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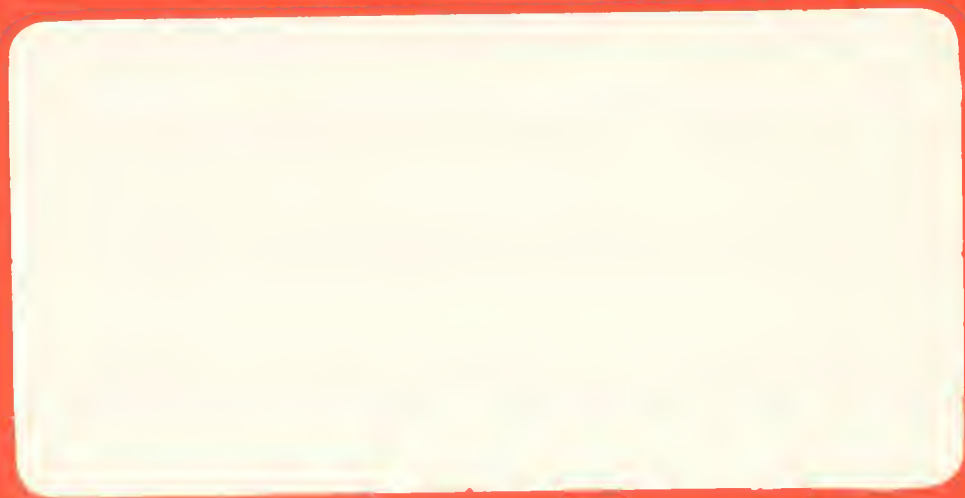
Faculty Working Papers

CAPITAL MARKET EQUILIBRIUM WITH DIVERGENT
HOLDING PERIOD LENGTH ASSUMPTIONS

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Summary:

    This paper uses Black's zero beta portfolio concept to demonstrate
the linearity of the capital market model despite divergent investor
holding period length assumptions. This linearity will exist for any
holding period length assumption for which the market model may be tested.
Tests based on different holding period lengths will result in different
levels of relative portfolio performance and may explain the negative
relationship between performance and risk frequently observed in the
literature.

CAPITAL MARKET EQUILIBRIUM WITH
DIVERGENT HOLDING PERIOD LENGTH ASSUMPTIONS

Gressis, Philappatos and Hayya (1976) (GPH) have pointed out that
as the holding period length assumption is changed, the Capital Market
Line (CML) will intersect the Efficient Frontier (EF) at different points
causing different investors to hold different efficient portfolios. GPH
assert that these divergent portfolio holdings will result in an ineffi-
cient market portfolio--and dire consequences for the capital market
model.

This paper shows that Black's (1972) zero beta portfolio concept
can be used to demonstrate the linearity of the CAPM, despite divergent
investor holding period length assumptions for _any_ holding period length
assumption for which it may be tested.

The paper points out that CAPM tests based on different holding
period length assumptions will result in different CML slopes and inter-
cepts. This phenomena is suggested as a possible explanation for the
negative relationship between performance and systematic risk frequently
observed in the literature.

Theoretical Foundation

Tobin (1965) points out that under the assumptions of stationarity
and independence of successive portfolio returns the N period expected
return, u_N, of a portfolio is:

$$u_N = (1 + u_1)^N - 1 \qquad (1)$$

here u_1 is the single period expected return. The n period variance
V_N is:

$$V_N = ((V_1 + (1 + u_1)^2)^N - (1 + u_1)^{2N} \qquad (2)$$

where V_1 is single period variance.[1]

These equations show that a portfolio which has the minimum single period variance at a given level of single period expected return will also have the minimum n period variance at the corresponding n period expected return. The equations also show that N period variance is a positive function of single period variance and single period expected return. The variance of high expected return portfolios will therefore increase faster with increasing holding period length than the variance of low expected return portfolios. Together, these equations indicate that as the holding period assumption is lengthened the shape of the efficient frontier changes but the composition does not.

GPH point out that these changes in shape cause the CML to intersect the efficient frontier at different intersection points for different holding period length assumptions. GPH point out that these different intersection points can be more than one corner portfolio apart and that linear combinations of such portfolios are not generally on the efficient frontier. This will result in an inefficient market portfolio and disastrous consequences for the capital market model.[2]

Holding Period and the Zero Beta Portfolio

The zero beta portfolio concept was developed by Brennen (1971) and Black (1972). They used the concept to demonstrate the linearity of the CAPM despite divergent borrowing and lending rates. Their proof is too lengthy (and well known) to be repeated here but an important

point should be noted: Resolution of the divergent borrowing and lending rate problem actually has little (directly) to do with borrowing and lending. The zero beta concept actually demonstrates the linearity of the CAPM in any situation where different investors wish to hold different efficient portfolios (if unrestricted short selling can be assumed). Divergent borrowing and lending rates are merely one possible reason for investors to hold different efficient portfolios.

GPH point out that divergent holding period length assumptions will also cause different investors to hold different portfolios. Equations (1) and (2) show that all of these portfolios will be efficient for all holding period length assumptions. Therefore, the zero beta portfolio concept can be used to demonstrate the linearity of the CAPM despite divergent holding period length assumptions. Moreover, since all of these portfolios are efficient for all holding period lengths, the CAPM will be linear for any holding period length assumption for which it may be tested (i.e., monthly data? daily data?).

Tests based on different holding period lengths will produce different slopes and intercepts for the CML. GPH point out (and Table 1 confirms) that the longer the holding period assumption the lower the risk and return of the CML/EF tangency portfolios. This indicates that if the holding period length assumption used to test the CAPM is longer (shorter) than the holding period length implied by the overall (market) clearing portfolio, the expected return axis intercept of the CML will be higher (lower) than the risk free rate and low risk portfolios will appear to be overvalued (undervalued) relative to high risk portfolios.

Friend and Blume (1970), Miller and Scholes (1972) and Black, Jensen and Scholes (1972) have found that low risk portfolios outperform high risk portfolios. This is consistent with the idea that the horizon periods used by these researchers (monthly, annual and monthly periods respectively) were longer than the holding period appropriate to the market portfolio.[3]

The Unlimited Short Selling Assumption

The zero beta portfolio concept will only work _if_ unlimited short selling can be assumed. It is therefore necessary to investigate the validity of this assumption in resolving the divergent holding period length problem.

At first glance, the unlimited short selling assumption seems hopelessly inappropriate for any purpose. In reality, short selling is not an immediate source of funds. In fact, the short seller is required to put up considerable collateral for the privilege of short selling. Moreover, actual computations (i.e., see Alexander (1977)) indicate that the zero beta portfolio involves preposterous amounts of short selling.

Blume (1973, p. 30) points out that the picture is not as bleak as it appears:

> ... the capital asset pricing theory may be robust to
> violations of the short sales assumption if it so happens
> that each investor's optimal portfolio involved no negative
> or short holdings. In this case, one could think of an
> investor's portfolio as consisting of a linear combination
> of the market portfolio and a zero beta portfolio. Such a
> zero beta portfolio might require short sales if it were
> actually to be held. However, if in combination with the
> market portfolio there were no net short positions, no
> actual short sales need to have taken place. Thus it is

theoretically possible that the short sales assumption
may be less restrictive than the usual risk-free rate
assumption.

Unfortunately, GPH point out that the CML/EF intersection points
resulting from different holding period length assumptions are generally
more than one corner portfolio apart. This shows that some portfolios
would involve short selling if it were permitted. Needless to say,
the assumptions which underlie a theory need not be perfectly valid for
the theory to be useful. It is probably not possible to make a defin-
itive statement as to the amount of short selling which can be permitted
before the zero beta approach becomes invalid. However, a generalization
is possible: For efficient portfolios with expected returns below the
market expected return, the further the portfolio is from the market,
the more unlikely it is to involve short selling. Similarly, for effi-
cient portfolios above the market; the higher the expected return of
the portfolio the more likely it is that short selling will be a
problem.[4]

These generalizations make it possible to compare the problems
created by divergent borrowing and lending rates with the problems
created by divergent holding period lengths. The literature has long
accepted (implicitly) the validity of the unlimited short selling as-
sumption in the borrowing and lending rate case. This paper points
out that the problems created by divergent holding period lengths are
similar.[5]

GPH calculate efficient frontiers for holding period lengths
ranging from 1 to 30 months.[6] They present 6 points on each efficient
frontier for each holding period length. This paper uses polynomial

TABLE 1

EXPECTED RETURN OF THE "MARKET" PORTFOLIO FOR
DIFFERENT RISK FREE RATE AND HOLDING PERIOD LENGTH ASSUMPTIONS

Holding Period Length (months)	Annual Risk Free Rate (%)			
	5%	6%	7%	8%
1	15.69	15.77	15.87	15.99
10	15.66	15.74	15.83	15.95
30	15.63	15.69	15.77	15.87
60	15.59	15.64	15.71	15.79*
120	15.58	15.61	15.65	15.70*

*The longest holding period assumption of the GPH paper was 30 months.
The points on the 60 and 120 month efficient frontiers presented in
Appendix 1 were calculated using equations (1) and (2) based on monthly
data. This procedure is consistent with the way in which GPH generated
their data.

curve fitting to express these curves algebraically (see Appendix 1).[7]
Tangency points between these curves and the CML were calculated for
different risk free rate and holding period length assumptions (see
Appendix 2).

[Insert Table 1 about here]

This analysis suggests that the deviation in tangency points caused
by a reasonable range of holding period length assumptions is generally
comparable to the deviation of tangency points resulting from a reason-
able range of interest rate assumptions (see Table 1). For example, on
the 10 month efficient frontier, risk free rates of 5% and 8% (annual)
produced CML/EF tangency points with annualized expected returns of
15.66% and 15.95% respectively. On the other hand, at a 6% (annual)
risk free rate, holding period lenghts of 1 and 120 months produce tan-
gency points with annualized expected returns of 15.77% and 15.61%
respectively. Therefore, the unlimited short selling assumption is
likely to be as valid in resolving the divergent holding period length
problem as it is in resolving the divergent borrowing and lending rate
problem.[8]

Summary

GPH have pointed out that the efficient frontier changes shape from
one holding period length assumption to another; and that these changes
in shape cause the capital market line to intersect the efficient fron-
tier at different points for different holding period length assumptions.

This paper points out that:

1) Black's (1972) zero beta portfolio concept can probably be used
   to resolve the divergent holding period length problem.

2) If the zero beta portfolio approach is applicable, the CAPM
    will be linear for <u>any</u> holding period length assumption for
    which it may be measured.

3) Tests of the CAPM based on different holding period length
    assumptions will result in different CML slopes and intercepts.
    This observation may explain the negative relationship between
    systematic risk and performance frequently observed in the
    literature.

One final word of caution: These results (and GPH's) are dependent
on Tobin's assumption that successive returns are stationary and indepen-
dent. For a look at some of the effects of autocorrelation see Gilster
(1979) or Hawawini (1979).

Efficient Frontier Points and Equations for Various Holding Period Assumptions

Number of periods k	Points on Efficient Portfolios												Estimated Equations of Efficient Frontiers
	A		B		C		D		E		F		
	$\mu_{(k)}$	$\sigma_{(k)}$	$\mu_{(k)}$	$\sigma_{(k)}$	$\mu_{(k)}$	$\sigma_{(k)}$	$\mu_{(k)}$	$\sigma_{(k)}$	$\mu_{(k)}$	$\sigma_{(k)}$	$\mu_{(k)}$	$\sigma_{(k)}$	
1	0.01172	0.02940	0.01236	0.03096	0.01308	0.03584	0.01380	0.04293	0.01452	0.05133	0.01524	0.06050	$224.7020\,u_1^2 - 5.2637\,u_1 + 0.0317 = \sigma_1^2$
10	0.12354	0.10345	0.13076	0.10959	0.13882	0.12775	0.14694	0.15421	0.15511	0.18589	0.16334	0.22097	$24.0810\,u_{10}^2 - 5.9478\,u_{10} + 0.3380 = \sigma_{10}^2$
30	0.41830	0.22716	0.44579	0.24383	0.47696	0.28879	0.50878	0.35456	0.54126	0.43516	0.57441	0.52736	$9.3280\,u_{30}^2 - 7.8055\,u_{30} + 1.6845 = \sigma_{30}^2$
60*	1.01196	0.45858	1.08977	0.50195	1.18084	0.60882	1.27582	0.76666	1.37485	0.96697	1.47813	1.20643	$5.7910\,u_{60}^2 - 11.7490\,u_{60} + 6.1694 = \sigma_{60}^2$
120*	3.04800	1.32165	3.36714	1.50469	3.75608	1.91394	4.17924	2.53652	4.63993	3.37954	5.14110	4.47156	$4.3405\,u_{120}^2 - 26.8267\,u_{120} + 43.1899 = \sigma_{120}^2$

*The longest holding period assumption of the GPH paper was 30 months. The points on the 60 and 120 month efficient frontiers listed above were calculated using equations (1) and (2) based on monthly data. This procedure is consistent with the way GPH generated their data.

APPENDIX 2

The Computation of Expected Returns for CML/EF Tangency Portfolios

Given a risk free rate, I, the slope of the CML in $[u,\sigma]$ space is:

$$\frac{du}{d\sigma} = \frac{u_m - I}{\sigma_m} \tag{1}$$

where u_m and σ_m are the market expected return and standard deviation. An efficient frontier plotted in $[u,\sigma]$ space is a branch of the hyperbola:

$$(Au^2 + Bu + C)^{1/2} = \sigma \tag{2}$$

Taking derivatives:

$$\frac{du}{d\sigma} = \frac{2(Au^2 + Bu + C)^{1/2}}{2Au + B} \tag{3}$$

Substituting equation (2) into equation (1) and then equating equations (1) and (3) gives:

$$\frac{u_m - I}{(Au_m^2 + Bu_m + C)^{1/2}} = \frac{2(Au^2 + Bu + C)^{1/2}}{2Au + B} \tag{4}$$

At the point of tangency, $u = u_m$. Solving for u_m:

$$u_m = -(BI + 2C)/(2AI + B) \tag{5}$$

The u_m of a specific holding period length assumption is calculated from equation (5) using the A, B and C parameters of the efficient frontier appropriate to that holding period length (see Appendix 1) and the risk free rate appropriate to that holding period length.

FOOTNOTES

1. Tobin, in effect, assumes stationarity of portfolio returns. GPH assume stationarity of security returns and correctly point out that under this assumption, portfolio stationarity will, in general, require rebalancing. The reader is free to make whichever assumption he finds most convincing.

2. Ross (1977, p. 1125) points out that "... if the market portfolio is inefficient, the CAPM will not hold." His article presents a simple but elegant justification for this statement.

3. Black (1972) has pointed out that this problem can also result from divergent borrowing and lending rate assumptions. In reality investors probably have different interest rate and holding period length assumptions. The phenomena observed by Black, Jensen and Scholes probably results from a combination of the two effects.

4. This is a direct consequence of the fact that under the unlimited short selling assumption all efficient portfolios can be expressed as linear combinations of the market and zero beta portfolios. The market portfolio involves no net short selling. A security which is short sold (held long) in the zero beta portfolio will decline linearly as a proportion of non-market efficient portfolios as portfolio expected return decreases (increases). Ultimately this decline in representation results in net short selling.

5. Unfortunately a precise comparison may be impossible. Divergent interest rate assumptions will result in concentrations of portfolio holdings at tangency points corresponding to the borrowing and lending rates. Divergent holding period length assumptions will probably produce a more uniform distribution of portfolios but the distribution may include extreme values (i.e., a holding period of 100 years). It is not clear which type of distribution creates the greatest short selling problems for the capital market model.

6. GPH created their efficient frontiers using 21 securities randomly selected from the NYSE. Their data encompassed the 167 monthly returns for the period 1956-1970.

7. All curves were fitted to quadratic functions. This will only provide an exact fit if short selling is unrestricted. GPH do restrict short selling. Nevertheless, the equations presented in Appendix 1 provide an excellent fit to the data points. The reader is encouraged to verify this for himself.

8. Table 1 presents an optimistic picture of the applicability of the zero beta concept to combinations of divergent interest rate and holding period length assumptions. Even the worst combinations of divergent holding period lengths and interest rates (1 month and an annual rate of 8% vs. 120 months and an annual rate of 5%) fail to produce substantial differences in CML/EF tangency points.

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