals twice their difference. The sum is also 2 more than 3 times the difference. What are the numbers?

6. The difference between two numbers is 6. The sum of their squares is 50. What are the numbers?

7. Find 3 consecutive integers whose sum is 69.

8. Find 4 consecutive odd integers whose sum is 32.

9. The sum of 2 consecutive even integers equals the product of the lesser one and the odd integer between them. What are the 2 even integers?

10. Of the 3 angles of a triangle the first is twice the second and the third is 20° more than the second. Find the number of degrees in each angle.

11. I have $3.30 in dimes, nickels, and quarters, having 34 pieces of money in all. The number of dimes is 2 more than the number of quarters. How many pieces have I of each kind?

12. A certain number has 3 digits. The tens digit is 4 more than the hundreds digit and 1 more than the units digit. The number itself is 5 more than 20 times the sum of the digits. Find the number.

13. A certain number has 2 digits. The sum of the digits is 8. If 18 is subtracted from the number, the digits will be reversed. Find the number.

14. A boy riding a bicycle at the rate of 12 miles an hour started to overtake a man who had left the same place 4 hours earlier walking at the rate of 3 miles an hour. How long was it before the boy overtook the man?

15. A passenger train leaves New York at the rate of 45 miles an hour. Three hours earlier, a freight left New York traveling in the same direction at the rate of 24 miles an hour. When will the passenger train overtake the freight?

16. Two airplanes fly over the same course, one in 5 hours, the other in 3 hours. One can fly 40 miles an hour more than the other. At what rate does each fly over the course? What is the length of the course?

17. A and B start from the same place at the same time and travel in the same direction, A at the rate of 15 miles an hour and B at the rate of 25 miles an hour. How far apart are they in 2 hours? In t hours? When will they be 50 miles apart?

18. A and B of the last exercise are in airplanes. A flies 5 hours and is compelled to stop. B flies 7 hours at a rate which is twice that of A and lands at a point 450 miles beyond where A lands. At what rates do they fly?

19. If A and B start from the same place at the same time, but go in opposite directions, A at the rate of 10 miles an hour and B at the rate of 8 miles an hour, how far apart will they be in 2 hours? In t hours? How long will it be before they are 90 miles apart?

20. A and B start at the same time from places 100 miles apart and travel toward each other at the same rate. They meet in 2 hours. At what rate do they travel?

21. A and B start at the same time from places 100 miles apart, A at the rate of 15 miles an hour and B at the rate of 25 miles an hour. When will they meet?

22. A and B started at the same time from towns 140 miles apart and traveled toward each other at the same rate. B was delayed for 1 hour by engine trouble. They met in 4 hours. What was their rate of travel and where did they meet?

23. A and B started at the same time from two towns 300 miles apart and traveled toward each other. A could travel 5 miles an hour more than B. They met in 8 hours. How far from A's starting point did they meet?

24. A man found that he could row 2 miles downstream in 15 minutes, but that it took him an hour to row back. What was the rate of the current? Use two unknowns.

25. A freight train leaves a certain station 4 hours before a passenger train. If they travel in the same direction, the passenger train will overtake the freight in 33 hours. If they travel in opposite directions, they will be 308 miles apart 3 hours after the passenger train starts. Find the rate of each. Use two unknowns.

26. Two automobiles are 225 miles apart. If they travel toward each other, they will meet in 5 hours. If they travel in the same direction, the faster will overtake the slower in 45 hours. Find the rate of each.

27. The length of a rectangle is 4 feet more than twice its width. If the area is 30 square feet, find the dimensions.

28. The altitude of a triangle is 3 feet less than twice its base. If the area is 85 square feet, find the number of feet in base and altitude.

29. One base of a trapezoid is 2 feet more than the other. The shorter base is twice the altitude. If the area is 21 square feet, find the dimensions.

30. From each of the 4 corners of a square piece of tin a square 3 inches on a side is cut out. If the total area remaining is 540 square inches, find a side of the original square.

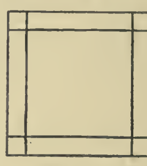

FIG. 46

31. A piece of tin is 20 inches on a side. How large a square must be cut from each corner that the area remaining shall be 364 square inches? (Fig. 46.)

32. A square 2 inches on a side is cut from each of the 4 corners of a certain square piece of tin. The sides are then turned up to form an open box. Find one side of the original square if the volume of the box is 242 cubic inches.

33. A certain garden is 50 by 40 feet (Fig. 47). It is divided into 4 parts by 2 walks running through the center parallel to the sides. If the total area of the walks is 261 square feet, find the width of the walk.

FIG. 47

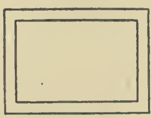

FIG. 48

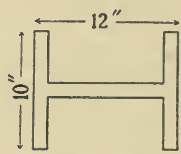

FIG. 49

34. A little park is 100 by 75 feet (Fig. 48). A walk is around the outside. Find the width of the walk if its total area is 1464 square feet.

35. A piece of tin is 15 by 25 inches. Squares are cut out of the corners so that the area remaining is 275 square inches. Find the size of the squares cut out.

36. A piece of iron of uniform width is of the shape and dimensions shown in Fig. 49. Find the width if the total area is 56 square inches.

37. A rectangle is 3 feet less than twice the width. If the area is 170 square feet, find the dimensions.

To find interest, multiply the principal by the rate and the result by the time expressed in years. This rule is stated as a formula thus: $I = PRT$.

Use the preceding formula for solving the following problems:

38. Find the interest on $520 for 3 years and 4 months at 4 per cent.

39. How much money must you have in the bank at 4 per cent interest to receive $100 each year as interest.

40. What principal put at interest at 5 per cent for three years will yield $1200 interest?

41. What is the rate of interest when a bank pays you $10 interest every six months on a deposit of $500?

42. How long will it take $2500 to yield $450 if put at interest at 6 per cent?

The **amount** is the principal plus the interest. The formula for the amount is, then,

$$\begin{aligned} A &= P + I \\ &= P + PRT \\ &= P(1 + RT) \end{aligned}$$

Use the above formula for the following problems:

43. How long will it take $1000 to amount to $1300 if put at interest at 6 per cent?

44. Solve the equation $I = PRT$ for P, R, and T.

45. What sum of money will amount to $1800 in 3 years at 4 per cent?

46. At what rate will $200 amount to $235 in 5 years?

47. In what time will $450 amount to $540 at 4 per cent?

48. A man has $1200; part of it is at interest at 4 per cent and the remainder at 6 per cent. The sum of the incomes is $56. Find each principal.

49. A man has $1500; part of it is at interest at 5 per cent and the remainder at 6 per cent. The interest on the part at 6 per cent is $2 more than the interest on the part at 5 per cent. Find each principal.

50. A man has \$7700 at interest, part at 5 per cent and part at 6 per cent. How much has he at interest at each rate, if the interest from one portion is equal to the interest from the other portion?

51. Solve the equation $A=P+PRT$ for P; solve for R; solve for T.

An angle is said to be the **complement** of another angle if their sum is 90°. What is the complement of an angle of 30°, 40°, 70°, 55°? Express in algebraic language the complement of an angle x.

An angle is said to be the **supplement** of another angle if the sum is 180°. What is the supplement of the angle 30°, 80°, 110°, 135°? Express in algebraic language the supplement of the angle x.

52. The supplement of a certain angle is 26° more than 3 times its complement. Find the angle.

53. Three times the complement of a certain angle added to twice its supplement is 360°. Find the angle.

54. Find an angle such that its supplement is 3 times its complement.

55. Find an angle such that its supplement shall be twice its complement.

56. Find an angle such that its supplement shall be n times its complement.

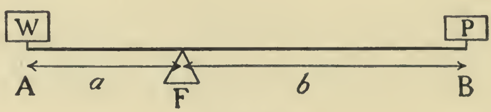

FIG. 50

Fig. 50 represents a teeter board. It is called in physics a **lever**. There are other kinds of levers. F, the point of support, is called the fulcrum. The distances a and b from A and B to F are called the lever arms. W and P are two weights placed at A and B. The teeter will balance if

$$a \cdot W = b \cdot P$$

If $a=5$ feet and $b=5$ feet, what can be said about W and P?

If $a=8$ feet, $b=4$ feet, and $W=50$, $P=$?

If a is greater than b, which is the heavier, W or P, when the teeter just balances? If F is at the middle of the board, what will be the relation of the two weights? Assign values to W or P and arrange on board so that they will just balance.

57. If $a=5$ feet, $b=10$ feet, and $P=30$ pounds, what is the weight of W?

58. If $W=45$ pounds, $P=30$ pounds, and $a=6$ feet, how far is b from the fulcrum?

59. Two children, W and P, just balance each other on a 16-foot teeter board. W weighs 45 pounds, P weighs 75. Where is the support placed?

60. A 16-foot teeter is balanced at the center. A boy weighing 45 pounds is at one end. How far from the support must a boy weighing 60 pounds sit in order just to balance?

61. A boy weighing 45 pounds sits at the end of the teeter of Exercise 60. Where shall a boy of 30 pounds sit in order to balance the first boy?

62. A 20-foot teeter is supported at a point 4 feet from center. A 42-pound boy sits at the end of the longer arm and just balances a boy at the other end. How much does the second boy weigh?

CHAPTER IX

Fractions

151. Definitions. The result of an inexact division always contains a fraction. The arithmetical fraction $\frac{2}{3}$ is the result of dividing 2 by 3. The fraction $\frac{7}{3}$ or its equal, the mixed number $2\frac{1}{3}$, is the result of dividing 7 by 3.

Any division may be so expressed whether the division be exact or inexact.

$\frac{10}{5}$ is in fractional form, though the result of the division is a whole number or integer.

In the same way $\frac{x-1}{x}$ is an algebraic fraction expressing the division of $x-1$ by x.

This suggests that the most satisfactory way of defining a fraction is to say that it is a certain way of expressing division; namely, $\frac{a}{b}$, which means that the upper number (numerator) is to be divided by the lower number (denominator).

152. Historical. We do not know when men first used fractions, but it must have been long before the time of the Babylonians, for we find fractions on clay tablets that have been dug up in Babylonian cities buried for thousands of years. The fractions used by the Babylonians all had the same denominator, 60. The early Egyptians used unit fractions; the numerators were 1, the denominators different, though they did have a special symbol for $\frac{2}{3}$. The Greeks used fractions of a more general kind with any numerator and any denominator. None of these peoples wrote fractions as we do. (See Fig. 51.)

The Babylonians wrote only the numerator, the denominator being understood, and the fraction being indicated by placing the numerator a little to the right of the ordinary position of the number in the line. The Greeks placed one

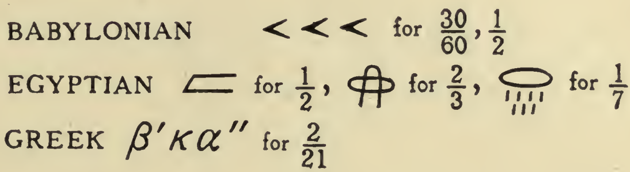

FIG. 51

stroke ′ after the numerator, and two strokes ″ after the denominator. The Hindus as early as 300 A.D. wrote the numerator over the denominator, but with no line between:

$$\begin{matrix}5\\8\end{matrix} \quad \text{for} \quad \frac{5}{8}$$

The line between was introduced later by the Arabs or possibly by the Hindus. It was certainly used before the time of Columbus, as we find in a manuscript on algebra written at that time the expression

$$\frac{100}{1 \text{ ding}} \quad \text{for} \quad \frac{100}{x}$$

153. Laws governing the use of fractions. In arithmetic we combine numbers whose exact values are known; in algebra we combine numbers without knowing their special values. The rules that hold in arithmetic must also hold in algebra. Algebraic fractions are handled according to the same rules of operation as are arithmetical fractions. The fundamental ideas are the same, but the forms of the fractions are often much more complicated.

154. The fundamental principle. Fractions may be put into various forms without having their values altered. A certain fundamental principle must be kept in mind in making such changes; namely, **if both terms of a fraction are divided (or multiplied) by the same number, the value of the fraction is unchanged.**

Illustration:

$$\frac{3}{4}=\frac{3\cdot 5}{4\cdot 5}=\frac{15}{20} \qquad \frac{2a}{3b}=\frac{2ac}{3bc}$$

$$\frac{12}{28}=\frac{12\div 4}{28\div 4}=\frac{3}{7} \qquad \frac{3ax}{5bx}=\frac{3a}{5b}$$

This principle may be put into symbols:

$$\frac{an}{bn}=\frac{a}{b}$$

The second fraction is the first fraction with its terms divided by n; the first fraction is the second with its terms multiplied by n. Verify this relation by using numbers for a, b, and n.

155. Reduction of a fraction to its lowest terms. How is a fraction reduced to its lowest terms in arithmetic? (See Art. **1.**) In algebra a fraction is reduced to its lowest terms in exactly the same way as in arithmetic. Divide both terms by the same number. The number used as a divisor must be a factor of the numerator and of the denominator.

$$\frac{49}{161}=\frac{\cancel{7}\cdot 7}{\cancel{7}\cdot 23}=\frac{7}{23}$$

(1) $$\frac{ab}{ac}=\frac{\cancel{a}b}{\cancel{a}c}=\frac{b}{c}$$

(2) $$\frac{ab+ac}{ad}=\frac{\cancel{a}(b+c)}{\cancel{a}d}=\frac{b+c}{d}$$

(3) $$\frac{(x+1)x}{a(x+1)}=\frac{\cancel{(x+1)}x}{a\cancel{(x+1)}}=\frac{x}{a}$$

EXERCISES

Reduce to lowest terms:

1. $\frac{35}{77}$
2. $\frac{2\cdot 2\cdot 7\cdot 2}{2\cdot 3\cdot 2}$
3. $\frac{186}{306}$
4. $\frac{3ax}{5ay}$
5. $\frac{3ax^2}{15ax}$
6. $\frac{5ax^2}{15bx}$

7. $\frac{2^4 \cdot 9}{2^3 \cdot 3^2}$

8. $\frac{7x^2y}{x^2 - xy}$

9. $\frac{a^2 - ab}{a^2 + ab}$

10. $\frac{3x^2}{5x^2 - 2x}$

11. $\frac{a^2 - ab}{ab - b^2}$

12. $\frac{288}{256}$

13. $\frac{t^3 - t^2}{2at^2}$

14. $\frac{5ax - 5a}{15a^2x + 20a}$

15. $\frac{2^4 \cdot 3 \cdot 11}{2^3 \cdot 7}$

16. $\frac{a^2 - b^2}{a^2 + b^2}$

17. $\frac{t^3 + 2t^2}{3t^6 + 6t^4}$

18. $\frac{r^2 - 2r + 1}{r^2 - 1}$

19. $\frac{189}{4 \cdot 81}$

20. $\frac{nx^2y - nxy^2}{nx^4y - nx^2y^3}$

21. $\frac{x^2 + 2x + 1}{1 - x^2}$

22. $\frac{x^2 - 2x + 1}{1 - x^2}$

23. $\frac{a^2 - b^2}{b - a}$

24. $\frac{x^2 - 4x + 4}{x^2 - 5x + 6}$

25. $\frac{Rr^2 - R^3}{\pi R^2 - \pi rR}$

26. $\frac{18a^4b^2 - 27a^6}{10b^2 - 15a^2}$

27. $\frac{t^3 - 4t}{3t - 6}$

156. Multiplication of fractions. How is the product of two fractions found in arithmetic? (See Art. **3.**) The product of two fractions in algebra is found in exactly the same way:

In arithmetic, $\frac{3}{7} \cdot \frac{5}{4} = \frac{15}{28}$

In algebra, (1) $\frac{a}{b} \cdot \frac{x}{y} = \frac{ax}{by}$

(2) $\frac{3x}{x-1} \cdot \frac{2x}{x+1} = \frac{6x^2}{x^2-1}$

(3) $\frac{a}{b} \cdot x = \frac{ax}{b}$

(4) $\frac{3x}{x-1} \cdot (x+1) = \frac{3x(x+1)}{x-1}$

EXERCISES

Multiply:

1. $\frac{3}{4}$ by 5

2. $\frac{7}{9}$ by 8

3. $\frac{3x}{2}$ by 5

4. $\frac{2}{4}$ by x

5. $\frac{2+x}{3}$ by 5

6. $\frac{x-1}{x^2+2}$ by x

7. $\frac{x-1}{x^3+3}$ by $x+1$ 8. $y+2$ by $\frac{y}{y+1}$ 9. $y-3$ by $\frac{y-3}{y+2}$

10. $\frac{7}{5}\cdot\frac{2}{3}$ 11. $\frac{22}{7}\cdot\frac{2}{3}$ 12. $\frac{7x}{3}\cdot\frac{5}{2}$

13. $\frac{2}{3}\cdot\frac{5}{7}\cdot\frac{4}{11}$ 14. $\frac{2a}{3}\cdot\frac{4b}{5}$ 15. $\frac{3a}{2b}\cdot\frac{5a}{7x}$

16. $\frac{2nx}{3t}\cdot\frac{7nx^2}{5t^3}$ 17. $\frac{x}{x+2}\cdot\frac{3x^3}{x-2}$ 18. $\frac{2x}{x+1}\cdot\frac{2x+1}{x-2}$

19. $\frac{ax}{x+1}\cdot\frac{bx}{x-1}\cdot\frac{c}{x}$ 20. $\frac{x-1}{x-2}\cdot\frac{x+2}{x+1}$ 21. $\frac{x^2+1}{x-3}\cdot\frac{x}{x+2}$

22. $\frac{3an^2}{a+n}\cdot 2n^3$ 23. $\frac{5}{x-1}\cdot\frac{x+1}{x^2+x+1}$ 24. $\frac{x-1}{x+1}\cdot\frac{2x}{x-1}$

It is usually desirable to reduce the result of a multiplication of a fraction to its lowest terms:

$$\frac{x-1}{x+1}\cdot\frac{2x}{x-1}=\frac{\cancel{(x-1)}2x}{(x+1)\cancel{(x-1)}}=\frac{2x}{x+1}$$

25. $\frac{x^2-y^2}{a}\cdot\frac{b}{x-y}$ 26. $\frac{3x}{x+y}\cdot\frac{7x+7y}{9}$

27. $\frac{x^2-y^2}{x^2+y^2}\cdot\frac{x^2+y^2}{x-y}$ 28. $\frac{x^2-2x}{x^2+x-2}\cdot\frac{x^2+2x-3}{x^2+x-6}$

29. $\frac{hr^2}{r^2-1}\cdot\frac{r^2-r}{hr(r+1)}$ 30. $\frac{nh}{n+h}\cdot\frac{n^2-h^2}{n^2h^2}$

The dividing out of the factors common to numerator and denominator may be done before the multiplication of the two fractions if desired. This is often the better plan:

$$\frac{\cancel{x-1}}{x+1}\cdot\frac{2x}{\cancel{x-1}}=\frac{2x}{x+1}$$

31. $\frac{2x}{4y}\cdot\frac{2x}{9y}$ 32. $\frac{3n}{5t}\cdot\frac{10nt}{6}$ 33. $\frac{5s^2}{7nt}\cdot\frac{21n^2s}{15t^3}$

34. $\frac{a^4}{b}\cdot\frac{bc}{a}$ 35. $\frac{63}{25}\cdot\frac{45}{14}$ 36. $\frac{a+b}{2}\cdot\frac{3}{a+b}$

37. $\frac{2x^2}{5}\cdot -\frac{15a}{x}$ 38. $\frac{x}{y}\cdot\frac{a}{x}\cdot\frac{y}{2}$ 39. $\frac{3x^2}{x-1}\cdot -\frac{x^2-1}{2x}$

40. $\frac{\pi r}{2}\cdot\frac{4r}{3\pi}$

41. $\frac{3a+3b}{5(a-b)}\cdot\frac{a^2-ab}{ba+b^2}$

42. $\frac{2x(x-2)}{x+3}\cdot\frac{5x(x+3)}{2(x-2)}$

43. $\frac{x^2+2x+3}{x-3}\cdot\frac{x^2-2x-3}{x-2}$

44. $\frac{R^2-r^2}{t^3}\cdot\frac{t^2-t}{R^2-2rR+r^2}$

45. $\frac{4K}{\pi r^4}\cdot\frac{1}{2r}\cdot\frac{\pi r^3}{a}$

46. $\frac{3x(x-5)}{25-x^2}\cdot\frac{x^2-25}{x^4+2x^3-35x^2}$

47. $\frac{x^4-5x^3+6x^2}{x^2-9}\cdot\frac{x^2-2x-15}{x^3-4x}$

48. $\frac{p^2+p-2}{p^2-7p}\cdot\frac{p^2-13p+42}{p^2+2p}$

49. $\frac{N^3-8N+15}{N^2-12N+35}\cdot\frac{N^2-15N+56}{N^2-17N+72}$

50. $\frac{e-f}{e^2-ef}\cdot\frac{ef}{e^2+f^2}$

51. $\frac{x-1}{x^2-2x-3}\cdot\frac{(x+1)^2}{2x-2}\cdot(x-3)$

157. Division of fractions. How is one fraction divided by another in arithmetic? (See Art. **5.**) In algebra the process is just the same.

$$\frac{a}{b}\div\frac{x}{y}=\frac{a}{b}\times\frac{y}{x}=\frac{ay}{bx}$$

$$\frac{x^2}{x^2-1}\div\frac{x}{x+1}=\frac{x^2}{x^2-1}\times\frac{x+1}{x}=\frac{x}{x-1}$$

EXERCISES

Perform the indicated operations:

1. $\frac{5}{8}\div\frac{3}{4}$

2. $\frac{63}{32}\div\frac{27}{16}$

3. $\frac{x}{2a}\div\frac{b}{y}$

4. $\frac{n}{x}\div\frac{3}{x^2}$

5. $\frac{2a}{3c}\div\frac{3c}{4a}$

6. $\frac{x}{x-1}\div\frac{2x}{x^2-1}$

7. $\frac{3ax}{5b}\div\frac{6a^2y}{10bx^2}$

8. $\frac{a+b}{a^2-ab}\div\frac{ab+a^2}{ab-b^2}$

9. $\frac{x^2+7x+12}{x^2-4x+4}\div\frac{x+4}{x-2}$

10. $\frac{x^2+3x}{x+4}\div\frac{x+3}{x^3+4x^2}$

11. $\frac{2(a-1)}{3(x+1)}\div\frac{4(1-a)}{5(x-1)}$

12. $\frac{2a}{a-b}\div\frac{3b}{a+b}$

13. $\frac{a}{a-b} \div (a+b)$

14. $(a+b) \div \frac{a}{a-b}$

15. $(x^2-9x+20) \div \frac{x-4}{x+5}$

16. $\frac{9}{16} \div \frac{27}{20} \div \frac{7}{12}$

17. $\frac{11}{25} \div \frac{22}{35} \times \frac{20}{13}$

18. $\frac{5y^2}{7a^3} \div \frac{21c^2}{4ax} \div \frac{35cy}{7a^2x}$

19. $\frac{n^2-n-20}{n^2-25} \div \frac{n+1}{n^2-25} \div \frac{n^2+2n-8}{n^2-n-2}$

20. $\frac{7x}{3a} \div \frac{21x^2}{2b}$

21. $\frac{t^2-11t+30}{t^2-6t+9} \div \frac{t-5}{t^2-3t} \times \frac{t^2-9}{t^2-36}$

22. $\frac{x^2-1}{3x} \div -\frac{x-1}{2a}$

158. Addition and subtraction of fractions. Just as in arithmetic two cases must be considered:

(*a*) Fractions with the same denominator

(*b*) Fractions with different denominators

Exactly the same methods are used in algebra as in arithmetic.

159. Fractions having the same denominators.

In arithmetic, $\frac{4}{3}+\frac{5}{3}-\frac{2}{3}=?$

In algebra, $\frac{a}{n}+\frac{b}{n}-\frac{c}{n}=\frac{a+b-c}{n}$

State as a rule.

EXERCISES

1. $\frac{9}{7}+\frac{2}{7}+\frac{1}{7}-\frac{5}{7}$

2. $\frac{1}{3}+\frac{2}{3}-\frac{5}{3}$

3. $\frac{n}{3}+\frac{2n}{3}+\frac{5n}{3}$

4. $\frac{2x}{a}+\frac{5x}{a}-\frac{3x}{a}$

5. $\frac{7a}{n}+\frac{a}{n}-\frac{4a}{n}$

6. $\frac{1}{x}-\frac{a}{x}+\frac{2a}{x}$

7. $\frac{3a-b}{c}+\frac{b}{c}$

8. $\frac{3-b}{n}+\frac{b}{n}-\frac{2}{n}$

9. $\frac{2R-3r}{R-r}+\frac{2r-R}{R-r}$

10. $\frac{3p-q}{2p-q}+\frac{p-q}{2p-q}$

11. $\frac{2a}{9}-\frac{2a-b-c}{9}$

12. $\frac{x^2-1}{x^2+1}-\frac{x^2+3}{x^2+1}$

13. $\frac{x^2-3x+2}{x^3+1}+\frac{2x^2+5x-7}{x^3+1}$

14. $\frac{4x^2-9}{x^2+1}-\frac{x^2-5}{x^2+1}+\frac{5-2x^2}{x^2+1}$

160. Fractions with different denominators. If the fractions to be added have different denominators, they must be reduced to equivalent fractions having the same denominator before they can be added. In many instances the required common denominator can be found by inspection. (See Art. **6.**)

In arithmetic, $\frac{2}{3}+\frac{1}{2}=\frac{4}{6}+\frac{3}{6}=\frac{7}{6}$

In algebra, (1) $\frac{a}{3}+\frac{2}{x}=\frac{ax}{3x}+\frac{6}{3x}=\frac{ax+6}{3x}$

(2) $\frac{a}{b}+\frac{m}{n}=\frac{an}{bn}+\frac{bm}{bn}=\frac{an+bm}{bn}$.

Evidently in this case the required denominator is the product of the denominators.

bn is a multiple of b, why?
bn is a multiple of n, why?
bn is called a common multiple of b and n.

Any number that can be divided exactly by a given number is called a **multiple** of that number. The common denominator required is simply a common multiple of the denominators. Any common multiple of the denominators of the fractions can be used, such as the product of all the denominators. In the illustration given, the fraction $\frac{a}{b}$ is reduced to an equivalent fraction whose denominator is bn by the multiplication of both terms of the fraction by n.

Thus, $\frac{a}{b}=\frac{an}{bn}$

Illustration:

$$\frac{x}{x-1}-\frac{1}{x+1}=\frac{x(x+1)}{x^2-1}-\frac{(x-1)1}{x^2-1}=\frac{x(x+1)-(x-1)}{x^2-1}=$$

$$\frac{x^2+x-x+1}{x^2-1}=\frac{x^2+1}{x^2-1}$$

EXERCISES

1. $\frac{7}{6}+5$

2. $\frac{9}{15}+\frac{3}{7}$

3. $\frac{7}{4}+\frac{2}{5}-\frac{1}{3}$

4. $\frac{2a}{3}+\frac{3b}{2}$

5. $\frac{1}{a}+\frac{1}{b}-\frac{1}{c}$

6. $\frac{3a}{x}-\frac{2b}{y}+\frac{1}{x}$

7. $\frac{1}{x+3}+\frac{1}{x-2}$

8. $\frac{3}{x-5}-\frac{2}{x+3}$

9. $\frac{a+b}{2}-\frac{2ab}{a+b}$

10. $\frac{a+b}{a-b}-\frac{a-b}{a+b}$

11. $\frac{1}{m+n}+\frac{1}{m-n}$

12. $2x+\frac{1}{1+x}$

13. $a-\frac{1}{a-1}$

14. $\frac{b}{a}-\frac{a}{b}-1$

15. $\frac{a}{bc}+\frac{b}{ca}+\frac{c}{ab}$

16. $\frac{K}{1-e}-\frac{K}{1+e}$

17. $\frac{7a-b-2}{a+4}+\frac{2a-b+5}{a-3}$

18. $\frac{7}{9}+\frac{2}{5}-\frac{1}{3}$

19. $\frac{x}{x^2+1}-\frac{1}{x+1}$

20. $\frac{x^2+1}{x^2-1}-\frac{x}{x-1}$

21. $\frac{x}{x-1}-\frac{x}{x^2+1}$

161. Lowest common denominator. Although the product of the denominators of all the fractions to be added will do for the common denominator, it often saves much labor if one uses the smallest denominator that will accomplish the purpose sought. An example will best illustrate the point.

First, using the product of all the denominators as the common denominator:

$$\frac{x}{x^2-1}-\frac{1}{x+1}=\frac{x(x+1)}{(x^2-1)(x+1)}-\frac{x^2-1}{(x^2-1)(x+1)}=$$
$$\frac{x^2+x-x^2+1}{(x^2-1)(x+1)}=\frac{x+1}{(x^2-1)(x+1)}=\frac{1}{x^2-1}$$

Second, if it is noticed that x^2-1 is a multiple of $x+1$, the work would proceed thus:

$$\frac{x}{x^2-1}-\frac{1}{x+1}=\frac{x}{x^2-1}-\frac{x-1}{x^2-1}=\frac{x-x+1}{x^2-1}=\frac{1}{x^2-1}$$

In this case x^2-1 is called the **lowest common denominator** of the fractions. It is the **lowest common multiple** of the denominators.

162. The lowest common multiple. The lowest common multiple (L.C.M.) of two or more expressions is the product of all the different prime factors of the expression. Each factor is used the greatest number of times that it appears in any one expression.

Find L.C.M. of 15, 12, 20.

$$15=3\cdot 5$$
$$12=2^2\cdot 3$$
$$20=2^2\cdot 5$$

L.C.M. is $3\cdot 5\cdot 2^2=60$

Find L.C.M. of x^2-1, x^2+2x+1, and $x-1$.

$$x^2-1=(x-1)(x+1)$$
$$x^2+2x+1=(x+1)^2$$
$$x-1=x-1$$

L.C.M. is $(x+1)^2(x-1)$

The lowest common multiple of several expressions can be found by inspection of the factored forms of the expressions. The factors should be picked out according to the principle given in the definition.

EXERCISES

Find the L.C.M.:

1. 36, 42, 12, 27
2. $5a^2b^3c^3$, $4abc^5$
3. a^2c, bc^2, cb^2
4. $7a^2$, $2ab$, $3b^2$
5. $2xy$, $3x^2y$, $4xy^2$
6. $24a^5b^2c^3$, $42a^6b^3c^2$
7. $(x-y)^2$, x^2-y^2, x^2
8. $x-3$, $x-4$, $x^2-7x+12$
9. $1-2a$, $1+2a$, $4a^2-1$
10. x^2+5x+6, x^2+3x+2, x^2+4x+3
11. n^2-b^2, $n+b$, $(n^2-b^2)^2$
12. $6h^2+54$, $3h-9$, $3h^2-27$
13. $12x^2+3x-42$, $12x^2+30x+12$, $32x^2-40x-28$
14. $ab-b^2$, $ab-a^2$, $(a-b)$, a
15. x^2-1, $2x-2$, $1-x$, $x-1$, x
16. $3(n-2)$, n^2-4n+4, n^2-2n

163. Addition of fractions. Add the following fractions. Be careful to remember that it is desirable to reduce the resulting fractions to lowest terms.

1. $\frac{3}{4}+\frac{5}{12}$
2. $\frac{5}{21}-\frac{3}{14}$
3. $\frac{7}{8}+\frac{5}{12}$
4. $\frac{8}{3}-\frac{3}{4}+\frac{5}{12}$
5. $\frac{2}{x}-\frac{3}{x^2}$
6. $\frac{2}{x}-\frac{5}{3x}$
7. $-\frac{2}{n}+\frac{7}{2n}$
8. $\frac{2}{n^2}+\frac{n}{3}$
9. $\frac{2}{x-1}+\frac{3x}{(x-1)(x+1)}$
10. $\frac{2a}{(a-b)a}+\frac{3b}{b(a-b)}$
11. $\frac{3x}{x^2-1}-\frac{2x}{x+1}$
12. $\frac{x-1}{(x+2)(x+1)}-\frac{x+3}{(x+1)(x-3)}$
13. $\frac{x+1}{x^2+5x+6}+\frac{x+3}{x^2+3x+2}$
14. $\frac{x+a}{x^2-ax}-\frac{x+3a}{x^2-a^2}$
15. $\frac{3}{n}+\frac{2}{5n^2}-\frac{3}{2n}$
16. $\frac{4}{x}-\frac{3}{x+1}-\frac{2}{(x+1)^2}$
17. $\frac{2}{(x-3)^3}-\frac{5}{(x-3)^2}+\frac{1}{x-3}$
18. $\frac{3x}{4x^2-9}+\frac{7}{2x+3}-\frac{2}{2x-3}$
19. $\frac{2a+1}{a+1}-\frac{3a-1}{a-1}+\frac{a(a+3)}{a^2-1}$
20. $\frac{2(x+3)}{x^2-2x}+\frac{3x-6}{x^2-3x}-\frac{4(x-1)}{x^2-5x+6}$
21. $\frac{x-4}{x^2-x}-\frac{x-8}{x^2+3x}$
22. $\frac{1}{a-x}-\frac{2x}{a^2-x^2}+\frac{2ax}{a+x}$
23. $\frac{2ax}{x^2-a^2}-\frac{x+a}{x-a}$
24. $\frac{x+y}{x-y}-\frac{x-y}{x+y}+\frac{2x^2-6y^2}{x^2-y^2}$
25. $\frac{1}{a}-\frac{1}{a+b}-\frac{1}{a^2b-ab^2}$
26. $\frac{2}{x}+\frac{5}{3x-3}-\frac{1}{6+3x}$
27. $1+n-\frac{1}{1-n}$
28. $a-\frac{Aa+Bb}{A+B}$
29. $1+x+x^2+\frac{1}{1+x}$
30. $\frac{y^2}{x+y}+x-y$
31. $1-\frac{2}{1+x}+x$
32. $1+x+x^2+x^3+\frac{x^4}{1-x}$
33. $x^2+3-\frac{3}{x-2}-2x$
34. $\frac{n(n-1)}{2}+\frac{n(n-1)(n-2)}{6}$

35. $\frac{n(n-1)}{2}+\frac{n(n+1)(n-1)}{6}$

36. $\frac{n(n-1)(n-2)}{2\cdot 3}+\frac{n(n-1)(n-2)(n-3)}{24}$

37. $\frac{1}{2g^2-g-1}-\frac{1}{2g^2+g-3}$

38. $\frac{hR}{4(R-r)}-\frac{3}{4}\cdot\frac{r^2h}{R^2-r^2}$

39. $\frac{T^2-2a^2T+a^4}{4a^2}-\frac{a^2}{4}$

40. $a-\frac{b(a-b)}{2(b-c)}-\frac{c(2c-a-b)}{2(b-c)}$

Find a short way of adding:

41. $\frac{1}{x-1}+\frac{1}{x+1}+\frac{1}{x-2}+\frac{1}{x+2}$

42. $\frac{2}{x+1}-\frac{1}{x-1}+\frac{1}{x+2}-\frac{1}{x-2}$

43. $\frac{1}{x-3}-\frac{3}{x-1}+\frac{3}{x+1}-\frac{1}{x+3}$

44. $\frac{1}{1-n}+\frac{1}{1+n}+\frac{1}{1+n^2}+\frac{4}{1+n^4}$

45. $\frac{a}{a-b}+\frac{b}{a+b}-\frac{a}{a-b}$

46. $\frac{a}{a-b}+\frac{b}{b-a}$

164. Signs of fractions. Exercise 46 of the last article can be worked in a simpler way if one takes advantage of a certain peculiarity in it.

$$\frac{a}{a-b}+\frac{b}{b-a}$$

You will notice that the denominator of the second fraction is the negative of the denominator of the first fraction. (See Art. **66.**) We can make the two denominators the same by multiplying both terms of the second fraction by -1.

Thus, $$\frac{a}{a-b}+\frac{-b}{-b+a}$$

or $$\frac{a}{a-b}+\frac{-b}{a-b}.$$

The sum is thus $$\frac{a-b}{a-b}=1$$

This illustration leads to the consideration of the changes in sign that can be made in the terms of a fraction.

We know that $$\frac{an}{bn}=\frac{a}{b}$$
is true for all values of n except zero.

It is true, then, for $n=-1$

$$\frac{a\cdot -1}{b\cdot -1}=\frac{-a}{-b}$$

or $$\frac{a}{b}=\frac{-a}{-b},\ \frac{2}{3}=\frac{-2}{-3}$$

and again $$\frac{-a\times -1}{b\times -1}=\frac{a}{-b}$$

That is, $$\frac{-a}{b}=\frac{a}{-b}$$

Hence one can say the signs of both terms of a fraction can be changed without altering the value of the fraction. Furthermore, we know by the law of signs for division that the quotient of a negative number and a positive number is a negative number:

$$\frac{-a}{b}=-\frac{a}{b}$$

and also $$\frac{a}{-b}=-\frac{a}{b}$$

and we may then write the very interesting identities

$$\frac{a}{-b}=\frac{-a}{b}=-\frac{a}{b}=-\frac{-a}{-b}$$

The significance of these identities will be seen more clearly if the + signs are introduced where they might properly be placed:

$$+\frac{+a}{-b}=+\frac{-a}{+b}=-\frac{+a}{+b}=-\frac{-a}{-b}$$

Speaking of the three signs, the sign of the numerator, the sign of the denominator, the sign of the fraction, we may say that if any two of them are changed the value of the fraction is unchanged.

What would be true if only one of these three signs were changed?

These changes of sign in a fraction are very convenient in altering the forms of fractions so that they may be more readily combined.

$$\frac{1-x}{x^2-1}$$

can be put in the more convenient form

$$-\frac{x-1}{x^2-1}=-\frac{1}{x+1}$$

The fractions $$\frac{1}{x-1}-\frac{4}{1-x}$$

can be more readily combined if we change the form to

$$\frac{1}{x-1}+\frac{4}{x-1}$$

The usefulness of these changes in sign, of course, depends upon your ability to recognize instances where such a change can be applied.

EXERCISES

1. $\dfrac{3x}{x-2}-\dfrac{2x}{2-x}$

2. $\dfrac{2}{x^2-1}-\dfrac{1}{1-x}$

3. $\dfrac{2}{1-x^2}-\dfrac{1}{1-x}-\dfrac{1}{x-1}$

4. $\dfrac{a}{a-b}+\dfrac{a+1}{b-a}-\dfrac{1}{a-b}$

5. $\dfrac{1}{a-1}-\dfrac{4}{1-a}-\dfrac{6}{1+a}$

6. $\dfrac{x+y}{x-y}+\dfrac{x}{y-x}$

7. $\dfrac{a}{a^2-3a-4}+\dfrac{1}{4-a}$

8. $\dfrac{a}{a-b}+\dfrac{a^2}{ba-a^2}$

165. Complex fractions. When the division of fractions or expressions containing fractions is written in the fractional form, the whole expression is called a **complex fraction.** Several of the examples of Art. **157** can be written in this form.

$\left(\dfrac{a^2}{b^2}+1\right)\div\left(\dfrac{a^2}{b^2}-1\right)$ may be written in one form $\dfrac{\dfrac{a^2}{b^2}+1}{\dfrac{a^2}{b^2}-1}$

This latter form is a complex fraction.

Complex fractions can often be reduced to the form of simple fractions most readily if we make use of the fundamental principle of the fraction that both terms of a fraction may be multiplied by the same number without any change in the value of the fraction.

In the case given above multiply both terms by b^2:

$$(1)\qquad \frac{\frac{a^2}{b^2}+1}{\frac{a^2}{b^2}-1}=\frac{\left(\frac{a^2}{b^2}+1\right)b^2}{\left(\frac{a^2}{b^2}-1\right)b^2}=\frac{a^2+b^2}{a^2-b^2}$$

So also,

$$(2)\qquad \frac{2+\frac{1}{a}}{\frac{1}{b}-1}=\frac{ab\left(2+\frac{1}{a}\right)}{ab\left(\frac{1}{b}-1\right)}=\frac{2ab+b}{a-ab}$$

and

$$(3)\qquad \frac{\frac{1}{2}+\frac{1}{3}}{2-\frac{1}{6}}=\frac{\left(\frac{1}{2}+\frac{1}{3}\right)6}{\left(2-\frac{1}{6}\right)6}=\frac{3+2}{12-1}=\frac{5}{11}$$

The number used as a multiplier is the L.C.D. of all fractions that appear in the terms of the complex fraction.

EXERCISES

Reduce to simple fractions:

1. $\dfrac{1+\frac{1}{x}}{1-\frac{1}{x}}$

2. $\dfrac{\frac{1}{a}-3}{3+\frac{2}{a}}$

3. $\dfrac{\frac{a+2}{a}+\frac{6}{a-3}}{\frac{a^2-a+2}{a-3}+1}$

4. $\dfrac{\frac{a}{b}-\frac{b}{a}}{\frac{a}{b}+\frac{b}{a}}$

5. $\dfrac{1-\frac{x-1}{2}-x}{1-\frac{1-x}{2}-x}$

6. $\dfrac{\frac{1}{2}-\frac{1}{3}}{\frac{1}{2}+\frac{1}{3}}$

7. $\dfrac{\dfrac{1}{5}+\dfrac{1}{3}}{\dfrac{3}{5}-\dfrac{2}{3}}$

8. Evaluate $\dfrac{a-b}{1+ab}$ for $a=\dfrac{2}{3}$, $b=\dfrac{5}{6}$

9. Evaluate $\dfrac{V}{\dfrac{1}{A}+\dfrac{1}{B}+\dfrac{1}{C}}$ for $V=231$, $A=2$, $B=3$, $C=4$.

10. $\dfrac{\dfrac{1}{1-x}+1}{\dfrac{1}{1-x}-x}$

11. $\dfrac{\dfrac{1-a}{1+a}-1}{\dfrac{1-a}{1+a}+1}$

12. $\dfrac{\dfrac{1-a}{1+a}-1}{\dfrac{1+a}{1-a}-1}$

13. $\dfrac{1}{1+\dfrac{q+1}{n-q}}$

14. $\dfrac{1}{3(x-1)}+\dfrac{-\frac{1}{3}x+\frac{1}{3}}{x^2+x+1}$

15. Evaluate $\dfrac{R^2+2Rr+r^2}{R^2r^2}$ for $R=\frac{1}{3}$, $r=\frac{1}{5}$

16. Evaluate $\dfrac{x^2+7}{x^2-5x+6}$ for $x=\frac{1}{3}$

17. Evaluate $\dfrac{1}{b-a}+\dfrac{1}{b-c}$ for $b=\dfrac{2ac}{a+c}$

18. Evaluate $\dfrac{1}{x+6}+\dfrac{1}{x+1}+\dfrac{1}{2x}-\dfrac{1}{x}$ for $x=\frac{2}{3}$

19. Evaluate $i=\dfrac{E}{R+\dfrac{r}{2}}$ for $E=30$, $R=7$, $r=3$

20. Evaluate $P=\dfrac{1+\dfrac{v}{V}}{1+\dfrac{v}{2V}}$ for $v=3$, $V=5$

21. Evaluate $\dfrac{n-m}{1-nm}$ for $n=\frac{1}{2}$, $m=\frac{1}{3}$

22. Find value of $\dfrac{2-x}{x+1}$ when $x=\dfrac{2-a}{a+1}$

23. Find value of $\dfrac{x-3}{x-5}-\dfrac{x-2}{x}$ when $x=\dfrac{5}{2}$

24. Find value of $\dfrac{x+a}{y}$ when $x=\dfrac{abc+ac}{b-c}$, $y=\dfrac{ac+a}{b-c}$

166. Graphs. The graphs of fractions have interesting peculiarities. For example, consider the fraction $\frac{12}{x}$.

Make a table of the values of $\frac{12}{x}$ from $x = -12$ to $x = 12$.

x	$\frac{12}{x}$	x	$\frac{12}{x}$	x	$\frac{12}{x}$	x	$\frac{12}{x}$
12	1.0	6	2.0	−1	−12	−7	−1.7
11	1.1	5	2.4	−2	−6	−8	−1.5
10	1.2	4	3.0	−3	−4	−9	−1.3
9	1.3	3	4.0	−4	−3	−10	−1.2
8	1.5	2	6.0	−5	−2.4	−11	−1.1
7	1.7	1	12.0	−6	−2.0	−12	−1.0

As we cannot divide by 0, $\frac{12}{x}$ has no value when $x = 0$.

In the figure the graph is drawn for the points on the right of the vertical axis. The other points should be joined in a similar way.

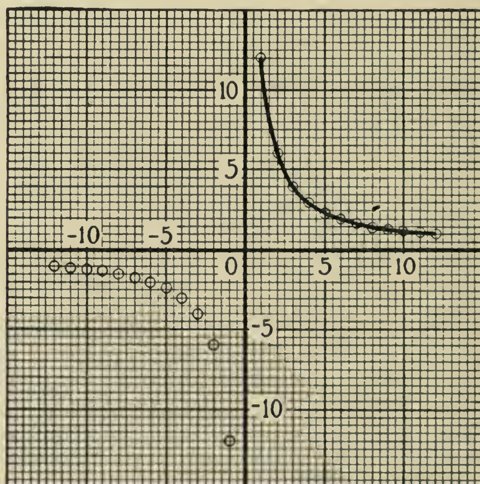

FIG. 52

Starting with $x = 12$, what is true of the value of $\frac{12}{x}$ as x gets smaller? What about $\frac{12}{x}$ when x gets very small?

EXERCISES

1. Plot $\frac{12}{x^2}$ in a similar way.

In drawing the graphs for the following fractions, draw smooth curves between the points plotted. Use integral values for x, but if in some places the points do not come near enough together use a fractional value of x in between. For the most part let 1 inch on the axis represent 1.

2. x^2+4 3. $\frac{10}{x^2+4}$ 4. $\frac{10x}{x^2+4}$

5. $\frac{x^2+1}{x}$ 6. $\frac{10}{x^2-4}$ 7. $\frac{x^2-1}{x}$

It is important to notice that the graphs of fractions are very different from the graphs of integral expressions. We have seen that the graphs of expressions of the first degree are all straight lines. The graphs of expressions of the second degree are all curves, but are all of the same kind. The graphs of fractions include a great variety of curves, some of which are shown in the exercises given here.

RATIO

167. Definitions. The quotient of two numbers or of two algebraic expressions is often called the **ratio** of the two numbers or expressions. (See Art. **9.**)

The quotient $\frac{a}{b}$ is called the ratio of a to b and is sometimes written in the form $a:b$, though generally the fractional notation is to be preferred. The numerator a is often called the **antecedent,** the denominator the **consequent.** The reason for these names is evident from the old notation $a:b$, the one that goes before and the one that follows.

As a ratio is simply a quotient, it has all the properties of a quotient and is to be treated as such. (See Art. **9.**) The term ratio is used so frequently in mathematical problems that you should become thoroughly acquainted with its meaning.

EXERCISES

1. Express the ratio of 4 to 6.

2. Express the ratio of 3 inches to 2 feet.

3. What is the ratio of the length to the width of a rectangle of dimensions 3 feet by 7 feet?

4. Is the ratio of 14 feet to 2 feet, 7 feet?

5. Express the ratio of 3 to 12 in the form of a decimal?

6. Find the ratio of 2 to 7 correct to 3 decimal places.

7. Measure the length of a piece of paper in both English and metric standards, and find the number of centimeters in 1 inch. What is the ratio of an inch to a centimeter?

8. What is the ratio of a mile to a kilometer? Of a kilometer to a mile? A kilometer is 1000 meters. A meter is 39.37 in.

9. In 1900 the population of the United States was 76,304,799, while in 1890 it was 62,622,250. Find the ratio of the population in 1900 to that in 1890 correct to .01.

10. The density of population is the ratio of the population to the area; that is, it is the number of people to the square mile. Find the density of population correct to one decimal place for the following countries, and show the results graphically by means of a series of straight lines of the proper length:

Country	Area	Population
United States....	3,574,658	91,972,266
Great Britain....	121,633	45,516,259
France	207,054	39,602,258
India	1,802,629	315,156,396
China	3,913,560	320,650,000

11. In a lot of 7 dozen eggs the ratio of good to bad is 2 to 3. How many good eggs are there in the lot?

12. Find two numbers in the ratio of 7 to 8 whose sum is 90.

13. Which is the greater ratio, $\frac{2}{3}$ or $\frac{3}{4}$?

14. What is the ratio of $\frac{a}{b}$ to $\frac{c}{d}$?

15. What is the ratio of $\frac{1}{x}$ to $\frac{1}{y}$?

16. Find the ratio of br^3 to ar^2.

17. What is the ratio of $\frac{a}{x-a}$ to $\frac{3a}{(x-a)^2}$?

18. Find the ratio of $\frac{x^2}{n^2-1}$ to $\frac{x^3}{(n-1)^2}$.

19. A rectangle is 3 by 5; squares are constructed on one end and on one side. What is the ratio of the areas of the squares?

20. Find the ratio of $\frac{n(n-1)}{2}$ to $\frac{n(n-1)(n-2)}{2 \cdot 3}$.

21. If $3x-4y=5x+6y$, find the ratio of x to y.

22. In what ratio should nuts at 13 cents a pound and 27 cents a pound be mixed to make a mixture worth 20 cents a pound?

23. In what ratio should two kinds of coffee worth 20 cents and 30 cents a pound respectively be mixed to make a blend worth 26 cents a pound?

VARIATION

168. Definitions. What is the circumference of a circle if its diameter is 2 feet? 3 feet? 4 feet? n feet? What is the ratio of the circumference to the diameter in each case? If the diameter is doubled, what is the effect on the ratio of circumference to diameter? If the diameter is halved, what is the effect? What effect will any change in the diameter have on the ratio? If the diameter is changed, what will be the effect on the circumference?

Any quantity that may take on various values is called a **variable** quantity. The diameter of a circle, the speed of an automobile, the distance a train travels, the height of a boy, the price of coal, are illustrations of **variable** quantities.

The number of cents in a dollar, the distance between two mile posts on a railroad, the number of days in January, the number 9 or 6, are quantities which do not change. Such numbers are called **constants.**

When one number changes as another number changes so that the ratio of two variable quantities is constant, that is, is always the same number, we often say that one **varies as** the other, or that the one **is proportional to** the other.

We write the statement x varies as y in the form

$$\frac{x}{y}=c$$

where c is the unknown constant. The more common way of writing the statement is without the use of fractions:

$$x=cy$$

which indicates another way of wording the statement. If one quantity varies as another, it is some constant number times the other. We say the circumference of a circle varies as its diameter. In this case we happen to know the constant and write

$$C=\pi D$$

PROBLEMS

1. The price paid for eggs at 35 cents a dozen varies as the number of dozen purchased. What is the constant in this case? State the relation in symbols, using c for total cost and d for number of dozen.

2. A train traveling 50 miles an hour leaves Chicago for New York. Using d for distance from Chicago, and t for number of hours, state the relation between d, t, and 50 in terms of variation and write the statement in symbols. What is the constant ratio?

3. The circumference of a circle varies as its radius. What is the constant in this case?

4. The diameter of a circle varies as the circumference. What is the constant in this case?

5. The amount of money I receive for 12 dozen eggs will vary as the price per dozen. State in symbols.

6. The area of a circle is given by the formula $A=\pi r^2$. The area varies as what?

7. The volume of a rectangular box varies as the product of its three dimensions. State in algebraic terms and state what value the constant has in this case.

8. The volume of a cube varies as what? What is the constant?

CHAPTER X

Square Roots and Quadratic Equations

169. Square root. One of the two equal factors of a number is called the **square root** of the number.

$$3 \cdot 3 = 9$$

3 is the square root of 9,

but $$-3 \cdot -3 = 9$$

Hence -3 is also a square root of 9.

9 has two square roots, $+3$ and -3, which are numerically equal, but opposite in sign.

We use the symbol $\sqrt{9}$ to represent the positive square root and $-\sqrt{9}$ to represent the negative square root:

$$\sqrt{9} \text{ stands for } +3$$
$$-\sqrt{9} \text{ stands for } -3$$
$$\pm\sqrt{9} \text{ stands for both } \pm 3$$

EXERCISES

1. $\sqrt{16}$
2. $-\sqrt{25}$
3. $+\sqrt{49}$
4. $2\sqrt{36}$
5. $\sqrt{a^2}$
6. $-\sqrt{x^2}$
7. $\pm\sqrt{x^2}$
8. $+\sqrt{4x^2}$
9. $-\sqrt{9x^2}$
10. $\pm\sqrt{16a^2}$
11. $\sqrt{121}$
12. $-\sqrt{100}$
13. $\pm\sqrt{169y^2}$
14. $\sqrt{144a^2}$
15. $\sqrt{169}$

The symbol $\sqrt{\ }$, called the **square root** sign, or the **radical** sign, has been used since the early part of the sixteenth century, when it was introduced by Riese and Rudolff. Before that time a number of different symbols were used to denote a square root.

170. Finding the square root of a number. The square root of a number may be found in several ways. Two methods will be suggested here:

(*a*) The division method used in arithmetic (see Art. **220**).

(*b*) The use of tables and a trial method.

The following table of squares can be easily calculated:

No.	Square	No.	Square	No.	Square	No.	Square
1	1	9	81	16	?	24	?
2	4	10	100	17	?	25	?
3	9	11	121	18	?	26	?
4	16	12	144	19	?	27	?
5	25	13	169	10	?	28	?
6	36	14	196	21	?	29	?
7	49	15	225	22	?	30	?
8	64			23	?		

Show how to find from the table the square roots of 81, 169, 196, 484, 625, 729.

171. Square root by trial. It is a simple matter to find the square root of any number that appears as a square in the table. But how shall we find the square root of a number that is not found as a square in the table?

How shall we find $\sqrt{6}$?

Recalling the fact that a square root of a number is one of the two equal factors of the number, we are simply to find a number which multiplied by itself will equal 6. A glance at the table will show that since 6 is between 4 and 9 the required square root will be between 2 and 3, for

$$2 \cdot 2 = 4 \text{ and } 3 \cdot 3 = 9$$

We may take for trial any number between 2 and 3, but since 6 is about midway between 4 and 9 it might be well to try 2.5.

Now $\qquad 2.5 \times 2.5 = 6.25$

As this product is more than 6 2.5 is a little too large; try 2.4.

Now $\qquad 2.4 \times 2.4 = 5.76$

As 5.76 is less than 6, 2.4 is a little too small.

The factor or square root sought must then lie between 2.4 and 2.5. This shows that its first two digits are 2.4. We may get a little nearer the required number by another similar trial. Try 2.45: $2.45 \times 2.45 = 6.0025$, which shows that 2.45 is a little too large. Try 2.44: $2.44 \times 2.44 = 5.9536$, which shows that 2.44 is too small. The root then lies between 2.44 and 2.45. The first three digits are certainly 2.44.

This process of trial can be carried out as far as one desires. It is a fact, the truth of which will be shown to you in a later course in algebra, that no matter how long the process is continued an end can never be reached of finding the square root of 6. The string of digits would go on forever. The exact $\sqrt{6}$ can never be found in the form of a decimal number. The number found at any stage of the calculation is called an **approximate value** of the root. For our purposes it is not necessary to find more than three digits of such roots.

If the process be applied to numbers that are squares, the work would come to an end, and an exact square root would be found. It will be seen upon reference to the table of squares on page 230 that most of the integers between 1 and 1000 are not squares. Only approximate square roots can be found for these integers.

Square roots of numbers that cannot be expressed exactly in decimal form are called **irrational roots.**

EXERCISES

1. Find the square roots of 1, 2, 3, 10, 21.

2. Form a table of the square roots of the integral numbers from 1 to 10 and preserve for future use.

172. Number of square roots. Every positive number has two square roots equal in numerical value, one positive and the other negative. What are the square roots of 25, 36, 49, 121, 5, 7?

173. Square root of a fraction. The square root of a fraction may be found thus:

Since $\frac{2}{3}\cdot\frac{2}{3}=\frac{4}{9}$, it follows that $\sqrt{\frac{4}{9}}=\frac{2}{3}$.

2 is the square root of 4. 3 is the square root of 9.

This suggests the rule

$$\sqrt{\frac{a}{b}}=\frac{\sqrt{a}}{\sqrt{b}}$$

Thus, $$\sqrt{\frac{5}{9}}=\frac{\sqrt{5}}{\sqrt{9}}=\frac{\sqrt{5}}{3}=\frac{2.23}{3}=.74+$$

$$\sqrt{\frac{4}{3}}=\frac{\sqrt{4}}{\sqrt{3}}=\frac{2}{1.73}=1.1+$$

Another way is to do the division first and then find the root.

Thus, $$\sqrt{\frac{3}{2}}=\sqrt{1.5}=1.2+$$

Other methods will be given in a later chapter. But what is given here is sufficient for any cases that may arise.

EXERCISE

Find the square roots of $\frac{36}{25}$, $\frac{121}{625}$, $\frac{81}{169}$, $\frac{49}{225}$, $\frac{5}{16}$, $\frac{7}{25}$, $\frac{3}{4}$, $\frac{23}{49}$, $\frac{4}{5}$, $\frac{16}{7}$, $\frac{49}{3}$, $\frac{25}{6}$, $\frac{2}{3}$, $\frac{5}{7}$, $\frac{3}{5}$, $\frac{2}{5}$.

174. Quadratic equations. An equation of the second degree in one unknown is called a quadratic equation. In previous chapters we have solved equations of the second degree by factoring. But there are equations that cannot be factored by any of the methods you have studied so far. For example, you cannot at this time solve the equation

$$x^2-4x+2=0$$

by factoring. It can be done, but the method is rather complicated. It is better to use a simpler method if one can be found.

175. The square-root method of solving a quadratic. Consider the equation

$$x^2-9=0$$

or, in a form better suited for our purpose,

$$x^2=9$$

This equation asks the question: What is the number whose square is 9? The required number is the square root of 9, or, in symbols,

$$x=\pm\sqrt{9}$$

$$x=3\pm, \text{ that is, } +3 \text{ and } -3$$

We have taken the square root of both sides of the equation.

It is unnecessary to use the double sign on both sides and write $\pm x=\pm 3$

for this would mean that

(1)	$+x=+3$	(3)	$-x=+3$
(2)	$+x=-3$	(4)	$-x=-3$

in which the third is the same as the second and the fourth is the same as the first. Why?

The work given above discloses another operation that can be applied to an equation; namely, **the square root of each side of an equation may be taken so long as both roots are used.**

176. Application of the square-root method. Equations of the type

$$x^2-49=0$$

may be solved equally well by the factoring or by the square-root method. The square-root method leads at once to the solution of an equation like $x^2-5=0$, which is not factorable in the sense in which you have understood factoring:

$x^2-49=0$	$x^2-5=0$
$x^2=49$	$x^2=5$
$x=\pm\sqrt{49}$	$x=\pm\sqrt{5}$
$x=7,\ x=-7$	$x=2.23,\ x=-2.23$

EXERCISES

Solve and check, using the square-root method:

1. $y^2=81$
2. $x^2=121$
3. $x^2-36=0$
4. $4x^2=64$
5. $2x^2-50=0$
6. $144-t^2=0$
7. $0=t^2-100$
8. $x^2=7$
9. $y^2-3=0$
10. $4x^2=5$
11. $9a^2-49=0$
12. $(x+1)^2=25$
13. $(x-2)^2=9$
14. $(x+5)^2=1$
15. $(n-1)^2=16$

177. Standard form. The standard form of the quadratic equation is

$$ax^2+bx+c=0$$

Write equations that come under this form by writing in values for a, b, and c. Quadratic equations can be reduced to this form by collecting the terms of the same degree and arranging them in the order of the powers. It is generally desirable to put the equation to be solved in the standard form. In some cases, however, this would be foolish. Would it be at all desirable to put the following into the standard form in order to solve them? Why?

$$(a-3)^2=\frac{1}{9}, \quad n(n+2)=0$$

EXERCISES

Put the following equations in standard form:

1. $2x^2-3x+2=5x-3$
2. $3-2x=5x-2+3$
3. $5(n-2)=3n(n-1)$
4. $7x-2+3x(x-2)=4x^2-3$
5. $3x^2-5=2x(3-x)+2x$
6. $7(n-2)^2-3(n+1)=7n+n^2$
7. $(n+2)(n-3)=(n-1)(n+3)$
8. $x(x-1)-4(x-3)=2x(x-2)$

178. Square-root method, completing the square. The same method may be applied to an equation in which the first-degree term is present.

For example, $x^2-6x-16=0$

Arrange the equation so that the terms containing the unknown shall be on one side and the known terms on the other:

$$x^2-6x=16$$

Make the side containing the unknown a trinomial square by adding the proper term. (Review Art. **118.**)

$$x^2-6x+?=16+? \tag{1}$$

In this case add 9, $$9=\left(\frac{6}{2}\right)^2 \tag{2}$$

$$x^2-6x+9=16+9 \tag{3}$$
$$=25$$

Show the square in another form·

$$(x-3)^2=25 \tag{4}$$
$$x-3=\pm 5 \tag{5}$$
$$x=\pm 5+3 \tag{6}$$
$$x=8 \text{ and } -2 \tag{7}$$

It is necessary to add the two numbers on the right-hand side before taking the square root.

This method is called the **square-root method of completing the square.**

If the coefficient of the term of second degree is some other number than one, the terms of the equation should be divided by that coefficient. This makes it easier to see what must be added to complete the square.

$$3x^2-2x=16 \tag{1}$$

Divide by 3, $$x^2-\frac{2}{3}x=\frac{16}{3} \tag{2}$$

Add $\left(\frac{1}{2}\cdot\frac{2}{3}\right)^2$, $$x^2-\frac{2}{3}x+\frac{1}{9}=\frac{16}{3}+\frac{1}{9} \tag{3}$$

Change form, $$\left(x-\frac{1}{3}\right)^2=\frac{49}{9} \tag{4}$$

Take square root, $$x-\frac{1}{3}=\pm\frac{7}{3} \tag{5}$$

Solve for x, $$x=+\frac{7}{3}+\frac{1}{3} \qquad x=-\frac{7}{3}+\frac{1}{3} \tag{6}$$

$$=\frac{8}{3} \qquad\qquad =-2 \tag{7}$$

RULE. **(1) Reduce equation to the standard form**

$$ax^2+bx=c$$

by collecting all terms containing the unknown on one side and the known terms on the other. And if necessary divide each side by the coefficient of the term of second degree.

(2) Complete the square of the side containing the unknowns by adding to both sides the square of half the coefficient of the term of first degree.

(3) Take the square root of both sides.

(4) Solve the two resulting linear equations for the unknown.

EXERCISES

Solve by completing the square:

1. $x^2-2x=48$
2. $y^2-6y-40=0$
3. $x^2+8x=9$
4. $x^2+10x-11=0$
5. $t^2+14t+24=0$
6. $x^2-3x=4$
7. $a^2+a-6=0$
8. $x^2-4x+3=0$
9. $5+x^2-6x=0$
10. $y^2+12y+27=0$
11. $x^2-5x+6=0$
12. $2x^2-7x=15$
13. $3y^2=14-19y$
14. $7x+5=6x^2$
15. $4x^2-2x=6-2x^2+3x$
16. $6x^2+6=13x$
17. $(5x+2)x=3(1-x^2)$
18. $5x^2-3x+10=9-10x^2-11x$

179. More difficult exercises. Let us try the square-root method on equations that cannot be factored.

$$x^2-4x+2=0$$

Change the form, $x^2-4x=-2$

Complete the square, $x^2-4x+4=2$

Put in square form, $(x-2)^2=2$

Take square root, $x-2=\sqrt{2}$, $x-2=-\sqrt{2}$

$x-2=+1.41$, $x-2=-1.41$

Solve for x, $x=3.41$ $x=.59$

Check for $x=3.41$:

$3.41^2-4(3.41)+2$	0
$11.6281-13.64+2$	0
$-.0119$	0

The two sides do not come out the same, but the error is very small and is due to the fact that 1.41 is only an approximate value for $\sqrt{2}$: consequently 3.41 is merely an approximate value for the number sought. Its exact value cannot be expressed in decimals.

Recalling Art. **15** on the use of approximate numbers, we might have done the checking thus:

	$3.41^2-4(3.41)+2$	0
	$11.6-13.6+2$	0
	0	0
and for .59,	$.59^2-4(.59)+2$	0
	$.35-2.36+2$	0
	$-.01$	0

Take another equation for illustration:

$$3x^2+5x=6$$

Divide both sides by 3, $\quad x^2+\frac{5}{3}x=2$

Complete square, $\quad x^2+\frac{5}{3}x+\left(\frac{5}{6}\right)^2=2+\left(\frac{5}{6}\right)^2$

Put in square form, $\quad \left(x+\frac{5}{6}\right)^2=\frac{97}{36}$

Take square root, $\quad x+\frac{5}{6}=+\sqrt{\frac{97}{36}}, \qquad x+\frac{5}{6}=-\sqrt{\frac{97}{36}}$

$$x+\frac{5}{6}=+\frac{\sqrt{97}}{6}, \qquad x+\frac{5}{6}=-\frac{\sqrt{97}}{6}$$

Solve for x, $\quad x=-\frac{5}{6}+\frac{\sqrt{97}}{6}, \qquad x=-\frac{5}{6}-\frac{\sqrt{97}}{6}$

$$=-\frac{5}{6}+\frac{9.8}{6}, \qquad =-\frac{5}{6}-\frac{9.8}{6}$$

$$=.8 \qquad\qquad =-2.4$$

EXERCISES

Solve by completing the square and check:

1. $x^2+4x=32$
2. $n^2-2n-1=0$
3. $x^2-4x=1$
4. $t^2+22-10t=0$
5. $2x^2-6x+3=0$
6. $3x^2+6x-4=0$
7. $36x^2+3=36x$
8. $2x^2-9x+9=0$
9. $9a^2-9a+2=0$
10. $9x^2-6x-80=0$
11. $(x+2)^2=4(x-1)^2$
12. $2t^2=7t+11$
13. $x^2-2x=a^2-1$
14. $x^2+14x+48=0$
15. $4x^2+16x+15=0$
16. $4x^2+2x-6=0$
17. $6x^2+5x-6=0$
18. $6x^2-5x-6=0$
19. $6x^2-29x+35=0$
20. $3n^2-10n-8=0$
21. $x^2-2x=2$
22. $2x^2+5x-1=0$
23. $6n^2-19n+10=0$
24. $5h^2+3=9h$

180. An important special case. Consider the quadratic equation

$$x^2-2x+3=0$$

Solve by completing the square:

$$x^2-2x=-3$$
$$x^2-2x+1=-2$$
$$(x-1)^2=-2$$

We are required to find the square root of -2. But as the product of two equal factors is never negative, but always positive, no such square root of -2 can be found. No real answer can be found for the equation. If an equation of this kind should arise from any problem, we should say that the problem is impossible. This can be handled by the invention of a new number. The method will not be considered in this book.

181. Summary of methods of solving a quadratic equation. The two methods used for solving quadratic equations are:

(*a*) Factoring.

(*b*) Square root or completing the square.

There is a third method which is not considered in this book. Each method has its advantages. In some cases it makes but little difference which method is used. In other cases one of the methods is greatly to be preferred to the other. In solving a quadratic you should select the method best adapted to the equation in hand.

Illustration. $3n^2-2n=0$, factoring, why?
$x^2=(a-1)^2$, square root, why?
$x^2-3x+2=0$, factoring, why?
$2x^2+8x-3=0$, completing square, why?

MISCELLANEOUS EXERCISES

Solve the following equations by the method best adapted to the equation in hand:

1. $3x^2-14x+8=0$
2. $8x^2-4=14x$
3. $3x^2-1=0$
4. $y^2+3y=0$
5. $2n^2=1-4n$
6. $24t=t^2$
7. $1+x^2-x=0$
8. $n^2+7n+3=0$
9. $(4x-1)x=8$
10. $6t^2+2t=5$
11. $a^2+22(a+5)=0$
12. $n^2+90=19n$
13. $21+R=2R^2$
14. $72=\frac{n}{2}[(48+(n-1)(-4)]$
15. $72(7-c^2)+16c^2=0$
16. $(4+\frac{K}{2})^2-(16+2K)=0$
17. $16(b-1)^2-80(b^2+6b+8)=0$
18. $21(1+6c-3c^2)+9(c+1)^2=0$
19. $11(11+s)(11-s)=792$
20. $5(n-1)(n+1)=3(3-n)(n+3)$
21. $\frac{x(x-1)}{4}=\frac{x(x+1)}{12}$
22. $(a-\frac{1}{3})^2=2(a-\frac{1}{3})$
23. $(a-\frac{1}{3})^2=\frac{1}{4}$
24. $(x-1)^2-4x^2=0$

182. Graphs. An interesting light is thrown upon the solution of quadratic equations if one considers the roots of an equation in connection with the graph of the left-hand side after the equation has been put in the standard form. Verify the results given below. State any conclusion you can derive from them concerning the roots and the graphs.

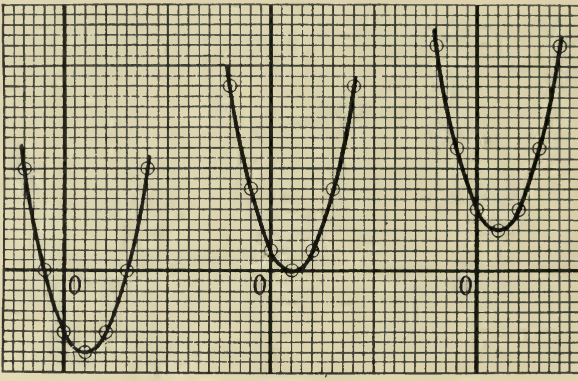

FIG. 53

$x^2-2x-3=0$	$x^2-2x+1=0$	$x^2-2x+3=0$
$(x-1)^2=4$	$(x-1)^2=0$	$(x-1)^2=-2$
$x=3, -1$	$x=1, 1$	

EXERCISES

Draw the graphs of the following expressions and determine from the graphs the roots of the corresponding equations when each is put equal to zero:

1. $x^2-2x-15$	2. x^2+2x-8	3. x^2-x-6
4. x^2+4x+3	5. x^2-9	6. x^2-4x+4
7. $x+3$	8. $(2x+6)^2$	9. $(x-4)^2$
10. $-x+4$	11. $3x-5$	12. x^2-8
13. x^2-4x+6	14. x^2+4x+4	15. x^2-7x+6

16. Show the roots of $x^2-3x+2=0$ by drawing the graph of x^2-3x+2.

183. Problems. Solve the following problems:

1. The sum of the squares of two consecutive integers is 265. What are the numbers?

2. The sum of the squares of two consecutive integers is 150. What are the numbers?

3. The sum of the squares of three consecutive integers is 50. What are the numbers?

4. A tinner wishes to make a square box 3 inches deep that will contain a cubic foot. How large a piece of tin is needed if he cuts a 3-inch square out of each corner?

5. How large a piece of tin will be needed if a square box is to be made that will contain one-half a cubic foot?

6. The tinner desires to make a box 6 inches longer than it is wide and 4 inches deep to contain 160 cubic inches after cutting out square corners. What will be the dimensions of the piece from which it is made?

7. Change the capacity of the box in the last exercise to 200 cubic inches.

8. It is desired to lay off a rectangular field that will contain 340 square rods if one side is 8 rods shorter than the other. Find the dimensions to be used.

9. What must be the dimensions of a rectangular field that is to contain 3 acres and have a 100-rod fence around it?

10. After laying a concrete drive I have enough concrete to lay 200 square feet of surface. How wide a walk can I lay about a lily pond 20 by 15 feet?

11. How wide a strip must a farmer plow around a field 80 by 60 rods to plow 15 acres?

12. A stockman stated that he bought a number of horses for $1200. Four died and he sold the remainder for $10 a head more than they cost, thereby making $200 on the transaction. How many did he buy?

The area of a circle (Fig. 54) is given by the formula,

$$A = \pi r^2$$

In the following problems use $\pi = \frac{22}{7}$. Remember that this gives but three significant figures.

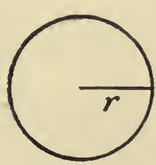

FIG. 54

13. Find area of a circle of radius 5 inches.

14. The area of a circle is 100. What is its diameter?

15. Two circles are drawn with the same center, one with radius 3 inches and one with radius 5 inches. What is the width of the ring between them? What is the area of the small circle? Of the larger? What is the area of the ring?

16. A button 2 inches in diameter is to be painted in two colors as indicated in Fig. 55. The area of the ring is to be the same as the area of the inside circle. The area of the inside circle will be what part of the area of the outside circle? Find the radius of the inside circle; also the width of the ring.

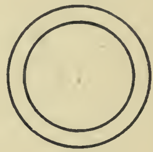

FIG. 55

17. How much must be added to the radius of a circle whose diameter is 4 inches to double its area?

18. What must be the diameter of a quart cup if its depth is to be 6 inches? 1 gallon = 231 cubic inches.

The volume of a cylinder of height h and radius r is $\pi r^2 h$.

19. A cylindrical box is to be made to fit between two shelves that are 8 inches apart. The box is to contain one cubic foot. What must be its diameter?

Right triangle problems:

The triangle in Fig. 56 has a right angle at C. It is called a right triangle and the side AB is called the hypotenuse. In the figure the sides are 3, 4, and 5. Squares are drawn on each side and each square is divided into unit squares. When the unit squares are counted it will be seen that

$$25 = 9 + 16$$
$$5^2 = 3^2 + 4^2$$

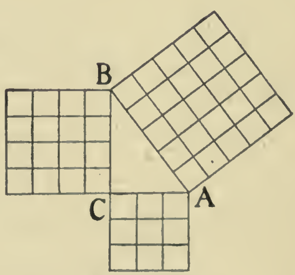

FIG. 56

That is, the square on the hypotenuse equals the sum of the squares on the two other sides.

It has been known for two thousand years or more that this fact is true for all right triangles.

If a and b are the sides of a triangle (Fig. 57) and c is the hypotenuse,

$$a^2+b^2=c^2$$

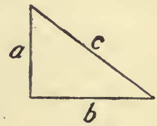

FIG. 57

20. Two sides of a right triangle are 5 and 12. What is the hypotenuse?

21. The hypotenuse of a right triangle is 25, one side is 7. The other is how much shorter than the hypotenuse?

22. One side of a right triangle is 9, the hypotenuse is 41. What is the other side?

23. Two sides of a rectangle are 1 and 2. What is the diagonal?

24. The hypotenuse of a right triangle is 10, one side is 5. What is the other side?

25. What is the diagonal of a rectangle if the sides are 60 inches and 11 inches?

26. If the diagonal of a rectangle is 26 inches and one side is 10 inches, find the other side.

27. What is the diagonal of a square if one side is S?

28. How much shorter is the diagonal path across a lot 100 by 175 than the sidewalk along its sides?

29. How large a square can you cut from a circular piece of tin of one foot diameter? (See Fig. 58.)

30. Find the ratio of the area of the square and the circle of the last problem, giving two decimal places.

31. The sum of the sides of a right triangle is 35 inches; the hypotenuse is 25 inches. Find the legs. Interpret the two answers.

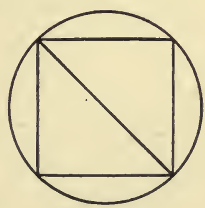

FIG. 58

32. One side of a right triangle is 4 feet longer than the other. Find the legs if the hypotenuse is 20 feet.

33. The difference between the sides of a right triangle is 5, the hypotenuse is 45. What are the sides?

34. Two sides of a rectangle are 3 and 4. If the shorter side remains unchanged, how much must be added to the longer side to increase the diagonal by 2?

35. Suppose the longer side of the rectangle in Ex. 34 were left unchanged. How much would be added to the shorter to increase the length of the diagonal by 2?

36. If the same amount were added to both sides of the rectangle in Ex. 34, how much should be added to increase the diagonal by 2?

37. The height of a certain flagstaff is unknown. But it is noticed that a flag rope fastened at the top is 4 feet longer than the pole and that when stretched tight the rope reaches the ground 20 feet from the base of the pole. How high is the pole?

38. A flagstaff, AB, 50 feet high, was broken off at a point C, the broken end rested on C and the top touched the ground at D, 30 feet from the base of the staff. Find the length of the part still standing.

CHAPTER XI

FRACTIONAL EQUATIONS IN ONE UNKNOWN

184. Definition. Equations frequently occur in which the unknown appears in the denominator of a fraction. Such equations are called **fractional** equations.

$$\frac{1}{x}=\frac{1}{2}+\frac{1}{3}, \quad \frac{5}{x^2+1}=\frac{3}{x}$$

are fractional equations.

185. Solution. A fractional equation is usually best solved by multiplying both sides by an expression that will remove all the denominators of the fractions. The least common denominator should be used for this purpose. Why?

This is called **clearing the equation of fractions.**

The resulting equation may be a linear equation, a quadratic equation, or an equation of still higher degree.

All the roots of the fractional equation will be roots of the new integral equation. But the reverse statement, that all the roots of the new equation are roots of the fractional equation, is not necessarily true. Some new roots might be introduced by the multiplication. In consequence of this fact it is necessary to test all roots found from the integral equation by substituting them in the original fractional equation.

Illustration 1:

Solve $\frac{3}{x}+5=\frac{1}{2}$

Multiply both sides by $2x$,

$$6+10x=x$$
$$9x=-6$$
$$x=-\frac{2}{3}$$

Check:

$\frac{3}{-\frac{2}{3}}+5$	$\frac{1}{2}$
$-\frac{9}{2}+5$	$\frac{1}{2}$
$\frac{1}{2}$	$\frac{1}{2}$

Illustration 2:

Solve $$\frac{4}{x-3}-\frac{x-1}{x+3}+\frac{x^2-1}{x^2-9}=0$$

Multiply both sides by x^2-9,

$$\begin{aligned}4(x+3)-(x-1)(x-3)+x^2-1&=0\\4x+12-(x^2-4x+3)+x^2-1&=0\\4x+12-x^2+4x-3+x^2-1&=0\\8x+8&=0\\x&=-1\end{aligned}$$

Check: $$\left.\begin{array}{r}\dfrac{4}{-1-3}-\dfrac{-1-1}{-1+3}+\dfrac{1-1}{1-9}\\-1+1+0\end{array}\right|\begin{array}{l}0\\0\end{array}$$

NOTE. Clearing of fractions applies to equations only and is not allowed in checking.

EXERCISE I

Apply to the following:

1. $\frac{2}{x}-5=\frac{1}{3}$
2. $\frac{1}{x}-\frac{1}{2}=\frac{2}{x}$
3. $7-\frac{1}{x}=4$
4. $\frac{4}{x}+5=\frac{6}{x}$
5. $\frac{2}{x}+6=1-x$
6. $\frac{5}{x}-x=\frac{1}{2}$

The essential difference in the character of a fractional and an integral equation can best be shown by the pictures or graphs of the left-hand number when all terms are put on that side. For instance, take the equations of Illustration 1 just given. (See Fig. 59.)

Put the fractional equation in the form $\frac{3}{x}+\frac{9}{2}=0$ (1)

The integral equation, $6+9x=0$ (2)

	−5	−4	−3	−2	−1	−.5	0	1	2	3	4	5
(1) $\frac{3}{x}+\frac{9}{2}$	3.9	3.8	3.5	3	1.5	−1.5		7.5	6	5.5	5.3	5.1
(2) $6+9x$	−39	−30	−21	−12	−3	1.5	6	15	24	33	42	51

The point where the graph cuts the horizontal axis indicates the root of the equation. The two graphs cut the

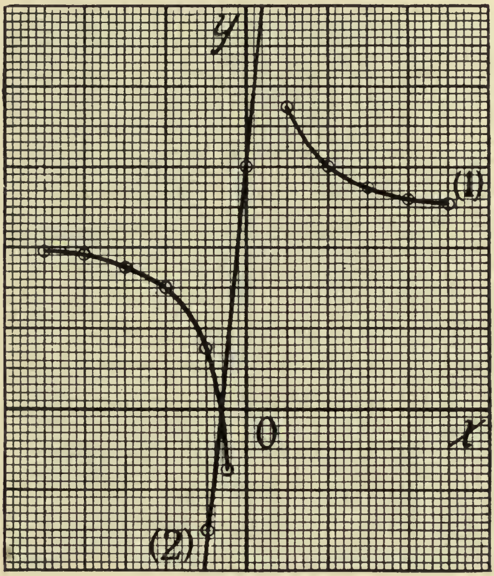

FIG. 59

x-axis at the same point, $-\frac{2}{3}$, which is the root of each equation.

Let the student draw in a similar way the graphs for Exercise 5, below.

EXERCISE II

Solve:

1. $x-\frac{2}{5}=\frac{1}{5}x+6$
2. $\frac{1}{4}+\frac{1}{6}=\frac{2}{x}$
3. $\frac{1}{40}+\frac{1}{d}=\frac{1}{10}$
4. $\frac{4}{m}+\frac{5}{m}+1=0$
5. $x+6=\frac{2}{x}$
6. $\frac{n+3}{4}=\frac{4}{n-3}$

7. $\frac{x}{5}-\frac{6}{x-1}=0$

8. $\frac{x^2}{x-1}=7$

9. $\frac{t}{2(1-t)}=\frac{1}{4}$

10. $\frac{x+1}{x-1}=\frac{x+7}{x+3}$

11. $\frac{x+2}{x-2}+\frac{x-2}{x+2}=\frac{40}{x^2-4}$

12. $\frac{6}{y+2}-\frac{y+1}{y-2}+\frac{y^2}{y^2-4}=0$

13. $\frac{x+3}{x-1}-\frac{x-1}{x+1}=\frac{7x-4}{x^2-1}$

14. $\frac{x-2}{x^2-4x}=\frac{4x-16}{x^2-3x}+\frac{1}{x}$

15. $\frac{5}{y}=3-\frac{2}{y+1}$

16. $\frac{1}{5}+\frac{x-5}{5x-20}=\frac{1}{3}$

17. $\frac{x}{x-6}=3+\frac{14}{x+6}$

18. $\frac{n-2}{n+3}=\frac{n+5}{n-4}+\frac{14-15n}{n^2-n-12}$

19. $\frac{n-1}{2n-6}-\frac{n-2}{3n-9}=\frac{n+1}{6}$

20. $\frac{x-5a}{5a}=\frac{1}{2}-\frac{5a}{2x}$

21. $\frac{x}{a}-1=\frac{6a}{x}$

22. $\frac{a-b}{x}=\frac{a}{b}$

23. $\frac{5b}{x}-\frac{b-5x}{b}=25$

24. $\frac{x-a}{ax}=\frac{1}{a}-\frac{ax-1}{x}$

25. $\frac{a}{bx}-a=\frac{b}{ax}-b$

26. $\frac{1}{b}+\frac{b}{ax}=\frac{x+2a}{ax}-\frac{a}{bx}$

27. $\frac{2a}{x}-\frac{ax+5a}{5x}+b=0$

28. $\frac{x-a}{b^2x}+\frac{x-b}{a^2x}=\frac{1}{ab}$

29. $\frac{m}{x+n}+\frac{n}{x+m}=\frac{2m}{x+m}$

30. $\frac{1}{a-b}+\frac{a-b}{x}=\frac{1}{a+b}+\frac{a+b}{x}$

31. $\frac{a}{bx-1}+\frac{b}{ax-1}=\frac{2b^2x}{(ax-1)(bx-1)}$

186. Transformation of formulas. In dealing with formulas that are solved for one letter it is often desirable to derive a formula solved for some other letter. When one is solving for any letter, all other letters are to be considered as known. Attention should be fixed on the letter for which the formula is to be solved, and the best methods of untangling it from the other numbers should be determined step by step.

Illustration 1:

Solve for n, $$k=\frac{n+i}{1+i}$$

Clear of fractions, $$k(1+i)=n+i$$
Subtract i from both sides, $k(1+i)-i=n$
It may be left in this form or expanded, as you wish,
$$n=k+ki-i$$

Illustration 2:

Solve the same formula for i.
Clear of fractions, $k(1+i)=n+i$
Expand, collect terms, and divide by the coefficient of i.
$$k+ki=n+i$$
$$ki-i=n-k$$
$$(k-1)i=n-k$$
$$i=\frac{n-k}{k-1}$$

You may test the correctness of your work by substituting your result in the original formula. Show how each line is obtained from the one above it.

$$k=\frac{n+\frac{n-k}{k-1}}{1+\frac{n-k}{k-1}}$$
$$=\frac{(k-1)n+n-k}{k-1+n-k}$$
$$k=\frac{kn-n+n-k}{k-1+n-k}$$
$$k=\frac{kn-k}{n-1}$$
$$k=\frac{k(n-1)}{n-1}$$
$$k=k$$

EXERCISES

Transform the formulas as indicated:

1. $m=-\frac{1}{n}$. Solve for n.
2. $\frac{y-b}{x}=m$. Solve for y.

3. $\frac{x}{a}+\frac{y}{b}=1$. Solve for y.

4. $1=\frac{m-3}{1+3m}$. Solve for m.

5. $1=\frac{m-n}{1+mn}$. Solve for m.

6. $r=\frac{2}{1+t}$. Solve for t.

7. $r=\frac{3e}{1-e}$. Solve for e.

8. $r=\frac{ep}{1+e}$. Solve for p.

9. $S=\frac{rt-a}{r-1}$. Solve for t, r.

10. $F=\frac{A(R-1)}{R+1}$. Solve for R.

11. $n=\frac{1}{6}\left(\frac{a-h}{h}\right)$. Solve for h.

12. $R=\frac{rr_1}{r+r_1}$. Solve for r, r_1.

13. $\frac{1}{M}-\frac{1}{E}=\frac{1}{S}$. Solve for each letter and find M when $E=365.25$, $S=29.53$.

14. $a=p\,\frac{n+1}{n-1}$. Solve for n.

15. $C=\frac{E}{R+\frac{r}{N}}$. Solve for E, R, r, N.

16. $g-\frac{T}{M}=\frac{T}{m}-g$. Solve for each letter.

17. $Q=\frac{W+R}{W+R+w}$. Solve for R.

18. $S=\frac{P}{A}+\frac{PLC}{I}$. Solve for P, for C.

19. $pf=\frac{St}{r+t}$. Solve for t.

20. $y=\frac{1}{p}\left(\frac{C}{2b}+a\right)$. Solve for each letter.

21. $I=\frac{En}{\frac{rn}{m}+R}$. Solve for R, for n.

187. Problems. Several types of problems lead naturally to fractional equations. A few illustrations are given in this article.

1. If 36 is divided by a certain number, the quotient is 5 more than the number. What is the number?

2. The quotient of 8 divided by a certain number exceeds 9 times the number by 6. What is the number?

3. Divide 60 into 2 parts such that their quotient is equal to $\frac{2}{3}$.

4. Divide 10 into 2 parts so that their ratio is twice one of the parts.

5. What number must be added to both numerator and denominator of $\frac{3}{11}$ in order that the resulting fraction shall equal $\frac{5}{6}$?

6. What number added to both numerator and denominator of the fraction $\frac{3}{4}$ will double the value of the fraction? What number will halve the value of the fraction?

7. What number must be added to the denominator of $\frac{7}{15}$ in order that the resulting fraction shall be equal to $\frac{1}{3}$?

8. What number must be added to the denominator of $\frac{12}{25}$ in order to get a fraction equal to $\frac{6}{7}$?

9. What number added to both numerator and denominator of the fraction $\frac{3}{5}$ will double the value of the fraction?

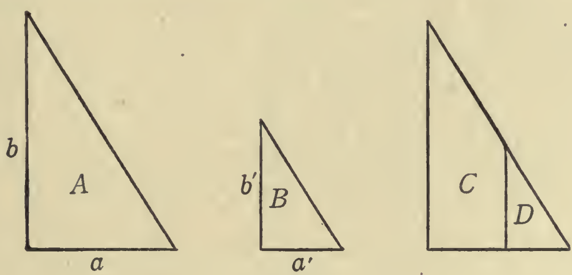

FIG. 60

Triangles of the same shape are called **similar triangles.**

The right triangles A and B are similar, as are also C and D (Fig. 60). It is shown in geometry that the ratios of corresponding sides of two similar triangles are equal, that is,

$$\frac{a}{b}=\frac{a'}{b'}$$

Verify this relation by measuring the sides in Fig. 60.

10. In two similar triangles if $a=7$, $b=12$, and $b'=36$, what will a' equal?

If $a=5$, $a'=12$, and $b'=40$, $b=?$
If $b=7$, $a=15$, and $a'=9$, $b'=?$
If $a=25$, $a'=40$, and $b'=9$, $b=?$

This relation between the sides of similar triangles may be used to estimate the height of a building or tree.

11. Suppose a spire casts a shadow of 60 feet while at the same time the shadow of a 6-foot post is 4 feet long. Find the height of the spire.

We have $$\frac{h}{60}=\frac{6}{4}$$

whence $$h=\frac{6\cdot 60}{4}=90$$

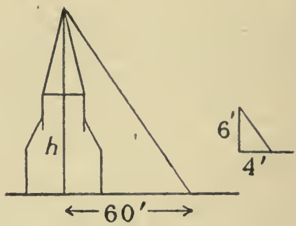

FIG. 61

12. It is desired to find the height of a flag pole on a cloudy day.

Let AB be the flag pole; CD be a yardstick or any rod; and E the eye of a boy lying on the ground so that he can just see the top of the flag pole over the end of the pole CD (Fig. 62). What lines must be measured to find the height of the flag pole? Using x for the height of the flag pole and a, b, c for the measured lengths, state a formula for x.

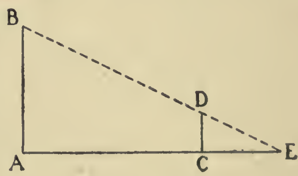

FIG. 62

Or, if the measuring rod is long enough, the pole and rod can be sighted in, while the boy is standing as in Fig. 63. Explain how the height of AB will be found then.

Assign values to these measured lines and solve for the other.

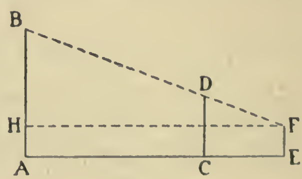

FIG. 63

What is the height of a tree when it was measured in this last way by a boy whose eye was 5 feet from the ground, an 8-foot pole being placed 6 feet from the boy? The distance from the boy to the tree was found to be 320 feet.

13. How far away is a tower which is known to be 150 feet high, if its top can just be seen over an 8-foot pole placed 20 feet from a man with his eye at the ground.

14. In Fig. 64 it is desired to find the distance AB. It is impossible to measure the distance directly because of the river between. AC, AE, and CD were measured. $AC=200$, $AE=100$, $CD=125$. How wide is the river?

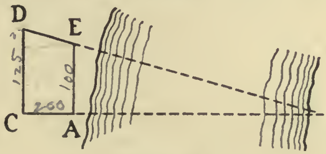

FIG. 64

The following are work problems:

15. A pipe delivers water to a 150-gallon tank at the rate of 20 gallons a minute. How long will it take to fill the empty tank? If a pipe can fill this tank in 10 minutes, how many gallons does it supply in one minute? What part of the tank will it fill in one minute? In 5 minutes?

16. A pipe can fill a certain tank in 12 minutes. What part of the tank can it fill in one minute? In 7 minutes?

17. Smith can do a piece of work in 3 days. What part of the work can he do in one day? In 2 days? In 3 days? Jones can do a piece of work in x days. How much of the work can he do in one day? In 4 days? In n days?

18. If one pipe will fill a cistern in 5 hours, what part of the cistern can it fill in 1 hour? If a second pipe can fill it in 9 hours, what part of the cistern can be filled in 1 hour if both pipes are open at the same time?

19. A tank containing 240 gallons is supplied by two pipes, one delivering 2 gallons a minute and the other 3 gallons a minute. How long will it take to fill the tank when both pipes are turned on?

The equality to be used here is:

$$\left.\begin{array}{c}\text{Total number of}\\ \text{gallons delivered}\\ \text{by first pipe}\end{array}\right\} + \left\{\begin{array}{c}\text{total number de-}\\ \text{livered by second}\\ \text{pipe}\end{array}\right\} = \left\{\begin{array}{c}\text{total num-}\\ \text{ber tank}\\ \text{can contain}\end{array}\right.$$

If t = the unknown number of minutes, the equation is

$$2t+3t=240$$

Therefore $$t=48$$

20. Suppose a tank of 240 gallons can be filled by one of two pipes in 20 minutes and by the other in 30 minutes. How long will it take to fill the tank when both pipes are turned on?

The same equality that was used in Exercise 5 can be used here:

$$\left.\begin{array}{c}\text{Total number of}\\ \text{gallons delivered}\\ \text{by first pipe}\end{array}\right\} + \left\{\begin{array}{c}\text{total number de-}\\ \text{livered by second}\\ \text{pipe}\end{array}\right\} = \left\{\begin{array}{c}\text{total num-}\\ \text{ber tank will}\\ \text{hold}\end{array}\right.$$

Let t = the unknown number of minutes to fill the tank

$\frac{240}{20}$ = number of gallons first pipe delivers in one minute

$\frac{240}{30}$ = number of gallons second pipe delivers in one minute

Hence the equation is $\frac{240}{20}t+\frac{240}{30}t=240$

Such work problems are often solved by the use of another equality, namely:

$$\left.\begin{array}{c}\text{Part of tank filled}\\ \text{by first pipe in}\\ \text{one minute}\end{array}\right\} + \left\{\begin{array}{c}\text{part of tank filled}\\ \text{by second pipe in}\\ \text{one minute}\end{array}\right\} = \left\{\begin{array}{c}\text{part of tank filled in}\\ \text{one minute by both}\\ \text{working together}\end{array}\right.$$

Here

$\frac{1}{20}$ = the part of tank filled by first pipe in one minute

$\frac{1}{30}$ = the part of tank filled by second pipe in one minute

$\frac{1}{t}$ = the part of tank filled by both pipes in one minute

Hence $$\frac{1}{20}+\frac{1}{30}=\frac{1}{t}$$

Solve for t.

It will be noticed that this solution takes no account of the capacity of the tank. In the first solution the capacity of the tank was used, but had no effect in determining the unknown time, for the 240 was divided out of the equation.

21. One pipe can fill a cistern in 42 hours; another can fill the cistern in 14 hours. How long will it take to fill the cistern when both pipes are turned on?

22. One pipe can fill a cistern in 2 hours; a pump can empty it in 5 hours. How long will it take to fill the cistern if both supply pipe and pump are working at the same time?

23. A tank of a certain capacity can be filled from one pipe in 30 minutes, from another in 60 minutes, and can be emptied by a pump in 40 minutes. How long will it take to fill the cistern when all three are working?

24. A cistern can be filled in 12 minutes by two pipes. One of them will fill it in 20 minutes. In what time would the other fill it?

25. A can do a piece of work in m hours and B in n hours. How long will it take them to complete the work if they work at the same time?

26. What will be the answer to Exercise 25 if (*a*) $m=3$, $n=15$; (*b*) $m=15$, $n=5$; (*c*) $m=7$, $n=5$.

27. One man takes 4 hours longer than another to saw a cord of wood and both working together can saw it in 3 hours. How long does it take each to saw a cord?

28. A tank can be filled in 3 hours by two faucets flowing at the same time. It will take one faucet 8 hours longer to fill it than it will the other. In what time can the tank be filled by each?

29. A certain number of letters are to be addressed. A can address them in 10 hours. B can do the work in 15 hours. How long will it take them to address the letters if they work at the same time?

30. A can do a piece of work in 6 days. It takes B three times as long to do the same work. How long will it take them to do the work when they work at the same time?

31. When working at the same time A and B can do a certain piece of work in 2 days. A when working alone can do it in 3 days. How long would it take B to do the work?

32. A and B can do a piece of work in 2 days. When working alone it will take A 3 days longer than B to do the work. How long will it take each?

33. A tank can be filled by one pipe in 6 hours, by another in 9 hours, and by a third in 18 hours. How long will it take to fill the tank when all are running?

34. A tank can be filled by three pipes in 6 hours, by one of them in 12 hours, by another in 36 hours. How long would it take the third one to fill the tank?

The following are motion problems:

35. A man who can row in still water at the rate of r miles an hour is rowing in a river whose current is 2 miles an hour. How fast can he go upstream? Downstream? How long would it take him to row 5 miles upstream? 6 miles downstream?

36. A boatman rowed 7 miles upstream and back in 8 hours. If the speed of the current was 3 miles an hour, what was his rowing rate in still water?

37. How far can a person who has 8 hours to spare ride in a wagon at 6 miles an hour so that he can return walking at the rate of 4 miles an hour and arrive home on time?

38. A certain river boat that can travel at the rate of 9 miles an hour takes 5 hours and 30 minutes to make a trip of 20 miles up the river and back, allowing a stop of 30 minutes. What is the average rate of the current?

39. Another boat made the same trip in 4 hours and 30 minutes. What was its rate of travel?

40. The boat of Exercise 38 is to take a pleasure party up the river to be gone just 2 hours. How soon after starting must the captain turn back? How far up can he go?

41. How much time should the captain of the first boat allow for a trip of 15 miles up the river and back with a stop of one hour?

CHAPTER XII

SETS OF EQUATIONS IN TWO UNKNOWNS

188. Introduction. In chapter vi we learned that it is often simpler to solve a problem by using two unknowns instead of but one. It was noticed that the use of two unknowns required the use of two equations and that these two equations must be furnished by the conditions of the problem.

For instance, the sum of two numbers is 27; their difference is 5. What are the numbers?

$$x+y=27$$
$$x-y=5$$

In the problems we have considered thus far both equations of the pair or set were of the first degree. But many problems give equations of a higher degree than the first, and others require fractional equations.

For instance, the sum of the squares of two numbers is 34; the difference between the numbers is 2. What are the numbers? The equations are

$$x^2+y^2=34$$
$$x-y=2$$

The study of all the different kinds of sets that may arise is beyond the scope of this book. We will consider only a few of the simpler types, namely: (1) both equations of the **first** degree; (2) one equation of the **first** degree and one of the **second**; (3) either or both equations **fractional.**

189. Both equations of the first degree. This type has been considered in chapter vi. Describe the three methods used in solving a set of two linear equations: by graphs, by addition, and by substitution.

190. Inconsistent equations. If an attempt is made to solve the two equations

$$3x+4y=9$$
$$3x+4y=6$$

by subtraction, we get the statement

$$0=3$$

which is absurd.

The two equations are **inconsistent**; that is, they cannot both be true for the same pair of values of x and y. Draw the graphs of these two equations and state what is their relation to each other.

191. Dependent equations. In the attempt to eliminate one of the unknowns from the set

$$2x+3y=5$$
$$4x+6y=10$$

all the terms disappear.

What is shown when the graphs of these two equations are drawn? One of these is really dependent upon the other and the equations are equivalent, and any solution of one will be a solution of the other. They determine no definite solution.

Two equations are **independent** if one cannot be derived from the other.

192. Solution. It follows from the preceding articles that a definite solution can be found for a set of two linear equations in two unknowns only when the equations are consistent and independent.

What can you say of the equations if their graphs are parallel or if they coincide?

EXERCISES

In the following problems use the method which seems to be best adapted to the equations at hand:

1. $4x+5y=40$
   $6x-7y=2$

2. $20=7p+2q$
   $55=20p+7q$

3. $3A=6$
$5A+B=5$

4. $5+a+2b=0$
$7+5a+b=0$

5. The sum of two numbers is 200; their difference is 79. What are the numbers?

6. In what proportions should two kinds of coffee worth 20 cents and 30 cents a pound be combined to make a mixture of 50 pounds worth 26 cents a pound?

7. $4x+3y=6000$
$x-2y=2000$

8. $5A+3B=19$
$2A-B=1$

9. $5y=3-x+3y$
$2x+3=9-4y$

10. $25=5\left(\frac{a+t}{2}\right)$
$t=a+(5-1)2$

11. $\frac{3}{4}w-\frac{1}{2}R=0$
$\frac{1}{2}w+\frac{3}{4}R=100$

12. $a+\frac{5}{2}d=\frac{5}{6}$
$a+7d=\frac{16}{3}$

13. $4(x-3y)=8$
$(x+y)=3(x-2y)$

14. $5(x-2y)-(x-y)=-24$
$11(2x+3y)+(2x-y)=200$

15. $9=k-k'$
$9=19k-361k'$

16. $3x+4y=25$
$3(3+x)+4(4+y)=0$

17. $3\left(\frac{x}{2}-\frac{y}{3}\right)=5, \quad 4-3\left(\frac{x}{4}+\frac{y}{6}\right)=2$

18. The sum of two numbers is s, and their difference is d. Find the numbers.

In the following four examples solve for x and y:

19. $2x+y=2a$
$2x-y=2b$

20. $ax=by$
$x+y=c$

21. $ax+by=1$
$bx+ay=1$

22. $\frac{x}{a+b}+\frac{y}{a-b}=2a$
$\frac{x-y}{4ab}=1$

23. Find a and b in terms of x and y:

$$x=3a+2b$$
$$y=a+5b$$

24. Solve for p and n:

$$180+(2p+2n)=R$$
$$180-(2p-2n)=R'$$

193. Fractional equations. In some instances fractional equations in two unknowns can be reduced to linear equations by clearing of fractions. The solution easily follows.

$$3x+4y=25 \qquad (1)$$

$$\frac{3}{4+y}=\frac{4}{3+x} \qquad (2)$$

Clearing of fractions, $3(3+x)=4(4+y)$ (3)

Let the pupil complete the solution.

EXERCISES

Solve for x and y:

1. $\frac{y}{3}-\frac{x}{4}=\frac{7}{12}$, $\frac{x-2}{y-3}=1$

2. $\frac{x-y-3}{x+y-4}=\frac{1}{2}$, $\frac{x}{y}=5$

3. $3x+4y=25$, $\frac{4}{y+8}=\frac{3}{x+6}$

4. $\frac{x-3}{y-1}=\frac{4}{5}$, $\frac{x-y}{3}=\frac{x}{6}-\frac{1}{2}$

5. $\frac{x-y+1}{x-y-1}=5$, $\frac{x+y+1}{x+y-1}=3$

6. $\frac{x+1}{y}=\frac{1}{a}$, $\frac{x}{y+1}=\frac{1}{b}$

7. $\frac{x+a}{y}=b$, $\frac{x}{y+a}=c$

8. $\frac{x-3y}{y}=\frac{1}{3}$, $\frac{x+y-3}{x}=\frac{1}{5}$

9. $\frac{x+y}{x}=1$, $\frac{x-y}{y}=2$

194. More complicated fractional equations. If an attempt is made to solve a set like

$$\frac{1}{x}+\frac{3}{y}=5 \qquad (1)$$

$$\frac{2}{x}-\frac{1}{y}=3 \qquad (2)$$

by clearing of fractions, we get

$$y+3x=5xy$$
$$2y-x=3xy$$

The resulting set is much too difficult for us to solve in this course. Do not clear such sets of fractions. Simply eliminate one of the fractions containing an unknown just as in linear sets.

Thus, $\frac{1}{x}+\frac{3}{y}=5$ (1)

Multiply (2) by 3, $\frac{6}{x}-\frac{3}{y}=9$ (3)

Add the members of the equations,

$$\frac{7}{x}=14$$

$$7=14x$$

$$\frac{1}{2}=x$$

Substitute in (1), $2+\frac{3}{y}=5$ Whence, $y=1$

Check in (1) and (2).

EXERCISES

Solve for x and y:

1. $\frac{1}{x}+\frac{1}{y}=6$, $\frac{2}{x}-\frac{1}{y}=3$

2. $\frac{2}{x}+\frac{1}{y}=10$, $\frac{4}{y}+\frac{3}{x}=20$

3. $\frac{4}{x}+\frac{2}{y}=-5$, $\frac{11}{y}-\frac{7}{x}=16$

4. $\frac{2}{x}+\frac{3}{y}=-5$, $\frac{8}{x}-\frac{5}{y}=31$

5. $\frac{15}{x}+\frac{21}{y}=10$, $\frac{20}{x}-\frac{6}{y}=2$

6. $\frac{1}{x}+\frac{1}{y}=a$, $\frac{1}{x}-\frac{1}{y}=b$

7. $\frac{1}{x}+\frac{1}{y}=\frac{1}{a}$, $\frac{1}{x}-\frac{1}{y}=\frac{1}{b}$

8. $\frac{1}{2x}-\frac{1}{3y}=-1$, $\frac{1}{3x}-\frac{1}{2y}=-1$

9. $\frac{3}{x}+\frac{a}{y}=m$, $\frac{2}{x}-\frac{b}{y}=n$

10. $\frac{m}{x}+\frac{n}{y}=1$, $\frac{n}{x}+\frac{m}{y}=1$

11. $\frac{a}{x}+\frac{b}{y}=1$, $\frac{a}{x}-\frac{b}{y}=5$

12. $\frac{a}{x}+\frac{b}{y}=n$, $\frac{a}{x}-\frac{b}{y}=m$

195. One equation of the first degree, one of the second. As an illustration take the set

$$2x+y=10 \qquad (1)$$
$$x^2+y^2=25 \qquad (2)$$

The most obvious plan to follow in solving this set is to eliminate one of the unknowns by substitution. Why not by addition?

It makes no real difference which unknown is chosen to be eliminated. In this case y is the better one to choose. Can you tell why? From the linear equation obtain y in terms of x.

$$y=10-2x \qquad (3)$$

Substitute this value of y in place of y in the quadratic equation

$$x^2+(10-2x)^2=25 \qquad (4)$$

Expand and solve,

$$x^2+100-40x+4x^2=25$$
$$5x^2-40x+75=0$$
$$x^2-8x+15=0$$
$$(x-3)(x-5)=0$$
$$x=3,\ x=5$$

It will be noticed that in the solution we replaced equation (2) by equation (4); that is, to solve the set (1) and (2), we solve the set

$$2x+y=10 \qquad (1)$$
$$x^2+(10-2x)^2=25 \qquad (4)$$

Consequently, to get the values of y that correspond to the values $x=3$ and $x=5$, we must substitute these values in succession in equation (1), the **linear** equation, or, what is the same thing, in (3).

Thus, for $x=3$, $y=10-6$, $=4$; and for $x=5$, $y=10-10$, $=0$.

there are two solutions to the set of equations, each composed of a pair of values, one for each unknown.

Check by substituting in both equations.

It is very important that the right values be paired.

$$\begin{cases} x=3, \\ y=4, \end{cases} \quad \begin{cases} x=5, \\ y=0, \end{cases} \quad \text{or,} \quad \begin{array}{c|c|c} x & 3 & 5 \\ \hline y & 4 & 0 \end{array}$$

It is perfectly evident that the same method of solution may be applied to any set of equations of this kind. The method may be stated in the form of a rule:

RULE. **(1) Using the linear equation, express one unknown in terms of the other.**

(2) Eliminate this unknown by substituting its value in the equation of second degree.

(3) Solve the resulting equation in one unknown.

(4) Find the values of the unknown that was eliminated by substituting the values found for the other unknown in the linear equation, keeping the pairs of corresponding values together.

(5) Check by substituting in both of the original equations.

EXERCISES

Solve:

1. $2x+y=0$ $y^2=16x$	2. $y=x^2-6x+8$ $y=2x+8$	3. $2y-x=5$ $x^2+y^2=25$
4. $x-y=3$ $xy=40$	5. $y^2=4x$ $2x+y=4$	6. $y+5x=x^2+4$ $3x+y=4$
7. $y-2x^2=5-5x$ $2x=4-y$	8. $y+5x=x^2+4$ $y=x-5$	9. $x^2+y^2=49$ $5x-4y=0$

196. Graphs. An interesting light may be thrown upon the solution of sets of equations of the kind we have just been considering by the following means: Draw the graphs of the two equations of the set on the same axis just as we did with a set of two linear equations, and then note their points of intersection. For illustration take set 5 of the last article:

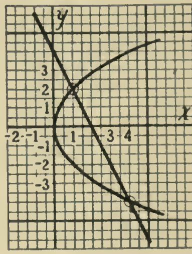

FIG. 65

$$y^2 = 4x \qquad (1)$$

$$2x + y = 4 \qquad (2)$$

Read off the coördinates of the intersection points of the two graphs (Fig. 65). Show that the points of the intersection of the two graphs correspond to the common solutions of the two equations of the set.

The following three sets of equations are treated in the same way:

(*a*)	(*b*)	(*c*)
$y = x^2 - 5x + 4$	$y = x^2 - 5x + 4$	$y = x^2 - 5x + 4$
$y = x - 4$	$y = x - 5$	$y = x - 6$

GRAPHS

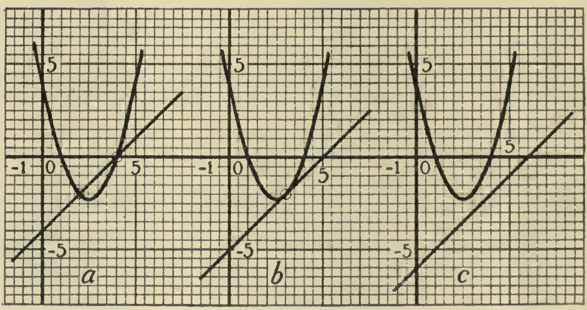

FIG. 66

SOLUTIONS

(*a*)

x	4	2
y	0	-2

(*b*)

x	3	3
y	-2	-2

(*c*) No solution

In (*a*) the graphs intersect; there are two different solutions.

In (*b*) the graphs touch each other; the two solutions are equal.

In (*c*) the graphs do not intersect; there are no real solutions.

197. Graphs of equations of the second degree. The graph of an equation of the first degree is always a straight line. But equations of the second degree have graphs of various kinds. It will be interesting to notice a few of them. In computing the table of pairs of values to be plotted it will be found most convenient to alter the form of the equation so that one letter is expressed in terms of the other.

Thus, $3x-y+x^2=5$

becomes $y=x^2+3x-5$

The equation $x^2+y^2=25$ is a little more difficult to handle. It becomes

$$y^2=25-x^2$$
$$y=\pm\sqrt{25-x^2}$$

For every value given to x there will be two values for y.

$y=+\sqrt{25-x^2}$			$-\sqrt{25-x^2}$	
x	y		y	
0	$\sqrt{25}$	$=5$	$-\sqrt{25}$	$=-5$
1	$\sqrt{24}$	$=4.9$	$-\sqrt{24}$	$=-4.9$
2	$\sqrt{21}$	$=4.6$	$-\sqrt{21}$	$=-4.6$
3	$\sqrt{16}$	$=4$	$-\sqrt{16}$	$=-4$
4	$\sqrt{9}$	$=3$	$-\sqrt{9}$	$=-3$
5	$\sqrt{0}$	$=0$	$-\sqrt{0}$	$=0$
6	$\sqrt{-11}=$		$-\sqrt{-11}=$	

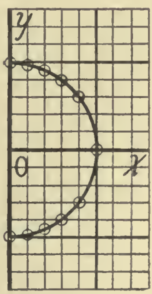

FIG. 67

What are the values of y when x is given negative values? Make such a table and plot all results. Notice that the point for $x=6$ cannot be plotted. Why?

Be sure to take a sufficient number of points to determine the position of the graph fairly well.

EXERCISE I

Plot:

1. $y-x^2=4$
2. $y^2=4x$
3. $x^2+y^2=16$
4. $x^2+y^2=36$
5. $x^2-y^2=36$
6. $4x^2+9y^2=36$
7. $(y-1)^2=x^2$
8. $x^2-y^2=-9$

The curves found by graphing the equations just given are of very great importance in the work of the world. The curves of Exercises 1 and 2 are called parabolas. Such curves are often used for the cables and arches of bridges. A ball thrown into the air, at a slant, rises and falls in a parabolic path. The curve of a paved street from curb to curb is frequently a parabola. The curves of Exercises 3 and 4 are circles; Exercises 5 and 8 give hyperbolas; Exercise 6 gives an ellipse (Fig. 68), which is frequently used for bridge arches. The earth moves about the sun in an elliptical path.

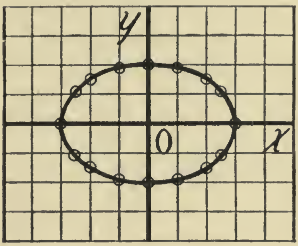

FIG. 68

EXERCISE II

Solve, drawing the graphs of the first three:

1. $y+2x=5$, $x^2+y^2=5$
2. $x^2-4x+y^2=0$, $x-y=0$
3. $x^2-y^2=16$, $3x=5y$
4. $45=\frac{n(a+9)}{2}$, $9=a+(n-1)$
5. $x^2+y^2=185$, $x+y=17$
6. $x+y=15$, $xy=36$
7. $x-y=12$, $xy=85$
8. $n+2p=5$, $n^2+2p^2=9$
9. $2x-y=5$, $x+3y=2xy$
10. $x+2y=7$, $\frac{3}{x}+\frac{6}{y}=5$
11. $x+3y-3=4y$, $3x^2+y^2=65+2x^2$
12. $x-\frac{x-y}{2}=4$, $y-\frac{x+3y}{x+2}=1$

13. $3x^2-2xy=15$
$2x+3y=12$

14. $xy=72$
$\frac{x}{y}=2$

15. $x^2-y^2=6$
$x-y=3$

16. $-\frac{3}{2}=\frac{n(\frac{1}{3}+l)}{2}$
$l=\frac{1}{3}+(n-1)(-\frac{1}{12})$

17. $4x^2+y^2=4$
$y+2x=1$

18. $y^2=2x+4$
$3x-2y=0$

19. $xy-y=5$
$x+y=3$

20. $p+q=100$
$\frac{1}{p}+\frac{1}{q}=\frac{1}{20}$

21. $2s=n(a+l)$
$l=a+(n-l)d$

22. In Ex. 21 determine n and l when $s=25$, $a=1$, $d=2$.

198. Problems. It will be found that in most of the following problems it is desirable to use two unknowns.

1. A farmer bought 160 acres of land for $16000. If part of it cost $80 an acre and part $120 an acre, find the number of acres bought at each price.

2. Two boys are 12 miles apart and walk toward each other, one at the rate of $2\frac{1}{2}$ miles an hour and the other at the rate of $3\frac{1}{2}$ miles an hour. How far will the first boy have walked when they meet. Give both algebraic and graphic solutions.

3. A dealer made the following offers: Six pounds of nuts and 4 pounds of candy for $4.50, or 4 pounds of nuts and 5 pounds of candy at $4.40. What is the price per pound of the candy and the nuts?

4. A grocer wishes to mix 2 brands of coffee worth 25 cents and 40 cents a pound respectively How many pounds of each must he use in making a mixture of 1(pounds worth 35 cents a pound?

5. A certain brand of coffee made up of two kinds in the ratio 1:2 formerly sold at 55 cents a pound. The first kind rose 20 per cent in price, the other 50 per cent. The same mixture now sells at 78 cents a pound. What is the present price of each kind of coffee used?

6. A crew that can row 10 miles an hour downstream finds that it takes them twice as long to row back the same distance. Find the rate of the current.

7. Suppose you were camping on a river. How could you find the rate of the current of the river and your own rate of rowing?

8. A man has $600 invested in Liberty bonds, a part at 4 per cent and the rest at $4\frac{1}{2}$ per cent. The annual income from the bonds is $26. How many dollars worth of each kind of bonds does he have?

9. Two children weighing 30 and 50 pounds respectively are riding on a 12-foot teeter board. How far from the lighter child should the support be placed so that they shall just balance?

10. Two children riding on a teeter 16 feet long just balance when the support is placed 10 feet from the end of the board. If the child on the long end weighs 40 pounds, what does the other weigh?

11. A teeter is supported at the middle. Two boys, one weighing 75 pounds and the other 45 pounds, wish to ride on it so that they will just balance. The lighter boy got on the end of the board which was 6 feet from the support. How far from the support should the heavier boy sit?

12. Two boys of unknown weights found that they balanced when one was twice as far from the fulcrum as the other. Can you find out how much they weighed? Can you find their relative weights? If the lighter boy was at the end of the 12-foot teeter, where was the heavier boy?

13. The heavier of the boys of the last exercise was rather bright. He remembered the 10-pound sack of flour he was taking home. He noticed that they just balanced when he was 3 feet from the fulcrum and the lighter boy was at the end of the teeter. The lighter boy took the flour in his lap and found that they just balanced when he moved $1\frac{1}{2}$ feet nearer the fulcrum. How much did each boy weigh?

14. Two objects of unknown weight just balanced when placed 20 inches and 12 inches from the middle of a rod which is balanced at the middle. If the objects are reversed in position it is necessary to add 4 pounds to the lighter object. Find the weights.

15. The sum of two numbers is 27, their product is 180. What are the numbers?

16. Two boys are carrying a weight of 150 pounds hanging from a 9-foot pole between them. The weight is $6\frac{1}{2}$ feet from one boy and $2\frac{1}{2}$ feet from the other. How much does each carry?

The principle of the lever can be applied in Problem 16 just as well as in teeter problems. The place where the weight is hung is to be taken as the fulcrum. If x is the part of the weight carried by one boy, how much will the other boy carry?

17. The sum of the squares of two numbers is 74 and the sum of the numbers is 12. What are the numbers?

18. The sum of two numbers is 9 and the difference between their squares is 135. What are the numbers?

19. The area of a rectangle is 84, while its perimeter is 38. What are its dimensions?

20. Divide 126 into two parts that shall be in the ratio of 2 to 7.

21. There are two numbers whose sum is 20. If the larger is divided by the smaller the quotient is 5 and the remainder is 2. Find the number.

22. There are two numbers whose difference is 6. If 4 times the larger is divided by 5 times the smaller the quotient is 1 and the remainder is 5. Find the numbers.

23. Find three consecutive integers such that the product of the first and third divided by the second gives a quotient of 24 and a remainder of 24.

24. There is a certain number of two digits such that if the number is divided by the difference between the digits the quotient is 21. The sum of the digits is 12. Find the number.

25. If a is added to both the numerator and the denominator of a certain fraction the result is equal to $\frac{3}{4}$. If b is subtracted from both the numerator and the denominator the result is equal to $\frac{1}{3}$. Find the fraction.

26. What will the answer to Exercise 25 become if (1) $a=5$, $b=3$; (2) $a=3$, $b=5$; (3) $a=4$, $b=1$?

27. Find three consecutive numbers such that the square of half of the middle one is $2\frac{1}{2}$ times the sum of the other two.

28. The numerator of a certain fraction is one less than the denominator. If 4 is subtracted from the numerator and 1 added to the denominator the value of the fraction is $\frac{1}{3}$. Find the fraction.

29. The sum of the numerator and the denominator of a certain proper fraction is 13. If 3 is added to the numerator and the denominator is multiplied by 2 the value of the fraction is $\frac{1}{2}$. Find the fraction.

30. If 2 is added to the numerator of a certain fraction and 2 subtracted from the denominator the value of the fraction is $\frac{7}{10}$. If 3 is added to the numerator and 4 is added to the denominator of the same fraction the value of the fraction is $\frac{1}{2}$. Find the fraction.

31. A and B can do a piece of work in 6 days. A works 2 days and B 3 days and in that time they do $\frac{2}{5}$ of the work. How long will it take each to do the work?

Solution. The equalities are:

$$\left.\begin{array}{l}\text{Part of work A can}\\ \text{do in 1 day}\end{array}\right\}+\left\{\begin{array}{l}\text{part of work B can}\\ \text{do in 1 day}\end{array}\right\}=\left\{\begin{array}{l}\text{part both can do}\\ \text{in 1 day}\end{array}\right.$$

Part A does in 2 days + part B does in 3 days = $\frac{2}{5}$ of the work

If x = no. of days it would take A to do the work

$$\frac{1}{x} = \text{Part of work A could do in 1 day}$$

$$\frac{2}{x} = \text{Part of work A could do in 2 days}$$

The equalities above expressed in algebraic terms are

$$\frac{1}{x}+\frac{1}{y}=\frac{1}{6}$$

$$\frac{2}{x}+\frac{3}{y}=\frac{2}{5}$$

32. A and B undertook to address a certain number of letters. They knew that they could do the work in 9 hours. After 4 hours A was taken sick. It took B 10 hours more to finish the job working alone. Which was the faster worker? How long would it take each to do the work alone?

33. The sum of two fractions having the numerators 2 and 5 is $\frac{3}{2}$. If the denominators are exchanged, the sum of the resulting fractions is 2. Find the fractions.

34. Water is being taken out of a tank by a pump B while the tank is being supplied by a pipe A. When both pipe and pump are running the tank can be filled in 12 hours. The tank has just been cleaned. The pump is started 2 hours after the pipe was turned on. The tank was filled in $3\frac{1}{3}$ hours. How long will it take the pipe to fill the tank?

35. A cistern can be filled by two pipes running together in 2 hours and 55 minutes; the larger pipe by itself will fill it 2 hours sooner than the smaller pipe by itself. How long would it take each pipe to fill the cistern?

CHAPTER XIII

RADICALS

199. Square root of a number. The square root of a number was defined as one of the two equal factors of a number. This was used in chapter x as the basis of the method of finding the square root of a number.

EXERCISES

Find the following square roots by inspection:

1. $\sqrt{49}$	2. $\sqrt{81}$	3. $\sqrt{16}$
4. $-\sqrt{25}$	5. $-\sqrt{a^2}$	6. $\sqrt{121}$
7. $\sqrt{225}$	8. $\sqrt{625}$	9. $-\sqrt{36}$
10. $-\sqrt{100}$	11. $\sqrt{x^2}$	12. $\sqrt{x^4}$
13. $-\sqrt{a^6}$	14. $-\sqrt{9a^2}$	15. $\sqrt{144}$

200. Square root of monomials. Finding the square root of a number is the reverse of finding the square of a number:

$$(3a^4)^2 = 3^2(a^4)^2 = 9a^8$$

To square a number, we square the numerical coefficient and multiply the exponent of each of the letters by 2.

To find the square root of a product, we reverse the operation; namely, take the square root of the numerical coefficient and divide the exponent of each letter by 2:

$$\sqrt{16a^2b^6} = \sqrt{16}\ a^{\frac{2}{2}}\ b^{\frac{6}{2}} = 4ab^3$$

EXERCISES

Find the following square roots:

1. $\sqrt{4a^2}$	2. $-\sqrt{9x^2}$	3. $\sqrt{81t^2}$
4. $\sqrt{25n^2}$	5. $-\sqrt{64}$	6. $\sqrt{16s^2}$
7. $\sqrt{25a^6}$	8. $-4\sqrt{9a^2x^4}$	9. $\sqrt{16x^2y^4}$
10. $\sqrt{625n^6}$	11. $-\sqrt{81h^4}$	12. $\sqrt{36R^2}$

13. $-\sqrt{196t^6}$	14. $\sqrt{324}$	15. $\sqrt{576}$
16. $\sqrt{25x^4}$	17. $\sqrt{(x+y)^4}$	18. $\sqrt{9n^4(w+1)^2}$
19. $-\sqrt{121n^6(a+b)^2}$	20. $-\sqrt{169a^2x^6}$	21. $\sqrt{36(x-1)^6}$

201. Kinds of numbers. In our study so far we have come upon several kinds of numbers, among which are:

Integers such as 2, 5, 27

Fractions such as $\frac{2}{3}, \frac{5}{7}, \frac{11}{5}$

Irrational square roots such as $\sqrt{2}$, $\sqrt{3}$, $\sqrt{18}$

Negative numbers such as $-3, -\frac{2}{3}, -\sqrt{3}$

The third kind belongs to a much larger class of numbers, called **irrationals,** including irrational cube roots, fifth roots, etc., such as $\sqrt[3]{3}$, $\sqrt[5]{7}$, and other more general irrationals such as $\pi = 3.14159+$.

It is proved in later courses in algebra that an irrational number cannot be expressed exactly as a fraction or as a decimal. We therefore use approximate values when using irrationals in arithmetical calculations as was done in chapter x.

It is the purpose of this chapter to show how we may combine certain irrationals without using their approximate values.

In this book we shall consider only square roots.

202. Definition. A root indicated by the use of the symbol $\sqrt{\ }$ is called a **radical.**

$$\sqrt{9}, \quad \sqrt{a^2+2ab+b^2}$$
$$\sqrt{3}, \quad \sqrt{a+x}$$

The square root operation can be completely carried out in the first two instances, the results being 3 and $a+b$. In the other instances the operation cannot be carried to completion; the expressions are really irrational.

203. Numerical calculation. Let it be required to find the numerical value of the following combinations:

(*a*) $\sqrt{3}+\sqrt{2}$, (*b*) $\sqrt{3}-\sqrt{2}$, (*c*) $\sqrt{3}\times\sqrt{2}$, (*d*) $\sqrt{3}\div\sqrt{2}$

The results, of course, are wanted in approximate decimal form and may be found by the use of approximate values for $\sqrt{2}$, $\sqrt{3}$. Carry the result out to four figures.

$$\sqrt{3}=1.732, \quad \sqrt{2}=1.414$$

(*a*)
$$\begin{array}{r} \sqrt{3}=1.732 \\ +\sqrt{2}=\underline{1.414} \\ 3.146 \end{array}$$

(*b*)
$$\begin{array}{r} \sqrt{3}=1.732 \\ -\sqrt{2}=\underline{1.414} \\ .318 \end{array}$$

(*c*)
$$\begin{array}{r} \sqrt{3}=1.732 \\ \times\sqrt{2}=\underline{1.414} \\ 2.449 \end{array}$$

(*d*)
$$\begin{array}{r} \sqrt{3}=1.732 \\ \div\sqrt{2}=\underline{1.414} \\ 1.224 \end{array}$$

Let the student compute in a similar way the following:

(*e*) $\sqrt{18}+\sqrt{8}$, (*f*) $\sqrt{18}-\sqrt{8}$, (*g*) $\sqrt{18}\times\sqrt{8}$, (*h*) $\sqrt{18}\div\sqrt{8}$

The values of $\sqrt{18}$ and $\sqrt{8}$ may be taken from the table of square roots. If your calculations have been made correctly, the results are for (*e*) 7.071, (*f*) 1.414, (*g*) 11.999 or 12.00, (*h*) 1.5000 or 1.500. All four of these can be calculated by simpler methods, which will be developed in the following articles.

204. Addition. There is no other way of calculating $\sqrt{2}+\sqrt{3}$ than that used in the last article. But $\sqrt{18}+\sqrt{8}$ presents possibilities. We can find the square root of a product by finding the square root of each factor and multiplying the square roots:

$$\sqrt{18}=\sqrt{9\cdot 2}=3\sqrt{2}$$

So also $\sqrt{8}=\sqrt{4\cdot 2}=2\sqrt{2}$

hence
$$\begin{aligned} \sqrt{18}+\sqrt{8} &= 3\sqrt{2}+2\sqrt{2} \\ &= 5\sqrt{2} \\ &= 5\times 1.414 = 7.070 \end{aligned}$$

Notice that 3 and 2 may be considered as the coefficient of $\sqrt{2}$.

This is not quite the same as the result found in the other way, but the error is no more than might be expected when one is working with approximate numbers.

There is, moreover, another aspect of the matter that is often of as much importance as the finding of the numerical value. The sum of the two radicals has been expressed in a more concise radical form:

$$5\sqrt{2} \text{ rather than } \sqrt{18}+\sqrt{8}$$

205. Simple radicals. The forms of radical expressions can be altered without change in their values. $5\sqrt{2}$ is an example of what is called a **simple radical.** $\sqrt{18}$ is not a simple radical. There is concealed in the 18 a square factor, $\sqrt{18}=\sqrt{9\cdot 2}=3\sqrt{2}$. Find $\sqrt{18}$ and $3\sqrt{2}$ and compare answers. The square root of an integer or an integral expression that has no square factor is called a **simple radical.** Two radicals that have the same number under the radical sign are called **like radicals.** The possibility of adding two radicals into one term depends upon whether or not they are like radicals.

EXERCISE I

Reduce to simple radicals. Verify 1–6 as you verified $\sqrt{18}=3\sqrt{2}$. In 22 to 28 assign values to the letters and work out.

1. $\sqrt{50}$	8. $\sqrt{72}$	15. $\sqrt{180}$	22. $\sqrt{5a^2}$
2. $\sqrt{12}$	9. $\sqrt{128}$	16. $\sqrt{24}$	23. $\sqrt{4a}$
3. $\sqrt{27}$	10. $\sqrt{162}$	17. $\sqrt{54}$	24. $\sqrt{a^2b}$
4. $\sqrt{8}$	11. $\sqrt{200}$	18. $\sqrt{96}$	25. $\sqrt{49x}$
5. $\sqrt{75}$	12. $\sqrt{48}$	19. $\sqrt{150}$	26. $\sqrt{81x^2y}$
6. $\sqrt{45}$	13. $\sqrt{147}$	20. $\sqrt{28}$	27. $\sqrt{3a^2x}$
7. $3\sqrt{98}$	14. $\sqrt{125}$	21. $\sqrt{90}$	28. $5\sqrt{32a^2}$

To work the following sums reduce to **simple radicals**, and find the numerical value of the results by use of the tables. Check each exercise by use of the tables without reducing to simple radicals.

Illustration:

$$\begin{aligned}\sqrt{50}-\sqrt{18}+\sqrt{3}&=5\sqrt{2}-3\sqrt{2}+\sqrt{3}\\&=2\sqrt{2}+\sqrt{3}\\&=2(1.414)+1.732\\&=2.828+1.732\\&=4.560\end{aligned}$$

Check:

$$\begin{aligned}\sqrt{50}&=7.071\\+\sqrt{3}&=\underline{1.732}\\&8.803\\-\sqrt{18}&=\underline{4.243}\\&4.560\end{aligned}$$

EXERCISE II

1. $2+\sqrt{3}$
2. $2\sqrt{3}-5\sqrt{3}+7\sqrt{3}$
3. $\sqrt{12}+\sqrt{27}$
4. $\sqrt{80}-\sqrt{45}$
5. $\sqrt{12}+\sqrt{75}-\sqrt{27}$
6. $\sqrt{18}-\sqrt{2}-\sqrt{8}$
7. $2\sqrt{3}-\sqrt{12}+\sqrt{48}$
8. $2\sqrt{2}-3\sqrt{3}+5\sqrt{3}-6\sqrt{2}$
9. $\sqrt{18}-\sqrt{2}+\sqrt{32}-5\sqrt{5}$
10. $\sqrt{98}-\sqrt{50}+\sqrt{32}$

Carry out the indicated operation whenever possible in the following:

11. $2\sqrt{a}-3\sqrt{a}-7\sqrt{a}$
12. $\sqrt{3a}-\sqrt{12a}+\sqrt{48a}$
13. $\sqrt{ab^2}+\sqrt{ab^2}$
14. $\sqrt{ab^3}+\sqrt{a^3b}$
15. $\sqrt{9x}-\sqrt{4x}+\sqrt{25x}$
16. $2\sqrt{3x}-3\sqrt{12x}+\sqrt{75x}$
17. $a\sqrt{b}-c\sqrt{b}$
18. $a-\sqrt{a^2b}+\sqrt{a^3b}$

206. Addition involving fractions. Add:

$$\sqrt{\frac{2}{3}}+\sqrt{24}$$

You can, of course, find $\sqrt{\frac{2}{3}}$ as in chapter x, but here we wish to see if we can combine these two radicals into one. There is a better way of proceeding. This is accomplished by a very simple device. Alter the fraction $\frac{2}{3}$ so that the denominator is a square by multiplying both terms by 3.

Thus, $$\sqrt{\frac{2}{3}}=\sqrt{\frac{2\times3}{3\times3}}=\sqrt{\frac{6}{9}}=\frac{\sqrt{6}}{3}=\tfrac{1}{3}\sqrt{6}$$

Hence $$\sqrt{\frac{2}{3}}+\sqrt{24}=\tfrac{1}{3}\sqrt{6}+2\sqrt{6}$$
$$=(\tfrac{1}{3}+2)\sqrt{6}$$
$$=\tfrac{7}{3}\sqrt{6}$$

The same method will work with any radical that has a fraction under the radical sign:

$$\sqrt{\frac{a}{b}}=\sqrt{\frac{ab}{b^2}}=\frac{\sqrt{ab}}{b}=\frac{1}{b}\sqrt{ab}$$

Supply values for a and b and work out the result by this method.

To be a simple radical, a radical must have neither a fraction nor a square factor under the radical sign.

EXERCISE I

Reduce to simple radicals and find numerical values where possible.

1. $\sqrt{\frac{1}{2}}$	7. $\frac{1}{2}\sqrt{\frac{5}{2}}$	13. $\sqrt{\frac{3}{5}}$	19. $\sqrt{\frac{x}{4}}$
2. $\sqrt{\frac{1}{3}}$	8. $\sqrt{\frac{3}{8}}$	14. $\sqrt{\frac{2}{3}a^2}$	20. $\sqrt{\frac{5x}{2}}$
3. $\sqrt{\frac{2}{5}}$	9. $\sqrt{\frac{2}{27}}$	15. $\sqrt{\frac{a}{x}}$	21. $\sqrt{\frac{a^2b}{c}}$
4. $3\sqrt{\frac{2}{7}}$	10. $\sqrt{\frac{1}{50}}$	16. $\sqrt{\frac{2}{3x}}$	22. $\sqrt{\frac{12x}{5}}$
5. $7\sqrt{\frac{3}{7}}$	11. $\sqrt{\frac{5}{7}}$	17. $a\sqrt{\frac{1}{a}}$	23. $\sqrt{\frac{3x^2}{2y}}$
6. $\sqrt{\frac{3}{2}}$	12. $\sqrt{\frac{1}{5}}$	18. $\sqrt{\frac{4}{x}}$	24. $\frac{1}{2}\sqrt{\frac{x^2}{2}}$

EXERCISE II

Complete the following additions and subtractions:

1. $\sqrt{\frac{1}{2}}-\sqrt{\frac{1}{8}}$
2. $\sqrt{\frac{2}{3}}+\sqrt{54}+\sqrt{\frac{3}{2}}$
3. $\sqrt{\frac{1}{5}}-\sqrt{\frac{5}{4}}+\sqrt{50}$
4. $\sqrt{\frac{1}{2}}+\sqrt{\frac{1}{8}}+\sqrt{\frac{1}{16}}+\sqrt{\frac{1}{4}}$
5. $\sqrt{\frac{1}{2}}-\sqrt{2}+\sqrt{\frac{9}{2}}$
6. $\sqrt{\frac{1}{2}}+\sqrt{\frac{1}{18}}-\sqrt{\frac{2}{9}}$
7. $\sqrt{\frac{5}{2}}-\sqrt{\frac{2}{5}}$
8. $\sqrt{\frac{3}{5}}+\sqrt{\frac{5}{3}}$

207. Multiplication. We have already used the general principle

$$\sqrt{ab}=\sqrt{a}\sqrt{b}$$

As long as a and b are positive numbers, the reverse of this is equally true:

$$\sqrt{a}\sqrt{b}=\sqrt{ab}$$

that is, $$\sqrt{2}\sqrt{3}=\sqrt{6}$$

Test this result by calculation, using the tables.

So $$\sqrt{18}\sqrt{8}=\sqrt{144}=12$$

Work out other illustrations of the use of this principle by supplying values for a and b.

Which method is to be preferred for calculation, the method used here or the method used in Art. **203?**

Find product of $2\sqrt{3}\times 7\sqrt{5}$

State a rule for multiplying radicals.

EXERCISES

Find the following products. In Ex. 13, 14, 15, 16, and 19 assign values for the letters and work out.

1. $\sqrt{2}\ \sqrt{5}$
2. $\sqrt{5}\ \sqrt{7}$
3. $\sqrt{4}\ \sqrt{3}$
4. $\sqrt{8}\ \sqrt{2}$
5. $\sqrt{3}\ \sqrt{27}$
6. $\sqrt{3}\ \sqrt{7}$
7. $\sqrt{6}\ \sqrt{15}\ \sqrt{10}$
8. $\sqrt{\frac{2}{3}}\ \sqrt{15}$
9. $\sqrt{\frac{3}{7}}\ \sqrt{\frac{7}{2}}$
10. $\sqrt{24}\ \sqrt{6}$

11. $\sqrt{\frac{2}{5}}\sqrt{45}$
12. $2\sqrt{7}\cdot 3\sqrt{2}$
13. $\sqrt{a}\sqrt{2a}$
14. $\sqrt{2a}\sqrt{3x}$
15. $\sqrt{3}\sqrt{2a}$
16. $\sqrt{3a}\sqrt{2a}$
17. $\sqrt{2a}\sqrt{3a}$
18. $\sqrt{2ab}\sqrt{2ax}$
19. $\sqrt{a+b}\sqrt{a-b}$
20. $\sqrt{x+y}\sqrt{x+y}$
21. $(2+\sqrt{3})(2-\sqrt{3})$
22. $(2+\sqrt{3})(2+\sqrt{3})$
23. $3\sqrt{2ab}\cdot\frac{1}{3}\sqrt{3ab}$
24. $(\sqrt{2}+\sqrt{3})(\sqrt{2}+\sqrt{3})$
25. $(\sqrt{3}+\sqrt{5})(\sqrt{3}-\sqrt{5})$
26. $(\sqrt{3}+\sqrt{2})(-\sqrt{3}-\sqrt{2})$
27. $(\sqrt{6}-\sqrt{3})(\sqrt{6}-\sqrt{2})$
28. $(\sqrt{15}-\sqrt{5})(\sqrt{15}+\sqrt{3})$

208. Division. We may obtain the numerical value of $\sqrt{3}\div\sqrt{2}$ as in Art. **203** by dividing 1.732 by 1.414. It will be remembered, however, that in Art. **173** we used the principle

$$\sqrt{\frac{a}{b}}=\frac{\sqrt{a}}{\sqrt{b}}$$

This principle is equally true if read in the reverse direction:

$$\frac{\sqrt{a}}{\sqrt{b}}=\sqrt{\frac{a}{b}}$$

Using this idea here, we have

$$\frac{\sqrt{3}}{\sqrt{2}}=\sqrt{\frac{3}{2}}=\sqrt{\frac{6}{4}}=\frac{1}{2}\sqrt{6}$$

which is very easily computed.

So also
$$\frac{\sqrt{18}}{\sqrt{8}}=\sqrt{\frac{18}{8}}=\sqrt{\frac{9}{4}}=\frac{3}{2}$$

or again
$$\frac{\sqrt{27}}{\sqrt{3}}=\sqrt{9}=3$$

Give other illustrations of the use of this principle by supplying values for a and b.

This method of division is very simple and direct in many cases. Another method is often used that in some cases is even more direct. The division $\sqrt{3} \div \sqrt{2}$ is regarded as a fraction $\frac{\sqrt{3}}{\sqrt{2}}$.

The form of this fraction is now changed to some form that is particularly desired. For some purposes it is desirable that there shall be no radical in the denominator. To secure this result multiply both terms by a number that will bring the desired form.

In the case $\frac{\sqrt{3}}{\sqrt{2}}$ it is easily seen that $\sqrt{2}$ is the proper multiplier:

$$\frac{\sqrt{3} \times \sqrt{2}}{\sqrt{2} \times \sqrt{2}} = \frac{\sqrt{6}}{2} \text{ is } \frac{1}{2}\sqrt{6}$$

For some purposes the numerator should be free of radicals. What multiplier should be used to secure that result?

We call this process of freeing a term of radicals **rationalizing** that term.

What multipliers will rationalize the denominators of the fractions $\frac{3}{\sqrt{3}}$, $\frac{3}{2\sqrt{3}}$, $\frac{\sqrt{2}}{\sqrt{5}}$, $\frac{\sqrt{3}}{\sqrt{8}}$, $\frac{\sqrt{2}}{\sqrt{50}}$?

EXERCISES

In each exercise use the method you think best suited to that particular exercise:

1. $\sqrt{50a} \div \sqrt{10a}$
2. $\sqrt{16} \div \sqrt{8}$
3. $\sqrt{5} \div \sqrt{2}$
4. $\sqrt{7a} \div \sqrt{2b}$
5. $3\sqrt{2} \div 2\sqrt{3}$
6. $\sqrt{3x} \div \sqrt{2x}$
7. $\sqrt{7} \div \sqrt{8}$
8. $\sqrt{6} \div \sqrt{12}$
9. $2\sqrt{5} \div \sqrt{6}$
10. $(2\sqrt{6} - 10\sqrt{2} + 6\sqrt{18}) \div \sqrt{3}$
11. $\sqrt{\frac{2}{3}} \div \sqrt{\frac{3}{5}}$
12. $(\sqrt{2} - \sqrt{3}) \div \sqrt{6}$

209. Division by a binomial. The same general procedure is to be followed when the divisor is a binomial. The division is to be expressed in the fractional form and the fraction reduced to the form desired. If it is desired to free the denominator of radicals, a satisfactory multiplier must be sought. A new problem is presented here.

Take, for instance, $$\frac{3}{\sqrt{3}+\sqrt{2}}$$

If the multiplier $\sqrt{3}+\sqrt{2}$ is used, we have

$$\frac{3(\sqrt{3}+\sqrt{2})}{3+2\sqrt{6}+2}$$

The desired end has not been accomplished. A familiar identity furnishes a clue to what we want:

$$(a+b)(a-b)=a^2-b^2$$

Can you follow the clue and determine the multiplier needed to free the denominator of radicals and reduce the fraction to the form required? Formulate a rule for reducing such fractions to equivalent fractions with denominators free of radicals.

Compute the value of the fraction $\frac{3}{\sqrt{3}+\sqrt{2}}$ by the direct use of the table and also by rationalizing the denominator. Get a result showing three significant figures and draw your own conclusion as to the two methods.

The related forms $\sqrt{a}+\sqrt{b}$ and $\sqrt{a}-\sqrt{b}$ are called **conjugates** of each other. Their product is rational.

$$(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b})=a-b$$

$$(\sqrt{5}+\sqrt{3})(\sqrt{5}-\sqrt{3})=5-3=2$$

What is the conjugate of $\sqrt{2}-\sqrt{3}$? of $\sqrt{7}+\sqrt{6}$?

Give other illustrations by supplying values for a and b.

EXERCISES

Reduce to equivalent fractions with rational denominators. Where possible find the numerical values. Supply values for letters in Ex. 6, 7, and 8.

1. $\dfrac{\sqrt{3}}{2+\sqrt{3}}$

2. $\dfrac{1}{\sqrt{3}-\sqrt{2}}$

3. $\dfrac{\sqrt{3}}{\sqrt{3}-2}$

4. $\dfrac{2\sqrt{2}}{2+\sqrt{2}}$

5. $\dfrac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}$

6. $\dfrac{\sqrt{x}-\sqrt{a}}{\sqrt{x}+\sqrt{a}}$

7. $\dfrac{x^2}{y+\sqrt{y^2-x^2}}$

8. $\dfrac{x-y}{\sqrt{x}-\sqrt{y}}$

9. $\dfrac{\sqrt{8}+2}{\sqrt{8}-2}+\dfrac{\sqrt{8}-2}{\sqrt{8}+2}$

10. $\dfrac{1-\frac{1}{2}\sqrt{2}}{1+\frac{1}{2}\sqrt{2}}$

11. $\dfrac{1}{2+\sqrt{2}}+\dfrac{1}{2-\sqrt{2}}$

12. $\dfrac{\sqrt{3}-1}{\sqrt{2}-1}$

210. Practical computation. After all the foregoing discussion what do you think would be the most feasible and common-sense way of computing the value of

$$\sqrt{\frac{198}{18}} \text{ or } \sqrt{\frac{257}{39}}$$

In a certain engineering problem the value of $\sqrt{1-\dfrac{2.56}{62.3}}$ is desired.

Find the value in the most satisfactory way you know.

211. Irrational equations. In the solution of problems it often happens that an equation appears with the unknown under a radical sign. Such equations are called **irrational** equations. A few simple equations of this kind will be considered here. One of the simplest is

$$\sqrt{x}=3 \qquad (1)$$

Inspection leads at once to the answer

$$x=9 \qquad (2)$$

In that very brief inspection you really squared the 3 and likewise the $\sqrt{x}$.

This brings to light another operation that can be applied to equations; namely, **both sides of the equation may be squared** and the relationship of the sides to each other remain undisturbed. Apply to the equation

$$\sqrt{x-3}-4=0$$

Squaring both sides, we have

$$x-3-8\sqrt{x-3}+16=0$$

an equation that still contains a radical and is even more complicated.

The object of squaring was to remove the radical. For this purpose the radical should stand alone. Alter the equation so that the radical will be alone:

$$\sqrt{x-3}=4$$

and then square,
$$x-3=16$$
$$x=19$$

Check:

$\sqrt{19-3}$	4
$\sqrt{16}$	4
4	4

Apply to $x-\sqrt{x+2}=0$ and check

Do both roots obtained from the derived integral equation satisfy the original irrational equation? How many roots does this particular irrational equation have?

Occasionally you find answers that will not check. These are not roots of the equation. This matter will be discussed in a later course.

Formulate a rule for solving irrational equations.

Apply rule to solving

$$x-\sqrt{x+2}=2$$

EXERCISES

Solve the following equations:

1. $\sqrt{x}-5=0$
2. $\sqrt{1-x}=5$
3. $\sqrt{x^2-9}-4=0$
4. $n+\sqrt{n+6}=14$
5. $2=3m+\sqrt{5m^2+11}$
6. $n-\sqrt{n+6}=14$
7. $t=\pi\sqrt{\frac{m}{g}}$. Find value for m when $t=1$, $\pi=\frac{22}{7}$, and $g=32.2$.
8. $C=\sqrt{1+\frac{1}{t^2}}$. Solve for t. Find t when $C=2$.
9. $x+\sqrt{x-2}=2$
10. $\sqrt{x}+7=0$
11. $\sqrt{x-2}=8$
12. $x+\sqrt{x-3}=15$
13. $\sqrt{x+5}-x=3$
14. $\sqrt{x-8}+10=x$

212. Checking solutions of quadratic equations. The roots of the equation

$$x^2-4x+2=0$$

are

$$2+\sqrt{2},\quad 2-\sqrt{2}$$

In chapter x these were checked by the use of approximate values, and of course the checking could not be exact. If, however, the radical forms are substituted, the checking is exact:

$$\begin{array}{r|l} (2+\sqrt{2})^2-4(2+\sqrt{2})+2 & 0 \\ 4+\cancel{4\sqrt{2}}+2-8-\cancel{4\sqrt{2}}+2 & 0 \\ 0 & 0 \end{array}$$

EXERCISES

Solve and check:

1. $4x^2-1=4x$
2. $5x^2+11=15x$
3. $x+3=\frac{11}{x}$
4. $4x^2-12x+9=0$

THE APPENDIXES

THE APPENDIXES

APPENDIX A

FACTORING

213. The sum of two cubes; the difference between two cubes. The following identities may be verified by doing the indicated multiplication:

$$(a^2-ab+b^2)(a+b)=a^3+b^3 \quad (1)$$
$$(a^2+ab+b^2)(a-b)=a^3-b^3 \quad (2)$$

Read from left to right, these give special types of multiplication. Read from right to left, they reveal types of expressions that can be easily factored.

When put in words, (1) as a guide to factoring reads: The sum of the cubes of two numbers equals the sum of the numbers times the square of the first number minus the product of the two numbers plus the square of the second number.

State (2) in words as a guide to factoring.

You should notice the ways in which the two identities are alike and also the ways in which they differ.

EXERCISES

Factor:

1. a^3-b^3
2. c^3+8
3. x^3-125
4. n^3-64
5. r^3+9
6. r^3+27
7. n^3+27p^3
8. $x^3y^3-t^3$
9. $1-125h^3$
10. s^3+1
11. t^3+64
12. $27n^3-8p^3$
13. $125+27h^3$
14. $64a^3+729$
15. $25x^3-200y^3$
16. n^4t+nt^4
17. $\pi R^3-\pi r^3$
18. $4c^5-32b^3c^2$
19. $1+16c^3$
20. $16+54n^3$

21. n^4-27n

22. $8s^4+s$

23. $\frac{a^3}{27}+\frac{b^3}{8}$

24. $\frac{1}{125}x^3-y^3$

25. $\frac{8}{27}a^3+\frac{1}{8}b^3$

26. $\frac{a^3}{8}-b^3$

Expand:

27. $(s-t)(s^2+st+t^2)$

28. $(y^2+5y+25)(y-5)$

29. $(2-n)(4+2n+n^2)$

30. $(t+4)(t^2-4t+16)$

31. $(n^2-2n+4)(n+2)$

32. $(x^2-x+1)(x+1)$

Fill in blanks:

33. $(x^2+?+49)(x-?)=x^3-?^3$

34. $(x^2+?x+9)(?)=x^3-?^3$

35. $(y+?)(y^2-?+36)=y^3+?^3$

36. $(x^2-x+?)(x+?)=x^3+1$

37. $(?+k)(25-?+k^2)=?^3+k^3$

38. $(7^2+7\cdot 9+9^2)(9-7)=?$

39. $(x^2-3xy+9y^2)(x+3y)=?$

40. $(2x-3y)(4x^2+6xy+9y^2)=?$

214. Factoring by grouping terms. Sometimes an expression that does not at first sight seem to conform to any of the factorable types known to you can be so altered by a judicious grouping of terms as to reveal a known type. There are two special cases of importance which will be considered in the next two articles.

215. A common binomial factor revealed. Consider the expression

$$ax+ay+bx+by$$

Grouping the first two terms and the last, we have

$$(ax+ay)+(bx+by)$$

Factoring each group separately, we have

$$a(x+y)+b(x+y)$$

In this form is revealed a binomial which is a common factor of the two terms of the expression.

The expression is seen to be of the common factor type and may be put in the form

$$(x+y)(a+b)$$

So also

$$\begin{aligned} t^3-2t^2-3t+6 &= (t^3-2t^2)-(3t-6) \\ &= t^2(t-2)-3(t-2) \\ &= (t-2)(t^2-3) \end{aligned}$$

EXERCISES

Factor:

1. $a(x+y)-b(x+y)$
2. $4(h-k)-a(h-k)$
3. $x(x-3)+5(x-3)$
4. $x(a-b)-4(a-b)$
5. $3a(y-z)+4b(y-z)$
6. $2x(x+2)-3(x+2)$
7. $a^2+ab+ac+bc$
8. $3x^3+2x^2+3x+2$
9. $x^2+2ax+3bx+6ab$
10. $ax^2-bx^2+ay^2-by^2$
11. x^3-2x^2+3x-6
12. $5x^3-x^2-5x+1$
13. $h^3+h^2k+hk^2+k^3$
14. $6t^2+3ts-2at-as$
15. $acn^2-bcn+adn-bd$
16. $10x^3+2x-25x^2-5$
17. $x^3-12-4x+3x^2$
18. $x^4-a^2b-a^2x^2+x^2b$
19. $xy-ay-bx+ab$
20. $3x^3-4x^2-6x+8$
21. $y^3-4y-3y^2+12$
22. $18a^3+12a^2-15a-10$
23. $15x^3-12x^2+35x-28$
24. $9x^3-x^2-9x+1$
25. x^3-x^2+x-1
26. $a^3-7a^2-4a+28$
27. $abx+3x-10ab-30$
28. $2a^3-3a^2-4a+6$

216. The difference between two squares revealed. Consider the expression

$$a^2+2ab+b^2-c^2$$

The first three terms are at once recognized as a trinomial square:

$$(a^2+2ab+b^2)-c^2$$

The expression may thus be put in the form

$$(a+b)^2-c^2$$

which is the difference between two squares, and we have

$$[(a+b)+c][(a+b)-c]$$

or

$$(a+b+c)(a+b-c)$$

So also

$$\begin{aligned} x^2-10x+25-y^2 &= (x^2-10x+25)-y^2 \\ &= (x-5)^2-y^2 \\ &= (x-5+y)(x-5-y) \end{aligned}$$

So also

$$c^2-a^2+2ab-b^2$$

The last three terms have the form of a trinomial square with the single exception that the signs of the square terms are minus instead of plus. We can rectify this by putting them in parentheses, thus:

$$\begin{aligned} c^2-a^2+2ab-b^2 &= c^2-(a^2-2ab+b^2) \\ &= c^2-(a-b)^2 \\ &= [c+(a-b)]\,[c-(a-b)] \\ &= (c+a-b)(c-a+b) \end{aligned}$$

EXERCISES

Factor:

1. $(a-b)^2-4$
2. $9-(x-y)^2$
3. $y^2-(x+2)^2$
4. $a^2-(4x-5)^2$
5. $(2x-3)^2-b^2$
6. $(x-y)^2-(a+b)^2$
7. $(a-b)^2-25x^2$
8. $16a^2b^2-(x^2-a)^2$
9. $49a^4-(a-5)^2$
10. $(3a+8)^2-36a^4$
11. $(4x-2y)^2-16x^2y^2$
12. $4(a-b)^2-(x-y)^2$
13. $9(x+2y)^2-25(x+a)^2$
14. $(x+3)^2-9y^2$
15. $x^2-2x+1-y^2$
16. $x^2-a^2+2ab-b^2$
17. $1+2ab-a^2-b^2$
18. $x^2+y^2-z^2-2xy$
19. $4-n^2-p^2-2pn$
20. $a^2+b^2-c^2-d^2+2ab+2cd$
21. $4-4n+n^2-n^4$
22. $h^4+4h^3+4h^2-9$
23. $x^2+6xy+9y^2-25$
24. $x^2-14x+49-4y^2$
25. $y^2-4x^2-25-20x$
26. $y^2+4x^2-4xy-25$
27. $9x^2-30x+25-16x^4$
28. $a^2+b^2-c^2-d^2-2cd+2ab$
29. $a^2-c^2-d^2+4+2cd-4a$
30. $x^2+9y^2-a^2-b^2-6xy+2ab$

217. Factoring after expanding. Sometimes an expression may be reduced to a factorable form by expanding it first. Consider

$$\begin{aligned} (a+b)^2-4ab &= a^2+2ab+b^2-4ab \\ &= a^2-2ab+b^2 \\ &= (a-b)^2 \end{aligned}$$

The example just given illustrates the fact that it is not always possible to get the original expression by multiplying together its factors, although the product so obtained is always equal to the original expression, for

$$(a-b)^2=a^2-2ab+b^2$$

which is not the original expression.

EXERCISES

Factor:

1. $4ad+(a-d)^2$
2. $(x+1)^2-5x-20$
3. $a(a-24)+63$
4. $(x+7)^2-28x$
5. $(a+b)^2-4ab$
6. $25x(x-y)+5(5x-4y)$
7. $5x^2+1+3x(5x-3)$
8. $3x(3x+10)+25$
9. $4a(3a-8)+5$
10. $(n-7)(n+2)+n^2+11$
11. $4x(x-y)-5y(4x-7y)$
12. $k(k-1)-c(c+1)$
13. $6x+(x-3)^2-18$
14. $(x-3)^2-3(x-2)+5$
15. $2(x-1)(x+1)+3x$
16. $(x+2)(x-1)-(3x+1)$
17. $(x-8)(x-2)+2x^2-13$
18. $3x^2-(x-2)(x+2)+5x-7$
19. $x^3-1-(x-1)$
20. $(2x+1)^2-3x(x+2)$
21. $4x^2+(x^2-2)^2-5$
22. $(x^2+6)^2-25x^2$
23. $(x^2-2)(x^2+2)-3x^2$
24. $2x^2-x(x+1)-6$

APPENDIX B

218. Square roots of polynomials. It is possible to find the square root of many algebraic expressions, that is, to express them as the product of two equal factors. Only the simplest cases arise in common practice. The square roots of these can easily be found by inspection. These have already been considered in Art. **118.** One has but to recall the simple identity

$$(a+b)^2=a^2+2ab+b^2$$

$$\sqrt{4x^2-12xy+9y^2}=\sqrt{(2x-3y)^2}$$
$$=2x-3y$$

EXERCISES

Find by inspection the following square roots:

1. x^2-4x+4
2. $a^2-6ab+9b^2$
3. $y^2-10y+25$
4. $x^2+36+12x$
5. $49+n^2+14n$
6. $4x^2-12x+9$
7. $25+36x^2-60x$
8. x^4+8x^2+16

219. Square roots of polynomials by the division method. The method of inspection is not easily adapted to finding the square root of longer polynomials than those considered in the last article.

A method has been devised for finding the square root in such cases, but it is seldom required in mathematical work. It is the same method that is commonly used in arithmetic for finding the square root of a number.

The method is founded on the relation

$$a^2+2ab+b^2=(a+b)^2$$

or

$$\sqrt{a^2+2ab+b^2}=a+b$$

It is to be observed that the first term of the root is the square root of the first term of the number; that is, a is the square root of a^2. If the square of a is subtracted from the number, there remains

$$2ab+b^2$$

The first term of this remainder is the product of twice the first term of the root and the second term of the root. This suggests that we may find the second term of the root by dividing the remainder by twice the first term of the root as a trial divisor.

Now notice that the remainder may be factored:

$$2ab+b^2=(2a+b)b$$

This shows that the trial divisor $2a$ must be completed by the addition of b, which makes the complete divisor

$$2a+b$$

If this is then multiplied by b, the second term of the root, and subtracted from the remainder, nothing remains, and the root $a+b$ is obtained.

An illustration or two will show the process to the best advantage:

Illustration 1:

$$\begin{array}{lrl|l}
 & & & 7x-5 \\
 & & & 49x^2-70x+25 \\
 & a^2= & & 49x^2 \\
\text{Trial divisor} & 2a= & 14x & -70x+25 \\
\text{Complete divisor} & 2a+b= & 14x-5 & \underline{-70x+25}
\end{array}$$

$7x-5$ is the square root.

Illustration 2:

$$\begin{array}{lll|l}
 & & & x^2+2x+3 \\
 & & & x^4+4x^3+10x^2+12x+9 \\
a^2 & = & & x^4 \\
2a & = & 2x^2 & 4x^3+10x^2 \\
2a+b & = & 2x^2+2x & 4x^3+4x^2 \\
2a & = & 2x^2+4x & 6x^2+12x+9 \\
2a+b & = & 2x^2+4x+3 & \underline{6x^2+12x+9}
\end{array}$$

The fundamental process is repeated until the expression whose root is to be found is used up.

The following figures illustrate the reasons for the method from a graphic point of view.

I represents the expression.

II represents the first remainder.

III. The line $2a$ represents the trial divisor; the base of the rectangle represents the complete divisor.

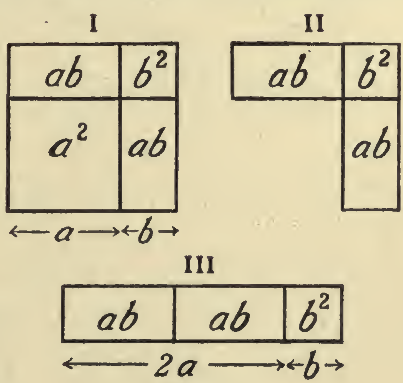

FIG. 69

EXERCISES

Find the square roots of the following polynomials:

1. $x^4+2x^3+3x^2+2x+1$
2. $4x^4+12x^3+13x^2+6x+1$
3. $n^6-2n^5+3n^4-4n^3+3n^2-2n+1$
4. $4y^4-12y^3-7y^2+24y+16$

220. Square root of arithmetical numbers. The square root of a number like 3721 can be found by the division method used in the last article. A simple modification is needed in order to determine what corresponds to the a^2, the first term of the number. Is it 3 or 37 in this case?

The square of a number of 1 digit has 1 or 2 digits.
The square of a number of 2 digits has 3 or 4 digits.
The square of a number of 3 digits has 5 or 6 digits.
The reverse statements are true:
A number of 1 or 2 digits has 1 digit in its square root.
A number of 3 or 4 digits has 2 digits in its square root.
A number of 5 or 6 digits has 3 digits in its square root.

Hence we mark off the numbers in pairs of digits beginning at the decimal point and marking each way.

$$\begin{array}{lrl}
 & & 60+1 \\
 & & 37\ 21 \\
a^2 = & & 36\ 00 \\ \hline
2a = & 120 & \vert\ 1\ 21 \\
b = & 1 & \vert \\ \hline
2a+b = & 121 & \vert\ 1\ 21 \\ \hline
\end{array}$$

The square root is 61.

$$\begin{array}{lrrl}
 & & & 200+30+2 \\
 & & & 5\ 38\ 24 \\
a^2 = & & & 4\ 00\ 00 \\ \hline
2a = & 400 & & \vert\ 1\ 38\ 24 \\
b = & 30 & & \vert \\ \hline
2a+b = & 430 & & \vert\ 1\ 29\ 00 \\ \hline
2a = & & 460 & \vert\ 9\ 24 \\
b = & & 2 & \vert \\ \hline
2a+b = & & 462 & \vert\ 9\ 24 \\ \hline
\end{array}$$

or, omitting the useless zeros,

```
           2  3  2                    3  5 .7
           5 38 24                   12 74 .49
           4                          9
     40  | 1 38                  60 | 3 74
      3  |                        5 |
     43  | 1 29                  65 | 3 25
     460  | 9 24                 700 | 49 49
       2  |                        7 |
     462  | 9 24                 707 | 49 49
```

If the figures of the root are placed above the proper group, there will be no difficulty in determining where the decimal point goes.

EXERCISES

Find the square roots of the following numbers:

1. 1156	2. 17424	3. 59049	4. .0841
5. 5329	6. 4489	7. 151.29	8. 4.9284

221. Approximate roots. The same method is used in finding approximate values for irrational roots. The only difference is that the process does not stop, but may be carried on as far as one chooses, zeros being annexed to the remainders as the work continues.

Find $\sqrt{3}$:

```
           1.7 3 2
           3.0000
           1
     20  | 200
      7  |
     27  | 189
    340    | 11 00
      3    |
    343    | 10 29
   3460      | 71 00        etc.
```

EXERCISES

Find the following square roots to four figures:

1. $\sqrt{2}$
2. $\sqrt{5}$
3. $\sqrt{18}$
4. $\sqrt{32}$
5. $\sqrt{7}$
6. $\sqrt{115}$

APPENDIX C

EXPONENTS

222. Positive integral exponents. We have been using a positive integer as an exponent to indicate the number of times a factor is to be used in a product.

$$a^5 \text{ means } aaaaa$$

We have worked with these exponents according to certain fixed rules or laws.

I. In multiplying,

$$a^5 \cdot a^3 = a^{5+3} = a^8$$

In multiplying we add the exponents of the same letter.

II. In finding a power,

$$(a^5)^2 = a^{2 \cdot 5} = a^{10}$$

In finding a power we multiply the exponent of the letter by the power to be found.

III. In dividing,

$$a^5 \div a^3 = a^{5-3} = a^2$$

In dividing we subtract the exponent of the divisor from the exponent of the same letter in the dividend.

IV. In finding a root,

$$\sqrt{a^6} = a^{\frac{6}{2}} = a^3$$

In finding a root we divide the exponent of the number by the number indicating the root.

223. Some peculiar expressions. In applying rules III and IV we come upon several very peculiar expressions.

$$a^5 \div a^5 = a^{5-5} = a^0$$
$$a^3 \div a^5 = a^{3-5} = a^{-2}$$
$$\sqrt{a^3} = a^{\frac{3}{2}}$$

224. Extension of exponent idea. If we are to be permitted to use Rules III and IV as in the last article, we must find some meaning for such expressions as a^0, a^{-2}, and $a^{\frac{3}{2}}$.

225. Meaning of zero exponent. The meaning of the zero exponent has been shown in Art. **141.**

$$a^5 \div a^5 = a^{5-5} = a^0$$

By ordinary division $\quad \dfrac{a^5}{a^5} = 1$

Therefore $\quad a^0 = 1$

EXERCISES

Find value of: 3^0, a^0, $2a^0$, $2xy^0$, ab^0, $a^2b^0 + 2ab + a^0b^2$, 27^0, 1921^0, 3×100^0.

226. Meaning of negative exponent.

$$a^3 \div a^5 = a^{3-5} = a^{-2}$$

But by reducing to lowest terms

$$\frac{a^3}{a^5} = \frac{1}{a^2}$$

Therefore $\quad a^{-2} = \dfrac{1}{a^2}$

A negative exponent means that the number over which it is placed is to be used as a divisor rather than as a factor.

EXERCISES

1. Find the value of: 2^{-1}, 3^{-2}, a^{-3}, a^{-5}, $3a^{-2}$, $2a^{-5}$, $2ab^{-2}$, 10^{-1}, 2×10^{-2}, 2.5×10^{-3}, 3×10^{-6}, $2^{-1} + 3^{-1}$, $2 \times 3^0 \times 5^{-2}$.

2. Express with negative exponents: $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{a}$, $\frac{1}{a^3}$, .00003, .00000062.

20

3. Express without negative exponents: a^{-1}, a^{-3}, ab^{-2}, $2ab^{-3}$, $3ab^{-1}x^{-2}$.

227. Meaning of a fractional exponent.

$$\sqrt{a^3}=a^{\frac{3}{2}}$$

Let us see what happens if we apply the multiplication law to $a^{\frac{3}{2}}$.

$$a^{\frac{3}{2}}\cdot a^{\frac{3}{2}}=a^{\frac{3}{2}+\frac{3}{2}}=a^{\frac{6}{2}}=a^3$$

Hence by definition $a^{\frac{3}{2}}$ is the square root of a^3.

The denominator of a fractional exponent indicates that a root is to be found, while the numerator denotes a power.

EXERCISES

1. Find the value of: $4^{\frac{1}{2}}$, $25^{\frac{1}{2}}$, $4^{\frac{3}{2}}$, $16^{\frac{3}{2}}$, $36^{\frac{1}{2}}$, $4^{\frac{2}{2}}$, $2^{\frac{1}{2}}$, $2^{\frac{3}{2}}$, $8^{\frac{1}{2}}$, $8^{\frac{3}{2}}$, $3^0\times2^{-1}\times49^{\frac{1}{2}}$, $2\times5^0\times10^{-2}\times36^{\frac{1}{2}}$, $8^{\frac{1}{3}}$, $8^{\frac{2}{3}}$, $8^{\frac{3}{3}}$, $4^{-\frac{1}{2}}$.

2. Express with fractional exponents: $\sqrt{5}$, $\sqrt{3}$, $\sqrt{a^5}$, $\sqrt{2a^3}$.

3. Express without fractional exponents: $a^{\frac{1}{2}}$, $a^{\frac{3}{2}}$, $a^{\frac{3}{2}}b^{\frac{1}{2}}$, $2a^{\frac{1}{2}}b^{\frac{1}{2}}$, $a^{-\frac{1}{2}}$, $a^{\frac{1}{2}}b^{-\frac{1}{2}}$.

228. Usefulness. Fractional and negative exponents arc used very frequently in higher mathematics. In fact, their invention made great advances in mathematics possible. If you go on to the further study of mathematics, you will find that there are still other kinds of numbers that can be used as exponents.

APPENDIX D

TABLE OF SQUARE ROOTS OF NUMBERS FROM 0 TO '99

	0	1	2	3	4	5	6	7	8	9
0	0.000	1.000	1.414	1.732	2.000	2.236	2.449	2.646	2.828	3.000
1	3.162	3.317	3.464	3.606	3.742	3.873	4.000	4.123	4.243	4.359
2	4.472	4.583	4.690	4.796	4.899	5.000	5.099	5.196	5.292	5.385
3	5.477	5.568	5.657	5.745	5.831	5.916	6.000	6.083	6.164	6.245
4	6.325	6.403	6.481	6.557	6.633	6.708	6.782	6.856	6.928	7.000
5	7.071	7.141	7.211	7.280	7.348	7.416	7.483	7.550	7.616	7.681
6	7.746	7.810	7.874	7.937	8.000	8.062	8.124	8.185	8.246	8.307
7	8.367	8.426	8.485	8.544	8.602	8.660	8.718	8.775	8.832	8.888
8	8.944	9.000	9.055	9.110	9.165	9.220	9.274	9.327	9.381	9.434
9	9.487	9.539	9.592	9.644	9.695	9.747	9.798	9.849	9.894	9.950

THE INDEX

[Bold-face figures indicate an extended treatment of the subject indexed. The numbers refer to pages.]