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EDITED BY

THOMAS S. FISKE AND HAROLD JACOBY
COMMITTEE OF PUBLICATION.

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PREFACE.

IN presenting this, the first volume of the BULLETIN OF THE NEW YORK MATHEMATICAL SOCIETY, the editors feel that they should express their deep sense of the obligation under which the encouragement and active assistance of the members of the Society have placed them. The interest which the Journal has excited, both in this country and abroad, shows how real is the need of a historical and critical journal, devoted to mathematical science, and published in the English language. It is to be hoped that in the future the BULLETIN may be able to extend its work, and ever the more adequately occupy the field which it is its aim to cover.

LIST OF PUBLISHED PAPERS

READ BEFORE THE NEW YORK MATHEMATICAL SOCIETY, TOGETHER WITH PLACE OF PUBLICATION.

BOCHER, Maxime. Collineation as a Mode of Motion. *Bulletin of the New York Mathematical Society*, vol. I., pp. 225–231, July, 1892.

ENGLER, Edmund A. A Geometrical Construction for Finding the Foci of the Sections of a Cone of Revolution. *Transactions of the Academy of Science of St. Louis*, vol. VI., pp. 49–55, April, 1892.

FISKE, Thomas S. Weierstrass's Elliptic Integral. *Annals of Mathematics*, vol. VI., pp. 7–11, June, 1891.
On the Doubly Infinite Products. *Bulletin of the New York Mathematical Society*, vol. I., pp. 61–66, Dec., 1891.

JACOBY, Harold. On the Determination of Azimuth by Elongations of Polaris. *Monthly Notices of the Royal Astronomical Society*, pp. 106–113, Dec., 1891.

MACFARLANE, Alexander. On Exact Analysis as the Basis of Language. *Bulletin of the New York Mathematical Society*, vol. I., pp. 189–193, May, 1892.

McCLINTOCK, Emory. On the Algebraic Proof of a Certain Series. *American Journal of Mathematics*, vol. XIV., pp. 67–71, Oct., 1891.
On Independent Definitions of the Functions log x and e^x. *American Journal of Mathematics*, vol. XIV., pp. 72–86, Oct., 1891.

MERRIMAN, Mansfield. Final Formulas for the Solution of Quartic Equations. *Bulletin of the New York Mathematical Society*, vol. I., pp. 202–205, June, 1892.

PUPIN, M. I. On a Peculiar Family of Complex Harmonics. *Transactions of the American Institute of Electrical Engineers*, vol. VIII., pp. 579–585, Dec., 1891.

STEINMETZ, Charles P. Multivalent and Univalent Involutory Correspondences in a Plane Determined by a Net of Curves of the n-th Order. *American Journal of Mathematics*, vol. XIV., pp. 39–66, Oct., 1891.
On the Curves which are Self-reciprocal in a Linear Nul-system, and their Configurations in Space. *American Journal of Mathematics*, vol. XIV., pp. 161–186, April, 1892.

STRINGHAM, Irving. A Classification of Logarithmic Systems. *American Journal of Mathematics*, vol. XIV., pp. 187–194, April, 1892.

BULLETIN OF THE

NEW YORK MATHEMATICAL SOCIETY

OCTONARY NUMERATION.

BY PROF. W. WOOLSEY JOHNSON.

THE comparatively small progress toward universal acceptance made by the metric system seems to be due not altogether to aversion to a change of units, but also to a sort of irrepressible conflict between the decimal and binary systems of subdivision.

Before the introduction of decimal fractions, about 1585, no connection would be felt to exist between the established scale of numeration and the method of subdividing physical units, and it would probably never occur to any one to subdivide a unit into tenths. The natural method is to bisect again and again. The mechanic prefers to divide the inch into halves, quarters, eighths, and sixteenths. The retailer of dry goods, whose unit is the yard, divides it into halves, quarters, and eighths, totally ignoring the inch. The mariner not only divides the horizontal angular space in which his course is laid down into quarters, thus recognizing the right angle as the natural unit,* but divides the space between the cardinal points of the compass into halves, quarters, and eighths. Where decimal money has been introduced quarters are insisted on in spite of their inconsistency with the decimal system. We are compelled to coin quarter dollars, and prices are very commonly quoted in eighths and even sixteenths of a dollar. Great Britain is compelled to coin eighths of a pound sterling, though half a crown contains a fraction of a shilling. The French divide the litre into quarters. The broker expresses prices in halves, quarters, and eighths of one per cent.

This irrepressible conflict would, of course, never have

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* The uncompromising advocate of the metric system will not content himself with the centesimal division of the degree, but insists upon the centesimal division of the quadrant, although it is difficult to see how the latter could possess any advantage in the way of facilitating numerical computations. But why do they not go further and advocate the centesimal division of the whole circumference?

existed, but all would have been harmony, if the radix of our system of numeration had been a power of two. Mr. Alfred B. Taylor published in 1887 a very interesting pamphlet on "Octonary Numeration," being a paper read before the American Philosophical Society. After an extended review of the question, with many interesting historical notes, he argues in favor of the octonary system, and then proceeds to "develop the scale of notation thus selected, and to derive from it an ideal system of weights and measures."

This is not the place to consider the merits of a system of weights and measures; we propose therefore to consider only the theoretical merits of the octonary system. We regret that in his ingenious paper Mr. Taylor has caused his system to wear an outlandish look by employing new names, not only for his units of weights and measures, but also for the numbers from one to eight, and even new characters for the seven digits. We see no necessity for changing the characters or the names of the digits, although it would be necessary, in order to avoid the use of an old name in a new meaning, to replace the suffix *-ty* by a new one to denote the second place (which Mr. Taylor, having changed the names of the digits, did not find necessary). We might use the suffix *-ate*—thus the octonary 40 would be read *fourate*, that is, four eights; 56 would be read *fivate-six*, that is five eights and six.

The only advantage of a large radix, *quâ* large, over a smaller one is in diminishing the number of figures required on the average to express a given number. The number of figures is inversely as the logarithm of the radix; and, in passing from the radix ten to the radix eight, it increases only in the ratio of 10 : 9. The ratio increases rapidly for smaller radices, until for the binary system it becomes 10 : 3, as compared with the denary, and 3 : 1 as compared with the octonary system. To set against this we have, in favor of the smaller radix, the simplicity due to dispensing with superfluous characters; but of far more importance is the simplifying of the multiplication table. For example, the octonary multiplication table stands thus:

1	2	3	4	5	6	7
2	4	6	10	12	14	16
3	6	11	14	17	22	25
4	10	14	20	24	30	34
5	12	17	24	31	36	43
6	14	22	30	36	44	52
7	16	25	34	43	52	61

In comparing the labor with which this table could be committed to memory with that required by the denary table, it would seem fair to disregard in both cases not only the line and column corresponding to 1 (although our German friends insist upon *ein mal eins*), but also those corresponding to 2 and to half the radix, on account of their simplicity. Thus the difficulty would be about as 6^2 to 4^2; indeed, it seems safe to say that the difficulty experienced by children in acquiring the multiplication table, and that of older people in retaining it in a condition fresh enough to be used without an effort of thought, would be reduced more than one-half even by this slight decrease in the magnitude of the radix.

For a further decrease of radix, the difficulty of the multiplication table decreases rapidly: for the binary system no multiplication table exists, but even for the radix four the difficulty has practically disappeared.

But this advantage of a very small radix is, as mentioned above, attended by a rapid increase in the number of figures required to express a given number; and the inconvenience arising from this source has, we think, been frequently underestimated. Binary arithmetic, in which the characters 1 and 0 alone are used, has even been proposed by some enthusiasts as a substitute for logarithmic computation. Mr. Taylor, in commenting upon this system, after mentioning the absence of tables to be committed to and retained in the memory, says: "Every form of calculation would be resolved into simple numeration and notation. In fact, calculation as an effort of mathematical thought might be said to be entirely dispensed with, and the labor of the brain to be all transferred to the eye and hand. A perfect familiarity with the notation of the scale, and with the simple rules of position, *would enable the operator to determine in every case by mere inspection, whether the next figure should be a* 1 *or a* 0. It follows that the only errors possible in such a work would be the merely clerical ones of the eye or hand; * * * and it may well be doubted whether, in all important and lengthy calculations, the binary system would not be found to afford a real economy of labor, instead of an increase as has been generally supposed."

Now it is to be remarked that the number of figures used in calculation would increase at a rate much greater than that of the number of figures used in expressing results. For example, in performing a multiplication in the binary notation, the number of figures to be written down (after making due allowance for the greater proportion saved by the occurrence of ciphers in the multiplier) would be about five times, instead of three times, the number occurring in the same operation performed in the octonary notation.

Again, whenever the columns to be added are of considerable length their summation, though executed by mere counting and the determination of the numbers "to carry," would require fixed attention, and involve liability to error; so much so, that the words we have italicized in the quotation appear hardly justified. The numbers to carry would be inconveniently large, especially if mentally expressed in the binary system. Indeed, counting in this system would obviously be very much more liable to error than in the denary (or in the octonary) system, which gives highly distinctive names to all such numbers as have to be carried in the mind in the course of calculation.

The same objection exists, though to a less degree, to the quaternary system, so that the labor of accurate calculation in this system, although perhaps less than in the denary system, would probably exceed that which would be required in the octonary system.

The conclusion appears to be inevitable that, considering only the two features which depend upon the mere size of the radix, ten is decidedly too large and four too small a radix, so that the ideal radix in this respect is about six or eight.

Passing now to the intrinsic character of the radix, it is desirable that the radix should be divisible by simple factors. Thus it is universally admitted that an uneven radix would be quite out of the question. It is indispensable for a multitude of purposes that even the least instructed persons should be familiar with the distinction between even and uneven numbers, and able to recognize at a glance to which class a given number belongs. It was formerly the custom to extol twelve as an ideal radix, because of its divisibility by two, three, four, and six. Divisibility by three, although incomparably less important than divisibility by two, would no doubt be a great convenience, much more so than divisibility by five; but it is doubtful whether much weight should be given to divisibility of the first power of the radix by four, so long as we do not adopt a purely binary system (that is, two or a power of two for radix). We ought rather to consider only the prime factors of the radix, so that six would possess all the advantages of twelve, and since on the other score twelve is far too large a radix, six would be far preferable to it. (The number of figures used to express a given number would for six exceed that for twelve only in the ratio 7:5, and would exceed that for ten only in the ratio 9:7.)

Against this advantage of divisibility by different prime factors we have to set the advantages of a purely binary system. Theoretical considerations here point in the same direction as the practical ones rehearsed in the first part of this paper. Owing to the unique character of the number two, it must be admitted that the expression of a given number in powers of two gives a better notion of its intrinsic character than expression in powers of any other number. Accordingly the binary system has always been regarded as theoretically the ideal system, although for practical purposes the great number of figures used in expressing numbers is an insuperable objection. Now it is to be noticed that if the radix is a power of two, we have virtually all the advantages of the binary system. For example, if we have a number expressed in the octonary system, we have only to substitute for the characters 0, 1, 2, ...7 their binary equivalents 000, 001, 010, ...111 to obtain the number in the binary system.

The digital expression of a number in the octonary system would be much more suggestive of its intrinsic nature than expression in any non-binary system, for the highest power of two contained as a factor in a number is its most important characteristic. Again the distinction between numbers of the form $4n + 1$ and those of the form $4n + 3$ is of great importance in the theory of numbers, and in the octonary system it would be obvious at a glance to which of these classes a given uneven number belongs. So also with the distinction between "evenly even" and "unevenly even" numbers. It is interesting also to note that the square of every uneven number would end in 1, the preceding figures expressing a triangular number. Thus the uneven squares in octonary notation are 1, 11, 31, 61, 121....

We have seen above that, if divisibility by another prime factor besides two be regarded as the paramount desideratum, six would be preferable to ten as a radix. But the tests of divisibility by small divisors (such as the familiar one for nine or three) would always to a great extent serve the same purpose as the divisibility of the radix. These tests depend upon the lowest value of n for which $r^n - 1$ or $r^n + 1$ (r being the radix) is divisible by the divisor in question; and they consist in reducing the given number to one of n places which will give the same remainder when divided by the given divisor. This is done in the first case by the addition of periods of n figures each, beginning with the units; and in the second case, by the addition of periods of $2n$ figures, followed by subtraction of the second period of n figures from the first. For example, with the radix ten we can test for each of the divisors seven, eleven, and thirteen, which are factors of $10^3 + 1$, by reducing to six places by addition of periods of six, and then to three places by subtraction of the figures representing thousands from the first or unit period of three figures.

Let us see how the matter would stand in the octonary system. For seven we should add all the digits, and for nine (and therefore for three) we should add by periods of two. Again since $8^2 + 1 = 5 \times 13$, we should test for five and thirteen (or oneate-five) by reducing to four figures by addition, and then to two figures by subtraction. Among small primes, eleven is the least adapted to the octonary system, but for this divisor we might convert the given number to the binary system, then reduce to ten figures by addition, and to five by subtraction (since $2^5 + 1 = 3 \times 11$), and finally reconvert into an octonary number of two digits.

As there is no doubt that our ancestors originated the decimal system by counting on their fingers, we must, in view of the merits of the octonary system, feel profound regret that they should have perversely counted their thumbs, although nature had differentiated them from the fingers sufficiently, she might have thought, to save the race from this error.

THE TEACHING OF ELEMENTARY GEOMETRY IN GERMAN SCHOOLS.

Inhalt und Methode des planimetrischen Unterrichts. Eine vergleichende Planimetrie. Von DR. HEINRICH SCHOTTEN. Leipzig, B. G. Teubner, 1890. 8vo, pp. iv. + 370.

WHOEVER has followed the efforts of the Association for the Improvement of Geometrical Teaching in England in the course of the last ten years will have been struck by the slowness of the progress made and the paucity of the practical results attained. In Germany there exists no such society; but a powerful agitation for the reform of geometrical teaching has been in progress there for at least sixty years, and with particular force during the last two decades. And yet, even from Germany, with its well developed and highly centralized system of education, comes the complaint that progress is slow and much remains to be done.

Recent statistics have shown, in particular, that the most widely used text-books are far from being the best. Thus, while Hubert Müller's Geometry, which may be regarded as the best representative of the "modern school," reached its third edition in 1889, after a lapse of fifteen years from its first appearance, Kambly's very inferior text-book, whose faults and mistakes have frequently been exposed and complained of, appeared in 1884 in its 74th edition.

This book of Kambly's easily leads in the list of text-books used in various schools; it is adopted in 217 schools, the next in order being another rather inferior book, by Koppe, introduced in 51 schools ; then follow Mehler's, used in 44, Reidt's in 29, etc., while there are 55 mathematical text-books used in but one school each. Similar statistics for our American schools would be both interesting and instructive.

Still the signs of improvement are not wanting. Some very good text-books of geometry have been published in recent years and are making, though slowly, their way to the front; preparatory ("propædeutic") courses in "intuitive" geometry in connection with geometrical drawing have been introduced in many schools, and are generally recommended by the school boards; excellent classified and graded collections of problems have appeared and are in actual use ; and, above all, the whole subject of the improvement of geometrical teaching has been ventilated and discussed with great thoroughness and completeness.

In this last respect the work done by Hoffmann's *Zeitschrift** cannot be estimated too highly. The volumes of this journal, specially devoted to the discussion of scientific instruction in the secondary schools, are replete with material for the study of this question, the editor himself being one of the principal contributors. It is to be hoped that the *Bulletin of the New York Mathematical Society* may, in the course of time, perform a similar service towards the improvement of mathematical instruction in this country.

On the other hand, the custom of many German schools of publishing scientific and educational essays in connection with the school calendar ("*Programm*") has given a welcome opportunity to many experienced teachers to express their ideas on the subject and to propose improvements.

The material that has grown up in this way is somewhat bewildering in extent, and, moreover, not very ready of access to American students. A full set of Hoffmann's *Zeitschrift* is probably to be found in but very few libraries in this country; many of the older "*Programme*" are hard to obtain; and of the legion of German text-books of geometry that have appeared during the present century only a very small number, of course, have found their way into American libraries.

The attempt made by Dr. Schotten to sum up the results of the various efforts of reform in geometrical teaching in the secondary schools and to give a critical survey of the literature of this subject, will therefore be welcomed by all interested in elementary geometrical instruction.

The title of Dr. Schotten's work is perhaps somewhat misleading, as it does not indicate that his study is confined

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* J. C. V. Hoffmann's *Zeitschrift für mathematischen und naturwissenschaftlichen Unterricht*, published by Teubner, Leipzig.

entirely to *German* books and papers. Nor is there any indication on the title-page that the present volume is only a first instalment of the work; a second volume is announced in the preface and on the last page of the book, but even this would not seem to exhaust the subject.

There is no table of contents, and no *general* index, a serious defect in a work of this kind, which may perhaps be remedied in the second volume. The book is of course made up in a large measure of quotations, interspersed with critical remarks by the author; unfortunately, the arrangement is far from convenient, and in some instances very awkward. In general, however, the author has well accomplished his exceedingly laborious task. He shows a thorough acquaintance with the literature of his subject, as well as good judgment and discrimination in making use of it.

In an introductory essay, Dr. Schotten briefly states his views on what is desirable in the way of reform. He sustains these views, which are not over radical, not so much by argument as by a large array of quotations from various sources. They may therefore be taken to fairly represent the better thought of the day on the subject, at least in Germany.

It may be of interest to give a short account of these fundamental principles in teaching geometry which have found the approval of so large a body of experienced and well-trained teachers.

The study of mathematics in the *Gymnasium* should begin with geometry (in *Tertia*, i. e., in the fourth and fifth years of the whole nine-year course), being followed by algebra in *Secunda* (also two years), while the last two years (*Prima*) are reserved for trigonometry and a thorough review of the whole course. There is no urgent demand for increasing the *extent* of mathematical instruction; but what is taught should be taught well, that is, with thoroughness and accuracy. The object of mathematical teaching in the *Gymnasium* is not to form mathematicians, but to improve the mind, not only by training in logical thinking, but by accustoming the student to precision of language in writing and speaking, by awakening his self-activity through the solution of problems, and in the case of geometry in particular, by forming and practising the power of mental intuition (*"Anschauung"*).

These objects, however, cannot be attained by the so-called Euclidean method of teaching geometry. While Euclid's arrangement of the propositions has long been abandoned in German text-books, his synthetic method of proof is still retained in many books. Here reform is most peremptory.

The "genetic" method should pervade the whole course; that is to say, the student should be led up in a natural way to each proposition, so as to see clearly its connection with what precedes, and finally conceive of it, not as a single artificial experiment in reasoning, but as an essential member of an organic whole.

The proof of a proposition should be obtained by what the Germans call the "heuristic" method, i.e., by the process that would naturally be adopted by any one trying to *find* the proof himself anew. A "synthetic" reconstruction of the proof may finally be added in some cases.

Frequent reviews are of course required to keep the student constantly alive to the conviction that he is studying a well-connected system, and not a mass of detached single facts.

The introduction of some of the ideas of modern projective geometry (symmetry, dualism, theory of rows and pencils, correspondences, etc.) will be found a great help in building up a natural system of geometry. But wherever used, these ideas must be closely interwoven with the whole system; it is decidedly objectionable to merely put these matters into an appendix at the end of the book as is sometimes done.

There is however a fundamental difficulty in introducing the ideas of projective geometry into elementary teaching. It lies in the fact that the circle is the only curved line considered in ordinary elementary geometry, while in modern geometry the circle appears as a very special case of a conic section.* This circumstance will indicate how far we may go in applying the methods of modern geometry to an elementary course, provided the study of the conic sections be excluded.

Let us now turn to the main body of Dr. Schotten's work. It is divided into five chapters: (1) Space, (2) Geometry, (3) The Space-Forms (solid, surface, line, point), (4) The Plane, (5) The Straight Line. In a second volume the author promises to treat in a similar way of (1) Direction and Distance; position of points, lines, and circles in their mutual relations; metrical relations; (2) The Axiom of Parallel Lines (Eucl. XI.); (3) The Angle; (4) Auxiliary Geometrical Ideas, such as equality, motion, dimension, concept, definition, proof, explanation, postulate, theorem, axiom, form, magnitude, position, figure, locus, symmetry, etc.; (5) Method.

The author prefaces each chapter by a brief statement of his own views, and then follow quotations from all those text-books or other works that express any original ideas on the subject of the chapter. Comments by the author on these quotations are usually given in foot-notes. But it must be

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* See O. RAUSENBERGER. *Elementargeometrie des Punktes, der Geraden und der Ebene, systematisch und kritisch behandelt.* Leipzig, Teubner, 1887, pp. 2–3.

said that the whole is not sufficiently well digested, and it requires some labor (which the author might have spared the reader) to get at the final results.

It will not be necessary to pass in review here the manifold and widely different views of the fundamental conceptions of geometry collated in Dr. Schotten's book.

The tendency in Germany seems to be at present to escape as far as possible the hidden dangers that await the teacher at the very threshold of geometry in the definitions of such ideas as space, geometry, the point, the plane, etc., by two means : (1) by requiring a preparatory course in geometrical drawing in which the student should become thoroughly familiar, in a practical way, with the fundamental geometrical ideas ; (2) by a strict adherence to Pascal's rules.

As these rules do not seem to be as widely known as they deserve to be,* they are here transcribed in full from Pascal's essay, "*De l'esprit géométrique.*" †

Rules for definitions.—"1. N'entreprendre de définir aucune des choses tellement connues d'elles-mêmes, qu'on n'ait point de termes plus clairs pour les expliquer. 2. N'omettre aucun des termes un peu obscurs ou équivoques, sans définition. 3. N'employer dans la définition des termes que des mots parfaitement connus, ou déjà expliqués."

Rules for axioms.—"1. N'omettre aucun des principes nécessaires sans avoir demandé si on l'accorde, quelque clair et évident qu'il puisse être. 2. Ne demander, en axiomes, que des choses parfaitement évidentes d'elles-mêmes."

Rules for demonstrations.—"1. N'entreprendre de démontrer aucune des choses qui sont tellement évidentes d'elles-mêmes qu'on n'ait rien de plus clair pour les prouver. 2. Prouver toutes les propositions un peu obscures, et n'employer à leur preuve que des axiomes très-évidents, ou des propositions déjà accordées ou démontrées. 3. Substituer toujours mentalement les définitions à la place des définis, pour ne pas se tromper par l'équivoque des termes que les définitions ont restreints."

Thus, conformably to Pascal's first rule on definitions, we find that some of the best German text-books do not try at all to define what is space, or what is a point, or even what is a straight line.

Strange as it may appear to some teachers, these text-books do not begin with several pages of definitions to be committed to memory, followed by a page of axioms again to be committed to memory. Nor are the demonstrations made

* Dr. Schotten, while quoting them somewhat inaccurately in translation, says that he does not know in what work of Pascal's they occur.

† PASCAL, *Pensées*, ed. Havet, Paris, Delagrave, 1883, pp. 555–556.

to cover exactly a whole page when they can be expressed in a line. Some of these authors, although well acquainted with synthetic, and even with non-Euclidean geometry, do not at all abhor the use of the expressions "direction" and "distance." Indeed, Dr. Schotten regards these two ideas as intuitively given in the mind and as so simple as not to require definition; he therefore bases the definition of the straight line on these two ideas, or rather recommends to elucidate the intuitive idea of the straight line possessed by any well-balanced mind by means of the still simpler ideas of direction and distance.

It is interesting to compare these views deduced by Dr. Schotten mainly with regard to their pedagogical value, and as a result of practical experience in teaching, with the conclusions arrived at by Prof. G. Peano* from a purely scientific point of view and based on the principles of mathematical logic.

A more philosophical discussion of the foundations of geometry is reserved in the German schools to the review course in the *Prima* of the *Gymnasium*. Then only will the student be able to appreciate to a certain degree the niceties involved in a careful treatment of the fundamental definitions and axioms of geometry.

It is to be hoped that Dr. Schotten will continue his studies in German "comparative planimetry," and that his second volume will not be deferred *ad calendas græcas*. It would also seem desirable that somebody should give us a similar account of what has been done in other countries in the same direction, in particular in England, France, and Italy.

In conclusion, the following two text-books might be mentioned, out of a large number of others, as giving a fair idea of the reform movement in Germany:

HUBERT MÜLLER, *Leitfaden der ebenen Geometrie*, Leipzig, Teubner, 1889;

HENRICI and TREUTLEIN, *Lehrbuch der Elementar-Geometrie*, ib., 1881.

For the more scientific study of the questions involved, the reader is referred to the following works, in which ample bibliographical references will be found:

OTTO RAUSENBERGER, *Die Elementargeometrie des Punktes, der Geraden und der Ebene*, Leipzig, Teubner, 1887.

BENNO ERDMANN, *Die Axiome der Geometrie*, Leipzig, 1877.

SCHMITZ-DUMONT, *Die mathematischen Elemente der Erkenntnistheorie*, Berlin, 1878.

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*See *Rivista di Matematica*, ed. by PEANO, Turin, vol. I. (1891), pp. 24-25.

J. C. BECKER, *Abhandlungen aus dem Grenzgebiete der Mathematik und Philosophie.* Zürich, 1870.

ALEXANDER ZIWET.

ANN ARBOR, *August* 1, 1891.

PICARD'S DEMONSTRATION OF THE GENERAL THEOREM UPON THE EXISTENCE OF INTEGRALS OF ORDINARY DIFFERENTIAL EQUATIONS.

TRANSLATED BY DR. THOMAS S. FISKE.

THE cardinal proposition in the theory of algebraic equations, that every such equation has a root, holds a place in mathematical theory no more important than the corresponding proposition in the theory of differential equations, that every differential equation defines a function expressible by means of a convergent series. This proposition was originally established by Cauchy, and was introduced, with a somewhat simplified demonstration, by Briot and Bouquet in their treatise on doubly periodic functions.* A new demonstration remarkable for its simplicity and brevity has been published by M. Emile Picard in the *Bulletin de la Société Mathématique de France* for March,† and reproduced on account of its striking character in the *Nouvelles Annales des Mathématiques* for May. This demonstration requires no auxiliary propositions, and depends upon no preceding part of the theory, except the simple consideration, that any ordinary differential equation is equivalent to a set of simultaneous equations of the first order.‡ The following is a translation of Picard's demonstration.

1. Consider the system of n equations of the first order

$$\frac{du}{dx} = f_1(x, u, v, \ldots, w),$$

$$\frac{dv}{dx} = f_2(x, u, v, \ldots, w),$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad ,$$

$$\frac{dw}{dx} = f_n(x, u, v, \ldots, w),$$

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* *Théorie des fonctions elliptiques*, p. 325.
JORDAN. *Cours d'analyse*, vol. III., p. 87.

† *Bulletin de la Société Mathématique de France*, Vol. XIX., p. 61.

‡ JORDAN. *Cours d'analyse*, vol. III., p. 4.

in which the functions f are continuous real functions of the real quantities $x, u, v, \ldots, w$ in the neighborhood of x_0, u_0, $v_0, \ldots, w_0$, and have determinate values as long as $x, u, v, \ldots, w$ remain within the respective intervals

$$\begin{gathered}(x_0 - a, x_0 + a),\\ (u_0 - b, u_0 + b),\\ (v_0 - b, v_0 + b),\\ \cdots\cdots,\\ (w_0 - b, w_0 + b),\end{gathered}$$

a and b denoting two positive magnitudes.

Suppose that n positive quantities $A, B, \ldots, L$ can be determined in such a manner that

$$\begin{gathered}| f(x, u', v', \ldots, w') - f(x, u, v, \ldots, w) |\\ < A\,|u'-u| + B\,|v'-v| + \ldots + L\,|w'-w|,\end{gathered}$$

in which $|\alpha|$ denotes as usual the absolute value of α, and $x, u, v, \ldots, w$ are contained in the indicated intervals. This will evidently be the case when the functions f have finite partial derivatives with respect to $u, v, \ldots, w$.

Starting with these very general hypotheses we will demonstrate that *there exist functions $u, v, \ldots, w$ of x, continuous in the neighborhood of x_0, satisfying the given differential equations, and reducing, for $x = x_0$, respectively to u_0, v_0, $\ldots, w_0$.*

2. We proceed by successive approximations. Taking first the system

$$\begin{gathered}\frac{du_1}{dx} = f_1(x, u_0, v_0, \ldots, w_0),\\ \cdots\cdots\cdots\cdots,\\ \frac{dw_1}{dx} = f_n(x, u_0, v_0, \ldots, w_0),\end{gathered}$$

we obtain by quadratures the functions $u_1, v_1, \ldots, w_1$, determining them in such a manner that they take for x_0 the values $u_0, v_0, \ldots, w_0$. Forming then the equations

$$\begin{gathered}\frac{du_2}{dx} = f_1(x, u_1, v_1, \ldots, w_1),\\ \cdots\cdots\cdots\cdots,\\ \frac{dw_2}{dx} = f_n(x, u_1, v_1, \ldots, w_1),\end{gathered}$$

we determine $u_2, v_2, \ldots, w_2$ by the condition that they take for x_0 the values $u_0, v_0, \ldots, w_0$ respectively. We continue this process indefinitely, the functions $u_m, v_m, \ldots, w_m$ being connected with the preceding $u_{m-1}, v_{m-1}, \ldots, w_{m-1}$ by the relations

$$\frac{du_m}{dx} = f_1(x, u_{m-1}, v_{m-1}, \ldots, w_{m-1}),$$

$$\cdots\cdots\cdots\cdots,$$

$$\frac{dw_m}{dx} = f_n(x, u_{m-1}, v_{m-1}, \ldots, w_{m-1}),$$

and, for $x = x_0$, satisfying the equations

$$u_m = u_0, \; v_m = v_0, \ldots, w_m = w_0.$$

We will now prove that when m increases indefinitely, $u_m, v_m, \ldots, w_m$ *tend toward limits which represent the integrals sought provided that x remains sufficiently near x_0.*

Let M be the maximum absolute value of the functions f when the variables upon which they depend remain between the indicated limits. Denote by ρ a quantity at most equal to a. If now x remains in the interval

$$(x_0 - \rho, x_0 + \rho),$$

we have

$$|u_1 - u_0| < M\rho, \ldots, |w_1 - w_0| < M\rho.$$

Hence, provided $M\rho < b$, the quantities $u_1, v_1, \ldots, w_1$ remain within the desired limits, and it is evident that the same is true of all the other sets of values $u, v, \ldots, w$. Denoting by δ a quantity at most equal to ρ, suppose that x remains in the interval

$$(x_0 - \delta, x_0 + \delta),$$

and write

$$u_m - u_{m-1} = U_m, \ldots, w_m - w_{m-1} = W_m;$$

we have, placing $m = 2, 3, \ldots, \infty$,

$$\frac{dU_m}{dx} = f_1(x, u_{m-1}, \ldots, w_{m-1}) - f_1(x, u_{m-2}, \ldots, w_{m-2}),$$

$$\cdots\cdots\cdots\cdots,$$

$$\frac{dW_m}{dx} = f_n(x, u_{m-1}, \ldots, w_{m-1}) - f_n(x, u_{m-2}, \ldots, w_{m-2}).$$

Since

$$| U_1 | < M\delta, \ldots, | W_1 | < M\delta,$$

the preceding equations, for $m = 2$, show that $| U_2 |$, $| V_2 |$, $\ldots$, $| W_2 |$ are less than

$$(A + B + \ldots + L) M \delta^2.$$

Continuing step by step it may be shown that $| U_m |, \ldots,$ $| W_m |$ are less than

$$M\delta (A + B + \ldots + L)^{m-1} \delta^{m-1}.$$

Now since

$$u_m = u_0 + U_1 + U_2 + \ldots + V_m,$$

u_m, and also $v_m, \ldots w_m$, will tend toward finite limits if

$$(A + B + \ldots + L) \delta < 1.$$

This condition will be fulfilled by making δ sufficiently small.

We see then that $u_m, v_m, \ldots, w_m$ tend toward determinate limits, $u, v, \ldots, w$, which are continuous functions of x in the interval

$$(x_0 - \delta, x_0 + \delta),$$

δ being the smallest of the three quantities

$$a, \frac{b}{M}, \frac{1}{A + B + \ldots + L},$$

and that $u, v, \ldots, w$ are represented by series which converge after the manner of a decreasing geometrical progression.

Moreover we have

$$u_m = \int_{x_0}^{x} f_1 (x, u_{m-1}, \ldots, w_{m-1}) \, dx + u_0,$$

and, since $u_m, v_m, \ldots, w_m$ differ from their limits as little as we please, whatever the value of x in the indicated interval, when m is sufficiently great, we have in the limit

$$u = \int_{x_0}^{x} f_1 (x, u, v, \ldots, w) \, dx + u_0,$$

Hence

$$\frac{du}{dx}=f_1\,(x,\ u,\ v,\ \ldots,\ w).$$

Similar results hold for the other functions. *The functions u, v, . . . , w are consequently the functions sought.*

CALCUL DES PROBABILITÉS. Par J. BERTRAND, de l'Académie Française, Secrétaire perpétuel de l'Académie des Sciences. Paris, Gauthier-Villars, 1889. 8vo., LVII + 332 pp.

THERE is possibly no branch of mathematics at once so interesting, so bewildering, and of so great practical importance as the theory of probability. Its history reveals both the wonders that can be accomplished and the bounds that cannot be transcended by mathematical science. It is the link between rigid deduction and the vast field of inductive science. A complete theory of probability would be a complete theory of the formation of belief. It is certainly a pity then, that, to quote M. Bertrand, "one cannot well understand the calculus of probabilities without having read La Place's work," and that "one cannot read La Place's work without having prepared himself for it by the most profound mathematical studies."

Though not otherwise is thorough knowledge to be gained, yet an exceedingly useful amount of knowledge is to be had without such effort. In fact, M. Bertrand's forty odd pages of preface on "The Laws of Chance" give an insight into the theory without the use of so much as a single algebraic symbol.

Listen to this *reductio ad absurdum* of Bernoulli's theory of moral expectation:

"'If I win,' says poor Peter, proposing a game of cards to Paul, 'you must pay three francs for my dinner.' 'Meal for meal,' replies Paul, 'you should pay twenty francs in case you lose, for that is the price of my supper.' 'If I lose twenty francs,' cries Peter, frightened out of his wits, 'I cannot dine to-morrow: without coming to that you might lose a thousand; put them up against my twenty. According to Daniel Bernoulli, you will still have the advantage.'"

Even more complete is the upsetting of Condorcet's calculation as to the probability of the sun's rising.

"Paul would wager that the sun rises to-morrow. The theory fixes the stakes. Paul shall receive a franc if the sun rises and will give a million if it fails to do so. Peter accepts the wager. Each morning he loses his franc and pays it. As the sun rises from morning to morning his chance daily diminishes. Paul conscientiously increases his stake; Peter as conscientiously pays his franc. The obligations remain equitable. The bettors travel through twenty countries from West to East. Peter always loses; he pursues his fortune however and takes Paul to the North; they cross the arctic circle; the sun stays a month below the horizon; Paul loses 30 millions and thinks the order of nature overturned."

Even La Place does not escape M. Bertrand's pleasant raillery, and M. Quetelet has his ideal average man dismissed as follows:

"In the body of the average man the Belgian author places an average soul. . . . The typical man would be without passions and without vices, neither foolish nor wise, neither ignorant nor learned, forever dozing: this is the mean between sleeping and waking; answering neither yes nor no; mediocre in everything. After having eaten for 38 years the average ration of a healthy soldier, he would die, not of old age, but of some average sickness that statistics would reveal to him."

But I must hasten on to the main body of the work: suffice it to say that in these few introductory pages is packed this variety of topics:

The Petersburg paradox, D'Alembert and Bernoulli's dispute as to the benefits of inoculation for small-pox, Bernoulli's theorem, the ruin of players, inverse probabilities; Poisson's law of large numbers, the application of the theory of probability to statistics, the theory of errors of observation, the probability of decisions.

Having seen so much done without any algebra at all, we are prepared to accept M. Bertrand's statement, as to the entire work, that "very few pages could embarrass a reader familiar with the elements of mathematics."

Scarce an integration sign, never a generating function, really it is charmingly simple and direct: and everywhere illuminated too by the common sense and mother wit that are so conspicuously displayed in the preface.

It is characteristic of the method of treatment that all lurking dangers, all insidious snares, are carefully pointed out by means of numerous concrete examples.

A good instance of this, though merely one of half a dozen bearing upon the same point, is the following problem to show the absurdity of trying to reckon probabilities when the favorable and possible cases are each infinite in number.

In a circle a chord is drawn at random. What is the probability that it shall be longer than the side of the inscribed equilateral triangle?

You might say: The probability is unchanged by fixing one end of the chord. The probability that it shall be long enough is then merely the probability that it does not lie without the angle made by the two chords of 120° meeting at the point. This probability is $\frac{1}{3}$.

Or, you might say: The direction matters not, provided the chord is not too far from the centre, viz., not more than half the radius of the circle. The probability is $\frac{1}{2}$.

Yet again: To choose a chord at random is to choose its middle point. In order that the chord shall be long enough its middle point must not be without a circle concentric with, and of half the radius of, the given circle. This gives the probability $\frac{1}{4}$.

Similarly, in a chapter on total and compound probabilities, are a number of problems showing how essential it is, in computing the probability of a compound event, that the probability to be given to the second composing event shall be that which it has when the first event is *known to have happened.*

Even Clerk Maxwell has violated this principle in obtaining the formula

$$\varphi(x) = Ge^{-k^2x^2}$$

in which x is the x-component of the velocity of a molecule of gas and $\varphi(x)$ its probability. He assumed that the x, y, and z components had independent probabilities. That they are not independent is plain enough from this: if x is the maximum velocity of a molecule, the movement is parallel to the x-axis, and y and z are nothing.

In the chapter on expectation the consideration of the Petersburg paradox gives an opportunity to again ridicule Daniel Bernoulli's moral expectation. I quote a few passages:

"You play, that is the hypothesis. Are you foolish or wise to do so? The question is not put."

"Peter, whose whole fortune is 100,000 francs, wishes a chance to gain 100 millions. 'Nothing is easier,' coolly replies the geometer whom he consults. 'If the game is equitable you will have 999 chances in a thousand of losing your 100,000 francs.'"

"The theory of moral expectation has become classic, never was the word more exactly used: it has been studied, it has been commented upon, it has been developed in works justly famous. Its success stops there, no one ever has made or ever can make any use of it."

Two chapters are devoted to James Bernoulli's famous theorem that no possible event can be so improbable but that there is as great a chance as you please of its happening some time or other, if only you wait long enough.

Three distinct demonstrations are given, of which the first is the most straightforward and complete. I give a sketch :

The probabilities of two contrary events being p and q, the most probable combination in μ trials is that in which the first event happens μp times and the second μq times. By Sterling's theorem its probability is

$$1 / \sqrt{2\pi\mu pq};$$

and the probability that it happens $\mu p \pm h$ times is

$$e^{-h^2/2\mu pq} / \sqrt{2\pi\mu pq}.$$

Approximately true for small values of h, it does to take this formula true for large and even infinite values, because then both the true values and those given by the formula are so small as to be negligible ; e. g. :

Putting $\mu = 1000$, $h = 100$, $p = q = \frac{1}{2}$, we get for the corresponding probability

$$e^{-20}\sqrt{2} / \sqrt{1000\,\pi} = 0.000\,000\,000\,520\,06.$$

By a simple artifice, we substitute for (1), giving the probability of an error h, the formula

$$e^{-z^2/2\mu pq}dz / \sqrt{2\pi\mu pq},$$

for the probability of an error between z and $z + dz$.

To test this formula, notice that the sum of all possible probabilities is certainty and that we should have and do have

$$\int_{-\infty}^{+\infty} e^{-z^2/2\mu pq}dz / \sqrt{2\pi\mu pq} = 1.$$

Other such tests can be given.

The probability then that an error shall be smaller than α is

$$2\int_0^{\alpha/2\mu pq} e^{-t^2} dt = \Theta(\alpha / 2\mu pq).$$

If α has a determinate value, however large, the probability that it shall not be surpassed approaches zero as μ is increased without limit, which proves Bernoulli's theorem.

The *relative* error grows smaller and smaller as the *absolute* error grows larger and larger.

To fix the meaning of this, consider how many times a coin would need to be tossed in order that the probability of obtaining heads at least a million more times than tails shall exceed 0.01. We get, putting μ for the required number,

$$0.99 = \Theta\left(1\,000\,000\ \sqrt{2}\,/\,\sqrt{\mu}\right) = \Theta\,(1.83)$$

and thence

$$\mu = 597\,211 \text{ millions.}$$

Not only when the probability of an event is known can the number of its happenings in a given number of trials be predicted almost with certainty, but when the probability is unknown, Bernoulli's theorem can be used inversely to find it. The ratio of the number of happenings of an event to the number of trials certainly approaches this unknown probability as the number of trials is increased.

Nevertheless, two conditions are necessary: the probability must not change during the trials, and it must have a determinate value.

"The King of Siam is forty years old, what is the probability that he shall live ten years? It is different for us than for those who have asked his doctor, different for the doctor than for those who have received his confidences; very different indeed for the conspirators who have taken steps to strangle him to-morrow."

In a word, Bernoulli's theorem applies to *objective*, not to *subjective*, probability.

An immediate consequence of the theorem is the inevitable ruin of any gambler who plays long enough at a fair game. But the number of games before ruin occurs may be enormous. Thus, in tossing coins at one franc a toss it requires 624 000 tossings to insure a probability of 0.9 that one player or the other shall lose 100 francs. Truly a courageous gambler would hardly be frightened at such a prospect.

If in this very game either party had started in with only a franc and was to receive a franc a game so long as he played without losing it, his mathematical expectation would be infinite.

Let us return to the inverse use of Bernoulli's theorem. Consider this problem:

An urn contains μ balls, some white, some black, in an unknown proportion. k drawings are made, the ball being replaced after each drawing. There are obtained m white and n black balls. What is the most probable composition of the urn?

Before the trial is made, all sorts of hypotheses are possible as to the composition of the urn. Suppose all compositions are equally probable. It follows that the probable ratio of white to black balls is $m : n$; and, that there should be a deviation ϵ from this ratio, the probability is

$$Ge^{-\epsilon^2 (m+n)/2pq},$$

where G is independent of ε.

The hypothesis that all compositions are *a priori* equally probable is rarely realized. Suppose that the balls were put into the urn by lot with a probability $\frac{1}{2}$ for each color.

We then get for the probable proportion of white balls

$$(\mu + 2m)/2(\mu + m + n),$$

which lies between $\frac{1}{2}$ and $m/(m + n)$.

If μ is very large this will approach $\frac{1}{2}$, no matter what the numbers m and n are; if, on the contrary, it is m and $m + n$ that are large, the fraction is very near to $m/(m + n)$, no matter what μ is.

The probability of causes is always thus affected by *a priori* probabilities.

To what is commonly known as the theory of least squares three chapters are devoted. In fact, a fourth chapter, on "Errors in the Position of a Point," is really an extension of Gauss's law of errors.

Very interesting is the criticism of Gauss's reasoning.

To begin with, can it be strictly maintained that the probability of an error Δ is $\varphi(\Delta)$?

"Does it not depend upon the quantity measured ?"

"If you take a weight, if you measure an angle, is there not a greater chance of a correct estimate if the weight is an exact number of milligrams, if the angle contains an exact number of seconds, than if it is necessary to add a fraction ? If this fraction, not given by the instrument, is exactly $\frac{1}{2}$, is there not a less chance of error in evaluating it than if it is 0.27 ?"

There is a case where the postulate is rigorously demonstratable, but the conclusion is nevertheless only approximate. Suppose, in fact, that the quantity to be measured is the proportion of white balls in an urn of unknown composition.

Of μ balls drawn, m are white.

The fraction m/μ is a measure of the ratio sought. The measure is the more precise as the number of balls drawn is greater. The operation repeated n times gives the n successive measures.

$$m_1/\mu, m_2/\mu, \ldots, m_n/\mu.$$

The most probable value of the ratio deduced from the drawings is

$$\Sigma m/n\mu = \Sigma (m/\mu)/n,$$

the arithmetical mean of n equally trustworthy measures.

Now, if in μ drawings from an urn we get m white balls, the probability that the ratio of white to the whole number of balls shall be m/μ is indeed *approximately*

$$\mu e^{-z^2\mu^2/2m(\mu-m)}/\sqrt{2\pi m(\mu - m)},$$

which is of the form

$$ke^{-k^2h^2}/\sqrt{\pi},$$

and precisely what Gauss's law would give; but if the law were rigorous the formula should be *exact.*

The hypothesis that the arithmetical mean of several quantities is the most probable value leads to inconsistencies. It requires, for example, that the most probable value of the square of the quantity shall be the arithmetical mean of the squares. Nor can the objection be avoided by making a distinction between measures directly observed and those resulting from calculation. A mechanic could easily attach to a balance a needle to indicate the square or the logarithm of a weight.

The same objection applies to expressing the probability of an error as a function of the error alone.

The problem is proposed, "If Gauss had adopted, instead of the mean, another mode of combination of the measures, what law of errors would he have deduced?"

The problem is not solved, but it is shown that if

$$f(x_1, x_2, \ldots, x_n)$$

is the most probable value of a quantity of which $x_1, x_2, \ldots, x_n$ are measurements, then, in order that the probability of an error shall be a function of the error, f must be the arithmetical mean of the x's increased by some function of their differences. In other words, it must be such that if all the x's are increased each by α, it shall also be increased by α.

In spite, however, of all theoretical objections to Gauss's law, constantly accumulating experience completely justifies its adoption. As to the arithmetical mean, Ferrero has shown (see Charles S. Peirce's review in *Am. Journ. Math.* I., 59) that all functions of the measurements that it would not be absurd to take for the most probable value of the quantity measured, will, if the measurements are good, agree in their results; while, of course, if the measurements are bad, no treatment can be expected to give good results.

It is gratifying to find these careful definitions of precision and weight.

"The precision of one measure is said to be α times that of another measure when the probability that an error is contained between z and $z + dz$ for the one is the same as that it shall be contained between αz and $\alpha\,(z + dz)$ for the other."

"The weight of one observation is said to be β times that of another, when the consequences that can be deduced as to the value of a magnitude measured by an observation in the first system are equivalent to those that can be deduced from β observations in the second system, all giving the same result."

"If β is a fraction m/n, it is necessary that m concordant observations of the first system can be replaced by n concordant observations of the second."

"The system of observations that gives to the error z the probability

$$ke^{-k^2z^2}/\sqrt{\pi}$$

has k for its precision and k^2 for its weight, if we take for units of precision and weight those of one observation in the system for which the probability of an error z is proportional to e^{-z^2}."

M. Bertrand argues at some length for the rejection of doubtful observations. He gives no criterion, however, as Peirce has done, to determine when they shall be rejected. This is left to the judgment of the computer, with the caution that the number of retained observations must be large. The probable value of the square of an error smaller than λ, when those larger than λ have been rejected, is

$$\frac{1}{2nk^2} \cdot \frac{\Theta\,(k\lambda) - 2k\lambda e^{-k^2\lambda^2}/\sqrt{\pi}}{[\Theta\,(k\lambda)]^2}.$$

As for Gauss's attempt, in his last memoirs, to break away from all hypotheses of a law of errors in establishing the method of least squares, it is shown that neglecting the squares and powers of the errors is equivalent to assuming the exponential law.

The equating of the probable value of a function to its true value is not unobjectionable. The following example shows this.

Five angles l_1, l_2, l_3, l_4, l_5, have been measured. The geometrical conditions require

$$l_4 + l_1 - l_3 = 0,$$

$$l_5 + l_2 - l_3 = 0.$$

It is found, however, that

$$l_4 + l_1 - l_3 = h_1,$$

$$l_5 + l_2 - l_3 = h_2.$$

Designating the errors really committed by e_1, e_2, e_3, e_4, e_5, we have:

$$e_4 + e_1 - e_5 = h_1,$$

$$e_5 + e_2 - e_3 = h_2.$$

No matter what the multipliers λ_1, λ_2, λ_3 may be, the trinomial

$$\lambda_1 h_1^2 + \lambda_2 h_2^2 + \lambda_3 h_1 h_2$$

is known.

This trinomial is a homogeneous function of the second degree in the errors; and calling m_2 the probable value of the square of one of the errors, that of the product of two of them being nothing, we shall find for the probable value of the trinomial, calculated before the measures are taken,

$$m^2(3\lambda_1 + 3\lambda_2 + \lambda_3).$$

Equating this to the true value gives

$$m^2 = (\lambda_1 h_1^2 + \lambda_2 h_2^2 + \lambda_3 h_1 h_2)/(3\lambda_1 + 3\lambda_2 + \lambda_3),$$

an infinite number of different values for m^2.

This does not furnish, however, the most serious reason why the chances of error cannot be precisely evaluated.

"It is supposed, *a priori*, that all the measurements of a system are equally precise; it is impossible in the vast majority of cases to believe in such equality: it is from lack of knowing reasons for preference that the results are accepted as equivalent. But, known or unknown, these reasons, if they exist, must have an influence upon the error really committed and of which it is pretended to give the probability."

"After having, with immense labor, discussed the transit of Venus observations of 1761, Encke found for the parallax of the sun 8''.49 with a probable error 0''.06. He could therefore bet 300000 to one that the error would not reach 0''.42. Nevertheless, astronomers have just accepted the parallax 8''.91 corresponding exactly to the error 0''.42."

"We can simply affirm, and this is the important point, that if the sum of the squares of the corrections are small, there is great likelihood that the observations have been well made."

The extension of Gauss's law to errors in the situation of a point gives for the most probable position of a point the centre of gravity of the observed positions supposed equally weighted.

Restricting ourselves to a variation in two coordinates, "the probability that an error shall be comprised between u and $u+du$ for x and between v and $v+dv$ for y is

$$Ge^{-k^2u^2-2\lambda uv-k'^2v^2}\,du\,dv.$$

Points of equal probability are upon the same ellipse having for its equation

$$k^2u^2+2\lambda uv+k'^2v^2=H,$$

u and v designating the differences between the coordinates of the point considered and the true position, the common centre of all the similar ellipses whose dimensions are proportional to $\sqrt{H}$.

A comparison is made between the theory and the results of firing 1000 shots at a target.

The law has been partially guessed by Galton in his *Discussion on the Data of Stature,* and more fully worked out by Mr. Hamilton Dickson. (See *Natural Inheritance,* p. 100 *et seq.*)

It seems a pity that in the chapter on the laws of statistics some slight reference at least should not have been made to Galton's investigations.

A point liable to be overlooked in applying the laws of probability to statistics is well stated.

"There are plenty of ways of consulting chance that will give the same mean without giving the same probabilities of error. Instead of drawing balls from an urn of a given composition, we could draw in order from many urns of various compositions. The average results would be the same as for drawings from an urn of average composition, the chances of error would not be."

"If, to take an extreme case, instead of drawing 10,000 times from an urn containing one white and one black ball, we draw alternately from two urns, one containing a white the other a black ball, we shall certainly get white 5000 times, the error will be nothing."

To represent tables of mortality, the substitution of several urns for a single one, seems, *a priori*, very plausible. Among individuals of the same age it is impossible not to find classes in which the chances of life are unequal."

The book ends very pleasantly with a chapter on the misapplications of the theory of probabilities to judicial decisions.

Apropos of this matter, does not the great probability that attaches to the results of concurrent independent judgments furnish the strongest possible argument for cultivating independence, for ridding ourselves of the systematic errors imposed by education and fashion, by party and sect?

I have very inadequately sketched this most admirable introduction to the science of probability: the life and vigor of the original cannot be reproduced in a brief review.

ELLERY W. DAVIS.

COLUMBIA, *August* 14, 1891.

THE NUMBER-SYSTEM OF ALGEBRA, Treated Theoretically and Historically. By Professor H. B. FINE. Boston and New York; Leach, Shewell & Sanborn, 1891. 8vo, pp. IX. + 131.

AT the present time we frequently find mathematical researches preceded by an historical account of the question under discussion : and this is but another proof of the increasing importance of the study of mathematical history. On the other hand, it is necessary that the history of any branch of the science form part of such books as are intended for students. For pedagogic reasons the historical part of a treatise ought to be placed at the end of the volume, or at least at the ends of the various chapters.

Mr. Fine's recent book takes its place among the not very numerous works combining a systematic treatment with an historical account. It may be regarded as an introduction to the theory of functions of one variable. In this short review I shall only refer to the historical part of the work, which occupies the latter part of the book (pp. 79–131). The author begins by noticing the symbols and systems of numeration, and then passes to the history of fractions and irrational quantity among the Ancients. He then summarizes the progress of algebra, from the earliest times down to Descartes, and finishes with the development of the fundamental notions of algebra, from Newton to Weierstrass and G. Cantor. For the ancient history, that of the Middle Ages and down to 1600, Mr. Fine has chiefly followed the well-known works of Moritz Cantor and Hankel. For modern history he has generally had recourse to original sources.

One or two improbable or inexact statements may be noticed. For instance, Regiomontanus is mentioned as the author of the *Algorithmus Demonstratus* (1534).* Again, the year 1630 is given as the date of the introduction of the sign ÷, and it is said to have been first used by Pell.† These inaccuracies are, however, of slight importance, and Mr. Fine's book will doubtless be found of much assistance to students of mathematics.

G. ENESTRÖM.

STOCKHOLM.

Translated from *Bibliotheca Mathematica*, 1891, No. 2, with additions from the author, by HAROLD JACOBY.

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* There exists a copy of this work, antedating the birth of Regiomontanus, and attributed to Jordanus Nemorarius.

† The sign is really due to Rahm, and the date is 1659. Compare BEMAN, *Bibl. Math.*, 1887, p. 96.

WEST AFRICAN LONGITUDES.

Telegraphic Determinations of Longitudes on the West Coast of Africa. From observations by Commander T. F. PULLEN, R. N., and W. H. FINLAY, Esq., M. A., F. R. A. S., made and reduced under the direction of DAVID GILL, Esq., F. R. S., Her Majesty's Astronomer at the Cape of Good Hope. London, Hydrographic Department, Admiralty, 1891 ; pp. 82.

IN this volume are recorded the last observations of the late Commander Pullen, who lost his life from malarial fever contracted while making night observations at Bonny, on the West African coast. The results have been worked out under the supervision of Dr. Gill, and the book contains not a few suggestions and remarks of interest to astronomers. The instrument employed was an altazimuth by Troughton and Simms, having a 14-inch vertical circle, read by four microscopes. This was selected as the most appropriate instrument available for the purpose; for it was decided to determine time by altitudes, after a careful consideration of the relative merits of meridian observations, and those in the vertical of the pole star. Dr. Gill expresses a very favorable opinion of the latter method, which has so long been strongly advocated by Döllen of Pulcova. It was abandoned chiefly because there is no bright star near the Southern pole.

The results afterwards proved the wisdom of not depending on meridian transits : indeed, the conditions of the climate on the West African coast are so unfavorable that there would be an excessive loss of time if meridian observations only were employed. Throughout all the observations with the altazimuth a mean time chronometer was used, without a chronograph.

Before the commencement of the campaign, the two observers, Pullen and Finlay, met at the Royal Observatory, Cape Town, and their relative personal equations were carefully determined by simple but accurate methods. As a result of these determinations, the correction + $0^s.085$ was afterwards applied to the differences of longitude obtained for the various stations. The time observations with the altazimuth were made with " circle right " and " circle left," and pairs of stars were taken at nearly equal altitudes near the Eastern and Western prime verticals. The mean from any such pair was then regarded as a complete time determination; and in this way the results came out very satisfactorily. The time determinations at the Cape were made by Mr. Finlay with the large meridian circle : and in the exchange of signals Thomson deadbeat galvanometers were used with success.

Several interesting remarks, due to Dr. Gill, occur in the book. The method of carrying chronometers (much affected by navy quartermasters) by means of a strap passed through the handles and over the top, is condemned. Indeed, it is possible to stop a chronometer, temporarily, when so carried, by a peculiar twist of the arm. Dr. Gill recommends holding the chronometer with both hands in front of the body, the elbows being pressed against the sides. The spring of the arms is then a great safeguard.

In another place, having called attention to the very high accuracy attained by Commander Pullen after comparatively little practice, Dr. Gill refers to an interesting remark of Professor Winnecke's to the effect that "the best training for an astronomical observer is a long course of *accurate work* on land with the sextant."

The ordinary method of circummeridian altitudes was used in measuring the latitudes of the stations. Stars were observed both North and South of the zenith, and certain systematic differences in the resulting latitudes are explained as the result of a looseness of the web-frame in the tube. The experience gained is summarized (p. 48) for the benefit of future observers with the portable altazimuth, and any one would do well to consult Dr. Gill's remarks before beginning work with this somewhat difficult instrument.

The positions of the various astronomical stations are carefully described, and the bearings of many surrounding permanent objects are set down. The places of the stars used are almost all taken from the Ephemerides and the Cape Catalogue. The volume concludes with several appendices containing various details and examples of observations and reductions. HAROLD JACOBY.

COLUMBIA COLLEGE, New York; 1891, *September*.

SOUTH AMERICAN LONGITUDES.

Telegraphic Determination of Longitudes in Mexico, Central America, the West Indies, and on the North Coast of South America, with the Latitudes of the Several Stations. By Lieutenants J. A. NORRIS and CHARLES LAIRD, U. S. N., published by order of Commodore F. M. RAMSAY, U. S. N., Chief of Bureau of Navigation, Navy Department. Washington, Government Printing Office, 1891; pp. 189.

THE above volume contains the results of longitude determinations executed by order of the U. S. Navy Department in the years 1888, 1889, and 1890. As will be seen from the title, the observations have been made in very unfavorable locations, so far as the comfort and health of the observers were concerned. It is therefore an evidence of great endurance and skill that so much was accomplished during the short time many of the stations were occupied. We read how one of the observing parties was compelled to proceed with its entire observing equipment, including instruments, a hundred miles in canoes up the Coatzacoalcos River, poling against the current. And afterwards another hundred miles by mule-train through the "tangled intricacies of a tropical forest." This trip took fourteen days.

But we must here occupy ourselves chiefly with the methods and results of the expedition, from a scientific point of view. Eleven longitude stations were occupied altogether, and the careful way in which the observation spots have been described, and referred by exact measurements to local permanent landmarks, is very much to be commended (pp. 16–19). When possible, the sites occupied by previous observers were again used. The instruments employed were two prismatic transits made in 1874, by Stackpole of New York, for the Transit of Venus Commission. Six break-circuit chronometers and two chronographs were used. The values of the instrumental constants were very carefully determined in the field, and afterwards verified at Washington. Whenever it was found possible the exchange of longitude signals was made automatically, the distant observer's clock recording on the local chronograph. When this could not be done, in consequence of the weak current through long cable lines, a mirror galvanometer was used at each end. This was found to work quite satisfactorily. The mirror galvanometers were set up in the cable offices, and the sending and receiving times of the signals were recorded chronographically by the observer at each end of the line. For this purpose wires were run from the cable offices to the observing huts or tents, which were usually quite close.

No allowance was made for personal equation, as circumstances would not allow of any adequate determination of that quantity. In the reductions, the method of least squares was used throughout the longitude work. The observations were first preliminarily reduced, and normal equations were then formed for the determination of the minute corrections required by the preliminary values of the instrumental constants. The polar stars were weighted for declination, but it is not stated what formula was used in assigning the weights. The adopted clock-corrections, however, are *not* those derived from the least square solution, but the means of the results from the separate time-stars, the latter being again reduced with the azimuth and collimation constants derived from the least square adjustment. The polar stars were excluded in this last process. The adopted values of the clock-correction, however, are always very nearly equal to those obtained from the least square reduction, the greatest difference being $0^s.035$. (Vera Cruz, 1889, January 17; p. 69.)

The latitude work was all done by Talcott's method, the star-places being derived from the *American Ephemeris*, the *Jahrbuch*, and the Catalogues of Newcomb, Safford, the Coast Survey, and Greenwich Observatory.

The volume contains excellent maps showing the surroundings of the various astronomical stations, and closes with an appendix giving the results of the many valuable magnetic observations made by the members of the Expedition.

HAROLD JACOBY.

COLUMBIA COLLEGE, New York; 1891, *September*.

NOTES.

THE officers of Section A at the Washington Meeting of the American Association for the Advancement of Science were: Vice-President, E. W. Hyde of Cincinnati; Secretary, F. H. Bigelow of Washington. The following papers were read: The evolution of algebra, by E. W. Hyde; On a digest of the literature of the mathemathical sciences, by Alex. S. Christie; Latitude of the Sayre Observatory, by C. L. Doolittle; The secular variation of terrestrial latitudes, by George C. Comstock; Groups of stars, binary and multiple, by G. W. Holley; Description of the great spectroscope and spectrograph constructed for the Halsted Observatory, Princeton, N. J., and Note on some recent photographs of the reversal of the hydrogen lines of solar prominences, by J. A. Brashear; Standardizing photographic films without the use of a standard light, by Frank H. Bigelow; On a modified form of zenith telescope for determining standard declinations, and On the application of the "photochronograph" to the automatic record of stellar occultations, particularly dark-limb emersions, by David P. Todd; Principles of the algebra of physics, by A. Macfarlane; The zodiacal light as related to terrestrial temperature variation, by O. T. Sherman; On the long-period terms in the motion of Hyperion, by Ormond Stone; Exhibition and description of a new scientific instrument, the aurora-inclinometer, by Frank H. Bigelow; The tabulation of light-curves: description, explanation, and illustration of a new method, and Stellar fluctuations: distinguished from variable stars: investigation of their frequency, by Henry M. Parkhurst; On certain space and surface integrals, by Thomas S. Fiske; The fundamental law of electromagnetism, by J. Loudon; Method of controlling a driving clock, by F. P. Leavenworth; On the bitangential of the quintic, by Wm. E. Heal; Parallax of α Leonis, by Jefferson E. Kershner. The officers elected for the Rochester Meeting are: Vice-President, J. R. Eastman of Washington; Secretary, W. Upton of Providence.

THE first volume of a work entitled "*Synopsis der Höheren Mathematik*," by the Rev. J. G. Hagen, Director of the Observatory of Georgetown College, Washington, D. C., has appeared from the press of Felix L. Dames, Berlin. Its 400 pages treat of Arithmetical and Algebraic Analysis. The contents are as follows: Part I., Theory of Numbers.—Part II., Theory of Complex Quantities.—Part III., Theory of Combinations.—Part IV., Theory of Series.—Part V., Theory of Infinite Products and Factorials.—Part VI., Theory of Continued Fractions.—Part VII., Theory of Finite Differences.—Part VIII., Theory of Functions.—Part IX., Theory of Determinants.—Part X., Theory of Invariants.—Part XI., Theory of Groups.—Part XII., Theory of Equations. The subject of the second volume will be Analytical and Synthetic Geometry. The entire work is to be contained in four volumes, which are promised at the rate of one a year.

THE deaths of a number of distinguished mathematicians have occurred since the beginning of the present calendar year. Among them may be recorded John Casey, died January 3; Sophie Kowalevski, February 10; Maximilien Marie, May 8; Wilhelm Matzka, June 9; and Wilhelm Eduard Weber, June 23. On another occasion we hope to give the readers of the *Bulletin* some account of their work and lives.

T. S. F.

WE learn from Hoffmann's *Zeitschrift* * that on the 5th and 6th of October a meeting will be held at Braunschweig, Germany, for the purpose of organizing an "association for the improvement of the teaching of mathematics and the natural sciences." At a preliminary meeting held at Jena, September 28 and 29, 1890,† and attended by about 90 teachers from all parts of Germany, the desirability of such an organization was discussed and fully established, and a provisional constitution

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* *Zeitschrift für den mathematischen und naturwissenschaftlichen Unterricht*, vol. 22 (1891), pp. 316–318 and pp. 397–398.

† *ib.*, vol. 21 (1890), pp. 561–574 and pp. 611–632.

was drawn up. The association is evidently intended to represent mainly the teachers employed in the *Gymnasium* and *Realschule,* although there is, of course, no class-restriction of membership, anybody interested in the object of the society being invited to join, and university professors in particular. The term "natural sciences" is understood to embrace physics, chemistry, mineralogy, botany, zoology, and geography.

The formation of this association is a significant fact in connection with the general movement for the reform of the so called higher schools that has been going on in Germany for many years. The sensation created by the Emperor's opening address to the committee called to consider the reform of the higher schools was somewhat abated by the rather conservative final report made by this committee. But the strength of the popular movement is not broken by any means. It is the avowed object of the new association to promote and strengthen the teaching of the exact sciences in the schools of Germany. The activity of the society is to bear mainly on the following points:

(1) The improvement and more ample use of scientific apparatus and other mechanical aids to instruction (the very general term "*Lehrmittel*" may be interpreted to include also text-books).

(2) The better preparation of teachers for their calling, by the establishment of special courses and "seminaries" for elementary teachers at the universities, lectures on the teaching of elementary mathematics, etc.

(3) The application of the recent advances in science and the arts to elementary instruction in the exact sciences.

A full account of this year's meeting will probably be published in Hoffmann's *Zeitschrift*. A. Z.

PROFESSOR W. H. ECHOLS, JR., recently Director of the Missouri School of Mines, has been called to a chair at the University of Virginia.

Professor M. W. Harrington, for a number of years Professor of Astronomy at the University of Michigan, is now Chief of the Weather Bureau in the Department of Agriculture at Washington, D. C.

Professor A. S. Hathaway has resigned his post at Cornell, to become Professor of Mathematics at the Rose Polytechnic Institute.

Professor A. L. Baker, of the Stevens School, Hoboken, N. J., has accepted a call to the University of Rochester.

Professors A. S. Hardy, of Dartmouth, and Fabian Franklin, of Johns Hopkins, have gone abroad to remain during the present academic year. T. S. F.

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Articles for insertion should be addressed to the

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CATALOGUE OF THE ASTRONOMISCHE GESELLSCHAFT.

Catalog der Astronomischen Gesellschaft. Erste Abtheilung. Catalog der Sterne bis zur neunten Grösse zwischen 80° nördlicher und 2° südlicher Declination für das Aequinoctium 1875. Drittes Stück, Zone + 65° bis + 70°, beobachtet auf der STERNWARTE CHRISTIANIA. Viertes Stück, Zone + 55° bis + 65°, beobachtet auf den STERNWARTEN HELSINGFORS und GOTHA. Vierzehntes Stück, Zone + 1° bis + 5°, beobachtet auf der STERNWARTE ALBANY. Leipzig, 1890. 3 vols., 4to.

ASTRONOMERS frequently need the positions of so-called fixed stars. They are wanted when a clock is to be regulated to true sidereal or true mean time; when, again, the astronomer is on his travels and desires to fix his latitude and longitude, and the direction on the earth of his meridian; or when he is observing some planet's or comet's course, and wishes to settle the various right ascensions and declinations it occupies from day to day and hour to hour, in order from them to calculate its orbit, predict its future course. and test the law of gravitation.

Thus accurate star places are the basis in one sense of all astronomy of position; but they have an interest of their own which is more prominent now than it ever has been, and is yearly increasing.

For no star is absolutely fixed; and the small motions of the stars which have been long detected are slowly accumulating their effects, and giving evidence that will in time throw much light on the structure of the universe.

The star which has by some been called the "runaway" (Groombridge 1830) moves over half a degree, as seen from the earth, in about two centuries and a half; so that a sharp-sighted observer in a dry mountain region (where the air is transparent enough) could readily detect its motion with the naked eye in about a life-time. Other stars move so slowly, in appearance, that a hundred years of the closest telescopic observation are necessary to detect any slight deviation from their former position. The constellations exhibit to the eye substantially the same appearance from century to century; it is only in very small details that they seem to alter during fifty years.

The study of these trifling motions (as they seem to us) is extremely fascinating to those who have undertaken it; and a vast amount of human effort has been spent in the acquisition of this form of knowledge.

The Alexandrine Greeks did something in mapping and listing the stars, by such simple devices as they possessed;

the Tartar prince Ulugh Beigh had an observatory at Samarcand devoted to the same object; Tycho Brahe the Dane and Hevelius of Dantzic added to the stock of good observations of star-places; but the earliest work which is now of much scientific value was done at Greenwich by the English Astronomers Royal. Flamsteed, who labored in the latter part of the seventeenth, and Bradley, who held his office near the middle of the eighteenth century, were eminent in their time; and Bradley especially did more to make precise observations than any one who had preceded him. With instruments of clumsy build, and inferior in power to those which are now carried from place to place over our Far West to fix a basis for our maps, he succeeded in putting this branch of astronomy upon a solid foundation. It was not till quite lately, however, that his observations were made fully available; for Bessel, whose immortal *Fundamenta Astronomiæ* appeared in 1818, reduced only a part of them, and the results which he obtained lacked the minute accuracy which is now needful and possible. With all these drawbacks Bessel's study of a part only of Bradley's work gave the science a prodigious impulse.

Since Bradley's time the art of observation has progressed very greatly. He was hampered by the intractable qualities of matter; his telescopes were small and unachromatic, his divisions roughly made, his means of reading them crude; but his capacity for practical astronomy unrivalled; *incomparabilis* Bessel calls him.

The instrument-makers Graham and Bird were succeeded in England by Ramsden and Troughton; in Germany by Reichenbach and the eldest Repsold. The opticians who made single lenses were distanced by Dollond with the achromatic object-glass, which Fraunhofer improved and nearly perfected. The telescope-tubes became shorter, retaining the same power; object-glasses of greater and greater aperture became practicable with moderate dimensions in the machinery. The micrometer-microscope was used both in dividing and reading off divisions, so that a circle of 10 inches in diameter is now as good as or better than a quadrant of 8 feet radius; and the spirit-level of a few inches in length is far more accurate and trustworthy than the plumb-line running through two stories of a house. In fact the mechanical construction of astronomical instruments has become a true fine art.

Astronomical observation of the first order now requires a subtle psychological analysis; the human brain and body have become tools for the mind to investigate and employ; just as we search into the minute errors of division which the finest workmen leave in our instruments—errors thought too large if a line is misplaced by a twenty-thousandth of an inch—so must we inquire most scrupulously and carefully into the possibility that our senses may lead us astray when we are under the completest strain of attention.

After 1790 efforts were made, in several directions, to catalogue the stars. Piazzi at Palermo, and Bradley's successors at Greenwich, spent most of their labors upon the brighter stars; those, some ten thousand in number, which are visible to the naked eye, or approach such visibility. Lalande at Paris undertook the gigantic task of observing all that his telescopes would reach. Fortunately for him, he had but small instruments. He succeeded in observing over forty thousand. Bessel, after completing his reduction of Bradley in 1818, obtained in 1819 a meridian circle capable of dealing with all ninth-magnitude stars not below or too near his horizon; and did all that was in one man's power towards cataloguing the stars, nearly 150,000 in number, which come into this category. He confined himself to less than one-half the sphere, or about two-thirds of his range of visibility, and took as many stars into his catalogue as he could observe. His scholar, Argelander, was at first his assistant at Königsberg; but was promoted to the observatory at Abô in Finland, afterwards removed to Helsingfors, and after excellent service there in another direction went to Bonn on the Rhine about 1840. There he extended Bessel's Zones in substantially Bessel's way to the neighborhood of the North pole (the immediate vicinity of this point had already been surveyed by Schwerd at Speyer), and afterwards from the southern point reached at Königsberg towards the southern horizon. This work has since been continued by our eminent countryman Gould to the South pole, on a truly gigantic scale, and with a still greater approach to completeness.

But in 1852 Argelander took a new departure. Up to this time astronomers had generally used their instruments of observation in searching for the stars. They are somewhat ill-adapted to this purpose. They bear high powers, and their fields are small, and illuminated, so as to make visible the spider-lines on which the bisections are made. All these circumstances cause the loss of many stars which are missed in the process of sweeping. So Lalande, with a small meridian instrument, picked up many stars afterwards overlooked by Bessel and Argelander; while they found many which Lalande ought to have been able to see. Nay more: Argelander had spent his leisure at Bonn, while his observatory was in building, before he had even a temporary shed for his transit instrument, in making a catalogue of naked-eye stars. In this there were some 40 of the fifth and sixth magnitude, which had never been seen through any telescope, so far as records indicated.

His new plan was to make a working list of stars to the ninth magnitude inclusive, by a survey of the heavens (the celebrated Bonner Durchmusterung), to include stars to the tenth. In this survey no special pains were taken to get accurate places ; the aim simply was to locate every such star nearly enough to identify it and map it somewhat roughly on a chart. A small telescope of three inches aperture was used with a field lighted only by the stars themselves, and a painted scale visible in this manner to replace the fine spider-lines of the more accurate instruments. With this apparatus a star could be roughly observed every four seconds, with a student-assistant to watch the clock-timing of passage ; while in the whole work the number of stars averaged seven or eight to the minute.

This observation was done twice for every part of the Northern hemisphere, and down to 2° south of the equator, between the years 1852 and 1859; and gave a catalogue of 324,198 stars, accurate enough to find them. Later Argelander's assistant, Schoenfeld, who did a great share of the actual work, extended it to the parallel of 23° south: leaving the continuation to the South pole to be effected by Dr. Gill at the Cape of Good Hope.

From this greatest of all star catalogues in size the stars whose magnitude was 9.0 or brighter were selected for more deliberate and precise observation. There were more than 100,000 of them ; and the observations have been now almost entirely completed.

The work was accomplished according to a plan formulated by Argelander in 1868. It was to be done, as Lalande, Bessel, and Argelander himself had previously worked, in zones bounded by parallels of declination. The rule was made that each star was to be *twice* observed, and with more pains and less hurry than had been possible in the previous zone observations. Thus the whole labor was beyond the powers of one astronomer, and it was divided among a number. The first year or two saw beginnings in Helsingfors (Finland), where Krueger, another collaborator and son-in-law of Argelander, labored ; in Kazan and Dorpat (Russia), in Christiania (Norway), in several German observatories, at Cambridge in England, Chicago, and Cambridge in Massachusetts. Various circumstances interrupted, the Chicago fire and consequent financial ruin of the establishment, and the call upon some astronomers in Europe for service thought more practical. The final arrangement of zones has been as follows :

80° to 75°,	Kazan.
75 to 70,	Dorpat.
70 to 65,	Christiania.
65 to 55,	Helsingfors and Gotha.
55 to 50,	Cambridge, Mass.
50 to 40,	Bonn.
40 to 35,	Lund, Sweden.
35 to 30,	Leyden.
30 to 25,	Cambridge, England.
25 to 15,	Berlin.
15 to 5,	Leipsic.
5 to 1,	Albany.
1 to —2,	Nicolaief.

The space around the North pole had been, meanwhile, again surveyed in the most thorough manner by Carrington and others, so that the limits here given covered the necessity in the Northern hemisphere. German observatories have done nearly half the work, the remainder being divided in somewhat unequal proportions between Russia, America, Scandinavia, England, and Holland. The great Russian observatory of Pulkova furnished the indispensable fundamental catalogue. Of course with equal breadth the polar zones are the smaller; thus of the three catalogues already published, Christiania (5° wide) contains 3,949 stars, Helsingfors (10° wide) 14,680, and Albany (only 4° wide) 8,241. This makes, in less than one-fifth of the Northern hemisphere, or one-tenth of the whole heavens, a grand total of 26,870; but some hundreds of these may be duplicates (between Helsingfors and Christiania), as the zones are made to overlap at the edges.

The limits of the present article are too narrow to enter into the technical details of the observations and reductions. The admirable introduction by Professor Boss to the Albany zone (printed in English) can be referred to as the best account of Argelander's plan in our language; the original instructions are given in the *Vierteljahrschrift der Astronomischen Gesellschaft*, Vol. II. The whole undertaking is in fact almost the original cause of the formation of the Society, which has since undertaken many other serious problems, and has become the leading astronomical society of the world. The results, reduced to 1875, are extremely accurate. Of course more time spent on every single observation would have rendered them still better; but all indications show that, in the great majority of cases, each coördinate of a star's place will be found accurate within a second of a great circle; so that if a star changes place but two or three seconds in a century, its motion can be detected before the year

2000. And as all Bessel's and Lalande's stars, however faint, have been reobserved, there is a vast mass of material now ready for the study of proper motions. A few years now will see the printing of the row of stately volumes which will contain the results of several centuries (where all the work is combined as if done by one astronomer) of human labor.

A continuation to the declination $-23°$ is now in progress in America, Algeria, and Austria; Gould's great work, about the same time, defers the necessity of going farther, although it does not render it superfluous. Photography will doubtless be called in to make this problem easier; or, rather, the Southern zones will be included in the present photographic survey, and perhaps repeated later by the same method.

The comparison of the three volumes mentioned at the head of this article is in many respects instructive. The astronomers were of different nations, employed widely varying instruments, and in one respect a different method. Fearnley of Christiania, and Krueger of Helsingfors and Gotha were pupils of Argelander, and employed the old "eye and ear" method (elaborated by the Greenwich astronomers of the last century). In this the transits of the stars across the meridian are watched by the astronomer, who continually counts by the ear the beats of his clock. If this makes too little sound, he can reinforce it by an electro-magnet. He notes where the star is at the integral second (or half-second) before it passes the wire, and where at the second or half-second after; and estimates the tenths by comparing a second or two afterwards what psychologists call the "traces" on his memory. The method is not always the most precise possible, as it requires long training so to regulate the mental processes that uniform results shall be obtained; but in high declinations, where the stars appear to move much slower, it has certain advantages, and is always free of the annoyance that the sheets or tapes of the chronographic method must be read off. Argelander himself never used the more modern American method, which is, other things being equal, the more accurate, but is not always the one which produces equally accurate results with the least labor.

The American is the telegraphic method. The star is seen approaching the wire, and the observer touches a telegraph key when he estimates that it has reached it. This instant is mechanically recorded on a chronograph. In one point this seems to be less accurate than the other; a very faint star is usually misplaced by the fact that the observer lingers in his judgment that the phenomenon has taken place when the effect is hard to see; so that the right ascensions of faint stars are too large when chronographically determined.

The Albany zone was so observed. Professor Boss of course determined how much each star was delayed in observation by this process; using an ingenious method invented by Bessel of artificially diminishing the light of the stars as seen through the telescope without altering the character of the image, and so found that his own mental processes delay his judgment by about a hundredth of a second per magnitude; that is, he would observe a star of the eighth magnitude seven-hundredths of a second later than one of the first in the same place; and so put it forward a second of arc and a small fraction in right ascension.

On the other hand, the Albany observations of right ascension are rather better, one by one, than those made at Helsingfors. This was probably in part due to Krueger's anxiety about his declinations, which gave him more trouble, owing to the weakness of his instrument in that respect. Fearnley, on the other hand, had a zone so far north (65° to 70°) that with the old method he was able to equal the quality of Boss' work in right ascension with the new, while his employment of verniers instead of reading microscopes has somewhat impaired his declinations.

But, all told, the uniformity of the three catalogues, due to the excellent plan formulated by Argelander, is more sensible and far more important than the trifling discrepancies in execution. The plan is in fact the quintessence of modern practical astronomy in the subject with which it deals. That it has been so warmly welcomed and so thoroughly executed by astronomers over the whole civilized globe is at once a proof of the excellence of their training and of the great advance which has been made in giving the human mind control over its own processes and over material objects.

TRUMAN HENRY SAFFORD.

A PROBLEM IN LEAST SQUARES.

BY PROF. MANSFIELD MERRIMAN.

To determine, by the method of least squares, the most probable values of a and b in the formula $y = ax + b$ when the observed values of both y and x are liable to error.

I. LET x_1 and y_1, x_2 and y_2, ... x_n and y_n be n pairs of observed values of two variables known to be connected by the relation

$$y = ax + b.$$

If the observed values of x were free from error, the most probable values of a and b would be deduced by the application of the common rules of the method of least squares. There would then be n observation equations of the form

$$ax + b - y = 0,$$

from which would result two normal equations

$$[x^2]\,a + [x]\,b - [xy] = 0,$$
$$[x]\,a + nb - [y] = 0,$$

whose solution gives for a the value

$$a_1 = \frac{n\,[xy] - [x]\,[y]}{n\,[x^2] - [x]^2}.$$

II. If however the observed values of y are free from error the formula should be written

$$\frac{1}{a}y - \frac{b}{a} - x = 0\,;$$

then by forming the normal equations and solving, there is found for a the value

$$a_2 = \frac{n\,[y^2] - [y]^2}{n\,[xy] - [x]\,[y]},$$

which in general is quite different from that given in I.

III. How shall the most probable value of a be found when the observed values of both x and y are subject to error? The following is the solution which I made in February, 1891, when considering the problem at the request of the Director of the Observatory of Harvard College:

Let the weight of each observed value of y be unity, and let the weight of each observed value of x be g. Then let a_1 and a_2 be computed by the formulas in I. and II. The most probable value of a is then one of the roots of the equation

$$a^2 + \left(\frac{g}{a_1} - a_2\right) a - g = 0.$$

IV. The demonstration of the last formula will be given in full in a paper which is to appear in the report of the U. S. Coast and Geodetic Survey for 1890. Here there is only space to illustrate its application by one or two numerical examples.

When a has been computed the most probable value of b is directly found from

$$b = \frac{[y] - a\,[x]}{n}.$$

V. An interesting corollary is applicable to the case where a is known *a priori* and a_1 and a_2 are derived from observations. Then from III. the value of g is

$$g = \frac{a - a_2}{\frac{1}{a} - \frac{1}{a_1}}$$

VI. As an example of the application of III. and IV. let the following be simultaneous observations of two thermometers having the same exposure :

No. :	1	2	3	4	5	6	7	8	9
y :	9°	10°	10°	11°	11°	11°	12°	12°	13°
x :	10°	10°	11°	10°	11°	12°	11°	12°	12°

It is required to find the relation between the scales, or the values of a and b in the formula $y = ax + b$, regarding the weights of the two series as equal.

Here $g = 1$, $n = 9$, $[y] = 99$, $[x] = 99$, $[x^2] = 1095$, $[y^2] = 1101$, $[xy] = 1095$. These inserted in I. give $a_1 = 1$ and inserted in II. give $a_2 = 2$. Then from III. there results :

$$a^2 + (1 - 2)\,a - 1 = 0.$$

from which $a = +\,1.618$ and $a = -\,0.618$. The former of these is the value required (since it makes the sum of the squares of the residual errors a minimum, the latter making that sum a maximum). From IV. the value of b is now found to be $-\,6.798$. Thus,

$$y = 1.618x - 6.798$$

is the most probable relation resulting from the given observations. The common rules of the method of least squares would give $y = x$ if observed values of x be taken without error and $y = 2\,x - 11$ if observed values of y be without error.

VII. As an illustration of the use of V. let the following be estimations of the magnitudes of stars by two observers :

No.	:	1	2	3	4	5	6
y	:	8°	9°	10°	10°	10°	11°
x	:	9°	9°	11°	9°	10°	9°

It is required to find the weight g, it being known *a priori* that $a = 1$. Here, from I. there is found $a_1 = \frac{22}{29}$, and from II. $a_2 = \frac{32}{22}$; then from V. there results

$$g = \frac{22 - 32}{22 - 29} = \frac{10}{7},$$

or the weight of the first series of observations is to that of the second as 7 is to 10.

VIII. If the equation between the variables be of a degree higher than the first, as $z^2 = aw^2 + b$, values of a and b may be deduced by following the above method, regarding z^2 and w^2 as observed values corresponding to y and x. Since, however, the real observed values are z and w I am not prepared to say that the results deduced for the parameters a and b will be strictly the most probable ones according to the principles of the method of least squares.

LEHIGH UNIVERSITY, *October*, 1891.

A NEW ITALIAN MATHEMATICAL JOURNAL.

Rivista di Matematica, diretta da G. PEANO. Torino, Fratelli Bocca, 1891.

ALMOST simultaneously with the *Bulletin of the New York Mathematical Society,* a new journal of a somewhat similar character has been founded in Italy. Like the *Bulletin,* the *Rivista di Matematica* is a monthly of at least sixteen pages 8vo. According to the prospectus "its scope is essentially didactic, its principal object being the improvement of the methods of teaching." The *Rivista* will contain "articles and discussions concerning the fundamental principles of the science and also the history of mathematics." "The review of text-books and all publications having reference to the teaching of mathematics will form an important feature." Questions and inquiries about mathematical subjects sent to the editor will be either answered directly or published in the journal. Articles intended for the journal may be written in any of the principal languages and will be translated if necessary. Subscriptions (7 francs per annum) are to be sent to the publishers, Fratelli Bocca, Turin.

The editor, Prof. Giuseppe Peano of the University of Turin, is well known through his original investigations in Mathematical Logic and in Grassmann's Geometrical Calculus, as well as through his rigorous and elegant treatment of the Infinitesimal Calculus. His own contributions to the *Rivista* so far (the first number appeared in January, 1891) relate mainly to the fundamental logical principles of the science of mathematics.

Among the longer articles by other contributors we find an interesting paper (pp. 42–66) by Professor Segre, of Turin, addressed to his students, in which he points out some of the distinctive features of modern mathematics and gives wholesome advice to the young mathematician who wishes to engage in original research. The author is evidently inspired by what may be called the modern Göttingen school (Riemann, Clebsch, and in particular Felix Klein), insisting as he does on the organic unity of the whole of mathematics, warning against excessive specialization, and recommending that the young mathematician should make it his object to bring to bear as far as possible all branches of mathematical science on the particular subject of his investigation. It is curious to note that, in the opinion of Prof. Segre, there exists a very pronounced preference for the study of pure geometry, to the injury of analytical studies, among the younger generation of Italian mathematicians. Some remarks in this paper as to mathematical rigor and the use of hyperspace gave rise to an interesting discussion between the author and the editor (pp. 66–69, and pp. 154–159). Other contributors are A. Favaro, G. M. Testi, E. Novarese, C. Burali-Forti, G. Vivanti, etc.

Among the reviews, the very full account given by Gino Loria of R. de Paolis' theory of geometrical groups* is most prominent (pp. 105–120). E. W. Hyde's Directional Calculus finds a competent and appreciative critic in the editor (pp. 17–19).

ALEXANDER ZIWET.

ANN ARBOR, *August* 10, 1891.

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* R. DE PAOLIS, *Teoria dei gruppi geometrici e delle corrispondenze che si possono stabilire tra i loro elementi. Memorie della Societa Italiana delle Scienze detta dei* XL., vol. VII. series III.

THE PHOTOCHRONOGRAPH.

The photochronograph, and its application to star transits. By J. G. HAGEN, S. J., and G. A. FARGIS, S. J., Georgetown College Observatory. Georgetown, D. C., 1891. 4to, pp. 36.

THE authors of the above publication are the first to lay before the astronomical world a solution, or at least a partial solution, of the very important problem of meridian transit photography. The instrument they have employed consists essentially of an electromagnetic shutter or "occulting bar," which can be attached to the eye-end of a transit instrument or meridian circle. The apparatus is so arranged that the current from a break-circuit clock moves the occulting bar every second in such a way that the image of a star in transit is impressed for a moment upon a photographic plate mounted behind the bar. A line of "star-dots" can afterwards be developed upon the plate. In order to refer the dots to the collimation axis of the instrument, a glass reticle plate, ruled with one vertical reference line, is permanently fixed in the tube, directly in front of the sensitized surface, and in contact with it. After the star transit is over, it is easy to impress the line upon the sensitized plate, by allowing the light of a lantern to fall for a moment upon the object-glass of the telescope. While this is being done, the line of star dots is shielded from the light by the occulting bar, now permanently interposed between the dots and the light. This method of impressing the reference line upon the plate is excellent, and is further improved by ruling the line with a break in the middle, so that *none* of the dots can possibly be "occulted" by the line itself. The plates are measured with a micrometric apparatus, by means of which it is easy to determine the instant of the passage of the star across the reference line.

The process thus very briefly outlined is given by the authors with all possible detail; even the preliminary apparatus, subsequently discarded as imperfect, being carefully described. Other experimenters in the same field should therefore be greatly aided by the present work. In this connection it is proper to refer to the earlier observations of L. M. Rutherfurd, of New York, who successfully employed an arrangement essentially similar to the photochronograph many years ago.* In the collection deposited by Mr. Ruther-

* B. A. GOULD, *Memoirs of the National Academy of Sciences*, vol. iv., p. 175.
L. M. RUTHERFURD, *American Journal of Science and Arts*, vol. iv., Dec., 1872.

furd at Columbia College* are many negatives showing lines of star-dots, together with micrometric measures of the same.

We shall now enter into a somewhat more careful examination of some of the statements contained in the book, taking them up in order. It is difficult to see why only one vertical line has been used on the reticle plate. The authors refer to their reason for this (p. 12), but without anywhere definitely stating it. One would think the presence of several vertical lines would offer a valuable control of possible irregular expansions of the film during development. Nor would there be any compensating disadvantage, for the admirable device of breaking the lines in the middle would prevent any interference with the star-dots. The effect of *irregular* distortion of the film would not be eliminated by the method of measurement (p. 24). It is gratifying to find (p. 13) that no trouble was experienced from a jarring of the instrument by the regular beats of the occulting bar.

Probably the most important difficulty of the method is touched upon by the authors in speaking of collimation (p. 17). In fact, it may safely be said that the photographic transit instrument will not be applicable to the finest fundamental work, until it becomes possible to determine the collimation and level constants photographically; without reversal in the Y's, and without the use of the hanging level. In all the observations so far made, the collimation constant has been determined from reversals alone, and the hanging level has always been employed. A very interesting remark occurs (p. 18) in connection with personal equation. By watching the occulting bar through an eye-piece while a star is in transit, the existence and effect of the observer's personal equation become very obvious.

We now come to Part II. of the book, which treats of the reduction of the observations. This part is the work of J. G. Hagen, S.J.; the first part, in which the instrument and methods are described, being by G. A. Fargis, S.J. The screw of the measuring micrometer has been examined for both periodic and progressive errors, according to the usual methods. The author very justly concludes that it is advisable to determine the screw value separately from each plate, though errors due to an oblique mounting of the plate in the tube would not be eliminated thereby, as seems to be implied in the text (p. 22, c). The adopted method of measurement, by which the dots are taken in corresponding pairs at nearly equal distances on both sides of the central line, has much in its favor. With regard to the example of a series of transits

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* J. K. Rees, *Annals of the New York Academy of Sciences*, vol. vi., June, 1891.

(p. 34) it may be said that the data are not sufficient to draw conclusions of a very definitive character, beyond the fact that the method gives results of very satisfactory accuracy. The azimuth constants for the evening (two in number) and the collimation constant have been derived from the fifteen observations themselves. Their values are stated to be the "most probable" ones. If they have been obtained by a least-square reduction in which the clock-rate was ignored, it is not remarkable that the final residuals show no evidence of a clock-rate (p. 35).

In conclusion, we may accord to the authors of this book the credit of having invented and made public a photographic method by which meridian transits may be observed with high accuracy, and with a complete freedom from personal equation. If there is a weak point, it will be found in the determination of the instrumental constants. The many other important purposes for which the photochronograph is very well adapted we shall not touch upon in this place. Some of them have already been described in print, and many others will doubtless shortly come into prominence.

HAROLD JACOBY.

COLUMBIA COLLEGE, NEW YORK, 1891, *October*.

NOMENCLATURE OF MECHANICS.

BY T. W. WRIGHT, PH.D.

THE nomenclature of mechanics is in a somewhat confused condition. There is some excuse for this because the science is one of the oldest, and at the same time one of the most progressive, as it certainly is the most comprehensive. New terms are being introduced, others are being suggested to take the place of old ones; but the naturally conservative cling to the old, and hence we have a duplication, and in some cases a triplication of names for the same thing. At the threshold we are met by a difficulty. How shall we define mechanics? Originally the science of machines, it is by some defined as the science of matter and motion. By others the term dynamics is applied to the science of matter and motion, and the term mechanics is discarded. The tendency at present seems to be in the direction of the latter method. The science is founded on three principles or laws laid down by Newton. These laws were originally enunciated in Latin, and the number of translations is very great. Here is a source of confusion. With a new translation come in new terms or a change in the meaning of old ones. For example, Newton's first law is called by some the *law of inertia*. What is inertia? Is it inertness, a mere negative property, or is it a property admitting of measurement, a quantitative property? When we come to the second law we have the idea of mass prominently brought forward. Since the second law includes the first, why introduce the term inertia at all? Is not mass sufficient? Call the first law the *law of mass* and the second the *law of mass-acceleration*. The reformers who drop inertia in the first law would have us call centre of gravity centre of mass, and moment of inertia moment of mass. The first of these changes, centre of mass for centre of gravity, is well under way and will probably prevail. The change from moment of inertia to moment of mass meets with less favor. Indeed, the new name seems as objectionable as the old, for the moment is not a simple moment, but a second moment.

Next in importance to a proper translation of the laws of motion is the settlement of the question of how *weight* shall be defined. One school use it in the sense of mass; another in the sense of force, it being the attractive force of the earth on mass; while a third contend for its use in both senses. The question was debated by some of the ablest physicists in England two or three years ago but no definite conclusion was reached. This and the relation

$$W = m\,g$$

form probably the center of greatest confusion in elementary mechanics. The perplexity of a beginner as to whether in a given problem he shall multiply or divide by g is extreme, and the mournful thing is that this is not owing to his own stupidity. The pit has been dug for him and is persistently kept open waiting for new victims.

The nomenclature is deficient in several respects. We have no single term for the unit of velocity, the *foot per second*, nor for the unit of acceleration, *the foot per second per second;* but must use these long phrases where a monosyllable ought to suffice. The most satisfactory suggestion I have seen is to use *f.s.* for unit velocity and *f.s.s.* for unit acceleration. Nor have we any name for the absolute unit of force in the British system. It is true that some recent writers use Prof. James Thomson's term the *poundal* for unit force. If we say poundal shall we say ouncal, tonal, etc.? Consistency would seem to force us to do so. The terms sound odd enough. Is the gain in simplicity in the dynamical formulas expressed in absolute units over that of the gravitation system a sufficient excuse for introducing terms that will probably never be used outside of the lecture room? What engineer would use foot-poundal for example? The nomenclature is also redundant. A single instance will suffice. Shall we say vis-viva, living force, or kinetic energy? All three are used to denote the same thing to the mystification of the beginner. All three can be found in text books of recent date. To my mind there is no doubt but that kinetic energy is the proper term.

Now, the confusion, deficiency, and redundancy being granted, what can be done? No one writer can do much to effect a change. But an association such as the *New York Mathematical Society* can do much. Expressions of opinion through the pages of this journal would probably lead to some more definite understanding than now exists. At least some of the more glaring absurdities and contradictions of our present system might be abated. Besides, it might tend to curb the ambition of writers to introduce ill-considered terms such as "heaviness" or "centre of weight" for centre of gravity and the like.

UNION COLLEGE, 1891, *October* 10.

A TREATISE ON LINEAR DIFFERENTIAL EQUATIONS. Vol. I. Equations with uniform coefficients. By THOMAS CRAIG, Ph.D. New York; John Wiley & Sons, 1889. 8vo, pp. ix. + 516.

THE appearance of Fuchs's two memoirs in 1866 and 1868 respectively, gave an impetus to research on linear differential equations which has resulted in the development of an enormous literature on the subject, consisting of articles and memoirs scattered through mathematical journals and the proceedings of learned societies. The systematization and presentation in a body of the principal methods and results developed in these isolated papers, is the work which has been undertaken by Professor Craig, and which has successfully issued in the first volume of the most advanced treatise on pure mathematics ever published by an American author. Whilst the presentation of the subject as a whole must prove of advantage to those few mathematicians who have access to the memoirs whence it draws, upon the many to whom the original sources are not open it confers an inestimable boon. To the English-reading student further it manifests in his own language the substance of what is for the most part in the original in French or German. Praise is due the author for the scrupulous care with which he credits every writer quoted, and for the fulness of his references, which give an added value to the volume. A glance at these references cannot fail to impress upon the reader a sense of the overwhelming influence which the continental element has had in shaping the development of modern differential equations. In fact, an analysis shows that of the sixty-odd names quoted in the volume more than three-fourths divide themselves about equally between the French and Germans, and of the remainder some eight may be claimed by the English speaking peoples: so that if this showing in relation to the populations of the countries concerned could be fairly considered as furnishing a criterion relative to the generality of interest manifested among the several peoples in the development of the subject, such interest in America and England as compared with that in France and Germany might be averaged as 1 to 7. The dropping of the average in the comparison, it may be frankly owned, would not advantage the showing of America.

The reader in his progress through this treatise will constantly have to do with the modern theory of functions, and will meet with some simple applications of the theory of substitutions. Both of these departments, with their numerous applications and possibilities of further development, offer a field whose successful cultivation on the continent shows a productive power giving as yet no sign of exhaustion. Professor Craig's book will have accomplished a useful mission if it helps to awaken American students to a sense of the work that is being done in Europe, and, as a consequence, rouses them to a realization of what is being left undone in America. There seem, however, at present to be definite tendencies making for the elevation of mathematics in America, and it may not perhaps be idle to indulge a hope that America will yet contribute in a fitting proportion to the development of the science. The preliminary knowledge of the theory of functions necessary to the reading of Professor Craig's book may be obtained from Hermite's *Cours*. To the student who desires an acquaintance with the theory of substitutions one can recommend Netto's *Substitutionentheorie* and Serret's *Cours d'Algebre Superieure,* though so far as is necessary for understanding the applications of the latter theory in the volume under consideration, a very partial reading of its treatment in either of the works mentioned will prove sufficient, and, in fact, a few words of explanation from one familiar with the substitution notation would probably suffice. The American student of mathematics who acquires a knowledge of these branches will in general do so by his own unaided efforts, for courses in them are offered by but a small number of our universities, and further, as regards unassisted study, it may unfortunately be said that few of our colleges and universities give a course in mathematics whose discipline prepares a man for such study. The fault lies perhaps not so much with the higher institutions of learning as with the preparatory and high schools, into whose hands our potential young mathematicians first fall, and which as a general rule allot to the study of algebra and geometry a time utterly inadequate to the laying of a basis on which the college can satisfactorily build. On the other hand, almost all our college professors, among whom we find, of course, the great majority of our mathematicians, are overworked. Teaching absorbs the energy and spontaneity which should be spent upon private study and research. For the latter scanty allowance is made, except in a few of our larger universities, conspicuous among which is that university in which the author of our treatise is a teacher. The lack of stimulus and encouragement due to the isolation in which the American mathematical professor has been wont to live, may (it is not an unreasonable anticipation) be remedied in some degree by the founding of a mathematical society of national scope with the publication of a bulletin. Thus may be fostered among American mathematicians a fellow interest in their science, to illustrate the advantages of which we might cite the subject of the work before us, which has been developed since the publication of Fuchs's memoirs, only by the cross-working of scores of European mathematicians.

Before the appearance of these two memoirs the only general class of linear differential equations for which a solution had been found was that in which all the coefficients are constant, but with the application of the modern theory of functions a new field opened up. In this theory the critical points of a function play an all-important role, and, as can be readily shown in the case of the equation which constitutes the theme of the volume under review, the critical points of the integrals of the equation are included among those of its coefficients. This property evidently gives us some hold upon the integrals and is, when combined with the fact that the general integral is a linear function of the particular integrals, more fruitful of results than would readily be anticipated, results of which but a few can here be hinted at.

The work opens with a recapitulation of the general properties of linear differential equations, followed by an extended modern treatment of the equation with constant coefficients. It then takes up the theory of the differential equation

$$(1)\ P(y) = \frac{d^n y}{dx^n} + p_1 \frac{d^{n-1}y}{dx^{n-1}} + p_2 \frac{d^{n-2}y}{dx^{n-2}} \cdots\cdots + p_n y = 0.$$

where the coefficients $p_1, p_2, \ldots, p_n$ are uniform functions of x, having only poles as critical points. Let $y_1, y_2, \ldots y_n$ denote a system of fundamental integrals of (1). If now the imaginary variable x make the circuit of a critical point in the plane, returning by any path to its point of departure, the coefficients, since they are uniform, will return into themselves, and the equation will be unaltered. Any integral of the original equation, then, necessarily remains such, and can at most have transformed into a linear function of the n fundamental integrals. It is now shown that among such transformed integrals, there will be at least one which will transform into itself multiplied by some constant s which is determined as the root of an equation of the nth degree in s called in reference to the critical point in question, the characteristic equation for the system of fundamental integrals $y_1, y_2, \ldots, y_n$.

There will be as many such integrals as there are solutions to the characteristic equation; and, in fact, corresponding to a λ-multiple root s_1 of this equation there will be a group of λ integrals $u_1, u_2, \ldots, u_\lambda$ which, when the variable x completes the closed circuit, may respectively be shown to transform into

$$\begin{array}{l} s_1 u_1, \\ s_{21} u_1 + s_1 u_2, \\ s_{31} u_1 + s_{32} u_2 + s_1 u_3, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ s_{\lambda 1} u_1 + s_{\lambda 2} u_2 + \ldots + s_{\lambda,\lambda-1} u_{\lambda-1} + s_1 u_\lambda. \end{array}$$

where the coefficients s are all constants, and the aggregate of such groups corresponding to the different roots of the characteristic equation will constitute a system of fundamental integrals of the differential equation. The theory is given for the point $x = 0$ considered as the typical critical point, the reasoning for any other critical point a being obtained by substituting $(x-a)$ for x wherever it may appear in our formulæ.

The group of integrals given above are now shown to be of the following forms:

$$(2) \quad \begin{cases} u_1 = x^{r_1} \varphi_{11} \\ u_2 = x^{r_1} \{\varphi_{21} + \varphi_{22} \log x\} \\ u_3 = x^{r_1} \{\varphi_{31} + \varphi_{32} \log x + \varphi_{33} \log^2 x\} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ u_\lambda = x^{r_1} \{\varphi_{\lambda_1} + \varphi_{\lambda_2} \log x + \ldots\ldots\ldots + \varphi_{\lambda\lambda} \log^{\lambda-1} x\} \end{cases}$$

where the φ's are uniform in the region of our critical point

$x = 0$, and such that anyone of them can be expressed in terms of those whose second subscript is 1; $\varphi_{11}, \varphi_{22}, \ldots\ldots, \varphi_{\lambda\lambda}$ differing from one another only by constant factors and $2\pi i r_1$ being equal to $\log s$; we find that this group (2) may be replaced by a number of sub-groups possessing precisely the properties just enumerated, and can further show that the transformation effected by a circuit of the critical point may be represented thus:

$$S \equiv \left| \begin{array}{l} y_1, \ y_2, \ \ldots; \ s_1 y_1, \ s_1(y_2 + y_1), \ \ldots, \ s_1(y_\alpha + y_{\alpha-1}) \\ y'_1, \ y'_2, \ \ldots; \ s_1 y'_1, \ s_1(y'_2 + y'_1), \ \ldots, \ s_1(y_{\alpha'} + y_{\alpha'-1}) \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ z_1, \ z_2, \ \ldots; \ s_2 z_1, \ s_2(z_2 + z_1), \ \ldots, \ s_2(z_\beta + z_{\beta-1}) \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{array} \right|$$

the interpretation of this notation being that a certain system of n independent integrals of our equation represented by the symbols $y_1, y_2 \ldots y'_1 \ldots z_1 \ldots$ on the left, are by circuit of the critical point transformed into the expressions corresponding in position on the right, where $s_1, s_2, \ldots$ are solutions of the *characteristic* equation. The aggregate of transformations thus effected is indicated by the letter S and is called the substitution for the point in question. As presented here it is said to be in its canonical form. There will be such a substitution corresponding to a single circuit of any critical point, to a multiple circuit of the same, or to a circuit including any combination of critical points, all substitutions, by the way, being reducible to successive applications of the substitutions for different individual points.

The aggregate of all possible substitutions is defined as the *group* of the equation. In a later chapter the author by the aid of the canonical form just given goes into the investigation of what are called function-groups, these being groups of functions which under all possible applications of a substitution-group transform into one another. The integrals of our equation (1) evidently constitute such a group and include, it may be, smaller function-groups formed by the linear functions of linearly independent integrals less than n in number. With the consideration of these the chapter just referred to concerns itself.

Reverting now to formulæ (2), if all the φ's entering into any one of the integrals u contain only finite negative powers of x, the integral is called *regular* in the region of the point $x = 0$, and with proper choice of r can be written

$$(3) \qquad F \equiv x^r \left\{ \varphi_0 + \varphi_1 \log x + \ldots + \varphi_k \log^k x \right\},$$

where the φ's are uniform, and $x^{-r}F$ becomes infinite for $x = 0$ in the same manner as $\alpha + \beta \log x + \ldots\ldots +$ $\lambda \log^k x$, α, β, ... being constants. In order that equation (1) should have a system of linearly independent regular integrals in the region of the point $x = 0$, it is shown to be necessary and sufficient that every coefficient p_j shall have $x = 0$ as an ordinary point or a pole of multiplicity not greater than j. Denoting by w_i the degree of x in the denominator of p_i, the value of i for which $w_i + n - i \equiv g$ is a maximum is called the *characteristic index* of equation (1), and by substitution in the differential quantic $P(y)$ of x^ρ for y, we will find that $x^{-\rho}P(x^\rho)$ developed in ascending powers of x has as its first term $G(\rho)\, x^{-g}$ where $G(\rho)$ is an integral function of ρ of degree $n - i \equiv \gamma$. $G(\rho) = 0$ is called the *indicial equation*, and it may be shown that the number of linearly independent regular integrals of (1) is not greater than the degree of this equation. The conditions that it should be equal to this degree are also determined, and in particular its degree is observed to be equal to n when all the integrals are regular. The exponent r in (3), where F is supposed to be a regular integral, is given by the indicial equation; and the coefficients of the φ's developed in positive powers of x are determined by substitution of F in the differential equation.

An extended application of the general theory is made to differential equations of the second order, particularly to the equation which has all its integrals regular and possesses but three critical points. This equation is shown to be transformable to one in which the critical points are 0, 1, ∞, an equation of which the hypergeometric series $F(\alpha, \beta, \gamma, x)$ is an integral. A complete translation of Goursat's memoir on this equation is embodied in the work, filling some 150 pages. An exhaustive discussion is given of its twenty-four integrals, which divided into six groups of four each, are connected by some twenty linear relations between integrals selected from the six groups taken three at a time. The portions of the plane in which the several integrals have a meaning are also indicated. An investigation is made of the transformations admitted by the series when all three quantities α, β, γ are not arbitrary; and an extended list of such transformations, with formulæ derived therefrom, is given. The theory of irreducible equations is briefly touched upon; as is also, at greater length, the theory of the decomposition of a linear differential equation into prime factors, with its application in the case of equations possessing regular integrals.

In equation (1) we can by a simple transformation readily get rid of its second term; and, as is shown in one of the later chapters of the book, by a transformation $z = \varphi(x)$, $y = z'^{-\frac{1}{2}(n-1)} u$, where the form of z is dependent on a differential equation of second order, we may still further rid ourselves of its third term, the equation so reduced being said to be in its *canonical form*. There are also certain *associate equations* $(n - 2)$ in number, the solutions of each of which consist in a set of variables dependent upon the integrals of equation (1) and possessing relative to the transformation mentioned, the invariantive property of returning into themselves multiplied by a power of z', among these equations being found the well-known equation of the n^{th} order on which depends the determination of an integrating factor for (1).

The volume concludes with a short chapter on equations with uniform doubly-periodic coefficients, a subject which the author expresses his intention of resuming in his second volume. Supposing w and w' to be the periods of our coefficients, by the substitution of $x + w$ or $x + w'$ for x, they will remain unaltered and the integrals will transform into linear functions of one another. By analogy the general theory already given suggests that the characteristic equations corresponding to these substitutions may give us constants s and s', by which the respective transformations multiply some integral u. When the general integral happens to be uniform such proves to be the case, there being at least one integral u which by the substitutions $x + w$ and $x + w'$ for x respectively transforms into $s\,u$ and $s'u$, and for the determination of such integrals, as also of the other integrals of the equation, methods are given.

J. C. FIELDS.

NOTES.

AT the meeting of the NEW YORK MATHEMATICAL SOCIETY held Saturday afternoon, October 3d, at half-past three o'clock, the Council announced that Professor Henry B. Fine had been appointed to fill the vacancy in their body. The following persons having been duly nominated, and being recommended by the Council, were elected to membership: Professor Thomas Craig, Johns Hopkins University; Dr. A. V. Lane, Dallas, Texas; Professor L. A. Wait, Cornell University; Professor George Egbert Fisher, University of Pennsylvania; Mr. William H. Metzler, Clark University; Professor Ellen Hayes, Wellesley College; Professor George A. Miller, Eureka College; Mr. Charles Nelson Jones, Milwaukee, Wisconsin; Dr. J. Woodbridge Davis, New York; Mr. Charles H. Rockwell, Tarrytown, N. Y.; Professor J. Burkitt Webb, Stevens Institute of Technology.

The following original papers were read: The Determination of Azimuth by Elongations of Polaris, by Mr. Harold Jacoby; On Powers of Numbers whose Sum is the Same Power of Some Number, by Dr. Artemas Martin; A Classification of Logarithmic Systems, by Professor Irving Stringham.

Professor Stringham's paper will be published in the *American Journal of Mathematics*, and Mr. Jacoby's has been communicated to the *Royal Astronomical Society of London*.

T. S. F.

IN the course of his paper mentioned above Dr. Martin presented the following very remarkable series of numbers recently found by him:

$$4^5+5^5+6^5+7^5+9^5+11^5 = 12^5$$

$$5^5+10^5+11^5+16^5+19^5+29^5 = 30^5$$

$$12^5+13^5+15^5+16^5+17^5+\ldots.+23^5+25^5+27^5+28^5+29^5+\ldots+35^5 = 50^5$$

$$1^6+2^6+4^6+5^6+6^6+7^6+9^6+12^6+13^6+15^6+16^6+18^6+20^6+21^6+22^6+23^6 = 28^6$$

The paper will be published elsewhere *in extenso*.

IN connection with Professor Merriman's article, it may be of interest to note that Professor Wright, also a contributor to the present number, gives a different treatment of the same problem in his "Treatise on the Adjustment of Observations," p. 206.

WILLIAM FERREL, the eminent meteorologist, died on Friday, September 18, at Maywood, Wyandotte County, Kansas. He was born in Bedford County, Pennsylvania, January 29, 1817. He studied at Franklin and Marshall College, and at Bethany College, being graduated from the latter in 1844. In 1857 he became an assistant in the office of the *American Ephemeris and Nautical Almanac*, and held that position for ten years. Thereafter, until 1882, he held a special appointment in the United States Coast Survey. In that year he was made assistant, with the rank of professor, in the Signal Service Bureau, where he remained until October, 1886, when he made his home in Kansas City, Missouri. He invented the maxima and minima tide predicting machine, which is now used by the Coast Survey in predicting the tides. Professor Ferrel received honorary elections to the Austrian, English, and German meteorological societies, and in 1868 was elected to membership in the National Academy of Sciences. Some of his principal works are "Motions of Fluids and Solids Relative to the Earth's Surface," published in 1859; "Determinations of the Moon's Mass from Tidal Observations," 1871; "Converging Series Expressing the Ratio between the Diameter and the Circumference of a Circle," 1871; "Tidal Researches," 1874; "Tides of Tahiti," 1874; "Meteorological Researches," in three parts, published consecutively, in 1875, 1878, and 1881; "Recent Advances in Meteorology," 1883, and "Temperature of the Atmosphere and the Earth's Surface," 1884.

IT is with regret that we learn of the death of our member, Asher Benton Evans. He died at Lockport, the place of his late residence, September 24, 1891. He was an alumnus of Madison (now Colgate) University of the class of 1860. He was widely known as an educator, and had been a contributor to the *Mathematical Monthly*.

THE Cambridge University Press announces: *Catalogue of Scientific Papers Compiled by the Royal Society of London*, new series for the years 1874–1883; *The Collected Mathematical Papers of Arthur Cayley, Sc. D., F. R. S., Sadlerian Professor of Mathematics in the University of Cambridge*, Vol. IV.; *A History of the Theory of Elasticity and of the Strength of Materials*, by the late I. Todhunter, F. R. S., edited and completed by Carl Pearson, Professor of Applied Mathematics, University College, London, Vol. II.

The Clarendon Press promises: *Mathematical Papers of the late Henry J. S. Smith, Savilian Professor of Geometry in the University of Oxford*, with portrait and memoir, 2 vols.; *A Treatise on Electricity and Magnetism*, by G. Clerk Maxwell, new edition.

THE October number of the *American Journal of Mathematics* begins the fourteenth volume. It contains as a frontispiece an excellent likeness of Professor Felix Klein of Göttingen.

JOHN WILEY & SONS have in preparation "A New Elementary Synthetic Geometry, Plane and Solid, especially adapted to high-school work, with numerous examples," by George Bruce Halsted, Professor of Mathematics in the University of Texas. T. S. F.

THE *Inland Press* (The Register Publishing Company, Ann Arbor, Mich.) has just issued: "Practical astronomy," by W. W. Campbell, a short treatise mainly intended for the use of surveyors and civil engineers; also "Logarithmic and other mathematical tables" (to five places), by W. J. Hussey. The same house announces as in preparation two translations from the German: O. Dziobek's "Mathematical theories of planetary motions," translated by Prof. M. W. Harrington; and E. Netto's "Theory of substitutions and its applications to algebra," translated by Dr. F. N. Cole. It is to be noticed that Dr. Netto has not only authorized the present translation, but has furnished the translator with a large amount of new material in the form of corrections and additions, so that some of the chapters of the original are almost entirely re-written, and the whole work will be considerably increased. The work will appear early in 1892.

PROFESSOR M. W. HARRINGTON having been appointed chief of the U. S. Weather Bureau, the astronomical observatory of the University of Michigan is temporarily in charge of the newly appointed instructor in astronomy, Mr. W. J. Hussey. The former instructor, Mr. W. W. Campbell, has accepted a position as assistant at the Lick Observatory, Mt. Hamilton, Cal. A. Z.

PROFESSOR CLARENCE A. WALDO, recently of the Rose Polytechnic Institute, is now at De Pauw University, Greencastle, Indiana. T. S. F.

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ON THE DOUBLY INFINITE PRODUCTS.*

BY DR. THOMAS S. FISKE.

THE familiar singly infinite products for the sine and cosine, due to Euler,†

$$\sin x = x\left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{4\pi^2}\right)\left(1-\frac{x^2}{9\pi^2}\right)\cdots,$$

$$\cos x = \left(1-\frac{4x^2}{\pi^2}\right)\left(1-\frac{4x^2}{9\pi^2}\right)\left(1-\frac{4x^2}{25\pi^2}\right)\cdots,$$

or in another form,

$$\sin x = x\prod_{-\infty}^{+\infty}\left[1-\frac{x}{m\pi}\right],$$

$$\cos x = \prod_{-\infty}^{+\infty}\left[1-\frac{x}{(m+\frac{1}{2})\pi}\right],$$

were first generalized by Abel. By a brilliant stroke of genius he obtained for the elementary doubly periodic functions the remarkable expressions ‡

$$\operatorname{sn} x = \frac{x\Pi\Pi\left[1-\frac{x}{m\omega+m'\omega'}\right]}{\Pi\Pi\left[1-\frac{x}{(m+\frac{1}{2})\omega+(m'+\frac{1}{2})\omega'}\right]},$$

$$\operatorname{cn} x = \frac{\Pi\Pi\left[1-\frac{x}{(m+\frac{1}{2})\omega+m'\omega')}\right]}{\Pi\Pi\left[1-\frac{x}{(m+\frac{1}{2})\omega+(m'+\frac{1}{2})\omega'}\right]},$$

$$\operatorname{dn} x = \frac{\Pi\Pi\left[1-\frac{x}{m\omega+(m'+\frac{1}{2})\omega'}\right]}{\Pi\Pi\left[1-\frac{x}{(m+\frac{1}{2})\omega+(m'+\frac{1}{2})\omega'}\right]},$$

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* Résumé of a Lecture delivered before the Society at the meeting of November 7, 1890.

† *Introductio in Analysin Infinitorum* (1748), lib. I. cap. IX.

‡ *Œuvres. Nouvelle édition* (1881), t. I., p. 343.
Journal für die reine u. angewandte Math. (CRELLE), Bd. II., p. 172.

in which m and m' are independent of each other and assume successively all integral values from $-\infty$ to $+\infty$, the simultaneous system $m = m' = 0$ alone being excluded in the numerator of the first fraction. Abel, however, did not make a complete and rigorous investigation as to the convergency of these products nor as to their identity with the functions of Jacobi. Cayley made the four doubly infinite products contained in the above expressions the starting point of a series of investigations.* He found for them a complete theory, based in part upon a geometrical interpretation, and upon it he built up the whole theory of the elliptic functions. Almost immediately afterwards, Eisenstein † discussed in a very elaborate manner, and by purely analytic methods, the general doubly infinite product

$$\Pi\,\Pi\left[1 - \frac{x}{m\alpha + n\beta + \gamma}\right],$$

and arrived at results which, when supplemented by the more recent theory of primary factors, due to Weierstrass,‡ have given to the subject a permanent and classical form.

The path which the student naturally follows in the study of the periodic functions, leads him directly to the consideration of these products and, at the same time, indicates their paramount importance. A theorem of Jacobi § shows him that no more general periodic functions of a single variable are possible than the doubly periodic or elliptic functions. He learns that such functions are but the ratios of single valued functions of another class, the so-called theta-functions; and these, it is soon seen, are nothing more or less than doubly infinite products. There is no doubt that the theory of the theta-functions of a single variable forms the natural introduction to that of the elliptic functions.

Before taking up the general products, the limit of the single product ‖

$$u = x \prod_{-p}^{q}\left[1 - \frac{x}{m\pi}\right],$$

* *Camb. Math. Journ.*, vol. IV., 1845, pp. 257–277.
Journ. des Math. (LIOUVILLE), t. X., 1845, pp. 385–420.
Collected Math. Papers, vol. I., nos. 24 and 26.

† *Mathematische Abhandlungen*, Berlin, 1847, pp. 213–334.
Journal für die reine u. angewandte Math. (CRELLE), Bd. XXXV., 1847, pp. 153–247.

‡ *Abhandlungen der Königl. Akad. der Wissenschaften zu Berlin vom Jahre* 1876.
Abhandlungen aus der Functionenlehre, von KARL WEIERSTRASS. Berlin, 1886, pp. 1–52.

§ *Gesammelte Werke*, Bd. I., p. 262.

‖ Cf. HERMITE, *Cours à la Sorbonne, Quatrième édition*, p. 89.

when p and q both become infinitely great, should be considered. It will be found to be indeterminate. In fact, if we have in the limit

$$\log \frac{p}{q} = a,$$

a being a given constant, then

$$u = e^{ax} \sin x.$$

A similar result holds for the infinite product representing $\cos x$.

In the investigations of Cayley corresponding results were developed in connection with the double products, for example

$$u = x \, \Pi \Pi \left[1 - \frac{x}{m\omega + m'\omega'} \right],$$

by the introduction of an auxiliary geometrical construction. The periods ω and ω' being always assumed respectively real and imaginary, a pair of rectangular axes were drawn, and corresponding to every factor in the product a point was set down, the coefficient of the real period being the abscissa and that of the imaginary period the ordinate. The entire finite portion of the plane was thus covered with a series of points forming the vertices of a net-work of squares constructed on the linear unit. These points were all enclosed within a contour of infinite dimensions, the form of which depended upon the relations between the infinite limits of the products. The value of the product was shown to depend upon the form of the contour, and in Cayley's memoirs the bounding contour is regarded successively as a square, a circle, and an infinite horizontal ribbon, and an infinite vertical ribbon.

By the application of logarithms one obtains

$$\log u = - x \Sigma\Sigma \frac{1}{m\omega + m'\omega'} - \frac{x^2}{2} \Sigma\Sigma \frac{1}{(m\omega + m'\omega')^2}$$

$$- \frac{x^3}{3} \Sigma\Sigma \frac{1}{(m\omega + m'\omega')^3} - \ldots .$$

Making the contour symmetrical with respect to the origin, the terms containing odd powers vanish, or

$$\log u = - \frac{x^2}{2} \Sigma\Sigma \frac{1}{(m\omega + m'\omega')^2} - \frac{x^4}{4} \Sigma\Sigma \frac{1}{(m\omega + m'\omega')^4} - \ldots .$$

For another contour, similarly

$$\log u' = - \frac{x^2}{2} \Sigma\Sigma \frac{1}{(m\omega + m'\omega')^2} - \frac{x^4}{4} \Sigma\Sigma \frac{1}{(m\omega + m'\omega')^4} - \ldots .$$

Hence

$$\log\frac{u}{u'} = \frac{x^2}{2}\Sigma\Sigma\frac{1}{(m\omega+m'\omega')^2}+\frac{x^4}{4}\Sigma\Sigma\frac{1}{(m\omega+m'\omega')^4}+\ldots,$$

the sums extending to the region enclosed between the two contours. All the terms except the first being infinitely small, we have

$$u' = ue^{-Bx^2}$$

where

$$B = \tfrac{1}{2}\Sigma\Sigma\frac{1}{(m\omega + m'\omega')^2} = \tfrac{1}{2}\iint\frac{dm\, dm'}{(m\omega + m'\omega')^2},$$

from which is readily seen the relation between two different systems of theta-functions. The system of theta-functions corresponding to the infinite horizontal ribbon is identical with that given by Jacobi.*

In Eisenstein's researches we have

$$u = \Pi\Pi\left[1 - \frac{x}{m\alpha + n\beta + \gamma}\right],$$

whence

$$\log u = -x\Sigma\Sigma\frac{1}{m\alpha + n\beta + \gamma} - \frac{x^2}{2}\Sigma\Sigma\frac{1}{(m\alpha + n\beta + \gamma)^2}$$

$$-\frac{x^3}{3}\Sigma\Sigma\frac{1}{(m\alpha + n\beta + \gamma)^3} - \ldots.$$

The whole theory is thus dependent upon that of the very general series.

$$\Sigma\Sigma\frac{1}{(m\alpha + n\beta + \gamma)^\mu}.$$

Eisenstein's elegant investigation as to the convergency of this series has been recognized as fundamental and has found its way into the text-books.† He deduced as the necessary condition for convergence

$$\mu > 2.$$

It follows that in the expansion of $\log u$ the coefficients of all the powers of x except the first two, have fixed sums indepen-

* JACOBI, *Fundamenta Nova* (1829), cap. 61.
† Cf. JORDAN, *Cours d'Analyse*, t. I., p. 165.

dent of the arrangement of their elements. Since however the first two coefficients may alter their values with a change in the arrangement of the factors of u, two functions which are related to each other in this way will be connected by an equation of the form

$$u' = u\, e^{-px - \frac{1}{2}qx^2}.$$

One finds in Eisenstein's memoir a very elaborate investigation as to the nature and value of the quantities p and q, and the results are applied to a general theory of the elliptic functions. In spite of the great interest of these further developments, it is unnecessary for our present purpose to enter into details upon them on account of the wonderful simplification brought about through Weierstrass's theory of primary factors.*

This theory enables us to express any continuous function which does not become infinite for finite values of the variable in a factorized form. It shows us, however, that the simplest factors of such a transcendental function, should differ from the linear factors of a rational entire algebraic function, in that each should have an exponential associated with it. Thus we find, according to this theory,

$$\sin x = x \prod_{-\infty}^{+\infty} \left[1 - \frac{x}{m}\right] e^{\frac{x}{m}}$$

an expression from which every element of indetermination has been eliminated. Now it is evident, after the investigations of Eisenstein, that we can remove the indetermination from the product

$$u = \Pi\,\Pi \left[1 - \frac{x}{m\alpha + n\beta + \gamma}\right]$$

by introducing the exponential factor

$$e^{x \Sigma\Sigma \frac{1}{m\alpha + n\beta + \gamma} + \frac{x^2}{2} \Sigma\Sigma \frac{1}{(m\alpha + n\beta + \gamma)^2}}.$$

The result may be exhibited as a product of the form

$$\Pi\,\Pi \left[1 - \frac{x}{m\alpha + n\beta + \gamma}\right] e^{\frac{x}{m\alpha + n\beta + \gamma} + \frac{1}{2}\frac{x^2}{(m\alpha + n\beta + \gamma)^2}}.$$

* WEIERSTRASS, *loc. cit.*
Cf. also JORDAN, *Cours d'Analyse*, t. II., pp. 315–317.

This product consequently denotes a function of unique character possessing all the essential properties of an ordinary theta-function.

The special case given by the formula *

$$u = x \,\Pi\, \Pi \left(1 - \frac{x}{w}\right) e^{\frac{x}{w} + \frac{1}{2}\left(\frac{x}{w}\right)^2},$$

in which

$$w = 2\mu\omega + 2\mu'\omega',$$

has been called by Weierstrass the sigma-function $\sigma(x)$, and is the basis of his beautiful theory of elliptic functions.

EARLY HISTORY OF THE POTENTIAL.

BY PROF. A. S. HATHAWAY.

THE object of the present article is to correct an error that occurs in Todhunter's "History of the Theories of Attraction" (vol. II., arts. 789, 1007, and 1138), and that is repeated, doubtless on Todhunter's authority, in various encyclopædias. This error consists in assigning to Laplace, instead of Lagrange, the honor of the introduction of the Potential into dynamics, an honor that the Encyclopædia Britannica makes the basis of a eulogy to Laplace (art. *Laplace*) in the words: "The researches of Laplace and Legendre on the subject of attractions derive additional interest and importance from having introduced two powerful engines of analysis for the treatment of physical problems, Laplace's Coefficients and the Potential Function. The expressions for the attraction of an ellipsoid involved integrations which presented insuperable difficulties; it was, therefore, with pardonable exultation that Laplace announced his discovery that the attracting force in any direction could be obtained by the direct process of differentiating a single function. He thereby translated the forces of nature into the language of analysis and laid the foundations of the mathematical sciences of heat, electricity, and magnetism."

The announcement here referred to was made by Laplace

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* BIERMANN, *Theorie der analytischen Functionen*, Leipzig, 1887, p. 834.

SCHWARZ. *Formeln und Lehrsätze zum Gebrauche der elliptischen Functionen*, Göttingen, 1885.

in the course of a memoir by Legendre between 1783 and 1785 : Encyclopædia Britannica (art. *Laplace*),—"* * * Legendre in a celebrated paper entitled *Recherches sur l'attraction des sphéroïdes homogènes*, printed in the tenth volume of the *Divers Savans*, 1783, * * * *"; Todhunter, Hist. Th. Attr., vol. II., p. 20, "A very important memoir by Legendre is contained in the tenth volume of the *Mémoires* * * * *présentés par divers Savans* * * * . The date of publication of the volume is 1785. The memoir, however, must have been communicated to the Academy at an earlier period ; for, in the treatise *De la Figure des Planètes*, which was published in 1784, Laplace refers to the researches of Legendre, which constitute the present memoir: see p. 96 of Laplace's treatise."

Todhunter continues, in art. 789: "In this memoir we meet for the first time the function V which we now call the *Potential*, and which denotes the sum of the elements of a body divided by their distances from a fixed point. The introduction of this function Legendre expressly assigns to Laplace. The following are the circumstances :

A point is situated outside a solid of revolution. Legendre has to determine the attractions of the solid at the point, along the radius vector which joins the point to the centre of the solid, and at right angles to this direction. He has found a series for the former; and he says the latter might be determined by similar investigations ; then he adds : '* * * * *mais on y parvient bien plus facilement à l'aide d'un Théorême que M. de la Place a bien voulu me communiquer: voici en quoi il consiste.*'

Then follows the theorem, which is enunciated and immediately demonstrated. The theorem is that the attraction along the radius vector is $-\frac{dV}{dr}$, and the attraction at right angles to the radius vector is $-\frac{dV}{rd\theta}$; where r is the radius vector and θ the angle which it makes with the axis of the solid : these attractions being estimated towards the centre, and the pole respectively."

As V is the notation used by Laplace in this announcement, it is plain, I think, where he found this method of differentiation to get the forces ; for that is the notation used by Lagrange in the last of several memoirs previous to 1783 in which he made use of the Potential. On this account it may be well to notice this last memoir first : *Theorie de la Libration de la Lune. Mémoires de l'Académie royale des Sciences et Belles-Lettres de Berlin, Année* 1780. *Œuvres*, t. V., p. 5.

"The memoir is divided into five sections. The first is designed for the exposition of a general analytical method for resolving all the problems of dynamics. This method, which I employed in my first memoir on the libration of the moon, has the singular advantage of requiring no construction and no geometrical or dynamical reasoning, but only analytical operations subjected to a process that is simple and uniform. * * * *

Let $m, m', m'', \ldots$ be the masses of the bodies, $P, Q, R, \ldots$ the accelerative forces that attract the body m towards centres whose distances are $p, q, r, \ldots, P', Q', R', \ldots$ the accelerative forces that attract the body m' towards centres whose distances are $p', q', r', \ldots$. * * * *

Taking into consideration the mutual disposition of the bodies, one will have several equations of condition among the variables x, y, z, etc. All these are expressed in terms of some one or more variables $\varphi, \psi, \ldots$ that are independent. By substitution and differentiation, one will have the general equation $\Phi d\varphi + \Psi d\psi + \ldots = 0$: thus $\Phi = 0$, $\Psi = 0, \ldots$, give as many equations as there are undetermined variables, by means of which these variables are determined. We shall show how to abridge the calculations necessary to reduce * * * * to functions of $\varphi, \psi, \ldots$ * * * * In regard to the terms $P\delta p + Q\delta q + R\delta r + \ldots$ and similar terms, we note that in the case of nature the forces $P, Q, R, \ldots$ are ordinarily functions of the distances $p, q, r, \ldots$, so that the terms of which they consist are all integrable. This also furnishes a means of simplifying very much the calculation of these terms; for it is only necessary, in the first place, to integrate the quantity $P\delta p + Q\delta q + R\delta r + \ldots$ in the ordinary way, and then differentiate it according to the characteristic δ. * * * *

Put for abridgment

$$T = \frac{1}{2}\left\{ m\,\frac{dx^2 + dy^2 + dz^2}{dt^2} + m'\frac{dx'^2 + dy'^2 + dz'^2}{dt^2} + \ldots \right\};$$

$$V = m\int(P\delta p + Q\delta q + R\delta r + \ldots) + m'\int(P'\delta p' + Q'\delta q' + R'\delta r' + \ldots) + \ldots,$$

and suppose $x, y, z, x', y', z', \ldots$ expressed in terms of other variables $\varphi, \psi, \ldots$; then substituting these values in T and V and differentiating according to the characteristic δ, regarding $\varphi, \psi, \ldots, d\varphi, d\psi, \ldots$, as the corresponding variables (d referring to the time) the above equation becomes

$$\left(d\frac{\delta T}{\delta d\varphi}-\frac{\delta T}{\delta\varphi}+\frac{\delta V}{\delta\varphi}\right)\delta\varphi+\left(d\frac{\delta T}{\delta d\psi}-\frac{\delta T}{\delta\psi}+\frac{\delta V}{\delta\psi}\right)\delta\psi+\ldots=0,$$

wherein $\frac{\delta T}{\delta\varphi}$ denotes the coefficient of $\delta\varphi$ in the differential of T, and $\frac{\delta T}{\delta d\varphi}$ the coefficient of $\delta d\varphi$ in the same differential, and so for the rest."

This investigation appears also in the *Méchanique Analytique;* but, as we shall see by another example, Todhunter did not recognize that the *Méchanique Analytique,* like the *Méchanique Céleste* of Laplace, was largely a compilation from preceding memoirs. Theories of Attraction, vol. II., p. 153, art. 994: "The first edition of a famous work by Lagrange, appeared in 1788 in one volume, entitled *Méchanique Analytique.* There is nothing in this edition which bears explicitly on our subject. But on his page 474 Lagrange gives, in fact, an integral in the form of a series of the partial differential equation

$$\frac{d^2V}{da^2}+\frac{d^2V}{db^2}+\frac{d^2V}{dc^2}=0;$$

and from this integral, as we shall see hereafter, Biot drew important inferences with respect to the attraction of a body." The solution here referred to was given by Lagrange in 1781: *Œuvres,* t. IV., p. 695, *Théorie du Mouvement des Fluides.*

The idea of differentiating in order to obtain the forces first appeared in Lagrange's memoir of 1763: *Œuvres,* t. VI., p. 5, *Recherches sur la Libration de la Lune, Prix de l'Académie Royale des Sciences de Paris,* t. IX., 1764. The kinetic energy is differentiated to obtain the accelerations, forming the first part of Lagrange's celebrated generalized equations of motion given first in complete form in 1780. The potential is used to obtain the forces for the first time by Lagrange in the memoir *Sur l'Equation Séculaire de la Lune; L'Académie Royale des Sciences de Paris,* t. VII., 1773; *Prix pour l'année* 1774; *Œuvres,* t. VI., p. 335.

"If a point A attract another point B with any force whatever F, and if $\varDelta$ be the distance between the two bodies and $d\varDelta$ the increment of this distance in supposing that A attracts B an infinitely small space $d\alpha$, then $-F\frac{d\varDelta}{d\alpha}$ is that part of the force F which acts in the direction $d\alpha$; and if one proposes to decompose this force in three mutually perpendicular directions $d\alpha$, $d\beta$, $d\gamma$, $-F\frac{d\Delta}{d\beta}$, $-F\frac{d\Delta}{d\gamma}$ are the remaining components. If F is proportional to $\frac{1}{\Delta^2}$, which is the case of celestial attraction, then

$$Fd\Delta = \frac{d\Delta}{\Delta^2} = -d\frac{1}{\Delta},$$

and consequently, the three forces are represented by the coefficients of $d\alpha$, $d\beta$, $d\gamma$, in the differential of $\frac{1}{\Delta}$. In short, it suffices to find the value of $\frac{1}{\Delta}$ and differentiate it by ordinary methods.

If the point B is attracted at the same time towards different points A, A', A'', . . , whose distances from B are Δ, Δ', Δ'', . . . , and if the attractions are $\frac{M}{\Delta^2}$, $\frac{M'}{\Delta'^2}$, $\frac{M''}{\Delta''^2}$, . . . , it is plain that one has only to seek the value of the quantity

$$\frac{M}{\Delta} + \frac{M'}{\Delta'} + \frac{M''}{\Delta''} + \ldots$$

and to differentiate it as a function of α, β, γ, when the coefficients of $d\alpha$, $d\beta$, $d\gamma$, in this differential immediately give the forces sought.

In general, if the point B is attracted by a body of any figure whatever, whose mass is M, then, considering each element, dM, of the body as an attracting point, it is only necessary to find the sum of all the quantities $\frac{dM}{\Delta}$, found by making the quantities that relate to the position of dM vary and regarding α, β, γ as constant; then, denoting this sum by Σ, and making it vary as to the quantities α, β, γ, that relate to the position of B, one has $\frac{d\Sigma}{d\alpha}$, $\frac{d\Sigma}{d\beta}$, $\frac{d\Sigma}{d\gamma}$ for the three forces in the directions $d\alpha$, $d\beta$, $d\gamma$, to which the total attractive force of the body M on B reduces." Lagrange then goes on to apply this method to his discussion of the moon.

In October, 1777, Lagrange read a paper that is devoted wholly to the potential and its applications to the dynamics of a system of bodies : *Remarques générales sur le Mouvement de plusieurs corps qui s'attirent mutuellement en raison inverse des carrés des distances. L'Académie royale des Sciences et Belles-Lettres de Berlin, année* 1777. *Œuvres*, t. IV., p. 402.

"Let M, M', M'', . . . be the masses of bodies which compose a given system, x, y, z the rectangular coordinates of the body M in space, x', y', z' those of the body M', and so on.

Put

$$\Omega = \frac{MM'}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}}$$

$$+ \frac{MM''}{\sqrt{(x-x'')^2 + (y-y'')^2 + (z-z'')^2}}$$

$$+ \frac{M'M''}{\sqrt{(x'-x'')^2 + (y'-y'')^2 + (z'-z'')^2}} + \ldots,$$

and let $\frac{d\Omega}{dx}, \ldots, \frac{d\Omega}{dx'}, \ldots$ denote, as usual, the coefficients of $dx, \ldots, dx', \ldots$ in the differential of Ω, regarded as a function of $x, \ldots, x', \ldots$.

One has $\frac{1}{M}\frac{d\Omega}{dx}, \frac{1}{M}\frac{d\Omega}{dy}, \frac{1}{M}\frac{d\Omega}{dz}$, for the forces with which the body M is attracted by the other bodies M', M'', in the directions of the coordinates x, y, z, and so on. It is easy to be convinced of this by performing the indicated differentiation: for that will give the same expressions as the decomposition of the forces that act upon each body in virtue of the attraction of each of the other bodies, supposed proportional to the mass divided by the square of the distance. This manner of representing the forces is, as one sees, extremely convenient, both for its simplicity and for its generality; and it has the further advantage that one distinguishes by it, clearly, the terms due to the different attractions of the bodies, for each of the attractions gives in the quantity Ω a term consisting of the product of the masses of the two bodies divided by their distance apart."

Lagrange goes on to give the equations of motion

$$M\frac{d^2x}{dt^2} = \frac{d\Omega}{dx}, \quad M\frac{d^2y}{dt^2} = \frac{d\Omega}{dy}, \quad M\frac{d^2z}{dt^2} = \frac{d\Omega}{dz}, \ldots,$$

multiplies them by $dx, dy, dz, \ldots$, adds and integrates, finding the equation of conservation of energy,

$$\Omega = \tfrac{1}{2} M \left\{\frac{dx^2 + dy^2 + dz^2}{dt^2}\right\}$$

$$+ \tfrac{1}{2} M' \left\{\frac{dx'^2 + dy'^2 + dz'^2}{dt^2}\right\} + \ldots + \text{constant}.$$

Since Ω does not change when the x-coordinates change by equal increments, he finds

$$\frac{d\Omega}{dx} + \frac{d\Omega}{dx'} + \frac{d\Omega}{dx''} + \ldots = 0$$

with similar equations in the y- and z-coordinates.

Substituting

$$\frac{d\Omega}{dx} = M\frac{d^2x}{dt^2}, \quad \frac{d\Omega}{dy} = M\frac{d^2y}{dt^2}, \quad \frac{d\Omega}{dz} = M\frac{d^2z}{dt^2}, \ldots,$$

and using

$$X = \frac{Mx + M'x' + \ldots}{M + M' + \ldots}, \quad Y = \frac{My + M'y' + \ldots}{M + M' + \ldots}, \ldots,$$

he has

$$\frac{d^2X}{dt^2} = 0, \quad \frac{d^2Y}{dt^2} = 0, \quad \frac{d^2Z}{dt^2} = 0.$$

In words, the centre of gravity moves uniformly in a straight line. The equations of motion are then shown to be unchanged when the centre of gravity is taken as the origin.

Since Ω does not change to the first order in α when

$$\frac{dy}{z} = \frac{dy'}{z'} = \ldots = -\frac{dz}{y} = -\frac{dz'}{y'} = \ldots . = \alpha,$$

Lagrange concludes that

$$\left(y\frac{d\Omega}{dz} - z\frac{d\Omega}{dy}\right) + \left(y'\frac{d\Omega}{dz'} - z'\frac{d\Omega}{dy'}\right) + \ldots = 0$$

with similar results for the axes of y, z.

Substituting the accelerations for the forces according to the equations of motion, and integrating, he finds the equation of conservation of areas

$$M\frac{ydz - zdy}{dt} + M'\frac{y'dz' - z'dy'}{dt} + \ldots = \text{constant};$$

and so on.

The article closes as follows:

"These theorems upon the movement of the centre of gravity have already been given in part by D'Alembert; but the manner in which I have demonstrated them is new, and, it appears to me, merits the attention of geometers by the utility with which it can be used. One perceives by the same principles that these theorems will be equally true if the bodies act upon each other by forces mutually proportional to any function whatever of the distance; for, calling $f(x)$ the force of attraction at the distance x, and putting

$$F(x) = \int f(x)\,dx,$$

one has only to change the value of Ω above into

$$\Omega = -MM'F(\sqrt{(x-x')^2+(y-y')^2+(z-z')^2})$$
$$-MM''F(\sqrt{(x-x'')^2+(y-y'')^2+(z-z'')^2}) - \ldots,$$

to easily obtain the same results."

The next memoir in which Lagrange uses the potential is that of 1780, already referred to, in which he completes his generalized equations of motion, and uses the notation V for the potential, which Laplace adopts.

Todhunter was not without warning of these facts, for he says, vol. II., p. 221: "I must cite another sentence from Biot's memoir; he says on page 208, after introducing the function V,

M. Lagrange a démontré que les coefficients différentiels

$$\frac{dV}{da}, \frac{dV}{db}, \frac{dV}{dc},$$

pris négativement expriment les attractions exercées par le sphéroïde sur ce même point, parallèlement aux trois axes rectangulaires. M. Laplace a fait voir ensuite que la fonction V est assujetie à l'équation différentielle partielle

$$\frac{d^2V}{da^2} + \frac{d^2V}{db^2} + \frac{d^2V}{dc^2} = 0.$$

I do not know on what authority the above expressions for component attractions are assigned to Lagrange; to me they appear due to Laplace: see art. 789, and also pages 70 and 133 of Laplace's *Figure des Planètes.*"

Todhunter attempts a defense, vol. II., p. 160: "Lagrange now proceeds to consider the attraction of the ellipsoid on an external particle. He introduces what we call the potential function, and denotes it by V. If f, g, h, denote the coordinates of the attracted particle, the attractions in the corresponding directions are $\frac{dV}{df}, \frac{dV}{dg}, \frac{dV}{dh}$. Lagrange does not claim these expressions for himself; and we know that they are really due to Laplace: see art. 789."

The argument is equally good if it be made to refer to Laplace's first announcement, given in art. 789, with "Laplace" and "Lagrange" interchanged. Moreover, Lagrange had claimed these expressions in the Berlin memoir of 1777, twenty years previous to the memoir Todhunter is describing.

Todhunter knew that Laplace constantly embodied the work of others in his own without credit (see preface vol. I.); and he cites a breach of etiquette towards Legendre in these very memoirs in the matter of Legendre's Coefficients, vol. II., p. 43:

"We will first reproduce a note bearing on the history of the subject which occurs at the beginning of the memoir. * * * * Legendre says:

'*La proposition qui fait l'objet de ce mémoire, étant démontrée d'une manière beaucoup plus savante et plus générale dans un mémoire que M. de la Place a déjà publié dans le volume de* 1782, *je dois faire observer que la date de mon mémoire est antérieure, et que la proposition qui paroît ici, telle qu'elle a été lûe en juin et juillet* 1784, *a donné lieu à M. de la Place, d'approfondir cette matière, et d'en présenter aux Géomètres, une théorie complète.*'"

This refers to Laplace's memoir *Figure des Planètes*, contained in the Paris *Mémoires* for 1782, published in 1785, and is the one of which Todhunter says (p. 56) "in this article we have for the first time the partial differential equation with respect to the coordinates of the attracted particle which the potential V must satisfy: it is expressed by means of polar coordinates," etc.

Nor did Todhunter neglect foreign memoirs alone, bearing on his subject; for if he had read the valuable Report on Dynamics by Cayley, *Brit. Ass. Rep.*, 1862, p. 184, he would have found the potential function properly credited to Lagrange, with a reference to the memoir, *Sur l'Équation Séculaire de la Lune*, of 1773.

In conclusion I ought to say that a sentence in Sir William Thomson's Baltimore lectures (1884), led me to investigate this subject, Lectures, p. 112: "I took the liberty of asking Professor Ball two days ago whether he had a name for this symbol ∇^2; and he has mentioned to me *nabla*, a humorous suggestion of Maxwell's. It is the name of an Egyptian harp which was of that shape. I do not know that it is a bad name for it. *Laplacian* I do not like for several reasons both historical and phonetical."

ROSE POLYTECHNIC INSTITUTE,
Terre Haute; 1891, *October* 22.

THE THEORY OF LIGHT.

The Theory of Light. By THOMAS PRESTON, M.A., Lecturer in Mathematics and Mathematical Physics, University College, Dublin. London and New York, Macmillan & Co., 1890. 8vo.

UNTIL within a very few years it has been a matter of considerable difficulty for American students interested in higher theoretical optics to pursue this study with advantage, for want of access to the original memoirs and the absence of any adequate presentation of their contents in any of the American text-books. For the students of the English universities, Airy's Undulatory Theory of Optics and Loyd's Wave Theory of Light have been the chief English helps until the publishing of Glazebrook's admirable Physical Optics. It has been a great pity that the clear and beautiful presentation of the subject given by President Barnard in 1862, and printed in the *Smithsonian Report* for that year, was not long since published in separate book form, as it would be to-day one of the very best books on the subject were it printed in a form accessible to college students having a fair command of elementary mathematics. I have been greatly surprised to find it so little known even among American students who have made a special study of the higher optics in European universities. Besides the English books referred to above, the admirable report of Professor Stokes on Double Refraction in the *British Association Report* for 1862, and the equally admirable later one on Optical Theories by Glazebrook in 1865, together with such books as Beer's Introduction to Higher Optics, Knochenhauer's Undulatory Theory of Light, Lord Rayleigh's articles in the Encyclopædia Britannica and the *Philosophical Magazine,* Verdet's *Leçons d'Optique Physique,* Poincaré, Briot, Sir William Thomson, and Professor Tait have heretofore supplied the special advanced student with most of his needs. It has been very desirable, however, that a treatise should be written for English-speaking students, similar to those which Verdet and Poincaré have written for the French, dealing both with profound theory and experimental facts. This seems to have been done by Professor Preston of Dublin.

The work in some respects resembles Verdet's *Optique Physique.* It begins, however, from a more elementary basis, and includes the more recent work of Lord Rayleigh and Professors Rowland and Hertz. In this latter respect as well as in some others, it is better than Glazebrook's Physical Optics. It lacks the splendid bibliography of Verdet, and is on the whole much more elementary.

It begins with the simplest principles of wave motion, treated both with and without mathematics. The difference between wave and group velocity is explained at the end of chap. II. after the manner of Lord Rayleigh's Sound. Huygens's principal and secondary waves are well explained, and Stokes's law of the intensity of the light at any point of a secondary wave is given together with a reference to his celebrated paper on the Dynamical Theory of Diffraction, 1849. At the end of chap. IV., explaining reflection upon the wave theory, we have this admirable remark: "In dealing with problems in the reflection of light we may therefore consider the light propagated in rays *if it facilitates the solution. Yet we must carefully bear in mind that rays have no physical existence, for it is waves that are propagated and not rays.*" (The italics are ours.) In chap. V., in discussing the energy equation, the density of the ether is referred to as "that property of it which *corresponds* to the density of ordinary matter, and *by which it possesses energy when in motion.*" We have here a hint of ether inertia that may be due to other causes than mere mass, e. g. rotatory inertia. The chapter on determination of refractive indices, gives a very full account of the usual methods, with references to the original memoirs. The section on gases is particularly good.

Chap. IX. is on diffraction, and it is at this point that the really mathematical part of the book may be said to begin. The treatment is clear and thorough, and nothing is slurred over that can possibly help a student to a competent understanding of the subject, although the treatment sometimes seems a little too concise. So vast is the ground covered, however, that this seems almost a necessity.

This is the only book so far as we know which gives a pretty full discussion of the theory and use of Rowland's concave gratings and an account of the new determination of wave lengths by Rowland and Bell.

The entire treatment of diffraction is very full and satisfactory, although the remarkable results of Lord Rayleigh and Professor Michelson on the distribution of the light from sources other than points are not given. Cornu's graphical methods of dealing with diffraction problems however are quite fully given. The recent applications of these researches of Rayleigh and Michelson to spectroscopy could hardly have had a place in the book, which has been out of press for more than a year.

An admirable but brief presentation of the views of MacCullagh and Neumann on the relation of the plane of vibration to the plane of polarization in polarized light, is given, together with the different suppositions involved as to the changes of density and rigidity of the ether in crystals on which diverse views on this point are founded. One might wish that some of these more purely physical or dynamical questions had been discussed at greater length. Reference is however made to the admirable report of Glazebrook on optical theories in the *British Association Report* for 1885. The papers of Sir William Thomson, Willard Gibbs, Ketteler, and Glazebrook, in the *Philosophical Magazine, American Journal of Science,* and *Wiedemann's Annalen,* since that time, however, are particularly important, especially the extraordinary speculation of Sir William Thomson on a "labile" ether and Willard Gibbs' able discussion of the same. The notion of a medium capable of transmitting transverse vibrations in virtue of a quasi-rigidity imparted to it by motion, especially rotatory motion, is evidently becoming more and more important and Sir William Thomson, although apparently still an adherent of the elastic solid theory, has himself shown how the rotation of the plane of polarization in a magnetic field can be explained by the assumption of such an ether. In the last chapter (XXI.) the author of this treatise has given a sketch, clear and as simple as the nature of the subject will allow, of the present state of the electro-magnetic theory of light and the experimental researches of Dr. Hertz on electro-magnetic waves. Oliver Heaviside has done so much work in this direction that it seems as if some mention might have been made of it. So also Professor Rowland.

The whole treatment of the subject of polarized light is full and satisfactory, while, on the whole, also very concise.

There are a large number of examples added to each chapter. These are well selected and the sources indicated from whence they are derived. The advanced student has thus pointed out to him what authorities on each part of the subject may be best for him to consult. One of the very best features of the book, in our opinion, is the impression it leaves on one's mind of its being, above all, an able, clear, and accurate presentation of the subject as it was left by Fresnel —the Newton of the undulatory theory of light. As Fresnel left it, so, except that Maxwell and Lorenz have shown that the vibrations are probably electro-magnetic in character, it *essentially* is to-day. Questions as to the ultimate structure and constitution of the ether are related to the undulatory theory of light, just as questions as to the *mechanism* of gravitation are to the theory of gravitation, as ordinarily treated. The ordinary mathematical theory and its confirmation by observation and experiment, rest intact, no matter what may prove to be the physical *mechanism* by which their results are brought about. The famous Baltimore lectures of Sir William Thomson, the British Association reports

(1862 and 1885) of Stokes and Glazebrook, and the later papers alluded to before, are the natural sources of information for those who wish to go into these matters.

For advanced students in colleges and all who wish to acquire a thorough knowledge of the existing state of the undulatory theory of light, we recommend this admirable treatise. The type and illustrations are also models of clearness and elegance and reflect credit upon the publishers as well as the author. JOHN E. DAVIES.

UNIVERSITY OF WISCONSIN,
Madison, *October* 12, 1891.

NOTES.

A REGULAR meeting of the NEW YORK MATHEMATICAL SOCIETY was held Saturday afternoon, November 7, at half-past three o'clock, the vice-president in the chair. The following persons having been duly nominated, and being recommended by the Council, were elected to membership: Professor Simon Newcomb, Navy Department and Johns Hopkins University; Dr. Oskar Bolza, Clark University; Mr. Charles Riborg Mann, Columbia College; Professor Ludovic Estes, University of North Dakota; Mr. Herbert Armistead Sayre, Montgomery, Alabama; Professor James Harrington Boyd, Macalester College; Dr. Asaph Hall, Jr., U. S. Naval Observatory; Dr. Percy F. Smith, Yale University; Mr. Edwin H. Lockwood, Yale University; Professor Robert Judson Aley, Indiana University; Professor Joseph V. Collins, Miami University; Dr. Charles H. Chapman, Johns Hopkins University; Professor Albert Munroe Sawin, University of Wyoming; Mr. Frank Gilman, Lowell, Massachusetts; Professor Henry Parker Manning, Brown University; Mr. Charles S. Peirce, Milford, Pennsylvania.

Mr. Charles P. Steinmetz read an original paper entitled "On the curves which are self-reciprocal in a linear nul-system, and their configurations in space."

Dr. Edward L. Stabler made some remarks upon the theory of errors which are equally probable between given limits.

THE *nul-system* in space, which formed the subject of Mr. Steinmetz's paper, is a one-to-one correspondence between points and planes such that any point lies in its conjugate plane, and conversely. A linear nul-system is one in which all the planes conjugate to the points of any straight line intersect one another in a second straight line, so that there exists a one-to-one correspondence between the lines in space.

T. S. F.

THE 64th meeting of the *Gesellschaft deutscher Naturforscher und Ärzte* (a German association corresponding to the American Association for the Advancement of Science) was held this year at Halle a. S., September 21 to 25. If the list of papers announced in advance as to be read in the different sections can be taken as an indication of what was actually done, it appears that the section for mathematics and astronomy is by far the strongest of all sections, not only numerically—25 papers, the next in order being the section of physics with 13 papers, then the section for instruments of precision (*Instrumentenkunde*) with 8 papers, etc. —but in particular considering the weight of the names represented. It is worthy of notice that astronomy has hardly any share in this programme, the subjects belonging almost exclusively to pure higher mathematics. The association has a special section (only recently organized) for elementary mathematics and natural sciences and the allied educational questions. Professor Georg Cantor, of the University of Halle, was president of the section of mathematics and astronomy; Dr. H. Wiener, of the same university, was secretary.

The following is a list of the papers announced to be read in this section:

L. Kronecker of Berlin, Opening address; K. Neumann of Leipzig, On a question in electrodynamics; L. Koenigsberger of Heidelberg, On the theory of systems of partial differential equations; F. Klein of Göttingen, Account of recent English investigations in mechanics; F. Meyer of Clausthal, Review of the present state of the theory of invariants; M. Noether of Erlangen, The fundamental proposition on the intersection of three surfaces; Rohn of Dresden, On rational twisted quartics; E. Papperitz of Dresden, The general system of the mathematical sciences; Worpitzky of Berlin, On the axioms of geometry; H. Wiener of Halle, On the foundations and the system of geometry; F. Kraft of Zürich, The meaning and value of Grassmann's *Ausdehnungslehre* for the whole domain of mathematics and mechanics; V. Eberhard of Königsberg, Elements of a systematic exposition of the forms of polyhedra; F. Müller of Berlin, On literary enterprises adapted to facilitate the study of mathematics; A. Pringsheim of Munich (subject not announced); Finsterwalder of Munich, The images in dioptric systems of larger aperture and larger field of vision; W. Dyck of Munich (subject not announced); H. Schubert of

Hamburg, On a question in enumerative (*abzählende*) geometry; M. Simon of Strasburg, On a question in absolute geometry; C. Reuschle of Stuttgart, A fundamental system of identities of the algebraic functions; R. Mehmke of Darmstadt, Description of mechanisms for the mechanical solution of equations; Hilbert of Königsberg, On complete (*volle*) systems of invariants; Stäckel, Wangerin, G. Cantor, of Halle (subjects not announced).

A more detailed account of the meeting may be given later.

On October 23 the Mathematical Society of the University of Michigan held its first meeting this fall. Professor F. C. Wagner read a paper on the mathematical principles of thermodynamics. The society was founded in November, 1890, and has held seven meetings in the course of the last academic year. Professor W. W. Beman is president; Dr. F. N. Cole is secretary. A. Z.

At the meeting of the National Academy of Sciences held at Columbia College, November 10 to 12, the following papers of a mathematical nature were read: Certain new methods and results in optics, by Professor Charles S. Hastings; New pendulum apparatus, by Professor T. C. Mendenhall; Astronomical methods of determining the curvature of space, by Professor C. S. Peirce; Variation of latitude, by Professor S. C. Chandler; Color system, by Professor O. N. Rood; Reduction of Rutherfurd's photographs, by Professor J. K. Rees; Measurement of Jupiter's satellites by interference, by Professor A. A. Michelson.

Professor Hastings's paper contained some new and very simple demonstrations of optical formulæ already known, as well as certain important formulæ altogether new, including a general expression for magnifying power applicable to *both* telescopes and microscopes. Professor Peirce presented astronomical evidence tending to show that space possesses a negative curvature, and called attention to various methods of conducting an investigation of this property of space. Professor Chandler exhibited curves showing that the recently discovered variation of latitude could be made to explain certain hitherto unaccountable discordances in older observations. His paper was followed by considerable discussion among the astronomers present; Professors Young, E. C. Pickering, C. S. Peirce, Abbe, and Dr. Gould taking part. The chief question debated was whether the variation has a *terrestrial* or *celestial* origin. The investigations are being published in the *Astronomical Journal*. Professor Michelson described his recent measurements of Jupiter's satellites at the Lick Observatory, and thought that we may hope to measure the angular diameters of some of the brighter stars, if they be as great as the hundredth part of a second of arc. His paper was perhaps the most important one of the session. In it was presented a new method of measuring the angular diameters of luminous discs by means of the interference phenomena produced by them. The experiments made at the Lick Observatory have been described in the *Publications of the Astronomical Society of the Pacific*. The 12-inch telescope was used, but a telescope is by no means indispensable for these observations, the chief requisite being a very favorable condition of atmosphere. It is to be hoped that these very promising researches will be continued. H. J.

THE series of lectures given last winter at Johns Hopkins University to teachers and those intending to become teachers was so successful that a similar series is to be given this winter. Among the lectures promised we note one on the teaching of mathematics by Professor Simon Newcomb.

La Nature announces the death of Édouard Lucas, Professor of Mathematics at the Lycée Saint-Louis. His death was due to injuries received from a mishap at Marseilles during the meeting of the French Association for the Advancement of Science, at which he presided over the section of mathematics. He was the author of many papers, but was most widely known through his *Récréations Mathématiques*. The second volume of his more recent work *Théorie des Nombres* is still in press.

IN an article entitled "Twelve *versus* Ten," which appears in the November number of the *Educational Review*, Professor W.B. Smith strongly advocates duodenary numeration.

JOHN WILEY & SONS have in preparation a new work on "The Theory of Errors and Method of Least Squares," by Professor W. Woolsey Johnson. T. S. F.

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ON LISTS OF COVARIANTS.

BY DR. EMORY MCCLINTOCK.

I USE the term covariants to include invariants, and I write particularly concerning lists of covariants (groundforms) of the binary quintic and sextic, those of quantics of lower degree being few and well known. When the weight of a covariant is spoken of in this article, it must be understood to mean the weight of its first term or "source." The symbol 5_n will denote that covariant of the quintic whose weight is n, and 6_m that covariant of the sextic whose weight is m. Thus, for example, 5_2 and 6_2 represent the hessians (weight 2) of the quintic and sextic respectively. The only case of ambiguity is 6_{15}, for which weight there are two covariants: one of these may be denoted by 6_{15a}, the other by 6_{15b}.

The table printed on the next page exhibits the terminology of different writers. Professor Cayley's* superb collection of the covariants of the quintic, in which each is designated by a letter of the alphabet, is arranged, as will be observed, first according to the degree in the coefficients, and secondly according to the order in the variables. Thus 5_0, of the first degree, is called A, and 5_2 and 5_4, of the second degree, come next; but 5_4 being of order 2 while 5_2 is of order 6, the letter B is assigned to 5_4, and so on. The small italics contained in the column headed by the name of Dr. Salmon † are the symbols used in a table at the end of his work, illustrative of transvection, and denote the seminvariants which form the sources of covariants of the quintic and of higher quantics as well. The letters a, g, h, i, j, k, are therefore used by him also for the sextic, together with l, m, n, q, representing respectively 6_8, 6_{14}, 6_{20}, 6_{10}. Clebsch ‡ and Gordan § differ but slightly in their nomenclature. Faà de Bruno ‖ designates invariants by the letter I, with subscripts indicating degree, and other covariants by the letter C, with subscripts indicating order and degree. The column headed by the name of Professor Sylvester contains his table ¶ of *germs* for the quintic, each source having its distinguishing germ, *i. e.*, the coefficient in it of the highest power of the final coefficient of the quintic. Thus, the quintic being

* *Mathematical Papers*, II, 273–309; Cambridge, 1889.

† *Modern Higher Algebra*, 4th Edition.

‡ *Theorie der Binären Algebraischen Formen*, Leipzig, 1872.

§ *Invariantentheorie*, herausgegeben von Kerschensteiner, Leipzig, 1887.

‖ *Théorie des Formes Binaires*, Turin, 1876.

¶ *American Journal of Mathematics*, V, 89.

COVARIANTS OF THE QUINTIC.

By Weight.	Deg-Order.	Cayley.	Salmon.	Clebsch.	Gordan.	Faà de Bruno.	Sylvester.
5_1	1–5	A	U, a	f	f	$C_{5,1}$	a
5_2	2–6	C	H, h	H	φ	$C_{6,2}$	(c)
5_3	3–9	F	g	T	t	$C_{9,3}$	(d)
5_4	2–2	B	S; i	i	i	$C_{2,2}$	(e)
5_5	3–5	E	k			$C_{5,3}$	a^2f
5_6	3–3	D	T, j	j	j	$C_{3,3}$	(e')
5_7	4–6	I				$C_{6,4}$	$a(c)f$
5_8	4–4	H	e			$C_{4,4}$	$(d)f$
5_9	5–7	L				$C_{7,5}$	$(c)^2f$
5_{10}	4–0	G	J	A	A	I_4	a^2f^2
5_{11}	5–3	K				$C_{3,5}$	$3a(e')f-2(c)(e)f$
5_{12}	5–1	J	α	α	α	$C_{1,5}$	$a(c)f^2$
5_{13}	6–4	N				$C_{4,6}$	$a(d)f^2$
5_{14}	6–2	M	τ	τ	τ	$C_{2,6}$	$(c)^2f^2$
5_{15}	7–5	P				$C_{5,7}$	$(c)(d)f^2$
5_{17}	7–1	O		β	β	$C_{1,7}$	$a^2(c)f^3$
5_{19}	8–2	R	ϑ	ϑ	ϑ	$C_{2,8}$	$a(c)^2f^3$
5_{20}	8–0	Q	K	B	B	I_8	$(c)(d)f^3$
5_{21}	9–3	S				$C_{3,9}$	$(c)^3f^3$
5_{27}	11–1	T		γ	γ	$C_{1,11}$	$(c)^2(d)f^4$
5_{30}	12–0	U	L	C	C	I_{12}	$(c)^2\Delta f^4$
5_{32}	13–1	V	·	δ	δ	$C_{1,13}$	$(c)^3(e')f^4$
5_{45}	18–0	W	R, I	R	R	I_{18}	$a(c)^5f^7$

$$ax^5 + 5bx^4y + 10cx^3y^2 + 10dx^2y^3 + 5exy^4 + fy^5,$$

the germ of any covariant is the coefficient of the highest power of f appearing in its source. In the column in question,

$$\begin{aligned}
(c) &= ac - b^2, \\
(d) &= a^2d - 3abc + 2b^3, \\
(e) &= ae - 4bd + 3c^2, \\
(e') &= ace - ad^2 + 2bcd - c^3 - b^2e,^* \\
\Delta &= a^2d^2 + 4ac^3 + 4db^3 - 3b^2c^2 - 6abcd.
\end{aligned}$$

This germ-theory of Professor Sylvester will doubtless lead in future to important results. We may even now make some practical use of it as an aid in reducing covariants to their simplest forms.

The collection of covariants of the quintic lately made by Professor Cayley from his past publications is not likely to be superseded for many years. It appears in that great series of volumes, not yet complete, which will endure as the noblest monument of their illustrious author. It gives each covariant in the fullest detail, with all the terms arranged in the most complete order, and with the numerical coefficients verified, in every instance, as perfectly as that mode of verification can accomplish it, by calculations printed at the foot of the columns. The covariants as published are free from any inaccuracy which I have been able to discover,† with the single exception of the one (5_8) called I. In this the third and fifth columns should each be multiplied throughout by 5, and in the second column *abcf*—10 should read ac^2e—10. Yet, perfect as this collection is, it does not profess to give, and in fact does not always give, each covariant in its simplest form. An instance in point may be seen as the result of an examination of the germs. The germ of V as printed is the coefficient of f^5, namely, in Professor Sylvester's notation, $a(c)^2(d)$. If we suppose that note has been taken, as in our column headed "Sylvester," of the germs of the preceding covariants tabulated by Professor Cayley, we see that the germ of V is the product of the respective germs of J and Q. In fact, the addition of $2\,JQ$ to V as printed would simplify it materially, both by eliminating from the "source" all terms in f^5 and otherwise. The germ of V, thus modified, is the coefficient of f^4, $(c)^2(e')$, as in the "Sylvester" column. Yet it does not follow necessarily that the simplest ground-

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* Printed erroneously ad^3 in the paper cited. The germ of 5_{30} is also printed incorrectly.

† As regards the quintic. The last column of 4_3, No. 9 of the quartic, is incorrect.

form may not have a compound germ. The case of 6_{14} is an instance to the contrary.

The synoptical tables of Clebsch and Gordan do not give the covariants, but merely symbolic expressions indicating how the covariants may be computed. Let no one, however, undertake to compute covariants as directed by the symbolic analysis. The expressions resulting from the application of the Clebsch-Gordan formulæ are often highly complicated. For instance, their formula for the covariant of weight 15 gives the complicated function $86BK + 7EG - 252\phi$, and that for weight 21 gives $252\psi + 29GK - 59BO$, where ϕ means a form of 5_{15} which I think simpler than P as tabulated, and ψ means a form of 5_{21} in some respects simpler than S. Yet of course these complicated expressions are true covariants, of the right weights, degrees, and orders. I mention them merely to illustrate the necessity, for those engaged in computing and tabulating covariants, of a simple method.

I am unable to prove that the method which I prefer will in every case produce the simplest form of covariant, and it will not apply to all covariants, but I have not yet known it to fail when applied, and so I give it for what it may be worth. If we call by the name of "simple transvection" that form of transvection (*Ueberschiebung*) in which one of the two covariants concerned is the quantic itself, my plan is to produce any desired covariant, when possible, by simple transvection from the nearest available covariant of lower weight. Simple transvection increases the degree by 1 and the weight (in the case of the quintic) by from 1 to 5, and it cannot be performed when the desired increase of weight exceeds the order of the covariant operated upon. Observing these limitations, it is not difficult to pick out a succession of available operations, for instance for the quintic, by referring to the table of weights, degrees, and orders of possible independent covariants. Representing by $[n]$ the operation of simple transvection which is to increase the weight by n, we shall have, successively,

$[2]\ 5_0 = 5_2$,	$[3]\ 5_8 = 5_{11}$,
$[1]\ 5_2 = 5_3$,	$[4]\ 5_8 = 5_{12}$,
$[4]\ 5_0 = 5_4$,	$[1]\ 5_{12} = 5_{13}$,
$[1]\ 5_4 = 5_5$,	$[3]\ 5_{11} = 5_{14}$,
$[4]\ 5_2 = 5_6$,	$[1]\ 5_{14} = 5_{15}$,
$[1]\ 5_6 = 5_7$,	$[4]\ 5_{13} = 5_{17}$,
$[2]\ 5_6 = 5_8$,	$[4]\ 5_{15} = 5_{19}$,
$[1]\ 5_8 = 5_9$,	$[5]\ 5_{15} = 5_{20}$,
$[5]\ 5_5 = 5_{10}$,	$[2]\ 5_{19} = 5_{21}$.

Although, as I have said, I cannot prove that simple transvection, applied to the nearest, will always produce the simplest possible result, it seems not unreasonable that this should be the case, since the operand is presumably in the simplest form, and the operation is of the simplest character. The operation just indicated for producing 5_8 yields an expression simpler than H, that for 5_{18} an expression simpler than P, that for 5_{21}, an expression simpler than S* ; the others produce the corresponding covariants tabulated by Professor Cayley, which are therefore, in all probability, the simplest attainable forms.

Another principle appears to be even more important than that of simple transvection from the nearest. It is that if for any quantic a groundform is wanted for any degree-weight for which one exists for a lower quantic, the same "source" should be employed. This principle enables us to use for 6_2, 6_3, 6_4, 6_5, 6_6, 6_7, 6_{10}, and 6_{11}, the sources of corresponding degree-weight for the quintic. Yet for some reason unknown to me this principle appears uniformly to be disregarded in the formation of 6_{10}. Even the "germ-table for the sextic" of Professor Sylvester assigns for 6_{10} a less simple form than 5_{10}. That it is less simple may be seen from an examination of the numerical coefficients:

6_{10} by Faà de Bruno's table,† ±186, ±330, ±549, ±330, ±186
6_{10} from 5_{10},.............. ±142, ±168, ±263, ±168, ±142
In fact, Faà de Bruno's 6_{10} is really $2 \cdot 6_6 \cdot 6_4 - 3 \cdot 6_{10}$.

Tables for the sextic are needed, as complete, correct, and well printed as those of Professor Cayley for the quintic. If any member of the Society, undeterred by the great labor which the task will involve, will undertake to compute such a set of tables for the sextic, to be published, say, in the *American Journal of Mathematics*, I shall be glad to contribute towards it my own computations of the first seventeen of the twenty-six groundforms, complete, with those of the simple forms of 5_8, 5_{18}, and 5_{21} already mentioned, which might usefully be published with the sextic tables. The utility of such printed tables consists largely in their availability for reference in case of need, and for this purpose they should be published, not singly or in small numbers as computed from time to time, but in masses. It is for this reason that I have not thought of publishing the computations just mentioned. I have made them, indeed, not intending publication, but in order to verify to the greatest extent my idea that the easiest way to find the simplest forms is, wherever practicable, to apply simple transvection as already explained.

* $H = 5_8 + B^2$; $P = 5_{18} - BK$; $S = \frac{1}{3}(GK - BO) - 5_{21}$.

† Corrected. As Professor Sylvester points out (*loc. cit.*), the tables printed by Faà de Bruno, useful as they are, contain many errors. The last column of this 6_{10} table is nearly all wrong, and only one column of the five is quite right.

Confining my attention to this question of simplicity, I have not even made search among mathematical journals to see what has already been done towards the computation of the more difficult covariants of the sextic, but will do so at the instance of any member of the Society willing to undertake the work of completing the series, who may not himself have access to a large library.

The class of cases to which I have referred as unsuitable for the application of simple transvection are those in which there is no groundform near enough upon which to operate. For instance, to produce by simple transvection the invariant 6_{18}, of the sixth degree in the coefficients, we should need as a basis of operation a covariant of degree 5 whose weight should not be less than 12, and whose weight and order combined should exceed 17. The only groundforms of degree 5, are, however, 6_{13} and 6_{14}, the former of order 4, the latter of order 2, and neither of them can be used to produce 6_{18}. It is of course possible in such cases to apply simple transvection to a complex covariant—as, for instance, to $6_4 . 6_8$, of order 6, for producing 6_{16}—but that will not usually produce the best results, and it is doubtless preferable to employ transvection (no longer "simple") of groundforms other than the quantic itself, in accordance with the recommendations of the text-books. Of the four text-books already cited which supply formulæ for computing the groundforms of the quintic and sextic, the formulæ collected by Salmon are apparently the best. So far as my observation has gone, the application of Salmon's formulæ has given simple results in most cases. Among the exceptions to this remark are 5_{21}, 6_{10}, 6_{14}.

After once applying simple transvection to produce 6_{15}, for which weight there are two groundforms of the same degree in the coefficients and order in the variables, we cannot again employ satisfactorily the rule of the nearest for producing the other form. Thus, [1] 6_{14} gives 6_{15a}, and we cannot again use [1] 6_{14} for producing 6_{15b}; nor can we profitably use [2] 6_{13}, perhaps because it is not only not so "near" as [1] 6_{14}, but even not so "near" as any combination of [2] 6_{13} and [1] 6_{14}. In this case the usual symbolic formula—Jacobian of 6_6 and 6_8—is the best for practical application.*

The nine groundforms of the sextic which remain to be computed or collected (if in simplest form) from other publications—6_{18}, for instance, is well known—are, as to weight, degree, and order, as follows, the weight being denoted by the subscript: 6_{18}, 6, 0; 6_{19}, 7, 4; 6_{20}, 7, 2; 6_{22}, 8, 2; 6_{23},

* I have not tested [4] 6_{11}.

9, 4; 6_{29}, 10, 2; 6_{30}, 10, 0; 6_{33}, 12, 2; 6_{45}, 15, 0. Of these nine, five may be derived by simple transvection, viz., [4] $6_{15} = 6_{19}$, [5] $6_{15} = 6_{20}$, [3] $6_{20} = 6_{23}$, [2] $6_{23} = 6_{25}$, [4] $6_{25} = 6_{29}$. I have written in two places 6_{15} for 6_{15a} or 6_{15b}, not known which is to be preferred, a matter to be settled in either case most easily by computing a few terms upon each basis. The three of higher weights, to which simple transvection will not apply, may probably be derived most simply by means of the formulæ given by Salmon.

To illustrate the process of simple transvection, which, although sufficiently implied, is not usually illustrated in the books, I give [4] $6_4 = 6_8$ in detail in the form of a table:

Operand.	Multipliers.		
6_4.	For (1).	For (2).	For (3).
	$\div 2$		$\div 2$
$ae - 4bd + 3c^2$	e	f	g
$2af - 6be + 4cd$	$-d$	$-e$	$-f$
$ag - 9ce + 8d^2$	c	d	e
$2bg - 6cf + 4de$	$-b$	$-c$	$-d$
$cg - 4df + 3e^2$	a	b	c

$$\left.\begin{array}{l}(1) = acg - 3adf + 2ae^2 - b^2g + 3bcf - bde - 3c^2e + 2cd^2 \\ (2) = -bcg - 8bdf + 9be^2 + 9c^2f - 17cde + adg + 8d^3 - aef \\ (3) = aeg - 3bdg + 2c^2g - af^2 + 3bef - cdf - 3ce^2 + 2d^2e\end{array}\right\} 6_8$$

The multipliers in this instance are extremely simple. The coefficient of a is always 1, as in this case, but in general those of the other multipliers are other integers. The rule which I find best for determining the integers forming the coefficients of the multipliers for simple transvection is given in another paper, as a special case of a broader rule for transvection in general. The paper in question, "On the Computation of Covariants by Transvection," to be read before the Society on January 2, 1892, will be printed elsewhere, the pages of the *Bulletin* being intended rather for critical and historical notes than for original investigations.

A FRENCH ANALYTICAL GEOMETRY.

Leçons de Géométrie Analytique. Par MM. BRIOT et BOUQUET. Revue et annotée par M. APPELL, professeur à la Faculté des Sciences. Paris, Ch. Delagrave, 1890. 8vo, pp. iii. + 722.

THIS popular French text-book reached its fourteenth edition in 1890. At that time, as we learn from the preface, changes in the programmes of the schools and improved methods of teaching had made a revision of the book advisable. This piece of work was done by M. Appell, a mathematician, whose name is as familiar to American students as to Frenchmen. The bare list of the articles in the book which he has touched covers a page and a half, and it is safe enough to say that "*nihil quod tetigit non ornavit.*" A treatise of this kind is of course more interesting to teachers of elementary mathematics than to any one else; to them even a slight account of a school book which has achieved great and lasting popularity in a nation where pure mathematics has flourished so splendidly and so long, can not fail to prove interesting by virtue of its subject.

The book opens with a concise notice of the different systems of plane coordinates, beginning with rectilinear coordinates in general and the particular case of rectangular axes; then passing rapidly over polar and bi-polar systems, and finally giving a notion of coordinates in general. These notions are all simple enough when presented in the transparent style of the authors; in fact plane coordinates are so much simpler than curves drawn on a sphere that it is a wonder that school books on geography should not give an account of them before taking up the subject of latitude and longitude which almost always proves difficult to young pupils. The writer was once explaining rectangular coordinates at a teachers' institute when one of the members rose and thanked him for *inventing* them; he had been trying to teach latitude and longitude without any of the preliminary ideas necessary to an understanding of the matter. At the close of the first chapter we read, "The representation of figures by equations is the basis of analytic geometry; it allows us to apply the processes of algebra to the study of figures. In analytic geometry we are concerned with three fundamental questions: when a figure is defined geometrically, to find its equation; reciprocally, when the equation is given, to construct the figure; finally, to study the relations which exist between the geometrical properties of the figures and the analytical properties of the equations."

Chapter II. takes up the first problem; various loci, including nearly all the simple curves whose names are familiar, are defined geometrically and their equations written down directly, as a mere statement of the definition in the language of algebra. The curves are drawn and sufficiently described. In this way the student not only gets a notion of what a locus is, but, what is far from easy, he comes to see how an equation, so different in its nature and belonging to quite another realm of thought, can represent a geometrical figure, and to look upon equations in x and y as brief statements of the truths of geometry. The mind is put in a condition to understand why the manipulation of an equation may lead to new facts. Without some such preparation it can hardly be very profitable to try to prove geometrical theorems with equations; there can be nothing in the student's mind corresponding to them, and a gulf which he can not bridge will exist between the proof and the conclusion. That the radius of a circle has a constant length is expressed algebraically by the equation

$$\rho = r;$$

that the sum of the distances of any point on the ellipse from the foci is constant, by

$$u + v = 2a,$$

and so on; the polar equations from their simplicity being first written down. The chapter closes with a list of exercises of which this is a specimen: "To construct the curve whose equation in bi-polar coordinates is $uv = a^2$; the distance between the poles being $2a$." Of course this lemniscate would be constructed by drawing circles with their centers at the poles. The student has been told previously how to find points on the ellipse in the same way. The problems are mostly too difficult for a beginner.

Chapter III. treats of the fundamental idea of homogeneity. A function $f(a, b, c, \ldots)$, we are told, is homogeneous and of degree m when $f(ka, kb, \ldots) = k^m f(a, b, \ldots)$; the sum, difference, product or quotient of any two homogeneous functions is homogeneous; and the same is true of any power, root or transcendental function of $f(a, b, c, \ldots)$; but the transcendental function must be of degree 0. Thus $\sin\left(\frac{ab}{a^2 + b^2}\right)$ is homogeneous, while $\sin(a + \sqrt{bc})$ is not. All this is sufficiently clear, and what follows shows its vital importance at the threshold of analytic geometry.

"When we seek the relations which exist between the lengths of the various lines $A, B, C, \ldots$ of a figure, we imagine these lines referred to a unit of length which is usually not specified and remains quite arbitrary." Hence the reasoning which leads to a relation among the lengths of these lines is independent of any particular unit, and the relation must subsist whatever be the unit. In particular it subsists if the unit be divided by k, that is if the number expressing each length is multiplied by k; hence the relation is homogeneous, or at any rate, if not, then it must break up into several relations which are each homogeneous. An apparent exception occurs when some line of the figure is taken as the unit of length, but the exception is explained and the homogeneity reestablished. "The equations which the theorems of elementary geometry lead to directly, are homogeneous. . . . The principle of homogeneity can be used at every step to verify the algebraic transformations which have been effected."

The above is a too brief account of a part of this very elementary and most interesting chapter. No part of it is difficult even for a young student, while it opens up to him a line of thought which he must follow throughout his scientific studies in whatever direction he turns. This is the kind of work which makes mathematicians and scientists, while the student whose analytic geometry consists only in manipulating a few equations of which the meaning is but dimly seen, finds it a barren and useless subject.

Some considerations follow, still of the simplest kind, which lead to the conclusion that "all rational expressions, and all irrational expressions containing only square roots, can be constructed by means of a limited number of right lines and circles." It is added, but not proved, that no others can. A little reading of this character would turn the attention of a goodly number of bright young men who are still at work upon some of the impossible problems of antiquity, to subjects more worthy of their abilities.

Book II. opens the study of the right line and circle. The treatment of these loci is similar to that in our familiar text books and is limited to the needs of beginners. The next chapter treats of geometric loci in general. In Book I. the equations of many loci were obtained simply by writing down the definition in the language of algebra; here we obtain the equations of loci by eliminating one or more variable parameters. The coordinates of a point may be explicitly given as functions of the parameter a, but more usually they are only implicitly given in two equations

$$(1)\quad f_1(x, y, a) = 0,$$
$$(2)\quad f_2(x, y, a) = 0.$$

Each value of a gives a pair of curves intersecting in a point of the locus, and "the equation of the locus is obtained by eliminating the parameter a between the equations (1) and (2)."

Why this should give the equation of the locus is a difficult thing for students to see; but it is less difficult when properly stated. The result is not a single equation in x and y, but a pair of equations

$$(3)\quad f(x, y, a) = 0,$$
$$(4)\quad F(x, y) = 0.$$

which are equivalent to (1) and (2). Any system of values of x, y, a, which satisfies (1) and (2) must satisfy (3) and (4); hence equation (4) is satisfied by the coordinates of every point on the locus. Conversely, every system of values of x, y, a which satisfies (3) and (4) must satisfy (1) and (2); it follows that (4) is the equation of the locus.

The teacher who reads this book will be everywhere delighted by the careful way in which the raw edges of thought are hemmed down; no loose threads are left to ravel out and destroy the fabric. It can hardly be doubted that this brave honesty in their elementary school books has much to do with that precision of thought and clearness of expression which makes the works of French mathematicians a perpetual refreshment to the reader. It would be an inquiry worth making, whether students in high schools and normal schools who never intend to enter college do not get more benefit from the study of geometry than any others. With them it is not merely a thing to be crammed for an entrance examination, but a subject to be studied for its educational value. It seems especially disastrous to make analytic geometry, a subject where the preliminary notions are so delicate and beautiful, a thing to be asked questions about when entering college; inasmuch as the amount of knowledge required can at best be but small, and will almost certainly be acquired under conditions likely to blight future results.

The remainder of the book, while deeply interesting, is more advanced; and it is not the purpose of this sketch to do more than call attention to those parts whose study may possibly be useful to teachers of classes which are beginning the subject.

C. H. CHAPMAN.

JOHNS HOPKINS UNIVERSITY, *December* 14, 1891.

THE ANNUAL MEETING OF GERMAN MATHEMATICIANS.

THROUGH the courtesy of the secretary, Dr. H. Wiener, of the University of Halle, who kindly sent us advance sheets of the *Proceedings*, we are now enabled to give a more detailed account of the papers read before the section for mathematics and astronomy of the German *Naturforscher-Versammlung* held at Halle, September 21 to 25, 1891. The meetings of this section constitute at the same time the annual meeting of the German Mathematical Union (*Deutsche Mathematiker-Vereinigung*). The section had seven meetings; the total number of members registered as present was 70.

1. The first paper read was a report by Prof. FELIX KLEIN, of Göttingen, *On recent English investigations in mechanics.* The following abstract of this paper is translated from the *Proceedings.*

"The distinguishing characteristic of the English work in mechanics in comparison with that of continental writers lies in its being based on a thorough grasp of physical reality and in the resulting graphical lucidity (*durchgängige Anschaulichkeit*) of the investigations. For this very reason the English work in mechanics proves particularly interesting and instructive to the mathematician accustomed to a purely abstract train of reasoning. The usual lack of that methodical treatment and mathematical rigor which the continental mathematician is wont to expect cannot be regarded as a serious objection ; in fact, it adds to the interest.

Among the matters of detail discussed by the speaker, his remarks on the history of the discovery of Hamilton's method of integrating the equations of dynamics may be of general interest. The matter seems to be entirely unknown, although Hamilton distinctly states the facts at various places in his writings, in particular in his first paper on systems of rays (1824). At the time when Hamilton began writing, the emission theory was still prevalent so that the determination of a ray of light passing through any non-homogeneous (but isotropic) medium was considered as a special case of the ordinary mechanical problem as to the motion of a material particle. It may be noticed in passing that the distinction between this special case and the general problem is not an essential one : by proceeding to higher spaces, any mechanical problem may be reduced to the determination of a ray of light traversing a properly selected medium. Now *Hamilton's discovery*, according to which *the integration of the differential equations of dynamics is made to depend upon the integration of a certain* *partial differential equation of the first order*, was simply the result of the fact that Hamilton, following the great movement just then taking place in physics, undertook to derive, from the point of view of the undulatory theory, the results in geometrical optics already known in the form of the corpuscular theory. Hamilton's method for integrating the differential equations of dynamics is, primarily, nothing but the general analytical expression for the relation between *ray* and *wave*, a distinction which in its physical form was well known at the time. Considered in this new light it is readily understood why Hamilton gave to his investigations that unnecessarily specialized form in which he published them and which was removed only later by Jacobi. In his investigations on systems of rays Hamilton had originally in view certain entirely practical questions relating to the construction of optical instruments. This is the reason why he operates throughout with waves of light issuing from single *points*. The real meaning of Jacobi's generalization is that *any other waves of light may be used to determine a ray.* The general wave is constructed in optics from the special waves by means of the so-called principle of Huygens. This construction is an exact equivalent to the analytical process by which we ascend in the theory of partial differential equations of the first order from any 'complete' solution to the 'general' solution."

2. The paper of Mr. PAPPERITZ, of Dresden, *On the system of the mathematical sciences*, it is announced, will be published elsewhere.

3. Mr. MAX SIMON, of Strassburg, read a paper *On the axiom of parallels.*

4. Mr. FRANZ MEYER, of Clausthal, presented an elaborate report *On the progress of the projective theory of invariants during the last twenty-five years*, which will probably be published *in extenso* by the German Mathematical Union.

5. Mr. FINSTERWALDER, of Munich, read a paper *On the images of dioptric systems of somewhat large aperture and field* which is published in the Transactions of the Bavarian Academy of Sciences (*Abhandlungen*, Class II, Vol. 17, Abth. 3, pp. 517–588).

6. Mr. ROHN, of Dresden, spoke *On rational twisted quartics*, illustrating his remarks with the aid of models.

7. Dr. H. WIENER, of Halle, read a paper *On the foundations and the systematic development of geometry*. The following abstract is given in the *Proceedings*.

"To be rigorous we may demand that the proof of a mathematical proposition should make use of those assumptions only on which the proposition really depends. The simplest conceivable assumptions are the existence of certain objects and the possibility of certain operations by which said objects may be connected. If it be possible, without further assumptions, to connect such objects and operations so as to produce propositions, these propositions will form a *self-sustaining* (*in sich begründet*) *domain* of science. Such, for instance, is algebra.

In geometry it is of interest to go back to the simplest objects and operations, since starting from these it is possible to build up an abstract science which will be independent of the axioms of geometry while its propositions run parallel to those of geometry.

The projective geometry of the plane offers an example. Let the objects be *points* and *lines*, the operations those of *joining* and *cutting*, and let objects as well as operations be restricted to a finite number. Throwing off the geometrical dress we shall have elements of two kinds and two kinds of operations such that the connection of any two elements of the same kind produces an element of the other kind. The geometrical propositions obtained on these assumptions (apart from combinatory propositions involving the number of elements) are *closing propositions* (*Schliessungssätze*), if this term be taken to mean propositions about eertain lines and points such that every one of the lines contains at least three of the points and every one of the points lies at least on three of the lines. Such are for instance : (1) Desargues's theorem of perspective triangles, and (2) Pascal's hexagram theorem applied to two lines.

The proof of such propositions cannot be obtained from the given objects and operations; in other words, this domain of geometry is not self-sustaining. If however the proof for any one such proposition (or for several) be taken from some other domain, then, by its repeated application a closed domain of plane geometry may be obtained. Thus, proving Desargues's theorem by means of solid geometry, we obtain the domain embracing all propositions usually derived by means of geometrical addition of vectors or points. The attempts at deriving proposition (2) above from (1) have not been successful. Another possibility would be its derivation by projection from a space of three or more dimensions, or else (which is easily done) by introducing the idea of continuity. *These two "closing propositions,"* however, *are sufficient to prove, without further considerations of continuity or infinite processes, the fundamental proposition of projective geometry*, and thus to develop the whole domain of linear projective plane geometry.

Similarly it is possible to build up a solid geometry resting on the point, line, and plane as fundamental elements, or objects. But in this case we obtain a self-sustaining domain.

These considerations can be extended to higher spaces. It will however be more important to descend from the plane to the *geometry of the line*. The only element we here have is the point; there can be neither joining nor cutting. It thus becomes necessary to borrow an operation from another domain; as such we may use constructions executed in the plane but concerning only points lying on our line, especially constructions of projective, involutory, and harmonic groups of points. It appears that the construction of harmonic groups is sufficient as the following proposition can be proved: *If in a line two pairs of points of an involution, or three pairs of corresponding points of a projective system be given, it is possible to construct the corresponding point to any other given point by a* FINITE *number of constructions of harmonic points.*

Other domains are obtained by introducing other assumptions. Thus the *geometry of order* presupposes the proposition that on a closed line four points can be divided in a definite way into two pairs that separate each other. Still other domains depend on the assumption of the *continuity* of the elements, which may be either the analytical continuity of the method of limits, or the geometrical continuity that finds its expression in the necessary meeting of points moving in a certain way in a line."

8. Mr. SCHUBERT, of Hamburg, read a paper *On the enumerative geometry of p-dimensional spaces of the first and second degrees.*

9. Mr. EBERHARD, of Königsberg: *Elements of the theory of forms of polyhedra.* An elaborate work by the author on this subject has just been published (*Zur Morphologie der Polyëder*, Leipzig, Teubner, 1891).

10. Mr. BOLTZMANN, of Munich: *On some points in Maxwell's theory of electricity;* will be published in the *Proceedings* of the section for physics.

11. Mr. HENSEL, of Berlin: *On the fundamental problem of the theory of algebraic functions;* see the same author's paper in the *Journal für Mathematik*, vol. 108, p. 142.

12. Mr. FELIX MÜLLER, of Berlin: *On literary enterprises adapted to facilitate the study of mathematics.* The speaker pointed out the desirability of an introduction to the bibliography of mathematics; complained of the want of subject-indexes in the most prominent mathematical journals; gave an account of the progress of recent bibliographical works; and expressed a regret that the continuation to Poggendorf's Dictionary of Authors has not yet appeared. He also laid before the Section a plan for a new Mathematical Dictionary for which he has been collecting the material for the last 20 years; it contains about 4000 mathematical terms and over 1200 names.

13. Mr. DYCK, of Munich : *On the forms of the systems of curves defined by a differential equation of the first order, in particular on the arrangement of the curves of principal tangents to an algebraic surface;* will be published elsewhere.

14. Mr. DAVID HILBERT, of Königsberg: *On full systems of invariants.*

"Let $J_1, J_2, \ldots, J_{n-2}$ be integral rational invariants of a binary ground-form of the nth order, of the degrees $\nu_1, \nu_2, \ldots \nu_{n-2}$, respectively, in the coefficients of the ground-form; and let these invariants be so selected that all other integral rational invariants of the ground-form are *integral* algebraic functions of those $n-2$ invariants. Then the integral rational invariants of the ground-form form the integral functions of a body (*Körper*) of algebraic functions; let g be the degree of this body. Then the following formula can be shown to hold:

$$\frac{g}{\nu_1 \nu_2 \ldots \nu_{n-2}} = -\frac{1}{2.\, n\,!}\left\{\left(\frac{n}{2}\right)^{n-2} - \binom{n}{1}\left(\frac{n}{2}-1\right)^{n-2} + \ldots \pm \binom{n}{\frac{1}{2}n-1}\right\} \text{ for even } n,$$

$$= -\frac{1}{4.\, n\,!}\left\{\left(\frac{n}{2}\right)^{n-3} - \binom{n}{1}\left(\frac{n}{2}-1\right)^{n-3} + \ldots \pm \binom{n}{\frac{1}{2}n-1}\right\} \text{ for odd } n.$$

15. Mr. SCHOENFLIES, of Göttingen : *On Configurations that can be derived from given space-elements by the operations of cutting and joining alone.* Referring to Dr. Wiener's paper (7) the speaker showed that prop. (2) (Pascal's hexagram applied to two lines) cannot be derived by the operations of joining and cutting alone.

16. Mr. MINKOWSKI, of Bonn : *On the geometry of numbers.* The author gives the name *number-frame* (*Zahlengitter*) to the totality of all those points of space whose rectangular coordinates are three *integral* numbers and considers certain solids and their relation to the frame. The two most important cases are as follows. (1) Solids having the origin of coordinates as centre and bounded by a surface that appears at no point concave from without. For such solids it can be shown that if the volume be $\geqq 2^3$ the solid must contain other points of the frame besides the origin. (2) Solids containing the origin and bounded by a surface which as seen from the origin shows no double point. If the

volume of such a solid be $\leqq 1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \ldots$, it is always possible to indicate deformations of the solid for which the volume remains constant, the origin remains fixed and all straight lines of the solid remain straight while all points of the frame excepting the origin are found outside the solid after deformation.

17. Mr. FRITZ KÖTTER, of Berlin : *On the problem of rotation treated by Mrs. Kovalevsky.* The paper develops somewhat farther the formulæ given by Mrs. Kovalevsky in the 12th volume of the *Acta Mathematica* for a certain integrable case of the problem of rotation of a heavy body about a fixed point.

18. Mr. PILITZ, of Jena : *A question in the theory of numbers.* After an introductory discussion of the necessity for a new calculus, or at least of a new way of conceiving of the combination of elements in the problems of the theory of numbers and the theory of functions, the speaker gave a proof of the proposition announced by Riemann as probably true : that the complex 0-points of the function $\zeta(s)$ all have $\frac{1}{2}$ as their real part.

19. Mr. F. STÄCKEL, of Halle : *On the bending of curved surfaces under certain conditions.*

20. Mr. A. WANGERIN, of Halle : *On the development of surfaces of rotation with constant negative curvature on each other.*

21. Mr. WILTHEISS, of Halle : *On some differential equations of the theta functions of two variables.*

22. Mr. G. CANTOR, of Halle : *On an elementary question in the theory of manifoldnesses.*

23. Mr. GORDAN, of Erlangen : *Remarks on a proposition of Mr. Hilbert.*

ALEXANDER ZIWET.

NOTES.

A REGULAR meeting of the NEW YORK MATHEMATICAL SOCIETY was held Saturday afternoon, December 5, at half-past three o'clock, the president in the chair. Mr. Wiley, Mr. Snook, and Dr. Pupin were appointed a committee to report at the annual meeting, on December 30, nominations for the officers and other members of the council for the calendar year 1892.

Dr. Pupin read an original paper entitled "On a peculiar family of complex harmonics," in which he deduced several useful properties of certain complex harmonic curves, explaining briefly their application to polyphasal and continuous current generators. Mr. Steinmetz and Dr. Webster made remarks upon the physical side of the paper. Mr. Steinmetz said that he had actually obtained in experiments with dynamos, curves which very closely resembled those given in Dr. Pupin's paper. T. S. F.

THE following courses of lectures, extending through the first half year, are being delivered at Clark University, Worcester, Mass.:

By Professor Story: (1) Enumerative geometry and the theory of coloring maps; (2) Historic development of arithmetic and algebra.

By Dr. Bolza: (1) Klein's icosahedron theory; (2) Definite integrals and calculus of variations.

By Dr. Taber: Modern higher algebra.

By Dr. White: (1) Theta-functions of three and four variables; (2) Modern synthetic geometry and higher plane curves.

By Mr. de Perrott: Application of analysis and group theory to the theory of numbers.

By Professor Michelson: Optical theories.

By Dr. Webster: Dynamics.

At the weekly mathematical conferences conducted by Professor Story, non-euclidean geometry has been the subject of systematic discussion, together with less extended consideration of topics of interest from other departments of mathematics. At a recent meeting the new Amsler's planimeter which has been added to the mathematical apparatus of the University was exhibited and its theory explained by Professor Story. Careful measurements on known areas give results accurate to within one-twentieth of one per cent.

J. W. A. Y.

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CHAVE Y CASTILLA (J.). Ensayo de una nueva Teoría de la Proporcionalidad de las Líneas rectas. Madrid 1891. 4. 91 pg. av. 3 planches. M. 2.50

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KLEIN'S MODULAR FUNCTIONS.

FELIX KLEIN, *Vorlesungen über die Theorie der elliptischen Modulfunctionen*, ausgearbeitet und vervollständigt von Dr. ROBERT FRICKE. Erster Band. Grundlegung der Theorie. Leipzig, Teubner, 1890. 8vo, pp. xix + 764.

THE mathematical public is under great obligation to Professor Klein's former pupil, Dr. Robert Fricke, for his able presentation of the theory of the modular functions. His clearness of treatment and skillful grouping of the many intricate features of the subject have rendered this theory now thoroughly accessible. Beside the work of arrangement, in itself a labor of no small magnitude, Dr. Fricke has contributed many of the intermediate steps necessary to the symmetry and completeness of the subject. His task has been performed throughout with a highly creditable degree of conscientiousness and ability.

The theory of the modular functions and the allied branches has been one of the chief series of investigations to which Professor Klein has devoted himself in the period of some twenty years over which his scientific activity now extends. It is characteristic of these investigations that they are not included as a subordinate part in any of the great mathematical theories heretofore commonly so recognized. Their distinctive tendency is in the direction of the combination and unification of the latter into a broader method of research. This idea has been developed by Klein to an extent and with an elaboration which have long since entitled it to recognition as an independent, and in the highest degree productive mathematical point of view. In the present paper some attempt is made to sketch the general outlines of the new method, so far as it concerns the modular functions, and to illustrate it more definitely by the consideration of some of the more important details.

Historically, Klein's work has developed accurately along the lines of a thoroughly predigested plan, the bolder features of which are already sharply defined in his earliest publications.* On this ground, then, a brief semi-biographical, semi-scientific sketch of his career may properly find place here. It is to be observed that this sketch makes no pretension to completeness. It confines itself mainly on the scientific side to the development of the theory of the regular bodies and of the modular functions.

Klein's first productive activity dates from his relation to

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* Cf. the preface to the "*Ikosaeder,*" and the *Eintrittsprogramm* mentioned on the following page.

Julius Plücker, as the latter's assistant in physics at Bonn. On Plücker's death the preparation of the second volume of his posthumous work on line geometry * was entrusted to Klein, who was then at the age of nineteen. The first volume was edited by Clebsch. Having completed this task, and having taken the doctor's degree at Bonn, Klein studied in Berlin and in Paris until the outbreak of the Franco-German war, which compelled his return to Germany. Soon afterward he was appointed *Privat-Docent* at Göttingen, where Clebsch was approaching the close of his brilliant career. In 1872 he was called to the *ordinarius* professorship of mathematics at Erlangen. His *Eintrittsprogramm* † prepared on the occasion of assuming this chair is certainly a most remarkable production for a young man of twenty-three, containing, as it does, not merely a foreshadowing, but actually a systematic program, conceived with perfect maturity and definiteness, of the scientific work to which he has since devoted himself. It is with the theory of operations that he is here concerned; not the formal theory of operations in themselves, but entirely with reference to the *content* to which the operations are conceived to be applied, in particular when this content is a geometrical configuration. Two such theories were already in existence: the theory of invariants and covariants, which deals with the effect of the entire system of linear transformations of two or more homogeneous variables, and the theory of substitutions, in which the operations are the permutations of a finite system of elements. These two theories can be regarded as extreme types, between which an infinite series of others can be inserted. A definite complex of these intermediate types has furnished the field to which Klein's labors have thus far mainly been devoted. The *Eintrittsprogramm* appears as a preliminary survey of the general doctrine of operations, with reference to geometrical configurations. It involves not only the discontinuous operations, within which Klein's specific work has been included, but also the continuous systems, which belong with differential equations, and the theory of which has been mainly developed by his friend and fellow-student, Professor Sophus Lie.

In Erlangen Klein formed the acquaintance of Gordan, to whose personal friendship and scientific cooperation a high tribute is paid in the preface to the *Ikosaeder*. It was here and at Munich, to which city Klein was called in 1875, that

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* *Neue Geometrie des Raumes, gegründet auf die Betrachtung der geraden Linie als Raumelement*, von Dr. JULIUS PLÜCKER. Leipzig, Teubner, 1869.

† *Vergleichende Betrachtungen über neuere geometrische Forschungen*, von Dr. FELIX KLEIN, o. ö Professor der Mathematik an der Universität Erlangen.

the theory of the icosahedron and the other regular bodies was gradually developed in a series of papers in the *Mathematische Annalen*, of which Klein had become an editor in 1873. It may be noted that nearly all of his writings are published in this journal, which is, indeed, distinctively the organ of the school of which its editor is the leader.

In 1881 Klein was called from Munich to Leipzig, where he remained until 1886, when he was appointed to the chair in Göttingen, vacated by the death of Enneper, which he still holds. At Munich he numbered among his students Hurwitz, Rohn, and Dyck, all of whom have made a name among mathematicians. In Leipzig the first American students were admitted to his *Seminar*, Messrs. Irving Stringham and Henry B. Fine, now professors of mathematics at the University of California and at Princeton, the precursors of a numerous throng, among whom the writer had the good fortune to be included. During the stay in Leipzig negotiations were at one time pending toward inviting Klein to Sylvester's vacated chair at Johns Hopkins. Various considerations, relating mainly to his health, never very robust, led him to decide in favor of remaining in Germany. A circumstance which must have contributed greatly to his decision was the fact that he had already gathered about him a band of talented and mature young mathematicians the direction of whose vigorous development was a most gratifying task. Among the members of his *Seminar* in 1884–5 were Pick now at Prague, Hölder now at Göttingen, Study at Marburg, and Fricke the editor of the *Modulfunctionen*.

The management of the *Seminar* has always been exceptionally efficient, even among the German models. It is Klein's custom to distribute among his students certain portions of the broader field in which he himself is engaged, to be investigated thoroughly under his personal guidance and to be presented in final shape at one of the weekly meetings. An appointment to this work means the closest scientific intimacy with Klein, a daily or even more frequent conference, in which the student receives generously the benefit of the scholar's broad experience and fertility of resource, and is spurred and urged on with unrelenting energy to the full measure of his powers. When the several papers have been presented, the result is a symmetric theory to which each investigation has contributed its part. Each member of the *Seminar* profits by the others' points of view. It is a united attack from many sides on the same field. In this way a strong community of interest is maintained in the *Seminar*, in addition to the pleasure afforded by genuine creative work.

The theory of the icosahedron appeared in book form in

1884.* The investigations included under this title traverse a definite, self-limited field, identified on the one hand with the groups of rotations of the regular bodies and the corresponding finite groups of linear transformations of a complex variable, and on the other with the theory of the algebraic equations of the first five degrees and certain other special types. The close relation of the subject to the theory of the modular functions is also so far touched upon as to indicate the direction of the more extended theory which has culminated in Dr. Fricke's book. From the point of view of this relation the *Ikosaeder* appears as a first step in the systematic treatment of the modular functions, for which it is also to serve as a model. In the meantime Klein's lectures were forecasting the coming theory, already developed in many of its features in articles in the *Annalen*. Beginning with the general theory of functions, he treated in successive semesters the elliptic functions, the elliptic modular functions, the new geometry, the hyperelliptic functions of deficiency 2, and the Plücker line geometry. Of these lectures the first three, together with the later lectures on algebra at Göttingen, relate largely to the present theory, while the others were to a considerable extent preliminary to the theory of the general equations of the sixth and seventh degrees.

In all these investigations it is again the theory of operations that furnishes the guiding principle and the general outline of the subject. But the field of research being once mapped out, it is characteristic of Klein to bring to bear on it every instrument that modern mathematics can provide. The theory of functions, invariants and covariants, differential equations, modern geometry, in short every method is put under requisition and made to render its contribution to the symmetry of the result. No advantageous point of view is neglected, and not until the subject is traversed in every direction, and until its external and internal relations are clearly pictured to the mind, is the investigation to be regarded as complete. For example the *Ikosaeder* discusses in successive chapters the rotations of the regular bodies, the corresponding groups of linear transformations and their invariants, the actual solution of the problem by the aid of a class of differential equations, the algebraic phases of the subject, based on the theory of substitutions, the general position of the theory in reference to other correlated fields, its historical development, the coordinated geometrical problems, etc. The same plan obtains in the lectures and the *Seminar*.

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* F. KLEIN, *Vorlesungen über das Ikosaeder und die Auflösung der Gleichungen vom fünften Grade*. Leipzig, Teubner, 1884.
English translation by G. C. MORRICE. London, Trübner, 1888.

The value to the student of this breadth of treatment is simply inestimable. No one can study long under Klein without obtaining an intelligent comprehension of most of the great tendencies in mathematics. This is at present particularly desirable in view of the extreme degree of specialization which has come to prevail among mathematicians. Of all the great services which Klein has rendered to mathematics, there is none more valuable than his successful unification of its heretofore rapidly diverging branches. Lately he has turned his attention to mechanics, in which new field, under the application of the same principle, we have every reason to expect from him another series of brilliant results.

Turning to the *Modulfunctionen,* one cannot but admire the simplicity and perfect proportion with which Dr. Fricke has developed the subject. So far as possible, everything is traced from first principles. The network of interwoven theories is constructed with a painstaking elaboration of details and a rare geniality of method. A clearer and more scholarly presentation than that before us could hardly be imagined.

The work divides into three principal investigations: (I.) the theory of the modular functions in the narrower sense as a specific class of elliptic transcendents; (II.) the formal definition of the general problem, as based on the doctrine of groups of operations; (III.) the union of these two methods and the further development of the subject in connection with a class of Riemann's surfaces.

We turn our attention for the present to the second division of the subject. The operations with which we have to deal belong to that fertile field of modern mathematics, the linear transformations. The characteristic system on which the theory of the modular functions turns is composed of all the linear transformations of a complex variable

$$(1) \qquad z' = \frac{\alpha z + \beta}{\gamma z + \delta},$$

or, in homogeneous form, of the binary linear transformations

$$(2) \qquad \begin{aligned} z'_1 &= \alpha z_1 + \beta z_2, \\ z'_2 &= \gamma z_1 + \delta z_2, \end{aligned}$$

for which the constants α, β, γ, δ are real integers, subject to the further condition that

$$\alpha\delta - \beta\gamma = +1.$$

The equations (1) and (2), and other similar types, must be regarded throughout, as already indicated, as defining *operations;* namely, the operation of passing in each case from the initial values z to the transformed values z'. Restricting ourselves for the present to the general linear transformations of a single complex variable, $z' = \dfrac{\alpha z + \beta}{\gamma z + \delta}$, if the values of z are represented in the ordinary manner by points in the complex plane, the transformation is to be conceived as carrying every point z to the corresponding position z'. The result of the transformation is therefore to effect a rearrangement of the position of the points of the plane, and this geometric conception, to be presently more fully developed, not only serves to picture the corresponding analytic formula, but may often with great advantage entirely replace it.

If now any transformation $z' = \dfrac{\alpha z + \beta}{\gamma z + \delta}$ is followed by a second $z'' = \dfrac{\alpha_1 z' + \beta_1}{\gamma_1 z' + \delta_1}$, the relation of the points z'' to the original points z is directly defined by the equation

$$(3) \qquad z'' = \frac{\alpha_1\left(\dfrac{\alpha z + \beta}{\gamma z + \delta}\right) + \beta_1}{\gamma_1\left(\dfrac{\alpha z + \beta}{\gamma z + \delta}\right) + \delta_1}$$
$$= \frac{(\alpha_1\alpha + \beta_1\gamma)z + \alpha_1\beta + \beta_1\delta}{(\gamma_1\alpha + \delta_1\gamma)z + \gamma_1\beta + \delta_1\delta} = \frac{\alpha_2 z + \beta_2}{\gamma_2 z + \delta_2},$$

which is again linear. The combination, or "product," of two linear transformations of a complex variable is therefore itself a linear transformation of a complex variable. The total system of these transformations accordingly forms a "group," this name being applied to any system of operations of whatever kind such that the product of any two of them is itself an operation of the system. If, furthermore, we confine our consideration to those transformations for which α, β, γ, δ are real integers, this more limited system is clearly still a group, which in reference to the including general group just considered is designated as a "subgroup" of the latter. Again we obtain a subgroup of this subgroup by selecting from the latter all those operations for which the determinant $\alpha\delta - \beta\gamma = +1$. For, on referring to (3), we have at once for the product of any two of these operations

$$(4) \qquad \alpha_2\delta_2 - \beta_2\gamma_2 = (\alpha_1\alpha + \beta_1\gamma)(\gamma_1\beta + \delta_1\delta)$$
$$-(\alpha_1\beta + \beta_1\delta)(\gamma_1\alpha + \delta_1\gamma) = (\alpha_1\delta_1 - \beta_1\gamma_1)(\alpha\delta - \beta\gamma) = +1.$$

By way of contrast we may observe that those transformations with real integral coefficients for which

$$\alpha\delta - \beta\gamma = -1$$

do not form a group, since the product of any two of them has for its determinant $(-1)(-1) = +1$. If however we combine the two systems $\alpha\delta - \beta\gamma = \pm 1$, the result is again a group.

The group composed of the transformations (1) or (2) is called simply the modular group, and is denoted by Γ. The two forms (1) and (2) are distinguished as the non-homogeneous and the homogeneous groups Γ respectively. We note that under the condition $\alpha\delta - \beta\gamma = +1$, a simultaneous change of sign of all the coefficients is admissible, but that the coefficients cannot otherwise be multiplied by a common factor. The change of sign is of no effect on the form (1), but alters the form (2). It appears therefore that the operations of (2) are precisely twice as numerous as those of (1).

The modular group has itself a great variety of subgroups, and it is precisely the theory of these subgroups which determines the formal character of the entire theory of the modular functions. The problem of establishing all these subgroups presents extreme difficulties and is not yet solved. Much is to be hoped, however, from the powerful general method of attacking the subject, devised by Klein and based on the theory of Riemann's surfaces.* The known subgroups are, with a few elementary exceptions, the "congruence groups," and their theory is exhaustively developed in the *Modulfunctionen*. These groups are defined by the additional condition that

$$(5) \quad \alpha \equiv \delta \equiv \pm 1, \quad \beta \equiv \gamma \equiv 0, \quad (\text{mod. } n),$$

where n is any integer.† Under this condition we have, referring again to (3),

$$\begin{matrix} \alpha_1\alpha + \beta_1\gamma \equiv \gamma_1\beta + \delta_1\delta \equiv \pm 1, \\ \alpha_1\beta + \beta_1\delta \equiv \gamma_1\alpha + \delta_1\gamma \equiv 0, \end{matrix} \quad (\text{mod. } n),$$

from which the group character is verified. In the case of the non-homogeneous transformations it is plainly sufficient to employ only the upper algebraic sign of ± 1. It is also a fact of great interest that those substitutions of the homoge-

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* *Modulfunctionen*, II., 5.

† More correctly, this is the definition of the *Hauptcongruenzgruppen*. For other cases *cf. Modulfunctionen*, II., 7, § 6.

neous congruence group for which the upper sign holds form a subgroup of the latter, which therefore agrees operation for operation with the non-homogeneous group.

Of the various characteristics of a group its *order*, *i.e.* the number of operations which it contains, is of prime importance. In the present case both the modular group and the congruence subgroups are of infinite order, and the question therefore presents itself here in a modified form, *viz.* it requires the determination of the ratio of the order of the entire group to that of the respective subgroups. This ratio is termed the "index" of the subgroup and for the modular n is denoted by $\mu(n)$, the corresponding subgroup being designated by $\Gamma_{\mu(n)}$. The value of the function $\mu(n)$ is deducible from purely arithmetical considerations. If we define as congruent (mod. n) all those transformations for which either of the relations hold

$$(6)\quad \begin{matrix} \alpha' \equiv \alpha, & \beta' \equiv \beta, & \gamma' \equiv \gamma, & \delta' \equiv \delta, \\ \alpha' \equiv -\alpha, & \beta' \equiv -\beta, & \gamma' \equiv -\gamma, & \delta' \equiv -\delta, \end{matrix} \quad (\text{mod. } n),$$

the value of $\mu(n)$ is equal to the number of incongruent (mod. n) systems of solutions of

$$\alpha\delta - \beta\gamma = +1.$$

If n is a prime number p, it is readily found that

$$\mu(n) = p\,\frac{(p^2 - 1)}{2}.$$

For a compound $n = q_1^{\nu_1} \cdot q_2^{\nu_2} \cdot q_3^{\nu_3} \ldots$ the calculation is more complicated. The result is found to be*

$$(7)\qquad \mu(n) = \frac{n^3}{2}\left(1 - \frac{1}{q_1^2}\right)\left(1 - \frac{1}{q_2^2}\right)\left(1 - \frac{1}{q_3^2}\right)\ldots$$

Leaving the specific theory of the modular group at this point for the moment, we have next to consider the position of the present investigations relatively to the general theory of linear transformation.† If we regard n elements z_1, z_2, ... z_n as coordinates in an $(n-1)$-dimensional space or manifoldness, the projective geometry of this space is identical on the formal side with the theory of the general group of linear transformations

* *Modulfunctionen*, II., 7, § 4.
† *Cf. Ikosaeder*, Chap. V.

$$(8)\qquad \begin{aligned} z'_1 &= a_{11} z_1 + a_{12} z_2 + \ldots + a_{1n} z_n, \\ z'_2 &= a_{21} z_1 + a_{22} z_2 + \ldots + a_{2n} z_n, \\ &\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ , \\ z'_n &= a_{n1} z_1 + a_{n2} z_2 + \ldots + a_{nn} z_n. \end{aligned}$$

It is a principal problem of this theory to determine the full system of invariant, covariant, and other concomitant forms belonging to any configuration of the space, defined by any given set of equations

$$\begin{aligned} f_1(z_1, z_2, \ldots z_n) &= 0, \\ f_2(z_1, z_2, \ldots z_n) &= 0, \\ &\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ , \\ f_n(z_1, z_2, \ldots z_n) &= 0. \end{aligned}$$

Prominence is also given to the determination of the identities which may exist among these concomitants. The theory of the subgroups of the general group of transformations is, however, not usually considered.

On the other hand, the theory of substitutions deals with the permutations (substitutions) of n given elements $z_1, z_2, \ldots z_n$. Such a substitution is commonly and most conveniently written in the cycle notation

$$(z_a\, z_b\, z_c \ldots)\ (z_\alpha\, z_\beta\, z_\gamma \ldots) \ldots$$

the effect of the substitution being precisely to permute the elements of each parenthesis cyclically. Written, however, in the form

$$z'_1 = z_{i_1},\quad z'_2 = z_{i_2},\ \ldots\ z'_n = z_{i_n},$$

(where the subscripts $i_1, i_2, \ldots i_n$ are identical, apart from their order, with $1, 2, \ldots n$), the substitution is obviously interpretable as a collineation of an $(n-1)$–dimensional space. From this point of view, we may regard the theory of substitutions as a special field within a general projective geometry; in other words, the groups of substitutions of n elements may be considered as subgroups of the general group of linear transformations of n coordinates. An important characteristic of the substitution groups is the fact that they are all of finite order, the latter being, in fact, always a divisor of $n!$ On the other hand, every substitution group, like the general linear group, possesses a system of invariants. These are the "functions belonging to the group," *i.e.* such rational integral functions of the n elements as are unchanged in value by all and by only the substitutions of the group. The invariants belonging to any substitution group G are all rational integral functions of any arbitrary one among them, with coefficients which are symmetrical in the n elements z. Every such group possesses, therefore, only a single independent invariant.

Again suppose H to be any subgroup of G with an invariant ψ. If all the substitutions of G are applied to ψ, the latter will take a series of values ψ_1 $(= \psi)$, ψ_2, ψ_3, ... ψ_k, their number k being the (always integral) ratio of the order of G to that of H. To every one of these values belongs a group of the same order as H and similar to H. These k groups are the "conjugates" of H with respect to G. In special cases they may all coincide; H is then a "self-conjugate" group (*ausgezeichnete Untergruppe*), and every ψ is a rational function of every other. A case of especial importance is that for which H reduces to the identical operation alone. H is then obviously self-conjugate, since it is the only group of order 1. In the general case, if we apply the substitution of G to the k values ψ, the effect is simply a permutation of these values. In the particular case where $H = 1$ we have then on the one hand a group of substitutions of the ψ's of the same order as G, and on the other, as an algebraic equivalent, an equal number of rational processes by which the ψ's proceed from one another. These processes also form a group. By a proper choice of ψ, it happens in certain cases that these rational relations become linear. The group G, which originally represented a system of collineations in an $(n - 1)$-dimensional space, appears then under a new form as identified with an equal group of linear transformations in the complex plane. Such a reduction in the dimension (as it may be called) of the group is obviously a step of the greatest importance. In the *Ikosaeder* and the *Modulfunctionen* this problem is reversed, the groups being taken at the start in their reduced form.*

Within the general theory of groups of linear transformations the projective geometry and the theory of substitutions appear as special cases, which possess the advantage of historical precedence and a correspondingly high degree of elaboration. To these Klein has now added in the *Modulfunctionen* a third system of certainly comparable interest and importance. There still remain an unlimited number of other special types which will undoubtedly in the future furnish one of the most fertile fields of mathematical research. The general problem has hardly yet been touched upon. In systematic form it requires the determination of all the binary, then the ternary, quaternary, and higher groups. In each dimension the groups of finite order naturally attract immediate attention. The

* *Cf. Ikosaeder*, I., 4, §§ 3–4, and I., 5, § 5; also *Math. Ann.* XV.

researches of Poincaré, Jordan, and Klein have shown that the number of finite groups is surprisingly small, and this field is accordingly narrowly limited. The problem of the finite binary groups is completely solved in the *Ikosaeder*. Of the finite ternary groups which are not reducible to binary forms, one of order 432 belongs with the theory of the points of inflection of the plane cubic, and has been repeatedly discussed from this point of view. Another of order 168 is treated in the *Modulfunctionen.** The Borchardt quaternary group belongs with the theory of the general equations of the sixth and seventh degrees, as the icosahedron group does with that of the fifth degree.† On the other hand, of the infinite binary groups, which naturally succeed the finite cases, the simplest instance is precisely the present modular group.

Returning to the modular group in particular, we can now, from analogy with the theory of substitutions, state briefly the nature of the problem involved. It requires the determination of all the subgroups of the modular group and of the corresponding invariants, together with the systematic examination of the functional relations between their invariants, particularly when these relations are algebraic. An important distinction from the theory of substitutions lies in the fact that, as the groups under consideration are for the most part of infinite order, their invariants are no longer rational, but belong to the family of integral transcendental functions with linear transformations into themselves. It must also be noted here that the *homogeneous* groups here considered have in every case not one invariant, but three, which are then connected by an identity, precisely as in the *Ikosaeder*. In regard to the entire theory we observe further that from Klein's point of view it is not to be regarded as an isolated subject, but is to be connected as closely as possible with other mathematical fields. It is to be examined from every possible side, and, in particular, it is to be made tangible by the aid of geometric representation.

At the outstart the theory of the elliptic functions is put under requisition. To obtain the invariant of the modular group the direct construction is not necessary. Such an invariant is already at hand. It is known that the periods of the elliptic integral of the first species, which we write throughout in the Weierstrass form

$$u = \int \frac{dz}{\sqrt{4z^3 - g_2 z - g_3}},$$

* *Modulfunctionen*, III., 7.

† KLEIN, *Math. Ann.* XXVIII., *Zur Theorie der allgemeinen Gleichungen sechsten und siebenten Grades.*

are all linear combinations of two among them, ω_1 and ω_2,

$$\omega'_1 = \alpha\omega_1 + \beta\omega_2,$$
$$\omega'_2 = \gamma\omega_1 + \delta\omega_2.$$

Again ω_1 and ω_2 can be expressed in the same way in terms of ω'_1 and ω'_2 if and only if $\alpha\delta - \beta\gamma = \pm 1$. Accordingly all the systems

$$\begin{matrix} \alpha\omega_1 + \beta\omega_2 \\ \gamma\omega_1 + \delta\omega_2 \end{matrix} \qquad (\alpha\delta - \beta\gamma = \pm 1)$$

furnish primitive period pairs. If we consider the ratio of such a pair, all such ratios are expressed in terms of any one of them by

$$\omega' = \frac{\alpha\omega + \beta}{\gamma\omega + \delta} \qquad \left(\alpha\delta - \beta\gamma = \pm 1,\ \omega = \frac{\omega_1}{\omega_2},\ \omega = \frac{\omega'_1}{\omega'_2}\right).$$

On the other hand, we have at once in the three invariants g_2, g_3, and the discriminant $\Delta = g_2^3 - 27g_3^2$ of the binary biquadratic form $4z_1^3z_2 - g_2z_1z_2^3 - g_3z_2^4$ the three invariants of the homogeneous modular group, while the absolute invariant $J = \frac{g_2^3}{\Delta}$ is the invariant of the non-homogeneous group. The periods ω_1, ω_2 and their ratio ω are also invariants of the biquadratic form, the latter being like J an absolute invariant. In reference to the modular group these quantities are again invariants belonging to the identical subgroup. Their calculation from g_2, g_3 and from J respectively are already furnished by the theory of the elliptic functions. Of the congruence subgroups (5) an invariant is also directly known in the case of modulus 2. This is the anharmonic ratio λ of the roots of the biquadratic form, or for the homogeneous group the three finite roots e_1, e_2, e_3 (where, in agreement with the general principle above stated, $e_1 + e_2 + e_3 = 0$). Under the operation of the modular group λ assumes the six familiar values

$$\lambda,\ \frac{1}{\lambda},\ 1-\lambda,\ \frac{\lambda}{\lambda-1},\ \frac{\lambda-1}{\lambda},\ \frac{1}{1-\lambda},$$

which furnish again a linear group. The latter is in fact a dihedron group. The six values of λ are connected with J by the equation

$$J : J-1 : 1 = 4(\lambda^2 - \lambda + 1)^3 : (\lambda + 1)^2(\lambda - 2)^2(2\lambda - 1)^2 : 27\lambda^2(\lambda - 1)^2.$$

To obtain a comprehension of functional relations which in the present state of advancement can be regarded as in any way satisfactory, recourse must be had to Riemann's surfaces. Perhaps no more brilliant exemplification of the value of this geometric instrument exists than the theory of the modular functions, as Klein has created it.* Beginning with the period ratio ω, regarded as a function of J, we suppose the values of ω to be laid off in one complex plane and those of J in another. Since however to every value of J correspond an infinite number of primitive period pairs, and consequently an infinite number of values of ω, we must, in order to secure a one to one correspondence between the ω and the J points, suppose the J plane to consist of an infinite number of leaves which are connected in cycles about certain junctions (*Verzweigungspunkte*). In the present case, as in the *Ikosaeder*, these junctions are three in number and lie at $J = 0$, $J = 1$, and $J = \infty$. The leaves are joined at these points in cycles of three, two, and an infinite number respectively. If now we suppose the J point to pass along the upper side of the real axis from $- \infty$ to $+ \infty$, the corresponding ω points describe in every case three circular arcs bounding a curvilinear triangle. To every upper half leaf of the J plane corresponds the interior of one of these triangles, in the sense that if the ω point takes successively every position in such a triangle, the corresponding J point will take successively every position in an upper half leaf. The ω triangles then produced fill just half of the upper half leaf of the ω plane. Between them lie an infinite number of empty spaces, of the same triangular form, and these new triangles correspond to the lower half leaves of the J plane. The triangles become infinitely small and are crowded infinitely close together as we approach the real axis, which is in fact a "natural boundary" beyond which the function $\omega(J)$ cannot be extended.

An immediate connection presents itself here with the theory of the modular group.† The effect of the latter on the systems of triangles is obviously merely to interchange the two sets corresponding to upper and lower half leaves each among themselves. Given any one of the triangles we can obtain every other belonging to the same system by applying to the former all the modular operations. We observe that it is only those operations

$$\omega' = \frac{\alpha\omega + \beta}{\gamma\omega + \delta}$$

for which $\alpha\delta - \beta\gamma = + 1$ that are here admissible; those

* *Cf. Modulfunctionen*, II. † *Ibid.*, II., 2.

for which $\alpha\delta - \beta\gamma = -1$ simply convert the upper half leaf of the ω plane into the similarly divided lower half leaf. This again agrees with the fact that the product of two of the latter operations belongs not to these but to the modular group.

Having now obtained a means of generating all the triangles of either system from a single one among them, the question naturally presents itself how the one system can be obtained from the other. We have seen that the real axis of the J plane corresponds to the circular arcs bounding an ω triangle. By a linear transformation $\omega' = \dfrac{a\omega + b}{c\omega + d}$ we can convert any circle in the ω plane, for example, one of the sides of the ω triangle into the real ω axis. The function $\omega'(J)$ has then an infinite series of real values corresponding to real values of J. Consequently conjugate imaginary values of J correspond to conjugate imaginary values of $\omega'(J)$. The ω' triangle corresponding to the lower J half leaf is therefore the reflection of that corresponding to the upper J half leaf on the ω real axis. Retransforming now to the original ω triangle, the reflection becomes the operation of ***inversion*** on the corresponding circular boundary; *i.e.* given the one triangle, if we construct the inverse of every point within it with respect to one of its sides we have a triangle of the second system. In fact every triangle of the ω plane can be obtained from any one by a series of such reflections on bounding lines.

The preceding considerations furnish an entirely new point of departure for the present and for a more general theory. We may suppose any triangle bounded by circular arcs to be *a priori* given, as corresponding to a half plane of the companion Riemann's surface, the analytic connection of the two variables being for the present purpose left out of immediate consideration. From the given triangle we then construct all possible others by the operation of reflection. In order that these may just fill the complex plane without overlapping, the angles of the given triangle must all be submultiples of 2π: $\dfrac{2\pi}{\nu_1}, \dfrac{2\pi}{\nu_2}, \dfrac{2\pi}{\nu_3}$. Three cases are distinguished according as

$$\frac{2\pi}{\nu_1} + \frac{2\pi}{\nu_2} + \frac{2\pi}{\nu_3} \lesseqgtr 2\pi, \; i.e. \; \frac{1}{\nu_1} + \frac{1}{\nu_2} + \frac{1}{\nu_3} \lesseqgtr 1.$$

In the case $\dfrac{1}{\nu_1} + \dfrac{1}{\nu_2} + \dfrac{1}{\nu_3} > 1$ only a small number of systems of integral values ν_1, ν_2, ν_3 are possible. These lead to the ***finite*** linear groups. The number of triangles is in this case also finite, and they cover the entire plane. On the other hand, if

$\frac{1}{\nu_1} + \frac{1}{\nu_2} + \frac{1}{\nu_3} < 1$ an infinite number of solutions are possible. The three circular boundaries of the given triangle have in this case a common real orthogonal circle. This circle is moreover the orthogonal circle of every triangle of the system. The latter all lie within this circle and are crowded more and more closely together as they approach the circumference. In the case of the modular group the circumference is exactly the real axis. This group is distinguished among other types by the criterion that $\nu_1 = 2$, $\nu_2 = 3$, $\nu_3 = \infty$. The remaining case, where the sum of the three angles of the triangle is 2π, leads to the theory of the periods of the elliptic function.*

Turning now to the subgroups of the modular group, we observe that these too have in each case a "fundamental domain" (*Fundamental-Bereich*). This is composed of a system of the ω triangles equal in number to the index of the group. This fundamental region, like the double ω triangle, has the property that from its points every other point in the complex plane can be obtained by the operations of the corresponding subgroup, and that it is the smallest region which has this property. Every ω triangle can be converted by the subgroup into one and only one triangle of the fundamental region. If now we suppose every ω triangle to be represented by its "equivalent" triangle in the fundamental region, the effect of all the operations of the modular group is simply to permute these representative triangles among themselves. These permutations again form a group, the group $G_{\mu(n)}$.† In accordance with the entire tendency of the subject, the question at once presents itself whether quantities referred to the several triangles of the fundamental region can be found such that the group $G_{\mu(n)}$ transforms them linearly. This is actually the case, and in fact for $n = 2, 3, 4, 5$ the corresponding linear groups are identical with the dihedron group of order 6, the tetrahedron, the octahedron, and the icosahedron groups respectively.

For these cases, which exhaust the possibility of binary groups, the "deficiency" of the fundamental region is 0. For the next important case $n = 7$, the deficiency is 3 and the corresponding linear group is ternary. It is in fact the group repeatedly treated by Klein in connection with the Gordan plane curve of the fourth order ‡

$$x_1^3 x_2 + x_2^3 x_3 + x_3^3 x_1 = 0.$$

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* *Cf.* throughout *Ikosaeder*, I., 5 and *Modulfunctionen*, I., 8.
† *Modulfunctionen*, II., 4, §6.
‡ *Ibid.*, III., 7.

With this brief and imperfect account we must now regretfully leave the subject, consoling ourselves with the reflection that Dr. Fricke's book contains in itself that which will most certainly attract deserved attention to this most beautiful of Klein's creations.

F. N. COLE.

ANN ARBOR, *December* 30, 1891.

PERTURBATIONS OF THE FOUR INNER PLANETS.

Periodic Perturbations of the Longitudes and Radii Vectores of the Four Inner Planets of the First Order as to the Masses. Computed under the direction of SIMON NEWCOMB. Washington, Navy Department, 1891; 4to, pp. 180.

THIS work forms the concluding part of volume III. of a series of astronomical researches, published under the general title, "*Astronomical Papers, prepared for the use of the American Ephemeris and Nautical Almanac.*"

During the past twelve years, one of the principal works which has been in progress at the office of the Nautical Almanac is that of collecting and discussing data for new tables of the planets. The most recent existing tables, which are now used in all European Ephemerides, are those of Leverrier, the construction of which was the greatest work ever undertaken by that celebrated astronomer. The first tables published, those of the Sun, were issued in 1858; those of Uranus and Neptune appeared about 18 years later. The whole work probably took about 25 years in preparation and publication. Yet the number of observations on which the tables were actually based was only a few hundred in the case of each planet, about 500 being used for Venus, 800 for Mars, and probably yet fewer in the cases of the other planets. The results were not completely discussed, and, in consequence, different data were employed in different tables, making it extremely difficult for future astronomers to derive the results of comparing them with future observations. None except those of the Sun and Mercury, which were the first issued, have shown a satisfactory agreement with subsequent observations. The error in the geocentric place of Venus at the time of the recent transit was surprisingly great, amounting to no less than nine seconds in longitude.

The actual number of observations now available for each of the principal planets is several thousand. The recent ones are of course better than those available thirty years ago. It therefore seemed desirable to undertake the construction of tables founded on all these observations which could be of value, and on uniform values of the masses of the planets and other elements.

As it was necessary to determine the masses from the periodic perturbations, the first requisite was a determination of the coefficients of these perturbations which should be beyond doubt. Although Leverrier's computations of these coefficients were carried out more fully than those of any of his predecessors, some doubts of their entire accuracy had been expressed. In such intricate computations, which necessarily proceed by successive approximations, and can never pretend to mathematical rigor, the possibility of sensible quantities being omitted can be avoided only by independent computations by different investigators using different methods. The present paper is entirely devoted to the computation of these coefficients. The adopted developments are so radically different from those of Leverrier that there can be no source of error common to the two. The agreement throughout may be called perfect, when compared with the probable error of the best observations. Rarely does a discrepancy amount to the hundredth of a second of arc.

The principal point in which the development differed from that of Leverrier is, that the eccentric anomaly is used, in the beginning, as the independent variable. In this way the series are made, in the first place, more rapidly convergent, and it is thus more easy to be sure of including all sensible terms. The use of this method requires, however, that the eccentric anomalies be changed to mean anomalies by the Besselian transformation. It was supposed that this transformation was one which could be effected with ease and rapidity. But in practice it proved so laborious that it is now doubtful whether the terms saved in the development will compensate for the labor of applying it.

A more radical change from Leverrier's method is, that the perturbations are computed by direct integration of the differential equations of motion, instead of employing the method of variation of elements. Notwithstanding the theoretical elegance of the latter method as developed by Lagrange, it becomes excessively prolix when we attempt to compute the periodic perturbations by it. But when the equations are directly integrated, the coefficients admit of being found with great facility, when once the development of two derivatives of the perturbative function in terms of the mean anomalies is effected. Altogether the method is a combination of the purely numerical process of development employed by Hansen, and the purely analytic one employed by Leverrier.

It is still a question whether the adopted method was actually the shortest, and whether much labor would not have been saved by employing the purely numerical development from the beginning.

The volume of which the above paper forms a part is wholly devoted to the developments of celestial mechanics. The opening paper is the development of the perturbative function in sines and co-sines of multiples of the eccentric anomaly which was employed in computing the perturbations.

This is followed by a determination of the inequalities of the Moon's motion due to the figure of the Earth, prepared by G. W. Hill. This is the most elaborate determination of these difficult inequalities that has ever been made, no less than 165 terms in the Moon's longitude, and yet more in the latitude, being computed. Nearly half the computed terms are, however, entirely insensible, even in the fourth place of decimals of seconds.

The third paper is on the motion of Hyperion, the seventh satellite of Saturn. In it is developed the theory of the curious relation between the mean motions of Hyperion and Titan, which H. Struve has since extended to one or more of the inner satellites.

This is followed by another paper by Mr. Hill, being a computation of certain lunar inequalities due to the action of Jupiter. The inequality in question was first discovered empirically from observations, and was traced by Mr. Neison to a sort of evection due to the action of Jupiter. Mr. Hill's coefficient is, however, only 0''.90, while observations gave 1''.50. Probably the theory is more nearly correct, as the uncertainty of observations of the Moon is much greater than in the case of other heavenly bodies, and it is difficult to separate the effects of an inequality of this kind from those of numerous other causes affecting the observations.

It is now expected that the tables of the four inner planets which are founded on the theories developed in the Astronomical Papers, and on the great mass of observations made since 1750, will be ready for the press in a little more than two years.

S. N.

MATHEMATICAL PROBLEMS.

Solution of Questions in the Theory of Probability and Averages. Appendix II. to *Mathematical Questions and Solutions from the Educational Times*, Vol. LV. By Professor G. B. ZERR, M.A.

THIS pamphlet of fifty-six pages contains solutions of more than forty problems in geometrical probability and mean values and of some other interesting mathematical problems. The solver was also the proposer of most of the problems. His solutions show skill and perseverance in evaluating many complicated definite multiple integrals.

Several problems relate to mean values of magnitudes determined by choosing random points with certain restrictions in a circle. For example, No. 11,153 is to find the average area of the dodecagon formed by joining twelve points taken at random in a circle, three in each quadrant. The expression found for this area is the quotient of two multiple integrals of twelve variables each. This is finally reduced to

$$\frac{2^{14} r^2}{15\pi^6}\left(\frac{\pi^2}{16} - \frac{704}{1575}\right)\left(2\pi - \frac{409}{105}\right) + \frac{2^7 r^2}{\pi^3}\left(\frac{5\pi^2}{64} - \frac{29}{45}\right) + \frac{2^{12} r^2}{3\pi^6}\left(\frac{\pi^2}{32} - \frac{86}{315}\right)\left(\frac{7\pi}{2} - \frac{853}{105}\right)$$

Problem 11,037 is as follows :—"Two points are taken at random in the surface of a given circle. An ellipse is described on the distance between the two points as major axis. If a point be taken at random in the left-hand half of this major axis, and with this point as a centre a circle is described at random, but so as to lie wholly within the ellipse, find the average area of the ellipse described on that portion of the major axis between the right-hand extremity and the circumference of the random circle." The result obtained is

$$\frac{\pi r^2}{1280}\left(\frac{2205\pi + 2012}{15\pi + 17}\right).$$

This solution involves the assumption that the major axis a of an ellipse being given all possible ellipses should be included by taking the unknown minor axis as an independent variable with the limits 0 and a. All possible ellipses might with equal propriety be included in other ways; as, by taking the eccentricity as the independent variable with the limits 0 and 1. The result would be altered by such a change.

Problem 11,130 implies an assumption of the same nature.

"A chord is drawn at random across a circle, and two points are taken at random within the circle; find the chance that both points lie on the same side of the random chord." The result $1 - \frac{128}{45\pi^2}$, is obtained by treating the distance of the chord from the centre as the independent variable. Why would it not be equally proper to take the arc subtended by the chord as the independent variable?

These problems, as stated, are indeterminate. The modes of including all possible ellipses and of choosing random chords must be fixed before the problems become definite.

It seems strange that an incorrect construction should be given for so elementary and easy a geometrical problem as No. 10,512. "Given four lines (in magnitude), construct two similar triangles each of which shall have two of the given lines as sides."

EDWARD L. STABLER.

NEW YORK, *December* 10, 1891.

NOTES.

THE annual meeting of the NEW YORK MATHEMATICAL SOCIETY was held Wednesday afternoon, December 30, at four o'clock, Professor Van Amringe presiding. The following persons having been duly nominated, and being recommended by the council, were elected to membership: Mr. Edwin Mortimer Blake, Columbia College; Professor Mary E. Byrd, Smith College; Professor Susan J. Cunningham, Swarthmore College; Mr. A. E. Kennely, Edison Laboratory; Mr. Alexander Kinseley, Lafayette, Ind.; Professor Anthony T. McKissick, Alabama Polytechnic Institute; Professor George D. Olds, Amherst College; Professor M. L. Pence, State College of Kentucky; Miss Amy Rayson, New York, N. Y.; Professor Benjamin Sloan, South Carolina College. The secretary reported that the membership of the Society was 210, of whom 37 lived in New York city and the immediate vicinity and were able to attend the meetings regularly. The treasurer's report having been read, an auditing committee was appointed to examine his accounts.

The nominating committee reported the following ticket for the officers and council of the Society for the ensuing year:—President, Dr. Emory McClintock; Vice-President, Professor Henry B. Fine; Treasurer, Mr. Harold Jacoby; Secretary, Dr. Thomas S. Fiske; other Members of Council, Professor J. K. Rees, Professor W. Woolsey Johnson, Professor

J. E. Oliver, Professor J. H. Van Amringe, Professor Thomas Craig. A ballot being taken this ticket was unanimously elected. At the invitation of the chair, Mr. R. S. Woodward briefly addressed the Society, referring to its work and aims, and expressing his hopes for its success and prosperity.

CONDENSED REPORT OF THE TREASURER FOR THE YEAR 1891.

Receipts.		*Expenditures.*	
Balance from 1890	$20.80	Printing Constitution......	$41.52
Net receipts from members and subscribers.........	974.24	" Bulletin...........	368.19
		Circulars, stationery, etc....	195.55
		Postage and miscellaneous..	118.78
		Balance.................	271.00
	$995.04		$995.04

HAROLD JACOBY, *Treasurer*.

We have examined the treasurer's accounts, and found the same correct.

G. L. WILEY,
T. E. SNOOK,
JAS. MACLAY, } *Auditing Committee.*

A REGULAR meeting of the NEW YORK MATHEMATICAL SOCIETY was held Saturday afternoon, January 2, at half-past three o'clock, the president in the chair. Professor D. S. Jacobus, of the Stevens Institute of Technology, having been duly nominated, and being recommended by the council, was elected a member. The following original papers were read: "Application of least squares to the development of functions," by Mr. Frank Gilman; "On the computation of covariants by transvection," by Dr. Emory McClintock. In Mr. Gilman's paper a method was given for finding a rational entire algebraic function of the n-th degree with numerical coefficients, which should approximately represent the value of a given function between certain limits of the variable, and which should furnish, in general between these limits, more accurate results than the first $n+1$ terms of its ordinary expression as a power-series. The numerical coefficients of the approximation were determined from the true values of the function calculated for values of the variable uniformly distributed between the limits, by the principle that the sum of the squares of its residuals should be a minimum. Dr. McClintock's paper contained an account of the general method for the computation of covariants of which a simple example, illustrating a special case, was given at the end of his article "On lists of covariants," published in No. 4 of the *Bulletin*, pp. 85–91. T. S. F.

In connection with Professor A. S. Hathaway's article "Early history of the potential" in No. 3, pp. 66–74, it may be remarked that the mistake of ascribing the discovery of the fundamental property of the force-function, or potential, to Laplace instead of Lagrange is a common one. In addition to the places mentioned by Professor Hathaway it is retained in the second edition of Maxwell's Electricity and Magnetism, vol. I. (1881), p. 14, and in the new edition of Thomson and Tait's Natural Philosophy, vol. I., part II. (1883), p. 28. Attention was called to this mistake by R. Baltzer in his note "*Zur Geschichte des Potentials,*" in Crelle-Borchardt's Journal, vol. 86 (1879), p. 216. The matter is also discussed in E. Heine's *Kugelfunctionen,* second edition, vol. II. (1881), p. 342, and in M. Bacharach's *Geschichte der Potentialtheorie* (1883), pp. 4–6.

None of these authors, however, mention the memoirs of Lagrange preceding that of October 2, 1777; so that it is of no little interest to see the first idea of the property of the force-function traced back in his writings to as early a date as 1763. Professor Hathaway's reference to Cayley's British Association Report for 1862 must be due to some oversight. The matter is not discussed there, nor is there any reference to Lagrange's memoir *Sur l'équation séculaire de la lune,* of 1773. A. Z.

In regard to the preceding note I have to state that Cayley's report on dynamics to which I intended to refer is in the *British Association Report* for 1857, p. 3. Besides the reference to Maxwell given by A. Z. there is another to page 74, where the error is repeated. A note just received from Professor P. G. Tait with reference to *nabla,* which is the quaternion vector-operation ∇, and not $-\nabla^2 = \frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2}$, encloses a copy of his address "On the importance of quaternions in physics," *Philosophical Magazine,* January, 1890, p. 92. We quote: "Hamilton did not, so far as I know, suggest any name. Clerk Maxwell was deterred by their vernacular signification, usually ludicrous, from employing such otherwise appropriate terms as *sloper* or *grader;* but adopted the word *nabla,* suggested by Robertson Smith from the resemblance of ∇ to an ancient Assyrian harp of that name." A. S. H.

John Wiley & Sons have in preparation "An elementary course in the theory of equations" by Dr. C. H. Chapman of Johns Hopkins University.

Leach, Shewell & Sanborn have just published "A treatise on plane and spherical trigonometry" by Professor E. Miller of the University of Kansas. T. S. F.

NEW PUBLICATIONS.

COMPILED BY B. WESTERMANN & CO., NEW YORK.

ALASIA (C.). Elementi della Teoria generale delle Equazioni ed in particolare delle Equazioni di 3. e 4. grado e delle Equazioni indeterminate. Napoli 1891. 8. 160 pg. M. 3.00

BOCHER (M.). Ueber die Reihenentwicklungen der Potentialtheorie. Göttingen 1891. 4. 66 pg.—Gekrönte Preisschrift. M. 2.00

BRAINARD (F. C.). The Sextant and other reflecting mathematical Instruments. New York 1891. 12mo. 120 pp. cloth. M. 2.50

BRISSE (Ch.). Recueil de Problèmes de Géométrie analytique à l'usage des classes de Mathématiques spéciales. 2. édition. Paris 1892. 8. 226 pg. av. 62 figures. M. 4.30

CHICOURAS (F.). Etude sur la solution du problème de la Quadrature du Cercle. Montpellier 1891. 8. 39 pg. M. 2.00

COLLET (A.). Navigation astronomique simplifiée. Paris 1891. 4to. 33 pp. and 9 plates. M. 8.50

DAVIS (J. W.). Theoretical Astronomy. Dynamics of the Sun. New York 1891. 4. w. illustrations. M. 16 00

DEMARTRES. Cours d'Analyse. Partie I. Fonctions de variables réelles. Lille 1891. 4. 194 pg. lithographiées. M. 9.00

DIVIC (F.). Die sieben Rechnungsoperationen mit allgemeinen Zahlen. Wien 1891. gr. 8. 168 pg. m. 33 Holzschnitten. M. 3.60

DUHEM (P.). Cours de Physique mathématique et de Cristallographie de la Faculté des Sciences de Lille. Leçons professées en 1890–91. Tome II. Paris 1891. 4. 4 et 310 pg. lithographiées. M. 12.00

DYER (J. M.) and WHITCOMBE (R. H.). Elementary Trigonometry. Cr. 8vo. pp. 260. Bell and Sons. 4s. 6d.

EVRARD (J.). Mémoire sur l'Interprétation des Symboles dits imaginaires ou Théorie des Acceptions (extrait principalement des travaux inédits de feu l'abbé George). Paris 1891. gr. in-8. 22 et 239 pg. av. figures.

FENKNER (H.). Lehrbuch der Geometrie. Mit einem Vorworte von W. Krumme. 2., verbesserte u. vermehrte Auflage. Theil I. Braunschweig 1892. gr. 8. 8 u. 178 pg. m. Figuren. M. 2.00

FIALKOWSKI (N.). Kurzgefasste praktische Geometrie. 2. Auflage. Wien 1891. gr. 8. 8 u. 134 pg. m. 183 Holzschnitten. M. 2.40

FISCHER (W.). Netze zur Herstellung geometrischer Körper für das perspectivische und projectivische Zeichnen. Dresden 1891. gr. quer fol. 39 Tafeln auf Carton. In Mappe. M. 10.00

GRAVELIUS (H.). Vierstellige logarithmisch-trigonometrische Tafeln für die Decimaltheilung des Quadranten. Berlin 1891. gr. 8. 64 pg. M. 1.50

HOUEL (G. J.). Fünfstellige Logarithmentafeln der Zahlen und der trigonometrischen Functionen, nebst den Gaussischen Additions und Subtractions logarithmen und verschiedenen Hülfstafeln. Neue Ausgabe. Berlin 1891. 8vo. pg. 46 + 118. M. 2.00

JACOBI (C. G. J.). Gesammelte Werke. Herausgeg. auf Veranlassung der Kgl. Preuss. Academie der Wissenschaften. Bd. VII : Geometrie, Astronomie, Abhandlungen historischen Inhalts. Briefe an Bessell und Gauss. Verzeichniss sämmtl. Abhandlungen Jacobi's. Herausgeg. von K. Weierstrass. Berlin 1891. 4to. 440 pp. M. 14.00

MANSION (P.). Theorie der partiellen Differentialgleichungen erster Ordnung. Vom Verfasser durchgesehene u. vermehrte deutsche Ausgabe, herausgegeben von H. Maser. Berlin 1892. gr. 8. 21 u. 489 pg. M. 12.00

MÉRAY (C.). Théorie des radicaux fondée exclusivement sur les propriétés générales des séries entières. Dijon 1890. 8vo. 75 pg. M. 1.80

MÜLLER (E. R.). Lehrbuch der planimetrischen Constructionsaufgaben, gelöst durch geometrische Analysis. Nach System Kleyer bearbeitet. Theil II. Stuttgart 1892. gr. 8. 6 u. 160 pg. m. 174 Figuren. M. 4.00

PEANO (G.). Die Grundzüge des geometrischen Calculs. Deutsche Ausgabe von A. Schepp. Leipzig 1891. 8. 38 pg. M. 1.20

SCHEFFLER (H.). Beiträge zur Zahlentheorie, insbesondere zur Kreis- und Kugel-theilung. Leipzig 1891. gr. 8. 11 u. 285 pg. m. 8 Tafeln. M. 6.00

SCHLÖMILCH (O.). Fünfstellige logarithmische und trigonometrische Tafeln. 4. Auflage. Braunschweig 1891. 8. 26 u. 171 pg. M. 2.00

SCHMIDT (A.). Die Strahlenbrechung auf der Sonne. Ein geometrischer Beitrag zur Sonnenphysik. Stuttgart 1891. 8. m. Abbildungen. M. 1.00

SOUILLAGUET (F.). Tables de point auxiliaire pour trouver rapidement la hauteur et l'azimuth estimés. Paris 1891. gr. in-8. M. 11.00

STERNER (M.). Principielle Darstellung des Rechenunterrichts auf historischer Grundlage. Theil I. München 1891. 8. 12 u. 533 pg. M. 6.00

STOURDZA (le prince Grigori). Les Lois fondamentales de l'univers. Gr. in-8. Baudry. 8 fr.

VILLICUS (F.). Die Geschichte der Rechenkunst vom Alterthume bis zum 18. Jahrhundert mit besonderer Rücksicht auf Deutschland und Oesterreich. 2., vermehrte u. verbesserte Auflage. Wien 1891. Lex. 8. 112 pg. m. zahlreichen Abbildungen. M. 2.80

VONDERLINN (J.). Lehrbuch des Projectionszeichnens. Nach System Kleyer bearbeitet. Theil III. 2. Hälfte. Stuttgart 1892. gr. 8. 8 u. 235 pg. m. 210 Figuren. M. 5.00

WALTER (B.). Ueber die lichtverzögernde Kraft gelöster Salzmoleküle und ein Verfahren zur genaueren Bestimmung von Brechungsexponenten. Jena 1891. 8. 36 pg. M. 1.20

CONTENTS.

Articles for insertion should be addressed to the
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THE MECHANICAL AXIOMS OR LAWS OF MOTION.

BY PROF. W. WOOLSEY JOHNSON.

THE three laws of motion were called by Newton *Axiomata, sive Leges Motus*. Professors Thomson and Tait in their *Natural Philosophy*, section 243, say: "An axiom is a proposition, the truth of which must be admitted as soon as the terms in which it is expressed are clearly understood. But, as we shall show in our chapter on 'Experience,' physical axioms are axiomatic to those only who have sufficient knowledge of the action of physical causes to enable them to see their truth." They then proceed to give Newton's three laws, remarking that "these laws must be considered as resting on convictions drawn from observation and experiment, *not* on intuitive perception."

Whether this be accepted as a proper definition of a physical axiom or not, it is at least desirable to include among the axioms of mechanics the smallest basis of postulated principles upon which it is possible to construct the science by rigid mathematical reasoning.

The laws of motion, in the classic form given them in the *Principia*, admirably express such a basis of postulated principles, although the charge of redundancy has been brought against the first and second laws; but, in the case of the third law, the tendency has been, on the other hand, to make it include too much, and to assume, either under its authority or directly, as axiomatic, principles like the "impossibility of perpetual motion" which ought rather to be shown to follow directly from the three laws of motion.

It is here proposed to discuss the question of the mechanical axioms, beginning with an examination of the laws as presented by Newton.

The first law is that "Every body keeps in its state of rest or of moving uniformly in a straight line, except to the extent in which it is compelled by forces acting on it to change its state."*

Newton had already defined *vis impressa* as an action from without changing a body's state of rest or of moving uniformly in a straight line, † and in the scholium to the *Definitiones* had pointed out that our measure of time depends upon the assumption of the first law of motion; so

* Corpus omne perseverare in statu suo quiescendi vel movendi uniformiter in directum, nisi quatenus illud a viribus impressis cogitur statum suum mutare.

† Definitio IV. Vis impressa est actio in corpus exercita, ad mutandum ejus statum vel quiescendi vel movendi uniformiter in directum.

that with respect to a single body there is a logical "vicious circle." We define those intervals of time as equal in which equal spaces are described when there is no action: we say there is no action when equal spaces are described in equal intervals of time. But the physical truth expressed is that *all* bodies undisturbed by action from without describe equal spaces in the successive intervals in which any one such body describes equal spaces. We then define these intervals as equal, and the action, which in any other body causes a departure from this normal state, as a force.

The second law is that "Change of motion is proportional to the moving force acting, and takes place in the direction of the straight line in which the force acts." *

The simplest form of the physical truth involved is this: — *A given force acting upon a given body produces the same acceleration in its own direction, in equal intervals of time, no matter how the body may be moving, nor what other forces may be acting at the same time.* The law as expressed by Newton involves the obvious deductions that the force is proportional to the acceleration produced in a given interval, and that to produce a given acceleration the force must be proportional to the mass; for motion had been defined as proportional to mass and velocity conjointly. The *vis motrix* is the whole force acting on the body, and is defined as "proportional to the motion which it produces in a given time," † in distinction from *vis acceleratrix* which is defined as proportional to the velocity produced in a given time. These definitions imply the second law, just as those of *vis insita* and *vis impressa* imply the first law. But the essential point is the constancy and independence of the effects of force, which effects are therefore suitable to be taken as measures of the force.

It has been objected that the first law is unnecessary because it is included in the second which implies that *no* force produces *no* change of motion. It appears inevitable that the expression of the second law should thus include the first; but it is nevertheless fitting that the normal state of the body suffering no action from without, a departure from which constitutes the "change of motion" when action takes place, should be stated in a separate axiom. The first law is, in fact, far more axiomatic than the second, or, in the language of the definition quoted above, requires a much smaller knowledge of the action of physical causes to enable one

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* Mutationem motus proportionalem esse vi motrici impressæ, & fieri secundum lineam rectam qua vis illa imprimitur.

† Definitio VIII. Vis centripetæ quantitas motrix est ipsius mensura proportionalis motui, quem dato tempore generat.

to see its truth. It is only necessary to get a clear notion of the absence of force to be in the mental state to admit the truth of the first law: but it requires a considerable familiarity with geometrical notions even to apprehend the manner in which the effect of a force upon a moving body is to be compared with its effect when the body is at rest, and the manner in which the effects of forces acting at an angle with one another are to be separated. Yet clear conceptions in these matters must be obtained before an intelligent assent can be given to the second law of motion.

The most axiomatic proposition involved in the second law is that two opposite equal forces acting upon a body at rest do not produce motion, which in some old treatises is taken as the first proposition in statics, and in others as the definition of the equality of forces.

The third law is that "There is always a reaction opposite and equal to an action: or the actions of two bodies upon one another are always equal and oppositely directed." *

The physical truth expressed is that *every force acting upon a body is the action upon it of another body which in turn is acted upon equally by the first body, the action taking place in the straight line which joins the two bodies.* In other words, the law asserts that no forces exist which consist simply of tendencies in certain directions, as the ancients supposed in the case of the "gravity" of certain bodies and the "levity" of others. This granted, the equality of the two phases of the action follows readily, by the aid of the notion of the transmissibility of force which is intimately connected with this law. Compare Newton's remarks on attraction, quoted below. But the language of some writers concerning the "numerous applications of this law" indicates the view that something more than the equality of pressures is implied in it.

Of the illustrations which in the *Principia* immediately follow the third law, the first is that of a stone pressed by the finger, when in turn the finger is pressed by the stone.† Here there is no intervening body between the two bodies in question. The resistance of the stone to the finger is, by the second law, equal to the force communicated to the finger tip by the muscles because it prevents its motion. By the third law, this is the same as the force communicated by the finger to the stone. If we go a step further and consider the equilibrium of the stone, the resistance of the support upon which it rests is equal to the last named force, and is the reaction of the

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* Actioni contrariam semper & æqualem esse reactionem: sive corporum duorum actiones in se mutuo semper esse æquales & in partes contrarias dirigi.

† Si quis lapidem digito premit, premitur & hujus digitus a lapide.

same force regarded as the action of the stone upon the support. Thus the force is transmitted through the substance of the stone.

In the next illustration, that of a horse drawing a stone attached to a rope, there is a body through which the force is transmitted. "The rope stretched both ways will by the same endeavour to relax itself urge the horse toward the stone and the stone toward the horse; and will impede the progress of the one as much as it promotes the progress of the other." *

The next illustration is from impact. "If any body impinging upon another body, by its own force in any manner changes the motion of that body, it will also in turn suffer in its own motion (on account of the equality of mutual pressure) the same change in the contrary direction." † Here the third law is cited in the clause in parenthesis, and the equality of the actions is indicated by the effects which are produced in accordance with the second law. Newton proceeds in fact to say, "By these actions equal changes are made not of velocities but of motions." These comments on the third law close with the words: "This law holds also in attractions, as will be proved in the next scholium."

The passage in the scholium here alluded to is as follows:—"In attractions I thus briefly show the matter. Any two bodies *A* and *B* mutually attracting each other, conceive some obstacle to be interposed, by which their approach is prevented. If either one of the bodies *A* is drawn more toward the other body *B* than the other *B* toward the first *A*, the obstacle will be urged more by the pressure of the body *A* than by the pressure of the body *B*, and hence will not remain in equilibrium. The stronger pressure will prevail, and will cause the system of the two bodies and the obstacle to move in a straight line in the direction toward *B*, and by an ever accelerated motion in free space to pass to infinity. Which is absurd and contrary to the first law." ‡

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* Funis utrinque distentus eodem relaxandi se conatu urgebit equum versus lapidem, ac lapidem versus equum; tantumque impediet progressum unius quantum promovet progressum alterius.

† Si corpus aliquod in corpus aliud impingens, motum ejus vi sua quomodocunque mutaverit, idem quoque vicissim in motu proprio eandem mutationem in partem contrariam vi alterius (ob æqualitatem pressionis mutuæ) subibit.

‡ In attractionibus rem sic breviter ostendo. Corporibus duobus quibusvis *A*, *B* se mutuo trahentibus, concipe obstaculum quodvis interponi, quo congressus eorum impediatur. Si corpus alterutrum *A* magis trahitur versus corpus alterum *B*, quam illud alterum *B* in prius *A*, obstaculum magis urgebitur pressione corporis *A* quam pressione corporis *B*; proindeque non manebit in æquibrio. Prævalebit pressio fortior, facietque ut systema corporum duorum & obstaculi moveatur in directum in partes versus *B*, motuque in spatiis liberis semper accelerato abeat in infinitum. Quod est absurdum & legi primæ contrarium.

In this passage Newton, so far from making the third law imply anything more than equality of pressures, shows that this equality of the two phases of an action follows from the simple assumption that in a system of bodies preserving their relative positions the mutual actions of any two cannot result in a tendency to motion. The axiomatic portion of the law consists in this assumption. A tendency to motion is in Newton's system always the action of an external body.

We find, however, in Thomson and Tait the following passage: "Of late there has been a tendency to split the second law into two, called respectively the second and third, and to ignore the third entirely, though using it *directly* in every dynamical problem; but all who have done so have been forced *indirectly* to acknowledge the completeness of Newton's system, by introducing as an axiom what is called D'Alembert's principle, which is really Newton's rejected third law in another form. Newton's own interpretation of his third law directly points out not only D'Alembert's principle, but also the modern principles of work and energy."*

In support of this the authors remark further on,† after commenting upon the third law, "In the scholium appended, he makes the following remarkable statement, introducing another description of actions and reactions subject to his third law, the full meaning of which seems to have escaped the notice of commentators:"—[Here follows the passage from the scholium, of which the authors give the following translation, in which "activity" and "counter-activity" are put for *actio* and *reactio.*]

"*If the activity of an agent be measured by its amount and its velocity conjointly; and if, similarly, the counter-activity of the resistance be measured by the velocities of its several parts and their several amounts conjointly, whether these arise from friction, cohesion, weight, or acceleration;—activity and counter-activity, in all combinations of machines, will be equal and opposite.*" ‡

Again Professor Tait in the article *Mechanics* in the Encyclopædia Britannica, 9th edition, quotes the passage and remarks: "This may be looked upon as a fourth law. But, in strict logic, the first law is superfluous. . . . Hence there are virtually only three laws, so far as Newton's system is concerned."

The "scholium to law III." is afterward referred to as

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* *Natural Philosophy*, Section 242. † Section 263.

‡ Nam si æstimetur agentis actio ex ejus vi & velocitate conjunctim; & similiter resistentis reactio æstimetur conjunctim ex ejus partium singularum velocitatibus & viribus resistendi ab earum attritione, cohæsione, pondere, & acceleratione oriundis; erunt actio & reactio, in omni instrumentorum usu, sibi invicem semper æquales.

giving us the principle of the "transference of energy from one body or system to another."

We shall be better prepared to estimate the import of the passage last quoted if we briefly consider the connection in which it stands in the *Principia.* The comments on the third law, quoted nearly in full above, are followed by six corollaries and a scholium, which complete the chapter entitled *Axiomata sive Leges Motus,* and immediately precede the treatise *De Motu Corporum.* Cor. I. gives the parallelogram of forces as deduced from laws II. and I. Cor. II. states the composition and resolution of forces. "Which composition and resolution is abundantly confirmed by the theory of machines."* The resolution of forces is then applied to prove that the efficiency of a force to turn a wheel is the product of the force and its arm—"the well known property of the balance, the lever, and the wheel"—and so on for the other simple machines, which are thus cited to confirm the truth of the laws of motion. Cor. III. proves by laws III. and II. that the quantity of motion "directed toward the same parts" is not altered by internal actions between the bodies of a system. Cor. IV. derives the conservation of the motion of the center of gravity. Cor. V. shows that the relative motions of a system of bodies enclosed in a given space are the same, whether the space be at rest or moving uniformly in a straight line, and cor. VI. extends this to the case in which the bodies are also acted upon by equal accelerating forces in the direction of parallel lines.

The scholium which follows is not a scholium to the third law exclusively, but is occupied with the experimental verifications of the laws; beginning with the discovery by Galileo, by means of the first two laws, that "the descent of heavy bodies is in the duplicate ratio of the time, and that the motion of projectiles takes place in a parabola, experiment confirming, except so far as these motions are somewhat retarded by the resistance of the air."† Then follows an account of experiments on the impact of bodies, showing that experience agrees with deductions drawn from the three laws, which ends with the words, "And this being established, the third law so far as impacts and reflexions are concerned is confirmed by a theory which plainly agrees with experience."‡

* Quæ quidem compositio & resolutio abunde confirmatur ex mechanica.

† Descensum gravium esse in duplicata ratione temporis, & motum projectilium fieri in parabola; conspirante experientia, nisi quatenus motus illi per aëris resistentiam aliquantulum retardantur.

‡ Atque hoc pacto lex tertia quoad ictus & reflexiones per theoriam comprobata est, quæ cum experientia plane congruit.

The paragraph concerning attractions, quoted above, comes next, and is followed by a statement of Newton's own experiments with magnetic attraction, directly confirming the third law.

The paragraphs concerning attraction are followed in the scholium by one opening with these words: "As, in impacts and reflexions, those bodies have the same efficiency of which the velocities are reciprocally as the innate forces: so in mechanical instruments for producing motion, those agents have the same efficiency, and by opposite endeavours sustain one another, of which the velocities estimated in the direction of the forces are reciprocally as the forces."* The *vires insitæ* or *vires inertiæ* are proportional to the masses, as explained in *Definitio* III., so that the meaning is this:—Just as bodies having equal momenta are of equal efficiency in the case of impact, so, in machines, agents are of equal power (and if opposed produce equilibrium) when the products of the force and the velocity of the point of application in the direction of the force are the same for each.

This is nothing more nor less than the "principle of virtual velocities,"—a succinct statement of "the whole theory of machines diversely demonstrated by various authors," already cited in the second corollary as a confirmation of the laws of motion, because on the one hand deducible from them, and, on the other hand, in agreement with experience.

After applying this principle of virtual velocities in detail to the several simple machines, Newton continues: "But to treat of mechanism does not belong to the present design. I wished only to show by these things how widely extends and how certain is the third law of motion." † Then follows the passage quoted by Thomson and Tait (see above) in which the only new idea involved is the inclusion of the resistance to acceleration among the "reactions."

The scholium shows indeed that Newton had a clear conception of what we now know as "D'Alembert's principle" ‡ as well as the "principle of virtual velocities," but does not, as it seems to me, indicate any intention to postulate a new axiom.

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* Ut corpora in concursu & reflexione idem pollent, quorum velocitates sunt reciproce ut vires insitæ: sic in movendis instrumentis mechanicis agentia idem pollent & conatibus contrariis se mutuo sustinent, quorum velocitates secundum determinationem virium æstimatæ, sunt reciproce ut vires.

† Cæterum mechanicam tractare non est hujus instituti. Hisce volui tantum ostendere, quam late pateat quamque certa sit lex tertia motus.

‡ It must be remembered, however, that we call such propositions "principles," not when they are presented simply as demonstrated theorems, but when they are made the basis of a systematic method of applying analysis to the solution of problems.

Professor Tait* regards the first words of the scholium—"Up to this, I have laid down principles received by mathematicians and confirmed by experiments in great number"†—as claiming for Newton the discovery of what, as stated above, he regards as a fourth law: as if Newton were about to proceed to some new axiom not yet known to the men of science of the day. Yet we have seen that the scholium treats of a variety of topics at great length, before coming to what is alleged to be the new axiom. The context rather shows that the matter new to mathematicians, to which Newton implicitly refers in the words quoted, is the body of the treatise *De Motu Corporum*, which immediately follows the introductory chapters—the *Definitiones*, and the *Axiomata* and Corollaries.

At the close of the article *Mechanics* Professor Tait summarizes the third law proper thus—"Every action between two bodies is a stress." He subsequently points out in the simple instance of a falling stone how force may be regarded either as "the *space-rate at which energy is transformed*," or "the *time-rate at which momentum is generated*," and says (§ 294) that these are "merely particular cases of Newton's two interpretations of action in the third law." He then proceeds to connect them analytically as follows:—"if s be the space described, v the speed of a particle,

$$\ddot{s} = \dot{v} = \frac{dv}{dt} = \frac{dv}{ds} \cdot \frac{ds}{dt} = v\frac{dv}{ds}.$$

Hence the equation of motion (formed by the second law)

$$m\ddot{s} = m\dot{v} = f,$$

which gives f as the time-rate of increase of momentum, may be written in the new form

$$mv\frac{dv}{ds} = \frac{d}{ds}\left(\tfrac{1}{2}mv^2\right) = f,$$

giving f as the space-rate of increase of kinetic energy."

Is it not equally true in the general case that the so-called two interpretations of action, so far from being the subjects of separate axioms, are demonstrably equivalent by virtue of the equality of the two phases of a stress and the second law of motion?

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* Article *Mechanics*, Encyclopædia Britannica, § 12.

† Hactenus principia tradidi a mathematicis recepta & experientia multiplici confirmata.

Another instance of the unnecessary assumption of physical axioms occurs in the *Natural Philosophy*. The principle that "the perpetual motion is impossible" is introduced as an axiom to prove that "If the mutual forces between the parts of a material system are independent of their velocities, whether relative to one another, or relative to any external matter, the system must be dynamically conservative. For if more work is done by the mutual forces on the different parts of the system in passing from one particular configuration to another, by one set of paths than by another set of paths, let the system be directed, by frictionless constraint, to pass from the first configuration to the second by one set of paths and return by the other, over and over again forever. It will be a continual source of energy without any consumption of materials, which is impossible."*

Again in Williamson and Tarleton's *Dynamics* (p. 397), the same demonstration is given, closing with the words "This process may be repeated forever, and thus an inexhaustible supply of work can be obtained from permanent natural causes without any consumption of materials. The whole of experience teaches us that this is impossible."

Thus we find these authors appealing to the general principle of the conservation of energy in proof of what is really but its simplest form, namely the equivalent transference of energy from body to body of a material system, and from the kinetic to the potential form, a proposition which is easily shown to be a consequence of Newton's laws of motion. The appeal to experience is in fact only necessary to establish the hypothesis laid down in the above quotation from Thomson and Tait, namely, that the forces do not in any way depend upon velocities, or, let us say, that the stress between two bodies depends only upon the distance between them.

We conclude this examination of the physical axioms with a brief sketch of the steps by which the conservation of energy, in its mechanical forms of kinetic and potential energy of masses, may be established directly from the axioms of 'the independent accelerative action of force,' 'the duality of stress,' and 'the dependence of its intensity solely upon distance.' The steps 4 and 5, which establish the right to deal with the kinetic energy o. relative motion, are developed more in detail, because the point does not seem to be sufficiently developed in the usual text-books.

1. Let the conservation of the motion of the centre of gravity be deduced, as in the *Principia*, from the second and third laws of motion.

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* *Natural Philosophy*, Art. 272.

2. The kinetic energy of a body may be decomposed into parts corresponding respectively to its component velocities in two *rectangular* directions.

3. A moving body acted upon by a force, directed to or from a fixed point and in magnitude a function of the distance of the body from that point, experiences a gain or loss of kinetic energy equal to the loss or gain of potential energy relative to the fixed point.

4. Although fixed centres of force do not exist, yet when a stress exists between two bodies (in magnitude a function of the distance), the centre of gravity being fixed, and dividing the distance between them in a fixed ratio, the actual change of potential energy is equal to the sum of the changes in the potential energy of the two bodies, each with reference to the centre of gravity as if it were a centre of force. Hence the sum of the potential energy and the kinetic energies of the two bodies is constant.

5. The total kinetic energy of a system of bodies may be decomposed as follows :—First, decompose the energy of each body into parts corresponding to the velocities perpendicular to and along the line in which the centre of gravity is moving. Put u for the first of these components, v for the second taken relatively to the centre of gravity, and V for the velocity of the centre of gravity. Then the total kinetic energy is

$$\tfrac{1}{2}\Sigma mu^2 + \tfrac{1}{2}\Sigma m(v + V)^2$$

From the property of the centre of gravity $\Sigma mv = 0$: therefore the total kinetic energy is

$$\tfrac{1}{2}\Sigma m(u^2 + v^2) + \tfrac{1}{2}\Sigma m \,.\, V^2$$

The first term is the sum of the kinetic energies corresponding to the motions relative to the centre of gravity, and the second is the kinetic energy corresponding to the total mass as if situated at and moving with the centre of gravity. Thus the total kinetic energy is equal to the *internal* kinetic energy of the system relative to the centre of gravity as a fixed point, and the *external* energy of the system due to the motion of the centre of gravity.

6. When a stress exists between two bodies whose centre of gravity is in motion, the stress causes at every instant the same gain of kinetic energy in one as loss in the other, if the distance is unchanged. But, when the distance is changing, we find by considering the external and internal energy of the system, that the former is unchanged by 1, and that the change in the latter is, by 4, compensated for by that in the potential energy connected with the stress, so that in either case the sum

of the two kinetic energies and the mutual potential energy is unchanged.

7. Hence in any system of bodies, between pairs of which stresses exist whose intensities depend solely upon the distances, the sum of the kinetic energies and the potential energy due to their relative positions is constant.

EIGHT-FIGURE LOGARITHM TABLES.

Tables des Logarithmes a huit décimales des nombres entiers de 1 à 120000, et des sinus et tangentes de dix secondes en dix secondes d'arc dans le système de la division centésimale du quadrant. Publiées par ordre du Ministre de la guerre. Paris, Imprimerie Nationale, 1891. 4to., pp. iv. + 628.

ADVOCATES of the decimal subdivision of the quadrant will be much pleased by the appearance of the above work, which contains the most extensive set of tables of the kind as yet issued. It is not intended in the present notice to enter upon the respective merits of the several systems of dividing the circle, but to consider the volume as a table of logarithms simply. As such it presents marked points of difference from the usual types. These differences are found almost exclusively in the trigonometric portion of the tables, that containing the logarithms of numbers being similar to the customary form. The logarithms of the four trigonometric functions appear on each double page in four separate tables, instead of the usual arrangement in parallel vertical columns. The interval of the argument is the same throughout the entire quadrant, no diminution being found near the beginning of the table. The auxiliary quantities for obtaining sines and tangents of small angles by means of the table of number logarithms are given; but they are placed upon the pages devoted to the trigonometric functions. It would probably be more convenient to find them as usual at the bottom of the pages containing the number logarithms.

The decimal progression of the argument allows the trigonometric tables to be arranged in the form usually adopted for the logarithms of numbers. But instead of ten columns headed with the digits 0 to 9, we find *eleven* columns, of which the first ten are headed 0 to 9. The eleventh, which has no heading, contains a repetition of the column headed 0. This makes it unnecessary to look back along the horizontal line, when we wish the difference between column 9 and the next one. Yet the size of the volume is somewhat increased by this system, and the tables containing the number logarithms are rather widely spaced in consequence. In fact the volume is more ponderous than might be expected in comparison with other logarithm books, notwithstanding that the adopted interval of ten decimal seconds is much smaller than that of ten ordinary seconds of arc. This is made plain if we compare the dimensions and weights of the following well-known logarithmic tables (*bound*):

NAME.	LENGTH.	BREADTH.	THICKNESS.	WEIGHT.
	mm.	mm.	mm.	grams.
Gauss 5-fig.	241	159	14	395
Bruhns 7-fig.	257	175	30	1139
French 8-fig.	370	298	48	3813
Vega 10-fig.	341	238	36	2600

Negative characteristics are given throughout for the logarithms of the trigonometric functions, when the corresponding numbers are less than unity. This departure from the usual custom of increasing the logarithms by 10 can hardly be regarded as an improvement. The greatest possible care has been taken to secure the accuracy of the tables; and in this respect they may be greatly commended. The actual numbers have been copied from the great manuscript tables of Prony, which are preserved in the archives of the Paris observatory. The typographical work, which is excellent, was executed at the *Imprimerie Nationale.*

HAROLD JACOBY.

SPHERICAL AND PRACTICAL ASTRONOMY.

An Introduction to Spherical and Practical Astronomy. By D'ASCOM GREENE, Professor in the Rensselaer Polytechnic Institute, Troy. Boston, Ginn & Co., 1891. 8vo, pp. viii. + 158.

PROFESSOR GREENE has written this work to supply the needs of those students who wish to begin the study of spherical and practical astronomy, and have but very little time to give to such study. The work is a stepping stone to Doolittle's and Chauvenet's books. The author deals only with "those practical methods which can be carried out by the use of portable instruments," and in describing those methods he is very brief, frequently altogether too brief.

The order of subjects is as follows: definitions; spherical problems; conversion of time; hour angles; the transit instrument; the sextant; finding time by observation, which includes time by transit observations, by equal altitudes and by single altitudes; finding differences of longitude, which includes the methods by the electric telegraph, by transportation of chronometers and by moon culminations; finding the latitude of a place by a circumpolar star, by a meridian altitude, by a zenith instrument, by a prime vertical instrument, by a single altitude and the corresponding time, and by circummeridian altitudes; finding the azimuth of a given line, by the elongation of a circumpolar star, by observing a body (*sic*) at a given instant, by observing a body at a given altitude, and by observing a body at equal altitudes.

These subjects are discussed in the first 95 pages. Then follows a very short (20 pages) treatment of the figure and dimensions of the earth, in which the author gives some of the fundamental formulæ of the spheroid, the elements of the spheroid as determined by measurement, the polyconic projection, spherical excess of triangles on the earth's surface, and geodetic determinations of latitudes, longitudes, and azimuths. The book has an appendix (pp. 115–150) on the method of least squares—and three tables on (I.) the correction for refraction, (II.) equation of equal altitudes of the sun, and (III.) for computing the reduction to the meridian. The simple mention of the contents will show how inadequate the treatment must be. It seems to the writer that it is far better to use always a book that encourages the student to study thoroughly a given subject rather than one that tempts him to be satisfied with very brief statements. In a work of this kind the discussion of the method of least squares is hardly appropriate.

The equations are not numbered consecutively throughout the book but the numbering begins anew with each chapter. In the discussion of equatorial interval no account is taken of the formula used when the declinations are 80° and over. On page 48 in the third paragraph there is a confusion of index correction with index error. In (1) on page 147 the rule should show that both the measured sum *and* each of the measured magnitudes should be adjusted.

For a work on practical astronomy it seems to me that the examples given are too few and insufficient, as in the chapter on the transit instrument. However, the author's idea seems to be that the instructor should supply such details. To colleges and technical schools where spherical and practical astronomy are given but little time, this work may prove quite acceptable as a basis of study.

J. K. REES.

NOTES.

A REGULAR meeting of the NEW YORK MATHEMATICAL SOCIETY was held Saturday afternoon, February 6, at half-past three o'clock, the president in the chair. The following persons having been duly nominated, and being recommended by the council, were elected to membership: Mr. Ernest William Brown, Haverford College; Dr. William S. Dennett, New York; Mr. Armin O. Leuschner, University of California; Professor Oscar Schmiedel, Bethany College; Professor Laenas Gifford Weld, State University of Iowa. Notice was given by the council that it was proposed to amend Article III. of the Constitution so as to read: *The officers of the Society shall be a President, a Vice-President, a Secretary, a Treasurer, a Librarian, and a Committee of Publication, which shall consist of two members, either or both of whom may at the same time hold any other office or offices.* An original paper on the "Transformation of a system of independent variables," by Professor J. C. Fields, was read. This paper has been transmitted to the *American Journal of Mathematics* for publication.

WE have to record the deaths of Leopold Kronecker at Berlin, December 29, in his sixty-eighth year; of George Biddle Airy at Greenwich, January 2, in his ninety-first year, and of John Couch Adams at Cambridge, January 21, in his seventy-second year.

THE paper "On a peculiar family of complex harmonics," read by Dr. Pupin before the NEW YORK MATHEMATICAL SOCIETY, December 5, 1891, has been published in full in the *Transactions of the American Institute of Electrical Engineers* for December, 1891, in connection with another paper, "On polyphasal generators," by the same author.

T. S. F.

THE suggestion made in the article "On lists of covariants" (*Bulletin*, No. 3, last paragraph of p. 89), has been superseded in the best possible way. Professor Cayley writes that he has all but five or six of the forms of the sextic complete, and adds: "I think of giving these tables in my volume V."

E. M.

Dr. Artemas Martin desires to call attention to two errors in Degen's *Canon Pellianus*.

On page 88 of Degen's Tables, in the line of denominators of partial fractions for square root of 853, *for* "15" *read* 14; so that the line will be

29, 4, 1, 5, 1, 2, 4, 1, 1, 14, 19, (2, 2)

instead of

29, 4, 1, 5, 1, 2, 4, 1, 1, 15, 19, (2, 2).

Page 98, square root of 929, *for*

"30, 2, 11, 1, 2, 3, 1, 5, 2, 1, 6, 1, 14, 2, 1, (2, 2)
1, 29, 5, 40, 19, 16, 40, 10, 20, 38, 8, 50, 4, 22, 32, (20, 20)"

read

30, 2, 11, 1, 2, 3, 2, 7, 5, (2, 2)
1, 29, 5, 40, 19, 16, 25, 8, 11, (23, 23)

Degen's values of x and y in both cases are correct. H. J.

MACMILLAN & Co. have in press a treatise on the "Applications of elliptic functions" by A. G. Greenhill. They have in preparation a work on "Hydrostatics" by the same author, and one on the "Theory of heat" by Thomas Preston.

T. S. F.

NEW PUBLICATIONS.

COMPILED BY B. WESTERMANN & CO., NEW YORK.

ACKLAND & HARDY. Graduated Exercises and Examples, with Solutions, for the "Institute of Actuaries' Text Book." London 1892. 8vo. cloth. M. 11

CANTOR (M.). Vorlesungen über Geschichte der Mathematik. II. I. 1891. 8vo, 499 pp. M. 14

CAYLEY (A.). The Collected Mathematical Papers. Vol. 4. 4to. Cambridge University Press. 25s.

CHAPPUIS & BARGET. Leçons de physique générale. Tome III. Acoustique, Optique and Electro-optique. 1892. 8vo. 10 fr.

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WIMMER (B.). Ueber eine allgemeine Classe von ein-zweideutigen Raumtransformationen. Erlangen 1891. 8. 19 pg. M. 1.00

CONTENTS.

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1892.

SOME RECENT ELEMENTARY WORKS ON MECHANICS. I.

The Laws of Motion, an elementary treatise on dynamics. By W. H. LAVERTY, late fellow of Queen's College, Oxford. London, Rivingtons. 1889. 8vo, pp. 212.

IN a recent number of the *Bulletin* (No. 2, pp. 48–50) Professor T. W. Wright complains of the confusion existing in the nomenclature of elementary mechanics. It would be easy to answer his questions from a purely theoretical point of view; indeed, in theoretical mechanics no difficulty is encountered in this respect. But it must be admitted that in elementary works, particularly in those of a more "applied" character, the confusion is great, both as to the use of terms and the way of presenting the fundamental laws.

By reviewing somewhat at length a few of the better recent works on elementary mechanics it may perhaps be possible to "fix the ideas" and arrive at some conclusions, at least as to what is the best modern usage in treating the subject.

Mr. Laverty's little work is rather different from the ordinary English text-book. There is no reference in the preface to the "examinations of the Science and Art Department for the elementary stage," nor any gentle hint to the reader that "most of the examples are taken from actual recent examination papers."

"The object of this treatise," says the author (p. v.), "is to put the subject of dynamics on a thoroughly sound basis, avoiding unsatisfactory illustrations and definitions which do nothing towards defining, and to endeavour to give the student such an accurate idea of the subject that he may be able *e.g.* to give explanations and illustrations of the laws without just merely copying these from the book."

The author's objections to definitions that do not define, to inadequate illustrations of the fundamental laws, and to the loose and confused ways of stating these laws found so often in elementary works are certainly well taken. The book is evidently the result of careful independent thinking and treats a well-worn subject in a fresh and original way. Newton's laws are given in good English and in modern scientific language; the discussion of their meaning and interdependence is noteworthy in many respects.

The outward appearance of the book is pleasing; the little volume is neatly printed and furnished with an alphabetical index in addition to an ample table of contents. The matter is well arranged and distributed into sections of convenient size; every subject is illustrated by a few "worked" examples followed by a large number of exercises for which the answers are given at the end of the volume.

Before discussing the points of principal importance a few minor matters might be mentioned which could readily be corrected in a second edition.

In art. 9, the terms "standard" and "unit" are used as if they meant the same thing. It is preferable to make a distinction. Thus, the standard of mass in the C. G. S. system is the kilogramme, that is a certain bar of platinum preserved in Paris, while the unit of mass is a gramme, that is any mass equal to a one thousandth part of that kilogramme.—The statement of art. 267 that "the laws of friction between bodies, as found by experiment, are surprisingly simple," gives a surprisingly optimistic view of the case.—The factor 2 in the first expression for $n-n'$ on p. 170 is a misprint; it should be dropped.—In example E, pp. 110–111, the factor g should be inserted in the expression for the work, or rather in the problem itself "42400000 ergs" should read "42400000g ergs."—

The numerical data in the exercises are usually so selected as to lead to answers expressible in round numbers. This method has obvious advantages for class work and examinations; it saves time and allows a certain display of ingenuity in arranging the numerical work conveniently for cancelling. But it accustoms the student to methods that are far from being the best in examples as they occur in actual practice. If the working of numerical examples is to be of any value it should lead the student to understand the bearing that every quantity involved in the formula has on the final result. The beginner should in particular learn to select for any constant the proper number of decimal places necessary in order to obtain the required accuracy of the result; he should also determine from the data the accuracy obtainable with the data of the problem. Thus, on p. 7 we find the problem: "How many metres are there in a mile, if there is .305 of a metre in a foot?" The answer is correctly given as 1610.4. But actually there are 1609.3 metres in a mile; the given constant .305 is not sufficiently exact to give the result correct within a decimetre. Would it not be better to refer the student to the more exact value of the constant given on a previous page (p. 5) and require him to select the proper number of decimal places?

It must be said, in general, that the author has an excessive fondness for such merely speculative problems as the following: "If the unit of area and time be 10 acres and 10 seconds; what is the unit of velocity expressed in miles per hour?" (p. 27.) Such meaningless problems occur in great number throughout the book. With this exception the exercises are very well selected and constitute a valuable feature of the book.

In the matter of symbols and names for the units Mr. Laverty is unusually radical. He manufactures them without the slightest compunction. The British unit of velocity (foot a second) is called *fas*, the C. G. S. unit (centimetre a second) *cas*; similarly the unit of acceleration are *sfas*, *scas*; those of momentum (a *fas* in a pound): *fasp* and *casgram*; of kinetic energy: *faspen* and *casgrammen*; of impulse: *bim* and *cim*; of force: *sfasp* and *scasgram*. This new notation is as ingenious as it is simple; *bim* for impulse strikes one as particularly happy. But will it be possible to bring this brilliant new coinage into circulation? And before this is accomplished, what is the poor student to do as soon as he leaves Mr. Laverty's class-room? Nobody will understand him when he begins to talk of *casgrammen* and *sfasp*, and he will have difficulty in understanding the old-fashioned rest of the world.

A new notation of this kind is entirely out of place in an elementary text-book. Originality is no doubt a good thing; but in a work for beginners it is to be used with moderation; an over-dose may become fatal. It is another question whether the notation is in itself good and its acceptation desirable.

It may be seriously questioned whether there is any actual need for special names and symbols for all these units. The British Association Committee on Units suggests the name *kine* for "a speed of 1 cm. per sec.;" J. B. Lock uses *vel* and *cel* for the units of velocity and acceleration; the term "quickening" has been proposed for unit acceleration. Mr. Laverty's scheme has the advantage over these separate efforts of being methodical and comprehensive; it also lends itself readily to farther extension. A "mile an hour" might be called a *mah*, a "yard a minute" a *yam*, etc. But the fact of the matter is that these numerous symbols and names can be of use almost exclusively in the elementary text-book. Later on we can get along without them. In most cases the unit can be understood from the context, as when the physicist says that the acceleration of gravity at a certain place is 981, meaning "centimetres per second," or when the engineer gives the angular velocity of his fly-wheel as 25, meaning "revolutions per minute." It is mere pedantry to require the unit to be stated explicitly under such circumstances. In other cases it is best to state the unit completely.

Mr. Laverty says, in regard to his notation (preface, p. x.): "These words should be looked upon simply as abbreviations (perhaps in some cases as aids to the memory); I have no desire to add new words to the language." But if they are not to become new words of the language, what is their use?

Are they to be learned only to be forgotten as soon as possible? And does not Mr. Laverty himself use them throughout as if they were new words of the language? Let us give the student in the elementary text-book nothing but the most approved notation of the science and the less perfect equivalents used in the applications; he will have enough to do in mastering these.

All these slight strictures, however, do not detract materially from the value of Mr. Laverty's work, which gives an admirable presentation of Newton's three laws of motion. After explaining the ideas of velocity and acceleration in the simplest cases, the idea of mass is introduced; the fundamental equations $v = \alpha t$, $\frac{1}{2}v^2 = \alpha x$ are multiplied by m and the quantities mv, $\frac{1}{2}mv^2$, $m\alpha$ are given the names *momentum, kinetic energy,* and *mass-acceleration,* respectively. There is no good reason why the term *force* should not be used here instead of mass-acceleration. If force were thus defined, the fundamental relations

$$mv = m\alpha \cdot t, \qquad \frac{1}{2}mv = m\alpha \cdot x$$

would at once show that force *is* the rate of change of momentum with the time, or the rate of change of kinetic energy with the distance.

The author prefers to call force that which *produces* change of momentum. At the same time he objects to calling this a definition of force. If this be not what the logicians call a *definitio realis,* it certainly is a *definitio nominalis:* we observe in nature a change of momentum, and to the cause of this change we give the name force.

Newton's first law is stated very clearly in the following terms (p. 46): "The momentum in a mass (or system of masses) cannot be increased or diminished except by the action of external force." This becomes a self-evident truth with the above definition of force as the cause of change of momentum; for when there is no cause there can be no effect. But unfortunately Mr. Laverty neglects to give a definition of force. And yet what concept needs definition more than force? In ordinary language the term is used in a variety of meanings; and on the other hand, force itself cannot be directly observed in nature (excepting the case of muscular force with which we are not concerned here), it is only its effects, *i.e.* changes of momentum, that can be directly measured. In all other respects Mr. Laverty's explanations and illustrations of the first law can only be commended.

The second law is given in this form (p. 68): "When momentum is produced, it is by the action of force; and the amount of momentum produced in a given time is proportional to and in the direction of the force."

It will be noticed that the first clause is but a re-statement of the first law; and the author very justly remarks (preface, p. VII., foot-note) that the first law might be dropped and the science of mechanics be based on only two laws, "the law of force (or momentum), and the law of work (or energy)."

While the first law merely states that we shall give the name force to the cause of any observed change of momentum, the second law defines force more accurately by saying that this cause is proportional to the effect produced and that the direction of the force shall mean the direction of the momentum produced. It also implies the independence of the action of two or more forces applied at the same point. Thus it follows that the parallelogram law applies to forces just as it applies to velocities and accelerations.

The third law is expressed as follows (p. 86): "The work done by a force (or any agent) on any mass (or system of masses) has its equivalent in the kinetic energy exhibited, and in the work done against molecular forces, gravity, and friction." The usual short form "action and reaction are equal and opposite" is rejected as meaningless as long as action and reaction are not carefully defined. "The fact is," says the author, p. VIII., "that, if by 'action' and 'reaction' are meant force and resistance, the third law is but an easy deduction from the second; while if d'Alembert's principle is really to be ultimately deduced from the law, it is better to enunciate it at once in proper form, and not in the usual indefinite and undefined terms." Thus, the third law in the simplest case is expressed by the equation

$$m\alpha \cdot x = \tfrac{1}{2}mv^2,$$

while in the most general case it leads to d'Alembert's principle (p. 92): "The internal pressures of any system of rigid bodies are in equilibrium amongst themselves."

After discussing each law for itself the author devotes several sections to illustrations and applications of the laws; these embrace the theory of the pendulum, Atwood's machine, the inclined plane, collision, projectiles, and circular motion. Only the most elementary mathematics are used throughout the book.

As a point not usually touched upon in elementary textbooks it may be mentioned that Mr. Laverty calls special attention to the fact that the parallelogram law would not hold for forces if they were not defined as they are by the second law, *viz.* as the time-rate of *momentum*, but *e.g.* as the time-rate of *kinetic energy*. It is well known that on this point turned the long controversy on the nature of force and energy between Descartes, Leibnitz, and their followers.*

The closing section contains some interesting general remarks on the nature of the three laws and the ways of testing their truth.

ALEXANDER ZIWET.

ANN ARBOR, MICHIGAN, *January* 1, 1892.

WEIERSTRASS AND DEDEKIND ON GENERAL COMPLEX NUMBERS.

WEIERSTRASS †—*Zur Theorie der aus n Haupteinheiten gebildeten complexen Grössen. Göttingen Nachrichten*, 1884.

DEDEKIND—*Zur Theorie der aus n Haupteinheiten gebildeten complexen Grössen. . Göttingen Nachrichten*, 1885.

DEDEKIND—*Erläuterungen zur Theorie der sogenannten allgemeinen complexen Grössen. Göttingen Nachrichten*, 1887.

IN closing his second memoir on biquadratic residues ‡ Gauss makes this remark: "Our general arithmetic, which goes so far beyond the limits of the geometry of the ancients, is entirely the creation of recent times. Starting with the notion of whole numbers its field has widened little by little. To whole numbers came fractions, to rational numbers the irrational ones; to the positive came the negative and to the real came the imaginary."

Once convinced that $\sqrt{-1}$ was properly an algebraical quantity and that it had a meaning, mathematicians began to look for other quantities of a similar nature. "Why," they asked themselves, "should algebra yield an imaginary unit which makes it possible to represent two dimensions of space analytically; and fail to yield a second imaginary unit which can be used to represent the third dimension?" The thing needed only to be sought for apparently, and at first they looked amongst the functions of $\sqrt{-1}$. Unfortunately it turned out that even the most promisingly irrational of these could all be broken up into a real part and $\sqrt{-1}$ times a second real quantity; algebra had done her best; if mathematicians wanted more imaginaries they must invent them. From the time of Gauss, then, until the present day the architects and the masterbuilders have turned occasionally

* See for instance E. MACH, *Die Mechanik in ihrer Entwickelung*, Leipzig, Brockhaus, 1889, pp. 254-259.

† Extract from a letter to Schwarz.

‡ *Werke*, II., p. 175.

from their labors upon the theory of functions, that monument which of all that human hands have built will rise the highest and stand the longest, to try their skill in constructing systems of imaginary, or complex, numbers.

Gauss himself was of the opinion that no complex numbers except those of type $x + \sqrt{-1}\,y$ would be found admissible into arithmetic,* but does not state his reason for the opinion. The occasion of the articles cited above was an inquiry into his most probable reason, an inquiry which involved a fundamental investigation into the properties of the hyper-complex [*über-complex*] numbers, as Dedekind calls them. After full and interesting researches, of which this paper aims to give a sketch, these great mathematicians came to opposite conclusions. The fact that in the field of complex numbers the product xy may vanish when neither x nor y is zero, a fact made public by Peirce long before,† seemed to Weierstrass so unlike anything in ordinary mathematics that he concluded this must have been Gauss's reason for excluding hyper-complex quantities from arithmetic. On the other hand Dedekind asserts that it is quite a common thing in ordinary arithmetic for such a product to vanish, and concludes that Gauss's reason for excluding quantities of a nature different from $x + iy$ ‡ was the fact that such quantities, conditioned as they must be, do not exist.

To construct a complex number Weierstrass writes down a system of n units $e_1, e_2, \ldots e_n$ and multiplies each by an ordinary real number ξ_r; then the expression $x = \xi_1 e_1 + \ldots + \xi_n e_n$ is a number of the kind considered. His first undertaking is so to define the fundamental operations of arithmetic for quantities of this kind that $x + y$, $x - y$, xy, x/y may all be linear expressions of the same form as x; and that the commutative, associative and distributive laws of addition and multiplication may hold good for them. It appears that the multiplication table for the units may be constructed in an infinite number of ways so as to satisfy all these requirements. Of course the fundamental condition is the first one, which comes to the same thing as this, that *every rational function of the units shall be expressible in the form*

$$\xi_1 e_1 + \ldots + \xi_n e_n.$$

Division is defined by the equation

$$\frac{a}{b} = \gamma_1 e_1 + \ldots + \gamma_n e_n = \gamma.$$

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* *Werke*, II., p. 178.

† *Am. Journ. Math.*, vol. IV. (1881), p. 97.

‡ x and y real; $i = \sqrt{-1}$.

Multiplying both members by b and equating the coefficients of $e_1, \ldots e_n$ on both sides, a set of n equations is obtained, linear in $\gamma_1, \ldots \gamma_n$. If their determinant vanishes identically, it is impossible to determine $\gamma_1, \ldots \gamma_n$, and therefore all multiplication tables are excluded which would bring this to pass. But even then there will be certain values of b for which this determinant will vanish. Suppose such a value chosen; we can then find a value of γ such that $b\gamma$ shall vanish, both b and γ being different from zero; for $b\gamma = 0$ leads to a system of n equations linear and homogeneous in $\gamma_1, \ldots \gamma_n$ whose determinant vanishes. The quantities b having this unique and wonderful property are called by Weierstrass "divisors of zero" [*Theiler der Null*].

It turns out that when b is a divisor of zero there are an infinite number of quantities γ such that $b\gamma = 0$, and thence it is an easy inference that the equation

$$ka + kbx + kcx^2 + \ldots + klx^m = 0$$

has an infinite number of roots if k is a divisor of zero. We have, in fact, only to make

$$a + bx + \ldots + lx^m = g$$

where g is any one of the infinite number of quantities satisfying the relation $kg = 0$.

"The existence of these divisors of zero which are not themselves zero, seems," says Weierstrass, "to make a real distinction between ordinary arithmetic and the arithmetic of hyper-complex* numbers"; but ordinary algebraic equations exist which have an infinite number of roots, namely those whose coefficients are all zero. As to this point then there is a good enough correspondence between the numbers of our common arithmetic and hyper-complex numbers.

The author now obtains a multiplication table of beautiful simplicity by the following process. He expresses the first, second, . . . , $(n + 1)$-th powers of x, where

$$x = \xi_1 e_1 + \ldots + \xi_n e_n$$

linearly in terms of $e_1, \ldots e_n$; then, excluding the case when the determinant of the right members of the first n equations vanishes, we can express $e_1, \ldots e_n$ in terms of the first n powers of x; and substituting these values in the last equation, obtain a relation among the powers of x of the form

$$\Delta_0 x^{n+1} + \ldots + \Delta_n x = 0$$

where Δ_0 is the determinant just mentioned.

* Weierstrass does not use this term.

Dividing by $\Delta_0 x$ this becomes, if we replace x by the particular value g,

$$g^n + \varepsilon_1 g^{n-1} + \dots + \varepsilon_n e_0 = 0 = f(g).$$

Here e_0 is a quantity which satisfies the conditions

$$e_0 z = z e_0 = z$$

for any number of the system. Its value is in fact g/g; which is determinate so long as g is not a divisor of zero. We are now in a position to put every number a in the form

$$a = a_0 e_0 + a_1 g + a_2 g^2 + \dots + a_n g^{n-1} = a(g)\ *$$

and the product of any two numbers takes the same form.

Consider now the algebraic equation $f(\xi) = 0$ formed by replacing g in $f(g)$ by ξ; form also the function $a(\xi)$ by replacing g by ξ in $a(g)$. There is no difficulty in seeing that the product $a(\xi) \cdot b(\xi)$ will vanish if it contains the factor $f(\xi)$. If $f(\xi) = 0$ has a root of multiplicity λ, it can be indicated by writing

$$f(\xi) = f_1^{\lambda}(\xi) \cdot F(\xi)$$

and the arbitrary function $\varphi(\xi) = f_1(\xi) \cdot F(\xi) \cdot \varphi_1(\xi)$ will be of such a nature that $\varphi^{\lambda}(\xi)$ is divisible by $f(\xi)$ and therefore vanishes; but if ξ be replaced by g in $\varphi(\xi)$ we obtain a hyper-complex quantity x whose λ-th power is obtained by replacing ξ by g in $\varphi^{\lambda}(\xi)$. The λ-th power of x will therefore vanish. Hence, if $f(\xi)$ has a multiple root, the equation

$$x^{\lambda} = 0$$

can be satisfied in as many ways as there are different choices of the function $\varphi(\xi)$; but this number is infinite. It is our intention, however, to allow an algebraic equation an infinite number of roots only when each of its coefficients is a multiple of the same divisor of zero †; matters must consequently be so arranged that $f(\xi) = 0$ shall have no multiple roots. To effect this, the original multiplication table must be so constituted that the discriminant of $f(\xi)$ shall not vanish. This imposes another restriction upon the freedom of choice of the coefficients ε_{ijk} in the equations

$$e_i e_j = \sum_1^n{}_k\, \varepsilon_{ijk} e_k \,. \quad (i, j = 1, 2, \dots n).$$

The simplified multiplication table is now in sight. Take any function $\varphi(\xi)$ of degree $n - 1$ and with real coefficients

* This is a departure from the notation of Weierstrass.

† Weierstrass, *loc. cit.*, p. 399.

and break up into partial fractions the quotient of $\varphi(\xi)$ by $f(\xi)$. This yields the equation

$$\frac{\varphi(\xi)}{f(\xi)} = \frac{A_1}{\xi - b_1} + \frac{A_2}{\xi - b_2} + \ldots + \frac{C_1 + D_1 \xi}{\xi^2 + 2h_1\xi + k_1} + \ldots;$$

the quadratic denominators corresponding to pairs of conjugate imaginary roots of $f(\xi) = 0$.* The quantity $\frac{A_1 f(\xi)}{\xi - b_1}$ is a polynomial in ξ of degree $n - 1$ and may be changed into a hyper-complex quantity, c_1, by replacing ξ by g as above. In the same way $\frac{A_2 f(\xi)}{\xi - b_2}$ leads to another quantity c_2. The product c_1c_2 is obtained by replacing ξ by g in $A_1A_2\frac{f(\xi)f(\xi)}{(\xi - b_1)(\xi - b_2)}$; but this product vanishes and, in consequence, $c_1c_2 = 0$. If then $f(\xi) = 0$ has the m real roots $b_1, \ldots b_m$, we may construct m hyper-complex quantities $c_1, c_2, \ldots c_m$ such that the product of any two of them vanishes. Moreover we can obtain

$$\left(\frac{A_1 f(\xi)}{\xi - b_1}\right)^2 = \vartheta(\xi) f(\xi) + B_1 \frac{f(\xi)}{\xi - b_1},$$

where B_1 is a constant. This reduces to $B_1 \frac{f(\xi)}{\xi - b_1}$ and we infer that c_1^2 is equal to c_1 times a real quantity. Moreover this real multiplier cannot be 0.

Again the product $\frac{(C_1 + D_1\xi)f(\xi)}{\xi^2 + 2h_1\xi + k_1}$ will, when ξ is replaced by g, yield two hyper-complex quantities, c_{m+1}, c'_{m+1} since C_1 and D_1 are both arbitary. These quantities form the doubly extended manifoldness $C_1c_{m+1} + D_1c'_{m+1}$; and each pair of conjugate imaginary roots of $f(\xi) = 0$ enables us to form a similar manifoldness. Repeating the reasoning already given we find that the product of any two quantities belonging to different manifoldnesses vanishes; thus

$$(C_r c_{m+r} + D_r c'_{m+r})(C_s c_{m+s} + D_s c'_{m+s}) = 0$$

whether D_r and D_s be different from zero or not; and that the product of two quantities belonging to the same manifoldness also belongs to that manifoldness. Suppose the whole number of partial fractions to be r; each fraction yields a simple or complex quantity $a_1, \ldots a_r$ and any hyper-complex quantity whatever can be expressed in the form

$$x = \xi_1 a_1 + \ldots + \xi_r a_r.$$

* The notation of Weierstrass is here altered for simplicity.

If y be any other quantity

$$y = \eta_1 a_1 + \ . \ . \ . \ + \eta_r a_r$$

then the rule for multiplication is

$$xy = \xi_1 \eta_1 a_1^2 + \xi_2 \eta_2 a_2^2 + \ . \ . \ . \ + \xi_r \eta_r a_r^2$$

If now in x the coefficients $\xi_1, \xi_2, \ . \ . \ . \ \xi_k$ all vanish, and in y the coefficients $\eta_k, \eta_{k+1}, \ . \ . \ . \ \eta_r$ all vanish, then xy will vanish while neither x nor y is zero.

An equation of the form

$$\alpha + \beta x + \gamma x^2 + \ . \ . \ . \ + \omega x^\lambda = 0$$

breaks up into r equations of the form

$$(B) \qquad \alpha_\mu + \beta_\mu x_\mu + \ . \ . \ . \ + \omega_\mu x_\mu{}^\lambda = 0$$

where $\alpha_\mu, \beta_\mu, \ . \ . \ . \ , x_\mu$ are ordinary quantities. Equation (B) can have an infinite number of roots—only in case $\alpha_\mu, \beta_\mu, \ . \ . \ . \ \omega_\mu$ all vanish. Suppose they do vanish : then

$$\alpha = \alpha_1 a_1 + \ . \ . \ . \ + \alpha_{\mu-1} a_{\mu-1} + \alpha_{\mu+1} a_{\mu+1} + \ . \ . \ . \ + \alpha_r \mu_r.$$

Taking any quantity

$$k = k_1 a_1 + \ . \ . \ . \ + k_{\mu-1} a_{\mu-1} + k_{\mu+1} a_{\mu+1} + \ . \ . \ . \ + k_r \mu_r,$$

we can put α in the form

$$\alpha = k\alpha' \text{ where } \alpha'_1 e_0 = \frac{\alpha_1 a_1}{k_1 a_1} = \frac{\alpha_1}{k_1} e_0 \text{ or } \alpha_1' = \frac{\alpha_1}{k_1};$$

similarly for α'_2, and so on. But $\alpha'_\mu = 0/0$; that is it may be anything we please. Proceeding in this way, the equation can be put in the form

$$k\alpha' + k\beta' x + \ . \ . \ . \ + k\omega' x^\lambda = 0$$

where k, having one coefficient zero, is a divisor of zero. Equation (B) having an infinite number of roots, of course x, of which each root of (B) forms a part, has an infinite number of values. We thus see why it is that in this system an equation must have an infinite number of roots when each coefficient is a multiple of the same divisor of zero.

Closing this section of his letter the distinguished author remarks that very likely Gauss's only reason for excluding from arithmetic these hyper-complex quantities was that he regarded the vanishing of xy when neither x nor y is zero as an insurmountable difficulty ; otherwise " it could hardly have escaped him that an arithmetic of these quantities can be constructed in which all the theorems are identical with those concerning ordinary complex quantities, or at least analogous to them." "In fact," he continues, "the arithmetic of hyper-complex quantities can lead to no result which could not be reached by processes known in the theory of ordinary complex quantities."

The views of Dedekind upon this last point quite coincide with those of Weierstrass; but for an account of his beautiful method of generating systems of complex quantities, the reader is for the present referred to the memoirs cited above.

C. H. CHAPMAN.

JOHNS HOPKINS UNIVERSITY, *February* 8, 1892.

EMILE MATHIEU, HIS LIFE AND WORKS.*

If it were asked what tyranny in this world has least foundation in reason and is at the same time most overbearing and capricious, none could be found to answer better to this description than *fashion;* that fashion which makes us admire to-day what but yesterday would have excited astonishment, and which may provoke ridicule to-morrow. We all know that this sovereign whose iron rule is so much more keenly felt on account of its injustice governs the thousand and one details of every-day life; that it is supreme in literature and in the arts. But those who have not watched closely the life of the scientific world may perhaps be surprised to hear that even there if you would please you must bend the knee to fashion. What? might exclaim the stranger to the world of science, can it be true that the mathematician knows other laws than the inflexible rules of logic? Does he care to obey other orders than the invariable commands of reason?—Well, yes. Of course, to have a mathematical production accepted as *correct*, it is sufficient that it conform to the precepts of logic; but to have it admired as beautiful, as interesting, as of importance, to gain honor and success by it, more is required: it must then satisfy the manifold and varying exactions imposed by the prevailing taste of the day, by the preferences of prominent men, by the preoccupations of the public.

Thus it comes to pass that, in mathematics as elsewhere, fashion will sometimes award the laurels to those who have not deserved the triumph and make victims of men whose lack of success is an injustice. In every country there are such victors and such victims; but nowhere perhaps are they

* Translated from the MS. of the author by Professor ALEXANDER ZIWET.

more numerous than in France. In this country where centralization is carried to an extreme, nothing is accepted unless it receive the sanction of Paris, or rather of certain constituted bodies, of certain official persons residing in Paris. Those who have been so fortunate as to have their work noticed by these persons and approved by these bodies, who have been granted admission to the chairs of the capital, form in the opinion of the French public the only men of science worthy of honor. The others, relegated to the provinces, are left to oblivion, almost like those *seigneurs* in the age of Louis XIV., whom a caprice of the monarch relegated to their country estates. Such are the reflections suggested to my mind by the contemplation of the life and works of Emile Mathieu. An indefatigable and productive worker he leaves behind him the results of a lifework, partly as newly acquired possessions of science, partly as suggestions that will open new paths to the seeker after truth. After a life full of disappointments, he died at a time when the official men of science hardly had begun to suspect that somewhere in the provinces, far away from the capital, there lived a mathematician whose works were an honor to his country. These works had one defect: the subjects they treated, the methods they employed, were not in fashion!

Emile Mathieu was born at Metz, on the 15th of May, 1835.* From early youth he showed a taste for study. While attending the *lycée* at Metz he was year after year awarded, by the consent of his fellow-pupils, the prize for scholarship and conduct. His uncle Aubertin, colonel of artillery and director of the gun foundries at Metz, was there to point him the way to the *Ecole Polytechnique*. But at this period it was not in mathematics he excelled, but in the study of the classics; the prizes he took again and again in those early years at the *lycée* were for Latin and Greek compositions. However, his special aptitude for the abstract sciences soon developed itself. From the time he reached the higher grades at the *lycée*, he continued to rank first in mathematics. He entered the Polytechnic School at an early age. There he devoted himself exclusively to mathematical studies; and a few months after leaving this institution he resigned his commission in the army to give himself entirely to scientific work. While yet at the Polytechnic School he had published an interesting paper † in which he extended to finite differences the algebra-

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* The biographical data contained in this article are for the most part taken from the *Notice sur E. Mathieu, sa vie et ses travaux*, prepared by his colleague G. Floquet for the *Bulletin de la Société des sciences de Nancy*.

† "*Nouveaux théorèmes sur les équations algébriques*," in *Nouv. Ann. de math.*, vol. 15 (1856), pp. 409–430.

ical theorems of Descartes and Budan regarding derivatives and differentials. This paper proved of some service to Mathieu when he presented himself for the degree of Bachelor of Science, which he had neglected to do before. As Duhamel began examining him in algebra the candidate presented to him a copy of his pamphlet, and the professor after glancing through its pages declared the examination finished.

Scarcely eighteen months had elapsed since this first examination when Mathieu, who had not yet reached the age of twenty-four, took the degree of Doctor of the Mathematical Sciences. On the 28th of March, 1859, he defended before the Sorbonne his thesis *On the number of values a function can assume, and on the formation of certain multiply transitive functions.* This thesis was very favorably received by the Faculty. The theory of substitutions which formed the subject of this thesis furnished the young mathematician material for two otherimportant papers,* which were published in Liouville's *Journal* between the years 1859 and 1862. In these papers Mathieu investigates more fully the idea of multiply transitive functions. He studies in particular the various classes of multiply transitive functions whose degree is a power of a prime number or such a power increased by one. In the course of this study he discovered the curious fivefold transitive function of 12 elements. This function and the fourfold transitive function of 11 elements which he also investigated form two entirely isolated cases in the domain of transitive functions as was shown by C. Jordan.†

In another memoir,‡ published in 1862 in the *Annali di Matematica*, Mathieu undertakes to apply to the solution of equations whose degree is a power of a prime a resolvent function which stands in the same relation to these equations as does Lagrange's resolvent to the equations whose degree is a prime number. These important researches concerning the most difficult parts of algebra and appearing within so brief a period could not fail to attract the attention of the scientific world to the young mathematician; and this attention soon manifested itself in the most flattering manner. In April, 1862, the Paris Academy of Sciences had to elect a

* "*Mémoire sur le nombre de valeurs que peut acquérir une fonction quand on y permute ses variables de toutes les manières possibles,*" in Liouville's *Journ. de math.*, 2 series, vol. 5 (1860), pp. 9–42; and "*Mémoire sur l'étude des fonctions de plusieurs quantités, sur la manière de les former et sur les substitutions qui les laissent invariables,*" ib., vol. 6 (1861), pp. 241–323.

† "*Recherches sur les substitutions,*" in Liouville's *Journ.*, 2 ser., vol. 17, p. 351.

‡ "*Mémoire sur la résolution des équations dont le degré est une puissance d'un nombre premier,*" in Tortolini's *Ann. di mat.*, vol. 4, pp. 113–132.

member in the section of geometry. Lamé who at the time was dean of the section asked that the name of E. Mathieu be placed on the list of candidates. Nor was Lamé alone with his opinion in the Academy ; Liouville fully approved it. This honor conferred upon a young man of not yet twenty-seven years of age who had not taken any steps to solicit such distinction was indeed a brilliant promise for the future. Who would then have predicted that he who so early received this promise was to die at the age of fifty-five, after a life wholly consecrated to the advance of science without being admitted by the Academy even among the number of its correspondents? Mathieu at that time held no official appointment. While engaged in the profound researches which gained him the favor of Lamé he was compelled to make a living by devoting himself to the exhausting and thankless work of a private tutor. Prouhet who was examiner (*répétiteur*) at the Polytechnic School procured him employment as assistant, or "quiz-master," in the *lycée* St. Louis, the *lycée* Charlemagne, and various private schools. These unremitting labors brought on a serious illness from which he at length recovered thanks to the care of his devoted mother.

In 1863 Mathieu first entered upon the study of mathematical physics. In a note *On the flow of liquids through tubes of very small diameter*, published in the *Comptes rendus* of the Academy of Sciences,* he shows that the adhesion of a very thin layer of liquid to the walls of the tube is sufficient to account for the results of Poiseuille's experiments. A few years later, in 1866, he published an important paper *On the dispersion of light.*† In the same year Lamé, whom ill-health prevented from continuing his course at the Sorbonne on mathematical physics and the theory of probability, presented him as his substitute to Duruy, then minister of public instruction. He was however not appointed as the minister had already made his selection. This chair of mathematical physics at the Sorbonne was to remain for Mathieu the never-attained goal of his ambition. In 1867, at the Congress of the Scientific Societies, a gold medal was awarded him for his fruitful researches. At the same time J. Bertrand published his well-known *Report on the progress of mathematical analysis.* The following passage is found in this report:

"M. E. Mathieu has studied far more fully than had been done before, the idea of transitive functions, first introduced by Cauchy, and his memoir deserves quite special mention,

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* "*Sur le mouvement des liquides dans les tubes de très-petit diamètre,*" in *Comptes rendus*, vol. 57 (1863), pp. 320–324.

† "*Mémoire sur la dispersion de la lumière,*" in Liouville's *Journ.*, 2 ser., vol. 11 (1866), pp. 49–102.

on account both of the importance of the new results it contains and of the ingenious form of the proofs. Other memoirs by M. Mathieu, relating to mathematical physics, give evidence, like his algebraical researches, of acute penetration and broad learning. An account of these memoirs will be given in another report, whose author, I trust, will heartily join me in calling attention to a young man who truly possesses the gifts of a mathematician, but has so far, in spite of the estimation in which he is held by all, remained outside the sphere to which his remarkable investigations ought to gain him ready access."

Toward the end of the year 1867, M. Duruy, influenced no doubt by the high reputation attained by the name of E. Mathieu, offered him the complementary course in mathematical physics just then created at the Sorbonne. This was an entrance to public instruction : the young mathematician accepted with eagerness. He published later, in 1872, the substance of this complementary course in a work to which we shall have to return. This work shows him thoroughly imbued with the teachings of the great masters, Fourier, Laplace, Poisson, Lamé. He proves himself fully conversant with their methods of integration and knows how to use them for the treatment of questions as yet unapproached, such as the difficult problem of the cooling of a planetary ellipsoid. Mathieu had already turned his attention to mathematical physics, having published a memoir on the theory of light, when this appointment determined him to devote his main efforts to the applications of analysis to mechanics and physics. He did not, however, completely abandon the pursuit of pure mathematics. Thus, in 1867, he published an important paper *On the theory of biquadratic remainders.** Gauss had found by induction that the biquadratic character of a prime number depends on its decomposition into the sum of two squares ; but he did not succeed in discovering the law of this dependence of which he says : "*At lex hujus distributionis abstrusior videtur, etiamsi quædam generalia prompte animadvertantur.*" Mathieu in his memoir actually discovers and proves this law. Later, in 1873, we see him return to the theory of substitutions and investigate the relations of his fourfold transitive function of 11 elements to Kronecker's function of 11 elements.†

But these investigations in pure analysis must henceforth be regarded as constituting only an incidental part of Ma-

* "*Mémoire sur la théorie des résidus biquadratiques,*" in Liouville's *Journ.*, 2 ser., vol. 12 (1867), pp. 377–438.

† "*Sur la fonction cinq fois transitive de 24 quantités,*" in Liouville's *Journ.*, 2 ser., vol. 18 (1873), pp. 25–46.

thieu's work. Researches in analytical mechanics, in celestial mechanics, in mathematical physics become the constant object of his meditations. In spite of the importance of the results obtained by him in the domain of the theory of substitutions and the theory of numbers, it can therefore be said that what characterizes his scientific individuality is his work in applied mathematics. "Not having found the encouragement I had expected for my researches in pure mathematics, I gradually inclined toward applied mathematics, not for the sake of any gain that I might derive from them, but in the hope that the results of my investigations would more engage the interest of scientific men." In this hope he was deceived. The death of Lamé resulted in finally bringing mathematical physics into discredit in France. D'Alembert, Clairaut, Lagrange, Laplace, Legendre, Fourier, Poisson, Cauchy, Navier. Fresnel, Ampère, Sadi Carnot, Clapeyron, Lamé, accumulated in the course of a century the discoveries that had grown out of the fruitful union of mathematical speculation and the observation of nature. Reactions are abrupt and extreme in the country that had brought forth this succession of men of genius. Suddenly, the path they had laid open was forgotten; the results of their researches were no more known to their successors; the problems that had occupied their minds were regarded as futile and childish; and while the higher minds took refuge in the realm of mathematical combinations devoid of all reality, the great mass of students turned to the ascertainment of facts, to experimentation without theory, without idea.

It is this forgotten, despised, and scorned tradition of the great mathematical physicists that E. Mathieu had the ambition and the honor to follow, in the face of the indifference of his time. All these great authors he studies with passion, he expounds and compares them, he corrects their errors, he elucidates and complements the most rigorous propositions they had obtained. He is imbued with their spirit; he fully appreciates whatever in their ideas is imperishable, and once in a while, as in the preface to his *Course of mathematical physics*, he has a smile of pity for those who pretend to despise that with which they are unacquainted. There is one among these masters to whom he has a particular affinity; it is Poisson,—Poisson who is too fertile in resource, too powerful in genius, to be appreciated at his full value by those, so numerous to-day, who dread long memoirs and difficult analytical processes. Mathieu had studied him thoroughly; he might be said to be his successor. Such tendencies were not calculated to secure Mathieu in the good graces of his contemporaries. His researches might require great intellectual qualities; they might be fraught with beautiful results;

what of it? He was the champion of a science that was out of fashion.

The complementary course to which he had been appointed did not promise him a very stable position. The future seemed so little assured that when the chair of pure mathematics at Besançon became vacant, Mathieu did not hesitate to apply for it. The scientific men who constituted the Council for the Improvement of the Polytechnic School unanimously recommended him for the position, and he received it without difficulty. Four years later, in 1873, he was transferred in the same capacity to Nancy. From this time on he finds himself relegated to profound and undeserved oblivion. Several times chairs become vacant at the Sorbonne, at the *Collège de France*; by his memoirs and his books, he is just the man to fill the place; and yet nobody thinks of seriously considering his candidacy. The Academy of Sciences forgets that, somewhere in France, there lives a man who through the whole of his scientific work, through the advance produced by it in physics, is fully entitled to its rewards and honors. And only a few months before his death is he at last allowed to adorn his button-hole with that decoration which is so stingily bestowed upon those who honor their country, and so profusely on those who reap honor and profit from their fatherland.

It had been Poisson's desire to give, in a series of works, a connected view of all that is rigorously known of mathematics as applied to the study of nature; but he had not time to publish more than four volumes of this gigantic undertaking: his *Treatise on mechanics*, his *Theory of heat*, and his *Theory of capillary action*. Mathieu conceived this same idea whose execution, owing to the broad advances made in all branches of mathematical physics since Poisson's time had assumed far wider dimensions though, on the other hand, the task had perhaps become more simple and easier to accomplish on account of the comprehensiveness and generality of modern analytical methods. More fortunate than Poisson, he was able to carry the work much farther than his predecessors; but he, too, died before accomplishing his purpose. Eight of the eleven volumes that this work was to comprise have been published. The first of these volumes appeared in 1873. It bears the title *Course of mathematical physics*;* but as the author himself remarked later on, it should have been called: On the methods of integration in mathematical physics. It represents the final form of the analytical introduction to the physical sciences that Mathieu had given his students at the Sorbonne in 1867–1868. In 1878 appeared the *Analytical*

* *Cours de physique mathématique*. Paris, Gauthier-Villars, 1874. 4to.

dynamics,* a kind of introduction to celestial mechanics. In 1883, Messrs. Gauthier-Villars published the *Theory of capillarity*,† in sumptuous typographical execution. This was followed, in 1885–1886, by the two volumes on the *Theory of the potential and its applications to electrostatics and magnetism* ‡; in 1888, by a volume on the *Theory of electrodynamics* §; finally, in 1890, by two volumes on the *Theory of elasticity of solid bodies.* ‖

At the time of his death Mathieu was actively engaged on the theory of the elasticity of the ether, that is on optics. We have examined with painstaking care the papers left by the indefatigable worker when the chain of his meditations was broken forever. But the hope of obtaining at least some fragments of the work he had planned was not realized; the notes we had in our hands did not bear the stamp of the author's genius. We could only trace out from them the general plan of this treatise which was intended to give an exposition of the traditional science of optics as elaborated, after Fresnel, by Green, MacCullagh, Newman, Lamé, and G. Kirchhoff. Within the narrow limits of this article it would be impossible to give a full account of the numerous new results dispersed throughout the memoirs and books published by Mathieu. We shall only try to sketch the general tendencies which mark his character and individuality as distinct from that of the army of mathematical physicists.

Like Poisson and Lamé, Mathieu is skilful in treating particular problems of mathematical physics, in integrating the partial differential equations in certain special cases. Let us here mention for illustration a few of the more difficult among the questions of this nature that he succeeded in solving. As early as in 1868 we find him engaged on a problem presenting great difficulties; it is the theory of the oscillatory motions of a homogeneous elliptic membrane subjected to an equal tension in all directions.¶ He succeeded in determining completely the sounds it produces and the shape of its nodal lines. The only cases whose solution was known before this

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* *Dynamique analytique*, Gauthier-Villars, 1878. 4to.

† *Théorie de la capillarité*, ib., 1883. 4to.

‡ *Théorie du potentiel et ses applications à l'électrostatique et au magnétisme. Ire partie: Théorie du potentiel*, ib., 1885. *IIe partie: Electrostatique et magnétisme*, ib., 1886. 4to.

§ *Théorie de l'électrodynamique*, ib., 1888. 4to.

‖ *Théorie de l'élasticité des corps solides. Ire partie: Considérations générales sur l'élasticité; emploi des coordonnées curvilignes; problèmes relatifs à l'équilibre de l'élasticité; plaques vibrantes*; ib., 1890. *IIe partie: Mouvements vibratoires des corps solides; équilibre de l'élasticité du prisme rectangle*; ib., 1890. 4to.

¶ "*Mémoire sur le mouvement vibratoire d'une membrane de forme elliptique*," in Liouville's *Journ.*, 2 ser., vol. 13 (1868), pp. 137–208.

are the vibrations of rectangular and triangular membranes whose theory was shown by Lamé to be intimately connected with certain delicate questions in the theory of numbers, and the vibrations of circular membranes whose properties depend on Bessel's functions. In his *Course of mathematical physics* Mathieu treats another difficult problem which had been pointed out to him by Lamé and which is somewhat allied to the preceding question, viz. the cooling of a planetary ellipsoid. There is a problem to which Lamé attached great importance; it is the theory of the deformations of a rectangular parallelopipedon whose six faces are subjected to forces distributed in any way whatever. After a long and unsuccessful study of the question he made it repeatedly the subject of one of the prizes of the Academy of Sciences, but without result. Mathieu succeeded in solving, if not the general problem, at least a rather comprehensive special case: the extremities of the prism pressing against two fixed walls and the external force being the same along a generating line of the prism.

We may also mention among the special problems solved by Mathieu the difficult question of the distribution of the electric currents in a rectangular prism or a rectangular lamina. But in spite of the analytical power displayed in the solution of such particular cases, they do not constitute the most important and characteristic part of Mathieu's work, the part that distinguishes his work from that of Fourier, of Poisson, of Cauchy. In the first place it must be said that, while full of respect for the tradition of these men of genius, Mathieu does not allow this reverence to become a superstition; he knows where to depart from their views. In the theory of elasticity he does not hesitate to abandon Poisson's favorite theory of molecular attraction to follow the more rigorous ideas of Lamé. In optics he shows that the calculations by which Cauchy thought to have unfolded the nature of the ether and its relations to ponderable matter lead to inadmissible conclusions; and he boldly modifies the differential equations by which the great master had represented the motion of light in an absorbing medium. In the second place, Mathieu is far more mindful of the generality of the methods he uses than was the custom with the great mathematicians of the beginning of the century. In the very preface to his *Course of mathematical physics* we see him proclaim his ideas on this point with perfect distinctness: "As the domain of science broadens and expands it becomes more and more necessary to expound its principles with clearness and conciseness and to substitute for artificial processes, however skilful, the transformations that can be accounted for by the nature of the subject. This is clearly illustrated in comparing the *Mécanique analytique* of Lagrange with the *Vorlesungen* of

Jacobi on the same subject. Examining the treatment of certain problems in each of these works the results obtained will often be found to be the same; the difference consists in the fact that in the latter work the calculations are performed according to rules laid down in advance."

The ideas expressed in this passage are everywhere kept in view in Mathieu's treatises. In his *Analytical dynamics* he introduces at the very beginning the general methods due to Hamilton and Jacobi. In his *Theory of capillarity* he lays aside the direct consideration of the capillary forces employed by Poisson, and follows Gauss in establishing the equations for the various problems by seeking to determine the minimum of the potential of the active forces. In the *Theory of the elasticity of solid bodies* he invariably uses the principle of virtual velocities to throw the problems of equilibrium into equations. This care for generality is also Mathieu's guide in the solution of problems requiring the use of hypotheses that are uncertain or only approximately true. Following a method which in our opinion could not be too much recommended, he always begins by establishing the equations of the problem and treating them as long as possible without making use of those hypotheses so as to introduce them only at the end. In this way he has treated the motion of a projectile in the air and the equilibrium of rods. General methods have the advantage of bringing into clear perspective the principles that serve to solve the problems, and in this way they will frequently lead one to recognize the possibility of attacking a problem which might seem to escape the treatment by more special processes. Mathieu has thus succeeded in throwing a clear light on certain theories not hitherto approached. We may mention two examples.

The oscillatory motion of a plane lamina had been treated before; but not that of a curved lamina. Without solving the problem of the vibrations of an absolutely general curved plate, Mathieu prepared at least the way for the solution of the general problem, reconnoitring, so to speak, the ground in two important directions, by studying the vibrations of a cylindrical plate of any cross-section, or *curved lamina*, and those of a plate of revolution, or *bell*. Let us briefly examine the results of his memoir *On the oscillatory motion of bells.** The thickness of ordinary bells is not generally the same throughout. Hence, to obtain a theory applicable to ordinary bells, the thickness must be assumed to vary in passing along any meridian from the top of the bell to the base. There is an essential distinction between the vibratory motion of a bell and that of a plane

* "*Mémoire sur le mouvement vibratoire des cloches,*" in *Journ. de l'École Polytechn.*, *Cah.* 51, pp. 177–247.

lamina. In the latter, as is well known, the longitudinal or tangential motion and the transversal or normal motion are given by different equations. In a bell, the normal and tangential vibrations are given by three equations which are not independent of each other. Another distinction from the case of a plane lamina lies in the fact that the pitch of a bell does not change if its thickness be varied throughout in the same ratio, since the terms depending on the square of the thickness in the differential equations are generally very small and may be neglected. This, at least, will be the case if only the deepest sounds produced by the bell are taken into consideration. When a bell vibrates under the strokes of the clapper the tangential vibrations are generally of the same order of magnitude as the normal vibrations. The author has examined whether it be possible to so select the meridian of a bell as to give it a purely tangential vibratory motion; and he has shown that this is only possible in a spherical bell of constant thickness. Although the differential equations of the most general vibratory motion of a spherical bell present themselves under a rather complicated form, Mathieu has succeeded in integrating them by means of formulæ of remarkable simplicity.

In connection with this theory of the vibrations of a bell which Mathieu owes to the generality of his methods we may mention another result which, though of a more special nature, deserves to remain classic on account of its intrinsic importance as well as of the elegance of the demonstration. This is the complete determination of the action produced by the capillary forces on a solid body partly immersed in a liquid. Poisson had only succeeded, by very complicated though skilful processes, in determining the vertical upward pressure produced on a solid of revolution whose axis is vertical.

But we must now speak of what is perhaps the most noteworthy part of Mathieu's work. Most problems of mathematical physics depend not only on one or more partial differential equations but also on so-called *boundary conditions* adapted to determine the arbitrary functions introduced by the integration of the equations. This science requires therefore the investigation of partial differential equations, not taken by themselves, but in connection with such boundary conditions and taking into account the form of any such conditions that may be given. This method has proved a remarkably fruitful source of beautiful results in analysis. It will be sufficient to mention the theory of the potential and the numerous theories connected with the principle of Dirichlet. If investigations of this kind have for some time been neglected by pure mathematicians, they have always called forth the efforts of the physicists. Poisson, Helmholtz, Kirchhoff have accumulated the results in the study of the equations that occur in the theories of sound and light. They have thus prepared the way for the resumption of those researches which, owing to the labors of H. A. Schwarz and E. Picard, are at the present day again coming into general favor. And in the work of maintaining this now triumphant tradition only gross injustice could refuse to recognize the important part taken by E. Mathieu. He has devoted a large number of memoirs to researches of this nature concerning the equations of sound, of elasticity, and of heat.

As early as in 1868, in his memoir on the vibrations of an elliptic membrane, he indicates or foresees a part of the results established later on by H. A. Schwarz and E. Picard in their beautiful researches on the equation

$$\frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} + pu = 0.$$

At a later period he rediscovers and systematizes the results obtained by Helmholtz in studying the equation

$$\Delta u + k^2 u = 0.$$

Similar considerations he extends to the equation

$$\Delta u = k \frac{\delta u}{\delta t}.$$

But his most noteworthy researches in this field are those on the partial differential equation of the fourth order

$$\Delta\Delta u = 0,$$

which governs the components of the pressures and the components of the displacements at the interior of an isotropic body in the state of elastic equilibrium.* Designating as *first potential* the function usually denoted simply as potential, the author considers under the name of *second potential* another analytical expression differing from the former in having the distance of two points substituted for the inverse of the distance; and he develops the entirely new theory of this second potential. Concerning the partial differential equation of the fourth order which expresses the equilibrium of elasticity he proves the following theorem : Every function that satisfies this equation at the interior of a surface and is there continuous itself as well as its derivatives of the first three orders is

* "*Mémoire sur l'équation aux différences partielles du quatrième ordre* $\Delta\Delta u = 0$, *et sur l'équilibre d'élasticité d'un corps solide*," in Liouville's *Journ.*, 2 ser., vol. 14 (1869), pp. 378–421.

the sum of the first potential of a layer covering the bounding surface and of the second potential of another layer spread over the same surface.

I here conclude this exposition of Mathieu's work, not for want of material, for I have said nothing of his researches in the theory of perturbations and regarding the problem of three bodies, but in order not to exceed the limits of this article. Besides, whoever desires to obtain accurate information on the state in which his predecessors had left the science of mathematical physics and on the advances made in it by himself can do better than read me by reading *him*. I have known E. Mathieu only as a man of science; and I have spoken of him only as such. To describe the man I must borrow the testimony of one of those who have best known him, one of his colleagues at the Faculty of Sciences of Nancy *: "Of an essentially straightforward, sincere, and generous nature, he was kindness itself. He possessed the devotion that seeks to be ignored. In July, 1890, when the fatal disease had already attacked him, he succeeded in concealing his ill-health from his colleagues, being unwilling to leave to them the burden of his examinations. In September, on his deathbed the same anxiety agitated his mind with respect to the October examinations. His loyal and trustworthy character made him esteemed and beloved by all; in the Faculty at Nancy he had none but friends. Sensitive to any kindness, touched by the slightest mark of sympathy, he belonged to those who are most easily satisfied. He lived in simple style dividing his time between his lectures and his mathematical researches."

P. DUHEM,
Lecturer in Mathematical Physics and Crystallography at the Faculty of Sciences of Lille.

* G. Floquet, *loc. cit.*

NOTES.

A REGULAR meeting of the NEW YORK MATHEMATICAL SOCIETY was held Saturday afternoon, March 5, at half-past three o'clock, the president in the chair. The following persons having been duly nominated, and being recommended by the council, were elected to membership: Professor Arthur Cayley, Cambridge University, England; Professor J. de Mendizábal Tamborrel, Military College, Mexico; Professor Truman Henry Safford, Williams College; Professor Edmund A. Engler, Washington University. The secretary read letters from Professor Cayley and Professor Sylvester expressing interest in the work of the Society.—The following original papers were read: "On exact analysis as the basis of language," by Professor Alexander Macfarlane; "A geometrical construction for finding the foci of the sections of a cone of revolution," by Professor Edmund A. Engler. Mr. Maclay made some remarks upon the locus of the centers of curvature of parallel sections of a ruled surface at points upon the same generatrix.

PROFESSOR J. J. SYLVESTER has been compelled to apply for leave of absence from Oxford on account of ill health. Mr. J. Griffiths of Jesus College, Oxford, will lecture on the "Recent geometry of the circle and triangle" for the professor.

WE learn from *Nature* that a memorial is to be presented to the University of Oxford by the council of the Association for the Improvement of Geometrical Teaching in regard to the Pass Examination papers in geometry. These generally consist entirely of propositions enunciated without any variation from the ordinary text of Euclid, and scarcely any attempt is made to discover whether a student's answers are other than the outcome of a mere effort of memory. The Association is of the opinion that such papers have the effect of a direct incentive to unintelligent teaching, and respectfully asks for the introduction of simple exercises and of simple questions suited to promote the rational study of geometry.

THE Register Publishing Co., Ann Arbor, announces that it has completed the publication of Professor Cole's translation of the "Theory of substitutions and its applications to algebra," by Professor E. Netto.

Naturæ Novitates announces the death of Dr. H. E.

Schroeter, Professor of Mathematics at the University of Breslau, on January 3, in his sixty-third year. T. S. F.

SIR ROBERT STAWELL BALL, Astronomer Royal for Ireland, and Professor of Astronomy at Trinity College, Dublin, has been elected Lowndean Professor of Astronomy at Cambridge University, to succeed the late Professor John Couch Adams. H. J.

NEW PUBLICATIONS.

COMPILED BY B. WESTERMANN & CO., NEW YORK.

AMANZIO (D.). Trattato di Algebra elementare. Napoli 1892. 12. 513 pg. M. 4.20

BALL (W. W. R.). Mathematical Recreations and Problems of past and present times. London and New York, Macmillan, 1892. 12mo, pp. 252. $2.25

BOURDON (J.). Tables pour le tracé des Courbes de raccordement par Angles successifs. Issoudun 1891. 12. 32 pg. av. figures. M. 1.20

CELS (J.). Sur les Equations Différentielles linéaires ordinaires. Paris 1891. 4. 80 pg. M. 4

ENGEL (G.). Die Bedeutung der Zahlenverhältnisse für die Tonempfindung. Dresden 1892. gr. 8. 59 pg. M. 1.50

FISCHER (A.). Ueber die Invarianten der linearen homogenen Differentialgleichungen sechster Ordnung. Halle 1891. 8. 32 pg. M. 120

GAERTNER (R.). Theilungen. Halle 1891. 8. 43 pg. M. 1.20

GRAY (John). Les Machines électriques à influence. Exposé complet de leur histoire et de leur théorie, suivi d'instructions pratiques sur la manière de les construire. Traduit de l'anglais et annoté par Georges Pellissier. In-8 avec figures. Gauthier-Villars. 5 fr.

GUYOU. Note sur les Approximations numériques. 2. édition. Paris 1891. 8. 51 pg. M. 0.80

KÖNIG (A.). Ueber den Helligkeitswerth der Spectralfarben bei verschiedener absoluter Intensität. Hamburg 1892. gr. 8. 80 pg. m. 4 Tafeln. M. 4

LOHSE (O.). Beobachtungen des Planeten Mars. Leipzig 1891. 4. 42 pg. m. 4 Tafeln. M. 5

MENZEL (H.). Ueber die Bewegung einer starren Geraden, welche mit mehreren von ihren Punkten in festen Ebenen oder auf festen Geraden gleitet. Münster 1891. 8. 64 pg. M. 1.80

NEW Departure in Astronomy: The Revolution of the Solar System. By E. H. Cr. 8vo, sd. Chapman and Hall. 2s.

NOUVEAU SYSTÈME de Calcul Intégral et Différentiel naturel direct, exposé à l'aide de l'algèbre élémentaire seulement. De la mesure des courbes qui en résulte. Par. S. G. F. 2. édition. Paris 1892. gr. in-8. M. 1.50

PETERSEN (G.). Teoria delle Equazioni algebriche. Versione italiana da G. Rozzolino e G. Sforza. Volume II. Napoli 1891. 8. 182 pg. M. 4.80

POULAIN (Aug.). Principes de la nouvelle géométrie du triangle. Gr. in-8. Croville-Morant. 2 fr. 50

PREYER (W.). Ueber den Ursprung des Zahlbegriffs aus dem Tonsinn und über das Wesen der Primzahlen. Hamburg 1892. gr. 8. 36 pg. M. 1.50

RICHARD. Notice sur les Coordonnées rectangulaires. Règles pratiques, exemples d'application. Bray-sur-Seine 1891. 8. 48 pg. av. figures. M. 2

SIMON (M.). Zu den Grundlagen der nicht-euklidischen Geometrie. Strassburg 1891. 4. 32 pg. m. 1 Tafel. M. 1.50

STIEDER (L.). Ueber die Anwendung der Wahrscheinlichkeitsrechnung in der anthropologischen Statistik. 2. Auflage. Braunsweig 1892. gr. 8. 6 u. 26 pg. M. 1.20

TUMLIRZ (O.). Théorie électromagnétique de la lumière. Traduit par G. van der Mensbrugghe. Enrichi d'additions faites par l'auteur. Paris 1892. gr. in-8. 12 et 157 pg. avec figures. M. 7

VIOLLE (J.). Cours de Physique. Tome II : Acoustique et Optique. Partie 2 : Optique géométrique. Paris 1892. gr. in-8. 356 pg. avec 276 figures. M. 9

WALTER (B.). Ueber die lichtverzögernde Kraft gelöster Salzmolekühle und ein Verfahren zur genaueren Bestimmung von Brechungsexponenten. Hamburg 1891. Lex. 8. 36 pg. M. 1.50

WEBER (R.). Problèmes sur l'Électricité. Recueil gradué contenant toutes les parties de la science électrique. 2. édition considérablement augmentée. Paris 1892. 12. avec figures. M. 5

WINKELMANN (A.). Handbuch der Physik. Herausgegeben unter Mitwirkung von F. Auerbach, F. Braun, S. Czapski, K. Exner u. A. (3 Bände in ca. 20 Lieferungen.). Breslau 1892. gr. 8. m. Holzschnitten.—Liefg. 11 : pg. 257—384 (v. Band 3). Jede Liefg. M. 3.60

WORTHINGTON (A. M.). Dynamics of Rotation : An Elementary Introduction to Rigid Dynamics. Cr. 8vo, pp. 162. Longmans. 3s. 6d.

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KRONECKER AND HIS ARITHMETICAL THEORY OF THE ALGEBRAIC EQUATION.

LEOPOLD KRONECKER, one of the most illustrious of contemporary mathematicians, died at Berlin on the 29th of last December in his 68th year.

For many years he had been one of the famous mathematicians of Germany and at the time of his death was senior active professor of the mathematical faculty of the University of Berlin and editor in chief of the *Journal für reine und angewandte Mathematik* (Crelle).

He was the last of the great triumvirate—Kummer, Weierstrass, Kronecker—to be lost to the university. Kummer retired nearly ten years ago because of sickness and old age, and recently Weierstrass followed him; but younger than the other two, Kronecker was overtaken by death in the midst of the work to which his life had been devoted. Despite his years he was much too early lost to science. The genius which had enriched mathematical literature with so many profound and beautiful researches showed no signs of weakness or weariness.

Kronecker was born at Liegnitz near Breslau in 1823. While yet a boy at the Gymnasium of his native town his fine mathematical talents attracted the notice of his master, Kummer, whose distinguished career was then just beginning. Kummer's persuasions rescued him from the business career for which he was preparing and brought him to the university.*

He studied at Breslau, whither in the meantime Kummer had been called, Bonn, and Berlin, making his degree at Berlin in 1845 with a dissertation of great value: *De unitatibus complexis.*

Of his instructors besides Kummer he was most influenced by Dirichlet, owing in part to Dirichlet's commanding abilities, in part to the strong arithmetical bent of Kronecker himself. As long as Dirichlet lived Kronecker's relations with him, as with Kummer, were those of the closest personal intimacy.

From the university Kronecker returned for a number of years to business and the management of his estates. But

* Kronecker makes this graceful acknowledgment of his debt to Kummer in the dedication of the *Festschrift* with which he honored Kummer's Doctor *Jubeläum* (*Grundzüge einer arithmetischen Theorie der algebraischen Grössen*): "In Wahrheit verdanke ich Dir mein mathematisches Dasein; ich verdanke Dir in der Wissenschaft die Du mich früh zugewendet wie in der Freundschaft die Du mir früh entgegengebracht hast, einen wesentlichen Theil des Glücks meines Lebens."

his mathematical activity was continuous and his fame grew apace. In 1853 he communicated to the Berlin Academy the solution of the problem: to determine all abelian equations belonging to any assigned "domain of rationality," and in 1857 the first of his famous memoirs on the complex multiplication of the elliptic functions. His letter to Hermite: *sur la résolution de l'équation du 5me degré* in which his solution of the equation is indicated appeared in 1858.

In 1861 he was made a member of the Berlin Academy of Sciences and in 1867 corresponding member of the Paris Academy.

His election to the Berlin Academy was an event of the first importance for his subsequent career, inasmuch as it was the occasion of his resuming the academic life. As member of the Academy he had the right to lecture at the University, and of this right—following the example of such men as the brothers Grimm, Kiepert, Jacobi and Borchardt—he forthwith availed himself, beginning in the winter of 1861–62 those lectures on Algebra which have for many years been one of the chief glories of Berlin. In 1883 his relations with the University were made closer still through the appointment "*Professor ordinarius*" and director—with Kummer and Weierstrass—of the mathematical *Seminar*.

The range of Kronecker's productive activity was very great. Besides distinguished work in the theory of definite integrals, he did work of the first importance in no less than three great departments of mathematics: the theory of numbers, algebra, and elliptic functions. As an arithmetician his name is associated with the great names of Gauss, Dirichlet, and Eisenstein; as an algebraist with those of Abel and Galois.

Some idea of the scope of Kronecker's contributions to mathematical literature may be conveyed by the following incomplete list of his more important memoirs: *Dissertatio de unitatibus complexis* (1845); *Zwei Sätze über Gleichungen mit ganzzahligen Coefficienten* (1857); *Ueber die algebraisch auflösbaren Gleichungen* (1853, 1856); *Sur les facteurs irréductibles de l'expression* $x^n - 1$ (1854); *Ueber elliptische Functionen für welche complexe Multiplication stattfindet* (1857, 1862); *Ueber complexe Einheiten* (1857); *Sur la résolution de l'équation du 5me degré* (1858); *Ueber lineare Transformationen* (1858); *Ueber die Theorie der algebraischen Functionen* (1861); *Ueber die verschiedenen Factoren der Discriminanten von Eliminationsgleichungen* (1865); *Ueber den Affect der Modulargleichungen* (1865); *Ueber bilineare Formen* (1868); *Ueber Systeme von Functionen mehrer Variabeln* (1869, 1878); *Ueber die verschiedenen Sturmschen Reihen und ihre gegenseitigen Beziehungen*

(1873); *Zur Theorie der Elimination einer Variabeln aus zwei algebraischen Gleichungen* (1881); *Zur Theorie der Abelschen Gleichungen* (1882); *Zur arithmetischen Theorie der algebraischen Formen* (1882); *Ueber die Bernouilli'schen Zahlen* (1883); *Ueber bilineare Formen mit vier Variabeln* (1883); *Grundzüge einer arithmetischen Theorie der algebraischen Grössen* (Kummer *Jubcläum* 1882); *Zur Theorie der elliptischen Functionen* (1883–1891); *Ueber den Zahlbegriff* (Zeller *Jubeläum* 1887). Most of his writings were published in the *Berichte der Berliner Akademie* or in the *Journal für reine und angewandte Mathematik.*

Among the finest of Kronecker's achievements were the connections which he established among the various disciplines in which he worked: notably that between the theory of quadratic forms of negative determinant and elliptic functions, through the singular moduli which give rise to the complex multiplication of the elliptic functions, and that between the theory of numbers and algebra, by his arithmetical theory of the algebraic equation.

He discovered * that to each class of quadratic forms corresponds a singular modulus which allows of complex multiplication; to the aggregate of classes of the same determinant, an algebraic equation with rational coefficients which he showed to be irreducible; and, in fine, that the theory of quadratic forms was an anticipation of the theory of elliptic functions, the two theories being so closely related that one could have derived the notions of class and order and other fundamental properties of the quadratic forms by investigation of the properties of the elliptic function.

He was above all things the great arithmetician and nowhere does this appear more clearly than in his algebraic writings. It is not merely that the purely arithmetical problems growing out of algebra were attractive to him—he "arithmetized" algebra itself. In the Zeller *Festschrift*, after declaring his allegiance in the words of Gauss: "Die Mathematik sei die Königin der Wissenschaften und die Arithmetik die Königin der Mathematik," he writes "Und ich glaube auch, dass es dareinst gelingen wird den gesammten Inhalt aller dieser mathematischen Disciplinen (Algebra and Analysis) zu 'arithmetisen', d. h. einzig und allein auf den im engsten Sinne genommenen Zahlbegriff zu gründen, also die Modificationen und Erweiterungen dieses Begriffs wieder abzustreifen, welche zumeist durch die Anwendungen auf die Geometrie und Mechanik veranlasst worden sind."

Kronecker arrived at the conception of an arithmetical theory of the algebraic numbers and functions very early.

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* Cf. Hermite: *Note sur M. Kronecker, Comptes Rendus,* Jan. 4, 1892.

There are indications of it even in a letter to Dirichlet written in 1856. As its various salient concepts and theorems were discovered they were announced in the *Berichte* of the Academy or developed in his lectures. But he did not arrange the whole into a consecutive and complete body of doctrine until 1882 in his *Grundzüge einer arithmetischen Theorie der algebraischen Grössen.* Within the limits of this brief sketch it would be impossible to convey any adequate notion of this monumental work. I can attempt only to indicate the salient points of the first and more elementary of the two parts into which it is divided.

The "domain of rationalty" $(R' R''..)$ of any system of quantities $R' R''$ embraces all rational functions of the R's with integral coefficients.

These R's may be quantities of any sort whatsoever, algebraic or transcendental constants or variables. In particular all the R's may equal 1 when the domain is that of rational numbers in the ordinary sense, or all the R's may be independent variables. In either of these cases the domain is said to be bounded naturally.

An integral function of one or several variables is *irreducible* in the domain $(R' R''..)$ when it contains no factor having coefficients which belong to this domain.

Every root of an irreducible algebraic equation of the n^{th} degree with coefficients which belong to the domain $(R'R''..)$ is called an algebraic function of the n^{th} order of the R's, the n roots of the same equation being called *conjugate* functions.

If a single such root G be "adjoined" to the R's the domain $(G,R',R''..)$ is the domain of the "genus" (Gattung) G, the genus itself embracing those functions of the domain which are, like G, functions of the n^{th} order.

If G and G' be algebraic functions of different genera, but such that all functions of the genus G belong to the domain of G', the genus G is said to be contained in the genus G'; and the order of G is a divisor of that of G'.

More than one G may of course be adjoined to the R's, but it is shown that any number of G's may be replaced by a single such function which indeed is but a linear function of the given G's, with integral coefficients. In terms of this G and the R's all functions of the domain $(G', G''.. : R', R''..)$ can be expressed rationally: or the domain $(G', G''..: R', R''..)$ is equivalent to a domain $(G, R', R'',..)$. This is a theorem of fundamental importance. For from it follows that in the discussion of all algebraic questions there may be selected as "elements" $R', R,''..$ of any domain of rationality whatsoever, a number of variables or indeterminates and a *single* algebraic function of them.

A quantity x is called an *integral* algebraic function of the

R's when it satisfies an equation in which the coefficient of the highest power of x is 1 and the remaining coefficients are integral functions of the R's with integral coefficients. It is a fundamental theorem of the theory that for every genus G there exists a finite number of such integral functions $x', x'', .. x^{(n+m)}$ in terms of which all other integral functions of the genus can be expressed linearly; *i.e.* in the form

$$\varphi' x' + \varphi'' x'' + \ . \ . \ . \ + \varphi^{(n+m)} x^{(n+m)}$$

where the φ's are integral functions of the R's with integral coefficients. Such a system of functions $x', x'', \ldots x^{(n+m)}$ is called a *fundamental system* of the genus. In special and important cases m can equal 0.

The square of the determinant of any set of n of the functions $x', x'', \ldots x^{(n+m)}$ and their conjugate functions is called the *discriminant* of these n functions. The aggregate of the discriminants of every set of n of the functions $x', x'', \ldots x^{(n+m)}$ constitutes a system of rational functions of the R's, such that whatever properties are common to them all belong also to the discriminant of *every* set of n functions of the genus and are thus characteristic of the genus itself, forming a complex of "invariants" of the genus in a higher sense of that word.

If there exist no algebraic relations among the R's, *i.e.* if the domain of rationality be the natural domain, there exists always an integral function of the R's with integral coefficients which is a common divisor of all the discriminants of the fundamental system of the genus and may therefore be appropriately called the *discriminant of the genus* itself. If m equal 0 the discriminant of the n elements $x', x'', \ldots x^{(n)}$ is itself the discriminant of the genus.

The discriminant of the genus is a divisor of the discriminant of every equation of the genus, *i.e.* of every equation a root of which is a function belonging to the genus, and the greatest common divisor of the discriminants of all these equations is a divisor of the $\frac{1}{2}n\,(n-1)$th power of the discriminant of the genus.

Again if the genus G' be contained in the genus G its discriminant will be a factor of the discriminant of G.

And finally the discriminant of the genus to which a set of functions belong which are defined by a *system* of equations $F_1 = 0$, $F_2 = 0$, . . . $F_n = 0$ is identical with the discriminant of this system of equations.

The demonstration of this last theorem as well as the further development of the theory necessitates a general investigation of elimination, the principal outcome of which is that the complete "resolvent" of a system of m equations in n quantities $x', x', x'', \ldots x^{(n)}$ is an equation of the form

$$F_1(x, x', \ldots x^{(n-1)})\, F_2(x, x', \ldots x^{(n-2)}) \ldots F_n\, (x) = 0$$

where $x = u_1 x_1 + n_2 x_2 + \ldots + u_n x_n$, the u's being indeterminates.

The system of equations or "partial resolvents" $F_1 = 0$, $F_2 = 0, \ldots F_n = 0$ is the complete equivalent of the given system. Each partial resolvent $F_k = 0$ represents a manifoldness of $n - k$ dimensions, so that speaking geometrically a given system of equations in n quantities may define simultaneously systems of points, lines, surfaces, etc.

Furthermore every divisor of the product $F_1 . F_2 \ldots F_n$ set equal to 0 constitutes the entire resolvent of a certain system of $n + 1$ equations. Whence the important theorem: the total content of every divisor of the resolvent of a system of equations in n quantities can be represented by a system of only $n + 1$ equations, and therefore also a system of any number of equations can be replaced by one of only $n + 1$. Any algebraic curve of double curvature, for instance, can be represented by a system of four algebraic equations.

Another most important result of this investigation of elimination is the demonstration that the concept of the algebraic function does not require any extension when systems of equations instead of single equations are brought under discussion.

This doctrine of elimination brings out the true significance of Galois' theory of algebraic equations.

Let $c_1, c_2, \ldots c_n$ be quantities belonging to the domain $(R', R'', \ldots)$ and

$$(A)\ x^n - c_1 x^{n-1} + c_2 x^{n-2} - \ldots \pm c_n = 0$$

an irreducible equation with the roots $\xi_1, \xi_2, \ldots \xi_n$.

Further let $f_1\ (x_1, x_2 \ldots x_n)$, $f_2\,(x_1, x_2, \ldots x_n), \ldots f_n\,(x_1, x_2, \ldots x_n)$ be the elementary symmetric functions defined by the identical equation

$$(x - x_1)\ (x - x_2) \ldots (x - x_n) = x^n - f_1 x^{n-1} + f_2 x^{n-2} - \ldots \pm f_n.$$

Then the n quantities $\xi_1, \xi_2, \ldots \xi_n$ may as well be regarded as determined by the system of n equations

$$(B)\ f_k\,(x_1, x_2 \ldots x_n) = c_k \qquad (k = 1, 2, \ldots n)$$

as by the single equation (A).

If $F\,(x) = 0$ be the resolvent of this system (B) or an irreducible part thereof, and, as above,

$$x = u_1 x_1 + u_2 x_2 + \ldots + u_n x_n,$$

the coefficients of $F\,(x)$ are integral functions of the indeterminates u and rational functions of the R's. And since the equations (B) can be satisfied only by systems of values such as

$$x_1 = \xi_{r_1},\ x_2 = \xi_{r_2},\ \ldots\ x_n = \xi_{r_n}$$

where $r_1\ r_2 \ldots r_n$ is some permutation of the numbers $1\ 2\ \ldots n$, $F(x)$ is simply the product

$$\Pi\ (x - u_1\ \xi_{r_1} - u_2\ \xi_{r_2} - \ldots - u_n\ \xi_{r_n})$$

extended over certain of these permutations.

If now

$$G\ (x, f_1, f_2, \ldots f_n) = 0$$

be the "Galois equation" whose $n\,!$ roots are the $n\,!$ functions $u_1 x_{i_1} + u_2 x_{i_2} + \ldots + u_n x_{i_n}$ gotten by forming the $n\,!$ permutations $i_1\ i_2 \ldots i_n$ of $1\ 2 \ldots n$, the coefficients of $x, f_1 \ldots f_n$ in $G\,(x, f_1, f_2, \ldots f_n)$ are integral functions of the u's with integral coefficients, and one of the irreducible factors of $G\,(x, c_1, c_2, \ldots c_n)$ must be the same with $F(x)$. Such a factor is therefore an integral function of x and the indeterminates $u_1, u_2, \ldots u_n$ with coefficients belonging to the domain of rationality $(R', R'' \ldots)$ and may be represented by $g\,(x, u_1, u_2 \ldots u_n)$.

$$g\ (x, u_1, u_2, \ldots u_n) = \Pi\,(x - u_1\xi_{r_1} - u_2\xi_{r_2} - \ldots - u_n\xi_{r_n})$$

or, if the terms of each factor be arranged with reference to the ξ's instead of the u's

$$g\ (x, u_1, u_2, \ldots u_n) = \Pi(x - u_{r_1}\xi_1 - u_{r_2}\xi_2 - \ldots - u_{r_n}\xi_n)\,;$$

that is to say $g\ (x_1, u_1, u_2, \ldots u_n)$, regarded as a function of the indeterminates u, is a function which remains unchanged for certain permutations of these u's, those represented by $r_1, r_2, \ldots r_n$.

In this manner, starting with any special equation (A) one is led to general functions of indeterminates which are characteristic of the equation and have the property of maintaining their values unchanged for certain permutations of these indeterminates.

The true significance of Galois' principle thus lies in the fact that it takes as basis for the investigation of an equation the system of equations which define its conjugate roots simultaneously.

The functions g to which it leads may themselves be made the starting point of the discussion. The problem then is when one replaces the indeterminates $u_1, u_2, \ldots u_n$ by $x_1, x_2, \ldots x_n$: the investigation of integral functions of n indeterminates $x_1, x_2, \ldots x_n$ with respect to the changes which they experience when the x's are permuted in all possible ways, the investigation taking its place in the general arithmetical theory when one regards the x's as algebraic functions of the n elementary symmetric functions f_r.

If the f's take the place of the R's so that the domain of rationality is $(f_1, f_2, \ldots f_n)$ every rational function of the x's is an algebraic function of this domain and as such belongs to a definite genus, called simply *genus of functions of* $x_1, x_2, \ldots x_n$.

The order of the genus of any single one of the x's is n, that of $u_1 x_1 + u_2 x_2 + \ldots + u_n x_n$, the "Galois genus," $n!$ This genus contains all others and therefore their orders are all divisors of $n!$ If ρ be the order of a genus g and $\frac{n!}{\rho}$ be r, r is the "number of permutations of the genus g," *i.e.* the number of permutations of the x's for which any function of the genus g remains unchanged.

A genus g is said to be a genus "proper" if after it is adjoined to the domain of rationality the equation

$$x^n - f_1 x^{n-1} + \ldots \pm f_n = 0$$

remains irreducible. When such a genus g is adjoined, so that the domain of rationality becomes $(f_1, f_2, \ldots f_n, g)$, the algebraic character of a function defined by $x^n - f_1 x^{n-1} + \ldots \pm f_n = 0$ is changed, it falls into a special "class" of algebraic functions. All algebraic equations belong to the same class which go over into each other by rational transformation and for which the functions of the roots belonging to a definite genus g belong also to the given domain of rationality $(R', R'' \ldots)$

This characteristic property of the class of an algebraic equation and the function which it defines may be called its *affect*. An irreducible equation

$$x^n - c_1 x^{n-1} + \ldots \pm c_n = 0$$

whose coefficients belong to the domain $(R', R'', \ldots)$ is said therefore to have a special *affect* where there exists a special function of its roots, which may be called the *affect*-genus, which likewise belongs to the given domain. The group of permutations of this genus is called the *Galois group* of the equation.

The *affect*-genus being $g(x_1, x_2, \ldots x_n)$, it is the system of $n + 1$ equations

$$g = c_0, f_k = c_k, (k = 1, 2, \ldots n)$$

which by Galois' principle takes the place of the single given equation. This system is satisfied only by the r systems of values

$$x_1 = \xi_{r_1}, x_2 = \xi_{r_2}, \ldots x_n = \xi_{r_n}$$

which correspond to the r permutations of the genus g. Its order is therefore r and it constitutes the irreducible part of the system $f_k = c_k$, $(k = 1, 2, \ldots n)$ whose order is $n\,!$

The $n\,!$ functions

$$x_1^{h_1} x_2^{h_2} \ldots x_{n-1}^{h_{n-1}} (h_k = 0, 1, \ldots n - k;\ k = 1, 2, \ldots n - 1)$$

are the elements of a "fundamental system" of the Galois genus; but the number of elements can be reduced to ρ, the order of the genus, if fractional numerical coefficients be allowed.

If the discriminant of the genus x_k, *i.e.*

$$\Pi\,(x_i - x_k)\ \ (i, k = 1, 2, \ldots n\,;\ i \gtrless k),$$

be D, the discriminant of the Galois genus is $D^{\frac{1}{2}n!}$. Therefore, since the discriminant of every other genus is a divisor of that of the Galois genus and D is irreducible, the discriminant of every genus is a power of D. From this fact it follows that for any given set of values of $f_1, f_2, \ldots f_n$ for which D does not vanish, an infinite number of special functions of each genus can be determined all of whose conjugates differ from one another, and in terms of which every other function of the same genus can be expressed rationally. Moreover this theorem leads to a remarkably simple demonstration of the "arithmetical existence" of the roots of algebraic equations.

Upon the profound researches of the second part of the *Grundzüge* we cannot now enter, though this contains the heart of the arithmetical theory. Here, by aid of the "*Modul-systeme*" and the principle of "association" the distinctively "arithmetical" properties of the integral algebraic functions are developed, their properties, namely, when considered with respect to their divisibility by other integral functions of the same genus; and the final step is taken in the "reduction" of the domain of rationality, whereby the entire theory of the algebraic functions is reduced to a theory of the integral functions of variables and indeterminates with integral coefficients.

Thus Kronecker's theory completes that of Galois. For it carries the general theory of equations back to a theory of indeterminates, which, before Galois, it was always assumed to be in the superficial and false sense, that the coefficients, and therefore the roots, of any equation may be treated as indeterminates.

The fine quality of Kronecker's work is even more notable than its range or the importance of its results. It possesses the rigor and elegance of the theory of numbers.

Early in the *Grundzüge*, when defining an irreducible function, Kronecker remarks: "Die Definition der Irreductibilität entbehrt so lange einer sicheren Grundlage als nicht eine Methode angegeben ist, mittels deren bei einer bestimmten vorgelegten Function entschieden werden kann, ob dieselbe der aufgestellten Definition gemäss irreductibel ist oder nicht,"* and proceeds therewith to supply the missing test.

This criterion, according to which no definition may be considered justified, no theorem established, until a method is supplied for determining in every given concrete case whether the definition or theorem actually applies or not, he everywhere insisted upon, scrupulously meeting its requirements in his own work and sharply criticising all failures to meet them in the works of others. A definition which did not stand this test he denominated the invention of a mere fiction, an artificial abstraction for which there should be no place in mathematics.

This is the rigor of the ancient Greek geometry—in rejecting hypothetical constructions Euclid recognized a similar criterion—and though far enough from being always realized in the modern analysis, must characterize every mathematical theory in its finite form. For until it has been attained, either the ultimate elements of the theory have not been reached or the artificial concepts with which it has aided itself in its growth have not been set aside and the theory deduced directly from these elements.

Closely related to this fine conception of mathematical rigor are the other salient traits of Kronecker's work.

It possesses that high artistic merit which consists in the perfect adaptation of means to ends. His methods are always pure, fit, direct, and the simplest which the requirements of absolute rigor will allow. Writing to Dirichlet in 1856 he says of a method which he has discovered for deducing the properties of solvable equations of prime degree that it meets all the proper requirements of simplicity and rigor, "denn die Methode verlangt keinen irgend höheren Standpunkt mathematischen Fassungsvermögens als das Problem selbst, welches dadurch erledigt wird."† And again for his principle of "association" he claims: "Sie gewahrt den 'einfachsten' erforderlichen und hinreichenden Apparat, um die arithmetischen Eigenschaften der allgemeinsten algebraischen grössen 'vollständig' und 'auf die einfachste Weise' darzulegen,"‡ adopting the phrases which are quoted from the first proposition of Kirchhoff's Mechanics. This "*Einfachheit*," to be sure, is of a kind which it oftentimes requires

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* *Grundzüge*, etc., p. 11.
† *Göttinger Nachrichten*, 1885, p. 364.
‡ *Grundzüge*, etc., p. 93.

much reflection to appreciate. He was a foe not only of artificial concepts but of all artificial methods and of all artificial or purely formal tendencies in mathematics. He would have rid mathematics of the artificial numbers and of its "symbolic" methods, and the devising of new functions seemed to him a foolish waste of energy. "God created numbers and geometry," I once heard him say, "but man the functions."

It was his boast that he was the most practical of mathematicians. He said whimsically to me one day last summer: "It is a pity that you, Americans, do not know me better. You would surely appreciate me, I am so practical." And in a somewhat transcendental sense of the word, to be sure, he was profoundly practical. He sought to avoid all mere abstractions and to give his theories concrete form. Thus in the Galois theory he replaced the abstraction, a group of substitutions, by concrete functions which remain unchanged for the substitutions of the group. Neither definition, theorem, nor method had value in his eyes which could not be applied to concrete cases, which could not be made to yield concrete results. On this account he did not set great store by the services of the theory of substitutions to algebra. With all its beauty, he would urge, it is only formal, it does not show how to construct the group of a given equation.

Kronecker influenced the mathematical thinking of Germany as much through his lectures as through his published writings. He was a very stimulating and interesting lecturer. To an unusual degree he took his hearers into his confidence and allowed them the privilege of watching the actual evolution of his thoughts. His lectures were not overprepared, but the details of even important demonstrations were left to take their chances in the lecture room. Occasionally there would be a disastrous slip in the reckoning or argument, or the outcome would be the discovery that the theorem sought to be established was false. But that only afforded opportunity to see the marvellous quickness with which he would run an error down and recover himself.

His lectures were always fresh. The principal courses were on determinants, theory of numbers, algebra, and definite integrals, and one of these in its turn he delivered each semester. But he never merely repeated himself. If a lecture did not differ from all its predecessors in content, it surely did in point of view or method. It was always the most recent product of his mathematical thinking.

In his lecturing, moreover, he avoided the excessive conciseness, which is the chief cause of the difficulty of his published writings.

Personally, Kronecker was most charming and amiable, a polished gentleman and man of the world. He was very generous with his time and thoughts, loving to talk to an appreciative listener of some favorite doctrine, or of the famous mathematicians with whom he had been associated.

He was a man of rare genius, a mathematician of the first rank in this century of great mathematicians.

HENRY B. FINE.

PRINCETON COLLEGE, *April* 20, 1892.

MULTIPLICATION OF SERIES.

BY PROF. FLORIAN CAJORI.

THE salient feature of the new era which analysis entered upon during the first quarter of this century is vividly illustrated in the history of infinite series. Extending from that time back to Newton we have a *formal* period which gave rise to general theorems, the validity of which was not thoroughly tested. Thus, in series, there were put forth during that epoch the binomial theorem, the theorems of Taylor, Maclaurin, John Bernoulli, and Lagrange. Infinite series were used by Newton, Leibnitz, and Euler in the study of transcendental functions. As a rule, the convergency of expressions was not ascertained, and the confusion which prevailed in the theory of series gave rise to curious paradoxes. But with the advent of Gauss, Cauchy, and Abel, began the new era which combined dexterity in form with *rigor of demonstration.*

In the multiplication of series, mathematicians of the earlier period considered simply the form of the products and hardly ever thought of inquiring further into the validity of the operation. Reliable tests for convergency were unknown. The product of any two infinite series was accepted with nearly the same degree of confidence as was the product of finite expressions. Thus, De Moivre * extended the binomial formula to infinite series and deduced the following formula:

$$(az + bz^2 + \ldots)^m$$
$$= a^m z^m + \frac{m}{1} a^{m-1} b z^{m+1} + \frac{m}{1} \cdot \frac{m-1}{2} a^{m-2} b^2 z^{m+2} + \ldots$$

This was accepted as true without any limitations whatever.

* A method of raising an infinite multinomial to any given power, or extracting any given root of the same. *Philosophical Transactions,* No. 230, 1697.

The first to cry "halt" to these reckless proceedings was Baron Cauchy. He instituted for the first time a painstaking examination of the principles of series and strove to introduce absolute rigor. He is the founder of the theory of convergency and divergency. He pointed out that if two series are convergent, their product is not necessarily so.

Thus,

$$1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \ldots$$

is convergent, but its square

$$1 - \frac{2}{\sqrt{2}} + \left(\frac{2}{\sqrt{3}} + \frac{1}{2}\right) - \left(\frac{2}{\sqrt{4}} + \frac{2}{\sqrt{6}}\right) + \ldots$$

is divergent. Not only did he discriminate between convergent and divergent series, but also between what we now call "absolutely convergent series" which are convergent even if all the terms are made positive, and "semi-convergent series" which cease to be convergent when the terms are all made to have like signs. In his *Cours d'analyse algébraique* (1821) Cauchy proved rigorously the following celebrated theorem: *If* Σu_n *and* Σv_n *converge* ABSOLUTELY *to values* U *and* V *respectively, then the series* $\Sigma(u_0v_n + u_1v_{n-1} + \ldots + u_nv_0)$ *converges to the value* UV. So far as the researches of Cauchy went, two absolutely convergent series appeared to be the only ones which could be multiplied by one another with absolute safety. This same theorem was proved also by Abel in course of his demonstration of the binomial formula,* but in the same article he took a giant step in advance by establishing the following theorem: *If the series* Σu_n *and* Σv_n *converge to the limits* U *and* V *respectively, then if the series* $\Sigma(u_0v_n + u_1v_{n-1} + \ldots + u_nv_0)$ *be convergent, it will converge to the product* UV. The beauty of this theorem lies in the fact that all three series in question may be semi-convergent. Strange to say, this result, so remarkable for its simplicity and generality and put forth by so prominent a mathematician as Abel, was for nearly half a century almost universally overlooked. Schlömilch's *Compendium der höheren Analysis* knows it not, nor does Bertrand's *Traité de calcul différentiel.*

Abel's theorem would dispose of the whole problem of multiplication of series, if we had a universal practical criterion of convergency for semi-convergent series. Since we do not

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* Crelle's *Journal*, Bd. 1, 1827; also *Œuvres complètes de N. H. Abel*, Tome 1, p. 66 *et seq.*

possess such a criterion, theorems have been established which remove in certain cases the necessity of applying tests of convergency to the product-series. Such a theorem is that of Mertens * who in 1875 demonstrated that Cauchy's theorem still holds true if, of the two convergent series to be multiplied together, only one is absolutely convergent. Thus, if the absolutely convergent series

$$1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \ldots$$

is multiplied by the semi-convergent series

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \ldots,$$

the product will surely converge to the value $2 \log 2$. A still more comprehensive but more complex theorem was given by Mr. A. Pringsheim, in 1882: † *If* $U = \sum_0^\infty {}_\nu u_\nu$, $V = \sum_0^\infty {}_\nu v_\nu$ *be convergent series of which one, say* U, *has the property that its terms, arranged in certain groups containing always a finite number of terms, constitute an absolutely convergent series, i.e. that*

$$U = (u_0 + u_1 + \ldots + u_{m_1}) + (u_{m_1+1} + \ldots + u_{m_2}) + \ldots$$

be absolutely convergent, then we have

$$UV = \sum_0^\infty {}_\nu w_\nu = W, \textit{ where } w_\nu = \sum_0^\nu {}_x u_x v_{\nu-x},$$

provided that the series $\sum_0^\infty {}_\nu u_\nu v_\nu$ *be absolutely convergent and remain so when any number of factors* u_ν, v_ν *is replaced by other factors of higher indices.* That is, in any number of terms, $u_\nu v_\nu$, the factors v_ν (or u_ν) may be erased and any other factors $v_{\nu+n}$ (or $u_{\nu+n}$) put in their places, with the single restriction that none of the indices be repeated in the series. Whenever applicable, the above theorem excels Cauchy's in this, that the often difficult determination of the convergency of the product-series is replaced by the easier determination of the absolute convergency of $\sum_0^\infty {}_\nu u_\nu v_\nu$. In illustration of this theorem I give the following example. Let

$$U = \sum_0^\infty {}_\nu \left\{ \frac{1}{4\nu+1} - \frac{1}{4\nu+2} + \frac{1}{\sqrt{4\nu+3}} - \frac{1}{\sqrt{4\nu+4}} \right\}.$$

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* Crelle's *Journal*, Bd. 79. Proofs of this theorem and of Abel's theorem will be found in Chrystal's Algebra, Part II., p. 127 and p. 135.

† *Mathematische Annalen*, Bd. 21, p. 327.

This semi-convergent series becomes absolutely convergent when its terms are grouped thus,

$$U = \sum_{0}^{\infty}{}_{\nu} \left\{ \frac{1}{4\nu+1} - \frac{1}{4\nu+2} \right\} + \sum_{0}^{\infty}{}_{\nu} \left\{ \frac{1}{\sqrt{4\nu+3}} - \frac{1}{\sqrt{4\nu+4}} \right\}.$$

The series

$$V = \sum_{0}^{\infty}{}_{\nu} \left\{ \frac{1}{(4\nu+1)^{\frac{1}{2}}} - \frac{1}{(4\nu+2)^{\frac{1}{2}}} + \frac{1}{4\nu+3} - \frac{1}{4\nu+4} \right\}$$

is semi-convergent. Each fraction in the first series stands for a term u_ν and each fraction in the second for a term v_ν. Hence

$$\sum_{0}{}_{\nu} u_\nu v_\nu = \sum_{0}^{\infty}{}_{\nu} \left\{ \frac{1}{(4\nu+1)^{\frac{3}{2}}} + \frac{1}{(4\nu+2)^{\frac{3}{2}}} + \frac{1}{(4\nu+3)^{\frac{3}{2}}} + \frac{1}{(4\nu+4)^{\frac{3}{2}}} \right\},$$

which is absolutely convergent and remains so if any number of terms, u_ν or v_ν, be replaced by others occurring later in the series. Hence the product, W, of the two series converges toward UV.

The importance of inquiring whether $\sum_{0}^{\infty}{}_{\nu} u_\nu v_\nu$ remains absolutely convergent after the substitution of higher terms in place of the lower, is brought out by Pringsheim in the following example. Take the series U, given above, and

$$V' = \sum_{0}^{\infty}{}_{\nu} \left\{ \frac{1}{\sqrt{4\nu+1}} - \frac{1}{\sqrt{4\nu+2}} + \frac{1}{4\nu+3} - \frac{1}{4\nu+4} \right\}.$$

In this case

$$\sum_{0}^{\infty}{}_{\nu} u_\nu v_\nu = \sum_{0}^{\infty}{}_{\nu} \frac{1}{(\nu+1)^{\frac{3}{2}}}$$

which is absolutely convergent, while

$$\sum_{0}^{\infty}{}_{\nu} u_\nu v_{\nu+2} = \sum_{0}^{\infty}{}_{\nu} \left\{ \frac{1}{(4\nu+1)(4\nu+3)} + \frac{1}{(4\nu+2)(4\nu+4)} + \frac{1}{\sqrt{(4\nu+3)(4\nu+6)}} + \frac{1}{\sqrt{(4\nu+4)(4\nu+6)}} \right\}$$

is divergent! It is indeed found that in this case the product UV' cannot be represented by the series W.

In proving his theorem, Pringsheim shows in the first place that an obviously *necessary* condition for the convergency of W, namely $\lim_{\nu=\infty} w_\nu = 0$, is satisfied. His theorem, like that of Cauchy and of Mertens, offers *sufficient* conditions for the applicability of the rule of multiplication, but they are not at the same time *necessary* conditions. He shows that Cauchy's and Mertens' theorems are included in his own.

Pringsheim then considers the multiplication and convergency of special classes of semi-convergent series, of which we shall mention one. He shows that if U and V are convergent series and one of them, say U, is so constituted that

$$U = \tfrac{1}{2}u_0 + \tfrac{1}{2}\sum_0^\infty {}^\nu (u_\nu + u_{\nu+1})$$

is absolutely convergent (as is the case when the terms u_ν never increase and have alternating signs), then

$$\lim_{\nu=\infty} w_\nu = 0$$

is a necessary and sufficient condition for the convergency of W. Mr. A. Voss* has treated similarly the more general case when the series U, expressed in the form

$$U = (u_0 + u_1) + (u_2 + u_3) + \cdots$$

is absolutely convergent and has shown that in this case the necessary and sufficient condition for the convergency of W lies in the two following relations:

$$\lim_{n=\infty}(u_0v_{2n} + u_2v_{2n-2} + \ldots + u_{2n}v_0) = 0;$$

$$\lim_{n=\infty}(u_1v_{2n-1} + u_3v_{2n-3} + \ldots + u_{2n-1}v_1) = 0.$$

Mr. Pringsheim reaches the following interesting conclusions: The product of two semi-convergent series can never converge absolutely, but a semi-convergent series, or even a divergent series, multiplied by an absolutely convergent series *may* yield an absolutely convergent product. Thus, the product of the absolutely convergent series

$$2 \log 2 = 1 + \frac{1}{1.2} - \frac{1}{2.3} + \frac{1}{3.4} - \ldots$$

and the semi-convergent series

$$\log 2 = \tfrac{1}{1} - \tfrac{1}{2} + \tfrac{1}{3} - \ldots \text{ is}$$

$$2(\log 2)^2 = 1 + 2\sum_1^\infty {}^\nu \left\{ (-1)^\nu \cdot \frac{1}{(\nu+2)(\nu+3)} \sum_1^\nu {}^x \frac{1}{1+x} \right\},$$

which series converges absolutely. Again, the absolutely convergent series

$$-1 + \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \ldots,$$

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* *Math. Annalen*, Bd. 24, p. 42.

multiplied by the divergent series

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \ldots$$

gives an absolutely convergent product. The strangeness of this last conclusion is removed when we consider that the series

$$-1 + \frac{1}{1.2} + \frac{1}{2.3} + \ldots$$
$$= -1 + (1 - \tfrac{1}{2}) + (\tfrac{1}{2} - \tfrac{1}{3}) + \ldots = 0.$$

Since one of the factor-series is zero, we may well have a product-series with a definite limiting value. This value in this case is itself zero, as is seen from the following expression for the product-series

$$W = -c_1 + \sum_1^{\infty}{}_{\nu} (c_\nu - c_{\nu+1}), \text{ where } c = \sum_1^{\nu}{}_{x} \frac{1}{x(\nu + 1 - x)}.$$

COLORADO COLLEGE, *March* 23, 1892.

ON EXACT ANALYSIS AS THE BASIS OF LANGUAGE.*

BY A. MACFARLANE, SC.D., LL.D.

Abstract.

A SCHEME for an artificial language was published in the Philosophical Transactions of the Royal Society for 1668 by Bishop Wilkins. Since, however, it presupposes a complete enumeration of all that is or can be known, it would be overthrown by every considerable advance in knowledge. The mathematician and philosopher Leibnitz devoted much thought to what he called a *specieuse générale*, which he hoped would be an aid in reasoning and invention; but he died without publishing even an outline of his system. The new universal language Volapük, invented by J. M. Schleyer of Constance, is built upon a purely linguistic basis, being derived from a comparative study of the chief natural languages. In this paper it is proposed to show that the proper and necessary basis for an artificial language is scientific analysis and classification, and two specimens of language

* Abstract of a paper presented to the Society at the meeting of March 5, 1892.

so constructed will exhibit the great complexity of the problem.

In the notation for numbers in Volapük we observe serious defects. As regards the digits there is no word to express 0. As regards the expressions for the denominations, an arbitrary use of the affix for the plural denotes the denomination *ten*: thus we have *tel*, two; *tels*, twenty; and the other names for the denominations are no more systematic than the English words. There is the usual jump from thousand to million; we are not told whether *telion* means thousand million or million million; and no words are provided to express fractional denominations. In physical works we meet with the highest development of the notation for number; it consists of a series of significant figures, and of a positive or negative power of ten. To vocalize this notation we require an elementary word for each of the elementary numbers, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9; and a series of words for the integer powers of ten, and for the fractional powers of ten. As there are five elementary vowels, ten words for the digits may be obtained by prefixing the consonants *b* and *l*.

Thus	0,	1,	2,	3,	4,	5,	6,	7,	8,	9,
	ba,	*be*,	*bi*,	*bo*,	*bu*,	*la*,	*le*,	*li*,	*lo*,	*lu*.

The word for a higher number is formed by taking the appropriate monosyllables in succession; for example: 11, *bebe*; 23, *bibo*; 105, *bebala*. The integer denominations may be expressed by affixing *p* to the number for the place or power of ten, while the fractional denominations may be expressed by adding *n* instead of *p*, thus:—

10,	10^2,	10^3,	10^4,	10^5,	10^6,	10^7,	10^8,	etc.
bep,	*bip*,	*bop*,	*bup*,	*lap*,	*lep*,	*lip*,	*lop*,	etc. and

$\frac{1}{10}$,	$\frac{1}{10^2}$,	$\frac{1}{10^3}$,	$\frac{1}{10^4}$,	$\frac{1}{10^5}$,	$\frac{1}{10^6}$,	$\frac{1}{10^7}$,	etc.
ben,	*bin*,	*bon*,	*bun*,	*lan*,	*len*,	*lin*,	etc.

For example, one hundred and twenty-three thousand would be vocalized by *bebipo bop*, and forty-five hundredths by *bula bin*.

Some years ago in a series of papers on "An analysis of the relationships of consanguinity and affinity," * the author devised a system of notation both literal and graphical, and indicated a corresponding nomenclature. On this analysis may be constructed another specimen of a scientific language, and by the system of words it provides for such relationships the efficiency of Volapük may be tested.

* *Proc. Roy. Soc. Edinb.*, Vol. X., p. 224; Vol. XI., pp. 5 and 162; Phil. Mag. June 1881; and Journal of the Anthrop. Inst. of London for 1882.

Let *a* denote the relationship of parent and *e* the reciprocal relationship of child; by forming the different permutations of these letters we get expressions for the several compound relationships. Those of the second order are :—

NOTATION.	GENERAL MEANING.	IRREDUCIBLE MEANING.
aa	parent of parent.	grandparent.
ae	parent of child.	consort.
ea	child of parent.	brother or sister.
ee	child of child.	grandchild.

The meaning given in the third column may not coincide exactly with that given in second; where a reduction of the expression is possible, that is, where *a* is followed by *e* or *e* by *a* ; the special or reduced meaning is excluded. Thus *ae* and *ea* each in its most general meaning includes *self;* when the special meaning of *self* is excluded, the *parent of child* becomes *consort*, and the *child of parent* becomes *brother* or *sister*.

Similarly the relationships of the third order are :

NOTATION.	GENERAL MEANING.	IRREDUCIBLE MEANING.
aaa	great grandparent.	great grandparent.
aae	grandparent of child.	parent-in-law.
aea	parent of child of parent.	step-parent.
aee	parent of grandchild.	child-in-law.
eaa	child of grandparent.	uncle or aunt.
eae	child of parent of child.	step-child.
eea	grandchild of parent.	nephew or niece.
eee	great grandchild.	great grandchild.

In the case of all these relationships, excepting the first and the last, the general meaning includes a simpler relationship to which it may reduce; for example, grandparent of child includes the simpler relationship of parent. In the same manner the relationships expressed by four, five or any number of elements may be exhibited.

To change this notation into a nomenclature, all that is necessary is to insert some consonant as *d* between the vowels; for then each combination can be easily pronounced. In the systematic language so derived *ada* means grandparent, *ade* consort, *eda* brother or sister, *ede* grandchild, *adada* great grandparent, *adade* parent-in-law, *adeda* step-parent, and so on.

Each genus of relationship is divided into species by introducing the distinction of sex. Let the consonants *m* and *f* denote male and female respectively, then the species of the first order are *ma* father, *fa* mother, *me* son, *fe* daughter. If we introduce the distinction of sex after the vowel we obtain such relationships as *mam* father of man, *maf* father of woman, *mef* son of woman. The species of the second order, obtained by introducing the distinction of sex before the first vowel only, are, *e.g.*; *mada* grandfather, *feda* sister, *fede* granddaughter. If the distinction of sex is introduced before the second vowel also, we may obtain: *mama* paternal grandfather, *mame* father of son, *fema* sister-german, *fefe* daughter of daughter, etc. Thirty-two species may be formed by introducing the distinction of sex after the last vowel, but four of these species reduce necessarily to the relationship of *self*; for example *mamem*. The double relationship involved in full brother may be denoted by *memfa*, that of full sister by *femfa*, and that of full brother or sister by *emfa*. If, on the other hand, we wish to express that the brothership is only half, we may replace *d* by *t*; thus *meta*, half-brother; *feta*, half-sister; and *eta*, half brother or sister. These principles suffice to supply a word for every possible relationship of consanguinity or affinity. The nomenclature is based on a notation which serves as the basis for a calculus,* and it seems to me that this is a developed specimen of the kind of language which Leibnitz had in his thoughts.

If we test Volapük by the vocabulary which it provides for these relationships we find that the words supplied are not founded on a scientific analysis, and indeed are far inferior to the terms supplied by the English language. Almost all the stem words, as *son*, son, *blod* brother, involve the masculine gender, the corresponding feminines being formed by prefixing *ji*. Thus daughter is expressed by *ji-son* and sister by *ji-blod*. There are no words to express the relationships which are independent of sex. The confusion on the subject of the more involved relationships is very great, no distinction being made for example, between step-brother and half-brother, both of which are denoted by *lafa-blod*. The derived relationships are not expressed by general rules for combining the elementary relationships, but on the contrary a few words are obtained in an arbitrary manner by attaching to the stems comparatively meaningless prefixes and affixes. It has been pointed out by several scholars,† that the inventor of Vola-

* Problems in Relationship. *Proc. Roy. Soc. Edinb.*, 1888.

† Dr. D. G. Brinton—"Aims and Traits of a World-language;" Dr. Horatio Hale—"An International Language," *Proc. A. A. A. S.*, Vol. XXXVII.

pŭk makes a fundamental error in proceeding synthetically instead of analytically, and in this matter of terms for relationship we have an example of that fundamental mistake.

NOTES.

A REGULAR meeting of the NEW YORK MATHEMATICAL SOCIETY was held Saturday afternoon, April 2, at half-past three o'clock, the president in the chair. The following persons having been duly nominated, and being recommended by the council, were elected to membership: Mr. B. S. Annis, Johns Hopkins University; Professor Samuel Marx Barton, Emory and Henry College; Dr. Maxime Bôcher, Harvard University; Mr. William H. Butts, Pontiac, Michigan; Dr. T. Proctor Hall, Clark University; Professor S. W. Hunton, Mount Allison University; Mr. W. F. King, Ottawa, Canada; Mr. B. M. Roszel, Johns Hopkins University; Dr. Arthur Schultze, New York. The proposed amendment to the Constitution (*Bulletin*, No. 6, p. 142.) was unanimously adopted, and the By-Laws were amended by striking out section 2 of by-law IX., and altering the number of the following section. The following original papers were read: "The cubic-projection and rotation of a tessaract," by Dr. T. Proctor Hall; "On final formulas for the algebraic solution of quartic equations," by Professor Mansfield Merriman.

A *tessaract* is a geometrical figure generated by the motion of a cube in the direction of the common perpendicular to its edges and faces, bearing exactly the same relation to a cube that a cube bears to a square. It is bounded by eight cubes, and has twenty-four faces, thirty-two edges, and sixteen vertices. Dr. Hall presented the Society with a wire model representing the projection of a tessaract into space of three dimensions.

THE Cambridge University Press has in preparation "A treatise on the mathematical theory of elasticity," by A. E. H. Love, fellow of St. John's College, Cambridge. The first volume of the work, which is to be in two volumes, is in press.

MACMILLAN & Co. have nearly ready a work on the "Theory of functions," by Professor Morley of Haverford College, Pa., and Professor Harkness of Bryn Mawr College, Pa.

AT the meeting of the *Académie des Sciences* at Paris on

March 7, committees were appointed to award the mathematical prizes of the current year. For the *Grand prix des sciences mathématiques,* The determination of the number of primes inferior to a given limit, the committee is composed of MM. Jordan, Poincaré, Hermite, Darboux, Picard. For the *Prix Bordin,* The application of the general theory of abelian functions to geometry, the committee consists of MM. Hermite, Poincaré, Darboux, Jordan, Picard. For the Prix Bordin of 1890, To study the surfaces whose linear element can be reduced to the form

$$ds^2 = [f(u) - \varphi(v)]\,(du^2 + dv^2),$$

the time of competition for which was extended until 1892, the committee is MM. Poincaré, Darboux, Picard, Hermite, Jordan.

THE memoir of M. Painlevé, which won for its author the *Grand prix* of 1890, To perfect in an important respect the theory of the differential equation of the first order and first degree, has just been published in full in the *Annales de l'École Normale,* while that of his competitor, M. Autonne, which was awarded an honorable mention, is in course of publication in the *Journal de l'École Polytechnique.*

WE learn from *Naturae Novitates* that Professor H. A. Schwartz, of Göttingen, has been called to Berlin as the successor of the late Professor Kronecker, and that Professor Rudolph Sturm has been invited to the professorship of mathematics at the University of Breslau.

THE second number of the current volume of the *American Journal of Mathematics* was delayed through the occurrence of a fire in the printing office. In future the new volumes will begin in January instead of in October. T. S. F.

AT Johns Hopkins University during the academic year 1892–93, the following graduate courses in mathematics will be given: by Professor Craig, (1) Theory of functions of one and two variables, (2) Mathematical seminary, (3) Partial differential equations, (4) Linear differential equations, (5) Elliptic and abelian functions; by Dr. Franklin, (6) A general course for graduate students on the elements of modern mathematics, (7) Theory of invariants, (8) Metrical theory of surfaces; by Dr. Chapman, (9) Mechanics and hydrodynamics, (10) Projective geometry, (11) History of mathematics.

T. C.

During the coming year at Clark University, Professor Story will lecture on the following subjects: (1) History of algebra during the Renaissance, (2) Advanced course on the geometry of surfaces and twisted curves, (3) Applications of quaternions, (4) Hyperspace and non-euclidean geometry, (5) Introductory courses on calculus of finite differences, probability, and theory of errors. Dr. Webster will lecture on Theory of functions according to Cauchy, Riemann, and Weierstrass, with applications to functions defined by certain differential equations. Besides, introductory courses will be given in: Theory of numbers, Modern higher algebra, Higher plane curves, General theory of surfaces and twisted curves, Quarternions, and Modern synthetic geometry. O. B.

NEW PUBLICATIONS.

COMPILED BY B. WESTERMANN & CO., NEW YORK.

Air-Temperature. Harmonic analysis of hourly observations of Air Temperature and Pressure at British Observatories. London 1892. fol. 12s

Cauchy (A.). Oeuvres complètes. Publiées sous la direction scientifique de l'Académie des Sciences. (29 volumes en 2 séries.) Série 1. Tome VII: Notes et articles extraits des Comptes-rendus hebdomadaires des séances de l'Académie des Sciences, suite. Paris 1892. gr. in-4. 446 pg. Prix de souscription, 20f

Dyter (C. G. J.). Repertorium der Differential- und Integralrechnung. 2. Auflage. Berlin 1892. 8. 118 pg. m. Figuren. M. 1.50

Faraday.—Jerrold (W.). Michael Faraday, the man of science. New York 1892. 8. 160 pg. cloth. $0.75

Flammarion (C.). La Pluralité des Mondes habités. Étude où l'on expose les conditions d'habitabilité des terres célestes, discutées au point de vue de l' Astronomie, de la Physiologie et de la Philosophie naturelle. 34. mille. Paris 1892. 8. 6 et 483 pg. av. planches coloriées et noires.

Gee (W. H.). Four-Place Logarithms and Tangent Tables. Manchester 1892. 8. cloth. 9d

Greenwich. Astronomical, Magnetical and Meteorological Observations made at the Royal Observatory, Greenwich, in the year 1890, under the direction of W. H. M. Christie. London 1892. Imp. 4. cloth. 20s

Lange (G.). Ueber die linearen homogenen Differentialgleichungen, denen die Periodicitätsmodule Abelscher Integrale genügen für den Fall, dass die Irrationalität 3. Grades ist. Halle 1892. 8. 45 pg. M. 1.20

LAVES (K.). Beiträge zur Bestimmung und Verwerthung der Bewegung der Erde um den Schwerpunkt des Systems Erde-Mond. Berlin 1891. 8. 46 pp. M. 1.20

LEVETT AND DAVISON. The Elements of Plane Trigonometry. 12mo. pp. xvi—520. Macmillan. $1.60

LOCK (J. B). The First Book of Euclid's Elements, arranged for beginners. Fcap. 8vo. pp. viii.—167. Macmillan. $0.60

MÜLLER (A.). Untersuchungen über den Brocard'schen Kreis und das verallgemeinerte Brocard'sche Dreieck. Tübingen 1891. 8. 70 pg. M. 1.20

OHM (G. S.). The Galvanic Circuit Investigated Mathematically. Translated by W. Francis. Edited by T. D. Lockwood. New York 1892. 8. 269 pp. Cloth. $0.50

PADÉ (H.). Premières Leçons d'Algèbre élémentaire. Nombres positifs et negatifs. Opérations sur les Polynômes. Avec une préface de J. Tannery. Paris 1892. 8. 23 et 81 pg. M. 2.20

RIEMANN.—Burkhardt (H.). Bernhard Riemann. Vortrag, gehalten zur Feier des 25-jährigen Todestages. Göttingen 1892. 8. M. 0.50

SCHNELL (H.). Ueber Schaaren unter einander perspectiver Tetraeder. Giessen 1891. 8. 25 pg. M. 1.20

TAIT (P. G.). Heat. New ed. Cr. 8vo. 368 pp. Macmillan. 6s

UHRIG (K.). Ueber trilineare und tetraedrale Collineation. Giessen 1891. 8. 40 pg. m. 1 Tafel. M. 1.50

VACQUANT (CH.). Cours de Géométrie. 4. edition. Paris 1892. 8. M. 6.60

WENSKE (O.). Behandlung einiger Aufgaben der Variationsrechnung, welche sich auf Raumcurven von constanter erster Krümmung beziehen. Göttingen 1892. 8. 60 pg. M. 1.50

WOGAN (F.). Bewegung zweier materieller Punkte, welche durch einen gewichtslosen Faden mit einander verbunden sind, im Raume und in der Ebene unter Einwirkung der Schwerkraft und beliebig gegebener Anfangsgeschwindigkeiten. Memel 1891. 4. 28 pp. mit 2 Tafeln. M. 1.50

CONTENTS.

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TOPOLOGY OF ALGEBRAIC CURVES.

In the *Mathematische Annalen*, Vol. 38 (1891), Mr. David Hilbert of Königsberg has a very interesting and suggestive article on the real branches of algebraic curves. The simplicity of the method which Mr. Hilbert employs, and the possibility of its being made to yield further important results seem sufficient reasons for presenting here, in some detail, that portion of the article which treats of plane curves. It has seemed to the present writer advisable to amplify portions of Mr. Hilbert's article, with the view of making his method more intelligible, and also to make some changes in the proof of the principal theorem, in order to avoid some slight inaccuracies that have crept into his demonstration.

The first part of the article in question is devoted to the determination of the maximum number of *nested branches* possible to a plane algebraic curve of order *n*, and of maximum deficiency. By *nested branches* is meant a group of even branches so arranged that the first lies entirely within the second, the second within the third, and so on, like a series of concentric circles.* It should be observed that some or all of the non-nested branches may, in perfect accord with this definition, lie within the ring-shaped regions formed by the nested branches. A single even branch, which neither encloses another branch nor is enclosed by one, may be looked upon as a nested branch or not, according to the nature of the question under discussion. For reasons that will presently appear, Hilbert does not consider the even branches of the conic and cubic as nested. Hilbert bases some of his investigations upon results previously obtained by A. Harnack,† and his method is entirely analogous to that of the latter.

Harnack had proved, in the article referred to, that a plane algebraic curve, without singularities, of order n and of deficiency p, can not have more than $p + 1$, that is, $\frac{1}{2}(n-1)(n-2) + 1$ *real* branches; and, further, that, for every positive integral value of n, a non-singular curve with $\frac{1}{2}(n-1)(n-2) + 1$ real branches actually exists. Setting out from this result of Harnack's, Hilbert shows first that a non-singular curve can have no more than $\frac{1}{2}(n-2)$ or $\frac{1}{2}(n-3)$ nested branches, according as n is even or odd; for, if it had more, a right line could be drawn meeting the curve in more

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* This definition is not scientific but it serves the present purpose. To make it rigorous Mr. Hilbert needs only to define accurately what is meant by *inside* and *outside* of a closed branch. Such a definition has virtually been given by VON STAUDT, *Geometrie der Lage*, § 1, 16.

† *Mathematische Annalen*, Bd. 10, *Ueber die Vieltheiligkeit der ebenen algebraischen Curven*.

than n points.* He proves further the following theorem: *For every positive integral value of n, a non-singular curve of order n exists, having the maximum number of real branches, $\frac{1}{2}(n-1)(n-2)+1$, and $\frac{1}{2}(n-2)$ or $\frac{1}{2}(n-3)$ nested branches, according as n is even or odd.*

We shall, for sake of brevity, designate an even branch by the term "*oval.*" It is evident that all nested branches are ovals. Moreover, we consider that case only where all the nested ovals are grouped in a single nest. We first assume the theorem true for a curve of n^{th} order, C_n, whose equation may be written $f=0$, and we assume further that an ellipse, E_2, whose equation we write $h=0$, can be constructed enclosing one or more of the nested ovals, and cutting a non-nested oval, b, in $2n$ points, whose order of succession shall be the same upon b as upon E_2. It is evident that E_2 and C_n have no other common point. The ellipse E_2 and the branch b form, by their intersections, $2n$ regions, each completely bounded by a *single* segment of E_2 and a *single* segment of b. Within one of these regions there exists one or more nested ovals. Whether this region, which we call R, contains the nested ovals interior to E_2 or exterior to it,† depends upon the nature of b, and its position with respect to E_2. (When E_2 encloses all the nested ovals, it may occur that none of these $2n$ regions contains a nested oval; in that case one of these regions will be all the plane exterior to E_2 and b, and this we designate by R.) Let s be any segment of E_2 determined by the intersections of E_2 and b, *except that segment which forms a portion of the boundary of R.* Upon s we choose $2(n+2)$ points, none of them coincident with the extremities of s, and join by right lines the first and second, the third and fourth, ... , and the $(2n+3)$th and $(2n+4)$th. Let the product of the equations of these $n+2$ right lines be $l=0$. Then for very small values of δ,

$$F \equiv fh \pm \delta l = 0$$

is the equation of a curve, C_{n+2}, of order $n+2$, lying very near the degenerate curve $fh=0$. This C_{n+2} passes through the points common to C_n and the right lines, and through the points common to E_2 and these lines, but not through the intersections of E_2 with C_n.

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* This theorem is not true for curves of order lower than the fourth. Moreover, it must be borne in mind that every non-singular curve with the maximum number of real branches has at least *one non-nested oval*, because $\frac{1}{2}(n-2)$ and $\frac{1}{2}(n-3)$ are each less than $\frac{1}{2}(n-1)(n-2)+1$.

† A nested oval exterior to E_2, since it encloses those interior to E_2, must also enclose E_2 itself. Therefore, when, among a number of isolated ovals, we have to consider *a single one* as nested, we choose as such, one that lies in the interior of E_2.

We proceed now to prove:

(1) that C_{n+2} has $p' + 1$ real branches, p' being the deficiency of C_{n+2};

(2) that C_{n+2} has the maximum number of nested branches; and

(3) that the ellipse, E_2, encloses one or more of the nested ovals of C_{n+2}, and cuts one of its non-nested ovals in $2(n + 2)$ points, whose order of succession upon C_{n+2} is the same as upon E_2.

1. Ignoring the branch b for the moment, it appears, from the form of the equation $F = 0$, that in the immediate vicinity of every other branch of C_n, there exists a *similar* branch of C_{n+2}. The C_n has by hypothesis $\frac{1}{2}(n-1)(n-2)$ real branches, exclusive of b. These give rise, therefore, to $\frac{1}{2}(n-1)(n-2)$ real branches of C_{n+2}. Furthermore, under proper choice of the sign of δ, there exists, in the vicinity of the complete boundary of each of the $2n$ regions formed by E_2 and b, an oval of C_{n+2}. The latter curve has no real branch save those already enumerated. Therefore *C_{n+2} has $\frac{1}{2}(n-1)(n-2) + 2n = \frac{1}{2}(n+1)n + 1 = p' + 1$ real branches.*

2. Each of the nested ovals of C_n gives rise to a nested oval of C_{n+2}. Moreover, the oval of C_{n+2} engendered by the boundary of R is itself a nested oval of C_{n+2}. The latter has, therefore, *one more* nested oval than does C_n. Since increasing n by 2, increases the functions $\frac{1}{2}(n-2)$ and $\frac{1}{2}(n-3)$ by 1, it follows that *C_{n+2} has the maximum number of nested branches.*

3. In the vicinity of that region, a portion of whose boundary is S, there exists an oval of *C_{n+2} which cuts the ellipse in the $2(n+2)$ points already determined upon s, and the order of succession of these $2(n+2)$ points is the same upon C_{n+2} as upon s.*

Hence, if our assumptions concerning C_n and E_2 are valid, the curve C_{n+2} has the maximum number of real branches, and also the maximum number of nested branches. And furthermore—and this is a very important point—the ellipse E_2 has the same position with respect to C_{n+2} that it was assumed to have with respect to C_n. It follows, then, that we may in like manner derive from the C_{n+2} a C_{n+4} having the same properties, and so on. If, then, we can prove our assumption valid for one even value, and for one odd value of n, we may conclude that our theorem is true for all values of n.

That these assumptions are valid when $n = 4$ can be demonstrated as follows: Let $f = 0$ be the equation of a given ellipse C_2, and $h = 0$ that of the auxiliary ellipse E_2. Let E_2 intersect C_2 in 4 real points; and upon any segment, s, of E_2 determined by two successive points of intersection, choose the 8 successive points, 1, 2, 3, . . . 8. Join by right lines, 1 with 2, 3 with 4, . . . , and 7 with 8. Let the product of the equa-

tions of these 4 right lines be $l = 0$. Then, for very small values of δ,

$$fh \pm \delta l = 0$$

represents a non-singular quartic, C_4, and, by proper choice of the sign of δ, this quartic has four ovals, one of which intersects E_2 in the eight points upon s. Moreover, within E_2 there lie one or two ovals of C_4, one if s is exterior to C_3, and two if s is within C_3. Now a quartic can have no more than $\frac{1}{2}(4-2) = 1$ nested oval. We choose as such, an oval in the interior of E_2. We have then a C_4 with the maximum number of real branches, viz., $\frac{1}{2}(4-1)(4-2)+1 = 4$; with the maximum number of nested ovals, 1; and the ellipse E_2 encloses this nested branch, and cuts a non-nested oval in $2(2+2) = 8$ real points, whose order of succession upon C_4 is the some as upon E_2. Hence *our assumption is valid when $n = 4$.*

That this is true also when $n = 5$ is similarly proved. Let $f = 0$ represent a straight line. Draw the ellipse, E_2, not cutting $f = 0$ in any real point. Upon E_2 choose six points and, as before, join alternate pairs by right lines. Let the product of the equations of these three right lines be $l = 0$. Then, when δ is very small,

$$fh \pm \delta l = 0$$

represents a non-singular cubic, C_3, the oval of which intersects E_2 in the six points whose order of succession upon E_2 and the oval is the same. Proceeding one step further, let the equation of C_3 be $f = 0$. Upon any segment of E_2 choose $2(3+2) = 10$ points, and join alternate pairs by right lines, the product of whose five equations is $l = 0$. Then, for sufficiently small values of δ,

$$fh \pm \delta l = 0$$

represents a non-singular quintic, C_5, and, upon proper choice of the sign of δ, this C_5 has six ovals, one of which intersects E_2 in ten points. Within E_2 lie two ovals of C_5, one of which we consider a nested oval. Moreover, C_5 has an odd branch in the vicinity of the odd branch of C_3. We have then a quintic with the maximum number of real branches, $\frac{1}{2}(5-1)(5-2)+1 = 7$; with the maximum number of nested branches, $\frac{1}{2}(5-3) = 1$; and with a non-nested oval cut by E_2 in $2(3+2) = 10$ real points; E_2 also encloses the nested branch. Hence, *our assumptions are valid when $n = 5$. The theorem is therefore true in general.*

Readers of Hilbert's article in the *Annalen* will notice some minor errors in his proof. He states, for instance, that the auxiliary ellipse may lie wholly within the innermost nested oval (see *Annalen,* vol. 38, p. 117). This is impossible, for the ellipse could not then be made to intersect a non-nested oval. Again, he allows the ellipse to cut *any* of the non-nested branches. If the ellipse be drawn to enclose all the nested branches and to intersect in $2n$ points an *odd* branch, the derived C_{n+2} will have indeed the maximum number of real branches, but one fewer than the maximum number of nested branches. And, lastly, Hilbert chooses the $2(n+2)$ points of E_2, through which the lines $l=0$ are to pass, upon any segment of E_2. If, however, these be taken upon that segment of E_2 which forms part of the boundary of R, the branch of C_{n+2} which has these points in common with E_2 will be a *nested* oval, and, though the C_{n+2} will then have $p+1$ real branches, and the maximum number of nested ovals as required, it will be impossible to carry the process further.

It will be observed that Hilbert's results apply only to curves of maximum deficiency, and of the maximum number of real branches, n being given. It by no means follows that a curve of order n and of maximum deficiency, but with fewer than the maximum number of real branches, cannot have more than $\frac{1}{2}(n-2)$ or $\frac{1}{2}(n-3)$ nested branches. For instance, in the case of the cubic discussed above, if δ be given the opposite sign to the one there chosen, the equation

$$fh \pm \delta\, l = 0$$

will represent a non-singular quintic, having but three real branches, two of which are nested.

And, in general, it is easily seen that a non-singular curve of even order, and possessing but $\frac{1}{2}n$ real branches, may have them all nested. Similarly, a curve of odd order having only $\frac{1}{2}(n+1)$ real branches, may have $\frac{1}{2}(n-1)$ of them nested. Hilbert leaves untouched also the case of singular curves, and thus excludes from his investigations a large class of curves. It would be interesting to know under what conditions, and in what way, the branches of a singular curve can be nested.

Lack of space prevents any discussion of the second part of Hilbert's article, in which the author determines some of the properties of curves in three-fold space. I give only the results of these investigations. By a method entirely analogous to that presented above, Hilbert proves the theorem: *An irreducible twisted curve of order n, with the maximum number of real branches* [$\frac{1}{4}(n-1)^2+1$ *when n is even, and* $\frac{1}{4}(n-1)(n-3)+1$ *when n is odd*] *can have no more than* $2\nu-2$, $2\nu-1$, $2\nu-1$ *odd branches, according as* $n=4\nu$,

$4\nu + 1$, $4\nu + 3$. *When* $n = 4\nu + 2$, *no odd branch can exist. Exceptional are the cases when* $n = 3, 4, 5$, *the maximum number of odd branches being* 1, 2, 3, *respectively*. Then, by applying Abel's theorem for elliptic functions, he proves, for every value of n, the existence of curves with the maximum number of real odd branches.

L. S. HULBURT.

WORCESTER, MASS., *April* 5, 1892.

FINAL FORMULAS FOR THE ALGEBRAIC SOLUTION OF QUARTIC EQUATIONS.*

BY MANSFIELD MERRIMAN, PH.D.

I. FINAL formulas for the algebraic solution of quadratic and cubic equations are well known. Such formulas exhibit the roots in their true typical forms, and lead to ready and exact numerical solutions whenever the given equations do not fall under the irreducible case. But for the quartic, or biquadratic, equation the books on algebra do not give similar final formulas. The solution of the quartic has been known since 1540, and numerous methods have been deduced for its algebraic resolution, yet in no case does this appear to have been completed in final practical shape. It is the object of this paper to state the final solution in the form of definite formulas.

II. The expression of the roots of the quartic is easily made in terms of the roots of a resolvent cubic, and the cubic itself is solved without difficulty. Yet great practical difficulty exists in treating a numerical equation on account of the presence of imaginaries in the roots of the resolvent. Witness the following example which is generally given to illustrate the method in connection with Euler's resolvent:

"Let it be required to determine the roots of the biquadratic equation,

$$x^4 - 25x^2 + 60x - 36 = 0.$$

By comparing this with the general form the cubic equation to be resolved is,

$$y^3 - 50y^2 + 729y - 3600 = 0$$

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* Abstract of a paper presented to the Society at the meeting of May 7, 1892.

the roots of which are found, by the rules for cubics, to be 9, 16 and 25, so that $\sqrt{y_1} = 3$, $\sqrt{y_2} = 4$, and $\sqrt{y_3} = 5$. Therefore, since the coefficient of x is positive,

$$\begin{aligned} x_1 &= \tfrac{1}{2}(-3-4-5) = -6 \\ x_2 &= \tfrac{1}{2}(-3+4+5) = +3 \\ x_3 &= \tfrac{1}{2}(+3-4+5) = +2 \\ x_4 &= \tfrac{1}{2}(+3+4-5) = +1 \end{aligned}$$

which are the four roots of the proposed equation."

III. Now all that can be said of this numerical work is that it is a verifying instance. It is not an algebraic solution in any sense of the word, for, as both the quartic and its cubic resolvent have real roots, this is the irreducible case where the numerical solution fails. In Euler's Algebra, 1774, where this example was first given, the roots of the cubic are obtained by the use of trigonometrical tables, but in subsequent quotations it is usually merely stated that they are found "by the rules for cubics." This numerical example has certainly no place in the exemplification of the algebraic solution of the quartic equation, and yet it is so given in most mathematical dictionaries and it may also be seen in the article Algebra in the last edition of the Encyclopædia Britannica.

As this *Bulletin* is intended for historical and critical remarks rather than for original investigations it will not be well to here set forth the method whereby I have brought the solution into such shape as to produce final practical formulas. But the results may perhaps be allowed place, as their statement is very brief.

IV. The following are final formulas for the algebraic solution of the quartic equation,

$$x^4 + 4ax^3 + 6bx^2 + 4cx + d = 0.$$

First, let m and n be determined by

$$\begin{aligned} m &= a^2d - 2abc + b^3 - bd + c^2 \\ n &= (b^2 + \tfrac{1}{3}d - \tfrac{4}{3}ac)^3. \end{aligned}$$

Secondly, let s and t be found from

$$\begin{aligned} s &= \tfrac{1}{2}(m + \sqrt{m^2 - n})^{\frac{1}{3}} \\ t &= \tfrac{1}{2}(m - \sqrt{m^2 - n})^{\frac{1}{3}}. \end{aligned}$$

Thirdly, let u, v, and w be derived by,

$$u = (a^2 - b) - (s + t)$$
$$v = 2\,(a^2 - b) - (s + t)$$
$$w = v^2 + 3\,(s - t)^2.$$

Then the four roots of the given quartic are expressed by the formulas,

$$x_1 = -a + \sqrt{u} + \sqrt{v + \sqrt{w}}$$
$$x_2 = -a + \sqrt{u} - \sqrt{v + \sqrt{w}}$$
$$x_3 = -a - \sqrt{u} + \sqrt{v - \sqrt{w}}$$
$$x_4 = -a - \sqrt{u} - \sqrt{v - \sqrt{w}}$$

in which the signs before the square roots are to be used as written provided $2a^3 - 3ab + c$ is negative, but if this is positive all radicals except $\sqrt{w}$ are to be reversed in sign.

V. As a numerical example, let the equation to be solved be the complete quartic,

$$x^4 - 8x^3 - 10x^2 + 56x + 192 = 0.$$

Here, by comparing the coefficients with the given form,

$$a = -2, \quad b = -\tfrac{5}{3}, \quad c = +14, \quad d = +192.$$

From these are first computed,

$$m = \tfrac{32023}{27}, \qquad n = \tfrac{822656953}{729},$$

and next in order are found,

$$s = 5.983, \qquad t = 4.350.$$

Accordingly $s + t$ is 10.333, and $s - t$ is 1.633, and then

$$u = 16, \qquad v = 1, \qquad w = 9.$$

Now as $2a^3 - 3ab + c$ has a negative value, the formulas give

$$x_1 = 2 + 4 + \sqrt{1 + 3} = +8$$

$$x_2 = 2 + 4 - \sqrt{1 + 3} = + 4$$

$$x_3 = 2 - 4 + \sqrt{1 - 3} = - 2 + \sqrt{-2}$$

$$x_4 = 2 - 4 - \sqrt{1 - 3} = - 2 - \sqrt{-2}\,.$$

which are the simplest expressions for the four roots.

As a second example let the proposed quartic equation be $x^4 + 7x + 6 = 0$. Here $a = 0$, $b = 0$, $c = \frac{7}{4}$, and $d = 6$. Then $m = \frac{49}{16}$ and $n = 8$. Next in order, $s = 0.8091$ and $t = 0.6180$, whence $u = + 1.427$, $v = - 1.427$ and $w = 2.146$. Now, c being positive, the roots are

$$x_1 = - 1.194 - \sqrt{- 1.427 + 1.465} = - 1.388$$

$$x_2 = - 1.194 + \sqrt{- 1.427 + 1.465} = - 1.000$$

$$x_3 = + 1.194 - \sqrt{- 1.427 - 1.465}$$

$$x_4 = + 1.194 + \sqrt{- 1.427 - 1.465}$$

which closely satisfy the given equation.

VI. The above formulas for the algebraic solution of the quartic equation are final in the sense that, like those so well known for the quadratic and cubic, they exhibit true symbolic representations of the roots in terms of the given coefficients, and that they are not capable of further essential simplification. They furnish the means of the discussion of all the circumstances concerning the occurrence of equal roots in the quartic, as well as of cases where the roots are connected by a known relation. They will be found to embrace the solution of all special and critical cases. For instance, applied to the binomial $x^4 - 1 = 0$ they give the roots $x_1 = + 1$, $x_2 = - 1$, $x_3 = + \sqrt{-1}$ and $x_4 = - \sqrt{-1}$. Again, if applied to the form $x^4 + 6bx^2 + d = 0$, they give the same solution as that by quadratics, for u becomes zero, v becomes $- 3b$ and w reduces to $9b^2 - d$. Lastly, they furnish ready and exact numerical solutions whenever the proposed equation has two real and two imaginary roots, or when two or more roots are equal. If there be either four unequal real roots or four unequal imaginary roots, the irreducible case arises where $m^2 - n$ becomes negative, and the formulas, although correctly representing the roots, fail to furnish numerical solutions in as simple forms as desired.

LEHIGH UNIVERSITY, *March*, 1892.

POINCARÉ'S MÉCANIQUE CÉLESTE.

Les Méthodes nouvelles de la Mécanique Céleste. Par H. POINCARÉ. Tome I. Paris, Gauthier-Villars, 1892. 8vo.

THE publication of this new work on Celestial Mechanics, embodying some of the results of the labors of mathematicians in that direction during the last fifteen years, comes as a welcome addition to our knowledge of this subject. Until lately, nearly all treatises have been written with a special object, that of obtaining expressions which can be used by the practical astronomer; the mathematical aspects of the problems solved have been almost entirely neglected. These latter have an interest of their own apart from any use which can be made of them, and it is to the study of such questions that M. Poincaré largely devotes himself. At the same time he points out where they can be applied usefully in the case of the problem of three bodies. But this is not all. Most of the results obtained can be applied equally to the general problems of dynamics where there is a force function, and by the use of a dissipation function could doubtless be applied to any natural problem whatever.

The applications are, however, more particularly made to a satellite system, in the special case when the three bodies move in one plane, as well as in the general case. The limitation generally imposed consists in making the ratios of the masses of two of the bodies to that of the third a small quantity, an assumption which, nevertheless, does not limit greatly the usefulness of the results. M. Poincaré says, "Le but final de la Mécanique céleste est de résoudre cette grande question de savoir si la loi de Newton explique à elle seule tous les phénomènes astronomiques," and for this end to be attained it is absolutely necessary to know whether the developments of the expressions for the position of any heavenly body do mathematically represent that position. In general, the series obtained must be convergent, and it is to the questions on the convergence of such series that M. Poincaré has been able to give some definite answers.

In his introduction, the author points out that the starting point of the present developments of the lunar theory, was the publication in Vol. I. of the *American Journal of Mathematics* of a paper by Dr. Hill entitled, "Researches in the lunar theory." It is true that in this memoïr, Dr. Hill has largely occupied himself in obtaining exact numerical and algebraical values for certain inequalities in the motion of the moon; but the general considerations involved at the beginning and end of it are of a far-reaching nature. In particular, a superior limit to the radius vector of the moon is found, and a general study of the surfaces of equal velocity is made. His consideration in a particular case of the moons of different lunations with respect to the primary, will be mentioned below.

M. Poincaré's book is principally based on his own memoir, "*Sur le problème des trois corps et les équations de la dynamique.*"* The arrangement is not quite the same. In the treatise, many of the demonstrations are more completely explained, the applications are more numerous, and much matter that is entirely new has been added. In what follows, I have not in any sense attempted to give a complete account of the book. Much that is given there is outside the scope of an article such as this; the results that are mentioned are chiefly noticed either because they can be given in a few words, or because from their peculiar interest they merit a somewhat longer treatment.

The first chapter deals with some general well-known theorems with respect to differential equations. Two types are selected. The general form which it is necessary to consider is shown by the system

$$\frac{dx_i}{dt} = X_i \qquad (i = 1, 2, \ldots n).$$

The X_i are analytic and single-valued functions of the x_i and may or may not contain the time explicitly. This type includes the system of canonical equations

$$\frac{dx_i}{dt} = \frac{dF}{dy_i}, \qquad \frac{dy_i}{dt} = -\frac{dF}{dx_i},$$

which possess a set of properties special to themselves. Some space is devoted to the consideration of these properties, and special attention is directed to changes of variables for which the system still remains canonical. The proofs for these theorems are sketched very briefly in cases where they are well-known.

In all the particular cases of the applications of canonical equations to the problem of three bodies, M. Poincaré works out the results with some detail. The masses are taken to be m_1, m_2, m_3; m_1 is the mass of the primary while m_2, m_3, satisfy

$$\beta\mu = \frac{m_1 m_2}{m_1 + m_2}, \qquad \beta'\mu = \frac{(m_1 + m_2)\, m_3}{m_1 + m_2 + m_3},$$

* *Acta Mathematica*, Vol. XIII.

such that μ is small while β, β' remain finite quantities. It is then possible to expand F in a series arranged in ascending powers of μ:

$$F = F_0 + \mu F_1 + \mu^2 F_2 + \ldots$$

In general F_0 will be independent of one system of canonical elements, say, the y_i.

The canonical equations given above correspond to n degrees of liberty. If we know an integral of the system, this number can be lowered by one unit. In general, if we know q integrals, Poisson's conditions must be fulfilled between these integrals taken two and two, in order that the number of degrees of liberty may be lowered by q units. The application of this to the general problem of three bodies is immediate. The three integrals for the motion of the centre of mass of the system being known and fulfilling the conditions, we can reduce the number of degrees of liberty from *nine* to *six*. The three known integrals of areas are also integrals of the system thus reduced, and by using two combinations of these latter, it is possible to reduce the system to *four* degrees of liberty; also in the case when the bodies move in one plane, the system can be reduced to *three* degrees of liberty. The usual transformations are then effected so as to leave the equations still in the canonical form and to carry only the smallest number of degrees of liberty.

The form of the disturbing function is also discussed, and it is considered under what circumstances we can develop it in ascending powers.

The second chapter deals with the general conditions for integration in series, and in particular with the conditions that these series may be convergent. It is here that M. Poincaré's penetrative genius especially shows itself. The complicated forms which appear in the lunar problem render it an almost impossible task to attack directly the question of convergence of the series obtained. But by going back to the differential equations themselves, and considering the disturbing function, he is able to obtain definite results, with respect to the problem of three bodies, for the convergence of those series which may be taken to represent certain particular solutions.

The notation introduced by M. Poincaré a short time back for dealing with questions of convergence is an especially happy one. It is as follows:—If we have two functions φ, ψ, expanded in ascending powers of x, y,

$$\varphi \ll \psi \qquad (\text{arg. } x, y)$$

denotes that the coefficient of every term in ψ is greater in absolute value than the corresponding term in φ, the "arguments" in terms of which the expansion is made being written as above. This can of course be used for any number of arguments. An extension of this notation is given at the end of the chapter. The coefficients, instead of being constants, are supposed to be periodic functions of the time; then, if every coefficient of ψ in its expansion according to powers of x, y, $e^{\pm u}$ is real, positive, and greater in absolute value than the corresponding coefficient in φ,

$$\varphi \ll \psi \qquad (\text{arg. } x, y, e^{\pm u}).$$

Cauchy's general theorems on convergence are quoted and extended to the case in which the function is expanded in terms of several variables. If we have a system of differential equations

$$\frac{dx}{dt} = \theta(x, y, z, \mu), \frac{dy}{dt} = \varphi(x, y, z, \mu), \frac{dz}{dt} = \psi(x, y, z, \mu)$$

where θ, φ, ψ are expanded in powers of x_0, y_0, z_0, and μ, t, there will exist three series expanded in powers of x_0, y_0, z_0 and μ, t which will satisfy these equations and reduce respectively to x_0, y_0, z_0 when $t = 0$. For these to be convergent it is necessary that $|x_0|$, $|y_0|$, $|z_0|$, $|\mu|$, $|t|$ should be sufficiently small. The restriction $|t|$ sufficiently small is evidently inconvenient, and Poincaré is able to get rid of it and to say that the series are convergent if t lies between given limits provided that $|\mu|$ be sufficiently small.

In most cases, however, expansion is not made in powers of the time, but in trigonometrical functions of it, and it therefore becomes necessary in the first instance to examine a system of differential equations,

$$\frac{dx_i}{dt} = \varphi_{i,1}x_1 + \varphi_{i,2}x_2 + \ldots + \varphi_{i,n}x_n \quad (i = 1, 2, \ldots n)$$

where the φ are all periodic functions of the time. The general solution found is,

$$x_i = c_1 e^{a_1 t} \lambda_{1,i} + c_2 e^{a_2 t} \lambda_{2,i} + \ldots + c_n e^{a_n t} \lambda_{n,i}$$

the λ being periodic functions of the time only, the a_i dependent on the roots of a determinantal equation, and the c_i arbitrary constants.

These a_i are called the characteristic exponents (*exposants caractéristiques*) of the solution. On them depends the whole nature of the various solutions. Thus if two of the exponents are equal, the time appears as a factor; if they are all pure imaginaries, the general solution contains periodic terms only, and so on. Also, on them depend the "asymptotic solutions." Chapter IV. is devoted to the consideration of these exponents.

Chapter III., which deals with periodic solutions, is perhaps the most interesting from the point of view of its immediate application to some of the problems in the lunar theory. In this connection, a periodic solution is defined as being such that the system at the end of a finite time T comes into the same *relative* position as at the beginning of that time. The period is then T. Thus if $\varphi(t)$ represent a periodic solution of period T

$$\varphi(t + T) = \varphi(t);$$

also if

$$\varphi(t + T) = \varphi(t) + 2k\pi \qquad (k = \text{whole number})$$

$\varphi(t)$ is still said to be a periodic solution. These two types are analogous to linear and angular coördinates, respectively. In the canonical system of coördinates as applied to dynamical problems, one set of elements belongs to the first type, and the conjugate set in general to the second type. It is to be noted that by defining a periodic solution in this way, the system can, so to speak, be separated from its external relations. The motions both of rotation and translation of the system as a whole can be detached, and those of its various parts amongst themselves considered.

The question which is put forward for examination is as follows: If for $\mu = 0$ we have a periodic solution, what are the conditions necessary in order that the solution shall still remain periodic when μ is not zero but a small quantity? It must be remembered that in this and in what follows, the term "periodic" has the meaning which has just been given to it. In order to answer the question, M. Poincaré considers the system

$$\frac{dx_i}{dt} = X_i$$

where the X_i are functions of the time periodic and of period 2π, as well as of the x_i. Space will not permit me to reproduce the argument, which finally reduces the answer to the consideration of the properties of a certain curve in the neighborhood of the origin. This curve is examined in certain particular cases and notably in the case where there are an infinite number of periodic solutions for μ zero, *i.e.* when the period is an arbitrary constant of the general solution. Generally, it is found that in these cases the equations *do* admit periodic solutions. In another particular case, the equations when $\mu = 0$ admit a solution of period 2π, and when μ is small but not zero, save in an exceptional case, the equations admit a solution of period $2k\pi$ (k being a whole number) which is different from the solution of period 2π, and is only not distinct from this latter when μ becomes zero.

If the X_i are periodic with respect to the time, the solution in general if periodic must have the same period. When however the time does not enter into the X_i explicitly, the period of the solution can be anything whatever. Suppose that the period selected when $\mu = 0$ be T. The question resolves itself into finding under what circumstances a solution of period $T + \tau$ is possible when μ is small. The argument proceeds in a somewhat similar manner as in the first case and similar results follow.

To apply these results to the problem of three bodies, suppose $\mu = 0$. Then two of the bodies describe ellipses about the third. At the end of a certain period measured by the difference of their mean motions, the system is found in the same relative position as at the beginning of the period. The solution for $\mu = 0$ is then periodic. Will periodic solutions be still possible when μ, instead of being zero, has a small positive value? From what has been proved above, we can say that such solutions *are* in general possible. M. Poincaré distinguishes three classes :—(1) when the inclinations and eccentricities are zero, (2) when the inclinations only are zero, (3) when the latter are not zero. He then examines these in detail.

Under (1) comes, as a particular case, Dr. Hill's now classic solution, where the mass of one body is supposed to be infinitely great and at an infinite distance, but to have a finite mean motion, and the mass of the other is infinitely small. The solutions are referred to axes moving with the infinitely distant body which takes a circular orbit. The period is one of the arbitraries and can be anything whatever. When $\mu = 0$ the motion is circular, and when μ is small, the curve does not differ much from a circle, and is somewhat elliptical in shape with its shorter axis directed constantly towards the sun. [If the sun be not infinitely distant, the only change in the curve is a loss of symmetry with regard to the line joining the earth and the sun.] Dr. Hill calculated the various shapes which the curve takes for different values of the arbitrary period, corresponding to gradually decreasing values of the constant of vis viva. As this latter constant diminishes, the ratio of the magnitude of the axes becomes greater, until for one particular value of it a cusp appears at each end of the greater axis. This gives what Dr. Hill calls, "the moon of maximum lunation." At the cusp and therefore in quadrature, the moon becomes for a moment stationary with respect to the sun.

He did not pursue the calculations beyond this point. It was stated however that any member of this class of satellites if prolonged beyond the moon of maximum lunation would oscillate to and fro about a mean place in syzygy, never being in quadrature. M. Poincaré points out an inaccuracy in this statement. The satellites which are never in quadrature, are indeed possible but belong to a different class of solution, and are not the analytical continuation of those studied by Dr. Hill. He shows that if we prolonged them beyond the critical orbit, they would cross the line of quadratures six times, cutting their own orbits twice and forming a curve with three closed spaces. The class to which the moons without quadrature belong has, as a limiting case, a moon which is stationary with respect to the sun and which is always either in conjunction or opposition.

M. Poincaré next goes on to consider the canonical system when F_0 is supposed independent of the y_i. This is the general problem of dynamics where the forces depend on the distances only and where we proceed by successive approximations. The first approximation is

$$x_i = const = a_i, \qquad \frac{dy_i}{dt} = const = n_i.$$

If the solution is to be periodic and of period T, all the $n_i T$ must be multiples of 2π. It is then shown that unless the Hessian of F_0 with respect to the x_i vanish, we can have a periodic solution of period T or differing little from T when μ is small. If this Hessian vanish we can sometimes find a function of F_0 whose Hessian does not vanish. If we cannot do this the case must be otherwise examined. Such an examination shows, that when the Hessian of F_0 vanishes, if the mean value R of F_1, with respect to t, admits of a maximum or a minimum, periodic solutions are still possible.

In the problem of three bodies, F_1 corresponds to the disturbing function, and we are led to periodic solutions of the second and third kinds. Here R does admit of a maximum or a minimum, and hence such periodic solutions are always possible. The periodic solutions of the first kind only cease to exist when n' is a multiple of $n - n'$. When, however, this ratio $n' : n - n'$ is nearly a whole number, as happens in several cases in the solar system, a large inequality will exist and its principal part can be calculated suitably by the help of these periodic solutions.

In the next chapter M. Poincaré passes on to the consideration of the characteristic exponents. One solution of a system of differential equations being known, it is required to find a solution differing little from it. The equations of variations are formed in the usual way, and these bring in the equations given above, which involve the characteristic exponents. As an example of the use of these equations, Dr. Hill's work on the motion of the lunar perigee is quoted, where he obtains the principal part of it accurately to a large number of places of decimals.*

It is then considered under what circumstances one or more of the exponents become zero, and their effect on the existence of a periodic solution. The argument and result depend chiefly on two things: first, the presence in, or absence from, the X_i of the time explicitly, and, secondly, the existence or non-existence of single-valued integrals of the system. If canonical equations be used, the exponents are equal and opposite in pairs. With the limitation that F_0 does not depend on the y_i, two exponents will be zero, and unless certain conditions be fulfilled, two exponents only will be zero. In the periodic solutions of the problem of three bodies, whether in one plane or not, two exponents and two only are zero. The solutions corresponding to these exponents are called "*solutions dégénérescentes,*" and are of the form

$$\xi_i = S_i'', \quad \eta_i = T_i'',$$

$$\xi_i = S_i^* + t\,S_i'', \quad \eta_i = T_i^* + t\,T_i'',$$

in which the S, T are periodic.

The canonical system given above has an integral which is known, namely the integral of vis viva. The author devotes himself in Chapter V. principally to prove that, save in certain exceptional cases, there does not exist any single-valued algebraic or transcendental integral other than that of vis viva. For this a function Φ is supposed to be analytic and single-valued for all values of x, y, μ within a certain region, and within this region to be developable according to powers of μ, thus:

$$\Phi = \Phi_0 + \mu\,\Phi_1 + \mu^2\,\Phi_2 + \,.\,.$$

As long as Φ_0 is not a function of F_0, it is proved that $\Phi =$ *const.* cannot be an integral of the system. If Φ_0 be a function of F_0, it is possible to find another integral which is distinct from F, and which does not reduce to F_0 when μ is zero. In case, however, the Hessian of F_0 be zero, an exceptional case arises, and it is in this exceptional case that the

* *Acta Mathematica*, Vol. VIII. See also a note by the writer in *Mo. Not. R. A. S.*, Vol. XVII. No. 6.

importance of the principle applied to problems in dynamics is seen. A general set of conditions is found, necessary but not sufficient for the existence of another integral of the equations. These conditions take the form of relations between the co-efficients in the development of F.

Applying these to the problem of three bodies, the author arrives at the conclusion, *that there cannot exist any new transcendental or algebraic single-valued integral of the problem of three bodies other than the well-known ones, whether we consider the particular cases of two, three, or the general case of four degrees of liberty mentioned above.* This important result is of course applicable here to the case only when μ is small, a restriction which nevertheless occurs in most problems of celestial mechanics. It is pointed out, however, that M. Bruns has demonstrated that there cannot exist any other *algebraic* single-valued integral for *any* values of the masses. In actual application M. Poincaré's theorem will be found the more useful, since he includes transcendental as well as algebraic forms in his demonstration.

The most interesting example given to illustrate the general theorem is that of the motion of a solid suspended from a fixed point and acted on by gravity only. The distance of the centre of mass of the body from the point of suspension is supposed small. Two integrals are known : is it possible that a third can exist ?* When the conditions are applied it is found that there is nothing to prevent the existence of a third integral, but since the conditions are necessary and not sufficient nothing proves that it does exist ; such an integral however cannot be algebraic.

Chapters VI. and VII. treat of the disturbing function and M. Poincaré's asymptotic solutions, respectively. In the consideration of the latter a series appears which is divergent in a manner analogous to Sterling's series.

ERNEST W. BROWN.

HAVERFORD COLLEGE, PA., *April*, 1892.

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* For an elementary discussion of this problem, see Routh's Rigid Dynamics (4th ed.) Vol. II., Chaps. IV., V.

NOTES.

A REGULAR meeting of the NEW YORK MATHEMATICAL SOCIETY was held Saturday afternoon, May 7, at half-past three o'clock, the president in the chair. The council announced that Professor D. A. Murray had been appointed as *librarian* to hold office during the remainder of the current year. The following persons having been duly nominated, and being recommended by the council, were elected to membership: Professor Louis Duncan, Johns Hopkins University; Lieutenant George Owen Squier, U. S. A. and Johns Hopkins University; Mr. Joseph Moody Willard, Johns Hopkins University; Miss Ella C. Williams, New York; Professor Robert Woodworth Prentiss, Rutgers College. The following papers were read: "The fundamental formulas of analysis generalized for space," by Professor A. Macfarlane; "A simple and direct method of separating the roots of ordinary equations," by Professor J. W. Nicholson.

THE prize of one thousand marks offered by the Prince Jablonowski Society in the department of mathematics and natural science for the year 1893, has for its subject: The determination of an extensive class of invariants of ordinary differential equations in accordance with the notation and methods of Lie.

AT the meeting of the London Mathematical Society on April 14, the following six mathematicians were elected honorary members: Messrs. Poincaré, Hertz, Schwartz, Mittag-Leffler, Beltrami, and Willard Gibbs.

DR. KURT HENSEL, *privat-docent* at the University of Berlin, has been appointed to a professorship of mathematics at that university. T. S. F.

PROFESSOR H. WEBER of Marburg has accepted a call to Göttingen to fill the post vacated by Professor H. A. Schwartz. Professor Frobenius of Zurich has accepted a call to Berlin.
M. BÔ.

A MEETING was held on Saturday, February 20, in the combination room of St. John's College, for the purpose of taking steps to place a memorial of the late Professor Adams, in Westminster Abbey, in recognition of his brilliant discoveries in astronomical science. H. J.

NEW PUBLICATIONS.

COMPILED BY B. WESTERMANN & CO., NEW YORK.

AUTONNE (L.). Sur la théorie des équations différentielles du premier ordre et du premier degré. Paris 1892. gr. in-8. 120 pg. M. 7.50

CARRERAS (T.). Traité analytique du Problème d'Echecs. Paris 1892. 8. 8 et 447 pg. av. figures. M. 10.50

CARTE PHOTOGRAPHIQUE DU CIEL. Réunion du comité international permanent pour l'exécution de la carte photographique du ciel à l'Observatoire de Paris en 1891. gr. in-4. 8 et 138 pg. M. 4

DINI (U.). Grundlagen für eine Theorie der Functionen einer veränderlichen reellen Grösse. Deutsch bearbeitet von J. Lüroth u. A. Schepp. Leipzig 1892. 8. 18 u. 552 pg. M. 12

DÖLLEN (W.). Stern-Ephemeriden auf d. J. 1892 zur Bestimmung von Zeit u. Azimut mittelst des tragbaren Durchgangsinstrumentes im Verticale des Polarsterns. Berlin 1891. Lex. 8. 39 pg. M. 1.50

EDWARDS (J.). An Elementary Treatise on the Differential Calculus. 2d ed., revised and enlarged. 8vo, pp. 528. Macmillan. 14s.

GARNIER (Ch.). Monographie de l'Observatoire de Nice. Paris 1892. Fol. avec 36 planches gravées. M. 50

GERLAND (E.). Geschichte der Physik. Leipzig 1892. 8. 6 u. 356 pg. m. 72 Abbildungen. Leinenband. M. 4

HAMILTON.—Graves (R. P.). Life of Sir W. R. Hamilton. Addendum. London 1892. 8. M. 1

NEWMAN (F. W.). The Higher Trigonometry. Super-rationals of Second Order. 8vo. Macmillan and Bowes (Cambridge). Macmillan. 6s. 6d.

RUDIO (F.). Ueber den Antheil der mathematischen Wissenschaften an der Cultur der Renaissance. Hamburg 1892. gr. 8. 33 pg. M. 60

SOMMERFELD (A.). Die willkürlichen Funktionen in der mathematischen Physik. Königsberg 1891. 8. 76 pg. M. 1.50

WÜST (A.). Leichtfassliche Anleitung zum Feldmessen und Nivelliren. für praktische Landwirthe und landwirthschaftliche Lehranstalten bearbeitet. 3. Auflage. Berlin 1892. 8. 16 u. 154 pg. m. 114 Abbildungen. Leinenband. M. 2.50

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BULLETIN

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EDWARDS' DIFFERENTIAL CALCULUS.

An Elementary Treatise on the Differential Calculus, with applications and numerous examples. By JOSEPH EDWARDS, M.A., formerly Fellow of Sidney Sussex College, Cambridge. Second edition, revised and enlarged. London and New York, Macmillan & Co. 1892. 8vo, pp. xiii + 521.

WHEN a mathematical text book reaches a second edition, so much enlarged as this, we know at once that the book has been received with some favour, and we are prepared to find that it has many merits. We are at once struck by Mr. Edwards' lucid and incisive style; his expositions are singularly clear, his words well chosen, his sentences well balanced. In the text of the book we meet with various useful results, notably in the chapter on "some well known curves," and moreover the arrangement is such that these results are easy to find; and in addition to these, numbers of theorems are given among the examples, and, this being a feature for which we are specially grateful, in nearly every case the authority is cited. Recognizing these merits, however, we notice that the book has many defects, some proper to itself, some characteristic of its species; and just because it is so attractive in appearance, it seems worth while examining it in detail, and pointing out certain specially vicious features.

A book of this size may fairly be required to serve as a preparation for the function theory; at all events, the influence of recent Continental researches should be evident to the eyes of the discerning. Mr. Edwards' preface strengthens this reasonable expectation, for he promises us "as succinct an account as possible of the most important results and methods which are up to the present time known." But we soon find that the "important results and methods" are those of the Mathematical Tripos; and in our disappointment we utter a fervent wish that instead of the "large number of university and college examination papers, set in Oxford, Cambridge, London, and elsewhere," Mr. Edwards had consulted an equally large number of mathematical memoirs published, principally, elsewhere. The Mathematical Tripos for any given year is not intended for a *Jahrbuch* of the progress of mathematics during the past year; and as long as so many will insist on regarding it in that light, text books of this type will continue to be published.

Nothing in this book indicates that Mr. Edwards is familiar with such works as Stolz's *Allgemeine Arithmetik*, Dini's *Fondamenti per la teorica delle funzioni di variabili reali*, or Tannery's *Théorie des fonctions d'une variable.* In support of our contention we may instance the definitions of function, limit, continuity, etc. On page 2, Lejeune Dirichlet's definition of a function is adopted. According to this very general definition, there need be no analytical connection between y and x; for y is a function of x even when the values of y are arbitrarily assigned, as in a table. That Mr. Edwards does not adhere to this definition is evident from his tacit assumption that *every* function $\varphi(x)$ can be represented by a succession of continuous arcs of curves. Whatever definition is adopted for a continuous function y of x, it is evident that to small increments of x must correspond small increments of y; but Weierstrass has proved that there exist functions which have this property, but which have nowhere differential coefficients. The well known example of such a function is

$$f(x) = \sum_{n=0}^{\infty} b^n \cos (a^n x\pi),$$

where a is an odd integer, b a positive constant less than 1, and ab greater than $1 + 3\pi/2$. According to the accepted definition, this function of x is continuous; according to Mr. Edwards' definition, it is not continuous, inasmuch as it cannot be represented by a curve $y = f(x)$ with a tangent at every point.

We acknowledge that Mr. Edwards displays a considerable degree of consistency in his view of the meaning of a continuous function, but we insist that after the adoption of the curve definition he should have been at some pains to prove that the numerous series of the type $\sum_1^{\infty} f_n(x)$ scattered throughout the book give rise to curves with tangents, whereas he never even takes the trouble to prove that they are continuous functions of x in any sense of the term. No more damaging charge can be brought against any treatise laying claim to thoroughness than that of recklessness in the use of infinite series; and yet Mr. Edwards has everywhere laid himself open to this charge. One of the most difficult things to teach the beginner in mathematics is to give proper attention to the convergency of the series dealt with. All the more need, then, that a text book of this nature should set an example of consistent, even *aggressive* carefulness in this respect. We do, it is true, find an occasional mention of convergence (pp. 9, 81, 454, etc.), but as a rule it is ignored. Mr. Edwards rearranges the terms of infinite and doubly infinite series, applying the law of commutation without pointing out that his series are unconditionally convergent; he differentiates $f(x) = \sum_1^{\infty} f_n(x)$ term by term, and gets $f'(x) = \sum_1^{\infty} f'_n(x)$, implying that the process is universally valid (*e.g.* p. 84); or, at all events, giving no hint that there are cases in which the differential coefficient of the sum of a convergent series is different from the sum of the differential coefficients of the individual terms. We find no formal recognition of the importance of uniform convergence in modern analysis, nothing even to suggest that he has ever heard of the distinction between uniform and non-uniform convergence. We begin to suspect that he has never looked into Chrystal's Algebra.

The unreasoning mechanical facility thus acquired in performing operations unhampered by any doubts as to their legitimacy, naturally leads Mr. Edwards to view with favour "the analytical house of cards, composed of complicated and curious formulæ, which the academic tyro builds with such zest upon a slippery foundation," *—and to build up many a one. A curious and interesting specimen is

$$f(x) = x^{x^{x^{\cdot^{\cdot^{\cdot}}}}}$$

to be continued to infinity. This expression has been examined by Seidel,† who points out that Eisenstein's paper in *Crelle*, vol. 28, requires correction. Before such an expression can be differentiated, a definite meaning must be assigned to it; but Seidel's conclusion is that, denoting x^x by x_1, x^{x_1} by x_2, x^{x_2} by x_3, and so on, then as x varies from 0 to $1/e^e$, $\underset{n=\infty}{L}\ x_{2n}$ increases from 0 to $1/e$, while $\underset{n=\infty}{L}\ x_{2n+1}$ decreases from 1 to $1/e$; beyond these limits for x, the case is different. In particular when $x > e^{1/e}$, the expression diverges. Our objection is not to the non-acceptance of Seidel's conclusions, but to the unnecessary use of a function of this doubtful character. Examples can be found to illustrate every point that ought to be brought up in an elementary treatise on the differential calculus without ranging over examination papers in search of striking novelties.

Feeling now somewhat familiar with Mr. Edwards' point of view, we examine his proofs of the ordinary expansions with a tolerably clear idea of what we are to expect. We find, of course, "the time-honoured short proof of the existence of the exponential limit, which proof is half the real proof plus a *suggestio falsi*"; we find in the chapter on expansions a general disregard of convergency considerations; we find throughout the book the assumption that

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* Professor CHRYSTAL, in *Nature*, June 25, 1891.

† *Abhandlungen der k. Ak. d. Wiss.* Bd. xi.

$\varphi(a) = \underset{x=a}{L}\ \varphi(x)$, and that $\varphi(0, 0) = \underset{x=0,\, y=0}{L}\ \varphi(x, y)$ *; we find the usual assumptions as to expansibility in series proceeding by integral powers, with disastrous results further on. We find the usual dread of the complex variable, though Mr. Edwards has given one or two examples involving it, without however explaining what is meant by $f(x + iy)$. We can hardly regard these examples, even with § 190, as a sufficient recognition of the complex variable in a treatise of this size. We must notice also the thoroughly faulty treatment of the inverse functions. For example, no explanation is given of the signs in $\frac{dy}{dx}$, when $y = \cos^{-1} x$ or $\sin^{-1} x$. Mr. Edwards' attitude towards many valued functions is simple enough; as a rule, he ignores the inconvenient superfluity of values. He does, it is true, give in § 54 a note, clear and correct, on this point; but he is very careful to confine this within the limits of the single section, and to indicate, by choice of type, that it is quite unimportant.

We pass on now to the second part, applications to plane curves; and here we must object emphatically to the introduction of so many detached and disconnected propositions relating to the theory of higher plane curves. From Mr. Edwards' point of view this is doubtless justified; we are quite ready to acknowledge that we know of no book that would enable a candidate to answer more questions on subjects of whose theory he is totally ignorant. The deficiency of a curve, *e.g.*, is a conception entirely independent of the differential calculus; but probably this single page will obtain many marks for candidates in the Mathematical Tripos; these we should not grudge if we thought an equivalent would be lost by a reproduction of Mr. Edwards' treatment of cusps. Our spirits rose when we remarked the italicised phrase on p. 224, that there is "*in general a cusp*" when the tangents are coincident. But three pages further on we find that the exception here indicated is simply our old friend, the conjugate point, whose special exclusion from the class in which it appears must be a perpetual puzzle to a thoughtful student with no better guidance than a book of this kind. Such a student, probably already familiar with projection, knows that the real can be projected into the imaginary, and the imaginary into the real. If then the acnode, appearing as a cusp, has to be specially excluded, why not the crunode? But here Mr. Edwards reproduces the now well established

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* See *e.g.* p. 122; and on this page note also the assumption that the relation between h, k, while $x + h$, $y + k$, tend to the limits x, y exerts no influence on the result.

error, calling tacnodes, formed by the contact of real branches, double cusps of the first and second species, and excluding those formed by the contact of imaginary branches; he even goes further astray, introducing Cramer's osculinflexion as a cusp that changes its species.

This matter of double cusps is a fundamentally serious one, and not a mere question of nomenclature. This persistent misnaming effectually disguises the essential characteristic of the cusp. It is *not the coincidence of the tangents* that makes a cusp. From the geometrical point of view it is the turning back of the (real) tracing point, expressed by the French and German names, {*point de rebroussement, Rückkehrpunkt*}; from the point of view of algebraical expansions (of y in terms of x, $y = 0$ being the tangent) the essential characteristic of a single cusp is that at some stage in the expansion there shall be a fractional exponent with an even denominator, so that the branch changes from real to imaginary *along its tangent;* from the point of view of the function theory, which is really equivalent to the last, the simple cusp is characterised by the presence of a *Verzweigungspunkt* combined with a double point. The simple cusp, that is, presents itself as an evanescent loop. A double cusp, then, in the sense in which Mr. Edwards uses the term, does not exist. There cannot be two consecutive cusps, vertex to vertex; for the branch if supposed continued through the cusp, changes from real to imaginary; and two *distinct* cusps, brought together to give a point of this appearance, produce a quadruple point.

While on this subject, we must mention Mr. Edwards' rule for finding the nature of a cusp. Find the two values of $\frac{d^2y}{dx^2}$; these by their signs determine the direction of convexity (§ 296). How does this apply *e.g.* to $y^2 = x^3$?

This confusion regarding cusps is made worse by the assumption already noticed that when $f(x, y) = 0$ is the equation of the curve, y can be expanded in a series of integral powers of x. This error is repeated on p. 258, where to obtain the branches at the origin, this being a double point, we are told to expand y by means of the assumption $y = px + \frac{qx^2}{2!} +$ etc. The whole exposition of this theory of expansion is most inadequate. In § 382 there is no hint that the terms obtained are the beginning of an infinite series, giving the expansion of (say) y in powers, not necessarily integral, of x; there is no hint what to do when the first terms of the expansion are found; there is no suggestion of the interpretation of the result when two expansions begin with the same terms. A thoughtful student *may* by a happy comparison of scattered examples (p. 200, and ex. 3, p. 230) arrive at the correct theory; but he surely deserves better guidance.

One or two more points must be noticed. The theory of asymptotes, when two directions to infinity coincide, cannot be satisfactorily developed without assuming a knowledge of double points; and the only way of giving the true geometrical significance is to introduce the conception of the line infinity, and to consider the nature of the intersections of the curve by this line. A tangent lying entirely at infinity does *not* "count as one of the *n* theoretical asymptotes"; if counted among the asymptotes at all, it has to be counted as the equivalent of two out of the *n*. This is one of the strongest arguments against including the line infinity in enumerating the asymptotes. The various expressions for the radius of curvature involve an ambiguity in sign; what is the meaning of this? The omission of this explanation causes obscurity, notably in § 330. The equation of a curve, referred to oblique axes, being $\varphi(x, y) = 0$, what is the condition for an inflexion? As a matter of fact it is the same as in the case of rectangular axes, given on p. 264; but as this is obtained from a formula for the radius of curvature, the investigation is not applicable. Throughout Mr. Edwards displays an almost exclusive preference for rectangular axes, and seems to regard the metric properties so obtained as of equal importance with descriptive properties. For instance, in the case of an ordinary double point (p. 224) instead of the *three* cases usually distinguished, we have *four*, the additional one being that of perpendicular tangents.

In the third part we notice that in the chapter on "undetermined forms" there is no discussion of the case of two variables, though it is on this that we have to rely for a rigorous proof of the theorem $\frac{\delta^2\varphi}{\delta x\delta y} = \frac{\delta^2\varphi}{\delta y\delta x}$. We recognize an old friend, the discussion of the limit of ∞/∞, in which it is first assumed, and then proved, that the limit exists. The statement of ex. 17, p. 457, is somewhat misleading; the formula there given for the expansion of $(x + a)^m$ is true when m is a positive integer; but when $m = -1$, it is evidently not true for $x = -b, -2b$, etc.* The treatment of maxima and minima of functions of two variables (§§ 497–501) is incomplete and incorrect. The geometrical illustration, as given on p. 424, omits the case of a section with a cusp, which is the simplest case that can occur when $rt = s^2$; of the more complicated cases Mr. Edwards attempts no discrimination; he does not even state correctly the principles that must guide us in this discrimination. The inexactness of the ordinary

* LAURENT, *Traité d'Analyse*, iii., 386.

criteria (given in § 498) appears at once from the example $u = (y^2 - 2px)(y^2 - 2qx)$ [Peano]. The origin is a point satisfying the preliminary conditions; taking then for x, y, small quantities h, k, the terms of the second degree are positive for all values except $h = 0$; when $h = 0$, the terms of the third degree vanish, and the terms of the fourth degree are positive; nevertheless the point does not give a minimum, which it should do by the test of § 498. For we can travel away from O in between the two parabolas, so coming to an adjacent point at which u has a small negative value, while for points inside or outside both parabolas the value of u is positive. The truth is, the nature of the value a of the function u at a point (x_0, y_0) at which $\frac{\delta\varphi}{\delta x}$ and $\frac{\delta\varphi}{\delta y}$ vanish, depends on the nature of the singularity of the curve $u = a$ at this point. If this curve has at (x_0, y_0) an isolated point of any degree of multiplicity, we have a true maximum or minimum of u; but if through (x_0, y_0) pass any number of real non-repeated branches of the curve, we have not a maximum or minimum; in Peano's example the branches coincide in the immediate neighbourhood of the origin, but then they separate, and therefore we have not a minimum value for u.

We object, then, to Mr. Edwards' treatise on the Differential Calculus because in it, notwithstanding a specious show of rigour, he repeats old errors and faulty methods of proof, and introduces new errors; and because its tendency is to encourage the practice of cramming "short proofs" and detached propositions for examination purposes.

CHARLOTTE ANGAS SCOTT.

BRYN MAWR, PA., *May* 18, 1892.

NOTE ON RESULTANTS.

BY PROF. M. W. HASKELL.

ON page 151 of Prof. Gordan's lectures on determinants* is to be found the theorem

$$R_{f,\phi} = R_{f+\phi.\psi,\phi}$$

where $R_{f,\phi}$ denotes the resultant of two functions f and ϕ of a single variable x of degree m and n respectively. This

* *Vorlesungen über Invariantentheorie*, herausgegeben von KERSCHENSTEINER. Erster Band. Leipzig, 1885.

theorem is proved under the restriction that n is not greater than m and that the degree of the arbitrary function ψ shall not be greater that $m - n$. The author goes on to say: "es wäre eine schätzenswerthe Arbeit, auch den Fall zu untersuchen, in welchem $m < n$ ist, d. h. allgemein die Frage zu behandeln: Wie hängen die Resultanten $R_{f+\phi.\psi,\phi}$ und $R_{f,\phi}$ zusammen, wenn wir über den Grad der diesbezüglichen Functionen keinerlei Voraussetzung machen?"

This statement is somewhat remarkable on account of the ease with which it may be shown that the theorem is true in general in exactly the form given above.

I. Let $F = f + \phi . \psi$. Then, no matter what the degree of the functions involved may be, if the degrees of F and ϕ be m and n respectively, n is certainly not greater than m, and the degree of ψ cannot be greater than $m - n$. Hence, by Gordan's result as quoted,

$$R_{F,\phi} = R_{F-\phi.\psi,\phi} = R_{f,\phi}.$$

That is, the theorem is true without restriction.

II. Suppose the resultant $R_{f,\phi}$ to be found in the usual way by the method of greatest common divisor. Two functions A and B of x, of degree $n - 1$ and $m - 1$ respectively can be found to satisfy the relation:

$$1 = A . f + B . \phi.$$

The coefficients of A and B are rational, but not integral, functions of the coefficients of f and ϕ, whose least common denominator is the resultant $R_{f,\phi}$.

It follows that

$$1 = A(f + \phi . \psi) + (B - A\psi)\phi$$

and the resultant $R_{f+\phi.\psi,\phi}$ is the least common denominator of the coefficients of A and $(B - A\psi)$. But the coefficients of ψ will evidently not occur in the denominators at all, and the least common denominator is therefore identical with that of the coefficients of A and B, viz. $R_{f,\phi}$.

BERKELEY, CAL., *May* 12, 1892.

COLLINEATION AS A MODE OF MOTION.*

BY MAXIME BÔCHER, PH.D.

IN the following paper I have attempted to give an account of some very simple matters, which, although familiar to many, appear to have attracted but little attention in this country. The subject, however, has never, as far as I know, been presented from precisely the point of view here adopted.

Perhaps the most important difference between the old and the new geometry lies in the extended use made during the present century of geometric transformations.† The change which has come about in this direction is due in part to the influence of certain branches of applied mathematics in which one has to deal not merely with geometric configurations but also with certain changes which these configurations are forced to undergo. There are however two distinct ways of looking at a transformation. First we may consider the original and the transformed figure as standing side by side, or even as occupying portions of the same space, the latter being in a certain sense a picture of the former; or secondly, we may consider the original figure to be gradually deformed according to a given law into the transformed figure. Each of these points of view can be traced to a physical origin. Perspective and allied subjects strikingly illustrate the first, while the second will most naturally be adopted in hydrodynamics, the theory of elasticity, etc. Now while the first of the above mentioned ways of looking at a transformation has the advantage of introducing no unnecessary element into the consideration, the second in turn has the advantage of making the idea of a transformation lose much of its abstractness, for by its aid we are enabled to see the points of the original figure rearrange themselves by a gradual motion into the transformed figure.

I wish to illustrate this way of looking at a transformation as a mode of motion by considering one of the simplest of transformations, the so-called linear transformation or collineation,‡ and for the sake of simplicity I will confine myself to two dimensions.

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* Lecture delivered June 4, 1892, before the New York Mathematical Society.

† The following remarks should be understood to apply only to point transformations, *i.e.*, to transformations which carry points over into points.

‡ The word collineation seems to be by far the best name for this transformation, not only because it is as applicable in synthetic as in analytic geometry, but also because the ambiguity which arises in speaking of a

Using any system of trilinear coordinates (x_1, x_2, x_3), a collineation will be expressed by the linear formulæ:

$$\begin{aligned} \rho x_1' &= a_1 x_1 + a_2 x_2 + a_3 x_3, \\ \rho x_2' &= b_1 x_1 + b_2 x_2 + b_3 x_3, \\ \rho x_3' &= c_1 x_1 + c_2 x_2 + c_3 x_3, \end{aligned} \qquad (1)$$

(ρ being an undetermined factor of proportionality).

It is however well known that in general a collineation leaves three points of the plane fixed while all other points are carried over into new positions. If now these three fixed points be taken as the vertices of the triangle of reference, the collineation will evidently be expressed by the very simple formulæ:

$$\rho x_1' = a x_1, \qquad \rho x_2' = b x_2, \qquad \rho x_3' = c x_3. \qquad (2)$$

These formulæ tell us into what position each point of the plane is carried over by the transformation; they give us, however, no clue as to what path it is advisable to regard as traversed by each point in passing from its original to its final position.

To determine this, let us first consider the case in which two of the fixed points are the circular points at infinity, the third (finite) fixed point being denoted by the letter O. This collineation may be shown by a simple calculation, to consist of a rotation of the plane as a whole about the point O combined with a uniform stretching of the plane away from (or contraction of the plane towards) this same point. In the case of a rotation, however, each point will naturally be regarded as moving from its original to its final position along the arc of a circle whose centre is at O; in the case of expansion or contraction on the other hand, the lines of motion will be the straight lines through O, the motion taking place away from O in the case of expansion and towards O in the case of contraction; the amount of the motion in any case being proportional to the distance from O. If, then, we have a combination of rotation and expansion (or contraction) the lines of motion will evidently be equal logarithmic spirals with pole at O. Taking O as the origin of a system of polar coordinates, the equation of this family of logarithmic spirals will be:

$$r = Ae^{k\phi},$$

linear transformation without specifying what system of coordinates we use is a very real objection, as there are other coordinates besides trilinear (for example, Darboux's tetracyclic coordinates) in which linear transformations are actually considered. The term "homographic transformation," introduced by Chasles, is not as expressive as the term collineation used some years before by Möbius. It does not seem as though Chasles' ignorance of the German language could justify us in adopting his poorer names in place of the original better ones.

where k is a constant determining the shape of the spirals, while A is a parameter varying from one member of the family to another. We shall find it more convenient to write in place of k the quotient k_1/k_2.

Let us now introduce a system of trilinear coordinates in which the vertices of the triangle of reference are the circular points at infinity and the point O. We will denote these coordinates by the letters (ξ, η, t), the first two referring to the imaginary sides of the triangle of reference through O, while the last refers to the line at infinity. In this system of homogeneous "circular" coordinates the equation of the above mentioned family of logarithmic spirals is readily found to be:

$$\xi^{k_2+ik_1}\,\eta^{k_2-ik_1}\,t^{-2k_2} = A^{2k_2},$$

or since k_1, k_2, A are any constants:

$$\xi^{\alpha}\,\eta^{\beta}\,t^{\gamma} = C,$$

provided that $\alpha + \beta + \gamma = 0$.

We can now write out at once the equations of the lines of motion in the general case where we have as fixed points any three points of the plane, for we have merely to project the point O and the circular points at infinity in the special case we have just considered into any other three points in order that the logarithmic spirals should go over into the lines of motion of a general collineation. The equation of the family of lines of motion, referred to the triangle of reference whose vertices are the fixed points, is then:

$$x_1^{\alpha}\,x_2^{\beta}\,x_3^{\gamma} = C,$$

where C is the variable parameter of the family while the constants α, β, γ (which are connected by the relation $\alpha + \beta + \gamma = 0$) depend upon the coefficients a, b, c of the linear transformation (2) above.*

The fixed points of a collineation may be all real, or one of them may be real and the other two conjugate imaginary. The last of these two possibilities need not detain us long, as it may be obtained by a real projection from the special case considered above where two of the fixed points were the circular points at infinity. In it we shall have in our triangle of reference one vertex and the side opposite real, while the remaining vertices and sides are imaginary. The lines of motion will have a spiral form, each consisting of an infinite

* It is easily found that α, β, γ are proportional respectively to $\log \frac{c}{b}$, $\log \frac{a}{c}$, $\log \frac{b}{a}$.

number of coils about the real fixed point. As these coils become larger they will become more and more elongated in the direction farthest from the real side of the triangle of reference, until each coil finally assumes a hyperbolic form, running out on the side of the fixed point farthest from the real side of the triangle of reference through infinity, and completing itself on the other side of the real line in question. These hyperbolic coils ultimately approach the real side of the triangle of reference asymptotically.

When all of the fixed points are real, however, the lines of motion will have completely lost their spiral character. Here again there are two cases to consider, according as the three coefficients a, b, c of the transformation have all the same sign, or one of them a different sign from the other two. The three indefinitely extended sides of the triangle of reference divide the plane into four parts, one finite and the other three infinite. We may speak of each of these parts as "triangles," in spite of the fact that each of the three infinite triangles appears to be divided into two distinct portions by the line at infinity. Using this terminology we may say that when all three coefficients a, b, c have the same sign, the interior of each of these four triangles is transformed into itself; but when one of the three coefficients has a different sign from the other two the triangles are interchanged in pairs. We will begin with the simpler of the two cases, in which each triangle is transformed into itself. The lines of motion in this case will be found to lie as follows :—*

Within the finite triangle the lines of motion all start from the vertex corresponding to the smallest of the three coefficients,† and run without singularity to the vertex corresponding to the largest of them ; at each of their extremities these curves are tangent to the side of the triangle joining that extremity with the vertex corresponding to the coefficient which lies in magnitude between the other two. The side of the triangle joining the vertices which correspond to the greatest and the smallest coefficient is, of course, itself a line of motion, and the same is true of the broken line consisting of the other two sides of the triangle.

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* One way of seeing this is to consider first the special case in which the triangle of reference consists of two lines at right angles to one another and the line at infinity, and then to project this into the general case. In the special case just mentioned we have to deal with the same transformation which occurs in the theory of small irrotational strains (see for instance MINCHIN, Uniplaner Kinematics, chap. v.). It is interesting to notice that this is a case in which the idea of lines of motion is naturally suggested by a physical application.

† The coefficients a, b, c are said to correspond to the vertices opposite the sides $x_1 = 0$, $x_2 = 0$, $x_3 = 0$ respectively.

The lines of motion within each of the other three triangles will be precisely like those just described, the difference in appearance being due to the fact that these triangles themselves extending through infinity, some of the lines of motion in one of these three triangles and all of the lines of motion in the other two will run through infinity on their way from the vertex corresponding to the smallest of the coefficients to the one corresponding to the largest.

It should be noticed that while these curves are in general transcendental and extend only between two fixed points of the collineation where they suddenly stop, we can find special collineations for which the curves are algebraic and all of any degree we please. The *family* of curves will not look particularly different in these cases from what it does when the curves are transcendental, but the curves themselves will have a different shape. They will now no longer stop at the two fixed points just mentioned, but will continue beyond them into another triangle, having singularities in these points when their degree is higher than the second (in the case of cubics, a cusp in one point, and a point of inflection in the other). The case where the lines of motion are conics all tangent at the extremities of one of the sides of the triangle of reference to the other two sides deserves special mention owing to its frequent occurrence in projective geometry.*

Coming now to the case where one of the coefficients of the transformation has a different sign from the other two, it is readily seen that the lines of motion are here imaginary although each contains an infinite number of real points. Every point of the plane is therefore carried over from its original to its final position through an imaginary path. We are therefore unable to follow the motion of the points of the plane. It is however possible to break up the transformation into two parts, one very simple, the other more complicated but having real lines of motion. Thus for instance we can break up the collineation:

$$\rho x_1' = -2x_1, \quad \rho x_2' = 3x_2, \quad \rho x_3' = 5x_3,$$

into the two collineations:

$$\rho \bar{x}_1 = -x_1, \quad \rho \bar{x}_2 = x_2, \quad \rho \bar{x}_3 = x_3;$$
$$\rho x_1' = 2\bar{x}_1, \quad \rho x_2' = 3\bar{x}_2, \quad \rho x_3' = 5\bar{x}_3.$$

* We may of course have conics as lines of motion when two of the fixed points of the collineation are imaginary, rotation of the plane about a point being a special case of this. In fact, whenever the lines of motion are conics, whether the fixed points are real or imaginary, the collineation will be merely a non-euclidian rotation if we take one of these conics as the absolute.

The second of these has as its lines of motion the real transcendental curves discussed above, while I may perhaps be allowed to describe the first as a "projective reflection" with regard to the side $x_1 = 0$ and the opposite vertex. The nature of the transformation brought about by this projective reflection is so simple that it need not be discussed here, and that we do not need the assistance of lines of motion to get a perfectly clear idea of it.* It is of course merely the projective generalization of ordinary reflection; reflection with regard to the axis of X, for instance, in a system of rectangular coordinates, being merely a projective reflection with regard to this line and the infinitely distant point on the axis of Y.

It remains to mention some of the literature connected with this subject. The transcendental curves, which we have called the lines of motion of the collineation, occur incidentally in a paper by Clebsch and Gordan in the *Mathematische Annalen*, vol. I. They were however first systematically considered by Klein and Lie in vol. IV. of the same journal (1871). The reader is referred to this paper for the modification of the lines of motion which occur in the various special cases (when the collineation has two coincident fixed points, etc.). The very brief indications there given can readily be amplified as has been done in this paper for the general case. The reader will also find in this beautiful paper an account of some of the remarkable properties of these curves, which thus gain an interest far above that attaching as yet to most other transcendental curves, owing to the fact that their properties form to some extent a systematic whole, not a mass of facts more or less ingeniously proved. More important still however is the connection of these lines of motion with Lie's now famous theory of differential equations,† some of the very earliest of Lie's investigations in this direction being contained in the paper just mentioned. By the introduction of infinitesimal transformations it is possible to obtain the lines of motion directly without first considering the special case in which the circular points at infinity are two of the fixed points. We thus find the equation of the lines of motion as the solution of a differential equation.

In still another way must Klein's name be connected with

* It should however be noticed that a projective reflection (and therefore any ordinary reflection) *may* be regarded as having real lines of motion, viz., conics. This will be most readily seen if we consider that the projective reflection with regard to the line at infinity and a finite point is equivalent to a rotation through an angle of 180° about that point.

† See Lie's recently published book on this subject edited by Scheffers.

this subject. A few preliminary remarks are necessary to explain this. The linear transformation of a single straight line into itself may be studied from precisely the same point of view as we adopted above in the case of two dimensions. Three cases would again present themselves: one in which the two fixed points are imaginary, and two in which they are real. In one of these last the transformation cannot be regarded as a real motion, while in the other two it can. Now the extension of our theory which suggests itself to us here depends upon the fact that the complex points of a straight line can be conveniently represented in a plane of which the line is the axis of reals. The linear transformation of the line will then give us a corresponding transformation of the plane which of course should not be confounded with the collineation discussed above. The coefficients here need no longer be real to give us a real transformation. This new transformation of the plane may also be regarded as a mode of motion and has been so treated by Klein in his lectures for a number of years (see an article by Prof. Cole in the *Annals of Mathematics* for June, 1890, and part II. chap. I. of the recently published *Modulfunctionen* of Klein-Fricke). The idea cannot fail to suggest itself that the transformation of the plane which we have called collineation should be generalized in a similar way by representing the complex as well as the real points of the plane. I do not know of this subject having been treated; it would of course lead us into four dimensional space.

HARVARD UNIVERSITY, *June*, 1892.

NOTES.

A REGULAR meeting of the NEW YORK MATHEMATICAL SOCIETY was held Saturday afternoon, June 4, at half past two o'clock, the president in the chair. The following persons having been duly nominated, and being recommended by the council, were elected to membership: Dr. James Whitbread Lee Glaisher, Trinity College, Cambridge, England; Mr. Ferdinand Shack, New York, N. Y. The following papers were read: "An expression for the total surface of an ellipsoid in terms of σ- and $\wp$- functions, including an application to the surface of a prolate spheroid," by Professor J. H. Boyd; "On collineation as a mode of motion," by Dr. Maxime Bôcher; "On Peters' formula for probable error," by Professor W. Woolsey Johnson.

THE meeting of the *Deutsche Mathematiker-Vereinigung*, which will be held this summer as usual in conjunction with that of the *Gesellschaft deutscher Naturforscher und Aerzte*, will take place at Nurenberg, September 12 to 18. Special interest attaches to the meeting this year on account of the organization by the Union of an exhibition of medals, charts, apparatus and instruments used in pure and applied mathematics. The Bavarian government will lend its aid to the enterprise, which has already secured the co-operation of several eminent mathematicians, of the leading publishers, instrument makers etc., and of a large number of high-schools and polytechnic institutes. The object of the exhibition is "to extend the use of the various auxiliaries in the shape of models, apparatus and instruments, which are of advantage for instruction and investigation in pure and applied mathematics, and to forward the interests of this kind of scientific work." A recent prospectus contains a preliminary classification of articles, giving as the main heads: (1) geometry and theory of functions, (2) arithmetic, algebra and integral calculus, (3) mechanics and mathematical physics.

PROFESSOR PEANO, the editor of the *Rivista di Matematica*, has undertaken a very interesting work, the parts of which will appear as supplements to his journal. It is an extended collection of the formulas and results of mathematics, expressed throughout in the language or notation of symbolic logic. The first signature of the work accompanies the number of the *Rivista* for April, 1892 (vol. II., No. 4).

THE publication of the collected works of the late Professor Weber has been undertaken by the Göttingen Academy of Sciences. The collection will probably fill six large octavo volumes, and it is to be completed by 1894. T. S. F.

DR. ARTHUR SCHÖNFLIES, *privatdocent* at the University of Göttingen, has been appointed professor *extraordinarius* at the same university.

HARVARD UNIVERSITY. Besides the more elementary courses, the class-room work in which will amount to twenty-three hours a week throughout the year, the following mathematical courses are offered for the year 1892–93:

By Professor J. M. Peirce; Algebraic plane curves; Quaternions (second course); Theory of functions (first course); Linear associative algebra, and the algebra of logic.

By Professor C. J. White; Planetary theory.

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87	25	(5_6),	(5_7).
123	4	G. B. ZERR,	G. B. M. ZERR.
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