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CALCULUS

MODERN MATHEMATICAL TEXTS

EDITED BY CHARLES S. SLICHTER

CALCULUS

BY

HERMAN W. MARCH, PH. D.
ASSOCIATE PROFESSOR OF MATHEMATICS
UNIVERSITY OF WISCONSIN

AND

HENRY C. WOLFF, PH. D.
PROFESSOR OF MATHEMATICS
DREXEL INSTITUTE

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PREFACE

One of the purposes of the elementary working courses in mathematics of the freshman and sophomore years is to exhibit the bond that unites the experimental sciences. "The bond of union among the physical sciences is the mathematical spirit and the mathematical method which pervade them." For this reason, the applications of mathematics, not to artificial problems, but to the more elementary of the classical problems of natural science, find a place in every working course in mathematics. This presents probably the most difficult task of the text-book writer,—namely, to make clear to the student that mathematics has to do with the laws of actual phenomena, without at the same time undertaking to teach technology, or attempting to build upon ideas which the student does not possess. It is easy enough to give examples of the application of the processes of mathematics to scientific problems; it is more difficult to exhibit by these problems, how, in mathematics, the very language and methods of thought fit naturally into the expression and derivation of scientific laws and of natural concepts.

It is in this spirit that the authors have endeavored to develop the fundamental processes of the calculus which play so important a part in the physical sciences; namely, to place the emphasis upon the mode of thought in the hope that, even though the student may forget the details of the subject, he will continue to apply these fundamental modes of thinking in his later scientific or technical career. It is with this purpose in mind that problems in geometry, physics, and mechanics have been freely used. The problems chosen will be readily comprehended by students ordinarily taking the first course in the calculus.

A second purpose in an elementary working course in mathematics is to secure facility in using the rules of operation which must be applied in calculations. Of necessity large numbers of drill problems have been inserted to furnish practice in using the rules. It is hoped that the solution of these problems will be regarded by teacher and student as a necessary part but not the vital part of the course.

While the needs of technical students have been particularly in the minds of the authors, it is believed that the book is equally adapted to the needs of any other student pursuing a first course in calculus. The authors do not believe that the purposes of courses in elementary mathematics for technical students and for students of pure science differ materially. Either of these classes of students gains in mathematical power from the type of study that is often assumed to be fitted for the other class.

In agreement with many others, the book is not divided into two parts, Differential Calculus and Integral Calculus. Integration with the determination of the constant of integration, and the definite integral as the limit of a sum, are given immediately following the differentiation of algebraic functions and before the differentiation of the transcendental functions. With this arrangement many of the most important applications of the calculus occur early in the course and constantly recur. Further, with this arrangement, the student is enabled to pursue more advantageously courses in physics and mechanics simultaneously with the calculus.

The attempt has been made to give infinitesimals their proper importance. In this connection Duhamel's Theorem is used as a valuable working principle, though the refinements of statement upon which a rigorous proof can be based have not been given.

The subjects of center of gravity and moments of inertia have been treated somewhat more fully than is usual. They are particularly valuable in emphasizing the concept of the definite integral as the limit of a sum and as a mode of calculating the mean value of a function. Sufficient solid analytic geometry is given to enable students without previous knowledge of this subject to work the problems involving solids. In the last chapter simple types of differential equations are taken up.

The book is designed for a course of four hours a week throughout the college year. But it is easy to adapt it to a three-hour course by suitable omissions.

The authors are indebted to numerous current text-books for many of the exercises. To prevent distracting the student's attention from the principles involved, exercises requiring complicated reductions have been avoided as far as possible.

The book in a preliminary form has been used for two years with students in the College of Engineering of the University of Wisconsin. Many improvements have been suggested by our colleagues, Professor H. T. Burgess, Messrs. E. Taylor, T. C. Fry, J. A. Nyberg, and R. Keffer. Particular acknowledgment is due to the editor of this series, Professor C. S. Slichter, for suggestions as to the plan of the book and for suggestive criticism of the manuscript at all stages of its preparation.

The authors will feel repaid if a little has been accomplished toward presenting the calculus in such a way that it will appeal to the average student rather as a means of studying scientific problems than as a collection of proofs and formulas.

University of Wisconsin,
November 6, 1916.

Herman W. March,
Henry C. Wolff.

CONTENTS

CHAPTER I

Derivative

CHAPTER II

Limits

CHAPTER III

The Power Function

CHAPTER IV

Differentiation of Algebration Functions

CHAPTER V

Second Derivative. Point of Inflection

CHAPTER VI

Applications

CHAPTER VII

Infinitesimals. Differentials. Definite Integrals

CHAPTER VIII

Circular Functions. Inverse Circular Functions

CHAPTER IX

Exponential and Logarithmic Functions

CHAPTER X

Maxima and Minima

CHAPTER XI

Polar Coördinates

CHAPTER XII

Integration

CHAPTER XIII

Applications of the Process of Integration. Improper Integrals

CHAPTER XIV

Solid Geometry

CHAPTER XV

Successive Integration. Center of Gravity. Moment of Inertia

CHAPTER XVI

Curvature. Evolutes. Envelopes

CHAPTER XVII

Series. Taylor's and Maclaurin's Theorems. Indeterminate Forms

CHAPTER XVIII

Total Derivative. Exact Differential

CHAPTER XIX

Differential Equations

CALCULUS

INTRODUCTION

1. Constant. Variable. Function. 1. A symbol of number or quantity, as a, to which a fixed value is assigned throughout the same problem or discussion is called a *constant.*

2. A symbol of number or quantity, as x, to which a succession of values is assigned in the same problem or discussion is called a *variable.*

Example. The mass or weight of mercury in a thermometer is constant. The number that results from measuring this quantity (weight) is a constant.

The volume of the mercury in the thermometer is variable. The number that results from measuring this quantity (volume) is a variable.

3. The variable y is said to be a *function* of the variable x if, when x is given, one or more values of y are determined.

4. x, the variable to which values are assigned at will is called the *independent variable,* or the *argument of the function.*

5. y, whose values are thereby determined, is called the *dependent variable.*

6. y is said to be a function of several variables $u, v, w, \cdots$ if, when $u, v, w, \cdots$ are given, one or more values of y are determined.

7. The variables $u, v, w, \cdots$, to which values are assigned at will are called the *independent variables,* or the *arguments of the function.*

Functions of a single variable or argument are represented by symbols such as the following: $f(x)$, $F(x)$, $\phi(x)$, $\psi(x)$. Functions of several arguments are represented by symbols such as $f(u, v, w)$, $F(u, v, w)$, $\phi(u, v, w)$.

2. The Power Function. 8. The function x^n, where n is a constant, is called the *power function.*

If n is positive the function is said to be of the *parabolic type*, and the curve representing such a function is also said to be of the parabolic type. If $n = 2$, the curve, $y = x^2$, is *a parabola.*

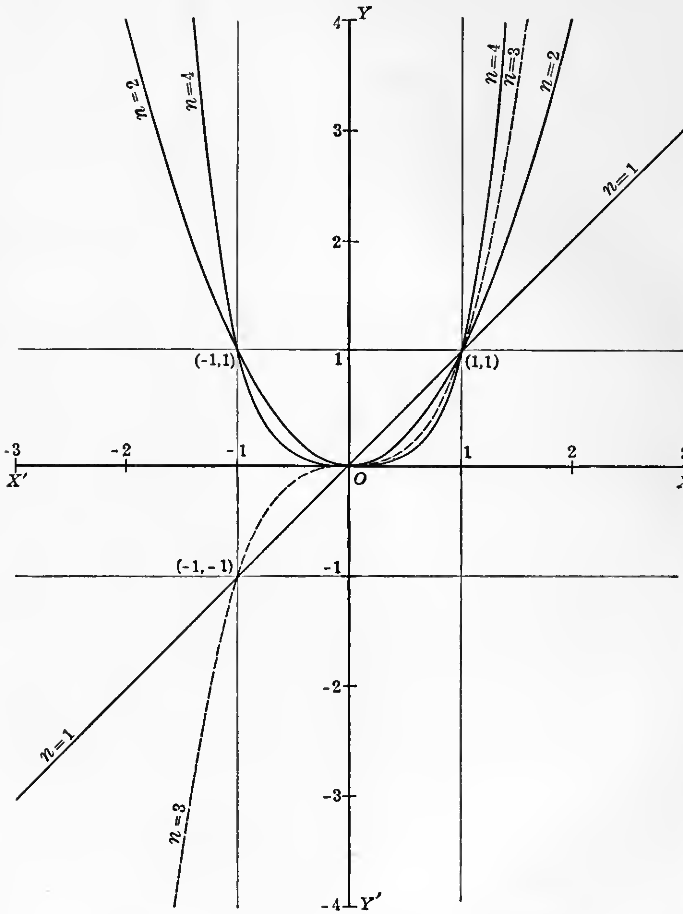

FIG. 1.—Curves for $y = x^n$, $n = 1, 2, 3$, and 4.

If n is negative the function x^n is said to be of the *hyperbolic type*, and the curve representing such a function is also said to be of the hyperbolic type. If $n = -1$, the curve, $y = x^{-1}$, is *an equilateral hyperbola.*

In Figs. 1, 2, 3, and 4, curves representing $y = x^n$ for different values of n are drawn. In Fig. 1, n has positive integral values; in Fig. 2, positive fractional values; in Fig. 3, negative integral values; and in Fig. 4, negative fractional values. The curves for

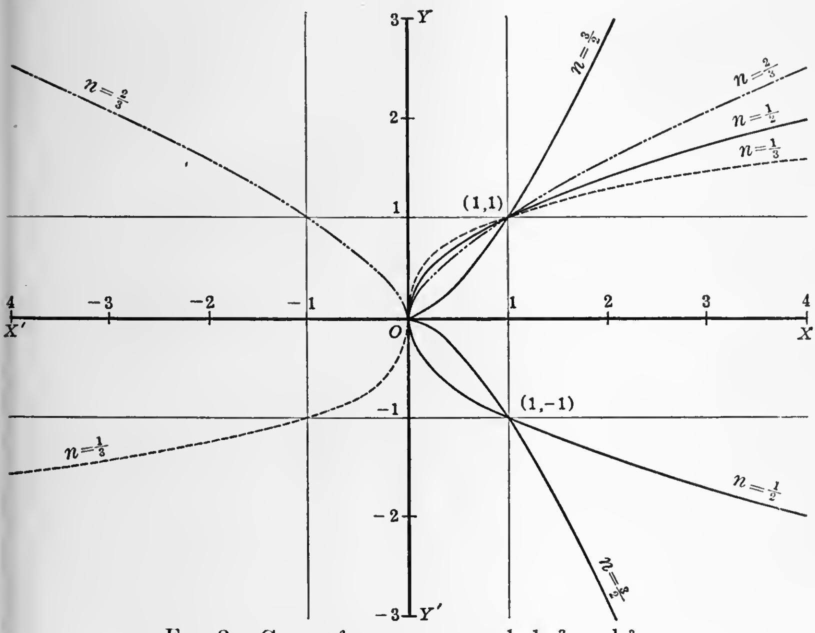

FIG. 2.—Curves for $y = x^n$, $n = \frac{1}{2}, \frac{1}{3}, \frac{2}{3}$, and $\frac{3}{2}$.

$y = x^n$ all pass through the point (1, 1). They also pass through the point (0, 0) if n is positive. If n is negative, they do not pass through (0, 0). In the latter case the coördinate axes are asymptotes to the curves.

3. The Law of the Power Function. 9. In any power function, if x changes by a fixed multiple, y also changes by a fixed multiple.

The same law can be stated as follows:

10. In any power function, if x increases by a fixed percent, y also increases by a fixed percent.

The preceding statements are also equivalent to the following:

11. In any power function, if x runs over the terms of a geometrical progression, then y also runs over the terms of a geometrical progression.

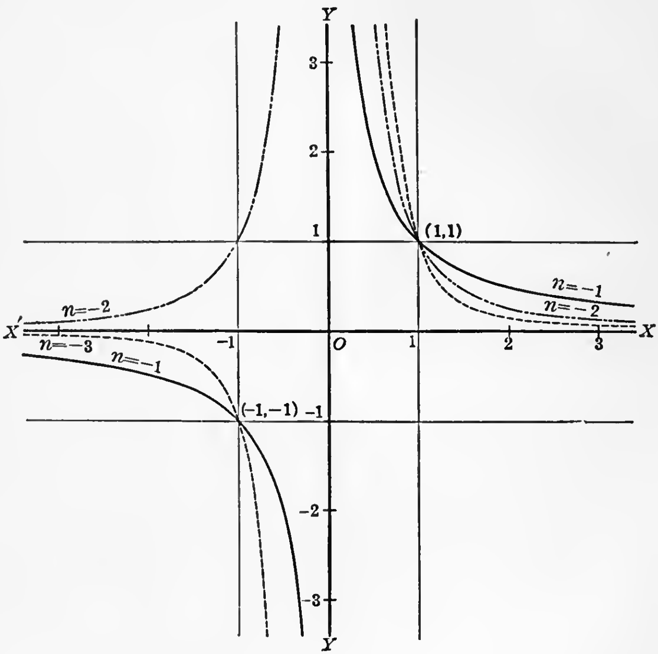

FIG. 3.—Curves for $y = x^n$, $n = -1$, -2, and -3.

4. Polynomials. Algebraic Function. 12. A polynomial in x is a sum of a finite number of terms of the form ax^n, where a is a constant and n is a positive integer or zero. For example:

$$ax^3 + bx^2 + cx + d.$$

13. A polynomial in x and y is a sum of a finite number of

terms of the form ax^my^n, where a is a constant and m and n are positive integers or zero. For example:

$$ax^2y^3 + bxy^2 + cx^2 + dy + e.$$

14. Functions of a variable x which are expressed by means of a finite number of terms involving only constant integral and

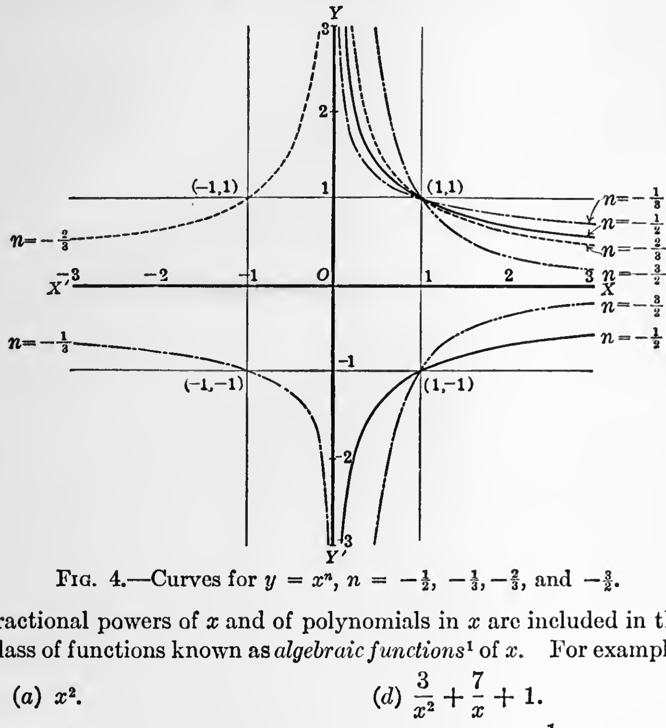

Fig. 4.—Curves for $y = x^n$, $n = -\frac{1}{2}, -\frac{1}{3}, -\frac{2}{3}$, and $-\frac{3}{2}$.

fractional powers of x and of polynomials in x are included in the class of functions known as *algebraic functions*[1] of x. For example:

(a) x^2.

(b) $x^{\frac{1}{3}} + (2x - 3)^{\frac{1}{4}}$.

(c) $\sqrt{x^2 + 4x + 7} + 4x + 5$.

(d) $\dfrac{3}{x^2} + \dfrac{7}{x} + 1$.

(e) $x + 5 + \dfrac{1}{\sqrt{x - 7}}$.

(f) $\dfrac{3x^2 + 5x + 7}{x^3 - 3x + 2}$.

[1] A function of x defined by the equation $F(x, y) = 0$, where $F(x, y)$ is a polynomial in x and y, is an *algebraic function* of x. For example, $y = \sqrt{x^2 + 2}$ is an algebraic function of x. For by squaring and transposing we obtain

$$y^2 - x^2 - 2 = 0,$$

in which the first member is a polynomial in x and y.

15. An algebraic function is said to be *rational* if it can be expressed by means of only integral powers of x together with constants.

Rational algebraic functions are divided into two classes: rational integral functions and rational fractional functions.

16. A *rational integral* function of x is a polynomial in x.

17. A rational fractional function is a quotient of two polynomials in x.

It is usually desirable to reduce rational fractional functions of x to a form in which the numerator is of lower degree than the denominator. This can always be done by performing long division.

Thus $y = \frac{x+3}{x+1}$ is equivalent to $y = 1 + \frac{2}{x+1}$, and $y = \frac{3x^2+5x+7}{x^2-3x+2}$ is equivalent to $y = 3 + \frac{14x+1}{x^2-3x+2}$.

5. Transcendental Functions. The circular (or trigonometric), the logarithmic, and the exponential functions are included in the class of functions known as *transcendental*[1] *functions.*

6. Translation. If, in the equation of a curve

$$f(x, y) = 0,$$

x is replaced by $(x - \alpha)$, the resulting equation,

$$f(x - \alpha, y) = 0,$$

represents the first curve translated parallel to the axis of x a distance α; to the right if α is positive; to the left if α is negative.

If y is replaced by $(y - \beta)$ the resulting equation,

$$f(x, y - \beta) = 0,$$

represents the original curve translated parallel to the axis of y a distance β; up if β is positive; down if β is negative. Thus $y = (x + 3)^2 - 4$ is the parabola $y = x^2$ translated three units to the left and four units down. See Fig. 5.

[1] All functions which are not algebraic functions as defined by the footnote on p. 5 are *transcendental functions.*

7. Elongation or Contraction, or Orthographic Projection, of a Locus. The substitution of $\frac{x}{a}$ for x in the equation of any locus multiplies all of the abscissas by a.

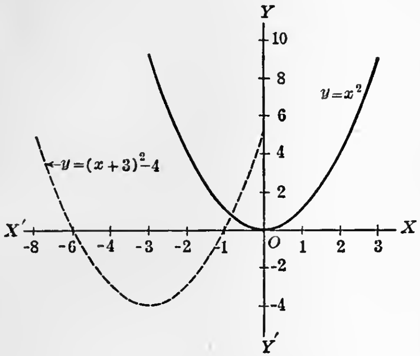

FIG. 5.

This transformation can be considered as the orthographic projection of a curve lying in one plane upon another plane, the two planes intersecting in the axis of y. If $a < 1$ the second curve is the projection of the former curve upon a second plane through

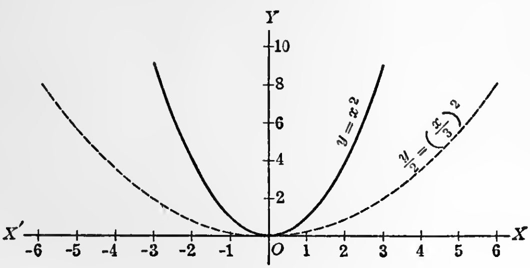

FIG. 6.

the Y-axis and making an angle α, whose cosine is equal to a, with the first plane. If $a > 1$, the first curve is the projection of the second when the cosine of the angle between their planes is $\frac{1}{a}$.

Similarly the substitution of $\frac{y}{a}$ for y in the equation of a locus multiplies the ordinates by a. The interpretation from the standpoint of orthographic projection is evident from what has just been said. See Figs. 6 and 7.

8. Shear. The curve $y = f(x) + mx$ is the curve $y = f(x)$ sheared in the line $y = mx$ in such a way that the y-intercepts remain unchanged. Every point on the curve $y = f(x)$ to the right of the Y-axis is moved up (down if m is negative) a distance proportional to its abscissa; and every point to the left of the Y-

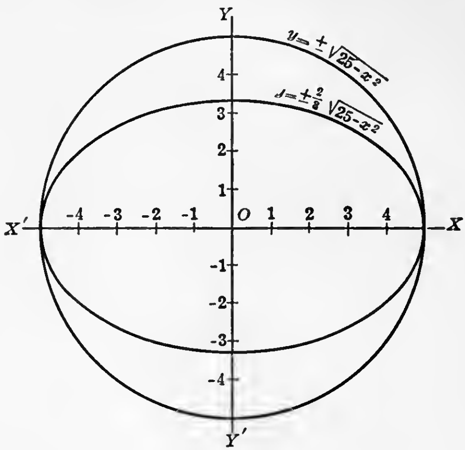

FIG. 7.

axis is moved down (up if m is negative) a distance proportional to its abscissa. The factor of proportionality is m.

In general a curve is changed in shape by shearing it in a line. The parabola is an exception to this rule.

Thus $y = ax^2$ sheared in the line $y = mx$ becomes

$$y = ax^2 + mx,$$

or

$$y = a\left(x + \frac{m}{2a}\right)^2 - \frac{m^2}{4a}.$$

This may also be considered as the result of translating the original curve by the amounts $-\frac{m}{2a}$ and $-\frac{m^2}{4a}$ in the x and y directions, respectively. Hence, by shearing, the parabola $y = ax^2$ is merely translated.

9. The Function a^x. In Fig. 8 are given the graphs of $y = a^x$, for the values $a = 1, 2$, and 3. By reflecting these curves in the line $y = x$ we have the corresponding curves for $y = \log_a x$.

The exponential function $y = a^x$ has the property that if x is given a series of values in arithmetical progression the corresponding values of y are in geometrical progression.

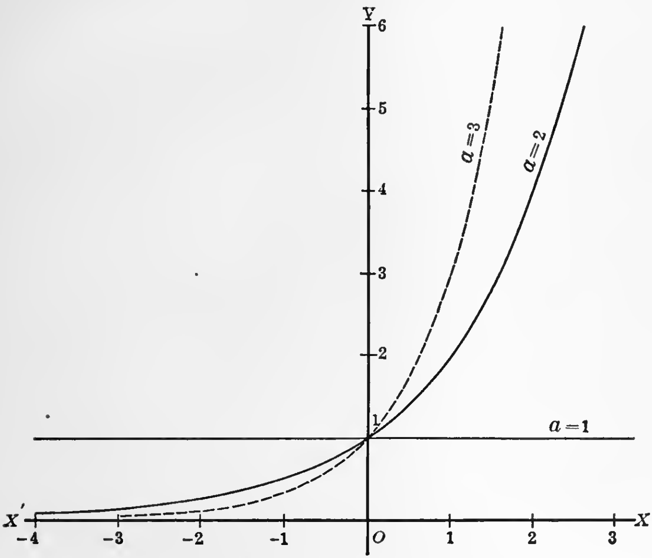

FIG. 8.—Curves for $y = a^x$, $a = 1, 2$, and 3.

10. The Function $\sin x$. The function $y = \sin x$ is represented in Fig. 53.

11. The Functions $\rho = a \cos \theta$, $\rho = b \sin \theta$, and $\rho = a \cos \theta + b \sin \theta$. The function $\rho = a \cos \theta$ is the circle OA, Fig. 9, and $\rho = b \sin \theta$ is the circle OB, Fig. 9. The function $\rho = a \cos \theta + b \sin \theta$ can be put in the form $\rho = R \cos (\theta - \alpha)$, where

$R = \sqrt{a^2 + b^2}$, and where $\cos \alpha = \frac{a}{R}$ and $\sin \alpha = \frac{b}{R}$. This function is represented by a circle, Fig. 9, passing through the pole, with diameter equal to R, and with the angle AOC equal to α. The maximum value of the function is R and the minimum value is $-R$.

12. Fundamental Transformations of Functions. It is valuable to formulate the transformations of simple functions, that most commonly occur, in terms of the effect that these transformations have upon the graphs of the functions. The following list of theorems on loci contains useful facts concerning these transformations:

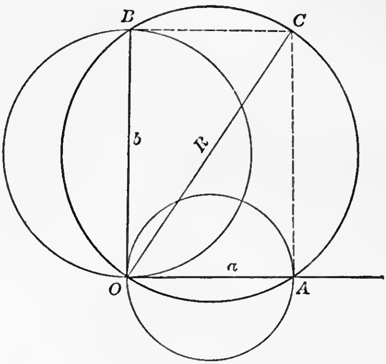

Fig. 9.

THEOREMS ON LOCI

I. If x be replaced by $(-x)$ in any equation containing x and y, the new graph is the reflection of the former graph in the Y-axis.

II. If y be replaced by $(-y)$ in any equation containing x and y, the new graph is the reflection of the former graph in the X-axis.

III. If x and y be interchanged in any equation containing x and y, the new graph is the reflection of the former graph in the line $y = x$.

IV. Substituting $\left(\frac{x}{a}\right)$ for x in the equation of any locus multiplies all abscissas by a.

V. Substituting $\left(\frac{y}{b}\right)$ for y in the equation of any locus multiplies all ordinates of the curve by b.

VI. If $(x - a)$ be substituted for x throughout any equation, the locus is translated a distance a in the x-direction.

VII. If $(y - b)$ be substituted for y in any equation, the locus is translated the distance b in the y-direction.

VIII. The addition of the term mx to the right side of $y = f(x)$ shears the locus $y = f(x)$ in the line $y = mx$.

IX. If $(\theta - \alpha)$ be substituted for θ throughout the polar equation of any locus, the curve is rotated about the pole through the angle α.

X. If the equation of any locus is given in rectangular coördinates, the curve is rotated through the positive angle α by the substitutions

$$x \cos \alpha + y \sin \alpha \quad \text{for } x$$

and

$$y \cos \alpha - x \sin \alpha \quad \text{for } y.$$

Exercises

1. Translate the curves

(*a*) $y = 2x^2$,	(*e*) $y = e^x$,	(*i*) $y = \frac{1}{x}$,
(*b*) $y = -3x^2$,	(*f*) $y = x^3$,	(*j*) $y = \frac{1}{x^2}$,
(*c*) $y = \log x$,	(*g*) $y = \sin x$,	(*k*) $y = x^{\frac{3}{2}}$,
(*d*) $y = e^{-x}$,	(*h*) $y = \cos x$,	(*l*) $y = x^{\frac{2}{3}}$,

two units to the right; three units to the left; five units up; one unit down; two units to the left and one unit down. Sketch each curve in its original and translated position on a sheet of squared paper.

2. Shear each curve given in Exercise 1 in the line $y = \frac{1}{2}x$; $y = -\frac{1}{2}x$; $y = x$; $y = -x$. Sketch each curve in its original and sheared position.

3. Write the equation of each curve given in Exercise 1 when reflected in the X-axis; in the Y-axis; in the line $y = x$; in the line $y = -x$. Sketch each curve before and after reflection.

4. Rotate the curves

(a) $\rho = a \sin \theta$, (b) $\rho = a \cos \theta$, (c) $\rho = a \cos \theta + b \sin \theta$, (d) $\rho = a\,(1 - \cos \theta)$, (e) $\rho = a(1 + \cos \theta)$, (f) $\rho = a(1 - \sin \theta)$, (g) $\rho = a(1 + \sin \theta)$, (h) $\rho = a\theta$,

about the pole through an angle $\frac{\pi}{3}, \frac{\pi}{6}, \frac{\pi}{2}, \pi, -\frac{\pi}{2}$. Sketch each curve in its original and rotated position.

5. Sketch the following pairs of curves on squared paper:

(a) $y = x^2$ and $y = x^2 + x$.
(b) $y = x^3$ and $y = (x - 3)^3 + 2$.
(c) $y = x^2$ and $y = -x^2 - 2x$.
(d) $y = x^4$ and $y = x^4 - 4x^3 + 6x^2 - 4x$.
(e) $y = -2x^2$ and $y = \frac{1}{2}x^2$.
(f) $y = x^2$ and $y = \frac{1}{4}x^2$.
(g) $y = \sin x$ and $y = \sin 2x$.
(h) $y = \sin x$ and $y = 2 \sin x$.
(i) $y = \cos x$ and $y = \sin\left(\frac{\pi}{2} - x\right)$.

6. Rotate the following curves about the origin through the angle indicated.

(a) $x^2 - y^2 = a^2$ through $45°$.
(b) $x^2 - y^2 = a^2$ through $-45°$.
(c) $x^2 - y^2 = a^2$ through $90°$.
(d) $x^2 - y^2 = a^2$ through $-90°$.
(e) $x^2 + y^2 = a$ through α.
(f) $y = mx^2$ through α.

CHAPTER I

DERIVATIVE

In Elementary Analysis the student investigated the dependence of a function upon one or more variables with the help of algebra and geometry.

He is now to study a very powerful method of investigating the behavior of functions, the method of *the infinitesimal calculus*, which was discovered by Newton and Leibnitz in the latter part of the 17th century. This method has made possible the great development of mathematical analysis and of its applications to problems in almost every field of science, particularly in engineering and physics.

13. Increments. Let us consider the following examples which illustrate the principles of the calculus:

Example 1. A steel bar, subjected to a tension, will stretch, and the amount of stretching, or the elongation, will continue to increase as the intensity of the force applied increases, until rupture occurs. The elongation is a function of the applied force. In fact, if the force is not too great, so that the elastic limit is not exceeded, experiment has shown that the elongation is proportional to the applied force (Hooke's Law). If we denote the elongation by y and the force by x, the functional relation between them will be expressed by the simple equation

$$y = kx,$$

where k is a constant. This relation is represented graphically by a straight line through the origin, Fig. 10.

Suppose that after the bar has been stretched to a certain length, the force is changed. This change in the force produces a corresponding change in the elongation, an increase if the force is increased, a decrease if the force is decreased. Evidently, from the law connecting the elongation and the force, this change in the elongation is directly proportional to the change in the force. We shall call the change in the force x, *the increment of the force, or the increment of* x, *and shall denote it by the symbol* Δx (read "increment of x" or "delta x"). The corresponding change in the elongation we call the *increment of the elongation, or the increment of* y, and denote it by Δy.

In Fig. 10 let P be any point on the line $y = kx$. If x takes on an increment Δx, y takes on an increment Δy. We see that the ratio of these increments, *i.e.*, the quotient $\frac{\Delta y}{\Delta x}$ is entirely independent of the magnitude and sense of Δx and of the position of P on the line. Indeed this ratio is the slope of the line. Here the increment of y is everywhere k times the increment of x.

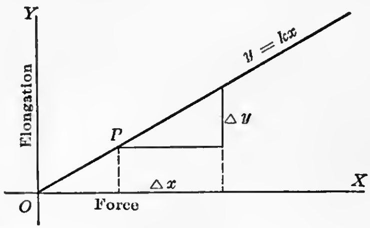

Fig. 10.

The relation between Δy and Δx can be shown without the use of the figure as follows: If x is given the increment Δx, y takes on an increment Δy so that

$$y + \Delta y = k(x + \Delta x).$$

On subtracting

$$y = kx,$$

$$\Delta y = k\Delta x.$$

Hence

$$\frac{\Delta y}{\Delta x} = k,$$

a quantity independent of x and of Δx.

Example 2. A train is moving along a straight track with a constant velocity, *i.e.*, it passes over equal distances in equal intervals of time. Denoting by s the distance measured, say in miles, from a fixed point, and by t the time measured, say in hours, which has elapsed since the train passed this point, the functional relation between s and t is expressed by

$$s = ct,$$

where c is a constant denoting the velocity of the train. This function is represented graphically by a straight line, Fig. 11. If we take an increment of time Δt following an instant t and measure the distance Δs passed over in this time, the quotient $\frac{\Delta s}{\Delta t}$ represents the velocity of the train, since we have assumed the velocity of the train to be uniform. Furthermore, the quotient $\frac{\Delta s}{\Delta t}$ will

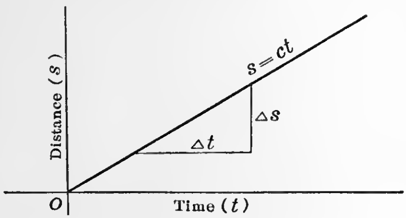

Fig. 11.

be independent of the length of Δt and of the time t to which the increment was given. Δs is everywhere c times Δt. This is evident from the graph.

In these two examples the functions were both linear functions of the independent variable. We have seen in these cases (and clearly the same is true for any linear function, $y = ax + b$) that the ratio of Δy to Δx is constant. Δy is everywhere equal to a constant times Δx, no matter how large Δx is taken and no matter at what point (x, y) on the graph the ratio is computed.

Example 3. Let us now take an example in which the functional relation is no longer a linear one. We shall find that the ratio of the increment of the function to the increment of the variable is no longer constant. Suppose that the train of Example 2 is not moving with constant velocity. *Then the quotient* $\frac{\Delta s}{\Delta t}$

is called the average velocity of the train during the interval of time Δt. Evidently this quotient will approximate more and more closely to a fixed value the smaller the interval of time Δt, is chosen. *The limiting value of the quotient* $\frac{\Delta s}{\Delta t}$ *as* Δt *approaches zero is called the velocity at the time t.*

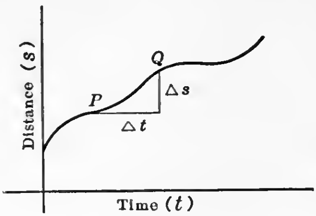

FIG. 12.

Let the curve of Fig. 12 represent graphically the relation between s and t. The ratio $\frac{\Delta s}{\Delta t}$ calculated at any point P on the curve is no longer constant as in Example 2, but varies with Δt and also with the position of the point P.

14. The Function $y = x^2$. Consider the power function $y = x^2$. Let us find the ratio of Δy to Δx at a certain point of the curve, say (0.2, 0.04), for different values of Δx. The results are given in the adjoining table.

Δx	Δy	$\frac{\Delta y}{\Delta x}$
0.4	0.32	0.8
0.2	0.12	0.6
0.1	0.05	0.5
0.05	0.0225	0.45
0.02	0.0084	0.42
0.01	0.0041	0.41
0.005	0.002025	0.405
0.002	0.000804	0.402
0.001	0.000401	0.401

We observe that as Δx is taken smaller and smaller the ratio $\frac{\Delta y}{\Delta x}$ approaches more and more closely a value in the vicinity of 0.4.

The value of $\frac{\Delta y}{\Delta x}$ will now be calculated for any point P, (x, y), on the curve $y = x^2$. From this value, which is a function of x and Δx, the limiting value as Δx approaches zero will be found. The point P, Fig. 13, has the abscissa x. If we give to x an increment Δx, we have corresponding to the abscissa, $x + \Delta x$, the point Q on the curve. Its ordinate is

$$y + \Delta y = (x + \Delta x)^2.$$

Δy is equal to the difference between the ordinates of P and Q, or

$$\Delta y = (x + \Delta x)^2 - x^2$$
$$= 2x\,\Delta x + (\Delta x)^2.$$

Then

$$\frac{\Delta y}{\Delta x} = 2x + \Delta x.$$

As Δx approaches zero the first term in the second member remains unchanged, while the second term approaches zero. It follows that the limiting value of $\frac{\Delta y}{\Delta x}$ as Δx approaches zero is $2x$. This result is expressed by the equation

$$\lim_{\Delta x \doteq 0} \frac{\Delta y}{\Delta x} = 2x,$$

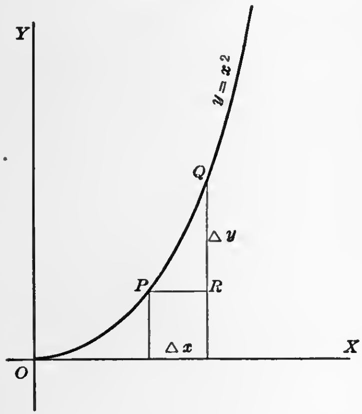

Fig. 13.

(read "limit of $\frac{\Delta y}{\Delta x}$ as Δx approaches zero"). When $x = 0.2$, $\lim_{\Delta x \doteq 0} \frac{\Delta y}{\Delta x} = 0.4$. This is the limiting value which the ratio tabulated in the last column of the table above is approaching. When $x = 3$, $\lim_{\Delta x \doteq 0} \frac{\Delta y}{\Delta x} = 6$. When $x = \frac{1}{2}$, $\lim_{\Delta x \doteq 0} \frac{\Delta y}{\Delta x} = 1$. Thus the formula just obtained enables us to calculate very easily, for any value of x, the limit of $\frac{\Delta y}{\Delta x}$ as Δx approaches zero.

15. Slope of the Tangent.—The curve of Fig. 14 is the graph of the function $y = f(x)$. On this curve take the point P with coördinates x and y, and a second point Q with coördinates $x + \Delta x$ and $y + \Delta y$. Draw the secant PQ making the angle ϕ with the X-axis and the tangent PT' making the angle τ with the X-axis. From the figure, $\frac{\Delta y}{\Delta x}$ is the slope of PQ, or

$$\frac{\Delta y}{\Delta x} = \tan \phi.$$

As Δx is taken smaller and smaller the secant PQ revolves about the point P, approaching more and more closely as its limiting position the tangent PT', and $\tan \phi$ approaches $\tan \tau$. (The

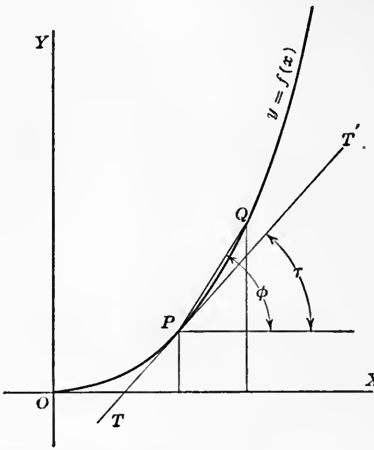

FIG. 14.

student will recall that the tangent to a curve at a point P is defined as the limiting position of the secant PQ as the point Q approaches P.) Hence

$$\lim_{\Delta x \doteq 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \doteq 0} \tan \phi = \tan \tau.$$

Hence $\lim\limits_{\Delta x \doteq 0} \frac{\Delta y}{\Delta x}$ *is equal to the slope of the tangent to the curve* $y = f(x)$ *at the point for which this limit is computed.*

In the case of the parabola, $y = x^2$, the slope of the tangent at the point (x, y) is $2x$. This shows that the curve becomes steeper and steeper for larger positive and negative values of x and that at $x = 0$ the slope is zero.

In Fig. 15, let the X-axis be divided uniformly and let the Y-axis be divided in such a way that distances measured from O on a uniform scale are equal to the squares of the numbers affixed to the points of division. Draw lines parallel to the Y-axis through equidistant points on the X-axis and lines parallel to the X-axis through points on the Y-axis whose affixed numbers on the non-uniform scale are equal to the numbers affixed to the points on the X-axis through which lines were drawn.

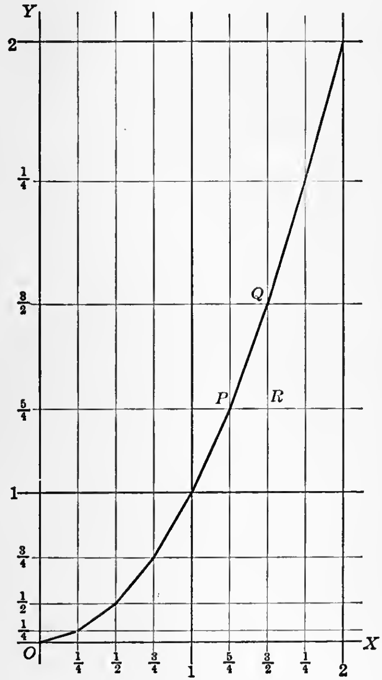

Fig. 15.

On the cross section paper thus constructed, any point at the intersection of a horizontal and a vertical line bearing the same number is a point on the curve $y = x^2$ which would be constructed in the usual way by using the uniform scale on the Y-axis as well as on the X-axis.

Join the consecutive points thus located by straight lines. These lines are the diagonals of the rectangles on the cross section paper and they are secants of the parabola $y = x^2$. Let PQ be such a diagonal and let $PR = \Delta x$. Then $RQ = \Delta y$ and $\frac{RQ}{PR} = \frac{\Delta y}{\Delta x}$, the slope of the secant PQ. The diagonals give an approximate idea of the slope of the curve. The construction shows why the slope increases so rapidly with x.

As more and more horizontal and vertical lines are inserted, the diagonals approach more and more nearly the direction of the tangent lines.

The fact that the slope of the tangent to the parabola $y = x^2$ is $2x$ furnishes an easy way of constructing the tangent at any point P (x, y). We have only to draw from P in the direction of the positive X-axis, a line PK of unit length, and from the extremity of this line, a line KT parallel to the Y-axis, whose length is twice the abscissa of P. The line joining PT is the tangent to the parabola at P. When the abscissa is negative the line KT is to be drawn downward.

16. Maxima and Minima. The algebraic sign of $\lim_{\Delta x \doteq 0} \frac{\Delta y}{\Delta x}$ enables us to tell at once where the function y is increasing and where it is decreasing as x increases. For, if the slope is positive at a point, the function is increasing with x at that point and the greater the slope the greater the rate of increase. Similarly if the slope is negative, the function is decreasing as x increases. Hence the function $y = x^2$ is a decreasing function when $x < 0$ and an increasing function when $x > 0$, since the slope is equal to $2x$. When $x = 0$ the slope is zero and the tangent is parallel to the X-axis. Since the function is decreasing to the left of $x = 0$ and increasing to the right of this line, it follows that the function decreases to the value zero when $x = 0$ and then increases. This value zero is a minimum value of the function $y = x^2$. In general we define minimum and maximum values of a function as follows:

Definition. Let $y = f(x)$, where $f(x)$ is any function of a single argument. *If y decreases to a value m as x increases and then begins to increase, m is called a minimum value of the function. If y increases to the value M as x increases and then begins to decrease, M is called a maximum value of the function.*

Thus in Fig. 16, if $ABDFHI$ is the graph of $y = f(x)$, the function increases to the value represented by the ordinate bB and then begins to decrease. bB is then a maximum value of the function. Similarly fF is another maximum value. dD and hH are minimum values of the function.

In referring to the graph of a function, points corresponding to maximum and points corresponding to minimum values of the function will be called, respectively, the maximum and minimum points of the curve. Thus B and F, Fig. 16, are maximum points and D and H are minimum points of the curve.

Thus, zero is a minimum value of $y = x^2$ or (0, 0) is a minimum point on the curve $y = x^2$.

It will be noticed that a maximum value, as here defined, is not necessarily the largest value of the function, nor is a minimum value the smallest value of the function. A maximum value may even be less than a minimum value.

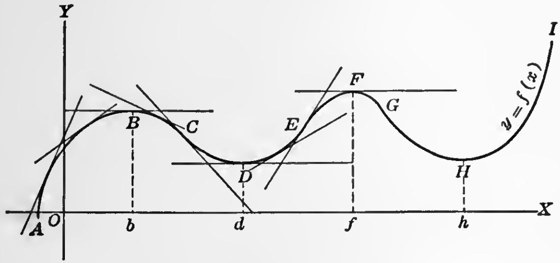

FIG. 16.

17. Derivative. We see that the limit of the ratio of the increment of the function to the increment of the independent variable as the latter increment approaches zero, is very useful in studying the behavior of the function. This limit is called *the derivative of the function with respect to the variable.* Hence the following definition:

The derivative of a function of a single independent variable with respect to that variable is the limit of the ratio of the increment of the function to the increment of the variable as the latter increment approaches zero. The derivative of a function y with respect to a variable x is denoted by the symbol $\frac{dy}{dx}\cdot$ This symbol will not be considered at present as representing the quotient of two quantities but as a symbol for a single quantity. Later it will be interpreted as a quotient. (See § 61.) It is read, "the derivative of y with respect to x." The process of finding the derivative is called differentiation.

18. Velocity of a Falling Body. As a further illustration of the application of the derivative let us attempt to find the velocity of a falling body at any instant. The law of motion has been experimentally determined to be

$$s = \tfrac{1}{2}gt^2,$$

where s is the distance through which the body falls from rest in time t. If s is measured in feet and t in seconds, the constant g is 32.2 feet per second per second. s is plotted as a function of the time in Fig. 17. At any time t, let t take on an increment Δt. s will take on an increment Δs, represented in the figure by the line RQ. Since $s = \frac{1}{2}gt^2$,

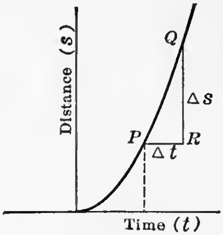

Fig. 17.

$$s + \Delta s = \tfrac{1}{2}g(t + \Delta t)^2. \tag{1}$$

Hence

$$\Delta s = \tfrac{1}{2}g(t + \Delta t)^2 - \tfrac{1}{2}gt^2,$$

or

$$\Delta s = gt\Delta t + \tfrac{1}{2}g(\Delta t)^2. \tag{2}$$

This is the distance through which the body falls in the interval Δt counted from the time t. The quotient $\frac{\Delta s}{\Delta t}$ is the average velocity for the interval Δt. The velocity at t has been defined as $\lim\limits_{\Delta t \doteq 0} \frac{\Delta s}{\Delta t}$, *i.e.*, as the derivative of s with respect to t. To find this limit divide (2) by Δt and obtain

$$\frac{\Delta s}{\Delta t} = gt + \tfrac{1}{2}g\Delta t,$$

the average velocity for the interval Δt. From which

$$\lim_{\Delta t \doteq 0} \frac{\Delta s}{\Delta t} = gt, \tag{3}$$

or

$$\frac{ds}{dt} = gt, \tag{4}$$

the velocity at t. Thus the velocity at the end of three seconds is 96.6 feet per second; at the end of four seconds, 128.8 feet per second.

19. Illustration. As an example of the use of the derivative in studying the behavior of a function, let us consider the power function

$$y = x^3.$$

$$y + \Delta y = (x + \Delta x)^3,$$

$$y + \Delta y = x^3 + 3x^2(\Delta x) + 3x(\Delta x)^2 + (\Delta x)^3,$$

$$\Delta y = 3x^2(\Delta x) + 3x(\Delta x)^2 + (\Delta x)^3,$$

$$\frac{\Delta y}{\Delta x} = 3x^2 + 3x(\Delta x) + (\Delta x)^2.$$

Then

$$\lim_{\Delta x \doteq 0} \frac{\Delta y}{\Delta x} = 3x^2.$$

$$\frac{dy}{dx} = 3x^2.$$

For $x = 0$ the derivative is equal to zero and consequently the tangent at (0, 0) is horizontal and coincides with the X-axis. For all other values of x the derivative is positive. This shows that the function is an increasing function for all these values of x. Where is the slope of the curve equal to 1? Equal to $\sqrt{3}$?

20. Illustration. The solution of the following problem will further illustrate the use of the derivative.

Find the dimensions of the gutter with the greatest possible carrying capacity and with rectangular cross section, which can be made from strips of tin 30 inches wide by bending up the edges to form the sides. See Fig. 18.

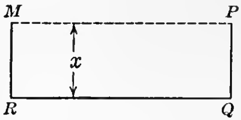

Fig. 18.

If the depth MR is determined, the width is also determined, since the sum of the three sides MR, PQ, and RQ is 30 inches. We seek to express the area of the cross section as a function of the depth. Denote the depth by x and the area by A. The width RQ is $30 - 2x$. Hence

$$A = (30 - 2x)x.$$

In Fig. 19, A is plotted as a function of x. A first increases with x

and then decreases. The value of x for which A reaches its greatest value can be determined from a graph with a high degree of approximation. The derivative can be used to calculate accurately this value of x and this saves construction of an accurate graph.

From O to H, A is an increasing function. Its derivative is therefore positive for this part of the curve. From H to N the function A is decreasing. Its derivative is therefore negative for this part of the curve. At the point H the derivative changes sign, passing from positive values through zero to negative values. The abscissa of the point H can then be found by finding the

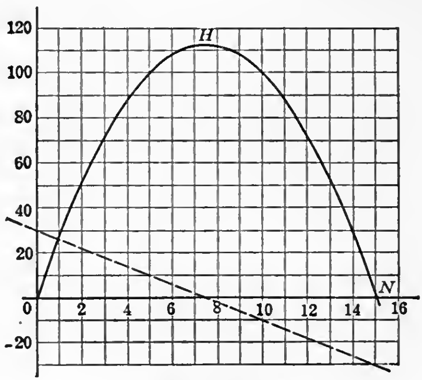

Fig. 19.

derivative of A with respect to x and determining where it changes sign. In this case the change of sign occurs where the derivative is equal to zero. We find by the method of increments

$$\frac{dA}{dx} = 30 - 4x = 4(7.5 - x).$$

$\frac{dA}{dx} = 0$ when $x = 7.5$. If $x < 7.5$, $\frac{dA}{dx}$ is positive and A is an increasing function. If $x > 7.5$, $\frac{dA}{dx}$ is negative and A is a decreasing function. This shows that A increases up to a certain value at $x = 7.5$ and then begins to decrease. Hence the gutter will have the greatest cross section if its depth be made 7.5 inches.

It is interesting to plot the derivative as a function of x on the same axes. See the dotted line, Fig. 19. The statements made concerning the derivative are verified in the graph.

Exercises

1. Consider the function $y = f(x)$ whose graph is given in Fig. 20. In what portions of the curve is the derivative positive? In what portions negative? Where is the derivative equal to zero?

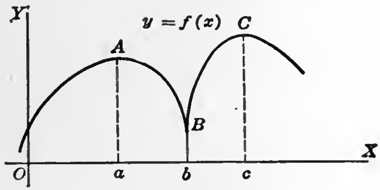

FIG. 20.

2. Find $\dfrac{dy}{dx}$, if $y = 3x^2$. For what values of x is the function increasing? For what values decreasing? At what point does the tangent line drawn to the curve representing the function, make an angle of 45° with the positive direction of the axis of x? Find the coördinates of the maximum or minimum points on the curve.

3. Answer questions asked in Exercise 2, if $y = x^3$.

4. Answer questions asked in Exercise 2, if $y = x^4$.

5. Answer questions asked in Exercise 2, if $y = x^5$.

6. Answer questions asked in Exercise 2, if $y = x^2 - 2x + 3$.

7. Answer questions asked in Exercise 2, if $y = \dfrac{x^3}{3} - \dfrac{3x^2}{2} + 2x - 6$.

8. Find the derivative of $\sqrt{x}$.

SOLUTION. Let $y = \sqrt{x}$.

Then

$$y + \Delta y = \sqrt{x + \Delta x}$$

and

$$\Delta y = \sqrt{x + \Delta x} - \sqrt{x}$$

$$\frac{\Delta y}{\Delta x} = \frac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta x}.$$

Rationalize the numerator:

$$\frac{\Delta y}{\Delta x} = \frac{1}{\sqrt{x + \Delta x} + \sqrt{x}}.$$

As Δx approaches zero the right-hand side of this equation approaches $\dfrac{1}{\sqrt{x} + \sqrt{x}}$. Then

$$\lim_{\Delta x \doteq 0} \frac{\Delta y}{\Delta x} = \frac{1}{2\sqrt{x}},$$

or

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}.$$

9. Find the derivative of $\sqrt{x - 2}$.

10. Find the derivative of $\sqrt{x^2 - 4}$.

CHAPTER II

LIMITS

In §17 the derivative was defined as the limit of a certain ratio. The word limit was used without giving its precise definition, as the reader was supposed to have a fair conception of the meaning of this term from previous courses in mathematics. However, since the entire subject of the calculus is based on limit processes it is well to review the precise definition and to state certain theorems from the theory of limits.

21. Definition. *If a variable changes by an unlimited number of steps in such a way that, after a sufficiently large number of steps, the numerical value of the difference between the variable and a constant becomes and remains, for all subsequent steps, less than any preassigned positive constant, however small, the variable is said to approach the constant as a limit, and the constant is called the limit of the variable.*

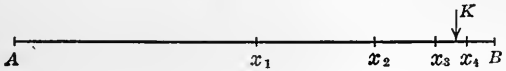

FIG. 21.

Illustration 1. Let AB, Fig. 21, be a line two units in length, and let x be the distance from A to a point on this line. Suppose that x increases from 0 by steps such that any value of x is greater than the preceding value by one-half of the difference between 2 and this preceding value, *i.e.*, by $\frac{2 - x}{2}$. $x_1, x_2, x_3, x_4, \cdots$ are the end points of the portions of the line representing the successive values of x. Then the lengths $x_1B = 1$, $x_2B = \frac{1}{2}$, $x_3B = (\frac{1}{2})^2$, $x_4B = (\frac{1}{2})^3, \cdots, x_nB = (\frac{1}{2})^{n-1}$ are the successive differences between the constant length 2 and the variable length x. This difference becomes and remains less than any preassigned length KB after a sufficient number of steps has been taken.

This is true however small the length KB is chosen. Therefore, by the definition of the limit of a variable, 2 is the limit of the variable x.

Illustration 2. Consider the variable $x^2 - 2$. Give to x the values $0, \frac{3}{2}, \frac{9}{4}, \frac{21}{8}, \frac{45}{16}, \frac{93}{32}, \cdots, \frac{3(2^n - 1)}{2^n} = 3\left[1 - \frac{1}{2^n}\right]$, which are chosen by starting with $x = 0$ and giving to it successive increments which are one-half the difference between 3 and the

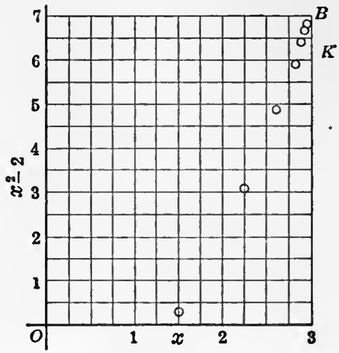

FIG. 22.

preceding value of x. The corresponding values of $x^2 - 2$ are given in the adjoining table. The corresponding points, excepting $(0, -2)$, are plotted in Fig. 22.

x	$x^2 - 2$
0.0	−2.00
1.5	0.25
2.25	3.06
2.62	4.89
2.81	5.91
2.91	6.45
2.95	6.72
2.98	6.86
2.99	6.93

From the table and the expression $x^2 - 2$ it is readily seen that the difference between 7 and the variable $x^2 - 2$ becomes and remains less than any previously assigned quantity (such as KB, Fig. 22) after a sufficiently large number of steps. Therefore 7 is the limit of the variable $x^2 - 2$ as x approaches 3.

Illustration 3. By giving x values nearer and nearer 2, the value of $\frac{1}{x - 2}$ becomes numerically larger and larger. Indeed its numerical value can be made greater than any preassigned positive number however large by choosing x sufficiently near 2. The variable $\frac{1}{x-2}$ does not approach a limit as x approaches 2. Instead of doing so it increases without limit.

If a variable changes by an unlimited number of steps in such a way that after a sufficiently large number of steps its numerical value becomes and remains, for all subsequent steps, greater than any preassigned positive number however large, the variable is said to become infinite. Illustration 3 of this section is an example of a variable which becomes infinite, or approaches infinity.

22. Notation. If in any limit process, the variable, say y, is a function of another variable, say x, the successive steps by which y changes are determined by those by which x changes. *If y approaches a limit A, as x approaches a limit a, we say that the limit of y as x approaches a is A, and write*

$$\lim_{x \doteq a} y = A.$$

After what has just been said, the meaning of the two following expressions will be clear:

$$\lim_{x \doteq \infty} y = A, \qquad \lim_{x \doteq a} y = \infty.$$

In the second case a limit does not really exist. The form of expression is only a convenient way of saying that if x is taken sufficiently near a, the value of y can be made to become and remain greater in numerical value than any preassigned positive number however large.

From the illustrations of the preceding section we have:

1. $\lim_{n \doteq \infty} x = 2$, where n is the number of steps taken.
2. $\lim_{x \doteq 3} (x^2 - 2) = 7.$
3. $\lim_{x \doteq 2} \frac{1}{x-2} = \infty.$

23. Infinitesimal. In the particular case where the limit of a variable is zero, the variable is said to be an *infinitesimal. An infinitesimal is a variable whose limit is zero.* Thus Δy and Δx which were used in §§**13**, **14**, and **15** are thought of as approaching zero and are infinitesimals. Hence the derivative, **§17,** is defined as the limit of the quotient of two infinitesimals. Infinitesimals are of fundamental importance in the Calculus. Indeed the subject is often called the Infinitesimal Calculus.

24. Theorems on Limits. The following theorems concerning limits are stated without proof:

Theorem I. *If two variables are always equal and if one approaches a limit, the other approaches the same limit.*

Theorem II. *The limit of the sum of two variables, each of which approaches a limit, is equal to the sum of their limits.*

Theorem III. *The limit of the difference of two variables, each of which approaches a limit, is equal to the difference of their limits.*

Theorem IV. *The limit of the product of two variables, each of which approaches a limit, is equal to the product of their limits.*

Theorem V. *The limit of the quotient of two variables, each of which approaches a limit, is equal to the quotient of their limits, provided the limit of the divisor is not zero.*

If the limit of the divisor is zero, the quotient of the limits in Theorem V has no meaning, since division by zero is an impossible operation. For, the quotient Q of two numbers A and B is defined as the number such that when it is multiplied by the divisor B, the product is the dividend A. Now if B is zero while A is not zero, there clearly is no such number.

25. The Indeterminate Form $\frac{0}{0}$. If, in the quotient considered above, A is also zero, any number will satisfy the requirement, so that Q is not determined. One encounters exactly this difficulty in seeking the value of $\frac{x^2 - 4}{x - 2}$ at $x = 2$. Its value is not determined at this point but it is determined for all finite values of x different from 2. We define its value at $x = 2$ as the limit of its value as x approaches 2. The student should construct a graph of this function. Usually we proceed as follows to find the desired limit.

$$\lim_{x \doteq 2} \frac{x^2 - 4}{x - 2} = \lim_{x \doteq 2} (x + 2) = 4.$$

The expression $\frac{x^2 - 4}{x - 2}$ is said to be *indeterminate* at $x = 2$, since any one of an infinite number of values can be assigned to it. The determination of its limiting value as x approaches the value 2 is called the *evaluation of the indeterminate form.* Indeterminate forms of this and other types are frequently found in the Calculus. Thus $\frac{\Delta y}{\Delta x}$ is an indeterminate form for $\Delta x = 0$. We have already seen in several cases how it can be evaluated. Exactly as in the example just given, we have sought the limit of the quotient as Δx approaches zero and not the quotient when $\Delta x = 0$, because the latter quotient has no meaning.

Exercises

1. Determine the following limits, if they exist.

(*a*) $\lim\limits_{x \doteq \frac{\pi}{2}} \cos x$. (*b*) $\lim\limits_{x \doteq 0} \cot x$.

(*c*) $\lim\limits_{x \doteq 0} \sin \frac{\pi}{x}$. Draw the curve for values of x between $-\pi$ and $+\pi$.

(*d*) $\lim\limits_{x \doteq 0} x \sin \frac{\pi}{x}$.

2. Evaluate the following indeterminate forms:

(*a*) $\left.\frac{x^2 - 9}{x - 3}\right|_{x \doteq 3}$ (*b*) $\left.\frac{x^4 + 6x^2}{3x^3 + x^2}\right|_{x \doteq 0}$

3. Find $\lim\limits_{x \doteq \infty} \frac{3x^2}{x}$; $\lim\limits_{x \doteq \infty} \frac{4x}{5x^2}$, $\lim\limits_{x \doteq \infty} \frac{4x^2}{5x^2}$, $\lim\limits_{x \doteq \infty} \frac{4x^2 + 3}{5x^2}$. Discuss the symbol $\frac{\infty}{\infty}$. Show that it is an indeterminate form.

26. Continuous and Discontinuous Functions. Draw the graphs of the following functions:

1. $y = \frac{1}{x}$. 2. $y = x^2$. 3. $y = 7x + \frac{1}{x}$.

4. $y = \tan x$. 5. $y = \sin x$. 6. $y = 3^{\frac{1}{x}}$.

Hint. In 6, values in the vicinity of $x = 0$ should be carefully determined. Take a set of values of x approaching 0 from the left and another set approaching it from the right.

7. $y = 3^x$. 8. $y = \frac{3^{\frac{1}{x}} + 2}{3^{\frac{1}{x}} + 1}$.

Study the vicinity of $x = 0$. See 6.

The functions 2, 5, and 7 are said to be *continuous* while 1, 3, 4, 6, and 8 are *discontinuous*. The meaning of these terms is obvious from the graphs that have been drawn. A precise definition follows: *A function $f(x)$ is said to be continuous at $x = a$ if* $\lim\limits_{x \doteq a} f(x) = f(a)$.

In 1, 3, 4, 6, and 8, this condition is not satisfied at $x = 0$, 0, $\frac{\pi}{2}$, 0, and 0, respectively. In these examples the functions either become infinite for the values of x in question or approach different limits as the value of x is approached from larger or smaller values. *A function $f(x)$ is said to be continuous in an interval (c, d), i.e., the interval $c \overline{\gtrless} x \overline{\gtrless} d$, if it is continuous at every point in this interval.* Thus the functions 2, 5, and 7 are continuous in any finite interval. The remaining functions are continuous in any interval not containing the points to which attention has been called.

CHAPTER III

THE POWER FUNCTION

27. In Chapter I the derivative of a function was found by what may be called the fundamental method, viz., by giving to the independent variable an increment, calculating the corresponding increment of the dependent variable, and finding the limit of the ratio of these increments as the increment of the independent variable approaches zero. This method is laborious and since it will be necessary to find derivatives in a large number of problems, rules will be established by means of which the derivatives of certain functions can be written down at once. *The process of finding the derivative of a function is called differentiation.*

In this chapter we shall find the derivative of the power function, and study the function by means of this derivative.

The graphs of $y = x^n$, for various values of n, appear in Figs. 1, 2, 3, and 4. If n is positive, the curves go through the points (0, 0) and (1, 1), and are said to be of the parabolic type. In this case x^n is an increasing function of x in the first quadrant. If n is negative, the curves go through the point (1, 1) but do not go through the point (0, 0). They are asymptotic to both axes of coördinates. These curves are said to be of the hyperbolic type. In this case x^n is a decreasing function of x in the first quadrant.

The law of the power function, as stated in **§3**, should be reviewed at this point.

28. Derivative of $\mathbf{x^n}$. Let $y = x^n$, where n is at first assumed to be a positive integer.

$$y + \Delta y = (x + \Delta x)^n.$$

$$y + \Delta y = x^n + nx^{n-1}\Delta x + \frac{n(n-1)}{\underline{|2}} x^{n-2} (\Delta x)^2 + \cdots + (\Delta x)^n.$$

$$\Delta y = nx^{n-1}\Delta x + \frac{n(n-1)}{\underline{|2}} x^{n-2} (\Delta x)^2 + \cdots + (\Delta x)^n.$$

$$\frac{\Delta y}{\Delta x} = nx^{n-1} + \frac{n(n-1)}{\underline{|2}} x^{n-2}\Delta x + \cdots + (\Delta x)^{n-1}.$$

$$\lim_{\Delta x \doteq 0} \frac{\Delta y}{\Delta x} = nx^{n-1},$$

or

$$\frac{dy}{dx} = nx^{n-1}. \tag{1}$$

This proof holds when n is a positive integer. In §§**33** and **42** it will be shown that the formula obtained holds for fractional and negative exponents. For the present we shall assume the formula true for these exponents.

Illustrations.

1. $\dfrac{d(x^3)}{dx} = 3x^2.$

2. $\dfrac{d\left(\dfrac{1}{x^2}\right)}{dx} = \dfrac{d(x^{-2})}{dx} = -2x^{-3} = -\dfrac{2}{x^3}.$

3. $\dfrac{d(t^4)}{dt} = 4t^3.$

4. $\dfrac{d\left(v^{\frac{1}{2}}\right)}{dv} = \frac{1}{2}v^{-\frac{1}{2}} = \dfrac{1}{2\sqrt{v}}.$

Exercises

Find $\dfrac{dy}{dx}$ in each of the following fifteen exercises:

1. $y = x^2.$
2. $y = x^4.$
3. $y = x^5.$
4. $y = x^{10}.$
5. $y = \dfrac{1}{x}.$
6. $y = x.$
7. $y = x^{\frac{1}{2}}.$
8. $y = x^{\frac{2}{3}}.$
9. $y = x^{\frac{3}{2}}.$
10. $y = \dfrac{1}{\sqrt{x^3}}.$
11. $y = x^{\frac{3}{4}}.$
12. $y = x^{\frac{5}{2}}.$
13. $y = x^{-2}.$
14. $y = x^{-\frac{1}{2}}.$
15. $y = \dfrac{1}{x^4}.$

16. Find the slope of each of the curves of Exercises 1–15 at the point (1, 1); also at the point whose abscissa is $\frac{1}{2}$ and whose ordinate is positive.

17. By making use of the derivative, find for what values of x each function given in Exercises 1–15 is increasing; is decreasing.

18. How does the slope of $y = x^n$ change with increasing x, if x is positive and if n is positive and less than 1? If n is positive and greater than 1?

19. Find where the slope of each curve given in Exercises 1–15 is equal to zero; equal to 1.

20. Find $\frac{ds}{dt}$ if:

(*a*) $s = t^2$. (*c*) $s = \sqrt{t}$. (*e*) $s = \sqrt[3]{t^2}$.

(*b*) $s = \frac{1}{t^3}$. (*d*) $s = \frac{1}{t^{\frac{2}{3}}}$. (*f*) $s = \frac{1}{\sqrt{t}}$.

21. Find $\frac{dy}{dt}$ if:

(*a*) $y = t^3$. (*c*) $y = \sqrt[3]{t}$. (*e*) $y = t^2$.

(*b*) $y = \frac{1}{t^4}$ (*d*) $y = \frac{1}{t^{\frac{1}{3}}}$. (*f*) $y = \frac{1}{t^2}$.

29. The Derivative of ax^n. In case the power function is written in the more general form ax^n, it is easy to see that the constant multiplier a will appear as a coefficient in all terms on the right-hand side of the equations in the proof in §28, and the derivative of $y = ax^n$ is, therefore,

$$\frac{dy}{dx} = nax^{n-1}, \tag{1}$$

or

$$\frac{d(ax^n)}{dx} = nax^{n-1}. \tag{2}$$

The proof of the formula is for positive integral values of n only, but as in §28 will be assumed for all commensurable exponents.[1]

Since ax^{n-1} is the given power function $y = ax^n$ divided by x, formula (1) may be written

$$\frac{dy}{dx} = n\frac{y}{x} \tag{3}$$

[1] The relation of formula (2) to that of §28 is at once evident when it is recalled that the curve $y = ax^n$ can be thought of as obtained from the curve $y = x^n$ by stretching all ordinates in the ratio $1:a$. Then the slope of the tangent at a point of $y = ax^n$ is a times the slope of the tangent to $y = x^n$ at the corresponding point; *i.e.*,

$$\frac{d(ax^n)}{dx} = a\frac{d(x^n)}{dx} = anx^{n-1}.$$

The geometrical meaning of formula (3) is shown by Fig. 23. The fraction $\frac{y}{x}$ is the slope of the radius vector OP from the origin to the point P on the curve. Formula (3) states that the slope, at any point of the graph of the function, $y = ax^n$, is n times the slope of the radius vector OP. Thus, if $n = 1$, $y = ax^n$ reduces to a straight line through the origin, and the line has the same slope as OP. If $n = 2$ the curve is the parabola $y = ax^2$, and the slope of the curve is always twice that of OP. If $n = -1$ the curve is the rectangular hyperbola, $y = \frac{a}{x}$, and the slope of the curve is the negative of the slope of OP.

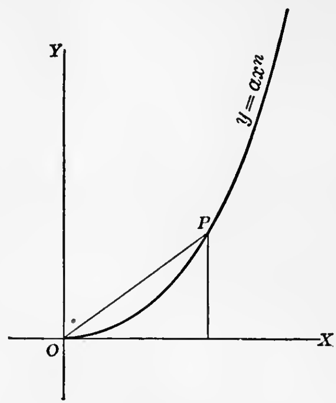

FIG. 23.

Illustrations.

1. $\frac{d(7x^2)}{dx} = 14x.$

2. $\frac{d\left(\frac{5}{x^2}\right)}{dx} = \frac{d(5x^{-2})}{dx} = 5\frac{d(x^{-2})}{dx} = -10x^{-3} = -\frac{10}{x^3}.$

3. $\frac{d(6\sqrt{t^3})}{dt} = 6\frac{d(t^{\frac{3}{2}})}{dt} = 9t^{\frac{1}{2}}.$

4. If $y = 5x^3$, $\frac{dy}{dx} = 3\frac{y}{x}.$

5. If $y = \frac{6}{x^2}$, $\frac{dy}{dx} = -2\frac{y}{x}$.

Exercises

Find $\frac{dy}{dx}$ in each of the following fifteen exercises.

1. $y = 4x^3$.
2. $y = 3\sqrt{x}$.
3. $y = 5x^4$.
4. $y = 3x$.
5. $y = \frac{2}{x^3}$.
6. $y = 2x^{\frac{2}{3}}$.
7. $y = 4x^{\frac{3}{2}}$.
8. $y = \frac{2}{3}x^{10}$.
9. $y = 10\sqrt[3]{x}$.
10. $y = -\frac{4}{x^2}$.
11. $y = -3x^2$.
12. $y = x$.
13. $y = -x$.
14. $y = -3\sqrt{x}$.
15. $y = -\frac{4}{x^3}$.

16. Find $\frac{ds}{dt}$ in each of the following:

(*a*) $s = 2t^2$. (*b*) $s = 3\sqrt{t}$. (*c*) $s = -4t^3$.

17. Find $\frac{dy}{dt}$ in each of the following:

(*a*) $y = 4\sqrt[3]{t^2}$. (*b*) $y = -4t$. (*c*) $y = -3t^2$.

18. Find the slope of each of the curves given in Exercises 1–15, at the point whose abscissa is 1; at the point whose abscissa is $\frac{1}{2}$.

19. For what values of x is each function given in Exercises 1–15 increasing? Decreasing? Where, if at all, is the slope of each of these curves zero?

20. Draw the curves $y = \frac{1}{3}x^2$, $y = \frac{2}{x}$, $y = x^3$, $y = \sqrt{x}$; and draw tangent lines to them at the points for which the abscissas are 1, 2, 3, and 4. Make a table showing the slope of the radius vector and the tangent line for each of these points.

30. Rate of Change of ax^n. Let $y = ax^n$, where x is a function of the time t. Since x is a function of t, y is a function of t. For example, $y = 3x^2$, where $x = t - 1$.

Let Δx and Δy be the increments of x and y, respectively, corresponding to the increment Δt of t. $\frac{\Delta y}{\Delta t}$ is the average rate of change of y during the interval Δt. $\frac{dy}{dt}$ is the rate of change of y at the instant t.

At any time, t

$$y = ax^n.$$

At the time $t + \Delta t$,

$$y + \Delta y = a(x + \Delta x)^n.$$

$$y + \Delta y = a\left[x^n + nx^{n-1}\Delta x + \frac{n(n-1)}{\lfloor 2}x^{n-2}(\Delta x)^2 + \cdots + (\Delta x)^n\right].$$

$$\Delta y = a\left[nx^{n-1} + \frac{n(n-1)}{\lfloor 2}x^{n-2}\Delta x + \cdots + (\Delta x)^{n-1}\right]\Delta x.$$

$$\frac{\Delta y}{\Delta t} = a\left[nx^{n-1} + \frac{n(n-1)}{\lfloor 2}x^{n-2}\Delta x + \cdots + (\Delta x)^{n-1}\right]\frac{\Delta x}{\Delta t}.$$

As Δt approaches zero, the expression within the brackets approaches nx^{n-1}, and $\frac{\Delta x}{\Delta t}$ approaches $\frac{dx}{dt}$.

Hence

$$\frac{dy}{dt} = anx^{n-1}\frac{dx}{dt},$$

or

$$\mathbf{\frac{d(ax^n)}{dt} = anx^{n-1}\frac{dx}{dt}}.$$

The rate of change of the function ax^n is expressed in terms of x and of $\frac{dx}{dt}$, the rate of change of x. If then the rate of change of x for a given value of x is known, the rate of change of the function for that value of x can be calculated.

Illustration 1. The side of a square is increasing at the uniform rate of 0.2 inch per second. Find the rate at which the area is increasing when the side is 10 inches long.

Let x be the length of the side, and y the area of the square. Then $\frac{dx}{dt} = 0.2$ and $\frac{dy}{dt}$ is the rate of increase of the area. To find this rate of increase, differentiate the function $y = x^2$.

$$\frac{dy}{dt} = 2x\frac{dx}{dt}.$$

Since

$$\frac{dx}{dt} = 0.2,$$

$$\frac{dy}{dt} = 0.4x.$$

When $x = 10$, $\frac{dy}{dt} = 4$. The area is increasing at the rate of 4 square inches per second. When $x = 13$, $\frac{dy}{dt} = 5.2$, the rate of change of the area at this instant.

Illustration 2. A spherical soap bubble is being inflated at the rate of 0.2 cubic inch per second. Find the rate at which the radius is increasing when it is 1.5 inches long.

Let r be the radius, and V the volume of the bubble. $\frac{dV}{dt} = 0.2$ and $\frac{dr}{dt}$, the rate of increase of the radius, is to be found.

$$V = \tfrac{4}{3}\pi r^3.$$

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

From which

$$\frac{dr}{dt} = \frac{1}{4\pi r^2}\frac{dV}{dt}.$$

Since

$$\frac{dV}{dt} = 0.2, \text{ and } r = 1.5,$$

$$\frac{dr}{dt} = \frac{1}{20\pi(1.5)^2} = 0.0071 \text{ inch per second.}$$

Exercises

1. Find the rate at which the surface of the soap bubble of Illustration 2 is increasing when $r = 1.5$ inches.

2. If each side of an equilateral triangle is increasing at the rate of 0.3 inch per minute, at what rate is the area of the triangle increasing when the side is 6 inches long?

3. Water is flowing at a uniform rate of 10 cubic inches per minute into a right circular cone whose semi-vertical angle is 45°, whose apex is down, and whose axis is vertical. At what rate is the surface of the water in the cone rising, and at what rate is the area of this surface increasing when the water in the cone is 25 inches deep?

31. The Derivative of the Sum of a Function and a Constant. Sketch, on the same set of axes, the graphs of the functions:

$y = x^2$; $y = x^2 - 5$; $y = x^2 + 3$; $y = x^2 + 10$. Find $\frac{dy}{dx}$ for each of the functions. Find $\frac{dy}{dx}$, if $y = x^2 + C$, where C is any constant.

Sketch a graph of any function $y = f(x)$, and on the same set of axes, graphs of $y = f(x) + C$ for several values of the constant C. What relation exists between $\frac{dy}{dx}$ for the different functions corresponding to the same value of x?

From these illustrations it is clear that *the derivatives of all functions which differ only by an additive constant are the same.* The reason for this is geometrically evident. For, the addition of a constant to a function has the effect of merely translating the graph of the function parallel to the Y-axis. The slope corresponding to any given abscissa is clearly not changed by this translation. Hence,

$$\frac{d[f(x) + C]}{dx} = \frac{d[f(x)]}{dx} \tag{1}$$

In particular

$$\frac{d[ax^n + C]}{dx} = \frac{d[ax^n]}{dx} = anx^{n-1} \tag{2}$$

Thus, if

$$y = 5x^3 + 7,$$

$$\frac{dy}{dx} = \frac{d(5x^3)}{dx} = 15x^2.$$

Exercises

1. Prove formula (2) above by the increment process.

Differentiate:

2. $y = 3x^2 + 2$.

3. $y = 5\sqrt{x} + 4$.

4. $y = 2x^3 - 3$.

5. $y = \frac{3}{x^2} + 5$.

6. $y = 2t^4 + 7$.

7. $s = 16t^2 + 5$.

8. $s = 2\sqrt{t^3} + 6$.

9. $s = \frac{2}{x^3} - 4$.

10. $x = 4t^3 - 2$.

11. $y = -4x^5 + 6$.

12. $y = -3x^4 + 2$.

13. $y = 7x^2 - 3$.

14. $y = 4x^3 + 5$.

15. $y = \frac{1}{x} + 2$.

16. $y = -\frac{1}{6}x^5 + 3$.

17. $y = \frac{1}{5}x^5 + 2$.

18. $y = \frac{3}{7}x^4 - 5$.

32. The Derivative of au^n. If $y = au^n$, where u is a function of x and n is a positive integer, the student will prove, as in §30, that

$$\frac{dy}{dx} = anu^{n-1}\frac{du}{dx},$$

or

$$\mathbf{\frac{d(au^n)}{dx} = anu^{n-1}\frac{du}{dx}.}$$

Illustrations.

1. $$\frac{d[5(x^2+3)^3]}{dx} = 5{\cdot}3(x^2+3)^2\frac{d(x^2+3)}{dx} = 15(x^2+3)^2\,2x = 30x(x^2+3)^2.$$

2. $$\frac{d[2(x^2+4)^4+10]}{dx} = \frac{d[2(x^2+4)^4]}{dx} = 2{\cdot}4(x^2+4)^3\frac{d(x^2+4)}{dx} = 2{\cdot}4(x^2+4)^3\,2x = 16x(x^2+4)^3.$$

3. If $y = (2x^2+1)^2$, find $\frac{dy}{dt}$.

$$\frac{dy}{dt} = \frac{d(2x^2+1)^2}{dt} = 2(2x^2+1)\frac{d(2x^2+1)}{dt}$$

$$= 2(2x^2+1){\cdot}2{\cdot}2x\frac{dx}{dt} = 8x(2x^2+1)\frac{dx}{dt}.$$

4. If $y = (x^2+1)^{\frac{3}{2}}$,

$$\frac{dy}{dx} = \tfrac{3}{2}(x^2+1)^{\frac{1}{2}}\,2x = 3x(x^2+1)^{\frac{1}{2}},$$

and

$$\frac{dy}{dt} = \tfrac{3}{2}(x^2+1)^{\frac{1}{2}}\,2x\frac{dx}{dt} = 3x(x^2+1)^{\frac{1}{2}}\frac{dx}{dt}.$$

Exercises

Find $\frac{dy}{dx}$:

1. $y = (4x^2 - 2)^3$.

2. $y = 5(2x^2 - 5)^3$.

3. $y = 2(3 - 4x^2)^3$

4. $y = \sqrt{x^2+1}$.

5. $y = (5 - x^2)^3$.

6. $y = \sqrt{9 - x^2}$.

7. $y = (3x^2 + 7)^2$.

8. $y = \sqrt{x^2 - 5}$.

9. $y = \sqrt{2x^2 + 3}$.

10. $y = \sqrt[3]{x^2 + 1}$.

11. $y = \dfrac{1}{\sqrt{3x - 2}}$.

12. $y = \dfrac{7}{\sqrt{5 - 2x^2}}$.

13. $y = \sqrt{(x^2 + 4)}$.

14. $y = \sqrt{(x + 1)}$.

15. $y = \dfrac{7}{\sqrt[3]{x^2 + 1}}$.

16. $y = \dfrac{5}{(x^2 - 1)^2}$.

17. $y = \dfrac{2}{(x^2 + 4)^2}$.

18. $y = \dfrac{3}{(5 - x^3)^2}$.

33. The Derivative of u^n, n a Positive Fraction. We are now in a position to prove that the rule for the derivative of u^n holds when n is a positive fraction of the form $\frac{p}{q}$, where p and q are integers. Let

$$y = u^{\frac{p}{q}}.$$

Raise each member to the power q:

$$y^q = u^p.$$

Since u is a function of x, y is a function of x. Hence each member is a function of x raised to a positive integral power. Then each member can be differentiated by the rule of §32 which was proved for positive integral exponents. We find

$$qy^{q-1}\frac{dy}{dx} = pu^{p-1}\frac{du}{dx}.$$

From which

$$\frac{dy}{dx} = \frac{p}{q}\frac{u^{p-1}}{y^{q-1}}\frac{du}{dx}.$$

Substitute for y in the second member and obtain

$$\frac{dy}{dx} = \frac{p}{q}\frac{u^{p-1}}{\left[u^{\frac{p}{q}}\right]^{q-1}}\frac{du}{dx} = \frac{p}{q}\frac{u^{p-1}}{u^{p-\frac{p}{q}}}\frac{du}{dx}.$$

Then

$$\frac{dy}{dx} = \frac{p}{q}u^{\frac{p}{q}-1}\frac{du}{dx},$$

and the rule is proved that

$$\frac{du^n}{dx} = nu^{n-1}\frac{du}{dx},$$

where n is a positive fraction whose numerator and denominator are integers. This rule has already been used in the solution of numerous exercises.

34. The Derivative of a Constant. Let $y = c$, where c is a constant. Corresponding to any Δx, $\Delta y = 0$, and consequently

$$\frac{\Delta y}{\Delta x} = 0,$$

and

$$\lim_{\Delta x \doteq 0} \frac{\Delta y}{\Delta x} = 0,$$

or

$$\frac{dy}{dx} = 0.$$

The derivative of a constant is zero.

Interpret this result geometrically.

35. The Derivative of the Sum of Two Functions. Let

$$y = u + v,$$

where u and v are functions of x. Let Δu, Δv, and Δy be the increments of u, v, and y, respectively, corresponding to the increment Δx.

$$y + \Delta y = u + \Delta u + v + \Delta v$$
$$\Delta y = \Delta u + \Delta v$$
$$\frac{\Delta y}{\Delta x} = \frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta x}$$
$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx},$$

or

$$\frac{\mathbf{d(u + v)}}{\mathbf{dx}} = \frac{\mathbf{du}}{\mathbf{dx}} + \frac{\mathbf{dv}}{\mathbf{dx}}.$$

The derivative of the sum of two functions is equal to the sum of their derivatives.

The student will observe that the proof given can be extended to the sum of three, four, or any finite number of functions.

Illustrations.

1. $\frac{d(6x + 15x^2)}{dx} = \frac{d(6x)}{dx} + \frac{d(15x^2)}{dx} = 6 + 30x.$

2. $\frac{d(2\sqrt{x} + 3x^2 + 4)}{dx} = \frac{d(2\sqrt{x})}{dx} + \frac{d(3x^2)}{dx} + \frac{d(4)}{dx} = \frac{1}{\sqrt{x}} + 6x.$

3. $\frac{d(t^2 + 2t^3 + 3)}{dt} = 2t + 6t^2$

Exercises

Differentiate the following functions with respect to x, also with respect to t:

1. $3x^4 - 2x^2 + 6.$

2. $5x^3 - 7x^2 - 2x - 10.$

3. $\frac{3}{5}x^5 - \frac{1}{3}x^3 + x - 7.$

4. $x^{\frac{1}{2}} + 2x^{\frac{1}{3}}.$

5. $\frac{3}{\sqrt{x}} - \frac{2}{\sqrt[3]{x}}.$

6. $3x^7 - 6x^8 + 9.$

7. $x^{\frac{2}{3}} - x^{-\frac{2}{3}}.$

8. $-\frac{1}{3}x^3 + \frac{1}{2}x^2 - x + 2.$

9. $x^{-\frac{1}{2}} + x^{-\frac{1}{3}}.$

10. $ax^2 + bx + c.$

11. $y = \sqrt{x^2 + 4x - 5}.$

12. $y = \frac{1}{\sqrt{x^2 - 5x + 7}}.$

13. $y = \sqrt{3x^2 - 2x + 5}.$

14. $y = \sqrt{6 - 3x - x^2}.$

15. $s = \sqrt{t + 1} + \sqrt[3]{2t - 3}$

16. $y = \frac{1}{x^2 - 7x - 6}.$

17. $y = \sqrt{x^2 - 5x + 4}.$

18. $y = (3x^2 + 2x + 2)^3.$

36. Differentiation of Implicit Functions. The derivative of one variable with respect to another can be found from an equation connecting the variables without solving the equation for either variable. For, if the variables are x and y, y is a function of x, even though its explicit form may not be known, and the usual rules for finding the derivative of functions can be applied to each member of the equation.

The following example will illustrate the process.

Illustration. Let $x^2 + y^2 = a^2$. Find $\frac{dy}{dx}$.

The left-hand member of the given equation is the sum of two functions of x, since y is a function of x. Further, the derivative of the left-hand member is equal to the derivative of the right-hand member. The derivative of the latter is in this case zero, since the right-hand member is constant. On differentiat-

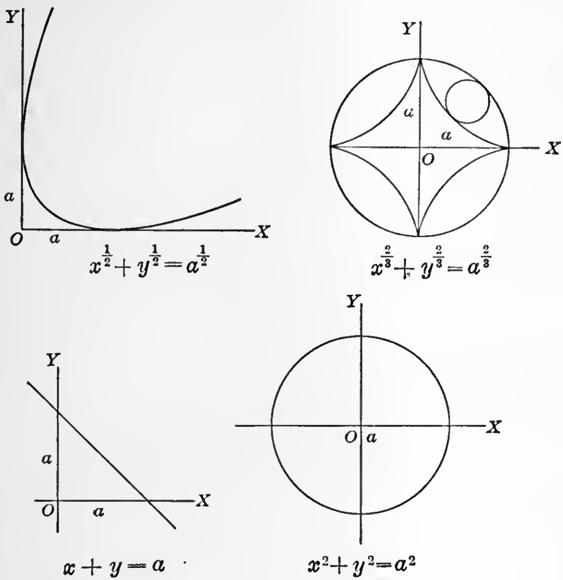

Fig. 24.

ing the left-hand member as the sum of two functions, we obtain

$$2x + 2y\frac{dy}{dx} = 0.$$

Solving for $\frac{dy}{dx}$,

$$\frac{dy}{dx} = -\frac{x}{y}.$$

When the derivative is found by differentiating each member of an equation in the implicit form, as in the foregoing illustration, the operation is called implicit differentiation.

Exercises

1. Draw the circle $x^2 + y^2 = a^2$ and show geometrically that the slope of the tangent at the point (x, y) is $-\frac{x}{y}$.

2. Solve the equation of Exercise 1 for y and find $\frac{dy}{dx}$.

From the following equations find $\frac{dy}{dx}$ by implicit differentiation:

3. $3x^2 + 4y^2 = 12.$

4. $x^2 - y^2 = a^2.$

5. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$ (Do not clear of fractions.)

If y is an implicit function of x expressed by an equation of the form

$$x^n + y^n = a^n, \tag{1}$$

differentiation gives

$$nx^{n-1} + ny^{n-1}\frac{dy}{dx} = 0,$$

or

$$\frac{dy}{dx} = -\left[\frac{x}{y}\right]^{n-1} \tag{2}$$

The equation (1) includes a number of important special cases. The graphs corresponding to the following values of n are shown in Fig. 24. For

$n = \frac{1}{2}$, $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$, a parabola,

$n = \frac{2}{3}$, $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$, an important hypocycloid,

$n = 1$, $x + y = a$, a straight line,

$n = 2$, $x^2 + y^2 = a^2$, a circle.

The graph of (1) passes through the points $(0, a)$ and $(a, 0)$ if n is positive.

6. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}.$

7. $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}.$

8. $x^3 + y^3 = a^3.$

9. $x^{\frac{3}{2}} + y^{\frac{3}{2}} = a^{\frac{3}{2}}.$

37. Anti-derivatives. Integration. Let it be required to find the equation of a curve whose slope at any point is equal to twice the abscissa of that point.

This means that at every point of the curve $\frac{dy}{dx} = 2x$. We seek then a function whose derivative is $2x$. $y = x^2$ is such a function. But $y = x^2 + C$, where C is a constant, is also a function having the same derivative. Hence there is an infinite number of functions whose derivatives are all equal to $2x$. The problem as proposed has then an infinite number of solutions, viz., the system of parabolas $y = x^2 + C$, corresponding to the infinitely many values of C.

If now we add to the statement of the problem the requirement that the curve shall pass through a given point, say (1, 2), it is geometrically evident that but one of the curves $y = x^2 + C$ will pass through the point. In other words there is but one value of C for which the latter requirement is satisfied. This value is determined by substituting the coördinates of the point in the equation $y = x^2 + C$, since they must satisfy this equation for some value of C, if the problem has a solution. On making the substitution we have

$$2 = 1 + C,$$

from which $C = 1$. Hence $y = x^2 + 1$ is the equation of the curve whose slope at any point is equal to twice the abscissa of the point and which passes through the point (1, 2).

The nature of the problem which has just been solved can be further explained by the following geometrical solution. Draw, Fig. 25, at the vertices of each small square on a sheet of coördinate paper on which a set of axes has been chosen, short lines whose slopes are equal to two times the abscissas of the respective vertices. A curve is to be drawn which at each of its points is tangent to a line such as those which have been drawn. Now it is impossible in the figure to draw lines through every point in the plane, but if the points through which the lines are drawn are sufficiently thick, the lines will serve to indicate the direction which the curve takes at nearby points. The lines may be regarded as pointers indicating the stream lines in flowing water. Then a point tracing the curve would move as a small cork would in water having the stream lines indicated by the figure.

Thus, to get the curve that goes through (1, 2), start from this point and, guided by the direction lines, sketch in as accurately as possible the curve to the right of this point. Do the same thing to the left, noting that here it is necessary to go against the stream lines instead of with them.

In Fig. 25 it should be noted that all lines through points having the same abscissa are parallel. This fact is of great

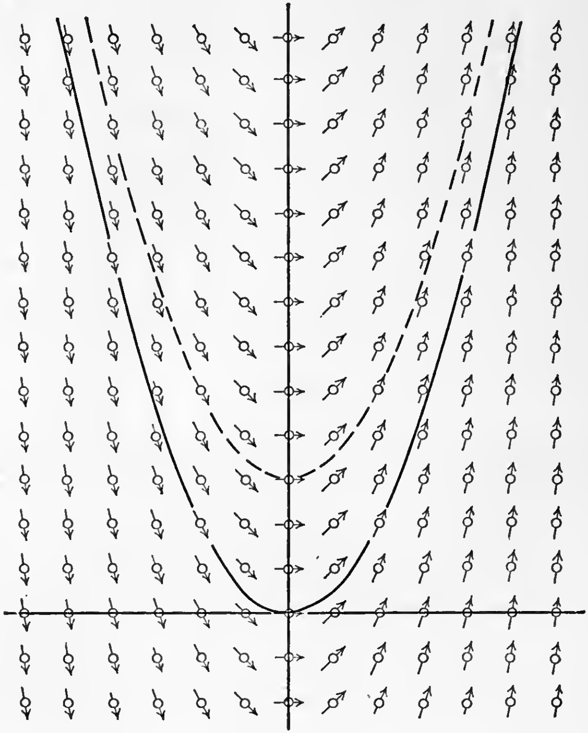

Fig. 25.

assistance in drawing. The squares on the coördinate paper can be used to advantage in drawing lines when the slope is known.

If the derivative which was given had been any other function of x, a geometrical solution could have been obtained by the same method.

The foregoing illustration introduces a new type of problem, viz., that of finding a function whose derivative is given. A function whose derivative is equal to a given function is called an *anti-derivative,* or *integral,* of the given function. From the illustration it is clear that any given function which has one anti-derivative, has an infinite number of anti-derivatives which differ from each other only by an additive constant. This latter fact is indicated in obtaining the anti-derivative of a given function by writing down the variable part of the anti-derivative and adding to it a constant C which is undetermined or "arbitrary." In a given application this constant will be determined by supplementary conditions as in the illustration at the beginning of this section.

The process of finding the anti-derivative of a given function is called *integration.*

Illustrations.

1. If $\frac{dy}{dx} = 3x^2$, $\quad y = x^3 + C.$
2. If $\frac{dy}{dx} = x^2$, $\quad y = \frac{1}{3}x^3 + C.$
3. If $\frac{dy}{dx} = 3x^2 + 2x$, $\quad y = x^3 + x^2 + C.$
4. If $\frac{dy}{dx} = 3x^2 + 2x + 7$, $\quad y = x^3 + x^2 + 7x + C.$
5. If $\frac{dy}{dx} = x^2 + x + 7$, $\quad y = \frac{x^3}{3} + \frac{x^2}{2} + 7x + C.$

If in Illustration 1 the curve is to pass through the point $(3, -2)$ we must have $-2 = 3^3 + C$, or $C = -29$. Hence the equation of the curve is $y = x^3 - 29$.

Exercises

Integrate the following ten functions:

1. $\frac{dy}{dx} = 3x^2.$
2. $\frac{dy}{dx} = 4x^3.$
3. $\frac{dy}{dt} = 4x^3 \frac{dx}{dt}.$
4. $\frac{dy}{dt} = 3x^2 \frac{dx}{dt}.$
5. $\frac{dy}{dx} = 3x^2 + 2x - 6.$
6. $\frac{dy}{dt} = (3x^2 + 2x + 6) \frac{dx}{dt}.$
7. $\frac{dy}{dt} = (ax + b) \frac{dx}{dt}.$
8. $\frac{dy}{dx} = 3x^2 - 2x^{\frac{1}{2}} + 7.$
9. $\frac{dy}{dx} = 10x^{-2} + 2x^{-3} - x + 7.$
10. $\frac{dy}{dx} = x^{\frac{1}{3}} + x^{-\frac{1}{3}}.$

Illustrations.

6. $\dfrac{dy}{dx} = 3(x^2 + 2)^2\, 2x.$

The right-hand side is in the form, $nu^{n-1}\dfrac{du}{dx}$, where n is 3, and u is $(x^2 + 2)$. Since the integral of $nu^{n-1}\dfrac{du}{dx}$ is $u^n + C$,

$$y = (x^2 + 2)^3 + C.$$

7. $\dfrac{dy}{dx} = (x^2 - 5)^3\, 2x = \frac{1}{4}[4(x^2 - 5)^3\, 2x].$

$$y = \tfrac{1}{4}(x^2 - 5)^4 + C.$$

8. $\dfrac{dy}{dx} = x(x^2 - 1)^5 = \frac{1}{12}[6(x^2 - 1)^5\, 2x].$

$$y = \tfrac{1}{12}(x^2 - 1)^6 + C.$$

9. $\dfrac{dy}{dt} = x^2(3 - x^3)^5\, \dfrac{dx}{dt} = -\frac{1}{18}\left[6(3 - x^3)^5\left(-3x^2\,\dfrac{dx}{dt}\right)\right].$

$$y = -\tfrac{1}{18}(3 - x^3)^6 + C.$$

10. $\dfrac{dy}{dt} = (x^2 - 2x + 3)^{-3}(x - 1)\dfrac{dx}{dt}$

$$= -\tfrac{1}{4}\left[-2(x^2 - 2x + 3)^{-3}(2x - 2)\frac{dx}{dt}\right].$$

$$y = \frac{-1}{4(x^2 - 2x + 3)^2} + C.$$

Exercises

Integrate:

11. $\dfrac{dy}{dx} = x\sqrt{x^2 - 1}.$ *Ans.* $y = \frac{1}{3}(x^2 - 1)^{\frac{3}{2}} + C.$

12. $\dfrac{dy}{dx} = (2x^3 + 3x^2)^{\frac{1}{3}}(x^2 + x).$ *Ans.* $y = \frac{1}{8}(2x^3 + 3x^2)^{\frac{4}{3}} + C.$

13. $\dfrac{dy}{dx} = (x + 1)^{\frac{1}{3}}.$ *Ans.* $y = \frac{3}{4}(x + 1)^{\frac{4}{3}} + C.$

14. $\dfrac{dy}{dx} = (2 - x^2)^2\, x.$

15. $\dfrac{dy}{dx} = x\sqrt{4 - x^2}.$

16. $\dfrac{dy}{dx} = (x^2 - 3)^2\, x.$

17. $\dfrac{dy}{dx} = x^2 + 3x.$

18. $\dfrac{dy}{dx} = (x^2 + 7)^3\, x.$

19. $\dfrac{dy}{dx} = \dfrac{x}{(x^2 + 4)^3}.$ *Ans.* $y = \dfrac{-1}{4(x^2 + 4)^2} + C.$

20. $\dfrac{dy}{dx} = (x^2 + 2x + 1)^3(x + 1)$.

21. $\dfrac{dy}{dx} = \dfrac{x}{\sqrt{x^2 + 4}}$.

22. $\dfrac{dy}{dx} = \dfrac{x}{\sqrt{5 - x^2}}$.

23. $\dfrac{dy}{dx} = (2 + x^3)^2 x^2$.

24. $\dfrac{dy}{dx} = \dfrac{x^2}{(4 - x^3)^4}$.

25. $\dfrac{dy}{dx} = (3x^2 + 2)^3 x$. *Ans.* $y = \frac{1}{24}(3x^2 + 2)^4 + C$.

26. $\dfrac{dy}{dt} = (2 - 3x^2)^2 x \dfrac{dx}{dt}$.

27. $\dfrac{dy}{dx} = (2x + 1)^2$. *Ans.* $y = \frac{1}{6}(2x + 1)^3 + C$.

28. $\dfrac{dy}{dt} = (3x - 2)^4 \dfrac{dx}{dt}$.

29. $\dfrac{dy}{dt} = (3 - 4x)^2 \dfrac{dx}{dt}$.

30. $\dfrac{dy}{dx} = \sqrt{x + 1}$. *Ans.* $y = \frac{2}{3}(x + 1)^{\frac{3}{2}} + C$.

31. $\dfrac{dy}{dx} = \sqrt{2x + 5}$.

32. $\dfrac{dy}{dt} = \sqrt{1 - x}\dfrac{dx}{dt}$.

33. $\dfrac{dy}{dx} = \sqrt{1 - 2x}$.

34. $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{x + 1}}$.

35. $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{3x + 2}}$.

36. $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{3 - 5x}}$.

37. $\dfrac{dy}{dx} = x\sqrt{4x^2 - 5}$.

38. $\dfrac{dy}{dx} = x\sqrt{9 - x^2}$.

39. $\dfrac{dy}{dx} = \dfrac{x}{\sqrt{4 - 3x^2}}$.

40. $\dfrac{dy}{dx} = \sqrt{(2x + 3)^3}$.

41. $\dfrac{dy}{dx} = \sqrt{(4 - 3x)^3}$.

42. $\dfrac{dy}{dx} = \sqrt{x^3 + 3x - 7}\,(x^2 + 1)$.

43. $\dfrac{dy}{dx} = (x + 4)\sqrt{x^2 + 8x + 9}$.

44. $\dfrac{dy}{dx} = (x - 3)\sqrt{7 - 6x + x^2}$.

45. $\dfrac{dy}{dx} = \dfrac{1 - 5x}{\sqrt{6 + 4x - 10x^2}}$.

46. $\dfrac{dy}{dx} = \dfrac{3x - 2}{\sqrt{3x^2 - 4x + 5}}$.

47. $\dfrac{dy}{dx} = \sqrt{4 - x}$.

48. $\dfrac{dy}{dx} = x(2 - x^2)^3$.

49. Find the equation of the curve whose slope at any point is equal to the square of the abscissa of that point and which passes through the point (2, 3).

50. Find the equation of the curve whose slope at any point is equal to the square root of the abscissa of that point and which passes through the point (2, 4).

51. Find the equation of the curve whose slope at any point is equal to the negative reciprocal of the square of the abscissa of that point and which passes through the point (1, 1).

38. Acceleration. The velocity of a body moving in a straight line may be either uniform or it may vary from instant to instant. In the latter case its motion is said to be accelerated, and this applies both to the case where there is an increase in velocity and the case where there is a decrease in velocity.

Thus it is a fact of common knowledge that the velocity of a body falling to the ground from a height increases with the distance through which the body has fallen, or with the time since the body started to fall. The time rate of change of the velocity of a moving body is an important concept in mechanics and physics.

If s denotes the distance passed over in time t, the velocity has been defined as the rate of change of s with respect to t. The notion of the velocity at a given instant was derived from that of the average velocity for an interval Δt. The *average velocity* was obtained by dividing the change in s, Δs, in a time Δt by Δt (*i.e.*, by dividing the distance passed over in time Δt by Δt). The limiting value of this quotient as Δt approaches zero was defined as the *velocity* at the beginning of the interval Δt.

In the same way if the velocity, v, changes by an amount Δv in the time Δt counted from a certain time t, the average rate of change of v for this interval is $\frac{\Delta v}{\Delta t}\cdot$ It is the *average linear acceleration*[1] for this interval. The *acceleration* at the time t is defined as the *limit of the average acceleration* as Δt approaches zero. It is then $\frac{dv}{dt}\cdot$ *The acceleration is the time rate of change of velocity.* In the case of a falling body it is known experimentally that for bodies falling from heights that are not too great, the velocity changes uniformly, due to the action of the force of gravity, *i.e.*,

[1] We suppose here that the body is moving in a straight line. If the path is curved, it will be seen later that the total acceleration is to be thought of as the resultant of two components, one of which produces a change in the direction of the velocity and the other a change in the magnitude of the velocity.

the time rate of change of the velocity is a constant. This constant is called the acceleration due to gravity and is usually denoted by g. In *F.P.S.* (foot-pound-second) units it is equal to 32.2 feet per second per second. That the unit of acceleration is 1 foot *per second per second* is explained by the fact that accelera-

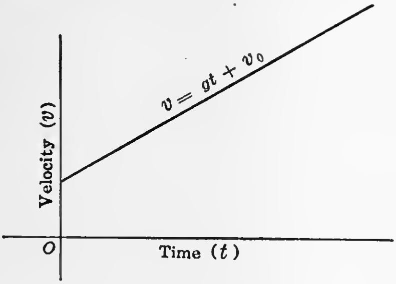

Fig. 26.

tion is the change *per second* of a velocity of a certain number of feet *per second.*

The differential equation of motion of the falling body can be written

$$\frac{dv}{dt} = g. \tag{1}$$

From which on integrating,

$$v = gt + C. \tag{2}$$

If it is given that the body starts falling from rest, we have as the condition for determining C, that $v = 0$ when $t = 0$. Equation (2) shows that C must be equal to zero. Then,

$$v = gt. \tag{3}$$

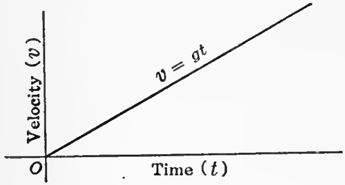

Fig. 27.

The graph, Fig. 27, of $v = gt$ is a straight line whose slope is g.

If the body had had an initial speed of v_0 feet per second, *i.e.*, if it had been projected downward instead of being dropped, the constant C would have been determined from the condition that

$v = v_0$ when $t = 0$. It follows from (2) that $C = v_0$, and the equation for v would have been

$$v = gt + v_0. \tag{4}$$

The graph of this function is shown in Fig. 26. It is again a straight line but it cuts the Y-axis at the point $(0, v_0)$.

The foregoing discussion evidently applies equally well to any *uniformly accelerated motion, i.e.*, to any motion where the rate of change of the velocity is constant. In all such cases the graph of v as a function of the time is a straight line.

Since $v = \frac{ds}{dt}$, equation (4) gives

$$\frac{ds}{dt} = gt + v_0.$$

Integrating,

$$s = \tfrac{1}{2}gt^2 + v_0t + C_2. \tag{5}$$

If t is measured from the instant the body begins to move and s from the position of the body at that instant, $s = 0$ when $t = 0$. From this condition $C_2 = 0$. Then the distance of the body from its initial position is given by

$$s = \tfrac{1}{2}gt^2 + v_0t.$$

If a body is thrown vertically upward, it is convenient to count distances measured upward, and upward velocities, as positive. Then, since the acceleration due to gravity diminishes v, equation (1) becomes

$$\frac{dv}{dt} = -g. \tag{1'}$$

The formulas (4) and (5) then become

$$v = -gt + v_0 \tag{4'}$$

$$s = -\tfrac{1}{2}gt^2 + v_0t. \tag{5'}$$

If a body falls from rest it is easy to express the speed as a function of the distance traversed. In this case, $v_0 = 0$. Then (4) and (5) become,

$$v = gt \tag{4''}$$

$$s = \tfrac{1}{2}gt^2. \tag{5''}$$

Elimination of t between these equations gives:

$$v = \sqrt{2gs}. \tag{6}$$

Exercises

1. If a body falls from rest how far will it fall in 10 seconds?

2. If a body is thrown vertically downward with a velocity of 10 feet per second, how far will it have moved by the end of 10 seconds? What will its velocity be?

3. If a body is thrown vertically upward with a velocity of 64.4 feet per second, what will the velocity be at the end of 10 seconds? What will be the position of the body? How far will it have moved?

4. Find the laws of motion if the acceleration is equal to $2t$ and if (1°) $s = 0$ and $v = 0$ when $t = 0$; (2°) $s = 3$ and $v = -2$ when $t = 0$.

5. If the acceleration is proportional to the time and if $v = v_0$ and $s = s_0$ when $t = 0$, show that

$$s = \frac{kt^3}{6} + v_0t + s_0.$$

CHAPTER IV

DIFFERENTIATION OF ALGEBRAIC FUNCTIONS

39. The Derivative of the Product of a Constant and a Variable. Let

$$y = cu,$$

where c is a constant and u and y are functions of x. Let Δu and Δy be the increments of u and y, respectively, corresponding to the increment Δx. Then

$$y + \Delta y = c(u + \Delta u)$$

$$\Delta y = c\Delta u$$

$$\frac{\Delta y}{\Delta x} = c\frac{\Delta u}{\Delta x}$$

$$\frac{dy}{dx} = c\frac{du}{dx},$$

or

$$\mathbf{\frac{d(cu)}{dx} = c\frac{du}{dx}.}$$

The derivative of the product of a constant and a function is equal to the constant times the derivative of the function.

Illustrations.

1. $\frac{d(3x^2)}{dx} = 3\frac{d(x^2)}{dx} = 6x.$

2. $\frac{d[4(x-2)^2]}{dx} = 4\frac{d(x-2)^2}{dx} = 8(x-2).$

3. $\frac{d[-\frac{2}{3}(x^2-5)^{\frac{1}{2}}]}{dx} = -\frac{2}{3}\frac{d(x^2-5)^{\frac{1}{2}}}{dx}$

$$= -\frac{2}{3}\,\frac{1}{2}(x^2-5)^{-\frac{1}{2}}\frac{d(x^2-5)}{dx}$$

$$= -\frac{2}{3}\frac{x}{(x^2-5)^{\frac{1}{2}}}.$$

40. The Derivative of the Product of Two Functions. Let

$$y = uv,$$

where u and v are functions of x.

$$y + \Delta y = (u + \Delta u)(v + \Delta v)$$

$$y + \Delta y = uv + u\Delta v + v\Delta u + \Delta u\Delta v$$

$$\Delta y = u\Delta v + v\Delta u + \Delta u\Delta v$$

$$\frac{\Delta y}{\Delta x} = u\frac{\Delta v}{\Delta x} + v\frac{\Delta u}{\Delta x} + \Delta u\frac{\Delta v}{\Delta x}.$$

Since Δu approaches zero as Δx approaches zero,

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx},$$

or

$$\mathbf{\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}.} \qquad (1)$$

The derivative of the product of two functions is equal to the first times the derivative of the second plus the second times the derivative of the first.

Illustrations.

1. $$\frac{d(x + 2)(x + 3)}{dx} = (x + 2)\frac{d(x + 3)}{dx} + (x + 3)\frac{d(x + 2)}{dx}$$
$$= (x + 2) + (x + 3) = 2x + 5.$$

2. $$\frac{d(x^2 + 3x)(x - 2)}{dx} = (x^2 + 3x)\frac{d(x - 2)}{dx} + (x - 2)\frac{d(x^2+3x)}{dx}$$
$$= (x^2 + 3x) + (x - 2)(2x + 3)$$
$$= 3x^2 + 2x - 6.$$

3. If $x^2 + xy^3 + y = 10$,

$$2x + x\frac{d(y^3)}{dx} + y^3 + \frac{dy}{dx} = 0,$$

$$2x + 3xy^2\frac{dy}{dx} + y^3 + \frac{dy}{dx} = 0,$$

whence

$$\frac{dy}{dx} = -\frac{2x + y^3}{3xy^2 + 1}.$$

Exercises

Differentiate the following:

1. $(x + 1)(x - 1)$.
2. $(x^2 + 2x)(x - 3)$.
3. $x(x^2 + 2x - 6)$.
4. $(x - 1)^2(x^2 + 1)$.
5. $(x^2 + 2x - 3)(x + 1)^2$.
6. $x\sqrt{x - 1}$.
7. $(x - 1)\sqrt{x}$.
8. $x(x - 1)^{\frac{1}{3}}$.
9. $y = (x + 1)\sqrt{x^2 - 2}$.
10. $y = (2 - x)\sqrt[3]{x^2 - 4}$.
11. $y = (2x + 3)\sqrt{4 - x^2}$.
12. $y = x\sqrt{1 - x^2}$.
13. $x^3y^3 + 3x - 7y = 15$
14. $x^2y - 3xy^2 = 10$.
15. $y\sqrt{x} + x\sqrt{y} = 3$.
16. $xy - x^2y = 0$.

41. The Derivative of the Quotient of Two Functions. Let

$$y = \frac{u}{v},$$

where u and v are functions of x. Then

$$yv = u.$$

Differentiating by the rule of §40,

$$y\frac{dv}{dx} + v\frac{dy}{dx} = \frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{\frac{du}{dx} - y\frac{dv}{dx}}{v}.$$

Replacing y by its value, $\frac{u}{v}$,

$$\frac{d\left(\frac{u}{v}\right)}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}. \qquad (1)$$

The derivative of the quotient of two functions is equal to the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the square of the denominator.

Illustration.

$$\frac{d\left[\dfrac{x^2+1}{x-2}\right]}{dx} = \frac{(x-2)\dfrac{d(x^2+1)}{dx} - (x^2+1)\dfrac{d(x-2)}{dx}}{(x-2)^2}$$

$$= \frac{(x-2)(2x) - (x^2+1)}{(x-2)^2}$$

$$= \frac{x^2-4x-1}{(x-2)^2}.$$

Exercises

Differentiate the following:

1. $\dfrac{x+1}{x-1}$. **4.** $\dfrac{\sqrt{x-1}}{1+x}$. **7.** $\dfrac{1}{x-\sqrt{x^2-1}}$.

2. $\dfrac{x^2-3}{x-2}$. **5.** $\dfrac{x^2-1}{\sqrt{x}}$. **8.** $\dfrac{x^2+4}{x-2}$.

3. $\dfrac{x+x^2}{1-x}$. **6.** $\dfrac{x}{(1-x)^{\frac{3}{2}}}$. **9.** $\dfrac{x^3+8}{x-2}$.

42. The Derivative of u^n, n Negative. In Chapter III the formula

$$\frac{du^n}{dx} = nu^{n-1}\frac{du}{dx}$$

was proved for n positive and commensurable. The formula was assumed for negative exponents. We are now in a position to give a proof of the formula for this case. Let

$$y = u^{-s},$$

where s is a positive commensurable constant. Then

$$y = \frac{1}{u^s},$$

or

$$yu^s = 1.$$

Differentiate by the formula of §40,

$$ysu^{s-1}\frac{du}{dx} + u^s\frac{dy}{dx} = 0,$$

$$\frac{dy}{dx} = -\frac{sy}{u}\frac{du}{dx} = -su^{-s-1}\frac{du}{dx}. \qquad (1)$$

This completes the proof that

$$\frac{du^n}{dx} = nu^{n-1}\frac{du}{dx},$$

if n is a commensurable constant.[1]

We see from equation (1) that

$$\frac{d\left(\frac{c}{u^n}\right)}{dx} = -\frac{nc}{u^{n+1}}\frac{du}{dx}. \tag{2}$$

Illustration.

$$\frac{d}{dx}\frac{3}{(x^2-1)^2} = -\frac{6}{(x^2-1)^3}\frac{d(x^2-1)}{dx} = -\frac{12x}{(x^2-1)^3}.$$

Exercises

Differentiate the following:

1. $\frac{2}{x^2}$.
2. $\frac{4}{1-x}$.
3. $\frac{2}{\sqrt{x}}$.
4. $-\frac{3}{x-1}$.
5. $\frac{5}{(x+1)^2}$.
6. $\frac{3}{\sqrt{x-1}}$.
7. $\frac{6}{x^2+1}$.
8. $\frac{7}{(x^2+1)^{\frac{1}{2}}}$.
9. $\frac{1}{(1-x^2)^{\frac{1}{2}}}$.

43. Maximum and Minimum Values of a Function. In Chapter I it was shown that the derivative of a function with respect to its argument is equal to the slope of the tangent drawn to the curve representing the function. The derivative is positive where the function is increasing and negative where the function is decreasing. These facts enable us to determine the maximum and minimum values of a function.

Additional exercises in finding maximum and minimum values of a function will be given in this section.

Illustration. Let

$$y = 2x^3 + 3x^2 - 12x - 10.$$

$$\frac{dy}{dx} = 6x^2 + 6x - 12 = 6(x+2)(x-1).$$

[1] It can be shown that the formula also holds for incommensurable exponents.

If x is less than -2, both factors of the derivative are negative. Then for all values of x less than -2, the derivative is positive and the function is increasing. If x is greater than -2 and less than 1, the first factor of the derivative is positive and the second negative. Hence, if $-2 < x < 1$, the derivative is negative and the function is decreasing. If x is greater than 1 the derivative is positive and the function is again increasing.

The function changes from an increasing to a decreasing function when x passes through the value -2, and changes from a

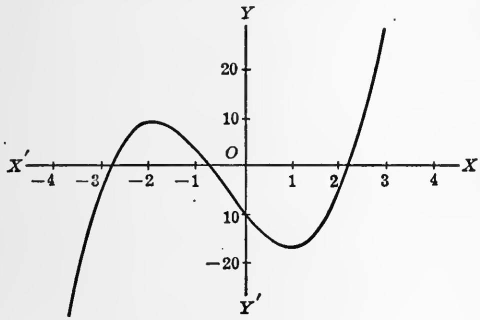

FIG. 28.

decreasing to an increasing function when x passes through the value 1. Hence the function has a maximum value when x equals -2, and a minimum value when x equals 1. These values, 10 and -17, respectively, are obtained by substituting -2 and 1 for x in the function. (See Fig. 28.) The more important results of the above discussion are put in tabular form below.

x	$x + 2$	$x - 1$	$\frac{dy}{dx}$	Function
$x < -2$	$-$	$-$	$+$	Increasing.
$-2 < x < 1$	$+$	$-$	$-$	Decreasing.
$1 < x$	$+$	$+$	$+$	Increasing.
$x = -2$	0	$-$	0	Max. value = 10.
$x = 1$	$+$	0	0	Min. value = -17.

It is to be observed that -2 and $+1$ are the only values of x at which the derivative can change sign and that these are the values that need to be examined in finding the maximum and minimum values of the function.

Exercises

Find where each of the following functions is increasing; decreasing. Find the maximum and minimum values if there are any. Sketch the curve representing each function.

1. $y = x^2$.

2. $y = x^3$.

3. $y = -2x^4$.

4. $y = x^3 + 3x - 2$.

5. $y = 3x^3 - 2x^2 - 6$.

6. $y = (x + 2)(x - 3)$.

7. $y = 2x^3 - 9x^2 + 12x - 10$

8. $y = x^3 - 3x + 7$.

9. $y = x^3 + x^2 - x - 1$.

10. $y = \frac{1}{x^2}$.

11. A sheet of tin 24 inches square has equal squares cut from each corner. The rectangular projections are then turned up to form a tray with square base and rectangular vertical sides. Find the side of the square that must be cut out from each corner in order that the tray may have the greatest possible volume.

Hint. Show that the function representing the volume of this tray is $4x(12 - x)^2$, where x is the side of the square cut out.

12. In a triangle whose sides are 10, 6, and 8 feet is inscribed a rectangle the base of which lies in the longest side of the triangle. Express the area of the rectangle as a function of its altitude. Find the dimensions of the rectangle of maximum area.

13. A ship A is 50 miles directly north of another ship B at a certain instant. Ship B sails due east at the rate of 5 miles per hour, and ship A sails due south at the rate of 10 miles per hour. Show that the distance between the ships is expressed by the function $\sqrt{125t^2 - 1000t + 2500}$, where t denotes the number of hours since the ships were in the position stated in the first sentence. At what time are the ships nearest together? At what rate are they separating or approaching when $t = 3$? When $t = 5$? When $t = 6$?

14. The stiffness of a rectangular beam varies as its breadth and as the cube of its depth. Find the dimensions of the stiffest beam which can be cut from a circular log 12 inches in diameter.

Miscellaneous Exercises

Differentiate the following twenty-five functions with respect to x:

1. $\frac{1}{x}$.
2. $\frac{1}{x^2}$.
3. $\frac{1}{x^3}$.
4. $(x - 3)^2$.
5. $(x - 2)^{\frac{1}{2}}$.
6. $(x^2 - 1)^2$.
7. $(x^3 - 2x^2 + x - 6)^2$.
8. $(x - 2)^{-3}$.
9. $(3 - x)^3$.
10. $(2 - x^2)^3$.
11. $(3 - x^3)^{-2}$.
12. $(2 - x^2)^{-3}$.
13. $(x + 1)(x - 2)$.
14. $\sqrt{x}$.
15. $x^{\frac{3}{2}}$.
16. $\frac{1}{\sqrt{x}}$.
17. $x(x - 1)^2$.
18. $x^2(1 - x)$.
19. $x(1 - x^2)^3$.
20. $(1 - x^3)^2(x - 2)^2$.
21. $\frac{x - 1}{x^2 + 1}$.
22. $\frac{1 - x}{1 + x^2}$.
23. $(x - 1)^{\frac{1}{3}}$.
24. $\frac{x}{\sqrt{a^2 - x^2}}$.
25. $x\sqrt{1 - x^2}$.

Integrate the following twenty expressions:

26. $\frac{dy}{dx} = x^3$.
27. $\frac{dy}{dx} = x^3 + x^2$.
28. $\frac{dy}{dx} = x^3 + x^2 + x + 1$.
29. $\frac{dy}{dx} = x^{\frac{1}{2}}$.
30. $\frac{dy}{dx} = \frac{1}{\sqrt{x}}$.
31. $\frac{dy}{dx} = x^{\frac{3}{2}} + x^{\frac{1}{2}}$.
32. $\frac{dy}{dx} = (x - 1)^{\frac{1}{2}}$.
33. $\frac{dy}{dx} = x(1 - x^2)^2$.
34. $\frac{dy}{dx} = x^2(x^3 - 2)^{\frac{1}{2}}$.
35. $\frac{dy}{dx} = \frac{1}{(1 - x)^2}$.
36. $\frac{dy}{dx} = \frac{1}{x^2}$.
37. $\frac{dy}{dx} = \frac{3}{x^3}$.
38. $\frac{dy}{dx} = \frac{2}{(1 - x)^3}$.
39. $\frac{dy}{dx} = \frac{x}{(1 - x^2)^2}$.
40. $\frac{dy}{dx} = \frac{1 - x}{(2x - x^2)^2}$.
41. $\frac{dy}{dx} = \frac{1}{\sqrt{x + 1}}$.
42. $\frac{dy}{dx} = \frac{1}{\sqrt{x - 1}}$.
43. $\frac{dy}{dx} = \frac{1}{\sqrt{1 - x}}$.
44. $\frac{dy}{dx} = x^{-\frac{1}{3}}$.
45. $\frac{dy}{dx} = x - 1$.

46. Find $\frac{dy}{dx}$ if $x^2 - y^2 = a^2$.

47. Find $\frac{dy}{dx}$ if $\frac{x^2}{16} + \frac{y^2}{9} = 1$.

48. A ladder 20 feet long leans against the vertical wall of a

building. If the lower end of the ladder is drawn out along the horizontal ground at the rate of 2 feet per second, at what rate is its upper end moving down when the lower end is 10 feet from the wall?

HINT. Let AC, Fig. 29, be the wall and let CB be the ladder. Let $AB = x$ and $AC = y$. Then

$$y = \sqrt{400 - x^2}$$

and

$$\frac{dy}{dt} = \frac{-x}{\sqrt{400 - x^2}} \frac{dx}{dt}.$$

But, since $\frac{dx}{dt} = 2$,

$$\frac{dy}{dt} = \frac{-2x}{\sqrt{400 - x^2}}.$$

The negative sign of the derivative indicates that the upper end of the ladder is moving down.

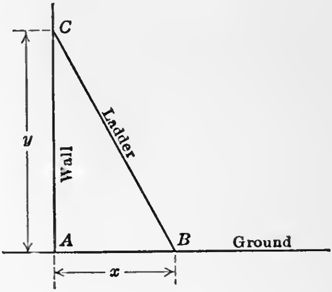

FIG. 29.

49. Answer the question of Exercise 48, if $x = 0$; $x = 2$; $x = 15$; $x = 20$.

50. With the statement of Exercise 48, find the rate at which the area of the triangle ABC, Fig. 29, is increasing when the lower end of the ladder is 5 feet from the wall.

51. With the statement of Exercise 48, find the position of the ladder when the area of the triangle ABC, Fig. 29, is a maximum.

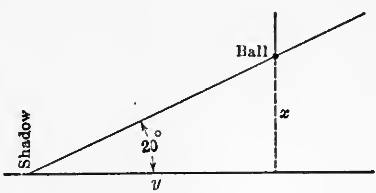

FIG. 30.

52. A ball is dropped from a balloon at a height of 1000 feet. Express the velocity of its shadow along the horizontal ground as a function of the time, if the altitude of the sun is 20°.

HINT. Let x, Fig. 30, be the distance of the falling body above the earth. Let y be the distance of the shadow from a point on the earth directly under the falling body.

53. With the statement of Exercise 52, find the velocity of the shadow when the ball leaves the balloon; when it is half way to the earth; when it reaches the earth.

54. A man standing on a dock is drawing in a rope attached to a boat at the rate of 12 feet per minute. If the point of attachment of the rope is 15 feet below the man's hands, how fast is the boat moving when 13 feet from the dock?

55. The paths of two ships A and B, sailing due north and east, respectively, cross at the point C. A is sailing at the rate of 8 miles per hour, and B at the rate of 12 miles per hour. If A passed through C 2 hours before B, at what rate are the two ships approaching or separating 1 hour after B passed through C? 3 hours after B passed through C? When are the two ships nearest together?

56. Two bodies are moving, one on the axis of x, the other on the axis of y, and their distances from the origin are given by

$$x = 3t^2 - 3t + 1,$$
$$y = 6t - 12,$$

the units of distance and time being feet and minutes, respectively. At what rate are the bodies approaching or separating when $t = 2$? When $t = 5$? When are they nearest together?

57. A ship is anchored in 35 feet of water and the anchor cable passes over a sheave in the bow 15 feet above the water. The cable is hauled in at the rate of 30 feet a minute. How fast is the ship moving when there are 80 feet of cable out?

58. A gas in a cylindrical vessel is being compressed by means of a piston in accordance with Boyle's law, $pv = C$. If the piston is moving at a certain instant so that the volume is decreasing at the rate of 1 cubic foot per second, at what rate is the pressure changing if at this instant the pressure is 5000 pounds per square foot and the volume is 10 cubic feet?

59. Water is flowing from an orifice in the side of a cylindrical tank whose cross section is 100 square feet. The velocity of the water in the jet is equal to $\sqrt{2gh}$, where h is the height of the surface of the water above the orifice. If the cross section of the jet is 0.01 square foot, how long will it take for the water to fall from a height of 100 feet to a height of 81 feet above the orifice?

60. At a certain instant the pressure in a vessel containing air is 3000 pounds per square foot; the volume is 10 cubic feet, and it is increasing in accordance with the adiabatic law, $pv^{1.4} = c$, at the rate of 2 cubic feet per second. At what rate is the pressure changing?

61. Water flows from a circular cylindrical vessel whose radius is 2 feet into one in the shape of an inverted circular cone whose vertical angle is 60°. (*a*) If the level of the water in the cylinder is falling uniformly at the rate of 0.5 foot a minute, at what rate is the water flowing? (*b*) At this rate of flow, at what rate will the level of the water in the cone be rising when the depth is 4 inches? When it is 20 inches?

62. A toboggan slide on a hillside has a uniform inclination to the horizon of 30°. A man is standing 300 feet from the top of the slide on a line at right angles to the slide. How fast is the toboggan moving away from the man 3 seconds after leaving the top? 10 seconds after leaving the top? (Use formula for speed of a body sliding down an inclined plane. Neglect friction.)

If the man is approaching the top of the slide at the rate of 10 feet a second, answer the same questions, it being supposed that the man is 300 feet away from the top of the slide when the toboggan starts.

44. Derivative of a Function of a Function. If $y = \phi(u)$ and $u = f(x)$, y is a function of x. The derivative of y with respect to x can be found without eliminating u. For any set of corresponding increments, Δx, Δy, and Δu,

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u}\,\frac{\Delta u}{\Delta x}.$$

Hence

$$\lim_{\Delta x \doteq 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \doteq 0} \frac{\Delta y}{\Delta u} \lim_{\Delta x \doteq 0} \frac{\Delta u}{\Delta x}.$$

Since Δu approaches zero as Δx approaches zero,

$$\mathbf{\frac{dy}{dx} = \frac{dy}{du}\,\frac{du}{dx}}. \tag{1}$$

This is the formula for the derivative of a function of a function.

Illustration. Let

$$y = u^3 + 5$$

and

$$u = 3x^2 + 7x + 10.$$

$$\frac{dy}{du} = 3u^2$$

and

$$\frac{du}{dx} = 6x + 7.$$

Then

$$\frac{dy}{dx} = 3u^2(6x + 7)$$
$$= 3(3x^2 + 7x + 10)^2 (6x + 7).$$

45. Inverse Functions. If $x = \phi(y)$, $\frac{dy}{dx}$ can be found by the rule

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}},$$

which is easily proved.

$$\frac{dy}{dx} = \lim_{\Delta x \doteq 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \doteq 0} \frac{1}{\frac{\Delta x}{\Delta y}} = \frac{1}{\frac{dx}{dy}}.$$

Illustration. If $x = 5y^6 + 7y^2 + 3$,

$$\frac{dx}{dy} = 30y^5 + 14y,$$

and

$$\frac{dy}{dx} = \frac{1}{2y(15y^4 + 7)}.$$

Exercises

1. Find $\frac{dy}{dx}$ in terms of x if:

(a) $y = \sqrt{u^2 + 7}$ and $u = 3x + 10$.
(b) $y = 2u^3 + 5u$ and $u = x^2 - 2x$.
(c) $y = \frac{1}{\sqrt{u^2 - 5}}$ and $u = x^2 - 2$.

2. Find $\frac{dy}{dx}$ if:

(a) $x = \sqrt{y^2 + 7}$.
(b) $x = \frac{1}{(y^2 + 2)^2}$.
(c) $x = \frac{1}{\sqrt{y^3 - 3}}$.
(d) $x = y$.

3. Find $\frac{dy}{dx}$ if:

(a) $y^4 + x^4 - 7xy = 15$.
(b) $3xy^3 + 6x^3y + 4x^2 = 15$.

46. Parametric Equations. If the equation of a curve is given in parametric form, $x = f(t)$, $y = \phi(t)$, it is important to be able to find the derivative of y with respect to x without eliminating t

between the given equations. A rule for doing this can be derived by the method used in §§**13** and **17.**

If t is given an increment Δt, x and y take on the increments Δx and Δy, respectively. Then

$$\frac{\Delta y}{\Delta x} = \frac{\frac{\Delta y}{\Delta t}}{\frac{\Delta x}{\Delta t}},$$

and

$$\lim_{\Delta t \doteq 0} \frac{\Delta y}{\Delta x} = \frac{\lim_{\Delta t \doteq 0} \frac{\Delta y}{\Delta t}}{\lim_{\Delta t \doteq 0} \frac{\Delta x}{\Delta t}},$$

or

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}.$$

Exercises

1. Find the slope at (6,1) of the curve whose parametric equations are

$$x = t^2 + t,$$
$$y = t - 1.$$

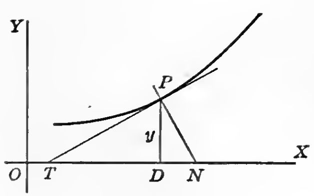

Fig. 31.

Find $\frac{dy}{dx}$ for each of the following:

2. $x = t^n,$

$y = t^2 + 1.$

3. $x = u^3 + 3,$

$y = \frac{1}{\sqrt{u}}.$

47. Lengths of Tangent, Normal, Subtangent, and Subnormal. In Fig. 31, PT is the tangent and PN is the normal at P. The lengths of the lines PT, PN, TD, and DN are called the tangent, the normal, the subtangent, and the subnormal, respectively, for the point P. Show that the lengths of these lines are:

$$TD = \frac{y}{\frac{dy}{dx}}, \tag{1}$$

$$DN = y\frac{dy}{dx}, \tag{2}$$

$$PT = \frac{y}{\frac{dy}{dx}}\sqrt{1 + \left(\frac{dy}{dx}\right)^2}, \tag{3}$$

$$PN = y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}. \tag{4}$$

Exercises

1. Obtain the length of the tangent, normal, subtangent, and subnormal for the point (1, 2) on the curve $y^2 = 4x$. Show that for points on this curve the subnormal is of constant length.

2. Write the equation of the tangent to $y^2 = 4x$ at the point (1, 2). Write the equation of the normal at the same point. It is to be noted that in this exercise the equations of the tangent and normal lines are to be found, and not the lengths of the tangent and normal as in the preceding exercise.

3. Write the equation of the tangent to

$$\frac{x^2}{25} + \frac{y^2}{9} = 1$$

at the point (3, 2.4). Use implicit differentiation.

4. Find the equation of the curve whose subnormal is of constant length 4 and which passes through the point (1, 3).

5. Find the length of the tangent to $y = \frac{2}{x}$ at the point where $y = 1$.

6. Find the length of the normal to the curve $y = \frac{1}{x - 1}$ at the point where $x = 3$.

7. Find the equation of the curve passing through the point (1, 3) and having a subtangent equal to the square of the ordinate.

CHAPTER V

SECOND DERIVATIVE. POINT OF INFLECTION

48. Second Derivative, Concavity. Since the first derivative of a function of x is itself a function of x, we can take the derivative of the first derivative. *The derivative of the first derivative is called the second derivative.* In the case of a function y of x, it is denoted by the symbol $\frac{d\left(\frac{dy}{dx}\right)}{dx}$, or $\frac{d}{dx}\left(\frac{dy}{dx}\right)$, or more commonly by $\frac{d^2y}{dx^2}$. $\frac{d^2y}{dx^2}$ is read *"the second derivative of y with respect to x."* It must here again be remembered that $\frac{d^2y}{dx^2}$ is not a fraction with a numerator and a denominator, but is only a symbol representing the derivative of the first derivative.

If $y = f(x)$, the first derivative of y with respect to x is sometimes written y' and very commonly $f'(x)$. Similarly the second derivative is indicated by $f''(x)$.

The derivative of the second derivative is called the *third derivative.* It is designated by $\frac{d^3y}{dx^3}$, or if $y = f(x)$, by $f'''(x)$. The nth derivative is designated by $\frac{d^ny}{dx^n}$, or by $f^{(n)}(x)$.

Between the points A and C, Fig. 16, where the curve is concave downward, the slope of the tangent decreases from large positive values near A to negative values near C. This means that the tangent revolves in a clockwise direction as the point of tangency moves along the curve from A toward C. Clearly this will always happen for any portion of a curve that is concave downward. (See Fig. 32, a, b, and c.) The slope decreases as the point of tangency moves to the right.

On the other hand, if a portion of a curve is concave upward, the slope of the tangent increases as the point of tangency moves to the right. Thus in Fig. 16 the slope of the tangent is negative at C and increases steadily to positive values at E. The same thing is evidently true for any portion of a curve that is concave upward. In this case the tangent line revolves in a counter-clockwise direction.

Since the first derivative of a function is equal to the slope of the tangent to the curve representing the function, what has just been said can be stated concisely as follows:

If an arc of curve is concave upward the first derivative is an increasing function, while if the curve is concave downward, the first derivative is a decreasing function.

If the second derivative of a function is positive between certain values of the independent variable x, the first derivative is an increasing function, the tangent line revolves in a counter-clock-

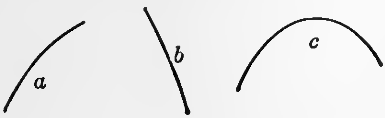

FIG. 32.

wise direction, and consequently the curve representing the function is concave upward between the values of x in question. If the second derivative is negative, the first derivative is a decreasing function and the curve is concave downward. Thus in Fig. 16 the second derivative is negative between A and C, and between E and G. It is positive between C and E, and between G and I.

49. Points of Inflection. Points at which a curve ceases to be concave downward and becomes concave upward, or *vice versa*, are called *points of inflection.*

At such points the second derivative changes sign. C, E, and G, Fig. 16, are points of inflection. At C, for instance, the second derivative changes from negative values to positive values.

Illustration 1. Study the curve $y = \frac{1}{6}x^3$ by means of its derivatives.

Differentiating,

$$\frac{dy}{dx} = \tfrac{1}{2}x^2,$$

$$\frac{d^2y}{dx^2} = x.$$

When $x < 0$, $\frac{d^2y}{dx^2} < 0$, $\frac{dy}{dx} = \frac{1}{2}x^2$ is a decreasing function, and the curve $y = \frac{1}{6}x^3$ is concave downward. When $x > 0$, $\frac{d^2y}{dx^2} > 0$, $\frac{dy}{dx}$ is an increasing function, and the curve $y = \frac{1}{6}x^3$ is concave upward.

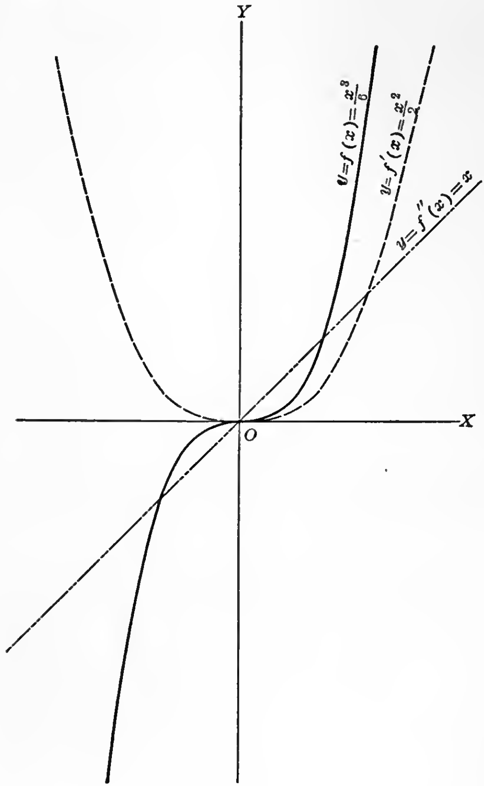

FIG. 33.

At the point where $x = 0$, $\frac{d^2y}{dx^2}$ changes sign from negative to positive, and the curve changes from being concave downward to being concave upward. Hence the point (0, 0) is a point of inflection.

Since $\frac{dy}{dx}$ is positive except when $x = 0$, $y = \frac{1}{6}x^3$ is an increasing function excepting when $x = 0$. When $x = 0$ the curve has a horizontal tangent.

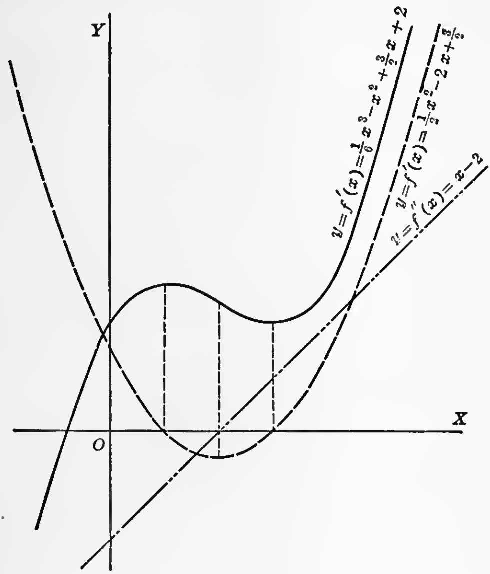

Fig. 34.

In Fig. 33 the graphs of the functions $y = \frac{1}{6}x^3$ and of its first and second derivatives are drawn on the same axes. Trace out in this figure all the properties mentioned in the discussion.

Illustration 2. Let

$$y = \tfrac{1}{6}x^3 - x^2 + \tfrac{3}{2}x + 2.$$

Differentiating,

$$\frac{dy}{dx} = \tfrac{1}{2}x^2 - 2x + \tfrac{3}{2}$$

$$= \tfrac{1}{2}(x - 1)(x - 3).$$

$$\frac{d^2y}{dx^2} = x - 2.$$

At $x = 2$, $\frac{d^2y}{dx^2}$ changes sign from minus to plus. Hence the curve is concave downward to the left, and concave upward to the right of the line $x = 2$. The point on the curve whose abscissa is 2 is then a point of inflection. The value of the function corresponding to $x = 1$ is a maximum value, and the value of the function corresponding to $x = 3$ is a minimum value. See Fig. 34 for a sketch of the function and its first and second derivatives. Trace out in the figure what has been given in the discussion.

The more important properties of the function are put in tabular form below.

x	$\frac{d^2y}{dx^2}$	$\frac{dy}{dx}$	Curve
$x < 2$	$-$	Decreasing	Concave downward.
$x > 2$	$+$	Increasing	Concave upward.
$x = 2$	0		Point of inflection ($y = 2\frac{1}{3}$).
$x < 1$		$+$	Increasing.
$1 < x < 3$		$-$	Decreasing.
$x > 3$		$+$	Increasing.
$x = 1$		0	Maximum point ($y = 2\frac{2}{3}$).
$x = 3$		0	Minimum point ($y = 2$).

Exercises

Find the maximum and minimum points and points of inflection of the following curves. Sketch the curves.

1. $y = x^3 - 3x^2$.

2. $y = x^3 + 3x^2$.

3. $y = 2x^3 + 3x^2 + 6x + 1$.

4. $y = 3x^4 - 4x^3 - 1$.

5. $y = x^3$.

6. $y = 2x^4 - 4x^3 - 9x^2 + 27x + 2$.

7. $y = 6x^4 - 4x^3 + 1$.

CHAPTER VI

APPLICATIONS

50. Area under a Curve: Rectangular Coördinates. An important application of the anti-derivative is that of finding the area under a plane curve.

Let $APQB$, Fig. 35, be a continuous curve between the ordinates $x = a$ and $x = b$. Further, between these limits, let the curve lie entirely above the X-axis. Our problem is to find the area, A, bounded by the curve, the X-axis, and the ordinates $x = a$ and $x = b$.

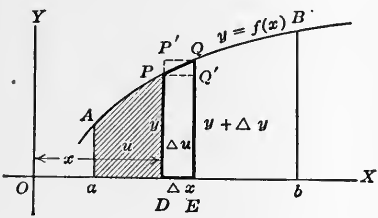

Fig. 35.

The area can be thought of as generated by a moving ordinate starting from $x = a$ and moving to the right to a position DP where the abscissa is x. This ordinate sweeps out the variable area u, which becomes the desired area A when $x = b$. On moving from the position DP to the position EQ where the abscissa is $x + \Delta x$, the ordinate to the curve takes on an increment Δy and the area u an increment Δu. By taking Δx small enough the curve is either ascending or descending at all points between P and Q. It follows at once from the figure that

$$y\Delta x < \Delta u < (y + \Delta y)\Delta x, \tag{1}$$

or

$$y < \frac{\Delta u}{\Delta x} < y + \Delta y.$$

(If the curve descends between P and Q the signs of inequality in (1) are reversed. The argument which follows will not be affected.)

As Δx approaches zero, Δy approaches zero and $y + \Delta y$ approaches y. Hence $\frac{\Delta u}{\Delta x}$, which lies between y and $y + \Delta y$, approaches y. Thus

$$\lim_{\Delta x \doteq 0} \frac{\Delta u}{\Delta x} = y,$$

or

$$\frac{du}{dx} = y. \qquad (2)$$

If the equation of the curve is $y = f(x)$,

$$\mathbf{\frac{du}{dx} = f(x).} \qquad (3)$$

Let $F(x)$ be a function whose derivative is $f(x)$. Then

$$u = F(x) + C.$$

C is determined by the condition that $u = 0$ when $x = a$. Then

$$C = -F(a)$$

and

$$u = F(x) - F(a), \qquad (4)$$

an expression for the variable area measured from the ordinate $x = a$ to the variable ordinate whose abscissa is x. A, the area sought, is obtained by putting $x = b$ in equation (4).

$$\mathbf{A = F(b) - F(a)} \qquad (5)$$

Illustration. Find the area A bounded by $y = x^2$, the X-axis, and the ordinates $x = 2$ and $x = 4$.

$$\frac{du}{dx} = x^2.$$

$$u = \tfrac{1}{3}x^3 + C.$$

When $x = 2$, $u = 0$, and $C = -\frac{8}{3}$. Then

$$u = \tfrac{1}{3}x^3 - \tfrac{8}{3},$$

and

$$A = \tfrac{64}{3} - \tfrac{8}{3} = \tfrac{56}{3}.$$

Exercises

1. Find the area bounded by the X-axis, the lines $x = 1$ and $x = 2$, and

(a) $y = mx$.

(b) $y = x^3$.

(c) $y = 2x^2 + 3x + 1$.

2. Find the area between the curves $y = x^2$ and $x = y^2$; $y^2 = a(a - x)$ and $y = a - x$; $y^2 = 4x$ and $y = 2x$; $y^2 = x^3$ and $y = x$.

3. Find the area bounded by $y = \sqrt{x + 1}$, the X-axis, and the ordinates $x = 0$ and $x = 2$.

51. Work Done by a Variable Force. In this section there is given a method of finding the work done by a variable force whose line of action remains unchanged.

Illustrations of such variable forces are:

1. The force of attraction between two masses, m and M, is given by the Newtonian law

$$f(s) = \frac{kMm}{s^2},$$

where s is the distance between the masses and k is a factor of proportionality. Note that the equation is of the form

$$f(s) = \frac{a}{s^2}.$$

2. The force exerted by the enclosed steam on the piston of a steam engine is, after cut-off, a function of the distance of the piston from one end of the cylinder.

3. The force necessary to stretch a bar is a function of the elongation of the bar.

Let AB, Fig. 36, represent a bar of length l, held fast at the left end, A. A force f is applied at its right end and the bar is stretched. It is shown experimentally that up to a certain limit the elongation, s, is proportional to the force applied (Hooke's Law), *i.e.*,

$$f = ks,$$

where k is a constant depending upon the length of the bar, its cross section, and the material of the bar.

The work done by a constant force in producing a certain displacement of its point of application in its line of action is defined as the product of the force by the displacement. In the problem which we are considering the force varies with the displacement. The work cannot be found by multiplying the displacement by the force. Instead it will be found by integrating an expression for the derivative of the work with respect to the displacement.

Let w denote the work done in producing the displacement from $s = a$ to a variable position $s = s$. Let Δw denote the work done in producing the additional displacement Δs. Let f denote the force acting at s, and $f + \Delta f$ the force acting at $s + \Delta s$. Δf

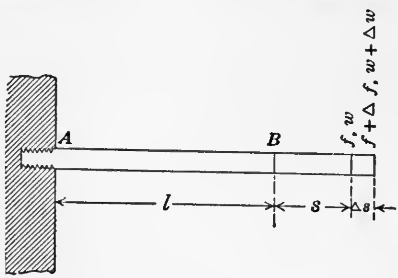

FIG. 36.

may be positive or negative according as the force increases or decreases with distance. For definiteness suppose Δf positive.

In producing the displacement Δs the force varies from f to $f + \Delta f$, and hence the work Δw lies between $f\Delta s$ and $(f + \Delta f)\Delta s$, which represent the work which would have been done had the forces f and $f + \Delta f$, respectively, acted through the distance Δs.

Hence

$$f\Delta s < \Delta w < (f + \Delta f)\Delta s,$$

or

$$f < \frac{\Delta w}{\Delta s} < f + \Delta f.$$

As Δs approaches zero, Δf approaches zero, and we obtain,

$$\frac{dw}{ds} = f. \qquad (1)$$

Integration gives

$$w = F(s) + C, \tag{2}$$

where $F(s)$ is a function whose derivative is f.

When $s = a$, $w = 0$. (2) gives $C = -F(a)$ and,

$$w = F(s) - F(a). \tag{3}$$

This represents the work done in the displacement from $s = a$ to $s = s$. The work, W, done in the displacement from $s = a$ to $s = b$ is obtained by substituting b for s in (3).

$$\mathbf{W = F(b) - F(a).} \tag{4}$$

Illustration 1. Find the work done in stretching a spring from a length of 20 inches to a length of 22 inches, if the length of the spring is 18 inches when no force is applied and if a force of 30 pounds is necessary to stretch it from a length of 18 inches to a length of 19 inches.

Denote the elongation of the spring by s. In accordance with Hooke's Law,

$$f = ks.$$

Since $s = 1$ when $f = 30$, $k = 30$ and $f = 30s$
Substituting in equation (1),

$$\frac{dw}{ds} = 30s.$$

$$w = 15s^2 + C.$$

The problem is to find the work done in changing the elongation from $s = 2$ to $s = 4$. When $s = 2$, $w = 0$. Hence $C = -60$, and

$$w = 15s^2 - 60.$$

The required work, W, is found by giving to s the value 4.

$$W = 240 - 60 = 180.$$

Thus the work done is 180 inch-pounds, or 15 foot pounds.

Illustration 2. Two masses M and m are supposed concentrated at the points A and B, respectively. Find the work done against the force of attraction in moving the mass m along the line AB from a distance a to a distance b from the mass M, the latter mass being fixed.

If f is the force of attraction between the two masses,

$$f = \frac{kmM}{s^2}.$$

Then

$$\frac{dw}{ds} = \frac{kmM}{s^2}.$$

From which

$$w = -\frac{kmM}{s} + C.$$

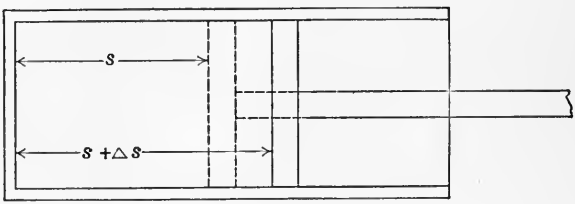

FIG. 37.

When $s = a$, $w = 0$. Hence

$$C = \frac{kmM}{a},$$

and

$$w = kmM\left[\frac{1}{a} - \frac{1}{s}\right].$$

To find the required work, W, let $s = b$.

$$W = kmM\left[\frac{1}{a} - \frac{1}{b}\right].$$

Illustration 3. Gas is enclosed in a cylinder, one end of which is closed by a movable piston. Find the work done by the gas in expanding in accordance with the law $pv^{1.4} = K$, from a volume of 3 cubic feet at a pressure of 15,000 pounds per square foot to a volume of 4 cubic feet.

Let A be the area of the cross section of the cylinder. Then pA is the force on the piston, Fig. 37, and

$$\frac{dw}{ds} = pA,$$

or

$$\frac{dw}{ds} = \frac{AK}{v^{1.4}} = \frac{AK}{(As)^{1.4}} = \frac{K}{A^{0.4}}\frac{1}{s^{1.4}}.$$

Integration gives

$$w = -\frac{1}{0.4}\frac{K}{A^{0.4}}\frac{1}{s^{0.4}} + C$$

$$= -\frac{1}{0.4}\frac{K}{v^{0.4}} + C.$$

When $v = 3$, $w = 0$. Hence

$$C = \frac{1}{0.4}\frac{K}{3^{0.4}},$$

and

$$w = \frac{K}{0.4}\left[\frac{1}{3^{0.4}} - \frac{1}{v^{0.4}}\right],$$

$$W = \frac{K}{0.4}\left[\frac{1}{3^{0.4}} - \frac{1}{4^{0.4}}\right].$$

When $v = 3$, $p = 15{,}000$. Hence $K = (15{,}000)(3^{1.4})$, and

$$W = 150{,}000[0.75 - (0.75)^{1.4}]$$
$$= 12{,}230.$$

Exercises

1. A spring is 12 inches long and a force of 120 pounds is necessary to stretch it from its original length, 12 inches, to a length of 14 inches. Find the work done in stretching the spring from a length of 13 inches to a length of 15 inches.

2. In the case of a bar under tension, Fig. 36, the relation between the stretching force, f, the original length of the bar, l, and the elongation of the bar, s, is given by

$$f = \frac{EAs}{l},$$

where E is the modulus of elasticity of the material of the bar and A is the area of the cross section of the bar. Find the work done in stretching a round iron rod $\frac{1}{2}$ inch in diameter and $4\frac{1}{2}$ feet long to a length of 54.5 inches, given that $E = 3 \cdot 10^7$ pounds per square inch.

3. A spherical conductor, A, is charged with positive electricity and a second spherical conductor, B, with negative electricity. The force of attraction between them varies inversely as the square of the distance between their centers. If the force is 10 dynes when the centers are 100 centimeters apart, find the work done by the force of attraction in changing the distance between the centers from 140 centimeters to 120 centimeters.

4. Find the work done by a gas in expanding in accordance with the law $pv^{\frac{4}{3}} = C$ from a volume of 5 cubic feet to one of 6 cubic feet, if $p = 70$ pounds per square inch when $v = 5$ cubic feet.

5. Find the work done in compressing a spring 6 inches long to a length of $5\frac{1}{2}$ inches if a force of 2000 pounds is necessary to compress it to a length of 5 inches.

6. The work done by a variable force can be represented graphically as the area under a curve whose ordinates represent the force. Construct the figures and prove this fact for Illustrations 1, 2, and 3.

52. Parabolic Cable. Suppose a cable, AOB, Fig. 38, a, is loaded uniformly and continuously along the horizontal, *i.e.*, so that any segment of the cable sustains a weight proportional to

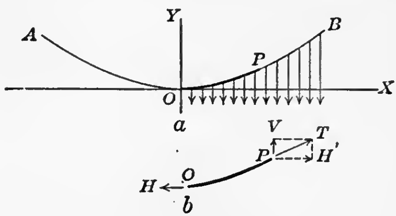

FIG. 38.

the projection of the segment upon a horizontal line. Let k be the weight carried by a portion of the cable whose horizontal projection is one unit of length.

Choose O, the lowest point of the cable, as origin and a horizontal line through O as axis of x. Let P be any point on the cable. Suppose the portion OP of the cable cut free, Fig. 38, b. To keep this portion in equilibrium a horizontal force H and an inclined force T must be introduced at the points O and P, respectively. The force H must be equal in magnitude to the tension in the cable at O, and it must act in the direction of the tangent line at that point. Similarly, the force T must be equal to the tension in the cable at the point P and act in the direction of the tangent line. The force T can be resolved into its vertical and horizontal components V and H', respectively. Now H and H' are the only horizontal components of the forces acting on OP and, since OP is in equilibrium, they must balance each other. Therefore,

$$H = H'. \tag{1}$$

Hence the horizontal component of the tension in the cable is independent of the point P, *i.e.*, it is a constant.

In like manner the only vertical components of the forces acting on OP are the weight kx supported by OP, acting downward, and V, the vertical component of T. They must balance one another. Hence

$$V = kx. \tag{2}$$

The slope of the tangent line to the curve at the point P is $\frac{V}{H'}$. Then

$$\frac{dy}{dx} = \frac{V}{H'} = \frac{kx}{H}. \tag{3}$$

This is the slope of the curve at any point. On integrating we obtain the equation of the curve apart from the arbitrary constant C.

$$y = \frac{kx^2}{2H} + C. \tag{4}$$

C is determined by the condition that $y = 0$ when $x = 0$. Then $C = 0$, and (4) becomes

$$y = \frac{kx^2}{2H}. \tag{5}$$

This is the equation of a parabola with its vertex at the origin.

53. Acceleration.[1] In §38 acceleration was defined as the time rate of change of velocity, *i.e.*, as the derivative of the velocity with respect to the time. But velocity is the derivative of distance with respect to time. Hence the acceleration is the second derivative of the distance with respect to the time. If s denotes the distance and t the time, the acceleration is expressed by $\frac{d^2s}{dt^2}$.

In the case of a freely falling body

$$\frac{d^2s}{dt^2} = g. \tag{1}$$

[1] The statements in this section refer to motion in a straight line.

The relation between s and t can be found from this differential equation by integrating twice, as follows:

The first integration gives

$$\frac{ds}{dt} = gt + C_1, \tag{2}$$

and the second

$$s = \tfrac{1}{2}gt^2 + C_1t + C_2. \tag{3}$$

Two arbitrary constants of integration are introduced. They can be determined by two conditions. If

$$s = s_0 \tag{4}$$

and

$$v = \frac{ds}{dt} = v_0 \tag{5}$$

when $t = 0$, (2) gives $C_1 = v_0$, and (3) gives $C_2 = s_0$. Then

$$s = \tfrac{1}{2}gt^2 + v_0t + s_0. \tag{6}$$

This result was found in §38 by essentially the same method, where the symbol $\frac{dv}{dt}$ was used instead of $\frac{d^2s}{dt^2}$.

Exercises

1. Solve Exercise 5, §38, by the method used above.

2. Obtain the relation $v = \sqrt{2gs}$ (see §38) directly from the equation

$$\frac{d^2s}{dt^2} = g.$$

HINT. Multiply by $2\frac{ds}{dt}$:

$$2\frac{ds}{dt}\frac{d^2s}{dt^2} = 2g\frac{ds}{dt}.$$

The first member is the derivative of $\left(\frac{ds}{dt}\right)^2$ with respect to t and the second that of $2gs$. We then have

$$\left(\frac{ds}{dt}\right)^2 = 2gs + C.$$

Determine C by the condition that $v = 0$ when $s = 0$.

54. The Path of a Projectile. An interesting application of integration is to find the equation of the path of a projectile, a baseball for instance, thrown with a given velocity at a given inclination to the horizontal.

Let O, Fig. 39, be the point from which the ball is thrown. Take this point as the origin of a system of rectangular coördinates. Let the ball be thrown so that its direction at the instant of leaving the hand makes an angle α with the horizontal, and let the initial velocity of the ball be v_0. Then the horizontal component of the initial velocity is $v_0 \cos \alpha$, while its vertical component is $v_0 \sin \alpha$. That is, at the instant the ball is thrown its x-coördinate is increasing at the rate of $v_0 \cos \alpha$ feet per second. Similarly the initial rate of change of the y-coördinate is $v_0 \sin \alpha$.

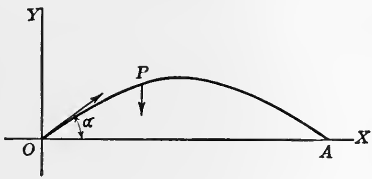

Fig. 39.

At the end of t seconds after the ball was thrown it is at the point P whose coördinates are x and y. *If the resistance of the air is neglected* there is no force acting on the ball tending to change the component of its velocity parallel to the X-axis. Hence the x-component of the velocity is at all times the same as at the beginning, viz., $v_0 \cos \alpha$. The x-component of the velocity is also $\frac{dx}{dt}$, viz., the time rate of change of the abscissa of the ball. Therefore we can write

$$\frac{dx}{dt} = v_0 \cos \alpha. \tag{1}$$

From which on integration

$$x = (v_0 \cos \alpha)t + C. \tag{2}$$

Time is counted from the instant the ball was thrown. The condition for determining C is then that $x = 0$ when $t = 0$. It follows that $C = 0$, and (2) becomes

$$x = (v_0 \cos \alpha)t. \tag{3}$$

This equation gives the x-coördinate of the ball at any time t.

In the vertical direction, the force of gravity acts to change the y-component of the velocity.

Then

$$\frac{d^2y}{dt^2} = -g. \tag{4}$$

The negative sign is used since the force of gravity causes the velocity in the direction of the positive Y-axis to decrease. Integration gives

$$\frac{dy}{dt} = -gt + C_2. \tag{5}$$

C_2 is determined by the condition that $\frac{dy}{dt} = v_0 \sin \alpha$ when $t = 0$. Then $C_2 = v_0 \sin \alpha$ and (5) becomes

$$\frac{dy}{dt} = -gt + v_0 \sin \alpha. \tag{6}$$

Integrating again,

$$y = -\tfrac{1}{2}gt^2 + (v_0 \sin \alpha)t + C_3. \tag{7}$$

Since $y = 0$ when $t = 0$, $C_3 = 0$, and (7) becomes

$$y = -\tfrac{1}{2}gt^2 + (v_0 \sin \alpha)t. \tag{8}$$

This is the y-coördinate of the ball at any time t. Equations (3) and (8) are the parametric equations of the path of the ball. The elimination of t between these equations gives the rectangular equation of the path,

$$y = -\frac{g}{2v_0^2 \cos^2 \alpha}x^2 + x \tan \alpha. \tag{9}$$

This is the equation of a parabola with its vertex at the point,

$$\left[\frac{v_0^2 \sin 2\alpha}{2g}, \frac{v_0^2 \sin^2 \alpha}{2g}\right].$$

It is to be remembered that in the solution of this problem the resistance of the air was neglected. Consequently the results obtained can be regarded only as approximations. Experimentally it has been shown that the resistance of the air increases with the velocity of the moving body. For low velocities the resistance is assumed to vary as the first power of the velocity, but for higher velocities, such as are attained by rifle balls, the resistance is assumed to vary with the second power, and the results obtained above cannot be considered to be even approximations.

Exercise

1. Find the angle of elevation, α, at which the ball must be thrown to make the range, OA, Fig. 39, a maximum.

CHAPTER VII

INFINITESIMALS, DIFFERENTIALS, DEFINITE INTEGRALS

55. Infinitesimals. *In* §23 *an infinitesimal was defined as a variable which approaches the limit zero.* Thus, $x^2 - 1$, as x approaches 1, is an infinitesimal.

It is to be noted that a variable is thought of as an infinitesimal only when it is in the state of approaching zero. Thus $x^2 - 1$ is an infinitesimal only when x approaches $+1$ or -1. An infinitesimal has two characteristic properties: (1) It is a variable. (2) It approaches the limit zero; *i.e.*, the conditions of the problem are such that the numerical value of the variable can be made less than any preassigned positive number, however small.

This meaning of the word infinitesimal in mathematics is entirely different from its meaning in everyday speech. When we say in ordinary language that a quantity is infinitesimal, we mean that it is very small. But it is a constant magnitude and not one whose numerical measure can be made less than any preassigned positive number, however small. Thus, 0.000001 of a milligram of salt might be spoken of as an infinitesimal quantity of salt, but the number 0.000001 is clearly not an infinitesimal in the sense of the mathematical definition. On the other hand, if we have a solution containing a certain amount of salt per cubic centimeter and allow pure water to flow into the vessel containing the solution while the solution flows off through an overflow pipe, the quantity of salt per cubic centimeter constantly diminishes. The amount of salt left in solution after a time t is then an infinitesimal, as t becomes infinite.

Infinitesimals are of fundamental importance in the Calculus. The derivative, which we have already used in studying functions, is the limit of the ratio of two infinitesimals, Δy and Δx.

56. $\lim\limits_{\alpha \doteq 0} \dfrac{\sin \alpha}{\alpha}$. Let the arc AB, Fig. 40, subtend an angle α

at the center, O, of a circle of radius r. The angle α is measured in radians. Let AT be tangent to the circle at A, and let BC be perpendicular to OA. The area of the triangle OCB is less than the area of the circular sector OAB, and this in turn is less than the area of the triangle OAT.

$$\tfrac{1}{2}(BC)(OC) < \tfrac{1}{2}\alpha r^2 < \tfrac{1}{2}(AT)r$$

$$\frac{BC}{r}\frac{OC}{r} < \alpha < \frac{AT}{r}$$

$$\frac{OC}{r}\sin\alpha < \alpha < \tan\alpha \qquad (1)$$

$$\frac{OC}{r} < \frac{\alpha}{\sin\alpha} < \frac{1}{\cos\alpha}. \qquad (2)$$

As the angle α approaches zero, OC approaches r and $\dfrac{OC}{r}$ approaches 1, and further, $\cos\alpha$ approaches 1. Hence the first and last members of the inequalities (2) approach the same limit, 1. Then the second member, $\dfrac{\alpha}{\sin\alpha}$, which lies between them, must approach the same limit, 1. Therefore

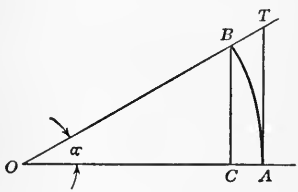

Fig. 40.

$$\lim_{\alpha \doteq 0} \frac{\sin\alpha}{\alpha} = \lim_{\alpha \doteq 0} \frac{1}{\dfrac{\alpha}{\sin\alpha}} = 1. \qquad (3)$$

57. $\displaystyle\lim_{\alpha \doteq 0} \frac{\tan\alpha}{\alpha}, \quad \lim_{\alpha \doteq 0} \frac{\tan\alpha}{\sin\alpha}.$

$$\lim_{\alpha \doteq 0} \frac{\tan\alpha}{\alpha} = \lim_{\alpha \doteq 0} \left(\frac{\sin\alpha}{\alpha}\,\frac{1}{\cos\alpha}\right)$$

$$= \left(\lim_{\alpha \doteq 0} \frac{\sin\alpha}{\alpha}\right)\left(\lim_{\alpha \doteq 0} \frac{1}{\cos\alpha}\right)$$

$$= 1. \qquad (1)$$

$$\lim_{\alpha \doteq 0} \frac{\tan\alpha}{\sin\alpha} = \lim_{\alpha \doteq 0} \frac{1}{\cos\alpha} = 1. \qquad (2)$$

58. $\displaystyle\lim_{\alpha \doteq 0} \frac{1 - \cos\alpha}{\alpha}.$

Since

$$\frac{1-\cos\alpha}{\alpha} = \frac{2\sin^2\frac{\alpha}{2}}{\alpha} = \frac{\sin\frac{\alpha}{2}}{\frac{\alpha}{2}}\sin\frac{\alpha}{2},$$

$$\lim_{\alpha\doteq 0}\frac{1-\cos\alpha}{\alpha} = \left(\lim_{\alpha\doteq 0}\frac{\sin\frac{\alpha}{2}}{\frac{\alpha}{2}}\right)\left(\lim_{\alpha\doteq 0}\sin\frac{\alpha}{2}\right).$$

Hence

$$\lim_{\alpha\doteq 0}\frac{1-\cos\alpha}{\alpha} = 0. \tag{1}$$

In Fig. 40, AB, AC, AT, BT, and BC are infinitesimals as α approaches zero. Then,

from (3), §56, $\lim_{\alpha\doteq 0}\frac{BC}{AB} = 1,$

from (1), §57, $\lim_{\alpha\doteq 0}\frac{AT}{AB} = 1,$

from (2), §57, $\lim_{\alpha\doteq 0}\frac{AT}{BC} = 1,$

from (1), §58, $\lim_{\alpha\doteq 0}\frac{AC}{AB} = 0.$

59. Order of Infinitesimals. Consider the infinitesimals x^2 and x as x approaches zero. The ratio of x^2 to x is x, which is itself an infinitesimal. The infinitesimals x^2 and x are represented, Fig. 41, by the ordinates MP and MN, to the curves $y = x^2$ and $y = x$. The quotient

$$\frac{x^2}{x} = \frac{MP}{MN}$$

is a measure of the relative magnitude of these infinitesimals as they approach zero. It shows that MP becomes small so much more rapidly than MN that the limit of their quotient is zero.

On the other hand, the infinitesimals $2x$ and x behave very differently. Their quotient is $\frac{2x}{x} = 2$, and the limit of this quotient is 2. In this case the limit of the ratio of the infinitesimals is not zero. (See Fig. 42.)

Again,

$$\lim_{\alpha \doteq 0} \frac{1 - \cos \alpha}{\alpha} = 0,$$

while

$$\lim_{\alpha \doteq 0} \frac{\sin \alpha}{\alpha} = 1.$$

These illustrations of the comparison of two infinitesimals lead to the following definitions of the order of one infinitesimal with respect to another.

Two infinitesimals, α and β, are said to be of the same order if the limit of $\frac{\alpha}{\beta}$ is a finite number not zero.

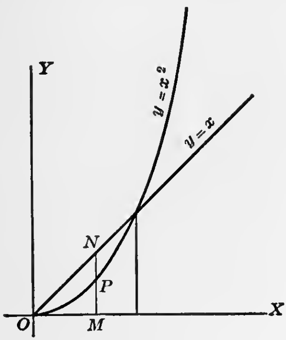

FIG. 41.

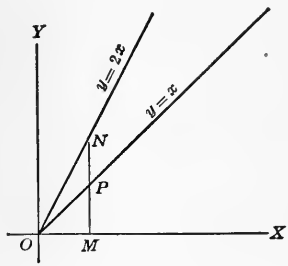

FIG. 42.

If the limit of $\frac{\alpha}{\beta}$ is zero, α is said to be of higher order than β.

Thus, $2x$ and x are of the same order; x^2 is an infinitesimal of higher order than x; $\sin \alpha$ and α, or CB and AB, Fig. 40, are of the same order; $\tan \alpha$ and α, or AT and AB, Fig. 40, are of the same order; $\tan \alpha$ and $\sin \alpha$, or CB and AT, Fig. 40, are of the same order; $1 - \cos \alpha$ is of higher order than α, or CA, Fig. 40, is of higher order than AB.

Let ACB, Fig. 43, be a right angle inscribed in a semicircle. Let BD be a tangent line, and let CE be perpendicular to BD. If the angle CAB approaches zero, BC, CD, BE, ED, CE, and arc BC are infinitesimals. From similar triangles

$$\frac{AB}{AC} = \frac{BC}{BE} = \frac{BD}{BC}.$$

Since $\lim \frac{AB}{AC} = 1$, it follows that

$$\lim \frac{BC}{BE} = \lim \frac{BD}{BC} = 1.$$

Hence BC and BE, and BD and BC are infinitesimals of the same order.

Again,

$$\frac{BC}{AB} = \frac{CE}{BC} = \frac{CD}{BD}.$$

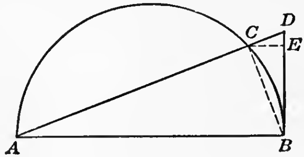

FIG. 43.

Since $\lim \frac{BC}{AB} = 0$,

$$\lim \frac{CE}{BC} = \lim \frac{CD}{BD} = 0.$$

Hence CE is an infinitesimal of higher order than BC, and CD is an infinitesimal of higher order than BD.

Again,

$$\frac{CB}{AC} = \frac{CE}{BE} = \frac{CD}{CB}.$$

Since $\lim \frac{CB}{AC} = 0$,

$$\lim \frac{CE}{BE} = \lim \frac{CD}{CB} = 0.$$

Hence CE is an infinitesimal of higher order than BE, and CD is an infinitesimal of higher order than CB.

Exercises

1. Show that $x - 2x^2$ and $3x + x^3$ are infinitesimals of the same order as x approaches zero.

2. Show that $1 - \sin\theta$ and $\cos^2\theta$ are infinitesimals of the same order as θ approaches $\frac{\pi}{2}$.

3. Show that $1 - \sin\theta$ is an infinitesimal of higher order than $\cos\theta$ as θ approaches $\frac{\pi}{2}$.

4. Show that $\sec\alpha - \tan\alpha$ is an infinitesimal as α approaches $\frac{\pi}{2}$.

5. Show that $1 - \sin\alpha$ is an infinitesimal of higher order than $\sec\alpha - \tan\alpha$ as α approaches $\frac{\pi}{2}$.

6. Show that $1 - \cos\theta$ is an infinitesimal of the same order as θ^2 as θ approaches zero.

7. Show that $\lim\limits_{\theta \doteq 0} \dfrac{\sin\theta - \theta}{\theta} = 0.$

8. Show that $\lim\limits_{\theta \doteq 0} \dfrac{\sin\theta - \theta}{\sin\theta} = 0.$

9. Show that $\lim\limits_{\theta \doteq 0} \dfrac{\tan\theta - \theta}{\theta} = 0.$

10. Show that $\lim\limits_{\theta \doteq 0} \dfrac{\tan\theta - \theta}{\tan\theta} = 0.$

11. Show that $\lim\limits_{\alpha \doteq 0} \dfrac{\sin\alpha - \tan\alpha}{\tan\alpha} = 0.$

12. Show that $\lim\limits_{\alpha \doteq 0} \dfrac{\sin\alpha - \tan\alpha}{\sin\alpha} = 0.$

60. Theorem. *The limit of the quotient of two infinitesimals, α and β, is not altered if they are replaced by two other infinitesimals, γ and δ, respectively, such that lim $\frac{\alpha}{\gamma} = 1$ and lim $\frac{\beta}{\delta} = 1$.*

Proof:

$$\frac{\alpha}{\beta} = \frac{\gamma \frac{\alpha}{\gamma}}{\delta \frac{\beta}{\delta}},$$

$$\lim \frac{\alpha}{\beta} = \frac{\lim \frac{\alpha}{\gamma}}{\lim \frac{\beta}{\delta}} \lim \frac{\gamma}{\delta} = \lim \frac{\gamma}{\delta},$$

since

$$\lim \frac{\alpha}{\gamma} = \lim \frac{\beta}{\delta} = 1.$$

It is evident from the proof that the limit of the quotient is unaltered if only one of the infinitesimals, say α, is replaced by another infinitesimal γ, such that $\lim \frac{\gamma}{\alpha} = 1$.

Illustrations.

1. Since

$$\lim_{\alpha \doteq 0} \frac{\sin \alpha}{\alpha} = 1,$$

$$\lim_{\alpha \doteq 0} \frac{1 - \cos \alpha}{\sin \alpha} = \lim_{\alpha \doteq 0} \frac{1 - \cos \alpha}{\alpha} = 0.$$

2. Since

$$\lim_{\alpha \doteq 0} \frac{\tan \alpha}{\alpha} = 1,$$

$$\lim_{\alpha \doteq 0} \frac{1 - \cos \alpha}{\tan \alpha} = \lim_{\alpha \doteq 0} \frac{1 - \cos \alpha}{\alpha} = 0.$$

3. In Fig. 40,

$$\lim_{\alpha \doteq 0} \frac{CA}{AB} = \lim_{\alpha \doteq 0} \frac{CA}{BC} = \lim_{\alpha \doteq 0} \frac{CA}{AT} = 0.$$

Exercises

1. Show that $\lim_{\alpha \doteq 0} \frac{1 - \cos \alpha}{\alpha^2} = \frac{1}{2}$.

Hint. $1 - \cos \alpha = 2 \sin^2 \frac{\alpha}{2}$.

2. Show that $\lim_{\alpha \doteq 0} \frac{\sin \alpha (1 - \cos \alpha)}{\alpha^3} = \frac{1}{2}$.

3. Show that $\lim_{\alpha \doteq 0} \frac{(\alpha - 5)^2 \sin \alpha}{\alpha} = 25$.

4. Show that $\lim_{\alpha \doteq 0} \frac{1 - \cos \alpha}{\cos \alpha \sin^2 \alpha} = \frac{1}{2}$.

5. Show that $\lim_{x \doteq 0} \frac{3x^2 - 4x^3}{2x^2 - 5x^4} = \frac{3}{2}$.

Hint. Replace numerator by $3x^2$ and denominator by $2x^2$.

6. Show that $\lim_{x \doteq \infty} \frac{\frac{2}{x^2} + \frac{3}{x^4}}{\frac{1}{x^2} + \frac{4}{x^3}} = \lim_{x \doteq \infty} \frac{\frac{2}{x^2}}{\frac{1}{x^2}} = 2$.

61. Differentials. Let PT, Fig. 44, be a tangent line drawn to the curve $y = f(x)$ at the point P. Let $DE = \Delta x$, $RQ = \Delta y$, and let angle $RPT = \tau$.

From the figure,

$$\frac{RM}{\Delta x} = \tan \tau = f'(x),$$

or

$$RM = f'(x) \cdot \Delta x.$$

This is the increment which the function would take on if it were to change uniformly at a rate equal to that which it had at P.

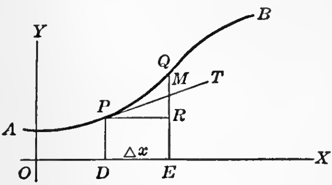

FIG. 44.

This quantity, $f'(x)\Delta x$, is called differential y, and is denoted by dy. Its defining equation is

$$dy = f'(x)\Delta x. \qquad (1)$$

Δx, *the increment of the independent variable*, is called differential x and is denoted by dx, *i.e.*, $\Delta x = dx$. Equation (1) becomes

$$dy = f'(x)dx.^{1} \qquad (2)$$

In Fig. 44, $RM = dy$ and $DE = PR = dx$.

In general, dy is not equal to Δy, the difference being MQ, Fig. 44. However, it will be shown that $\lim\limits_{\Delta x \doteq 0} \dfrac{\Delta y}{dy} = 1$.

$$\lim_{\Delta x \doteq 0} \frac{RQ}{PR} = f'(x),$$

or

$$\lim_{\Delta x \doteq 0} \left[\frac{RQ}{RM}\,\frac{RM}{PR}\right] = f'(x). \qquad (3)$$

But, since $\dfrac{RM}{PR}$ is constant and equal to $f'(x)$, equation (3) becomes

$$\lim_{\Delta x \doteq 0} \frac{RQ}{RM} = 1,$$

[1] In the expression (2) for the differential of the function $f(x)$, the first derivative is the coefficient of the differential of the argument, and for this reason it is sometimes called the *differential coefficient.*

or

$$\lim_{\Delta x \doteq 0} \frac{\Delta y}{dy} = 1. \tag{4}$$

It is to be noted that dx is an arbitrary increment and that dy is then determined by this increment and the value of the derivative, *i.e.*, by the slope of the tangent at the point for which the differential is computed. dx and dy are then definite quantities and we can perform on them any algebraic operation. Thus we can divide (2) by dx and obtain

$$\frac{dy}{dx} = f'(x), \tag{5}$$

where dy and dx denote the differentials of y and x, respectively. Thus from the definition of differentials the first derivative may be regarded as the quotient of the differential of y by the differential of x.

It is to be observed, however, that this statement gives no new meaning to the derivative, since the derivative was used in the definition of the differential.

62. Formulas for the Differentials of Functions. In accordance with equation (5) of the preceding section, any formula involving first derivatives can be regarded as a formula in which each first derivative is replaced by the quotient of the corresponding differentials. Thus,

$$\frac{d\left(\frac{u}{v}\right)}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}.$$

Each derivative being considered as a fraction whose denominator is dx, we can multiply by dx, and obtain

$$d\left(\frac{u}{v}\right) = \frac{vdu - udv}{v^2}.$$

In words, *the differential of a fraction is equal to the denominator times the differential of the numerator minus the numerator times the differential of the denominator, all divided by the square of the denominator.* It will be noted that the wording is the same as that for the derivative of a fraction except that throughout the word *differential* replaces the word *derivative.*

The other formulas for derivatives which have been developed are expressed below with the corresponding formulas for differentials.

Formulas

1. $\dfrac{dc}{dx} = 0.$ $\qquad dc = 0.$

2. $\dfrac{d(cu)}{dx} = c\dfrac{du}{dx}.$ $\qquad d(cu) = c\,du.$

3. $\dfrac{d(u+v)}{dx} = \dfrac{du}{dx} + \dfrac{dv}{dx}.$ $\qquad d(u+v) = du + dv.$

4. $\dfrac{du^n}{dx} = nu^{n-1}\dfrac{du}{dx}.$ $\qquad du^n = nu^{n-1}\,du.$

5. $\dfrac{d(uv)}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}.$ $\qquad d(uv) = u\,dv + v\,du.$

6. $\dfrac{d\left(\dfrac{u}{v}\right)}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}.$ $\qquad d\left(\dfrac{u}{v}\right) = \dfrac{v\,du - u\,dv}{v^2}.$

7. $\dfrac{d\left(\dfrac{c}{v}\right)}{dx} = -\dfrac{c\dfrac{dv}{dx}}{v^2}.$ $\qquad d\left(\dfrac{c}{v}\right) = -\dfrac{c\,dv}{v^2}.$

8. $\dfrac{d\left(\dfrac{c}{v^n}\right)}{dx} = -\dfrac{cn\dfrac{dv}{dx}}{v^{n+1}}.$ $\qquad d\left(\dfrac{c}{v^n}\right) = -\dfrac{cn\,dv}{v^{n+1}}.$

9. $\dfrac{du^{\frac{1}{2}}}{dx} = \dfrac{\dfrac{du}{dx}}{2u^{\frac{1}{2}}}.$ $\qquad du^{\frac{1}{2}} = \dfrac{du}{2u^{\frac{1}{2}}}.$

The formula for the differential of $y = cu^n$ can be put in the following convenient form:

10. $\dfrac{dy}{y} = n\dfrac{du}{u},$

which is obtained directly by dividing $dy = cnu^{n-1}du$ by $y = cu^n$.

The process of finding either the derivative or the differential of a function is called differentiation.

The process of finding a function when its derivative or differential is given is called integration.

We have no symbol representing integration when applied to derivatives. The symbol for integration when applied to differentials is $\int$. Thus $\int 3x^2dx = x^3 + C$. The origin of this symbol will be explained later. It is read "*integral of.*"

Illustrations.

1. If $y = \sqrt{1 - x^2}$,

$$dy = \tfrac{1}{2}(1 - x^2)^{-\frac{1}{2}}(-2xdx)$$
$$= -\frac{xdx}{\sqrt{1 - x^2}}.$$

By formula 10, where $u = 1 - x^2$,

$$\frac{dy}{y} = \frac{1}{2}\,\frac{-2x\,dx}{(1 - x^2)}$$
$$= -\frac{x\,dx}{1 - x^2}.$$

2. If $y = \dfrac{x}{x^2 - 1}$,

$$dy = \frac{(x^2 - 1)dx - x\,d(x^2 - 1)}{(x^2 - 1)^2}$$
$$= \frac{(x^2 - 1)\,dx - x\,(2x\,dx)}{(x^2 - 1)^2}$$
$$= -\frac{(x^2 + 1)dx}{(x^2 - 1)^2}.$$

3. If $dy = x\,dx$,

$$y = \int x\,dx$$
$$= \tfrac{1}{2}\int 2x\,dx$$
$$= \frac{x^2}{2} + C.$$

4. If $dy = x\sqrt{1 - x^2}\,dx$,

$$y = \int x\,(1 - x^2)^{\frac{1}{2}}dx$$
$$= -\tfrac{1}{2}\,\tfrac{2}{3}\int \tfrac{3}{2}\,(1 - x^2)^{\frac{1}{2}}\,(-2x\,dx)$$
$$= -\frac{(1 - x^2)^{\frac{3}{2}}}{3} + C.$$

5. If $\frac{dy}{y} = \frac{dx}{x-1}$,

$$y = C(x-1),$$

by formula 10.

6. If $\frac{dy}{y} = \frac{x\,dx}{x^2-1}$,

$$\frac{dy}{y} = \frac{1}{2}\frac{2x\,dx}{x^2-1}$$

$$y = C\sqrt{x^2-1}.$$

Exercises

Find dy in the following ten exercises:

1. $y = x^2 - 3x - 2.$

2. $y = \frac{x}{x-1}.$

3. $y = x^{\frac{1}{3}} - x^{-\frac{2}{3}} - 3x.$

4. $y = (x-2)^{\frac{1}{2}}.$

5. $y = (x^2-2)^{\frac{1}{3}}.$

6. $y = \frac{\sqrt{x}}{(x-1)^2}.$

7. $y = (x-1)(x^2-1)^2.$

8. $y = (x^2+x-2)^3.$

9. $y = (x-1)^{-\frac{1}{2}}.$

10. $y = (x^2-1)^{-\frac{3}{2}}.$

Integrate the following:

11. $\int x^2\,dx.$

12. $\int (x^2-1)\,x\,dx.$

13. $\int (x^3-3x+5)\,(x^2-1)\,dx.$

14. $\int (x^2-2x-6)^3\,(x-1)\,dx.$

15. $\int \frac{dx}{x^2}.$

16. $\int \sqrt{x}\,dx.$

17. $\int \frac{dx}{\sqrt{x}}.$

18. $\int x^{\frac{1}{3}}dx.$

19. $\dfrac{dy}{y} = \dfrac{dx}{x}.$

20. $\dfrac{dy}{y} = \dfrac{x\,dx}{x^2 - 1}.$

21. $\dfrac{dy}{y} = \dfrac{(x^2 - 2x + 4)dx}{x^3 - 3x^2 + 12x - 2}.$

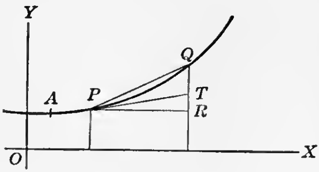

Fig. 45.

63. Differential of Length of Arc: Rectangular Coördinates. Let PR, Fig. 45, $= \Delta x$, $RQ = \Delta y$, the chord $PQ = \Delta c$, and the arc $PQ = \Delta s$. (s represents the length of arc measured from some point A.) PT is the tangent at P.

We have

$$(\Delta c)^2 = (\Delta x)^2 + (\Delta y)^2$$

$$\lim_{\Delta x \doteq 0} \left(\frac{\Delta c}{\Delta x}\right)^2 = 1 + \lim_{\Delta x \doteq 0} \left(\frac{\Delta y}{\Delta x}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2.$$

Since[1]

$$\lim_{\Delta x \doteq 0} \frac{\Delta c}{\Delta s} = 1,$$

[1] When Δx is taken so small that the curve has no point of inflection between P and Q, the chord $PQ <$ arc $PQ < PT + TQ$, or $\Delta c < \Delta s < PT + TQ$. Whence,

$$1 < \frac{\Delta s}{\Delta c} < \frac{PT}{\Delta c} + \frac{TQ}{\Delta c}. \qquad (1)$$

$$\left(\frac{PT}{\Delta c}\right)^2 = \frac{(dx)^2 + (dy)^2}{(\Delta x)^2 + (\Delta y)^2} = \frac{1 + \left(\frac{dy}{dx}\right)^2}{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}.$$

Therefore

$$\lim_{\Delta x \doteq 0} \left(\frac{PT}{\Delta c}\right)^2 = 1.$$

$$\lim_{\Delta x \doteq 0} \frac{TQ}{\Delta c} = \lim_{\Delta x \doteq 0} \frac{\Delta y - dy}{\Delta y} \frac{\Delta y}{\Delta c}$$

$$= \left[\lim_{\Delta x \doteq 0}\left(1 - \frac{dy}{\Delta y}\right)\right]\left[\lim_{\Delta x \doteq 0} \frac{\Delta y}{\Delta c}\right] = 0,$$

since

$$\lim_{\Delta x \doteq 0} \frac{dy}{\Delta y} = 1$$

Then from (1),

$$\lim_{\Delta x \doteq 0} \frac{\Delta s}{\Delta c} = 1.$$

Δc can be replaced by Δs (§60).

$$\lim_{\Delta x \doteq 0} \left(\frac{\Delta s}{\Delta x}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2,$$

or

$$\left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2.$$

$$(ds)^2 = (dx)^2 + (dy)^2 \qquad (1)$$

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \qquad (2)$$

$$ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy \qquad (3)$$

Equation (1) shows that the line PT, Fig. 45. represents ds. If τ denotes the angle made by the line PT with the positive X-axis,

$$dx = \cos\tau\, ds$$

$$dy = \sin\tau\, ds.$$

Illustration. Find the length of the curve $y = \frac{2}{3}x^{\frac{3}{2}}$ between the points whose abscissas are 3 and 8.

$$\frac{dy}{dx} = x^{\frac{1}{2}}$$

$$\left(\frac{dy}{dx}\right)^2 = x.$$

Substituting in formula (2),

$$ds = \sqrt{1 + x}\, dx.$$

Integrating,

$$s = \tfrac{2}{3}(1 + x)^{\frac{3}{2}} + C.$$

When $x = 3$, $s = 0$. Hence $C = -\frac{16}{3}$, and

$$s = \tfrac{2}{3}(1 + x)^{\frac{3}{2}} - \tfrac{16}{3}.$$

This formula gives the length of the curve measured from the point whose abscissa is 3 to the point whose abscissa is x. On placing $x = 8$ we obtain $s = \frac{38}{3}$, the length of the curve from the point corresponding to $x = 3$ to the point corresponding to $x = 8$.

Exercises

Find the differentials of the length of the following curves:

1. $y = x^3$.
2. $x^2 + y^2 = 4$.
3. $y = x^2$.
4. $y^2 = x$.
5. $3x^2 + 4y^2 = 12$.
6. $xy = 1$.
7. $xy^2 = 1$.
8. $y = x^{-\frac{3}{2}}$

64. The Limit of $\Sigma f(x)\Delta x$. Let $y = f(x)$ be a continuous function between $x = a$ and $x = b$. In §50 it was shown that the area bounded by the curve, the X-axis, and the ordinates $x = a$ and $x = b$ is given by the formula

$$A = F(b) - F(a), \tag{1}$$

where $F(x) = \int f(x)dx$. A second expression will now be found for the area. Divide the interval $b - a$, Fig. 46, into n equal parts and at each point of division erect an ordinate. Complete the rectangles as indicated in the figure.

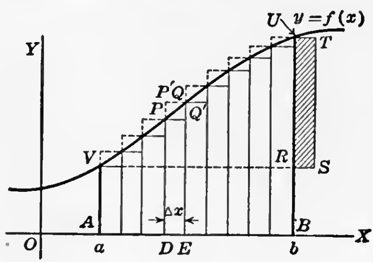

FIG. 46.

The sum of the rectangles of which $DEQ'P$ is a type, is approximately equal to the area $ABUV$. The greater n, the number of rectangles, *i.e.*, the smaller Δx, the closer will the sum of the rectangles approximate the area $ABUV$. We say then that

$$A = \lim_{n \doteq \infty} \Sigma\, DEQ'P,$$

or

$$A = \lim_{\Delta x \doteq 0} \sum_{x=a}^{x=b} f(x)\Delta x. \tag{2}$$

The above expression (2) represents the *actual* area and *not* an approximation to it, as can be shown by finding the greatest possible error corresponding to a given number of rectangles and then proving that this error approaches zero as the number of rectangles becomes infinite. Thus it is easily seen that the difference between the true area A and the sum of the rectangles is less than the area of the rectangle $RSTU$. The altitude, $f(b) - f(a)$, of this rectangle is constant while the length of the base, Δx, approaches zero. Hence the area of $RSTU$ approaches zero. Therefore the limit of the sum of the rectangles is the area sought.

On equating the two expressions for A, given by (1) and (2), we have

$$\mathbf{\lim_{\Delta x \doteq 0} \sum_{x=a}^{x=b} f(x)\Delta x = F(b) - F(a),} \tag{3}$$

where

$$F(x) = \int f(x)dx.$$

This equation is *the important result* of this section. It gives a means of calculating

$$\lim_{\Delta x \doteq 0} \sum_{x=a}^{x=b} f(x)\Delta x.$$

For, to calculate this limit we need only to find the integral of $f(x)dx$ and take the difference between the values of this integral at $x = a$ and $x = b$. The result of this section will be restated and emphasized in the next section.

65. Definite Integral. The expression

$$\lim_{\Delta x \doteq 0} \sum_{x=a}^{x=b} f(x)\Delta x$$

which was introduced in the preceding section is of such great importance that it is given a name, "*the definite integral of* $f(x)$ *between the limits a and b*," *and is denoted by the symbol*

$$\int_a^b f(x)dx.$$

Equation (3), §**64**, gives a means of calculating the value of the definite integral.

The function $F(x)$, the integral of $f(x)dx$, is called the *indefinite integral* of $f(x)dx$ in order to distinguish it from the definite integral which is defined independently of it, viz., as the limit of a certain sum.

We have then the following definition and theorem:

Definition. *Let* $f(x)$ *be a continuous function in the interval from* $x = a$ *to* $x = b$*, and let this interval be divided into n equal parts of length* Δx *by points* $x_1, x_2, x_3, \ldots, x_{n-1}$*. The "definite integral of* $f(x)$ *between the limits a and b" is the limit of the sum of the products* $f(x_i)\,\Delta x$ *formed for all of the points* $x_0 = a, x_1, x_2, \ldots, x_{n-1}$*, as the number of divisions becomes infinite.*

Theorem. *The definite integral of* $f(x)$ *between the limits a and b is calculated by finding the indefinite integral,* $F(x)$*, of* $f(x)dx$ *and forming the difference* $F(b) - F(a)$.

The symbol for the definite integral,

$$\int_a^b f(x)dx,$$

is read "*the integral from a to b of* $f(x)dx$." As we have seen, it means

$$\int_a^b f(x)dx = \lim_{\Delta x \doteq 0} \sum_a^b f(x)\Delta x.$$

Many problems, such as finding the work done by a variable force, the volume of a solid, the coördinates of the center of gravity, lead to definite integrals. But, no matter how a definite integral may have been obtained and no matter what other meaning it may have, it can always be regarded as representing the area included by the curve $y = f(x)$, the X-axis, and the ordinates $x = a$ and $x = b$, provided that $f(x)$ is a function which can be represented by a continuous curve. This fact, that

$$\int_a^b f(x)dx$$

can be regarded as representing an area, enables us to calculate its value. For the area in question is equal to $F(b) - F(a)$, where $F(x)$ is the indefinite integral of $f(x)dx$. Consequently we have, in all cases,

$$\int_a^b f(x)dx = F(b) - F(a).$$

This is often written

$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a),$$

to show how the result is to be calculated. Thus

$$\int_1^2 x^2dx = \frac{x^3}{3}\Big|_1^2 = \frac{2^3}{3} - \frac{1^3}{3} = \frac{7}{3}.$$

Exercises

Evaluate the following definite integrals:

1. $\int_1^3 (2x + 3)dx.$ **2.** $\int_1^4 \frac{dx}{x^2}.$ **3.** $\int_0^a \sqrt{a^2 + x^2}\, xdx.$

66. Duhamel's Theorem. *If $\alpha_1, \alpha_2, \alpha_3, \cdots, \alpha_n$ are n infinitesimals of like sign, the limit of whose sum is finite as n becomes infinite, and if $\beta_1, \beta_2, \beta_3, \cdots, \beta_n$ are a second set of infinitesimals such that*

$$\lim_{n \doteq \infty} \frac{\beta_i}{\alpha_i} = 1,$$

where $i = 1, 2, 3, \cdots, n$, then

$$\lim_{n \doteq \infty} \sum_1^n \alpha_i = \lim_{n \doteq \infty} \sum_1^n \beta_i.$$

PROOF. Let $\frac{\beta_i}{\alpha_i} = 1 + \epsilon_i.$

Since

$$\lim_{n \doteq \infty} \frac{\beta_i}{\alpha_i} = 1,$$

$$\lim_{n \doteq \infty} \epsilon_i = 0.$$

At first let it be assumed that the α's are positive. Let E be the numerical value of the largest ϵ, *i.e.*,

$$E \geqq |\epsilon_i|, \qquad i = 1, 2, 3, \cdots, n.$$

Then, since $\beta_i = \alpha_i + \epsilon_i\alpha_i, \qquad i = 1, 2, 3, \cdots, n,$

$$\alpha_1 - E\alpha_1 \leqq \beta_1 \leqq \alpha_1 + E\alpha_1$$
$$\alpha_2 - E\alpha_2 \leqq \beta_2 \leqq \alpha_2 + E\alpha_2$$
$$\alpha_n - E\alpha_n \leqq \beta_n \leqq \alpha_n + E\alpha_n.$$

Adding, we get

$$(1 - E)\sum_{i=1}^{i=n} \alpha_i \leqq \sum_{i=1}^{i=n} \beta_i \leqq (1 + E)\sum_{i=1}^{i=n} \alpha_i.$$

Since

$$\lim_{n \doteq \infty} E = 0,$$

$$\lim \sum_{i=1}^{i=n} \alpha_i = \lim \sum_{i=1}^{i=n} \beta_i$$

and the theorem is proved.

If the α's are negative, it will be necessary to change the proof just given, only by reversing the signs of inequality.

Section **64** furnishes an illustration of this theorem. In this example the limit of the sum of the infinitesimal trapezoidal areas *DEQP* is finite as n becomes infinite, since it is the area sought.

$$DEQ'P < DEQP < DEQP',$$

(see Fig. 46), or

$$y\Delta x < DEQP < (y + \Delta y)\Delta x,$$

or

$$1 < \frac{DEQP}{DEQ'P} < \frac{y + \Delta y}{y}.$$

This shows that the limit of the ratio of the trapezoidal area to the area of the corresponding rectangle is 1 as n becomes infinite. Then by Duhamel's Theorem,

$$\lim_{n \doteq \infty} \Sigma\, DEQ'P = \lim_{n \doteq \infty} \Sigma\, DEQP = A.$$

Since we are able to replace the infinitesimals $DEQP$ by the infinitesimals $DEQ'P$, we may calculate the area which is the sum of these infinitesimals by means of the definite integral. This is a characteristic process in the use of the definite integral. The quantity sought is subdivided into n portions which are infinitesimals as n becomes infinite. These are replaced by n other infinitesimals of the form $f(x_i)\Delta x$. The limit of the sum of the latter infinitesimals is a definite integral.

Since the limits of the two sums are equal by Duhamel's Theorem, the definite integral is equal to the quantity sought.

Illustrations of the applications of Duhamel's Theorem to obtain definite integrals representing work, force, volume, etc., follow.

67. Work Done by a Variable Force. In §**51** there was found the work done by a variable force, $f(s)$, in producing a displacement

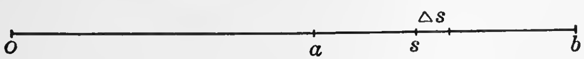

Fig. 47.

from $s = a$ to $s = b$. We shall now obtain the same result by building up the definite integral which represents the work. Divide the total displacement $b - a$, Fig. 47, into n equal parts of length Δs. The force acting at the left end of one of these parts is $f(s)$, while that acting at the right end is $f(s + \Delta s)$. The total work done in producing the displacement, $b - a$, is approximately

$$\sum_{s=a}^{s=b} f(s)\Delta s.$$

The actual work is the limit of this sum as Δs approaches zero.[1]

[1] This step can be justified by using Duhamel's Theorem. Let Δw represent the work done in producing the displacement Δs. Then

Illustration 1. The solution of the problem of Illustration 1, **§51,** is expressed by

$$w = \int_2^4 30s\,ds = 15s^2 \Big|_2^4 = 180.$$

Illustration 2. The solution of the problem of Illustration 2, **§51,** is expressed by

$$w = kmM \int_a^b \frac{ds}{s^2} = -\,kmM\,\frac{1}{s}\Big|_a^b = -\,kmM\left[\frac{1}{b} - \frac{1}{a}\right]$$
$$= kmM\left[\frac{1}{a} - \frac{1}{b}\right].$$

Illustration 3. In solving the problem of Illustration 3, **§51,** we can write

$$w = \lim_{\Delta s \doteq 0} \sum_{s=a}^{s=b} (pA)\Delta s = \lim_{\Delta s \doteq 0} \sum_{s=a}^{s=b} pA\,\Delta s$$
$$= \lim_{\Delta v \doteq 0} \sum_{v=v_1}^{v=v_2} p\Delta v = \int_{v_1}^{v_2} p\,dv,$$

where v_2 and v_1 are the volumes corresponding to $s = a$ and $s = b$, respectively. Since $pv^k = C$, $p = \frac{C}{v^k}$, and

$$w = C\int_{v_1}^{v_2} \frac{dv}{v^k} = \frac{C}{1-k}\,v^{1-k}\Big|_{v_1}^{v_2} = \frac{C}{1-k}\,(v_2^{1-k} - v_1^{1-k}).$$

The student will complete the numerical work.

$$W = \lim_{n \doteq \infty} \Sigma\,\Delta w.$$

But
$$f(s)\Delta s < \Delta w < f(s + \Delta s)\Delta s,$$
or
$$1 < \frac{\Delta w}{f(s)\Delta s} < \frac{f(s + \Delta s)}{f(s)}.$$
Then
$$\lim_{\Delta s \doteq 0} \frac{\Delta w}{f(s)\Delta s} = 1.$$
Hence by Duhamel's Theorem
$$\lim_{\Delta s \doteq 0} \sum_a^b \Delta w = \lim_{\Delta s \doteq 0} \sum_a^b f(s)\Delta s = \int_a^b f(s)ds.$$
$$W = \lim_{\Delta \doteq 0} \sum_{s=a}^{s=b} f(s)\Delta s = \int_a f(s)ds.$$

Exercises

1. Set up and evaluate definite integrals representing the work sought in Exercises 1–5, §**51,** Chapter VI.

2. Water is pumped from a round cistern whose median section is a parabola. The cistern has a diameter of 8 feet at the top and it is 16 feet deep. The water is 10 feet deep. Find the work done in pumping the water from the cistern if the discharge of the pump is 3 feet above the top of the cistern and if the friction in the pump and the friction of the water in the pipes are neglected.

3. Find the work done by a gas in expanding in accordance with the law $pv^{1.4} = C$ from a volume of 10 cubic feet to one of 12 cubic feet, if when $v = 9$ cubic feet $p = 100$ pounds per square inch.

4. Find the work done in stretching a spring whose original length was 15 inches from a length of 16 inches to a length of 18 inches if a force of 40 pounds is required to stretch it to a length of 16 inches.

5. Find the work done in compressing a spring of original length 5 inches to a length of $3\frac{1}{2}$ inches, if a force of 900 pounds is required to compress it to a length of 4 inches.

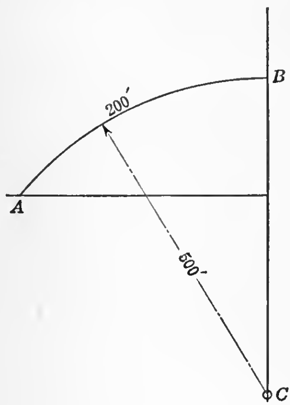

Fig. 48.

6. The force due to friction is proportional to the component of force normal to the surface over which a body is being moved. Find work done in dragging a body weighing 100 pounds from the base to the top of a slide in the form of a segment of a sphere, Fig. 48, if the distance $AB = 200$ feet and the radius of the sphere is 500 feet. Express the result in terms of μ, the coefficient of friction.

68. Volume of a Solid of Revolution. The area bounded by the curve $y = f(x)$, Fig. 46, the ordinates $x = a$ and $x = b$, and the X-axis, is revolved about the X-axis. Find the volume of the solid generated.

Divide the interval $AB = b - a$ on the X-axis into n equal parts of length Δx and pass planes through the points of division perpendicular to the X-axis. These planes divide the volume into

n portions, Δv. A typical portion can be regarded as generated by revolving $DEQP$, Fig. 46, about the base DE in the X-axis. Replace the volume of this slice by that of the cylinder generated by the revolution of $DEQ'P$ about the X-axis. Its volume is $\pi y^2 \Delta x$. The total volume is then

$$V = \lim_{\Delta x \doteq 0} \sum_a^b \pi y^2 \Delta x,$$

or

$$V = \pi \int_a^b [f(x)]^2 dx.$$

Illustration. Find the volume between the planes $x = 1$ and $x = 3$ of the solid generated by revolving the curve $y = x^2 + x$ about the X-axis.

$$V = \pi \int_1^3 (x^2 + x)^2 dx = \pi \int_1^3 (x^4 + 2x^3 + x^2) dx$$
$$= \pi \left[\tfrac{1}{5}x^5 + \tfrac{1}{2}x^4 + \tfrac{1}{3}x^3\right]_1^3 = \pi x^3 \left[\tfrac{1}{5}x^2 + \tfrac{1}{2}x + \tfrac{1}{3}\right]_1^3$$
$$= 27\pi\left(\tfrac{9}{5} + \tfrac{3}{2} + \tfrac{1}{3}\right) - \pi\left(\tfrac{1}{5} + \tfrac{1}{2} + \tfrac{1}{3}\right) = \tfrac{1456}{15}\pi.$$

Exercises

1. Find the volume between the planes $x = 0$ and $x = 3$ of the solid generated by revolving the parabola $y^2 = 6x$ about the X-axis.

2. Find the volume of a sphere of radius r.

3. Find the volume of the ellipsoid of revolution generated by revolving the ellipse

$$\frac{x^2}{16} + \frac{y^2}{9} = 1$$

about the X-axis; about the Y-axis.

4. Find the volume between the planes $x = 0$ and $x = 4$ of the solid generated by revolving $y^2 = x^3$ about the X-axis.

5. Find the volume of the solid generated by revolving $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ about the X-axis.

6. Find the volume generated by revolving $y^2 = 2ax - x^2$ about the X-axis.

7. Find the volume generated by revolving the oval of $y^2 = x(x - 1)(x - 2)$ about the X-axis.

69. Length of Arc: Rectangular Coördinates. In **§63** the length of arc of a curve was found by integrating its differential. We shall now express the length of arc by means of a definite integral.

To find the length of arc $APQB$, Fig. 49, divide CH into n equal parts of length Δx each. At the points of division erect ordinates dividing the arc AB into n parts of which PQ is one. The length of arc AB is defined by

$$s = \lim_{n \doteq \infty} \sum \Delta c,$$

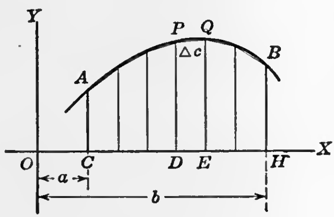

FIG. 49.

where Δc is the length of the chord PQ. Then

$$s = \lim_{n \doteq \infty} \sum \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

$$= \lim_{n \doteq \infty} \sum \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2} \Delta x.$$

Since

$$\lim_{n \doteq \infty} \frac{\sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}} = 1,$$

it follows by Duhamel's Theorem that

$$s = \lim_{n \doteq \infty} \sum_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \Delta x.$$

Hence,

$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

Exercises

1. Find the length of the curve $y = x^{\frac{3}{2}}$ between the points (0, 0) and (1, 1).

2. Find the entire length of $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

3. Find the entire length of $x^2 + y^2 = a^2$.

4. Find the length of $y^2 = 4x^3$ between the points (0, 0) and (4, 16).

70. Area of a Surface of Revolution. The portion AB, Fig. 49, of the curve $y = f(x)$, between the ordinates $x = a$ and $x = b$, is revolved about the X-axis. Find the area, S, of the surface generated.

Pass planes as in §69 perpendicular to the X-axis through the equidistant points of division of the interval $CH = b - a$. Denote the convex surface of the frustum of the cone generated by the revolution of $DEPQ$ by ΔF. The area, S, of the surface of revolution will be defined as the limit of the sum of the convex surfaces, ΔF, of these frusta as n becomes infinite, *i.e.*, as Δx approaches zero. Then,

$$S = \lim_{\Delta x \doteq 0} \sum \Delta F = \lim_{\Delta x \doteq 0} \sum 2\pi \frac{y + (y + \Delta y)}{2} \Delta c$$

$$= \lim_{\Delta x \doteq 0} \sum 2\pi \left(y + \frac{\Delta y}{2}\right) \Delta c.$$

By Duhamel's Theorem we can replace $y + \dfrac{\Delta y}{2}$ by y, since $\lim\limits_{\Delta x \doteq 0} \dfrac{y + \frac{1}{2}\Delta y}{y} = 1.$ Hence,

$$S = \lim_{\Delta x \doteq 0} \sum 2\pi y \,\Delta c = \lim_{\Delta x \doteq 0} \sum 2\pi y \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2} \Delta x.$$

Since

$$\lim_{\Delta x \doteq 0} \frac{\sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}}{\sqrt{1 + \left(\frac{dy}{dy}\right)^2}} = 1,$$

$$S = \lim_{\Delta x \doteq 0} \sum_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \Delta x.$$

Therefore

$$S = 2\pi \int_a^b y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

$$= 2\pi \int_{x=a}^{x=b} y ds,$$

where ds is the differential of the length of arc. The latter form is easily remembered since $2\pi y ds$ is the area of the strip of surface generated by revolving ds, the differential of arc, about the X-axis at a distance y from it. If it is more convenient to integrate with respect to y, ds can be replaced by

$$\sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy,$$

and the limits are the values of y corresponding to $x = a$ and $x = b$. Thus

$$S = 2\pi \int_{y=y_1}^{y=y_2} y \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy = 2\pi \int_{y=y_1}^{y=y_2} y\, ds.$$

Exercises

1. Find the surface between the planes $x = 0$ and $x = 5$ of the paraboloid of revolution obtained by revolving $y^2 = 4x$ about the X-axis.

2. Find the surface of the sphere generated by revolving $x^2 + y^2 = a^2$ about the X-axis.

3. Find the surface of the right circular cone whose altitude is 10 feet and the radius of whose base is 5 feet.

4. Find the surface of the solid generated by the revolution of $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ about the X-axis.

71. Element of Integration. The first step in setting up a definite integral is to break up the area, volume, work, length, or whatever it is desired to calculate, into convenient parts which are infinitesimals as their number approaches infinity. These parts are then replaced by other infinitesimals of the typical form $f(x)dx$, which must be so chosen that the limit of the ratio of each infinitesimal of the second set to the corresponding infinitesimal of the first set is one. $f(x)dx$ is called the "*element*" of the integral or of the quantity which the integral represents. Thus the element of volume is $\pi y^2 dx$, that of area is $y dx$, that of work is $F dx$.

If the magnitude which it is desired to calculate is broken up into suitable parts, the expressions for the elements can be written down at once. The best way of retaining in mind the formulas of §§**68**, **69**, and **70** is to understand thoroughly how the elements are chosen. The process of writing down the element of integration at once becomes almost an intuitive one.

72. Water Pressure. The pressure at any given point in a liquid at rest is equal in all directions. The pressure per unit area at a given depth is equal to the pressure on a horizontal surface of unit area at that depth, *i.e.*, to the weight of the column of liquid supported by this surface. This weight is proportional to the depth. Hence the pressure at a depth x below the surface of the liquid is given by the formula $p = kx$. If the liquid is water and the depth x is expressed in feet, $k = 62.5$ pounds per cubic foot.

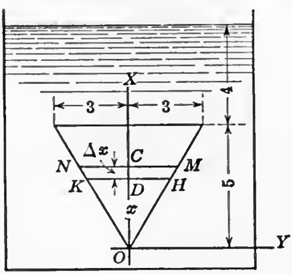

FIG. 50.

The method to be used in finding the water pressure on any vertical surface is illustrated in the solution of the following problems:

1. Find the pressure on one side of a gate in the shape of an isosceles triangle whose base is 6 feet and whose altitude is 5 feet, if it is immersed vertically in water with its vertex down and its base 4 feet below the surface of the water.

Take the origin at the vertex of the triangle, the axis of x vertical, and the axis of y horizontal, as in Fig. 50. The altitude is supposed to be divided into n equal parts and through the points of division horizontal lines are supposed to be drawn dividing the surface into strips. The trapezoid $KHMN = \Delta A$ is a typical strip. Denote the pressure on this strip by ΔP. The abscissa of the lower edge of the strip is x and the pressure at this lower edge is $k(9 - x)$. Then the total pressure is

$$P = \lim_{n \doteq \infty} \sum k(9 - x)\Delta A. \tag{1}$$

In accordance with Duhamel's Theorem we can replace ΔA by $2y\Delta x$.

$$P = \lim_{n \doteq \infty} \sum_{x=0}^{x=5} 2k(9 - x)y\Delta x, \tag{2}$$

or

$$P = 2k\int_0^5 (9 - x)y\,dx. \tag{3}$$

Since

$$y = \frac{3x}{5},$$

$$P = \frac{6k}{5}\int_0^5 (9 - x)x\,dx$$

$$= \frac{6k}{5}\left[\frac{9x^2}{2} - \frac{x^3}{3}\right]_0^5 = 5312.5 \text{ pounds}. \tag{4}$$

In general, if u denotes the depth below the surface of the liquid and z denotes the width, at the depth u, of the vertical surface on which the pressure is to be computed,

$$P = k\int_a^b uz\,du, \tag{5}$$

where a and b are the depths of the highest and lowest points, respectively, of the surface. For,

$$P = \lim_{n \doteq \infty} \sum ku\Delta A = \lim_{\Delta u \doteq 0} \sum_{u=a}^{u=b} kuz\Delta u = k\int_a^b uz\,du.$$

2. Find the total pressure on a vertical semi-elliptical gate whose major axis lies in the surface of the water, given that the semi-axes of the ellipse are 8 feet and 6 feet. Take the origin at

the center, the axis of x horizontal and the axis of y positive downward. The element of pressure is

$$2kyx\,dy$$

and the total pressure is

$$P = 2k\int_0^6 yx\,dy.$$

x is expressed in terms of y by means of the equation of the ellipse,

$$\frac{x^2}{64} + \frac{y^2}{36} = 1.$$

Then

$$P = 2k\,\tfrac{4}{3}\int_0^6 y\sqrt{36 - y^2}\,dy.$$

Exercises

1. Find the pressure on the vertical parabolic gate, Fig. 51: (*a*) if the edge AB lies in the surface of the water; (*b*) if the edge AB lies 5 feet below the surface.

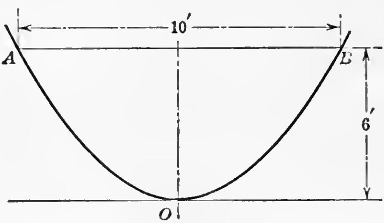

FIG. 51.

2. Find the pressure on a vertical semicircular gate whose diameter, 10 feet long, lies in the surface of the water.

73. Arithmetic Mean. The arithmetic mean, A, of a series of n numbers, $a_1, a_2, a_3, \cdots, a_n$, is defined by the equation

$$nA = a_1 + a_2 + a_3 + \cdots + a_n,$$

or

$$\mathbf{A} = \frac{\mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3 + \cdots + \mathbf{a}_n}{\mathbf{n}}.$$

That is, A is such a number that if each number in the sum

$a_1 + a_2 + a_3 + \cdots + a_n$ be replaced by it, this sum is unaltered.

74. Mean Value of a Function. We can extend the idea involved in the arithmetic mean to other problems.

Illustration 1. Suppose a body moves with uniform velocity a distance of 1 foot during the first second, a distance of 2 feet during the second second, a distance of 3 feet during the third second, and so on for 10 seconds. At the end of 10 seconds the body would have moved $1 + 2 + 3 + \cdots + 10 = 55$ feet. The mean, or average, velocity of the body is the constant velocity with which the body would describe this distance in the same time. It is equal to 5.5 feet per second.

If the velocity of the body instead of changing abruptly as indicated above were changing continuously in accordance with the law $v = t$, the total distance s traversed in 10 seconds would be

$$s = \int_0^{10} v\,dt = \int_0^{10} t\,dt = 50.$$

The mean velocity, V, the constant velocity which a body must have in order to traverse the same distance in the same length of time, is $50 \div 10 = 5$ feet per second. This can be expressed by the formula

$$\int_0^{10} V dt = \int_0^{10} v\,dt.$$

From this equation

$$V = \frac{\int_0^{10} v\,dt}{10}.$$

In general, if $v = f(t)$, the mean velocity, V, of the body in the interval of time between $t = a$ and $t = b$ is expressed by the equation

$$\int_a^b V dt = \int_a^b f(t)dt,$$

or, since V is a constant,

$$V = \frac{\int_a^b f(t)dt}{b - a}.$$

V is the constant velocity, which replacing the variable velocity, $v = f(t)$, at every instant in the interval between $t = a$ and $t = b$, gives the same distance traversed, *i.e.*, leaves the value of the integral, $\int_a^b v\,dt$, unchanged.

Illustration 2. Consider the work done by a variable force f acting in a straight line, the X-axis, and producing a displacement from $x = a$ to $x = b$. If the law of the force is $f = \phi(x)$, the mean force F in the interval from $x = a$ to $x = b$, or the constant force which would do the same work while producing the displacement $b - a$, is given by the equation

$$\int_a^b F dx = \int_a^b \phi(x)dx,$$

or

$$F = \frac{\int_a^b \phi(x)dx}{b - a}.$$

F is a constant such that if, in the integral $\int_a^b \phi(x)dx$, the function $\phi(x)$ be replaced by it, the value of the integral remains unchanged.

Illustration 3. Let a unit of mass be situated at each of the points on the X-axis whose abscissas are $x_1, x_2, x_3, \cdots, x_n$. The X-axis is taken horizontal and the masses are acted upon by gravity. We shall find the distance, $\bar{x}$, from the origin at which the n masses must be concentrated in order that the sum of the moment about the origin of the forces acting on the masses shall be unchanged.

Clearly $\bar{x}$ must satisfy the equation

$$gn\bar{x} = g(x_1 + x_2 + x_3 + \cdots + x_n),$$

or

$$\bar{x} = \frac{x_1 + x_2 + x_3 + \cdots + x_n}{n}.$$

If there are $m_1, m_2, m_3, \cdots, m_n$ units of mass concentrated at $x_1, x_2, x_3, \cdots, x_n$, respectively, the mean moment arm, $\bar{x}$, the distance from the origin at which the masses must be concentrated in order that the sum of the moments about the origin of the forces acting on the masses shall be unchanged, is given by the equation

$$(m_1 + m_2 + \cdot \cdot \cdot m_n)\bar{x} = m_1x_1 + m_2x_2 + \cdot \cdot \cdot + m_nx_n,$$

or

$$\bar{x} = \frac{\Sigma m_ix_i}{\Sigma m_i}, \ (i = 1, 2, 3, \cdot \cdot \cdot, n). \tag{1}$$

$\bar{x}$ is a constant such that if in the sum Σm_ix_i each of the numbers $x_1, x_2, \cdot \cdot \cdot, x_n$ be replaced by $\bar{x}$ this sum is not changed.

Now let there be a continuous distribution of matter along the X-axis from $x = a$ to $x = b$. Divide the interval $b - a$ into n segments each of length Δx. An expression for the approximate sum of the moments about the origin of the forces acting on the mass is $\Sigma gx\Delta m_i$, where Δm_i is the mass of the segment Δx_i. An expression for the approximate force is $\Sigma g\Delta m_i$. Hence an expression for the approximate $\bar{x}$ is $\frac{\Sigma gx\Delta m_i}{\Sigma g\Delta m_i}$. It is readily seen that as Δx approaches zero, the numerator approaches the total moment and the denominator approaches the total mass. Hence

$$\bar{x} = \frac{\lim\limits_{\Delta m \doteq 0} \Sigma gx\Delta m}{\lim\limits_{\Delta m \doteq 0} \Sigma g\Delta m} = \frac{\int_a^b x\,dm}{\int_a^b dm}. \tag{2}$$

$\bar{x}$ is a constant such that if in the integral, $\int^b x\,dm$, x is replaced by $\bar{x}$, the value of the integral is unchanged.

For example, if the density is proportional to x^2, *i.e.*, is equal to kx^2, the element of mass, dm, is kx^2dx, and we have

$$\bar{x} = \frac{\int_a^b xkx^2dx}{\int_a^b kx^2dx} = \frac{\int_a^b x^3dx}{\int_a^b x^2dx} = \frac{3}{4}\frac{b^4 - a^4}{b^3 - a^3}.$$

The mean value, M, of the function $f(x)$ with respect to the *magnitude u, which is a function of x, is defined by the equation*

$$\int_{x=a}^{x=b} M du = \int_{x=a}^{x=b} f(x) du, \tag{3}$$

where M is a constant, or

$$M = \frac{\int_{x=a}^{x=b} f(x) du}{\int_{x=a}^{x=b} du}. \tag{4}$$

M is a constant such that if the function $f(x)$ is replaced by it in $\int_{x=a}^{x=b} f(x) du$, the value of the integral is not changed.

In (2), $\bar{x}$ is the mean value of x with respect to the magnitude, m.

A particular case of (4) is that in which $u = x$. Then (4) becomes

$$\mathbf{M} = \frac{1}{b - a} \int_a^b f(x)\, dx. \tag{5}$$

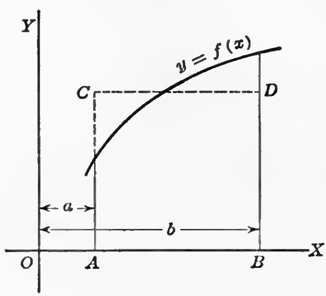

Fig. 52.

Illustrations 1 and 2 are cases of this type.

When $u = x$, as in equation (5), M can be interpreted as the altitude, AC, of a rectangle with base $AB = b - a$, Fig. 52, whose area is equal to the area bounded by the curve $y = f(x)$, the X-axis, and the ordinates $x = a$ and $x = b$. From this standpoint M is called the mean ordinate of the curve $y = f(x)$ in the interval from $x = a$ to $x = b$.

Illustration 4. Find the mean ordinate of the curve $y = x^2$ between the ordinates $x = 0$ and $x = 2$.

$$M = \frac{1}{b - a} \int_a^b f(x) dx = \tfrac{1}{2} \int_0^2 x^2 dx = \tfrac{1}{6} x^3 \Big|_0^2 = \tfrac{4}{3}.$$

Illustration 5. Find the mean with respect to u of x between the limits $u = 1$ and $u = 9$, if $u = x^2$.

$$\int_{u=1}^{u=9} M\,du = \int_{u=1}^{u=9} x\,du,$$

$$M\int_1^3 2x\,dx = \int_1^3 2x^2\,dx,$$

$$8M = \tfrac{52}{3},$$

$$M = \tfrac{13}{6}.$$

Exercises

Find the mean ordinates for the following curves:

1. $y = x^{\frac{1}{2}}$ between $x = 0$ and $x = 3$.

2. $y = x^{\frac{1}{2}}$ between $x = 2$ and $x = 4$.

3. $y = 3x^3$ between $x = 0$ and $x = 2$.

4. $y = 3x^3$ between $x = 1$ and $x = 3$.

5. $y = x^{\frac{1}{3}}$ between $x = 0$ and $x = 1$.

6. Find the radius of the right circular cylinder of altitude 3 whose volume is equal to the volume between the planes $x = 2$ and $x = 5$ of the solid generated by revolving $y = x + x^2$ about the X-axis.

7. Find the radius of the right circular cylinder of altitude $b - a$ whose volume is equal to the volume between the planes $x = a$ and $x = b$ of the solid generated by revolving $y = f(x)$ about the X-axis.

8. The density cf a thin straight rod 10 inches long and of uniform cross section is proportional to the distance from one end. Find the mean density of the rod.

9. Find the mean velocity of a freely falling body between the time $t = 1$ second and $t = 3$ seconds.

10. The density of a rod is given by $\rho = 3x^2$, where x is the distance from one end. Find the mean density if the rod is 10 inches long.

11. Find the mean moment arm in the case of the rods of Exercises 8 and 10, about a horizontal axis through the end of the rod ($x = 0$). The rods are horizontal, and perpendicular to the axis about which moments are taken. The rods are supposed to be acted upon by forces due to gravity alone.

12. Find the mean ordinate of a semicircle, the ends of which are upon the X-axis.

CHAPTER VIII

CIRCULAR FUNCTIONS. INVERSE CIRCULAR FUNCTIONS

Up to this point only functions have been discussed which are simple algebraic combinations of powers of the dependent variable. Many interesting applications of the calculus to the study of these functions have been given. We shall now take up the study of the application of the methods of the calculus to another very important class of functions, the circular functions. It is apparent that the principles developed in the preceding chapters are equally applicable to the circular functions and to the algebraic functions.

As the student has already learned, the circular functions occur very frequently in the study of the physical sciences and their applications, because by means of them periodic phenomena can be studied.

75. Derivative of sin u.

Let

$$\begin{aligned} y &= \sin u. \\ y + \Delta y &= \sin (u + \Delta u), \\ \Delta y &= \sin (u + \Delta u) - \sin u \\ &= \sin u \cos \Delta u + \cos u \sin \Delta u - \sin u, \\ \frac{\Delta y}{\Delta u} &= \frac{\cos u \sin \Delta u}{\Delta u} - \frac{\sin u(1 - \cos \Delta u)}{\Delta u}. \end{aligned}$$

Then

$$\lim_{\Delta u \doteq 0} \frac{\Delta y}{\Delta u} = \cos u \lim_{\Delta u \doteq 0} \frac{\sin \Delta u}{\Delta u} - \sin u \lim_{\Delta u \doteq 0} \frac{1 - \cos \Delta u}{\Delta u}.$$

Hence by **§56** and **§58**

$$\frac{dy}{du} = \cos u. \tag{1}$$

Whence

$$\frac{dy}{dx} = \cos u \frac{du}{dx}. \tag{2}$$

The corresponding formula for dy is

$$dy = \cos u du. \tag{3}$$

It has thus been shown that

$$\frac{\mathbf{d}(\sin \mathbf{u})}{\mathbf{dx}} = \cos \mathbf{u} \frac{\mathbf{du}}{\mathbf{dx}} \tag{4}$$

and

$$\mathbf{d}(\sin \mathbf{u}) = \cos \mathbf{udu}.$$

Well known properties of the function $y = \sin u$ can be verified by formula (1). Thus $\sin u$ is an increasing function between $u = 0$ and $u = \frac{\pi}{2}$, and between $u = \frac{3\pi}{2}$ and $u = 2\pi$, and decreasing between $u = \frac{\pi}{2}$ and $u = \frac{3\pi}{2}$. The same facts are shown by

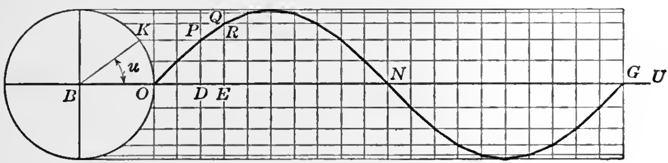

FIG. 53.

the derivative, $\cos u$, which is positive between $u = 0$ and $u = \frac{\pi}{2}$, and between $u = \frac{3\pi}{2}$ and $u = 2\pi$, and negative between $u = \frac{\pi}{2}$ and $u = \frac{3\pi}{2}$. Further, $\sin u$ has maximum and minimum values for $u = \frac{\pi}{2}$ and $u = \frac{3\pi}{2}$, respectively. The same facts are shown by the derivative, $\cos u$, which becomes zero at these points and changes sign at $\frac{\pi}{2}$ from plus to minus, and at $\frac{3\pi}{2}$ from minus to plus.

The slope of the sine curve is approximately the slope of the diagonal PQ of a rectangle in Fig. 53. The greater the number of equal parts into which the circumference of the circle is divided and hence the smaller the subdivisions of the arc, the closer do the slopes of these diagonals approach the slopes of the tangents.

76. Derivatives of cos u, tan u, cot u, sec u, csc u. The derivatives of the remaining circular functions can be obtained from that of the sine.

Let $y = \cos u$. Then

$$y = \sin\left(\frac{\pi}{2} - u\right)$$

and

$$\frac{dy}{dx} = \cos\left(\frac{\pi}{2} - u\right)\frac{d\left(\frac{\pi}{2} - u\right)}{dx}$$

$$= \cos\left(\frac{\pi}{2} - u\right)\left(-\frac{du}{dx}\right)$$

$$= -\sin u\,\frac{du}{dx}.$$

Hence

$$\frac{\mathbf{d(\cos u)}}{\mathbf{dx}} = -\mathbf{\sin u}\,\frac{\mathbf{du}}{\mathbf{dx}} \tag{1}$$

and

$$\mathbf{d(\cos u) = -\sin u\ du.}$$

By writing

$$\tan u = \frac{\sin u}{\cos u},$$

$$\cot u = \frac{\cos u}{\sin u},$$

$$\sec u = \frac{1}{\cos u},$$

and

$$\csc u = \frac{1}{\sin u},$$

the student will show that

$$\frac{\mathbf{d(\tan u)}}{\mathbf{dx}} = \mathbf{\sec^2 u\,\frac{du}{dx}}, \quad \text{or} \quad \mathbf{d(\tan u) = \sec^2 u\ du} \tag{2}$$

$$\frac{\mathbf{d(\cot u)}}{\mathbf{dx}} = \mathbf{-\csc^2 u\,\frac{du}{dx}}, \quad \text{or} \quad \mathbf{d(\cot u) = -\csc^2 u\ du} \tag{3}$$

$$\frac{\mathbf{d(\sec u)}}{\mathbf{dx}} = \mathbf{\sec u \tan u\,\frac{du}{dx}}, \quad \text{or} \quad \mathbf{d(\sec u) = \sec u \tan u\ du} \tag{4}$$

$$\frac{\mathbf{d(\csc u)}}{\mathbf{dx}} = \mathbf{-\csc u \cot u\,\frac{du}{dx}}, \quad \text{or} \quad \mathbf{d(\csc u) = -\csc u \cot u\ du} \tag{5}$$

Illustration 1. Find the first and second derivatives of $3 \sin (2x - 5)$.

$$\frac{d[3 \sin (2x - 5)]}{dx} = 3 \frac{d[\sin (2x - 5)]}{dx}$$

$$= 3 \cos (2x - 5) \frac{d(2x - 5)}{dx}$$

$$= 6 \cos (2x - 5).$$

Differentiating again,

$$\frac{d^2[3 \sin (2x - 5)]}{dx^2} = 6 \frac{d[\cos (2x - 5)]}{dx}$$

$$= -6 \sin (2x - 5) \frac{d(2x - 5)}{dx}$$

$$= -12 \sin (2x - 5).$$

Illustration 2. If $y = \sin 2x \cos x$, find $\frac{dy}{dx}\cdot$ Since $\sin 2x \cos x$ is the product of two functions, apply formula (1) §40.

$$\frac{dy}{dx} = \sin 2x(-\sin x) \frac{dx}{dx} + (\cos x)(\cos 2x) \frac{d2x}{dx}$$

$$= 2 \cos x \cos 2x - \sin x \sin 2x.$$

Illustration 3. If $y = 3 \sin x + 4 \cos x$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}\cdot$

$$\frac{dy}{dx} = 3 \cos x - 4 \sin x \tag{6}$$

$$\frac{d^2y}{dx^2} = -(3 \sin x + 4 \cos x) = -y. \tag{7}$$

From (6)

$$\frac{dy}{dx} = 4 \cos x \left(\tfrac{3}{4} - \tan x\right).$$

When $0 < x < \frac{\pi}{2}$, $\cos x$ is positive. The second factor, $\frac{3}{4} - \tan x$, is positive when $x < \tan^{-1} (\frac{3}{4})$, and negative when $x > \tan^{-1} (\frac{3}{4})$. Thus, when x is in the first quadrant the function has a maximum value corresponding to $x = \tan^{-1} (\frac{3}{4})$.

When $\frac{\pi}{2} < x < \pi$, $\frac{dy}{dx}$ is negative.

When $\pi < x < \frac{3\pi}{2}$, $\cos x$ is negative, and $\frac{3}{4} - \tan x$ is negative when $x < \tan^{-1}(\frac{3}{4})$, and positive when $x > \tan^{-1}(\frac{3}{4})$. Thus when x is in the third quadrant the function has a minimum value corresponding to the value $x = \tan^{-1}(\frac{3}{4})$.

When $\frac{3\pi}{2} < x < 2\pi$, $\frac{dy}{dx}$ is positive.

The same facts can be seen directly from the function, for it can be put in the form

$$y = 5(\tfrac{4}{5}\cos x + \tfrac{3}{5}\sin x).$$

Let $\cos\alpha = \frac{4}{5}$ and $\sin\alpha = \frac{3}{5}$. Then

$$y = 5(\cos x \cos\alpha + \sin x \sin\alpha),$$

or

$$y = 5\cos(x - \alpha).$$

In polar coördinates this represents a circle passing through the origin, with a diameter of 5. (See Fig. 54.) x is the vectorial angle and y the radius vector. The diameter OB makes an angle α with the polar axis. As x varies from 0 to π the circle is described, and as x varies from π to 2π, y is negative and the circle is described a second time.

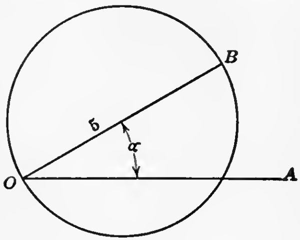

Fig. 54.

Hence y has a maximum value 5 when x is equal to α, and a minimum value -5 when x is equal to $\alpha + \pi$.

Illustration 4. If $y = \tan^3 3x = (\tan 3x)^3$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

The function is of the form $y = u^n$. Hence

$$\begin{aligned}\frac{dy}{dx} &= 3(\tan 3x)^2 \frac{d(\tan 3x)}{dx}\\ &= 3\tan^2 3x \sec^2 3x \frac{d3x}{dx}\\ &= 9\tan^2 3x \sec^2 3x.\end{aligned}$$

$$\frac{d^2y}{dx^2} = 9\left[\tan^2 3x \frac{d(\sec^2 3x)}{dx} + \sec^2 3x \frac{d(\tan^2 3x)}{dx}\right]$$

$$= 9\left[\tan^2 3x \cdot 2 \sec 3x \frac{d(\sec 3x)}{dx} + \sec^2 3x \cdot 2 \tan 3x \frac{d(\tan 3x)}{dx}\right]$$

$$= 18\left[\tan^2 3x \sec 3x \sec 3x \tan 3x \frac{d\,3x}{dx} \right.$$
$$\left. + \sec^2 3x \tan 3x \sec^2 3x \frac{d\,3x}{dx}\right]$$

$$= 54(\tan^3 3x \sec^2 3x + \tan 3x \sec^4 3x)$$
$$= 54 \tan 3x \sec^2 3x(\tan^2 3x + \sec^2 3x).$$

Illustration 5. If $\frac{dy}{dx} = \cos x$, find y.

$$\frac{dy}{dx} = \cos x.$$

$$y = \sin x + C.$$

Illustration 6. If $\frac{dy}{dx} = \cos 3x$, find y.

$$\frac{dy}{dx} = \tfrac{1}{3}\left[\cos 3x \frac{d\,3x}{dx}\right].$$

The expression within the bracket is the derivative of $\sin 3x$, hence,

$$y = \tfrac{1}{3} \sin 3x + C.$$

Illustration 7. If $\frac{dy}{dx} = \sin 3x$, find y.

$$\frac{dy}{dx} = -\tfrac{1}{3}\left[-\sin 3x \frac{d\,3x}{dx}\right].$$

Hence

$$y = -\tfrac{1}{3} \cos 3x + C.$$

Illustration 8. If $\frac{dy}{dx} = \sec^2 2x$, find y.

$$\frac{dy}{dx} = \tfrac{1}{2}\left[\sec^2 2x \frac{d\,2x}{dx}\right],$$

Hence

$$y = \tfrac{1}{2} \tan 2x + C.$$

Illustration 9. If $\frac{dy}{dx} = \sec 5x \tan 5x$, find y.

$$\frac{dy}{dx} = \tfrac{1}{5}\left[\sec 5x \tan 5x \frac{d\,5x}{dx}\right].$$

Hence

$$y = \tfrac{1}{5} \sec 5x + C.$$

Illustration 10. If $dy = \cos 3x\, dx$, find y.

$$\begin{aligned} y &= \int \cos 3x dx \\ &= \tfrac{1}{3} \int \cos 3x\, d(3x) \\ &= \tfrac{1}{3} \sin 3x + C. \end{aligned}$$

Illustration 11. If $dy = \sin^2 2x \cos 2x\, dx$, find y.

$$\begin{aligned} y &= \int (\sin 2x)^2 \cos 2x\, dx \\ &= \tfrac{1}{3}\,\tfrac{1}{2} \int 3(\sin 2x)^2 \cos 2x\, d(2x) \\ &= \tfrac{1}{6} \int 3(\sin 2x)^2 d(\sin 2x). \end{aligned}$$

Hence

$$y = \tfrac{1}{6}(\sin 2x)^3 + C.$$

Illustration 12. If $dy = \tan^3 5x \sec^2 5x\, dx$, find y.

$$\begin{aligned} y &= \int \tan^3 5x \sec^2 5x\, dx \\ &= \tfrac{1}{4}\,\tfrac{1}{5} \int 4(\tan 5x)^3 \sec^2 5x\, d(5x) \\ &= \tfrac{1}{20} \int 4(\tan 5x)^3\, d(\tan 5x). \end{aligned}$$

Hence

$$y = \tfrac{1}{20}(\tan 5x)^4 + C.$$

Illustration 13.

$$\begin{aligned} \int \sin 5x \cos 3x\, dx &= \int \tfrac{1}{2}[\sin(5x + 3x) + \sin(5x - 3x)]dx \\ &= \tfrac{1}{2} \int \sin 8x\, dx + \tfrac{1}{2} \int \sin 2x\, dx \\ &= -\tfrac{1}{16} \cos 8x - \tfrac{1}{4} \cos 2x + C. \end{aligned}$$

Illustration 14.

$$\begin{aligned} \int \cos 7x \sin 3x\, dx &= \int \tfrac{1}{2}[\sin(3x + 7x) + \sin(3x - 7x)]dx \\ &= \tfrac{1}{2} \int \sin 10x\, dx - \tfrac{1}{2} \int \sin 4x\, dx \\ &= -\tfrac{1}{20} \cos 10x + \tfrac{1}{8} \cos 4x + C. \end{aligned}$$

Illustration 15.

$$\int \cos 4x \cos 7x\, dx = \int \tfrac{1}{2}[\cos(7x + 4x) + \cos(7x - 4x)]\, dx$$
$$= \tfrac{1}{2}\int \cos 11x\, dx + \tfrac{1}{2}\int \cos 3x\, dx + C$$
$$= \tfrac{1}{22}\sin 11x + \tfrac{1}{6}\sin 3x + C.$$

Illustration 16.

$$\int \sin 4x \sin 2x\, dx = -\tfrac{1}{2}\int [\cos(4x + 2x) - \cos(4x - 2x)]\, dx$$
$$= -\tfrac{1}{2}\int \cos 6x\, dx + \tfrac{1}{2}\int \cos 2x\, dx$$
$$= -\tfrac{1}{12}\sin 6x + \tfrac{1}{4}\sin 2x + C.$$

Exercises

In Exercises 1 to 10, verify the differentiation.

1. $y = \sin 5x,\quad \dfrac{dy}{dx} = 5\cos 5x,\quad \dfrac{d^2y}{dx^2} = -25y.$

2. $y = \cos 3x,\quad \dfrac{dy}{dx} = -3\sin 3x,\quad \dfrac{d^2y}{dx^2} = -9y.$

3. $y = \tan 2x,\quad \dfrac{dy}{dx} = 2\sec^2 2x,$

$\dfrac{d^2y}{dx^2} = 8\sec^2 2x \tan 2x.$

4. $y = \sin x \cos 2x,\quad \dfrac{dy}{dx} = \cos 2x \cos x - 2\sin 2x \sin x$

5. $y = \sin \dfrac{3x - 2}{5},\quad \dfrac{dy}{dx} = \tfrac{3}{5}\cos \dfrac{3x - 2}{5},$

$\dfrac{d^2y}{dx^2} = -\tfrac{9}{25}y.$

6. $y = \tan^3 5x,\quad dy = 15\tan^2 5x \sec^2 5x\, dx.$

7. $y = \sec^4 3x,\quad dy = 12\sec^4 3x \tan 3x\, dx.$

8. $\begin{bmatrix} y = a(1 - \cos\theta), & dy = a\sin\theta\, d\theta. \\ x = a(\theta - \sin\theta), & dx = a(1 - \cos\theta)d\theta. \end{bmatrix}$

9. $y = a\sin\left(\dfrac{2\pi t}{T} - e\right),\quad \dfrac{dy}{dt} = \dfrac{2a\pi}{T}\cos\left(\dfrac{2\pi t}{T} - e\right).$

10. $y = x\sin x,\quad dy = (x\cos x + \sin x)\, dx.$

11. From the results of Exercise 8, show that $\dfrac{dy}{dx} = \cot\dfrac{\theta}{2}.$

Find dy in Exercises 12–20.

12. $y = \tan 2x \sin 2x$.

13. $y = \dfrac{\sin 2x}{\cos^2 2x}$.

14. $y = \sin (x^2 + 3x - 2)$.

15. $y = \cos (3 - x)^2$.

16. $y = \sqrt{\sin 2x}$.

17. $y = x \cos 2x - \tan 2x$.

18. $y = \tan^{\frac{3}{2}} (x - 1)$.

19. $y = \cos^3 (1 - x^2 - 2x)$.

20. $y = \sin^{\frac{1}{2}} (2x - 1) \cos^{\frac{3}{2}} (2x - 1)$.

Integrate:

21. $dy = \sin 2x\, dx$.

22. $dy = \cos 2x\, dx$.

23. $dy = \sec^2 4x\, dx$.

24. $dy = \sec 5x \tan 5x\, dx$.

25. $dy = \sin x \cos x\, dx$.

26. $dy = \tan x \sec^2 x\, dx$.

27. $dy = \sqrt{\sin 2x} \cos 2x\, dx$.

28. $dy = \cos^5 x \sin x\, dx$.

29. $dy = \sec^4 x \tan x\, dx = \sec^3 x \sec x \tan x\, dx$.

30. $dy = \sec^5 (x - 1) \tan (x - 1)\, dx$.

31. Find the area under one arch of the sine curve.

32. Find the area under one arch of the curve $y = 2a^2 \sin^2 x$.

HINT. $\sin^2 x = \dfrac{1 - \cos 2x}{2}$.

33. The equations of Exercise 8 are the parametric equations of the cycloid. Find the length of one arch of the cycloid.

HINT. $ds = \sqrt{(dx)^2 + (dy)^2}$. Express ds in terms of θ and $d\theta$.

34. Find the area under one arch of the cycloid.

35. $x = a \cos \theta$, $y = a \sin \theta$. Find $\dfrac{dy}{dx}$. Find the length of the curve. Find the area bounded by the curve.

36. $x = a \cos \theta$, $y = b \sin \theta$. Find $\dfrac{dy}{dx}$. Find the area bounded by the curve.

37. $x = a \cos^3 \phi$, $y = a \sin^3 \phi$. Find $\dfrac{dy}{dx}$. Find the length of the curve.

38. Find the volume bounded by the surface obtained by revolving $y = \sin x$ about the X-axis.

39. A man walks at the constant rate of 4 feet per second along the diameter of a semicircular courtyard whose radius is 50 feet. The sun's rays are perpendicular to the diameter. How fast is the man's shadow moving along the semicircular wall of the courtyard when he is 30 feet from the end of the diameter?

40. A drawbridge 25 feet long is raised by chains attached to the end of the bridge and passing over a pulley 25 feet above the hinge of the bridge. The chain is being drawn in at the rate of 6 feet a minute. Horizontal rays of light fall on the bridge and it casts a shadow on a vertical wall. How fast is the shadow moving up the wall when 13 feet of the chain have been drawn in?

41. Find $\frac{dy}{dx}$ if $x = y\sqrt{y - 1}$.

42. Find $\frac{dy}{dx}$ if $x = \sqrt{1 - \sin y}$.

43. If $\rho^2 = a^2\cos\ 2\theta$, show by implicit differentiation that $\frac{d\rho}{d\theta} = -\frac{a^2 \sin 2\theta}{\rho}$.

44. If $\rho^2 \cos\theta = a^2 \sin 3\theta$, find $\frac{d\rho}{d\theta}$.

45. $\int \sin 6x \cos 2x\, dx$.

46. $\int \cos 4x \cos 3x\, dx$.

47. $\int \cos 5x \sin 2x\, dx$.

48. $\int \sin 8x \sin 3x\, dx$.

49. $\int \sin 4x \cos 7x\, dx$.

50. $\int \cos 5x \cos 9x\, dx$.

51. $\int \sin \omega t \cos at\, dt$.

52. $\int \cos \omega t \cos at\, dt$.

53. Find the mean ordinate of the curve $y = \sin x$ between the limits $x = 0$ and $x = \pi$.

77. Derivatives of the Inverse Circular Functions.[1] The formulas for the derivatives of the inverse circular functions are readily obtained from those of §§**75** and **76.**

[1] The student will recall that $\sin^{-1} u$ is defined for values of u between -1 and $+1$ only, and that it is a many valued function. To a given value of u there correspond infinitely many angles whose sines are equal to u. This will be seen to be the case on sketching the curve $y = \sin^{-1} u$. In this and future discussions of this function it will be made single valued by considering only those values of $y = \sin^{-1} u$ which lie between $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$, inclusive.

The positive sign of the radical in the final formula (1) is chosen because $\cos y = \sqrt{1 - u^2}$ is positive when y lies between $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$.

Of the functions occurring in (2), (3), (4), (5), and (6), $y = \cos^{-1} u$, and $y = \sec^{-1} u$ are made single valued by choosing y between 0 and π, while the remaining functions, $y = \tan^{-1} u$, $y = \cot^{-1} u$, $y = \csc^{-1} u$, are made single valued by choosing y between $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$. Show that the proper sign has been chosen for the radicals in the formulas (2), (5), and (6).

Let $y = \sin^{-1} u$. Then $\sin y = u$. Differentiation gives

$$\cos y \frac{dy}{dx} = \frac{du}{dx}.$$

$$\frac{dy}{dx} = \frac{1}{\cos y} \frac{du}{dx}$$

$$= \frac{1}{\sqrt{1 - \sin^2 y}} \frac{du}{dx}.$$

Hence

$$\frac{dy}{dx} = \frac{\frac{du}{dx}}{\sqrt{1 - u^2}}.$$

Therefore,

$$\frac{d(\sin^{-1} u)}{dx} = \frac{\frac{du}{dx}}{\sqrt{1 - u^2}}, \quad \text{or} \quad d(\sin^{-1} u) = \frac{du}{\sqrt{1 - u^2}} \tag{1}$$

The student will show that

$$\frac{d(\cos^{-1} u)}{dx} = -\frac{\frac{du}{dx}}{\sqrt{1 - u^2}}, \quad \text{or} \quad d(\cos^{-1} u) = -\frac{du}{\sqrt{1 - u^2}} \tag{2}$$

$$\frac{d(\tan^{-1} u)}{dx} = \frac{\frac{du}{dx}}{1 + u^2}, \quad \text{or} \quad d(\tan^{-1} u) = \frac{du}{1 + u^2} \tag{3}$$

$$\frac{d(\cot^{-1} u)}{dx} = -\frac{\frac{du}{dx}}{1 + u^2}, \quad \text{or} \quad d(\cot^{-1} u) = -\frac{du}{1 + u^2} \tag{4}$$

$$\frac{d(\sec^{-1} u)}{dx} = \frac{\frac{du}{dx}}{u\sqrt{u^2 - 1}}, \quad \text{or} \quad d(\sec^{-1} u) = \frac{du}{u\sqrt{u^2 - 1}} \tag{5}$$

$$\frac{d(\csc^{-1} u)}{dx} = -\frac{\frac{du}{dx}}{u\sqrt{u^2 - 1}}, \quad \text{or} \quad d(\csc^{-1} u) = -\frac{du}{u\sqrt{u^2 - 1}} \tag{6}$$

Illustration 1. If $y = \sin^{-1}(x^2 - 2x - 3)$, find dy. By formula (1)

$$dy = \frac{d(x^2 - 2x - 3)}{\sqrt{1 - (x^2 - 2x - 3)^2}}$$

$$= \frac{2(x - 1)dx}{\sqrt{1 - (x^2 - 2x - 3)^2}}.$$

Illustration 2. If $y = \tan^{-1} 3x$, find dy. By formula (3)

$$dy = \frac{d(3x)}{1 + (3x)^2}$$

$$= \frac{3dx}{1 + 9x^2}.$$

Illustration 3. If $dy = \dfrac{dx}{1 + x^2}$, find y.

$$y = \int \frac{dx}{1 + x^2},$$

or

$$y = \tan^{-1} x + C.$$

Illustration 4. If $dy = \dfrac{dx}{1 + 9x^2}$, find y.

$$y = \int \frac{dx}{1 + 9x^2}$$

$$= \tfrac{1}{3} \int \frac{3dx}{1 + (3x)^2}.$$

The expression under the integral sign is now of the form $\dfrac{du}{1 + u^2}$ whose integral is $\tan^{-1} u$. Hence

$$y = \tfrac{1}{3} \tan^{-1} (3x) + C.$$

Illustration 5. If $dy = \dfrac{dx}{4 + 9x^2}$, find y.

$$y = \int \frac{dx}{4 + 9x^2}$$

$$= \tfrac{1}{4} \int \frac{dx}{1 + (\tfrac{3}{2} x)^2}$$

$$= \tfrac{1}{4} \cdot \tfrac{2}{3} \int \frac{\tfrac{3}{2} dx}{1 + (\tfrac{3}{2} x)^2}.$$

Hence

$$y = \tfrac{1}{6} \tan^{-1} (\tfrac{3}{2}x) + C.$$

Illustration 6. If $dy = \dfrac{dx}{\sqrt{4 - 9x^2}}$, find y.

$$y = \int \frac{dx}{\sqrt{4 - 9x^2}}$$

$$= \tfrac{1}{2} \int \frac{dx}{\sqrt{1 - (\tfrac{3}{2}x)^2}}$$

$$= \tfrac{1}{2} \cdot \tfrac{2}{3} \int \frac{\tfrac{3}{2}\,dx}{\sqrt{1 - (\tfrac{3}{2}x)^2}}.$$

The expression under the integral sign is now of the form $\dfrac{du}{\sqrt{1 - u^2}}$ whose integral is $\sin^{-1} u$. Hence

$$y = \tfrac{1}{3} \sin^{-1} (\tfrac{3}{2}x) + C.$$

Exercises

Find $\dfrac{dy}{dx}$ in Exercises 1–10.

1. $y = \sin^{-1} (x^2)$.
2. $y = \sin^{-1} (x - 1)$.
3. $y = \tan^{-1} (x^2)$.
4. $y = \tan^{-1} (x - 1)$.
5. $y = \sin (\sin^{-1} x)$.
6. $y = \sin^{-1} (\sin x)$.
7. $y = \tan^{-1} \dfrac{x}{x - 1}$.
8. $y = \sin^{-1} (1 - x)^2$.
9. $y = \sec^{-1} (x^2 - 3)$.
10. $y = x \sin^{-1} x$.

Integrate:

11. $dy = \dfrac{dx}{4 + x^2}$.
12. $dy = \dfrac{dx}{9 + x^2}$.
13. $dy = \dfrac{dx}{25 + 16x^2}$.
14. $dy = \dfrac{dx}{a^2 + x^2}$. *Ans.* $y = \dfrac{1}{a} \tan^{-1} \dfrac{x}{a} + C$.
15. $dy = \dfrac{x\,dx}{1 + x^4}$.

16. $dy = \dfrac{dx}{\sqrt{1 - 4x^2}}.$

17. $dy = \dfrac{dx}{\sqrt{9 - 4x^2}}.$

18. $dy = \dfrac{dx}{\sqrt{a^2 - x^2}}.$ *Ans.* $y = \sin^{-1}\dfrac{x}{a} + C.$

19. $dy = \dfrac{x\,dx}{\sqrt{1 - x^4}}.$

20. $dy = \dfrac{dx}{x\sqrt{x^2 - a^2}} = \dfrac{1}{a}\,\dfrac{\dfrac{dx}{a}}{\dfrac{x}{a}\sqrt{\left(\dfrac{x}{a}\right)^2 - 1}}.$ *Ans.* $y = \dfrac{1}{a}\sec^{-1}\dfrac{x}{a} + C.$

Using the results of Exercises 14, 18, and 20 as formulas, evaluate the following integrals:

21. $\displaystyle\int\frac{dx}{\sqrt{16 - x^2}}.$

22. $\displaystyle\int\frac{dx}{\sqrt{16 - 9x^2}}.$

23. $\displaystyle\int\frac{dx}{\sqrt{1 - 9x^2}}.$

24. $\displaystyle\int\frac{dx}{25 + x^2}.$

25. $\displaystyle\int\frac{dx}{25 + 16x^2}.$

26. $\displaystyle\int\frac{dx}{1 + 16x^2}.$

27. $\displaystyle\int\frac{dx}{x\sqrt{x^2 - 25}}.$

28. $\displaystyle\int\frac{dx}{x\sqrt{9x^2 - 25}}.$

29. $\displaystyle\int\frac{dx}{x\sqrt{9x^2 - 1}}.$

30. $\displaystyle\int\frac{dx}{x^2 + 17}.$

31. $\displaystyle\int\frac{dx}{\sqrt{13 - x^2}}.$

32. $\displaystyle\int\frac{dx}{x\sqrt{x^2 - 19}}.$

33. $\displaystyle\int\frac{dx}{5x^2 + 8}.$

34. $\displaystyle\int\frac{dx}{x^2+4x+5} = \int\frac{dx}{(x+2)^2+1}.$

35. $\displaystyle\int\frac{dx}{x\sqrt{3x^2 - 14}}.$

36. $\displaystyle\int\frac{dx}{\sqrt{9 - x^2}}.$

37. Find the area between the ordinates $x = 0$, $x = \frac{1}{2}$, the X-axis and the curve $y = \dfrac{1}{\sqrt{1 - x^2}}.$

38. Find the area under the curve $y = \dfrac{4a^3}{x^2 + 4a^2}$, above the X-axis, and between the ordinates $x = 0$ and $x = b$. Find the limit of the area as b increases without limit.

39. Find the mean ordinate of the curve $y = \frac{1}{1 + x^2}$, between the limits $x = 0$ and $x = \frac{\pi}{4}$.

78. Velocity and Acceleration. If a particle is moving in a curved path, its velocity at any point is represented by a vector laid off along the tangent with its length equal to the magnitude of the velocity, $\frac{ds}{dt}$. Thus the velocity at the point P, Fig. 55, is represented by the vector PT. It can be resolved into the components PK and PM, parallel to the X- and Y-axes, respectively. These components represent the time rates of change of the coördinates of the moving point P, *i.e.*,

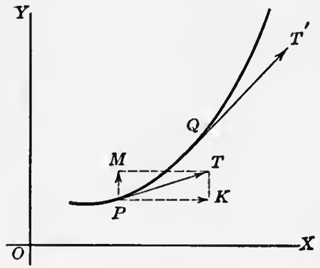

FIG. 55.

$$PK = \frac{dx}{dt}$$

and (1)

$$PM = \frac{dy}{dt}.$$

Since

$$PT = \sqrt{(PK)^2 + (PM)^2},$$

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}. \qquad (2)$$

This relation can be obtained directly from (2), **§63,** if we consider x and y functions of t. For, we can divide by dt and obtain the equation (2).

In Fig. 55, let PT be the velocity at P, and QT' that at Q. Draw from a common origin, o, Fig. 56, the vectors op and oq equal to the vectors PT and QT', respectively. Then pq equals the vector increment, Δv. The average acceleration for the interval Δt is equal to $\frac{\Delta v}{\Delta t}$ directed along pq. Lay off, on pq, pm

equal to $\frac{\Delta v}{\Delta t}$. As Δt approaches zero, Q approaches P, and q approaches p as indicated by the dotted line, Fig. 56; pm approaches a vector pt directed along the tangent to the arc pq at p. This vector, the limit of $\frac{\Delta v}{\Delta t}$, represents the acceleration of the particle moving in the curved path. Let us calculate its x and y components. In Fig. 56, denote:

op by v and its components by v_x and v_y,
oq by v' and its components by v'_x and v'_y,
pq by Δv and its components by $\Delta v_x = v'_x - v_x$ and $\Delta v_y = v'_y - v_y$,
pt by j and its components by j_x and j_y.

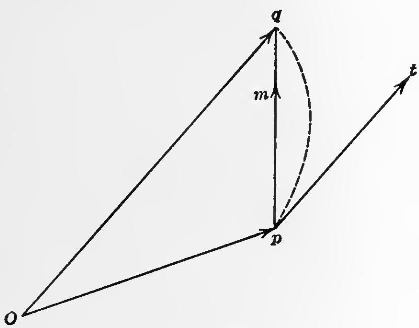

FIG. 56.

Then

$$j_x = \lim_{\Delta t \doteq 0} \frac{\Delta v_x}{\Delta t} = \frac{dv_x}{dt} = \frac{d\left(\frac{dx}{dt}\right)}{dt} = \frac{d^2x}{dt^2} \tag{3}$$

$$j_y = \lim_{\Delta t \doteq 0} \frac{\Delta v_y}{\Delta t} = \frac{dv_y}{dt} = \frac{d\left(\frac{dy}{dt}\right)}{dt} = \frac{d^2y}{dt^2}. \tag{4}$$

The magnitude and direction of the vector j are given by:

$$j = \sqrt{\left(\frac{d^2x}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2}\right)^2} \tag{5}$$

and

$$\tan \phi = \frac{\frac{d^2y}{dt^2}}{\frac{d^2x}{dt^2}},$$

where ϕ is the angle made by pt, Fig. 56, with the positive direction of the X-axis.

Again we can resolve the acceleration j into components along the tangent and normal. In Fig. 57, PL is the tangential component and PJ is the normal component. The tangential component clearly produces the change in the magnitude of the velocity, and the normal component the change in its direction.

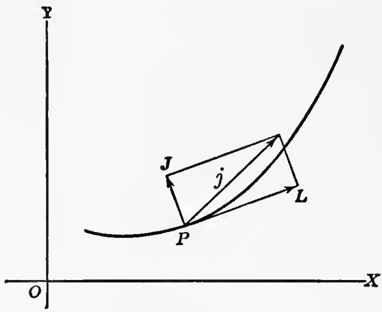

FIG. 57.

79. Angular Velocity and Acceleration. If a body is rotating about an axis, the amount of rotation is given by the angle θ through which a line in the body turns which intersects the axis and is perpendicular to it. Thus in the case of a wheel the rotation is measured by the angle θ through which a spoke turns. θ is a function of the time t. The rotation is uniform if the body rotates through equal angles in equal intervals of time. If the uniform rate of rotation is ω radians per second, the body rotates through $\theta = \omega t$ radians in t seconds. If the rotation is not uniform the rate at which the body is rotating at any instant, the angular velocity, is

$$\omega = \lim_{\Delta t \doteq 0} \frac{\Delta \theta}{\Delta t} = \frac{d\theta}{dt}.$$

Similarly, the angular acceleration α is the time rate of change of the angular velocity. Then,

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}.$$

If we consider a particle at a distance r from the axis of rotation, its linear velocity v is

$$v = \omega r$$

and is directed along the tangent to the circle described by the particle. The tangential acceleration is

$$j_t = \alpha r.$$

Exercises

1. The following formulas have been established for linear motion, with constant acceleration:

$$v = v_o + jt.$$

$$s = v_o t + \tfrac{1}{2}jt^2.$$

$$\frac{v^2}{2} - \frac{v_o^2}{2} = js. \qquad \text{(See §38.)}$$

Show that the corresponding formulas for rotation are:

$$\omega = \omega_o + \alpha t.$$

$$\theta = \omega_o t + \tfrac{1}{2}\alpha t^2.$$

$$\frac{\omega^2}{2} - \frac{\omega_o^2}{2} = \alpha\theta.$$

2. A flywheel 10 feet in diameter makes 25 revolutions a minute. What is the linear velocity of a point on the rim?

3. Find the constant acceleration, such as the retardation caused by a brake, which would bring this wheel to rest in 30 seconds. How many revolutions would it make before coming to rest?

4. A resistance retards the motion of a wheel at the rate of 0.5 radian per second per second. If the wheel is running at the rate of 10 revolutions a second when the resistance begins to act, how many revolutions will it make before stopping?

5. A wheel of radius r is rotating with the uniform angular velocity ω. Find the direction and magnitude of the acceleration of a point on the rim.

HINT. The coördinates of the point can be written $x = r \cos \omega t$, $y = r \sin \omega t$. Find $\frac{d^2x}{dt^2}$ and $\frac{d^2y}{dt^2}$.

6. A wheel of radius r is rolling with the uniform angular velocity ω along a horizontal surface without slipping. How fast is the axle moving forward? The parametric equations of a point P on the rim are:

$$x = r(\omega t - \sin \omega t)$$
$$y = r(1 - \cos \omega t).$$

Find the magnitude and the direction of the velocity of P at any instant. What is the velocity of a point at the top of the wheel? At the bottom?

7. If a particle moves in such a way that its coördinates are $x = a \cos t + b$, $y = a \sin t + c$, where t denotes time, find the equation of the path and show that the particle moves with constant tangential velocity.

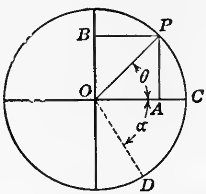

FIG. 58.

80. Simple Harmonic Motion. Let the point P, Fig. 58, move upon the circumference of a circle of radius a feet with the uniform velocity of v feet per second, so that the radius OP rotates at the rate of $\frac{v}{a} = \omega$ radians per second. The projection, B, of P on the vertical diameter moves up and down. If the point P was at C when $t = 0$, the displacement, $OB = y$, is given by

$$y = a \sin \theta = a \sin \omega t.$$

If the point P was at D when $t = 0$, we have

$$y = a \sin (\omega t - \alpha). \tag{1}$$

Any motion such that the displacement at time t is given by (1) is called a simple harmonic motion. Thus the point B, Fig. 58, describes simple harmonic motion. The abbreviation "*S.H.M.*" will be used for "*simple harmonic motion.*"

From (1) it follows that the velocity of a point describing *S.H.M.* is

$$\frac{dy}{dt} = a \omega \cos (\omega t - \alpha) \tag{2}$$

and that the acceleration is

$$\frac{d^2y}{dt^2} = -a\,\omega^2 \sin(\omega t - \alpha). \tag{3}$$

The second member is $-\omega^2 y$, by equation (1). Hence

$$\frac{d^2y}{dt^2} = -\,\omega^2 y, \tag{4}$$

or

$$\frac{d^2y}{dt^2} + \omega^2 y = 0. \tag{5}$$

Equation (4) shows that the acceleration of a particle describing *S.H.M.* is proportional to the displacement and oppositely directed. That the acceleration is oppositely directed to the displacement is to be expected from the character of the motion, which is an oscillation about a position of equilibrium. Thus if the body is above this position the force is directed downward, and *vice versa*. In Fig. 58, the point B has a positive acceleration when below O and a negative acceleration when above O. The acceleration is zero at O, a maximum at the lower end of the diameter, and a minimum at the upper end.

In accordance with (2) the velocity is zero at the two ends of the diameter. The velocity has its greatest numerical value when B passes through O in either direction.

Equation (4), or (5), is called the differential equation of *S.H.M.* The proportionality factor ω^2 is connected with the period T by the relation $T = \frac{2\pi}{\omega}\cdot$ The equation (4) was obtained from (1). Frequently it is desired to solve the converse problem, viz., to find the motion of a particle whose acceleration is proportional to the displacement and oppositely directed. In other words, a relation between y and t is sought which satisfies equation (4). Clearly (1) is such a relation. However, it will be instructive to obtain this relation directly from (4).

First, a differential equation equivalent to (4) will be obtained in the solution of the problem of the motion of the simple pendulum.

81. The Simple Pendulum. Let P, Fig. 59, be a position of the bob of a simple pendulum at a given instant and let it be moving to the right. If s denotes the displacement considered positive on the right of the position of equilibrium, $\frac{d^2s}{dt^2}$ is the acceleration in the direction of the tangent PT, for $\frac{ds}{dt}$ is the velocity along the tangent. This acceleration must be equal to the tangential component of the acceleration due to gravity, if the resistance of the air be neglected. This component is equal to $-g \sin\theta$. Since it acts in a direction opposite to that in which s is increasing, it must be taken with the negative sign, *i.e.*, the acceleration diminishes the velocity. We have then

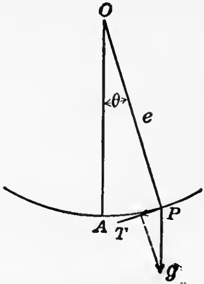

FIG. 59.

$$\frac{d^2s}{dt^2} = -g \sin\theta. \tag{1}$$

If the angle through which the pendulum swings is small, $\sin\theta$ can be replaced by θ. Then (1) becomes

$$\frac{d^2s}{dt^2} = -g\theta. \tag{2}$$

Since $s = l\theta$,

$$\frac{d^2\theta}{dt^2} = -\frac{g\theta}{l}. \tag{3}$$

Putting $\frac{g}{l} = \omega^2$ for convenience in writing,

$$\frac{d^2\theta}{dt^2} = -\omega^2\theta. \tag{4}$$

Multiplying by $2\frac{d\theta}{dt}$ and integrating,

$$\left(\frac{d\theta}{dt}\right)^2 = -\omega^2\theta^2 + C^2.$$

The arbitrary constant is written for convenience in the form C^2. The constant must be positive. Otherwise the velocity would be imaginary. Extracting the square root,

$$\frac{d\theta}{dt} = \sqrt{C^2 - \omega^2\theta^2},$$

or

$$\frac{d\theta}{\sqrt{C^2 - \omega^2\theta^2}} = dt. \tag{5}$$

Integration gives

$$\frac{1}{\omega}\sin^{-1}\frac{\omega\theta}{C} = t + C_1.$$

$$\sin^{-1}\frac{\omega\theta}{C} = \omega t + \omega C_1$$

$$= \omega t + C_2,$$

where the constant ωC_1 is replaced by the constant C_2. Then

$$\frac{\omega\theta}{C} = \sin(\omega t + C_2),$$

$$\theta = \frac{C}{\omega}\sin(\omega t + C_2)$$

$$= C_3 \sin(\omega t + C_2),$$

where $\frac{C}{\omega}$ has been replaced by C_3. Therefore

$$\theta = C_3 \sin(\omega t + C_2) \tag{6}$$

is the equation of the angular displacement of the pendulum.

The form of (6) shows that the motion is of period $\frac{2\pi}{\omega} = 2\pi\sqrt{\frac{l}{g}}$. It is a *S.H.M.* and contains two arbitrary constants. They can be determined by two conditions, *e.g.*, the displacement and velocity at a given instant. Suppose the bob drawn aside to the right so that the string makes an angle θ_o with the vertical. The bob is then released without being given an impulse; *i.e.*, with an initial velocity zero. The time will be counted from the instant of release. The conditions are then

$$\theta = \theta_o \tag{7}$$

and

$$\frac{d\theta}{dt} = 0 \tag{8}$$

when $t = 0$. From (6),

$$\frac{d\theta}{dt} = \omega C_3 \cos(\omega t + C_2).$$

The condition (8) gives

$$0 = \omega C_3 \cos C_2,$$

or $\cos C_2 = 0$. Whence $C_2 = \frac{\pi}{2}$. Then (6) becomes

$$\theta = C_3 \cos \omega t.$$

The condition (7) gives

$$\theta_o = C_3.$$

Hence

$$\theta = \theta_o \cos \omega t. \tag{9}$$

Multiplying by l and recalling that $l\theta = s$, and denoting $l\theta_o$ by s_o, we have as the equation for the displacement s,

$$s = s_o \cos \omega t. \tag{10}$$

The period is $T = \dfrac{2\pi}{\omega} = 2\pi \sqrt{\dfrac{l}{g}}$. When is the velocity of the bob greatest? When least, numerically?

Equation (6), the solution of (4), shows that, if the acceleration of a particle is proportional to its displacement and oppositely directed, the particle describes *S.H.M.*

Exercises

1. Write the differential equations of the following simple harmonic motions. Find the period in each case.

$$y = 5 \sin 3t.$$

$$y = 6 \sin \left(3t + \frac{\pi}{3}\right).$$

$$y = 5 \cos 3t.$$

$$y = 4 \sin 2t + 3 \cos 2t.$$

$$y = 7 \sin (8t + \alpha).$$

2. Write the equation of a *S.H.M.* which satisfies the equations:

$$\frac{d^2y}{dt^2} + 9y = 0.$$

$$\frac{d^2y}{dt^2} + 3y = 0.$$

$$\frac{d^2y}{dt^2} + a^2y = 0.$$

CHAPTER IX

EXPONENTIAL AND LOGARITHMIC FUNCTIONS

82. Derivative of the Exponential and Logarithmic Functions. Let

$$y = a^x. \qquad (1)$$

Then

$$y + \Delta y = a^{x + \Delta x}$$

$$\Delta y = a^x(a^{\Delta x} - 1)$$

$$\frac{\Delta y}{\Delta x} = a^x\left(\frac{a^{\Delta x} - 1}{\Delta x}\right)$$

$$\frac{dy}{dx} = a^x \lim_{\Delta x \doteq 0} \frac{a^{\Delta x} - 1}{\Delta x}. \qquad (2)$$

Since $\frac{a^{\Delta x} - 1}{\Delta x}$ is independent of x, $\lim_{\Delta x \doteq 0} \frac{a^{\Delta x} - 1}{\Delta x}$ is a constant for a given value of a. Call this constant K, so that

$$K = \lim_{\Delta x \doteq 0} \frac{a^{\Delta x} - 1}{\Delta x}. \qquad (3)$$

Then from (2),

$$\frac{dy}{dx} = Ka^x. \qquad (4)$$

Equation (4) shows that the slope of the curve $y = a^x$ is proportional to the ordinate of the curve. In other words, the rate of increase of the exponential function is proportional to the function itself.

When $x = 0$, it follows from (2) and (3) that

$$\frac{dy}{dx} = \lim_{\Delta x \doteq 0} \frac{a^{\Delta x} - 1}{\Delta x} = K.$$

Consequently the constant K introduced above is the slope of the curve $y = a^x$ at the point (0, 1). This slope depends upon the value of a. Let e be that particular value of a for which the corresponding curve, $y = e^x$, has a slope equal to 1 at the point where it crosses the Y-axis.

If, then,

$$y = e^x, \tag{5}$$

equation (4) becomes

$$\frac{dy}{dx} = e^x,$$

since $K = 1$ in this case. Or

$$\frac{de^x}{dx} = e^x. \tag{6}$$

Then

$$\frac{\mathbf{de^u}}{\mathbf{dx}} = \mathbf{e^u} \frac{\mathbf{du}}{\mathbf{dx}} \tag{7}$$

and

$$\mathbf{de^u = e^u du.} \tag{8}$$

Equation (6) shows that the slope of the curve $y = e^x$ is equal to the ordinate of the curve. The number e is the base of the natural system of logarithms. It is sometimes called the Naperian base. Its value, 2.71828 · · · · , will be calculated later in the course.

The formula for the derivative of the natural logarithm of a function is now easily obtained. *In calculus if no base is indicated, the natural base is understood.* Thus $\log u$ means $\log_e u$.

If

$$y = \log u,$$

$$u = e^y$$

and by (7)

$$\frac{du}{dx} = e^y \frac{dy}{dx}.$$

Whence

$$\frac{dy}{dx} = \frac{1}{e^y} \frac{du}{dx}$$

$$= \frac{1}{u} \frac{du}{dx}.$$

That is

$$\frac{\mathbf{d\,(log\,u)}}{\mathbf{dx}} = \frac{\mathbf{1}}{\mathbf{u}} \frac{\mathbf{du}}{\mathbf{dx}},$$

or (9)

$$\mathbf{d(log\,u)} = \frac{\mathbf{du}}{\mathbf{u}}.$$

Since[1]

$$\log_a u = \log_a e \log u, \tag{10}$$

$$\frac{d(\log_a u)}{dx} = \log_a e \frac{1}{u}\frac{du}{dx},$$

or (11)

$$d(\log_a u) = \log_a e \frac{du}{u}.$$

If $y = a^u$,

$$\log y = u \log a$$

$$\frac{1}{y}\frac{dy}{dx} = \log a \frac{du}{dx}$$

$$\frac{dy}{dx} = y \log a \frac{du}{dx}$$

$$= a^u \log a \frac{du}{dx}.$$

That is

$$\frac{da^u}{dx} = a^u \log a \frac{du}{dx}$$

or (12)

$$da^u = a^u \log a \, du$$

Illustrations.

1. If $y = e^{x^2}$, $dy = e^{x^2}d(x^2) = 2xe^{x^2}dx.$
2. If $y = e^{\sin x}$, $dy = e^{\sin x}d(\sin x) = \cos x \, e^{\sin x}\,dx.$
3. If $y = \log_{10}(x+1)$, $dy = \log_{10} e \frac{d(x+1)}{x+1} = \log_{10} e \frac{dx}{x+1}.$
4. If $y = \log(x+1)$, $dy = \frac{dx}{x+1}.$

[1] Let

$$z = \log u$$

Then

$$e^z = u$$

Taking logarithms to the base a,

$$z \log_a e = \log_a u$$

That is

$$\log u \log_a e = \log_a u$$

5. If $y = e^{\tan^{-1}x}$, $\quad dy = e^{\tan^{-1}x}\,d(\tan^{-1}x) = e^{\tan^{-1}x}\dfrac{dx}{1+x^2}$.

6. If $y = \log\dfrac{(1+x)^2}{(1-x)^3}$, $\quad y = 2\log(1+x) - 3\log(1-x)$,

and

$$dy = \frac{2dx}{1+x} + \frac{3dx}{1-x} = \frac{5+x}{1-x^2}dx.$$

7. If $y = e^x \sin x$, $\quad \dfrac{dy}{dx} = e^x(\cos x + \sin x)$

and

$$\frac{d^2y}{dx^2} = 2e^x \cos x.$$

Exercises

Find the first derivative of each of the following:

1. $y = e^{x^2}$. **6.** $y = \log(1 - x^2)$. **11.** $y = e^{\sin x}$.

2. $y = e^{x^4}$. **7.** $y = e^x \cos x$. **12.** $y = e^{\tan x^2}$.

3. $y = \log(x^2)$. **8.** $y = e^{3x}$. **13.** $y = \log\sqrt{x^2 - 1}$.

4. $y = \log(x^3)$. **9.** $y = e^{3x^2}$. **14.** $y = e^{-x}\sin x$.

5. $y = \log(x^2 - 1)$. **10.** $y = \log\dfrac{(1+x^2)^3}{(1-x^2)^4}$. **15.** $y = 10^x$.

16. $y = \log_{10} x$.

17. $y = 5^x$.

18. $y = x^5\, 5^x$.

19. Show that the subtangent for the curve $y = a^x$ is a constant. What is this length when $a = e$?

Illustrations.

8. If $dy = e^x dx$, $\quad y = \int e^x dx = e^x + C$.

9. If
$$dy = xe^{x^2}dx.$$
$$y = \int xe^{x^2}dx$$
$$= \tfrac{1}{2}\int e^{x^2}d(x^2)$$
$$= \tfrac{1}{2}e^{x^2} + C.$$

10. If
$$dy = \frac{dx}{x},$$
$$y = \log x + C$$
$$= \log x + \log K$$
$$= \log Kx.$$

11. If
$$dy = \frac{dx}{x+1},$$
$$y = \log(x+1) + C$$
$$= \log K(x+1).$$

12. If
$$dy = \frac{xdx}{x^2+1},$$
$$y = \int \frac{xdx}{x^2+1}$$
$$= \tfrac{1}{2}\int \frac{2xdx}{x^2+1}$$
$$= \tfrac{1}{2}\log(x^2+1) + C$$
$$= \log\sqrt{x^2+1} + C$$
$$= \log K\sqrt{x^2+1}.$$

13. If
$$dy = e^{\sin x}\cos x\,dx,$$
$$y = \int e^{\sin x}\cos x\,dx$$
$$= e^{\sin x} + C.$$

14. If
$$dy = \frac{(x+1)dx}{x^2+2x+3},$$
$$y = \int \frac{(x+1)dx}{x^2+2x+3}$$
$$= \tfrac{1}{2}\int \frac{2(x+1)dx}{x^2+2x+3}$$
$$= \tfrac{1}{2}\log(x^2+2x+3) + C$$
$$= \log\sqrt{x^2+2x+3} + C$$
$$= \log K\sqrt{x^2+2x+3}.$$

15. If
$$dy = \tan x\,dx,$$
$$y = \int \tan x\,dx$$
$$= -\int -\frac{\sin x}{\cos x}\,dx$$
$$= -\log\cos x + C$$
$$= \log\sec x + C.$$

16. If

$$dy = \cot x dx,$$

$$y = \int \cot x\, dx$$

$$= \int \frac{\cos x}{\sin x} dx$$

$$= \log \sin x + C.$$

17. If

$$dy = \sec x\, dx,$$

$$y = \int \sec x\, dx + C$$

$$= \int \frac{(\sec x + \tan x) \sec x\, dx}{\sec x + \tan x}$$

$$= \int \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x} dx$$

$$= \log (\sec x + \tan x) + C.$$

18. Find

$$\int \frac{dx}{\sqrt{x^2 \pm a^2}}.$$

Let

$$v = \sqrt{x^2 \pm a^2}.$$

$$v^2 = x^2 \pm a^2$$

$$2v\, dv = 2x\, dx$$

whence

$$\frac{dv}{x} = \frac{dx}{v},$$

and

$$\frac{dv}{x} = \frac{dx}{v} = \frac{dx + dv}{x + v}.$$

Then, since $v = \sqrt{x^2 \pm a^2}$,

$$\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \int \frac{dx + dv}{x + v}$$

$$= \log (x + v) + C$$

$$= \log (x + \sqrt{x^2 \pm a^2}) + C.$$

19. If

$$dy = \frac{e^x dx}{e^x + 1},$$

$$y = \int \frac{e^x dx}{e^x + 1}$$

$$= \log (e^x + 1) + C.$$

20. If
$$\frac{dy}{y} = \frac{dx}{x},$$
$$\log y = \log x + \log C$$
$$\log y = \log Cx$$
$$y = Cx.$$

21. If
$$\frac{dy}{y} = n\frac{dx}{x},$$
$$\log y = n \log x + \log C$$
$$= \log x^n + \log C$$
$$= \log Cx^n,$$
$$y = Cx^n.$$

Exercises

The results of Illustrations 15, 16, 17, and 18 are to be used as formulas of integration.

In the following exercises find y:

20. $dy = x^2 e^{x^3} dx.$

21. $dy = e^{\tan x} \sec^2 x\, dx.$

22. $dy = \dfrac{dx}{x + 1}.$

23. $dy = \dfrac{dx}{1 - x}.$

24. $dy = \dfrac{x\, dx}{1 + x^2}.$

25. $dy = \dfrac{x\, dx}{1 - x^2}.$

26. $dy = \dfrac{(e^x + 1)dx}{e^x + x}.$

27. $dy = \tan 2x\, dx.$

28. $dy = \cot 2x\, dx.$

29. $dy = \sec 2x\, dx.$

30. $dy = (\log x)^2 \dfrac{dx}{x}.$

31. $dy = \dfrac{\cos x\, dx}{\sin x + 3}.$

32. $dy = \dfrac{(e^x - e^{-x})dx}{e^x + e^{-x}}.$

33. $dy = \dfrac{\sec^2 x\, dx}{\tan x}.$

34. $dy = \dfrac{(e^{2x} - 1)dx}{e^{2x} + 1}.$

35. Find the area between the equilateral hyperbola $xy = 10$, the X-axis, and the lines $x = 1$ and $x = 2$.

36. The volume of a gas in a cylinder of cross section A expands from volume v_1 to volume v_2. If it expands without change in temperature the pressure, p, on the piston varies inversely as the volume (Boyle's Law, $pv = K$). Show that the work done by the expansion is

$$K \int_{v_1}^{v_2} \frac{dv}{v} = K \log \frac{v_2}{v_1}.$$

37. The subtangent of a curve is of constant length, k. Find the equation of the curve.

38. For what value of x is the rate of change of $\log_{10} x$ the same as the rate of change of x?

39. $\int \csc x \, dx$. (See Illustration 17.)

40. $\int \tan 3\theta \, d\theta$.

41. $\int e^{\cos\theta} \sin\theta \, d\theta$.

42. $\int \cot \frac{\theta}{3} \, d\theta$.

43. $\int x \sec^2(x^2 + 1) dx$.

44. $\int \sec \frac{x}{3} \, dx$.

45. $\int_1^x \frac{dx}{x}$.

46. $\int_0^1 \frac{xdx}{\sqrt{1 + x^2}}$.

47. $\int \frac{dx}{\sqrt{1 - x^2}}$.

48. $\int \frac{xdx}{\sqrt{1 - x^2}}$.

49. $\int 10^x dx$.

50. $\int \frac{dx}{\cos^2 (5x - 4)}$.

51. $\int_0^{\frac{\pi}{2}} \frac{(1 - \sin x)dx}{x + \cos x}$.

52. $\int e^{(1 + \cos\theta)} \sin\theta \, d\theta$.

53. $\int \frac{x^2 dx}{a^2 + x^3}$.

54. $\int \frac{x^2 dx}{\sqrt{a^2 + x^3}}$.

55. $\int \frac{x^2 dx}{(a^2 + x^3)^2}$.

56. $\int (\tan 2\theta - 1)^2 d\theta = \frac{1}{2} \tan 2\theta + \log \cos 2\theta + C$.

57. $\int \csc(7x + 5) \, dx$.

58. $\int \frac{(x + 2)dx}{\sqrt{x^2 + 4x + 7}}$.

59. $\int e^{\frac{1}{x}} \frac{dx}{x^2}$.

60. $\int \frac{(3x + 2)dx}{3x^2 + 4x + 9}$.

61. $\int \frac{(x + 2)dx}{(x^2 + 4x + 7)^2}$.

62. Show that $y = ae^{-kt} \cos \omega t$ satisfies the differential equation

$$\frac{d^2y}{dt^2} + 2k \frac{dy}{dt} + (\omega^2 + k^2)y = 0.$$

63. Find the mean ordinate of the curve $y = \frac{1}{x}$ between the limits $x = 1$ and $x = 2$.

83. Logarithmic Differentiation. It is often advantageous in finding the derivative of $y = f(x)$ to take the logarithm of each member before differentiating. A number of examples will be solved to illustrate the process.

Illustration 1. Find the derivative of $\frac{(x - 1)^{\frac{2}{3}}}{(x + 1)^{\frac{3}{5}}}$. Let

$$y = \frac{(x - 1)^{\frac{2}{3}}}{(x + 1)^{\frac{3}{5}}}$$

and take the logarithm of each member.

$$\log y = \tfrac{2}{3} \log (x - 1) - \tfrac{3}{5} \log (x + 1).$$

Differentiating,

$$\frac{1}{y}\frac{dy}{dx} = \frac{2}{3(x - 1)} - \frac{3}{5(x + 1)}$$

$$= \frac{x + 19}{15(x^2 - 1)}.$$

$$\frac{dy}{dx} = \frac{x + 19}{15(x^2 - 1)} y$$

$$= \frac{x + 19}{15(x^2 - 1)} \frac{(x - 1)^{\frac{2}{3}}}{(x + 1)^{\frac{3}{5}}}$$

$$= \frac{x + 19}{15(x - 1)^{\frac{1}{3}}(x + 1)^{\frac{8}{5}}}.$$

Illustration 2. Find the derivative of $\frac{\sqrt{1 - x^2}}{\sqrt[3]{x^2 + 1}}$.

$$y = \frac{\sqrt{1 - x^2}}{\sqrt[3]{x^2 + 1}}$$

$$\log y = \tfrac{1}{2}\log(1 - x^2) - \tfrac{1}{3}\log(x^2 + 1)$$

$$\frac{1}{y}\frac{dy}{dx} = -\frac{x}{1 - x^2} - \frac{2x}{3(x^2 + 1)}$$

$$= -\frac{x(5 + x^2)}{3(1 - x^4)}$$

$$\frac{dy}{dx} = -\frac{x(5 + x^2)}{3(1 - x^4)}\frac{\sqrt{1 - x^2}}{\sqrt[3]{x^2 + 1}}$$

$$= -\frac{x(5 + x^2)}{3\sqrt{1 - x^2}\,(x^2 + 1)^{\frac{4}{3}}}.$$

This method is manifestly shorter and simpler than that of differentiating by the rule for the derivative of a quotient.

Illustration 3. Find the derivative of $(x^2 + 1)^{3x+2}$.

$$y = (x^2 + 1)^{3x+2}$$

$$\log y = (3x + 2)\log(x^2 + 1)$$

$$\frac{1}{y}\frac{dy}{dx} = (3x + 2)\frac{2x}{x^2 + 1} + 3\log(x^2 + 1)$$

$$\frac{dy}{dx} = \left[(3x + 2)\frac{2x}{x^2 + 1} + 3\log(x^2 + 1)\right](x^2 + 1)^{3x+2}.$$

$\dfrac{1}{y}\dfrac{dy}{dx}$ is called the logarithmic derivative of y with respect to x. It will be considered further in a later article.

Exercises

Find the derivative in Exercises 1–8.

1. $y = \dfrac{(x + 1)^{\frac{3}{7}}}{(x - 7)^{\frac{2}{5}}}$.

2. $y = \dfrac{(x + 3)^2}{(x - 4)^3 (x - 5)^4}$.

3. $y = (x + 1)^{\frac{2}{3}}(2x + 5)^{\frac{3}{4}}$.

4. $y = x(1 + x)\sqrt{1 - x}$.

5. $y = x^n n^x$. (Solve by two methods.)

6. $y = x^{\sin x}$.

7. $s = (7t + 3)10^{3t-2}$.

8. $y = x^{\sqrt{x}}$.

In Exercises 9–16 find the logarithmic derivative.

9. $y = e^{7x}$.

10. $y = x^2$.

12. $y = x^n$.

13. $y = cx^n$.

11. $y = \frac{1}{x^2}$. **14.** $y = e^{kx+L} = ce^{kx}$.

15. $y = 10^{kx+L}$.

16. $y = uv$, where u and v are functions of x.

17. $y = uvw$, where u, v, and w are functions of x. Find $\frac{dy}{dx}$.

Find y if its logarithmic derivative is:

18. $6x + 7$. *Ans.* $y = Ce^{3x^2+7x}$.

19. $\frac{3}{x}$. **20.** $\frac{n}{x}$. **21.** $F(x)$. **22.** k.

84. Compound Interest Law. If

$$y = Ce^{kt}, \tag{1}$$

$$\frac{dy}{dt} = Cke^{kt} = ky. \tag{2}$$

Equation (2) expresses the fact already noted in **§82,** as a characteristic property of the exponential function, viz., that the function increases at a rate proportional to itself. We can show, conversely, that if a function increases at a rate proportional to itself, it is an exponential function.

Thus, let it be given that

$$\frac{dy}{dt} = ky. \tag{3}$$

Then

$$\frac{dy}{y} = kdt$$

$$\log y = kt + C$$

$$y = e^{kt+C} = e^C e^{kt}.$$

Hence

$$y = C_1 e^{kt}. \tag{4}$$

When a function varies according to this law it is said to follow the "compound interest law." For, if a sum of money be placed at compound interest, its rate of increase, for any interest term, is proportional to the amount accumulated at the beginning of that term. The more frequently the interest is compounded the more nearly does the amount accumulated increase according to the exponential law.

In many cases in nature the function decreases at a rate proportional to itself. The compound interest law appears in this case in the form Ce^{-kt}, where k is a positive constant. For, if

$$\frac{dy}{dt} = -ky,$$

it follows that

$$y = Ce^{-kt}.$$

Illustration 1. Newton's law of cooling states that the temperature of a heated body surrounded by a medium of constant temperature decreases at a rate proportional to the difference in temperature between the body and the medium. Let θ denote the difference in temperature. Then

$$\frac{d\theta}{dt} = -k\theta. \tag{5}$$

and

$$\theta = Ce^{-kt}. \tag{6}$$

The meaning of the constant C is seen at once on setting $t = 0$. It is the difference in temperature between the body and the medium at the time $t = 0$. If this initial difference in temperature is known, (6) gives the temperature of the body at any later instant. Call the initial difference in temperature θ_0. (6) becomes

$$\theta = \theta_0 e^{-kt}. \tag{7}$$

The time which is required for the difference in temperature to fall from θ_1 to θ_2 can be found from (7). Thus

$$\theta_1 = \theta_0 e^{-kt_1}$$

$$\theta_2 = \theta_0 e^{-kt_2}$$

$$\frac{\theta_1}{\theta_2} = e^{-k(t_1 - t_2)}$$

whence

$$t_2 - t_1 = \frac{1}{k}\log\frac{\theta_1}{\theta_2}. \tag{8}$$

This result could have been obtained directly from the differential equation (5). Thus

$$\frac{d\theta}{dt} = -k\theta,$$

$$dt = -\frac{1}{k}\frac{d\theta}{\theta}. \tag{9}$$

Integrating the left-hand member between the limits t_1 and t_2 and the right-hand member between the limits θ_1 and θ_2,

$$\int_{t_1}^{t_2} dt = -\frac{1}{k}\int_{\theta_1}^{\theta_2}\frac{d\theta}{\theta}$$

$$= -\frac{1}{k}\log\theta\Big|_{\theta_1}^{\theta_2}$$

$$= \frac{1}{k}\log\frac{\theta_1}{\theta_2}$$

$$t_2 - t_1 = \frac{1}{k}\log\frac{\theta_1}{\theta_2}.$$

Illustration 2. Find the law of variation of the atmospheric pressure with height.

Consider a column of air of unit cross section (Fig. 60). Denote height above sea level by h and the pressure on unit cross section at this height by p. The difference in pressure at C and D is the weight of the gas within the element of volume of height Δh.

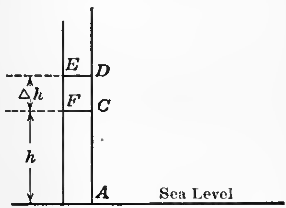

FIG. 60.

Thus

$$\Delta p = -g\rho\,\Delta h,$$

where ρ is the average density of the air in the volume $CDEF$.

Then

$$\frac{\Delta p}{\Delta h} = -g\rho \qquad (1)$$

and

$$\frac{dp}{dh} = -g\rho,$$

where ρ is the density at C. If the temperature is assumed constant, the air obeys Boyle's Law, $pv = c$, where v denotes the volume occupied by unit mass of air. Since

$$\rho = \frac{\text{mass}}{\text{volume}} = \frac{1}{v} = \frac{p}{c},$$

$$\frac{dp}{dh} = -kp,$$

where

$$k = \frac{g}{c}.$$

Integration gives

$$\log p = -kh + \log C_1,$$

or

$$p = C_1 e^{-kh}.$$

When $h = 0$, $p = p_0$, the pressure at sea level, and $C_1 = p_0$. Hence

$$p = p_0 e^{-kh}. \tag{2}$$

If h is measured in meters and p in millimeters of mercury, $k = \frac{1}{8000}$, (2) becomes

$$p = 760e^{-\frac{h}{8000}}. \tag{3}$$

Exercises

1. A law for the velocity of chemical reactions states that the amount of chemical change per unit of time is proportional to the mass of changing substance present in the system. The rate at which the change takes place is proportional to the mass of the substance still unchanged. If q denotes the original mass, find an expression for the mass remaining unchanged after a time t has elapsed.

2. Assuming that the retardation of a boat moving in still water is proportional to the velocity, find the distance passed over in time t after the engine was shut off, if the boat was moving at the rate of 7 miles per hour at that time. *Ans.* $s = \frac{7}{k}(1 - e^{-kt})$.

3. The number of bacteria per cubic centimeter of culture increases under proper conditions at a rate proportional to the number present. Find an expression for the number present at the end of time t. Find the time required for the number per cubic centimeter to increase from b_1 to b_2. Does this time depend on the number present at the time $t = 0$?

4. A disk is rotating about a vertical axis in a liquid. If the retardation due to friction of the liquid is proportional to the angular velocity ω, find ω after t seconds if the initial angular velocity was ω_0.

5. If the disk of Exercise 4 is rotating very rapidly, the retardation is proportional to ω^2. Find ω after t seconds if the initial angular velocity was ω_0.

85. Relative Rate of Increase. If the rate of change of a function is divided by the function itself, the quotient is the rate of change of the function per unit value of the function. This quotient has been called the *relative rate of increase* of the function. If a function varies according to the compound interest law, its relative rate of increase is constant, *i.e.*,

$$\frac{1}{y}\frac{dy}{dt} = k.$$

One hundred times the relative rate of increase is the percent rate of increase. Thus if

$$\frac{1}{y}\frac{dy}{dt} = 0.02,$$

the percent rate of increase is 2. This means that y increases 2 percent per unit time. Any of the Exercises 1–5 might have been stated in terms of the relative rate of increase of the function concerned.

Exercises

1. Given that the intensity of light is diminished 2 percent by passing through one millimeter of glass, find the intensity I as a function of t, the thickness of the glass through which the light passes.

2. The temperature of a body cooling according to Newton's Law fell from 30° to 18° in 6 minutes. Find the percent rate of decrease of temperature per minute.

86. Hyperbolic Functions. The engineering student is likely to meet in his reading certain functions called the hyperbolic functions. These functions present analogies with the circular functions and they are called hyperbolic sine, written sinh, hyperbolic cosine, written cosh, and so on.

These functions are defined by the equations:

$$\cosh x = \frac{e^x + e^{-x}}{2}, \qquad \operatorname{sech} x = \frac{1}{\cosh x},$$

$$\sinh x = \frac{e^x - e^{-x}}{2}, \qquad \operatorname{csch} x = \frac{1}{\sinh x},$$

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \qquad \coth x = \frac{1}{\tanh x}.$$

$\cosh x$ and $\operatorname{sech} x$ are even functions, while the remaining four are odd functions

Exercises

1. By making use of the definitions the student will show that the following identities hold. They are analogous to those satisfied by the circular functions.

$$\cosh^2 x - \sinh^2 x = 1.$$

$$1 - \tanh^2 x = \operatorname{sech}^2 x.$$

2. Show by the use of the defining equations that:

$$\frac{d \cosh x}{dx} = \sinh x.$$

$$\frac{d \sinh x}{dx} = \cosh x.$$

$$\frac{d \tanh x}{dx} = \operatorname{sech}^2 x.$$

$$\frac{d \coth x}{dx} = -\operatorname{csch}^2 x.$$

$$\frac{d \operatorname{sech} x}{dx} = -\operatorname{sech} x \tanh x.$$

$$\frac{d \operatorname{csch} x}{dx} = -\operatorname{csch} x \coth x.$$

3. Sketch the curves $y = \cosh x$, $y = \sinh x$, and $y = \tanh x$.

87. Inverse Hyperbolic Functions. The logarithms of certain functions can be expressed in terms of inverse hyperbolic functions

Let

$$y = \sinh^{-1} x.$$

$$x = \sinh y = \frac{e^y - e^{-y}}{2},$$

or

$$e^{2y} - 2xe^y - 1 = 0,$$

whence

$$e^y = x \pm \sqrt{x^2 + 1}.$$

The minus sign cannot be taken since e^y is always positive. Hence

$$e^y = x + \sqrt{x^2 + 1},$$

and

$$y = \sinh^{-1} x = \log (x + \sqrt{x^2 + 1}).$$

Exercises

1. Show that

$$\cosh^{-1} x = \log (x \pm \sqrt{x^2 - 1}).$$

Since

$$x - \sqrt{x^2 - 1} = \frac{1}{x + \sqrt{x^2 - 1}},$$

$$\log (x - \sqrt{x^2 - 1}) = -\log (x + \sqrt{x^2 - 1}).$$

Therefore

$$\cosh^{-1} x = \pm \log (x + \sqrt{x^2 - 1}).$$

The inverse hyperbolic cosine is then not single valued. Two values of $\cosh^{-1} x$, equal numerically but of opposite sign, correspond to each value of x greater than 1.

2. Show that:

$$\tanh^{-1} x = \tfrac{1}{2} \log \frac{1 + x}{1 - x}, \qquad \text{if} \quad x^2 < 1;$$

$$\coth^{-1} x = \tfrac{1}{2} \log \frac{x + 1}{x - 1}, \qquad \text{if} \quad x^2 > 1;$$

$$\operatorname{sech}^{-1} x = \pm \log \frac{1 + \sqrt{1 - x^2}}{x}, \qquad \text{if} \quad 0 < x \leqq 1;$$

$$\operatorname{csch}^{-1} x = \log \frac{1 + \sqrt{x^2 + 1}}{x}, \qquad \text{if} \quad x > 0;$$

and

$$\operatorname{csch}^{-1} x = \log \frac{1 - \sqrt{x^2 + 1}}{x}, \qquad \text{if} \quad x < 0.$$

The student is not advised to memorize the formulas of this and the preceding sections at this stage in his course, but to acquire sufficient familiarity with the hyperbolic functions to enable him to operate with these functions by referring to the definitions and formulas given here and to others that he will find in mathematical tables.

88. The Catenary. Let AOB, Fig. 61 a, be a cable suspended from the points A and B and carrying only its own weight. Let us find the equation of the curve assumed by the cable, considering it homogeneous. We shall assume that the curve has a vertical line of symmetry, OY, and that the tangent line drawn to the curve at O is horizontal.

Take OY as the Y-axis. Imagine a portion of the curve, OP, of length s, cut free. To hold this portion in equilibrium the forces H and T, Fig. 61 b, must be introduced at the cut ends.

H and T are, respectively, equal to the tension in the cable at the points O and P and they act in the direction of the tangent lines drawn to the curve at these points. The portion of the cable OP, Fig. 61 b, is in equilibrium. Hence H', the horizontal component of T, is equal to H.

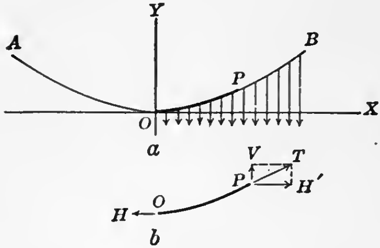

FIG. 61.

V, the vertical component of T, must balance the weight of the portion OP of the cable. Hence

$$V = sw,$$

where w is the weight of a unit length of the cable.

From Fig. 61 b, it is seen that

$$\frac{dy}{dx} = \frac{V}{H'} = \frac{V}{H} = \frac{ws}{H}.$$

Let

$$\frac{w}{H} = \frac{1}{a}.$$

Then

$$\frac{dy}{dx} = \frac{s}{a}. \tag{1}$$

This differential equation involves three variables, viz., x, y, and s. s may be eliminated by differentiating and substituting for $\frac{ds}{dx}$ its value,

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2}.$$

Thus

$$\frac{d\left(\frac{dy}{dx}\right)}{dx} = \frac{1}{a}\frac{ds}{dx} = \frac{1}{a}\sqrt{1 + \left(\frac{dy}{dx}\right)^2}.$$

The equation now involves only two variables and may be written

$$\frac{d\left(\frac{dy}{dx}\right)}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}} = \frac{1}{a}dx. \tag{2}$$

If we look upon $\frac{dy}{dx}$ as the variable u, the left-hand side of equation (2) is

$$\frac{du}{\sqrt{1+u^2}}$$

whose integral is $\log (u + \sqrt{1+u^2})$. (See Illustration 18, §82.) Integrating (2),

$$\log\left[\frac{dy}{dx} + \sqrt{1+\left(\frac{dy}{dx}\right)^2}\right] = \frac{x}{a} + C. \tag{3}$$

When $x = 0$, $\frac{dy}{dx} = 0$.

Hence $C = 0$ and (3) becomes

$$\frac{dy}{dx} + \sqrt{1+\left(\frac{dy}{dx}\right)^2} = e^{\frac{x}{a}}. \tag{4}$$

From the symmetry of the curve $\frac{dy}{dx}$ changes sign when x is replaced by $-x$. Then from (4),

$$-\frac{dy}{dx} + \sqrt{1+\left(\frac{dy}{dx}\right)^2} = e^{-\frac{x}{a}}. \tag{5}$$

Subtracting (5) from (4),

$$2\frac{dy}{dx} = e^{\frac{x}{a}} - e^{-\frac{x}{a}}, \tag{6}$$

or

$$\frac{dy}{dx} = \sinh\frac{x}{a}. \tag{7}$$

Integrating (7),

$$y = a\cosh\frac{x}{a} + C_2. \tag{8}$$

If the origin is taken a units below the point O, Fig. 61 a, $y = a$ when $x = 0$, and $C_2 = 0$. Hence

$$y = a\cosh\frac{x}{a}. \tag{9}$$

This is the equation of the curve assumed by the cable. It is called the catenary.

Equation (9) can be written

$$Y = \cosh X, \tag{10}$$

where

$$Y = \frac{y}{a} \quad \text{and} \quad X = \frac{x}{a}.$$

The constant a depends upon the tautness of the cable. Equation (10) shows that the curve $y = \cosh x$ if magnified the proper number of diameters will fit *any* cable hanging under its own weight.

The length of OP can be found by substituting in formula 2, §63, the value of $\frac{dy}{dx}$ given by (7), and integrating.

$$\begin{aligned} ds &= \sqrt{1 + \sinh^2 \frac{x}{a}}\, dx \\ &= \cosh \frac{x}{a}\, dx. \\ s &= a \sinh \frac{x}{a} + C_3. \end{aligned}$$

Since s is measured from the point where the curve crosses the Y-axis, $s = 0$ when $x = 0$. Hence $C_3 = 0$ and

$$s = a \sinh \frac{x}{a}. \tag{11}$$

Exercises

1. If the two supports A and B, Fig. 61, are on a level, L feet apart, and if the sag is d feet, show that the tension, T, in the cable at the points A and B is

$$T = w(a + d).$$

2. Beginning with equation (6) find expressions for y and s without making use of hyperbolic functions.

3. If the cable, Fig. 61, is drawn very taut, show that the equation of its curve is approximately

$$y = \frac{x^2}{2a},$$

if the origin of coördinates is taken at the lowest point of the cable.

Hint. Begin with equation (2) and note that $\left(\frac{dy}{dx}\right)^2$ is small compared with 1.

CHAPTER X

MAXIMA AND MINIMA

In previous chapters maximum and minimum values of functions have been found by making use of the derivative. Besides this method several others which do not involve the use of the derivative may, at times, be used to advantage.

89. The Maximum or Minimum of $y = \alpha x^2 + \beta x + \gamma$. In elementary analysis the student learned that $y = \alpha x^2 + \beta x + \gamma$ represents a parabola with its axis parallel to the Y-axis, and that the equation can be put in the form $y = \alpha(x - p)^2 + q$. The point (p, q) is the vertex of the parabola. If α is positive, the vertex is a minimum point, if negative, a maximum point of the curve. Let

$$y = 3x^2 - 12x + 19.$$
$$y = 3(x - 2)^2 + 7.$$

The last equation shows at once that the minimum value of the function is 7 and that it occurs when $x = 2$.

Exercises

Find the maximum or minimum values of the following:

1. $y = 3x^2 - 2x + 1.$

2. $y = 3x - 2x^2 + 1.$

3. $y = 3x^2 + 7x.$

4. If a body is thrown vertically upward with an initial velocity of a feet per second, its height h in feet at the end of t seconds is given by

$$h = at - 16.1t^2.$$

To what height will the body rise if thrown with an initial velocity of 32.2 feet per second? When will it reach this height?

90. The Function $a \cos x + b \sin x$. The function $a \cos x + b \sin x$ is of frequent occurrence. If it is put in the form of the product of a constant by the cosine of a variable angle, the maximum and minimum values can be found at once. Thus

$$a \cos x + b \sin x = \sqrt{a^2+b^2}\left[\frac{a}{\sqrt{a^2+b^2}}\cos x + \frac{b}{\sqrt{a^2+b^2}}\sin x\right]$$

Now, $\frac{a}{\sqrt{a^2+b^2}}$ and $\frac{b}{\sqrt{a^2+b^2}}$ may be regarded as the cosine and sine, respectively, of an angle α. For if P, Fig. 62, be the point (a, b) and the angle POX be α,

$$\cos \alpha = \frac{a}{\sqrt{a^2+b^2}},$$

and

$$\sin \alpha = \frac{b}{\sqrt{a^2+b^2}}.$$

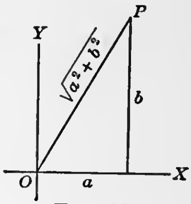

FIG. 62.

Hence

$$a \cos x + b \sin x = \sqrt{a^2+b^2}\,(\cos x \cos \alpha + \sin x \sin \alpha),$$

or

$$a \cos x + b \sin x = \sqrt{a^2+b^2}\cos (x - \alpha). \tag{1}$$

The quadrant in which α lies will be determined by the signs of a and b.

α is in the first quadrant if a is positive and b is positive.
α is in the second quadrant if a is negative and b is positive.
α is in the third quadrant if a is negative and b is negative.
α is in the fourth quadrant if a is positive and b is negative.

In polar coördinates equation (1) shows that the function $a \cos x + b \sin x$ is represented by a circle passing through the pole, of diameter $\sqrt{a^2+b^2}$, and with its center on the line making an angle α with the polar axis.

The right-hand side of equation (1) shows that the function is represented graphically in rectangular coördinates by a cosine curve of amplitude $\sqrt{a^2+b^2}$. Thus, the maximum value of $a \cos x + b \sin x$ is $\sqrt{a^2+b^2}$ and occurs when $x = \alpha$. The minimum value of the function is $-\sqrt{a^2+b^2}$ and occurs when $x = \alpha + \pi$.

Two examples giving rise to this function are solved below.

Illustration 1. The weight W, Fig. 63, rests upon the horizontal surface AB. P is the force, inclined at an angle θ with the horizontal, which will just cause the weight to slide over the plane. The problem is to find the angle θ for which P will be a minimum. The coefficient of friction is denoted by μ.

The normal pressure, N, between the weight and the plane is $(W - P \sin \theta)$, the difference between W and the vertical component of P. The force of friction, F, is then $\mu (W - P \sin \theta)$. The horizontal component of P equals F. Hence

$$\mu (W - P \sin \theta) = P \cos \theta,$$

or

$$\frac{\mu W}{P} = \cos \theta + \mu \sin \theta. \tag{2}$$

Since μ and W are constants, $\frac{\mu W}{P}$ is a maximum when and only when P is a minimum. Hence to find the minimum value of P we may find the maximum value of $\frac{\mu W}{P}$ and multiply its reciprocal by μW

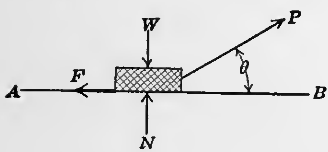

Fig. 63.

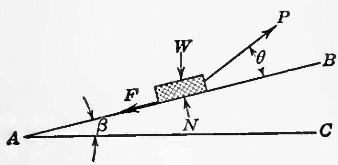

Fig. 64.

From (2),

$$\frac{\mu W}{P} = \sqrt{1 + \mu^2} \cos (\theta - \alpha),$$

where α is an angle in the first quadrant whose tangent is μ, the coefficient of friction. Therefore, when θ is acute and equal to $\tan^{-1} \mu$, P is a minimum and equal to $\frac{\mu W}{\sqrt{1 + \mu^2}}$.

Illustration 2. A weight, W, Fig. 64, rests upon the inclined plane AB. Find θ so that P, the force which will just cause W to move up the plane, shall be a minimum.

The normal pressure, N, between the weight and the plane AB

is equal to $W \cos \beta - P \sin \theta$. Then F, the force of friction, is equal to

$$\mu (W \cos \beta - P \sin \theta),$$

where μ is the coefficient of friction. We have then

$$F = \mu (W \cos \beta - P \sin \theta).$$

Since the force of friction must balance the components of P and W parallel to the plane AB, we have

$$\mu(W \cos \beta - P \sin \theta) = (P \cos \theta - W \sin \beta).$$

Hence

$$\frac{W(\mu \cos \beta + \sin \beta)}{P} = \cos \theta + \mu \sin \theta,$$

or

$$\frac{W(\mu \cos \beta + \sin \beta)}{P} = \sqrt{1 + \mu^2} \cos (\theta - \alpha). \qquad (3)$$

where α is the acute angle whose tangent is μ. Thus, the left-hand side of (3) is a maximum, and P a minimum, when θ is acute and equal to $\tan^{-1} \mu$.

The minimum value of P is then $\dfrac{W(\mu \cos \beta + \sin \beta)}{\sqrt{1 + \mu^2}}$.

Exercise

In Fig. 64, find the minimum force, P, and the angle between its line of action and AB, which will just prevent the weight W from sliding down the plane.

91. The Function $\mathbf{mx \pm \sqrt{a^2 - x^2}}$. Frequently problems in maxima and minima lead to functions of the form $mx \pm \sqrt{a^2 - x^2}$. The curve for

$$y = mx \pm \sqrt{a^2 - x^2} \qquad (1)$$

can be obtained by shearing the circle $y = \pm \sqrt{a^2 - x^2}$ in the line $y = mx$. Every ordinate of the circle to the right of the Y-axis is increased (or decreased if m is negative) by an amount proportional to the distance from the Y-axis. To the left of the Y-axis the ordinates are decreased if m is positive and increased if m is negative.

The maximum value of y is easily found by placing

$$x = a \cos t. \qquad (2)$$

Then from (1)

$$y = a(m \cos t + \sin t) \qquad (3)$$

$$= a\sqrt{1 + m^2} \cos (t - \alpha), \qquad (4)$$

where $\alpha = \tan^{-1} \dfrac{1}{m}$. The maximum value of y occurs when

$$x = a \cos \alpha = \frac{am}{\sqrt{1 + m^2}}.$$

92. Maxima and Minima by Limits of Curve. In case $f(x, y) = 0$ is of the second degree in x and y, and in a few other cases, the maximum and minimum values of y can be found by determining when x changes from real to complex values.

The method will be illustrated by an example.

Let

$$y = \frac{x^2 + 6}{2x + 1}.$$

Then

$$x = y \pm \sqrt{(y + 3)(y - 2)}. \qquad (1)$$

From equation (1) it is seen that for values of y greater than 2 or less than -3, x has two distinct real values. When $y = 2$ or -3, x has two equal real values. When $-3 < y < 2$, x is imaginary. This shows that the line $y = c$ meets the curve in two distinct points if it is more than two units above or more than three units below the X-axis; that it is tangent to the curve when two units above and when three units below the X-axis, and that it does not cut the curve when it falls within the limits two units above and three units below the X-axis. Hence the function has a minimum value 2 and a maximum value -3.

Exercises

1. Find the maximum and minimum values of

$$\frac{x^2 - 2x + 19}{2x + 5}.$$

2. Find the maximum rectangle which can be inscribed in a circle of radius 10.

93. Maxima and Minima Determined by the Derivative. The first derivative has been used to determine the value of the argument corresponding to maxima and minima of functions. Immediately to the left of a maximum point the function is increasing with x and consequently the first derivative is positive. On the other hand, immediately to the right of such a point the function is decreasing as x increases and the first derivative is negative. Similarly it follows that the first derivative is negative immediately to the left and positive immediately to the right of a minimum point. In both cases the first derivative changes sign as the independent variable passes through the value for which the function has a maximum or a minimum value. This change of sign may take place in a number of ways.

Illustration 1. Thus, in the case of the function

$$y = x^2 - 2x + 7,$$

the derivative,

$$\frac{dy}{dx} = 2x - 2 = 2(x - 1),$$

is negative to the left and positive to the right of the line $x = 1$. When $x = 1$, $\frac{dy}{dx} = 0$ and the curve has a horizontal tangent. In the vicinity of this point the curve has the shape shown in Fig. 65.

FIG. 65.

At first thought it might appear that if the first derivative is negative to the left and positive to the right of a certain point, it certainly must become zero at this point. This is, however, by no means the case, as the next illustration will show.

Illustration 2. $y = 4 + (x - 1)^{\frac{2}{3}}$. Although the minimum value of this function can be determined at once by noting that it represents a semi-cubical parabola with its vertex at (1, 4), the problem will be worked by the method of the calculus for illustrative purposes.

The derivative,

$$\frac{dy}{dx} = \frac{2}{3(x - 1)^{\frac{1}{3}}},$$

is negative when $x < 1$ and positive when $x > 1$. Hence the function is decreasing to the left and increasing to the right of

$x = 1$. When $x = 1$, $y = 4$. This value is a minimum value of the function. For $x = 1$ the derivative does not exist, as the denominator becomes zero. Let us see what really happens in the vicinity of $x = 1$. As x approaches 1 from the left, $\frac{dy}{dx}$ takes on larger and larger negative values. The form of the curve to the left and in the immediate vicinity of the point (1, 4) is something like that shown in Fig. 66. The line $x = 1$ is tangent to the curve at this point.

As x approaches 1 from the right, *i.e.*, through decreasing values, the derivative becomes larger and larger. The form of the curve to the right of the line $x = 1$ is also shown in Fig. 66. The line $x = 1$ is also tangent to the portion of the curve obtained by allowing x to approach 1 from the right.

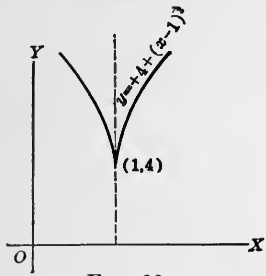

FIG. 66.

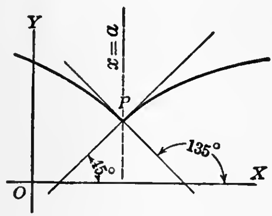

FIG. 67.

It is now apparent that the first derivative may change sign without passing through zero. In the above illustration it changes sign by becoming infinite.

The first derivative may change sign in still another way as illustrated by the curve of Fig. 67. Let us suppose that the derivative approaches -1 as x approaches a from the left, and the value $+1$ as x approaches a from the right. The function has a minimum value at the point P, for the derivative changes from minus to plus as x increases through the value a and consequently the function is decreasing to the left and increasing to the right of $x = a$.

The essential thing at a minimum point is that the derivative changes sign from minus to plus, and at a maximum point that it changes sign from plus to minus.

A derivative which is continuous at a maximum or a minimum point changes sign by passing through zero. But it may change sign by becoming infinite, as the second illustration shows, or by becoming otherwise discontinuous as explained above. This last type is of rare occurrence and will not be referred to again.

Illustration 3. $y = x^3 + 3$. The derivative of this function, $\frac{dy}{dx} = 3x^2$, is positive for all values of x except $x = 0$, when it is zero. The function is increasing for all these values of x. At this point, (0, 3), there is a horizontal tangent but the function has neither a maximum nor a minimum at the point, for it increases up to the value 3 for $x = 0$ and then continues to increase to the right. This illustration brings out clearly the fact that there is no reason for assuming that a function has a maximum or a minimum value at a point where the first derivative is zero. What kind of a point is the point (0, 3)?

94. Second-Derivative Test for Maxima and Minima. In the first of the three types of maximum or minimum points considered in §**93**, the first derivative changes continuously from positive to negative values or *vice versa*. For a maximum point of this type the curve is concave downward and the second derivative is negative at such a point. For a minimum point the curve is concave upward and the second derivative is positive. A convenient test for the behavior of a function at a point *where the first derivative is zero* is then, to substitute the abscissa of this point in the expression for the second derivative. If the second derivative is positive the point is a minimum point; if negative, a maximum point. If the second derivative is zero, the test fails. This test *also fails* for maximum or minimum points where the *first derivative is discontinuous.*

Examine the curves

$$y = x^3,$$
$$y = x^4,$$
$$y = x^5.$$

95. Study of a Function by Means of its Derivatives. The following is a summary of the application of the first and second derivatives to tracing the curve representing a function:

1. The function is increasing if the first derivative is positive, and decreasing if it is negative.

2. To find maximum and minimum points find the values of x for which the first derivative becomes zero or infinite. If the derivative changes sign at any of these points, the corresponding point is a maximum or minimum point according as the change is from plus to minus or *vice versa.*

Points at which the *first derivative is equal to zero* can also be tested by substituting the abscissa of the points in the second derivative. If the second derivative is positive, the point is a minimum point, if negative, a maximum point.

3. Points of inflection are found by determining where the second derivative changes sign. As in the case of the first derivative, the change in sign can take place through zero or infinity. If the change is from positive to negative values the curve changes from being concave upward to being concave downward.

The abscissas of the points at which the first and the second derivatives become zero or infinite we shall call the critical values. These values and these alone need be tested in studying the behavior of an ordinary curve. The investigation of a curve by means of its derivatives can be put in the tabulated form shown in the following illustrative examples:

1. $y = \frac{1}{6}x^3$. (See Fig. 33.)

$$\frac{dy}{dx} = \frac{1}{2}x^2.$$

$$\frac{d^2y}{dx^2} = x.$$

$$\frac{dy}{dx} = 0 \text{ when } x = 0.$$

$$\frac{d^2y}{dx^2} = 0 \text{ when } x = 0.$$

x	$\frac{d^2y}{dx^2}$	$\frac{dy}{dx}$	y
$x < 0$	< 0	> 0	Concave downward, increasing
$x > 0$	> 0	> 0	Concave upward, increasing

Here (0, 0) is a point of inflection. There is neither a maximum nor a minimum point.

2. $y = \frac{1}{6}x^3 - x^2 + \frac{3}{2}x + 2$. (See Fig. 34.)

$$\frac{dy}{dx} = \frac{1}{2}x^2 - 2x + \frac{3}{2}$$
$$= \frac{1}{2}(x-1)(x-3).$$
$$\frac{d^2y}{dx^2} = x - 2.$$
$$\frac{dy}{dx} = 0 \text{ when } x = 1, 3.$$
$$\frac{d^2y}{dx^2} = 0 \text{ when } x = 2.$$

x	$\frac{d^2y}{dx^2}$	$\frac{dy}{dx}$	y
$x < 2$	< 0	Decreasing	Concave downward
$x > 2$	> 0	Increasing	Concave upward
$x < 1$		> 0	Increasing
$1 < x < 3$		< 0	Decreasing
$x > 3$		> 0	Increasing
1			$\frac{8}{3}$
2			$\frac{7}{3}$
3			2
0			2

$(2, \frac{7}{3})$ Point of inflection.
$(1, \frac{8}{3})$ Maximum point.
$(3, 2)$ Minimum point.

Apply second derivative test for $x = 1$ and $x = 3$.

96. Applications of Maxima and Minima. In solving problems involving maxima and minima the first step is to set up from the conditions of the problem the function whose maximum or minimum value is sought. Frequently the function will be expressed, at first, in terms of two or more variables. Usually, however, there is a relation between these variables, and the function can be expressed in terms of a single variable. After this has been done the maximum or minimum values can be found.

Exercises

1. Equal squares are cut from the corners of a rectangular piece of tin 30 by 20 inches. The rectangular projections are then turned up forming the sides of an open box. Find the size of the squares cut out if the volume of the box is a maximum.

2. A man who is in a boat 3 miles from the nearest point, A, of a straight shore wishes to reach, in the shortest possible time, a point B on the shore which is 6 miles from A. Find the point of the shore toward which he should row, if he can row at the rate of 3 miles per hour and walk at the rate of 5 miles per hour.

3. The horizontal component of the tension in the guy wire BC, Fig. 68, is to balance the horizontal pull P. If the strength of the wire varies as its cross section, and if its cost varies as its weight, find the angle θ such that the cost of the guy wire shall be a minimum.

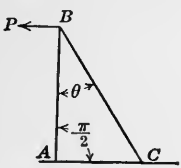

Fig. 68.

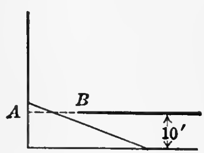

Fig. 69.

4. Find the length of the shortest beam that can be used to brace a wall if the beam passes over a second wall 6 feet high and 8 feet from the first.

5. A steel girder 30 feet long is moved on rollers along a passageway 10 feet wide, and through the door AB, Fig. 69, at the end of the passageway. Neglecting the width of the girder, how wide must the door be in order to allow the girder to pass through?

6. A sign 10 feet high is fastened to the side of a building so that the lower edge is 25 feet from the ground. How far from the building should an observer on the ground stand in order that he may see the sign to the best advantage, *i.e.*, in order that the angle at his eye subtended by the sign may be the greatest possible? The observer's eye is $5\frac{1}{2}$ feet from the ground.

7. A man in a launch is m miles from the nearest point A of a straight shore. Toward what point on the shore should he head his boat in order to reach, in the shortest possible time, an inland point whose distance from the nearest point B of the shore is n miles? The man can run the boat v_1 miles per hour and can walk v_2 miles per hour. The distance AB is p miles.

Ans. Toward a point such that

$$\frac{\sin \theta_1}{v_1} = \frac{\sin \theta_2}{v_2},$$

where θ_1 and θ_2 are the angles made by the paths of the man with the normal to the beach. It will be noticed that the path taken by the man is similar to that followed by a ray of light in passing from one medium to another with a different index of refraction.

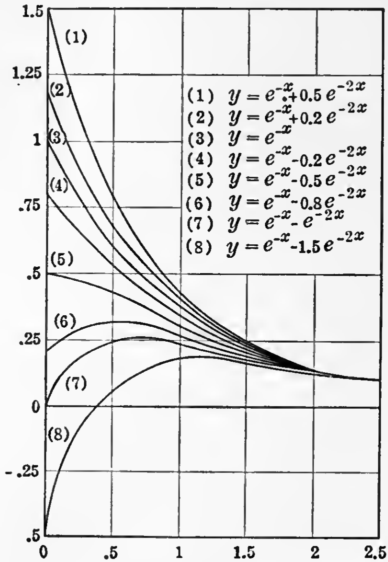

Fig. 70.

8. A man in a launch is m miles from the nearest point A of a straight shore. He wishes to touch shore and reach, in the shortest possible time, a second point on the lake whose distance from the nearest point B on shore is n miles. In what direction must he head his boat if the distance AB is p miles?

The path taken by the man is similar to the path of a ray of light reflected by a plane surface.

9. It is desired to make a gutter, whose cross section shall be a segment of a circle, by bending a strip of tin of width a. Find the radius of the cross section of maximum carrying capacity.

10. A sector is cut from a circular piece of tin. The cut edges of the remaining portion of the sheet are then brought together to form a cone. Find the angle of the sector to be cut out in order that the volume of the cone shall be a maximum.

11. The stiffness of a rectangular beam varies as its breadth and as the cube of its depth. Find the dimensions of the stiffest beam which can be cut from a circular log 12 inches in diameter.

12. The strength of a rectangular beam varies as its breadth and as the square of its depth. Find the dimensions of the strongest beam which can be cut from a circular log 12 inches in diameter.

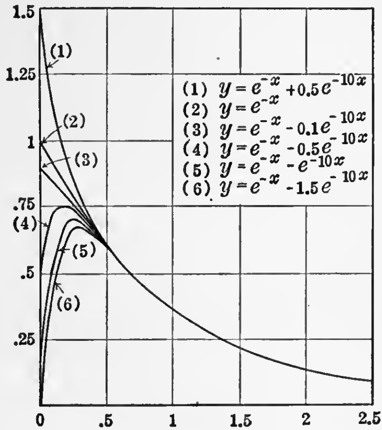

FIG. 71.

13. Consider the sum

$$y = ax^n + bx^r$$

for positive values of x only. First, if n and r are of like sign, show that: (1) a maximum or a minimum value exists if a and b are of unlike sign; (2) neither a maximum nor a minimum value exists if a and b are of like sign. Second, discuss the same cases if n and r have opposite signs.

14. Determine the exact values of the maxima shown in Figs. 70 and 71.

HINT. Consider first the general case

$$y = e^{-x} - ae^{-bx}.$$

CHAPTER XI

POLAR COÖRDINATES

97. Direction of Curve in Polar Coördinates. Let BPQ, Fig. 72, be a curve referred to O as pole and OA as polar axis. Let P be any point of the curve and let PT be a tangent to the curve at this point. Let PS be the radius vector of the point P, produced.

A point describing the curve, when at P, moves in the direction PT. This direction is given by the angle ψ through which the radius vector produced must rotate in a positive direction about P, in order to become coincident with the tangent line.

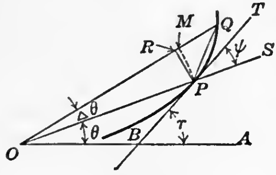

FIG. 72.

An expression for $\tan\psi$ will now be found. Let Q, Fig. 72, be a second point of the curve. PR is perpendicular to OQ, and PM is a circular arc with O as center and radius $OP = \rho$.

$$\tan\psi = \lim_{\Delta\theta \doteq 0} \tan RQP = \lim_{\Delta\theta \doteq 0} \frac{PR}{RQ}. \tag{1}$$

The infinitesimals PR and RQ can be replaced by PM and MQ, respectively, if (see §60)

$$\lim_{\Delta\theta \doteq 0} \frac{PR}{PM} = 1 \tag{2}$$

and

$$\lim_{\Delta\theta \doteq 0} \frac{RQ}{MQ} = 1. \tag{3}$$

Equation (2) is true by equation (3), §56. The proof of equation (3) follows:

$$\begin{aligned}\lim_{\Delta\theta \doteq 0} \frac{RQ}{MQ} &= \lim_{\Delta\theta \doteq 0} \frac{RM + MQ}{MQ} \\ &= \lim_{\Delta\theta \doteq 0} \frac{\rho(1 - \cos\Delta\theta) + \Delta\rho}{\Delta\rho} \\ &= 1 + \lim_{\Delta\theta \doteq 0} \frac{\rho(1 - \cos\Delta\theta)}{\Delta\theta}\, \frac{\Delta\theta}{\Delta\rho}.\end{aligned}$$

Hence

$$\lim_{\Delta\theta \doteq 0} \frac{RQ}{MQ} = 1,$$

since

$$\lim_{\Delta\theta \doteq 0} \frac{1 - \cos \Delta\theta}{\Delta\theta} = 0.$$

From (1), (2), and (3) it follows that

$$\tan \psi = \lim_{\Delta\theta \doteq 0} \frac{PM}{MQ} = \lim_{\Delta\theta \doteq 0} \frac{\rho\Delta\theta}{\Delta\rho}.$$

Hence

$$\tan \psi = \rho \frac{d\theta}{d\rho}.$$

$$\tan \psi = \frac{\rho}{\frac{d\rho}{d\theta}}. \tag{4}$$

This formula[1] can be easily remembered if the sides of the triangular figure MQP, Fig. 72, are thought of as straight lines, and

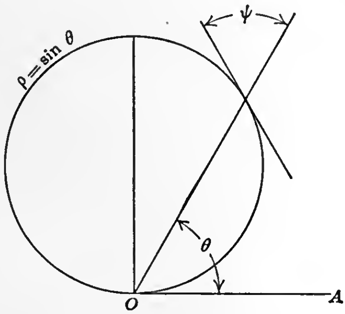

Fig. 73.

[1] This formula enables us to give another proof for $\frac{d \sin \theta}{d\theta}$. In polar coördinates $\rho = \sin \theta$ represents a circle, Fig. 73. By geometry, $\psi = \theta$. Then

$$\tan \psi = \tan \theta = \frac{\rho}{\frac{d\rho}{d\theta}} = \frac{\sin \theta}{\frac{d\rho}{d\theta}},$$

or

$$\frac{d\rho}{d\theta} = \cos \theta,$$

$$\frac{d \sin \theta}{d\theta} = \cos \theta.$$

the angle MQP as equal to ψ. Then the tangent of ψ would be

$$\rho\frac{d\theta}{d\rho} = \frac{\rho}{\frac{d\rho}{d\theta}}.$$

Formula (4) corresponds to $\frac{dy}{dx} = \tan\tau$ in rectangular coördinates.

Illustration 1. If $\rho = e^{a\theta}$,

$$\frac{d\rho}{d\theta} = ae^{a\theta}$$

and $\tan\psi = \frac{1}{a}$, a constant.

Illustration 2. Find the equation of the family of curves for which the angle between the radius vector produced and the tangent line is a constant.

$$\tan\psi = k.$$

$$\frac{\rho}{\frac{d\rho}{d\theta}} = k,$$

or

$$\frac{\frac{d\rho}{d\theta}}{\rho} = \frac{1}{k}.$$

$$\frac{d\rho}{\rho} = \frac{1}{k}d\theta.$$

Integrating

$$\log\rho = \frac{\theta}{k} + C$$

$$\rho = e^{\frac{\theta}{k}+C}$$

$$= e^{C}e^{\frac{\theta}{k}},$$

or

$$\rho = Ke^{\frac{\theta}{k}},$$

where K is an arbitrary constant.

Exercises

Find $\tan \psi$ for each of the following curves:

1. $\rho = \dfrac{a}{\theta}$.

2. $\rho = a\theta$.

3. $\rho = e^{a\theta}$.

4. $\rho = a(1 - \cos\theta)$.

5. $\rho = \dfrac{a}{1 - \cos\theta}$.

6. $\rho = a\cos(\theta - \alpha)$.

98. Differential of Arc: Polar Coördinates. We shall now find an expression for $\dfrac{ds}{d\theta}$ in polar coördinates. From Fig. 72,

$$(\text{chord } PQ)^2 = (PR)^2 + (RQ)^2.$$

From which

$$\lim_{\Delta\theta \doteq 0}\left(\frac{\text{chord } PQ}{\Delta\theta}\right)^2 = \lim_{\Delta\theta \doteq 0}\left(\frac{PR}{\Delta\theta}\right)^2 + \lim_{\Delta\theta \doteq 0}\left(\frac{RQ}{\Delta\theta}\right)^2.$$

Replacing chord PQ by arc $PQ = \Delta s$, PR by $PM = \rho\Delta\theta$, and RQ by $MQ = \Delta\rho$,

$$\lim_{\Delta\theta \doteq 0}\left(\frac{\Delta s}{\Delta\theta}\right)^2 = \lim_{\Delta\theta \doteq 0}\left(\frac{\rho\Delta\theta}{\Delta\theta}\right)^2 + \lim_{\Delta\theta \doteq 0}\left(\frac{\Delta\rho}{\Delta\theta}\right)^2.$$

Therefore

$$\left(\frac{ds}{d\theta}\right)^2 = \rho^2 + \left(\frac{d\rho}{d\theta}\right)^2 \qquad (1)$$

and

$$\mathbf{ds} = \sqrt{\rho^2 + \left(\frac{\mathbf{d}\rho}{\mathbf{d}\theta}\right)^2}\,\mathbf{d}\theta. \qquad (2)$$

This formula can be written

$$(\mathbf{ds})^2 = \rho^2(\mathbf{d}\theta)^2 + (\mathbf{d}\rho)^2. \qquad (3)$$

It corresponds to $(ds)^2 = (dx)^2 + (dy)^2$ in rectangular coördinates. It can be remembered easily by the help of the triangle MQP, Fig. 72.

The length of the curve can be expressed as a definite integral. Thus: (See Fig. 74)

$$s = \lim_{\Delta\theta \doteq 0} \sum_{\theta = \alpha}^{\theta = \beta} PQ$$

$$= \lim_{\Delta\theta \doteq 0} \sum_{\theta=\alpha}^{\theta=\beta} \sqrt{(PR)^2 + (RQ)^2}$$

$$= \lim_{\Delta\theta \doteq 0} \sum_{\theta=\alpha}^{\theta=\beta} \sqrt{\left(\frac{PR}{\Delta\theta}\right)^2 + \left(\frac{RQ}{\Delta\theta}\right)^2}\, \Delta\theta$$

$$= \lim_{\Delta\theta \doteq 0} \sum_{\theta=\alpha}^{\theta=\beta} \sqrt{\rho^2 + \left(\frac{d\rho}{d\theta}\right)}\, \Delta\theta$$

$$= \int_\alpha^\beta \sqrt{\rho^2 + \left(\frac{d\rho}{d\theta}\right)^2}\, d\theta.$$

Illustration. Find the entire length of the curve $\rho = a(1 - \cos\theta)$. This curve is symmetrical with respect to the polar axis. The length of the upper half will be found and multiplied by 2.

$$\frac{d\rho}{d\theta} = a \sin\theta.$$

$$\frac{s}{2} = \int_0^\pi \sqrt{\rho^2 + \left(\frac{d\rho}{d\theta}\right)^2}\, d\theta$$

$$= \int_0^\pi \sqrt{a^2(1 - \cos\theta)^2 + a^2 \sin^2\theta}\, d\theta$$

$$= 2a \int_0^\pi \sqrt{\frac{1 - \cos\theta}{2}}\, d\theta$$

$$= 2a \int_0^\pi \sin\frac{\theta}{2}\, d\theta$$

$$= -4a \cos\frac{\theta}{2}\Big|_0^\pi = 4a.$$

$$s = 8a.$$

Exercises

1. Find the entire length of the curve $\rho = 2a \sin\theta$.

2. Find the entire length of the curve $\rho = a(1 - \sin\theta)$.

3. Find the entire length of the curve $\rho = a \sin^3 \frac{\theta}{3}$.

4. Find the length of $\rho = e^{a\theta}$ between the points corresponding to $\theta = 0$ and $\theta = \pi$. Also between the points corresponding to $\theta = 0$ and $\theta = \frac{\pi}{3}$.

5. Prove formula (3) directly from

$$x = \rho \cos \theta,$$
$$y = \rho \sin \theta,$$

and

$$ds = \sqrt{(dx)^2 + (dy)^2}.$$

99. Area: Polar Coördinates. Find the area bounded by the curve $\rho = f(\theta)$ and the radii vectores $\theta = \alpha$ and $\theta = \beta$. We seek the area BOC, Fig. 74. Draw radii vectores dividing the angle BOC into n equal parts $\Delta\theta$. Let POQ be a typical one of the n portions into which the area is divided by these radii. The angle POQ is $\Delta\theta$. The line OP makes an angle θ with the initial line OA, and its length is $\rho = f(\theta)$. Denote the area of BOC by A.

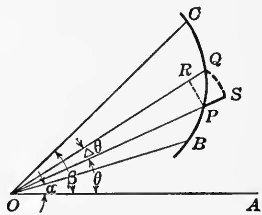

FIG. 74.

$$A = \lim_{n \doteq \infty} \sum POQ. \tag{1}$$

Replace[1] POQ by the circular sector POR whose area is $\frac{1}{2}\rho^2\Delta\theta$. Then

$$A = \lim_{\Delta\theta \doteq 0} \sum_{\theta=\alpha}^{\theta=\beta} \tfrac{1}{2}\rho^2\Delta\theta,$$

$$A = \tfrac{1}{2}\int_\alpha^\beta \rho^2 d\theta,$$

or

$$A = \tfrac{1}{2}\int_\alpha^\beta [f(\theta)]^2\, d\theta. \tag{3}$$

Exercises

1. Find the area bounded by the curve $\rho = 2a \sin \theta$.

2. Find the area bounded by the cardioid $\rho = 2a(1 - \cos \theta)$.

HINT. $\cos^2 \theta = \dfrac{1 + \cos 2\theta}{2}$.

[1] Let $\Delta A = OPQ$ (Fig. 74). PR and QS are arcs of circles. Then

$$OPR < \Delta A < OSQ,$$

i.e.,

$$\tfrac{1}{2}\rho^2\Delta\theta < \Delta A < \tfrac{1}{2}(\rho + \Delta\rho)^2\Delta\theta.$$

3. Find the area bounded by $\rho = 2a(1 + \sin\theta)$.

4. Find the area bounded by *one* loop of $\rho = 10\cos 2\theta$.

5. Find the area bounded by *one* loop of $\rho = 10\sin 2\theta$.

6. Find the area bounded by *one* loop of $\rho = a\cos 3\theta$.

7. Find the area bounded by *one* loop of $\rho^2 = 10\cos 2\theta$.

8. Find the area swept out by the radius vector of the curve $\rho = a\theta$, as θ varies from 0 to 2π.

9. Find the area bounded by the radii vectores $\theta = \frac{\pi}{4}$, $\theta = \pi$, and the curve $\rho = \frac{10}{\theta^2}$.

10. Find the area bounded by the radii vectores $\theta = 0$, $\theta = \frac{\pi}{2}$ and the curve $\rho = 5\theta^2$.

CHAPTER XII

INTEGRATION

100. Formulas. In Chapters III, VI and VII the following formulas of integration, with the exception of (19), have been used. They are collected here for reference, and should be memorized by the student.

1. $\int u^n\,du = \frac{1}{n+1}u^{n+1} + C$, if $n \neq -1$.

2. $\int \frac{du}{u} = \log u + C.$

3. $\int e^u\,du = e^u + C.$

4. $\int a^u\,du = \frac{1}{\log_e a}a^u + C.$

5. $\int \sin u\,du = -\cos u + C.$

6. $\int \cos u\,du = \sin u + C.$

7. $\int \sec^2 u\,du = \tan u + C.$

8. $\int \csc^2 u\,du = -\cot u + C.$

9. $\int \sec u \tan u\,du = \sec u + C.$

10. $\int \csc u \cot u\,du = -\csc u + C.$

11. $\int \tan u\,du = \log \sec u + C.$

12. $\int \cot u\,du = \log \sin u + C.$

13. $\int \sec u\,du = \log(\sec u + \tan u) + C.$

14. $\int \csc u\,du = -\log(\csc u + \cot u) + C.$

15. $\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\frac{u}{a} + C.$

16. $\displaystyle\int \frac{du}{a^2 + u^2} = \frac{1}{a} \tan^{-1} \frac{u}{a} + C.$

17. $\displaystyle\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a} \sec^{-1} \frac{u}{a} + C.$

18. $\displaystyle\int \frac{du}{\sqrt{u^2 \pm a^2}} = \log (u + \sqrt{u^2 \pm a^2}) + C.$

19. $\displaystyle\int \frac{du}{u^2 - a^2} = \frac{1}{2a} \log \frac{u - a}{u + a} + C, \text{ if } u > a$

$$= \frac{1}{2a} \log \frac{a - u}{a + u} + C, \text{ if } u < a.$$

Formula (19) is proved as follows:

$$\frac{1}{u^2 - a^2} = \frac{1}{2a}\left[\frac{1}{u - a} - \frac{1}{u + a}\right]$$

$$\int \frac{du}{u^2 - a^2} = \frac{1}{2a}\int \left[\frac{1}{u - a} - \frac{1}{u + a}\right] du$$

$$= \frac{1}{2a}\int \frac{du}{u - a} - \frac{1}{2a}\int \frac{du}{u + a}$$

$$= \frac{1}{2a} \log (u - a) - \frac{1}{2a} \log (u + a) + C.$$

$$= \frac{1}{2a} \log \frac{u - a}{u + a} + C.$$

This formula leads to the logarithm of a negative number if $u < a$. To obtain a formula for this case write

$$\frac{1}{u^2 - a^2} = \frac{1}{2a}\left[-\frac{1}{a - u} - \frac{1}{a + u}\right].$$

Then

$$\int \frac{du}{u^2 - a^2} = \frac{1}{2a} \log \frac{a - u}{a + u} + C.$$

Exercises

1. $\displaystyle\int \frac{x\,dx}{\sqrt{16 - x^2}}.$

2. $\displaystyle\int \frac{dx}{\sqrt{16 - x^2}}.$

3. $\displaystyle\int \frac{dx}{\sqrt{x^2 - 16}}.$

4. $\displaystyle\int \frac{dx}{x^2 + 16}.$

5. $\int \frac{dx}{x^2 - 16}$.

6. $\int \frac{dx}{16 - x^2}$.

7. $\int \frac{x\,dx}{x^2 - 16}$.

8. $\int \frac{dx}{x\sqrt{x^2 - 16}}$.

9. $\int \cot 7t\,dt$.

10. $\int \frac{(x + a)\,dx}{x^2 + 2ax}$.

11. $\int (x^2 - 16)^{\frac{3}{2}} x\,dx$.

12. $\int \sin(2x - 3)\,dx$.

13. $\int \sec^2(5\alpha + 2)\,d\alpha$.

14. $\int \sec(2\theta + 4)\tan(2\theta + 4)\,d\theta$.

15. $\int \csc^2(3 - 2\phi)\,d\phi$.

16. $\int e^{-x^2} x\,dx$.

17. $\int e^{\cos\theta}\sin\theta\,d\theta$.

18. $\int \frac{3t^2\,dt}{30t^3 + 13}$.

19. $\int (\sqrt{c} - \sqrt{x})^3\,dx$.

20. $\int \sqrt{3 + 4x}\,dx$.

21. $\int \frac{dy}{\sqrt{5 - 3y}}$.

22. $\int e^{\frac{y}{3}}\,dy$.

23. $\int e^{\tan(2x+3)}\sec^2(2x + 3)\,dx$.

24. $\int \frac{x^2 + 2}{x + 1}\,dx$. Divide numerator by denominator.

25. $\int \left[e^{\frac{x}{2}} + e^{-\frac{x}{2}}\right] dx$.

26. $\int a^{5x}\,dx$.

27. $\int (2x + 4)^{0.52}\,dx$.

28. $\int \frac{t^3 + 3}{t + 1}\,dt$.

29. $\int \frac{dx}{7x^2 + 11}$.

30. $\int \frac{dx}{\cos^2(3x - 2)}$.

31. $\int e^{3x}\,dx$.

32. $\int \frac{dx}{4x^2 - 9}$.

33. $\int \frac{\cos x\,dx}{4 + 3\sin x}$.

34. $\int \frac{dx}{\sqrt{16 - 9x^2}}$.

35. $\int \tan(3\alpha + 4)\,d\alpha$.

36. $\int (\tan\theta + \cot\theta)^2\,d\theta = \tan\theta - \cot\theta + C$.

37. $\int \left(\sin\frac{\theta}{5} - \cos 5\theta\right) d\theta$.

38. $\int \cot(5t - 8)\,dt$.

39. $\int \cos(3t - 4)\,dt$.

40. $\int \frac{y\,dy}{9 - 4y^2}$.

41. $\int \frac{dz}{\sqrt{3z+7}}$.

42. $\int \frac{dx}{7-5x^2}$.

43. $\int \tan(2x-5)\,dx$.

44. $\int \sec(2y+4)\,dy$.

45. $\int \csc(2y-7)\,dy$.

46. $\int \cot(3t+11)\,dt$.

47. $\int \sec^2\left(\frac{x}{3}-5\right)dx$.

48. $\int \cos(3-2x)\,dx$.

49. $\int \frac{2x+5}{x^2+5x+41}\,dx$.

50. $\int \frac{dt}{9t^2+4}$.

51. $\int \frac{dt}{9t^2-4}$.

52. $\int \frac{dt}{\sqrt{9t^2-4}}$.

53. $\int \frac{dt}{\sqrt{4-9t^2}}$.

54. $\int \frac{dt}{t\sqrt{9t^2-4}}$.

55. $\int \frac{t\,dt}{\sqrt{9t^2-4}}$.

56. $\int \frac{t\,dt}{9t^2-4}$.

57. $\int \sec 5x\,dx$.

58. $\int \sin(\omega t+\alpha)\,dt$.

59. $\int \cos^2 4x \sin 4x\,dx$.

60. $\int \sin^4(x+3)\cos(x+3)\,dx$.

61. $\int \cos^5(3x-2)\sin(3x-2)\,dx$.

62. $\int \sec^2(9-7x)\,dx$.

63. $\int e^{\tan\frac{x}{3}}\sec^2\frac{x}{3}\,dx$.

64. $\int \tan^4 x \sec^2 x\,dx$.

65. $\int \cos^2 3x \sin 3x\,dx$.

66. $\int \tan^3 5x \sec^2 5x\,dx$.

67. $\int \sec^4 x \tan x\,dx$.

68. $\int \csc^5 x \cot x\,dx$.

69. $\int x\tan(2x^2-5)\,dx$.

70. $\int \frac{\sin x\cos x\,dx}{4+\sin^2 x}$.

71. $\int \frac{\cos x\,dx}{4+\sin^2 x}$.

72. $\int e^{\frac{1}{x}}\frac{1}{x^2}\,dx$.

73. $\int (a^{\frac{2}{3}}-x^{\frac{2}{3}})^3\,dx$.

74. $\int \frac{dt}{4-25t^2}$.

75. $\int t\sqrt{6t^2-17}\,dt$.

76. $\int \frac{x\,dx}{\sqrt{9-x^2}}$.

77. $\int \frac{\sin 5x\,dx}{3\cos 5x+11}$.

78. $\int e^{\sin 3x}\cos 3x\,dx$.

79. $\int e^{x^2+6x+7}(x+3)\,dx$.

80. $\int \cos 5x \sin 3x\,dx$.

81. $\int \cos 3x \cos 5x \, dx.$

82. $\int \sin 7x \sin 4x \, dx.$

83. $\int \sin 3t \cos 4t \, dt.$

84. $\int_0^{\pi} \sin^2 5t \, dt.$

85. $\int_0^{\pi} \sin mt \cos nt \, dt,$ where m and n are integers. What is the value if $m = n$?

86. $\int \cos (3\omega t + \alpha) \sin (3\omega t + \alpha) \, dt.$

87. $\int \frac{2x + 3}{x^2 + 9} \, dx.$

88. $\int \frac{x^3 + 3x^2 + 7}{x^2 + 9} \, dx$

89. $\int \frac{x + 1}{3x^2 - 11} \, dx.$

90. $\int \frac{3x + 2}{4x^2 - 16} \, dx.$

91. $\int \sin^3 5\theta \cos 5\theta \, d\theta.$

92. $\int \sin^2 x \, dx.$

93. $\int x\sqrt{16 - x^2} \, dx.$

94. $\int \sec^2 \left(\frac{x}{3} + 2\right) dx.$

95. $\int \frac{t^2}{5t^3 + 7} \, dt.$

96. $\int \sec^3 4x \tan 4x \, dx.$

97. $\int \frac{e^{\sqrt{x + 2}}}{\sqrt{x + 2}} \, dx.$

98. $\int \frac{dx}{\sqrt{5 - 7x^2}}.$

99. $\int \frac{dx}{\sqrt{3x^2 - 5}}.$

100. $\int \sin 4x \cos 6x \, dx.$

101. $\int (\sqrt{a} - \sqrt{x})^2 \, dx.$

102. $\int \frac{2x + 3}{2x + 7} \, dx.$

103. $\int \frac{dx}{\sqrt{3x + 2}}.$

104. $\int \sin^4 2x \cos 2x \, dx.$

105. $\int \sqrt{\sin x} \cos x \, dx.$

106. $\int e^{-3t} \, dt.$

107. $\int (2x - 5)^{\frac{1}{3}} \, dx.$

108. $\int \sec \frac{x}{2} \, dx.$

109. $\int \sec (3\phi - 2) \tan (3\phi - 2) \, d\phi.$

110. $\int \tan^5 (2x - 1) \sec^2 (2x - 1) \, dx.$

111. $\int \frac{\sec^2 3x}{1 + \tan 3x} dx.$

112. $\int \sqrt{2 - 3x}\, dx.$

113. $\int \tan (5 - 2x)\, dx.$

114. $\int \frac{x\, dx}{5 - 3x^2}.$

115. $\int \frac{dy}{y^2 - 7}.$

116. $\int \frac{x + 2}{x + 5} dx.$

117. $\int \frac{x + 4}{x^2 - 9} dx.$

118. $\int \frac{x\, dx}{3x^2 + 4}.$

119. $\int \frac{dy}{y\sqrt{3y^2 - 7}}.$

120. $\int \frac{y\, dy}{\sqrt{3y^2 - 7}}.$

121. $\int \sec^5 \theta \tan \theta\, d\theta.$

122. $\int \frac{dy}{\sqrt{5y^2 - 13}}.$

123. $\int \sin^3 \frac{x}{3} \cos \frac{x}{3} dx.$

124. $\int \cos 2x \sin \frac{x}{2} dx.$

101. Integration of Expressions Containing ax² + bx + c, by Completing the Square.

Illustration 1.

$$\int \frac{dx}{x^2 + 4x + 9} = \int \frac{dx}{(x + 2)^2 + 5} = \frac{1}{\sqrt{5}} \tan^{-1} \frac{x + 2}{\sqrt{5}} + C.$$

Illustration 2.

$$\int \frac{dx}{\sqrt{3 + 4x - x^2}} = \int \frac{dx}{\sqrt{3 - (x^2 - 4x)}} = \int \frac{dx}{\sqrt{3 + 4 - (x - 2)^2}}$$

$$= \int \frac{dx}{\sqrt{7 - (x - 2)^2}} = \sin^{-1} \frac{x - 2}{\sqrt{7}} + C.$$

Illustration 3.

$$\int \frac{dx}{7x^2 + 3x + 11} = \frac{1}{7} \int \frac{dx}{x^2 + \frac{3}{7}x + \frac{9}{196} + \frac{11}{7} - \frac{9}{196}}$$

$$= \frac{1}{7} \int \frac{dx}{(x + \frac{3}{14})^2 + \frac{299}{196}}$$

$$= \frac{1}{7} \frac{14}{\sqrt{299}} \tan^{-1} \frac{x + \frac{3}{14}}{\frac{\sqrt{299}}{14}} + C$$

$$= \frac{2}{\sqrt{299}} \tan^{-1} \frac{14x + 3}{\sqrt{299}} + C.$$

Illustration 4.

$$\int \frac{dx}{\sqrt{6+2x-3x^2}} = \frac{1}{\sqrt{3}}\int \frac{dx}{\sqrt{2+\frac{2}{3}x-x^2}} = \frac{1}{\sqrt{3}}\int \frac{dx}{\sqrt{2-(x^2-\frac{2}{3}x)}}$$

$$= \frac{1}{\sqrt{3}}\int \frac{dx}{\sqrt{\frac{19}{9}-(x-\frac{1}{3})^2}} = \frac{1}{\sqrt{3}}\sin^{-1}\frac{x-\frac{1}{3}}{\sqrt{\frac{19}{9}}} + C$$

$$= \frac{1}{\sqrt{3}}\sin^{-1}\frac{3x-1}{\sqrt{19}} + C.$$

Exercises

1. $\int \frac{dx}{\sqrt{3+2x-x^2}}$.

2. $\int \frac{dx}{x^2+6x+25}$.

3. $\int \frac{dx}{x^2-6x+5}$.

4. $\int \frac{dx}{\sqrt{2x^2+2x-3}}$.

5. $\int \frac{dx}{2x^2+5x-3}$.

6. $\int \frac{dx}{3x^2-4x+5}$.

7. $\int \frac{dt}{\sqrt{2+3t-2t^2}}$.

8. $\int \frac{dx}{5x^2-8x+1}$.

9. $\int \frac{dt}{\sqrt{1+2t+2t^2}}$.

10. $\int \frac{2x-5}{x^2+6x+25}\,dx = \int \frac{2x+6-11}{x^2+6x+25}\,dx$

$$= \int \frac{(2x+6)dx}{x^2+6x+25} - 11\int \frac{dx}{x^2+6x+25}.$$

11. $\int \frac{4x+7}{\sqrt{2x^2+2x-3}}\,dx$.

12. $\int \frac{4x+11}{x^2+2x+5}\,dx$.

13. $\int \frac{2x+9}{\sqrt{5-4x-x^2}}\,dx$.

14. $\int \frac{3x-5}{\sqrt{2+3x-2x^2}}\,dx$.

15. $\int \frac{dx}{x\sqrt{2x^2+3x-2}}$. Substitute $x = \frac{1}{z}$.

16. $\int \frac{dx}{x\sqrt{3+6x+5x^2}}$.

17. $\int \frac{dx}{\sqrt{6x-x^2}}$.

18. $\int \frac{dy}{\sqrt{4y^2+12y-7}}$.

19. $\int \frac{dx}{9x^2-24x-9}$.

20. $\displaystyle\int \frac{dx}{\sqrt{8 + 12x - 4x^2}}.$

21. $\displaystyle\int \frac{2x + 5}{\sqrt{8 + 12x - 4x^2}}\, dx.$

22. $\displaystyle\int \frac{dx}{16x^2 - 24x + 24}.$

23. $\displaystyle\int \frac{dx}{x\sqrt{4 + 12x - 7x^2}}.$

24. $\displaystyle\int \frac{dx}{x\sqrt{8x^2 + 12x - 4}}.$

102. Integrals Containing Fractional Powers of x or of a + bx.
Illustration 1.

$$\int \frac{x^{\frac{1}{2}} - x^{\frac{1}{3}}}{x^{\frac{1}{3}} + 4}\, dx.$$

Let $x = z^6$. Then $dx = 6z^5\, dz$, and

$$\int \frac{x^{\frac{1}{2}} - x^{\frac{1}{3}}}{x^{\frac{1}{3}} + 4}\, dx = 6 \int \frac{z^3 - z^2}{z^2 + 4}\, z^5\, dz = 6 \int \frac{z^8 - z^7}{z^2 + 4}\, dz.$$

The integration can be performed after dividing the numerator by the denominator until the degree of the remainder is less than 2. After integration replace z by $x^{\frac{1}{6}}$.

Illustration 2.

$$\int \frac{(x + 2)^{\frac{3}{4}} + 4}{(x + 2)^{\frac{1}{2}} - 3}\, dx.$$

Let $x + 2 = z^4$. Then

$$\int \frac{(x + 2)^{\frac{3}{4}} + 4}{(x + 2)^{\frac{1}{2}} - 3}\, dx = 4 \int \frac{(z^3 + 4)z^3}{z^2 - 3}\, dz.$$

Divide the numerator by the denominator. The integration can readily be performed. After integration replace z by $(x + 2)^{\frac{1}{4}}$.

In general if fractional powers of a single linear expressior, $a + bx$, occur under the integral sign, let $a + bx = z^n$, where n is the least common denominator of the exponents of $a + bx$. The linear expression $a + bx$ reduces to x when $a = 0$ and $b = 1$. See, for example, Illustration 1.

Exercises

1. $\displaystyle\int \frac{x^2\, dx}{(2x + 3)^{\frac{2}{3}}}.$

2. $\displaystyle\int \frac{(x + 1)^{\frac{1}{4}}}{1 + (x + 1)^{\frac{1}{2}}}\, dx.$

3. $\int \frac{x+2}{x\sqrt{x+3}}\,dx.$

4. $\int \frac{2x+3}{x\sqrt{x-2}}\,dx.$

5. $\int \frac{3x-4}{x\sqrt{x+1}}\,dx.$

6. $\int \frac{\sqrt{x+1}+1}{\sqrt{x+1}-1}\,dx.$

7. $\int \frac{dx}{x^{\frac{2}{3}}+3x^{\frac{3}{4}}}.$

8. $\int x\sqrt[4]{x+4}\,dx.$

9. $\int \frac{dx}{x\sqrt{x+4}}.$

10. $\int \frac{3x-2}{x\sqrt{2x+3}}\,dx.$

11. $\int (x^{\frac{1}{4}}+x^{\frac{1}{3}})\,dx.$

12. $\int \frac{2x+5}{3x\sqrt{2x-3}}\,dx.$

13. $\int \frac{(x+2)^{\frac{1}{2}}+1}{1+(x+2)^{\frac{1}{3}}}\,dx.$

14. $\int \frac{\sqrt{x-3}\,dx}{x+4}.$

15. $\int \frac{x^{\frac{1}{5}}+x^{\frac{1}{2}}}{x(1+x^{\frac{1}{5}})}\,dx.$

16. $\int \frac{\sqrt{2x+3}\,dx.}{3x-2}.$

103. Integrals of Powers of Trigonometric Functions.

(*a*) $\int \sin^m x \cos^n x\,dx$ where at least one of the exponents is an odd positive integer. This includes $\int \sin^m x\,dx$ and $\int \cos^n x\,dx$ where the exponents are odd.

Illustration 1.

$$\begin{aligned}\int \sin^3 x \cos^2 x\,dx &= \int (1-\cos^2 x)\cos^2 x \sin x\,dx\\ &= \int \cos^2 x \sin x\,dx - \int \cos^4 x \sin x\,dx\\ &= -\frac{\cos^3 x}{3}+\frac{\cos^5 x}{5}+C.\end{aligned}$$

Illustration 2.

$$\begin{aligned}\int \cos^3 x\,dx &= \int (1-\sin^2 x)\cos x\,dx\\ &= \int \cos x\,dx - \int \sin^2 x \cos x\,dx\\ &= \sin x - \frac{\sin^3 x}{3}+C.\end{aligned}$$

It is seen that the process consists in combining one of the functions $\sin x$ or $\cos x$ with dx to form the differential of $-\cos x$ or of $\sin x$, respectively, and of expressing the remaining factors of the function to be integrated in terms of $\cos x$ or $\sin x$, respectively.

Exercises

1. $\int \sin^5 x\, dx.$
2. $\int \sin^2 x \cos^3 x\, dx.$
3. $\int \cos^3 x \sin^3 x\, dx.$
4. $\int \sin^3 x\, dx.$
5. $\int \sqrt{\sin x} \cos^3 x\, dx.$
6. $\int \cos^4 x \sin^3 x\, dx.$
7. $\int \frac{\cos^3 x}{\sqrt{\sin x}}\, dx.$
8. $\int \frac{\sin^5 \theta}{(\cos \theta)^{\frac{1}{3}}}\, d\theta$
9. $\int \sin^2 \alpha \cos^3 \alpha\, d\alpha.$
10. $\int \cos^2 (2x+3) \sin^3 (2x+3)\, dx.$

(*b*) $\int \sin^m x \cos^n x\, dx$ when m and n are both even positive integers. In this case make use of the relations:

$$\sin^2 x = \tfrac{1}{2}(1 - \cos 2x).$$
$$\cos^2 x = \tfrac{1}{2}(1 + \cos 2x).$$
$$\sin x \cos x = \tfrac{1}{2} \sin 2x.$$

Illustration 1.

$$\int \sin^2 x\, dx = \int \tfrac{1}{2}(1 - \cos 2x)\, dx = \tfrac{1}{2}\int dx - \tfrac{1}{2}\int \cos 2x\, dx$$
$$= \frac{x}{2} - \frac{\sin 2x}{4} + C.$$

Illustration 2.

$$\int \sin^2 x \cos^2 x\, dx = \tfrac{1}{4}\int \sin^2 2x\, dx = \tfrac{1}{8}\int (1 - \cos 4x)\, dx$$
$$= \frac{x}{8} - \frac{\sin 4x}{32} + C.$$

Illustration 3.

$$\int \cos^4 x\, dx = \tfrac{1}{4}\int (1 + \cos 2x)^2\, dx$$
$$= \tfrac{1}{4}\int (1 + 2\cos 2x + \cos^2 2x)\, dx$$
$$= \tfrac{1}{4}x + \tfrac{1}{4}\sin 2x + \tfrac{1}{8}\int (1 + \cos 4x)\, dx$$
$$= \tfrac{3}{8}x + \tfrac{1}{4}\sin 2x + \tfrac{1}{32}\sin 4x + C.$$

Illustration 4.

$$\int \sin^2 x \cos^4 x\, dx = \int (\sin x \cos x)^2 \cos^2 x\, dx$$
$$= \tfrac{1}{8}\int \sin^2 2x\,(1 + \cos 2x)\, dx$$
$$= \tfrac{1}{16}\int (1 - \cos 4x)\, dx + \tfrac{1}{8}\int \sin^2 2x \cos 2x\, dx$$
$$= \tfrac{1}{16}x - \tfrac{1}{64}\sin 4x + \tfrac{1}{48}\sin^3 2x + C.$$

Exercises

1. $\int \sin^2 x\,dx.$
2. $\int \cos^4 2x\,dx.$
3. $\int \sin^4 x \cos^2 x\,dx.$
4. $\int \sin^4 x\,dx.$
5. $\int \sin^2 3x\,dx.$
6. $\int \cos^4 5x\,dx.$

(*c*) $\int \tan^n x\,dx$ and $\int \cot^n x\,dx.$

Illustration 1.

$$\int \tan^4 x\,dx = \int \tan^2 x\,(\sec^2 x - 1)\,dx = \frac{\tan^3 x}{3} - \int \tan^2 x\,dx$$
$$= \frac{\tan^3 x}{3} - \int (\sec^2 x - 1)\,dx$$
$$= \frac{\tan^3 x}{3} - \tan x + x + C.$$

Illustration 2.

$$\int \cot^5 x\,dx = \int (\csc^2 x - 1)^2 \cot x\,dx$$
$$= \int \csc^4 x \cot x\,dx - 2\int \csc^2 x \cot x\,dx + \int \cot x\,dx$$
$$= -\tfrac{1}{4}\csc^4 x + \csc^2 x + \log \sin x + C.$$

(*d*) $\int \sec^n x\,dx$ and $\int \csc^n x\,dx$, n an even integer.

Illustration 1.

$$\int \sec^4 x\,dx = \int (1 + \tan^2 x)\sec^2 x\,dx = \tan x + \tfrac{1}{3}\tan^3 x + C.$$

When n is odd this method fails. (See **§106.**)

(*e*) $\int \tan^m x \sec^n x\,dx$ and $\int \cot^m x \csc^n x\,dx$ when n is a positive even integer, or when m and n are both odd.

Illustration 1.

$$\int \tan^4 x \sec^4 x\,dx = \int \tan^4 x(1 + \tan^2 x)\sec^2 x\,dx$$
$$= \tfrac{1}{5}\tan^5 x + \tfrac{1}{7}\tan^7 x + C.$$

Illustration 2.

$$\int \tan^3 x \sec^3 x\,dx = \int \tan^2 x \sec^2 x \sec x \tan x\,dx$$
$$= \int (\sec^2 x - 1)\sec^2 x \sec x \tan x\,dx$$
$$= \int (\sec^4 x - \sec^2 x)\sec x \tan x\,dx$$
$$= \tfrac{1}{5}\sec^5 x - \tfrac{1}{3}\sec^3 x + C.$$

If m is even and n is odd the methods of **§106** must be used, for the integral reduces in this case to the integral of odd powers of the secant.

Exercises

1. $\int \tan^5 x\, dx.$
2. $\int \csc^4 x\, dx.$
3. $\int \tan^2 x \sec^6 x\, dx.$
4. $\int \tan^3 x \sec^5 x\, dx.$
5. $\int \cot^4 x\, dx.$
6. $\int \csc^6 x\, dx.$
7. $\int \tan^4 x \sec^2 x\, dx.$
8. $\int \sec^6 x\, dx.$
9. $\int \tan^5 x \sec^3 x\, dx.$
10. $\int \tan^6 x \sec^4 x\, dx.$
11. $\int \tan^{\frac{2}{3}} x \sec^4 x\, dx.$
12. $\int (\tan^2 x + \tan^4 x)\, dx.$
13. $\int \sec^6 x \tan^{\frac{3}{2}} x\, dx.$
14. $\int (\tan\theta + \cot\theta)^3\, d\theta.$
15. $\int \tan^2 \theta\, d\theta.$
16. $\int \sec^6 \theta \tan^{-4} \theta\, d\theta.$

104. Integration of Expressions Containing $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$, $\sqrt{x^2 - a^2}$ by Trigonometric Substitution. The methods of **§103** find frequent application in the integration of expressions which result from the substitution of a trigonometric function for x in integrals containing radicals reducible to one of the forms $\sqrt{a^2 + x^2}$, $\sqrt{a^2 - x^2}$, or $\sqrt{x^2 - a^2}$.

Illustration 1. $\int \sqrt{a^2 - x^2}\, dx$. Let $x = a \sin\theta$. Then $dx = a\cos\theta\, d\theta$, and

$$\begin{aligned}\int \sqrt{a^2 - x^2}\, dx &= \int a^2 \cos^2\theta\, d\theta = \tfrac{1}{2}a^2(\theta + \tfrac{1}{2}\sin 2\theta) + C\\ &= \tfrac{1}{2}a^2(\theta + \sin\theta\cos\theta) + C\\ &= \tfrac{1}{2}a^2\left[\sin^{-1}\frac{x}{a} + \frac{x}{a^2}\sqrt{a^2 - x^2}\right] + C\\ &= \tfrac{1}{2}a^2 \sin^{-1}\frac{x}{a} + \tfrac{1}{2}x\sqrt{a^2 - x^2} + C.\end{aligned}$$

Illustration 2. $\int \sqrt{a^2 + x^2}\, x^3\, dx$. Let $x = a\tan\theta$. Then

$$\begin{aligned}\int \sqrt{a^2 + x^2}\, x^3\, dx &= a^5 \int \tan^3\theta \sec^3\theta\, d\theta\\ &= a^5 \int \tan^2\theta \sec^2\theta \tan\theta \sec\theta\, d\theta\\ &= a^5 \int (\sec^4\theta - \sec^2\theta) \tan\theta \sec\theta\, d\theta\end{aligned}$$

$$= a^5(\tfrac{1}{5}\sec^5\theta - \tfrac{1}{3}\sec^3\theta) + C$$

$$= a^5\left[\frac{\left(1+\dfrac{x^2}{a^2}\right)^{\frac{5}{2}}}{5} - \frac{\left(1+\dfrac{x^2}{a^2}\right)^{\frac{3}{2}}}{3}\right] + C$$

$$= \frac{(a^2+x^2)^{\frac{5}{2}}}{5} - \frac{a^2(a^2+x^2)^{\frac{3}{2}}}{3} + C.$$

Illustration 3. $\displaystyle\int \frac{\sqrt{x^2-a^2}}{x}\,dx$. Let $x = a\sec\theta$. Then $dx = a\sec\theta\tan\theta\,d\theta$, and

$$\int \frac{\sqrt{x^2-a^2}}{x}\,dx = a\int \frac{\sec\theta\tan^2\theta\,d\theta}{\sec\theta}$$

$$= a\int \tan^2\theta\,d\theta$$

$$= a\,(\tan\theta - \theta) + C$$

$$= a\sqrt{\frac{x^2}{a^2}-1} - a\sec^{-1}\frac{x}{a} + C$$

$$= \sqrt{x^2-a^2} - a\sec^{-1}\frac{x}{a} + C.$$

The integration can also be performed directly if the numerator is rationalized. Thus,

$$\int \frac{\sqrt{x^2-a^2}}{x}\,dx = \int \frac{(x^2-a^2)dx}{x\sqrt{x^2-a^2}}$$

$$= \int \frac{x\,dx}{\sqrt{x^2-a^2}} - a^2\int \frac{dx}{x\sqrt{x^2-a^2}}$$

$$= \sqrt{x^2-a^2} - a\sec^{-1}\frac{x}{a} + C.$$

The substitutions used in these illustrations are summarized in the following table:

Radical	Substitution	Radical becomes
$\sqrt{a^2-x^2}$	$x = a\sin\theta$	$a\cos\theta$
$\sqrt{a^2+x^2}$	$x = a\tan\theta$	$a\sec\theta$
$\sqrt{x^2-a^2}$	$x = a\sec\theta$	$a\tan\theta$

Expressions involving $\sqrt{ax^2 + bx + c}$ can frequently be integrated by completing the square under the radical sign and making a trigonometric substitution.

Illustration 1.

$$\int \frac{x\,dx}{\sqrt{3 + 2x - x^2}} = \int \frac{x\,dx}{\sqrt{4 - (x - 1)^2}}.$$

Let $x - 1 = 2 \sin \theta$. Then $x = 1 + 2 \sin \theta$ and $dx = 2 \cos \theta\, d\theta$. Hence

$$\begin{aligned}\int \frac{x\,dx}{\sqrt{3 + 2x - x^2}} &= 2\int \frac{(1 + 2 \sin \theta) \cos \theta\, d\theta}{2 \cos \theta}\\ &= \int (1 + 2 \sin \theta) d\theta\\ &= \theta - 2 \cos \theta + C\\ &= \sin^{-1} \frac{x - 1}{2} - \sqrt{3 + 2x - x^2} + C.\end{aligned}$$

Illustration 2.

$$\int \frac{dx}{\sqrt{(2ax - x^2)^3}} = \int \frac{dx}{[a^2 - (x - a)^2]^{\frac{3}{2}}}.$$

Let $x - a = a \sin \theta$. Then $x = a(1 + \sin \theta)$ and $dx = a \cos \theta\, d\theta$.

$$\begin{aligned}\int \frac{dx}{\sqrt{(2ax - x^2)^3}} &= \int \frac{a \cos \theta}{a^3 \cos^3 \theta}\, d\theta\\ &= \frac{1}{a^2}\int \sec^2 \theta\, d\theta\\ &= \frac{1}{a^2} \tan \theta + C\\ &= \frac{1}{a^2} \frac{\sin \theta}{\cos \theta} + C\\ &= \frac{1}{a^2} \frac{\frac{x - a}{a}}{\frac{1}{a}\sqrt{2ax - x^2}} + C\\ &= \frac{1}{a^2} \frac{x - a}{\sqrt{2ax - x^2}} + C.\end{aligned}$$

Exercises

1. $\int \frac{dx}{\sqrt{x^2 - a^2}}$.
4. $\int \frac{dx}{x\sqrt{a^2 + x^2}}$
7. $\int \frac{dx}{x^2(x^2 + 9)^{\frac{3}{2}}}$
2. $\int \frac{\sqrt{a^2 + x^2}}{x^2}\, dx$.
5. $\int \frac{dx}{x\sqrt{a^2 - x^2}}$.
8. $\int \frac{dx}{(1-x)\sqrt{1-x^2}}$.
3. $\int x^3\sqrt{1 + x^2}\, dx$.
6. $\int \frac{dx}{x\sqrt{7 - 3x^2}}$.
9. $\int \frac{dx}{(4x - 3 - x^2)^{\frac{3}{2}}}$.

10. $\int (a^{\frac{2}{3}} - x^{\frac{2}{3}})^{\frac{3}{2}}\, dx = 3a^2 \int \cos^4 \theta \sin^2 \theta \, d\theta$.

Hint. Let

$$x^{\frac{2}{3}} = a^{\frac{2}{3}} \sin^2 \theta,$$

or

$$x = a \sin^3 \theta.$$

11. $\int \frac{x^2\, dx}{\sqrt{a^2 - x^2}}$
13. $\int \frac{dx}{x^3\sqrt{x^2 - 9}}$.
15. $\int \frac{dx}{x(x^2 - 4)^{\frac{3}{2}}}$.
12. $\int \sqrt{9 - 5x^2}\, dx$.
14. $\int (9 - x^2)^{\frac{3}{2}}\, dx$.
16. $\int \frac{dx}{(16 - x^2)^{\frac{3}{2}}}$.
17. $\int \frac{dx}{(x^2 + 6x + 25)^{\frac{3}{2}}}$.
19. $\int \frac{dx}{(x^2 + 4x - 5)^{\frac{3}{2}}}$.
18. $\int \frac{x^2\, dx}{\sqrt{6x - x^2}}$.
20. $\int \sqrt{2 + 6x - x^2}\, dx$.

105. Change of Limits of Integration. In working the preceding exercises by substitution it was necessary to express the result of integration in terms of the original variable. In the case of definite integrals this last transformation can be avoided by changing the limits of integration.

Illustration 1. $\int_0^a x^2 \sqrt{a^2 - x^2}\, dx$. Let $x = a \sin \theta$. Then $dx = a \cos \theta \, d\theta$.

$$\text{When } x = 0, \sin \theta = 0 \text{ and } \theta = 0.$$

$$\text{When } x = a, \sin \theta = 1 \text{ and } \theta = \frac{\pi}{2}.$$

As x varies continuously from 0 to a, θ varies continuously from 0 to $\frac{\pi}{2}$. Hence we have

$$\int_0^a x^2\sqrt{a^2 - x^2}\,dx = a^4\int_0^{\frac{\pi}{2}} \sin^2\theta\cos^2\theta\,d\theta$$

$$= a^4(\tfrac{1}{8}\theta - \tfrac{1}{32}\sin 4\theta)\Big|_0^{\frac{\pi}{2}}$$

$$= \frac{\pi a^4}{16}.$$

Illustration 2. $\int_0^a \frac{x^3\,dx}{\sqrt{a^2 + x^2}}$. Let $x = a\tan\theta$. Then $dx = a\sec^2\theta\,d\theta$.

When $x = 0$, $\tan\theta = 0$ and $\theta = 0$.

When $x = a$, $\tan\theta = 1$ and $\theta = \frac{\pi}{4}$.

As x varies continuously from 0 to a, θ varies continuously from 0 to $\frac{\pi}{4}$. Hence we have

$$\int_0^a \frac{x^3\,dx}{\sqrt{a^2 + x^2}} = a^3\int_0^{\frac{\pi}{4}} \tan^3\theta\sec\theta\,d\theta = a^3(\tfrac{1}{3}\sec^3\theta - \sec\theta)\Big|_0^{\frac{\pi}{4}}$$

$$= \tfrac{1}{3}a^3(2 - \sqrt{2}).$$

Illustration 3. $\int_0^a \sqrt{a^2 - x^2}\,dx$. By using the substitution $x = a\sin\theta$ we obtain

$$\int_0^a \sqrt{a^2 - x^2}\,dx = a^2\int_0^{\frac{\pi}{2}} \cos^2\theta\,d\theta$$

$$= \tfrac{1}{2}a^2\,(\theta + \tfrac{1}{2}\sin 2\theta)\Big|_0^{\frac{\pi}{2}}$$

$$= \frac{\pi a^2}{4}.$$

The above integral is of frequent occurrence in the application of the calculus. The integrand, $\sqrt{a^2 - x^2}$, is represented graphically by the ordinates of a circle of radius a, center at the origin. The integral then represents the area of one-quarter of this circle. (See §§**64** and **65**.) The value of any integral of this form may be written down at once. Thus,

$$\int_5^7 \sqrt{4-(x-5)^2}\,dx = \int_0^2 \sqrt{4-u^2}\,du = \frac{\pi\, 2^2}{4} = \pi.$$

$$\int_0^{\sqrt{3+z^2}} \sqrt{3+z^2-x^2}\,dx = \frac{\pi(3+z^2)}{4}.$$

Exercises

1. $\int_0^3 (9-x^2)^{\frac{3}{2}}\,dx.$

2. $\int_0^a \frac{dx}{\sqrt{2ax-x^2}}.$

3. $\int_0^3 \frac{x^2\,dx}{\sqrt{9-x^2}}.$

4. $\int_2^5 \frac{dx}{\sqrt{5+4x-x^2}}.$

5. $\int_0^5 \sqrt{25-x^2}\,dx.$

6. $\int_0^3 \sqrt{9-x^2}\,dx.$

7. $\int_0^a \frac{dx}{(a^2+x^2)^{\frac{3}{2}}}.$

8. $\int_{2\sqrt{2}}^4 \frac{dx}{x(x^2-4)^{\frac{3}{2}}}$

9. $\int_0^4 \frac{dx}{(x^2+16)^2}.$

10. $\int_3^6 \frac{dx}{x^2(x^2-9)^{\frac{1}{2}}}.$

11. $\int_4^7 \sqrt{9-(x-4)^2}\,dx.$

12. $\int_0^{\sqrt{b^2-y^2}} \sqrt{b^2-y^2-x^2}\,dx.$

106. Integration by Parts. The differential of the product of two functions u and v is

$$d(uv) = u\,dv + v\,du. \tag{1}$$

Integrating we obtain

$$uv = \int u\,dv + \int v\,du$$

From which

$$\int \mathbf{u\,dv} = \mathbf{uv} - \int \mathbf{v\,du}. \tag{2}$$

This equation is known as the formula for *integration by parts.* It makes the integration of $u\,dv$ depend upon the integration of dv and of $v\,du$.

Illustration 1. $\int x \log x\, dx$. Let $\log x = u$ and $x\,dx = dv$. The application of (2) gives

$$\int x \log x\, dx = \tfrac{1}{2}x^2 \log x - \tfrac{1}{2}\int x^2 \frac{1}{x}\, dx$$
$$= \tfrac{1}{2}x^2 \log x - \tfrac{1}{4}x^2 + C.$$

Illustration 2. $\int xe^{3x}dx$. Let $e^{3x}dx = dv$ and $x = u$. The application of (2) gives

$$\int xe^{3x}dx = \tfrac{1}{3}xe^{3x} - \tfrac{1}{3}\int e^{3x}\, dx$$
$$= \tfrac{1}{3}xe^{3x} - \tfrac{1}{9}e^{3x} + C$$
$$= \tfrac{1}{9}e^{3x}(3x - 1) + C.$$

If we had let $x\,dx = dv$ and $e^{3x} = u$ we should have obtained a more complicated expression to integrate than that with which we started.

Exercises

1. $\int x^2 \log x\, dx$.
2. $\int x \cos x\, dx$.
3. $\int \sin^{-1} x\, dx$
4. $\int x^2 e^{4x}\, dx$. (Apply formula (2) twice in succession.)
5. $\int \tan^{-1} x\, dx$.
6. $\int x \sin x\, dx$.
7. $\int x^n \log x\, dx$.
8. $\int x^2 \tan^{-1} 2x\, dx$.
9. $\int x \sin^3 x\, dx$.
10. $\int \log x\, dx$.
11. $\int x^2 \sin 2x\, dx$.
12. $\int \sin x \log \cos x\, dx$.

107. The Integrals $\int e^{ax} \sin nx\, dx$, $\int e^{ax} \cos nx\, dx$. Let $u = \sin nx$ and $dv = e^{ax}dx$. Then

$$\int e^{ax} \sin nx\, dx = \frac{1}{a}e^{ax} \sin nx - \frac{n}{a}\int e^{ax} \cos nx\, dx.$$

A second integration by parts with $u = \cos nx$ and $dv = e^{ax}dx$ gives

$$\int e^{ax} \sin nx\, dx = \frac{1}{a}e^{ax} \sin nx - \frac{n}{a^2}e^{ax} \cos nx - \frac{n^2}{a^2}\int e^{ax} \sin nx\, dx.$$

The last term is equal to the integral in the first member multiplied by $\frac{n^2}{a^2}$. On transposing this term to the first member we obtain

$$\frac{a^2 + n^2}{a^2}\int e^{ax} \sin nx\, dx = \frac{e^{ax}}{a^2}(a \sin nx - n \cos nx) + C.$$

Then

$$\int \mathbf{e^{ax} \sin nx\, dx} = \frac{\mathbf{e^{ax}}}{\mathbf{a^2 + n^2}}(\mathbf{a \sin nx - n \cos nx}) + \mathbf{C'}$$

$$= \frac{\mathbf{e^{ax}}}{\sqrt{\mathbf{a^2 + n^2}}} \sin (\mathbf{nx} - \alpha) + \mathbf{C'}, \qquad (1)$$

where

$$\cos \alpha = \frac{a}{\sqrt{a^2 + n^2}}$$

and

$$\sin \alpha = \frac{n}{\sqrt{a^2 + n^2}}.$$

The student will show in a similar way that

$$\int \mathbf{e^{ax} \cos nx\, dx} = \frac{\mathbf{e^{ax}}}{\mathbf{a^2 + n^2}}(\mathbf{n \sin nx + a \cos nx}) + \mathbf{C}$$

$$= \frac{\mathbf{e^{ax}}}{\sqrt{\mathbf{a^2 + n^2}}} \cos (\mathbf{nx} - \alpha) + \mathbf{C}, \qquad (2)$$

where

$$\cos \alpha = \frac{a}{\sqrt{a^2 + n^2}}$$

and

$$\sin \alpha = \frac{n}{\sqrt{a^2 + n^2}}.$$

Exercises

The student will work exercises 1–5 by the method used in obtaining (1) and (2) above. In the remaining exercises he may obtain the results by substituting in (1) and (2) as formulas.

1. $\int e^{-5t} \sin 7t\, dt.$

2. $\int e^{-3t} \cos 8t\, dt.$

3. $\int e^{-0.5t} \sin 3t\, dt.$

4. $\int e^{-0.2t} \cos 4t\, dt.$

5. $\int e^{-x} \sin x\, dx.$

6. $\int e^{-2t} \cos 5t\, dt.$

7. $\int e^{-0.4t} \sin \omega t\, dt.$

8. $\int e^{-0.2t} \cos \omega t\, dt.$

9. $\int e^{-0.4t} \cos 5t\, dt.$

10. $\int e^{-0.2t} \sin 4t\, dt.$

11. Find α in exercises 1–10.

108. $\int \sec^3 x\, dx$. This integral can be evaluated by a method similar to that used in the last article.

$$\int \sec^3 x\, dx = \int \sec x \sec^2 x\, dx$$
$$= \sec x \tan x - \int \sec x \tan^2 x\, dx.$$

Since $\tan^2 x = \sec^2 x - 1$,

$$\int \sec^3 x\, dx = \sec x \tan x - \int \sec^3 x\, dx + \int \sec x\, dx.$$

Transposing the next to the last term to the first member, dividing by 2, and integrating the last term we have

$$\int \sec^3 x\, dx = \tfrac{1}{2}\left[\sec x \tan x + \log(\sec x + \tan x)\right] + C.$$

Exercises

1. $\int \csc^3 x\, dx.$
2. $\int \sec^5 x\, dx.$
3. $\int \dfrac{x^4}{\sqrt{x^2 - a^2}}\, dx.$
4. $\int_0^a \sqrt{a^2 + x^2}\, dx.$
5. $\int \sqrt{a^2 + x^2}\, dx.$
6. $\int \sqrt{x^2 - 4x + 11}\, dx.$
7. $\int_a^{2a} \dfrac{x^4}{\sqrt{x^2 - a^2}}\, dx.$
8. $\int_3^6 \sqrt{x^2 - 9}\, dx.$

109. Wallis' Formulas. Formulas will now be derived which make it possible to write down at once the values of the definite integrals:

$$\int_0^{\frac{\pi}{2}} \sin^n \theta\, d\theta,$$

$$\int_0^{\frac{\pi}{2}} \cos^n \theta\, d\theta,$$

and

$$\int_0^{\frac{\pi}{2}} \sin^m \theta \cos^n \theta\, d\theta,$$

where m and n are positive integers greater than 1.

$$\int_0^{\frac{\pi}{2}} \sin^n \theta\, d\theta = \int_0^{\frac{\pi}{2}} \sin^{n-1} \theta \sin \theta\, d\theta.$$

Integration by parts gives

$$\int_0^{\frac{\pi}{2}} \sin^n \theta \, d\theta = -\sin^{n-1}\theta \cos\theta \Big|_0^{\frac{\pi}{2}} + (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2}\theta \cos^2\theta \, d\theta$$

$$= (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2}\theta(1 - \sin^2\theta)\, d\theta$$

$$= (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2}\theta \, d\theta - (n-1)\int_0^{\frac{\pi}{2}} \sin^n \theta \, d\theta.$$

On transposing the last term and dividing by n we obtain

$$\int_0^{\frac{\pi}{2}} \sin^n \theta \, d\theta = \frac{n-1}{n}\int_0^{\frac{\pi}{2}} \sin^{n-2}\theta \, d\theta.$$

This equation can be regarded as a reduction formula for expressing

$$\int_0^{\frac{\pi}{2}} \sin^n \theta \, d\theta$$

in terms of an integral in which $\sin\theta$ occurs with its exponent diminished by 2. Applying this formula successively we obtain

$$\int_0^{\frac{\pi}{2}} \sin^n \theta \, d\theta = \frac{n-1}{n}\,\frac{n-3}{n-2}\int_0^{\frac{\pi}{2}} \sin^{n-4}\theta \, d\theta$$

$$= \frac{n-1}{n}\,\frac{n-3}{n-2}\,\frac{n-5}{n-4}\int_0^{\frac{\pi}{2}} \sin^{n-6}\theta \, d\theta$$

$$= \begin{cases} \dfrac{(n-1)(n-3)\cdots 4\cdot 2}{n(n-2)\cdots 5\cdot 3}\displaystyle\int_0^{\frac{\pi}{2}} \sin\theta \, d\theta & \text{if } n \text{ is odd.} \\[2ex] \dfrac{(n-1)(n-3)\cdots 3\cdot 1}{n(n-2)\cdots 4\cdot 2}\displaystyle\int_0^{\frac{\pi}{2}} d\theta & \text{if } n \text{ is even.} \end{cases}$$

$$\boldsymbol{\int_0^{\frac{\pi}{2}} \sin^n \theta \, d\theta = \begin{cases} \dfrac{(n-1)(n-3)\cdots 4\cdot 2}{n(n-2)\cdots 3\cdot 1} & \text{if } n \text{ is odd.} \\[2ex] \dfrac{(n-1)(n-3)\cdots 3\cdot 1}{n(n-2)\cdots 4\cdot 2}\,\dfrac{\pi}{2} & \text{if } n \text{ is even.} \end{cases}}$$

From the fact that the integrals

$$\int_0^{\frac{\pi}{2}} \sin^n x\, dx$$

and

$$\int_0^{\frac{\pi}{2}} \cos^n x\, dx$$

represent the areas under the curves $y = \sin^n x$ and $y = \cos^n x$, respectively, between the limits $x = 0$ and $x = \frac{\pi}{2}$, it is clear from the graphs that

$$\int_0^{\frac{\pi}{2}} \cos^n x\, dx = \int_0^{\frac{\pi}{2}} \sin^n x\, dx.$$

The results obtained can be expressed in the single formula

$$\int_0^{\frac{\pi}{2}} \cos^n \theta\, d\theta = \int_0^{\frac{\pi}{2}} \sin^n \theta\, d\theta = \frac{(n-1)(n-3)\cdots 2 \text{ or } 1}{n(n-2)\cdots 2 \text{ or } 1}\alpha, \quad (1)$$

where $\alpha = 1$ if n is odd, and $\alpha = \frac{\pi}{2}$ if n is even.

In a similar way we shall evaluate

$$\int_0^{\frac{\pi}{2}} \sin^m \theta \cos^n \theta\, d\theta.$$

$$\int_0^{\frac{\pi}{2}} \sin^m \theta \cos^n \theta\, d\theta = \int_0^{\frac{\pi}{2}} \sin^{m-1} \theta \cos^n \theta \sin \theta\, d\theta$$

$$= -\frac{\sin^{m-1}\theta \cos^{n+1}\theta}{n+1}\Bigg|_0^{\frac{\pi}{2}} + \frac{m-1}{n+1}\int_0^{\frac{\pi}{2}} \sin^{m-2}\theta \cos^{n+2}\theta\, d\theta$$

$$= \frac{m-1}{n+1}\int_0^{\frac{\pi}{2}} \sin^{m-2}\theta \cos^n \theta(1 - \sin^2\theta)\, d\theta$$

$$= \frac{m-1}{n+1}\int_0^{\frac{\pi}{2}} \sin^{m-2}\theta \cos^n \theta\, d\theta - \frac{m-1}{n+1}\int_0^{\frac{\pi}{2}} \sin^m \theta \cos^n \theta\, d\theta.$$

Transposing the last term to the left member of the equation

$$\left[1+\frac{m-1}{n+1}\right]\int_0^{\frac{\pi}{2}} \sin^m\theta\cos^n\theta\,d\theta = \frac{m-1}{n+1}\int_0^{\frac{\pi}{2}} \sin^{m-2}\theta\cos^n\theta\,d\theta$$

$$\int_0^{\frac{\pi}{2}} \sin^m\theta\cos^n\theta\,d\theta = \frac{m-1}{m+n}\int_0^{\frac{\pi}{2}} \sin^{m-2}\theta\cos^n\theta\,d\theta.$$

Apply this formula successively and obtain

$$\int_0^{\frac{\pi}{2}} \sin^m\theta\cos^n\theta\,d\theta = \frac{m-1}{m+n}\,\frac{m-3}{m+n-2}\int_0^{\frac{\pi}{2}} \sin^{m-4}\theta\cos^n\theta\,d\theta$$

$$= \begin{cases} \dfrac{(m-1)(m-3)\cdots 1}{(m+n)(m+n-2)\cdots(n+2)}\displaystyle\int_0^{\frac{\pi}{2}} \cos^n\theta\,d\theta & \text{if } m \text{ is even} \\[2ex] \dfrac{(m-1)(m-3)\cdots 2}{(m+n)(m+n-2)\cdots(n+3)}\displaystyle\int_0^{\frac{\pi}{2}} \sin\theta\cos^n\theta\,d\theta & \text{if } m \text{ is odd} \end{cases}$$

$$= \begin{cases} \dfrac{(m-1)(m-3)\cdots 1\cdot(n-1)(n-3)\cdots 1}{(m+n)(m+n-2)\cdots(n+2)(n)(n-2)\cdots 2}\,\dfrac{\pi}{2} & \text{if } n \text{ is even} \\[2ex] \dfrac{(m-1)(m-3)\cdots 1\cdot(n-1)(n-3)\cdots 2}{(m+n)(m+n-2)\cdots(n+2)(n)(n-2)\cdots 3} & \text{if } n \text{ is odd} \end{cases} \text{ and } m \text{ is even.}$$

$$ \dfrac{(m-1)(m-3)\cdots 2}{(m+n)(m+n-2)\cdots(n+3)(n+1)} \quad \text{if } n \text{ is either even or odd, and } m \text{ is odd.}$$

The right-hand member of the last formula of this group can be put in a form similar to the others by multiplying numerator and denominator by $(n-1)(n-3)\cdots 2$ or 1. It becomes

$$\frac{(m-1)(m-3)\cdots 2\cdot(n-1)(n-3)\cdots 2 \text{ or } 1}{(m+n)(m+n-2)\cdots(n+3)(n+1)(n-1)(n-3)\cdots 2 \text{ or } 1}$$

These formulas for $\int_0^{\frac{\pi}{2}} \sin^m\theta\cos^n\theta\,d\theta$ can all be expressed in the single formula

$$\int_0^{\frac{\pi}{2}} \sin^m\theta\cos^n\theta\,d\theta = \frac{(m-1)(m-3)\cdots 2 \text{ or } 1\cdot(n-1)(n-3)\cdots 2 \text{ or } 1}{(m+n)(m+n-2)\cdots 2 \text{ or } 1}\,\alpha, \qquad (2)$$

where $\alpha = 1$ unless m and n are both even, in which case $\alpha = \frac{\pi}{2}$.

Illustration 1. By formula (1)

$$\int_0^{\frac{\pi}{2}} \sin^9 \theta \, d\theta = \frac{8\cdot6\cdot4\cdot2}{9\cdot7\cdot5\cdot3\cdot1} = \frac{128}{315}.$$

Illustration 2.

$$\int_0^{\frac{\pi}{2}} \cos^4 \theta \, d\theta = \frac{3\cdot1}{4\cdot2}\,\frac{\pi}{2} = \frac{3\,\pi}{16}.$$

Illustration 3. By formula (2),

$$\int_0^{\frac{\pi}{2}} \sin^5 x \, \cos^3 x \, dx = \frac{4\cdot2\cdot2}{8\cdot6\cdot4\cdot2} = \frac{1}{24}.$$

Illustration 4.

$$\int_0^{\frac{\pi}{2}} \sin^5 x \, \cos^4 x \, dx = \frac{4\cdot2\cdot3\cdot1}{9\cdot7\cdot5\cdot3\cdot1} = \frac{8}{315}.$$

Illustration 5.

$$\int_0^{\frac{\pi}{2}} \sin^6 x \, \cos^4 x \, dx = \frac{5\cdot3\cdot1\cdot3\cdot1}{10\cdot8\cdot6\cdot4\cdot2}\,\frac{\pi}{2} = \frac{3\pi}{512}.$$

Exercises

1. $\int_0^{\frac{\pi}{2}} \sin^5 \theta \, d\theta.$

2. $\int_0^{\frac{\pi}{2}} \cos^{10} \theta \, d\theta.$

3. $\int_0^{\frac{\pi}{2}} \cos^9 \theta \, d\theta.$

4. $\int_0^{\frac{\pi}{2}} \sin^6 \theta \, d\theta.$

5. $\int_0^{\frac{\pi}{2}} \cos^6 \theta \, d\theta.$

6. $\int_0^{\frac{\pi}{2}} \sin^8 \theta \, d\theta.$

7. $\int_0^{\frac{\pi}{2}} \cos^{11} x \, dx.$

8. $\int_0^{\frac{\pi}{2}} \sin^7 \phi \, d\phi.$

9. $\int_0^{\frac{\pi}{2}} \sin^4 x \cos^5 x \, dx.$

10. $\int_0^{\frac{\pi}{2}} \sin^3 x \cos^5 x \, dx.$

11. $\int_0^{\frac{\pi}{2}} \sin^4 x \cos^6 x \, dx.$

12. $\int_0^{\frac{\pi}{2}} \sin^5 \phi \cos \phi \, d\phi.$

13. $\int_0^{\frac{\pi}{2}} \sin^2 x \cos^8 x\, dx.$

14. $\int_0^{\frac{\pi}{2}} \cos^4 x \sin^5 x\, dx.$

15. $\int_0^{a} (a^2 - x^2)^{\frac{3}{2}}\, dx.$

16. $\int_0^{a} x^2 (a^2 - x^2)^{\frac{3}{2}}\, dx.$

17. $\int_0^{a} (a^{\frac{2}{3}} - x^{\frac{2}{3}})^{\frac{3}{2}} dx.$

18. $\int_0^{a} x(a^{\frac{2}{3}} - x^{\frac{2}{3}})^{\frac{3}{2}} dx.$

19. $\int_0^{\pi} a^2(1 - \cos\theta)^2 d\theta = 4a^2 \int_0^{\pi} \sin^4 \frac{\theta}{2}\, d\theta.$

Let $\theta' = \frac{\theta}{2}$. Then $d\theta = 2d\theta'$ and $\theta' = \frac{\pi}{2}$ when $\theta = \pi$, and $\theta' = 0$ when $\theta = 0$. Hence

$$a^2 \int_0^{\pi} (1 - \cos\theta)^2 d\theta = 8a^2 \int_0^{\frac{\pi}{2}} \sin^4 \theta'\, d\theta'.$$

Wallis' formula can now be applied.

By transformations similar to the foregoing many integrals can be put into a form to which Wallis' formulas can be applied.

20. $\int_0^{\frac{\pi}{4}} \cos^2 2\theta\, d\theta = \frac{1}{2} \int_0^{\frac{\pi}{2}} \cos^2 \theta'\, d\theta'.$

21. $\int_0^{2a} (2ax - x^2)^{\frac{3}{2}} dx.$ (Substitute $x = 2a \sin^2 \theta$.)

22. $\int_0^{2a} x\sqrt{2ax - x^2}\, dx.$

110. Integration of $\int \frac{\mathbf{a \sin x + b \cos x}}{\mathbf{c \sin x + d \cos x}} \mathbf{dx}$. Integrals of this form can be reduced by the substitution, $z = \tan \frac{x}{2}$. In making this substitution it is necessary to express $\sin x$, $\cos x$, and dx in terms of z. This is easily done as follows. (The student is advised to observe the method carefully, but not to learn the results as he can readily obtain them whenever needed.) Since

$$z = \tan \frac{x}{2},$$

$$x = 2 \tan^{-1} z,$$

and

$$dx = 2\frac{dz}{1+z^2}.$$

Further,

$$\cos\frac{x}{2} = \frac{1}{\sec\frac{x}{2}} = \frac{1}{\sqrt{1+\tan^2\frac{x}{2}}} = \frac{1}{\sqrt{1+z^2}}$$

and

$$\sin\frac{x}{2} = \tan\frac{x}{2}\cos\frac{x}{2} = \frac{z}{\sqrt{1+z^2}}.$$

Then

$$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} = \frac{2z}{1+z^2}$$

and

$$\cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2} = \frac{1-z^2}{1+z^2}.$$

Illustration 1. $\int \frac{dx}{1+4\cos x}.$ On making the substitution $z = \tan\frac{x}{2}$ we obtain by using the values just found for $\cos x$ and dx in terms of z,[1]

$$\int \frac{\frac{2dz}{1+z^2}}{1+4\frac{1-z^2}{1+z^2}} = 2\int\frac{dz}{1+z^2+4(1-z^2)}$$

$$= 2\int\frac{dz}{5-3z^2}$$

$$= -\frac{2}{\sqrt{3}}\int\frac{\sqrt{3}\,dz}{3z^2-5}$$

$$= -\frac{2}{2\sqrt{3}\sqrt{5}}\log\frac{\sqrt{3}z-\sqrt{5}}{\sqrt{3}z+\sqrt{5}} + C$$

$$= \tfrac{1}{15}\sqrt{15}\log\frac{\sqrt{3}\tan\frac{x}{2}+\sqrt{5}}{\sqrt{3}\tan\frac{x}{2}-\sqrt{5}} + C.$$

[1] The student will derive these values in each problem worked in order to familiarize himself with the method.

Illustration 2. $\int \frac{dx}{5 - 3\sin x}$. Let $z = \tan\frac{x}{2}$. Then

$$\int \frac{dx}{5 - 3\sin x} = \int \frac{\frac{2dz}{1+z^2}}{5 - \frac{6z}{1+z^2}} = 2\int \frac{dz}{5 + 5z^2 - 6z}$$

$$= \frac{2}{5}\int \frac{dz}{z^2 - \frac{6}{5}z + 1} = \frac{2}{5}\int \frac{dz}{(z - \frac{3}{5})^2 + \frac{16}{25}}$$

$$= \frac{2}{5}\cdot\frac{5}{4}\tan^{-1}\frac{z - \frac{3}{5}}{\frac{4}{5}} + C = \frac{1}{2}\tan^{-1}\frac{5z - 3}{4} + C$$

$$= \frac{1}{2}\tan^{-1}\frac{5\tan\frac{x}{2} - 3}{4} + C.$$

Exercises

The student will find $\cos x$, $\sin x$, and dx in terms of the new variable in each of the exercises.

1. $\int \frac{dx}{3 + 5\cos x}$.
2. $\int \frac{dx}{5 - 3\cos x}$.
3. $\int \frac{dx}{4 - 5\sin x}$.
4. $\int \frac{\sin x\, dx}{2 + \sin x}$.
5. $\int \frac{\cos x}{3 + 2\cos x}\,dx$.
6. $\int \frac{\cos x}{1 - 3\sin x}\,dx$
7. $\int \frac{3 + 4\sin x}{1 + 2\sin x}\,dx$.
8. [1] $\int \frac{1 + \sin x}{\sin x(1 + \cos x)}\,dx$.
9. $\int \frac{dx}{5 - 3\sin 2x}$.
10. $\int \frac{dx}{4 - 5\cos 2x}$.

111. Partial Fractions. A rational fraction is the quotient of two polynomials, *e.g.*,

$$\frac{a_0x^m + a_1x^{m-1} + \cdots + a_{m-1}x + a_m}{b_0x^n + b_1x^{n-1} + \cdots + b_{n-1}x + b_n} = \frac{\phi(x)}{f(x)}. \qquad (1)$$

[1] The integrand is not in the form given in the heading of this article, but the substitution $z = \tan\frac{x}{2}$ enables us to transform any expression containing only integral powers of $\sin x$ and $\cos x$ into a rational function of z, *i.e.*, into a function containing only integral powers of z.

If the degree of the numerator, m, is greater than or equal to the degree of the denominator, n, the fraction can be transformed by division into the sum of a polynomial and a fraction whose numerator is of lower degree than the denominator. In this case the division is always to be performed before applying the methods of this section.

The integration of a rational fraction cannot in general be accomplished by the methods which have been given if the degree of the denominator is greater than 2. Illustrations will now be given of a process by which a rational fraction can be expressed as the sum of fractions whose denominators are either of the first or second degrees.

Illustration 1.

$$\int \frac{x^2+2}{x^3-2x^2-9x+18}\,dx.$$

Factoring the denominator

$$x^3-2x^2-9x+18=(x-2)(x-3)(x+3).$$

Assume

$$\frac{x^2+2}{x^3-2x^2-9x+18}=\frac{A}{x-2}+\frac{B}{x-3}+\frac{C}{x+3},$$

where A, B and C are to be so determined that this equation shall be satisfied for all values of x. Clearing of fractions

$$\begin{aligned} x^2+2 &= Ax^2-9A+Bx^2+Bx-6B+Cx^2-5Cx+6C \\ &= (A+B+C)x^2+(B-5C)x-9A-6B+6C. \end{aligned}$$

On equating the coefficients[1] of x^2, x, x^0, we obtain the following three equations for the determination of A, B and C.

$$\begin{aligned} A+B+C &= 1. \\ B-5C &= 0. \\ -9A-6B+6C &= 2. \end{aligned}$$

[1] In applying this process use is made of the fact that if two polynomials in x are identically equal, the coefficients of like powers of x are equal. Thus, given the identity

$$\alpha_0x^n+\alpha_1x^{n-1}+\cdots+\alpha_{n-1}x+\alpha_n=\beta_0x^n+\beta_1x^{n-1}+\cdots+\beta_{n-1}x+\beta_n,$$

then

$$\begin{aligned} \alpha_0 &= \beta_0 \\ \alpha_1 &= \beta_1 \\ \cdots &= \cdots \\ \alpha_n &= \beta_n. \end{aligned}$$

From these equations

$$A = -\tfrac{6}{5}.$$
$$B = \tfrac{11}{6}.$$
$$C = \tfrac{11}{30}.$$

Hence

$$\frac{x^2+2}{x^3-2x^2-9x+18} = \frac{-6}{5(x-2)} + \frac{11}{6(x-3)} + \frac{11}{30(x+3)}$$

and

$$\int \frac{x^2+2}{x^3-2x^2-9x+18}\,dx = -\tfrac{6}{5}\int\frac{dx}{x-2} + \tfrac{11}{6}\int\frac{dx}{x-3} + \tfrac{11}{30}\int\frac{dx}{x+3}$$

$$= -\tfrac{6}{5}\log(x-2) + \tfrac{11}{6}\log(x-3) + \tfrac{11}{30}\log(x+3) + C.$$

Short Method. The foregoing method of determining the values of A, B, $\cdot\cdot\cdot$, by equating coefficients of like powers of x, is perfectly general. However, a shorter method can sometimes be used. Thus in the illustration just given write the result of clearing of fractions in the form

$$x^2+2 = A(x-3)(x+3) + B(x-2)(x+3) + C(x-2)(x-3).$$

Since this relation is true for all values of x, it is true for $x = 2$. On setting $x = 2$, we obtain

$$6 = -5A.$$

Hence

$$A = -\tfrac{6}{5}$$

On setting $x = 3$, we obtain

$$11 = 6B.$$

Hence

$$B = \tfrac{11}{6}.$$

On setting $x = -3$, we obtain

$$11 = 30C.$$

Hence

$$C = \tfrac{11}{30}.$$

Illustration 2.

$$\int\frac{x^2+1}{(x+1)(x-1)^3}\,dx.$$

Let

$$\frac{x^2+1}{(x+1)(x-1)^3} = \frac{A}{x+1} + \frac{B}{(x-1)^3} + \frac{C}{(x-1)^2} + \frac{D}{x-1}.$$

On clearing of fractions,

$$x^2+1 = A(x-1)^3 + B(x+1) + C(x-1)(x+1) + D(x-1)^2(x+1),$$

or

$$x^2+1 = Ax^3 - 3Ax^2 + 3Ax - A + Bx + B + Cx^2 - C + Dx^3 - Dx^2 - Dx + D.$$

In the first form put $x = 1$. Then

$$B = 1.$$

In the first form put $x = -1$. Then

$$-8A = 2.$$

Hence

$$A = -\tfrac{1}{4}.$$

Equating coefficients of x^3 in the second form

$$A + D = 0.$$

Hence

$$D = -A = \tfrac{1}{4}.$$

Equating coefficients of x^2 in the second form,

$$-3A + C - D = 1.$$

Hence

$$C = 1 - \tfrac{3}{4} + \tfrac{1}{4} = \tfrac{1}{2}.$$

Consequently

$$\frac{x^2+1}{(x+1)(x-1)^3} = \frac{-1}{4(x+1)} + \frac{1}{(x-1)^3} + \frac{1}{2(x-1)^2} + \frac{1}{4(x-1)}$$

and

$$\int \frac{x^2+1}{(x+1)(x-1)^3}dx = -\tfrac{1}{4}\int\frac{dx}{x+1} + \int\frac{dx}{(x-1)^3} + \tfrac{1}{2}\int\frac{dx}{(x-1)^2} + \tfrac{1}{4}\int\frac{dx}{x-1}$$

$$= -\tfrac{1}{4}\log(x+1) - \frac{1}{2(x-1)^2} - \frac{1}{2(x-1)} + \tfrac{1}{4}\log(x-1) + C$$

$$= \log\sqrt[4]{\frac{x-1}{x+1}} - \frac{x}{2(x-1)^2} + C.$$

Illustration 3.

$$\int \frac{3x^2 - 2x + 2}{(x-1)(x^2 - 4x + 13)}\, dx.$$

Let

$$\frac{3x^2 - 2x + 2}{(x-1)(x^2 - 4x + 13)} = \frac{A}{x-1} + \frac{Bx + C}{x^2 - 4x + 13}.$$

Clearing of fractions,

$$3x^2 - 2x + 2 = A(x^2 - 4x + 13) + Bx(x-1) + C(x-1),$$

or

$$3x^2 - 2x + 2 = Ax^2 - 4Ax + 13A + Bx^2 - Bx + Cx - C.$$

In the first form put $x = 1$. We obtain

$$3 = 10A.$$

Hence

$$A = \tfrac{3}{10}.$$

Equating the constant terms in the second form,

$$13A - C = 2.$$

Hence

$$\tfrac{39}{10} - C = 2$$

and

$$C = \tfrac{19}{10}.$$

Equating the coefficients of x^2 in the second form,

$$A + B = 3.$$

Hence

$$B = 3 - \tfrac{3}{10} = \tfrac{27}{10}.$$

Consequently

$$\int \frac{3x^2 - 2x + 2}{(x-1)(x^2 - 4x + 13)}\, dx = \tfrac{3}{10}\int \frac{dx}{x-1} + \tfrac{1}{10}\int \frac{27x + 19}{x^2 - 4x + 13}\, dx$$

$$= \tfrac{3}{10}\log(x-1) + \tfrac{27}{20}\int \frac{(2x-4)\, dx}{x^2 - 4x + 13} + \tfrac{73}{10}\int \frac{dx}{(x-2)^2 + 9}$$

$$= \tfrac{3}{10}\log(x-1) + \tfrac{27}{20}\log(x^2 - 4x + 13) + \tfrac{73}{30}\tan^{-1}\frac{x-2}{3} + \text{C}.$$

Illustration 4.

$$\int \frac{2x\, dx}{(1+x)(1+x^2)^2}.$$

Let $$\frac{2x}{(1+x)(1+x^2)^2} = \frac{A}{1+x} + \frac{Bx+C}{(1+x^2)^2} + \frac{Dx+E}{(1+x^2)}.$$

In Illustrations 1 to 4 a fraction was broken up into "partial fractions." The denominators were the factors of the denominator of the given fraction. In Illustrations 1 and 2 the factors were all real linear factors, while in Illustrations 3 and 4 there were also factors of the second degree which could not be factored into two real linear factors. The method of procedure will be further indicated by the following examples. They will be grouped under the numbers I, II, III, and IV, corresponding to Illustrations 1, 2, 3, and 4.

I. Factors of denominator linear, none repeated.

(a) $$\frac{x^2+5}{(x-1)(x+1)(x-3)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x-3}.$$

(b) $$\frac{x^2+2x+7}{(x+4)(2x+3)(x-2)(3x+1)} = \frac{A}{x+4} + \frac{B}{2x+3} + \frac{C}{x-2} + \frac{D}{3x+1}.$$

II. Factors of denominator linear, some repeated.

(a) $$\frac{x^2+2x+5}{(x-2)^2(x-3)^3(x+1)} = \frac{A}{(x-2)^2} + \frac{B}{x-2} + \frac{C}{(x-3)^3} + \frac{D}{(x-3)^2} + \frac{E}{x-3} + \frac{F}{x+1}.$$

(b) $$\frac{x^3+4x-2}{(2x+1)^3(x+3)(x-4)^2} = \frac{A}{(2x+1)^3} + \frac{B}{(2x+1)^2} + \frac{C}{2x+1} + \frac{D}{x+3} + \frac{E}{(x-4)^2} + \frac{F}{x-4}.$$

III. Denominator contains factors of second degree, none repeated.

(a) $$\frac{x^2+7x+3}{(x^2+4)(x-2)} = \frac{Ax+B}{x^2+4} + \frac{C}{x-2}.$$

(b) $$\frac{x^3-3x+5}{(x^2+2)(x^2-4x+7)(x+3)} = \frac{Ax+B}{x^2+2} + \frac{Cx+D}{x^2-4x+7} + \frac{E}{x+3}.$$

(c) $$\frac{x^2+2x-5}{(x^2+7)(x-2)^2} = \frac{Ax+B}{x^2+7} + \frac{C}{(x-2)^2} + \frac{D}{x-2}.$$

IV. Denominator contains factors of second degree, some repeated.

(a) $$\frac{x^3 + 2x^2 + 5}{(x^2 + 2x + 10)^2(x^2 + 3)(x + 2)} = \frac{Ax + B}{(x^2 + 2x + 10)^2} + \frac{Cx + D}{x^2 + 2x + 10} + \frac{Ex + F}{x^2 + 3} + \frac{G}{x + 2}.$$

Exercises

1. $\displaystyle\int \frac{x^2 + x - 1}{x^3 + x - 10}\,dx.$

2. $\displaystyle\int \frac{2x^2}{(x + 1)^3}\,dx.$

3. $\displaystyle\int \frac{(x - 4)\,dx}{x^3 - 6x^2 + 9x}.$

4. $\displaystyle\int \frac{x\,dx}{(x + 1)(x^2 + 4)}.$

5. $\displaystyle\int \frac{(3 + 4x - x^2)\,dx}{(x - 1)(x^2 - 2x + 5)}.$

6. $\displaystyle\int \frac{5x^2 + 13x - 7}{(x + 4)(2x + 1)^2}\,dx.$

7. $\displaystyle\int \frac{x^3 + x^2 + 7x + 1}{(x^2 + 1)^2}\,dx.$

8. $\displaystyle\int \frac{x^4 + x^3 + 3}{x^3 - 9x}\,dx.$ (Divide numerator by denominator.)

CHAPTER XIII

APPLICATIONS OF THE PROCESS OF INTEGRATION. IMPROPER INTEGRALS

112. In this section a brief summary and review of the applications of the process of integration will be given.

1. *Area under a Plane Curve: Rectangular Coördinates.*

$$A = \int_a^b f(x)dx.$$

See §**64**, and Fig. 46.

2. *Area: Polar Coördinates.*

$$A = \tfrac{1}{2}\int_\alpha^\beta \rho^2\, d\theta.$$

See §**99**, and Fig. 74.

3. *Length of Arc of a Plane Curve: Rectangular Coördinates.*

$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_{y_1}^{y_2} \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy.$$

See §**69**, and Fig. 49.

4. *Length of Arc: Polar Coördinates.*

$$s = \int_\alpha^\beta \sqrt{\rho^2 + \left(\frac{d\rho}{d\theta}\right)^2}\, d\theta$$

$$= \int_{\rho_1}^{\rho_2} \sqrt{1 + \rho^2\left(\frac{d\theta}{d\rho}\right)^2}\, d\rho.$$

See §**98**, and Fig. 72.

5. *Volume of a Solid of Revolution.*

$$V = \int_a^b \pi y^2\, dx.$$

See §**68**, and Fig. 49.

6. *Surface of a Solid of Revolution.*

$$S = 2\pi \int_a^b y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$
$$= 2\pi \int_{x=a}^{x=b} y\, ds$$
$$= 2\pi \int_{y_1}^{y_2} y\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$
$$= 2\pi \int_{y=y_1}^{y=y_2} y\, ds.$$

See **§70,** and Fig. 49.

7. *Water Pressure on a Vertical Surface.*

$$P = k\int_a^b uz\, du,$$

where z denotes the width of the surface at depth u and $k = 62.5$ pounds per cubic foot if u and z are expressed in feet. See **§72,** and Fig. 50.

8. *Work Done by a Variable Force.* See **§67.**

Exercises

1. Find the area in the first quadrant between the circle $x^2 + y^2 = a^2$ and the coördinate axes.

The definite integral which occurs in the solution of this problem is of very frequent occurrence. See Illustration 3, **§105.**

2. Find the area bounded by the lemniscate, $\rho^2 = a^2 \cos 2\theta$.

3. Find the length of one quadrant of the circle $x^2 + y^2 = a^2$, or $x = a \cos \theta$, $y = a \sin \theta$.

4. Find the length of $\rho = 10 \cos \theta$.

5. Find the volume of a sphere of radius a.

6. A solid is generated by a variable square moving with its center on, and with its plane perpendicular to, a straight line. The side of this square varies as the distance, x, of its center from a fixed point on the line, and is equal to 2 when $x = 3$. Find the volume generated by the square when its center moves from $x = 2$ to $x = 7$.

7. Find the area of the surface of a sphere of radius a.

8. The unstretched length of a spring is 25 inches. Find the work done in stretching it from a length of 27 inches to a length of

29 inches, if a force of 400 pounds is necessary to stretch it to a length of 26 inches.

9. A trough 3 feet deep and 2 feet wide at the top has a parabolic cross section. Find the pressure on one end when the trough is filled with water.

10. Find the length of the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$, or $x = a\cos^3\theta$, $y = a\sin^3\theta$.

11. Show that the work done by the pressure of a gas in expanding from a volume v_1 to a volume v_2 is given by

$$\int_{v_1}^{v_2} p\,dv.$$

where p is the pressure per unit area.

HINT. Take a cylinder closed by a piston of area A forced out a distance Δx by the expanding gas. Denote by Δw the work done by the gas in expanding from a volume v to a volume $v + \Delta v$, Then,

$$W = \lim_{\Delta v \doteq 0} \sum \Delta w = \lim_{\Delta x \doteq 0} \sum pA\,\Delta x = \lim_{\Delta v \doteq 0} \sum p\,\Delta v$$
$$= \int_{v_1}^{v_2} p\,dv.$$

12. Find the area of one quadrant of the ellipse $x = a\cos\theta$, $y = b\sin\theta$.

13. Find the area of one loop of the curve $\rho = a\cos 2\theta$.

14. Find the length of the cardioid, $\rho = a(1 - \cos\theta)$.

15. Find the volume of the ellipsoid of revolution generated by revolving the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ about the X-axis; about the Y-axis.

16. A volume is generated by a variable equilateral triangle moving with its plane perpendicular to the X-axis. Find the volume of the solid between the planes $x = 0$ and $x = 2$, if a side of the triangle is equal to $2x^2$.

17. Find the area of the surface generated by revolving about the X-axis the portion of the arc of the catenary

$$y = \frac{a}{2}\left[e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right]$$

between $(0, a)$ and (x_1, y_1).

18. Find the area under one arch of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$.

19. Find the length of that portion of $9y^2 = x^3$ above the X-axis between $x = 0$ and $x = 3$.

20. Find the volume generated by revolving the portion of the catenary

$$y = \frac{a}{2}\left[e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right]$$

between $x = 0$ and $x = b$ about the X-axis; about the Y-axis.

21. Find the volume generated by revolving the hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$, or $x = a\cos^3\theta$, $y = a\sin^3\theta$, about the X-axis.

22. Find the area included between the parabolas $4y^2 = 25x$ and $5x^2 = 16y$.

23. Find the area between the X-axis, the curve $y = x^2 - 4x + 9$, and the ordinates $x = 1$ and $x = 7$.

24. Find the area between the curve $y = \sin x$, the X-axis, and $x = 0$ and $x = \pi$.

25. If a gas is expanding in accordance with Boyle's law, $pv = C$, find the work done in expanding from a volume v_1 to a volume v_2. Represent the work graphically by an area.

26. Find the work done if the gas is expanding in accordance with the adiabatic law, $pv^k = C$.

HINT. From the result of Exercise 11,

$$W = C\int_{v_1}^{v_2}\frac{dv}{v^k} = \frac{C}{1-k}\,v^{1-k}\Big|_{v_1}^{v_2} = \frac{C}{1-k}\,(v_2^{1-k} - v_1^{1-k}).$$

Now,

$$C = p_1v_1^k = p_2v_2^k.$$

Hence

$$W = \frac{1}{1-k}\,(p_2v_2 - p_1v_1).$$

Represent the work graphically by an area. Use the same scale as in Exercise 25.

27. Find the area of one quadrant of the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$. See Exercise 1.

28. Find the length of $\rho = e^{a\theta}$ from $\theta = 0$ to $\theta = 2\pi$.

29. Find the length of $\rho = e^{-a\theta}$ from $\theta = 0$ to $\theta = -\infty$, if a is assumed positive.

30. Find the area bounded by the cardioid $\rho = a(1 + \cos\theta)$.

31. Find the area bounded by $\rho = 10\sin\theta$.

32. Find the area bounded by the hypocycloid $x = a\cos^3\theta$, $y = a\sin^3\theta$.

33. Find the area between $y^2 = 4x$ and $y^2 = 8x - x^2$.

34. Find the work done by a gas in expanding isothermally from an initial volume of 2 cubic feet and pressure of 7000 pounds per square foot to a volume of 4 cubic feet.

35. Find the work done if the gas expands adiabatically. Take $k = \frac{17}{16}$, the value for steam. (See Exercise 26.)

36. Find the pressure on a trapezoidal gate closing a channel containing water, the upper and lower bases of the wet surface being 25 feet and 18 feet, respectively, and the distance between them being 10 feet.

37. Find the area between the catenary

$$y = \frac{a}{2}\left[e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right],$$

the X-axis, and the ordinates $x = 0$ and $x = a$.

38. Find the length of $\rho = a\theta$ from $\theta = 0$ to $\theta = 2\pi$.

39. Set up the integral representing the length of one quadrant of the ellipse $x = a \cos \theta$, $y = b \sin \theta$.

40. Find the volume generated by a circle of variable radius moving with its plane perpendicular to the X-axis, between the planes $x = 2$ and $x = 8$. The radius is proportional to x^3 and is equal to 54 when $x = 3$.

41. Find the volume generated by revolving one arch of the cycloid $x = a(\theta - \sin \theta)$, $y = a(1 - \cos \theta)$ about the X-axis; about the tangent at the vertex.

42. Find the area of the surface generated by revolving a quadrant of a circle about a tangent at one extremity.

43. If the density of a right circular cylinder varies as the distance from one base, find the mass of the cylinder if the altitude is h and the radius of the base is r.

44. The force required to stretch a bar by an amount s is given by

$$F = \frac{Eas}{L},$$

where E is the modulus of elasticity of the material of the bar, a is the area of the cross section, and L is the original length. Find the work that is done in stretching a bar whose unstretched length is 400 inches to a length of 401 inches, if $E = 30{,}000{,}000$ pounds per square inch and $a = 1.5$ square inches.

45. Find the area of one loop of $\rho = 10 \sin 3\theta$.

46. Find the length of

$$y = \frac{a}{2}\left[e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right],$$

from $(0, a)$ to (x_1, y_1).

47. Find the length of one arch of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$.

48. Find the volume of the anchor ring generated by revolving the circle $x^2 + (y - b)^2 = a^2$ about the X-axis, a being less than b.

49. Find the area of the small loop of $\rho = a \sin^3 \frac{\theta}{3}$.

50. Find the work done in pumping the water out of a cistern 20 feet deep, in which the water stands 8 feet deep, if the cistern is a paraboloid of revolution and the diameter at the surface of the earth is 8 feet.

51. Find the volume included between two equal right circular cylinders, radius a, whose axes intersect at right angles.

52. Find the area of the surface generated by revolving one arch of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$, about the X-axis; about a tangent at the vertex.

53. Find the area bounded by $\rho = 3 + 2\cos\theta$.

54. Find the area bounded by the small loop of $\rho = 2 + 3\cos\theta$.

55. Find the area of the surface generated by revolving the cardioid $\rho = a(1 + \cos\theta)$ about the polar axis.

56. Find the volume bounded by the surface of Exercise 55.

113. Improper Integrals. Since $\frac{1}{\sqrt{x-1}}$ becomes infinite at $x = 1$, the definite integral

$$\int_1^7 \frac{1}{\sqrt{x-1}}\,dx$$

must not be evaluated by the usual process. For, the assumption has been made that in the integral

$$\int_a^b f(x)\,dx$$

$f(x)$ is a continuous finite function at $x = a$ and $x = b$ as well as at all intermediate points, and the evaluation of this integral was based on the area under the curve $y = f(x)$. In this case

$$f(x) = \frac{1}{\sqrt{x-1}}$$

becomes infinite at the lower limit. The area under the curve

$$y = \frac{1}{\sqrt{x-1}}$$

between the ordinates $x = 1$ and $x = 7$ has no meaning. In fact the integral in question has no meaning in accordance with the definition of a definite integral already given. A new definition is necessary. We define

$$\int_1^7 \frac{1}{\sqrt{x-1}}\,dx$$

as

$$\lim_{\eta \doteq 0} \int_{1+\eta}^7 \frac{1}{\sqrt{x-1}}\,dx,$$

where η is a positive number, if this limit exists. Otherwise the integral has no meaning. Now,

$$\lim_{\eta \doteq 0} \int_{1+\eta}^7 \frac{1}{\sqrt{x-1}}\,dx = \lim_{\eta \doteq 0} \left(2\sqrt{x-1}\right)\Big|_{1+\eta}^7$$

$$= \lim_{\eta \doteq 0} \left(2\sqrt{6} - 2\sqrt{\eta}\right) = 2\sqrt{6}.$$

Since the limit exists we say that

$$\int_1^7 \frac{1}{\sqrt{x-1}}\,dx = 2\sqrt{6}.$$

Graphically this means the limit as η approaches zero of the area under the curve $y = \dfrac{1}{\sqrt{x-1}}$, between the ordinates $x = 1 + \eta$ and $x = 7$, exists and is equal to $2\sqrt{6}$.

Exercise 1. Show that

$$\int_1^7 \frac{dx}{(x-1)^n}$$

exists if $0 < n < 1$.

On the other hand, when $n = 1$,

$$\lim_{\eta \doteq 0} \int_{1+\eta}^7 \frac{1}{x-1}\,dx = \lim_{\eta \doteq 0} \log(x-1)\Big|_{1+\eta}^7$$

$$= \lim_{\eta \doteq 0} (\log 6 - \log \eta) = \lim_{\eta \doteq 0} \log \frac{6}{\eta}.$$

This limit does not exist and consequently we say that

$$\int_1^7 \frac{1}{x-1}\,dx$$

has no meaning or does not exist.

Graphically this means that the area under the curve

$$y = \frac{1}{x-1}$$

between the ordinates $x = 1 + \eta$ and $x = 7$ increases without limit as η approaches zero.

Exercise 2. Show that

$$\int_1^7 \frac{dx}{(x-1)^n}$$

does not exist if $n \geqq 1$. (Note that the case $n = 1$ has just been considered.) If $n < 0$ no question as to the meaning of the integral can arise. Why?

A definite integral in which the function to be integrated becomes infinite at the upper limit is treated in the same way. Thus

$$\int_0^1 \frac{dx}{\sqrt{1-x}}$$

is defined as

$$\lim_{\eta \doteq 0} \int_0^{1-\eta} \frac{dx}{\sqrt{1-x}}$$

where η is a positive number, if this limit exists.

Exercise 3. Show that

$$\int_0^1 \frac{dx}{(1-x)^n}$$

has a meaning *in accordance with this definition* if $0 < n < 1$, and that it has no meaning if $n \geqq 1$. If $n < 0$ no question can arise as to the meaning of the integral.

It is easy to see how to proceed in case the function under the integral sign becomes infinite at a point within the interval of integration. Thus

$$\int_0^7 \frac{dx}{(x-1)^n}, \text{ where } n \text{ is a positive integer,}$$

is defined as

$$\lim_{\eta \doteq 0}\left[\int_0^{1-\eta} \frac{dx}{(x-1)^n} + \int_{1+\eta}^7 \frac{dx}{(x-1)^n}\right]$$

where η is a positive number, if this limit exists. If not, the integral has no meaning. If $n < 0$ no limit process is necessary.

Exercises

Evaluate the following integrals if they have a meaning:

1. $\int_0^1 \frac{dx}{\sqrt{x}}$.
2. $\int_0^1 \frac{dx}{x^2}$.
3. $\int_0^1 \frac{dx}{x}$.
4. $\int_{-1}^{+1} \frac{dx}{\sqrt{x+1}}$.
5. $\int_{-1}^{+1} \frac{dx}{x^2}$.
6. $\int_0^a \frac{dx}{\sqrt{a-x}}$.
7. $\int_0^a \frac{dx}{\sqrt{a^2-x^2}}$.
8. $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$.
9. $\int_{\frac{1}{3}}^5 \frac{dx}{3x-4}$.
10. $\int_{-1}^{+1} \frac{dx}{x^{\frac{2}{3}}}$.
11. $\int_0^5 \frac{dx}{(x-2)^{\frac{2}{3}}}$.

12. Find the area between the curve $y^2 = \frac{x^3}{2a - x}$, its asymptote and the X-axis.

114. Improper Integrals: Infinite Limits. In §113, the interval of integration was finite. In other words neither of the limits of the integral

$$\int_a^b f(x)dx$$

was infinite.

The integral

$$\int_0^\infty \frac{dx}{x^2 + a^2}$$

will be defined as

$$\lim_{b \doteq \infty} \int_0^b \frac{dx}{x^2 + a^2},$$

if this limit exists. Now

$$\lim_{b \doteq \infty} \int_0^b \frac{dx}{x^2 + a^2} = \lim_{b \doteq \infty} \frac{1}{a} \tan^{-1} \frac{x}{a} \Big|_0^b = \lim_{b \doteq \infty} \frac{1}{a} \tan^{-1} \frac{b}{a} = \frac{1}{a} \frac{\pi}{2}.$$

$\lim_{b \doteq \infty} \int_0^b \frac{dx}{x^2 + a^2}$ represents graphically the limit of the area under the curve $y = \frac{1}{x^2 + a^2}$ between the ordinates $x = 0$ and $x = b$ as b increases indefinitely.

Consider

$$\int_1^\infty \frac{dx}{x}.$$

$$\lim_{b \doteq \infty} \int_1^b \frac{dx}{x} = \lim_{b \doteq \infty} \log x \Big|_1^b = \lim_{b \doteq \infty} \log b.$$

But $\log b$ increases without limit as b increases witnout limit. Hence $\int_1^\infty \frac{dx}{x}$ has no meaning.

Exercises

Evaluate the following integrals if they have a meaning:

1. $\int_0^\infty \frac{dx}{(1 + x)^2}.$

2. $\int_0^\infty e^{-x}\, dx.$

3. $\int_0^\infty x\, e^{-x}\, dx$

4. $\int_0^\infty x^2\, e^{-x}\, dx.$

5. Find the area between the witch, $y = \frac{8a^3}{x^2 + 4a^2}$, and the axis of x.

CHAPTER XIV

SOLID GEOMETRY

115. Coördinate Axes. Coördinate Planes. Just as the position of a point in a plane is given by two coördinates, for example by its perpendicular distances from two mutually perpendicular coördinate axes, the position of a point in space is given by three coördinates, for example by its perpendicular distances from three mutually perpendicular planes of reference, called the coördinate planes. Let the three coördinate planes be those represented in Fig. 75, viz., XOY, called the XY-plane, YOZ, called the YZ-plane, and ZOX, called the ZX-plane. Then the position of the point P whose perpendicular distances from the YZ-, ZX-, and XY-planes

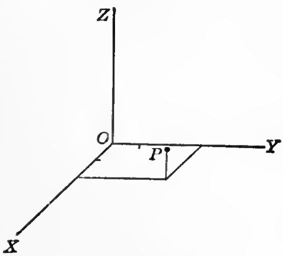

FIG. 75.

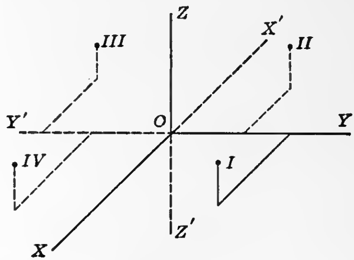

FIG. 76.

are 2, 3, and 1, respectively, is represented by the coördinates 2, 3, and 1. The lines of intersection of the planes of reference are called the axes. Thus $X'OX$, $Y'OY$, and $Z'OZ$, Fig. 76, are called the axes of x, y, and z, respectively. The coördinates of a point P measured parallel to these axes are known as its x, y, and z coördinates, respectively. Thus for the particular point P of Fig. 75, $x = 2$, $y = 3$, and $z = 1$. More briefly we say that the point P is the point (2, 3, 1). In general, (x, y, z) is a point whose coördinates are x, y, and z. If these coördinates are given the position of the point is determined, and if a point is given these coördinates are determined.

The relation between a function of a single independent variable and its argument can be represented in a plane by a curve, the ordinates of which represent the values of the function corresponding to the respective values of the abscissas. Thus, $y = f(x)$ is represented by a curve. To an abscissa representing a given value of the argument there correspond one or more points on the curve whose ordinates represent the values of the function. In like manner a function of two independent variables x and y can be represented in space. Choose the system of coördinate planes of Fig. 75. Assign values to each of the independent variables x and y. These values fix a point in the XY-plane. At this point erect a perpendicular to the XY-plane, whose length z represents the value of the function corresponding to the given values of the arguments. Thus a point P is determined. And for all values of x and y in a given region of the XY-plane there will, in general, correspond points in space. The locus of these points is a surface. The surface represents the relation between the function and its two independent arguments just as a curve represents the relation between a function and its single argument.

Thus if $z = \pm\sqrt{25 - x^2 - y^2} = f(x,y)$, $\pm\sqrt{12}$ are the values of the function corresponding to the values $x = 2$ and $y = 3$. Then the points $(2, 3, 2\sqrt{3})$ and $(2, 3, -2\sqrt{3})$ lie on the surface $z = \pm\sqrt{25 - x^2 - y^2}$. If $x = -3$ and $y = 1$, $z = \pm\sqrt{15}$. The corresponding points on the surface are $(-3, 1, \sqrt{15})$ and $(-3, 1, -\sqrt{15})$.

The coördinate planes divide space into eight octants. Those above the XY-plane are numbered as shown in Fig. 76. The octant immediately below the first is the fifth, that below the second is the sixth, and so on. The points $(2, 3, 2\sqrt{3})$ and $(2, 3, -2\sqrt{3})$ lie in the first and fifth octants, respectively. The points $(-3, 1, \sqrt{15})$ and $(-3, 1, -\sqrt{15})$ lie in the second and sixth octants, respectively.

The locus of points satisfying the equation

$$z = \pm\sqrt{25 - x^2 - y^2} \tag{1}$$

is a sphere of radius 5. For, this equation can be written in the form $x^2 + y^2 + z^2 = 25$, which states that for any point P on the surface (1), $OP = \sqrt{x^2 + y^2 + z^2} = 5$. The left member is the square of the distance, OP, of the point P (x, y, z), from O, since OP is the diagonal of a rectangular parallelopiped whose edges are x, y, and z. If then the coördinates of P satisfy (1), this point is at a distance 5 from the origin. It lies on the sphere, of radius 5, whose center is at the origin.

116. The Distance between Two Points. The student will show that the distance d between the two points (x_1, y_1, z_1) and (x_2, y_2, z_2) is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}. \tag{1}$$

See Fig. 77. If the point (x_1, y_1, z_1) is the origin, (0, 0, 0), the expression for d becomes

$$\rho = \sqrt{x_2^2 + y_2^2 + z_2^2}. \tag{2}$$

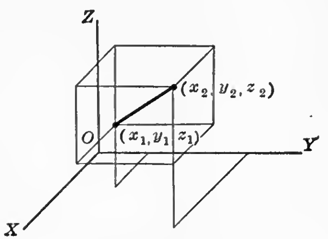

Fig. 77.

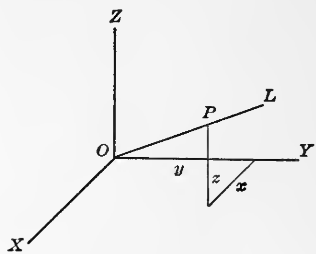

Fig. 78.

Exercises

Find the distance between the following points:

1. (1, 2, 3) and (3, 5, 7).
2. (1, −2, 5) and (3, −2, −1).
3. (0, −3, 2) and (0, 0, 0).
4. (0, 0, 3) and (0, 2, 6).
5. (0, 0, −5 and (2, 0, 6).
6. (−3, 2, −1) and (0, 0, 0).

117. Direction Cosines of a Line. Let OL, Fig. 78, be any line passing through the origin. Let α, β, and γ be, respectively, the angles, less than 180°, between this line and the positive directions of the X-, Y-, and Z-axes. These angles are called the direction angles of the line, and their cosines are called the *direction cosines of the line.* Let P, whose coördinates are x, y, and z, be any point on the line. Let $OP = \rho$. Then

$$x = \rho \cos \alpha,$$
$$y = \rho \cos \beta,$$

and

$$z = \rho \cos \gamma.$$

Squaring and adding the above equations we obtain

$$x^2 + y^2 + z^2 = \rho^2(\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma).$$

Since

$$x^2 + y^2 + z^2 = \rho^2,$$
$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1. \quad (1)$$

The direction cosines of any line are defined as the direction cosines of a parallel line passing through the origin. Then, *the sum of the squares of the direction cosines of any line is equal to unity.*

Exercises

Find the direction cosines of the lines passing through each of the following pairs of points.

1. (0, 0, 0) and (1, 1, 1).
2. (0, 0, 0) and (2, −3, 4).
3. (0, 0, 0) and (−1, 2, −3).
4. (1, 2, 3) and (5, 6, 7).
5. (−2, 3, −1) and (−3, −4, 3).

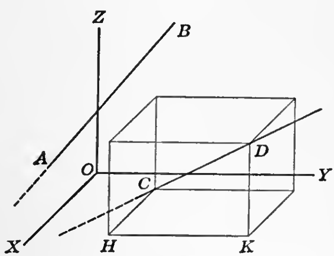

Fig. 79.

118. Angle between Two Lines. Let AB and CD, Fig. 79, be two lines, and let their direction cosines be $\cos \alpha_1$, $\cos \beta_1$, $\cos \gamma_1$, and $\cos \alpha_2$, $\cos \beta_2$, $\cos \gamma_2$, respectively. Denote the angle between the lines by θ. Let CH, HK, and KD be the edges of the parallelopiped formed by passing planes through C and D parallel to the coördinate planes. The projection of CD on AB is clearly equal to the sum of the projections of CH, HK, and KD on AB.

Hence

$$CD \cos \theta = CH \cos \alpha_1 + HK \cos \beta_1 + KD \cos \gamma_1.$$

Now

$$CH = CD \cos \alpha_2,$$
$$HK = CD \cos \beta_2,$$

and

$$KD = CD \cos \gamma_2.$$

Consequently

$$CD \cos \theta = CD(\cos \alpha_1 \cos \alpha_2 + \cos \beta_1 \cos \beta_2 + \cos \gamma_1 \cos \gamma_2).$$

Hence

$$\cos \theta = \cos \alpha_1 \cos \alpha_2 + \cos \beta_1 \cos \beta_2 + \cos \gamma_1 \cos \gamma_2. \quad (1)$$

Exercises

Find the cosine of the angle between the lines determined by the points of Exercises 1 and 2; 2 and 3; 3 and 4, of the preceding section.

119. The Normal Form of the Equation of a Plane.—Let ABC, Fig. 80, be a plane. Let ON, the normal from O, meet it in N. Let the length of ON be p and let its direction angles be α, β, and γ. If p, α, β, and γ are given the plane is determined.

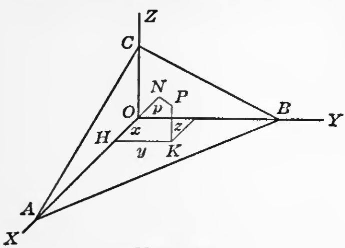

FIG. 80.

We seek to find the equation of the plane. Let P, with coördinates x, y, and z, be any point in the plane. The sum of the projections of $OH = x$, $HK = y$, $KP = z$, and PN upon ON is $ON = p$.

The projection of OH on ON is $x \cos \alpha$.
The projection of HK on ON is $y \cos \beta$.
The projection of KP on ON is $z \cos \gamma$.
The projection of PN on ON is 0.

Hence

$$\mathbf{x} \cos \alpha + \mathbf{y} \cos \beta + z \cos \gamma = \mathbf{p}. \quad (1)$$

If P does not lie in the plane ABC, the projection of PN on ON is not zero, and the coördinates of P do not satisfy (1). Hence the locus of a point satisfying (1) is a plane. Equation (1) is the normal form of the equation of the plane. p is taken to be positive. The algebraic signs of $\cos\alpha$, $\cos\beta$, and $\cos\gamma$ are determined by the octant into which ON extends.

Illustration 1. Find the equation of a plane for which $p = 2$, $\alpha = 60°, \beta = 45°$.

$$\cos\alpha = \tfrac{1}{2},$$
$$\cos\beta = \frac{1}{\sqrt{2}}.$$

Then by (1), §117,

$$\cos^2\gamma = 1 - \tfrac{1}{4} - \tfrac{1}{2}.$$

Hence

$$\cos\gamma = \pm\tfrac{1}{2}.$$

The equation of the plane is

$$\frac{x}{2} + \frac{y}{\sqrt{2}} \pm \frac{z}{2} = 2.$$

There are thus two planes satisfying the conditions of the problem, one forming with the coördinate planes a tetrahedron in the first octant, the other a tetrahedron in the fifth octant.

Exercises

1. Find the equation of a plane if $\alpha = 60°$, $\beta = 135°$, $p = 2$, and if the normal ON extends into the eighth octant.

2. If $\alpha = 120°$, $\beta = 60°$, $p = 5$ and if the normal ON extends into the sixth octant.

120. The Equation Ax + By + Cz = D. The general equation of the first degree in x, y, and z is

$$Ax + By + Cz = D, \tag{1}$$

where A, B, C, and D are real constants. D may be considered positive. For, if the constant term in the second member of an equation of the form (1) is not positive it can be made so by dividing through by -1.

Divide (1) by $\sqrt{A^2 + B^2 + C^2}$ and obtain

$$\frac{A}{\sqrt{A^2+B^2+C^2}}\,x + \frac{B}{\sqrt{A^2+B^2+C^2}}\,y + \frac{C}{\sqrt{A^2+B^2+C^2}}\,z = \frac{D}{\sqrt{A^2+B^2+C^2}}. \tag{2}$$

The coefficient of x is either equal to or less than unity in numerical value. It can then be considered as the cosine of some angle, say α. Similarly the coefficient of y may be considered as the cosine of some angle β, and that of z as the cosine of some angle γ. Further the sum of the squares of these coefficients is equal to 1. Hence α, β, and γ are the direction angles of some line. Then (2) is in the form

$$x \cos \alpha + y \cos \beta + z \cos \gamma = p, \tag{3}$$

where

$$p = \frac{D}{\sqrt{A^2 + B^2 + C^2}},$$

and $\cos \alpha$, $\cos \beta$, and $\cos \gamma$ are the coefficients of $x, y,$ and z, respectively, in equation (2). Hence (3) is the normal form of the equation of a plane. Equation (1) is the general equation of the first degree in the variables x, y, and z. Therefore every equation of the first degree in x, y, and z represents a plane.

Illustration 1. Put $3x - 2y - z = 6$ in the normal form. Divide by $\sqrt{A^2 + B^2 + C^2} = \sqrt{9 + 4 + 1} = \sqrt{14}$ and obtain

$$\frac{3x}{\sqrt{14}} - \frac{2y}{\sqrt{14}} - \frac{z}{\sqrt{14}} = \frac{6}{\sqrt{14}}.$$

The plane is $\frac{6}{\sqrt{14}}$ units distant from the origin, and forms, with the coördinate planes, a tetrahedron in the eighth octant.

Exercises

Transform each of the following equations to the normal form, find the distance of each plane from the origin, and state in which octant it forms a tetrahedron with the coördinate planes.

1. $3x - 2y - z = 1.$

2. $x + y + z = -1.$

3. $x - 3y + 2z = 3.$

4. $x - 2y + 3z + 2 = 0.$

5. $2x - y - z - 1 = 0.$

6. $x + 2y = 6.$

7. $x - z = 4.$

8. $x = 2.$

9. $x = -1.$

10. $z = y.$

121. Intercept Form of the Equation of a Plane. We seek the equation of a plane whose intercepts on the X-, Y-, and Z-axes are a, b, and c, respectively.

The general equation of a plane is

$$Ax + By + Cz = D. \tag{1}$$

The constants are to be so determined that the plane will pass through the points $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$.

On substituting the coördinates $(a, 0, 0)$, in (1), we obtain

$$Aa = D,$$

or

$$A = \frac{D}{a}.$$

Similarly, since (1) passes through $(0, b, 0)$,

$$Bb = D,$$

or

$$B = \frac{D}{b}.$$

And, since it passes through $(0, 0, c)$,

$$Cc = D,$$

or

$$C = \frac{D}{c}.$$

With these values of A, B, and C, (1) becomes

$$\frac{Dx}{a} + \frac{Dy}{b} + \frac{Dz}{c} = D,$$

or

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1. \tag{2}$$

Equation (2) is known as the intercept form of the equation of a plane.

Illustration. Transform the equation $3x - 2y - 5z = 4$ to the intercept form. Divide by 4 and obtain

$$\frac{x}{\frac{4}{3}} + \frac{y}{-2} + \frac{z}{-\frac{4}{5}} = 1.$$

The intercepts on the X-, Y-, and Z-axes are $\frac{4}{3}$, -2, and $-\frac{4}{5}$, respectively.

Exercises

Transform each of the following equations to the intercept form:

1. $x + y + z = 3.$
2. $2x - 3y + 4z = 7.$
3. $2x + y - z + 2 = 0.$
4. $2x + 7y - 3z = 1$
5. $x - y + 3z = -\frac{1}{4}.$
6. $y - 2x - 3z = 5.$

122. The Angle between Two Planes. The angle between two planes is the angle between the normals drawn to them from the origin. The cosine of the angle between the normals can be found by formula (1) **§118,** in which $\alpha_1, \beta_1, \gamma_1$ and $\alpha_2, \beta_2, \gamma_2$ are the direction angles of the normals.

Illustration. Find the angle between the planes

$$x + y + z = 1 \tag{1}$$

and

$$2x + y + 2z = 3. \tag{2}$$

Transform these equations to the normal form and obtain

$$\frac{x}{\sqrt{3}} + \frac{y}{\sqrt{3}} + \frac{z}{\sqrt{3}} = \frac{1}{\sqrt{3}} \tag{3}$$

and

$$\frac{2x}{3} + \frac{y}{3} + \frac{2z}{3} = 1. \tag{4}$$

The direction cosines of the normals to the first and second planes are $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$, and $\frac{2}{3}, \frac{1}{3}, \frac{2}{3}$, respectively. Then, if θ is the angle between the normals, formula (1), **§118,** gives

$$\cos\theta = \frac{2}{3\sqrt{3}} + \frac{1}{3\sqrt{3}} + \frac{2}{3\sqrt{3}} = \frac{5}{3\sqrt{3}}.$$

From which $\theta = 74.5°$.

Exercises

Find the angle between the following pairs of planes:

1. $x - 3y + 2z = 6$ and $x - 2y + z = 1.$
2. $x - 2y + 3z = 2$ and $2x + y - 2z = 3.$

123. Parallel and Perpendicular Planes. If two planes are parallel $\theta = 0$ and $\cos\theta = 1$. If they are perpendicular $\theta = 90°$ and $\cos\theta = 0$.

Let

$$A_1x + B_1y + C_1z = D_1 \tag{5}$$

and

$$A_2x + B_2y + C_2z = D_2 \tag{6}$$

be the equations of two planes. After writing these equations in the normal form it is found that

$$\cos\theta = \frac{A_1A_2 + B_1B_2 + C_1C_2}{\sqrt{A_1^2 + B_1^2 + C_1^2}\,\sqrt{A_2^2 + B_2^2 + C_2^2}}. \qquad (7)$$

If

$$A_1A_2 + B_1B_2 + C_1C_2 = 0, \qquad (8)$$

$\cos\theta = 0$ and the planes (5) and (6) are perpendicular.

If the planes (5) and (6) are parallel, the corresponding coefficients must be equal or proportional. For then and only then will their normals be parallel.

Exercises

From the following equations pick out pairs of parallel planes and pairs of perpendicular planes.

1. $x + y + z = 6.$
2. $x - y - z = 2.$
3. $2x + 2y + 2z = 7.$
4. $3x - 2y - z = 8.$
5. $2x - 3y + z = 1.$

124. The Distance of a Point from a Plane. Let (x_1, y_1, z_1) be any point and let

$$Ax + By + Cz = D$$

be the equation of a plane. We shall find the distance of the point from the plane.

Now

$$Ax + By + Cz = K,$$

where K is any constant, is the equation of a plane parallel to the given plane. (See §**123**.) Let us choose K so that this plane shall pass through the given point (x_1, y_1, z_1). To do this substitute the coördinates of the point in the equation and solve for K. This gives

$$K = Ax_1 + By_1 + Cz_1.$$

Placing the equation of each plane in the normal form we have

$$\frac{Ax + By + Cz}{R} = \frac{D}{R}$$

and

$$\frac{Ax + By + Cz}{R} = \frac{K}{R} = \frac{Ax_1 + By_1 + Cz_1}{R},$$

where $R = \sqrt{A^2 + B^2 + C^2}$.

The given plane is $\frac{D}{R}$ units distant from the origin, and the plane through the point (x_1, y_1, z_1) is $\frac{Ax_1 + By_1 + Cz_1}{R}$ units distant from the origin. Then the distance, d, between the two planes, and hence the distance of the given point from the given plane, is equal to the difference of these two distances, or

$$\mathbf{d} = \frac{\mathbf{Ax_1 + By_1 + Cz_1 - D}}{\sqrt{\mathbf{A^2 + B^2 + C^2}}}.$$

Illustration. Find the distance of the point $(1, 2, -1)$ from the plane $3x - y + z + 7 = 0$.

$$d = \frac{Ax_1 + By_1 + Cz_1 - D}{\sqrt{A^2 + B^2 + C^2}} = \frac{3 \cdot 1 - 1 \cdot 2 + 1 \cdot (-1) + 7}{\sqrt{3^2 + (-1)^2 + 1^2}} = \frac{7}{\sqrt{11}}.$$

Exercises

In each of the following find the distance of the given point from the given plane:

1. $(3,1,-2)$; $3x + y - 2z - 6 = 0$.

2. $(-1, 2, -3)$; $x - y - 2z + 1 = 0$.

3. $(0, 2, -3)$; $2x + 3y - 5z - 10 = 0$.

125. Symmetrical Form of the Equations of a Line. Let PP_1, Fig. 81, be a line passing through the given point P_1 (x_1, y_1, z_1), and having the direction cosines $\cos\alpha$, $\cos\beta$, $\cos\gamma$. In order to find the equations of the line, let P (x, y, z), be any point on the line and denote the distance PP_1 by d. Then

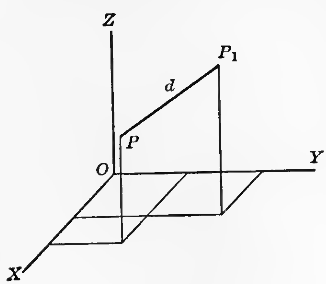

FIG. 81.

$$x - x_1 = d\cos\alpha,$$
$$y - y_1 = d\cos\beta,$$
$$z - z_1 = d\cos\gamma,$$

and therefore

$$\frac{x - x_1}{\cos\alpha} = \frac{y - y_1}{\cos\beta} = \frac{z - z_1}{\cos\gamma}. \qquad (1)$$

These equations are known as the symmetric equations of the straight line.

Frequently a straight line is represented by the equations of two planes of which it is the intersection.

Illustration 1.

$$3x - y + 1 = 0, \tag{2}$$

$$5x - z = 3. \tag{3}$$

From these equations the symmetrical form of the equations can readily be obtained. From (2) and (3) we obtain

$$x = \frac{y-1}{3} = \frac{z+3}{5},$$

or

$$\frac{x-0}{1} = \frac{y-1}{3} = \frac{z+3}{5}. \tag{4}$$

The denominators, 1, 3, and 5, of (4) are not the direction cosines of the line, but they are proportional to them. Upon dividing each by $\sqrt{35}$, the square root of the sum of their squares, they become the direction cosines. Then

$$\frac{x-0}{\frac{1}{\sqrt{35}}} = \frac{y-1}{\frac{3}{\sqrt{35}}} = \frac{z+3}{\frac{5}{\sqrt{35}}}$$

is the symmetrical form of the equations of the line.

The line therefore passes through the point (0, 1, −3) and has the direction cosines given by the denominators in the preceding equations.

Illustration 2. Consider the line which is the intersection of the planes

$$13x + 5y - 4z = 40,$$

$$-13x + 10y - 2z = 23.$$

On eliminating x we obtain

$$5y - 2z = 21,$$

and on eliminating y we obtain

$$13x - 2z = 19.$$

From the last two equations we find

$$z = \frac{5y-21}{2} = \frac{13x-19}{2},$$

or

$$\frac{x - \frac{19}{13}}{\frac{2}{13}} = \frac{y - \frac{21}{5}}{\frac{2}{5}} = \frac{z-0}{1}.$$

These are the equations of a line which passes through the point $(\frac{19}{13}, \frac{21}{5}, 0)$ and whose direction cosines are proportional to $\frac{2}{13}$, $\frac{2}{5}$, and 1. The student will find the direction cosines.

In Illustration 1, equation (2) represents a plane parallel to the Z-axis whose trace in the XY-plane is the line $3x - y + 1 = 0$. Equation (3) represents a plane parallel to the Y-axis whose trace in the ZX-plane is the line $5x - z = 3$.

In Illustration 2 the position of the two planes which intersect in the straight line is not so evident. By eliminating first x and then y, the equations of two planes passing through the same line are obtained, one of which is parallel to the X-axis and the other to the Y-axis.

Exercises

Put the equations of the following lines in the symmetrical form:

1. $x + 2y + 3z = 6,$
$x - y - z = 1.$

2. $x + y - z = 1,$
$x - 3y + 2z = 6.$

3. $x - y + 2z = 0,$
$x + 2y - 3z = 0.$

126. Surfaces of Revolution. Let

$$y^2 = 4z \tag{1}$$

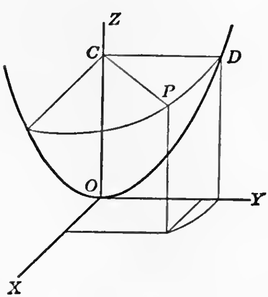

FIG. 82.

be the equation of a curve in the YZ-plane, Fig. 82, and let it be rotated about the Z-axis. The surface generated is a surface of revolution. Any point D on the curve describes a circle of radius CD, equal to the y-coördinate of the point D. During the revolution the z-coördinate does not change. Let P be any position taken by D in the revolution. Let the coördinates of P be (x, y, z).

$$x^2 + y^2 = (CP)^2 = (CD)^2 \tag{2}$$

But by (1), $(CD)^2 = 4z,$

where z is the common z-coördinate of D and P. Then (2) becomes

$$x^2 + y^2 = 4z, \tag{3}$$

an equation satisfied by any point on the surface of revolution. We note that (3) is obtained from (1) by replacing y^2 by $x^2 + y^2$, or y by $\sqrt{x^2 + y^2}$.

In general, if

$$f(y, z) = 0 \tag{4}$$

is the equation of a plane curve in the YZ-plane, the equation of the surface of revolution generated by revolving it about the Z-axis is obtained by writing $\sqrt{x^2 + y^2}$ for y, *i.e.*, the equation of the surface of revolution is

$$f(\sqrt{x^2 + y^2}, z) = 0. \tag{5}$$

This equation can also be regarded as the equation of the surface generated by revolving the curve $f(x, z) = 0$, lying in the XZ-plane, about the Z-axis.

Similarly,

$$f(y, \sqrt{x^2 + z^2}) = 0 \tag{6}$$

is the equation of the surface generated by revolving the plane curve $f(y, x) = 0$ about the Y-axis; and

$$\phi(x, \sqrt{y^2 + z^2}) = 0 \tag{7}$$

is the equation of the surface generated by revolving the plane curve $\phi(x, z) = 0$ about the X-axis.

Illustration 1. The equation of the surface generated by rotating $x^2 + (y - \beta)^2 = a^2$ about the X-axis is

$$x^2 + [\sqrt{y^2 + z^2} - \beta]^2 = a^2.$$

Exercises

Find the equation of the surface generated by rotating:

1. $y = x^2$ about the Y-axis.

2. $y = x^2 - a^2$ about the X-axis.

3. $b^2x^2 + a^2y^2 = a^2b^2$ about the X-axis.

4. $b^2x^2 - a^2y^2 = a^2b^2$ about the X-axis.

5. $b^2x^2 - a^2y^2 = a^2b^2$ about the Y-axis.

6. $x^2 + y^2 = a^2$ about the Y-axis.

7. $x^2 + y^2 = a^2$ about the X-axis.

8. $y = mx$ about the X-axis.

9. $y = mx$ about the Y-axis.

127. Quadric Surfaces. Any equation of the second degree between x, y, and z, of which

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Kz + L = 0 \tag{1}$$

is the general form, represents a surface which is called a *quadric surface*, or *conicoid*.

By a suitable rotation and translation of the axes, the equation of any quadric surface can be put in one of the following forms:

$$\pm\frac{x^2}{a^2} \pm \frac{y^2}{b^2} \pm \frac{z^2}{c^2} = 1, \tag{2}$$

$$\frac{x^2}{a^2} \pm \frac{y^2}{b^2} \pm \frac{z^2}{c^2} = 0, \tag{3}$$

$$\frac{x^2}{a^2} \pm \frac{y^2}{b^2} = \pm\, 2cz. \tag{4}$$

The particular form assumed by the equation depends upon the values of the coefficients in (1).

The quadric surface

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \tag{5}$$

is called the *ellipsoid*. To find the shape and properties of this surface, let

$$x = k, \tag{6}$$

where k is any real constant. This equation represents a plane perpendicular to the axis of x. Equations (5) and (6) considered as simultaneous equations represent the curve of intersection of the ellipsoid with the plane. If x is eliminated between (5) and (6) there results

$$\frac{y^2}{\left(\frac{b\sqrt{a^2 - k^2}}{a}\right)^2} + \frac{z^2}{\left(\frac{c\sqrt{a^2 - k^2}}{a}\right)^2} = 1, \tag{7}$$

the equation of the curve of intersection in the plane $x = k$. Equation (7) is the equation of an ellipse. The semi-axes of the ellipse are $\frac{b\sqrt{a^2 - k^2}}{a}$, and $\frac{c\sqrt{a^2 - k^2}}{a}$. These axes grow shorter as k increases in numerical value from 0 to a. When $k = \pm\, a$

the elliptical section reduces to a point. When $|k| > a$, the lengths of the axes of the ellipse become imaginary, *i.e.*, the plane $x = k$, ($|k| > a$), does not meet the surface (5) in real points. Hence the surface is included between the planes $x = \pm a$.

The above discussion shows that the surface represented by the equation (5) is included between the planes $x = \pm a$; is symmetrical with respect to the YZ-plane; and has elliptical sections made by planes perpendicular to the axis of x. These sections grow smaller as the cutting plane is moved away from the YZ-plane and at a distance $\pm a$ reduce to a point.

In a similar manner, by taking $y = k$, and then by taking $z = k$, the student will discuss plane sections of the ellipsoid (5) perpendicular to the Y-axis and to the Z-axis.

a, b, and c, are called the semi-axes of the ellipsoid.

It can be shown that any plane section of the ellipsoid is an ellipse.

The surface represented by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \tag{8}$$

will now be discussed. Let $z = k$. Then

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 + \frac{k^2}{c^2} \tag{9}$$

is the equation of the plane section made by $z = k$. It is an ellipse whose semi-axes are $\frac{a\sqrt{c^2 + k^2}}{c}$ and $\frac{b\sqrt{c^2 + k^2}}{c}$. They increase in length with the numerical value of k. The axes have a minimum length when $k = 0$. The surface represented by equation (8) is symmetrical with respect to the XY-plane, and every section parallel to this plane is an ellipse. The smallest elliptical section is that made by the XY-plane.

If $x = k$, equation (8) becomes

$$\frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 - \frac{k^2}{a^2}, \tag{10}$$

an hyperbola.

If $k < a$, the transverse axis of the hyperbola is parallel to the

Y-axis. If $k > a$, the transverse axis is parallel to the Z-axis. When $k = a$, equation (10) reduces to

$$\frac{y^2}{b^2} - \frac{z^2}{c^2} = 0,$$

or

$$\left(\frac{y}{b} + \frac{z}{c}\right)\left(\frac{y}{b} - \frac{z}{c}\right) = 0,$$

the equation of two straight lines.

The student will discuss the curves of intersection of the surface (8) with planes parallel to the XZ-plane.

The surface is called the hyperboloid of one sheet, or of one nappe.

Exercises

The student will discuss the following surfaces and make sketches of them:

1. $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1,$ the hyperboloid of two sheets.

2. $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 2cz,$ the hyperbolic paraboloid.

3. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 2cz,$ the elliptic paraboloid.

128. Cylindrical Surfaces. If the circle

$$x^2 + y^2 = 25 \tag{1}$$

be moved parallel to itself so that all of its points describe lines parallel to the Z-axis, it will generate a right circular cylinder. The equation of this cylinder is sought. In any plane $z = k$, the relation between x and y for points in the curve of intersection of this plane and this cylinder is the same as that for points in the plane $z = 0$, viz., $x^2 + y^2 = 25$.

Now, this equation is satisfied by all points on the surface for all values of z. Hence it is the equation of the surface.

The cylindrical surface just considered can be regarded as generated by a line moving parallel to the Z-axis and passing through points of the circle $x^2 + y^2 = 25$ in the plane $z = 0$.

In general *a cylindrical surface is a surface generated by a line moving parallel to itself.*

It is clear that the equation

$$f(x, y) = 0 \tag{2}$$

represents the cylindrical surface generated by a line moving parallel to the Z-axis and passing through points of the curve $f(x, y) = 0$ in the plane $z = 0$. The equation of a section of (2) made by any plane $z = k$ is $f(x, y) = 0$.

Thus

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{3}$$

represents an elliptical cylinder whose elements are parallel to the Z-axis.

$$x^2 + y^2 = 2ax \tag{4}$$

represents a circular cylinder whose elements are parallel to the Z-axis. The center of the section in the plane $z = 0$ is the point $(a, 0)$.

By the same reasoning

$$y^2 + z^2 = a^2 \tag{5}$$

represents a circular cylinder whose elements are parallel to the X-axis.

$$z^2 = 4x \tag{6}$$

represents a parabolic cylinder whose elements are parallel to the Y-axis.

The plane

$$x - 4y + 3 = 0 \tag{7}$$

can be regarded as a cylindrical surface whose elements are parallel to the Z-axis and which pass through the line

$$y = \frac{x}{4} + \frac{3}{4}$$

in the plane $z = 0$.

In general, an equation in which one of the letters x, y, z is absent, represents a cylindrical surface whose elements are parallel to the axis corresponding to the letter which does not appear in the equation.

Exercises

Describe the surfaces represented by the following equations:

1. $x^2 + y^2 = 16$.
2. $\frac{x^2}{4} + \frac{y^2}{16} = 1$.
3. $x^2 - y^2 = 1$.
4. $z^2 + y^2 = 25$.
5. $z^2 - x^2 = 25$.
6. $x + 3y = 10$.
7. $x^2 - y^2 = 0$.
8. $xy = 1$.
9. $xz = 2$.
10. $(x - 3)(x + z) = 0$.
11. $y^2 = 4x$.
12. $y^2 + z^2 = 2ay$.
13. $x^2 + y^2 = 10x$.

129. Partial Derivatives. Let $z = f(x, y)$ be a function of two independent variables, x and y. When x takes on an increment Δx, while y remains fixed, z takes on an increment which we shall denote by $\Delta_x z$. When y takes on an increment, Δy, while x remains fixed, z takes on an increment which we shall denote by $\Delta_y z$.

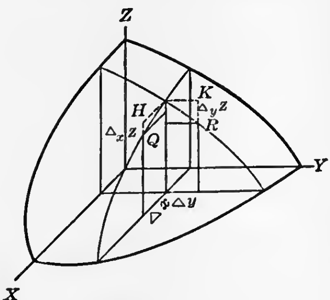

FIG. 83.

For example, if a gas be enclosed in a cylinder with a movable piston, the volume v of the gas is a function of the temperature T and of the pressure p which can be varied by varying the pressure on the piston. If the temperature alone be changed the volume will take on a certain increment $\Delta_T v$. If the pressure alone be changed the volume will take on the increment $\Delta_p v$.

If $z = f(x, y)$ be represented by a surface, Fig. 83, the increment of z obtained by giving x an increment, while y remains constant, is the increment in z measured to the curve cut out by a plane $y = k$, a constant. Thus $\Delta_x z = HQ$; similarly $\Delta_y z = KR$.

The limit of the quotient $\frac{\Delta_x z}{\Delta x}$ as Δx approaches zero is called the partial derivative of z with respect to x. It is denoted by the symbol $\frac{\partial z}{\partial x}$. Then

$$\frac{\partial z}{\partial x} = \lim_{\Delta x \doteq 0} \frac{\Delta_x z}{\Delta x}.$$

It is evidently calculated from $z = f(x, y)$ by the ordinary rules of differentiation, y being treated as a constant. Thus if $z = x^2y$,

$$\frac{\partial z}{\partial x} = 2xy.$$

Geometrically $\frac{\partial z}{\partial x}$ represents the slope of the tangent at the point (x, y, z) to the curve cut from the surface by the plane through this point parallel to the XZ-plane.

Similarly

$$\frac{\partial z}{\partial y} = \lim_{\Delta y \doteq 0} \frac{\Delta_y z}{\Delta y}$$

and it is calculated by differentiating $z = f(x, y)$, treating x as a constant. Geometrically it represents the slope of the tangent at the point (x, y, z) to the curve cut from the surface by the plane through this point, parallel to the YZ-plane. If $z = x^2y$,

$$\frac{\partial z}{\partial y} = x^2.$$

Illustration 1. If $z = \sin \frac{x}{y}$,

$$\frac{\partial z}{\partial x} = \cos \frac{x}{y} \frac{\partial}{\partial x}\left(\frac{x}{y}\right) = \frac{1}{y} \cos \frac{x}{y}$$

and

$$\frac{\partial z}{\partial y} = \cos \frac{x}{y} \frac{\partial}{\partial y}\left(\frac{x}{y}\right) = -\frac{x}{y^2} \cos \frac{x}{y}.$$

130. Partial Derivatives of Higher Order. If z is differentiated twice with respect to x, y being treated as a constant, the derivative obtained is called the second partial derivative of z with respect to x. It is denoted by the symbol $\frac{\partial^2 z}{\partial x^2}$. Similarly the second partial derivative of z with respect to y is denoted by the symbol $\frac{\partial^2 z}{\partial y^2}$.

If z is differentiated first with respect to x, y being treated as a constant, and then with respect to y, x being treated as a constant, the result is denoted by the symbol $\frac{\partial^2 z}{\partial y \partial x}$. If the differentiation takes place in the reverse order the result is denoted by the symbol

$\frac{\partial^2 z}{\partial x \partial y}$. The first is read "the second partial derivative of z with respect to x and y;" the second, "the second partial derivative of z with respect to y and x." In the case of functions usually occurring in Physics and Engineering, viz., functions which are continuous and which have continuous first and second partial derivatives, $\frac{\partial^2 \mathbf{z}}{\partial \mathbf{y} \partial \mathbf{x}} = \frac{\partial^2 \mathbf{z}}{\partial \mathbf{x} \partial \mathbf{y}}$. The order of differentiation is immaterial.

Illustration 1. $z = x^2 y$.

$$\frac{\partial z}{\partial x} = 2xy, \qquad \frac{\partial^2 z}{\partial x^2} = 2y, \qquad \frac{\partial^2 z}{\partial y \partial x} = 2x.$$

$$\frac{\partial z}{\partial y} = x^2, \qquad \frac{\partial^2 z}{\partial y^2} = 0, \qquad \frac{\partial^2 z}{\partial x \partial y} = 2x.$$

In this case

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y}.$$

Illustration 2. $z = \sin \frac{x}{y}$.

$$\frac{\partial z}{\partial x} = \frac{1}{y} \cos \frac{x}{y}.$$

$$\frac{\partial^2 z}{\partial x^2} = -\frac{1}{y^2} \sin \frac{x}{y}.$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{1}{y}\left(-\sin \frac{x}{y}\right)\left(-\frac{x}{y^2}\right) - \frac{1}{y^2} \cos \frac{x}{y}$$

$$= \frac{1}{y^3}\left(x \sin \frac{x}{y} - y \cos \frac{x}{y}\right).$$

$$\frac{\partial z}{\partial y} = -\frac{x}{y^2} \cos \frac{x}{y}.$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{2x}{y^3} \cos \frac{x}{y} - \frac{x^2}{y^4} \sin \frac{x}{y}.$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{x}{y^2}\left(\sin \frac{x}{y}\right)\left(\frac{1}{y}\right) - \frac{1}{y^2} \cos \frac{x}{y}$$

$$= \frac{1}{y^3}\left(x \sin \frac{x}{y} - y \cos \frac{x}{y}\right).$$

Here again, we notice that

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y}.$$

Exercises

1. Find $\frac{\partial^2 z}{\partial x^2}$, $\frac{\partial^2 z}{\partial y^2}$, $\frac{\partial^2 z}{\partial x \partial y}$, and $\frac{\partial^2 z}{\partial y \partial x}$, for each of the functions:

(*a*) $z = \frac{x}{y}$. (*b*) $z = xy^2$. (*c*) $z = x^3y$.
(*d*) $z = \sin xy$. (*e*) $z = e^x \sin y$.

2. Find $\frac{\partial^2 z}{\partial y \partial x}$ and $\frac{\partial^2 z}{\partial x \partial y}$ for each of the following functions:

(*a*) $z = x^2y$. (*b*) $z = x \sin^{-1} y$. (*c*) $z = x \cos y$.
(*d*) $z = y \log x$. (*e*) $z = e^y \sin x$. (*f*) $z = y \tan^{-1} x$.

It is seen that $\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y}$ in all of these cases.

In the above discussion z was considered to be a function of two independent variables only. The notion of partial derivatives can, however, be extended to functions of three or more variables.

Illustration 3. If $z = x^2yt$,

$$\frac{\partial z}{\partial x} = 2xyt,$$

$$\frac{\partial z}{\partial y} = x^2t,$$

$$\frac{\partial z}{\partial t} = x^2y,$$

$$\frac{\partial^2 z}{\partial x \partial t} = 2xy,$$

and

$$\frac{\partial^3 z}{\partial t \partial y \partial x} = 2x.$$

CHAPTER XV

SUCCESSIVE INTEGRATION. CENTER OF GRAVITY. MOMENT OF INERTIA

131. Introduction. In the preceding chapters there have been numerous examples of successive integration of functions of a single independent variable. Thus, to determine the law of motion of a falling body whose differential equation of motion is

$$\frac{d^2s}{dt^2} = g,$$

it is necessary to integrate twice. The result of the first integration is $\frac{ds}{dt} = gt + C_1$, and that of the second is $s = \frac{1}{2}gt^2 + C_1t + C_2$.

Exercises

1. If $\frac{d^2y}{dx^2} = 2x$, find y as a function of x, given that $\frac{dy}{dx} = 3$ when $x = 1$, and $y = 2$ when $x = 4$; given that $y = 4$ when $x = 2$, and that $y = 7$ when $x = 4$.

2. Find y if $\frac{d^3y}{dx^3} = 7x$. Assign suitable conditions to determine the constants of integration.

3. Find y if $\frac{d^4y}{dx^4} = 2x^2$.

The operation of finding the result of Exercise 2 can be written

$$\begin{aligned}\int\left[\int\left[\int 7x\,dx\right]dx\right]dx &= \int\left[\int\left[\tfrac{7}{2}x^2 + C_1\right]dx\right]dx \\ &= \int\left(\tfrac{7}{6}x^3 + C_1x + C_2\right)dx \\ &= \frac{7x^4}{24} + C_1\frac{x^2}{2} + C_2x + C_3.\end{aligned}$$

The first member can be written

$$\int\int\int 7x\,dx\,dx\,dx.$$

It is a triple integral and indicates that integration is to be performed three times in succession. An arbitrary constant of integration is introduced with each integration. If each integration is performed between limits the constants of integration do not appear. Thus,

$$\int_1^2\int_1^3\int_2^4 7x\,dx\,dx\,dx = \int_1^2\int_1^3 \frac{7x^2}{2}\bigg|_2^4 dx\,dx$$
$$= \int_1^2\int_1^3 42\,dx\,dx$$
$$= \int_1^2 42\,x\bigg|_1^3 dx$$
$$= \int_1^2 84\,dx$$
$$= 84\,x\bigg|_1^2 = 84.$$

132. Illustration of Double Integration. Let

$$\frac{\partial^2 z}{\partial x \partial y} = x^2 + y^2. \tag{1}$$

Integration with respect to y, treating x as a constant, gives

$$\frac{\partial z}{\partial x} = x^2 y + \frac{y^3}{3} + \phi\,(x),$$

where $\phi(x)$ is an arbitrary function of x. This arbitrary function of x takes the place of an arbitrary constant of integration in the case of a single independent variable. A second integration, this time with respect to x, gives

$$z = \frac{x^3 y}{3} + \frac{xy^3}{3} + \int \phi(x)\,dx + \psi(y), \tag{2}$$

where $\psi(y)$ is an arbitrary function of y.

The result contains an arbitrary function of x and an arbitrary function of y. Equation (2) represents a surface, but a very arbitrary one on account of the presence of the arbitrary functions $\int\phi(x)dx$ and $\psi(y)$. The process of finding (2) from (1) is indicated by a double integral sign. Thus,

$$\int\int (x^2 + y^2)\,dy\,dx,$$

which means

$$\int \left(\int (x^2 + y^2) dy \right) dx.$$

Upon performing the integration indicated, first with respect to y, then with respect to x, we obtain

$$\int\int (x^2 + y^2)\, dy\, dx = \int \left(x^2y + \frac{y^3}{3} + \phi(x) \right) dx$$
$$= \frac{x^3y}{3} + \frac{xy^3}{3} + \int \phi(x)\, dx + \psi\,(y).$$

Instead of an indefinite double integral such as the one just considered we may have a definite double integral. If the integration with respect to y is performed before that with respect to x, the limits of integration with respect to y may be functions of x. Thus,

$$\int_1^3 \int_x^{x^2} (x^2 + y^2)\, dy\, dx = \int_1^3 \left(x^2y + \frac{y^3}{3} \right) \Bigg|_{y=x}^{y=x^2} dx$$
$$= \int_1^3 \left(x^4 + \frac{x^6}{3} - \frac{4x^3}{3} \right) dx.$$

The last integration is readily performed. It is to be noted that in evaluating a double integral, x is treated as a constant when the integration with respect to y is performed.

If in a definite integral dx is written before dy, the integration with respect to x is to be performed first.[1]

Exercises

1. $\int_0^1 \int_2^4 xy\, dy\, dx.$

2. $\int_0^1 \int_0^x xy\, dy\, dx.$

3. $\int_0^1 \int_{2x}^{x^{\frac{3}{2}}} xy^2\, dy\, dx.$

4. $\int_0^{\pi} \int_0^{a(1 + \cos\theta)} r\, dr\, d\theta.$

5. $\int_0^a \int_{a-x}^{\sqrt{a^2-x^2}} dy\, dx$

6. $\int_0^5 \int_0^3 \int_2^4 x^2y^3z^2\, dz\, dy\, dx.$

[1] Usage varies on this point. The student will have to observe in every case the convention adopted in the book he is reading.

7. $\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2}} dz\, dy\, dx.$

8. $\int_0^1 \int_{x^2}^{\sqrt{x}} x\, dy\, dx.$

10. $\int_0^1 \int_{x^2}^{\sqrt{x}} dy\, dx.$

9. $\int_0^1 \int_{x^2}^{\sqrt{x}} (x^2 + y^2)\, dy\, dx.$

11. $\int_0^1 \int_{x^2}^{\sqrt{x}} y\, dy\, dx.$

12. $\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2 - x^2 - y^2}\, dy\, dx.$

HINT. To perform the integration in Exercise 12, let $\sqrt{a^2 - x^2} = b$ and make use of the result of Illustration 3, §**105**.

133. Area by Double Integration: Rectangular Coördinates. A plane area can be represented by a double integral. Thus, let it be required to find the area A between the curves $y = f_1(x)$, $y = f_2(x)$, and the lines $x = a$ and $x = b$. The area of the strip $IJKH$, Fig. 84, is approximately

$$\lim_{\Delta y \doteq 0} \sum_{y_1}^{y_2} \Delta y\, \Delta x = \Delta x \lim_{\Delta y \doteq 0} \sum_{y_1}^{y_2} \Delta y = \Delta x \int_{y_1}^{y_2} dy,$$

where y_1 and y_2 are the ordinates of the two curves $y = f_1(x)$ and $y = f_2(x)$, respectively. And the area sought is approximately

$$\sum_{x=a}^{x=b} \Delta x \int_{y_1}^{y_2} dy.$$

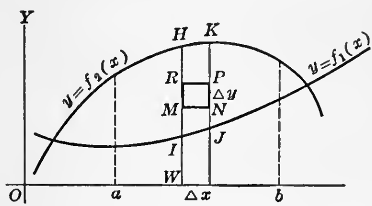

FIG. 84.

The smaller Δx is taken, the closer the approximation. The limit of this sum as Δx approaches zero is the area sought. Since y_1 and y_2 are functions of x, $\int_{y_1}^{y_2} dy$ is a function of x and consequently

$$\lim_{\Delta x \doteq 0} \sum_{x=a}^{x=b} \Delta x \int_{y_1}^{y_2} dy = \int_a^b \int_{y_1}^{y_2} dy\, dx = A.$$

It is to be noted that in setting up this integral the summation with respect to y was performed first, giving the area of a vertical strip for a particular value of x. Consequently, the integration with respect to y is to be performed first, x being treated as a constant. On performing the integration with respect to y we obtain

$$A = \int_a^b (y_2 - y_1)dx = \int_a^b [f_2(x) - f_1(x)]dx,$$

a single integral which might have been set up at once by considering the area as the sum of vertical strips of length $y_2 - y_1$, and of width Δx. It is, however, desirable to be able to set up a double integral over an area.

In choosing the limits for a double integral, the student should proceed systematically. The process of setting up the above integral with its limits is the following: The "element" is the rectangular element of area $dy\,dx$. The "summation" (integration) of this element, for a particular value of x, between the limits for y of WI and WH, the ordinates of the curves $y = f_1(x)$ and $y = f_2(x)$, gives the area of the typical strip $IJKH$. The "summation" (integration) of the strips of which this is a typical one, between the extreme values of x, $x = a$ and $x = b$, gives the area sought. Thus

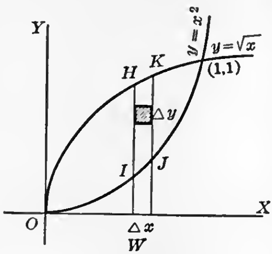

FIG. 85.

$$A = \int_a^b \int_{y_1}^{y_2} dy\,dx.$$

The procedure may be briefly summarized in the following concise directions. Write first the element $dy\,dx$, then the integral sign, then the limits $f_1(x)$, $f_2(x)$, then another integral sign with the limits a and b.

Illustration. Find by double integration the area between the parabolas $y^2 = x$ and $y = x^2$. The integral is set up as follows: Write the element $dy\,dx$, then an integral sign with the limits x^2 and $\sqrt{x}$. This represents the area of the typical strip, $IJKH$, Fig. 85, for a fixed x. All of the strips of which this is a typical one are to be summed from $x = 0$ to $x = 1$, the abscissas of the points of intersection of the curves. Then write the second integral sign preceding the first with the limits 0 and 1. Thus

$$A = \int_0^1 \int_{x^2}^{\sqrt{x}} dy\, dx = \tfrac{1}{3}.$$

Exercises

1. Find by double integration the area between the curves $y = x$ and $y^2 = x^3$.

2. Find the area of Exercise 1 by integrating first with respect to x and then with respect to y.

3. Find by double integration the area between $y^2 = a(a - x)$ and $x^2 + y^2 = a^2$.

4. Find the area between $y^2 = ax$ and $y^2 = 2ax - x^2$.

5. Find the area of Exercises 3 and 4 by integrating first with respect to x.

6. Find the area bounded by $y^2 = 4x$, $x + y = 3$, and the X-axis.

7. Find the area of a rectangle by double integration.

8. Find the smaller area between $x^2 + y^2 = 1$ and $y = x + \frac{1}{2}$.

134. Geometrical Meaning of the Definite Double Integral. Consider the definite double integral

$$\int_a^b \int_{y_1 = f_1(x)}^{y_2 = f_2(x)} f(x, y)\, dy\, dx. \tag{1}$$

In accordance with the definition of a definite single integral, §65, (1) can be written

$$\int_a^b \left[\lim_{\Delta y \doteq 0} \sum_{y_1}^{y_2} f(x, y)\ \Delta y \right] dx. \tag{2}$$

Here x is considered constant under the summation sign, and $f(x, y)$ is, for such a fixed x, a function of y alone.

$$\lim_{\Delta y \doteq 0} \sum_{y_1}^{y_2} f(x, y)\, \Delta y = \int_{y_1}^{y_2} f(x, y)\, dy$$

is a function of x, since x occurs as an argument of f and also in the limits of integration. Hence we can write (2) in the form

$$\lim_{\Delta x \doteq 0} \sum_a^b \Delta x \left[\lim_{\Delta y \doteq 0} \sum_{y_1}^{y_2} f(x, y)\, \Delta y \right]$$

$$= \lim_{\Delta x \doteq 0} \sum_a^b \lim_{\Delta y \doteq 0} \sum_{y_1}^{y_2} f(x, y)\, \Delta y\, \Delta x, \tag{3}$$

where Δx under the second summation sign is regarded as a constant multiplier.

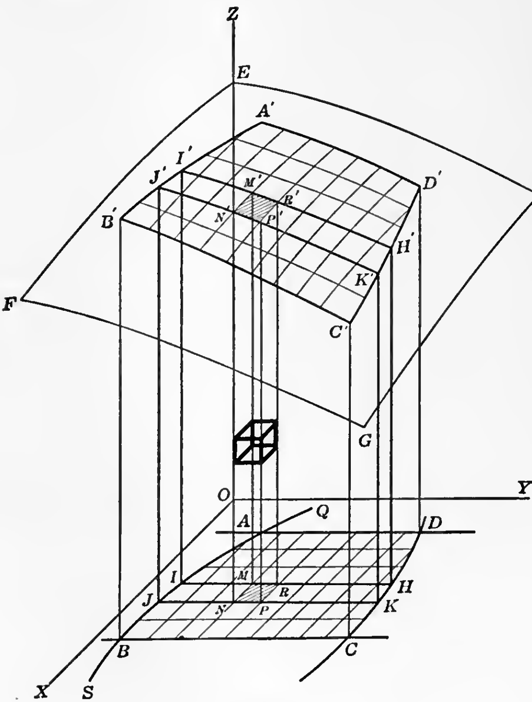

FIG. 86.

In Fig. 86, let $EFGL$ represent the surface $z = f(x, y)$; $QABS$, in the XY-plane, the curve $y = f_1(x)$; $DHKC$, the curve $y = f_2(x)$; AD the line $x = a$; BC the line $x = b$; and $A'B'C'D'$ the portion of the surface cut from $z = f(x, y)$ by the cylinders $y = f_1(x)$, $y = f_2(x)$, and the planes $x = a$ and $x = b$.

Divide $ABCD$ into small rectangles, as shown in the figure, by lines parallel to the X- and Y-axes, at intervals of Δy and Δx, respectively. Through these lines pass planes parallel to the XZ- and YZ-planes. These planes divide the solid bounded by the planes and surfaces of Fig. 86 into vertical columns of rectangular cross section $\Delta y \, \Delta x$. The column erected on $MNPR$ as a base is a typical one. $f(x, y)\Delta y \, \Delta x$ represents approximately the volume of the column whose base is $MNPR$ and whose top is $M'N'P'R'$, since the area of its base is $\Delta y \, \Delta x$ and its altitude is $MM' = f(x, y)$. Then the sum of the columns at a fixed distance from the YZ-plane,

$$\sum_{y_1}^{y_2} f(x, y) \, \Delta y \, \Delta x,$$

is approximately the volume of the slab between the planes $IHH'I'$ and $JKK'J'$, *i.e.*, between the planes $x = x$ and $x = x + \Delta \, x$. And

$$\sum_{a}^{b} \Delta x \left[\sum_{y_1}^{y_2} f(x, y) \, \Delta y \right],$$

the sum of the volumes of all the slabs, is approximately the volume of the solid $ABCDA'B'C'D'$. If Δy and Δx are each taken smaller and smaller this sum will eventually represent a very close approximation to the volume in question, and the limit of this sum as Δy and Δx approach zero is the volume. Hence the integral

$$\int_a^b \int_{y_1}^{y_2} f(x, y) \, dy \, dx,$$

which we have seen is equal to

$$\lim_{\Delta x \doteq 0} \sum_{x=a}^{x=b} \Delta x \left[\lim_{\Delta y \doteq 0} \sum_{y_1}^{y_2} f(x, y) \, \Delta y \right],$$

represents the volume bounded by the plane $z = 0$, the surface $z = f(x, y)$, the planes $x = a$ and $x = b$, and the cylinders $y = f_1(x)$ and $y = f_2(x)$.

Illustration. Find the volume contained in the first octant of the sphere $x^2 + y^2 + z^2 = a^2$. See Fig. 87. The equation of the surface is

$$z = \sqrt{a^2 - x^2 - y^2}.$$
$$y_1 = f_1(x) = 0$$
$$y_2 = f_2(x) = \sqrt{a^2 - x^2},$$

the trace of the sphere on the XY-plane. The volume of the column on $MNPR$ as a base is

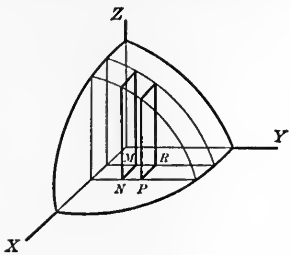

FIG. 87.

$$\sqrt{a^2 - x^2 - y^2}\,\Delta y\,\Delta x,$$

or, as we shall say in the future,

$$\sqrt{a^2 - x^2 - y^2}\,dy\,dx.$$

The summation of these columns for a fixed x gives

$$\int_0^{\sqrt{a^2-x^2}} \sqrt{a^2 - x^2 - y^2}\,dy\,dx,$$

the volume, expressed as a function of x, of the slab between the planes $x = x$ and $x = x + \Delta x$ or $x = x + dx$. The summation of all these slabs from $x = 0$ to $x = a$ gives

$$\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2 - x^2 - y^2}\,dy\,dx,$$

the volume of one octant of the sphere. This integral was evaluated in Exercise 12, §132.

Exercises

1. Find the volume of the segment of the paraboloid $y^2 + 2z^2 = 4x$, cut off by the plane $x = 5$.

2. Find the volume bounded by the cylinders $y = x^2$ and $y^2 = x$, and the planes $z = 0$ and $z = 1$.

3. Find the volume common to the cylinders $x^2 + y^2 = a^2$ and $y^2 + z^2 = a^2$.

4. Find the volume between the cylindrical surface $y^2 = x^3$, the plane $y = x$, and the planes $z = 0$ and $z = 1$.

135. Area: Polar Coördinates. Let it be required to find, by double integration, the area between the radii vectores $\theta = \alpha$, and $\theta = \beta$, and the curve $\rho = f(\theta)$. Divide the area as shown in Fig. 88, the radii making an angle of $\Delta\theta$ with each other and the radii of the concentric circles differing by $\Delta\rho$. The area of $MPQR$ is equal to

$$\tfrac{1}{2}(\rho + \Delta\rho)^2\,\Delta\theta - \tfrac{1}{2}\,\rho^2\,\Delta\theta = \rho\,\Delta\rho\,\Delta\theta + \tfrac{1}{2}\,\Delta\rho^2\,\Delta\theta$$

As $\Delta\rho$ approaches zero,

$$\lim_{\Delta\rho \doteq 0} \frac{\rho\,\Delta\rho\,\Delta\theta + \frac{1}{2}\Delta\rho^2\,\Delta\theta}{\rho\,\Delta\rho\,\Delta\theta} = 1.$$

Hence

$$\lim_{\Delta\rho \doteq 0} \sum_{\rho=0}^{\rho=f(\theta)} (\rho\,\Delta\rho\,\Delta\theta + \tfrac{1}{2}\Delta\rho^2\,\Delta\theta) = \lim_{\Delta\rho \doteq 0} \sum_{\rho=0}^{\rho=f(\theta)} \rho\,\Delta\rho\,\Delta\theta = \int_{\rho=0}^{\rho=f(\theta)} \rho\,d\rho\,\Delta\theta.$$

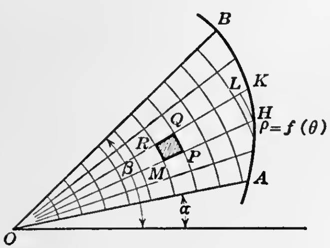

Fig. 88.

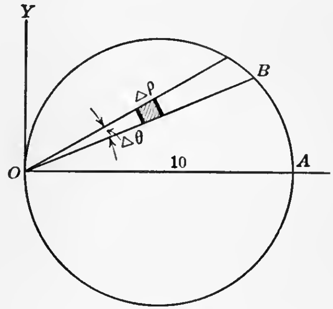

Fig. 89.

This sum represents the area of the sector OHL. The total area sought is the limit of the sum of these sectors as $\Delta\theta$ approaches zero, *i.e.*

$$A = \lim_{\Delta\theta \doteq 0} \sum_{\theta=\alpha}^{\theta=\beta} \Delta\theta \int_{\rho=0}^{\rho=f(\theta)} \rho\,d\rho = \int_{\alpha}^{\beta}\int_{0}^{f(\theta)} \rho\,d\rho\,d\theta.$$

This integral is to be set up as follows: The element of area is the approximately rectangular area $MPQR$ whose area is approximately $(MR)(MP) = \rho\,d\rho\,d\theta$. This element is to be summed from $\rho = 0$ to $\rho = f(\theta)$. This gives approximately the area of the typical sector OHK. These sectors are to be summed from $\theta = \alpha$ to $\theta = \beta$.

More briefly: Write down the element $\rho\, d\rho\, d\theta$, then an integral sign. Its limits are the extreme values of ρ for a given θ. Then write before this integral another integral sign. Its limits are to be chosen so as to sum up all the sectors such as OHK.

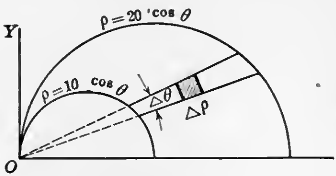

FIG. 90.

Illustration 1. Find the area of the circle $\rho = 10 \cos \theta$, Fig. 89. The area bounded by the semicircle above the initial line will be found and multiplied by two.

$$A = 2\int_0^{\frac{\pi}{2}}\int_0^{10\cos\theta} \rho\, d\rho\, d\theta.$$

Show that

$$A = 2\int_0^{10}\int_0^{\cos^{-1}\frac{\rho}{10}} \rho\, d\theta\, d\rho.$$

Illustration 2. Find by double integration the area between $\rho = 10 \cos \theta$ and $\rho = 20 \cos \theta$. See Fig. 90.

$$A = 2\int_0^{\frac{\pi}{2}}\int_{10\cos\theta}^{20\cos\theta} \rho\, d\rho\, d\theta.$$

Show that

$$A = 2\int_0^{10}\int_{\cos^{-1}\frac{\rho}{10}}^{\cos^{-1}\frac{\rho}{20}} \rho\, d\theta\, d\rho + 2\int_{10}^{20}\int_0^{\cos^{-1}\frac{\rho}{20}} \rho\, d\theta\, d\rho.$$

Which method is the simpler in this case?

Exercises

Find by double integration:

1. The area of the circle $\rho = 5 \sin \theta$.

2. The area of the cardioid $\rho = a(1 - \cos \theta)$.

3. The area of the lemniscate $\rho^2 = a^2 \cos 2\theta$.

4. The area outside $\rho = a\,(1 + \cos \theta)$ and inside $\rho = 3a \cos \theta$.

136. Volume of a Solid: Triple Integration. We shall now find the volume of the solid of Fig. 86 by triple integration. Suppose the solid further subdivided by planes parallel to the XY-plane and at a distance Δz apart, into rectangular parallelopipeds of volume $\Delta z\ \Delta y\ \Delta x$. Then the volume of the column on the base $MNPR$ is approximately

$$\sum_{z=0}^{z=f(x,y)} \Delta z\ \Delta y\ \Delta x.$$

Then

$$\sum_{y_1}^{y_2} \sum_{z=0}^{z=f(x,y)} \Delta z\ \Delta y \Delta x$$

is approximately the volume of the slab between the planes $IHI'H'$ and $JKJ'K'$, *i.e.*, between the planes $x = x$ and $x = x + \Delta x$.

$$\sum_{x=a}^{x=b} \sum_{y_1}^{y_2} \sum_{z=0}^{z=f(x,y)} \Delta z\ \Delta y\ \Delta x$$

is approximately the sum of the volumes of all of the slabs. If Δz, Δy, and Δx are each taken smaller and smaller, this sum will represent a very close approximation to the volume sought, and the limit of this sum as Δx, Δy and Δz approach zero is exactly this volume. Hence the integral,

$$\int_a^b \int_{y_1}^{y_2} \int_0^z dz\ dy\ dx,$$

represents the volume bounded by the plane $z = 0$, the surface $z = f(x, y)$, the planes $x = a$ and $x = b$, and the cylinders $y = f_1(x)$ and $y = f_2(x)$.

Illustration 1. Find by triple integration the volume of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

See Fig. 91.

$$V = 8 \int_0^a \int_0^{b\sqrt{1 - \frac{x^2}{a^2}}} \int_0^{c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}} dz\ dy\ dx$$

$$= \int_0^a \int_0^{b\sqrt{1 - \frac{x^2}{a^2}}} c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}\ dy\ dx.$$

The student will perform the integration.

Illustration 2. Find by triple integration the volume of the solid bounded by the cylinder $x^2 + y^2 = 2ax$, the plane $z = 0$, and the paraboloid of revolution $x^2 + y^2 = 4az$. Write the element of volume, $dz\,dy\,dx$. The integration with respect to z between the limits 0 and $\frac{x^2 + y^2}{4a}$ gives the volume of the typical vertical column of base $dy\,dx$, extending from the point (x, y) in the plane $z = 0$ to the surface of the paraboloid, $z = \frac{x^2 + y^2}{4a}$. Next, x being kept fixed, these columns are summed into a typical slab by integrating with respect to y from the X-axis,

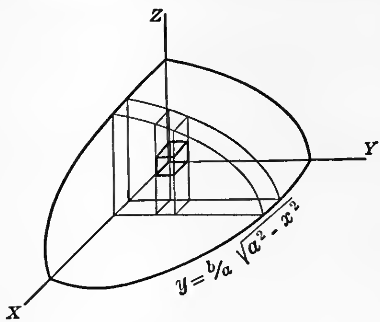

FIG. 91.

$y = 0$, to $y = \sqrt{2ax - x^2}$, the trace of the cylinder in the XY-plane. Finally the integration with respect to x from $x = 0$ to $x = 2a$ gives one-half of the total volume sought, viz., that lying in the first octant.

$$V = 2\int_0^{2a}\int_0^{\sqrt{2ax - x^2}}\int_0^{\frac{x^2+y^2}{4a}} dz\,dy\,dx.$$

The student will perform the integration.

Exercises

1. Find the volume common to the cylinders $x^2 + y^2 = r^2$ and $x^2 + z^2 = r^2$.

2. Find the volume of one of the wedges cut from the cylinder $x^2 + y^2 = r^2$ by the planes $z = 0$ and $z = mx$.

3. Find the volume in the first octant bounded by the coördinate planes and the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.

4. Set up the integral representing the volume bounded by the surface $x^{\frac{2}{3}} + y^{\frac{2}{3}} + z^{\frac{2}{3}} = a^{\frac{2}{3}}$.

5. Find the volume between $y^2 + z^2 = 4ax$ and $x - z = a$.

6. Find the volume between the planes $y = 0$, $z = 0$ and the surfaces $z = x^2 + 4y^2$, $y = 1 - x^2$.

7. Find the volume between $y^2 + 2z^2 = 4x$ and $z = x$.

137. Center of Mass, Centroid. Let there be a system of masses $m_1, m_2, m_3, \ldots, m_n$ situated at the points (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), $\ldots$, (x_n, y_n, z_n), respectively. The mean distance with respect to mass, of the system from the YZ-plane is

$$\bar{x} = \frac{\Sigma m_i x_i}{\Sigma m_i}. \tag{1}$$

The mean distances, with respect to mass, of the system from the ZX- and XY-planes are, respectively,

$$\bar{y} = \frac{\Sigma\, m_i y_i}{\Sigma\, m_i} \tag{2}$$

and

$$\bar{z} = \frac{\Sigma\, m_i z_i}{\Sigma\, m_i} \tag{3}$$

The point $(\bar{x}, \bar{y}, \bar{z})$ is called the *centroid*, or the *center of mass*, of the system of masses $m_1, m_2, \cdots, m_n$.

$m_i\, x_i$ is called the *moment*, and x_i the *moment arm*, of the mass m_i with respect to the YZ-plane.[1] Then $\bar{x}$ is the mean moment arm with respect to the YZ-plane of the masses $m_1, m_2, \cdots, m_n$. For, equation (1) shows that if all the masses were placed at the distance, $\bar{x}$, from the YZ-plane, the moment with respect to this plane would be the same as the sum of the moments of the masses. Hence we can say that the centroid of a system of masses is a point such that if all the masses were concentrated at this point, the moment with respect to each coördinate plane would be equal to

[1] The term moment of a mass with respect to a plane has evidently a different significance from the term moment as applied to a force.

the sum of the moments with respect to the corresponding planes of the masses in their given positions.

138. Centroid Independent of the Position of the Coördinate Planes. It will now be shown that the distance of the centroid, $(\bar{x}, \bar{y}, \bar{z})$, from any plane is the mean of the distances, with respect to mass, of the masses $m_1, m_2, \cdot \cdot \cdot, m_n$, from that plane. And thus it will be shown that $(\bar{x}, \bar{y}, \bar{z})$, the centroid, is a point whose position with reference to the masses is independent of the choice of the coördinate planes.

Let $ax + by + cz + d = 0$ be the equation of a given plane (see §**120**). The distance, $\bar{\rho}$, of the point $(\bar{x}, \bar{y}, \bar{z})$ from this plane is

$$\bar{\rho} = \frac{a\bar{x} + b\bar{y} + c\bar{z} + d}{R}, \tag{1}$$

where

$$R = \sqrt{a^2 + b^2 + c^2}.$$

(See §**124.**) On substituting the values of $\bar{x}$, $\bar{y}$, and $\bar{z}$ from (1), (2), and (3) §**137,** and reducing the absolute term, d, to the common denominator, we have

$$\bar{\rho} = \frac{a\Sigma m_i x_i + b\Sigma m_i y_i + c\Sigma m_i z_i + d\Sigma m_i}{R\Sigma m_i},$$

or

$$\bar{\rho} = \frac{\Sigma m_i \left[\frac{ax_i + by_i + cz_i + d}{R}\right]}{\Sigma m_i}. \tag{2}$$

But $\dfrac{ax_i + by_i + cz_i + d}{R} = \rho_i$ is the distance of the point (x_i, y_i, z_i) from the given plane. Hence (2) can be written in the form

$$\bar{\rho} = \frac{\Sigma\, m_i \rho_i}{\Sigma\, m_i}. \tag{3}$$

This proves the statement at the beginning of this section. In other words, if all the masses of the system were concentrated at the centroid, the moment with respect to any plane would be equal to the moment of the system of masses with respect to this plane.

139. Center of Gravity. Let the system of masses considered above be acted upon by gravity. It will be shown that the line of action of the resultant force passes through the center of mass, or the centroid.

Since the position of the centroid is independent of the choice of axes, choose the positive direction of the axis of z vertically upward and the axes of x and y horizontal. The force acting on m_1 is m_1g, that on m_2 is m_2g, etc. The resultant force is equal to $\Sigma\, m_ig$ and is directed vertically downward. Its line of action meets the XY-plane in a point $(\alpha, \beta, 0)$ such that its moment, $\alpha\Sigma m_ig$, about the Y-axis is equal to Σm_igx_i, the sum of the moments of the forces acting on the individual masses; and such that the moment, $\beta\Sigma m_ig$, about the X-axis, is equal to the sum of the moments, Σm_igy_i, of the forces about this axis.

$$\alpha\Sigma m_ig = \Sigma m_igx_i,$$

and

$$\beta\Sigma m_ig = \Sigma m_igy_i.$$

Whence

$$\alpha = \frac{\sum m_ix_i}{\sum m_i}$$

and

$$\beta = \frac{\sum m_iy_i}{\sum m_i}.$$

Consequently

$$\alpha = \bar{x} \text{ and } \beta = \bar{y}.$$

Hence the resultant passes through the centroid of the system of masses.

If the masses all lie in one plane, say the XY-plane, $\bar{z}$ is zero and the centroid is fixed by the two coördinates $\bar{x}$ and $\bar{y}$. The product m_ix_i is called the moment of the mass m_i with respect to the Y-axis. In this case $\bar{x}$ is the mean moment arm with respect to the Y-axis.

If the masses all lie upon a line, say the X-axis, the centroid is fixed by a single coördinate, $\bar{x}$.

140. Centroid of a Continuous Mass. If instead of discrete masses we have a continuous mass, the coördinates of the center of mass. or the centroid, are clearly,

$$\bar{x} = \frac{\lim\limits_{\Delta m \doteq 0} \sum x \, \Delta m}{\lim\limits_{\Delta m \doteq 0} \sum \Delta m} = \frac{\int x \, dm}{\int dm},$$

$$\bar{y} = \frac{\lim\limits_{\Delta m \doteq 0} \sum y \, \Delta m}{\lim\limits_{\Delta m \doteq 0} \sum \Delta m} = \frac{\int y \, dm}{\int dm},$$

$$\bar{z} = \frac{\lim\limits_{\Delta m \doteq 0} \sum z \, \Delta m}{\lim\limits_{\Delta m \doteq 0} \sum \Delta m} = \frac{\int z \, dm}{\int dm}.$$

The integration is to be extended throughout the entire mass, and the integrals considered may be single, double, or triple, depending on the form of the mass.

Illustration 1. Find the center of gravity of a bar, Fig. 92, of length L, whose linear density, ρ, may vary. Let the axis of x coincide with the bar, the origin being taken at one end. The

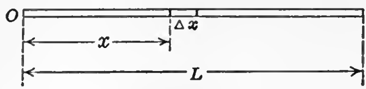

FIG. 92.

mass of an "element" of the bar of length dx is $\rho \, dx$, ρ being a function of x, the distance of the element from the origin. The moment of this element of mass, $dm = \rho \, dx$, about an axis through the origin perpendicular to the bar is

$$x \, dm = x\rho \, dx.$$

$\bar{x}$, the abscissa of the centroid, the only coördinate necessary to fix the centroid in this case, is given by

$$\bar{x} = \frac{\int_0^L \rho x \, dx}{\int_0^L \rho \, dx}.$$

The numerator represents the total moment, and the denominator the total mass. If the bar is of uniform density, ρ can be taken out from under the integral sign. Then

$$\bar{x} = \frac{\int_0^L x\,dx}{\int_0^L dx} = \frac{\left.\frac{x^2}{2}\right|_0^L}{\left.x\right|_0^L} = \frac{\frac{1}{2}L^2}{L} = \tfrac{1}{2}L.$$

If the linear density is proportional to the distance from one end, then $\rho = kx$ and we have

$$\bar{x} = \frac{k\int_0^L x^2\,dx}{k\int_0^L x\,dx} = \frac{\left.\frac{1}{3}x^3\right|_0^L}{\left.\frac{1}{2}x^2\right|_0^L} = \tfrac{2}{3}L.$$

Illustration 2. Let it be required to find the center of gravity of a plate of uniform thickness and of mass ρ per unit volume or of mass ρ per unit surface. Take a plate of the shape of Fig. 84. The mass of the element $MNPR$ is $\rho\,dy\,dx$. The moment of this element about the Y- axis is $x\rho\,dy\,dx$, and its moment about the X-axis is $y\,\rho\,dy\,dx$. Then

$$\bar{x} = \frac{\int x\,dm}{\int dm} = \frac{\int_a^b\int_{y_1}^{y_2} \rho\,x\,dy\,dx}{\int_a^b\int_{y_1}^{y_2} \rho\,dy\,dx},$$

$$\bar{y} = \frac{\int y\,dm}{\int dm} = \frac{\int_a^b\int_{y_1}^{y_2} \rho\,y\,dy\,dx}{\int_a^b\int_{y_1}^{y_2} \rho\,dy\,dx},$$

If ρ is constant,

$$\bar{x} = \frac{\int_a^b\int_{y_1}^{y_2} x\,dy\,dx}{\int_a^b\int_{y_1}^{y_2} dy\,dx},$$

and

$$\bar{y} = \frac{\int_a^b\int_{y_1}^{y_2} y\,dy\,dx}{\int_a^b\int_{y_1}^{y_2} dy\,dx},$$

The numerator of each of these expressions is the integral of the product of an element of area by its distance, x or y, from the Y-axis or X-axis, respectively. The denominator is the area. The mass does not enter into either of these formulas. We are thus led to speak of the centroid of an area, of a line or of a solid, without reference to its mass. This notion of the centroid of a geometrical figure, a line, an area, or a solid, without reference to its material composition is an important one. For, in many problems in mechanics one is interested in the centroid of a geometrical configuration as such. Thus in the study of the deflection of beams it is necessary to know the position of the centroid of the cross section of the beam.

Illustration 3. Find the centroid of the solid represented in Fig. 86. The element of mass, dm, is equal to $\rho\, dz\, dy\, dx$, and its moment with respect to the YZ-plane is $x\rho\, dz\, dy\, dx$. Then

$$\bar{x} = \frac{\int x\, dm}{\int dm} = \frac{\int_a^b \int_{y_1}^{y_2} \int_0^{f(x,y)} \rho x\, dz\, dy\, dx}{\int_a^b \int_{y_1}^{y_2} \int_0^{f(x,y)} \rho\, dz\, dy\, dx}.$$

Similarly:

$$\bar{y} = \frac{\int y\, dm}{\int dm} = \frac{\int_a^b \int_{y_1}^{y_2} \int_0^{f(x,y)} \rho y\, dz\, dy\, dx}{\int_a^b \int_{y_1}^{y_2} \int_0^{f(x,y)} \rho\, dz\, dy\, dx},$$

and

$$\bar{z} = \frac{\int z\, dm}{\int dm} = \frac{\int_a^b \int_{y_1}^{y_2} \int_0^{f(x,y)} \rho z\, dz\, dy\, dx}{\int_a^b \int_{y_1}^{y_2} \int_0^{f(x,y)} \rho\, dz\, dy\, dx}.$$

If the density is constant, ρ can be canceled from numerator and denominator.

If the solid has an axis or a plane of symmetry the centroid lies in this axis or in this plane.

Illustration 4. Find the centroid of the area in the first quadrant bounded by the circle $x^2 + y^2 = a^2$.

If we use double integration we have, in accordance with Illustration 2,

$$\bar{x} = \frac{\int_0^a \int_0^{\sqrt{a^2 - x^2}} x \, dy \, dx}{\frac{\pi a^2}{4}},$$

and

$$\bar{y} = \frac{\int_0^a \int_0^{\sqrt{a^2 - x^2}} y \, dy \, dx}{\frac{\pi a^2}{4}}.$$

Radicals could be avoided in the evaluation of the numerator of the expression for $\bar{x}$ if the integration were performed first with respect to x and then with respect to y. Thus

$$\bar{x} = \frac{\int_0^a \int_0^{\sqrt{a^2 - y^2}} x \, dx \, dy}{\frac{\pi a^2}{4}}.$$

The student will evaluate each expression given for $\bar{x}$.

From the symmetry of the figure, $\bar{x} = \bar{y}$, and it is not necessary to evaluate the integral for $\bar{y}$.

In finding the centroid in this case, and indeed in many cases, it is easier to use single integration than double integration. Thus if we choose as the element of area, the strip $y\,dx$ parallel to the Y-axis, the moment of this strip about the Y-axis is $xy\,dx$, and

$$\bar{x} = \frac{\int x \, dm}{\int dm} = \frac{\int_0^a xy \, dx}{\int_0^a y \, dx} = \frac{\int_0^a x\sqrt{a^2 - x^2} \, dx}{\frac{\pi a^2}{4}}.$$

Illustration 5. Find the centroid of the solid in the first octant bounded by the sphere $x^2 + y^2 + z^2 = a^2$.

The method of Illustration 3 gives

$$\bar{x} = \frac{\int_0^a \int_0^{\sqrt{a^2 - x^2}} \int_0^{\sqrt{a^2 - x^2 - y^2}} x \, dz \, dy \, dx}{\frac{\pi a^3}{6}}.$$

From considerations of symmetry, $\bar{y} = \bar{z} = \bar{x}$.

Here again it is simpler to use single integration. Choose as element a slab of thickness dx parallel to the YZ-plane. The base of such a slab is a quadrant of a circle of radius $\sqrt{a^2 - x^2}$, where x is the distance of the slab from the YZ-plane. The volume of this elementary slab is

$$\frac{\pi(a^2 - x^2)}{4}\,dx.$$

Hence

$$\bar{x} = \frac{\dfrac{\pi}{4}\displaystyle\int_0^a x\,(a^2 - x^2)\,dx}{\dfrac{\pi a^3}{6}}.$$

Exercises

Find the coördinates of the centroid of:

1. The area between $y = x^2$ and $y^2 = x$.

2. The areas of Exercises 1, 3, and 6, §**133**.

3. A triangular plate.

HINT. Draw lines parallel to the base, BC, Fig. 93, at intervals dx along the median AM. The mass of each strip is proportional to $AL = x$ and can be regarded as concentrated at its centroid on the line AM. Hence we can think of the triangular plate as replaced by the bar AM whose density is proportional to the distance from the end A In accordance with Illustration 1, its centroid is at a point I two-thirds of the way from A to M.

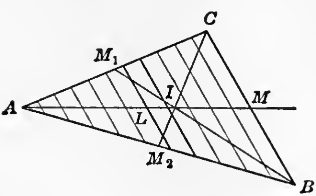

FIG. 93.

The centroid of a triangle can also be located without any calculation whatever. From Fig. 93 it follows that the centroid lies on the median AM. The same argument shows that it lies on the medians BM_1 and CM_2. Hence it lies at the point of intersection of the medians, *i.e.*, at a point two-thirds of the way from a vertex to the middle of the opposite side.

4. The area of a semicircular plate of radius r. (Single integration will be sufficient.)

5.[1] Let $OMKB$, Fig. 94, be a quadrant of a circle of radius r. Let $OMDB$ be a square. Denote by C_1, C_2, and C_3 the centers of gravity of the square, the quadrant of the circle, and the area $MDBKM$, respectively; and by A_1, A_2, and A_3 the corresponding areas. Then

$$A_2\bar{x}_2 + A_3\bar{x}_3 = A_1\bar{x}_1$$

$$\bar{x}_3 = \frac{A_1\bar{x}_1 - A_2\bar{x}_2}{A_3} = 0.223\,r.$$

$$DC_3 = 0.315\,r.$$

6. A circular arc of radius r and central angle 2α. See Fig. 95.

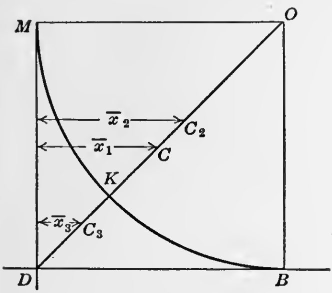

FIG. 94.

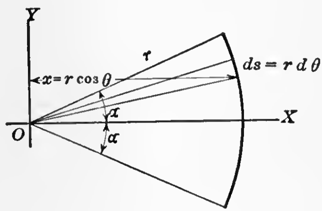

FIG. 95.

HINT. The centroid lies on the radius which bisects the central angle since this line is an axis of symmetry. Choose this radius as the axis of x and the center of the circle as the origin. Then $\bar{y} = 0$, and

$$\bar{x} = \frac{\int_{-\alpha}^{\alpha} xr\,d\theta}{2r\alpha} = \frac{\int_{-\alpha}^{\alpha} r\cos\theta\, r\,d\theta}{2r\alpha} = \frac{r^2\int_{-\alpha}^{\alpha}\cos\theta\,d\theta}{2r\alpha} = \frac{r\sin\alpha}{\alpha}.$$

This problem can also be solved by using rectangular coördinates. Thus

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \sqrt{1 + \frac{x^2}{y^2}}\,dx = \frac{r}{y}\,dx = \frac{r\,dx}{\sqrt{r^2 - x^2}}.$$

$$\bar{x} = \frac{2r\int_{r\cos\alpha}^{r}\frac{x\,dx}{\sqrt{r^2 - x^2}}}{2r\alpha} = \frac{2r^2\sin\alpha}{2r\alpha} = \frac{r\sin\alpha}{\alpha}.$$

7. The portion of the arc of the circle $x^2 + y^2 = r^2$ which lies in the

[1] Exercises 5, 9, 20, 21, and 22 taken from Technical Mechanics by Maurer.

first quadrant. Use the result of Exercise 6. Also find the result directly.

8. The parabolic segment of altitude a and base b. See Fig. 96.

HINT. Show that the equation of the parabola is $4ay^2 = b^2x$. *Ans.* $\bar{x} = \frac{3}{5}a$.

9. A conical or pyramidal solid of altitude a and base A.

HINT. Let $OMNO$, Fig. 97, represent the projection of the solid on the XY-plane. Divide the solid by planes parallel to the base into laminas or plates of thickness dx. Then the area of the lamina

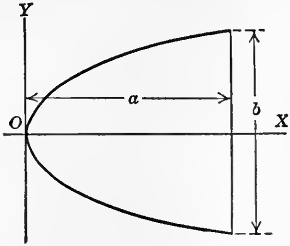

FIG. 96.

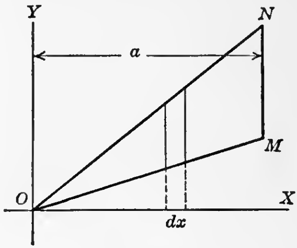

FIG. 97.

whose abscissa is x is $\frac{Ax^2}{a^2}$; and its volume is $\frac{Ax^2dx}{a^2}$. The volume of the solid is $\frac{Aa}{3}$. Hence:

$$\bar{x} = \frac{\int_0^a x\left(\frac{Ax^2\,dx}{a^2}\right)}{\frac{Aa}{3}} = \frac{3a}{4}.$$

Further the centroid of every lamina lies on the line joining the apex with the centroid of the base. Consequently the centroid of the solid lies on that line.

10. The hemisphere generated by revolving one quadrant of $x^2 + y^2 = r^2$ about the X-axis. Evidently $\bar{y} = \bar{z} = 0$ and

$$\bar{x} = \frac{\pi\int_0^r xy^2\,dx}{\frac{2}{3}\pi r^3}.$$

11. The surface of the hemisphere of Exercise 10.

12. The segment of a paraboloid of revolution of altitude h.

13. The semi-ellipsoid of revolution generated by revolving one quadrant of $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ about the X-axis.

14. The surface of the paraboloid of Exercise 12.

15. The surface of a right circular cone. Any conical or pyramidal surface.

16. The area of the cycloid $x = a(\theta - \sin \theta)$, $y = a(1 - \cos \theta)$.

17. The arc of the cycloid of Exercise 16.

18. The area in the first quadrant under $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

19. The arc of the curve of Exercise 18 in the first quadrant.

20. The segments of the ellipse indicated in Fig. 98. It will be found that the centroid of the segment $XAAX$ coincides with that of

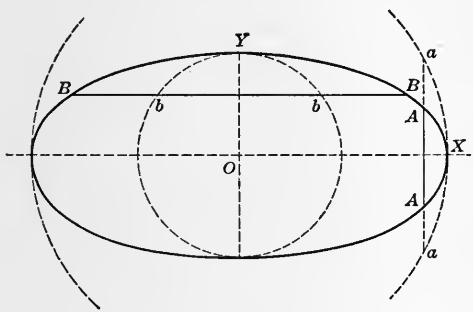

FIG. 98.

the segment $XaaX$ of the circumscribed circle, and that the centroid $YBBY$ coincides with that of the segment $YbbY$ of the inscribed circle.

21. C_1 and C_2 are the centers of gravity of the two portions of Fig. 99. Show that their distances from the sides of the enclosing rectangle, $a \times b$, are those marked in the figure. The curve OC is a parabola. See Exercise 5.

22. Find the centroid of the portion of a right circular cylinder shown in Fig. 100. C is the centroid. Its distance from the axis of the cylinder shown is $\frac{r^2 \tan \alpha}{4h}$, and from the base is $\frac{h}{2} + \frac{r^2 \tan \alpha}{8h}$. When the oblique top cuts the base in a diameter (lower part of Fig. 100) the distance of the centroid from the axis is $\frac{3\pi r}{16}$ and from the base $\frac{3\pi a}{32}$.

23. Find the centroid of the volume lying in the first octant and included between the cylinders $x^2 + y^2 = a^2$, $x^2 + z^2 = a^2$.

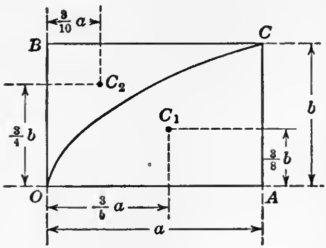

FIG. 99.

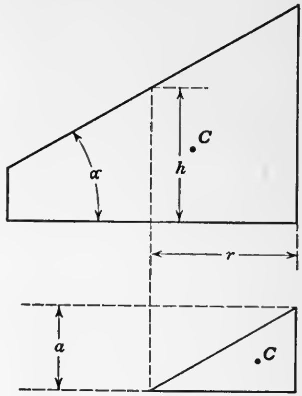

FIG. 100.

141. Theorems of Pappus. Theorem I. *The area of the surface generated by revolving an arc of a plane curve about an axis in its plane and not intersecting it is equal to the length of the arc multiplied by the length of the path described by its centroid.*

Theorem II. *The volume of the solid generated by revolving a plane surface about an axis lying in its plane and not intersecting its boundary is equal to the area of the surface multiplied by the length of the path described by its centroid.*

Proof of I. Let ABC, Fig. 101, be an arc of length L lying in the XY-plane. Then $\bar{y}$, the ordinate of its centroid, is given by the equation:

$$\bar{y} = \frac{\int y\,ds}{\int ds} = \frac{\int y\,ds}{L}. \qquad (1)$$

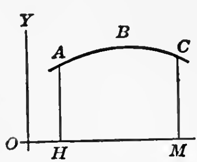

FIG. 101.

Whence

$$\int y\,ds = \bar{y}\,L. \qquad (2)$$

The surface generated by revolving the arc ABC about the X-axis is given by

$$S = 2\pi \int y\, ds. \tag{3}$$

It follows then from (2) and (3) that

$$S = 2\pi\, \bar{y}\, L. \tag{4}$$

But $2\pi\bar{y}$ is the length of the circular path described by the centroid of the arc ABC. Hence the theorem is proved.

Proof of II. Let ABC, Fig. 102, be a plane surface of area A. Then $\bar{y}$, the ordinate of its centroid, is given by

$$\bar{y} = \frac{\int y\, dA}{A}. \tag{5}$$

Whence

$$\int y\, dA = A\bar{y}. \tag{6}$$

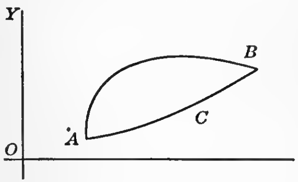

Fig. 102.

Now the volume of the solid generated by the revolution of the area ABC about the X-axis is

$$V = 2\pi \int\int y\, dy\, dx = 2\pi \int y\, dA. \tag{7}$$

It follows from (6) and (7) that

$$V = 2\pi A\bar{y}. \tag{8}$$

Hence the theorem is proved.

Exercises

1. Find the surface of the anchor ring generated by revolving the circle $x^2 + (y - b)^2 = a^2$, $a < b$, about the X-axis.

2. Find the volume of the anchor ring of Exercise 1.

3. Find, by using one of the theorems of Pappus, the centroid of a quadrant of a circular arc, radius a.

Hint. The rotation of the arc about the X-axis, which coincides with a radius drawn to one extremity, generates the surface $S = 2\pi a^2$. Then, by (4),

$$S = 2\pi a^2 = 2\pi\bar{y}L = 2\pi\,\bar{y}\,\frac{\pi a}{2}.$$

Hence

$$\bar{y} = \frac{2a}{\pi}.$$

4. Find, by using one of the theorems of Pappus, the centroid of a quadrant of a circular area.

142. Centroid: Polar Coördinates. The formulas are readily obtained for finding the coördinates of the centroid of an area bounded by a curve whose equation is given in polar coördinates. The area of the element $MPQR$, Fig. 88, is $\rho\, d\rho\, d\theta$, and its moment about the Y-axis is $\rho\, d\rho\, d\theta\, \rho \cos\theta = \rho^2 \cos\theta\, d\rho\, d\theta$. Hence

$$\bar{x} = \frac{\int\int \rho^2 \cos\theta\, d\rho\, d\theta}{\int\int \rho\, d\rho\, d\theta}.$$

Similarly,

$$\bar{y} = \frac{\int\int \rho^2 \sin\theta\, d\rho\, d\theta}{\int\int \rho\, d\rho\, d\theta}.$$

If it is advantageous, the integration with respect to θ can be performed first.

Exercises

Find the coördinates of the center of gravity of:

1. The area of $\rho = a(1 + \cos\theta)$. The area of the upper half of the same cardioid.

2. The area of one loop of $\rho = a \cos 2\theta$.

3. A circular sector of central angle 2α.

4. One quadrant of a circle. A semicircle. (Obtain directly and also use the result of Exercise 3.)

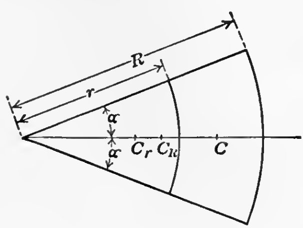

FIG. 103.

5. The area of a portion of a circular ring, Fig. 103, of radii R and r, and of central angle 2α. Denote by C_R the centroid of the sector of radius R, and by C_r that of the sector of radius r, and by C that of the given portion of the ring. Let the abscissas of these points be x_R, x_r, and $\bar{x}$, respectively, and let the corresponding areas be denoted by A_R, A_r, and A.

Then

$$A_r x_r + A\bar{x} = A_R x_R.$$

Hence

$$\bar{x} = \frac{A_R x_R - A_r x_r}{A} = \frac{2}{3}\frac{R^3 - r^3}{R^2 - r^2}\frac{\sin \alpha}{\alpha}.$$

Obtain this directly by integration.

6. A segment of a circle of radius r cut off by a chord of length c. Use the method of Exercise 5. The distance of the centroid from the center is

$$\frac{c^3}{12A} = \frac{2r^3 \sin^3 \alpha}{3A},$$

where A = area of the segment $= r^2 (2\alpha - \sin 2\alpha)$.

143. Moment of Inertia. Consider a system of masses, m_1, m_2 . . ., m_n, moving with linear accelerations, $j_1, j_2, \ldots, j_n$, respectively. The forces acting on these masses are then $m_1 j_1$, $m_2 j_2, \ldots, m_n j_n$, respectively; and the sum of the moments of these forces about an axis is equal to $\Sigma m_i j_i r_i$ where $r_1, r_2, \ldots r_n$, respectively, are the moment arms of these forces with respect to this axis. If now the masses are rigidly connected and rotate about an axis, they have a common angular acceleration. Let the common angular acceleration be denoted by α. Then $j_1 = \alpha r_1$, $j_2 = \alpha r_2, \ldots, j_n = \alpha r_n$, where $r_1, r_2, \ldots, r_n$ are the distances of the masses $m_1, m_2, \ldots, m_n$ from the axis of rotation. The sum of the moments, $\Sigma m_i j_i r_i$, becomes $\alpha \Sigma m_i r_i^2$. This is the moment necessary to produce the angular acceleration α. To produce unit angular acceleration a moment equal to $\Sigma m_i r_i^2$ is necessary. This moment, $\Sigma m_i r_i^2$, is called the moment of inertia, and is denoted by the symbol I. Thus

$$\mathbf{I} = \Sigma\, \mathbf{m_i r_i^2}. \tag{1}$$

The moment of inertia of the system would be unchanged if the n masses of the system were situated at a distance k from the axis of rotation such that

$$\Sigma m_i k^2 = \Sigma m_i r_i^2,$$

or

$$\mathbf{k^2} = \frac{\Sigma \mathbf{m_i r_i^2}}{\Sigma \mathbf{m_i}}.$$

k is called the *radius of gyration.* Its square is the mean of the squares of the distances $r_1, r_2, \ldots, r_n$ with respect to the mass.

The moment of inertia of a system with respect to an axis of rotation plays the same rôle in the discussion of a motion of rotation as the total mass in the discussion of a motion of translation. In the former case the moment necessary to produce an angular acceleration α is $\alpha\Sigma m_i r_i^2$. In the latter case the force necessary to produce a linear acceleration j is $j\Sigma m_i$.

The kinetic energy of a rotating system can be expressed in terms of its moment of inertia and its angular velocity. If a particle of mass m is rotating with angular velocity ω about an axis and at a distance r from it, its kinetic energy is equal to one-half the product of its mass by the square of its linear velocity, *i.e.*, to $\frac{1}{2}m\omega^2r^2$. And if there is a system of particles of masses $m_1, m_2, \ldots, m_n$, at distances $r_1, r_2, \ldots, r_n$, respectively, from the axis, all rotating with the angular velocity ω, the kinetic energy of the system is equal to $\frac{1}{2}\Sigma m_i\omega^2 r_i^2 = \frac{1}{2}\omega^2\Sigma m_i r_i^2 = \frac{1}{2}\omega^2 I$.

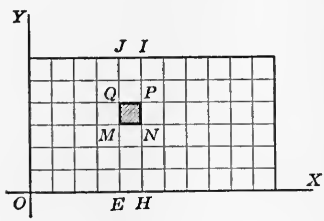

FIG. 104.

If a rectangular plate of uniform thickness ξ and composed of material of uniform density, ρ, rotate about an axis through one corner and perpendicular to its plane, its moment of inertia can be found by a process of double integration. Let the sides of the rectangle be a and b and take the origin at one corner, Fig. 104. The moment of inertia of the rectangle $MNPQ$ is approximately the product of its mass, $\rho\xi\,\Delta y\,\Delta x$, and the square of the approximate distance, $\sqrt{x^2 + y^2}$, of its mass particles from the origin. That is, the moment of inertia of $MNPQ$ is approximately

$$\rho\xi(x^2 + y^2)\Delta y\,\Delta x.$$

That of the strip $EHIJ$ is approximately

$$\rho\xi \lim_{\Delta y \doteq 0} \sum_{y=0}^{y=b}(x^2 + y^2)\,\Delta y\,\Delta x = \rho\xi\,\Delta x \int_0^b (x^2 + y^2)\,dy.$$

And the moment of inertia of the entire plate is obtained by taking the limit of the sum of the moments of inertia of these strips as Δx approaches zero, viz.,

$$I = \rho\xi \lim_{\Delta x \doteq 0} \sum_{x=0}^{x=a} \Delta x \int_a^b (x^2 + y^2)\,dy$$
$$= \rho\xi \int_0^a \int_0^b (x^2 + y^2)\,dy\,dx = \tfrac{1}{3}M(a^2 + b^2),$$

where $M = \rho\xi ab$, the mass of the plate.

We have obtained the moment of inertia of the plate by integrating over its area the product of the mass of the element, $\rho\xi\,dy\,dx$, by the square of its distance, $\sqrt{x^2 + y^2}$, from the axis of rotation.

If instead of a rectangular plate we consider a plate of any shape, say that of Fig. 84, its moment of inertia is given by

$$I = \rho\xi \int_a^b \int_{y_1}^{y_2} (x^2 + y^2)\,dy\,dx. \tag{2}$$

If the density, ρ, and the thickness, ξ, are variable the foregoing argument shows that they must be written under the integral sign. For, the element of integration is $\rho\xi(x^2 + y^2)\,dy\,dx$ and only when ρ and ξ are constant can they be taken out from under the integral sign. If ρ and ξ are variable we have

$$I = \int_a^b \int_{y_1}^{y_2} \rho\xi\,(x^2 + y^2)\,dy\,dx. \tag{3}$$

(2) and (3) can be written in a form easily remembered, viz.,

$$\mathbf{I} = \int \mathbf{r}^2\,\mathbf{dm}, \tag{4}$$

where dm is an element of mass, and r is its distance from the axis.

Sometimes in finding the moment of inertia of a body it is advantageous to choose the element of mass so that a single integral will suffice. See for example Illustration 2, below.

Illustration 1. Find I of a right-angled triangular plate whose thickness is 0.5 inch, and whose legs are 10 inches and 4 inches, about an axis through the vertex of the right angle and perpendicular to the plane of the triangle. The density of the material is 0.03 pound per cubic inch. See Fig. 105.

$$I = 0.03 \cdot 0.5 \int_0^{10} \int_0^{-\frac{2}{5}x + 4} (x^2 + y^2)\, dy\, dx.$$

The student will carry out the integration and find the radius of gyration.

Illustration 2. Find I of a circular plate about an axis through its center and perpendicular to its plane. The plate has a radius of 10 inches. It is 2 inches thick and its density is 0.04 pound per cubic inch.

Hint. Here it is convenient to divide the plate into concentric rings of inner radius r and of width dr. See Fig. 106. The

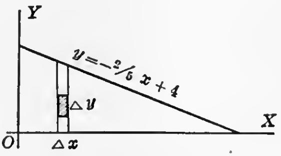

Fig. 105.

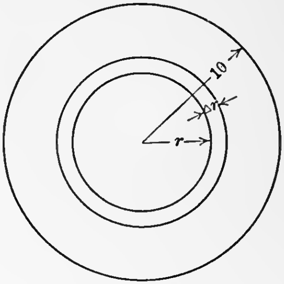

Fig. 106.

volume of such a ring is $2 \cdot 2\pi r \cdot dr$, and its mass is $0.04 \cdot 4\pi r \cdot dr$. The distance of this mass from the axis is r. Hence

$$I = 0.16\, \pi \int_0^{10} r^3\, dr.$$

Also find the radius of gyration.

144. Transfer of Axes. Theorem. *The moment of inertia of a body about any axis is equal to its moment of inertia about a parallel axis through the centroid, increased by the product of the mass by the square of the distance between the axes.*

Let AB, Fig. 107, be the axis about which the moment of inertia is desired. Choose a system of rectangular axes such that the origin, O, is at the centroid, such that the Z-axis is parallel to AB, and such that the ZY-plane contains the line AB. Consider an element of mass, dm, at P. The moment of inertia of the body about AB is then

$$I = \int (PB)^2\,dm = \int \left[(y-d)^2 + x^2 \right] dm,$$

or

$$I = \int (x^2 + y^2)\,dm - 2d \int y\,dm + d^2 \int dm. \qquad (1)$$

The first term of the right-hand side of (1) is the moment of inertia, I_o, of the body about the Z-axis, an axis through the centroid. The second term, $\int y\,dm$, is the moment of the body with respect to the XZ-plane, a plane passing through its centroid.

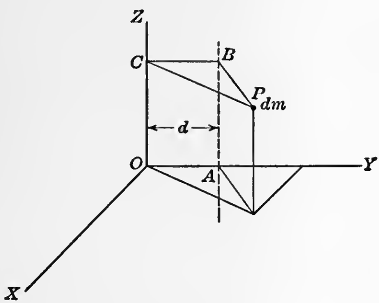

FIG. 107.

$$\bar{y} = \frac{\int y\,dm}{\int dm}.$$

Since $\bar{y} = 0$, $\int y\,dm = 0$. The last term, $d^2 \int dm$, is d^2M, where M is the mass of the body. Hence

$$\mathbf{I = I_o + Md^2}.$$

145. Moment of Inertia of an Area. We have spoken of the center of gravity of an area quite apart from any idea of mass and have stated that this is a useful conception in the study of mechanics. In the same way the solution of some problems in mechanics requires the moment of inertia of an area quite apart from any idea of mass.

The moment of inertia of a plane area about an axis through the origin and perpendicular to its plane is defined by the integral.

$$\mathbf{I} = \iint (\mathbf{x}^2 + \mathbf{y}^2)\, \mathbf{dy\, dx} = \iint (\mathbf{x}^2 + \mathbf{y}^2)\, \mathbf{dx\, dy}.$$

The theorem on the transfer of axes holds in this case if the word "area" is substituted for the word "mass."

Exercises

Find the moment of inertia of the following:

1. A rectangle of sides a and b about one corner. (See Fig. 104.) About the centroid. About one base. About a line parallel to one base and passing through the centroid.

2. A right triangle, legs a and b, about one of the legs. About a line through the centroid parallel to this leg.

3. The area of a circle about an axis through a point on its circumference and perpendicular to its plane. See Illustration 2, §143.

4. The area of a circle about a diameter.

HINT.

$$I = \int_{-a}^{a} y^2\, 2x\, dy.$$

5. The area of a circle about a tangent line.

6. The area between $y = x^2$ and $y^2 = x$ about an axis through the origin perpendicular to the XY-plane.

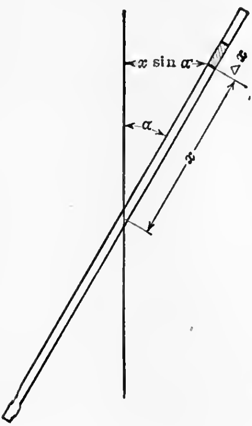

FIG. 108.

7. A uniform bar of length L and linear density ρ about an axis through one end perpendicular to the bar. Find I about a parallel axis through the middle point of the bar.

8. A bar of length L, whose density is proportional to the distance from one end, about an axis perpendicular to the bar through the end of least density.

9. A slender uniform rod, Fig. 108, about a line through its middle point and making an angle α with the rod.

Ans. $I = \frac{1}{12} mL^2 \sin^2 \alpha$, where m is the mass and L is the length of the rod.

HINT. Denote by ρ the linear density. Then

$$I = \rho \int x^2 \sin^2\alpha \, dx, \text{ with proper limits.}$$

10. The rod of Exercise 9 about a parallel axis through one end.

11. A wire bent into the form of a circular arc, Fig. 109, about the origin. Also find the moments of inertia, I_x and I_y about the X- and Y-axes, respectively.

$$I = \int_{-\alpha}^{\alpha} r^2 \, r \, d\theta;$$

$$I_x = \int_{-\alpha}^{\alpha} r^2 \sin^2\theta \, r \, d\theta;$$

$$I_y = \int_{-\alpha}^{\alpha} r^2 \cos^2\theta \, r \, d\theta.$$

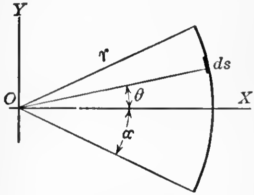

FIG. 109.

12. A triangle of base b and altitude h about an axis through the vertex parallel to the base. Divide the area into strips parallel to the base and of width dx. The axis of x is drawn from the vertex perpendicular to the base.

$$I = \int_0^h \frac{x^2 bx \, dx}{h}.$$

13. A triangle about a line through the center of gravity parallel to the base. Use the result of Exercise 12.

14. The area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ about the major axis. (Use single integration.) About the minor axis. About the origin.

146. Moment of Inertia: Polar Coördinates. The moment of inertia of the element $r \, dr \, d\theta$ about an axis through the origin is $r^2 r \, dr \, d\theta$. Hence the moment of inertia of an area is

$$I = \int\int r^3 \, dr \, d\theta,$$

with proper limits. If the moment of inertia of a plate is required, $r^3 \, dr \, d\theta$ is to be multiplied by ρ, the density per unit area.

Exercises

Find the moment of inertia of the following:

1. The area of the cardioid $\rho = a(1 + \cos\theta)$ about an axis through the origin perpendicular to the plane of the cardioid. About the initial line.

2. The area of one loop of $\rho = a \cos 2\theta$ about the initial line.

3. A circular sector of central angle 2α about the radius of symmetry.

4. The arc of the sector of Exercise 3 about the radius of symmetry.

5. The area of Exercise 3, about an axis through the center of the circle and perpendicular to the plane of the sector. About a parallel axis through the centroid.

147. Moment of Inertia of a Solid.—We wish to find the moment of inertia of the solid of Fig. 86, about the Z-axis. The moment of inertia of the element of mass, $\rho\, dz\, dy\, dx$, about the Z-axis is equal to $\rho(x^2 + y^2)\, dz\, dy\, dx$. The total moment of inertia of the solid about this axis is the integral of this element throughout the solid. Hence,

$$I_z = \int_a^b \int_{y_1}^{y_2} \int_{z=0}^{z=f(x,y)} \rho(x^2 + y^2)\, dz\, dy\, dx. \tag{1}$$

Similarly

$$I_x = \int_a^b \int_{y_1}^{y_2} \int_{z=0}^{z=f(x,y)} \rho(y^2 + z^2)\, dz\, dy\, dx, \tag{2}$$

$$I_y = \int_a^b \int_{y_1}^{y_2} \int_{z=0}^{z=f(x,y)} \rho(z^2 + x^2)\, dz\, dy\, dx. \tag{3}$$

If the solid be regarded as a geometrical volume of density 1, the ρ's disappear, and the formulas (1), (2), and (3) can be written

$$I_z = \int\int\int (x^2 + y^2)\, dz\, dy\, dx, \tag{4}$$

$$I_x = \int\int\int (y^2 + z^2)\, dz\, dy\, dx, \tag{5}$$

$$I_y = \int\int\int (z^2 + x^2)\, dz\, dy\, dx. \tag{6}$$

Let

$$I_{yz} = \int\int\int \rho x^2\, dz\, dy\, dx, \tag{7}$$

$$I_{zx} = \int\int\int \rho y^2\, dz\, dy\, dx, \tag{8}$$

$$I_{xy} = \int\int\int \rho z^2\, dz\, dy\, dx. \tag{9}$$

The quantities (7), (8), and (9) will be called the moments of inertia of the solid with respect to the YZ-plane, the XZ-plane, and the XY-plane, respectively. They are the integrals of the product of the element of mass by the square of its distance from the respective planes. They can very frequently be found by a single integration by taking as element a plane lamina between two planes parallel to the plane with respect to which the moment is computed. If this is the case the moment of inertia about the coördinate axes can easily be found by noting that from the equations (1), (2), (3); and (7), (8), (9):

$$\mathbf{I_z} = \mathbf{I_{yz}} + \mathbf{I_{xz}},$$
$$\mathbf{I_x} = \mathbf{I_{xz}} + \mathbf{I_{xy}},$$
$$\mathbf{I_y} = \mathbf{I_{xy}} + \mathbf{I_{yz}}.$$

That is, *the moment of inertia about the Z-axis is equal to the sum of the moments of inertia with respect to the YZ- and XZ-planes, and so on.*

In general, *the moment of inertia of a body about an axis is equal to the sum of its moments of inertia with respect to two perpendicular planes which intersect in that axis.*

In the same way it follows from the formula for the moment of inertia of an area, $I = \int\int (x^2 + y^2)\, dy\, dx$, that *the sum of the moments of inertia of an area (or a plate) about two perpendicular axes is equal to its moment of inertia about an axis perpendicular to the plane of these axes through their point of intersection.*

Illustration 1. Find the moment of inertia of the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$ about each of its axes.

First Method.

$$I_x = 8\int_0^a \int_0^{b\sqrt{1 - \frac{x^2}{a^2}}} \int_0^{c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}} (y^2 + z^2)\, dz\, dy\, dx.$$

Carry out the integration far enough to see that it is not simple and then note the relative simplicity of the

Second Method. Compute I_{xz}, the moment of inertia with respect to the XZ-plane. Take as element of integration the elliptical plate cut out by the planes $y = y$ and $y = y + \Delta y$. The equation of the intersection of the ellipsoid and the plane $y = y$ is

$$\frac{x^2}{a^2\left(1 - \frac{y^2}{b^2}\right)} + \frac{z^2}{c^2\left(1 - \frac{y^2}{b^2}\right)} = 1.$$

Now the area of an ellipse is π times the product of its semi-major and semi-minor axes. Hence the area of this ellipse is

$$\pi a\sqrt{1-\frac{y^2}{b^2}}\,c\sqrt{1-\frac{y^2}{b^2}} = \pi ac\left(1-\frac{y^2}{b^2}\right).$$

The volume of the elliptical plate in question is

$$\pi ac\left(1-\frac{y^2}{b^2}\right)dy$$

and its moment of inertia with respect to the XZ-plane is

$$\pi acy^2\left(1-\frac{y^2}{b^2}\right)dy.$$

The total moment of inertia of the ellipsoid with respect to this plane is then

$$\begin{aligned} I_{xz} &= \pi ac\int_{-b}^{b} y^2\left(1-\frac{y^2}{b^2}\right)dy \\ &= \pi ac\left(\frac{y^3}{3}-\frac{y^5}{5b^2}\right)\Bigg|_{-b}^{+b} \\ &= 2\pi ac\left(\frac{b^3}{3}-\frac{b^3}{5}\right) = \frac{4\pi ab^3c}{15} \end{aligned}$$

I_{xy} can be written down at once as

$$I_{xy} = \frac{4\pi abc^3}{15}.$$

Then

$$I_x = I_{xz} + I_{xy} = \frac{4\pi abc}{15}(b^2+c^2).$$

We can write down at once by interchanging letters:

$$I_y = \frac{4\pi abc}{15}(a^2+c^2).$$

$$I_z = \frac{4\pi abc}{15}(a^2+b^2).$$

Illustration 2. Find the moment of inertia of a right circular cone about a line through its vertex perpendicular to its axis, if the radius of the base is b and the altitude is h. Choose the vertex as origin and the axis of the cone as axis of x. Consider the plate of radius $\frac{bx}{h}$ cut out by the planes $x = x$ and $x = x + dx$. Its moment of inertia about a diameter parallel to the axis of rotation through the vertex is equal to $\frac{\pi\, b^4 x^4\, dx}{4\, h^4}$. (See expression for I of a circular plate about a diameter, Exercise 4, §**145**). Then the moment of inertia of this plate about the axis of rotation through the vertex is equal to this moment of inertia about an axis through the centroid increased by its volume (mass if density = 1) multiplied by the square of the distance between this axis and the parallel axis through the vertex (see §**144**) *i.e.*, to

$$\frac{\pi b^4 x^4 dx}{4h^4} + \frac{\pi b^2 x^2\, dx}{h^2} x^2.$$

And the total moment of inertia of the cone about the axis through the vertex is equal to the integral of this moment of inertia from $x = 0$ to $x = h$. That is

$$I = \int_0^h \left[\frac{\pi b^4 x^4}{4h^4} + \frac{\pi b^2 x^4}{h^2}\right] dx$$

$$= \frac{\pi b^4 h}{20} + \frac{\pi b^2 h^3}{5} = \frac{\pi b^2 h}{20}(b^2 + 4h^2).$$

Exercises

Find the moment of inertia of:

1. The cone of Illustration 2, about a parallel axis through the centroid of the cone. About a diameter of the base.

2. A right circular cylinder, the radius of whose base is r, and whose altitude is h, about a diameter of one base. About a parallel axis through the centroid.

3. A rectangular parallelopiped with edges a, b, and c, about an axis through the centroid parallel to one edge.

4. A right circular cylinder about its axis.

5. A hollow right circular cylinder of outer radius R, inner radius r, and altitude h, about its central axis. About a diameter of one base. About a diameter of the plane section through the centroid perpendicular to the axis.

6. A right rectangular pyramid of base $a \times b$ and of altitude h, about an axis through the centroid parallel to the edge a. About an axis through the vertex and the center of gravity.

Ans. $I_1 = \frac{abh}{60}(b^2 + \frac{3}{4}h^2)$. $I_2 = \frac{abh}{60}(a^2 + b^2)$.

7. A frustum of a right cone about its axis if the radius of the large base is R, that of the small base is r, and the altitude is h.

$$Ans. \quad I = \tfrac{1}{10}\pi h \frac{R^5 - r^5}{R - r}.$$

8. A hollow sphere about a diameter, if the outer radius is R and the inner radius is r.

9. A paraboloid of revolution, the radius of whose base is r and whose height is h, about the axis of revolution.

$$Ans. \quad I = \tfrac{1}{6}\pi r^4 h.$$

10. The anchor ring generated by revolving the circle $[x - R]^2 + y^2 = r^2$ about the Y-axis.

$$Ans. \quad I_x = \pi^2 R r^2 (R^2 + \tfrac{5}{4}r^2). \quad I_y = 2\pi^2 R r^2 (R^2 + \tfrac{3}{4}r^2).$$

11. A right circular cone about its axis.

12. A right elliptical cylinder of height L, and having the semi-major and semi-minor axes of its cross section equal to a and b, respectively, about an axis through the centroid parallel to b.

13. A quadrant of a circular plate about one of its bounding radii.

14. An equilateral triangle of side $2a$, about a median. About a line through a vertex perpendicular to one of the sides through that vertex.

CHAPTER XVI

CURVATURE. EVOLUTES. ENVELOPES

148. Curvature. Let PT and QT', Fig. 110, be tangents drawn to the curve APQ at the points P and Q, respectively. Denote the length of the arc PQ by Δs and the angles of inclination of PT and QT' to the positive direction of the X-axis by τ and $\tau + \Delta\tau$, respectively. $\Delta\tau$ gives a rough measure of the deviation from a straight line, of that portion of the arc of the curve between the points P and Q. The sharper the bending of the curve between the points P and Q the greater is $\Delta\tau$ for equal values of Δs. The *average curvature* of the curve between the points P and Q is defined by the equation

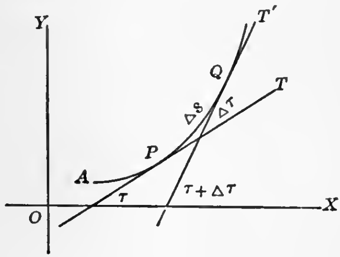

FIG. 110.

$$\textit{Average Curvature} = \frac{\Delta\tau}{\Delta s}. \tag{1}$$

The average curvature of a curve between two points P and Q is the average change between these points, per unit length of arc, of the inclination to the X-axis of the tangent line to the curve. Or, more briefly, the average curvature is the average change per unit length of arc, in the inclination of the tangent line.

The curvature at P is defined as the limit of the average curvature between the points Q and P as Q approaches P. On denoting the curvature by K, we have in accordance with the definition,

$$K = \lim_{\Delta s \doteq 0} \frac{\Delta\tau}{\Delta s} = \frac{d\tau}{ds}. \tag{2}$$

The curvature at a point P is then the rate of change at this point of the inclination of the tangent line per unit length of arc. The curvature is a measure of the amount of bending of a curve in the vicinity of a point.

149. Curvature of a Circle. It is clear that the average curvature of a circle, Fig. 111, is

$$\frac{\Delta\tau}{\Delta s} = \frac{\Delta\tau}{r\,\Delta\tau} = \frac{1}{r}.$$

Hence the average curvature is independent of Δs and consequently the curvature, the limit of the average curvature as Δs approaches zero, is

$$K = \frac{1}{r}. \qquad (1)$$

The curvature of a circle is constant and equal to the reciprocal of its radius.

150. Circle of Curvature. Radius of Curvature. Center of Curvature. Through any point P of a curve infinitely many circles can be drawn which have a common tangent with the curve at P and whose centers are on the concave side of the curve. Of these circles there is one whose curvature is equal to that of the curve at P, *i.e.*, one whose radius is equal to the reciprocal of the curvature at P. This circle is called the *circle of curvature* at the point P. The radius of this circle is called the *radius of curvature*, and its center the *center of curvature*, of the curve at the point P. The radius of curvature is denoted by R and, in accordance with (2), §148, its length is

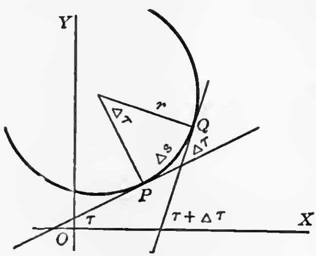

FIG. 111.

$$\mathbf{R} = \frac{\mathbf{1}}{\mathbf{K}} = \frac{\mathbf{ds}}{\mathbf{d\tau}}. \qquad (1)$$

151. Formulas for Curvature and Radius of Curvature: Rectangular Coördinates. For obtaining the curvature at any point on the curve $y = f(x)$, we shall now develop a formula involving the first and second derivatives of y with respect to x. The above formula for curvature K can be written

$$K = \frac{\frac{d\tau}{dx}}{\frac{ds}{dx}} = \frac{\frac{d\tau}{dx}}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}. \tag{1}$$

Since

$$\tau = \tan^{-1}\frac{dy}{dx},$$

$$\frac{d\tau}{dx} = \frac{\frac{d^2y}{dx^2}}{1 + \left(\frac{dy}{dx}\right)^2}.$$

Consequently

$$\mathbf{K} = \frac{\frac{d^2y}{dx^2}}{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}, \tag{2}$$

and, by (1), §**150**,

$$\mathbf{R} = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}. \tag{3}$$

We shall understand by K and R the numerical values of the right-hand members of (2) and (3), respectively, since we shall not be concerned with the algebraic signs of K and R.

Illustration. Find the curvature of $y = x^2$.

$$\frac{dy}{dx} = 2x, \quad \frac{d^2y}{dx^2} = 2.$$

Substitution in formula (2) gives

$$K = \frac{2}{(1 + 4x^2)^{\frac{3}{2}}}.$$

From this expression it is seen that the maximum curvature occurs when x is zero, and that the curvature decreases as x

increases in numerical value. When $x = 0$, $K = 2$. When $x = \pm 1$, $K = \frac{2\sqrt{5}}{25}$.

Exercises

Find the curvature and radius of curvature of each of the curves:

1. $y = 2x - x^2$.
2. $y = x^{\frac{8}{2}}$.
3. $y = \frac{1}{x}$.
4. $y = x^2 - x^3$.
5. $y = \frac{3}{x^2}$.
6. $y = \sqrt{x}$.
7. $y = 3x^{\frac{1}{3}}$.
8. $y = x^{-\frac{3}{2}}$.
9. $y = \frac{1}{\sqrt{x}}$.

10. If $\rho = f(\theta)$ is the equation of a curve in polar coördinates, show that

$$\mathbf{K} = \frac{\rho^2 + 2\left[\frac{d\rho}{d\theta}\right]^2 - \rho\frac{d^2\rho}{d\theta^2}}{\left[\rho^2 + \left(\frac{d\rho}{d\theta}\right)^2\right]^{\frac{3}{2}}}.$$

Hint.

$$K = \frac{d\tau}{ds} = \frac{\frac{d\tau}{d\theta}}{\frac{ds}{d\theta}}.$$

$$\tau = \theta + \psi. \qquad \text{(See Fig. 72.)}$$

$$\frac{d\tau}{d\theta} = 1 + \frac{d\psi}{d\theta}.$$

Obtain $\frac{d\psi}{d\theta}$ from the relation

$$\psi = \tan^{-1}\frac{\rho}{\frac{d\rho}{d\theta}}.$$

$\frac{ds}{d\theta}$ is given in §98.

152. Curvature: Parametric Equations. If the equation of a curve is expressed in parametric form, $x = f(t)$, $y = F(t)$, the curvature can be found by differentiating x and y and substituting in (2), §151. t can be eliminated from the result if desired.

Illustration 1. If $x = t$ and $y = t^2$,

$$\frac{dy}{dx} = 2t, \text{ and } \frac{d^2y}{dx^2} = \frac{d(2t)}{dt}\frac{dt}{dx} = 2.$$

Hence

$$K = \frac{2}{(1+4t^2)^{\frac{3}{2}}} = \frac{2}{(1+4x^2)^{\frac{3}{2}}}.$$

Illustration 2. Find the curvature of the ellipse $x = a \cos t$, $y = b \sin t$.

$$\frac{dy}{dx} = -\frac{b}{a}\cot t.$$

$$\frac{d^2y}{dx^2} = -\frac{b}{a}\frac{d}{dt}\cot t\frac{dt}{dx} = \frac{b}{a}\csc^2 t\left[-\frac{1}{a\sin t}\right] = -\frac{b}{a^2}\csc^3 t.$$

$$K = \frac{-\frac{b}{a^2}\csc^3 t}{\left[1+\frac{b^2}{a^2}\cot^2 t\right]^{\frac{3}{2}}} = \frac{-ab}{(a^2\sin^2 t + b^2\cos^2 t)^{\frac{3}{2}}} = \frac{-ab}{\left[\frac{a^2}{b^2}y^2 + \frac{b^2}{a^2}x^2\right]^{\frac{3}{2}}}$$

$$= -\frac{a^4b^4}{(b^4x^2 + a^4y^2)^{\frac{3}{2}}}.$$

Exercises

1. Find the curvature of the curve $x = a\cosh t$, $y = a\sinh t$.

2. Find the curvature of the curve $x = a(t - \sin t)$, $y = a(1 - \cos t)$.

153. Approximate Formula for the Curvature. If a curve deviates but little from a horizontal straight line, $\frac{dy}{dx}$ is small and $\left(\frac{dy}{dx}\right)^2$ in formula (2), **§151,** is very small compared with 1. Hence the denominator differs very little from 1 and the formula becomes approximately

$$K = \frac{d^2y}{dx^2}. \tag{1}$$

This approximate formula for K is frequently used in mechanics in the study of the flexure of beams. The slope of the elastic curve of a beam is so small that $\frac{d^2y}{dx^2}$ can be used for the curvature without appreciable error.

The approximate formula for the radius of curvature R is

$$R = \frac{1}{\frac{d^2y}{dx^2}}. \tag{2}$$

154. Center of Curvature. Evolute. Formulas will now be obtained for the coördinates of the center of curvature of a curve corresponding to any point P. Let the coördinates of P be x and y. Denote by α and β the coördinates of the center of curvature of the curve at this point. There are four cases to be considered. See Fig. 112, *a*, *b*, *c*, *d*.

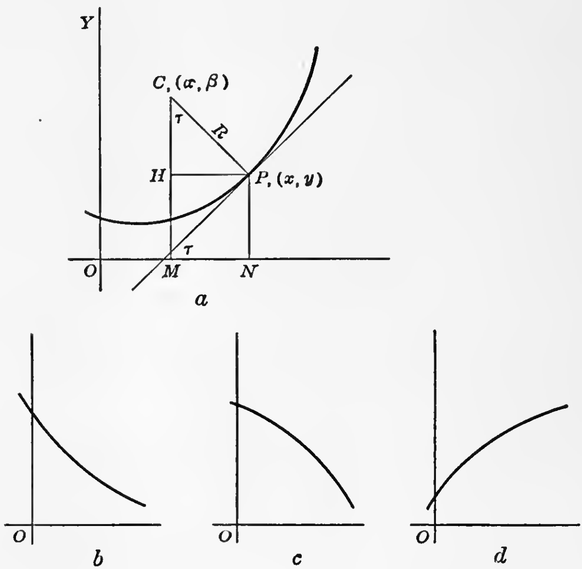

FIG. 112.

In Fig. 112, *a*,

$$\alpha = OM = ON - HP = x - R\sin\tau,$$
$$\beta = MC = NP + HC = y + R\cos\tau.$$

Since

$$\tan\tau = \frac{dy}{dx},$$

$$\cos\tau = \frac{1}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}; \quad \sin\tau = \frac{\frac{dy}{dx}}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}.$$

Consequently,

$$\alpha = x - \frac{dy}{dx}\,\frac{1 + \left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}, \tag{1}$$

and

$$\beta = y + \frac{1 + \left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}. \tag{2}$$

The student can show that, since $\frac{dy}{dx}$ is negative for a descending curve and positive for an ascending curve, and since $\frac{d^2y}{dx^2}$ is positive when a curve is concave upward and negative when a curve is concave downward, formulas (1) and (2) hold for the three curves represented in Fig. 112, *b*, *c*, *d*.

Illustration. Find the coördinates of the center of curvature corresponding to any point on the curve $y = \pm 2\sqrt{x}$. Only the positive sign will be used. If the negative sign is used it will only be necessary to change the sign of β.

$$\frac{dy}{dx} = \frac{1}{\sqrt{x}}$$

$$\frac{d^2y}{dx^2} = -\frac{1}{2x^{\frac{3}{2}}}.$$

$$\alpha = x + \frac{1}{\sqrt{x}}\,\frac{1 + \frac{1}{x}}{\frac{1}{2x^{\frac{3}{2}}}} = 3x + 2.$$

$$\beta = y - \frac{1 + \frac{1}{x}}{\frac{1}{2x^{\frac{3}{2}}}} = y - 2\sqrt{x}(x + 1) = y - y\left[\frac{y^2}{4} + 1\right] = -\frac{y^3}{4}.$$

The equation of the locus of the center of curvature is obtained by eliminating x and y from the equations for α and β and the equation of the original curve. Thus

$$x = \frac{\alpha - 2}{3};\ \ y = -(4\beta)^{\frac{1}{3}}.$$

Substituting in $y^2 = 4x$, we obtain

$$\beta^2 = \tfrac{4}{27}(\alpha - 2)^3,$$

the equation of the locus of the center of curvature. This is the equation of a semi-cubical parabola whose vertex is at the point (2, 0).

The locus of the center of curvature corresponding to points on a curve is called the evolute of that curve. Its equation is easily obtained in many cases by eliminating x and y from equations (1) and (2) and the equation of the original curve. Otherwise (1) and (2) constitute its parametric equations, α and β being expressed in terms of the parameters x and y.

Exercises

1. Find the evolute of $y = 4x^2$.

2. Find the evolute of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

HINT It will be found that

$$\alpha = \frac{(a^2 - b^2)x^3}{a^4}; \quad \beta = -\frac{(a^2 - b^2)y^3}{b^4}.$$

Whence

$$x = \left[\frac{a^4\alpha}{a^2 - b^2}\right]^{\frac{1}{3}}; \quad y = -\left[\frac{b^4\beta}{a^2 - b^2}\right]^{\frac{1}{3}}.$$

Elimination gives

$$(a\alpha)^{\frac{2}{3}} + (b\beta)^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}}.$$

3. Find the parametric equations of the evolute of the cycloid,

$$x = a(\theta - \sin\theta),$$
$$y = a(1 - \cos\theta).$$

Ans. $\alpha = a(\theta + \sin\theta)$, $\beta = -a(1 - \cos\theta)$. Show that the evolute is an equal cycloid with its cusp at the point $(-\pi a, -2a)$.

4. Find the equation of the evolute of

$$x = a(\cos\theta + \theta\sin\theta),$$
$$y = a(\sin\theta - \theta\cos\theta).$$

Ans. $\alpha = a\cos\theta$, $\beta = a\sin\theta$. Discuss.

155. Envelopes. If the equation of a curve contains a constant c, infinitely many curves can be obtained by assigning different values to c. Thus

$$(x - c)^2 + y^2 = a^2 \qquad (1)$$

is the equation of a circle of radius a whose center is at $(c, 0)$. By assigning different values to c we get a system of equal circles whose centers lie on the X-axis. A constant such as c, to which infinitely many values are assigned, is called a *parameter.* A constant such as a, which is thought of as taking on only one value during the whole discussion, is called an *absolute constant.* We say that equation (1) represents a *family of circles* or a *system of circles* corresponding to the parameter c.

The general equation of a family of curves depending upon a single parameter can be written in the form,

$$f(x, y, c) = 0. \tag{2}$$

Exercises

State the family of curves represented by the following equations containing a parameter:

1. $y = x^2 + c,$ c being the parameter.
2. $y = mx + b,$ b being the parameter.
3. $y = mx + b,$ m being the parameter.
4. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$ a being the parameter.
5. $y^2 = m(x + m),$ m being the parameter.
6. $x^2 + y^2 = a^2,$ a being the parameter.

Consider again the family of circles (1). Two circles of the family corresponding to the values, c and $c + \Delta c$, of the parameter intersect in the points Q and Q', Fig. 113. We seek the limiting positions of these points of intersection as Δc approaches zero. Clearly they are the points P and P', respectively, on the lines $y = \pm a$. Such a limiting position of the point of intersection of two circles of the family is called the point of intersection of two "consecutive" circles of the family. In general, the limiting position of the point of intersection of two curves, $f(x, y, c)$, $f(x, y, c + \Delta c)$, of a family, as Δc

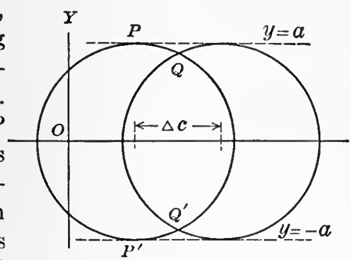

FIG. 113.

approaches zero, is called the point of intersection of "consecutive" curves of the family.

In the case of the family of circles (1) the locus of the points of intersection of "consecutive" circles is the pair of straight lines $y = \pm a$. This locus is called the *envelope* of the family of circles. In general, the *envelope of a family of curves* depending upon one parameter is the locus of the points of intersection of "consecutive" curves of the family. It will be shown in a later chapter that the envelope of a family of curves is tangent to every curve of the family.

Exercise

Draw a number of lines of the family

$$x \cos \theta + y \sin \theta = p,$$

where θ is the parameter, and sketch the envelope.

A general method of obtaining the envelope of a family of curves will now be given.

The equation of a curve of the family is

$$f(x, y, c) = 0, \tag{3}$$

where c has any fixed value. The envelope is the locus of the limiting position of the point of intersection of any curve (3) of the family with a neighboring curve, such as

$$f(x, y, c + \Delta c) = 0, \tag{4}$$

as the second curve is made to approach the first by letting Δc approach zero. The coördinates of the points of intersection of the curves representing equations (3) and (4) satisfy

$$f(x, y, c + \Delta c) - f(x, y, c) = 0. \tag{5}$$

Then they satisfy

$$\frac{f(x, y, c + \Delta c) - f(x, y, c)}{\Delta c} = 0, \tag{6}$$

since Δc does not depend on either x or y. Then the coördinates of the limiting positions of these points of intersection satisfy

$$\lim_{\Delta c \doteq 0} \frac{f(x, y, c + \Delta c) - f(x, y, c)}{\Delta c} = 0.$$

The first member of this equation is the derivative of $f(x, y, c)$ with respect to c. It may be written in the form,

$$\frac{\partial f(x, y, c)}{\partial c} = 0. \tag{7}$$

The differentiation is performed with respect to c, x and y being treated as constants. The point of intersection also lies on (3). Hence the equation of its locus is obtained by eliminating c between (3) and (7).

Illustration 1. Find the equation of the envelope of the family of circles, $(x - c)^2 + y^2 = a^2$, c being the parameter.

The equation of the curve written in the form $f(x, y, c) = 0$ is

$$(x - c)^2 + y^2 - a^2 = 0. \tag{I}$$

Differentiating with respect to c,

$$-2(x - c) = 0. \tag{II}$$

The elimination of c between (I) and (II) gives

$$y^2 = a^2,$$

or

$$y = \pm a,$$

as the envelope.

Illustration 2. Find the equation of the envelope of the family of lines, $x \cos \theta + y \sin \theta = p$, θ being the parameter.

On differentiating the first member of

$$x \cos \theta + y \sin \theta - p = 0 \tag{I}$$

with respect to θ we obtain

$$-x \sin \theta + y \cos \theta = 0. \tag{II}$$

The result of eliminating θ between (I) and (II) is

$$x^2 + y^2 = p^2,$$

a circle of radius p about the origin as center.

Exercises

1. Find the envelope of the family of straight lines $y = mx + \dfrac{p}{m}$, where m is the parameter. Draw figure.

2. Find the envelope of the family of lines $y = mx + a\sqrt{1 + m^2}$, where m is the parameter. Draw figure.

3. Find the envelope of the family of parabolas $y^2 = c(x - c)$, c being the parameter.

4. Find the envelope of the family of lines of constant length whose extremities lie in two perpendicular lines.

5. Find the envelope of $y = px - p^2$, p being the parameter. Draw figure.

6. Find the envelope of the family of curves $(x - c)^2 + y^2 = 4pc$, c being the parameter. Draw figure.

7. The equation of the path of a projectile fired with an initial velocity v_0 which makes an angle α with the horizontal, is

$$y = x \tan \alpha - \frac{gx^2}{2v_0^2 \cos^2 \alpha}.$$

Find the envelope of the family of paths obtained by considering α a parameter.

Ans. $$y = \frac{v_0^2}{2g} - \frac{gx^2}{2v_0^2}.$$

8. The equation of the normal to $y^2 = 4x$ at the point P, whose cöordinates are x_1 and y_1, is

$$y - y_1 = -\frac{y_1}{2}(x - x_1).$$

Since $y_1^2 = 4x_1$, this may be written

$$y_1 x + 2y - \frac{y_1^3}{4} - 2y_1 = 0.$$

Find the equation of the envelope of the normals as P moves along the curve.

Hint. On differentiating with respect to the parameter y_1 we obtain

$$y_1 = \pm 2\frac{\sqrt{x - 2}}{\sqrt{3}}.$$

On substituting this value of y_1 in the equation of the normal and squaring we obtain

$$y^2 = \frac{4(x - 2)^3}{27}.$$

This is the evolute of the parabola as we have seen in **§154.**

156. The Evolute as the Envelope of the Normals. In Exercise 8 of **§155** it was seen that the evolute of a parabola is the envelope of its normals. This is true for any curve. The result is fairly evident from the examination of the curves of the exercises of

§155 and their evolutes. It will be shown that the normals to a curve are tangent to its evolute.

The parametric equations of the evolute are

$$\alpha = x - R \sin \tau, \qquad (1)$$

$$\beta = y + R \cos \tau. \qquad (2)$$

On differentiating with respect to the variable s, which is permissible since x, y, R, and τ are all functions of s, we obtain

$$\frac{d\alpha}{ds} = \frac{dx}{ds} - \frac{dR}{ds} \sin \tau - R \cos \tau \frac{d\tau}{ds},$$

$$\frac{d\beta}{ds} = \frac{dy}{ds} + \frac{dR}{ds} \cos \tau - R \sin \tau \frac{d\tau}{ds}.$$

Now

$$\frac{dx}{ds} = \cos \tau,$$

$$\frac{dy}{ds} = \sin \tau,$$

$$\frac{d\tau}{ds} = \frac{1}{R}.$$

Then the foregoing equations become

$$\frac{d\alpha}{ds} = - \frac{dR}{ds} \sin \tau,$$

$$\frac{d\beta}{ds} = \frac{dR}{ds} \cos \tau.$$

Hence the slope of the tangent to the evolute is

$$\frac{d\beta}{d\alpha} = - \cot \tau \qquad (3)$$

Therefore the tangent to the evolute is parallel to the normal to the curve at the point (x, y) to which (α, β) corresponds. But the normal to the curve at (x, y) passes through (α, β). Hence it is tangent to the evolute at this point.

It can also be shown that if C_1 and C_2, Fig. 114, are the centers of curvature corresponding to the points P_1 and P_2, the length of the arc C_1C_2 of the evolute is equal to the difference in the lengths of the radii of curvature, R_1 and R_2. For, from the above values of $d\alpha$ and $d\beta$ it follows that

$$\sqrt{d\alpha^2 + d\beta^2} = dR.$$

But $\sqrt{d\alpha^2 + d\beta^2}$ is the differential of the arc of the evolute. Call it $d\sigma$. Then $d\sigma = dR$, and hence on integrating $\sigma = R + C$. σ and R are functions of s, the arc of the given curve. Then corresponding to a change $\Delta s(= \text{arc } P_1P_2)$ in s, σ and R will take on the increments $\Delta\sigma$ and ΔR which are equal by the foregoing equation. But $\Delta\sigma = \text{arc } C_1C_2$, and $\Delta R = R_2 - R_1$. Hence arc C_1C_2 equals $R_2 - R_1$.

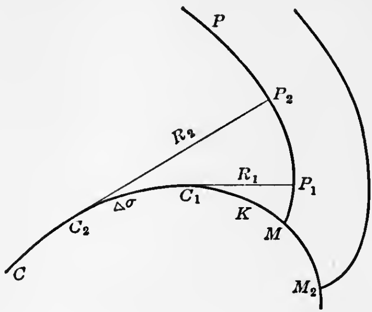

FIG. 114.

157. Involutes. In Fig. 114, suppose that one end of a string is fastened at C and that it is stretched along the curve CC_2C_1KM. If now the string be unwound, always being kept taut, the point M will, in accordance with the properties of the evolute, trace out the curve MP_1P_2P. This curve is called the *involute* of the curve KC_1C_2C. If longer or shorter lengths of string, such as CKM_2 be used, other involutes will be traced. In fact to a given curve there correspond infinitely many involutes. The given curve is the evolute of each of these involutes. We see that while a given curve has but one evolute it has infinitely many involutes.

In Exercise 4, **§154,** the circle $x = a \cos \theta$, $y = a \sin \theta$ was found as the evolute of the curve $x = a(\cos \theta + \theta \sin \theta)$, $y = a(\sin \theta - \theta \cos \theta)$. Then the latter curve is an involute of the circle. The student will draw a figure showing a position of the string as it would be unwound to generate the involute and indicate the angle θ.

CHAPTER XVII

SERIES. TAYLOR'S AND MACLAURIN'S THEOREMS. INDETERMINATE FORMS.

158. Infinite Series. The expression

$$u_1 + u_2 + u_3 + \cdots + u_n + \cdots \qquad (1)$$

where $u_1, u_2, u_3, \cdots, u_n, \cdots$ is an unlimited succession of numbers, is called an *infinite series.*

Let s_n denote the sum of the first n terms of the infinite series (1). Thus

$$s_n = u_1 + u_2 + u_3 + \cdots + u_n. \qquad (2)$$

If, as n increases without limit, s_n approaches a limit s, this limit is called the sum of the infinite series, and the series is said to be *convergent.*

Illustration 1. In Illustration (1), **§21**, AB, Fig. 21, is a line 2 units long. The lengths Ax_1, x_1x_2, x_2x_3, x_3x_4, $\cdots$ $x_{n-1}x_n$, $\cdots$ are $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots, \frac{1}{2^{n-1}}, \cdots$, respectively. For this series

$$s_n = 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \cdots + \frac{1}{2^{n-1}}. \qquad (3)$$

The limit of this sum as n increases without limit is 2, as the figure shows. Or, we may write,

$$1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \cdots + \frac{1}{2^{n-1}} + \cdots = 2. \qquad (4)$$

Illustration 2. The sum of the geometrical progression

$$1 + r + r^2 + r^3 + r^4 + \cdots + r^n \qquad (5)$$

does not approach a limit if $|r| \overline{\gtrless} 1$, but if $|r| < 1$ it approaches the limit $\frac{1}{1-r}$, when n becomes infinite.

The infinite series (1) is said to be *divergent* or to diverge if, as n increases without limit, s_n does not approach a limit.

Thus the series

$$1 + 2 + 3 + 4 + \cdots + n + \cdots,$$
$$1 - 1 + 1 - 1 + 1 - 1 + 1 - \cdots,$$
$$1 + R + R^2 + R^3 + R^4 + \cdots + R^n + \cdots \qquad (R > 1)$$

are illustrations of divergent series.

If the terms of an infinite series are functions of a variable x and if the series is convergent for any range of values for x, the series defines a function of x for that range of values. Thus, if $|x| < 1$, the series

$$1 + x + x^2 + x^3 + \cdots + x^n + \cdots \qquad (7)$$

defines the function $\frac{1}{1-x}$. On the other hand, if the series is divergent it does not define a function of x. Thus, if $|x| > 1$, the series (7) is divergent and does not define the function $\frac{1}{1-x}$, or any other function.

It may happen that the sum of a few terms of an infinite series representing a function is a very close approximation to the value of the function. As an illustration take the infinite geometrical progression (7), which when $|x| < 1$ represents the function $\frac{1}{1-x}$. If the terms after x^{k-1} are neglected, the error is

$$x^k + x^{k+1} + \cdots + x^n + \cdots = \frac{x^k}{1-x}.$$

The error, $\frac{x^k}{1-x}$ is very small compared with the value of the function, $\frac{1}{1-x}$, and decreases as k increases; *i.e.*, a better and better approximation is obtained the greater the number of terms retained.

Another infinite series is obtained by expanding $(1 + x)^{\frac{3}{2}}$ by the binomial theorem,

$$(1 + x)^{\frac{3}{2}} = 1 + \tfrac{3}{2}x + \tfrac{3}{8}x^2 - \tfrac{1}{16}x^3 + \cdots \qquad (8)$$

This series can be shown to be convergent when $|x| < 1$ and divergent when $|x| > 1$.

Just as the function $\frac{1}{1-x}$ is represented to a high degree of approximation by the first few terms of the series (7) when $|x|$ is small, the function $(1 + x)^{\frac{3}{2}}$ is represented approximately by the first few terms of (8) when $|x|$ is small. In both cases the functions are represented approximately by polynomials. A method will be developed in the succeeding articles which will enable us to determine polynomial approximations to other functions, such as $\sin x$, $\tan x$, e^x.

An infinite series of the form

$$a_0 + a_1x + a_2x^2 + a_3x^3 + \dots + a_nx^n + \dots \quad (9)$$

is called a *power series in x*. One of the form

$$a_0 + a_1(x - a) + a_2(x - a)^2 + a_3(x - a)^3 + \cdots + a_n(x - a)^n + \cdots \quad (10)$$

is called a *power series in* $(x - a)$. The series (7) and (8) are power series in x representing the functions $\frac{1}{1 - x}$ and $(1 + x)^{\frac{3}{2}}$, respectively. In the succeeding articles power series will be obtained representing the functions $\sin x$, $\tan x$, e^x, etc.

159. Rolle's Theorem. Let $f(x)$ be a single-valued continuous function between $x = a$ and $x = b$, having a continuous first

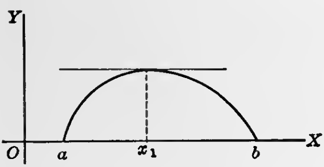

Fig. 115.

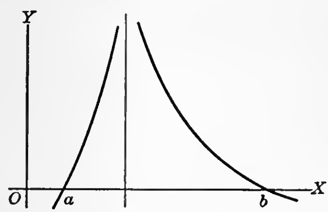

Fig. 116.

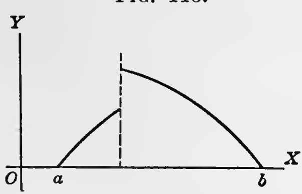

Fig. 117.

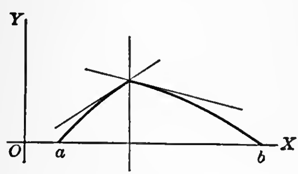

Fig. 118.

derivative, $f'(x)$, between the same limits. Further, let $f(a) = 0$ and $f(b) = 0$, *i.e.*, let the curve representing the function cross or touch the X-axis at $x = a$ and $x = b$. The curve may or may not cross or touch the X-axis at intermediate points. (See Fig. 115.) Since $f(x)$ is continuous it cannot have a vertical asymptote between $x = a$ and $x = b$ as shown in Fig. 116, nor can it have a finite discontinuity as shown in Fig. 117. Since $f'(x)$ is continuous between $x = a$ and $x = b$, the curve cannot change its direction abruptly between these limits, as shown in Figs. 118 and 119. Since cases such as are represented by Figs. 116, 117, 118, and 119 are excluded, the curve, Fig. 115, must have a horizontal tangent at some point $x = x_1$ between $x = a$ and $x = b$. Hence the

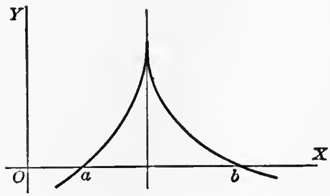

Fig. 119.

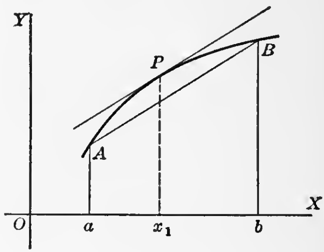

Fig. 120.

Theorem. *If $f(x)$ is a single-valued function from $x = a$ to $x = b$, and if $f(x)$ and $f'(x)$ are continuous between these limits, and further if $f(a) = 0$ and $f(b) = 0$, then $f'(x_1) = 0$, where $a < x_1 < b$.*

160. Law of the Mean. Let $f(x)$ be a single-valued function between $x = a$ and $x = b$. Further let $f(x)$ and $f'(x)$ be continuous between these limits, Fig. 120. It is then apparent from the figure that at some point P between A and B, the tangent line to the curve will be parallel to the secant line AB. Hence the

Theorem. *If $f(x)$ and $f'(x)$ are continuous between $x = a$ and $x = b$, then*

$$f'(x_1) = \frac{f(b) - f(a)}{b - a},$$

where $a < x_1 < b$, or

$$f(b) = f(a) + (b - a)f'(x_1). \tag{1}$$

An analytic proof of this law will also be given. Define a number S_1 by the equation

$f(b) = f(a) + (b - a)S_1$, or

$$f(b) - f(a) - (b - a)S_1 = 0. \tag{2}$$

It will be shown that $S_1 = f'(x_1)$, where $a < x_1 < b$. From the first member of (2) build up the function $\phi_1(x)$ by replacing a by x.

$$\phi_1(x) = f(b) - f(x) - (b - x)S_1. \tag{3}$$

Then

$$\phi'_1(x) = -f'(x) + S_1. \tag{4}$$

Since $f(x)$ and $f'(x)$ are continuous between $x = a$ and $x = b$, $\phi_1(x)$ and $\phi'_1(x)$ are continuous between the same limits. By (3) and (2), $\phi_1(a) = 0$, and by (3), $\phi_1(b) = 0$. Hence $\phi_1(x)$ satisfies the conditions of Rolle's Theorem and consequently

$$\phi'_1(x_1) = 0,$$

or

$$f'(x_1) - S_1 = 0,$$

or

$$S_1 = f'(x_1),$$

where $a < x_1 < b$. On substituting this value of S_1 in (2) we obtain

$$f(b) = f(a) + (b - a)f'(x_1),$$

which proves the theorem.

161. The Extended Law of the Mean. Let $f(x)$ be a function which with its first and second derivatives, $f'(x)$ and $f''(x)$, is continuous from $x = a$ to $x = b$. Define a number S_2 by the equation

$$f(b) = f(a) + (b - a)f'(a) + \frac{(b - a)^2}{2}S_2, \tag{1}$$

or

$$f(b) - f(a) - (b - a)f'(a) - \frac{(b - a)^2}{2}S_2 = 0.$$

From the first member of the latter equation, form the function $\phi_2(x)$ by replacing a by x:

$$\phi_2(x) = f(b) - f(x) - (b - x)f'(x) - \frac{(b - x)^2}{2}S_2. \tag{2}$$

Then

$$\begin{aligned}\phi'_2(x) &= -f'(x) - (b - x)f''(x) + f'(x) + (b - x)S_2\\ &= (b - x)\,[S_2 - f''(x)].\end{aligned}$$

Since $f(x)$, $f'(x)$, and $f''(x)$ are continuous, $\phi_2(x)$ and $\phi'_2(x)$ are continuous. Further by (2) and (1), $\phi_2(a) = 0$, and by (2),

$\phi_2(b) = 0$. Hence the conditions of Rolle's Theorem are satisfied, and

$$\phi'_2(x_2) = 0, \tag{3}$$

where $a < x_2 < b$.
Or

$$[b - x_2][S_2 - f''(x_2)] = 0,$$

or

$$S_2 = f''(x_2). \tag{4}$$

On substituting this value of S_2 in equation (1), we obtain

$$f(b) = f(a) + (b - a)f'(a) + \frac{(b-a)^2}{2} f''(x_2), \tag{5}$$

where $a < x_2 < b$.

162. Taylor's Theorem with the Remainder. Finally let $f(x)$ and its first n derivatives be continuous from $x = a$ to $x = b$. Define S_n by the equation

$$f(b) = f(a) + (b - a)f'(a) + \frac{(b-a)^2}{2!} f''(a) + \cdots + \frac{(b-a)^{n-1}}{(n-1)!} f^{(n-1)}(a) + \frac{(b-a)^n}{n!} S_n, \tag{1}$$

or

$$f(b) - f(a) - (b - a)f'(a) - \frac{(b-a)^2}{2!} f''(a) - \cdots - \frac{(b-a)^{n-1}}{(n-1)!} f^{(n-1)}(a) - \frac{(b-a)^n}{n!} S_n = 0.$$

Form the function $\phi_n(x)$ by replacing a in the last equation by x.

$$\phi_n(x) = f(b) - f(x) - (b - x)f'(x) - \frac{(b-x)^2}{2!} f''(x) - \cdots - \frac{(b-x)^{n-1}}{(n-1)!} f^{(n-1)}(x) - \frac{(b-x)^n}{n!} S_n. \tag{2}$$

Then

$$\phi'_n(x) = -\frac{(b-x)^{n-1}}{(n-1)!} f^{(n)}(x) + \frac{(b-x)^{n-1}}{(n-1)!} S_n. \tag{3}$$

Since $f(x), f'(x), \cdots, f^{(n)}(x)$ are continuous, $\phi_n(x)$ and $\phi'_n(x)$ are continuous. By (2) and (1) $\phi_n(a) = 0$, and by (2) $\phi_n(b) = 0$. Hence the conditions of Rolle's Theorem are satisfied, and

$$\phi'_n(x_n) = 0,$$

or

$$S_n = f^{(n)}(x_n) \tag{4}$$

where $a < x_n < b$. Hence the

Theorem: *If* $f(x)$, $f'(x)$, $f''(x)$, $\cdots$, $f^{(n)}(x)$ *are continuous from* $x = a$ to $x = b$,

$$f(b) = f(a) + (b-a)f'(a) + \frac{(b-a)^2}{2!}f''(a) + \cdots + \frac{(b-a)^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{(b-a)^n}{n!}f^{(n)}(x_n), \tag{5}$$

where $a < x_n < b$.

This theorem, which is only an extension of the theorem expressing the law of the mean, is called *Taylor's Theorem with the remainder.* The last term is called the *remainder.*

If b is replaced by x, (5) becomes

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{(x-a)^n}{n!}f^{(n)}(x_n), \tag{6}$$

where $a < x_n < x$. This inequality is sometimes written $x_n = a + \theta(x - a)$, where $0 < \theta < 1$.

Illustration 1. Let $f(x) = e^x$. Then

$$\begin{array}{ll} f(x) = e^x & f(a) = e^a \\ f'(x) = e^x & f'(a) = e^a \\ f''(x) = e^x & f''(a) = e^a \\ \cdots\quad\cdots & \cdots\quad\cdots \\ f^{(n)}(x) = e^x & f^{(n)}(a) = e^a \end{array}$$

Hence by (6)

$$e^x = e^a\left[1 + (x-a) + \frac{(x-a)^2}{2!} + \cdots + \frac{(x-a)^{n-1}}{(n-1)!}\right] + \frac{(x-a)^n}{n!}e^{x_n}. \tag{7}$$

If $a = 0$,

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{n-1}}{(n-1)!} + \frac{x^n}{n!}e^{x_n}. \tag{8}$$

If $a = 0$ and $x = 1$,

$$e = 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{(n-1)!} + \frac{1}{n!}e^{x_n}. \tag{9}$$

The remainder in (8) and (9) can be made as small as we please by choosing n sufficiently large.

Taylor's Theorem may be expressed in still another form by setting b in (5) equal to $a + h$.

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{\lfloor 2} f''(a) + \cdots + \frac{h^n}{\lfloor n} f^{(n)}(x_n) \qquad (10)$$

where $a < x_n < a + h$, or $x_n = a + \theta h$, $0 < \theta < 1$.

If the values of a function and its derivatives are known at a, then the values of the function at a point $a + h$ can be computed by this formula.

In (10), $f(a + h)$ is represented approximately by a polynomial of degree $n - 1$ in h. The coefficients are the derivatives of $f(x)$ at $x = a$. The error in the approximation is given by the last term. This term gives only a means of estimating the error, since x_n is not known. The maximum error can, however, be determined by substituting $M^{(n)}$, the greatest numerical value of $f^{(n)}(x)$ in the interval $(a, a + h)$ for $f^{(n)}(x_n)$. The numerical value of the error is therefore less than

$$M^{(n)} \frac{h^n}{\lfloor n}.$$

If $a = 0$, (6) becomes

$$\mathbf{f(x) = f(0) + f'(0)\,x + f''(0)\frac{x^2}{\lfloor 2} + \cdots + f^{(n-1)}(0)\frac{x^{(n-1)}}{\lfloor (n-1)}}$$

$$\mathbf{+ f^{(n)}(x_n)\frac{x^n}{\lfloor n}}, \qquad (11)$$

where $0 < x_n < x$, or $x_n = \theta x$, $0 < \theta < 1$.

In (11) it is assumed that the function $f(x)$ and its first n derivatives are continuous from $x = 0$ to $x = a$. (11) is known as *Maclaurin's Theorem with the remainder.*

Illustration 2. Expand $\sin x$ by Maclaurin's Theorem in powers of x as far as the term containing x^6.

$$\begin{array}{llll}
f(x) = \sin x & \qquad f(0) = 0 \\
f'(x) = \cos x & \qquad f'(0) = 1 \\
f''(x) = -\sin x & \qquad f''(0) = 0 \\
f'''(x) = -\cos x & \qquad f'''(0) = -1 \\
f^{IV}(x) = \sin x & \qquad f^{IV}(0) = 0 \\
f^{V}(x) = \cos x & \qquad f^{V}(0) = 1 \\
f^{VI}(x) = -\sin x & \qquad f^{VI}(0) = 0 \\
f^{VII}(x) = -\cos x & \qquad f^{VII}(x_n) = -\cos x_7.
\end{array}$$

Substitution in (11) gives

$$\sin x = x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \frac{x^7}{\lfloor 7} \cos x_7, \tag{12}$$

where $0 < x_7 < x$.

Since $|\cos x_7| < 1$, $\sin x$ differs from

$$x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5}$$

by a number less than $\frac{x^7}{\lfloor 7}$.

In general, since $\sin x$ and its derivatives are continuous,

$$\sin x = x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \cdot \cdot \cdot \pm \frac{x^n}{\lfloor n} (\sin x_n \text{ or } \cos x_n), \tag{13}$$

where $0 < x_n < x$. Thus the difference between $\sin x$ and

$$x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \cdot \cdot \cdot \pm \frac{x^{n-1}}{\lfloor n-1}$$

is less than $\frac{x^n}{\lfloor n}$, a number which for a given x can be made as small as we please by taking n sufficiently large. Hence the series (13) can be used in computing the value of $\sin x$. If x is small, only a few terms of the series need be used to obtain a very close approximation to $\sin x$. Thus in formulas in which $\sin x$ occurs, $\sin x$ is frequently replaced by x if the angle is small. Such a substitution was made in equation 1, **§81.** It must be remembered in making the substitution that x is expressed in radians.

163. Taylor's and Maclaurin's Series. If $f(x)$ and all of its derivatives are continuous within an interval, the number of terms in (6), (10), and (11), **§162,** can be increased indefinitely. These equations then become, respectively,

$$\mathbf{f(x) = f(a) + f'(a)(x - a) + f''(a)\frac{(x-a)^2}{\lfloor 2} + \cdot \cdot \cdot}$$
$$\mathbf{+ f^{(n)}(a)\frac{(x-a)^n}{\lfloor n} + \cdot \cdot \cdot.} \tag{1}$$

$$\mathbf{f(a+h) = f(a) + f'(a)h + f''(a)\frac{h^2}{\lfloor 2} + \cdot \cdot \cdot + f^{(n)}(a)\frac{h^n}{\lfloor n} + \cdot \cdot \cdot.} \tag{2}$$

$$\mathbf{f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{\lfloor 2} + \cdot \cdot \cdot + f^{(n)}(0)\frac{x^n}{\lfloor n} + \cdot \cdot \cdot.} \tag{3}$$

In (1), $f(x)$ and its derivatives are assumed to be continuous from a to x.

In (2), $f(x)$ and its derivatives are assumed to be continuous from a to $a + h$.

In (3), $f(x)$ and its derivatives are assumed to be continuous from 0 to x.

The series (1) and (2) are called *Taylor's Series* and (3) is called *Maclaurin's Series.*

If we denote the last term in each of the equations (6), (10), and (11), **§162**, by R_n, it is necessary that

$$\lim_{n \doteq \infty} R_n = 0$$

in order that (1), (2), and (3) shall represent $f(x)$, $f(a + h)$, and $f(x)$, respectively.

Such series represent a function only so long as they are convergent. Later in this chapter means of testing the convergence of series will be discussed. The series (1), (2), and (3), if convergent, represent $f(x)$ but do not give a means of estimating the error made by stopping with a given term. This can best be determined from the expression for the remainder R_n in Taylor's or Maclaurin's Theorem with the remainder.

Illustration 1. Represent $\sin x$ by a power series in $(x - a)$. Use formula (1).

$$\begin{aligned} f(x) &= \sin x & f(a) &= \sin a \\ f'(x) &= \cos x & f'(a) &= \cos a \\ f''(x) &= -\sin x & f''(a) &= -\sin a. \\ f'''(x) &= -\cos x & f'''(a) &= -\cos a \\ f^{\text{IV}}(x) &= \sin x & f^{\text{IV}}(a) &= \sin a \\ f^{\text{V}}(x) &= \cos x & f^{\text{V}}(a) &= \cos a \end{aligned}$$

Then by (1)

$$\sin x = \sin a + \cos a\,(x - a) - \sin a\,\frac{(x-a)^2}{\underline{|2}} - \cos a\,\frac{(x-a)^3}{\underline{|3}} + \sin a\,\frac{(x-a)^4}{\underline{|4}} + \cos a\,\frac{(x-a)^5}{\underline{|5}} - \cdots.$$

The corresponding Maclaurin's Series is obtained by letting $a = 0$.

$$\sin x = x - \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} - \frac{x^7}{\underline{|7}} + \cdots.$$

Illustration 2. Expand $\tan x$ in a power series in x.

$$\begin{aligned}
f(x) &= \tan x. & f(0) &= 0 \\
f'(x) &= \sec^2 x. & f'(0) &= 1 \\
f''(x) &= 2\sec^2 x \tan x. & f''(0) &= 0 \\
&= 2(\tan x + \tan^3 x). & & \\
f'''(x) &= 2(\sec^2 x + 3\tan^2 x \sec^2 x) & f'''(0) &= 2 \\
&= 2(1 + 4\tan^2 x + 3\tan^4 x) & & \\
f^{\text{IV}}(x) &= 16\tan x \sec^2 x + 24\tan^3 x \sec^2 x & f^{\text{IV}}(0) &= 0 \\
&= 16\tan x + 40\tan^3 x + 24\tan^5 x & & \\
f^{\text{V}}(x) &= 16\sec^2 x + 120\tan^2 x \sec^2 x + 120\tan^4 x \sec^2 x & f^{\text{V}}(0) &= 16
\end{aligned}$$

On substituting in (3) we obtain

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots.$$

The next two terms are $\frac{17}{315}x^7$ and $\frac{62}{2835}x^9$.

Exercises

1. Expand $\cos x$ in a power series in x.

2. Expand $\cos x$ in a power series in $(x - a)$.

3. Expand $\cos (a + h)$ in a power series in h.

4. Expand $\sin (a + h)$ in a power series in h.

5. Express the remainder after three terms in each of the series of Exercises 1, 2, 3, 4.

6. Expand e^x in a power series in x.

7. Expand e^{a+h} in a power series in h.

8. Expand e^x in a power series in $(x - a)$.

9. Expand $\log (1 + x)$ in a power series in x.

10. Expand $\log (1 - x)$ in a power series in x.

11. Expand $\tan^{-1} x$ in a power series in x.

12. By the use of the series already found, compute:

(*a*) $\sqrt[3]{e}$ to 5 decimal places.

(*b*) $\sqrt[10]{e}$ to 6 decimal places.

(*c*) $\sin 3°$ to 6 decimal places.

(*d*) cosine of 1 radian to 4 decimal places.

13. By the use of the result of Exercise 3, find $\cos 33°$ correct to 4 decimal places.

14. By the use of the result of Exercise 4, find $\sin 32°$ correct to 4 decimal places.

164. Second Proof for Taylor's and Maclaurin's Series. These series can be obtained very simply in another way if we make certain assumptions and do not attempt to justify them.

Assume that $f(x)$ can be represented by an infinite power series in $(x - a)$:

$$f(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots + a_n(x-a)^n + \cdots, \quad (1)$$

where a_0, a_1, a_2, $\cdots$, a_n, $\cdots$ are coefficients which are to be determined. Assume further that the result of differentiating the second member term by term any given number of times, is equal to the corresponding derivative of the first member. Then,

$$\left.\begin{aligned}
f'(x) &= a_1 + 2a_2(x-a) + 3a_3(x-a)^2 \\
&\qquad + \cdots + na_n(x-a)^{n-1} + \cdots \\
f''(x) &= 2a_2 + 6a_3(x-a) \\
&\qquad + \cdots + n(n-1)a_n(x-a)^{n-2} + \cdots \\
f'''(x) &= 6a_3 + \cdots + n(n-1)(n-2)a_n(x-a)^{n-3} + \cdots \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
f^{(n)}(x) &= \underline{|n}\,a_n + \cdots
\end{aligned}\right] \quad (2)$$

Put $x = a$ in (1) and (2).

$$\begin{aligned}
f(a) &= a_0, \\
f'(a) &= a_1, \\
f''(a) &= 2\,a_2, \\
f'''(a) &= \underline{|3}\,a_3, \\
\cdots &\quad \cdots \\
f^{(n)}(a) &= \underline{|n}\,a_n.
\end{aligned}$$

whence

$$\begin{aligned}
a_0 &= f(a), \\
a_1 &= f'(a), \\
a_2 &= \frac{f''(a)}{\underline{|2}}, \\
a_3 &= \frac{f'''(a)}{\underline{|3}}, \\
\cdots &\quad \cdots \\
a_n &= \frac{f^{(n)}(a)}{\underline{|n}}, \\
\cdots &\quad \cdots
\end{aligned}$$

Substituting in (1) we obtain

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{\underline{|2}}(x - a)^2$$
$$+ \frac{f'''(a)}{\underline{|3}}(x - a)^3 + \cdot \cdot \cdot + \frac{f^{(n)}(a)}{\underline{|n}}(x - a)^n + \cdot \cdot \cdot.$$

By setting $a = 0$, and $x = a + h$, we get (3) and (2), respectively, of the preceding section.

165. Tests for the Convergence of Series. Several tests will now be given for determining whether or not a series is convergent. They will be given without proof, though in most cases the proof is not difficult.

If a series $u_1 + u_2 + \cdot \cdot \cdot + u_n + \cdot \cdot \cdot$ is convergent,

$$\lim_{n \doteq \infty} u_n = 0.$$

The converse of this statement is not true. Thus the series

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \cdot \cdot \cdot + \frac{1}{n} + \cdot \cdot \cdot \qquad (1)$$

is divergent, although

$$\lim_{n \doteq \infty} \frac{1}{n} = 0.$$

That this series is divergent can easily be seen as follows:

$$\tfrac{1}{3} + \tfrac{1}{4} > \tfrac{1}{2}$$
$$\tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8} > \tfrac{1}{2}$$
$$\tfrac{1}{9} + \tfrac{1}{10} + \tfrac{1}{11} + \tfrac{1}{12} + \tfrac{1}{13} + \tfrac{1}{14} + \tfrac{1}{15} + \tfrac{1}{16} > \tfrac{1}{2}$$
$$\cdot\cdot\cdot \quad \cdot\cdot\cdot \quad \cdot\cdot\cdot \quad \cdot\cdot\cdot \quad \cdot\cdot\cdot \quad \cdot\cdot\cdot$$

The terms of the series can then be grouped into infinitely many groups such that the sum of the terms in each group is greater than $\tfrac{1}{2}$. But the series

$$1 + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} + \cdot \cdot \cdot$$

is divergent. Much more then is the series (1) divergent.

Test 1. *If* $\lim_{n \doteq \infty} u_n$ *is not zero the series is divergent.* This test is easy to apply and if it shows the series to be divergent, no further investigation is necessary.

Test 2. Alternating series. A series of decreasing terms whose signs are alternately plus and minus and for which

$$\lim_{n \doteq \infty} u_n = 0$$

is convergent.

Thus the series

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{1}{6} + \cdots \qquad (2)$$

is convergent.

The reason for the convergence of such an alternating series can be seen as follows. Denote by S_n the sum of the first n terms and suppose the $(n+1)^{\text{th}}$ term positive. (See Fig. 121.) Then, since the terms are constantly decreasing,

$$S_{n+1} > S_n;\ S_{n+2} < S_{n+1};\ S_{n+2} > S_n.$$

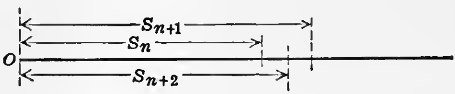

FIG. 121.

It is clear that as n increases S_n oscillates back and forth but always within narrower and narrower limits, owing to the fact that the terms are constantly decreasing. As n becomes infinite the amount of this oscillation approaches zero since

$$\lim_{n \doteq \infty} u_n = 0.$$

S_n therefore approaches a limit.

Test 3. Comparison Test. If the terms of a series are in numerical value less than or equal to the corresponding terms of a known convergent series of positive terms, the series is convergent. If the terms of a series of positive terms are greater than or equal to the corresponding terms of a divergent series of positive terms, the series is divergent.

A useful series for comparison is the geometrical series

$$a + ar + ar^2 + ar^3 + \cdots + ar^n + \cdots, \qquad (3)$$

which is convergent if $|r| < 1$ and divergent if $|r| \geqq 1$. See also the series (*a*) of Illustration 2 of this section.

Test 4. *The Ratio Test.* By comparison with the geometrical series it can be shown that *the series*

$$u_1 + u_2 + u_3 + \cdots + u_n + \cdots$$

is convergent if

$$\lim_{n \doteq \infty} \left| \frac{u_{n+1}}{u_n} \right| < 1,$$

divergent if

$$\lim_{n \doteq \infty} \left| \frac{u_{n+1}}{u_n} \right| > 1.$$

If

$$\lim_{n \doteq \infty} \left| \frac{u_{n+1}}{u_n} \right| = 1$$

the test fails. In this case other tests must be applied.

There are a great many tests for the convergence of series but only a few can be given here. It should be added that there is no test that can be applied to all cases.

Illustration 1. Test the series

$$1 - \tfrac{2}{1} + \tfrac{3}{2} - \tfrac{4}{3} + \tfrac{5}{4} - \tfrac{6}{5} + \cdots$$

for convergence.

Since

$$\lim_{n \doteq \infty} u_n$$

is not zero, the series is divergent (Test 1). It is to be noted that the terms of the series are alternately positive and negative and that they decrease, but they decrease to the limiting value 1 instead of 0. Hence test 2 does not apply.

Illustration 2. Test the series

$$1 + \frac{1}{2^t} + \frac{1}{3^t} + \frac{1}{4^t} + \cdots \qquad (a)$$

for convergence. This series is useful in testing the convergence of series by comparison.

If $t = 1$ we have seen that this series is divergent. (See (1).) If $t < 1$ each term of (a) is greater than the corresponding term of (1) and hence (a) is divergent. If $t > 1$ we can compare (a) with

$$1 + \frac{1}{2^t} + \frac{1}{2^t} + \frac{1}{4^t} + \frac{1}{4^t} + \frac{1}{4^t} + \frac{1}{4^t} + \cdots \qquad (b)$$

Each term of (a) is less than or equal to the corresponding term of (b). But (b) is convergent since it can be written

$$1 + 2\left(\frac{1}{2^t}\right) + 4\left(\frac{1}{4^t}\right) + 8\left(\frac{1}{8^t}\right) + \cdots$$

or

$$1 + \frac{2}{2^t} + \frac{4}{4^t} + \frac{8}{8^t} + \frac{16}{16^t} + \cdots$$

which is a geometric series whose ratio, $\frac{2}{2^t}$, is less than 1. Hence (a) is convergent when $t > 1$. Summing up:

(a) is divergent if $t \leq 1$.
(a) is convergent if $t > 1$.

Illustration 3. Test the series

$$1 + \frac{1}{\underline{|2}} + \frac{1}{\underline{|3}} + \frac{1}{\underline{|4}} + \cdots$$

for convergence.

Apply test 4.

$$u_n = \frac{1}{\underline{|n}}.$$

Hence

$$\lim_{n \doteq \infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n \doteq \infty} \frac{\dfrac{1}{\underline{|n+1}}}{\dfrac{1}{\underline{|n}}} = \lim_{n \doteq \infty} \frac{1}{n+1} = 0.$$

The series is therefore convergent.

Illustration 4. For what values of x, if any, is the series

$$x - \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} - \frac{x^7}{\underline{|7}} + \cdots$$

convergent?

$$|u_n| = \left|\frac{x^{2n-1}}{\underline{|2n-1}}\right|.$$

Then

$$\lim_{n \doteq \infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n \doteq \infty} \left|\frac{\dfrac{x^{2n+1}}{\underline{|2n+1}}}{\dfrac{x^{2n-1}}{\underline{|2n-1}}}\right| = \lim_{n \doteq \infty} \left|\frac{x^2}{2n(2n+1)}\right| = 0,$$

for all finite values of x. Hence the series is convergent for all finite values of x, positive or negative.

Illustration 5. For what values of x is

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + \cdots$$

convergent?

$$|u_n| = \left|\frac{x^n}{n}\right|.$$

$$\lim_{n \doteq \infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n \doteq \infty} \left|\frac{\dfrac{x^{n+1}}{n+1}}{\dfrac{x^n}{n}}\right| = \lim_{n \doteq \infty} \left| x \frac{n}{n+1} \right| = |x|.$$

The series is therefore convergent if $|x| < 1$. Furthermore it is convergent if $x = 1$ (Test 2), and divergent if $x = -1$. (See series (1).)

As has been stated there is no one test of convergence which can be applied with certainty of success to any given series. The tests which can be most frequently applied have been given. It is suggested that the following procedure be observed in general.

1. See if $\lim\limits_{n \doteq \infty} u_n = 0$.
2. If so, is test 2 applicable?
3. If not, try the ratio test, test 4. This will fail if

$$\lim_{n \doteq \infty} \left|\frac{u_{n+1}}{u_n}\right| = 1.$$

4. In this case, and in cases where the other tests fail or are difficult, try the comparison test.

Exercises

Test the following series for convergence.

1. $\frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \frac{1}{9} + \frac{1}{11} - \cdots$.

2. $\frac{1}{3} - \frac{2}{5} + \frac{3}{7} - \frac{4}{9} + \frac{5}{11} - \cdots$.

3. $\frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \cdots$.

4. $\frac{\underline{|2}}{10} + \frac{\underline{|3}}{10^2} + \frac{\underline{|4}}{10^3} + \frac{\underline{|5}}{10^4} + \cdots$.

5. $\frac{1}{\underline{|2}} + \frac{2}{\underline{|3}} + \frac{3}{\underline{|4}} + \frac{4}{\underline{|5}} + \cdots$.

6. $1 + \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \frac{1}{4\sqrt{4}} + \cdots$.

7. $1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \frac{1}{9^2} - \cdots$.

8. $1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \cdots$.

9. $\frac{1}{1\cdot 2} + \frac{1}{3\cdot 4} + \frac{1}{5\cdot 6} + \cdots$.

For what values of x are the following series convergent?

10. The Maclaurin's series for e^x? Exercise 6, §**163**.

11. The Maclaurin's series for $\cos x$? Exercise 1, §**163**.

12. The Maclaurin's series for $\sin x$? Illustration 1, §**163**.

13. The Maclaurin's series for $\log (1 - x)$? Exercise 10, §**163**.

14. The Maclaurin's series for $\tan^{-1}x$? Exercise 11, §**163**.

166. Computation of Logarithms. The series of Exercise 9, §**163**, for $\log (1 + x)$ is convergent only when $-1 < x \leq +1$, and that for $\log (1 - x)$, Exercise 10, §**163**, only when $-1 \leq x < +1$. It would appear then impossible to find the logarithm of a number greater than 2 by these formulas. By a very simple device it is, however, possible to obtain formulas for finding the logarithm of any number.

From the series of Exercises 9 and 10, §**163** it follows that

$$\log \frac{1+x}{1-x} = \log (1 + x) - \log (1 - x)$$

$$= 2\left[x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right], \quad (1)$$

where $|x| < 1$. Let $x = \frac{1}{2z+1}$. Then

$$\frac{1+x}{1-x} = \frac{z+1}{z}.$$

and

$$\log \frac{z+1}{z} = 2\left[\frac{1}{2z+1} + \frac{1}{3(2z+1)^3} + \frac{1}{5(2z+1)^5} + \cdots\right], \quad (2)$$

where $z > 0$, or

$$\log (z+1) = \log z + 2\left[\frac{1}{2z+1} + \frac{1}{3(2z+1)^3} + \frac{1}{5(2z+1)^5} + \cdots\right]. \quad (3)$$

By letting $z = 1$, $\log 2$ can be computed by this formula. The series is much more rapidly convergent than that for $\log (1 + x)$, $x = 1$. In fact, 100 terms of the latter series must be taken to obtain log 2 correct to two decimal places, while four terms of the new series (3) will give log 2 correct to four decimal places. After log 2 has been found, log 3 can be found by setting $z = 2$. The logarithm of 4 is found by taking twice log 2; log 5 by setting $z = 4$; log 6 by adding log 3 and log 2, and so on.

Exercise

Compute log 5 correct to four decimal places, given that log 4 = 1.38629. Here, as always in the Calculus, the base is understood to be e.

167. Computation of π. By letting $x = 1$ in the series for $\tan^{-1}x$, Exercise 11, **§163,** the following equation is obtained from which π can be computed:

$$\frac{\pi}{4} = \tan^{-1} 1 = 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \cdots.$$

This series converges very slowly. To obtain a more rapidly converging series make use of the relation

$$\tan^{-1} 1 = \tan^{-1} \tfrac{1}{2} + \tan^{-1} \tfrac{1}{3}.$$

Then

$$\begin{aligned}\frac{\pi}{4} &= \tfrac{1}{2} - \frac{1}{(3)(2^3)} + \frac{1}{(5)(2^5)} - \frac{1}{(7)(2^7)} + \cdots \\ &\quad + \tfrac{1}{3} - \frac{1}{(3)(3^3)} + \frac{2}{(5)(3^5)} - \frac{1}{(7)(3^7)} + \cdots.\end{aligned}$$

168. Relation between the Exponential and Circular Functions. If it be admitted that the Maclaurin's series expansion

$$e^z = 1 + z + \frac{z^2}{\underline{|2}} + \frac{z^2}{\underline{|3}} + \cdots, \tag{1}$$

which was proved for real values of z, is also true when z is imaginary, we obtain, on setting $z = ix$,

$$\begin{aligned}e^{ix} &= 1 + ix + \frac{(ix)^2}{\underline{|2}} + \frac{(ix)^3}{\underline{|3}} + \frac{(ix)^4}{\underline{|4}} + \frac{(ix)^5}{\underline{|5}} + \frac{(ix)^6}{\underline{|6}} + \frac{(ix)^7}{\underline{|7}} + \cdots \\ &= 1 + ix - \frac{x^2}{\underline{|2}} - \frac{ix^3}{\underline{|3}} + \frac{x^4}{\underline{|4}} + \frac{ix^5}{\underline{|5}} - \frac{x^6}{\underline{|6}} - \frac{ix^7}{\underline{|7}} + \cdots\end{aligned} \tag{2}$$

On separating real and imaginary parts this becomes

$$\begin{aligned}e^{ix} &= 1 - \frac{x^2}{\underline{|2}} + \frac{x^4}{\underline{|4}} - \frac{x^6}{\underline{|6}} + \cdots \\ &\quad + i\left(x - \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} - \frac{x^7}{\underline{|7}} + \cdots\right).\end{aligned} \tag{3}$$

Since (Exercise 1 and Illustration 1, §163),

$$\cos x = 1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} - \frac{x^6}{\lfloor 6} + \cdots$$

and

$$\sin x = x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \frac{x^7}{\lfloor 7} + \cdots$$

it follows that

$$\mathbf{e}^{ix} = \cos \mathbf{x} + \mathbf{i} \sin \mathbf{x}. \tag{4}$$

On changing the sign of x it results that

$$\mathbf{e}^{-ix} = \cos \mathbf{x} - \mathbf{i} \sin \mathbf{x}. \tag{5}$$

Solving equations (4) and (5) for $\cos x$ and $\sin x$,

$$\cos \mathbf{x} = \frac{\mathbf{e}^{ix} + \mathbf{e}^{-ix}}{2} \tag{6}$$

and

$$\sin \mathbf{x} = \frac{\mathbf{e}^{ix} - \mathbf{e}^{-ix}}{2\mathbf{i}} \tag{7}$$

These interesting relations between the circular and exponential functions are of very great importance.

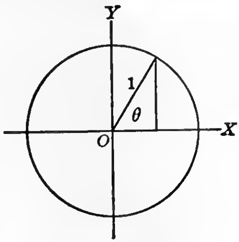

Fig. 122.

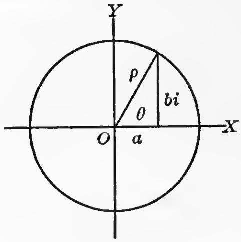

Fig. 123.

If θ represent the vectorial angle in the complex number plane, then it is clear from Fig. 122 that $e^{i\theta}$ represents a point on the unit circle (circle of radius 1 about the origin as center) in this plane. Further, any complex number $a + bi$ can be put in the form $\rho e^{i\theta}$. For (Fig. 123)

$$a + bi = \rho\,(\cos\theta + i \sin\theta) = \rho e^{i\theta},$$

where $\rho = \sqrt{a^2 + b^2}$.

Exercises

Represent by a point in the complex plane:

1. $3e^{\frac{i\pi}{3}}$.
3. $e^{\frac{i\pi}{4}}$.
5. $e^{i\pi}$.
7. $e^{2i\pi}$.
2. $2e^{-\frac{i\pi}{3}}$.
4. $e^{\frac{i\pi}{2}}$.
6. $e^{-i\pi}$.
8. $5e^{\frac{3i\pi}{4}}$.
9. Express the numbers of Exercises 1–8 in the form $a + bi$.

169. DeMoivre's Theorem. The interesting and important theorem, known as DeMoivre's Theorem,

$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta \tag{1}$$

can be easily established by the use of the relation (4) of §168. For,

$$(\cos \theta + i \sin \theta)^n = (e^{i\theta})^n = e^{in\theta} = \cos n\theta + i \sin n\theta.$$

Exercises

Find, by the use of (1),

1. The cube of $1 + i$.
2. The square of $\frac{1 + i\sqrt{3}}{2}$.
3. The cube of $\frac{1 + i\sqrt{3}}{2}$.
4. The cube of $\frac{-1 + i\sqrt{3}}{2}$.
5. The cube of $\frac{-1 - i\sqrt{3}}{2}$.
6. The cube of $\frac{5(1 + i\sqrt{3})}{2}$.

In (1), n may be a fraction as well as an integer. It will then indicate a root instead of a power. In this case we do not have simply one root:

$$(\cos \theta + i \sin \theta)^{\frac{1}{m}} = \cos \frac{\theta}{m} + i \sin \frac{\theta}{m},$$

(n having been placed equal to $\frac{1}{m}$, where m is an integer) but $m - 1$ additional roots. This follows from the fact that

$$e^{i\theta} = e^{i(\theta + 2p\pi)} \tag{2}$$

where $p = 0, 1, 2, 3, 4, \cdots, m, m + 1, \cdots$. Hence we can write

$$(\cos \theta + i \sin \theta)^{\frac{1}{m}} = [e^{i\theta}]^{\frac{1}{m}} = [e^{i(\theta + 2p\pi)}]^{\frac{1}{m}},$$

or

$$(\cos \theta + i \sin \theta)^{\frac{1}{m}} = e^{\frac{i(\theta + 2p\pi)}{m}}, (p = 0, 1, 2, \cdots) \tag{3}$$

It would appear at first sight as if there were infinitely many roots corresponding to the infinitely many values of p. But a little consideration shows that when $p \overline{\gtrless} m$, the roots already found by letting p take the values $0, 1, 2, \cdots, m-1$, repeat themselves, since $e^{2i\pi} = 1$. There are then exactly m mth roots of $e^{i\theta} = \cos\theta + i\sin\theta$,

$$e^{\frac{i(\theta+2p\pi)}{m}} = \cos\frac{\theta+2p\pi}{m} + i\sin\frac{\theta+2p\pi}{m}, \tag{4}$$

where $p = 0, 1, 2, \cdots, m-1$.

Illustration. Find the three cube roots of -1.

$$\begin{aligned}(-1)^{\frac{1}{3}} &= (e^{i\pi})^{\frac{1}{3}} \\ &= \left[e^{i(\pi+2p\pi)}\right]^{\frac{1}{3}} \qquad (p = 0, 1, 2) \\ &= e^{\frac{i(\pi+2p\pi)}{3}} \qquad (p = 0, 1, 2) \\ &= e^{\frac{i\pi}{3}}, \quad e^{i\pi}, \quad \text{and } e^{\frac{5i\pi}{3}}.\end{aligned}$$

Exercises

1. Show that the three cube roots of $a + bi = \rho e^{i\theta}$ are: $\sqrt[3]{\rho}\, e^{\frac{i\theta}{3}}$, $\sqrt[3]{\rho}\, e^{\frac{i(\theta+2\pi)}{3}}$, and $\sqrt[3]{\rho}\, e^{\frac{i(\theta+4\pi)}{3}}$. How would these roots be determined graphically?

2. Find the two square roots of $1 + i$.

3. Find graphically the two square roots of i.

4. Find graphically the three cube roots of 1.

170. Indeterminate Forms. It has already been shown that $\frac{0}{0}$ has no meaning. See §25. Thus

$$\frac{x^2-4}{x-2}$$

has no meaning at $x = 2$. Its value at $x = 2$ is defined as

$$\lim_{x \doteq 2} \frac{x^2-4}{x-2} = 4.$$

Similarly

$$\frac{\sin\alpha}{\tan\alpha}$$

has no meaning at $\alpha = 0$. Its value at $\alpha = 0$ *is defined as*

$$\lim_{\alpha \doteq 0} \frac{\sin\alpha}{\tan\alpha} = 1.$$

In general, if $\phi(a) = 0$ and $f(a) = 0$, the value of the function $\frac{\phi(x)}{f(x)}$ at $x = a$ *is defined as*

$$\lim_{x \doteq a} \frac{\phi(x)}{f(x)}.$$

The calculation of this limit is simplified in many cases by the application of the law of the mean. See §160.

Thus, let it be given that $\phi(a) = 0$ and $f(a) = 0$ and let $\phi(x)$ and $f(x)$ satisfy the conditions imposed in the statement of the law of the mean. Then

$$\lim_{x \doteq a} \frac{\phi(x)}{f(x)} = \lim_{x \doteq a} \frac{\phi(a) + (x-a)\,\phi'[a + \theta_1(x-a)]}{f(a) + (x-a)\,f'[a + \theta_2(x-a)]}$$

$$= \lim_{x \doteq a} \frac{\phi'[a + \theta_1(x-a)]}{f'[a + \theta_2(x-a)]} = \frac{\phi'(a)}{f'(a)}, \quad 0 < \begin{matrix}\theta_1\\ \theta_2\end{matrix} < 1.$$

If $\phi'(a)$ and $f'(a)$ are also zero, we make use of the extended law of the mean. Thus

$$\lim_{x \doteq a} \frac{\phi(x)}{f(x)} = \lim_{x \doteq a} \frac{\phi(a) + (x-a)\,\phi'(a) + \frac{(x-a)^2}{\underline{|2}}\,\phi''[a + \theta_1(x-a)]}{f(a) + (x-a)\,f'(a) + \frac{(x-a)^2}{\underline{|2}}\,f''[a + \theta_2(x-a)]}$$

$$= \lim_{x \doteq a} \frac{\phi''[a + \theta_1(x-a)]}{f''[a + \theta_2(x-a)]} = \frac{\phi''(a)}{f''(a)}.$$

The process is to be continued further if $f''(a)$ and $\phi''(a)$ are both zero.

Illustration 1.

$$\lim_{x \doteq 0} \frac{e^x - 1}{x} = \lim_{x \doteq 0} \frac{e^x}{1} = 1.$$

Illustration 2.

$$\lim_{x \doteq 0} \frac{e^x - e^{-x} - 2x}{x - \sin x} = \lim_{x \doteq 0} \frac{e^x + e^{-x} - 2}{1 - \cos x}$$

$$= \lim_{x \doteq 0} \frac{e^x - e^{-x}}{\sin x}$$

$$= \lim_{x \doteq 0} \frac{e^x + e^{-x}}{\cos x}$$

$$= 2.$$

The Form $\frac{\infty}{\infty}$. The same process is employed in evaluating the indeterminate form $\frac{\infty}{\infty}$. The proof is omitted.

Illustration 3.

$$\lim_{x \doteq \infty} \frac{x^2}{e^x} = \lim_{x \doteq \infty} \frac{2x}{e^x} = \lim_{x \doteq \infty} \frac{2}{e^x} = 0.$$

The Form $0\,\infty$. The indeterminate form $0\,\infty$ can be thrown into either of the forms $\frac{0}{0}$ or $\frac{\infty}{\infty}$. Thus

$$\lim_{x \doteq 0} x \cot x = \lim_{x \doteq 0} \frac{x}{\tan x} = \lim_{x \doteq 0} \frac{1}{\sec^2 x} = 1.$$

Other indeterminate forms are: $\infty - \infty$, 1^{∞}, 0^0, ∞^0.

Thus, if $\phi(x)$ and $f(x)$ become infinite for $x = a$, $\phi(a) - f(a)$ is *defined as*

$$\lim_{x \doteq a} [\phi(x) - f(x)].$$

This expression can be written

$$\lim_{x \doteq a} [\phi(x) - f(x)] = \lim_{x \doteq a} \frac{\frac{1}{f(x)} - \frac{1}{\phi(x)}}{\frac{1}{\phi(x) f(x)}},$$

an indeterminate form of the type $\frac{0}{0}$.

If $\phi(x)$ becomes infinite and $f(x)$ becomes 1 for $x = a$, $[f(a)]^{\phi(a)}$ is *defined as*

$$\lim_{x \doteq a} [f(x)]^{\phi(x)}.$$

This limit can be calculated as follows. Let $y = [f(x)]^{\phi(x)}$. Then

$$\log y = \phi(x) \log f(x) = \frac{\log f(x)}{\frac{1}{\phi(x)}},$$

an indeterminate form of the type $\frac{0}{0}$. If

$$\lim_{x \doteq a} \frac{\log f(x)}{\frac{1}{\phi(x)}}$$

is found to be c, then

$$\lim_{x \doteq a} y = e^c.$$

The two remaining forms are evaluated in a manner simiiar to the last.

Many indeterminate forms can be evaluated directly by simple algebraic transformations.

Exercises

Evaluate the following:

1. $\lim\limits_{x \doteq 1} \dfrac{\log x}{x - 1}$.
2. $\lim\limits_{\theta \doteq 0} \dfrac{1 - \cos\theta}{\cos\theta \sin^2\theta}$.
3. $\lim\limits_{x \doteq 0} \dfrac{x\cos x - \sin x}{x}$.
4. $\lim\limits_{x \doteq 0} \dfrac{\tan x - \sin x}{x - \sin x}$.
5. $\lim\limits_{x \doteq 1} \dfrac{x^n - 1}{x - 1}$.
6. $\lim\limits_{x \doteq 0} \dfrac{\sin 3x}{\sin 2x}$.
7. $\lim\limits_{x \doteq 0} \dfrac{\tan 3x}{x}$.
8. $\lim\limits_{x \doteq 0} \dfrac{x^3}{\sin x - x}$.
9. $\lim\limits_{x \doteq 3} \dfrac{x^2 - x - 6}{x^2 - 9}$.
10. $\lim\limits_{x \doteq \infty} \dfrac{3x^2 + 5}{4x^2 + 1}$.
11. $\lim\limits_{x \doteq 0} \dfrac{\log x}{\cot x}$.
12. $\lim\limits_{\phi \doteq \frac{\pi}{2}} \dfrac{\tan 2\phi}{\tan 5\phi}$.
13. $\lim\limits_{x \doteq \infty} \dfrac{x^n}{\log x}$.
14. $\lim\limits_{x \doteq \infty} \dfrac{x^n}{e^x}$.
15. $\lim\limits_{x \doteq \infty} \dfrac{e^x}{x^3}$.
16. $\lim\limits_{x \doteq \infty} e^x \tan \dfrac{1}{x}$.
17. $\lim\limits_{x \doteq 1} \left[\dfrac{2}{x^2 - 1} - \dfrac{1}{x - 1}\right]$.
18. $\lim\limits_{\theta \doteq \frac{\pi}{2}} \left[\left(\dfrac{\pi}{2} - \theta\right) \tan\theta\right]$
19. $\lim\limits_{x \doteq 1} \left[\dfrac{1}{\log x} - \dfrac{1}{x - 1}\right]$.
20. $\lim\limits_{x \doteq 0} (\cos x)^{\cot x}$.
21. $\lim\limits_{x \doteq 0} (1 - x)^{\frac{1}{x}}$.
22. $\lim\limits_{x \doteq 0} (\csc x)^{\tan x}$.
23. $\lim\limits_{x \doteq 0} (\sin x)^{\tan x}$.
24. $\lim\limits_{x \doteq 0} \dfrac{\sin x}{x}$.
25. $\lim\limits_{\theta \doteq 0} \dfrac{\tan\theta}{\sin\theta}$.

CHAPTER XVIII

TOTAL DERIVATIVE. EXACT DIFFERENTIAL

171. The Total Derivative. Let $z = f(x, y)$ and let x and y be functions of a third variable t, the time for example. We seek an expression for $\frac{dz}{dt}$, the derivative of z with respect to t, in terms of $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

As an illustration of what is meant, let z denote the area of a rectangle whose sides x and y are functions of t, and at a given instant let each side be changing at a certain rate. The rate at which the area is changing is sought.

Returning to the general problem let t take on an increment Δt. Then x takes on the increment Δx and y the increment Δy, and consequently z the increment Δz. We then have

$$z = f(x, y) \tag{1}$$

$$z + \Delta z = f(x + \Delta x, y + \Delta y)$$

$$\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y) \tag{2}$$

$$\Delta z = f(x+\Delta x, y+\Delta y) - f(x, y+\Delta y) + f(x, y+\Delta y) - f(x, y) \tag{3}$$

$$\frac{\Delta z}{\Delta t} = \frac{f(x + \Delta x, y + \Delta y) - f(x, y + \Delta y)}{\Delta x}\frac{\Delta x}{\Delta t} + \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}\frac{\Delta y}{\Delta t} \tag{4}$$

Taking the limits of both sides of (4) as Δt approaches zero, we have

$$\frac{dz}{dt} = \frac{\partial f(x, y)}{\partial x}\frac{dx}{dt} + \frac{\partial f(x, y)}{\partial y}\frac{dy}{dt}, \tag{5}$$

since Δx and Δy each approach zero as Δt approaches zero. Equation (5) can be written in the form

$$\mathbf{\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}}. \tag{6}$$

This states that the rate of change of z with respect to t is equal to the rate of change of z with respect to x, times the rate of change of x with respect to t, plus the rate of change of z with respect to y, times the rate of change of y with respect to t.

If $t = x$, (6) becomes

$$\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\frac{dy}{dx}.$$

This formula applies when $z = f(x, y)$ and y is a function of x, *e.g.*, $y = \phi(x)$.

Multiplying (6) by dt we obtain

$$\mathbf{dz} = \frac{\partial \mathbf{z}}{\partial \mathbf{x}}\,\mathbf{dx} + \frac{\partial \mathbf{z}}{\partial \mathbf{y}}\,\mathbf{dy}. \tag{7}$$

This defines dz, which is called the *total differential of z.*

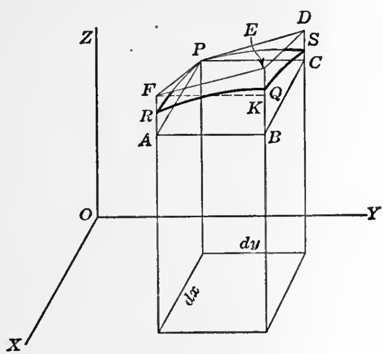

Fig. 124.

We shall now give a geometrical interpretation of dz. Let P, Fig. 124, be the point (x, y, z) on the surface $z = f(x, y)$. Let

$$PC = dy$$

and

$$PA = dx.$$

Then Q is the point $(x + dx, y + dy, z + \Delta z)$. Let $PDEF$ be the plane tangent to the surface at the point P. Then PF is tangent to the arc PR, and PD is tangent to the arc PS.

From F draw FK parallel to AB meeting BE in K.

$$BE = BK + KE$$

$$BK = AF = \frac{\partial z}{\partial x}\,dx.$$

Since $FK = PC$ and $PD = FE$, triangle KFE is equal to the triangle CPD, and

$$KE = CD = \frac{\partial z}{\partial y}\,dy.$$

Therefore

$$BE = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy.$$

Hence $BE = dz$. Consequently dz may be interpreted as the increment measured to the tangent plane to $z = f(x, y)$ at the point $P\ (x, y, z)$ when x and y are given the increments dx and dy respectively.

Illustration 1. If $z = xy$, the area of a rectangle of sides x and y, we obtain by using (7),

$$dz = y\,dx + x\,dy.$$

The first term on the right-hand side represents the area of the strip $BEFC$, Fig. 125. The second term the area of $DCGH$. The difference between Δz and dz is the area of the rectangle $CFLG$, which becomes relatively smaller, the smaller dx and dy become.

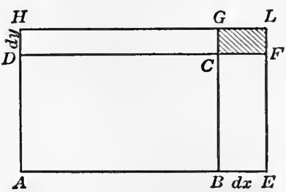

FIG. 125.

The above expression could have been obtained by the formula for the differential of the product of two variables.

Illustration 2. The base of a rectangular piece of brass is 15 feet and its altitude is 10 feet. If the base is increasing in length at the rate of 0.03 foot per hour and the altitude at the rate of 0.02 foot per hour, at what rate is the area changing?

Let x denote the base, y the altitude, and z the area. Then

$$z = xy$$

and
$$\frac{dz}{dt} = y\frac{dx}{dt} + x\frac{dy}{dt}$$
$$= (10)(0.03) + (15)(0.02).$$

Illustration 3. $z = \dfrac{x}{y}\cdot$

$$\frac{\partial z}{\partial x} = \frac{1}{y}, \qquad \frac{\partial z}{\partial y} = -\frac{x}{y^2}.$$

and, by (7),
$$dz = \frac{1}{y}\,dx - \frac{x}{y^2}\,dy,$$
or
$$dz = \frac{y\,dx - x\,dy}{y^2},$$
a result which could have been obtained by differentiating the quotient $\dfrac{x}{y}$ by the usual rule.

Exercises

Find by formula (7) the total differential of each of the following functions:

1. $z = x^2y.$ **4.** $z = \dfrac{x}{y^2}\cdot$ **7.** $z = x^2e^y.$

2. $z = xy^2.$ **5.** $z = x\log y.$ **8.** $z = e^x\sin y.$

3. $z = \dfrac{x^2}{y}\cdot$ **6.** $z = e^x\cos y.$ **9.** $z = e^{ay}\cos nx.$

Find $\dfrac{dz}{dt}$ if:

10. $z = x^2\cos y.$ **11.** $z = e^x\sin y.$

12. The radius of the base of a right circular cylinder is 8 inches and its altitude is 25 inches. If the radius of the base is increasing at the rate of 0.2 inch per hour and its altitude at the rate of 0.6 inch per hour, at what rate is the volume increasing?

13. Given the formula connecting the pressure, volume, and temperature of a perfect gas, $pv = Rt$, R being a constant. If $t = 523°$, $p = 1500$ pounds per square foot, and $v = 21.2$ cubic feet, find the approximate change in p when t changes to 525° and v to 21.4 cubic feet.

14. If with the data of Exercise 13, the temperature is changing at the rate of 1° per second, while the volume is changing at the rate of 0.4 cubic foot per second, at what rate is the pressure changing?

15. The edges of a rectangular parallelopiped are 6, 8, and 10 feet. They are increasing at the rate of 0.02 foot per second, 0.03 foot per second and 0.04 foot per second, respectively. At what rate is the volume increasing?

172. Exact Differential. An expression of the form

$$M\,dx + N\,dy,$$

where M and N are functions of x and y, may or may not be the differential of some function of x and y. If it is, it is called an *exact differential.* Thus

$$\sin y\,dx + x\cos y\,dy \tag{1}$$

is an exact differential, for it is the differential of $z = x\sin y$. The coefficient of dx is $\frac{\partial z}{\partial x} = \sin y$, and that of dy is $\frac{\partial z}{\partial y} = x\cos y$.

$$x^2\sin y\,dx + x\cos y\,dy \tag{2}$$

is not an exact differential. It is fairly evident from (1) that we cannot find a function $z = f(x, y)$ such that $\frac{\partial z}{\partial x} = x^2 \sin y$ and $\frac{\partial z}{\partial y} = x\cos y$.

We seek to find a test for determining whether or not an expression of the form

$$M\,dx + N\,dy \tag{3}$$

is an exact differential. If (3) is the exact differential of a function z, we must have,

$$\frac{\partial z}{\partial x} = M \tag{4}$$

and

$$\frac{\partial z}{\partial y} = N, \tag{5}$$

since

$$dz = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy. \tag{6}$$

Differentiate (4) with respect to y and (5) with respect to x and obtain

$$\frac{\partial^2 z}{\partial y\, \partial x} = \frac{\partial M}{\partial y} \tag{7}$$

and

$$\frac{\partial^2 z}{\partial x\, \partial y} = \frac{\partial N}{\partial x}. \tag{8}$$

Since, in general,

$$\frac{\partial^2 z}{\partial y\, \partial x} = \frac{\partial^2 z}{\partial x\, \partial y},$$

it follows that if (3) is an exact differential, we must have

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}. \tag{9}$$

The condition (9) must be satisfied if (3) is an exact differential. It does not follow, however, without further proof, that (3) is an exact differential if (9) is satisfied. It can, however, be shown that this is the case. The proof will be omitted. (3) cannot be an exact differential unless (9) is satisfied and is an exact differential if (9) is satisfied.

When an expression of the form (3) is given, the first step is to determine whether or not it is an exact differential by applying the test (9). If it is an exact differential, the next step is to find the function z of which it is the differential. This step will be illustrated by integrating several differentials for which the functions from which they were obtained by differentiation are known.

Illustration 1. If $z = x^3 + 2x^2y + y^2 + C$,

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$
$$= (3x^2 + 4xy)dx + (2x^2 + 2y)dy.$$

If then we are given the exact differential

$$dz = (3x^2 + 4xy)dx + (2x^2 + 2y)dy$$

and are required to find the function of which it is the differential, we note first that

$$\frac{\partial z}{\partial x} = 3x^2 + 4xy.$$

Then

$$z = x^3 + 2x^2y + a \textit{ function of } y \textit{ alone}.$$

And this function of y is to be so determined that

$$\frac{\partial z}{\partial y} = 2x^2 + 2y.$$

Clearly the term $2x^2$ is obtained by taking the derivative with respect to y of $2x^2y$, a term already found, and consequently it is not to be added. $2y$ is the derivative of y^2. y^2 is then the function of y which is to be added to the terms already found. Further an arbitrary constant is to be added since its differential will be zero. Then

$$z = x^3 + 2x^2y + y^2 + C$$

is the function whose differential was given. If, as is usually the case, it had been given that

$$(3x^2 + 4xy)dx + (2x^2 + 2y)dy = 0 \tag{10}$$

it would have been required to find a function of x and y such that its differential would be zero. Now the first member is, as we have seen, the differential of

$$z = x^3 + 2x^2y + y^2.$$

But, if $dz = 0$, $z = C$. Then

$$x^3 + 2x^2y + y^2 = C$$

is the relation between x and y which satisfies the given equation.

Illustration 2. If

$$z = e^x \cos y + x^2 + \sin y + y^3,$$

$$dz = (e^x \cos y + 2x)\,dx + (-\,e^x \sin y + \cos y + 3y^2)\,dy$$

Now let it be given that

$$(e^x \cos y + 2x)\,dx + (-\,e^x \sin y + \cos y + 3y^2)\,dy = 0. \tag{11}$$

From its derivation we know that the left-hand member is an exact differential, dz. Let us proceed to find z as if it were unknown.

$$\frac{\partial z}{\partial x} = e^x \cos y + 2x.$$

Then

$$z = e^x \cos y + x^2 + \textit{a function of y alone.} \tag{12}$$

The function of y is to be so determined that

$$\frac{\partial z}{\partial y} = -e^x \sin y + \cos y + 3y^2. \tag{13}$$

The first term is evidently obtained by differentiating $e^x \cos y$, a term already found in (12). The remaining two terms in (13) are obtained by differentiating $\sin y + y^3$. These are to be added to the terms already found in (12).

Then

$$z = e^x \cos y + x^2 + \sin y + y^3.$$

But, since $dz = 0$, $z = C$. Hence

$$e^x \cos y + x^2 + \sin y + y^3 = C$$

is a solution of (2).

Illustration 3. Integrate if possible the equation

$$(e^x y + \sin y + 2x)\, dx + (e^x + x \cos y + e^y + 2y - \sin y)\, dy = 0. \tag{14}$$

We have first to determine whether or not the first member is an exact differential. Apply the test (9).

$$\frac{\partial M}{\partial y} = e^x + \cos y.$$

$$\frac{\partial N}{\partial x} = e^x + \cos y.$$

Hence (9) is satisfied and the first member of (14) is an exact differential. On integrating the coefficient of dx with respect to x we obtain

$$e^x y + x \sin y + x^2.$$

To this we have to add

$$e^y + y^2 + \cos y,$$

the terms which arise from the integration of the coefficient of dy and which contain y alone. (The other terms in the coefficient of dy arise from the differentiation of terms already found by integrating the coefficient of dx.) Then the solution of (14) is

$$e^x y + x \sin y + x^2 + e^y + y^2 + \cos y = C.$$

173. Exact Differential Equations. *Equations involving differentials or derivatives are called differential equations.* Those of the type

$$M\, dx + N\, dy = 0 \tag{1}$$

where the first member is an exact differential, are called **exact differential equations.**

The equations (10), (11), and (14) of Illustrations 1, 2, and 3, **§172,** are exact differential equations. The process of finding the relation between y and x, which when differentiated gives a certain differential equation, is called the integration of the equation.

The procedure in dealing with an equation of type (1) is to determine first whether or not it is exact by applying the test (9), **§172.** If it is, integrate the coefficient of dx with respect to x and to this result add those terms which contain y only, which are obtained by integrating the coefficient of dy with respect to y.

Exercises

Are the following differential equations exact? Integrate those which are exact.

1. $3x^2y^2\,dx + 2x^3y\,dy = 0.$
2. $\dfrac{1}{y}\cos\left(\dfrac{x}{y}\right)dx - \dfrac{x}{y^2}\cos\left(\dfrac{x}{y}\right)dy = 0.$
3. $y\,e^{xy}\,(1 + x + y)\,dx + x\,e^{xy}\,(1 + x + y)\,dy = 0.$
4. $y\,e^{xy}\,dx + x\,e^{xy}\,dy = 0.$
5. $(x^3y + 2x)\,dx - (3x^2y - 5x)\,dy = 0.$
6. $\left(\dfrac{2x}{y^3} + 1\right)dx - \left(\dfrac{3x^2}{y^4} + 2y\right)dy = 0.$
7. $e^{\frac{x}{y}}\left(2 + \dfrac{x}{y}\right)dx - \dfrac{x}{y^2}\;e^{\frac{x}{y}}\left(2 + \dfrac{x}{y}\right)dy = 0.$

174. In **§155** the envelope of a family of curves was defined, and its parametric equations were found to be

$$f(x, y, c) = 0 \tag{1}$$

$$\frac{\partial f}{\partial c} = 0. \tag{2}$$

We shall now show that the envelope is tangent to each curve of the family of curves (1).

At a given point (x, y) of the curve determined by giving c a particular value in (1), the slope of the tangent is found from the equation

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0. \tag{3}$$

If the point also lies upon the envelope its coördinates satisfy (1) and (2). The equation of the envelope can be regarded as given by (1) where c is the function of x and y found by solving (2) for c. On differentiating (1) with respect to x, regarding c as a function of x and y, the slope, $\frac{dy}{dx}$, of the tangent to the envelope is given by

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial c}\frac{dc}{dx} = 0, \tag{4}$$

where

$$\frac{dc}{dx} = \frac{\partial c}{\partial x} + \frac{\partial c}{\partial y}\frac{dy}{dx}.$$

But on the envelope $\frac{\partial f}{\partial c} = 0$. Hence (4) becomes

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0. \tag{5}$$

Equations (3) and (5) show that the slope of the tangent line to the envelope at the point (x, y) is the same as the slope of the tangent line at the same point to a curve of the family (1). Hence the envelope is tangent to each curve of the family of curves (1).

CHAPTER XIX

DIFFERENTIAL EQUATIONS

175. Differential Equations. An equation containing derivatives or differentials is called a *differential equation.* If no derivative higher than the first appears it is called a differential equation of the *first order.* If the equation contains the second, but no higher derivative, the equation is said to be of the *second order.* And so on. Numerous differential equations have already occurred in this course. We shall now consider the solution of differential equations somewhat systematically.

176. General Solution. Particular Integral. Let

$$f(x, y, c) = 0 \tag{1}$$

be any equation between x, y, and the constant c. If (1) is differentiated with respect to x there results the equation

$$F(x, y, y', c) = 0. \tag{2}$$

Between (1) and (2) the constant c can be eliminated giving the differential equation of the first order

$$\phi(x, y, y') = 0. \tag{3}$$

Equation (3) follows for any value of the constant c.

Let

$$f(x, y, c_1, c_2) = 0 \tag{4}$$

be an equation involving two constants, c_1 and c_2. By differentiating (4) we obtain

$$F(x, y, y', c_1, c_2) = 0 \tag{5}$$

and

$$\phi(x, y, y', y'', c_1, c_2) = 0. \tag{6}$$

Between equations (4), (5), and (6), c_1 and c_2 may be eliminated giving the differential equation of the second order

$$\psi(x, y, y', y'') = 0. \tag{7}$$

From the equation (1) containing one arbitrary constant the differential equation of the first order (3) is obtained. From the equation (4) containing two arbitrary constants the differential equation of the second order (7) is obtained. In like manner from a relation between x and y containing n arbitrary constants a differential equation of the n^{th} order is obtained by differentiating, and eliminating the constants.

Equation (1) is a solution of equation (3). It is called the *general solution* and involves one arbitrary constant of integration, c. Equation (4) is called the *general solution* of (7). It involves two arbitrary constants, or constants of integration. It can be shown that the general solution, or general integral, of a differential equation contains a number of arbitrary constants, or constants of integration, equal to the order of the differential equation.

A *particular integral* is obtained from the general integral by giving particular values to the constants of integration.

177. Exact Differential Equations. This type of differential equation was discussed in **§173.**

178. Differential Equations; Variables Separable. The variables x and y are said to be separable in a differential equation which can be put in the form $f(x)\,dx + \phi(y)\,dy = 0$. The first member is equal to a function of x alone multiplied by dx plus a function of y alone multiplied by dy.

Illustration 1.

$$(1 + y^2)x\,dx + (1 + x^2)y\,dy = 0.$$

On dividing by $(1 + y^2)(1 + x^2)$ this equation becomes

$$\frac{x\,dx}{1 + x^2} + \frac{y\,dy}{1 + y^2} = 0.$$

Integration gives

$$\tfrac{1}{2}\log(1 + x^2) + \tfrac{1}{2}\log(1 + y^2) = C.$$

This reduces to

$$(1 + x^2)(1 + y^2) = e^{2C} = C_1,$$

or

$$y^2 = \frac{C_1}{1 + x^2} - 1.$$

Illustration 2.

$$\sqrt{1-y^2}\,dx + \sqrt{1-x^2}\,dy = 0.$$

Then

$$\frac{dx}{\sqrt{1-x^2}} + \frac{dy}{\sqrt{1-y^2}} = 0,$$

and the variables are separated. Integration gives

$$\sin^{-1} x + \sin^{-1} y = C.$$

Take the sine of each member, observing that the first member is the sum of two angles, and obtain

$$x\sqrt{1-y^2} + y\sqrt{1-x^2} = \sin C = C_1.$$

Exercises

Solve the following differential equations:

1. $(1-x)dy - (1+y)dx = 0.$ *Ans.* $(1+y)(1-x) = C.$

2. $\sin x \cos y\, dx = \cos x \sin y\, dy.$

3. $(x - \sqrt{1+x^2})\sqrt{1+y^2}\, dx = (1+x^2)dy.$

4. $\dfrac{dy}{dx} = 5y^2x.$

5. $\dfrac{dy}{dx} + \dfrac{y^2+4y+5}{x^2+4x+5} = 0.$

6. $(1+x)dy = y(1-y)dx.$ *Ans.* $y = c(1+x)(1-y).$

7. $(1-x)y\, dx + (1+y)x\, dy = 0.$

8. $\dfrac{dy}{dx} + e^x y = e^x y^2.$

9. $(x^2 + yx^2)dy - (y^2 - xy^2)dx = 0.$

10. $x\dfrac{dy}{dx} + 2y = xy\dfrac{dy}{dx}.$

11. $3e^x \sin y\, dx + (1 - e^x)\cos y\, dy = 0.$

12. $(xy + x^3y)dy - (1+y^2)dx = 0.$

Ans. $(1+x^2)(1+y^2) = cx^2.$

179. Homogeneous Differential Equations. The differential equation

$$M\,dx + N\,dy = 0 \qquad (1)$$

is said to be *homogeneous* if M and N are homogeneous functions of x and y of the same degree.

A function $f(x, y)$ of the variables x and y is said to be *homogeneous of degree n* if after the substitutions $x = \lambda x'$, $y = \lambda y'$ have been made,

$$f(x, y) = \lambda^n f(x', y').$$

Thus

$$ax^2 + bxy + cy^2$$

is homogeneous of degree 2. For, on making the substitutions indicated, it becomes

$$\lambda^2(ax'^2 + bx'y' + cy'^2).$$

The expression

$$ax^2\sqrt{x^2 + y^2} + bx^3 \tan^{-1}\left(\frac{y}{x}\right)$$

is homogeneous of degree 3. For, after the substitutions indicated above, it becomes

$$\lambda^3\left[ax'^2\sqrt{x'^2 + y'^2} + bx'^3 \tan^{-1}\left(\frac{y'}{x'}\right)\right].$$

A homogeneous differential equation of the form (1) is solved by placing $y = vx$, and thus obtaining a new differential equation in which the variables, v and x, are separable.

Illustration:

$$(x^2 + y^2)\,dx + 3xy\,dy = 0.$$

Let

$$y = vx.$$

Then

$$dy = v\,dx + x\,dv,$$

and

$$x^2(1 + v^2)\,dx + 3vx^2(v\,dx + x\,dv) = 0.$$

$$x^2(1 + 4v^2)\,dx + 3vx^3\,dv = 0.$$

Separating the variables

$$\frac{dx}{x} + \frac{3v\,dv}{1 + 4v^2} = 0.$$

$$\log\left[x(1 + 4v^2)^{\frac{3}{8}}\right] = C,$$

$$x(1 + 4v^2)^{\frac{3}{8}} = C_1.$$

On substituting $v = \frac{y}{x}$ we obtain as the solution of the given equation

$$x^{\frac{1}{4}}(x^2 + 4y^2)^{\frac{3}{8}} = C_1,$$

or

$$x^2(x^2 + 4y^2)^3 = C_2.$$

Exercises

Solve the following differential equations

1. $y^2 + x^2\frac{dy}{dx} = xy\frac{dy}{dx}.$

2. $x^2y\,dx - (x^3 + y^3)\,dy = 0.$

3. $(8y + 10x)\,dx + (5y + 7x)\,dy = 0.$

4. $(2\sqrt{xy} - x)\,dy + y\,dx = 0.$

5. $x\frac{dy}{dx} = y + \sqrt{x^2 + y^2}.$

6. $x\cos\frac{y}{x}\frac{dy}{dx} = y\cos\frac{y}{x} - x.$

7. $x\frac{dy}{dx} - y = \sqrt{x^2 - y^2}.$

8. $(y - x)\,dy + y\,dx = 0.$

180. Linear Differential Equations of the First Order. The equation

$$\frac{dy}{dx} + Py = Q, \tag{1}$$

where P and Q are functions of x only, is called a *linear differential equation.* It is of the first degree in y and its derivative. Multiply the equation by

$$e^{\int P\,dx}$$

and obtain

$$e^{\int P\,dx}\left[\frac{dy}{dx} + Py\right] = e^{\int P\,dx}\,Q. \tag{2}$$

The left-hand member is the derivative of

$$e^{\int P\,dx}\,y,$$

as may be confirmed by differentiating this product. The integration of (2) gives

$$e^{\int P\,dx}y = \int Qe^{\int P\,dx}dx + C.$$

Illustration 1.

$$\frac{dy}{dx} + xy = x^3. \tag{3}$$

Here $P = x$ and $Q = x^3$. Then

$$e^{\int P\,dx} = e^{\int x\,dx} = e^{\frac{x^2}{2}}.$$

Multiply both members of (3) by $e^{\frac{x^2}{2}}$.

$$e^{\frac{x^2}{2}}\left[\frac{dy}{dx} + xy\right] = e^{\frac{x^2}{2}}x^3.$$

Integration gives

$$ye^{\frac{x^2}{2}} = \int e^{\frac{x^2}{2}}x^3\,dx + C$$
$$= e^{\frac{x^2}{2}}x^2 - 2e^{\frac{x^2}{2}} + C.$$

Hence

$$y = x^2 - 2 + Ce^{-\frac{x^2}{2}}.$$

Illustration 2.

$$\frac{dy}{dx} + \frac{1}{x}y = x^2 + 3x + 4. \tag{4}$$

$$e^{\int P\,dx} = e^{\int \frac{dx}{x}} = e^{\log x} = x.$$

Multiply both members of (4) by x.

$$x\left[\frac{dy}{dx} + \frac{1}{x}y\right] = x^3 + 3x^2 + 4x.$$

Integration gives

$$xy = \frac{x^4}{4} + x^3 + 2x^2 + C,$$

or

$$y = \frac{x^3}{4} + x^2 + 2x + \frac{C}{x}.$$

This illustration is inserted to call attention to the well-known simple relation $e^{\log x} = x$, which there will be frequent occasion to use in solving equations of this type. It should be recalled that $e^{n \log x} = e^{\log (x^n)} = x^n$. Thus

$$e^{-\log x} = \frac{1}{x}.$$

Exercises

1. $\dfrac{dy}{dx} + 2xy = e^{-x^2}$.
2. $\dfrac{dy}{dx} + y \cos x = \sin 2x$.
3. $\cos^2 x \dfrac{dy}{dx} + y = \tan x$.
4. $(x^2 + 1) \dfrac{dy}{dx} + 2xy = 4x^2$.
5. $\dfrac{dy}{dx} + \dfrac{2y}{x+1} = (x+1)^3$.
6. $x(1 - x^2)\, dy + (2x^2 - 1)y\, dx = ax^3\, dx$.
7. $\dfrac{dy}{dx} - n\dfrac{y}{x} = e^x x^n$.
8. $(1 + x^2)\, dy + \left(xy - \dfrac{1}{x}\right) dx = 0$.
9. $\dfrac{dy}{dx} + \dfrac{1 - 2x}{x^2} y = 1$.
10. $(1 + y^2)\, dx = (\tan^{-1} y - x)\, dy$.

181. Extended Form of the Linear Differential Equation. An equation of the form

$$\frac{dy}{dx} + Py = Qy^n \tag{1}$$

is easily reduced to the linear form. For, on dividing (1) by y^n, we obtain

$$y^{-n} \frac{dy}{dx} + Py^{-n+1} = Q. \tag{2}$$

The first term of the left-hand member of (2) is, apart from a constant factor, the derivative of y^{-n+1}, which occurs in the second term. If we let $z = y^{-n+1}$ we obtain the linear differential equation

$$\frac{1}{1-n} \frac{dz}{dx} + Pz = Q,$$

or

$$\frac{dz}{dx} - (n-1)Pz = -(n-1)Q.$$

Illustration 1.

$$\frac{dy}{dx} + y \cos x = y^4 \sin 2x.$$

Dividing by y^4

$$y^{-4}\frac{dy}{dx} + y^{-3}\cos x = \sin 2x.$$

Let $z = y^{-3}$. Then

$$y^{-4}\frac{dy}{dx} = -\tfrac{1}{3}\frac{dz}{dx}$$

and the equation becomes

$$-\tfrac{1}{3}\frac{dz}{dx} + z \cos x = \sin 2x,$$

or

$$\frac{dz}{dx} - 3z\cos x = -3\sin 2x.$$

This equation can be readily solved by **§180**; y^{-3} is to be substituted for z in the result.

Exercises

1. $\frac{dy}{dx} + \frac{1}{x}y = x^2y^6.$
2. $\frac{dy}{dx} + y = xy^3.$
3. $3y^2\frac{dy}{dx} - 7y^3 = x + 1.$
4. $(1 - x^2)\frac{dy}{dx} - xy = axy^2.$
5. $\frac{dy}{dx} + \frac{2}{x}y = 3x^2y^{\frac{4}{3}}.$
6. $x\frac{dy}{dx} + y = y^2 \log x.$
7. $\frac{dy}{dx} + \frac{2}{x+1}y = \frac{x^3}{y^2}.$

182. Applications. Let there be an electric circuit, whose resistance is R, whose coefficient of self-induction is L, and which contains an electromotive force, which at first we shall suppose constant and equal to E. It is required to find the current i at any time t after the time $t = 0$, at which the circuit was closed. The equation connecting the quantities involved is readily set up. The applied *E.M.F.*, E, must overcome the resistance of the circuit and its self-induction. The former requires an *E.M.F.* equal to iR, and the latter an *E.M.F.* proportional to the time rate of change of current, viz., $\frac{di}{dt}$, and equal to $L\frac{di}{dt}$. The applied *E.M.F.*, E, must equal the sum of these two *E.M.F.*'s.

Hence
$$L\frac{di}{dt} + Ri = E. \tag{1}$$

The student will show that, if i equals zero when t equals zero, the solution of this linear equation is

$$i = \frac{E}{R}\left[1 - e^{-\frac{Rt}{L}}\right]. \tag{2}$$

If the battery or other source of $E.M.F.$ is suddenly cut out of the circuit, the current falls off in such a way that the differential equation

$$L\frac{di}{dt} + Ri = 0 \tag{3}$$

is satisfied. Show that the law at which the current falls off is

$$i = i_0 e^{-\frac{R}{L}(t - t_0)}, \tag{4}$$

if the instant at which the battery is cut out is the time $t = t_0$ and if the current at this instant is $i = i_0$.

If the $E.M.F.$ is variable, the relation between the quantities involved in the circuit is still governed by (1),

$$L\frac{di}{dt} + Ri = E, \tag{1}$$

in which E is now variable. Suppose $E = E_0 \sin \omega t$. This supposes that an alternating $E.M.F.$ is acting in the circuit. The differential equation to be solved is

$$L\frac{di}{dt} + Ri = E_0 \sin \omega t. \tag{5}$$

Show that

$$\begin{aligned} i e^{\frac{R}{L}t} &= \frac{E_0}{L}\frac{1}{\frac{R^2}{L^2} + \omega^2}\left[\frac{R}{L}\sin \omega t - \omega \cos \omega t\right]e^{\frac{R}{L}t} + C \\ &= \frac{E_0}{R^2 + \omega^2 L^2}(R \sin \omega t - \omega L \cos \omega t)e^{\frac{R}{L}t} + C \\ &= \frac{E_0}{\sqrt{R^2 + \omega^2 L^2}} \sin(\omega t - \phi)e^{\frac{R}{L}t} + C, \end{aligned}$$

where

$$\sin\phi = \frac{\omega L}{\sqrt{R^2 + \omega^2 L^2}}.$$

$$\cos\phi = \frac{R}{\sqrt{R^2 + \omega^2 L^2}}.$$

Then,

$$i = \frac{E_0}{\sqrt{R^2 + \omega^2 L^2}} \sin(\omega t - \phi) + Ce^{-\frac{R}{L}t}.$$

Since the last term becomes negligible after a short time because of the factor

$$e^{-\frac{R}{L}t},$$

it is scarcely necessary to determine C. On dropping out the last term as unimportant except in the immediate vicinity of $t = 0$, we have

$$i = \frac{E_0}{\sqrt{R^2 + \omega^2 L^2}} \sin(\omega t - \phi). \qquad (6)$$

The current, therefore, alternates with the same frequency as the *E.M.F.*, but lags behind it and differs from it in phase by ϕ. It is to be noted that the maximum value of the current is not $\frac{E_0}{R}$ but $\frac{E_0}{\sqrt{R^2 + \omega^2 L^2}}$. The quantity $\sqrt{R^2 + \omega^2 L^2}$ replaces, in alternating currents, the resistance R of the ordinary circuit. It is called the impedance of the circuit.

183. Linear Differential Equations of Higher Order with Constant Coefficients and Second Member Zero. A typical differential equation of this class is the following:

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + a_{n-1} \frac{dy}{dx} + a_n y = 0, \quad (1)$$

where $a_0, a_1, \cdots, a_n$ are constants. As the equations of this class which occur in the applications are usually of the second order we shall confine our discussion in this article to linear differential equations of the second order. Consider

$$a_0 \frac{d^2 y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = 0. \qquad (2)$$

Let us assume that

$$y = e^{mx} \tag{3}$$

and find, if possible, the values of m for which (3) is a solution of (2). The substitution of (3) in (2) gives

$$e^{mx}(a_0m^2 + a_1m + a_2) = 0. \tag{4}$$

The first factor cannot vanish. The second, equated to zero, gives a quadratic equation in m. Call its roots m_1 and m_2. Then (3) is a solution of (2) if m has either of the values m_1 or m_2, the roots of

$$a_0m^2 + a_1m + a_2 = 0. \tag{5}$$

The equation (5) in m, obtained from the given differential equation by writing m^2 for $\frac{d^2y}{dx^2}$ and m for $\frac{dy}{dx}$ is called the *auxiliary equation.*

Two solutions of (2) are

$$y = e^{m_1x} \quad \text{and} \quad y = e^{m_2x}.$$

Furthermore,

$$y = C_1e^{m_1x}$$

is a solution of (2). For, after the substitution of this value of y in (2), C_1 can be taken out as a common factor and the other factor vanishes in accordance with (4) or (5). In the same way,

$$y = C_2e^{m_2x}$$

is a solution of (2). And finally the sum of the two solutions

$$y = C_1e^{m_1x} + C_2e^{m_2x} \tag{6}$$

is a solution of (2). This can be seen by substituting in (2) and recalling that m_1 and m_2 are roots of (5). When m_1 is not equal to m_2, (6) is known as the *general solution* of the differential equation (2). It contains two arbitrary constants, the number which the general solution of a differential equation of the second order must contain.

The values of these constants are determined in a particular problem by two suitable conditions.

Illustration.

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0.$$

The auxiliary equation is

$$m^2 - 5m + 6 = 0,$$
$$(m - 2)(m - 3) = 0.$$

Hence $m_1 = 2$, $m_2 = 3$. The general solution is then

$$y = C_1e^{2x} + C_2e^{3x}.$$

Exercises

1. $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 0.$
2. $\frac{d^2y}{dx^2} - 4y = 0.$
3. $\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 10y = 0.$
4. $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 12y = 0.$
5. $\frac{d^2y}{dx^2} + 7\frac{dy}{dx} = 0.$

184. Auxiliary Equation with Equal Roots. The method just given fails when the auxiliary equation has equal roots, $m_1 = m_2$. For, equation (6), §**183**, becomes

$$y = C_1e^{m_2x} + C_2e^{m_2x}$$
$$= (C_1 + C_2)e^{m_2x}.$$

But $C_1 + C_2$ is an arbitrary constant and the solution contains only one arbitrary constant instead of two. When the auxiliary equation has equal roots, $m_1 = m_2$, equation (2) can be written in the form

$$\frac{d^2y}{dx^2} - 2m_1\frac{dy}{dx} + m_1^2y = 0.$$

Its general solution is

$$y = (C_1 + C_2x)e^{m_1x}.$$

This solution can be verified by direct substitution.

Illustration.

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0.$$

The auxiliary equation is

$$m^2 - 4m + 4 = 0.$$
$$m_1 = m_2 = 2.$$

The general solution is

$$y = (C_1 + C_2x)e^{2x}.$$

Exercises

1. $4\dfrac{d^2y}{dx^2} + 12\dfrac{dy}{dx} + 9y = 0.$

2. $\dfrac{d^2y}{dx^2} - 8\dfrac{dy}{dx} + 16y = 0.$

3. $\dfrac{d^2y}{dx^2} = 0.$

185. Auxiliary Equation with Complex Roots. If the auxiliary equation has complex roots the general solution can be written in a form different from (6), **§183.** The importance of the result will be evident at once when it is observed that it contains the harmonic functions sine and cosine. If the coefficients of the given differential equation (2), **§183** are real, and if m_1 and m_2 are complex, they must be conjugate imaginary numbers. Let $m_1 = a + ib$. Then $m_2 = a - ib$. Then (6) becomes

$$\begin{aligned} y &= C_1e^{ax + ibx} + C_2e^{ax - ibx} \\ &= e^{ax}(C_1e^{ibx} + C_2e^{-ibx}). \end{aligned}$$

Now, by (4) and (5), **§167,**

$$\begin{aligned} e^{ibx} &= \cos bx + i \sin bx \\ e^{-ibx} &= \cos bx - i \sin bx. \end{aligned}$$

Then

$$y = e^{ax}\left[(C_1 + C_2)\cos bx + i(C_1 - C_2)\sin bx\right]$$

On placing $C_1 + C_2 = A$ and $i(C_1 - C_2) = B$, we obtain

$$\begin{aligned} y &= e^{ax}(A \cos bx + B \sin bx) \\ &= e^{ax}C \cos (bx - \phi). \end{aligned}$$

In the last form the two arbitrary constants of integration are C and ϕ.

Illustration 1.

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 13y = 0.$$

The auxiliary equation is

$$m^2 + 4m + 13 = 0.$$

Hence

$$m = -2 \pm 3i.$$

Then

$$y = e^{-2x} (A \cos 3x + B \sin 3x).$$

Illustration 2.

$$\frac{d^2y}{dx^2} + 4y = 0.$$

$$m^2 + 4 = 0.$$

Whence

$$m = \pm 2i = 0 \pm 2i.$$

Then

$$y = A \cos 2x + B \sin 2x.$$

Exercises

1. $\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} + 5y = 0.$

2. $\dfrac{d^2y}{dx^2} + 9y = 0.$

3. $\dfrac{d^2y}{dx^2} + y = 0.$

4. $\dfrac{d^2\theta}{dt^2} + \omega^2\theta = 0.$

186. Damped Harmonic Motion. The resistance offered by the air to the motion of a body through it, is roughly proportional to the velocity, if the velocity is a moderate one. In §**81**, the differential equation of the motion of the simple pendulum was derived on the assumption that the force of gravity was the only force acting upon the bob of the pendulum. If the resistance of the air is also taken into account we shall have to add to the second member of the equation, $l\dfrac{d^2\theta}{dt^2} = -g \sin \theta$, a term, $-2kl\dfrac{d\theta}{dt}$ proportional to the velocity $l\dfrac{d\theta}{dt}$. See equation (1), §**81**. The differential equation of the motion is then

$$l\frac{d^2\theta}{dt^2} = -g \sin \theta - 2kl\frac{d\theta}{dt}. \tag{1}$$

The negative sign is used before the last term because the force due to the resistance of the air acts in a direction opposite to that of the motion. The advantage of choosing $2k$ as the proportionality factor instead of k will appear later. k is a positive constant.

From (1) we obtain

$$\frac{d^2\theta}{dt^2} + 2k\frac{d\theta}{dt} + \frac{g}{l}\sin\theta = 0. \tag{2}$$

As in **§81** assume that θ is small and replace $\sin\theta$ by θ. Also let $\frac{g}{l} = \omega^2$. Then (2) becomes

$$\frac{d^2\theta}{dt^2} + 2k\frac{d\theta}{dt} + \omega^2\theta = 0.$$

This is a linear differential equation of the second order with constant coefficients and can be solved by the method of **§185.** The auxiliary equation is

$$m^2 + 2km + \omega^2 = 0,$$

whence

$$m = -k \pm \sqrt{k^2 - \omega^2}.$$

When the velocity is not very great, as in the case of an ordinary pendulum, k is very small for air and is much less than ω. The expression under the radical sign is negative. We write then

$$m = -k \pm i\sqrt{\omega^2 - k^2},$$

$\omega^2 - k^2$ being positive.

The solution of (3) is

$$\theta = Ae^{-kt}\cos\left[t\sqrt{\omega^2 - k^2} - \epsilon\right],$$

or, multiplying both sides by l and replacing Al by B,

$$s = Be^{-kt}\cos\left[t\sqrt{\omega^2 - k^2} - \epsilon\right].$$

The motion is a damped harmonic motion. The amplitude decreases with the time. The period $\frac{2\pi}{\sqrt{\omega^2 - k^2}}$ is a little greater than $\frac{2\pi}{\omega}$, the period of the free motion.

Since k is very small in comparison with ω, we can, for an approximate solution of our problem, neglect k^2 in comparison with ω^2. Equation (4) becomes

$$s = Be^{-kt}\cos(\omega t - \epsilon). \tag{5}$$

This represents the motion with a high degree of approximation.

The arbitrary constants B and ϵ can be determined by suitable initial conditions. For example, let it be given that $s = s_0$ and $\frac{ds}{dt} = 0$ when $t = 0$. On differentiating (5) we obtain

$$\frac{ds}{dt} = Be^{-kt}[-k\cos(\omega t - \epsilon) - \omega\sin(\omega t - \epsilon)]. \tag{6}$$

For $t = 0$ we obtain from (5) and (6)

$$s_0 = B\cos\epsilon$$
$$0 = B(-k\cos\epsilon + \omega\sin\epsilon).$$

From the latter of these two equations

$$\tan\epsilon = \frac{k}{\omega}.$$

From the former

$$B = s_0\sec\epsilon = s_0\sqrt{1 + \frac{k^2}{\omega^2}} = s_0$$

to the degree of approximation used above. We have then as the approximate equation of motion

$$s = s_0e^{-kt}\cos(\omega t - \epsilon) \tag{7}$$

where

$$\epsilon = \tan^{-1}\frac{k}{\omega} = \frac{k}{\omega}, \text{ approximately.}$$

Since k is very small, ϵ is very small.

It follows from (5) and (7) that the period of the pendulum in the case just considered is very little different from that of the same pendulum swinging in a vacuum. The amplitude of the swing, however, is affected and diminishes continually with the time.

INDEX

(Numbers refer to pages)

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