THE BOSTON COLLOQUIUM

LECTURES ON MATHEMATICS

DELIVERED FROM SEPTEMBER 2 TO 5, 1903, BEFORE MEMBERS OF THE AMERICAN MATHEMATICAL SOCIETY IN CONNECTION WITH THE SUMMER MEETING HELD AT THE MASSACHUSETTS INSTITUTE OF TECHNOLOGY BOSTON, MASS.

BY

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TO

PROFESSOR JOHN MONROE VAN VLECK, LL.D.

THESE LECTURES ARE AFFECTIONATELY INSCRIBED BY

HIS FORMER PUPILS

HENRY SEELY WHITE

EDWARD BURR VAN VLECK

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PREFACE.

For a number of years the American Mathematical Society has held a Colloquium in connection with its Summer Meeting at intervals of two or three years. In the circular sent out prior to the first Colloquium, in 1896, the purpose and the plan of the undertaking were described as follows:[1] "The objects now attained by the Summer Meeting are two-fold: an opportunity is offered for presenting before discriminating and interested auditors the results of research in special fields, and personal acquaintance and mutual helpfulness are promoted among the members in attendance. These two are the prime objects of such a gathering. It is believed however that a third no less desirable result lies within reach. From the concise, unrelated papers presented at any meeting only few derive substantial benefit. The mind of the hearer is too unprepared, the impression is of too short duration to produce accurate knowledge of either the content or the method. . . . Positive and exact knowledge, scientific knowledge, is rarely increased in these short and stimulating sessions.

"On the other hand, the courses of lectures in our best universities, even with topics changing at intervals of a few weeks, do give exact knowledge and furnish a substantial basis for reading and investigation. . . .

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[1] *Cf. Bull. Am. Math. Soc.*, ser. 2, vol. 3 (1896), p. 49.

"To extend the time of a lecture to two hours, and to multiply this time by three or by six, would be practicable within the limits of one week. An expert lecturer could present, in six two-hour lectures, a moderately extensive chapter in some one branch of mathematics. With some new matter, much that is old could be mingled, including for example digests of recent or too much neglected publications. There would be time for some elementary details as well as for more profound discussions. In short, lectures could be made profitable to all who have a general knowledge of the higher mathematics."

As a forerunner of the Colloquia here outlined may be mentioned the Evanston Colloquium of 1893, which followed the Congress of Mathematics held in connection with the World's Fair in Chicago, Professor Klein, of Göttingen, being the sole speaker. But whereas that Colloquium covered, in a descriptive manner, a variety of topics, — it comprised twelve lectures, — the Colloquia of the Society have been characterized by close contact with the actual analytical development of the topic treated.

The following Colloquia have been held:

I. The Buffalo Colloquium, 1896.

(*a*) Professor Maxime Bôcher, of Harvard University: "Linear Differential Equations, and Their Applications."

This Colloquium has not been published, but several papers appeared at about the time of the Colloquium, in which the author dealt with topics treated in the lectures.*

(*b*) Professor James Pierpont, of Yale University: "Galois's Theory of Equations."

This Colloquium was published in the *Annals of Mathematics*, ser. 2, vols. 1 and 2 (1900).

* Two of these papers were: "Regular Points of Linear Differential Equations of the Second Order"; Harvard University, 1896; "Notes on Some Points in the Theory of Linear Differential Equations," *Annals of Math.*, vol. 12, 1898.

II. The Cambridge Colloquium, 1898.

(*a*) Professor William F. Osgood, of Harvard University: "Selected Topics in the Theory of Functions."

This colloquium was published in the *Bulletin of the Amer. Math. Soc.*, ser. 2, vol. 5 (1898), p. 59.

(*b*) Professor Arthur G. Webster, of Clark University: "The Partial Differential Equations of Wave Propagation."

III. The Ithaca Colloquium, 1901.

(*a*) Professor Oscar Bolza, of the University of Chicago: "The Simplest Type of Problems in the Calculus of Variations."

Published in amplified form under the title: *Lectures on the Calculus of Variations*, Chicago, 1904.

(*b*) Professor Ernest W. Brown, of Haverford College: "Modern Methods of Treating Dynamical Problems, and in Particular the Problem of Three Bodies."

IV. The Boston Colloquium, 1903.

(*a*) Professor Henry S. White, of Northwestern University: three lectures on "Linear Systems of Curves on Algebraic Surfaces."

(*b*) Professor Frederick S. Woods, of the Massachusetts Institute of Technology: three lectures on "Forms of Non-Euclidean Space."

(*c*) Professor Edward B. Van Vleck, of Wesleyan University; six lectures on "Selected Topics in the Theory of Divergent Series and Continued Fractions."

This colloquium is here published in full.

At Commencement, 1903, Professor John Monroe Van Vleck, M.A., LL.D., completed his fiftieth year of service at Wesleyan University, and retired shortly after from the chair of Mathematics and Astronomy. All three of the speakers at the Boston Colloquium were former students of his, one of them being his son and colleague in the department of mathematics. It is fitting that this volume of lectures held at that Colloquium be inscribed to him.

THOMAS S. FISKE,
WILLIAM F. OSGOOD,
Committee on Publication.

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Chapter 1.

Transition from Plane Curves to Surfaces.

The notion of equivalence as formulated in projective geometry has simplified greatly the study of algebraic curves and surfaces, particularly those of low order. The next step toward a wider survey is the admission of all birational transformations of the plane, or of space of three or more dimensions. In the plane, the theory of Cremona transformations is no longer new, and the elements are familiar to all students of geometry. Not so, however, in space of more than two dimensions; probably for the reason that nothing is known analogous to the theorem that a plane Cremona transformation is resolvable into a succession of quadric transformations and collineations. And even in plane geometry the intricacies of the transformations themselves have kept most students from the matter of higher importance, the properties of figures that remain invariant under all transformations of the group. Yet there does exist a body of doctrine under the accepted title of "Geometry on an algebraic curve," and a fair beginning has been made upon a similar theory, the "Geometry on an algebraic surface." * These titles are intended to cover

* Consult, for an outline of the geometry upon an algebraic curve, Pascal's *Repertorium der höheren Mathematik*, Part II, Chapter V, § 4; or the more extended articles: C. Segre, "Introduzione alla geometria sopra un ente algebrico semplicemente infinito"; E. Bertini, "La geometria delle serie lineari sopra una curva piana secondo il metodo algebrico," — both in *Annali di Matematica*, ser. 2, vol. 22 (1894). For the corresponding theories regarding surfaces, the best reference is to the comprehensive summary by Castelnuovo and Enriques: "Sur quelques récents résultats dans la théorie des surfaces algébriques," *Math. Annalen*, vol. 48 (1896). Supplementary results are summarized in a later paper by the same authors: "Sopra alcune questioni fondamentali nella teoria delle superficie algebriche," *Annali di matematica pura ed applicata*, ser. 3, vol. 6 (1901).

only such properties of a curve or surface as appertain to the entire class of curves or surfaces that can be related birationally to the fundamental form.

A plane algebraic curve may have its order changed by a Cremona transformation, but not its deficiency (genre, Geschlecht). As to sets of points on the curve, two sets which together make up a complete intersection of a second curve with the first do not lose that property by birational transformation, if we exclude from consideration fundamental points introduced by the transformation itself.* Mutually residual sets of points, and corresidual sets, preserve their relation. Hence the group of sets of points corresidual with any given set becomes of importance. If a given set of D points lies on a curve of deficiency p, and if a corresidual set can be found containing k arbitrary points, then these numbers are connected by the relation constituting the Riemann-Roch theorem

$$k = D - p + \rho,$$

where ρ is zero if $D > 2p - 2$.

The totality of all sets of D points corresidual to any one set is termed a group or *series*, and is denoted by a symbol g_D^k. Such a series is called *complete*. If by any algebraic restrictions a series is separated out from it, of course that would be called incomplete or partial. For example, on a plane nodal cubic a series g_3^2 is cut out by all straight lines, incomplete because any three arbitrary points of the curve are corresidual to any other three. Every series g_D^k can be cut out upon the fundamental curve by a linear system of auxiliary curves whose equation may be written, with k parameters:

$$F_0 + l_1 F_1 + l_2 F_2 + \cdots + l_k F_k = 0.$$

As on a single curve sets of points, so in a plane, linear systems of curves are studied. By every birational transformation, linear

* Or if we employ no auxiliary curves except such as are adjoint to that containing the point sets.

systems are carried over into linear systems. A complete linear system is defined most easily by specifying the multiplicity that a curve of the system must have in each point of a fundamental set, and by prescribing the *order* of the curves. Thus $\binom{a_1\ a_2 \cdots}{s_1\ s_2 \cdots}$ can indicate that in a_1 every curve is to have a multiple point of order at least s_1, etc. If the base points alone, with their respective multiplicities, determine a system under consideration, that system is termed *complete.* If the base points actually impose, for curves of order m, fewer conditions than would be expected from their several multiplicities, the system is *special;* otherwise it is *regular.* It is an important theorem that no set of r base points can be so *located* as to produce an $(r + 1)$th variable multiple point on the curves of the system; *i. e.*, the multiple points of the generic curve of a plane linear system lie all in the base points of the system.

Adjoint curves of a linear system are familiar to the student of function theory; they have in every multiple point of order s for the given system a multiplicity of order at least $s - 1$. The adjoints of order lower by 3 than the original system are important from the fact that they transform always into the corresponding system of adjoints to the transformed curves. On this account the term adjoint, as used ordinarily, implies a curve of order $m - 3$ unless differently specified. *Second adjoints* are adjoint to adjoints of the system, etc. The employment of successive adjoint systems as a means of investigation is due to S. Kantor and to G. Castelnuovo, the latter acknowledging the priority of the former.* On every curve its adjoints cut out a unique complete series g_{2p-2}^{p-1}, called the *canonical series.* The deficiency of the first or second adjoints of a linear system is denoted by P_1 or P_2, and may be termed first, or second, canonical deficiency. Aside from the canonical series upon curves of a system, the most important are the *characteristic series* of the system, that is the totality of sets of points in which two curves of the system intersect. If a plane linear system is complete, then the *characteristic series* on each

* See *Math. Annalen,* vol. 44 (1894), p. 127.

curve is a *complete series* upon that curve. So far the definitions and propositions refer to curves in a plane; the question is in order whether they can be transferred to systems of curves lying upon curved surfaces.

First, it is noticed that by means of a linear system of curves the plane may be related point for point to a surface in space of three or more dimensions.* If the system is k-fold infinite, $k + 2$ members of the system can be related arbitrarily to $k + 2$ hyperplanes in space of k-dimensions. Take $k = 3$ for ease; then a curve of the system

$$u_1 f_1 + u_2 f_2 + u_3 f_3 + u_4 f_4 = 0$$

may be assigned to a plane $(u_1 : u_2 : u_3 : u_4)$ in ordinary space. Curves through one point become then planes through one point, and the ∞^2 points of the plane become the ∞^2 points of some algebraic surface F. All such surfaces are called *rational*. Similarly a linear family of curves triply infinite upon any surface relate that surface point for point to another surface in threefold space, linear systems of curves in one giving rise to linear systems upon the other, and *the transformed system will lack fundamental or base points*. The value of such projectively related pictures of a linear system of curves was first emphasized by C. Segre.

Secondly, there are surfaces not rational. For example, there are irrational ruled surfaces. But for many purposes, ruled surfaces and rational surfaces are classed together and constitute, with their equivalents, a small, indeed an exceptional, class in the vast field of algebraic surfaces. Planes are also *regular surfaces*, that is, they have their geometrical and numerical (or arithmetical) deficiencies equal, as will be explained directly. On regular surfaces, most of the theorems upon linear systems of curves on rational surfaces retain their validity; not so on the irregular. New characters crop out in the systems of curves, characters which indicate the nature of the surface. But the linearity of a system

* Exceptional cases are discussed by Enriques: "Ricerche di geometria sulle superficie algebriche," *Torino Memorie*, ser. 2, vol. 44 (1893), p. 178.

of curves is still susceptible of precise definition, and that in two ways whose equivalence constitutes an important theorem.

If on any surface, rational or not, there exists a system of curves doubly infinite, such that two arbitrary points determine one and only one curve containing them, that may be termed a linear net upon the surface in question; and Enriques proves that the ∞^2 curves of such a system can be *projectively* related to the straight lines of a plane. If the series is ∞^3, and if three arbitrary points determine uniquely a curve of the system which shall contain them, then its curves are referable projectively to the planes of three-space, etc. Only *simply* infinite systems escape this far-reaching theorem, and thus give rise to a most interesting unsettled question, indicated by Castelnuovo.*

Definitions of residual and corresidual curves upon a surface are those which any one could formulate at once from the use of these terms for sets of points upon a curve; their significance upon a twisted curve is the same as upon its plane projection. So of complete systems, both of curves and of surfaces, the latter admitting of course multiple curves as well as base points. For a surface of order m, the adjoints invariantively related are of order $m-4$, containing as $(s-1)$-fold curve every s-fold curve of the given surface. If these first adjoint surfaces form a k-fold infinite linear system, the number k is an invariant of the surface and is termed its *geometric deficiency* (p_g). Attempting to express this number in terms of the order m of the surface, the order d and deficiency π of its double curve (if any), and of the number t of triple points on this double curve, one would find a second number

$$p_n = \tfrac{1}{6}(m-1)(m-2)(m-3) - d(m-4) + 2t + \pi - 1,$$

called the *numerical deficiency* of the surface. This number also is an invariant of the surface, as Noether first proved, and may

* Castelnuovo: "Alcuni risultati sui sistemi lineari di curve appartenenti ad una superficie algebrica." *Memorie di matematica e di fisica della Società Italiana delle Scienze*, ser. 3, vol. 10 (1896), pp. 82–102. See especially the close of his preface.

be *either equal to or less than* p_g, but never greater. Rational surfaces have $p_g = p_n = 0$; ruled surfaces have p_n negative. If $p_g = p_n$, then the above-mentioned theorem of Enriques concerning linearity holds true also for systems which are only simply infinite. Surfaces of the first adjoint system cut out upon a given surface a system of curves, each of deficiency $p^{(1)}$ or less. This invariant number $p^{(1)}$ we may call the *canonical deficiency* of the surface; the curves form an unique complete linear system, just as do the point sets of the canonical series on a plane curve.

The definitions here given are but a part of those found useful in this fascinating branch of geometry. The true way to learn something of the subject is not to master first all its definitions and distinctions, but to study the proofs of some few leading theorems. Such are Enriques's proof of the equivalence of two geometrical definitions of the linearity of a system (mentioned above), and the following less elementary propositions:

1. Surfaces whose plane or hyperplane sections are irreducible unicursal curves are either ruled or rational (Noether).*

2. So also surfaces whose plane or hyperplane sections are irreducible elliptic curves (Castelnuovo),† or hyperelliptic of any deficiency π (Enriques).‡ For plane sections, not hyperelliptic, of deficiency $\pi > 2$, the corresponding theorem is not yet fully known.§ The proof of this theorem I shall give in full.

3. Upon any algebraic surface $f(x, y, z, t) = 0$ a linear differential of first kind is said to exist (Picard), if an expression involving four rational functions P_1, P_2, P_3, P_4, of the coördinates:

$$\int [P_1(x, y, z, t) \cdot dx + P_2 \cdot dy + P_3 \cdot dz + P_4 \cdot dt]$$

is finite and determinate, independent of the path of integration,

* Noether's theorem is more general. See *Math. Annalen*, vol. 3 (1871): "Ueber Flächen, welche Schaaren rationaler Curven besitzen."

† "Sulle superficie algebriche," etc., *Lincei Rendiconti*, January, 1894.

‡ "Sui sistemi lineari," etc., *Math. Annalen*, vol. 46 (1895), pp. 179–199.

§ For full information, see the second paper, cited above, of Castelnuovo and Enriques. I regret that this paper had not come to my notice before giving these lectures.

when taken upon the surface between any two arbitrary points. If the surface $f = 0$ is a *cone*, such differentials exist, for they are the abelian differentials of first kind upon its plane sections. Picard proves* that if the surface $f = 0$ have no multiple points or curves, then no such differential can exist upon it. There are however surfaces of all orders above the third which contain (or admit) one such integral; others, from the sixth order upward, which admit two, and so on. These surfaces and the mode of discovering them and of defining them have been the occasion of some of the most interesting studies of *Picard* and *Humbert*. The elementary part of Picard's first paper upon this topic I shall give in some detail, indicating in conclusion certain points that might prove worthy of further study.

Chapter 2.

Linear Systems of Curves on an Algebraic Surface. The Two Geometric Definitions are Concordant.

In plane geometry a linear system of algebraic curves is defined analytically by an equation containing linearly and homogeneously two or more parameters; as for example:

$$\lambda_0\phi_0 + \lambda_1\phi_1 + \lambda_2\phi_2 + \cdots + \lambda_\kappa\phi_\kappa = 0,$$

the λ's being parameters, and the ϕ's a set of polynomials homogeneous of like degree in the current coördinates. This is called a κ-fold infinite (∞^κ) linear system. As we restrict our field to include only systems defined by fixed base points, the curves $\phi_i = 0$ must be supposed all to contain the base points of the system. In a plane such a system may be studied by means of its equation, but for other surfaces one must either assume an analytic representation as definition, or else take such geometric features of a plane linear system as seem most important and transfer them to sets of curves on surfaces in general. We follow

* Picard et Simart: *Théorie des fonctions algébriques de deux variables indépendantes*, vol. 1 (1897), pp. 119–120.

the latter plan, and two definitions naturally present themselves.

First, using an auxiliary hyper-space of as many dimensions as the system of curves exhibits, an ∞^r system of curves on an algebraic surface is called linear if its elements (the individual curves) can be put in correspondence one-to-one, projectively, with the hyper-planes of a space of r dimensions, S_r.

Second, using no auxiliary outside the points of the surface itself, an ∞^r system of curves on an algebraic surface is called linear if through r generic points of that surface there passes one and only one curve of the system. This definition is to be used only when $r > 1$. For if $r = 1$, the generators of a ruled surface would fall under this definition, and one sees immediately the impropriety of calling them a linear system.

Notice that a system linear under the first definition must also be linear under the second. For by relating curves to hyperplanes we relate the algebraic surface F to a new surface F' in S_r, as explained in the preceding chapter; and through r points on F' there will pass one hyperplane, hence through r points in F there will pass one curve of the system and no more. The first definition therefore includes the second; does the second include the first? We shall show that it does, so that the two definitions shall be proven equivalent for all cases except $r = 1$, that is, for all except linear sheaves or pencils. The proof is essentially that of Enriques * as presented by Segre.†

Two lemmas may well precede the theorem.

LEMMA 1. *Projectivity of two flat spaces.* Two flat spaces of n dimensions, S_n and S'_n, can be projectively related by assigning to any $n + 2$ generic hyperplanes or S_{n-1}'s of the first any $n + 2$ generic S'_{n-1}'s of the second, one to one, as corresponding forms. The proof is by mathematical induction; to gain a clear idea of it, state it for points instead of hyperplanes, and model the transition from S_n and S'_n to S_{n+1} and S'_{n+1} upon von Staudt's ‡ transition

* Enriques: "Una questione sulla linearità dei sistemi di curve appartenenti ad una superficie algebrica." Rome, *Lincei Rendiconti*, July, 1893.

† Segre: *Loc. cit.* in *Annali di Matematica*, ser. 2, vol. 22, § 27.

‡ von Staudt: *Geometrie der Lage*, p. 69.

from S_2 to S_3. In S_3 take any 5 points A, B, C, D, E, such that no four lie in a plane, and in S_3' similarly A', B', C', D', E', like letters denoting corresponding points. In the plane $C'D'E'$ or S_2' call P' the point of intersection with the line $A'B'$, and in CDE or S_2 let P be on the line AB. As P and P' must correspond, this gives 4 points in S_2 corresponding to 4 fixed points of S_2', and therefore by hypothesis fixes the projectivity between the two planes. The pencil of rays in S_2 through P corresponds projectively to that through P' in the other plane, S_2'. If now Q denotes any point of S_3, to find its corresponding point Q' in S_3' let Q be projected from A and from B into two points A_1 and B_1 of the plane S_2. These are collinear with P, and we can find their corresponding points A_1' and B_1' collinear with P' in S_2', and so, by using A' and B' as centers of projection, the point Q' desired. Points on the line AB itself have their corresponding points fixed by the assignment of 3 points A, B, P to the points A', B', P' respectively in the line $A'B'$.

Lemma 2. In an ∞^2 algebraic system of irreducible curves upon an algebraic surface, if the system is linear according to the second definition, then the points of the surface form sets of n (some finite number), such that if a curve of the system contains one point of any set it must necessarily contain also the other $n-1$ points of that set.

The proof rests on the algebraic characters of the system. Call the system (C) and any curve of the system C_i. Select any point A_1 of the surface. It does not determine a curve. Let C_1 and C_2 be any two irreducible curves through A_1. They intersect in $n-1$ other points $A_2, A_3, \cdots, A_n$, $(n \geqq 1)$. Since two of these points, *e. g.*, A_1 and A_2 lie on two curves, they must lie on an infinity of curves; *i. e.*, it will require at least one additional point to determine a single curve from among those that contain both A_1 and A_2. If P is a generic point not on all curves that contain A_1, then by hypothesis the two points A_1 and P determine one curve, which shall be denoted by C_3. Also among the curves that contain A_1 and A_2, at least one will contain the additional point P.

This can be none other than C_3, hence the curve which is determined by A_1 and the generic point P will contain also A_2. By parity of reasoning it must contain as well $A_3, \cdots, A_n$. But as P was any point, C_3 was any curve through A_1, consequently every curve of (C) that contains an arbitrary point A_1 must contain also $n-1$ other determinate points, as asserted by the lemma.

The principal theorem can now be proven if two facts are established. First the theorem should be found to subsist for the particular case $r = 2$, so that the base may be provided for a mathematical induction. Then, secondly, the mode of induction employed in Lemma 1 must be shown to be applicable to a system of curves conforming to the second definition.

Particular Theorem. *A doubly infinite algebraic system of irreducible algebraic curves upon any algebraic surface can be brought into a one-to-one relation with the system of all lines in a plane by assigning to four arbitrarily chosen curves of the system (no three through one point), four arbitrarily chosen lines of the plane (no three through one point), as corresponding lines, and by requiring further that to curves having a common point shall correspond lines with a point in common.*

To prove this, associate every set of m points, such as the $A_1, A_2, \cdots, A_m$, of Lemma 2, together as one element A. Then there is upon the surface an ∞^2 system of C's and a second system of A's related thus: Two generic C's determine one A and two A's determine one C. Now these are precisely the incidence relations upon which depends the familiar proof that four lines of one plane and four of another determine a projectivity of the two systems of lines; here the lines and points of the one plane are replaced by elements C and A. The requisite of continuity is provided for by the hypothesis that the system is of algebraic character. Therefore the lines of a plane and the curves of the system (C) stand in a one to one relation, as asserted by the theorem. This relation is called projective, meaning that it is independent of the particular four pairs, line and curve, that may be selected to determine the correspondence. Otherwise stated:

If the lines of two planes are related in the mode above described to the curves of a system, the planes are thereby projectively related to each other.

As to the second matter, it is needful to show that the elements used as auxiliaries in Lemma 1 have unique analogues in a system, triply infinite, of curves conforming to the second definition. What were called points there have become curves here, hence the lines and planes must be replaced by ∞^1 and ∞^2 systems of curves. We need only examine, accordingly, whether the postulate: a line and a plane intersect in one point, retains its validity. Let a "line" be given by two curves, a "plane" by three; or to adhere more closely to the definition, consider an S_1 given by two points, a and b, and an S_2 consisting of all the curves of the ∞^3 system S_3 that pass through a third point c. Then will S_1 and S_2 have in common one and only one curve. For in the S_3 there is an S_2' containing the point a; in this S_2' there is one curve C that contains the points b and c (and by the explanations of the above theorem we see that it must contain all the intersections of any two curves fixing the S_1). As containing a and b it lies in S_1; as containing c it lies in S_2, and as containing these three arbitrary points it is by the definition unique. Therefore, all the constructions of Lemma 1 have their unique analogues in the system S_3.

We conclude that the transition from an ∞^2 system to one ∞^3 is possible, and that for $r = 3$ the first and second definitions are equivalent. *Mutatis mutandis*, the induction from $r = m$ to $r = m + 1$ can be made by similar means. Recapitulating we have therefore the theorem:

An ∞^r algebraic system of irreducible algebraic curves upon any algebraic surface is linear if either (1) its elements can be put in a one-to-one correspondence, projectively, with the hyperplanes of an r-fold space; or (2) if through r generic points of the surface there passes one and only one curve of the system. For $r > 1$ these two defining properties can be inferred, each from the other.

Chapter 3.

Surfaces whose Plane Sections are Hyperelliptic Curves.

Plane curves of any deficiency above 1 may be hyperelliptic, and those of deficiency 2 are necessarily so. The specific feature of an hyperelliptic plane curve of order n is this, that its adjoint curves of order $n-3$, its "ϕ-curves," arrange its points in pairs. That is, if a ϕ-curve contains any one point P of a hyperelliptic curve C, it will of necessity contain a second determinate point Q of C; then P and Q form what is called a *conjugate pair*; each is the *conjugate point* of the other. It is well known that a ϕ-curve can be found which shall contain $p-1$ arbitrary points of C, where p denotes the deficiency of the curve C. These facts lead to interesting conclusions about any linear system of hyperelliptic curves in a plane, or in any rational surface.

In a plane, a linear system of hyperelliptic curves may be of the first or second kind. In a system of the first kind, a curve passing through any one point is not thereby necessitated to pass through a determinate second point; in a system of the second kind this compulsion does exist, and all curves of the system that contain a point P contain also Q, its conjugate point. Of the second kind, for example, is a certain family of plane sextics having double points in seven common points of three cubics: $\phi_1 = 0$, $\phi_2 = 0$, $\phi_3 = 0$. The equation

$$\sum C_{ik}\phi_i\phi_k = 0 \qquad (i, k = 1, 2, 3)$$

gives a linear system of sextics, the C_{ik} being arbitrary. Outside of the seven base points, let any point P be on both cubics:

$$\phi_1 = 0 \qquad \text{and} \qquad \phi_2 = 0.$$

Their ninth intersection, Q, is determined by the eighth, a familiar theorem; and sextics of the system which pass through P, being given by the equation (according to Noether's theorem)

$$C_{11}\phi_1^2 + 2C_{12}\phi_1\phi_2 + C_{22}\phi_2^2 + \phi_3(C_{13}\phi_1 + C_{23}\phi_2) = 0$$

must contain also the remaining intersection Q of $\phi_1 = 0$ and

$\phi_2 = 0$. Notice that ϕ_1, ϕ_2, ϕ_3 are adjoint ϕ's of all sextics of the system, so that Q is the conjugate point of P on every sextic that contains them.

We mention systems of this second kind, only in order to exclude them from further discussion here. Let (H) be an ∞^3 system of the first kind of hyperelliptic plane curves H_1, H_2, etc., of order n, and let (ϕ) be the system of adjoint curves of order $n-3$, *i. e.*, let the curves $\phi_1, \phi_2, \cdots$ have as $(i-1)$-fold points the i-fold base points of the system (H). Consider any point P of the plane. Its conjugate Q on any curve H of the system must lie, by definition, upon every ϕ-curve containing P. Since Q is a variable point, its locus must needs form a part of every ϕ-curve through P, and these ϕ-curves accordingly must be degenerate. By parity of argument every ϕ-curve must consist of $(p-1)$ distinct parts where p is the common deficiency of curve H, and each part must intersect every curve H in only two points, a conjugate pair, outside the multiple base points of the system (H). For an example of this, let the system (H) consist of all curves of order n having in a fixed point O a multiple point of order $n-2$. Any ϕ-curve must have in O an $(n-3)$-fold point, and is itself of order $n-3$, therefore it will consist of $n-3$ right lines through O. Every constituent right line has with any curve H $n-2$ intersections in O, and two outside that point; the latter two are conjugate points on the curve, which is consequently hyperelliptic. Another example, with the ϕ's compounded of conics, is the system of curves of order $2m+3$ with four fixed multiple points of order $m+1$. The fact that for these plane systems the points conjugate to a given point fill out a definite locus is the thing to which we shall wish to recur.

In space of three dimensions, let a surface F have all its plane sections hyperelliptic curves (C) of deficiency p. Can these be represented by a system of curves all in one plane? Is the surface F rational, *i. e.*, transformable into a plane, point-for-point, rationally? This question again may be approached by the aid of conjugate pairs of points. We should expect of course that analogues of ϕ-curves would be, in space, ϕ-surfaces, and that those that pass through a point P on any curve of the system would contain its conjugate Q; and further, that all points Q conjugate to P would lie on some determinate curve of the surface. This last supposition can be established by *reductio ad absurdum.*

The surface F and its plane sections (C) are algebraic. Each curve C containing a selected point P has, by hypothesis, one particular point Q conjugate to P. Therefore the Q's on the ∞^2 curves through P must suit one of the following three descriptions.

(a) They may be finite in number, $Q_1, Q_2, \cdots Q_k$. But then every plane section of the surface through P would need to contain the line PQ_1, or $PQ_2, \cdots$, or PQ_k. This is absurd.

(b) The Q's may be simply infinite, ∞^1 in number, filling one or more algebraic curves on the surface, or lastly —

(c) The Q's may fill all the ∞^2 points of the surface F. We shall reject this after showing that in this case the surface must be rational; *i. e.*, rationally and reversibly transformable into a plane, whereas on the contrary, in a plane or any other rational surface the Q's must be only a simple infinity, ∞^1.

Suppose, therefore, that every point Q of the surface is conjugate to a given P upon some one or more curves of the system. It cannot be so upon all plane sections through the secant PQ, for then must every point of any plane section be conjugate to P on that section, contrary to the hyperelliptic hypothesis. Through every ray PQ there lie then a finite number r of planes in which P and Q are conjugate. Any one of these determines all the others, for P and the plane through P fix Q, and the rest follow. Now such a grouping of the planes through P into sets of r planes, each set being determined by any one of its planes, is called an involution. Castelnuovo * proves the remarkable theorem, *that every involution of the planes about a point in space of three dimensions is rational;* *i. e.* its groups can be correlated univocally and reversibly to the points of a plane, each group to one point.

* *Math. Annalen,* vol. 44 (1894), pp. 125–155.

Thus through the involution every point Q of the surface can be related to some one point of an auxiliary plane, and *vice versa*. But if the surface F be transformed algebraically and univocally into a plane, then its plane sections will be transformed into a linear system of hyperelliptic curves in that plane, conjugate points going into conjugate points: whereas we have seen that in a plane the conjugates Q of P do not fill the whole plane, but only an ∞^1 locus. Supposition (c) is thus dismissed, and (b) alone is tenable.

We have then as a starting point this fact, that for a generic point P of the surface there is a definite curve p containing all its conjugates Q on the curves of the system (C); and this curve p can meet each curve (each plane) only once outside the point P itself. If then p is of order s, it must have in P an $(s-1)$-fold point. It must also be a plane curve; for a plane can be passed through P and two arbitrary points of p, and will contain $s-1+1+1=s+1$ points of a curve of order s, hence must contain the entire curve p. *This curve p can be shown to be either a straight line or a conic.*

If p is not a line or a conic, its order s must be at least 3, whence it must have in P a double point $(2 \leqq s-1)$ or multiplicity of higher order. As p is a plane curve, this means that its plane is tangent in P to the surface F; and so that every line joining P to a conjugate Q is a tangent in P to the surface, and by symmetry of the relation between P and Q, tangent also to F in Q. This is not possible unless either the curve p is a curve of plane contact (so that P would be an exceptional point of F), or else the curve p consists wholly of straight lines through P. This alternative is equally impossible, as no ruled surface has through every point three or more generators. Therefore the hypothesis $s \geqq 3$ leads to absurdity; and we have to examine the two possible cases: $s=1$ and $s=2$.

$s=1$. If p is a straight line, it does not contain P, since $s-1=0$. To P is associated one generator p of the ruled surface F, and conversely, to every point Q of p must be associated the generator q passing through P. *F is then a ruled surface*, of hyperelliptic section, with its generators arranged in pairs cutting conjugate points in every plane.

$s = 2$. If p is a conic, three cases can be distinguished. First, to every point Q on p may belong a conic q containing P but different from p, and these may be in themselves complete plane sections of the surface. If this were so, the surface would be a quadric. But the conics may not be complete plane sections of the surface, and this possibility it is convenient to divide into two parts, as follows: Secondly, the q's may be conics distinct from the p's. The surface F will contain in this second case a *doubly infinite system of decomposable or reducible plane sections.* Or thirdly (the only case not trivial), the conic q, while its corresponding point Q describes the conic p, may continually coincide with p. There is then only a simple infinity of conics (p) upon the surface. To show that this system is a rational sheaf, consider its section by an arbitrary plane: on the hyperelliptic section curve each conic p cuts two conjugate points P and Q, and either P or Q determines p completely, hence the system (p) is in one-to-one relation with the series of pairs of conjugate points upon a hyperelliptic plane curve — a linear series, and must therefore be rational.* Now these three alternatives lead to a single conclusion, through the application of well known theorems.

First if the surface F were quadric, it would be rational; but then it would be discussed as a surface with all its plane sections rational.† For the second case we adduce Kronecker's theorem ‡

* Indeed these planes form the developable of a twisted cubic curve, since no one of them counts twice; Castelnuovo shows that the immediate generalization of this remark holds for hyperspace.

† See paper by E. Picard: "Sur les surfaces algébriques dont toutes les sections planes sont unicursales," *Crelle's Journal*, vol. 100 (1885); and a correlated paper of E. H. Moore: "Algebraic surfaces of which every plane section is unicursal in the light of n-dimensional geometry," *Amer. Jour. of Math.*, vol. 10 (1888), p. 17.

‡ See the historical note and demonstration by Castelnuovo. "Sulle superficie algebriche che ammettono un sistema doppiamente infinito di sezioni piane riduttibili," *Lincei Rendiconti*, January, 1894.

that *a surface having a double infinity of plane sections that are decomposable curves is either a Steiner's quartic surface or a ruled surface.*—Steiner's "Roman Surface" is the quartic having three double lines through a triple point, and is rational, since it can be projected from its triple point upon a plane. The third case is decided by *Noether's* theorem * that *a surface containing a rational system of rational curves is rational.*

The conclusion can be condensed now into the form : *Every algebraic surface whose plane sections are hyperelliptic curves of deficiency 2 or more is either (1) a ruled surface or (2) a rational surface, and in the latter alternative it contains a rational sheaf of conics.* This latter phrase obviously rejects two of the alternatives of the preceding paragraph, and this is warranted by the rationality of the surface, the representative system of plane curves being therefore the criterion. For we recall that in a linear system of plane hyperelliptics the p-curves and q-curves discussed above are component parts of the degenerate ϕ-curves, and a p coincides with all its q's.

This highly general theorem allows us to study upon plane systems the geometry upon an extensive family of surfaces in space and in hyperspace as well, since the existence of a triply infinite linear system of hyperelliptic curves in a surface is equivalent to the hypothesis that we have used concerning plane sections in ordinary space. And for linear systems of the first kind in a plane reduced types have been found by Castelnuovo,† from which all others are derived by Cremona transformations. It remains to develop to the same extent the theory of systems of the second kind. This would demand an acquaintance with the work of Bertini on plane involutions of index 2, and of Clebsch and Noether on rational double planes.

An extension in another direction has been given by Castel-

† M. Noether : "Ueber Flächen, welche Schaaren rationaler Curven besitzen," *Math. Annalen*, vol. 3 (1871), pp. 173–4. The theorem is more general than that here cited.

* "Sulle superficie algebriche le cui sezioni piane sono curve iperellittiche." *Palermo Rendiconti*, vol. 4 (1890), pp. 73–88.

nuovo,* who discussed under only one specializing restriction the surfaces whose plane sections are of deficiency 3. These are of four kinds, so far as numerated, and not all rational.† In looking for other possible extensions, it should be remembered that there are other classes of highly specialized curves, differing from the hyperelliptic in the degree of the singular series of special groups which occurs upon them. Of such classes, individual curves have received some study, but linear systems little or none.

Chapter 4.

Linear Exact Differentials of the First Kind on an Algebraic Surface.

§ 1. *The Existence of Integrals on Given Surfaces.*

When the theory of integrals upon algebraic curves was extended to surfaces, the first step was the discussion of double integrals. These have been described already (Chapter 1), and attention has been called to two important numbers, characteristic of a surface, to which they give rise, the geometrical and the numerical deficiency. Every surface above the lowest orders possesses double integrals of the first kind, everywhere finite, unless its singularities have become too numerous. The increase of singular points and lines causes a diminution of the geometrical deficiency, p_g. Double integrals and their classification were introduced by Clebsch and Noether about 1870. Fifteen years later a different and even more interesting extension of curve theory to surfaces was made by Picard.‡ The new integrals that he introduced are simple integrals whose path of integration is restricted to

* "Sulle superficie algebriche le cui sezioni sono curve di genere 3." *Torino Atti*, vol. 25 (1890).

† If the surface is of order above the *fourth*, with plane sections all of deficiency 3, it is rational. See Castelnuovo and Enriques "Sopra alcune questioni fondamentali nella teoria delle superficie algebriche," *Annali di Matematica*, ser. 3, vol. 6 (1901), esp. Sec. V, § 16.

‡ "Sur les intégrales de différentielles totales algébriques de première espèce," *Jour. de math.*, ser. 4, vol. 1 (1885).

lie in the surface, while the integrals are further required to be functions of their limit points alone, not of the particular path of integration. The number of linearly independent, everywhere finite, integrals of this kind is a new invariant characteristic of the surface; and it is found that this number is zero when the surface is non-singular, but increases (according to a law not precisely known) with the multiplication of singularities. This is the theory of which I propose now to give a sketch, following very closely Picard's article cited above.*

Upon an algebraic surface

$$f=f(x, y, z)=0$$

a linear differential expression in dx, dy, dz can be reduced by the use of the relation:

$$f'_x dx + f'_y dy + f'_z dz = 0. \tag{1}$$

By this means any expression of the form

$$Adx + Bdy + Cdz$$

may take on either one of the three aspects:

$$\frac{Bf'_x - Af'_y}{f'_x}dy + \frac{Cf'_x - Af'_z}{f'_x}dz.$$

$$\frac{Cf'_y - Bf'_z}{f'_y}dz + \frac{Af'_y - Bf'_x}{f'_y}dx,$$

$$\frac{Af'_z - Cf'_x}{f'_z}dx + \frac{Bf'_z - Cf'_y}{f'_z}dy,$$

Let the first be chosen, and abbreviate it to

$$\frac{Qdx - Pdy}{f'_z}.$$

Concerning this expression two things are to be noted. First, if the surface be cut by an arbitrary plane, then by the adjunction of the equation of that plane this must become an abelian differential of the first kind upon the plane curve of section. Secondly,

* For details, see also the book of Picard and Simart.

either x or y could have been taken as dependent variable instead of z.

From the first observation it follows that both P and Q must be entire functions of x, y, z, of order $m-2$ when m denotes the order of the surface $f=0$.

From the second, converting the differential into its two equivalent forms:

$$\frac{\dfrac{Q\cdot f'_y + P\cdot f'_x}{f'_z}dx + Pdz}{f'_y} = \frac{-\dfrac{Qf'_y + Pf'_x}{f'_z}dy - Q\cdot dz}{f'_x}$$

we find further that the fractional form

$$\frac{Qf'_y + Pf'_x}{f'_z}$$

must reduce to an integral form upon the surface, *i. e.*, by virtue of the equation of the surface. Let $-R$ denote this integral form, and N a suitable polynomial of order $m-3$, so that we shall have identically:

$$(2) \qquad P\cdot f'_x + Qf'_y + Rf'_z + N\cdot f \equiv 0.$$

This gives us for equivalent differential expressions on the surface:

$$(3) \qquad \frac{Q\cdot dx - P\cdot dy}{f'_z} = \frac{Rdy - Q\cdot dz}{f'_x} = \frac{Pdz - Rdx}{f'_y}.$$

There is yet to apply the condition for an exact differential, in order that the integral between any limiting points may be a function of those limits independent of the path of integration. That condition in one form will be, upon the surface:

$$(4) \qquad \frac{d}{dy}\left(\frac{Q}{f'_z}\right) + \frac{d}{dx}\left(\frac{P}{f'_z}\right) = 0.$$

Performing these differentiations by the aid of (1), and multiplying by $(f'_z)^2$ we have for $f=0$:

$$\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = 0.$$

This must hold over the surface $f = 0$, hence using (2) and completing the algebraic identity by a term in $f(x, y, z)$ we find:

$$(5) \qquad \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial}{\partial z}\left(R + N \cdot \frac{f}{f'_z}\right) + L \cdot \frac{f}{(f'_z)^2} \equiv 0$$

where L denotes some integral function of x, y, z, of order $2m - 6$. Expanding in part the third term, we distinguish terms which on their face must contain a factor $f(x, y, z)$:

$$\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} + N\right) + f(x, y, z)\left(\frac{\partial}{\partial z} \cdot \frac{N}{f'_z} + \frac{L}{f'^2_z}\right) \equiv 0.$$

Since the first group of four terms are integral, and of order lower than m, they cannot contain the factor $f(x, y, z)$ otherwise than by vanishing identically. Thus we must have for all values of x, y, z the identity

$$(6) \qquad \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} + N \equiv 0.$$

Insert again the equivalent of $N \cdot f$ from (2):

$$(7) \qquad P \cdot f'_x + Q \cdot f'_y + R \cdot f'_z \equiv f(x, y, z) \cdot \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right).$$

The form of this identity invites us to write f homogeneously in (x, y, z, t), — and of course the other functions also, and to employ Euler's identity

$$mf = xf'_x + yf'_y + zf'_z + tf'_t,$$

so that equation (7) becomes:

$$(mP + xN)f'_x + (mQ + yN)f'_y + (mR + zN)f'_z + tN \cdot f'_t \equiv 0.$$

In this it will appear more simple to write

$$(8) \qquad \begin{gathered} mP + xN = t\theta_1, \quad mQ + yN = t\theta_2 \quad mR + zN = t\theta_3, \\ N = \theta_4. \end{gathered}$$

To show that $\theta_1, \theta_2, \theta_3$ are integral, recall that

$$\frac{Qdx - Pdy}{f'_z}$$

is a total differential nowhere infinite, and that in the plane $t = 0$ it is an abelian integral of the first kind, and so must have the form

$$\phi(x, y, z)\cdot\frac{ydx - xdy}{f'_z},$$

where ϕ is of order $m - 3$. For this reason we must have identically:

$$P = x\phi + t\phi_1, \quad Q = y\phi + t\phi_2, \quad R = z\phi + t\phi_3,$$

where ϕ is homogeneous of order $m - 3$ in x, y, z, and ϕ_1, ϕ_2, ϕ_3 are of order $m - 3$ in x, y, z, and t. Therefore

$$N = -\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right)$$

$$\equiv -3\phi - (m-3)\phi - t\left(\frac{\partial \phi_1}{\partial x} + \frac{\partial \phi_2}{\partial y} + \frac{\partial \phi_3}{\partial z}\right),$$

or

$$\equiv -m\phi + t\phi_4.$$

These expressions give for the θ's the integral forms:

$$\theta_1 = \frac{m(x\phi + t\phi_1) + x(-m\phi + t\phi_4)}{t} = m\phi_1 + x\phi_4$$

$$\theta_2 = m\phi_2 + y\phi_4, \quad \theta_3 = m\phi_3 + z\phi_4, \quad \theta_4 = -m\phi + t\phi_4.$$

Effecting the substitutions (8) in conditions (6) and (7), and using Euler's relation for N:

$$x\frac{\partial N}{\partial x} + y\frac{\partial N}{\partial y} + z\frac{\partial N}{\partial z} = (m-3)N - t\frac{\partial N}{\partial t},$$

we have the two relations which the θ's must satisfy:

(9) $$\frac{\partial \theta_1}{\partial x} + \frac{\partial \theta_2}{\partial y} + \frac{\partial \theta_3}{\partial z} + \frac{\partial \theta_4}{\partial t} \equiv 0,$$

(10) $$\theta_1\frac{\partial f}{\partial x} + \theta_2\frac{\partial f}{\partial y} + \theta_3\frac{\partial f}{\partial z} + \theta_4\frac{\partial f}{\partial t} \equiv 0.$$

These conditions are now symmetrical in the four homogeneous variables, and by the aid of four parameters c_1, c_2, c_3, c_4 we can bring the integral (3) into the symmetrical form used by Berry.*

(11)
$$u = \int \frac{\begin{vmatrix} c_1 & c_2 & c_3 & c_4 \\ \theta_1 & \theta_2 & \theta_3 & \theta_4 \\ x & y & z & t \\ dx & dy & dz & dt \end{vmatrix}}{c_1 f'_x + c_2 f'_y + c_3 f'_z + c_4 f'_t}.$$

This integral is subject of course to the further restriction of remaining finite in all singular points or lines of the surface. *Prima facie*, the presence of singular points or lines seems a restriction upon the number of linearly independent sets of polynomials θ. But the fact is, that if the surface have no singularities, it can have no such integrals. For by the identity (10) the $(m-1)^3$ intersections of three first polar surfaces:

$$f'_x = 0, \quad f'_y = 0, \quad f'_z = 0$$

must fall, either on the surface $f'_t = 0$, thus indicating a multiple point of the fundamental surface; or if not one fall on this fourth polar, then all must needs lie on the surface $\theta_4 = 0$. This last, however, is of order $m-3$, and its equation cannot be a linear combination of three equations of order $m-1$. And the conclusion cannot be escaped by supposing the polars: $f'_x = 0, f'_y = 0, f'_z = 0$ to have a curve in common, since then that curve must pierce in a number of points (or else lie wholly in) the fundamental surface. Hence the surface $f = 0$ must have at least multiple points, as was to be proven.

Whether a surface whose equation is given does or does not possess linear differentials of the first kind, and how many linearly independent, — this can be determined by first finding the multiple curves and points, and then counting the conditions imposed by them and by the identities (6) and (7), or (9) and (10).

Remark. The number and nature of singularities that a surface of given order must possess in space of three dimensions in

* *Trans. Cambridge Phil. Soc.*, vol. 18 (1900), p. 333–4.

order to admit one, two, $\cdots$, independent exact differentials of this first kind might prove an accessible and profitable subject for further inquiry. To extend this inquiry to surfaces in hyperspace would require a systematic preliminary study of curves and surfaces in such a space not yet completed.

§ 2. *The Existence of Surfaces of Given Character, in particular Hyperelliptic Surfaces.*

If a surface in threefold space, $f = 0$, possesses exactly two linearly independent exact differentials of the first kind

$$\frac{Qdx - Pdy}{f_z} = du \quad \text{and} \quad \frac{Q_1 dx - P_1 dy}{f'_z} = dv,$$

then every algebraic curve lying upon it has the same two independent abelian differentials of the first kind, and hence these integrals have four independent sets of periods. It can be proven easily that the geometrical deficiency of the surface is

$$p_g = 1,$$

and that the expression

$$\iint \frac{QP_1 - Q_1P}{f'_z} \cdot \frac{dxdy}{f'_z}$$

is the double integral of the first kind, finite for all boundary curves on the surface.

Conversely Picard shows, (*l. c.*) that if x, y, z are given as fourfold periodic functions of two independent variables, the locus of a point (x, y, z) is a surface of this sort. For a simplest illustration let the functions reduce to elliptic, and in the Weierstrass notation set

$$x = p(u), \quad y = p(v), \quad z = p'(u) + p'(v).$$

This gives an equation between x, y, and z:

$$z = \sqrt{4x^3 - g_2x - g_3} + \sqrt{4y^3 - g_2y - g_3},$$

or for brevity

$$z = \sqrt{R(x)} + \sqrt{R(y)}.$$

Accordingly the surface is of the sixth order:

$$4R(x)\cdot R(y) - [z^2 - R(x) - R(y)]^2 = f(x, y, z) = 0.$$

The two integrals on the surface, u and v, are represented as follows:

$$u = \int \frac{dx}{\sqrt{R(x)}} = \int \frac{z^2 - R(x) + R(y)}{f'_z}\cdot dx,$$

$$v = \int \frac{dy}{\sqrt{R(y)}} = \int \frac{z^2 + R(x) - R(y)}{f'_z}\cdot dy.$$

As to double integrals, the one of the first kind belonging to the surface degenerates into $\int du\cdot dv$, which is evidently finite. The double lines of the sextic surface may be perceived immediately, one of them being obviously the straight line $x = y$, $z = 0$; another a conic in the (x, y) plane; and three lines at infinity.

One linear differential of the first kind can exist on a surface of order as low as the fourth. There are five types of such quartic surfaces, found by Poincaré,* Berry† and de Franchis; the five types are projectively distinct, that is, collineations cannot transform one into another; but Berry has found that under birational transformations all five are equivalent to a cubic cone devoid of double line.

Of these five types, perhaps the easiest of derivation is the following. The quantities θ, being of order $m - 3$, are linear. Let their planes coincide with those of the tetrahedron of reference, viz.:

$$\theta_1 = x, \quad \theta_2 = y, \quad \theta_3 = -z, \quad \theta_4 = -t,$$

thus satisfying the condition (9). It remains to satisfy (10),

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} - z\frac{\partial f}{\partial z} - t\frac{\partial f}{\partial t} \equiv 0.$$

* *Comptes Rendus*, vol. 99 (Dec. 29, 1884).

† *Ibidem*, Sept. 2, 1899. See also his papers, cited above, in the *Trans. Cambridge Phil. Soc.*

Since also by Euler's identity for homogeneous functions

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} + t\frac{\partial f}{\partial t} \equiv 4f,$$

it follows that

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} \equiv z\frac{\partial f}{\partial z} + t\frac{\partial f}{\partial t} \equiv 2f.$$

Hence the form is homogeneous of order 2 in x and y, also in z and t. Symbolically

$$f = (a_1x^2 + 2a_2xy + a_3y^2)(b_1z^2 + 2b_2zt + b_3t^2),$$

each product a_ib_k denoting an arbitrary real quantity. This is a familiar ruled quartic surface with two double lines ($x = y = 0$ and $z = t = 0$). It is generated by taking for directrices these two double lines and any plane quartic which has nodes upon the two lines.

This suggests the interpretation of conditions (9) and (10) by a complex of lines. The connex

$$u_1\theta_1 + u_2\theta_2 + u_3\theta_3 + u_4\theta_4 \equiv \theta = 0$$

gives rise to a complex when every point (x) is joined to its corresponding point, and condition (10) is the requirement that the complex line originating in a point (x) of the surface $f = 0$, shall be tangent to that surface. Speaking of the line joining a point (x) to its corresponding point in the connex : $\theta = 0$ as a *trajectory* of that connex, we say : *A surface $f = 0$ of order m will possess a linear exact differential of the first kind if a complex* $(m - 3, 1)$ *exists such that the trajectories of all points on the surface are tangent to the surface.*

Remark. When one linear exact differential exists on the surface, and only one, it is invariantively related to the surface under a much larger group than that of the collineations and *a fortiori* under the latter group. Instead of seeking the integral when the surface is given, and finding it as an irrational covariant of the surface, one might attempt to determine the surface as a rational covariant of the forms $\theta_1, \theta_2, \theta_3, \theta_4$, occurring in the integral. But the surface may be not determinate. (In the above example it had still 8 free parameters.) Also the θ's depend on the choice of planes of reference. Hence more precisely one should seek to determine the mixed form $f(x, u)$ (connex) covariant with the connex

$$\theta = u_1\theta_1 + u_2\theta_2 + u_3\theta_3 + u_4\theta_4$$

such that every set of values (u) makes the surface $f(x, u) = 0$ a surface possessing the integral of the first kind represented in (11). In other words the connex θ is to satisfy the relation

$$\frac{\partial^2\theta}{\partial u_1 \partial x_1} + \frac{\partial^2\theta}{\partial u_2 \partial x_2} + \frac{\partial^2\theta}{\partial u_3 \partial x_3} + \frac{\partial^2\theta}{\partial u_4 \partial x_4} \equiv 0,$$

while the covariant $f(x, u)$, or f, is to be of order in the (x) higher by 3 than θ, and shall satisfy also identically the equation:

$$\frac{\partial\theta}{\partial u_1}\frac{\partial f}{\partial x_1} + \frac{\partial\theta}{\partial u_2}\frac{\partial f}{\partial x_2} + \frac{\partial\theta}{\partial u_3}\frac{\partial f}{\partial x_3} + \frac{\partial\theta}{\partial u_4}\frac{\partial f}{\partial x_4} \equiv 0.$$

Of course the chief interest in this problem would be found in the lower orders, 4, 5, 6. It might be possible to solve a similar problem of the theory of forms when the surface is to have two or more independent integrals of the first kind.

To return to surfaces with two independent exact differentials of the first kind, we note two theorems of Picard. *The existence of two such differentials is impossible upon any surface of order* $m < 6$. *If a surface have two such differentials, its plane sections are curves of deficiency at least* $p = 2$, *and its geometrical deficiency is* $p_g \geqq 1$.

Picard establishes directly the existence of a class of surfaces with two differentials, in brief as follows: Let the Cartesian coördinates of a point be given as uniform functions, quadruply periodic, of two independent variables. Let the relation be such that to every point (x, y, z) of the surface there corresponds one and only one pair of values of the two independent variables

u, v. Then the surface has exactly two linear differentials of the first kind.*

For if the surface equation

$$f(x, y, z) = 0$$

is satisfied identically by three uniform functions

$$x = F_1(u, v), \quad y = F_2(u, v), \quad z = F_3(u, v)$$

and the functions F_1, F_2, F_3 have four simultaneous systems of periods, then since

$$dx = \frac{\partial F_1}{\partial u} du + \frac{\partial F_1}{\partial v} dv$$

$$dy = \frac{\partial F_2}{\partial u} du + \frac{\partial F_2}{\partial v} dv$$

these partial derivatives

$$\frac{\partial F_1}{\partial u}, \ldots, \frac{\partial F_2}{\partial v}$$

must be likewise quadruply periodic uniform functions of u, v, and therefore rational functions of x, y, z. Accordingly the solutions of these two equations

$$du = Q_1 dx - P_1 dy$$

$$dv = Q_2 dx - P_2 dy$$

are differentials of the first kind upon the surface, and independent by hypothesis. But any third differential of this kind on the surface is necessarily a linear function of these two, with constant coefficients. If it be denoted by dw:

$$dw = Q_3 dx - P_3 dy$$

$$\equiv \left(Q_3 \frac{\partial x}{\partial u} du - P_3 \frac{\partial y}{\partial u} du\right) + \left(Q_3 \frac{\partial x}{\partial v} dv - P_3 \frac{\partial y}{\partial u} dv\right)$$

$$\equiv \phi(x, y, z) \cdot du + \psi(x, y, z) dv,$$

* *Liouville*, ser. 4, vol. 1 (1885).

then as the functions ϕ and ψ are assumed to remain finite throughout the surface $f = 0$, and are seen by the foregoing to be rational in x, y, z, they can be nothing but constants, as was to be proven. The double integral of first kind on the surface is $\int\int du \cdot dv$; the proof that it is unique is closely similar to the above. Functions of the properties required for F_1, F_2, F_3, are readily expressed by quotients of theta-functions of two variables.

Surfaces of this sort are called by Picard and Humbert *hyperelliptic surfaces.* They are to be distinguished carefully from surfaces whose *plane sections* are hyperelliptic, or which have a *linear* net of hyperelliptic curves upon them, for those we have seen to be rational ($p_g = 0$); while these, possessing one double integral everywhere finite, have $p_g = 1$.

Hyperelliptic surfaces of order lower than the sixth do not exist, as was said. This evokes recollections of Kummer's surface of the fourth order; but that, as Picard shows, is not of this class, because it has *two sets* of values (u, v) for every point (x, y, z). *Humbert* has discussed hyperelliptic surfaces *in extenso*,* in particular those of sixth order. He extends this mode of establishing their existence by theta-formulæ, so as to employ the next higher class of thetas, those in three independent variables. In this way he reaches surfaces containing three linearly independent exact linear differentials of the first kind and proves that their order must be higher than six. An example is given of the eighth order, but the order seven is left in doubt. Of such representation of these surfaces, the chief advantage is that every algebraic curve lying in the surface is given by the vanishing of some theta function, so that by the use of theorems more or less familiar in the theory of thetas, one obtains an exhaustive treatment of geometry upon a surface.

It is apparent that this line of investigation opens a prospect of a classification of surfaces based on properties much more general than those merely projective. As was indicated in a remark upon quartics, this calls for the projective study (for the sake of

* *Liouville*, ser. 4, vol. 5 (1889), vol. 9 (1893), and ser. 5, vol. 2 (1896).

models) of surfaces which become interesting under this more searching light. And the special classes — as those related to point-pairs on one curve, on two curves, those in which the *periods* of the arguments fall into some integral relation, etc.— those offer a field most inviting and likely to yield rich fullness of even the simpler geometric forms.

FORMS OF NON-EUCLIDEAN SPACE.

By FREDERICK S. WOODS.

By a non-euclidean geometry we shall mean any system of geometry which, while differing in essential particulars from that of Euclid, is nevertheless in accord with the facts of experience within the limits of the errors of observation. The space in which such a geometry is valid is a non-euclidean space. It is clear that the test of experience can be applied only within a restricted portion of space, so that non-euclidean spaces, while having essentially the same properties in such a restricted region, may differ widely when considered in their entirety. It is the purpose of the present lectures to present especially those non-euclidean spaces, investigated by Clifford, Klein and Killing, which have been named by the last author the *Clifford-Klein'sche Raumformen*.*

For the sake of clearness it is necessary to begin with the geometry of a restricted portion of space. Here the author has followed the development of his own article on "Space of Constant Curvature,"† to which the reader is referred for references to the literature and for fuller handling of some of the subject matter of the first five paragraphs of these lectures.

The point of view adopted is that objective space presents certain phenomena of form, position and magnitude, which demand explanation as do other physical phenomena. This explanation the geometrician gives by the assumption of certain hypotheses,

*Clifford, W. K., "A Preliminary Sketch of Biquaternions," *Mathematical Papers*, No. XX.

Klein, F., "Zur Nicht-Euklidischen Geometrie," *Math. Annalen*, vol. 37 (1890), p. 344. *Lectures on Mathematics*, Lecture XI, New York, 1894. "Zur ersten Vertheilung der Lobatchewsky Preise," *Math. Annalen*, vol. 59 (1898), especially pp. 591–592.

Killing, W., "Ueber Clifford-Klein'sche Raumformen," *Math. Annalen*, vol. 39 (1891). *Einführung in die Grundlagen der Geometrie*, vol. 1, Chap. 4; Paderborn, 1893.

† *Annals of Mathematics*, ser. 2, vol. 3 (1902), p. 71.

which he is free to make as he pleases, provided that they are self-consistent. The test of the validity of the hypotheses lies in their results. We make at first hypotheses which follow the ideas of Riemann's famous *Habilitationsschrift*.*

It is admitted that questions may be raised which lie back of these hypotheses, as, for example, the possibility of reducing them to simpler axioms, but the discussion of such questions lies outside our present province. The Riemann method has for us the double advantage of allowing the immediate use of analytic methods and of restricting the discussion at the outset to a small region of space.

A geometry having thus been developed in a restricted portion of space is extended to all space by means of new hypotheses, which are essentially those used by Killing in his *Grundlagen der Geometrie*. In the further development the ideas of the last named treatise have been largely followed.

1. The First Two Hypotheses.

As already said, we adopt in our investigations the method of Riemann by which our objective space is assumed to be an example of an extent (*Mannigfaltikeit*) of three dimensions in which an element may be determined by means of coördinates. We assert this explicitly in the following words:

First Hypothesis. *Space is a continuum of three dimensions in which a point may be determined by three independent real coördinates* (z_1, z_2, z_3). *If a properly restricted portion of space is considered, the correspondence between point and coördinate is one-to-one and continuous.*

Within our space, we may pick out at pleasure one-dimensional extents or lines. We shall restrict ourselves to lines which may be expressed by the equations

$$z_1 = f_1(t), \quad z_2 = f_2(t), \quad z_3 = f_3(t),$$

where t is an arbitrary parameter and f_1, f_2 and f_3 are continuous

* Riemann, B., "Ueber die Hypothesen, welche der Geometrie zu Grunde liegen," *Gesammelte Werke*, 1st ed. p. 254; 2d ed. p. 272.

functions possessing continuous first derivates, nowhere vanishing simultaneously. For such a line we may introduce the conception of length as follows. Consider a portion of the line corresponding to values of t lying between the values t_0 and T inclusive, and let this portion be divided into n segments to the extremities of which correspond the values $t_1, t_2, t_3, \cdots t_{n-1}, T$. Let further (z_1, z_2, z_3) and $(z_1 + \delta z_1, z_2 + \delta z_2, z_3 + \delta z_3)$ be the coördinates of the extremities of any segment, corresponding respectively to t_i and t_{i+1}. We may then assume arbitrarily a function

$$\phi(z_1, z_2, z_3;\ \delta z_1, \delta z_2, \delta z_3)$$

which has the following two properties: First, it shall become infinitesimal with δz_1, δz_2, δz_3, and consequently with $t_{i+1} - t_i$; and secondly, the sum of the n values of this function, computed for the n segments of the line, shall approach a limit as n is indefinitely increased and each of the n quantities $t_{i+1} - t_i$ approaches zero, this limit to be independent of the manner in which the segments of the line are taken. *This limit is defined as the length of the line.* If in particular we take

$$\phi = \sqrt{\Sigma a_{ik} \delta z_i \delta z_k}, \qquad (i, k = 1, 2, 3;\ a_{ki} = a_{ik})$$

the length of the line is expressed by the integral

$$s = \int_{t_0}^{T} \sqrt{\Sigma a_{ik} \frac{dx_i}{dt} \frac{dx_k}{dt}}\, dt.$$

The differential of this integral, namely,

$$ds = \sqrt{\Sigma a_{ik} dx_i dx_k},$$

we call the *line-element of the space.* We express these conventions in a new hypothesis as follows:

SECOND HYPOTHESIS. *The length of a line shall be determined by means of a line-element given by the equation*

$$ds = \sqrt{\Sigma a_{ik} dx_i dx_k}, \qquad (a_{ki} = a_{ik};\ i, k = 1, 2, 3)$$

where the a_{ik} are functions of z_1, z_2, z_3, possessing continuous derivatives of the first four orders, the determinant $|a_{ik}|$ does not vanish identically, and the expression under the radical sign is positive for *all values* $z_1, z_2, z_3, dz_1, dz_2, dz_3$, *provided that* (z_1, z_2, z_3) *is a point of space and that* dz_1, dz_2, dz_3 *are not all zero.*

2. Definitions.

We proceed now to develop the conceptions of a geodesic surface, a geodesic line, an angle and a direction, which shall correspond to the conceptions of a straight line, a plane, an angle and a direction in Euclidean space.

1. *Geodesic Line.* A geodesic line is defined roughly as the shortest distance between two points. To determine its equations, we have to find the conditions that the integral

$$s = \int_{t_0}^{T} \sqrt{\Sigma a_{ik} \frac{dx_i}{dt} \frac{dx_k}{dt}}\, dt$$

shall be a minimum. The Calculus of Variations gives as necessary conditions, the three equations

$$\frac{d}{dt} \frac{1}{\sqrt{R_t}} \left(a_{l1} \frac{dz_1}{dt} + a_{l2} \frac{dz_2}{dt} + a_{l3} \frac{dz_3}{dt} \right) = \tfrac{1}{2} \frac{1}{\sqrt{R_t}} \sum_{ik} \frac{\partial a_{ik}}{\partial z_l} \frac{dz_i}{dt} \frac{dz_k}{dt},$$

where $l = 1, 2, 3$, and

$$R_t = \Sigma a_{ik} \frac{dz_i}{dt} \frac{dz_k}{dt}.$$

If we take as the independent parameter the length s, as defined by the integral, these equations take the somewhat simpler form

$$\frac{d}{ds} \left[a_{11} \frac{dz_1}{ds} + a_{12} \frac{dz_2}{ds} + a_{13} \frac{dz_3}{ds} \right] = \tfrac{1}{2} \Sigma \frac{\partial a_{ik}}{\partial z_1} \frac{dz_i}{ds} \frac{dz_k}{ds},$$

$$\frac{d}{ds} \left[a_{12} \frac{dz_1}{ds} + a_{22} \frac{dz_2}{ds} + a_{23} \frac{dz_3}{ds} \right] = \tfrac{1}{2} \Sigma \frac{\partial a_{ik}}{\partial z_2} \frac{dz_i}{ds} \frac{dz_k}{ds},$$

$$\frac{d}{ds} \left[a_{13} \frac{dz_1}{ds} + a_{23} \frac{dz_2}{ds} + a_{33} \frac{dz_3}{ds} \right] = \tfrac{1}{2} \Sigma \frac{\partial a_{ik}}{\partial z_3} \frac{dz_i}{ds} \frac{dz_k}{ds},$$

which must be considered in connection with the identity

$$\Sigma a_{ik} \frac{dz_i}{ds} \frac{dz_k}{ds} = 1.$$

Conversely these conditions are sufficient if s is not too great.

More precisely: Let $(z_1^{(0)}, z_2^{(0)}, z_3^{(0)})$ be any point point of space, and (z_1, z_2, z_3) any second point such that $|z_i - z_i^{(0)}|$ does not exceed a suitably chosen positive quantity, h. Then the above equations admit one and only one solution which passes through the points $(z^{(0)})$ and (z) and has all its points lying in the region $|z_i^{(0)} - z_i| < h$; aud for the corresponding curve the integral s has a smaller value than for any other curve joining the points $(z^{(0)})$ and (z).

We take the equations accordingly as the defining equations of the geodesic lines and shall apply this name to the curves satisfying these equations, even if the curves have been so prolonged that the minimum property no longer holds.

2. *Direction.* In accordance with the theory of differential equations it is always possible to find one and only one solution of the above equations which takes on at an arbitrary point (z_1, z_2, z_3) any arbitrary values (not all zero) of the differential coefficients

$$\frac{dz_1}{ds}, \quad \frac{dz_2}{ds}, \quad \frac{dz_3}{ds}.$$

If these differential coefficients satisfy initially the condition

$$\Sigma a_{ik} \frac{dz_i}{ds} \frac{dz_k}{ds} = 1$$

this relation will be fulfilled for all values of s.

The geodesic lines which radiate from a point are hence distinguished from each other by the ratios of the values of the differential coefficients, which may consequently be regarded as fixing the direction of the line; the direction being, broadly, that property of the line which distinguishes it from all others through the same point. It will be convenient to denote dz_i/ds by ζ_i and to speak shortly of the direction $(\zeta_1, \zeta_2, \zeta_3)$, or ζ. These quantities satisfy the relation

$$\Sigma a_{ik}\zeta_i\zeta_k = 1.$$

3. *Angle.* The angle θ between two intersecting curves with the directions ζ' and ζ'' is defined by the equation

$$\cos\theta = \Sigma a_{ik}\zeta_i'\zeta_k''.$$

In particular two intersecting curves are perpendicular if

$$\Sigma a_{ik}\zeta_i'\zeta_k'' = 0.$$

4. *Geodesic Surface.* A geodesic surface is defined as a pencil of geodesic lines. More precisely: Take any two geodesic lines OA and OB intersecting at O, having at that point the directions α and β respectively, and making the angle ω with each other. Consider any other geodesic line OM with the direction

$$\zeta_i = \lambda\alpha_i + \mu\beta_i,$$

where λ and μ are parameters subject only to the condition

$$\lambda^2 + \mu^2 + 2\lambda\mu \cos\omega = 1,$$

which arises from substitution in

$$\Sigma a_{ik}\zeta_i\zeta_k = 1.$$

As λ, μ take all possible values, OM generates a pencil of lines, which is defined as a geodesic surface. It may be shown without difficulty that in this pencil there is one and only one line perpendicular to OA and that this may replace OB in defining the pencil. We shall then have

$$\zeta_i = \alpha_i \cos\theta + \beta_i \sin\theta,$$

where θ is the angle between OA and OM.

If now P is any point on OM and r is the length of OP, the coördinates z_i of P are determined by integrating the equations of the geodesic lines, choosing the solution which has at O the direction ζ, and substituting r for s. We have then

$$z_i = f_i(\zeta_1, \zeta_2, \zeta_3, r) = \phi_i(\theta, r),$$

the functions ϕ_i being continuous together with their partial derivatives of the first and second orders. By taking θ and r as independent parameters, we have the equations of the geodesic surfaces.

3. The Third Hypothesis.

The method of superposition, involving the assumption that a geometric figure may be moved from one position to another without altering its size or properties, is fundamental in the Euclidean geometry and would seem to be a necessity in any explanation of spatial phenomena. The hypotheses thus far made do not carry with them the necessity of any such superposition. This may be clearly seen by examples from the Euclidean geometry of a kind which we shall frequently employ in the following pages. In thus using the Euclidean geomety, we do not assume that it is objectively true, but that it is a self-consistent system which explains experience. Consider any surface on which a system of curvilinear coördinates (u, v) have been established. This surface is a two-dimensional space satisfying the first two hypotheses, the line element being of the form

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2.$$

Such surfaces, however, offer various possibilities in the matter of superposing one portion upon another. One needs only to consider the ellipsoid, the right circular cylinder, and the sphere as examples.

To bring the principle of superposition into our present discussion, we shall define a *displacement* as a transformation by which a continuous portion of space is brought into a continuous point for point correspondence either with itself or with another portion of space in such a manner that the lengths of corresponding portions of lines are the same. Let S be a portion of space in which the coördinates of a point P are (z_1, z_2, z_3), and let S' be a portion of space in which the coördinates at a point P' are (z_1', z_2', z_3'). Let the line-element in S be denoted by

$$ds = \sqrt{\Sigma a_{ik} dz_i dz_k}$$

and the line element in S' by

$$ds' = \sqrt{\Sigma a'_{ik} dz'_i dz'_k},$$

where a'_{ik} denotes the value of a_{ik} for (z'_1, z'_2, z'_3). In order that S may be displaced into S', it is necessary that

$$ds = ds'$$

by virtue of relations of the form

$$z_i = \psi_i(z'_1, z'_2, z'_3),$$

where the ψ_i are continuous functions of (z'_1, z'_2, z'_3), possessing continuous first derivatives, and establishing a one-to-one relation between the points of S and S'.

It is easy to show that by any displacement, geodesic lines are transformed into geodesic lines, geodesic surfaces into geodesic surfaces, and angles are left unchanged.

The existence of displacements in space is made the subject of a new hypothesis.

THIRD HYPOTHESIS. *If P is any point of space, it shall be possible to displace a restricted portion of space surrounding P upon itself in such a manner that any two geodesic lines through P shall correspond to any other two geodesic lines through P, provided only that the two latter lines make the same angle with each other as do the two former lines.*

The question of displacement of a surface is intimately connected with the quantity called by Gauss the measure of the curvature, or simply the *curvature*, of the surface. Under that term we understand a quantity K defined by the relation

$$K = \frac{1}{2\sqrt{EG - F^2}} \left(\frac{\partial}{\partial u} \left[\frac{F}{E\sqrt{EG - F^2}} \frac{\partial E}{\partial v} - \frac{1}{\sqrt{EG - F^2}} \frac{\partial G}{\partial u} \right] \right.$$
$$\left. + \frac{\partial}{\partial v} \left[\frac{2}{\sqrt{EG - F^2}} \frac{\partial F}{\partial u} - \frac{1}{\sqrt{EG - F^2}} \frac{\partial E}{\partial v} - \frac{F}{E\sqrt{EG - F^2}} \frac{\partial E}{\partial u} \right] \right).$$

With the geometric interpretation of the curvature as usually given on the hypothesis that the surface lies in Euclidean space we have nothing to do. For us the curvature is simply the above expression which is fully determined when the line-element of the surface is given, and may be shown to be an invariant of the surface, that is independent of the coördinates used to define a point upon the surface. When K is the same for all points of the surface, the surface is said to be one of constant curvature. The importance of the curvature lies in the two theorems:

A necessary condition that two portions of surfaces may be brought into point for point correspondence with preservation of distance is that they have the same curvature at corresponding points.

If the two portions of surfaces are of constant curvature, the condition is also sufficient.

The Gaussian measure of curvature of a surface is extended by Riemann to space of n dimensions. For three dimensions consider a point (z_1, z_2, z_3) and two directions $(\alpha_1, \alpha_2, \alpha_3)$ and $(\beta_1, \beta_2, \beta_3)$, taken from that point. Then the Riemann curvature is a function

$$K(z_1, z_2, z_3;\ \alpha_1, \alpha_2, \alpha_3;\ \beta_1, \beta_2, \beta_3)$$

which gives the Gaussian curvature of the geodesic surface determined by the point and the directions. The Riemann curvature of a general space is accordingly dependent both on the point of space for which it is reckoned and on the directions of the lines taken through that point to define a geodesic surface. But if the space satisfies our third hypothesis, the curvature is a function of the point only. For by this hypothesis, any two geodesic pencils with their vertices at the same point P may be brought into point for point correspondence with preservation of distance. Hence by the surface theorems above quoted, the two geodesic surfaces formed by the pencils must have the same curvature at corresponding points and in particular at P. Schur * has proved that when the curvature is thus constant at each point, it does not change as we pass from point to point. The space is then said to be of constant curvature. A new proof of Schur's theorem will be given in the following paragraph.

4. The Line-Element.

Take any point O at which the functions a_{ik} are single-valued and continuous. Then, as we have seen, there exists around O

* Schur, F., "Ueber den Zusammenhang der Räume constanten Riemann'schen Krümmungsmasses," *Math. Annalen*, vol. 27 (1886), p. 593.

a region of space such that any point P of the region can be joined to O by one and only one geodesic line lying in the region. We shall call this region T. Through O take in T three mutually perpendicular geodesic lines OA, OB, OC. This can be done by taking directions $(\alpha_1, \alpha_2, \alpha_3)$, $(\beta_1, \beta_2, \beta_3)$, $(\gamma_1, \gamma_2, \gamma_3)$ so as to satisfy the relations

$$\Sigma a_{ik}^{(0)}\alpha_i\alpha_k = 1, \quad \Sigma a_{ik}^{(0)}\beta_i\beta_k = 1, \quad \Sigma a_{ik}^{(0)}\gamma_i\gamma_k = 1,$$
$$\Sigma a_{ik}^{(0)}\alpha_i\beta_k = 0, \quad \Sigma a_{ik}^{(0)}\beta_i\gamma_k = 0, \quad \Sigma a_{ik}^{(0)}\gamma_i\alpha_k = 0,$$

where $a_{ik}^{(0)}$ signifies the value of a_{ik} at O. The direction of any geodesic line through O is then

$$\zeta_i = a_1\alpha_i + a_2\beta_i + a_3\gamma_i,$$

where a_1, a_2, a_3 are independent parameters subject only to the condition

$$a_1^2 + a_2^2 + a_3^2 = 1,$$

which arises from

$$\Sigma a_{ik}^{(0)}\zeta_i\zeta_k = 1.$$

The direction may accordingly be named by means of (a_1, a_2, a_3).

Let P be any point on this geodesic line and let the distance OP be denoted by r, where r is positive if measured in the direction a_i, and negative if measured in the opposite direction. We may take the quantities (a_1, a_2, a_3, r) as the coördinates of P. Then to any set of values of the coördinates corresponds only one point P, and to any point P correspond only the coördinates (a_1, a_2, a_3, r) or $(-a_1, -a_2, -a_3, -r)$. Between old the and new coördinates, there exist relations of the form

$$z_i = F_i(a_1, a_2, a_3, r),$$

where the functions F_i are continuous and possess continuous derivatives of the first two orders since they are the solutions of the differential equations of the geodesic lines.

By the substitution in

$$ds^2 = \Sigma a_{ik}dz_idz_k,$$

the form of the line-element is obtained as

$$ds^2 = \sum_{ik} A_{ik} da_i da_k + \sum_l A_{l4} da_l dr + A_{44} dr^2, \qquad (i, k, l = 1, 2, 3)$$

where

$$A_{mn} = \sum_{ik} a_{ik} \frac{\partial z_i}{\partial a_m} \frac{\partial z_k}{\partial a_n}.$$

The direct calculation of the values of the coefficients is difficult: but we shall prove by an indirect method that the proper form is

$$ds^2 = \frac{\sin^2 kr}{k^2} (da_1^2 + da_2^2 + da_3^2) + dr^2,$$

where k is a constant.

To do this, consider any curve C, defined by the equations

$$a_1 = f_1(t), \quad a_2 = f_2(t), \quad a_3 = f_3(t), \quad r = f_4(t).$$

If P_0 is any fixed point on C and θ is the angle between the geodesic lines OP_0 and OP, θ is a function of t and hence t is a function of θ, which for small portions of C is one valued. We may consequently write for the equations of C

$$a_1 = \phi_1(\theta), \quad a_2 = \phi_2(\theta), \quad a_3 = \phi_3(\theta), \quad r = \phi_4(\theta).$$

If from these four equations we omit the fourth, thus allowing r to take any value, we have the equations of a surface, which passes through the curve C, as is evident, and also contains the point O since the equations are satisfied by $r = 0$. The surface is analogous to a cone of the Euclidean geometry, for the lines $\theta =$ const. are geodesic lines radiating from O to the points of C. These lines form one of the systems of coördinate curves on the surface; the other system is composed of the lines $r =$ const., each of which is the locus of points equally distant from O. If we refer to the general form of the line-element of a surface

$$ds^2 = Edr^2 + 2Fdrd\theta + Gd\theta^2,$$

it is clear that in the present case, $E = 1$, since $s = r$ when $\theta =$ const.; and $F = 0$, since the curves $r =$ const. cut the geodesics $\theta =$ const. at right angles by a theorem of the Calculus of Variations.* We have therefore on the surface

$$ds^2 = Gd\theta^2 + dr^2,$$

* Cf. Kneser, A., *Variationsrechnung*, p. 48. Bolza, O., *Calculus of Variations*, p. 164.

and we proceed next to replace $d\theta$ by its value in terms of a_i. For that, we call $\delta\theta$ the angle between two neighboring geodesic lines OP and OQ, with directions a and $a + \delta a$, where

$$a_1^2 + a_2^2 + a_3^2 = 1,$$

$$(a_1 + \delta a_1)^2 + (a_2 + \delta a_2)^2 + (a_3 + \delta a_3)^2 = 1.$$

Then

$$\begin{aligned}\cos \delta\theta &= a_1(a_1 + \delta a_1) + a_2(a_2 + \delta a_2) + a_3(a_3 + \delta a_3)\\ &= 1 + a_1 \delta a_1 + a_2 \delta a_2 + a_3 \delta a_3\\ &= 1 - \tfrac{1}{2}(\delta a_1^2 + \delta a_2^2 + \delta a_3^2).\end{aligned}$$

so that

$$\sin^2 \frac{\delta\theta}{2} = \tfrac{1}{4}(\delta a_1^2 + \delta a_2^2 + \delta a_3^2).$$

From this follows in the differential notation

$$d\theta^2 = da_1^2 + da_2^2 + da_3^2,$$

So that the line-element of the suface is

$$ds^2 = G(da_1^2 + da_2^2 + da_3^2) + dr^2.$$

This is in particular the element of the length of the curve C, since C is on the surface. But C is any curve in space and hence the above expression is the line-element of the space.

We seek now to determine G. For that purpose consider

$$G = A_{11} = \Sigma a_{ik} \frac{\partial z_i}{\partial a_1} \frac{\partial z_k}{\partial a_2}$$

where (see p. 40)

$$z_i = F_1(a_1, a_2, a_3, r) = z_i^{(0)} + (a_1 \alpha_i + a_2 \beta_i + a_3 \gamma_i) r + \cdots.$$

Hence

$$G = r^2 \Sigma a_{ik} \alpha_i \alpha_k + \cdots.$$

and consequently

$$(\sqrt{G})_{r=0} = 0, \qquad \left(\frac{\partial \sqrt{G}}{\partial r}\right)_{r=0} = 1.$$

Thus far the discussion is applicable to any space which satisfies the first two hypotheses. We examine now the effect of introducing the third hypotheses. A geodesic surface formed by a pencil of lines with its vertex at O is a special case of the conical surfaces just discussed and its line-element is therefore

$$ds^2 = G d\theta^2 + dr^2.$$

The formula for the curvature K reduces to

$$K = \frac{1}{\sqrt{G}} \frac{\partial^2 \sqrt{G}}{\partial r^2}.$$

By the third hypothesis, any one of these surfaces may be brought into correspondence with any other by means of a displacement by which a point at the distance r from O on the one surface corresponds to any point at the same distance r from O on the other surface. Hence the curvature K is a function of r alone, that is

$$\frac{1}{\sqrt{G}} \frac{\partial^2 \sqrt{G}}{\partial r^2} = \phi(r).$$

From this and the conditions governing G when $r = 0$, it follows that G is a function of r only.

The exact form of G is obtained by the following considerations: The equations of the geodesic lines in the new coördinates are

$$G \frac{d^2 a_i}{ds^2} + 2G' \frac{da_i}{ds} \frac{dr}{ds} + \lambda a_i = 0, \qquad (i = 1, 2, 3)$$

$$\frac{d^2 r}{ds^2} + \frac{G'}{2G} \lambda = 0,$$

where

$$G' = \frac{dG}{dr}$$

and

$$\lambda = G \left[\left(\frac{da_1}{ds} \right)^2 + \left(\frac{da_2}{ds} \right)^2 + \left(\frac{da_3}{ds} \right)^2 \right] = 1 - \left(\frac{dr}{ds} \right)^2.$$

Take now the geodesic surface $a_3 = 0$, for which the line-element is

$$ds^2 = G(da_1^2 + da_2^2) + dr^2,$$

and apply the Calculus of Variations to determine the shortest line on this surface connecting any two points. Such a line exists if the points are not too remote, and its equations will be found to be exactly those obtained when a_3 is placed equal to 0 in the equations of the geodesic lines in space. It follows that any two points on the surface $a_3 = 0$ may be connected by a geodesic line lying wholly on the surface. In particular any point of the surface is the vertex of a pencil of geodesic lines which lies on the surface.

Take now P_1 any point in $a_3 = 0$, and choose on the geodesic line OP_1 the point M equidistant from O and P_1. This point M may be used as the vertex of a pencil which covers the surface. By the third hypothesis, there exists a displacement by which this pencil is self-corresponding, the point M being fixed and the geodesic line MP corresponding to MO. Hence the curvature of $a_3 = 0$ at P_1 equals that at O, and the surface is consequently one of constant curvature. But the surface $a_3 = 0$ may be brought into correspondence with any other geodesic surface formed by a pencil of lines with vertex O. Hence K is independent of r throughout and is consequently constant. We place

$$K = k^2,$$

and have the three cases of a space of constant positive curvature, a space of constant negative curvature, or a space of zero curvature, according as k is real, pure imaginary, or zero. To determine G, we have the differential equation

$$\frac{1}{\sqrt{G}} \frac{\partial^2 \sqrt{G}}{\partial r^2} = k^2$$

with the initial conditions

$$(\sqrt{G})_{r=0}, \quad \left(\frac{\partial \sqrt{G}}{\partial r}\right)_{r-0} = 1.$$

Hence

$$\sqrt{G} = \frac{\sin kr}{k}.$$

If k is real this determination of k is final.

If k is pure imaginary, we may place $k = ik'$ and

$$\sqrt{G} = \frac{\sinh k'r}{k'}.$$

If k is zero, we may place

$$\sqrt{G} = \operatorname*{Lim}_{k=0} \frac{\sin kr}{k} = r.$$

It will be more convenient to retain the general form for k, since the above changes are readily made. We have accordingly the line-element in the desired form,

$$ds^2 = \frac{\sin^2 kr}{k^2}(da_1^2 + da_2^2 + da_3^2) + dr^2.$$

It is to be emphasized that we have shown the existence of a displacement by which O is transferred into any other point P_1 and reciprocally. By the combination of two such displacements, a displacement may be found by which any point P_1 of T may be made to correspond to any other point P_2 of T.

5. Geometry in a Restricted Portion of Space.

We shall, for the present, confine our attention to the portion of space T already defined and introduce the coördinates *

$$\left.\begin{aligned} x_0 &= \cos kr, \\ x_i &= a_i \frac{\sin kr}{k}, \qquad (i = 1, 2, 3) \end{aligned}\right\} \tag{1}$$

where

$$x_0^2 + k^2(x_1^2 + x_2^2 + x_3^2) = 1. \tag{2}$$

The line-element is now

$$ds^2 = \frac{1}{k^2} dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2, \tag{3}$$

and the differential equations of the geodesic lines are

$$\frac{d^2x_i}{ds^2} + k^2 x_i = 0. \qquad (i = 0, 1, 2, 3) \tag{4}$$

* These coördinates are called by Killing the Weierstrassian coördinates, because they were first used by Weierstrass in seminar work in 1872.

The integrals of these equations are

$$x_i = A_i \sin ks + B_i \cos ks, \qquad (i = 0, 1, 2, 3) \quad (5)$$

where the constants must be so chosen as to satisfy the conditions

$$\left.\begin{aligned} A_0^2 + k^2(A_1^2 + A_2^2 + A_3^2) = 1, \\ B_0^2 + k^2(B_1^2 + B_2^2 + B_3^2) = 1, \\ A_0B_0 + k^2(A_1B_1 + A_2B_2 + A_3B_3) = 0, \end{aligned}\right\} \qquad (6)$$

which are necessary and sufficient in order that the conditions (2) and (3) may be satisfied. In fact the constants B_i are the coördinates of the point from which s is measured and the constants kA_i are the values of dx_i/ds at that point and consequently fix the direction of the line.

We may write the equations of a geodesic line in terms of any two points upon it. Let y_i and z_i be the two points, and let l be the distance between them measured on the geodesic line. If we measure s from z_i, we have from (5),

$$z_i = B_i, \quad y_i = A_i \sin kl + B_i \cos kl. \qquad (7)$$

From these follow, with aid of the relation (6),

$$y_0z_0 + k^2(y_1z_1 + y_2z_2 + y_3z_3) = \cos kl, \qquad (8)$$

an important formula which gives the distance between two points in T.

If x_i is any other point on the geodesic line, we have from (5) and (7)

$$x_i = \lambda y_i + \mu z_i, \qquad (9)$$

where

$$\lambda = \frac{\sin ks}{\sin kl}, \quad \mu = \frac{\sin k(l-s)}{\sin kl},$$

or, otherwise written,

$$\lambda \sin kl = \sin ks, \quad \lambda \cos kl + \mu = \cos ks.$$

Hence λ and μ must satisfy the condition

$$\lambda^2 + \mu^2 + 2\lambda\mu \cos kl = 1, \qquad (10)$$

which is also the necessary and sufficient condition that x_i may satisfy relation (2).

Conversely any equations of the form (9) for which conditions (10) and (8) hold represent a geodesic line, provided they are satisfied by points in T. For it is always possible to find an angle ν, such that

$$\sin k\nu = \lambda \sin kl,$$
$$\cos k\nu = \lambda \cos kl + \mu.$$

From the condition (3) it follows that $ds^2 = d\nu^2$. It can then be verified that the functions

$$x_i = \lambda y_i + \mu z_i$$

satisfy the differential equations (4).

We collect these important results in the following theorem :

Any geodesic line may be represented by the equations

$$x_i = \lambda y_i + \mu z_i, \qquad (i = 0, 1, 2, 3)$$

where y_i and z_i are any two points on the line, and λ and μ are parameters satisfying the relation

$$\lambda^2 + \mu^2 + 2\lambda\mu \cos kl = 1,$$

l being the distance between the two points y_i and z_i.

Conversely any equations of the above form represent a geodesic line if they are satisfied by points of T.

From this follows immediately :

Any two linear homogeneous equations in x_i represent a geodesic line if satisfied by coördinates of points in T; and conversely any geodesic line may be represented by two such equations.

As to the geodesic surfaces we have the theorem :

Any geodesic surface is represented by a linear homogeneous equation in x_i; and conversely any such equation represents a geodesic surface if it is satisfied by points in T.

To prove the last theorem, consider a pencil of geodesic lines determined by two lines through B_i with the directions A_i' and A_i'' respectively. It has the equations

$$x_i = (\lambda A_i' + \mu A_i'') \cos ks + B_i \sin ks,$$

where

$$\lambda^2 + \mu^2 + 2\lambda\mu \cos \theta = 1$$

θ being the angle between the two lines A'_i and A''_i. From this it readily follows that the coördinates of any point on the pencil satisfy an equation of the form

$$c_0 x_0 + c_1 x_1 + c_2 x_2 + c_3 x_3 = 0.$$

Conversely if this equation is given and y_i and z_i are any two points satisfying it, the point

$$x_i = \lambda y_i + \mu z_i, \qquad (\lambda^2 + \mu^2 + 2\lambda\mu \cos kl = 1)$$

will also satisfy it. Hence any points on the locus of the equation may be connected by a geodesic line lying wholly on the locus. The locus may therefore be considered as a pencil of geodesic lines and is therefore a geodesic surface.

Explicit formulas for the displacements in T may now be written. Since these displacements are continuous, one-to-one point transformations by which a geodesic line is transformed into a geodesic line and the expression for cos kl is invariant they will have the form :

$$\begin{aligned}
x'_1 &= \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_0 x_0,\\
x'_2 &= \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_0 x_0,\\
x'_3 &= \gamma_1 x_1 + \gamma_2 x_2 + \lambda_3 x_3 + \gamma_0 x_0,\\
x'_0 &= \delta_1 x_1 + \delta_2 x_2 + \delta_3 x_3 + \delta_0 x_0,
\end{aligned}$$

where

$$\begin{aligned}
\delta_0^2 + k^2(\alpha_0^2 + \beta_0^2 + \gamma_0^2) &= 1,\\
\delta_i^2 + k^2(\alpha_i^2 + \beta_i^2 + \gamma_i^2) &= k^2, && (k = 1, 2, 3)\\
\delta_i\delta_h + k^2(\alpha_i\alpha_h + \beta_i\beta_h + \gamma_i\gamma_h) &= 0. && (i, h = 0, 1, 2, 3;\ i \neq h)
\end{aligned}$$

From these conditions it follows that determinant $|\alpha_1\beta_2\gamma_3\delta_0| = \pm 1$. If we add to our definition of a displacement the condition that it may be reduced to the identical substitution by a continuous change of the coefficients, we shall have the new condition

$$|\alpha_1\beta_2\gamma_3\delta_0| = +1.$$

Conversely, any linear substitution in which the coefficients satisfy the above conditions represents a displacement in T, provided that it is satisfied by at least one pair of corresponding points in T.

We have now the full data for constructing a system of geometry in T. The following are some of the fundamental theorems which are readily proved.* In fact some have already been proved in the preceding discussions and the theorems are repeated here for completeness.

1. A geodesic line is completely and uniquely determined by any two points.

2. A geodesic surface is completely and uniquely determined by any three points not in the same geodesic line.

3. If two points on a geodesic surface are connected by a geodesic line, the line lies wholly on the surface.

4. Two geodesic lines, or a geodesic line and a geodesic surface, intersect in at most one point.

5. Two geodesic surfaces intersect in a geodesic line, if they intersect at all.

6. On a given geodesic surface, one and only one geodesic line can be drawn perpendicular to a given geodesic line at a given point.

7. If a geodesic line is perpendicular to each of two intersecting geodesic lines at their point of intersection, it is perpendicular to every line of the pencil defined by the two intersecting lines.

Such a line is said to be perpendicular to the geodesic surface defined by the pencil.

8. Through any point of a geodesic surface, one and only one geodesic line can be drawn perpendicular to the surface.

9. Through a given point on a geodesic surface, one geodesic line can in general be drawn perpendicular to a given geodesic line on the surface not passing through the given point, and never more than one.

10. Through a given point not on a geodesic surface, one

* Proofs of all these theorems may be found in the *Annals* article already cited.

geodesic line can in general be drawn perpendicular to the surface, and never more than one.

11. The sum of the angles of a triangle formed by three intersecting geodesic lines is equal to, greater than, or less than, π, according as k is zero, real, or pure imaginary.

It appears that the geodesic lines in T have all the properties of the straight lines of practical life or of the Euclidean geometry. In the endeavor to construct a material line which shall be "straight," we may proceed by attempting to realize the shortest distance between two points by stretching a string or otherwise. The result is simply a geodesic line by definition. Or we may look for a line which may be revolved upon itself when two points are fixed. This is also a property of the geodesic lines. A geodesic surface has the properties of a plane. The practical testing of a plane surface by the application of a straight edge has its full significance in T. The practical measurement of length and angle by the application of an assumed unit is also possible in T. We see then that the groundwork of experimental geometry is the same for all spaces which satisfy our three hypotheses. These spaces agree also in the first ten theorems above stated. A distinction appears first in the eleventh theorem, which appears to present a means for determining the curvature of our objective space. The test fails, however, owing to the impossibility of exact measurements. All we can discover is that the sum of the angles of a triangle does not differ very much from π and it is possible to show that if the sides of a triangle are sufficiently large compared with k the divergence of the sum of its angles from π is within the limits of the errors of observation.*

We may say then: *Any space which satisfies the three hypotheses is, as far as our present knowledge goes, in full accord with all facts of experience, provided suitable values are given to the constants involved.*

* See, for example, the calculation in Lobachevsky's *Zwei geometrische Abhandlungen*, translated by F. Engel, Leipzig, 1899, pp. 22–24.

6. The Fourth and Fifth Hypotheses.

In order to extend our system of geometry outside of the region T, new hypotheses are necessary. These hypotheses must be such that their verification transcends experience, but it lies close at hand to assume that certain properties which are true as far as experience extends are everywhere true. We accordingly frame our hypotheses as follows:

Fourth Hypothesis. *Any portion of space in which the greatest geodesic distance does not exceed some constant M, dependent on the nature of the space, may be so displaced that an arbitrary point of this portion of space may be made to coincide with any point whatever in space.*

Fifth Hypothesis. *A displacement of a portion of space is completely and uniquely determined by the displacement of any portion of space which forms a three-dimensional part of the first portion.*

The meaning of the fourth hypothesis may be illustrated by the plane and the cone of the Euclidean geometry, as examples of two dimensional spaces satisfying the first three hypotheses. The region corresponding to T may be taken indefinite in extent in the case of the plane, but for the cone must be so taken that no point of the cone shall be covered more than once. The size of this region on the cone depends then upon its nearness to the vertex of the cone. It is clear that the cone does not satisfy the fourth hypothesis, since by definition a displacement demands a one-to-one correspondence of two regions and no matter how small a region may be taken on a cone this region can not be moved indefinitely near the vertex of the cone without overlapping itself. A right circular cylinder in Euclidean space would satisfy the fourth hypothesis, the quantity M being then the circumference of the right section. Similarly a Euclidean sphere satisfies the fourth hypothesis.

In like manner the fourth hypothesis applied to a three dimensional space rules out singular points and involves the assumption that space is boundless. It does not however assert that space is infinite in any or all directions.

The fifth hypothesis asserts that if a definite displacement is applied to a region of space S_i, any other region S_k which is connected with S_i in a definite manner suffers at the same time a certain definite displacement determined by the displacement of S_i. It leaves it still possible, however, that the displacement of S_k may depend upon the manner in which S_k is connected with S_i. Take, for example, the Euclidean right circular cylinder, and consider two strips of the surface connecting the same two points but in such a way that one strip winds around the cylinder more times than does the other. The same motion imparted to the same end of each strip imparts a different motion to the other ends.

The fifth hypothesis also asserts that if by a continuous displacement S_i returns to its original position, so does also S_k.

7. The Extended Coördinate System.

We may now extend our coördinate system x_i from the region T, for which it has been defined, to all points of space. For that purpose, let us consider a region of space S_0 composed of the points whose geodesic distances from O are less than, or equal to a constant R, where R is less than the smaller of the two quantities ρ and $M/2$, ρ being the length of the shortest geodesic line which can be drawn from O in T and M being the constant mentioned in the fourth hypothesis. Analytically we have in S_0

$$x_i = a_i \frac{\sin ks}{k} \qquad (i = 1, 2, 3)$$

$$x_0 = \cos ks,$$

where

$$a_1^2 + a_2^2 + a_3^2 = 1, \quad |s| \leqq R, \quad R < \rho, \quad R < \frac{M}{2}.$$

We shall first prove that *any geodesic line can be indefinitely continued.* For consider any geodesic line OQ in S_0 of length R, and take O_1 a point on OQ such that $OO_1 = l < R$. There exists a displacement such that the point O corresponds to O_1 and a region T_0

around O corresponds to a region T_1 around O_1 in such a manner that the portion of the geodesic line OQ which lies in T_0 corresponds to the portion of the same line which lies in T_1 and extends in the same direction. Here T_0 and T_1 are both contained in S_0, but by virtue of the fifth hypothesis this displacement of T_0 into T_1 determines a displacement of S_0 into a new position S_1. The line OQ of length R goes then into a line O_1Q_1 of the same length; that is, the line OQ_1 has the length $R + l$. Now we can repeat this operation with the region S_1 by selecting on O_1Q_1 a point O_2 such that $O_1O_2 = l$, and displacing O_1 into O_2 in the proper manner. In this way the line OQ is extended indefinitely, but it is of course consistent with the theorem that the line should be a closed line.

Any point in space may be joined to O by a geodesic line. A rigorous proof of this statement may be given by means of the method introduced by Hilbert into the Calculus of Variations under the name of the "*Häufungsverfahren.*"* The details are too involved to be presented here. We content ourselves with noticing that since space is a continuum by our first hypothesis, any point P may be connected with O by a continuous curve. Now the Hilbert method consists in showing that among all the curves that can be drawn between O and P there is one such that no other has a greater length, and that this curve in sufficiently small portions is a geodesic line as we have defined it.

By virtue of the two theorems just proved, we may write

$$x_i = a_i \frac{\sin ks}{k}, \qquad (i = 1, 2, 3)$$

$$x_0 = \cos ks, \qquad (a_1^2 + a_2^2 + a_3^2 = 1),$$

where s is unrestricted, with the assurance that all values of x_i thus determined represent a point of space and that any point of space may be represented in this way. This is our generalized coördinate system.

Let us take now any point P. By the fourth hypothesis, the

* Consult for example the dissertation of Chas. A. Noble, "Eine neue Methode in der Variationsrechung," Göttingen, 1901.

region S_0 may be so displaced that O corresponds with P and S_0 with a congruent region S_n. There exist then relations between the coördinates of points in S_n and the coördinates of points in S_0. We shall show that *these relations have the same form as those which define a displacement in* S_0. For that purpose connect O and P with a geodesic line and take on this line the points O, O_1, O_2, $\cdots$, $O_n = P$, such that the distance $O_i O_{i+1}$ is less than R. If then O is displaced so as to coincide in succession with O_1, O_2, $\cdots$, P, there is determined a chain of congruent regions S_0, S_1, S_2, $\cdots$, S_n, each of which has points in common with the preceding one. The displacement of S_0 into S_1 however is fully determined by the fact that a region around O is displaced into a region around O_1, both regions lying in S_0. Hence all coördinates of all the points in S_1 are connected with those of S_0 by relations of the form given in paragraph 5. It follows that in S_1 the line element is the same as in S_0, that a linear equation represents a geodesic surface, that two such equations represent a geodesic line, and that a displacement of a portion of S_1 is represented by equations of the same form as in S_0. In like manner we can proceed from S_1 to S_2, and hence eventually to S_n, thus establishing the fact to be proved.

It is clear that if more than one geodesic line can be drawn from O to P, P will have more than one set of coördinates and more than one set of equations will connect the coördinates of S_n and S_0.

Let now any displacement be imparted to S_0. By the fifth hypothesis, a displacement is then imparted to S_n through the chain S_0, S_1, S_2, $\cdots$, S_n. It is easy to see that the analytic expression of this displacement of S_n will be found by substituting in the displacement defined for S_0 the coördinates of the points of S_n determined by the chain S_0, S_1, $\cdots$, S_n.

We may now establish the important proposition: *If k is a real quantity, every geodesic line is closed and has a length not exceeding* $2\pi/k$.*

* This theorem is due to Killing. His proof is essentially that of the text.

For proof consider a point Q at a distance $\pi/2k$ from O on the geodesic line $x_2=0$, $x_3=0$. The coördinates of Q are $(1/k, 0, 0, 0)$. Let a chain of congruent regions $S_0, S_1, S_2, \cdots, S_n$, be strung along the line OQ, the point Q lying in S_n, and each region being obtained from the preceding one by the substitution

$$\begin{aligned} x_1' &= x_1 \cos kl + x_0 \frac{\sin kl}{k}, \\ x_2' &= x_2, \\ x_3' &= x_3, \\ x_0' &= -x_1 k \sin kl + x_0 \cos kl, \end{aligned}$$

where $l < R$.

Apply now to S_0 the displacement

$$\begin{aligned} x_1' &= x_1 \cos \phi - x_2 \sin \phi, \\ x_2' &= x_1 \sin \phi + x_2 \cos \phi, \\ x_3' &= x_3, \\ x_0' &= x_0. \end{aligned}$$

This displacement will be transmitted to S_n through the chain $S_0, S_1, \cdots, S_n$. The distance D between the new and the original position of a point is given by

$$\begin{aligned} \cos kD &= x_0 x_0' + k^2(x_1 x_1' + x_2 x_2' + x_3 x_3') \\ &= x_0^2 + k^2 x_3^2 + k^2(x_1^2 + x_2^2) \cos \phi \\ &= \cos \phi + (x_0^2 + k^2 x_3^2)(1 - \cos \phi). \end{aligned}$$

Now the line $x_0 = 0$, $x_3 = 0$, a portion of which lies in S_n, is displaced into itself, each point being moved through a distance D where

$$\cos kD = \cos \phi.$$

Hence as ϕ varies from 0 to 2π, the point Q is moved on $x_0 = 0$, $x_3 = 0$ through a distance $2\pi/k$. But a continuous variation of ϕ from 0 to 2π restores S_0 and hence S_n to its original position. Hence the geodesic line $x_0 = 0$, $x_3 = 0$ cannot have a length greater than $2\pi/k$. The theorem is thus proved for a particular geodesic line; but by proper choice of the origin and coördinate axes, any geodesic line may be given the equations $x_0 = 0$, $x_3 = 0$ and hence the theorem holds universally. It may be explicitly noted that we have not proved that a geodesic line may not have a length less than $2\pi/k$, nor that all geodesic lines have the same length.

We are now prepared to prove the proposition:

To any set of values (x_0, x_1, x_2, x_3) *satisfying the fundamental relation*

$$x_0^2 + k^2(x_1^2 + x_2^2 + x_3^2) = 1$$

corresponds one and only one point of space.

In the proof, it will be convenient to separate the three cases of zero, negative, and positive curvature.

1. If $k = 0$, the coördinates of any point are

$$x_i = a_i r, \quad x_0 = 1. \qquad (i = 1, 2, 3.)$$

2. If $k = ik'$, the coördinates are

$$x_i = a_i \frac{\sinh k'r}{k'}, \quad x_0 = \cosh k'r \qquad (i = 1, 2, 3).$$

3. If k is real, the coördinates are

$$x_i = a_i \frac{\sin kr}{k}, \quad x_0 = \cos kr \qquad (i = 1, 2, 3).$$

It is now readily seen that if the quantities x_i are given, the quantities a_1, a_2, a_3, r are uniquely determined in cases 1 and 2, except for sign; while in case 3 multiples of $2\pi/k$ may be added to r and the signs are also ambiguous. The change of sign of all four quantities (a_i, r) does not alter the point determined by them and an addition of $2\pi/k$ to r in case 3 amounts simply to traversing the length of the geodesic line one or more times. Given the quantities x_i therefore, we lay off at O a definite direction a_i and measure on the geodesic line with this direction a definite distance r. We obtain in this way one and only one point.

8. The Auxiliary Space Σ.

The discussion of the following paragraphs will be clarified by making use of familiar propositions of the projective geometry. In so doing, we avail ourselves of theorems which are in essence analytic. Their geometric clothing lends vividness to their meaning and helps greatly in their application. We consider then a projective geometry in which a point is fixed by the homogeneous coördinates

$$\xi_0 : \xi_1 : \xi_2 : \xi_3.$$

A linear homogeneous equation defines a plane, two such equations a straight line. In this geometry we define a system of projective measurement, based upon the fundamental quadric

$$\xi_0^2 + k^2(\xi_1^2 + \xi_2^2 + \xi_3^2) = 0.$$

The distance Δ between two points is by definition given by the relation

$$\cos k\Delta = \frac{\xi_0\xi_0' + k^2(\xi_1\xi_1' + \xi_2\xi_2' + \xi_3\xi_3')}{\sqrt{\xi_0^2 + k^2(\xi_1^2 + \xi_2^2 + \xi_3^2)}\sqrt{\xi_0'^2 + k^2(\xi_1'^2 + \xi_2'^2 + \xi_3'^2)}}.$$

Any collineation which leaves the fundamental quadric invariant we shall call a movement of the projective space. Such a movement leaves distance and angle unaltered. The space in which this geometry prevails we shall call the auxiliary space Σ.

The points of Σ may be made to correspond to the points of S by placing

$$x_i = \frac{\xi_i}{\pm\sqrt{\xi_0^2 + k^2(\xi_1^2 + \xi_2^2 + \xi_3^2)}}, \qquad (i = 0, 1, 2, 3)$$

where the sign of the radical is the same for all values of i. It is clear that geodesic lines and surfaces in S correspond to straight lines and planes in Σ and conversely. Geodesic distances and angles in S correspond to projective lengths and angles in Σ and a displacement in S corresponds to a movement in Σ and conversely.

Now if k is zero or pure imaginary, x_0 is always positive, since

$x_0 = \cos ks$. Hence in these two cases, the sign of the radical is unambiguous. If k is real, however, x_0 may be either positive or negative, and hence either sign of the radical may be taken. Hence:

If k is zero or pure imaginary, any point of Σ corresponds to one and only one point of S; while if k is real, any point of Σ may correspond either to one or to two points of S according as x_i and $-x_i$ are the coördinates of the same or of different points of S.

On the other hand, any point of S corresponds to as many points of Σ as there are different sets of coördinates belonging to the point of S. To follow this more in detail, let us consider the point O which corresponds in Σ to the point $(0:0:\ 0:1)$. If O has other coördinates it must be possible to draw a geodesic line from O which shall again return to O. This follows from the expressions for the coördinates. Let us call this line g. Correspondingly, we have in Σ a straight line γ connecting two points o and o', each of which corresponds to O. The length of g, and hence of γ, most be less than the quantity R which occurred in the definition of S_0: for all lines of length R or less, radiating from O determine points in S_0, in which no closed line is possible. Since any point of space may be taken for O, we may say:

Two points in Σ which correspond to the same point in S can not be nearer together than a certain finite quantity.

9. Forms of Space Which Allow Free Motion as a Whole.

We are to examine in this paragraph the results of assuming that the displacement of S_n caused by a displacement of S_0 is independent of the manner in which S_n is connected with S_0; that is, it is independent of the chain of bodies $S_0, S_1, \cdots, S_n$. In this case any displacement of S_0 imparts a unique displacement to each and every point of space. We express this by saying that space allows free motion as a whole. We assert:

If S allows free motion as a whole, any point of S corresponds to one and only one point of Σ.

Consider a point P in S_0 and let us assume that P corresponds to two points Π and Π'. As shown in the last paragraph, if Π and Π' are connected by a straight line γ, there will correspond in S a line g which starts from P and returns to the same point. Along this line we may construct a chain of congruent regions S_0, $S_1, S_2, \cdots, S_n$, where S_n is the same region as S_0. Corresponding to this configuration, we have in Σ a chain of regions $\Sigma_0, \Sigma_1, \Sigma_2, \cdots, \Sigma_n$, where Σ_n is distinct from Σ_0. Now any displacement imparted to S_0 is transmitted through the chain $S_0, S_1, \cdots, S_n$ back to S_n. But this displacement of S_n must be the same as that of S_0, if space is movable as a whole. If, for example, S_0 is so moved that all points on a geodesic line l are fixed, S_n must be moved in the same manner. Correspondingly, we must have in Σ a displacement by which two straight lines λ and λ', one lying in Σ_0, the other lying in Σ_n are each point for point fixed. This, however, is impossible unless Σ_n coincides with Σ_0. Hence the assumption that P corresponds to two points Π and Π' is untenable.

Spaces of Zero Curvature.

If $k = 0$, the relation between points of S and those of Σ is one to one. In other words, to each point of S corresponds one and only one set of coördinates x_i and conversely. We have therefore a geometry in which the theorems of paragraph 5 hold universally. In addition all geodesic lines are infinite in length. We may consequently introduce the conception of parallel lines by the following definition: A line AB is parallel to CD when AB is the limit approached by a line AG intersecting CD, as the point of intersection recedes indefinitely. It may then be shown that through any point of space there goes one and only one geodesic line which is parallel to a given geodesic line not passing through the given point. The resulting geometry is the *Euclidean Geometry.*

Spaces of Constant Negative Curvature.

If k is pure imaginary, again the relation between the points of S and those of Σ is one to one. We have again a space in which the theorems of paragraph 5 hold universally and in which all geodesic lines are infinite in length. If parallel geodesic lines are defined as for $k = 0$, then through a given point there go two and only two geodesic lines parallel to a given geodesic line not passing through the given point. All other geodesic lines through the point and lying on the geodesic surface determined by the given point and the given geodesic line are separated by the parallel lines into two classes, consisting respectively of the lines which do, and of the lines which do not, intersect the given geodesic line. The geometry is the *Lobachevskian Geometry.*

Spaces of Constant Positive Curvature.

If k is real, two cases present themselves. In the first case, the relation between the points of S and those of Σ is two-to-one. Then to each point of S corresponds only one set of coördinates and conversely. In particular, the coördinates x_i and $-x_i$ belong to different points of space. The theorems of paragraph 5 hold only in a restricted portion of space in which the greatest geodesic distance is π/k. All geodesic lines are closed and of length equal to $2\pi/k$. Two intersecting geodesic lines intersect again at a distance π/k on each of them from the first point of intersection. There are no parallel lines in the sense of the definition given for $k = 0$. In fact any two geodesic lines on the same geodesic surface intersect. All geodesic lines perpendicular to the same geodesic surface intersect in two points which are distant $\pi/2k$ from the surface. The geometry is that called by Klein the *Spherical Geometry.*

In the second case, the relation between the points of S and those of Σ is one-to-one, in the sense that to each point of S belongs the two sets of coördinates x_i and $-x_i$. The theorems of paragraph 5 hold for a portion of space in which the greatest geodesic distance is π/k. All geodesic lines are closed and of a length π/k and any two intersecting geodesic lines return to the point of intersection without previously meeting. All geodesic lines perpendicular to the same geodesic surface meet in a point at a distance $\pi/2k$ from the surface. The geometry is called by Klein the *Elliptic Geometry.*

We may sum up as follows:

The only spaces satisfying our five hypotheses and allowing free motion as a whole are the Euclidean, Lobachevskian, Spherical and Elliptic spaces.

10. Forms of Space Which do Not Allow Free Motion as a Whole.

We consider next spaces in which the displacement of S_n caused by the displacement of S_0 is dependent upon the manner in which S_n is connected with S_0. These are called by Killing the Clifford-Klein space. They have been illustrated in paragraph 6.

From what has preceded, it is clear that in the Clifford-Klein spaces a point must have more than one set of x_i-coördinates. Consider then the region S_0 and let x_i be one set of coördinates of its points. Then if x'_i are also the coördinates of its points, x'_i may be obtained from x_i, as we have seen, by following out a chain of displacements by which S_0 takes in succession the positions S_0, S_1, S_2, $\cdots S_n = S_0$. That is x'_i and x_i are connected by relations which have the form of the displacement formulas. Suppose these relations denoted by D_1. Let now y_i be the coördinates of a point P lying outside of S_0. It may be connected with S_0 by a geodesic line and a chain of regions S_0, S'_1, S'_2, $\cdots$, S'_n constructed along this line. If the displacement D_1 is imposed upon S_0, it will be transmitted to S'_n; and since S_0 returns to its original position the same is true of S'_n, by the fifth hypothesis. That is the transformation D_1 gives a relation between two sets of coördinates of any point of space. Such a transformation is said by Killing to represent the coincidence of points.

It is clear that the inverse transformation D_1^{-1} also represents the coincidence of points, and if D_1 and D_2 each represents the coincidence of points, the transformation D_1D_2 does also, and this is true when D_2 is the same as D_1. That is, *the transformations which represent the coincidence of points in space form a group.* This group we shall call the *group of the space.*

The group of the space interpreted in Σ is a group of collineations by which the fundamental quadric is invariant and by which points that correspond to the same point in S are transformed into each other. Because of the theorems established in paragraph 8 it follows that the group of the space interpreted in Σ must not only be properly discontinuous but must be subject to the condition that the distance between corresponding points shall never be less than a certain finite quantity. In particular, *no transformation of the group may have a real fixed point.* If the region of discontinuity of the group in Σ is obtained, this region will correspond in a one-to-one manner to S, when k is zero or pure imaginary, and in either a one-to-one manner or a one-to-two manner to S when k is real. Conversely, the region of discontinuity of any properly discontinuous group in Σ, by which the distance between two corresponding points is never less than a finite quantity, will furnish an example of a space satisfying the five hypotheses. Hence the problem to determine the Clifford-Klein space is reduced to the problem to determine all groups with the required properties.

Before proceeding to the nearer discussion of the problem, we may note that our derivation of the group of the space is based upon the consideration of a three-dimensional region S_0 in which each point has different sets of coördinates. This region gives opportunity to apply the fifth hypothesis. There is still the possibility therefore that certain exceptional one-dimensional or two-dimensional regions may exist, upon which the same point may have sets of coördinates not connected by transformations of the group. The following two examples are given by Killing of a two-dimensional space of zero curvature having an exceptional line.

1. Consider a cylinder in Euclidian space standing upon a cubic curve with a double point. The geometry of the cylinder is that of the Euclidean plane except for the presence of the double line.

We call $2a$ the length of the loup of the cubic, and take as the origin of coördinates the point on the loup equidistant from the double point in each direction. Then if we take for one coördinate the length s of the cubic and for the other the length h of an element of the cylinder, the coördinates (s, h) correspond in a one-to-one manner to the points of the surface, except that the coördinates (a, h) and $(-a, h)$ correspond to the same point of the surface.

2. Consider a cylinder in Euclidean space standing on a lemniscate. Its geometry is the same as that of the Euclidean plane for restricted portions. We will take the origin at the double point of the lemniscate, define s as the length of the curve and h as the length of an element of the cylinder. Then if $2a$ is the entire length of the lemniscate, the group of the surface is

$$s' = s + 2na,$$
$$h' = h,$$

where n is an integer; that is, the coördinates (s, h) and $(s + 2na, h)$ refer to the same point of the surface. But the coördinates $(0, h)$ and (na, h) also refer to the same point of the surface, since they give points on the double line.

Examples of a similar kind may be formed for three dimensional spaces without difficulty as far as the analytic work is concerned. How far they are conceivable as an explanation of physical space, involving as they do the passing of space through itself without break in the continuity of each of the intersecting portions may be open to question. They have been examined by no one in detail and we shall rule them out of the following discussion.

We pass now to the special consideration of the three kinds of space.

Spaces of Zero Curvature.

If $k = 0$, Σ is the Euclidean space and its movements are the Euclidean movements. A rotation around an axis cannot be a transformation of the group of the space S since, as we have seen, no transformation of the group can have a real fixed point. We must form the group therefore by the use of translations and screw motions.

The use of translations alone lead to three and only three properly discontinuous groups, having for regions of discontinuity respectively:

(*a*) A parallelopiped with three finite edges.

(*b*) The limiting figure of a parallelopiped when one edge becomes infinite.

(*c*) The limiting figure of a parallelopiped when two edges become infinite.

The geometry in S may be readily constructed by operating with the Euclidean geometry in the regions (*a*), (*b*), (*c*), respectively. Whenever a straight line meets a bounding face of the region, it is continued from the corresponding point of the opposite face. For brevity we shall mention without proof some of the results in case (*a*).

Some geodesic lines are closed and some are infinite in length and those which are closed are not all of the same length. In fact geodesic lines can be drawn, having the finite length $la + mb + nc$, where a, b, c, are the lengths of the edges of the parallelopiped and l, m, n, are any three relatively prime integers. Geodesic surfaces are of three kinds. Some are indefinite in extent, possessing no points with more than one set of coördinates. On these the geometry is identical with the Euclidean geometry. Others are represented in Σ by a strip of a plane bounded by parallel lines and have in S the connectivity and geometry of a Euclidean cylinder. Other surfaces are represented in Σ by a plane parallelogram and have in S the connectivity of a ring surface.

No exhaustive study has been made of the Clifford-Klein spaces whose groups contain screw motions. In fact Klein says, without proof, that a screw motion is not allowable, but Killing gives the following two examples which seem valid:

(*a*) The group of the space is generated by a single screw motion:

$$x_1' = x_1 \cos\alpha - x_2 \sin\alpha,$$

$$x_2' = x_1 \sin\alpha + x_2 \cos\alpha,$$

$$x_3' = x_3 + h,$$

where α and h are constants. The region of discontinuity in Σ is then bounded by two parallel planes at a distance h units from each other.

(*b*) The group of the space is

$$x_1' = (-1)^l x_1 + ma,$$
$$x_2' = (-1)^l x_2 + nb,$$
$$x_3' = x_3 + lc,$$

where a, b, c, are constants and l, m, n, are arbitrary parameters.

We give without proof some of the striking peculiarities of S in the case (a).

There is a unique geodesic line of length h which we shall call the axis of the space. If α and π are incommensurable, this is the only closed geodesic line; if α and π are commensurable, all geodesic lines parallel to the axis are closed and of lengths equal to multiples of h. For all values of α there are geodesic lines with double points. Through any point of space there goes an infinite number of such geodesic lines having the given point for a double point; and for a given direction, not parallel or perpendicular to the axis, there exist an infinite number of geodesic lines with double points. Geodesic surfaces are of three kinds: (1) those perpendicular to the axis, (2) those parallel to or containing the axis, (3) those which have neither of these relations to the axis. On geodesic surfaces of the first kind, all geodesic lines are infinite in length and the geometry is that of the Euclidean plane. On geodesic surfaces of the second kind, there are no closed geodesic lines but a geodesic line may have a double point. On geodesic surfaces of the third kind, all kinds of geodesic lines lie. The last two kinds of surfaces present the peculiarities of cylinders with double lines mentioned on pp. 62–3.

Spaces of Constant Positive Curvature.

If k is real, there is a fundamental difference between spaces of an even and those of an odd number of dimensions. It is a simple matter to apply our foregoing discussion to space of two dimensions by dropping the coördinate x_3 and making necessary changes. It appears that any displacement has a real fixed point and consequently there can be no group of the space. If we rule out such special lines and points as occur on cylinders with double lines, we are then led to the discussion of paragraph 9. Hence the theorem:

A non-euclidean space of two dimensions and of constant positive curvature for which our hypotheses hold and in which no special points exist has the connectivity and the geometry of either the spherical surface or the elliptic plane.

Consider now a space of three dimensions. The study of the collineations which leave invariant the quadric

$$\xi_0^2 + k^2(\xi_1^2 + \xi_2^2 + \xi_3^2) = 0,$$

and which we call the movements of Σ, lead to the following results.* By any real movement in Σ two real lines G and H, reciprocal polars with respect to the fundamental quadric, are unaltered as a whole, each point on each of the lines being displaced through a distance which is constant for that line. If the displacement is different for the two lines G and H, these are the only fixed lines. If however the displacement is the same for G and H, then all lines of a certain line congruence are fixed, this congruence being made up of all lines which intersect the same two conjugate imaginary generators of the fundamental quadric. Any point of Σ is then displaced a constant distance along the line of the congruence which contains the point.

Such a transformation is the nearest analogy in a space of constant positive curvature to a translation in Euclidean space. It is accordingly called a *translation*, and the congruence of fixed lines are called *Clifford parallels*. The name parallels is suggested by the relation of these lines to a translation, but they have other properties analogous to those of the Euclidean parallels. For example, from any point in either of two Clifford

* Consult for the details of the geometry of this paragraph: Klein, "Zur Nicht-Euklidischen Geometrie," *Math. Annalen*, vol. 37 (1890), p. 544.

parallels a common perpendicular can be drawn to the two, and the portion of the perpendicular included between the two has always the same length. Again, if a line cut two Clifford parallels the corresponding angles are equal.

The Clifford parallels are of two kinds, according as the generators of the fundamental quadric which determine them belong to one or the other of the two sets of generators of the quadric. Similarly we must distinguish between two kinds of translations. Two translations of the same kind carried out in succession are equivalent to a translation of the same kind, but two translations of different kinds are not equivalent to a translation. Hence the translations of each kind form by themselves a group.

Let us consider first non-euclidean spaces whose groups are formed by translations alone. These translations must all be of the same kind. If we place $k = 1$, for convenience, and introduce λ and μ as the parameters of a point on the fundamental quadric, whereby

$$\lambda = -\frac{\xi_1 + i\xi_2}{\xi_3 + i\xi_0} = \frac{\xi_3 - i\xi_0}{\xi_1 - i\xi_2},$$

$$\mu = \frac{\xi_1 + i\xi_2}{\xi_3 - i\xi_0} = -\frac{\xi_3 + i\xi_0}{\xi_1 - i\xi_2},$$

then any translation of the one kind causes a substitution of the form

$$\lambda' = \frac{(d + ic)\lambda - (b - ia)}{(b + ia)\lambda + (d - ic)}, \qquad \mu' = \mu$$

and conversely.

On the other hand, if we interpret λ in the usual manner as a complex variable upon the unit sphere, the above substitution represents a rotation of the sphere. To any translation of the one kind in Σ corresponds then a rotation of the sphere, and in fact the angle of rotation of the sphere is equal to the distance by which the points of Σ are displaced along a system of Clifford parallels. The group of the space corresponds to a group of rotations of the sphere, and since the amount of displacement by any transformation of the group is never less than a finite quantity R it follows that the group of rotations can contain no infinitesimal rotation. This condition is met only by the groups of rotations by which a regular polyedron concentric with the sphere is transformed into itself. We have accordingly the theorem :*

If a Clifford-Klein space of constant positive curvature is transformed into itself by a group of translations, this group must be holoedric-isomorph with a group of the regular polyedra; and conversely, to any group of the regular polyedra correspond four spaces of constant positive curvature, according as the coördinates x_i *and* $-x_i$ *represent the same or different points of space and as the group of the space is made up of translations of one or the other kind.*

It remains to ask if groups of the space may contain displacements which are not translations. This question is answered in the negative by Killing (*l. c.*) but his proof is not satisfactory. He shows conclusively that if D is a displacement belonging to the group of the space and if G and H are the two fixed lines, then the smallest displacement along either line caused by the repetition of D must be π/q, where q is an integer, the same for both lines. But he errs in assuming that this minimum displacement is caused in both lines by the same transformation. For example, consider the displacement D

$$x_1' = x_1 \cos\frac{\pi}{5} - x_2 \sin\frac{\pi}{5},$$

$$x_2' = x_1 \sin\frac{\pi}{5} + x_2 \cos\frac{\pi}{5},$$

$$x_3' = x_3 \cos\frac{3\pi}{5} - x_0 \sin\frac{3\pi}{5},$$

$$x_0' = x_3 \sin\frac{3\pi}{5} + x_0 \cos\frac{3\pi}{5},$$

and the group $D, D^2, D^3, D^4, D^5 = 1$. The two fixed lines are

* This theorem is new as far as the author knows. Killing (*Grundlagen der Geometrie*, vol. 1, p. 341) notices that if the group of a space of $k = 1$ contains a translation, the amount of the translation must be an aliquot part of π, but he leaves the impression that any three such translations may be combined at pleasure to form a group of a space.

$G\,(x_1 = 0,\, x_2 = 0)$ and $H\,(x_3 = 0,\, x_0 = 0)$ and the smallest displacement along each is $\pi/5$. But this displacement is produced along G by D and D^4 and is produced along H by D^2 and D^3. By no substitution of the group, however, can the distance between two corresponding points fall below a definite finite quantity. Hence the group, which is not composed of translations, is allowable as the group of a non-euclidean space. The investigation of such groups is yet to be made.

Clifford's Surface of Zero Curvature.

It is of interest at this point to mention Clifford's surface of zero curvature and finite extent which first led to the conception of the Clifford-Klein spaces. This surface may be obtained by choosing on the fundamental quadric of the above space of constant positive curvature two conjugate imaginary lines from each set of generators. The quadric surface which passes through the quadrilateral thus formed is the surface required. It is clear that the surface contains two sets of Clifford parallels and is transformed into itself by two translations. If we take the four lines on the fundamental quadric as corresponding respectively to $\lambda = 0$, $\lambda = \infty$, $\mu = 0$, and $\mu = \infty$ in our previous notation, the equation of the surface is

$$\xi_1^2 + \xi_2^2 - a^2(\xi_3^2 + \xi_0^2) = 0,$$

where a is a real constant.

We may define the two sets of Clifford parallels on the surface by the parameters u and v, where

$$-\frac{\xi_1 + a\xi_3}{\xi_2 + a\xi_0} = \frac{\xi_2 - a\xi_0}{\xi_1 - a\xi_3} = u,$$

$$\frac{\xi_1 + a\xi_3}{\xi_2 - a\xi_0} = -\frac{\xi_2 + a\xi_0}{\xi_1 - a\xi_3} = v.$$

To obtain the line-element of the surface, we write first $\xi_i = \rho x_i$, where $\rho^2 = \Sigma\xi_i^2$. Then for the space

$$ds^2 = \Sigma dx_i^2 = \frac{\rho^2 \Sigma d\xi_i^2 - (\rho d\rho)^2}{\rho^4},$$

and this applied to the surface gives

$$ds^2 = \frac{du^2}{(1+u^2)^2} - 2\frac{a^2-1}{a^2+1}\frac{du\,dv}{(1+u^2)(1+v^2)} + \frac{dv^2}{(1+v^2)^2}.$$

If we take next as parameters σ and τ the lengths of the generators, by placing

$$\sigma = \int \frac{du}{1+u^2}, \quad \tau = \int \frac{dv}{1+v^2},$$

the line element takes the simpler form

$$ds^2 = d\sigma^2 - 2\frac{a^2-1}{a^2+1} d\sigma d\tau + d\tau^2.$$

From this it appears that the Gaussian curvature of the surface is zero.

The relation between a point of the surface and a value-pair (u, v) is one-to-one. Hence if v is kept constant and u varies from $-\infty$ to $+\infty$ the corresponding point describes a generator once. At the same time σ varies continuously from $-\pi/2$ to $\pi/2$. The total length of a generator is then π, a finite quantity.

The area of a portion of a surface of which the line element is

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2$$

is defined by the double integral

$$\iint \sqrt{EG - F^2}\, dudv$$

taken over the portion. Hence the total area of the Clifford surface is

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{2a}{a^2+1} d\sigma d\tau = \frac{2a\pi^2}{a^2+1}.$$

We have therefore an example of an unbounded surface of zero curvature upon which the connectivity and the geometry is that of a parallelogram on the Euclidean plane, the opposite sides of the parallelogram corresponding point for point.

Spaces of Constant Negative Curvature.

If k is pure imaginary, we may place $k=i$ for convenience. We have then in Σ the fundamental quadric

$$\xi_1^2+\xi_2^2+\xi_3^2-\xi_0^2=0,$$

points in the interior of which correspond to real values of x_i and hence to real points of S. Any collineation which leaves this quadric invariant determines a linear substitution of the parameters λ and $\bar{\lambda}$ where

$$\lambda=\frac{\xi_1+i\xi_2}{\xi_0-\xi_3}, \qquad \bar{\lambda}=\frac{\xi_1-i\xi_2}{\xi_0-\xi_3};$$

and conversely any pair of linear substitutions

$$\lambda'=\frac{\alpha\lambda+\beta}{\gamma\lambda+\delta}, \qquad \bar{\lambda}'=\frac{\bar{\alpha}\bar{\lambda}+\bar{\beta}}{\bar{\gamma}\bar{\lambda}+\bar{\delta}},$$

where the determinants $\alpha\delta-\beta\gamma$ and $\bar{\alpha}\bar{\delta}-\bar{\beta}\bar{\gamma}$ are not zero, determines such a collineation.* These collineations are the movements of Σ. A real movement occurs when and only when $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$ and $\bar{\delta}$, are conjugate imaginary to α, β, γ, δ, respectively. A real movement may consequently be determined by the single substitution

$$\lambda'=\frac{\alpha\lambda+\beta}{\gamma\lambda+\delta}.$$

Let us suppose first that the substitution in λ leaves two distinct values of λ unaltered. There correspond two fixed points on the fundamental quadric, and we may without loss of generality assume the coördinate system in such a way that these correspond to the values $\lambda=\pm 1$. The substitution may then be written

* Consult for proof and historical references: Fricke-Klein, *Vorlesungen über die Theorie der automorphen Functionen*, vol. 1, pp. 44–59.

$$\frac{\lambda' - 1}{\lambda' + 1} = e^{\alpha + i\beta} \frac{\lambda - 1}{\lambda + 1},$$

and is a loxodromic substitution when $\alpha \neq 0$, $\beta \neq 0$, an elliptic substitution when $\alpha = 0$, $\beta \neq 0$, and a hyperbolic substitution when $\alpha \neq 0$, $\beta = 0$. The corresponding substitution of x_i is readily computed to be

$$x_1' = x_1 \cosh \alpha - x_0 \sinh \alpha,$$

$$x_2' = x_2 \cos \beta - x_3 \sin \beta,$$

$$x_3' = x_2 \sin \beta + x_3 \cos \beta,$$

$$x_0' = - x_1 \sinh \alpha + x_0 \cosh \alpha,$$

and the distance l between two corresponding points is determined by the equation

$$\begin{aligned} \cosh l &= - x_1' x_1 - x_2' x_2 - x_3' x_3 + x_0' x_0 \\ &= - (x_2^2 + x_3^2) \cos \beta + (x_0^2 - x_1^2) \cosh \alpha \\ &= \cosh \alpha + (x_2^2 + x_3^2)(\cosh \alpha - \cos \beta). \end{aligned}$$

If $\alpha = 0$, every point on the line $x_2 = 0$, $x_3 = 0$ is fixed. Hence an elliptic substitution can not occur in the group of the space. If $\alpha \neq 0$, $l \geqq \alpha$. Hence hyperbolic or loxodromic substitutions may occur in the group.

Consider next a parabolic substitution of λ by which only one value of λ is unaltered. By proper choice of the coördinate system this substitution may take the form

$$\lambda' = \lambda + a$$

and the corresponding substitution of x_i is

$$x_1' = x_1 + \frac{\bar{a} + a}{2}(x_0 - x_3),$$

$$x_2' = x_2 + \frac{i(\bar{a} - a)}{2}(x_0 - x_3),$$

$$x_3' = x_3 + \frac{\bar{a} + a}{2} x_1 + \frac{i(\bar{a} - a)}{2} x_2 + \frac{a\bar{a}}{2}(x_0 - x_3),$$

$$x_0' = x_0 + \frac{\bar{a} + a}{2} x_1 + \frac{i(\bar{a} - a)}{2} x_2 + \frac{a\bar{a}}{2}(x_0 - x_3).$$

The distance l between two corresponding points is given by the equation

$$\cosh l = 1 + \frac{a\bar{a}}{2}(x_0 - x_3)^2.$$

There is no fixed point in finite space, for the assumption $x_0 = x_3$ carries with it the equality

$$x_1^2 + x_2^2 = -1.$$

We may however find corresponding points whose distance apart is less than any assigned quantity. For if we take y_i to represent any point, the coördinates

$$x_1 = \lambda y_1, \quad x_2 = \lambda y_2, \quad x_3 = \lambda y_3 + \mu, \quad x_0 = \lambda y_0 + \mu$$

represent a point for all values of λ and μ which satisfy the relation

$$\lambda^2 + 2\lambda\mu(y_0 - y_3) = 1.$$

The displacement l of the point x_i is determined by

$$\cosh l = 1 + \frac{a\bar{a}}{2}\lambda^2(y_0 - y_3)^2,$$

and l can be made as small as we please by taking λ sufficiently small. Hence a parabolic substitution can not occur in the group of the space.

We may have then as allowable groups of a Clifford-Klein space of constant negative curvature only those which correspond to groups of linear substitutions of λ which are properly discontinuous when interpreted in Σ and contain only hyperbolic and loxodromic substitutions.

The more minute discussion of the Clifford-Klein space depends therefore upon the knowledge of the groups called by Poincaré the Kleinian groups. It is worth noticing that whereas in the group theory the greatest attention has been paid to Kleinian groups with elliptic and parabolic substitutions, it is exactly these groups which are of no interest in the geometric problem before us. Geometry here waits for the development of the theory of groups.

SELECTED TOPICS IN THE THEORY OF DIVERGENT SERIES AND OF CONTINUED FRACTIONS.

By EDWARD B. VAN VLECK.

Part I.

Lectures 1–4. Divergent Series.

It may not be inappropriate for me to preface the first four lectures with a few words of a general character concerning divergent series. These will serve the double purpose of indicating the nature of the problems to be treated and of binding together the separate lectures.

The problem presented by any divergent series is essentially a functional one. When a divergent series of numbers is given, its genesis is usually to be found in some known or unknown function. The value which we attach to it is defined as the limit of a suitably chosen convergent process, and the elements of the process are the terms of the given series or are functions having these terms for their individual limits. Most commonly the given numerical series

$$a_0 + a_1 + a_2 + \cdots$$

is connected with the power series

$$a_0 + a_1x + a_2x^2 + \cdots, \tag{1}$$

and the question thus reduces to that of determining under what conditions or restrictions a value may be assigned to the latter series when x approaches 1. The primary topic therefore is the divergent power series, and to this we shall confine our attention exclusively.

This topic, if broadly considered, presents itself under at least four very different aspects. What is given is in every case a power series with a radius of convergence which is not infinite. Suppose first that the radius is greater than zero and that the circle of convergence is not a natural boundary. Then the series defines within this circle an analytic function. In the region of divergence without the circle the value of the function may be obtained by the familiar process of analytic continuation. Theoretically the determination of the function is a satisfactory one, for *Poincaré* * has shown that the function throughout the domain in which it is regular can be obtained by means of an enumerable set of elements, $P_n(x - a_n)$. Practically, however, when *Weierstrass'* process is employed for analytic continuation, the labor is so excessive as to render the process nearly valueless except for purposes of definition. Hence to-day a search is being made for a workable substitute. I may refer particularly in this connection to the investigations by *Borel* and *Mittag-Leffler*. As I consider the work of the former to be both suggestive and practical, I have taken it as the basis of my second lecture.

A second aspect of our topic, intimately connected with the continuation of the function defined by (1), is the determination of the position and character of its singularities in the region where the series diverges. This subject is treated in Lecture 3.

When the circle of convergence is a natural boundary, it does not appear to be impossible, despite the earlier view of *Poincaré* to the contrary,† to discover, at least in a certain class of cases, an appropriate, although a non-analytic mode of continuing the function across the boundary into other regions where it will be again analytic. The thesis of *Borel* and its recent continuation in the *Acta Mathematica*, together with some excellent remarks by *Fabry*,‡ appear to be about all that has been done in this direction. A very brief discussion of the subject will be given in the fourth lecture in connection with series of polynomials and of rational fractions.

Lastly, we have the conundrum of the truly divergent power series — the series which converges only when $x = 0$. It is upon

* *Rendiconti del Circolo Matematico di Palermo*, vol. 2 (1888), p. 197, or see Borel's *Théorie des fonctions*, p. 53.

† The conclusions of Poincaré and Borel are not actually inconsistent, but a new point of view is taken by the latter.

‡ *Compt. Rend.*, vol. 128 (1899), p. 78.

this interesting problem that our attention will be especially focused in the first two lectures. In applying henceforth the term divergent to power series, I shall restrict it to series having a zero-radius of convergence.

I shall offer no excuse for any irregularity or incompleteness of treatment. The admirable treatise by *Borel* on *Les Séries divergentes* (1901) and the masterly little book of *Hadamard, La Série de Taylor et son prolongement analytique* (1901), leave little or nothing to be desired in the line of systematic development. While it is impossible not to repeat much that is found in these books, I have also supplemented with other material and sought to give as fresh a presentation as possible.

Lecture 1. *Asymptotic Convergence.*

Few more notable instances of the difference between theoretical and practical mathematics are to be found than in the treatment of divergent series. After the dawn of exact mathematics with *Cauchy* the theoretical mathematician shrank with horror from the divergent series and rejected it as a treacherous and dangerous tool. The astronomer, on the other hand, by the exigencies of his science was forced to employ it for the purpose of computation. The very notion of convergence is said by *Poincaré** to present itself to the astronomer and to the mathematician in complementary or even contradictory aspects. The astronomer requires a series which converges rapidly at the outset. He cares not what the ultimate character may be, if only the first few terms, twenty for example, suffice to compute the desired function to the degree of accuracy required. Consequently he judges the series by these terms. If they increase, the series is for him non-convergent. To the mathematician the question is not at all concerning the nature of the series *ab initio*, but solely concerning its ultimate character.

Let me illustrate the difference by referring to *Bessel's* series

$$J_n = \frac{x^n}{2^n n!}\left(1 - \frac{x^2}{2(2n+2)} + \frac{x^4}{2.4(2n+2)\,(2n+4)} - \cdots\right),$$

* *Les méthodes nouvelles de la mécanique céleste,* vol. 2, p. 1.

which is a solution of the equation

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-n^2)y=0. \tag{2}$$

This is convergent for all values of x, but when x is very large the series is worthless for computation owing to the rapid and long-continued increase of the terms before the convergence finally sets in. The astronomer and physicist therefore have been driven to use for large values of x an expansion which is of the form *

$$Ax^{-\frac{1}{2}}\sin x\left(A_0+\frac{A_1}{x}+\frac{A_2}{x^2}+\cdots\right)$$
$$+\,Bx^{-\frac{1}{2}}\cos x\left(B_0+\frac{B_1}{x}+\frac{B_2}{x^2}+\cdots\right),$$

or, what is the same thing,

$$\begin{aligned} &Ce^{ix}x^{-\frac{1}{2}}\left(C_0+\frac{C_1}{x}+\frac{C_2}{x^2}+\cdots\right)\\ &\qquad+De^{-ix}x^{-\frac{1}{2}}\left(D_0+\frac{D_1}{x}+\frac{D_2}{x^2}+\cdots\right). \end{aligned} \tag{3}$$

Here the multipliers of C and D are only formal solutions of the differential equation (2). In respect to convergence they have a character exactly opposite to that of J_n, since for very large values of x the terms at first decrease rapidly but finally an increase begins. At this point the computer stops and obtains a good approximate value of J_n.

What is the significance of this? It is strange indeed that no attempt was made to study the question until 1886, when *Poincaré* † and *Stieltjes* ‡ simultaneously took it up. That so evident and important a problem should have been so long ignored by the mathematician emphasizes strongly the need of closer touch between him and the astronomer and the physicist. Both *Poincaré* and *Stieltjes* regarded the series as the asymptotic representation

* See, for example, Gray and Mathew's *Treatise on Bessel Functions*, chap. 4.

† *Acta Math.*, vol. 8, p. 295 ff.

‡ *Thesis, Ann. de l'Ec. Nor.*, ser. 3, vol. 3, p. 201.

of one or more functions. While the latter writer studied carefully certain divergent series of special importance with the object of obtaining from the series a yet closer approximation to the function by a species of interpolation, *Poincaré* developed the idea of asymptotic representation into a general theory.

To explain this theory * and at the same time to develop certain aspects scarcely considered by *Poincaré*, I shall start with the genesis of a Taylor's series. Take an interval $(0, a)$ of the positive real axis, and denote by $f(x)$ any real function which is continuous and has $n+1$ successive derivatives at every point within the interval. No hypothesis need be made concerning the character of the function at the extremities of the interval except to suppose that $f(x), f'(x), \cdots, f^{(n)}(x)/n!$ have limiting values $a_0, a_1, \cdots, a_n$ when x approaches the origin. Thus the function at any point within the interval will be represented by Taylor's formula:

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \frac{x^{n+1}}{(n+1)!} f^{(n+1)}(\theta x) \qquad (0 < \theta < 1).$$

If the function is unlimitedly differentiable and limiting values of $f^{(n)}(x)/n!$ exist for all values of n when x approaches 0, the number of terms in the formula can be increased to any assigned value. Thus the function gives rise formally to a series

$$(1) \qquad a_0 + a_1 x + a_2 x^2 + \cdots,$$

uniquely determined by the limiting values of the function and its derivatives.

The converse conclusion, that the series determines uniquely a function fulfilling the conditions above imposed in some small interval ending in the origin, can not, however, be drawn. This is not even the case when the series is convergent. Suppose, for example, that $a_n = 0$ for all values of n. Then in addition to

* Cf. Peano, *Atti della R. Accad. delle Scienze di Torino*, vol. 27 (1891), p. 40; reproduced as *Anhang III* ("Ueber die Taylor'sche Formel") in Genocchi-Peano's *Differential- und Integral-Rechnung*, p. 359.

$f(x) \equiv 0$ we have the functions $e^{-1/x}$, e^{-1/x^2}, $\cdots$, which fulfill the assigned conditions. They are, namely, unlimitedly differentiable within a positive interval terminating in the origin, and when x approaches the origin from within this interval, the functions and their derivatives have the limit 0. From this it follows immediately that if values other than zero be prescribed for the a_n, the function will not be uniquely determined, since to any one determination we may add constant multiples of $e^{-1/x}$, e^{-1/x^2}, $\cdots$.

Inasmuch as the correspondence between the function and the series is not reversibly unique, the series can not be used, in general, for the computation of the value of the generating function. Nevertheless, although this is the case, the series is not without its value. For consider the first m terms, m being a fixed integer. If x is sufficiently diminished in value, each of these terms can be made as small as we choose in comparison with the one which precedes it, and the series therefore at the beginning has the appearance of being rapidly convergent, even though it be really divergent. Evidently also as x is decreased, it has this appearance for a greater and greater number of terms, if not throughout its entire extent. Now by hypothesis the generating function was unlimitedly differentiable within the interval, and the successive derivatives are consequently continuous within $(0, a)$. Hence if the interval is sufficiently contracted, $f^{(m+1)}(x)/(m+1)!$ can be made as nearly equal to a_{m+1} throughout the interval as is desired. We have then for the remainder in Taylor's formula:

$$R_{m+1}(x) = \frac{f^{(m+1)}(\theta x)}{(m+1)!} x^{m+1} = a_{m+1} x^{m+1}(1+\zeta) \qquad (|\zeta| < \epsilon), \tag{4}$$

in which ϵ is an arbitrarily small positive quantity. Consequently if the first $m+1$ terms of the series should be used to compute the value of the generating function, the error committed would be approximately equal to the next term, provided x be taken sufficiently small.

In these considerations there is, of course, nothing to indicate when x is sufficiently small for the purpose. If the result holds simultaneously for a large number of consecutive values of m, the best possible value for the function consistent with our information would evidently be obtained by carrying the computation until the term of least absolute value is reached and then stopping. Herein is probably the justification for the practice of the computer in so doing.

Equation (4) which gave a *limit to the error in stopping with the* $(m+1)$*th term* shows also that *this limit grows smaller as* x *diminishes.* Since, furthermore, by increasing m sufficiently the $(m+2)$th term of (1) may be made small in comparison with the $(m+1)$th term, it is clear that on the whole, *as* x *diminishes, we must take a greater and greater number of terms to secure the best approximation to the function.* These two facts may be comprised into a single statement by saying that the approximation given by the series is of an asymptotic character. This will hold whether the series is convergent or divergent.

This notion can be at once embodied in an equation. From (4) we have

$$(5) \qquad \lim_{x=0+} \frac{f(x) - a_0 - a_1 x - \cdots - a_m x^m}{x^m}$$

$$= \lim_{x=0+} \frac{R_{m+1}(x)}{x^m} = 0 \qquad (m = 1, 2, \cdots).$$

This equation is an exact equivalent of the two properties just mentioned and is adopted by *Poincaré* * as the definition of asymptotic convergence. More explicitly stated, the series (1) is said by him to represent a function $f(x)$ asymptotically when equation (5) holds for all values of m.

It will be noticed that this definition omits altogether the assumptions concerning the nature of the function with which we started in deriving the series. Not only has the requirement of unlimited differentiability within an interval been omitted but the existence of right-hand limits for the derivatives as x approaches the origin is not even postulated. If the value a_0 be assigned to

* Loc. cit.

the function at the origin, it will have a first derivative, a_1, at this point but it need not have derivatives of higher order.*

The exclusion of the requirement of differentiability has undoubtedly its advantages. It enlarges the class of functions which can be represented asymptotically by the same series. It also simplifies the application of the theory of asymptotic representation, and this is perhaps the chief gain. The results of *Poincaré's* theory can readily be surmised. The sum and product of two functions represented asymptotically by two given series are represented asymptotically by the sum- and product-series respectively, and the quotient of the two functions will be represented correspondingly, provided the constant term of the divisor is not 0. Also if $f(x)$ is any function represented by the series (1), whether convergent or divergent, and

$$\phi(x) = b_0 + b_1 x + b_2 x^2 + \cdots$$

is a second series having a radius of convergence greater than $|a_0|$, the asymptotic representation of $\phi[f(x)]$ will be the series which is obtained from

$$b_0 + b_1(a_0 + a_1 x + \cdots) + b_2(a_0 + a_1 x + \cdots)^2 + \cdots$$

by rearranging the terms in ascending powers of x. Lastly, the integral of $f(x)$ will have for its asymptotic representation the term by term integral of (1). But the correspondence of the function and series may be lost in differentiation, for even if the function permits of differentiation, its derivative will not necessarily be a function having an asymptotic power series. Examples of this kind can be readily given.†

* The ordinary definition of an nth derivative is here assumed. If, however, we define the second derivative by the expression

$$f''(0) = \lim_{x=0} \frac{f(2x) - 2f(x) + f(0)}{x^2}$$

and the higher derivatives in similar fashion, the function must have derivatives of all orders.

† Cf. Borel, *Les Séries divergentes*, p. 35.

This failure is on many accounts an unfortunate one. If a further development of *Poincaré's* theory is to be made — and this seems to me both a possibility and a desirability — his definition probably should be restricted by requiring (a) that the function corresponding to the series shall be unlimitedly differentiable in some interval terminating in the origin, and (b) that the derivatives of the function should correspond asymptotically to the derivatives of the power series. These demands are satisfied in the case of an analytic function defined by a convergent series and seem to be indispensable for an adequate theory of divergent series.*

Thus far we have considered asymptotic representation only for a single mode of approach to the origin. Suppose now that an analytic function of a complex variable x is represented by (1) for all modes of approach to the origin, and let a_0 be the value assigned to the function at this point. Then if the function is one-valued and analytic about the origin, it must also be analytic at this point since it remains finite. Hence the series must be convergent.

The case which has an interest therefore is that in which the asymptotic representation is limited to a sector terminating in the origin. Suppose then that (1) is a given divergent series, and let a function be sought which fulfills the following conditions: (a) the function shall be analytic within the given sector for values of

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* These requirements are formulated from a mathematical standpoint with a view to extending the theory of analytic functions, and doubtless will be too stringent for various astronomical investigations. Prof. E. W. Brown suggests that for such investigations the conditions might perhaps be advantageously modified by making the requirements for only m derivatives, m being a number which varies with x and increases indefinitely upon approach to the critical point. He also points out the difficulties of an extension in the case of numerous astronomical series which have the form $f(x, t) = a_0 + a_1 x + a_2 x^2 + \cdots$, where a_i is a function of x and t, $\partial f / \partial t$ being a convergent series. Poincaré's definition is however still applicable.

Oftentimes in celestial mechanics the only information concerning the function sought is afforded in the approximation given by the asymptotic series. An objection to Poincaré's definition is that it presupposes a knowledge of the function sought, for example, that $\lim f(x) = a_0$, when $x = 0$. As a matter of fact the properties are often unknown. See in this connection p. 89 of these lectures.

x which are sufficiently near to the origin; (b) it shall be represented asymptotically by the given series within the sector, whether inclusive or exclusive of the boundary will remain to be determined; (c) the asymptotic representation shall not be valid if the angle of the sector is enlarged. So far as I am aware, the existence of a function or of functions which meet these requirements has never been demonstrated, though it seems likely that they in general exist. It is, however, very possible that the sector must be restricted in position as well as in magnitude. It may be found necessary to require that the interior of the sector shall not include certain arguments of x; for example, in the case of the series $\Sigma m!x^m$ * the argument 0, for which the terms have all the same sign. † If this be true, the sector will very probably have two such arguments for its boundaries. When there is a function which satisfies the conditions imposed, it can not be unique. For clearly $e^{-1/x}$, $e^{-1/x^{\frac{1}{2}}}$, $e^{-1/x^{\frac{1}{3}}}$, $\cdots$, within certain sectors of angle π, 2π, 3π, $\cdots$, have an asymptotic series in which each coefficient is 0. If, then, any function has been obtained satisfying the conditions stated, one or more of these exponentials, after multiplication by suitable constants, may be added to the function without destroying its properties. Hence if a divergent series is to represent a function uniquely, supplementary conditions must be imposed. The nature of these conditions has not yet been ascertained. ‡

In closing the general discussion a simple extension of the notion of asymptotic convergence should be mentioned which is necessary for the applications to follow. $F(x)$ is said to be represented asymptotically by

$$\Phi(x)\{a_0 + a_1x + a_2x^2 + \cdots\}$$

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* This series is discussed in the next lecture.

† Borel (*loc. cit.*, p. 36) in his exposition of Poincaré's theory seems to make the definite statement that there are arguments for which no corresponding function exists, but I am unable to find any proof of the statement.

‡ In this connection see pp. 89–92 of Borel's article, *Ann. de l'Ec. Nor.*, ser. 3, vol. 16 (1899).

when the series in parenthesis gives such a representation of $F(x)/\Phi(x)$.

The applications of *Poincaré's* theory have been made chiefly in the province of differential equations * where divergent series are of very common occurrence. We will take for examination the class of equations, of which the theory is perhaps the most widely known, the homogeneous linear differential equation with polynomial coefficients:

$$(6) \qquad P_n(x)\frac{d^ny}{dx^n} + P_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + P_0(x)y = 0.$$

This is, in fact, the class of equations to which *Poincaré* first applied his theory,† but his discussion of the asymptotic representation of the integrals was limited to a single rectilinear mode of approach to the singular point under consideration. The determination of the sectors of validity for the asymptotic series has been made by *Horn*,‡ who in a number of memoirs has carefully studied the application of the theory to ordinary differential equations.§

As is well known, the only singular points of (6) are the roots of $P_n(x)$ and the point $x = \infty$. For a regular singular point || we have the familiar convergent expressions for the integrals given by Fuchs. Consider now an irregular singular point. By a linear transformation this point may be thrown to ∞, the equation being still kept in the form (6). Suppose then that this has been done. If P_n is of the pth degree, the condition that $x = \infty$ shall be a regular singular point is that the degrees of $P_{n-1}, P_{n-2}, \cdots, P_0$ shall be at most equal to $p-1, p-2, \cdots, p-n$, respectively.

For an irregular singular point some one or more of the degrees must be greater. Let h be the smallest positive integer for which the degrees will not exceed successively

* In addition to the memoirs cited below Poincaré's *Les méthodes nouvelles de la mécanique céleste* and various memoirs by Kneser may be consulted.

† *Acta Math.*, vol. 8 (1886), p. 303. See also *Amer. Jour.*, vol. 7 (1885), p. 203.

‡ *Math. Ann.*, vol. 50 (1898), p. 525.

§ See various articles in *Crelle's Journal* and the *Mathematische Annalen.*

|| *Stelle der Bestimmtheit.*

$$p + (h - 1), \quad p + 2(h - 1), \quad p + 3(h - 1), \cdots.$$

The number h is called the rank of the singular point ∞, and the differential equation can be satisfied formally by the series of *Thomae* or the so-called normal series:

$$(7) \qquad S_i = e^{\frac{\alpha_i x^h}{h} + \alpha_{i,1} x^{h-1} + \dots + \alpha_{i,h-1} x} x^{\rho_i} \left(C_i + \frac{C_{i,1}}{x} + \frac{C_{i,2}}{x^2} + \cdots \right)$$
$$(i = 1, 2, \cdots, n).$$

Unless certain exceptional conditions are fulfilled, there are n of these expansions, and in general they are divergent. To simplify the presentation let us confine ourselves to the case for which $h = 1$. Then at least one of the polynomials succeeding P_n will be of the pth degree, and none of higher degree. Place

$$P_n = A_n x^p + B_n x^{p-1} + \cdots,$$
$$P_{n-1} = A_{n-1} x^p + B_{n-1} x^{p-1} + \cdots,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$P_0 = A_0 x^p + B_0 x^{p-1} \cdots,$$

and construct the equation

$$(8) \qquad A_n \alpha^n + A_{n-1} \alpha^{n-1} + \cdots + A_0 = 0.$$

The n roots of this equation are the n quantities α_i which appear in the exponential components of the S_i.

As a particular illustration of the class of equations under consideration, *Bessel's* equation (Eq. (2)) may be cited. Here the point ∞ is of rank 1, the characteristic equation is

$$A_0 \alpha^2 + A_1 \alpha^2 + A_2 \equiv \alpha^2 + 1 = 0,$$

with the roots

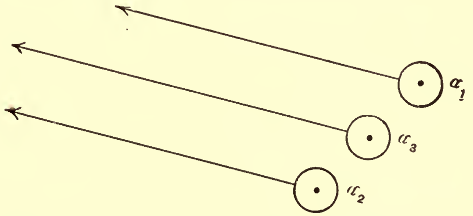
$$\alpha_1 = -i, \quad \alpha_2 = +i,$$

and the two *Thomaean* integrals are

$$(9) \qquad \begin{aligned} y_1 &= e^{ix} x^{\rho_1} \left(C_0 + \frac{C_1}{x} + \cdots \right), \\ y_2 &= e^{-ix} x^{\rho_2} \left(D_0 + \frac{D_1}{x} + \cdots \right), \end{aligned}$$

in which ρ_1, ρ_2 are yet to be ascertained. After this has been done, the coefficients of (9) can be determined by direct substitution in (2).

To avoid complications we will assume that the n roots of the characteristic equation (8) are all distinct, also that the real parts of no two roots are equal. Mark now in the complex plane the points $\alpha_1, \alpha_2, \cdots, \alpha_n$, and draw from them to infinity a series of parallel rays having such a direction that no one of the rays with its prolongation in the opposite direction shall contain two or more of these points. Finally surround the points α_i with small circles,



so that we shall have the familiar loop circuits for the paths of integration of the integrals which we now proceed to form. Put

$$\eta_i = \int e^{zx} v_i(z) dz \qquad (i = 1, \cdots, n), \tag{10}$$

in which $v_i(z)$ is a function to be subsequently fixed. In order that the integral may have a sense, x will be so restricted that the real part of zx shall be negative for the rectilinear parts of the loop circuits. We can then so determine $v_i(z)$ that η_i shall be a solution of (6).

For this purpose substitute η_i for y in (6). A reduction, based on the integration of (10) by parts,* gives for $v_i(z)$ the equation

$$(A_n z^n + A_{n-1} z^{n-1} + \cdots + A_0) \frac{d^p v}{dz^p} + \cdots + (\quad) v = 0, \tag{11}$$

This is known as *Laplace's* transformed equation. While the original equation was of the nth order with coefficients of the pth

* Cf. Picard's *Traité d'Analyse*, vol. 3, p. 383 ff., or Poincaré, *Amer. Jour.*, vol. 7 (1885), p. 217 ff.

degree, the transform is of the pth order with coefficients of the nth degree. Its singular points in the finite plane are the roots of the first coefficient of (11), which is identical with the left hand member of (8). Furthermore, an inspection of (11) shows immediately that each of these singular points α_i is regular, and the exponents which belong to it are

$$0,\ 1,\ 2,\ \cdots,\ p-2,\ \beta_i \equiv -(\rho_i + 1) \qquad (i = 1,\ 2,\ \cdots,\ n),$$

in which ρ_i is the exponent of x, hitherto undetermined in (7). Hence if β_i is not an integer, there is an integral of (11) having the form

$$(z-\alpha_i)^{\beta_i}(k_0 + k_1(z-\alpha_i) + k_2(z-\alpha_i)^2 + \cdots),$$

which, when continued analytically, can be taken as the function v_i. Thus for the solution of (6) we obtain

$$\eta_i = \int e^{zx}(z-\alpha_i)^{\beta_i}(k_0 + k_1(z-\alpha_i) + \cdots)dz.$$

If, finally, $\alpha_i + y/x$ is substituted for z the integral becomes

$$(12) \qquad \eta_i = e^{\alpha_i x} x^{-\beta_i - 1 = \rho_i} \int e^y y^{\beta_i}\left(k_0 + k_1\frac{y}{x} + k_2\frac{y^2}{x^2} + \cdots\right)dy,$$

where the transformed path of integration is a loop circuit which encloses the origin of the y-plane, the rectilinear portion of the path lying in the half plane for which the real part of y is negative.

We have thus reached a solution of the differential equation under the form of an improper integral of a convergent series. The integration of (12) term by term, which is a purely formal process, gives at once the normal integral S_i of (7), in which

$$C_{i,n} = k_n \int e^y y^{\beta_i + n} dy.$$

The asymptotic character of S_i can be quickly demonstrated.* For let $u^n R_n(u)$ denote the remainder after n terms of the series

$$k_0 + k_1 u + k_2 u^2 + \cdots.$$

Then

*Horn, *loc. cit.*, or *Acta Math.*, vol. 24 (1901), pp. 299 ff.

$$x^n\left\{\eta_i e^{-a_i x} x^{-\rho_i} - \left(C_i + \frac{C_{i,1}}{x} + \cdots + \frac{C_{i,n}}{x^n}\right)\right\}$$
$$= \frac{1}{x}\int e^y y^{\beta_i+n+1} R_{n+1}\left(\frac{y}{x}\right) dy.$$

Since the integral in the right hand member, taken along the loop circuit, can be shown to remain finite when $x = \infty$, we have

$$\lim_{x=\infty} x^n \left\{\eta_i e^{-a_i x} x^{-\rho_i} - \left(C_i + \frac{C_{i,1}}{x} + \cdots + \frac{C_{i,n}}{x^n}\right)\right\} = 0.$$

But this is the statement of *Poincaré's* definition of asymptotic convergence for $x = \infty$.

I have sketched this lengthy process in some detail because it is a thoroughly typical one and indicates the present status of the theory of asymptotic series. It will be observed that the following course is pursued :

1. First, it is discovered that the differential equation permits of formal solution by a certain divergent series.

2. By some independent process the existence of an actual solution is ascertained which permits formally of expansion into the series. Usually the solution is found under the form of an integral, and *Horn* has applied the theory chiefly in cases in which solutions of this form were known. (Lately, however, he has used solutions obtained from the differential equation by the process of successive approximation.*)

3. The asymptotic character of the series is then argued and, finally, the sector within which this representation is valid is determined.

The status of the theory thus exhibited seems to me an unsatisfactory and transitional one. It is to be hoped that ultimately the theory will be so developed that the mere existence of a divergent power series as a *formal* solution of the differential equation will be sufficient for the immediate affirmation of the existence of one or more solutions which are analytic functions with certain specified properties.

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* *Math. Ann.*, vol. 51 (1898), p. 346. In *Crelle's Journal*, vol. 118 (1897), still another method is used for obtaining the solutions.

It remains yet to fix the sectors within which the solutions η_i can be represented asymptotically by the normal integrals. These sectors have been specified by *Horn** in the following manner. Let straight lines be drawn from each singular point a_i to every other point and produce each joining line to infinity in both directions. A set of lines will be thus fixed, radiating from the point ∞. Let their arguments, taken in the order of decreasing magnitude, be denoted by

$$\omega_1, \omega_2, \cdots, \omega_r, \omega_{r+1} = \omega_1 - \pi, \cdots, \omega_{2r} = \omega_r - \pi.$$

Suppose now that the argument of the rectilinear part of the path of integration for η_i in the plane of z lies between $\omega_{\rho-1}$ and ω_ρ. Then η_i is represented asymptotically by S_i for values of the argument of x between $\pi/2 - \omega_{\rho-1}$ and $\pi/2 - \omega_{\rho+r}$.†

To the general solution of (6), $c_1\eta_1 + c_2\eta_2 + \cdots + c_n\eta_n$, there corresponds the divergent expansion

$$(13)\qquad \begin{aligned} c_1S_1 + \cdots + c_nS_n = c_1e^{a_1x}x^{\rho_1}\left(C_1 + \frac{C_{1,1}}{x} + \frac{C_{1,2}}{x^2} + \cdots\right) + \\ \cdots + c_ne^{a_nx}x^{\rho_n}\left(C_n + \frac{C_{n,1}}{x} + \frac{C_{n,2}}{x^2} + \cdots\right). \end{aligned}$$

Here the real parts of two exponents, a_ix and a_jx, are equal only when $arg\,(a_i - a_j)x$ is an odd multiple of $\pi/2$; that is, when $arg\,x$ is equal to $\pi/2 - \omega_i\,(i = 1, \cdots, 2r)$. Suppose then that for

$$\pi/2 - \omega_{\rho-1} < arg\ x < \pi/2 - \omega_{\rho+r}$$

we so assign subscripts to the a_i that

$$R(a_1x) > R(a_2x) > \cdots > R(a_nx).$$

Then all the integrals for which $c_1 \neq 0$ have in common the asymptotic series c_1S_1, while those for which $c_1 = c_2 = \cdots = c_{i-1}$,

* Horn, *Math. Ann.*, vol. 50 (1898), p. 531.

† In certain cases the asymptotic representation may be valid for a greater range of values of the argument of x, as in the case of Bessel's equation discussed below.

$c_i \neq 0$, are represented by $c_i S_i$. Thus it appears that between the arguments considered S_n is the only one of the n asymptotic series S_i which defines a solution of the differential equation (6) uniquely.

Changes in the asymptotic series representing a solution may occur from two causes, either because x passes through one of the critical values above mentioned for which there is a change in the dominant exponential in (13), or because of a sudden alteration in the values of the constants c_i for certain values of the argument. This can be made clear, in conclusion, by illustrating with *Bessel's* equation.* For this equation, as we saw,

$$\alpha_1 = -i, \quad \alpha_2 = +i,$$

and hence

$$\omega_1 = \frac{3\pi}{2}, \quad \omega_2 = \frac{\pi}{2}.$$

Also since *Laplace's* transform for the particular case before us is †

$$(z^2+1)\frac{d^2v}{dz^2} + 3z\frac{dv}{dz} + (\quad)v = 0,$$

the exponent ρ_i for either of the two singular points $z = \pm i$ has the value $-\frac{1}{2}$. Accordingly the series (13) for $c_1\eta_1 + c_2\eta_2$ may be written

$$Ce^{ix}x^{-\frac{1}{2}}\left(C_0 + \frac{C_1}{x} + \frac{C_2}{x^2} + \cdots\right)$$
$$+ De^{-ix}x^{-\frac{1}{2}}\left(D_0 + \frac{D_1}{x} + \frac{D_2}{x^2} + \cdots\right) = CU(x) + DV(x),$$

as previously given in (3). If the imaginary part of x is negative, $CU(x)$ is the dominant term in (3) and gives the asymptotic representation of the general solution, $c_1\eta_1 + c_2\eta_2$. On the other hand, if the imaginary part is positive, the dominant term is

* A brief but very interesting discussion is given in a letter of Stokes in the *Acta Math.*, vol. 26 (1902), pp. 393–397. Compare also § 3 of Horn's article, *Math. Ann.*, vol. 50 (1898), p. 525.

† *Math. Ann.*, vol. 50, p. 539, Eq. B'.

$DV(x)$. The changes in the values of C and D take place only when arg x passes through the values $(2n+1)\pi/2$. Then the coefficient of the dominant term remains unaltered, while the coefficient of the inferior term is altered by an amount proportional to the coefficient of the dominant term.† We conclude, therefore, that in general the asymptotic series for any solution of *Bessel's* equation changes abruptly for values of the argument congruent with $0 \pmod{\pi}$. Furthermore, the series can not be valid for a greater range of values of the argument unless when arg $x = 0$, either $D = 0$ or $C = 0$. In the former case we have a particular solution $C\eta_1$ which is represented by the series $CU(x)$ for

$$-\pi < \arg x < 2\pi,$$

and in the latter case a solution $D\eta_2$ represented by $DV(x)$ for

$$-2\pi < \arg x < \pi.$$

Thus from the infinitely many solutions of *Bessel's* equation having the common asymptotic representation $CU(x)$ and $DV(x)$ respectively, these two solutions can be singled out by the requirement that the asymptotic representation shall have the maximum sector of validity.

Lecture 2. *The Application of Integrals to Divergent Series.*

In the first lecture a divergent series was connected with a group of functions, for which it afforded a common asymptotic representation. In the present lecture I shall treat of methods which have been used to derive a function uniquely from the series. To establish, whenever possible, such a unique connection, to develop the properties of the function, and to determine the laws and conditions under which the series can be manipulated as a substitute for the function — this may be said to be the ultimate aim of the theory of divergent series.

Up to the present time this goal has been reached only for a restricted class of divergent series. Furthermore, the uniqueness

† Stokes, *loc. cit.*

of correspondence between the function and the series has been attained, not by a specification of the properties of the function, but by means of some algorithm which, when applied to the series, yields a single function. Unquestionably the instrument by which the greatest progress has been made thus far is the integral. The first successes, however, were reached by *Laguerre* * and *Stieltjes* † through the use of continued fractions, and very possibly in the end the continued fraction will prove to be the best, as it was the earliest tool. But as yet it has been applied only in cases in which the function can be represented under the form of an integral as well as of a continued fraction, although with greater difficulty.

To explain the use of integrals let us consider the familiar divergent series treated by *Laguerre*,

$$(1) \qquad 1 + x + 2!\,x^2 + 3!\,x^3 + \cdots.$$

This is, I believe, historically the first divergent series from which a functional equivalent was derived. ‡ Since

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* See No. 20 of the bibliography at the end of lecture 6.

† Bibliography, No. 26a.

‡ Laguerre (*loc. cit.*) gives the function first in the form of a continued fraction and later proves its identity with the integral which gives rise to the divergent series. Borel at the opening of the second chapter of *Les Series divergentes* remarks that "Laguerre parait avoir le premier montré nettement l'utilité qu'il peut y avoir à transformer une série divergente . . . en une fraction continue convergente." It seems almost to have escaped notice (see, however, p. 110 of Pringsheim's report, *Encyklopädie der Math. Wissenschaften*, I A 3), that Euler (Bibliography, No. 46) derived a continued fraction from the divergent series

$$1 + mx + m(m+n)x^2 + m(m+n)\,(m+2n)x^3 + \cdots,$$

of which Laguerre's series is a special case, and clearly realizes the utility of the continued fraction. Moreover, a close parallel to the course followed by Laguerre is found in the work of Laplace who derives from the expression

$$e^{x^2}\int_x^\infty e^{-x^2}dx$$

a divergent series and from this in turn a continued fraction, the convergents of which were stated by him and proved by Jacobi to be alternately greater and less than the expression. Had Jacobi proved also the convergence of the continued fraction, the work of Laguerre would have had an exact parallel for real values of x. Cf. No. 47 of the bibliography.

$$m! = \Gamma(m+1) = \int_0^\infty e^{-z} z^m dz,$$

the series may be written

$$\int_0^\infty e^{-z} dz + x \int_0^\infty e^{-z} z dz + x^2 \int_0^\infty e^{-z} z^2 dz + \cdots,$$

the path of integration being the positive real axis. If, then, by a merely formal process, the sum of the integrals is replaced by the integral of the sum, we obtain

$$\int_0^\infty e^{-z} (1 + xz + x^2 z^2 + \cdots)\, dz,$$

or a function

(2) $$f(x) \equiv \int_0^\infty e^{-z} F(zx) dz,$$

in which

$$F(zx) = \frac{1}{1 - zx}.$$

The function thus derived is an improper integral which has a significance for all values of x except those which are real and positive. It can be shown also to be analytic for all except the excluded values of x. One of the simplest proofs is as a corollary of the following exceedingly fundamental theorem of *Vallée-Poussin*,* which we shall have occasion to use again later: *If in the proper integral*

$$\int_a^b f(x, z) dz$$

the integrand is continuous in z and x for all values of z upon the path of integration and for all values of x within a region T; if, furthermore, for each of the above values of z it is analytic in x over the region T, the integral will also be an analytic function of x in the interior of T. By this theorem, if t is a point on the positive real axis,

$$\int_0^t \frac{e^{-z} dz}{1 - zx}$$

* *Ann. de la Soc. Scient. de Bruxelles*, vol. 17 (1892–3), p. 323.

will represent an analytic function of x over any closed region of the x-plane which excludes the positive real axis. If, now, t passes through any indefinitely increasing set of values,

$$t_1 < t_2, < t_3, \cdots,$$

we have in

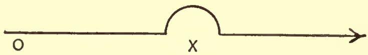
$$f_i(x) = \int_0^{t_i} \frac{e^{-z}dz}{1 - zx}$$

a series of analytic functions which is seen at once to converge uniformly over the region considered, since

$$|f_i(x) - f_j(x)| \equiv \left| \int_{t_j}^{t_i} \frac{e^{-z}dz}{1 - zx} \right| < \epsilon$$

for sufficiently great values of i and j. The limit (2) is therefore analytic.

By deforming the path of integration the same conclusion concerning the analytic character of the function (2) can be extended to all values of x upon the positive real axis excepting 0 and ∞, and when the deformation is made on opposite sides of a fixed point x, the two values of the integral will be found to differ by



(3)
$$2i\pi \frac{1}{x} e^{-\frac{1}{x}}.$$

The integral accordingly represents a multiple-valued function with the singular points 0 and ∞, the various branches of which differ from one another by multiples of the period (3). For the initial branch which was given in (2) the limit of $f^{(n)}(x)/n!$ will be the $(n + 1)$th coefficient of (1) if x approaches the origin along any rectilinear path except the positive real axis.

Let the process which has been adopted for the series of *Laguerre* be applied next to any other series

(I)
$$a_0 + a_1x + a_2x^2 + \cdots$$

having a finite radius of convergence. If we write the series in the form

$$a_0 + a_1 x + 2\,!\left(\frac{a_2}{2\,!}\right)x^2 + \cdots + n\,!\left(\frac{a_n}{n\,!}\right)x^n + \cdots,$$

then replace the factor $n\,!$ by its expression as a Γ-integral, and finally, by a step having in general only formal significance, bring all the terms under a common integral sign, we shall obtain

$$\int_0^\infty e^{-z}\left(1 + a_1 xz + \frac{a_2}{2\,!}x^2 z^2 + \cdots\right)dz$$

or

(4) $$\int_0^\infty e^{-z}F(zx)dz,$$

in which

(5) $$F(u) = 1 + a_1 u + \frac{a_2}{2\,!}u^2 + \cdots + \frac{a_n}{n\,!}u^n + \cdots \qquad (u = zx).$$

This integral is the expression upon which *Borel* builds his theory of divergent series, and may be regarded as a generalization of a very interesting theorem of *Caesaro.** The series (5) is called the *associated series* of (I).

Two cases are now to be distinguished according as the fundamental series (I) has, or has not, a radius of convergence R which is greater than 0. If the radius is not zero, the associated series has an infinite radius since

$$\lim_{n=\infty}\sqrt[n]{\frac{a_n}{n\,!}} = \lim_{n=\infty}\sqrt[n]{\frac{R^{-n}(1+\epsilon)^n}{n\,!}} = 0,$$

and it accordingly represents an entire function. It is a simple matter to prove that the integral (4) will have a sense if x lies within the circle of convergence of (I), and that the values of the integral and series are identical. But the integral may also have a sense for values of x which lie without the circle, and in this case the integral may be used to get the analytic continuation of (I).

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* Cf. Borel, *Les Séries divergentes*, pp. 88–98.

The series is said by *Borel* to be *summable* * at a point x when the integral (4) has a meaning at this point.

The second case is that in which the fundamental series is divergent. The associated series in this case may be either convergent or divergent. If it is convergent only over a portion of the plane of $u = zx$, we are to understand by $F(u)$ not merely the value of the associated series but of its analytic continuation. Let x for an instant be given a fixed value. Then when z describes the positive real axis, u in its plane describes the ray from the origin passing through the point x. If $F(u)$ is holomorphic along this ray, it is possible that the integral (4) will have a sense. Suppose that this holds good as long as x lies within a certain specified region of its plane. Then for this region a function will be obtained uniquely from the divergent series by the use of the integral, precisely as in the case of the series of *Laguerre*.

This method of treatment is obviously restricted to divergent series for which the associated series are convergent, and it will not always be applicable even to these. A divergent series in which there is an infinite number of coefficients of the same order of magnitude as the corresponding coefficients of

$$(6) \qquad 1 + x + (2\,!)^2x^2 + (3\,!)^2x^3 + \cdots + (n\,!)^2x^n + \cdots$$

can not be summed in this manner. It will be noticed, however, that the series just given is one whose first associated series is the series of *Laguerre*, and whose second associated series is consequently convergent.

The method of *Borel* can be readily extended so as to take account also of such series, or, more generally, of series that have an associated series of the nth order which is convergent. One mode of doing this is by the introduction of an n-fold integral. Suppose, for example, that in (6) one of the two factorials $n\,!$ is replaced by

$$\int_0^\infty e^{-z}z^n dz$$

*Some other term would be preferable since his definition refers only to one of many possible modes of summation. A series may be simultaneously "summable" at a point x by one method, and non-summable by another.

and the other by

$$\int_0^\infty e^{-t}t^n dt.$$

The $(n+1)$th term of the series becomes

$$x^n \int_0^\infty \int_0^\infty e^{-t-z} z^n t^n dz dt,$$

and we obtain the two-fold integral

$$\int_0^\infty \int_0^\infty \frac{e^{-t-z}}{1-tzx} dz dt$$

for the functional equivalent of the series. This is a function, the initial branch of which is analytic over the entire plane of x except at the points 0 and ∞.

We turn now to the consideration of the region of summability, in which x must lie in order that the integral shall have a sense. *Borel* has determined the shape of this region when the fundamental series (I) is convergent, but in so doing he restricts himself to what he calls the *absolutely summable* series. The series (I) is said to be *absolutely summable* for any value of x when the integral (4) is absolutely convergent and when, furthermore, the successive integrals

$$(7) \qquad \int_0^\infty e^{-z} \left| \frac{d^\lambda F(zx)}{dz^\lambda} \right| dz \qquad (\lambda = 1, 2, \cdots)$$

have also a sense.*

To fix the shape of the region *Borel* shows first that if a function defined by a convergent series (I) is absolutely summable at a point P, it is analytic within the circle described upon the line OP as diameter, connecting P with the origin O; conversely, if it is analytic within and upon a circle having OP as diameter, it must be absolutely summable along OP, inclusive of the point

* The condition (7) was not originally included in Borel's definition of absolute summability (*Ann. de l'Ec. Nor.*, ser. 3, vol. 16, 1899), and is superfluous in fixing the shape of the region. Cf. *Math. Ann.*, vol. 55 (1902), p. 74. The modification of the definition was introduced in the *Séries divergentes* and is needed for the developments explained below, p. 102. Chapters 3 and 4 of this treatise can be read in connection with the present lecture.

P. As P moves outward from the origin along any ray, the limiting position for the circle is one in which it first passes through a singular point S, and at this point SP and OS subtend a right angle. The region of absolute summability can therefore be obtained as follows: Mark on each ray from the origin the nearest singular point of the function defined by (I), if there is such a point in the finite plane. Then through this point draw a perpendicular to the line. Some or all of these perpendiculars will bound a polygon, the interior of which contains the origin and is not penetrated by any one of the perpendiculars. This region is called the *polygon of summability.* If the singularities of the function are a set of isolated points, the polygon will be rectilinear. For the extreme case in which the circle of convergence is a natural boundary, the polygon and circle coincide. In every other case the circle is included in the polygon. Thus by the use of (4) *Borel* effects an analytic continuation of the series over a perfectly definite region whenever an analytic continuation exists. On passing to the exterior of the polygon the series ceases to be absolutely summable. As an example of this result, take the series

$$x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots,$$

which is the familiar expansion of $\frac{1}{2} \log (1 + x)/(1 - x)$. The singular points of the function are $+ 1$ and $- 1$, the circle of convergence is the unit circle, and the polygon of summability is a strip of the plane included between two perpendiculars to the real axis through the points ± 1.

When the given series is divergent, the form of the domain of summability has not been determined with such precision. The only information which we have upon the subject is contained in a brief but important communication by *Phragmen* in the *Comptes Rendus,** published since the appearance of *Borel's* work. *Phragmen* considers here the domain, not of absolute, but of simple summability for *Laplace's* integral

* Vol. 132, p. 1396; June, 1901.

$$(8)\qquad \int_0^\infty e^{-z} f(zx)\,dz,$$

in which $f(zx)$ denotes an arbitrary function.

To adopt a term of *Mittag-Leffler*, the domain is a "star," which is derived as follows: Draw any ray from the origin. If the series is summable at a point x_0 of this line, *Phragmen* shows that it is summable at every point between x_0 and the origin O. There is therefore some point P of the line which separates the interval of summability from the interval of non-summability. If the function is summable for the entire extent of the ray, P lies at infinity. In any case let the segment OP be obliterated and then make a cut along the remainder of the line. When the same thing is done for every ray which terminates at the origin, there is left a region called a star, bounded by a set of lines radiating from a common center, the point at infinity.

Phragmen says that the proof of this result is so simple that it can be given "*en deux mots.*" For this reason I shall reproduce it here. We are to show that if the integral converges for any value $x = x_0$, it will also converge for $x = \theta x_0$, if $0 < \theta < 1$. Place

$$f(zx_0) = \phi(z) + i\psi(z).$$

For $x = x_0$ the real and imaginary components of the integrals,

$$(9)\qquad \int_0^\infty \phi(z) e^{-z} dz, \qquad i\int_0^\infty \psi(z) e^{-z} dz,$$

have a sense. We are to prove that the integrals

$$(10)\qquad \int_0^\infty \phi(z\theta) e^{-z} dz, \qquad \int_0^\infty \psi(z\theta) e^{-z} dz,$$

obtained by replacing x_0 by θx_0, also exist. Consider either integral, for example the former. Let $0 < a_1 < a_2 < \infty$, and put

$$J = \int_{a_1}^{a_2} \phi(z\theta) e^{-z} dz.$$

By the change of variable $w = \theta z$ this becomes

$$J = \frac{1}{\theta}\int_{\theta a_1}^{\theta a_2} \phi(w)e^{-\frac{w}{\theta}}\,dw = \frac{1}{\theta}\int_{\theta a_1}^{\theta a_2} e^{-w\left(\frac{1}{\theta}-1\right)}\,\phi(w)e^{-w}dw.$$

Since $e^{-w(1/\theta-1)}$ is a positive and decreasing function in the interval considered, the second mean-value theorem of the integral calculus* may be applied, giving

$$(11) \qquad J = \frac{e^{-a_1(1-\theta)}}{\theta}\int_{\theta a_1}^{\theta a} \phi(w)e^{-w}dw,$$

in which a designates an appropriate value between a_1 and a_2. This, as *Phragmen* says, proves the theorem, but a word or two of explanation additional to his "*deux mots*" may not be unacceptable to some of my hearers. The necessary and sufficient condition for the existence of the first of the two integrals given in (10) is that by taking two values a_1 and a_2 sufficiently small or two values sufficiently large, the integral J may be made as small as we choose. Now this is true of

$$\int_{a_2}^{a} \phi(w)e^{-w}dw$$

since the integrals (9) exist, and equation (11) show then that it must be true likewise of J because the factor $e^{-a_1(1-\theta)}/\theta$ has an upper limit for $0 < \theta_1 < \theta < 1$ and $0 < a_1 < \infty$. It follows therefore that the integrals (10) exist.

Two other facts stated by *Phragmen* are also of interest. The function of x defined by (8) is a monogenic function which is holomorphic at every point in the interior of a circle described upon OP as diameter. If, also, in place of $f(zx)$ we take the associated series $F(zx)$ of a convergent series (I), the star of convergence coincides with *Borel's* polygon of absolute summability. Thus the regions of absolute and non-absolute summability are the same, or differ at most only in respect to the nature of the boundary points.

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* Bonnet's form: *Encyklopädie der Math. Wiss.*, II A 2, § 35.

It might be thought that the result of *Phragmen* makes the concept of absolute summability useless. This is, however, in no wise the case. At any rate, *Borel* employs the concept to establish the important conclusion that a divergent series, if absolutely summable, can be manipulated precisely as a convergent series. Thus if two absolutely summable series, whether convergent or divergent, are multiplied together, the resultant series will also be absolutely summable, and the function which it defines will be the sum or product of the functions defined by the two former series. Or, again, if an absolutely summable series is differentiated term by term, another such series is obtained, and the latter yields a function which is the derivative of the one defined by the former series. Lastly, the function determined by an absolutely summable series can not be identically zero, unless all the coefficients of the series vanish.

These facts make possible the immediate application of *Borel's* theory to differential equations. If, in short,

$$P(x, y, y', \cdots, y^{(n)}) = 0$$

is a differential equation which is holomorphic in x at the origin and is algebraic in y and its derivatives, any absolutely summable series (I), which satisfies formally the equation, defines an analytic function that is a solution of the equation. For example, it will be found that the series of *Laguerre* satisfies formally the equation

$$x^2 \frac{dy}{dx} + (x - 1)\, y = -1$$

and hence the function

$$f(x) = \int_0^\infty \frac{e^{-z} dz}{1 - zx}$$

must be a solution of the equation.

These conclusions of *Borel* should be strongly emphasized. In any complete theory of divergent series it is an ultimatum that they shall in all essential points * permit of manipulation

* In an absolutely summable series it is not always legitimate to change the order of an infinite number of terms. Cf. Borel, *Journ. de Math.*, ser. 5, vol. 2 (1896), p. 111.

precisely as convergent series, this property being a requisite for satisfactory application to differential equations.

In our preceding exposition of *Borel's* theory, we have introduced his chief integral by a method which permits of expansion in various directions. *Le Roy* in his very excellent thesis* suggests a change of the function in *Laplace's* integral which greatly enlarges the applicability † of *Borel's* method without essentially changing its character. Let the initial series (I) be first written

$$a_0 + a_1\Gamma(p+1)\frac{x}{\Gamma(p+1)} + a_2\Gamma(2p+1)\frac{x^2}{\Gamma(2p+1)} + \cdots$$
$$+ a_n\Gamma(np+1)\frac{x^n}{\Gamma(np+1)} + \cdots,$$

and then replace the second factor in each term by

$$\Gamma(np+1) = \frac{1}{p}\int_0^\infty e^{-z^{1/p}}z^{1/p-1+n}dz.$$

This gives for the formal equivalent of the series the integral

$$\text{(12)} \qquad \frac{1}{p}\int_0^\infty e^{-z^{1/p}}z^{1/p-1}F(zx)dz,$$

in which the associated function is now

$$\text{(13)} \qquad F(zx) = 1 + \frac{a_1 zx}{\Gamma(p+1)} + \frac{a_2 z^2 x^2}{\Gamma(2p+1)} + \cdots.$$

The number p remains to be fixed. If the series (I) is divergent, there is a critical value of p such that any smaller value of p gives an associated series having a zero radius of convergence, while a larger value gives one with an infinite radius of convergence. This critical value p' may be said to gauge or measure

* *Annales de Toulouse*, ser. 2, vol. 2 (1900), p. 416.

† Since this was written, a very interesting application of Le Roy's idea to differential equations has been made by Maillet, *Ann. de l'Ec. Nor.*, ser. 3, vol. 20 (1893), p. 487 ff.

the degree of divergence of the series. For the divergent series treated by *Borel*, $p' \leqq 1$. If $p' = 0$, the series (I) has a finite radius of convergence. On the other hand, when $p' = \infty$, *Le Roy's* integral can not be applied, but it may be conjectured that such cases will be of very rare occurrence. *Le Roy* proposes to employ the integral when the associated series is convergent for $p = p'$ and when also its circle of convergence has a finite radius and is not a natural boundary. The function obtained from (12) will be unique, and he shows that the series which are summable by its use like the series of *Borel*, can be manipulated as convergent series. One might also inquire whether, in case (13) diverges for $p = p'$ and we take $p > p'$, we shall not get a unique result irrespective of the value of p.

Other forms of integrals may also be selected for the summation of the series, as for example,*

$$\int_0^\infty f(z) F(zx) dz,$$

in which

$$F(zx) = \beta_0 + \beta_1 zx + \beta_2 z^2 x^2 + \cdots.$$

To generate the given series (I) we must so select $f(x)$ and $F(zx)$ that

$$a_n = \beta_n \int_0^\infty f(z) \cdot z^n dz.$$

Borel chooses for $f(z)$ the exponential function, making in consequence $F(zx)$, his associated series, dependent only upon the given series. Hence his process is called very appropriately the exponential method of summation. *Stieltjes*,† on the other hand, with his continued fraction arrives at an integral in which $F(u)$ is the fixed function and $f(z)$ is the variable function dependent on the series given. For the fixed function he takes

$$F(zx) = \frac{1}{1 - zx} = 1 + zx + z^2x^2 + \cdots,$$

* Cf. Le Roy, *loc. cit.*, pp. 414–415.

† *Loc. cit.*

so that

(14) $$a_n = \int_0^\infty f(z) \cdot z^n dz.$$

At first sight this choice of functions would seem to be a very desirable one, for the function defined by the divergent series is obtained in the familiar form

(15) $$\phi(z) = \int_0^\infty \frac{f(z)dz}{1 - zx}.$$

Upon examination, however, it turns out to be otherwise. For suppose the divergent series to be given and $f(z)$ is to be found. The problem is then a very difficult one, that of the inversion of the integral (14) when a_n is given for all values of n. This is what *Stieltjes* terms "the problem of the moments." It does not admit of a unique solution, for *Stieltjes* himself * gives a function,

$$f(z) = e^{-\sqrt[4]{z}} \sin \sqrt[4]{z},$$

which will make $a_n = 0$ for all values of n. If the supplementary condition is imposed that $f(z)$ shall not be negative between the limits of integration, only a single solution $f(z)$ is possible, but the divergent series is thereby restricted to belong to that class which *Stieltjes* derives naturally and elegantly by the consideration of his continued fraction.

Thus far our attention has been confined exclusively to integrals in which one of the limits of integration is infinite. There are, however, advantages in using appropriate integrals having both limits finite, at least if the given series is convergent and the integral is used for the purpose of analytic continuation. In particular, the integral

(16) $$f(x) = \int_0^1 V(z) F(zx) dz$$

should be noted, to which *Hadamard* has drawn attention in his thesis.† This falls under *Vallée-Poussin's* theorem when $V(z)$ is

* *Loc. cit.*, § 55.

† *Journ. de Math.*, ser. 4, vol. 8 (1892), pp. 158–160.

continuous along the path of integration and when also $F(u)$ is analytic in $u = zx$ for all values of z upon the path of integration and for values of x in some specified region of the x-plane. If, as we suppose, the path is rectilinear, the values of x to be excluded are evidently those which lie on the prolongations of the vectors from the origin to the singular points of $F(x)$. The region of convergence of (16) is consequently a star, whose boundary consists of prolongation of these vectors.* Thus *Hadamard's* integral, when applied to the analytic continuation of a function, is superior to *Borel's* in the extent of its "region of summability." This is illustrated in *Le Roy's* thesis † with the very familiar series :

$$1 + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot \cdots \cdot (2n-1)}{2 \cdot 4 \cdot \cdots \cdot 2n} x^n.$$

Here the coefficient of x^n is

$$\frac{1}{\pi} \int_0^1 \frac{z^n dz}{\sqrt{z(1-z)}},$$

so that

$$f(x) = \frac{1}{\pi} \int_0^1 \frac{dz}{\sqrt{z(1-z)(1-zx)}}.$$

Since $F(zx) \equiv 1/(1 - zx)$, the region of summability is the entire plane of x with the exception of the part of the real axis between $x = 1$ and $x = \infty$. *Borel's* polygon of summability for the series, on the other hand, is only the half plane lying to the left of a perpendicular to the real axis through the point $x = 1$.

Much, it seems to me, can yet be done in following up the use of *Hadamard's* integral. One special case has been studied already by *Le Roy*, in which the $(n + 1)$th coefficient of (I) has the form

$$a_n = \int_0^1 z^n \phi(z) dz.$$

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* This conclusion also holds if only $\int_0^1 V(z)dz$ is an absolutely convergent integral, as is shown by Hadamard.

† p. 411.

The series therefore defines a function

$$\int_0^1 \frac{\phi(z)dz}{1 - zx}$$

which is analytic over the entire plane except along the real axis between $x = 1$ and $x = \infty$. The path of integration may also permit of deformation so as to show that the cut between the points is not an essential cut. It is interesting to note that if $\phi(z)$ is positive between 0 and 1, the primary branch of the function has only real roots which are, moreover, greater than 1.*

LECTURE 3. *On the Determination of the Singularities of Functions Defined by Power Series.*

Up to the present time comparatively little successful work has been done in determining the singularities of functions defined by power series, and the little which has been done relates mostly to singularities upon the circle of convergence. Work of this special nature I shall omit from consideration here, thus passing over the memoirs of *Fabry*, and I shall call your attention to the literature which treats of the singularities in a wider domain.

The most fundamental and practical result yet obtained is undoubtedly a brilliant theorem of *Hadamard*,† in the wake of which a number of other interesting memoirs have followed. This theorem is as follows:

If two analytic functions are defined by the convergent power series

(1) $$\phi(x) = a_0 + a_1x + a_2x^2 + \cdots,$$

(2) $$\psi(x) = b_0 + b_1x + b_2x^2 + \cdots,$$

the only singularities of the function

(3) $$f(x) = a_0b_0 + a_1b_1x + a_2b_2x^2 + \cdots$$

will be points whose affixes γ_{ij} are the product of affixes of the singular points α_i and β_j of the first two functions.

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* Le Roy, *loc. cit.*, pp. 330–331.

† *Acta Math.*, vol. 22 (1898), p. 55.

The possibility that $x = 0$ should, in addition, be a singular point has been pointed out since by *Lindelöf.*

Although *Hadamard's* proof of the theorem is not a complicated one, I shall present here a still simpler proof given by *Borel.** Let R and R' be the radii of convergence of (1) and (2) respectively, and take a number ρ such that $R/\rho > 1/R'$. If then $|zx| \leqq |\rho x| < R$ and $|x| > 1/R'$, the product of $\phi(zx)$ and $\psi(1/x)$ can be developed into a *Laurent's* power series which is valid in a circular ring in the x-plane, having its center at the origin and the outer and inner radii R/ρ and $1/R'$ respectively. In this product the absolute term is obviously

$$(4) \qquad f(z) = a_0 b_0 + a_1 b_1 z + a_2 b_2 z^2 + \cdots.$$

Consider now the integral

$$(5) \qquad \frac{1}{2i\pi} \int_c \frac{\phi(zx)\psi\left(\frac{1}{x}\right) dx}{x},$$

in which c is a closed path surrounding the origin and contained within the circular ring. As long as z in its plane lies within a circle of radius $\rho < RR'$, having its center in the origin, the integral will surely define a function of z, and this function is evidently equal to the residue of the integrand for $x = 0$, which is $f(z)$.

We shall now seek to extend this function by varying z and at the same time deforming appropriately the path of integration. By the theorem of *Vallée Poussin* quoted in Lecture 2, the integral will continue to represent an analytic function of z, provided at every stage the integrand remains analytic in x and z; x being any point upon the path of integration. Now the values to be avoided are clearly the singular points of the functions $\phi(zx)$ and $\psi(1/x)$; namely the points :

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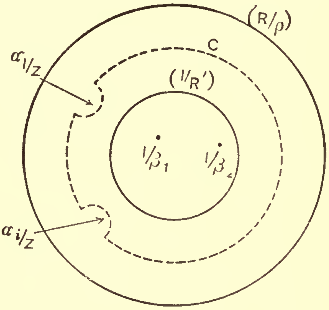
* *Bull. de la Soc. Math. de France,* vol. 26 (1898), pp. 238–248.

An interesting proof "*in multi case*" is given without the use of integrals by Pincherle in the *Rendiconto della R. Accad. delle Scienze di Bologna,* new ser., vol. 3 (1898–9), pp. 67–74.

$$zx = \alpha_i, \quad x = \frac{1}{\beta_j}.$$

The points $x = 1/\beta_j$ lie within the circle $(1/R')$ which is the inner circumference of the ring, while the points $x = \alpha_i/z$ before the variation of z lie without the outer circumference (R/ρ). For simplicity of presentation it may be convenient to assume at first that these points form an aggregate of isolated points. Suppose then that z follows any path in its plane emerging from the circle (ρ). Then the points α_i/z describe certain corresponding paths which we will mark in the x-plane. At the same time the contour c may be deformed continuously so as to recede before the points α_i/z without sweeping over any point $1/\beta_j$, provided merely that α_i/z never collides with a point $1/\beta_j$; that is, z must never pass through a point $\alpha_i\beta_j$. Now when z is held fixed, a deformation in the contour c, subject of course to the condition indicated, produces no change in the value of the integral $f(z)$, since the integrand is holomorphic between the initial and deformed paths. On the other hand, when the path is kept fixed and z is varied, we have the analytic continuation of $f(z)$ in accordance with the theorem of *Vallée Poussin*. By the two changes together $f(z)$ may be continued over the entire plane of z with the exception of the points $\alpha_i\beta_j = \gamma_{ij}$. To these should, of course, be added $z = \infty$, also $z = 0$ as a possible singular point for any branch of $f(z)$ except the initial branch.



It should be observed that γ_{ij} is shown to be a potential rather than an actual singular point. When, however, it is such a point, the character of the point depends in general solely upon the nature of the singularities α_i and β_j for (1) and (2) respectively. This fact was noticed by *Borel* and demonstrated in the following manner. Let

$$c_0 + c_1x + c_2x^2 + \cdots$$

be any convergent series defining a function $\phi_1(x)$ which is regular at α_i. Then $\phi_2(x) = \phi_1(x) + \phi(x)$ is a function which has at α_i the same singularity as $\phi(x)$. The combination of the series for $\phi_2(x)$ and for $\psi(x)$ by *Hadamard's* process gives the function

$$f_2(x) = (a_0 + c_0)b_0 + (a_1+c_1)b_1x + (a_2+c_2)b_2x^2 + \cdots = f(x) + f_1(x),$$

in which

$$f_1(x) = c_0b_0 + c_1b_1x + c_2b_2x^2 + \cdots.$$

Now since $\phi_1(x)$ is regular at α_i, when compounded with $\psi(x)$ it must give a function $f_1(x)$ which is regular at γ_{ij}. It follows that $f_2(x)$ and $f(x)$ have the same singularity at γ_{ij}. Thus the nature of this singular point is not altered by any change in $\phi(x)$ or $\psi(x)$ which does not affect the character of the points α_i and β_j. It depends therefore solely upon the character of the singularities compounded.

Complications arise only when there is a second pair of singularities α_k, β_l such that

$$\gamma_{ij} = \alpha_i\beta_j = \alpha_k\beta_l.$$

Clearly the resultant singularity is then dependent upon both pairs. Their effects may be so superimposed as to create an ugly singularity, or they may, on the other hand, so neutralize each other that γ_{ij} is a regular point. Very simple examples of the latter occurrence can be easily given. It seems probable that when γ_{ij} is but once a product of an α by a β, it must always be a singular point, but this has not yet been proved. Its demonstration will greatly enhance the value and applicability of *Hadamard's* theorem, for then it can be stated in numerous cases, not what the singular points of $f(x)$ may be, but what they actually are.

A detailed study of the nature of the dependence of the singularity γ_{ij} upon α_i and β_j would probably be both interesting and profitable. *Borel* examines the case in which α_i and β_j are poles of any orders, p and q, and shows that γ_{ij} is then a pole of order $p + q - 1$. It can, furthermore, be easily shown that whenever α_i is a pole of the first order, γ_{ij} is the same kind of singular point as β_j. For suppose that we put $\alpha_i = 1$, which may be done without loss of generality. The principal part of $\phi(x)$ at the pole α_i is then

$$\frac{A_i}{x-1} = -A_i(1 + x + x^2 + \cdots),$$

and the composition of this with $\psi(x)$ gives for the corresponding component of $f(x)$

$$-A_i(b_0 + b_1 x + b_2 x^2 + \cdots).$$

Hence the singularities γ_{ij} and β_j differ by a multiplicative constant.

Only one other general fact concerning the composition of singularities seems to be known. *Borel* proves that if the functions $\phi(x)$ and $\psi(x)$ are one-valued at α_i and β_j respectively, $f(x)$ is also one-valued at γ_{ij}. Thus when two one-valued functions are compounded, the resultant function is also one-valued. But this statement, as he himself points out, must be correctly construed and will not necessarily hold true when the singular points of the two given functions are not sets of isolated points but condense in infinite number along curves. To construct an example in which $f(x)$ in not one-valued, *Borel* makes use of the fact, now so well known, that the decision whether the circle of convergence is or is not a natural boundary of a given series depends upon the arguments of its coefficients. If, for instance, we take the series

$$1 + e^{i\theta_1}x + e^{i\theta_2}x^2 + \cdots,$$

which has a radius of convergence equal to 1, by a proper choice of the arguments θ_n the circle of convergence can be made a natural boundary. Put now

$$\sqrt{1-x} = c_0 + c_1 x + c_2 x^2 + \cdots, \tag{6}$$

in which the coefficients are necessarily real. Clearly the unit circle will be a natural boundary for

$$\phi(x) = c_0 + c_1 e^{i\theta_1}x + c_2 e^{i\theta_2}x^2 + \cdots.$$

and for

$$\psi(x) = 1 + e^{-i\theta_1}x + e^{-i\theta_2}x^2 + \cdots.$$

Yet the function $f(x)$ which is derived from these two one-valued functions by *Hadamard's* process is the two-valued function (6) which exists over the entire plane of x.

I have dwelt at some length upon *Hadamard's* theorem and its consequences because of their evident interest and importance. It is worthy of note that for analytic functions defined by power series the first great advance in the determination of the singularities over their entire domain has been made by methods that are roughly parallel to those currently employed in the consideration of their convergence. The convergence of series is indeed too difficult a question to be settled by any one rule or by any finite set of rules, but the methods of comparison with series known to be convergent have been found to be not only most efficient but also adequate for most practical purposes. In somewhat similar fashion *Hadamard's* theorem will determine the singular points of numerous functions by linking them with other series, of which the singularities are known.

One of the simplest applications of this theorem is obtained by compounding a given series

$$a_0 + a_1x + a_2x^2 + \cdots \tag{7}$$

with itself once, twice, $\cdots$, to m times. All the singularities of the resulting series

$$a_0^i + a_1^ix + a_2^ix^2 + \cdots \qquad (i = 1, 2, \cdots, m), \tag{8}$$

except possibly $x = 0$ and $x = \infty$, are included among the points obtained by multiplying i affixes of the singular points of (7) among themselves in all possible ways. If the m series (8) are multiplied each by a constant k_i and are then added, a new series

$$G(a_0) + G(a_1)x + G(a_2)x^2 + \cdots \tag{9}$$

is obtained, in which $G(u)$ denotes the polynomial $k_1u + \cdots + k_mu^m$.

This function has no singular points other than those which are possible for the m series from which it was derived. When r different series

$$a_0 + a_1x + a_2x^2 + \cdots,$$

$$b_0 + b_1x + b_2x^2 + \cdots,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$r_0 + r_1x + r_2x^2 + \cdots,$$

are used, a similar conclusion is reached for the series

$$G(a_0, b_0, \cdots, r_0) + G(a_1, b_1 \cdots, r_1)x + G(a_2, b_2, \cdots, r_2)x^2 + \cdots,$$

where G denotes a polynomial in which the constant term is lacking.

These results are of particular interest when applied to the series

$$(10) \qquad 1 + x + 2x^2 + \cdots + nx^n + \cdots$$

and

$$(11) \qquad 1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n} + \cdots,$$

which are the expansions of $1 + x/(1-x)^2$ and $\log(1+x)$. Since these functions have only one singular point, $x = 1$, in the finite plane, the only possible singularities of

$$\Sigma G\left(n, \frac{1}{n}\right)x^n$$

are $x = 0, 1, \infty$.*

The continued repetition of the above process for combining series leads naturally to a consideration of series of the form

$$(12) \qquad \Sigma P(a_n)x^n$$

in which a convergent power series $P(u)$ appears in place of the polynomial $G(u)$. Various theorems concerning cases of this

* Obviously a constant term can be included now in the polynomial $G(n, 1/n)$.

series have been given recently by *Leau*,* *Le Roy*,† *Desaint*,‡ *Lindelöf*,§ *Ford* ‖ and *Faber*,¶ though the proof of some of these theorems has no direct relation to *Hadamard's* theorem. The importance of such work is, however, apparent, inasmuch as numerous series which occur in analysis can be put into the form under consideration, as for example $\Sigma(\sin \pi/n)\, x^n$.

Three cases must be distinguished according as the radius of convergence of the initial series (7) is less than, equal to, or greater than 1. If the radius is less than 1, the singular point nearest to the origin has a modulus less than 1, and the continued multiplication of the affix of the point by itself gives a series of points which approach indefinitely close to the origin. The presumption, therefore, would naturally be that the series (12) is then divergent, but this is very far from being always true, as will be seen at once by referring to the series $\Sigma(x^n \sin a_n)$ and $\Sigma(x^n \cos a_n)$ in which a_n is real. The applicability of *Hadamard's* theorem consequently ceases.

The case in which the radius of convergence of (7) is greater than 1 has been investigated very recently by *Desaint*. In this case the expected theorem is obtained. If, namely, $P(u)$ is a convergent series without a constant term, $\Sigma P(a_n) x^n$ defines a function which can have no singular points, besides $x = 0$ and $x = \infty$, than those which result from the multiplication of the affixes of the singular points among themselves in all possible ways and to any number of times.** *Desaint's* proof is based upon the fact that $\Sigma P(a_n) z^n$, after the omission of a suitable number of terms, can be expressed in the form

* *Journ. de Math.*, ser. 5, vol. 5 (1899), p. 365.

† *Loc. cit.*

‡ *Journ. de Math.*, ser. 5, vol. 8 (1902), p. 433.

§ *Acta Societatis Scientiarum Fennicæ*, vol. 31 (1902).

‖ *Journ. de Math.*, ser. 5, vol. 9 (1903), p. 223.

¶ *Math. Ann.*, vol. 57 (1903), p. 369.

** This is a somewhat sharper statement of the result than that given by Desaint. In his theorem $x = 1$ is given as a possible singular point, but this, as appears from the proof to be given here, is due solely to the admission of a constant term into $P(u)$. He also fails to note that $x = 0$ may be a singular point.

$$z^N \sum_{k=N}^{\infty} c_n \int_{c'} \cdots \int_{c'} \frac{\dfrac{f(t_1)f(t_2)\cdots f(t_k)}{(2i\pi)^k (t_1 t_2 \cdots t_k)^{N+1}}}{1-\dfrac{z}{t_1 t_2 \cdots t_k}} dt_1 dt_2 \cdots dt_k,$$

in which $f(t)$ is the function defined by (7) for $x = t$, c' is an appropriately chosen contour, and c_n denotes the nth coefficient of

$$P(u) = c_1 u + c_2 u^2 + \cdots.$$

Although his proof is essentially simple in character, I shall give here a new and simpler proof, based directly upon *Hadamard's* theorem.

Place first

$$f_i(x) = a_0^i + a_1^i x + a_2^i x^2 + \cdots \qquad (i = 2, 3, \cdots),$$

and consider the expression

$$c_n f_n(x) + c_{n+1} f_{n+1}(x) + \cdots$$

in which n denotes some fixed integer. If $r > 1$ denotes the radius of convergence of the fundamental series (7), the radius of $f_i(x)$ will be r^i. Describe about the origin a circle (r') having a radius $r' < r^n$. If a sufficient number of initial terms be cut off in each of the series,

$$f_n(x), \quad f_{n+1}(x), \quad \cdots, \quad f_{2n}(x),$$

the maximum absolute values of the remainders within or upon the circle (r') can be made as small as is desired. Suppose then that after m terms of each have been removed, the remainders

$$(13) \qquad r_n(x), \quad r_{n+1}(x), \quad \cdots, \quad r_{2n}(x)$$

do not exceed

$$\epsilon^n, \quad \epsilon^{n+1}, \quad \cdots, \quad \epsilon^{2n}$$

respectively, in which ϵ is some small positive number. Let us now substitute in *Hadamard's* integral

$$F(z) = \frac{1}{2i\pi}\int_c \frac{\phi(zx)\psi\left(\frac{1}{x}\right)dx}{x},$$

any two of the functions (13) for ϕ and ψ.

Put for example

$$\phi(zx) = r_{n+i}(zx), \qquad \psi\left(\frac{1}{x}\right) = r_{n+j}\left(\frac{1}{x}\right),$$

and choose the unit circle as the path of integration. Then if $|z| \leqq r'$, the absolute values of the arguments of the series $\phi(zx)$ and $\psi(1/x)$ will be less than their radii of convergence since $|x| = 1$ and $r > 1$. The conditions for the existence of *Hadamard's* integral are therefore fulfilled. Since also

$$|r_{n+i}(zx)| < \epsilon^{n+i}, \qquad \left|r_{n+j}\left(\frac{1}{x}\right)\right| < \epsilon^{n+j},$$

we have

$$|F(z)| \leqq \frac{\epsilon^{2n+i+j}}{2\pi}\int \frac{|dx|}{|x|} = \epsilon^{2n+i+j}.$$

But by *Hadamard's* theorem $F(z) = r_{2n+i+j}(z)$, and hence

$$(14) \qquad |r_i(z)| < \epsilon^i \qquad (|z| \leqq r'),$$

for all values of i from $2n$ to $4n$ inclusive. The reasoning can now be repeated with $2n$ in place of n, and so on; therefore (14) is true for all values of $i \geqq n$.

Thus far the value of ϵ has remained arbitrary. Let its value now be taken less than the radius of convergence of $P(u)$. Then by (14) the series

$$(15) \qquad c_n r_n(x) + c_{n+1}r_{n+1}(x) + \cdots$$

will be uniformly convergent in (r'). Since, furthermore, all the component series $r_{n+i}(i = 0, 1, 2, \cdots)$ are likewise so convergent, by a fundamental and familiar theorem of *Weierstrass* * the terms of the collective series (15) may be rearranged into an ordinary series in ascending powers of x. But this rearrangement gives

* Harkness and Morley's *Introduction to the Theory of Analytic Functions*, p. 134.

$$\sum_{j=m}^{\infty}\left(\sum_{i=n}^{\infty} c_i a_j^i\right)x^j,$$

or the remainder after the $(m-1)$th power of x in

$$(16) \qquad \sum_{j=0}^{\infty} P(a_j)x^j - c_1 f(x) - c_2 f_2(x) - \cdots - c_{n-1} f_{n-1}(x).$$

Now the series (15) before its rearrangement was a uniformly convergent series of analytic functions and defined a function which was analytic within (r'). It follows that (16) is also analytic within this circle, and hence

$$\sum_{n=1}^{\infty} P(a_n)x^n$$

has no singularities within this circle except those of

$$f_1(x),\ f_2(x),\ \cdots,\ f_{n-1}(x).$$

But the radius of (r') was any quantity short of r^n, and this conclusion therefore holds within a circle having its center in the origin and a radius equal to r^n. By increasing n indefinitely, the theorem of *Desaint* results. It is evident also that if $f_1(x)$, and therefore $f_i(x)$, represents a one-valued function, $\Sigma P(a_n)x^n$ must also be such a function.

There remains yet for consideration the third class of cases in which the radius of convergence of the fundamental series is 1. If upon the circle of convergence there is any singular point with an incommensurable argument, the continued multiplication of its affix by itself gives a set of points everywhere dense upon the circle of convergence. It is therefore to be expected that this circle will be, in general, a natural boundary for $\Sigma P(a_n)x^n$, and accordingly the cases which will be of chief interest will be those in which all the singular points upon the circle have commensurable arguments. A simple case of this character is obtained when either (10) or (11) is chosen as the generating series. If the former be selected, the resulting series has the form $\Sigma P(n)x^n$. This has a special interest inasmuch as its study has proved to be of profit both for the theory of analytic continuation and of divergent series. The reason becomes apparent when the statement is made that it is possible to throw any Taylor's series, $\Sigma a_n x^n$, whether convergent or divergent, into the particular form $\Sigma P(n)x^n$, and in an infinite number of ways. This fact follows as a corollary of a very general theorem of *Mittag-Leffler*,* which, when restricted to the special case before us, establishes the existence of a function $P(x)$, which is holomorphic over the entire finite plane and assumes the pre-assigned values $a_0, a_1, a_2, \cdots$ in the points $x = 0, 1, 2, \cdots$. Consequently the character of the function defined by $\Sigma P(n)x^n$ is made to depend upon the behavior of $P(x)$ as x approaches ∞.

Inasmuch as $\Sigma P(n)x^n$ is perfectly general, limitations must be imposed upon $P(u)$ in any attempt to extend *Hadamard's* theorem to this series. But whenever the theorem is applicable, the only possible singularities of $\Sigma P(n)x^n$ are $x = 0, 1, \infty$. *Leau* † establishes the correctness of this result when $P(u)$ is an entire function of order less than 1,‡ giving also a more general theorem § concerning $\Sigma P(a_n)x^n$ of which this is a special case. The like conclusion holds concerning the singular points of $\Sigma P(1/n)x^n$, provided only that $P(x)$ is holomorphic at the origin.‖

Very recently these results of *Leau* have been proved more simply by *Faber*, but in a more restricted form, an artificial cut being drawn from $x = 1$ to $x = \infty$ to obtain a one valued function. In addition, *Faber* shows that if for any prescribed ϵ and for a sufficiently large r the inequality

$$(17) \qquad |P(re^{i\phi})| < e^{\epsilon r}$$

* *Acta Math.*, vol. 4 (1884), p. 53, theorem D. For a reference to this theorem I am indebted to Professor Osgood. Theorem 2 of Desaint's memoir (p. 438) is in contradiction with this, but his proof is here inadequate since r_k (p. 440) has not necessarily a lower limit.

† *Loc. cit.*, p. 418.

‡ He also shows that $\Sigma P(n)x^n$ is then a one-valued function.

§ *Loc. cit.*, p. 417. See also *Bull. de la Soc. Math. de France*, vol. 26 (1898), p. 267.

‖ *Loc. cit.*, p. 418; see also p. 407.

is fulfilled, the point $x = 1$ must be an essential singularity, and the function represented by $\Sigma P(n)x^n$ is consequently one-valued.* Conversely, if $f(x)$ is a one-valued function which has only one singular point, and if that point is an essential singularity, $f(x)$ can be expressed in the form $\Sigma P(n)x^n$, in which $P(u)$ is an entire function satisfying (17). More generally, if there are l essential singularities $x_1, \cdots, x_l$ and no other singular points in the finite plane, the coefficient of x^n must be

$$a_n = \frac{1}{x_1^n} P_1(n) + \cdots + \frac{1}{x_l^n} P_l(n) + a'_n,$$

in which $P_1(n), \cdots, P_l(n)$ are entire functions of the nature above specified and in which $\lim \sqrt[n]{a'_n} = 0$. This converse has an especial interest because as yet few theorems have been discovered giving the necessary form of the coefficients of a power series for an analytic function with prescribed functional properties.

Other theorems concerning $\Sigma P(n)x^n$ have recently been derived without requiring that $P(n)$ shall be holomorphic over the entire plane.

As a sample of these I shall cite in conclusion the following theorem of *Lindelöf:* †

If $P(n)$ represents a function fulfilling the following conditions:

1. $P(z)$ is analytic for every point of the complex plane $z = \tau + it$ for which $\tau \geqq 0$ (except possibly at the origin, for which $P(z)$ has a determinate value).

2. A number ϵ being arbitrarily given, it is possible to find another number R such that by putting $z = re^{i\phi}$ we will have for $r > R$

$$|P(z)| < e^{\epsilon r} \qquad \left(-\frac{\pi}{2} \leqq \phi \leqq \frac{\pi}{2}\right):$$

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* Le Roy three years earlier had noted this conclusion when $P(x)$ is an entire function whose "apparent order" is less than 1; *loc. cit.*, p. 348, footnote. Faber does not seem to be aware of Le Roy's statement. The difference between the two statements is slight but becomes important in formulating the new and interesting converse which Faber adds.

† *Loc. cit.*, § 13.

then the principal branch of the function $\Sigma P(n)z^n$ will be holomorphic throughout the complex plane excepting possibly on the segment $(1, +\infty)$ of the real axis. Furthermore, the function approaches 0 as a limit when x tends toward the point at infinity along any ray having an argument between 0 and 2π.

LECTURE IV. *On Series of Polynomials and of Rational Fractions.*

In the last two lectures I have spoken of the use of integrals for the study of analytic extension and of divergent series. The topic of to-day's lecture is the representation of functions by means of series of polynomials and of rational fractions. This subject forms a very natural transition to the succeeding lecture upon continued fractions, since an algebraic continued fraction is in reality nothing but a series of rational fractions advantageously chosen for the study of a corresponding function which, when known, is commonly given in the form of a power series.

The literature relating directly or indirectly to series of polynomials and of rational fractions is a vast one, with many ramifications. Thus in one direction there are various researches of importance upon the non-uniform convergence of series of continuous functions, and in this connection I may refer particularly to the recent work of *Osgood* and *Baire*, an excellent report of which is contained in *Schönflies' Bericht über die Mengenlehre.** Another part of the field comprises numerous memoirs devoted to special series of polynomials and rational fractions. Quite recently a more systematic and general study has been begun by *Borel*, *Mittag-Leffler*, and others, and it is to this that I am to call your especial attention.

Two very familiar facts, both discovered by *Weierstrass*, may be said to be the origin of this study. I refer, of course, to the theorem that any function which is continuous in a given finite interval of the real axis can be expressed in that interval as an

* *Jahresbericht der deutschen Mathematiker-Vereinigung*, vol. 8, pp. 224–241.

absolutely and uniformly convergent series of polynomials,* and, secondly, to the possibility that a single series of rational fractions may represent two or more distinct analytic functions in different portions of its domain of convergence. A notable advance upon the theorem first mentioned was made by *Runge*† in 1884, who proved that any one-valued analytic function throughout the domain of its existence can be represented by a series of rational functions; furthermore, this domain may be of any shape whatsoever, provided only it forms a two-dimensional continuum. *Runge's* proof of these important results is not only worthy of careful study, but contains also certain conclusions which were announced again by *Painlevé*‡ in 1898, though without proof. The conclusions reached were as follows :

Let D be a domain consisting of any number of separate pieces of the complex plane, in each of which we will suppose an analytic function to be defined. The functions thus defined can be, at pleasure, either distinct functions or parts of one or more functions. In any case a series of rational functions can be formed which will converge absolutely and uniformly in any region lying in the interior of D and represent in each separate piece the prescribed function. Furthermore, this representation can be made in an infinite number of ways. Let the ensemble of the points excluded from D be represented by E. When E consists of a single connected continuum of any sort, whether linear or areal, any point a of E can be arbitrarily selected, and the function can be expanded into the series

$$\sum_0^\infty G_n\left(\frac{1}{x-a}\right)$$

* "Ueber die analytische Darstellbarkeit sogenannter willkürlicher Functionen einer reellen Veränderlichen"; *Berliner Sitzungsberichte*, 1885, p. 633 or *Werke*, vol. 3, p. 1. Simple proofs of the theorem have been given by Lebesque, *Bull. des Sciences Math.*, ser. 2, vol. 22 (1898), p. 278, and by Mittag-Leffler, *Rendiconti di Palermo*, vol. 14 (1900), p. 217, with an extension to functions of two variables. In this connection see Painlevé's note in the *Compt. Rend.*, vol. 126 (1898), p. 459.

† *Acta Math.*, vol. 6, p. 229.

‡ *Compt. Rend.*, vol. 126, pp. 201 and 318.

in which $G_n[1/(x-a)]$ denotes a polynomial in $1/(x-a)$. If, in particular, the continuum E contains the point $x=\infty$, an ordinary series of polynomials, $\Sigma G_n(x)$, can be employed. When E consists of a finite number of separate pieces (or isolated points), the expansion can be put under the form

$$\sum_{n=1}^{\infty} G_n^{(1)}\left(\frac{1}{x-a_1}\right)+\sum_{n=1}^{\infty} G_n^{(2)}\left(\frac{1}{x-a_2}\right)+\cdots+\sum_{n=1}^{\infty} G_n^{(q)}\left(\frac{1}{x-a_q}\right),$$

in which $a_1, \cdots, a_q$ are points arbitrarily chosen in the separate pieces.

In the familiar case in which only a single analytic function

$$a_0+a_1(x-a)+a_2(x-a)^2+\cdots \tag{1}$$

is given, it is natural to seek a series of polynomials having the greatest possible domain of convergence. Unless the function is one-valued, the most convenient domain is in general the star of *Mittag-Leffler*. This is constructed for the series (1) by first marking on each ray which terminates in a the nearest singular point and then obliterating the portion of the ray beyond this point. The region which remains when this has been done is a star having a for its center. *Mittag-Leffler* * shows that within the star the given analytic function can be represented by a series of polynomials in which the coefficients of the polynomials depend only upon the value of the function and its derivatives at a, † or, in other words, upon the coefficients of (1). If, in short, we put

$$g_n(x)=\sum_{\lambda_1=0}^{n^2}\sum_{\lambda_2=0}^{n^4}\cdots\sum_{\lambda_n=0}^{n^{2n}}\frac{(\lambda_1+\lambda_2+\cdots+\lambda_n)!\,a_n}{\lambda_1!\,\lambda_2!\cdots\lambda_n!}\left(\frac{x-a}{n}\right)^{\lambda_1+\lambda_2+\cdots+\lambda_n}$$

and

$$G_n(x)=g_n(x)-g_{n-1}(x),$$

* *Acta Math.*, vol. 23 (1899), p. 43; vol. 24, pp. 183, 205; vol. 26, p. 353. A good summary is found in the *Proc. of the London Math. Soc.*, vol. 32 (1900), pp. 72–78.

† In this respect his work is superior to that of Runge and others. Runge, for example, presupposes a knowledge of the function at an infinite number of points.

then $\sum_{n=0}^{\infty} G_n(x)$ is a series which converges uniformly in any region lying, with its boundary, entirely in the interior of the star. The series may also converge outside the star. *Borel* * has shown, furthermore, that the series of *Mittag-Leffler* is not the only possible one, but there is an infinity of polynomial series sharing the same property within the star.

It will be noticed that the construction of the series of *Mittag-Leffler* is in no wise dependent upon the convergence of the initial power series. In certain cases, at least, the polynomial series converges when the given series (1) is itself divergent. It is natural therefore to look for a theory of divergent series based upon convergent series of polynomials. As yet, however, no such theory has been invented. One of the chief difficulties in the way is that the polynomial series do not afford a unique mode of representing an analytic function. Now the difference between any two series of polynomials for the same function in an assigned area is a third series which vanishes at every point of the area, though the separate terms do not. This is a decidedly awkward point, and occasions difficulty in proving or disproving the identity of two functions expressed by polynomial series. It is true, indeed, that this difficulty will scarcely present itself when we start with a convergent power series which is to be continued analytically, the polynomial series then giving continuations of a common function. But when the series (1) is divergent and there is no known function which it represents, it is an open question whether the different series of polynomials which are obtained from (1) by application of diverse laws will furnish the same or different functions. If different functions, is there any ground for preferring one series of polynomials to another?

Up to the present time two essentially different principles seem to have been followed in the formation of series of polynomials. In the work of *Runge, Borel, Painlevé* and *Mittag-Leffler* the coefficients in the polynomials vary with the character of the ana-

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* *Ann. de l'Ec. Nor.*, ser. 2, vol. 16 (1899), p. 132, or *Les Séries divergentes*, p. 171.

lytic function to be represented; for example, in the polynomials of *Mittag-Leffler* they are functions of the coefficients of the given element, $\sum a_n x^n$. By appropriately choosing the coefficients of the polynomials these writers obtain a very large region of convergence and at the same time are able to greatly vary its shape. On the other hand, the series which are met in the practical branches of mathematics—for instance, in the theory of zonal harmonics—have the form

$$c_0 G_0(x) + c_1 G_1(x) + c_2 G_2(x) + \cdots \tag{2}$$

in which the polynomials $G_n(x)$ are entirely independent of the function represented, while the c_i vary. The polynomials themselves are selected according to the shape of the region of convergence. Thus if the region is a circle, we may put

$$G_n(x) = (x - a)^n,$$

and we have then the ordinary Taylor's series. Or if it be an ellipse having the foci ± 1, we may take for our polynomials either the successive zonal harmonics or a second succession of polynomials (also called *Legendre's* polynomials) which are connected by the recurrent relation:

$$G_{n+2}(x) - 2x(2n + 3) G_{n+1}(x) + 4(n + 1)^2 G_n(x) = 0. \tag{3}$$

In a recent number of the *Mathematische Annalen* (July, 1903) *Faber* has considered this second class of polynomials from a somewhat general point of view and has demonstrated that any function which is holomorphic within a closed branch of a single analytic curve, as for example an ellipse or a lemniscate of one oval, can be expressed by a series of the form (2). The properties of his series are similar to those of Taylor's series. In the case of the latter, to ascertain whether $\Sigma a_n x^n$ converges in the interior of a circle having its center in the origin and a radius R, we have only to determine the maximum modulus of a point of condensation of the set of points $\sqrt[n]{a_n}$ $(n = 1, 2, 3, \cdots)$. If it is exactly equal to $1/R$, the circle (R) is the circle of convergence, and there is at least one singularity upon its circumference. If, on the other hand, it is greater or less than $1/R$, the series will have a smaller or a larger circle of convergence. So also to the given branch of the analytic curve there corresponds a certain critical value. When this is exactly equal to the upper limit of $|\sqrt[n]{c_n}|$ in *Faber's* series, the given analytic branch is the curve of convergence. At every point within, the series converges, while it diverges at every exterior point, and upon the curve there must lie at least one singular point of the function defined by (2). If, however, the upper limit is greater or less than the critical value, we consider a certain series of simple, closed analytic curves, (as for example a series of confocal ellipses), among which the given analytic branch must, of course, be included. The curve of convergence is then fixed by the reciprocal of the upper limit of $|\sqrt[n]{c_n}|$ provided this limit is not too large. Moreover, as in the case of Taylor's series, the function cannot vanish identically unless every $c_n = 0$, and in consequence the series vanishes identically. It is therefore impossible that the same function shall be represented by two different series of the given form.

In view of the last mentioned fact it might be of especial interest to apply this class of polynomial series to the study of divergent series.

In the most familiar and useful polynomial series the successive polynomials are connected by a linear law of recurrence,

$$(4) \qquad k_0 G_{n+m}(x) + k_1 G_{n+m-1}(x) + \cdots + k_m G_n(x) = 0,$$

in which the coefficients k_i are polynomials in x and n. Thus the zonal harmonics have as their law of recurrence

$$(n+1)G_{n+1}(x) - (2n+1)xG_n(x) + nG_{n-1}(x) = 0.$$

Many series of this nature are also included in the class considered by *Faber*. The form of the region of convergence has been determined by *Poincaré* * upon the hypothesis that equation

* *Amer. Journ. of Math.*, vol. 7 (1885), p. 243.

(4) has a limiting form for $n=\infty$. Let the equation be first divided through by k_0, and then denote the limits of the successive coefficients for $n=\infty$ by $k_1(x)$, $k_2(x)$, $\cdots$ $k_m(x)$. Construct next the auxiliary equation

$$(5) \qquad z^m + k_1(x)z^{m-1} + k_2(x)z^{m-2} + \cdots + k_m(x) = 0.$$

Except for particular values of x there will be one root of this equation which has a larger modulus than any other. Let $r(x)$ be that root. *Poincaré** shows that with increasing n the ratio $G_n(x)/G_{n-1}(x)$ will approach, in general, this root as its limit. The region of convergence is therefore confined by a curve of the form $C=|r(x)|$, and the value of C for the series (2) is to be taken equal to the radius of convergence of $\Sigma c_n y^n$.†

By way of illustration let us take the series $\Sigma c_n G_n(x)$ in which the polynomial obeys the law

* More specifically, Poincaré proves that if no two roots of (5) are of equal modulus, $G_n(x)/G_{n-1}(x)$ has always a limit, and this limit is equal to some root of (5), usually the one of greatest modulus.

† Poincaré has given no proof that the series (2) will converge at those points within the curve $|r(x)|=C$, for which there are two or more distinct roots of (5) having a common modulus greater than the moduli of the remaining roots. Thus in the example which is quoted below (p. 127), these are the points of the real axis which are included between $+1$ and -1. This gap in Poincaré's theory can be filled in by the following theorem which I have given in the *Transactions of the Amer. Math. Soc.*, vol. 1 (1900), p. 298: If the coefficients in the series $\Sigma A_n y^n$ are connected by a recurrent relation having the limiting form

$$A_n + k_1 A_{n-1} + \cdots + k_m A_{n-m} = 0,$$

the series will converge at the worst within a circle whose radius is the reciprocal of the greatest modulus of any root of the auxiliary equation

$$z^n + k_1 z^{n-1} + \cdots + k_m = 0.$$

Denote this maximum by r, *irrespective of the number of roots having this maximum modulus.* Then

$$|A_n| < M(r+\varepsilon)^n \qquad (n=1, 2, \ldots).$$

Hence if C is the radius of convergence of $\Sigma c_n y^n$, the series $\Sigma c_n A_n$ will converge when $C>r$. Suppose now that A_n depends upon x and put $A_n = G_n(x)$. It follows then from my theorem that $\Sigma c_n G_n(x)$ will *always* converge when $C>r$. But this is what was to be proved.

At the time of the publication of my work I was not aware of Poincaré's article, and I therefore failed to point out the relation of the two memoirs.

$$(n^2+1)G_{n+2}(x) - 2n^2xG_{n+1}(x) + (n^2+x^2)G_n(x) = 0.$$

For $n = \infty$ the limiting form of this equation is

$$G_{n+2}(x) - 2xG_{n+1}(x) + G_n(x) = 0,$$

or the same as the limiting form for the zonal harmonic. The auxiliary equation is

$$z^2 - 2xz + 1 = 0,$$

of which the roots are

$$z = x \pm \sqrt{x^2-1}.$$

The curves $|x \pm \sqrt{x^2-1}| = C$ are easily seen to be ellipses having the foci ± 1. Hence if R is the radius of convergence of $\Sigma c_n y^n$, the region of convergence of (2) is the interior of an ellipse,

$$|x \pm \sqrt{x^2-1}| = R.$$

Poincaré also examines such exceptional cases as that which is specified by relation (3), which has no proper limiting form. But upon this work we can not longer dwell. I wish, however, to emphasize its fundamental character, inasmuch as many previous, and even subsequent conclusions concerning the convergence of series of the form (2) are comprised in *Poincaré's* result.

Somewhat earlier in the lecture I set forth the arbitrary character of the function which could be represented by series of polynomials and rational fractions. We have seen also how this arbitrary element was entirely eradicated by confining ourselves to polynomials which obey a linear law of recurrence. In the remainder of this lecture I wish to develop the consequences of restricting a series of rational fractions in the manner supposed by *Borel* in his thesis * and its recent continuation in the *Acta Mathematica.*† *Borel* seeks to so restrict a series of rational fractions, $\Sigma P_n(x)/R_n(x)$, as to ensure a connection between the position of the poles of its separate terms and the position of the singular points of the function which the series collectively represents. On this account he assigns

* *Ann. de l'Ec. Nor.*, ser. 3, vol. 12 (1895), p. 1.

† Vol. 24 (1900), p. 309.

an upper limit to the degrees of $P_n(x)$ and $R_n(x)$. But this is not enough, and he proceeds therefore to limit the magnitude of the coefficients in the numerators. On the other hand, he allows any distribution whatsoever for the roots of the denominators, thus leaving himself at liberty to vary greatly the nature of the function represented.

In his thesis he develops the case

$$\phi(z) = \sum_{n=1}^{\infty} \frac{A_n}{(z-a_n)^{m_n}} \qquad (m_n \leqq m) \tag{6}$$

which had been previously considered by *Poincaré* * and *Goursat*.† To avoid semi-convergent series or, in other words, functions, of which the character depends not merely upon the position of the poles a_n and the values of A_n but also upon the order of summation, the condition is imposed that ΣA_n shall be absolutely convergent. Then if there is any area of the z plane which contains no poles, the series (6) must converge within this region. Since furthermore it is uniformly convergent in any interior sub-region, it defines an analytic function within the area. There may be several such areas separated by lines or regions in which the poles are everywhere dense. This is precisely the case to be considered now.

To simplify matters, let us suppose that the poles are everywhere dense along certain closed curves of ordinary character, but nowhere inside the curves. *Poincaré* and *Goursat* show that each curve is a natural boundary for the analytic function $\phi(z)$ defined by (6) in its interior. *Borel's* proof is as follows. Denote the component of (6) which corresponds to a_n by

$$\phi_1(z) = \frac{B_m}{(z-a_n)^m} + \frac{B_{m-1}}{(z-a_n)^{m-1}} + \cdots + \frac{B_1}{z-a_n}$$

and the remaining part by

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* *Acta Societatis Fennicæ*, vol. 12 (1883), p. 341, and *Amer. Journ. of Math.* vol. 14 (1892), p. 201.

† *Compt. Rend.*, vol. 94 (1882), p. 715.

$$\phi_2(z) = \sum_{i=1}^{r} \frac{A_i'}{(z-\beta_i)^{m_i}} + \sum_{i=r+1}^{\infty} \frac{A_i''}{(z-\beta_i)^{m_i}}.$$

It is evident that if a_n lies within any one of the curves considered, a_n is a pole of $\phi(z)$. Now when these interior poles condense in infinite number in the vicinity of any point of the curve, it must, of course, be a singularity of $\phi(z)$. Consider next any one of the points a_n which lies upon the boundary but is not a point of condensation of the interior poles, and let z approach this point along the normal. Describe a circle upon the line $z - a_n$ as diameter. If z is sufficiently near to a_n, the circle will exclude every one of the points a_i, excepting a_n which lies upon its boundary. Since also ΣA_n is absolutely convergent, by increasing r the second component of $\phi_2(z)$ may be made less in absolute value than $\epsilon/|z - a_n|^m$, in which ϵ is an arbitrarily small prescribed quantity. If, then, H denotes the maximum of the first component of $\phi_2(z)$ as z now moves up to a_n, we have

$$|\phi_2(z)| < H + \frac{\epsilon}{|z - a_n|^m}.$$

Consequently,

$$\lim_{z=a_n} \phi(z)\cdot(z-a_n)^m = \lim \phi_1(z)\cdot(z-a_n)^m + \lim \phi_2(z)\cdot(z-a_n)^m = B_m.$$

This shows that $|\phi(z)|$ increases indefinitely when z approaches any pole a_n of the mth order along a normal, and removes the possibility that the poles, because they are infinitely thick upon the curve, may so neutralize one another that the function can be carried analytically across the curve at a_n. As, moreover, we suppose the points a_n of order m to be everywhere dense upon the curve, it must be a natural boundary.

It is apparent now that the expression (6) continues the initial function $\phi(z)$ across a natural boundary into other regions where it defines in similar manner other analytic functions with natural boundaries. But, it may be asked, is there any proper sense in which these analytic functions may be regarded as a continuation of one another? Just here *Borel* steps in and, after imposing further conditions, shows that when the function defined by (6) within some one of the curves is zero, the functions defined within the other curves must also vanish.* Take $m = 1$, so that

$$\phi(z) = \Sigma \frac{A_n}{z - a_n}. \tag{7}$$

By a linear transformation

$$z' = \frac{az + b}{cz + d}$$

any interior point of one curve may be taken as the origin and any interior point of a second curve may be transformed simultaneously into the point at infinity without changing the character of the series to be investigated. Now at the origin the successive coefficients in the expansion of $\phi(z)$ into a Taylor's series are the negative of

$$\Sigma \frac{A_n}{a_n}, \quad \Sigma \frac{A_n}{a_n^2}, \quad \Sigma \frac{A_n}{a_n^3}, \cdots \tag{8}$$

while those in the expansion for $z = \infty$ are

$$\Sigma A_n, \quad \Sigma A_n a_n, \quad \Sigma A_n a_n^2, \cdots. \tag{9}$$

Borel proves that when

$$\lim_{n=\infty} \sqrt[n]{A_n} = 0,$$

the coefficients (9) must vanish if those given in (8) do. Any one of the analytic functions under discussion is therefore completely determined by any other, the expression (7) being the intermediary by which we pass from one to the other.

So far as yet appears, this method of continuing an analytic function across a natural boundary is of very limited applicability. Its significance has been made clearer by *Borel's* later memoir in the *Acta Mathematica*. Here the rational fractions are of a less highly specialized character, but the essential nature of the investigation can still be exhibited without abandoning the expression (6). Let $|A_n| < u_n^{m+1}$, where u_n denotes the nth term of a convergent series

* Cf. pp. 32–33 of his thesis or pp. 94–98 of his *Théorie des fonctions*.

of positive numbers. We shall suppose that the poles of the terms of (6) are everywhere dense over a large portion of the plane, leaving, however, at least one area free from poles, so that there shall be an analytic function to continue, though even this is not necessary. *Borel* proves that parallel to any assigned direction there will be an infinity of straight lines, everywhere dense throughout the plane, along which the series (6) will converge absolutely and uniformly. The function defined along these lines is therefore a continuous one.

The proof of this result is short and simple. Describe about the poles a_n as centers circles which have successively the radii $u_n (n = 1, 2, \cdots)$. If there is any point which lies outside all of these circles, the series (6) must there converge, since at such a point the absolute value of the nth term is

$$\left| \frac{A_n}{(z - a_n)^{n_m}} \right| < \frac{u_n^{m+1}}{u_n^m} = u_n, \qquad (n > N),$$

that is, less than the nth term of a convergent series of positive numbers. But are there points outside of all the circles? To settle this question, take any straight line perpendicular to the assigned direction and project orthogonally all the circles upon the line. The total sum of all the projections, $2\Sigma u_n$, will be convergent. Moreover, by cutting off a sufficient number of terms at the beginning of (6), the sum of the projections may be made less than any assigned segment ab of the line. Let N terms be cut off for this purpose. Take any point c of the segment which does not lie upon the projection of any circle nor coincide with the projection of one of the first N poles of (6). At c erect a perpendicular to ab. This will be a line parallel to the assigned direction which throughout its entire extent lies without all the circles, excepting possibly the first N. Hence the series (6) will converge absolutely and uniformly along the line, even though the line lie infinitesimally close to some set of poles in the system. Lastly, because ab was an interval of arbitrary length, these lines of convergence must be everywhere dense throughout the plane, obviously forming a non-enumerable aggregate.

Since the series is uniformly convergent, it can be integrated term by term. Clearly also the numerators A_i in (6) can be so conditioned that the term-by-term derivative of (6) shall be uniformly convergent. Then the derivative of $\phi(z)$ is coincident with the derivative of the series. It is even possible to so choose the A_i that the series will be unlimitedly differentiable.

I may add that in any region of the plane there will be an infinite or, more specifically, a non-enumerable set of points, through each of which passes an infinite number of lines of convergence. If a closed curve is given it will be possible to approximate as closely as desired to this curve by a rectilinear polygon, along whose entire length the series converges and defines a continuous function. Integration around such a polygon gives for the value of the integral the product of $2i\pi$ into the sum of the residues of those fractions whose poles lie in the interior of the polygon. Finally, if we take for axes of x and y two perpendicular lines of continuity of $\phi(z)$, all the lines of uniform continuity which meet at their intersection will give a common value for $\phi'(z)$, and the real and imaginary parts of $\phi(z)$ will satisfy *Laplace's* equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Thus we have in $\phi(z)$ a species of quasi-monogenic function. One question *Borel* has as yet found himself unable to resolve. If $\phi(z) = 0$ along a finite portion of any line, will the series in consequence vanish identically? If this question be answered in the affirmative, the analogy with an ordinary analytic function will be still more complete.

Let us now return to the case in which two or more functions with natural boundaries are defined by (7). The lines of continuity just described form an infinitely thick mesh-work along which $\phi(z)$ can be carried continuously from the one analytic function into the others. Suppose again that the origin is not a point of condensation of the poles a_n so that $\phi(z)$ can be expanded at the origin into a Maclaurin's series $\Sigma c_i z^i$. Now if a ray is drawn from the origin through the pole a_n and the portion of the ray between a_n and ∞ is retained as a cut, the mth term of (7) can be expanded into a series of polynomials

$$-\frac{A_m}{a_m}\sum_{n=1}^{\infty} G_n\left(\frac{z}{a_m}\right)$$

which converges over the plane so cut. The series (7) can therefore be resolved into a double series

$$\sum_m \sum_n \frac{-A_m}{a_m} G_n\left(\frac{z}{a_m}\right),$$

and this expression will be valid on an infinity of rays from the origin which do not pass through any of the poles. Since, moreover, the poles are an enumerable set of points, these rays will be infinitely dense between any two arguments which may be taken. By further conditioning the A_n, *Borel* is able to rearrange the terms of the double series so as to form a series of polynomials $\sum_n Q_n(z)$, in which

$$Q_n(z) = -\sum_{m=1}^{\infty}\frac{A_m}{a_m} G_n\left(\frac{z}{a_m}\right),$$

and in this way he obtains a series of polynomials which is convergent on a dense set of rays through the origin.

It also appears that the polynomial series $\Sigma Q_n(z)$ can be formed directly from $\Sigma c_i z^i$ without the intervention of (7). When, therefore a Maclaurin's series is given which corresponds to such an expression (7) as is now under discussion, the continuation of the function can be made along the above set of rays. Now the rays cut any curve upon which either (7) or $\Sigma Q_n(z)$ defines a continuous function in a set of points everywhere dense. The value of the function along the entire curve therefore depends only upon the coefficients c_i; *i. e.*, upon the value of the function and its derivatives at the origin. It is shown, moreover, that any point of the plane which is not a point of condensation of the poles a_n may be converted by transformation of axes into such an origin. Finally, *Borel* gives a case in which the poles may be everywhere dense over the entire plane, so that the function defined by (7) is nowhere analytic, and yet its value is determined along the lines of continuity by the value of the function and its derivatives at the origin. Here then is a class of non-analytic functions sharing a most fundamental property in common with the analytic functions! Is it not then possible, as *Borel* surmises, that there is a wider theory of functions, similar in its outlines to the theory of analytic functions and embracing this as a special case? If so, the conceptions of *Weierstrass* and of *Meray* are capable of generalization.

Part II. On Algebraic Continued Fractions.

Lecture 5. *Padé's Table of Approximants and its Applications.*

Both historically and prospectively one of the most suggestive and important methods of investigating divergent power series is by the instrumentality of algebraic continued fractions. It is for this reason that I have ventured to combine in a single course of lectures two subjects apparently so unrelated as divergent series and continued fractions. I shall not, however, confine myself to the consideration of the latter subject solely with reference to the theory of divergent series. It is rather my purpose to give some account of the present status of the theory of algebraic continued fractions. At the close of the next lecture a bibliography of memoirs connected with the subject is appended, to which reference is made throughout this lecture and the next by means of numbers enclosed in square brackets.

By the term *algebraic* continued fraction is understood, in distinction from a continued fraction with numerical elements, one in which the elements — *i. e.*, the partial numerators and denominators — are functions of a single variable x or of several variables [16, *a*, p. 4]. Although the term algebraic does not seem to me to be fortunately chosen, I shall nevertheless accept it and use it to indicate the class of continued fractions which it is proposed to consider here.

The first foundations of a theory of continued fractions were laid by *Euler*, who early employed them [1, *a*] to derive from a given power series

$$k_0 + k_1x + k_2x^2 + \cdots \qquad (k_n \neq 0)$$

a continued fraction of the form

$$(1) \qquad \frac{1}{b_0} + \frac{a_1x}{b_1x + d_1} + \frac{a_2x}{b_2x + d_2} + \cdots.$$

A second form, also introduced by *Euler** [46, *a*] is the more familiar one

$$(2) \qquad \frac{a_0}{1} + \frac{a_1x}{1} + \frac{a_2x}{1} + \frac{a_3x}{1} + \cdots,$$

which was later used by *Gauss* [34] in his celebrated continued fraction for $F(\alpha, \beta, \gamma, x)/F(\alpha, \beta + 1, \gamma + 1, x)$. From this time on still other forms were discovered so that it became impossible to speak of a unique development of a function into a continued fraction. Among these forms may be especially mentioned the continued fraction

$$(3') \qquad \frac{1}{a_1x + b_1} + \frac{1}{a_2x + b_2} + \frac{1}{a_3x + b_3} + \cdots,$$

used by *Heine*, *Tchebychef*, and others in approximating to series in descending powers of x. By the substitution of $1/x$ for x and a simple reduction this can be transformed, after the omission of a factor x, into

$$(3) \qquad \frac{1}{a_1 + b_1x} + \frac{x^2}{a_2 + b_2x} + \frac{x^2}{a_3x + b_3} + \cdots.$$

The reason for this variety of form and for the occurrence, in

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* Padé in his thesis (p. 38) traces it back to Lambert [2, *a*] and Lagrange, but Euler's use is earlier still.

particular, of the three types just given is discussed by *Padé* in his thesis [16, a]. As this thesis is the foundation for a systematic study of continued fractions, it will be necessary to give a recapitulation of its chief results.

Let

$$S(x) = c_0 + c_1 x + c_2 x^2 + \cdots \qquad (c_0 = 1) \tag{4}$$

be any given power series, whether convergent or divergent. If $N_p(x)/D_q(x)$ denotes an arbitrary rational fraction in which the numerator and denominator are of the pth and qth degrees respectively, there will be $p + q + 1$ parameters which can be made to satisfy an equal number of conditions. Let them be so determined that the expansion of N_p/D_q in ascending powers of x shall agree with (4) for as great a number of terms as possible. In general, we can equate to zero the first $p + q + 1$ coefficients of the expansion of $D_q S(x) - N_p$ in ascending powers of x, and no more. Hence, unless N_p and D_q have a common divisor, the series for N_p/D_q agrees with (4) for an equal number of terms, and the approximation is said to be of the $(p + q + 1)$th order. In exceptional cases the order of the approximation may be either greater or less. *Padé* examines these exceptional cases and proves strictly that among all the rational fractions in which the degrees of numerator and denominator do not exceed p and q respectively, there is, taken in its lowest terms, one and only one, the expansion of which in a series will agree with (4) for a greater number of terms than any other. Such a rational fraction I shall term an *approximant* of the given series.

The existence of approximants was, of course, well known before *Padé*, but no systematic examination of them had been made except by *Frobenius* [13], who determined the important relations which normally exist between them. *Padé* goes further, and arranges the approximants, expressed each in its lowest terms, into a table of double entry :

	$p=0$	$p=1$	$p=2$	$\ldots$
$q=0$	$\dfrac{N_{00}}{D_{00}} = c_0$	$\dfrac{N_{10}}{D_{10}} = c_0 + c_1x$	$\dfrac{N_{20}}{D_{20}} = c_0 + c_1x + c_2x^2$	$\ldots$
$q=1$	$\dfrac{N_{01}}{D_{01}}$	$\dfrac{N_{11}}{D_{11}}$	$\dfrac{N_{21}}{D_{21}}$	$\ldots$
$q=2$	$\dfrac{N_{02}}{D_{02}}$	$\dfrac{N_{12}}{D_{12}}$	$\dfrac{N_{22}}{D_{22}}$	$\ldots$
$q=3$	. . . . .	. . . . .	. . . . .	. .

When the order of approximation of a rational fraction, taken in its lowest terms, is exactly equal to the sum of the degrees of numerator and denominator, increased by 1, the fraction will be found once and only once in the table. If, conversely, a fraction N_{pq}/D_{pq} occurs but once in the table, the numerator and denominator are of degree p and q respectively, and the order of the approximation which the fraction affords is exactly $p+q+1$. The approximant is then said by *Padé* to be normal. We shall also call the table *normal* when it consists only of normal fractions, or, in other words, when no approximant occurs more than once in the table.

Obviously all approximants which lie upon a line perpendicular to the principal diagonal of the table correspond to the same value of $p+q+1$. Hence in a normal table they approximate to (4) in equal degree, and accordingly may be said to be *equally advanced* in the table. If $p+q+1$ increases in passing from one fraction to another, the latter is the more advanced.

Two approximants will be called *contiguous* if the squares of the table in which they are contained have either an edge or a vertex in common.

Consider now a normal table, and take any succession of approximants, beginning with one upon the border of the table and passing always from one approximant to another which is contiguous to it but more advanced. *Padé* shows that any such sequence of approximants makes a continued fraction of which the approximants are the successive convergents. * Thus a countless manifold of continued fractions can be formed, any one of which through its convergents gives the initial series to any required number of terms and hence defines the series and table uniquely. In all of *Padé's* continued fractions the partial numerators are monomials in x.

The continued fraction is called regular when its partial numerators are all of the same degree and likewise its denominators, certain specified irregularities being admitted in the first one or two partial fractions. These irregularities disappear when the continued fraction, as is most usual, commences with the corner element of the table. (Cf. the continued fractions (2) and (3).)

In a normal table a regular continued fraction can be obtained in any one of three ways. If we take for the convergents the approximants which fill a horizontal or vertical line, a continued fraction is obtained which — except for the irregularity permitted at the outset — is of the form (1) given above. If the approximants lie upon the principal diagonal or any parallel line, the continued fraction is of type (3). Lastly, if the convergents lie upon a stair-like line, proceeding alternately one term horizontally to the right and one term vertically downward, the continued fraction is of the familiar form (2).

When a table is not normal, the approximants which are identical with one another are shown by *Padé* to fill always a square, the edges of which are parallel to the borders of the table. When the square contains $(n + 1)^2$ elements, the irregularity may be said to be of the nth order. The vertical, horizontal, diagonal and stair-like lines give regular continued fractions as before, unless they cut into one or more of these square blocks of equal approximants. When this happens, certain irregularities appear in the continued fraction which give rise to various difficulties in the consideration of matters of convergence and other questions.

On this account it is natural to inquire first whether the continued fraction has or has not a normal character. If it has, the

* This is also tacitly implied in the relations given by Frobenius [13, p. 5].

existence of the three regular types of continued fractions is assured. The necessary and sufficient condition that the table shall be normal is that no one of the determinants

$$c_{\alpha\beta} = \begin{vmatrix} c_{\alpha-\beta+1} & c_{\alpha-\beta+2} & \cdots & c_{\alpha} \\ c_{\alpha-\beta+2} & c_{\alpha-\beta+3} & \cdots & c_{\alpha+1} \\ . & . & . & . \\ c_{\alpha} & c_{\alpha+1} & \cdots & c_{\alpha+\beta-1} \end{vmatrix} \qquad \begin{matrix}(\alpha,\ \beta \geqq 0; \\ \quad c_i = 0 \text{ if } i < 0)\end{matrix}$$

shall vanish [16, a, p. 35]. It will be noticed that the determinants are of the same sort as those which play so conspicuous a rôle in *Hadamard's* discussion of series representing functions with polar singularities.

So far as I am aware, the normal character of the table has been established as yet only in the following cases: (1) for the exponential series [37] and for $(1+x)^m$ when m is not an integer [35, d],† by *Padé*; and (2) for the series of *Stieltjes*, by myself [45].

The construction of *Padé's* table leads at once to a number of new and important questions. The numerators and the denominators of the approximants constitute groups of polynomials which it is only natural to expect will be characterized by common or kindred properties. The table then affords a suitable basis for the classification of polynomials. Thus, for example, the polynomials of

† At least half of the table for $F(a, 1, \gamma, x)$ has a normal character. This was proved incidentally in my thesis [76] by showing that the remainders corresponding to approximants on or above the diagonal of the table were all distinct. The method of conformal representation was there employed, but the same fact can also be demonstrated very simply by means of Gauss' *relationes inter contiguas* (formulas (19) and (20) of [34]). The approximants in the other half of my table (Cf. [76], p. 44) were constructed on different principles from Padé's, the approximation being made simultaneously with reference to *two* points, $x = 0$ and $x = \infty$, but the resulting continued fractions were of the same form as Padé's. It is noteworthy that the *relationes inter contiguas* lead to such a table rather than to the one of Padé's construction.

In the case of $F(-m, 1, 1, -x) \equiv (1+x)^m$ the half of Padé's table below the diagonal is also normal, since the reciprocal of the approximants in the lower half are the approximants in the upper half of the table for

$$F(m, 1, 1, -x) = (1+x)^{-m}.$$

The normal character of the table for e^x then follows since $e^x = \lim\limits_{g=\infty} F(g, 1, 1, x/g)$.

Legendre and similar polynomials are obtained from the series for $\log(1-x)/(1+x)$, while the numerators and denominators of the approximants for $(1+x)^m$ are the hypergeometric polynomials $F(-\mu, -\nu \pm m, -\mu+\nu, -x)$, in which μ and ν are integers, or the so-called polynomials of *Jacobi* [65]. In these, as in numerous other cases, the denominators of the convergents and the remainder-functions,* formed by multiplying each denominator into the corresponding remainder, are solutions of homogeneous linear differential equations of the 2nd order which have a common group, and the relations of recurrence between three successive denominators or remainder-functions are the *relationes inter contiguas* of *Gauss* and *Riemann*. (See in particular, [75, *d*] and [76].) The further study of such groups of polynomials will probably bring to light new and important properties. The position of the roots of the denominators should especially be ascertained, because the distribution of these roots has an intimate connection with the form of the region of convergence of the continued fraction and oftentimes also with the position and character of the function which the continued fraction defines.

Probably the most fundamental question concerning *Padé's* table is that of the convergence of the various classes of continued fractions or lines of approximants. The first investigation of the convergence of an algebraic continued fraction was made by *Riemann* [18] in 1863, followed by *Thomé* [19] a few years later.† Both writers investigated the continued fraction of *Gauss* by rather painful methods, not based absolutely upon the algorithm of the continued fraction but upon extraneous considerations. This is not surprising, for there were at that time no general criteria for the convergence of continued fractions with *complex* elements, and even now the number is astonishingly small.

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* In at least half of the table. See the preceding footnote.

† As Riemann's work appeared posthumously, Thomé's has the priority of publication (1866) but was itself preceded by Worpitzky's dissertation, to which reference is made in a subsequent footnote.

The two principal criteria for convergence correspond to the familiar tests for the convergence of a real continued fraction

$$(5)\qquad \frac{\mu_1}{\lambda_1} + \frac{\mu_2}{\lambda_2} + \frac{\mu_3}{\lambda_3} + \cdots,$$

in which either (1) all the elements are positive or (2) the partial denominators λ_i are positive and the partial numerators μ_i are negative. The latter class of real continued fractions is known to converge if $\lambda_i \geqq 1 - \mu_i$. *Pringsheim* [29] has shown that when the elements are complex, the condition $|\lambda_i| \geqq 1 + |\mu_i|$ is still sufficient for convergence. If, furthermore, the continued fraction has the customary normal form in which $\mu_n = 1$, the condition may be replaced by the less restrictive one [29, p. 320],

$$\left|\frac{1}{\lambda_{2n-1}}\right| + \left|\frac{1}{\lambda_{2n}}\right| \leqq 1.$$

The necessary and sufficient condition for the convergence of the first class of real continued fractions can be most easily expressed after it has been reduced to the form

$$\frac{1}{\lambda'_1} + \frac{1}{\lambda'_2} + \frac{1}{\lambda'_3} + \cdots \qquad (\lambda'_n > 0).$$

If then $\Sigma\lambda'_n$ is divergent, the continued fraction converges, while it diverges if $\Sigma\lambda'_n$ is convergent.* But in the latter case limits exist for the even and the odd convergents when considered separately. This result is included in the following theorem which I gave in the *Transactions* of 1901 for continued fractions with complex elements [31]: If in

$$\frac{1}{\alpha_1 + i\beta_1} + \frac{1}{\alpha_2 + i\beta_2} + \frac{1}{\alpha_3 + i\beta_3} + \cdots$$

the elements α_i have all the same sign and the β_i are alternately positive and negative,† the continued fraction will converge if $\Sigma|\alpha_n + i\beta_n|$ is divergent; on the other hand, if $\Sigma|\alpha_n + i\beta_n|$ is

*Seidel, *Habilitationsschrift*, 1846, and Stern, *Journ. für Math.*, vol. 37 (1848), p. 269.

† Zero values are permissible for either a_i or β_i.

convergent and *either* the α_i *or* the β_i fulfill the condition just stated concerning their signs, the even and the odd convergents have separate limits.

The most general criterion for the convergence of

$$\frac{b_1}{1} + \frac{b_2}{1} + \frac{b_3}{1} + \cdots$$

(b_i real or complex) seems to be the one which I gave in October, 1901 [32, b, § 5].

Two remarks of a general nature concerning the convergence of algebraic continued fractions may be of interest. In the consideration of numerical continued fractions a difficulty frequently encountered is that the removal of a finite number of partial fractions μ_i/λ_i at the beginning of (5) may affect its convergence or divergence. The convergence is therefore not determined solely by the ultimate character of the continued fraction, as is true of a series. *Pringsheim* [29] has proposed to call the convergence *unconditional* when it is not destroyed by the removal of the first n partial fractions of (5). The difficulties due to conditional convergence usually disappear from consideration in treating algebraic continued fractions. For let N_n/D_n now denote the nth convergent. If after the removal of the first n partial fractions the continued fraction converges uniformly in a given region and accordingly represents a function $F(z)$ which is holomorphic within the region, then after the restoration of the initial terms the continued fraction will define the function

$$\frac{N_n + F(z)N_{n-1}}{D_n + F(z)D_{n-1}}, \tag{6}$$

which must be either holomorphic or meromorphic within the given region [32, a or c]. An exception occurs only if the denominator of (6) vanishes identically in the region. This is impossible for the second and third types of continued fractions, since the development of a rational fraction $-D_n/D_{n-1}$ in either type (2) or (3) consists of a finite number of terms, whereas the development of $F(z)$, by hypothesis, continues indefinitely.

The second remark relating to convergence is that its discussion for a continued fraction is usually reduced to the corresponding question for an infinite series. The succession of convergents

$$\frac{N_n}{D_n}, \quad \frac{N_{n+1}}{D_{n+1}}, \quad \frac{N_{n+2}}{D_{n+2}}, \quad \cdots$$

is, in fact, obviously equivalent to the series

$$\frac{N_n}{D_n} + \left(\frac{N_{n+1}}{D_{n+1}} - \frac{N_n}{D_n}\right) + \left(\frac{N_{n+2}}{D_{n+2}} - \frac{N_{n+1}}{D_{n+1}}\right) + \cdots.$$

But the latter by means of the familiar relations connecting the denominators or the numerators of three consecutive convergents may be reduced to the form:

$$(7) \qquad \frac{N_n}{D_n} + (-1)^n \mu_1 \mu_2 \cdots \mu_n \left(\frac{\mu_{n+1}}{D_n D_{n+1}} - \frac{\mu_{n+1}\mu_{n+2}}{D_{n+1}D_{n+2}} + \frac{\mu_{n+1}\mu_{n+2}\mu_{n+3}}{D_{n+2}D_{n+3}} - \cdots \right).$$

We turn now from these general considerations to the questions of convergence connected with *Padé's* table. Under what conditions will the various lines of approximants converge; in particular, the three standard types of continued fractions obtained by following (1) the horizontal or vertical lines, (2) the stair-like lines, and (3) the diagonal lines? When they converge simultaneously, have they a common limit? If not, what are the mutual relations between the functions which they define? What is the form of the region of convergence?

These and other questions press upon us, and are of great interest. A complete investigation has been made only for the exponential series. *Padé* [37, *a*] finds that when p/q for any succession of approximants N_{pq}/D_{pq} converges to a value ω, the approximants converge toward the generating function e^x for all values of x. Furthermore, the numerators and denominators separately converge, the former to the limit $e^{\omega x/\omega+1}$, the latter to $e^{-x/\omega+1}$. This smooth result is not, however, a typical one, not even for entire functions. It is due at least in part to the fact that e^x is an entire function without zeros. This will be apparent after an examination has been made of the vertical and horizontal lines of *Padé's* table, which we now proceed to consider.

It is obvious that the first $p + q + 1$ terms of the given series (4) determine an equal number of terms of the series for its reciprocal. If, therefore, in the table each approximant is replaced by its reciprocal and the rows and columns are then interchanged, we shall obtain the table for the reciprocal series. The problems presented by the horizontal and vertical lines of the table are consequently of essentially the same character, and our attention may be confined henceforth to the horizontal lines alone.

By the interchange just described the zeros and poles of (4) become the poles and zeros respectively of the reciprocal function. In the case of the exponential function the reciprocal series has the same character as the initial series, each defining an entire function without zeros, and the simultaneous convergence of rows and columns for all values of x was therefore to be expected; but in general this does not hold.

In investigating the convergence of the horizontal lines the first case to be considered is naturally that of a function having a number of poles and no other singularities within a prescribed distance of the origin. It is just this case that *Montessus* [33, a] has examined very recently. Some of you may recall that four years ago in the Cambridge colloquium Professor *Osgood** took *Hadamard's* thesis † as the basis of one of his lectures. This notable thesis is devoted chiefly to series defining functions with polar singularities. *Montessus* builds upon this thesis and applies it to a table possessing a normal character. Although his proof is subject to this limitation, his conclusion is nevertheless valid when the table is not normal, as I shall show in some subsequent paper.

The first horizontal row of the table scarcely needs consideration, for it consists of the polynomials obtained by taking successively 1, 2, 3, $\cdots$ terms of the series. Consequently the continued fraction obtained from the first row,

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**Bull. of the Amer. Math. Soc.*, vol. 5, pp. 74–78.

† *Journ. de Math.*, ser. 4, vol. 8 (1892).

$$\frac{a_0}{1} - \frac{a_1 x}{a_1 x + a_0} - \frac{a_0 a_2 x}{a_2 x + a_1} - \frac{a_1 a_3 x}{a_3 x + a_2} - \cdots,$$

is identical with the series, and its region of convergence is a circle.

Let R_1 be the radius of this circle and q_1 the number of poles of (4) which lie upon its circumference. Suppose also that the next group of poles, q_2 in number, lie upon a circle of radius R_2, having its center in the origin; that q_3 poles lie upon the next circle (R_3); and so on indefinitely or until a circle is reached which contains a non-polar singularity. *Hadamard* (*l. c.*, § 18) has proved that the denominators D_{pq} of the approximants of the $(q_1 + 1)$th row, of the $(q_1 + q_2 + 1)$th row, and so on, approach a limiting form as we advance in the row, and that the limiting polynomials give the positions of the first q_1, $q_1 + q_2$, $\cdots$ poles respectively. Thus if, for example,

$$D_{p, q_1+1} = 1 + B_{pq_1}^{(1)} x + B_{pq_1}^{(2)} x^2 + \cdots + B_{pq_1}^{(q_1)} x^{q_1}$$

and

$$\lim_{p=\infty} B_{pq_1}^{(i)} = B_i,$$

the first group of poles are the roots of the polynomial $1 + B_1 x + \cdots B_{q_1} x^{q_1}$. Using this result of *Hadamard*, *Montessus* shows that in a normal table the approximants of the (q_1+1)th row converge at every point within the circle (R_2) — excepting, of course, at the q_1 poles — but not without this circle; that the approximants of the $(q_1 + q_2 + 1)$th row converge similarly within the circle (R_3) except at the included $q_1 + q_2$ poles; and so on.

In proving this *Montessus* makes use of an idea advanced in *Padé's* thesis ([16, *a*, p. 51], or [24]) which, though applicable in the present case, is possibly somewhat misleading. In *Padé's* continued fractions the partial numerators μ_i are monomials in x. This is due to the fact that there is a steady increase in the order of the approximation afforded by the successive convergents at $x = 0$. Consider now the series (7), and let T denote the region or set of points in the x-plane for which $|D_n|$, from and after some value of n, has both an upper and a lower limit. Then in T the continued fraction will converge or diverge simultaneously with the power series,

$$\mu_{n+1} - \mu_{n+1}\mu_{n+2} + \mu_{n+1}\mu_{n+2}\mu_{n+3} \cdots. \tag{8}$$

Call C the circle of convergence of (8). At all points of T within C the continued fraction converges, and at all exterior points of T it diverges. On this account *Padé* proposes to call C the "circle of convergence" of the continued fraction. In the case which we have just been discussing this concept is applicable because of the existence of limiting forms for the denominators of the rows considered. The region T comprises the entire finite plane with the exception of the roots of the limiting form, and the circle C is successively identical with (R_2), (R_3), ... Thus, as we pass down the rows of the table, we obtain continued fractions having an increasing region of convergence.

In introducing the term circle of convergence for a continued fraction *Padé* ignores all points not included in T. Call the excluded point-set T'. If $|D_n|$ increases indefinitely with increasing n over the whole or a part of T' the series (7) may converge, and this may happen even though (8) is a divergent series.* The term circle of convergence is therefore an inappropriate one, although the considerations upon which it is based are useful.

Nothing more of account seems to be known concerning the the convergence of the horizontal and vertical lines.† The more common and important continued fractions are obtained from diagonal and stair-like paths through the table. In many familiar continued fractions of the second type,

$$\frac{a_0}{1} + \frac{a_1 x}{1} + \frac{a_2 x}{1} + \frac{a_3 x}{1} + \cdots, \tag{2}$$

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*The coefficients in the continued fraction of Stieltjes (discussed later in the lecture) can be easily so determined as to give a case of this sort, the region of convergence of (7) being the entire plane with the exception of the negative half of the real axis. We suppose, with Padé that the absolute term of D_n is taken equal to 1.

†It is perhaps worth noting that the coefficients in the first type of continued fractions can not be selected arbitrarily if it is to be connected with such a table as Padé constructs. In the other two types the coefficients are entirely arbitrary.

a_n with increasing n approaches a limit, as for instance in the continued fraction of *Gauss* where $\lim a_n = -\frac{1}{4}$. The significance of the existence of such a limit I first pointed out for a comprehensive class of cases in 1901 [32, *a*], and since then I have shown by simpler methods [32, *c*] that the result is perfectly general. Let $\lim a_n = k$. Then the continued fraction converges, save at isolated points, over the entire plane of x with the exception of the whole or a part of a cut drawn from $x = -1/4k$ to $x = \infty$ in a direction which is a continuation of the vector from $x = 0$ to $x = -1/4k$. Within the plane thus cut the limit of the continued fraction is holomorphic except at the isolated points which (if they exist) are poles. When there is no limit for a_n but only an upper limit U for its modulus, the continued fraction (see [32, *b*]) is meromorphic or holomorphic at least within a circle of radius $1/4U$ having its center in the origin.* A special case is that in which $\lim a_n = 0$. The limit of the continued fraction is then a function which is holomorphic or meromorphic over the entire plane. A comparison of this last result with that of *Montessus* shows that a much greater region of convergence has now been obtained. This is doubtless, in general, a reason for preferring the second and third types of continued fractions to the first.

As another illustration of the second type of continued fraction I shall choose the celebrated continued fraction of *Stieltjes* [26, *a*]. In this each coefficient a_n is positive. By putting $x = 1/z$ in (2), the continued fraction, after dropping a factor z, can be thrown into the form

$$\frac{1}{a_1'z} + \frac{1}{a_2'} + \frac{1}{a_3'z} + \frac{1}{a_4'} + \frac{1}{a_5'z} + \cdots, \qquad (a_n' > 0), \tag{8'}$$

which is the form preferred by *Stieltjes*. To every such continued fraction there corresponds a series

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* A demonstration of this property within the circle $(1/4U)$ has been previously given in a dissertation by *Worpitzky* [18 bis], which has come to my notice for the first time during the examination of the proof-sheets of these lectures. This dissertation bears the date 1865 and appears to be the earliest published memoir treating of the convergence of algebraic continued fractions.

(9) $$\frac{c_0}{z} - \frac{c_1}{z^2} + \frac{c_2}{z^3} - \frac{c_3}{z^4} + \cdots$$

for which

(10)
$$A_n \equiv \begin{vmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-1} \\ c_1 & c_2 & c_3 & \cdots & c_n \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ c_{n-1} & c_n & c_{n+1} & \cdots & c_{2n-2} \end{vmatrix} > 0,$$

$$B_n \equiv \begin{vmatrix} c_1 & c_2 & c_3 & \cdots & c_n \\ c_2 & c_3 & c_4 & \cdots & c_{n+1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ c_n & c_{n+1} & c_{n+2} & \cdots & c_{2n-1} \end{vmatrix} > 0.$$

The correspondence is also a reciprocal one. To every series which fulfills these conditions there corresponds a continued fraction of the above type with positive coefficients. From the conditions (10) it follows that $c_i > 0$ and that $c_n/c_{n-1} > c_{n-1}/c_{n-2}$. If, therefore, the increasing ratio c_n/c_{n-1} has a finite limit, the series is convergent. On the other hand, if it increases without limit, the series is divergent.

In investigating the convergence of the continued fraction the especial skill of *Stieltjes* was shown. From the relation connecting three consecutive denominators (numerators) of the convergents it was shown easily that either set of alternate denominators (numerators) made a Sturm's series, whence it follows that all the roots of the denominators (numerators) lie upon the negative half of the real axis of z. This leads naturally to the conjecture that the region of convergence will be the entire plane of z with the exception of the whole or a part of the negative half axis, and that the functional limit will have no zeros exterior to this half of the axis. First the convergence is examined when z is real and positive. The criterion of *Seidel*, cited previously in this lecture, then applies. If, namely, $\Sigma a_n'$ is divergent, the continued fraction will converge along the positive axis, while if $\Sigma a_n'$ is convergent, the two sets of alternate convergents have limits which are distinct. The conclusion is next extended by *Stieltjes* to the half of the complex plane for which the real part of z is positive.

This brings him to the difficult part of his problem, the extension of the result to the other half-plane but with exclusion of the real axis. Here, particularly, *Stieltjes* [26, a, § 30] shows his ingenuity. He overcomes the difficulty by establishing first a preliminary theorem which is of vital importance for sequences of polynomials or rational fractions. The theorem is as follows. Let $f_1(z), f_2(z), \cdots$ be a sequence of functions which are holomorphic within a given region T, and suppose that $\Sigma_{n=1}^{\infty} f_n(z)$ is uniformly convergent in some part T'' of the interior of T. Then if $f_1(z) + f_2(z) + \cdots + f_n(z)$ has an upper limit *independent of* n in any arbitrary region T' which includes T'' but is contained in the interior of T, the series $\Sigma f_n(z)$ will converge uniformly in T' and therefore has as its limit a function which is holomorphic over the whole interior of T*.

In the application of this theorem *Stieltjes* decomposes each convergent $N_n(z)/D_n(z)$ into partial fractions,

$$\frac{M_1}{z + a_1} + \frac{M_2}{z + a_2} + \cdots + \frac{M_r}{z + a_r}$$

in which

$$M_i > 0, \qquad a_i \geqq 0, \qquad \sum_{i=1}^{r} M_i = c_0.$$

From this it follows that N_n/D_n has an upper limit independent of n in any closed region of the plane which does not contain a point of the negative half-axis. If now in either the sequence of the odd convergents or of the even convergents we denote the nth term of the sequence by N_n/D_n and place

$$f_1(z) + f_2(z) + \cdots + f_n(z) = \frac{N_n(z)}{D_n(z)},$$

the series $\Sigma_{n=1}^{\infty} f_n(z)$ converges uniformly in any portion of the plane

* For a further extension of this line of work, see Osgood, *Annals of Math.*, ser. 2, vol. 3 (1901), p. 25.

for which the real part of z is positive. All the conditions of the lemma of *Stieltjes* are now fulfilled, and the region of convergence may be extended over the entire plane with the exception of the negative half-axis.

On account of the uniform character of the convergence the limit of either sequence is holomorphic at every point exterior to the negative half-axis. When $\Sigma a_n'$ is divergent, the two limits coincide and the continued fraction itself is convergent. On the other hand, if $\Sigma a_n'$ is convergent, the two limits are distinct. *Stieltjes* shows also that in the latter case the numerators and the denominators of either sequence converge to holomorphic functions $p(z)$, $q(z)$ of *genre* 0, and the two pairs of functions are connected by the equation

$$q(z)p_1(z) - q_1(z)p(z) = 1,$$

which corresponds to the familiar relation

$$D_{2n}N_{2n-1} - D_{2n-1}N_{2n} = 1.$$

A more direct method [31] of demonstrating the convergence results of *Stieltjes* is by an extension * of the criterion previously cited for the convergence of continued fractions in which the partial fractions $1/(\alpha_n + i\beta_n)$ have an α_n of constant sign and a β_n of alternating sign. The introduction of the lemma of *Stieltjes* is consequently unnecessary, but I wish nevertheless to emphasize its fundamental importance. Other notable results which it will be impossible to reproduce here are also contained in his splendid memoir.

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* If namely, $\Sigma \mid \alpha_n + i\beta_n \mid$ is divergent and the condition concerning the signs either of the α_n or of the β_n is fulfilled, the continued fraction will converge provided $\mid \alpha_n \mid / \mid \beta_n \mid$ has a lower or an upper limit respectively. Put now $z = w^2$ in (8′) so that it becomes

$$\frac{1}{w}\left(\frac{1}{a_1'w+}\,\frac{1}{a_2'w+}\,\frac{1}{a_3'w+}\cdots\right).$$

When $\Sigma a_n'$ is divergent, this falls under the extended criterion if we put $a_n'w = \alpha_n + i\beta_n$, except when z is negative. On the other hand, when $\Sigma a_n'$ is convergent, the criterion applies without extension directly to (8′). In either case the uniform character of the convergence follows with the addition of a few lines.

It is interesting to bring this work of *Stieltjes* into connection with the table of *Padé* [44]. The odd convergents of the continued fraction of *Stieltjes* fill the principal diagonal of *Padé's* table, thus constituting by themselves a continued fraction of the third type, and the even convergents fill the parallel file immediately below, forming a similar continued fraction. The significance of distinct limits for the two sets of convergents is thus made clearer.

The series of *Stieltjes* has perhaps its greatest interest when treated in connection with the theory of divergent series. Although the continued fraction always converges if the series does, the converse is not true. For when the series (9) is divergent, two cases are to be distinguished according as $\Sigma a_n'$ is divergent or convergent. In the former case the continued fraction gives one and only one functional equivalent of the divergent series. *Le Roy* states,* though without proof, that the function furnished is identical with the one obtained from the series by the method of *Borel*, whenever the latter method is applicable also. When $\Sigma a_n'$ is convergent, two different functions are obtained from the continued fraction, the one through the even and the other through the odd convergents. And if there are two such functions which correspond to the series, there must be an infinite number. For if $\phi(x)$ and $\psi(x)$, when expanded formally, give rise to the same divergent series, so also will

$$\frac{\phi(x) + c\,\psi(x)}{1 + c},$$

in which c denotes an arbitrary constant. Special properties, however, attach themselves to the two functions picked out by the continued fraction of *Stieltjes*, upon which we can not linger here.

This result of *Stieltjes* seems to me to be especially significant, since it indicates a division of divergent series into at least two classes, the one class containing the series for which there is properly a single functional equivalent and the other comprising the

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* *Loc. cit.*, p. 428.

series which correspond to sets of functions. It is, of course, just possible that this distinction may be due to the nature of the algorithm employed in deriving the functional equivalent of the series, but it is far more probable that the difference is intrinsic and independent of the particular algorithm. If this view be correct, the method of *Borel* which gives a single functional equivalent, is limited in its application to series of the first class.

An extension of the work of *Stieltjes* has been sought in two distinct directions by modification of the conditions imposed upon his series. *Borel* [43] so modifies them as to make the series (when divergent) fulfill the requirement imposed in lecture 2 and permit of manipulation precisely as a convergent series. In the last number of the *Transactions* * [45] I began a study of series which are subject to only one of the two restrictions expressed in the inequalities (10), but was obliged to bring the work to a hurried close to prepare these lectures. In the main, the corresponding continued fractions have the same properties as the continued fraction of *Stieltjes*, but a considerable difference is shown in regard to convergence. Though the roots of the numerators and denominators of the convergents are still real, they are no longer confined to the negative half of the real axis, and may be infinitely thick along the entire extent of the axis. In certain cases the continued fraction converges in the interior of the positive and negative half planes, defining in each an analytic function which has the real axis as a natural boundary. The continued fraction therefore effects the continuation of an analytic function across such a boundary, and gives a natural instance of such a continuation † — natural in distinction from artificial examples set up with the express object of showing the possibility of a unique, non-analytic extension.

Padé [17, *a*] has suggested the foundation of a theory of diver-

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* July, 1903.

† Earlier instances of a natural continuation are also to be found, as, for example, that afforded by

$$\sum_m \sum_{m'} \frac{1}{(m + m'\omega)^4}$$

across the axis of reals.

gent series upon the continued fractions of his table. The difficulties of carrying out the suggestion are undoubtedly very great and have been pointed out by *Borel.** Not only must the convergence of the principal lines of approximants and the agreement of their limits be investigated, but the combination of two or more divergent series must also be considered. It is not enough to point out, as does *Padé*, that the approximants of given order for any two series, whether divergent or convergent, determine uniquely the approximants of the same or lower order for the sum- and product-series. For practical application of the theory it must be proved also that the function defined by the table corresponding to the new series is, under suitable limitations, the sum or product of the functions defined by the given divergent series. But great as are the difficulties of such an investigation, even for restricted classes of series, the reward will probably be correspondingly great.

So far as it has been yet investigated, the diagonal type of continued fractions seems to have accomplished nearly everthing that can fairly be asked of a sequence of rational fractions. Not only does it afford a convenient and natural algorithm for computing the successive fractions, but in every known instance the region of convergence is practically the maximum for a series of one valued functions. The continued fraction of *Halphen* [21, *a*], so frequently cited as an instance of a continued fraction which diverges though the corresponding series converges, might appear at first sight to be an exception. But this divergence occurs only at special points. In fact, the continued fraction not only converges at the center of the circle of convergence for the series, but, as *Halphen* himself says, continues the function over the entire plane with the exception of certain portions of a line or curve. If then, continued fractions offer such advantages for known series and classes of functions, is it too much to expect that in the future they will throw a powerful searchlight upon the continuation of analytic functions and the theory of divergent series?

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* *Les Séries divergentes*, p. 60.

LECTURE 6. *The Generalization of the Continued Fraction.*

In the last lecture the algebraic continued fraction was presented under the form of a series of approximants for a given function. An immediate generalization of this conception can be obtained either by increasing the number of points at which an approximation is sought or by requiring a simultaneous approximation to several functions. The latter generalization results at once from an attempt to increase the dimensions of the algorithm or, in other words, the number of terms in the linear relation of recurrence between the successive convergents or approximants. As this generalization is without doubt the more important, I shall make it the chief subject of this lecture. But a few words, at least, should be devoted to the former extension, which is worthy of a more careful and systematic study than it has received.

Denote again by $N_p(x)/D_q(x)$ a rational fraction with arbitrary coefficients. These can, in general, be so determined that its expansion at $x = 0$ shall agree for n_1 successive terms with a given series

$$c_0 + c_1x + c_2x^2 + \cdots$$

its expansion at $x = a_1$ for n_2 successive terms with

$$b_0 + b_1(x - a_1) + b_2(x - a_1)^2 + \cdots,$$

at $x = a_2$ for n_3 successive terms with

$$k_0 + k_1(x - a_2) + k_2(x - a_2)^2 + \cdots,$$

and so on, the total number of conditions thus imposed being equal to $p + q + 1$ or the number of parameters in the rational fraction. To each set of values for the n_i and q there corresponds an approximant, and the various approximants can be arranged into a table of multiple entry according to the values of these quantities. Continued fractions, at least in the case of a normal table, can then be obtained by following any path which passes successively from one approximant to another contiguous to it but more advanced in the table. As we proceed along the path, the degree of approximation for each of the points $0, a_1, a_2, \cdots$ must not decrease while at each step it is to increase for at least some one point. The partial numerators of the continued fraction are then either positive integral powers of $x, x-a_1, x-a_2, \cdots$, or the products of such powers. The degrees of the approximations obtained by stopping the continued fraction with any term can be inferred readily from the degrees of the partial numerators in $x, x-a_1, x-a_2, \cdots$. The details of the theory have not been worked out.*

The interest of such work can perhaps best be made apparent by referring to the developments for the simplest case in which each n_i is taken equal to 1. The rational fraction N_p/D_q is then completely determined by the requirement that at $p+q+1$ given points $a_1 = 0, a_2, a_3, \cdots$ it shall take an equal number of prescribed values, $A_1, A_2, A_3, \cdots$. If these are the values which a single function assumes at the points, we have the rational fractions which were introduced by *Cauchy* into the theory of interpolation [99, *a*] and which have been quite recently formed into a table and examined by *Padé* [112]. As $p+q+1$ increases, the number of points at which the approximation is sought likewise steadily increases.

When $q = 0$, the rational fraction becomes the familiar interpolation-polynomial of *Lagrange*,

$$\sum_{1}^{i=p+q+1} \frac{f(a_i)}{\phi'(a_i)} \cdot \frac{\phi(x)}{x-a_i},$$

in which

$$\phi(x) = (x-a_1)(x-a_2)\cdots(x-a_{p+q+1}).$$

This has been put into a very interesting form by *Frobenius* [95] which permits, without reconstruction,† of an indefinite increase in the number of its terms. Let us first take $1/(z-x)$ as the particular function of x for which an approximation is sought. From the equations

*The only investigation of this character is found in [76], but on account of the nature of the functions there considered certain variations were made in the construction of the table.

†Cf. also [99, *a*].

$$\frac{1}{z-x}=\frac{1}{(z-a_1)-(x-a_1)}=\frac{1}{z-a_1}+\frac{x-a_1}{z-a_1}\cdot\frac{1}{z-x}$$
$$=\frac{1}{z-a_1}+\frac{x-a_1}{z-a_1}\left(\frac{1}{z-a_2}+\frac{x-a_2}{z-a_2}\cdot\frac{1}{z-x}\right)=\cdots,$$

the series

$$(1)\quad \frac{1}{z-x}=\frac{1}{z-a_1}+\frac{x-a_1}{(z-a_1)(z-a_2)}+\frac{(x-a_1)(x-a_2)}{(z-a_1)(z-a_2)(z-a_3)}+\cdots,$$

is immediately derived, provided that the a_i are so distributed as to fulfill proper conditions for the convergence of the series. If now we take successively 1, 2, 3, $\cdots$ terms of the expansion, we obtain the series of polynomials,

$$N_0(x)=\frac{1}{z-a_1},\quad N_1(x)=\frac{1}{z-a_1}+\frac{x-a_1}{(z-a_1)(z-a_2)},\ \cdots,$$

and it is evident that $N_n(x)$ for the $n+1$ values $x=a_1, a_2, \cdots, a_{n+}$ agrees in value with $1/(z-x)$. By applying to (1) the well-known formula of *Euler* [1, a]* for converting any infinite series into a continuous fraction it follows immediately that these polynomials are the successive convergents of the continued fraction

$$\frac{\frac{1}{z-a_1}}{1}\;\frac{\frac{x-a_1}{z-a_2}}{-1+\frac{x-a_1}{z-a_2}}\;\frac{\frac{x-a_2}{z-a_3}}{-1+\frac{x-a_2}{z-a_3}-}\cdots.$$

The generalization of formula (1) can be made at once in the familiar manner by the use of *Cauchy's* integral. We get thus

$$f(x)=\frac{1}{2i\pi}\int\frac{f(z)dz}{z-x}=f(a_1)+\frac{(x-a_1)}{2i\pi}\int\frac{f(z)dz}{(z-a_1)(z-a_2)}+\cdots,$$

which by placing

$$\phi_n(x)=(x-a_1)(x-a_2)\cdots(x-a_n)$$

may be written

* Cf. *Encyklopädie der Math. Wiss.*, I A 3, p. 134, formula (104).

$$(2) \qquad f(x) = f(a_1) + (x - a_1)\left(\frac{f(a_1)}{\phi_2'(a_1)} + \frac{f(a_2)}{\phi_2'(a_2)}\right) + \cdots$$
$$\cdots + (x - a_1)(x - a_2)\cdots(x - a_n)\sum_{i=1}^{n+1}\frac{f(a_i)}{\phi_{n+1}'(a_i)} + \cdots.$$

For most interesting discussions of the convergence and properties of series having the form

$$A_0 + A_1(x - a_1) + A_2(x - a_1)(x - a_2) + \cdots$$

I may refer to memoirs by *Frobenius* [95] and *Bendixson* [99, *c*]. I shall content myself here with pointing out one simple application which is given implicitly by both writers but has been noted again recently by *Laurent* [103].

Let $f(x)$ be any analytic function the values of which are given at a series of points p_i having a regular point P as their limit. Describe about P as center any circle **C** within and upon which $f(x)$ is holomorphic, and denote the points p_i which fall within this circle by $a_1, a_2, \cdots$. Then $\lim a_i = P$. If now z describes the perimeter of the circle and x is a fixed interior point, the series (1) will be uniformly convergent and consequently permit of integration term by term. Equation (2) therefore gives an expression for $f(x)$ which is valid in the interior of **C**. This expression shows at once that an analytic function is determined uniquely when its values are known in a sequence of points having a regular point P as their limit. If, in particular, each $f(a_i) \equiv 0$, $f(x)$ must vanish identically. In other words, the zeros of an analytic function can not be infinitely dense in the vicinity of a non-singular point. Further, *Bendixson* points out that the convergence of the right hand member of (2) is not only the necessary but the sufficient condition that $f(a_1), f(a_2), f(a_3), \cdots$ shall be the values of some analytic function at a set of points a_i having a limit point P.

We turn now to the generalization of the algorithm of the continued fraction. The first investigation on this subject is found in a paper of *Jacobi*,* published posthumously in 1868. The developments of *Jacobi* were, however, of a purely numerical nature. On this side they have been perfected recently by *Fr. Meyer* [83]. The first example of a functional extension was given by *Hermite* in his famous memoir [84] upon the transcendence of e, and the theory has been developed since independently of each other by *Pincherle* and *Padé*.

To explain the nature of the generalization it will be desirable first to refer to the mode in which a continued fraction is commonly generated. Two numbers or functions, f_0 and f_1, are given, from which a sequence of other numbers or functions is obtained by placing

$$(3)\qquad \begin{aligned} f_2 &= \lambda_1 f_1 - f_0, \\ f_3 &= \lambda_2 f_2 - f_1, \\ f_4 &= \lambda_3 f_3 - f_2, \\ &\cdots\cdots \end{aligned}$$

in which the λ_i are determined in accordance with some stated law. For the quotient f_0/f_1, we obtain successively

$$\frac{f_0}{f_1} = \lambda_1 - \frac{1}{\dfrac{f_1}{f_2}} = \lambda_1 - \frac{1}{\lambda_2 - \dfrac{1}{\dfrac{f_2}{f_3}}} = \cdots,$$

and it therefore gives rise to the continued fraction

$$(4)\qquad \lambda_1 - \frac{1}{\lambda_2} - \frac{1}{\lambda_3} - \cdots.$$

By means of the equations (3) each f_{n+1} can be expressed linearly in terms of the initial quantities f_0, f_1. Thus

$$(5)\qquad f_{n+1} = A_{1,\,n+1} f_1 + A_{0,\,n+1} f_0,$$

in which $A_{0,\,n+1}$, $A_{1,\,n+1}$ are polynomials in the elements λ_i. It is easy to see that these polynomials both satisfy the same difference

* "Allgemeine Theorie der kettenbruchähnlichen Algorithmen, in welchen jede Zahl aus drei vorhergehenden gebildet wird." *Journ. für Math.*, vol. 69 (1868), p. 29.

equation as f_i,

$$f_{n+1} = \lambda_n f_n - f_{n-1};$$

and for their initial values we have

$$A_{1,1} = 1, \quad A_{0,1} = 0,$$

$$A_{1,2} = \lambda_1, \quad A_{0,2} = -1.$$

Consequently $A_{1,n}$ and $-A_{0,n}$ are the numerator and denominator of the $(n-1)$th convergent of (4).

When the generating relations have the form

$$f_0 = \lambda_1 f_1 + \mu_2 f_2,$$

$$f_1 = \lambda_2 f_2 + \mu_3 f_3,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot,$$

the resultant continued fraction is

$$(4') \qquad \lambda_1 + \frac{\mu_2}{\lambda_2 +} \frac{\mu_3}{\lambda_3 +} \cdots.$$

A distinction then appears between the system of functions $(A_{1,n+1}, -A_{0,n+1})$ and the system which consists of the numerator and denominator of the nth convergent. Though the quotient of the two functions of either system is the nth convergent, the former pair of functions satisfy the same relation of recurrence as the f_i, namely,

$$f_n = \lambda_{n+1} f_{n+1} + \mu_{n+2} f_{n+2};$$

while the corresponding relation for the other system is

$$g_n = \lambda_n g_{n-1} + \mu_n g_{n-2}.$$

The latter equation is called by *Pincherle* [77, a] the inverse of the former. In the continued fraction (4) we took $\mu_i = -1$ so that the two relations were coincident.

The immediate generalization of these considerations is obtained by taking $m+1$ initial quantities $f_0, f_1, \cdots, f_m$ in place of two. With a very slight change of notation we may write

$$
(6)\qquad
\begin{aligned}
&f_0 + \lambda_1 f_1 + \mu_2 f_2 + \cdots + \nu_m f_m = f_{m+1},\\
&f_1 + \lambda_2 f_2 + \mu_3 f_3 + \cdots + \nu_{m+1} f_{m+1} = f_{m+2},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&f_{n-m} + \lambda_{n-m+1} f_{n-m+1} + \mu_{n-m+2} f_{n-m+2} + \cdots + \nu_n f_n = f_{n+1}.
\end{aligned}
$$

Then by expressing f_n in terms of the $m + 1$ given quantities we have

$$
(7)\qquad f_n = A_{0,\,n} f_0 + A_{1,\,n} f_1 + \cdots + A_{m,\,n} f_m,
$$

in which $A_{i,\,n}$ is a polynomial in terms of the λ_i, μ_{i+1}, $\cdots$, ν_{i+m-1} $(i = 1,\ 2,\ \cdots,\ n - m)$. These $m + 1$ polynomials $A_{i,\,n}$ satisfy the same difference equation (6) as the f_n, and for their initial values we plainly have

$$
\begin{array}{lcccc}
 & A_{0,\,n} & A_{1,\,n} & \cdots & A_{m,\,n}\\
n = 0 & 1 & 0 & \cdots & 0,\\
n = 1 & 0 & 1 & \cdots & 0,\\
\cdot & \cdot & \cdot & \cdot & \cdot\\
n = m & 0 & 0 & \cdots & 1.
\end{array}
$$

Hence they constitute a complete system of independent integrals of (6). Furthermore, in analogy with the relation between two successive convergents of (4),

$$
\begin{vmatrix} D_n & N_n \\ D_{n-1} & N_{n-1} \end{vmatrix} = 1,
$$

we have [83, *a*, p. 170]

$$
(8)\qquad
\begin{vmatrix}
A_{0,\,n} & A_{1,\,n} & \cdots & A_{m,\,n}\\
A_{0,\,n+1} & A_{1,\,n+1} & \cdots & A_{m,\,n+1}\\
\cdot & \cdot & \cdot & \cdot\\
A_{0,\,n+m} & A_{1,\,n+m} & \cdots & A_{m,\,n+m}
\end{vmatrix} = (-1)^{nm}.
$$

The relation which is the inverse of (6) has the form

$$
(9)\qquad g_{n+m} + \lambda_n g_{n+m-1} + \mu_n g_{n+m-2} + \cdots + \nu_n g_n = g_{n-1}.
$$

To obtain a system of independent integrals of this equation, let

$P_{0,n}$ denote the minor of $A_{0,n}$ in (8), $P_{1,n}$ the minor of $A_{1,n}$ after the first column has been moved over the remaining columns so as to become the last, $P_{2,n}$ the minor of $A_{2,n}$ after the first two columns have been moved over the remaining columns so as to become the last two, and so on. It can be demonstrated easily that the desired system is obtained by placing $g_{i,n+m} \equiv P_{i,n}$ $(i = 0, 1, \cdots, m)$, and these new polynomials rather than the $A_{i,n}$ are the true analogues of the numerator and denominator of an ordinary continued fraction. The connection between the two systems of polynomials is, however, both an intimate and a reciprocal one, for not only is (9) the inverse of (6) but the converse is also true. On this account the two systems can be employed simultaneously with advantage in working with the generalized continued fraction.

For all except the very lowest values of n the new polynomials can be found from the equations*

$$(9') \quad P_{i,n} + \lambda_n P_{i,n-1} + \mu_n P_{i,n-2} + \cdots + \nu_n P_{i,n-m} = P_{i,\,n-m-1}.$$

In place of these relations it will be often found convenient to employ such a process as is indicated in the following equations for $m = 2$ [83, a, p. 180].†

$$\frac{P_{0,1}}{P_{1,1}} = q_{1,1}, \qquad \frac{P_{0,2}}{P_{1,2}} = q_{1,1} + \frac{q_{2,2}}{q_{2,1}}, \qquad \frac{P_{0,3}}{P_{1,3}} = q_{1,1} + \cfrac{q_{2,2} + \cfrac{1}{q_{3,1}}}{q_{2,1} + \cfrac{q_{3,2}}{q_{3,1}}},$$

$$\frac{P_{0,4}}{P_{1,4}} = q_{1,1} + \cfrac{q_{2,2} + \cfrac{1}{q_{3,1} + \cfrac{q_{4,2}}{q_{4,1}}}}{q_{2,1} + \cfrac{q_{3,2} + \cfrac{1}{q_{4,1}}}{q_{3,1} + \cfrac{q_{4,2}}{q_{4,1}}}}, \cdots, \qquad \begin{array}{l}(q_{i,1} = -\lambda_i)\\ (q_{i,2} = -\mu_i)\end{array}$$

---

*Cf. [83, a, p. 174, eq. X].

†Cf. E. Fürstenau, "Ueber Kettenbrüche höherer Ordnung"; *Jahresbericht über das königliche Realgymnasium zu Wiesbaden;* 1873 4. See also Scott's *Determinants*, Chap. 13, § 11–12.

We may therefore very properly call the system of values

$$\begin{bmatrix} \lambda_1 & \mu_1 & \cdots & \nu_1 \\ \lambda_2 & \mu_2 & \cdots & \nu_2 \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

the *norm* of a generalized continued fraction, which itself consists of the computation of the $P_{i,n}$ or their ratios.

To apply this generalization to the construction of algebraic continued fractions, it is only necessary to select as the $m+1$ initial functions $f_0, \cdots, f_m$ series in ascending powers or series in descending powers of x. The nature of the ensuing theory will be explained sufficiently by developing here the simplest case, in which three such series are given [77, *c*.] Take then

$$S_0 = k_0 + \frac{k_1}{x} + \frac{k_2}{x^2} + \cdots \qquad (k_0 \neq 0),$$

$$S_1 = \frac{l_0}{x} + \frac{l_1}{x^2} + \frac{l_2}{x^3} + \cdots \qquad (l_0 \neq 0),$$

$$S_2 = \frac{m_0}{x^2} + \frac{m_1}{x^3} + \frac{m_2}{x^4} + \cdots. \qquad (m_0 \neq 0)$$

If we next place

$$(10) \qquad S_0 + (a_0 x + a_0')S_1 + b_0 S_2 = S_3,$$

the coefficients a_0, a_0', b_0 can be so determined that S_3 shall begin with at least as high a power of $1/x$ as the third. Normally the degree is exactly 3, and similarly for each consecutive value of n we have

$$S_n + (a_n x + a_n')S_{n+1} + b_n S_{n+2} = S_{n+3}$$

in which S_n denotes a series beginning with the nth power of $1/x$. Hence unless certain specified conditions are satisfied, a regular continued fraction will be obtained having the norm :

$$\begin{Bmatrix} 1 & a_0 x + a_0' & b_0 \\ 1 & a_1 x + a_1' & b_1 \\ 1 & a_2 x + a_2' & b_2 \\ \cdot & \cdot & \cdot \end{Bmatrix}.$$

This norm will not be altered in any way by dividing (10) through by S_0. It is therefore determined uniquely by the ratios of S_0, S_1, S_2, and conversely the ratios by the norm.

Without loss of generality we may set $S_0 = 1$. Place also

$$S_{n+3} = A_{n+3} + B_{n+3}S_1 + C_{n+3}S_2, \tag{11}$$

$$P_n = \begin{vmatrix} B_n & C_n \\ B_{n+1} & C_{n+1} \end{vmatrix}, \quad Q_n = \begin{vmatrix} C_n & A_n \\ C_{n+1} & A_{n+1} \end{vmatrix}, \quad R_n = \begin{vmatrix} A_n & B_n \\ A_{n+1} & B_{n+1} \end{vmatrix}.$$

If then $n+3$ in (11), is replaced successively by n and $n+1$, and the two equations are solved for S_1 and S_2, we obtain

$$S_1 = \frac{Q_n + C_{n+1}S_n - C_n S_{n+1}}{P_n}$$

or

$$S_1 - \frac{Q_n}{P_n} = \frac{\lambda_n}{P_n} \qquad (\lambda_n = C_{n+1}S_n - C_n S_{n+1}), \tag{12}$$

and

$$S_2 - \frac{R_n}{P_n} = \frac{\mu_n}{P_n} \qquad (\mu_n = B_n S_{n+1} - B_{n+1}S_n). \tag{13}$$

An examination of P_n, Q_n, R_n, λ_n, μ_n will show that their degrees in x are

$$n-1,\ n-2,\ n-3,\ -r-1,\ -r. \qquad (n = 2r),$$

$$n-1,\ n-2,\ n-3,\ -r-1,\ -r-1 \qquad (n = 2r+1).$$

Hence the expansions of Q_n/P_n and R_n/P_n in descending powers of x, agree with S_1 and S_2 to terms of degree $3r-1$ and $3r-2$ inclusive if $n = 2r$, and of the $3r$th degree if $n = 2r+1$. The generalized continued fraction therefore affords a solution of the problem: to find two rational fractions with a common denominator which shall give as close an approximation to the given functions S_1 and S_2 as is consistent with the degrees prescribed for their numerators and denominators.

When three series in ascending powers of x,

$$\bar{S}_i = k_0^{(i)} + k_1^{(i)}x + k_2^{(i)}x^2 + \cdots \qquad (i = 1, 2, 3),$$

are chosen as the initial functions, a more comprehensive algorithm can be introduced. *Padé* [79, a] takes three polynomials $A_p^{(1)}(x)$, $A_{p'}^{(2)}(x)$, $A_{p''}^{(3)}(x)$ with undetermined coefficients, the degrees of which are indicated by their subscripts, and requires that their coefficients shall be so determined that the expansion of

$$A_p^{(1)}\bar{S}_1 + A_{p'}^{(2)}\bar{S}_2 + A_{p''}^{(3)}\bar{S}_3$$

in ascending powers of x shall begin with as high a power as possible. Ordinarily this is the $(p + p' + p'' + 2)$th power. To each set of values of p, p', p'' he shows that there corresponds uniquely a group of three polynomials which he terms the "associated polynomials," and these groups he arranges into a table of triple entry according to the values of p, p', p''. An exactly similar table can not be constructed for three series in descending powers of x, inasmuch as the substitution of $1/x$ for x in $A_p^{(1)}, \cdots, A_{p''}^{(2)}$ gives three rational fractions, with powers of x in the denominators which can not be thrown away unless

$$p = p' = p''.$$

The new table is handled by *Padé* in the same manner as the one previously constructed for a single series. In particular, he examines the relations

$$\alpha A_p^{(i)} + \beta A_q^{(i)} + \gamma A_r^{(i)} = A_s^{(i)} \qquad (i = 1, 2, 3),$$

which exist between four successive groups of associated polynomials, α, β, γ being rational functions of x which are independent of the value of i. When it is possible to so select a sequence $\cdots A_p^{(i)}, A_q^{(i)}, A_r^{(i)}, A_s^{(i)}, A_t^{(i)}, \cdots$ that α, β, γ, are polynomials of invariable degree for any four consecutive terms in the sequence, the sequence or continued fraction is said to be regular. In a normal table there are found to be four distinct types of such continued fractions. It is worth noting, however, that the diagonal type which was the best in an ordinary table, no longer exists since it is found that when the sequence fills a diagonal file of the table, α, β, and γ are no longer polynomials but rational fractions having a common denominator.

In one important respect *Padé's* investigation has a narrower reach than *Pincherle's* and needs completion. The existence of a second group of associated polynomials — the P_n, Q_n, R_n of *Pincherle* — is not brought to light. As has been already pointed out, it is this second group of polynomials which is the true analogue of the convergent of an ordinary continued fraction and which must take precedence in considering the convergence of the algorithm or the closeness of the approximation afforded to the initial functions. *Pincherle's* definition of convergence [82] is not, however, so framed as to require explicitly the introduction of these polynomials. If the given system of difference equations is

$$(14) \qquad f_{n+3} = c_n f_{n+2} + d_n f_{n+1} + f_n \qquad (n = 0, 1, 2, \cdots),$$

the continued fraction is said by him to be convergent when the two following conditions are fulfilled:

(1) There is a system of integrals F_n, F_n', F_n'' of (14) such that F_n'/F_n, F_n''/F_n have limits for $n = \infty$, and these limits are different from 0.

(2) There is also one particular integral — called by *Pincherle* the *integrale distinto* — the ratio of which to every other integral of (14) has the limit zero.

Pincherle's interest is evidently concentrated upon this *principal* integral. It seems to me, however, more natural to call the algorithm convergent when the ratios Q_n/P_n and R_n/P_n (cf. Equations 12 and 13) converge to finite limits for $n = \infty$. Under ordinary circumstances these limits will doubtless coincide with the ratios of the generating functions, f_1/f_0 and f_2/f_0.

In the case of an ordinary continued fraction the two definitions coalesce. For suppose that the nth convergent N_n/D_n of (4′) has the limit L. Then $N_n - LD_n$ is such an integral of the difference equation,

$$f_n = \lambda_n f_{n-1} + \mu_n f_{n-2},$$

that its ratio to any other integral, $k_1 N_n + k_2 D_n$, has the limit 0. Conversely, if the principal integral $N_n - LD_n$ exists, there must be a limit L for the continued fraction. Possibly the case in which the principal integral is D_n might be called an exception, since the continued fraction is then convergent by *Pincherle's* definition, but $\lim N_n/D_n = \infty$.

A study of the conditions of convergence, so far as I am aware, has at present been made in only two special cases. *Fr. Meyer* [83, *a*, § 7] has made a *partial* investigation when the coefficients $\lambda_n, \cdots, \nu_n$ in equations (6) are negative constants. *Pincherle* [82] has examined the case in which the coefficients of the recurrent relation

$$f_n + (a_n x + a'_n) f_{n+1} + b_n f_{n+2} = f_{n+3}$$

have limiting values and finds that the generalized continued fraction is convergent for sufficiently large values of x. Let the limits of the coefficients be denoted by a, a', and b respectively. To demonstrate the convergence he avails himself of the notable theorem of *Poincaré*, already cited in Lecture 4. If, namely, no two roots of the equation

$$z^3 - bz^2 - (ax + a')f - 1 = 0 \tag{15}$$

are of equal modulus, f_n/f_{n-1} will have a limit for $n = \infty$, and this limit will be one of the roots of the auxiliary equation (15), usually the root of greatest modulus. From this it follows directly that A_n/A_{n-1}, B_n/B_{n-1}, C_n/C_{n-1} as quotients of integrals of the difference equation last given, also P_n/P_{n-1}, Q_n/Q_{n-1}, R_n/R_{n-1} as integrals of the inverse equation, have each a definite limit. The existence of limits for Q_n/P_n and of R_n/P_n is then established for sufficiently great values of x, and the analytic character of these limits is finally argued. Let them be denoted by $U(x)$ and $V(x)$. Then $X_n = A_n + B_n U(x) + C_n V(x)$ is the principal integral of the difference equation, and has the following distinctive property: Its expansion in powers of $1/x$ begins with the highest possible power consistent with the degrees of A_n, B_n, C_n, and coincides with f_n for each successive value of n.

Bibliography of Memoirs relating to Algebraic Continued Fractions.

In the following bibliography only works in Latin, Italian, French, German, and English are included. In *Wölffing's Mathematischer Bücherschatz* (heading *Kettenbrüche*) several dissertations, etc., are mentioned which may possibly relate to algebraic continued fractions but which are not accessible to the writer. They are therefore not included here. The writer would be glad to have his attention called to any noteworthy omissions in the bibliography.

In many cases it has been extremely difficult to draw the line between inclusion and exclusion, especially under divisions VI–IX.

Any classification of the material which may be adopted will be open to objections, but even an imperfect classification will probably add greatly to the usefulness of the bibliography. Since much of the work relating to algebraic continued fractions appears elsewhere under other headings, it is believed that such a bibliography as is here given may be of service.

For a brief resumé of the theory of algebraic continued fractions the reader is referred to *Osgood's* section of the *Encyklopädie der Math. Wissenschaft,* II B 1, §§ 38–39.

I. On the Derivation of Continued Fractions from Power Series. General Theory.

A. Early Works.

1. **Euler.** (*a*) Introductio in analysin infinitorum, vol. 1, chap. 18, 1748.

   (*b*) De transformatione serierum in fractiones continuas. Opuscula analytica, vol. 2, pp. 138–177, 1785.

2. **Lambert.** (*a*) Verwandlung der Brüche. Beyträge zum Gebrauche der Mathematik und deren Anwendung, vol. 2_1, p. 54 ff., p. 161, 1770.

   (*b*) Mémoire sur quelques propriétés remarquables des quantités transcendentes circulaires et logarithmiques. Histoire de l'Acad. roy. des sciences et belles-lettres à Berlin, 1768.

3. **Trembley.** Recherches sur les fractions continues. Mém. de l'Acad. roy. de Berlin, 1794, pp. 109–142.

4. **Kausler.** (*a*) Expositio methodi series quascunque datas in fractiones continuas convertendi. Mém. de l'Acad. imp. des sciences de St. Pétersbourg, vol. 1, pp. 156–174, 1802.

(*b*) De insigni usu fractionum continuarum in calculo integrale. Ibid., vol. 1, pp. 181–194, 1803.

5. **Viscovatov.** (*a*) De la méthode générale pour reduire toutes sortes des quantités en fractions continues. Ibid., vol. 1, pp. 226–247, 1805.

(*b*) Essai d'une méthode générale pour réduire toutes sortes de séries en fractions continues. Nova Acta Acad. Scient. imp. Petropolitanæ, vol. 15, pp. 181–191, 1802.

6. **Bret.** Théorie générale des fractions continues. Gergonne's Annales de Math., vol. 9, pp. 45–49, 1818. Unimportant.

7. **Scubert.** De transformatione seriei in fractionem continuam. Mém. de l'Acad. imp. des sciences de St. Pétersbourg, vol. 7, pp. 139–158, 1820.

8. **Stern.** (*a*) Zur Theorie der Kettenbrüche und ihre Anwendung. Jour. für Math., vol. 10, pp. 241–265, 1833.

(*b*) Zur Theorie der Kettenbrüche. Jour. für Math., vol. 18, pp. 69–74, 1838.

9. **Heilermann.** (*a*) Ueber die Verwandlung der Reihen in Kettenbrüche. Jour. für Math., vol. 33, pp. 174–188, 1846; also vol. 46, pp. 88–95, 1853.

(*b*) Zusammenhang unter den Coefficienten zweier gleichen Kettenbrüche von verschiedener Form. Zeitschrift für Math. und Phys., vol. 5, pp. 362–363, 1860. Unimportant.

10. **Hankel.** Ueber die Transformation von Reihen in Kettenbrüche. Berichte der Sächischen Gesellschaft der Wissenschaft zu Leipzig, vol. 14, pp. 17–22, 1862.

11. **Muir.** (*a*) On the transformation of *Gauss*' hypergeometric series into a continued fraction. Proc. of the London Math. Soc., vol. 7, pp. 112–118, 1876.

(*b*) New general formulæ for the transformation of infinite series into continued fractions. Trans. of the R. Soc. of Edinburgh, vol. 27, pp. 467–471, 1876.

The general formulæ in these memoirs, which Muir supposed to be new, had been previously given by Heilermann in 9(*a*).

12. **Heine.** Handbuch der Kugelfunction, 2[te] Auflage, 1878; chap. 5, Die Kettenbrüche, pp. 260–297.

This gives a good idea of the state of the theory up to 1878.

B. The Modern Theory.

The beginnings of this theory are to be found in Nos. 110 and 111.

13. **Frobenius.** Ueber Relationen zwischen den Näherungsbrüchen von Potenzreihen. Jour. für Math., vol. 90, pp. 1–17, 1881.

    This fundamental memoir marks an important advance. See 16(a).

14. **Stieltjes.** Sur la réduction en fraction continue d'une série procédant suivant les puissances descendantes d'une variable. Ann. de Toulouse, vol 3, H, pp. 1–17, 1889.

15. **Pincherle.** Sur une application de la théorie des fractions continues algébriques. Comp. Rend., vol. 108, p. 888, 1889.

16. **Padé.** (a) Sur la représentation approchée d'une fonction par des fractions rationnelles. Thesis, published in the Ann. de l'Ec. Nor., ser. 3, vol. 9, supplement, pp. 1–93, 1892.

    This very fundamental memoir is the best one to read for the purpose of learning the elements of the theory of algebraic continued fractions. The same point of view is taken as by Frobenius in (13) and is more completely developed. The thesis was preceded by the two following preliminary notes:

    (a') Sur la représentation approchée d'une fonction par des fractions rationnelles. Comp. Rend, vol. 111, p. 674, 1890.

    (a'') Sur les fractions continues régulières relatives a e^x. Comp. Rend, vol. 112, p. 712, 1891.

    (b) Recherches nouvelles sur la distribution des fractions rationnelles approchées d'une fonction. Ann. de l'Ec. Nor., ser. 3, vol. 19, pp. 153–189, 1902.

    (c) Aperçu sur les développements récents de la théorie des fractions continues. Compte rendu du deuxième Congrès international des mathématiciens tenu a Paris, pp. 257–264, 1900.

    Only a restricted portion of the field is here reviewed, and in this portion the important work of Pincherle is overlooked.

17. **Padé.** (a) Sur les séries entières convergentes ou divergentes et les fractions continues rationelles. Acta Math., vol. 18, pp. 97–111, 1894.

    (a') Sur la possibilité de définir une fonction par une série entière divergente. Comp. Rend., vol. 116, p. 686, 1893.

See also No. 26*a*, 76.

II. On Convergence.

(For a résumé of the criteria for the convergence of continued fractions with real elements see Pringsheim's report in the *Encyklopädie der mathematischen Wissenschaften*, I A 3, p. 126, ff.)

18. **Riemann.** Sullo svolgimento del quoziente di due serie ipergeometriche in frazione continua infinita, 1863. Gesammelte mathematische Werke, pp. 400–406.

18, *bis.* **Worpitzky.** Untersuchung über die Entwickelung der monodromen und monogenen Functionen durch Kettenbrüche. Programm, Friedrichs Gymnasium und Realschule, Berlin, 1865.

This program and the two following memoirs of Thomé were published before Riemann's posthumous fragment.

19. **Thomé.** (*a*) Ueber die Kettenbruchentwickelung der Gauss'schen Function $F(\alpha, 1, \gamma, x)$. Jour. für Math., vol. 66, pp. 322–336, 1866.

(*b*) Ueber die Kettenbruchentwickelung des Gauss'schen Quotienten

$$\frac{F(\alpha, \beta+1, \gamma+1, x)}{F(\alpha, \beta, \gamma, x)}.$$

Ibid., vol. 67, pp. 299–309, 1867.

20. **Laguerre.** Sur l'integrale

$$\int_x^\infty \frac{e^{-x}}{x}\, dx.$$

Bull. de la Soc. Math. de France, vol. 7, pp. 72–81, 1879, or Oeuvres, vol. 1, p. 428.

Historically an important memoir because of its development of the connection between a divergent power series and convergent continued fraction. See the first footnote in lecture 4; also No. 102, p. 30.

21. **Halphen.** (*a*) Sur la convergence d'une fraction continue algébrique. Comp. Rend., vol. 100 (1885), pp. 1451–1454.

(*b*) Same subject. Ibid., vol. 106 (1888), pp. 1326–1329.

(*c*) Traité des fonctions elliptiques. Chap. 14. Fractions continues et intégrales pseudo-elliptiques.

22. **Pincherle.** Alcuni teoremi sulle frazioni continue. Atti delle R. Accad. dei Lincei, ser. 4, vol. 5_1, pp. 640–643, 1889.

The test for convergence given here is included in a more general criterion given later by Pringsheim, No. 29.

23. **Pincherle.** Sur les fractions continues algébriques. Ann. de l'Ec. Nor., ser. 3, vol. 6, pp. 145–152, 1889.

An incomplete result is here obtained. See No. 32*c* for the complete theorem.

24. **Padé.** Sur la convergence des fractions continues simples. Comp. Rend., vol. 112, p. 988, 1891. Also found in §§ 45–47 of No. 16*a*.

25. **Banning.** Ueber Kugel- und Cylinderfunktionen und deren Kettenbruchentwickelung. Dissertation, Bonn, 1894, pp. 1–33.

26. **Stieltjes.** (*a*) Recherches sur les fractions continues. Annales de Toulouse, vol. 8, J, pp. 1–122, and vol. 9, A, pp. 1–47. 1894–95. Published also in vol. 32 of the Mémoires présentés à l'Acad. des sciences de l'Institut National de France.

A rich memoir, developing particularly the connection between an important class of continued fractions and the corresponding integrals.

(a') Sur un développement en fraction continue. Comp. Rend., vol. 99, p. 508, 1884.

(a'') Same subject. Ibid., vol. 108 (1889), p. 1297.

(a''') Sur une application des fractions continues. Ibid., vol. 118 (1894), p. 1315.

(a^{iv}) Recherches sur les fractions continues. Ibid., vol. 118 (1894), p. 1401.

Markoff. (*b*) Note sur les fractions continues. Bull. de l'Acad. imp. des sciences de St. Petersbourg, ser. 5, vol. 2, pp. 9–13, 1895.

This gives a discussion of the relation of his work to that of Stieltjes.

27. **H. von Koch.** (*a*) Sur un théoréme de *Stieltjes* et sur les fonctions définies par des fractions continues. Bull. de la Soc. Math. de France, vol. 23, pp. 33–40, 1895.

(a') Sur la convergence des determinants d'ordre infini et des fractions continues. Comp. Rend., vol. 120, p. 144, 1895.

28. **Markoff.** Deux démonstrations de la convergence de certaines fractions continues. Acta Math., vol. 19, pp. 93–104, 1895.

Contained also in his Differenzenrechnung (deutsche Uebersetzung), chap. 7, § 21–22.

This discusses the convergence of the usual continued fraction for

$$\int_a^b \frac{f(y)dy}{z-y}$$

when $f(y) > 0$ between the limits of integration.

29. **Pringsheim.** Ueber die Convergenz unendlicher Kettenbrüche. Sitzungsberichte der math.-phys. Classe der k. bayer'schen Akad. der Wissenschaften, vol. 28, pp. 295–324, 1898.

The most comprehensive criteria for convergence yet obtained are found in 29, 31, and 32*b*.

30. **Bortolotti.** Sulla convergenza delle frazioni continue algebriche. Atti della R. Accad. dei Lincei, ser. 5, vol. 8_1, pp. 28–33, 1899.

31. **Van Vleck.** On the convergence of continued fractions with complex elements. Trans. Amer. Math. Soc., vol. 2, pp. 215–233, 1901.

32. **Van Vleck.** (*a*) On the convergence of the continued fraction of Gauss and other continued fractions. Annals of Math., ser. 2, vol. 3, pp. 1–18, 1901.

(*b*) On the convergence and character of the continued fraction

$$\frac{a_1 z}{1} + \frac{a_2 z}{1} + \frac{a_3 z}{1} + \cdots$$

Trans. Amer. Math. Soc., vol. 2, pp. 476–483, 1901.

(*c*) On the convergence of algebraic continued fractions whose coefficients have limiting values. Ibid., vol. 5, pp. 253–262, 1904.

33. **Montessus.** (*a*) Sur les fractions continues algébriques. Bull. de la Soc. Math. de France, vol. 30, pp. 28–36, 1902.

The content of this memoir was discussed in lecture 5.

(*b*) Same title. Comp. Rend., vol. 134 (1902), p. 1489.

See also 37*a′*, 41.

III. On Various Continued Fractions of Special Form.

A. The Continued Fraction of Gauss.

34. **Gauss.** Disquisitiones generales circa seriem infinitam

$$1 + \frac{\alpha \cdot \beta}{1 \cdot \gamma} x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1 \cdot 2 \cdot \gamma(\gamma+1)} x^2 + \cdots$$

Deutsche Uebersetzung von Simon, or Werke, vol. 3, pp. 134–138, 1812.

34, *bis.* **Vorsselman de Herr.** Specimen inaugurale de fractionibus continuis. Dissertation, Utrecht, 1833.

Numerous references are given here to the early literature upon continued fractions.

34, *ter.* **Heine.** Auszug eines Schreibens über Kettenbrüche von Herrn E. Heine an den Herausgeber. Jour. für Math., vol. 53, pp. 284–285, 1857.

See also 40*c*, p. 231.

35. **Euler.** (*a*) Commentatio in fractionem continuam in qua illustris *Lagrange* potestates binomiales expressit. Mémoires de l'Acad. imp. des sciences de St. Pétersbourg, vol. 6, pp. 3–11, 1818.

Pade. (*b*) Sur la généralisation des développements en fractions continues, donnés par *Gauss* et par *Euler*, de la fonction $(1+x)^m$. Comp. Rend., vol. 129, p. 753, 1899.

(*c*) Sur la généralisation des développements en fractions continues, donnés par *Lagrange* de la fonction $(1+x)^m$. Ibid., vol. 129, p. 875, 1899.

(*d*) Sur l'expression générale de la fraction rationnelle approchée de $(1+x)^m$. Ibid., vol. 132, p. 754, 1901.

See also Nos. 11, 32*a*, 65.

B. The Continued Fractions for e^x.

36. **Winckler.** Ueber angenäherte Bestimmungen. Wiener Berichte, Math.-naturw, Classe, vol. 72, pp. 646–652, 1875.

37. **Padé.** (*a*) Mémoire sur les développements en fractions continues de la fonction exponentielle, pouvant servir d'introduction à la théorie des fractions continues algébriques. Ann. de l'Ec. Nor., Ser. 3, vol. 16, pp. 395–426, 1899.

(*a′*) Sur la convergence des réduites de la fonction exponentielle. Comp. Rend., vol. 127, p. 444, 1898.

See also Nos. 16*a″*, 106, and pages 243–5 of 40*c*.

C. The Continued Fraction of Bessel.

38. **Günther.** Bemerkungen über Cylinder-Functionen. Archiv der Math. und Phys., vol. 56, pp. 292–297, 1874.

39. **Graf.** (*a*) Relations entre la fonction Bessélienne de 1[re] espèce et une fraction continue. Annali di Mat., ser. 2, vol. 23, pp. 45–65, 1895.

Giving references to earlier works where the continued fraction of Bessel is found.

Crelier. (*b*) Sur quelques propriétés des fonctions Besséliennes, tirées de la théorie des fractions continues. Annali di Mat., vol. 24, pp. 131–163, 1896.

See also Nos. 25, 32*a*.

D. The Continued Fraction of Heine.

40. **Heine.** (*a*) Ueber die Reihe

$$1+\frac{(q^\alpha-1)(q^\beta-1)}{(q-1)(q^\gamma-1)}x+\frac{(q^\alpha-1)(q^{\alpha+1}-1)(q^\beta-1)(q^{\beta+1}-1)}{(q-1)(q^2-1)(q^\gamma-1)(q^{\gamma+1}-1)}x^2+\cdots.$$

Jour. für Math., vol. 32, pp. 210–212, 1846.

(*b*) Untersuchung über die (selbe) Reihe. Ibid., vol. 34, pp. 285–328, 1847.

(*c*) Ueber die Zähler und Nenner der Näherungswerthe von Kettenbrüche. Ibid., vol. 57, pp. 231–247, 1860.

Christoffel (*d*) Zur Abhandlung "Ueber Zähler und Nenner" (u. s. w.) des vorigen Bandes. Ibid., vol. 58, pp. 90–91, 1861.

41. **Thomae.** Beiträge zur Theorie der durch die Heine'sche Reihe darstellbaren Funktionen. Jour. für Math., vol. 70, 1869. See pp. 278–281 where the convergence of Heine's continued fraction is proved.

See also 32*a*.

42. (On *Eisenstein's* continued fractions).

Heine. (*a*) Verwandlung von Reihen in Kettenbrüche. Jour. für Math., vol. 32, pp. 205–209, 1846.

See also vol. 34, p. 296.

Muir. (*b*) On Eisenstein's continued fractions. Trans. Roy. Soc. of Edinburgh, vol. 28, part 1, pp. 135–143, 1877.

Muir plainly was not aware of the preceding memoir by Heine.

E. The Continued Fraction of Stieltjes. (*See No. 26.*)

43. **Borel.** Les séries de *Stieltjes*, Chap. 5 of his Mémoire sur les séries divergentes. Ann. de l'Ec. Nor., ser. 3, vol. 16, pp. 107–128; and also chap. 2 of his treatise, Les Séries divergentes, pp. 55–86, 1901.

44. **Padé.** Sur la fraction continue de *Stieltjes*. Comp. Rend., vol. 132, p. 911, 1901.

45. **Van Vleck.** On an extension of the 1894 memoir of *Stieltjes*. Trans. Amer. Math. Soc., vol. 4, pp. 297–332, 1903.

See also Nos. 27, 102.

F. The Continued Fraction for

$$1 + mx + m(m+n)x^2 + m(m+n)(m+2n)x^3 + \cdots$$

and its special cases.

46. **Euler.** (*a*) De seriebus divergentibus. Novi commentarii Acad. scientiarum imperialis Petropolitanæ, vol. 5, pp. 205–237, 1754–5; in particular pp. 225 and 232–237.

(*b*) De transformatione seriei divergentis

$$1 - mx + m(m+n)x^2 - m(m+n)(m+2n)x^3 + \cdots$$

in fractionem continuam. Nova acta Acad. scientiarum imperialis Petropolitanæ, vol. 2, pp. 36–45, 1784.

Gergonne. (*c*) Recherches sur les fractions continues. Gergonne's Annales de Math., vol. 9, pp. 261–270, 1818.

47. **Laplace.** (*a*) Traité de mecanique celeste. Oeuvres, vol. 4, pp. 254–257, 1805.

Jacobi. (*b*) De fractione continua in quam integrale $\int_x^\infty e^{-x^2} dx$ evolvere licet. Jour. für Math., vol. 12, pp. 346–347, 1834, or Werke, vol. 6, p. 76.

See also p. 79 of No. 20, and the first note under lecture 2.

G. Periodic Continued Fractions, and Continued Fractions Connected with the Theory of Elliptic Functions.

48. **Abel.** (*a*) Sur l'intégration de la formule différentielle $\rho dx/\sqrt{R}$, R et ρ étant des fonctions entières. Jour. für Math., vol. 1, pp. 185–221, 1826, or Oeuvres, vol 1, p. 104 ff.

Dobnia. (*b*) Sur le développement de $\sqrt{R}$ en fraction continue. Nouvelles Ann. de Math., ser. 3, vol. 10, pp. 134–140, 1891.

49. **Jacobi.** (*a*) Note sur une nouvelle application de l'analyse des fonctions elliptiques à l'algébre. Jour. für Math., vol. 7, pp. 41–43, 1831, or Werke, vol. 1, p. 327.

Borchardt. (*b*) Application des transcendantes abéliennes à la théorie des fractions continues. Ibid., vol. 48, pp. 69–104, 1854.

50. **Tchebychef.** Sur l'intégration des différentielles qui contiennent une racine carrée d'un polynôme du troisième ou du quatrième degré. Mémoires de l'Acad. imp. des sciences de St. Pétersbourg, ser. 6, vol. 8, pp. 203–232, 1857.

51. **Frobenius und Stickelberger.** Ueber die Addition und Multiplication der elliptischen Functionen. Jour. für Math., vol. 88, pp. 146–184, 1880.

52. **Halphen.** Sur les intégrales pseudo-elliptiques. Comp. Rend., vol. 106 (1888), pp. 1263–1270.

53. **Bortolotti.** Sulle frazioni continue algebriche periodiche. Rendiconti del Circolo Mat. di Palermo, vol. 9, pp. 136–149, 1895.

See also Nos. 21, 26(*a*), 40.

H. Miscellaneous.

54. **Euler.** (*a*) Speculationes super formula integrali

$$\int \frac{x^n dx}{\sqrt{a^2 - 2bx + cx^2}}$$

ubi simul egregiæ observationes circa fractiones continuas occurrent. Acta Acad. scientiarum imperialis Petropolitanae, 1784, pars posterior, pp. 62–84, 1782.

(*b*) Summatio fractionis continuæ cujus indices progressionem arithmeticam constituunt. Opuscula Analytica, vol. 2, pp. 217–239, 1785.

55. **Spitzer.** (*a*) Darstellung des unendlichen Kettenbruchs

$$x+\frac{1}{x+1}+\frac{1}{x+2}+\frac{1}{x+3}+\cdots$$

in geschlossener Form, nebst anderen Bemerkungen. Archiv der Math. und Phys., vol. 30, pp. 81–82, 1858.

(*b*) Darstellung des unendlichen Kettenbruchs

$$2x+1+\frac{1}{2x+3}+\frac{1}{2x+5}+\frac{1}{2x+7}+\cdots$$

in geschlossener Form. Ibid., vol. 30, pp. 331–334, 1858.

(*c*) Note über eine Kettenbrüche. Ibid., vol. 33, pp. 418–420, 1859.

(*d*) Darstellung des unendlichen Kettenbruches

$$\Psi(x)=n(2x+1)+\frac{m}{n(2x+3)}+\frac{m}{n(2x+5)}+\cdots$$

in geschlossener Form. Ibid., vol. 33, pp. 474–475, 1859.

56. **Laurent.** (*a*) Note sur les fractions continues. Nouvelles Ann. de Math., ser. 2, vol. 5, pp. 540–552, 1866.

This treats the continued fraction

$$\frac{x}{1}+\frac{x}{1}+\frac{x}{1}+\cdots.$$

E. Meyer. (*b*) Ueber eine Eigenschaft des Kettenbruches $x-\frac{1}{x}-\frac{1}{x}-\cdots$. Archiv der Math. und Phys., ser. 3, vol. 5, p. 287, 1903.

Meyer's results will be found on p. 548 of Laurent's memoir and differs only in that x has been replaced by $-1/x^2$.

57. **Schlömilch.** (*a*) Ueber den Kettenbruch für tan z. Zeitschrift für Math. und Phys., vol. 16, pp. 259–260, 1871.

Glaisher. (*b*) A continued fraction for tan nx. Messenger of Math., ser. 2, vol. 3, p. 137, 1874.

(*c*) Note on continued fractions for tan nx. Ibid., ser. 2, vol. 4, pp. 65–58, 1875.

58. **Schlömilch.** Ueber die Kettenbruchentwickelung für unvollständige Gamma-function. Zeitschrift für Math. und Phys., vol. 16, pp. 261–262, 1871.

This gives the development of $\int_0^x t^{\mu-1} e^{-t} dt$.

59. **Schendel.** Ueber eine Kettenbruchentwickelung. Jour. für Math., vol. 80, pp. 95–96, 1875.

60. **Lerch.** Note sur les expressions qui, dans diverses parties du plan, représentent des fonctions distinctes. Bull. des sciences Math. ser. 2, vol. 10, pp. 45–49, 1886.

61. **Stieltjes.** (*a*) Sur quelques intégrales definies et leur développement en fractions continues. Quar. Jour. of pure and applied Math., vol. 24, pp. 370–382, 1890.

(*b*) Note sur quelques fractions continues. Ibid., vol. 25, pp. 198–200, 1891.

62. **Hermite.** Sur les polynomes de Legendre. Jour. für Math., vol. 107, pp. 80–83, 1891.

This connects $D_x^{(\gamma)} P^{(n)}(x)$ with a continued fraction.

IV. On the Connection of Continued Fractions with Differential Equations and Integrals.

A. Riccati's Differential Equation.

63. **Euler.** (*a*) De fractionibus continuis observationes. Commentarii academiæ scientiarum imperialis Petropolitanæ, vol. 11, see pp. 79–81, 1739.

(*b*) Analysis facilis æquationem Riccatianam per fractionem continuam resolvendi. Mémoires de l' Acad. imperiale des sciences de St. Pétersbourg, vol. 6, pp. 12–29, 1813.

64. **Lagrange.** Sur l'usage des fractions continues dans le calcul intégral. Nouveaux Mém. de l'Acad. roy. des sciences et belles-lettres de Berlin, 1776, pp. 236–264, or Oeuvres, vol. 4, p. 301 ff.

One of the few important early works.

See 54*b*; also No. 66*a* for work on differential equations of the 1st order.

B. Miscellaneous Differential Equations of the Second Order.

In a numerous class of continued fractions the denominators of the convergents satisfy allied (**Heun**, "*gleichgruppige*") differential equations of the second order. Early instances are found in works of *Gauss* (No. 114), *Jacobi* (No. 65) and *Heine* (No. 72). The theory, from two different aspects, is furthest developed in 66*a* and 76.

65. **Jacobi.** Untersuchung über die Differentialgleichung der hypergeometrischen Reihe. Nachlass. Jour. für Math., vol. 56, 1859; see in particular § 8, pp. 160–161, or Werke, vol. 6, p. 184.

66. **Laguerre.** (*a*) Sur la réduction en fractions continues d'une fraction qui satisfait à une équation différentielle linéaire du premier ordre dont les coefficients sont rationnels. Jour. de Math., ser. 4, vol. 1, pp. 135–165, 1885.

This is a comprehensive memoir which incorporates substantially all the following memoirs:

(*b*) Sur la réduction en fractions continues d'une classe assez étendue de fonctions. Comp. Rend., vol. 87 (1878), p. 923, or Oeuvres, vol. 1, p. 322.

(*c*) Same title as (*a*). Bull. de la Soc. Math. de France, vol. 8 (1880), pp. 21–27, or Oeuvres, vol. 1, p. 438.

(*d*) Sur la réduction en fraction continue d'une fraction qui satisfait à une équation linéaire du premier ordre à coefficients rationnels. Comp. Rend., vol. 98 (1884), pp. 209–212 or Oeuvres, vol. 1, p. 445.

67. **Laguerre.** (*a*) Sur l'approximation des fonctions d'une variable au moyen de fractions rationnelles. Bull. de la Soc. Math. de France, vol. 5 (1877), pp. 78–92 or Oeuvres, vol. 1, p. 277.

(*b*) Sur le développement en fraction continue de

$$e^{\operatorname{arc\,tan}\left(\frac{1}{x}\right) = \int \frac{dx}{1+x^2}}.$$

Ibid., vol. 5 (1877), pp. 95–99 or Oeuvres, vol. 1, p. 291.

(*c*) Sur la fonction $\left(\frac{x+1}{x-1}\right)^{\omega}$.

Ibid., vol. 8 (1879), pp. 36–52, or Oeuvres, vol. 1, p. 345.

(*d*) Sur la réduction en fractions continues de $e^{F(x)}$, $F(x)$ désignant un polynôme entier. Jour. de Math., ser. 3, vol. 6 (1880), pp. 99–110, or Oeuvres, vol. 1, p. 325.

(*d′*) Same subject. Comp. Rend., vol. 87 (1878), p. 820, or Oeuvres, vol. 1, p. 318.

68. **Humbert.** Sur la réduction en fractions continues d'une classe de fonctions. Bull. de la Soc. Math. de France, vol. 8, pp. 182–187, 1879–1880.

69. **Hermite et Fuchs.** Sur un développement en fraction continue. Acta Math., vol. 4, pp. 89–92, 1884.

See also No. 20, 34 *ter*, 71–76.

C. Differential Equations of Order Higher than the Second.

70. **Pincherle.** Sur la génération de systèmes recurrents au moyen d'une équation linéaire differentielle. Acta Math., vol. 16, pp. 341–363, 1892–3.

See also No. 15, 86, 87, 124*b*.

D. The integral $\int_a^b \frac{f(x)dx}{x - z}$.

71. **Heine.** (*a*) Ueber Kettenbrüche. Monatsberichte der k. preussischen Akad. der Wissenschaften zu Berlin, 1866, pp. 436–451.
(*a′*) Mittheilung über Kettenbrüche. Auszug aus dem Monatsberichte, u. s. w. Jour. für Math., vol. 67, pp. 315-326, 1867.

See also Nos. 12, 26*a*, 28, 45, 102, 113, 118*a*.

E. Hyperelliptic and Similar Abelian Integrals.

72. **Heine.** Die Lamé'schen Functionen verschiedener Ordnungen. Jour. für Math., vol. 60, 1862, pp. 252–303; in particular pp. 256, 275, 294–297. Or see his Handbuch, vol. 1 (2te Auf.), pp. 388–396 and 468.

73. **Laguerre.** Sur l'approximation d'une classe de transcendantes qui comprennent comme cas particulier les intégrales hyperelliptiques. Comp. Rend., vol. 84, pp. 643–645, 1877.
(Not found in vol. 1. of his Oeuvres.)

74. **Humbert.** Sur l'équation différentielle linéaire du second ordre. Jour. de l'Ec. Polytech., vol. 29, cahier 48, pp. 207–220, 1880.

75. **Heun.** (*a*) Die Kugelfunctionen und Lamé'schen Functionen als Determinanten. Dissertation, pp. 1–32, Göttingen, 1881.
(*b*) Ueber lineäre Differentialgleichungen zweiter Ordnung deren Lösungen durch den Kettenbruchalgorithmus verknüpft sind. Habilitationsschrift. 1881.
(*c*) Integration regulärer lineärer Differentialgleichungen zweiter Ordnung durch die Kettenbruchentwickelung von ganzen Abel'schen Integralen dritter Gattung. Math. Ann., vol. 30, pp. 553–560, 1887.
(*d*) Beiträge zur Theorie der Lamé'schen Functionen. Math. Ann., vol. 33, pp. 180–196, 1889.
The important group-properties of the continued fraction are here brought out and are further developed in No. 76.

76. **Van Vleck.** Zur Kettenbruchentwickelung hyperelliptischer und ähnlicher Integrale. Dissertation, Göttingen; published in the Amer. Jour. of Math., vol. 16 (1894), pp. 1–91.

After development first from an algebraic standpoint the subject is carried further by the method of conformal representation. The suggestion of this treatment is given in *Klein's* Differentialgleichungen, 1890–91, vol. 1, pp. 180–186.

V. Generalization of the Algebraic Continued Fraction.

A. General Theory.

So far as I have been able to ascertain, the first instance of the generalization is contained in *Hermite's* memoir, No. 84. The development of a general theory is due to *Pade* and *Pincherle.* Nos. 77*a*, 77*b*, and 79*a* are especially recommended.

77. **Pincherle.** (*a*) Saggio di una generallizzazione delle frazioni continue algebriche. Memoirie della R. Accad. delle Scienze dell' Istituto di Bologna, ser. 4, vol. 10, p. 513–538, 1890.

(*a'*) Di un'estensione dell' algorithmo delle frazioni continue. Rendiconti, R. Istituto Lombardo di Scienze e Lettere, ser. 2, vol. 22, pp. 555–558, 1889.

(*b*) Sulla generalizzazione delle frazioni continue algebrique. Annali di Mat., ser. 2, vol. 19, pp. 75–95, 1891.

78. **Hermite.** Sur la généralisation des fractions continues algébriques. Annali di Mat., ser. 2, vol. 21, pp. 289–308, 1893.

79. **Padé.** (*a*) Sur la généralisation des fractions continues algébriques. Jour. de Math., ser. 4, vol. 10, pp. 291–329, 1894.

(*a'*) Same subject. Comp. Rend., vol. 118, p. 848, 1894.

80. **Bortolotti.** Un contributo alla teoria delle forme lineari alle differenze. Annali di Mat., ser. 2, vol. 23, pp. 309–344, 1895.

81. **Cordone.** Sopra un problema fundamentale delle teoria delle frazioni continue algebriche generalizzate. Rendiconti del Circolo di Palermo, vol. 12, pp. 240–257, 1898.

Cordone seeks the regular algorithms which are similar to those of Padé but occur in connection with n series in descending powers of x.

B. Convergence of the Generalized Algorithm.

82. **Pincherle.** Contributo alla generalizzazione delle frazioni continue. Memoirie della R. Accad. delle Scienze dell' Istituto di Bologna, ser. 5, vol. 4, pp. 297–320, 1894.

83. **W. Franz Meyer.** (*a*) Ueber kettenbruchähnlichen Algorithmen. Verhand. des ersten internationalen Mathematiker-Kongresses in Zürich, pp. 168–181, 1898 ; see in particular § 7.

(a') Zur Theorie der kettenbruchähnlichen Algorithmen. Schriften der phys-ökonomischen Gesellschaft zu Königsberg, vol. 38, pp. 57–66, 1897.

C. Special Cases of the Algorithm.

84. **Hermite.** Sur la fonction exponentielle. Comp. Rend., vol. 77, pp. 18–24, 74–79, 226–233, 285–293, 1873.

This is the famous work proving the transcendence of e.

85. **Hermite.** (a) Sur l'expression $U \sin x + V \cos x + W$. Extrait d'une lettre à Monsieur Paul Gordan. Jour. für Math., vol. 76, pp. 303–311, 1873.

(b) Sur quelques approximations algébriques. Ibid., vol. 76, pp. 342–344, 1873.

(c) Sur quelques équations différentielles linéaires. Extrait d'une lettre à M. L. Fuchs de Gottingue. Ibid., vol. 79, pp. 324–338, 1875.

86. **Laguerre.** Sur la fonction exponentielle. Bull. de la Soc. Math. de France, vol. 8 (1880), pp. 11–18, or Oeuvres, vol. 1, p. 336.

87. **Humbert.** (a) Sur une généralisation de la théorie des fractions continues algébriques. Bull. de la Soc. Math. de France, vol. 8, pp. 191–196; vol. 9, pp. 24–30, 1879–1881.

(b) Sur la fonction $(x-1)^a$. Ibid., vol. 9, pp. 56–58, 1880–81.

88. **Pincherle.** Sulla rappresentazione approssimata di una funzione mediante irrazionali quadratici. Rendiconti, R. Istituto Lombardo di Scienze e Lettere, ser. 2, vol. 23, pp. 373–376, 1890.

89. **Pincherle.** (a) Una nuova estensione delle funzioni sferiche. Memoirie della R. Accad. delle Scienze dell'Istituto di Bologna, ser. 5, vol. 1, pp. 337–370, 1890.

(a') Sulla generalizzazione delle funzioni sferiche. Bologna Rendiconti, 1891–92, pp. 31–34.

(b) Un sistema d'integrali ellittici considerati come funzioni dell'invariante assoluto. Atti della R. Accad. dei Lincei, ser. 4, vol. 7_1, pp. 74–80, 1891.

90. **Bortolotti.** (a) Sui sistemi ricorrenti del 3° ordine ed in particolare sui sistemi periodici. Rendiconti del Circolo di Palermo, vol. 5, pp. 129–151, 1891.

(b) Sulla generalizzazione delle frazioni continue algebriche periodiche. Ibid., vol. 6, pp. 1–13, 1892.

VI. Series of Polynomials (Näherungsnenner).

The series

$$\frac{1}{v-u} = \sum_{n=0}^{\infty} (2n+1) Q^{(n)}(v) P^{(n)}(u)$$

was first given by *Heine* in Crelle's Jour., vol. 42 (1851), p. 72. See also his Handbuch, vol. 1, pp. 78–79, 197–200. Among the numerous works relating to expansions in terms of *Kugelfunctionen erster und zweiter Gattung* may be mentioned :

91. **Bauer.** Von den Coefficienten der Reihen von Kugelfunctionen einer Variablen. Jour. für Math., vol. 56, pp. 101–121, 1859.
92. **C. G. Neumann.** Ueber die Entwickelung einer Function mit imaginärem Argumente nach den Kugelfunctionen erster und zweiter Gattung, Halle, 1862.
93. **Thomé.** Ueber die Reihen welche nach Kugelfunctionen fortschreiten. Jour. für Math., vol. 66, pp. 337–343, 1866.
94. **Laurent.** Mémoire sur les fonctions de Legendre. Jour. de Math., ser. 3, vol. 1, pp. 373–398, 1875.

   See the comments by Heine in vol. 2, pp. 155–157, also by Darboux and Laurent in the same vol., pp. 240, 420.

Numerous memoirs relate to series in terms of the polynomials arising from the expansion of $(1-2ax+a^2)^\nu$. It suffices here to refer to the Encyklopädie der Math. Wissenschaften, I A 10, § 31.

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95. **Frobenius.** Ueber die Entwicklung analytischer Functionen in Reihen, die nach gegebenen Functionen fortschreiten. Jour. für Math., vol. 73, pp. 1–30, 1871.

   An interesting memoir.
96. **Darboux.** Sur l'approximation des fonctions de très-grands nombres et sur une classe étendue de développements en série, Part 2. Jour. de Math., ser. 3, vol. 4, pp. 377–416, 1878.
97. **Gegenbauer** Ueber Kettenbrüche. Wiener Berichte, vol. 80, Abth. 2, pp. 763–775, 1880.
98. **Poincaré.** (a) Sur les équations linéaires aux différentielles ordinaires et aux différences finies. Amer. Jour. of Math., vol. 7, pp. 243–257, 1885.

   This gives an important criterion for the convergence of series of polynomials. See lecture 4.

   (a') Sur les séries des polynomes. Comp. Rend., vol. 56, p. 637, 1883.
99. On the series $\Sigma A_n(x-a_1)(x-a_2)\cdots(x-a_n)$.

   A series of this form is employed in **Newton's** interpolation formula, Philosophiæ naturalis principia, book 3, lemma V. See the Encyklopädie der Math. Wissenschaften, I D 3, § 3. A similar use is made by

Cauchy. (*a*) Sur les fonctions interpolaires. Comp. Rend., vol. 11, pp. 775–789, 1841.

See next No. 95.

Peano. (*b*) Sulle funzioni interpolari. Atti della R. Accad. delle Scienze di Torino, vol. 18, pp. 573–580, 1883.

Bendixson. (*c*) Sur une extension à l'infini de la formule d'interpolation de Gauss. Acta Math., vol. 9, pp. 1–34, 1886.

(*c′*) Sur la formule d'interpolation de Lagrange. Comp. Rend., vol. 101 (1885), pp. 1050–1053 and 1129–1131.

Pincherle. (*d*) Sull'interpolazione. Memoirie della R. Accad. delle Scienze di Bologna, ser. 5, vol. 3, pp. 293–318.

(See a "note historique" by **Eneström**, Comp. Rend., vol. 103, p. 523, 1886).

See also No. 103.

100. **Pincherle,** Sur le développement d'une fonction analytique en série de polynomes. Comp. Rend., vol. 107, p. 986, 1888.

101. **Pincherle.** Résumé de quelques résultats relatifs à la théorie des systèmes recurrents de fonctions. Mathematical Papers, Chicago Congress, 1893, pp. 278–287.

102. **Blumenthal.** Ueber die Entwickelung einer willkürlichen Funktion nach den Nennern des Kettenbruches fur

$$\int_{-\infty}^{0} \frac{\phi(\xi)d\xi}{z-\xi}.$$

Dissertation, Göttingen, 1898.

The most advanced development of this subject is found in the work of *Blumenthal* and *Pincherle.*

103. **Laurent.** Sur les séries de polynomes. Jour. de Math., ser. 5, vol. 8, pp. 309–328, 1902.

104. **Stekloff.** Sur le développement d'une fonction donée en séries procédant suivant les polynomes de Tchébicheff et, en particulier, suivant les polynomes de Jacobi. Jour. für Math., vol. 125, pp. 207–236, 1903.

See also Nos. 20, 70, 71.

104 *bis.* **Rouché.** Mémoire sur le développement des fonctions en séries ordonnées suivant les dénominateurs des réduites d'une fraction continue. Jour. de l'Ec Polytech., cahier 37, pp. 1–34.

This memoir has a close connection with the work of Tchebychef.

VII. On the Roots of the Numerators and Denominators of the Convergents.

105. **Sylvester.** (*a*) On a remarkable modification of Sturm's theorem. Phil. Mag., ser. 4, vol. 5, pp. 446–456, 1853.

(*b*) Note on a remarkable modification of Sturm's theorem and on a new rule for finding superior and inferior limits to the roots of an equation. Ibid., vol. 6, pp. 14–20, 1853.

(*c*) On a new rule for finding superior and inferior limits to the real roots of any algebraic equation. Ibid., vol. 6, pp. 138–140, 1853.

(*d*) Note on the new rule of limits. Ibid., vol. 6, pp. 210–213, 1853.

(*e*) On a theory of the syzygetic relations of two rational integral functions, comprising an application to the theory of Sturm's functions, and that of the greatest algebraic common measure. Phil. Trans., 1853; see in particular p. 496 ff.

(*f*) Théorème sur les limites des racines réelles des équations algébriques. Nouvelles Ann. de Math., ser. 1, vol. 12, pp. 286–287, 1853.

(*g*) Pour trouver une limite supérieure et une limite inférieure des racines réelles d'une équation quelconque. Ibid., ser. 1, vol. 12, pp. 329–336, 1853.

106. **Laguerre.** Sur quelques propriétés des équations algébriques qui ont toutes les racines réelles. Nouvelles Ann. de Math., ser. 2, vol. 19 (1880), pp. 224–239, or Oeuvres, vol. 1, pp. 113–118.

Laguerre considers here the roots of the numerators and denominators of the approximants for $f(x)$ and $1/f(x)$ when $f(x)$ is a polynomial with real roots.

107. **Gegenbauer.** (*a*) Ueber algebraische Gleichungen welche nur reele Wurzeln besitzen. Wiener Berichte, vol. 84 (1882), Abt. 2, see in particular pp. 1106–1107.

(*b*) Ueber algebraische Gleichungen welche eine bestimmte Anzahl complexer Wulzeln besitzen. Ibid, vol. 87, pp. 264–270, 1883.

108. **Markoff.** Sur les racines de certaines équations. Math. Ann., vol. 27, pp. 143–150, 1886.

108 *bis*. **Hurwitz.** Ueber die Nullstellen der Bessel'schen Function. Math. Ann., vol. 33, pp. 246–266, 1889.

Although the functions considered in this memoir are of a special character, the memoir is mentioned here on account of the methods employed.

109. **Porter.** On the roots of functions connected by a linear recurrent relation of the second order. Annals of Math., ser. 2, vol. 3, pp. 55–70, 1902.

See also Nos. 20, 26*a*, 31, 32*a*, 45, 56, 71, 74, 76, 87*a*, 118*a*.

VIII. Approximation to a Function at More Than One Point. Connection of Continued Fractions with the Theory of Interpolation.

Under No. 99 have been already classified various works which relate to simultaneous approximation at several points. In addition, the following memoirs may also be consulted:

110. **Cauchy.** Sur la formule de Lagrange relativ à interpolation. Analyse Alg., p. 528, or Oeuvres, ser. 2, vol. 3, pp. 429–433.

111. **Jacobi.** Ueber die Darstellung eine Reihe gegebner Werthe durch eine gebrochene rationale Function. Jour. für Math., vol. 30, pp. 127–156, 1846, or Werke, vol. 3, p. 479.

112. **Padé.** Sur l'extension des propriétés des réduites d'une fonction aux fractions d'interpolation de Cauchy. Comp. Rend., vol. 130, p. 697, 1900.

See also Nos. 95, 99.

For general works upon interpolation which bring out the relation of the subject to continued fractions, see Heine's Handbuch der Kugelfunctionen, vol. 2, and Markoff's Differenzenrechnung (deutsche Uebersetzung), chap. 1, 6, 7 ; also the following memoir :

113. **Posse.** Sur quelques applications des fractions continues algébriques. Pp. 1–175, 1886.

114. **Gauss.** Methodus nova integralium valores per approximationem inveniendi. Werke, vol. 3, pp. 165–196, 1816.

115. **Christoffel.** Ueber die Gaussische Quadratur und eine Verallgemeinerung derselben. Jour. für Math., vol. 55, pp. 61–82, 1858.

116. **Mehler.** Bemerkungen zur Theorie der mechanischen Quadraturen. Ibid., vol. 63, pp. 152–157, 1864.

117. **Posse.** Sur les quadratures. Nouvelles Ann. de Math., ser. 2, vol. 14, pp. 49–62, 1875.

118. **Stieltjes.** (*a*) Quelques recherches sur la théorie des quadratures dites mécaniques. Ann. de l'Ec. Nor., ser. 3, vol. 1, pp. 409–426, 1884.

We find here the origin of his notable 1894 memoir, No. 26*a*.

(*a′*) Sur l'évaluation approchée des intégrales. Comp. Rend., vol. 97, pp. 740 und 798, 1883.

(*b*) Note sur l' intégrale $\int_a^b f(x)G(x)dx$.

Nouv. Ann. de Math., ser. 3, vol. 7, pp. 161–171, 1888.

119. **Markoff.** Sur la méthode de Gauss pour le calcul approché des intégrales. Math. Ann., vol. 25, pp. 427–432, 1885.

120. **Pincherle.** Su alcune forme approssimate per la rappresentazione di funzioni. Memoirie della R. Accad. delle Scienze dell'Istituto di Bologna, ser. 4, vol. 10, pp. 77–88, 1889.

121. **Tchebychef.** A brief sketch of the memoirs below will be found on pp. 17–20 of *Vassilief's* memoir on "P. L. Tchebychef et son oeuvre scientifique."

(*a*) Sur les fractions continues. Jour. de Math., ser. 2, vol. 3, pp. 289–323, 1858, or Oeuvres, vol. 1, p. 203–230.

(*b*) Sur une formule d'analyse. Bull. Phys. Math. de l'Acad. des sciences de St. Pétersbourg, vol. 13, pp. 210–211, 1854, or Oeuvres, vol. 1, pp. 701–702.

(*c*) Sur une nouvelle série. Ibid., vol. 17, pp. 257–261, 1858, or Oeuvres, vol. 1, pp. 381–384.

(*d*) Sur l'interpolation par la méthode des moindres carrés. Mém. de l'Acad. des sciences de St. Pétersbourg, ser. 7, vol. 1, pp. 1–24, 1859, or Oeuvres, vol. 1, pp. 473–498.

(*e*) Sur le développement des fonctions à une seule variable. Bull. de l'Acad. imp. des sciences de St. Pétersbourg, ser. 7, vol. 1, pp. 194–199, 1860, or Oeuvres, vol. 1, pp. 501–508.

IX. Miscellaneous.

122. **Tchebychef.** (*a*) Sur les fractions continues algébriques. Jour. de Math., ser. 2, vol. 10, pp. 353–358, 1865, or Oeuvres, vol. 1, pp. 611–614.

(*b*) Sur le développement des fonctions en séries à l'aide des fractions continues, 1866. Oeuvres, vol. 1, pp. 617–636.

(*c*) Sur les expressions approchées, linéares par rapport a deux polynomes. Bull. des sciences Math. et Astron., ser. 2, vol. 1, pp. 289, 382 ; 1877.

Hermite. (*d*) Sur une extension donnée à la théorie des fractions continues par M. Tchebychef. Jour. für Math., vol. 88, pp. 12–13, 1880.

123. **Tchebychef.** (*a*) Sur les valeurs limites des intégrales. Jour. de Math., ser. 2, vol. 19, pp. 157–160, 1874.

(*b*) Sur la representation des valeurs limites des intégrales par des residus integraux (1885). Acta. Math. vol. 9, pp. 35–56, 1887.

Markoff. (*c*) Démonstration de certaines inégalités de M. Tchebychef. Math. Ann., vol. 24, pp. 172–178, 1884.

(*d*) Nouvelles applications des fractions continues. Math. Ann., vol. 47, pp. 579–597, 1896.

124. **Laguerre.** (*a*) Sur le développement de $(x-z)^m$ suivant les puissances de (z^2-1). Comp. Rend., vol. 86 (1878), p. 956, or Oeuvres, vol. 1, p. 315.

(*b*) Sur le développement d'une fonction suivant les puissances d'une polynome. Jour. für Math., vol. 88 (1880); in particular, p. 37, or Oeuvres, vol. 1, p. 298.

(*c*) Same subject. Comp. Rend., vol. 86, (1878) p. 383, or Oeuvres, vol. 1, p. 295.

(*d*) Sur quelques théorémes de M. Hermite. Extrait d'une lettre addressée à M. Borchardt. Jour. für Math., vol. 89 (1880), pp. 340–342, or Oeuvres, vol. 1, p. 360.

125. **Sylvester.** Preuve que π ne peut pas être racine d'une équation algébrique à coefficients entiers. Comp. Rend., vol. 111, pp. 866–871, 1890.

A fundamental error in the proof has been pointed out by Markoff. See p. 386 of vol. 30 of the Fortschritte der Math.

126. **Gegenbauer.** Ueber die Näherungsnenner regulärer Kettenbrüche. Monatshefte für Math. und Phys., vol. 6, pp. 209–219, 1895.

127. **Bortolotti.** Sulla rappresentazione approssimata di funzioni algebriche per mezzo di funzioni razionale. Atti della R. Accad. dei Lincei, ser. 5, vol. 1_1, pp. 57–64, 1899.

Addendum to I A.

128. **Euler.** De fractionibus continuis dissertatio. Comment. Petrop., vol. 9, p. 129 ff., 1737.