OODE,

SCIENCE AND ART DEPARTMENT
OF THE COMMITTEE OF COUNCIL ON EDUCATION,
SOUTH KENSINGTON MUSEUM.

A CATALOGUE

OF A COLLECTION OF MODELS OF

RULED SURFACES,

CONSTRUCTED

BY M. FABRE DE LAGRANGE;

WITH AN APPENDIX, CONTAINING AN ACCOUNT OF THE APPLICATION OF ANALYSIS TO THEIR INVESTIGATION AND CLASSIFICATION,

BY C. W. MERRIFIELD, F.R.S.,

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The set of Models described in this Catalogue is deposited in the Educational Collection at the South Kensington Museum.

Catalogue of a Collection of Models of Ruled Surfaces, constructed by M. Fabre de Lagrange, with an Appendix, containing an account of the application of Analysis to their investigation and classification,

BY C. W. MERRIFIELD, F.R.S.,

Principal of the Royal School of Naval Architecture and Marine Engineering, and Superintendent of the Naval Museum at South Kensington.

INTRODUCTION.

This collection illustrates the principal types of the class of surfaces which can be traced out in space by the motion of a straight line.

These surfaces, on account of the facility with which they can be constructed and represented, and of the ease with which their intersections can be determined, are of more consequence than any others in the geometry of the Industrial Arts. It is only in small work, which can be put into the lathe, that the class of surfaces of revolution approaches them in respect of general utility. The most important surfaces of all, the plane, the right cylinder, the right cone, and the common screw, belong to both classes.

The representation of the surfaces by means of silk threads is of course only approximate; an approximation of the same character as the representation of a curve by a dotted or chain line, or by a series of right lines touching the actual curve.

FIG. 1.

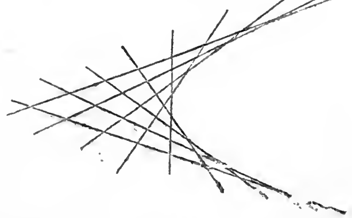


FIG. 2.

Fig. 1 is an example of the first, and Fig. 2 of the second. In both cases, the curve, although not actually drawn, is indicated with sufficient approximation for most practical purposes. Models Nos. 10 and 30 also afford illustrations of the principle exhibited in Fig. 2.

The models are constructed with especial reference to the possibility of changing their shape, by moving some of the supports of the strings, by altering the lengths or positions of certain parts, or by converting upright forms into oblique. This possibility of *deformation,* as the process is technically called, greatly enhances the value of the models, by allowing them to represent a much greater variety of surfaces than if they were fixed. They are, however, too delicate to be much pulled about, and, unless they are very cautiously handled, the strings are apt to become entangled or break. They should never be used except by a person who understands them, and they should not be shifted without some good reason.

In order to make this collection as useful as possible to the student of geometry, it has been thought advisable to give, in an appendix, a short account of the application of analysis to the investigation of these surfaces, and of their properties. The statement of these properties is scattered over a great number of treatises and tracts, and there exists no single work which gives a full account of ruled surfaces. The appendix, of course, requires some knowledge of analytical geometry of three dimensions. Any of the smaller modern treatises, such as Aldis or Leroy, contain more than is necessary as an introduction to the subject. The statements in the appendix have been chiefly taken from Monge's *Applications de l'Analyse à la Géométrie.*

Geometrical drawings of most of the surfaces represented by these models are contained in Bradley's *Practical Geometry* (2 vols., oblong folio, published by Chapman and Hall). Many of them will also be found in the French treatises on practical and descriptive geometry, such as Leroy, Adhémar, Lefebure de Fourcy, De la Gournerie, and in their treatises on *Stereotomy* and *Stone-cutting* (*coupe des pierres*). Many of them are also given in Sonnet's *Dictionnaire des Mathématiques Appliquées.*

CATALOGUE.

1. Hyperbolic Paraboloid generated by a single system of right lines.

Two bars each pierced with holes equally spaced. One bar is fixed, the other swings round an axis, which, moreover, can be inclined at different angles to the fixed bar.

When the bars are parallel the strings indicate a plane. When they are inclined to one another, but still in the same plane, the strings still indicate a plane; but when the bars are not in the same plane, the surface is the hyperbolic paraboloid.

This surface is sometimes called the *twisted plane*. But it must not be supposed that it can be made by bending a plane. On the contrary, when the surface *is* twisted, no two of the strings lie in the same plane, and, therefore, no part of the surface is plane. It can neither be flattened nor made from a plane, without stretching or contraction.

The hyperbolic paraboloid is the natural surface proper for a ploughshare.

2. Hyperbolic Paraboloid.

Two bars pierced with holes at equal distances, the holes being connected by two different systems of strings. The surface, as well as the arrangement, is very nearly the same as in No. 1, only that there are two paraboloids instead of one. As the movable bar swings round, the paraboloid opens out while the other closes up. If the bars are swung so as to be in the same plane, one system of strings describes a plane by parallel lines, and the other by lines radiating from a point. If one bar is now turned so as to be end for end, we still get a plane, the set of parallel lines now passing through a point, while the set which previously passed through a point has now become parallel.

The pair of paraboloids intersect in three right lines. There is also a fourth intersection on the "line at infinity."

3. Hyperbolic Paraboloid.

Two bars, equally spaced; each turns on an arm perpendicular to itself, and one arm swings on a pillar. These arms can be ranged in one plane, and also turned end for end.

4. Hyperbolic Paraboloid generated by two systems of right lines.

A skew quadrilateral with four equal sides, each pierced with the same number of holes, equally spaced. The model exhibits the double generation of the surface. The plane containing two of the sides turns about hinges connecting it with the plane of the other two sides. By closing or opening this hinge the paraboloid opens out or closes. When completely open, it forms a plane divided into *diamonds*. When completely closed it again forms a plane, but the division is no longer uniform. The strings then become tangents to a plane parabola.

5. Hyperbolic Paraboloid.

A skew quadrilateral turning upon four hinges with parallel axes or pins.

The difference between this and the last is not in the kind of surface or mode of generation, but in the manner of *deforming* the surface. In No. 4 the lengths of the strings alter; while in this model they remain unaltered. Moreover, although the surface flattens in two ways, yet in both ways the strings become tangents to a plane parabola instead of parallel.

This model is well adapted for showing the leading sections of the solid. All sections parallel to the pins of the hinges are plane parabolas, which degenerate into right lines when taken also parallel to the brass bars. Any other sections, whether perpendicular to the hinges or inclined to them, give hyperbolas, which degenerate into a pair of right lines when the plane of section is a tangent to the surface.

It may be worth while to remark that there is nothing absurd in the tangent plane to a surface cutting that surface, as a student unaccustomed to those subjects might at first think. On the contrary, when a surface is bent one way in one direction and the other way in the opposite direction, the tangent plane *must* cut it. In this case, the plane passing through any two intersecting strings is a tangent plane, and evidently cuts the surface along each string.

If we imagine two planes parallel to the hinge pins, and each bisecting a pair of opposite bars, we obtain the *asymptotic planes* of the paraboloid, each of which is the assemblage of the asymptotic lines of the hyperbolas parallel to the principal hyperbolic section. Their being asymptotic has reference to these hyperbolas, and not to the parabolic character of the surface.

6. Hyperbolic Paraboloid.

A skew quadrilateral, with its opposite sides equal in length, and pierced with holes at equal distances.

Nearly similar to No. 5, but differently mounted, and with the sides of different lengths, the alternate sides only being equal. It is virtually a slightly different aspect of the same surface as No. 5.

7. Hyperbolic Paraboloid.

A skew quadrilateral, with all its sides equal, and pierced holes at equal distances.

As far as the curved surface is concerned, the same as No. 5. But the hinges are altered in direction, and the model shows plans and elevations of the right line generators of the surface. The rings also show parabolic sections of the surface.

In consequence of the alteration in the direction of the hinges, the spacing of the inclined bars, although equidistant, is at a different pitch from that of the horizontal bars.

8. Hyperbolic Paraboloid.

A skew quadrilateral with all its sides equal, and pierced with holes at equal distances. It shows the plans and elevations of the right line generators. The rings show the parabolas of the *principal sections.*

No. 7 represents one quarter of what is shown in No. 8. The upper corners of Nos. 7 and 8 correspond; but the lower corner of No. 7 corresponds with the middle ring of No. 8.

9. Hyperbolic Paraboloid.

A skew quadrilateral with all its sides unequal. The surface is the same as Nos. 7 and 8, but the proportions and the portion of the surface chosen for representation are different. The quadrilateral base being irregular, the strings alter in length as the surface is deformed by closing the hinges.

10. Hyperbolic Paraboloid.

Skew quadrilateral, pivoting on a single hinge. Intended to show the construction of the parabola connecting two roads which meet obliquely. This construction is used by engineers in laying out roads.

11. Hyperboloid of one Sheet.

Two rings or circles in parallel planes are pierced with equally spaced holes. In a certain position the threads give, 1st, a cylinder, and 2ndly, a cone.

The upper ring turns round a pin at its centre. In turning it, the cylinder closes in and the cone opens out, each altering into a hyperboloid of one sheet. We can go on turning the ring until these coincide in one hyperboloid, of which we thus get both systems of generating lines.

If the rings are set on a slope the hyperboloid is elliptic. If the rings are horizontal the hyperboloid is one of revolution.

Sloping one ring so as not to be parallel with the other, gives rise to some curious ruled surfaces, but these are not in general hyperboloids.

12. Hyperboloid of one Sheet.

Two rings of different radius in parallel planes are divided into the same number of equal parts. The smaller and upper ring turns round a pin at its centre. In a particular position of the rings, the threads give two cones. Turning the ring transforms each of the cones into a hyperboloid, and when the two hyperboloids coincide, we get the two systems of right line generators.

The same stand also has a model of a hyperboloid with only one set of strings. By turning the upper ring either way it deforms into a cone, in the one case with its vertex between the rings, and in the other with its vertex at a considerable height above the rings.

Both these can have their upper rings moved along the top bar so as to incline the surfaces. We still get cones and hyperboloids, but it is only when the rings are horizontal and centre to centre, that we get surfaces of revolution.

13. Hyperboloid of one Sheet; with its asymptotic cone.

14. Hyperboloid of one Sheet; with its asymptotic cone.

The tangent plane to the cone is also drawn. It meets the hyperboloid in two parallel right lines.

One of these right lines is the line of contact of a hyperbolic paraboloid with the hyperboloid, and the tangent plane is one of the director planes of the paraboloid, both systems of generating lines of which are exhibited.

15. Hyperboloid of one Sheet.

A slight variation from No. 14. The paraboloid only shows one system of right line generators, and the tangent plane is made by parallel instead of radiating lines.

16. Hyperboloid of one Sheet, and its tangent paraboloid.

This shows the transformation of a cylinder and its tangent plane into a hyperboloid and its tangent paraboloid.

17. Conoïd with its director plane. The director curve is a plane curve.

By shifting the position of the brasses, the conoïds deform into different conoïds or other allied surfaces.

18. Conoïd with a director cone. The director curve is of double curvature.

By shifting the position of the brasses the conoïds deform into different conoïds or other allied surfaces.

19. Conoïd showing both sheets of the surface.

By shifting the position of the brasses the conoïds deform into different conoïds or other allied surfaces.

20. Conoïds. Model showing the transformation of a cylinder into a conoïd and back again. Also model showing the transformation of a cone into a conoïd and back again. It is to be noticed that the head-lines of the two conoïds, that is to say, the right line in which the two sheets of each conoïd meet, are perpendicular to one another.

The transformation is effected by making the upper semicircle turn through two right angles.

21. Conoïds.

Intersection of two equal conoïds having a common director plane. The horizontal intersection is a plane ellipse.

22. Conoïd, in contact with a hyperbolic paraboloid.

23. Conoïds. Two equal circles in parallel planes, divided equi-distantly, are connected by threads, so as to form four surfaces.

A cylinder.	A conoïd.
A cone.	A second conoïd.

The director planes, as well as the head lines, of these conoïds are at right angles to one another.

24. Conoïds.

Two equal circles in parallel planes are connected by threads so as to form four surfaces.

A cylinder.
A cone.
A conoïd.
A second conoïd, with its director plane and line at right angles to those of the former.

Same arrangement as No. 23, except that the lower ring is replaced by a plane of section a little higher up. The section gives,—

For the cone, a circle smaller than the upper ring.
For the cylinder, a circle of the same size as the upper ring.
For the conoïds, two ellipses turned crosswise.

25. Model exhibiting the simultaneous transformation of a conoïd into a cylinder, a cylinder into a conoïd, the paraboloid touching the conoïd into the tangent plane of a cylinder, and the tangent plane of a cylinder into the tangent paraboloid of a conoïd, and reciprocally.

The changes may be arranged as follows:—

From.	Into.
Conoïd.	Cylinder.
Tangent paraboloïd.	Tangent plane.
Cylinder.	Conoïd.
Tangent plane.	Tangent paraboloïd.

These changes are all effected simultaneously by one movement, which can be reversed.

26. Model exhibiting the transformation, first, of a conoïd into a cylinder. Second, of the tangent paraboloid of the conoïd into the tangent plane of the cylinder.

27. French Skew Arch (*biais passé*).

The inner drum, of yellow thread, represents this surface. It is a skew surface, with a right line director; and its faces, the planes of the two semicircles, are usually parallel, although the model permits them to be placed obliquely to one another. The horizontal line joining the centres of the two large semicircles is the right line director.

The construction for any one of the generating lines is as follows:—Draw a plane through the right line director at any selected obliquity. It will, of course, give the radii of the outside circles, and the line joining the points at which it cuts the inside semicircles will be a generator of the surface. This line will evidently pass through the director line, because it is in the same plane with it.

In stone or brickwork, the sides of the voussoirs will be given by the auxiliary plane in question. When the openings are parallel the voussoir joints are therefore plane, and the simplicity thus gained is the chief reason for adopting this form of skew arch. It is usual to take the right line director perpendicular to the openings, and symmetrical to them; that is to say, passing through the middle point of the parallelogram of the springing plane.

When the openings are not parallel, the voussoir joints shown by the model are deformed into hyperbolic paraboloids. This deformation, is, however, very slight, and in practical work would be avoided altogether by adhering to the principle of drawing a plane through the director line.

The spacing of the voussoirs is usually determined by dividing the outer semicircle into equal parts.

This form of arch is inconvenient when the obliquity, and the length of the barrel are excessive, for the generators are not generating lines of the cylinder containing the opening semicircles, but chords of it, and, therefore, at the middle, falling considerably inside it. The arch therefore droops in the middle, and this would be ugly and inconvenient if the proportions were excessive.

It is interesting to compare this surface with the skew vault of Marseille (*arrière voussure de Marseille*), an example of which is shown in the set of plaster models contributed by the "Brothers of the Christian Schools," and another, in imitation brickwork, among M. Schröder's models of furnaces. In this case the curvilinear directors do not tally with one another, although they remain parallel, and the right line director is a vertical line behind the smaller arch. The construction for the right line generators is the same for both, namely, to consider an auxiliary plane pivoting about the right line director.

28. Staircase Vault for a square well (*vis St. Gilles carrée*).

29. Staircase Vault. Model for exhibiting some properties of this ruled surface, by showing how it is obtained from the deformation of a cylinder (*douelle de la vis St. Gilles carrée*).

30. Cylinder with Helix and developable Helixoid.

The helix is simply a screw thread. The developable helixoid, shown by the purple threads, is the surface swept out by the right line tangents of the helix. If we consider that each gore can be turned a very little bit about the thread which separates it from the next gore, we see that the surface can be flattened out or developed into a plane, without any crumpling. This happens because every two consecutive generating lines meet one another on the helix. That is why its surface is called developable. Its section by a horizontal plane is the involute of the circle.

The model allows the pitch of the helix to be shortened by lowering the upper plate, and the cylinder can also be inclined. When oblique, however, the curve which replaces the helix is not such a screw thread as can be turned in the lathe.

31. Skew Helixoid.

This surface is described by a right line which always passes through the axis of a cylinder and makes a constant angle with that axis. It also passes through a helix or screw thread traced on the cylinder. The model only shows the surface, not the cylinder. It is the surface of what is known as the screw with a triangular thread. The section by a horizontal plane is the spiral of Archimedes.

This is not the commonest form of the skew helixoïd; that is best seen on the underside of a screw staircase, or on the driving face of a common screw propeller. In these, two generating lines are at right angles to the axis.

The surface may also be considered as generated by a line which makes a constant angle with a given fixed line, and moves up that line, and at the same time turns round it, at uniform rates.

32. Skew Surface with its tangent paraboloid, capable of transformation into another skew surface while the paraboloid deforms into a plane.

This is (for a certain position of the lower semicircle) a skew surface with a director plane, the plane being vertical. The director curves are: one of them a circle divided equidistantly, the other a semicircle divided so as to keep the strings parallel to the director plane.

Intersections of ruled Surfaces.

33. Intersection of two cones having double contact with one another, that is to say, having a pair of tangent planes in common.

The consequence of their having double contact is that their curve of intersection breaks up into two plane ellipses.

The vertices of the cones slide along a rule which turns on a universal joint. See also model No. 38.

34. Common groin. Intersection of two cylinders having a pair of common tangents. The model may be set square or oblique.

35. Intersection of two cylinders, one piercing the other so as to give two separate loops of intersection.

36. Intersection of two cylinders, having a common tangent, so as to give a curve having a double point at the point of contact.

37. Intersection of two cylinders, neither completely piercing the other, so as to give only one loop of intersection.

38. Intersection of two cones, having double contact, along a pair of plane ellipses.

39. Groin. Oblique intersection of two splayed vaults of the same spring.

40. A pair of intersecting planes, which, by pulling the brass ball so as to give simultaneous rotation to the two upper rods, deform into paraboloids first, and then into planes described by radiating strings.

41. Intersecting cylinder and plane. By pulling the brass ball the head brasses rotate together, and the cylinder deforms into, first, a hyperboloid, and then a cone, while the plane deforms into, first, a paraboloid, and then again into a plane with radiating lines.

42. A pair of intersecting cylinders on circular bases. By pulling the brass ball the head brasses rotate together, and the cylinders deform, first, into hyperboloids, and then into cones.

43. A pair of intersecting cylinders on irregular bases. By pulling the brass ball the head brasses rotate together, and the cylinders deform, becoming at last cones.

44. Groin.

Model showing the deformation of a common groin, both obliquely, and by splaying the vaults. The model shows not only the intersection, but the plans of the intersection and of the generating lines.

45. Helix or Screw-thread.

Model showing the transformation of the right line generators of a right cylinder into screw threads of various pitch or obliquity.

The pitch of a screw is the distance between two successive turns, measured in a direction parallel to the axis. When this distance is small, the screw is said to have a fine pitch, when great, a coarse pitch or high pitch.

APPENDIX.

By C. W. Merrifield, F.R.S., Principal of the Royal School of Naval Architecture and Superintendent of the Naval Museum at South Kensington.

This Appendix contains an account of the application of analysis to the investigation and classification of ruled surfaces.

It is not proposed to follow all the deformations which those surfaces can be made to undergo in the arrangements illustrated by the models. That would take a very large volume, and, even so, could hardly be given completely. The analysis has been kept as simple as possible, and has been written out in the form which appeared best adapted to the consideration of surfaces, not with a view to their general properties, but specially to the particular mode of generation by means of straight lines. For that reason, no mention has been made of the cones and cylinders of the second degree. These are treated with sufficient fulness in all the ordinary books.

The student may extend much of what is stated here by introducing the principles of elliptic deformation and oblique deformation. The latter is frequently equivalent to a change in the direction of coordinates. Both these transformations are applicable to nearly all that follows, and the student should bear this application always in mind. He will do well to work it out to its consequences in some of the simpler cases, for which the formulæ are not unmanageably long, as they are apt to become if used injudiciously.

Motion of a right Line.

A right line is completely defined by the condition that it shall meet four fixed curves in space, to the extent that there is not an infinite number of right lines which will satisfy this condition. The condition, in fact, gives rise to an equation, which may have more roots than one, but in which each root will have a definite and not a variable value. Moreover, the root of the equation may be imaginary, and it may happen that the geometrical line may also be imaginary. Admitting imaginary quantities, however, the line is definite.

If the line be only conditioned to meet three curves in space, it is free to move so as to trace out a curved surface, which is called a ruled surface. It may, as before, happen that the director curves are so chosen that no real line can meet all three in real points. If, therefore, there are three directors, one of them must be taken within certain limits of position or direction in order that the problem may be really possible instead of imaginary. The point is to see distinctly that a right line must have three directors, and three only, to trace out a surface. The surface so traced is called a ruled surface.

It is possible to replace one of the director curves by some equivalent condition. But the conditions taken altogether must be such as are equivalent to the restriction imposed by three director curves, neither more nor less. The classification depends upon the selection of these conditions. Passing by the common cone and cylinder, let us proceed to consider the next in order of simplicity.

The Hyperbolic Paraboloid.

This surface is traced out by a variable right line, which always meets three fixed right lines, which are parallel to one plane.*

Suppose that the plane in question is made to pass through one of the lines, which is taken as the axis of x, and that this plane is taken for the plane of (x, z). Suppose, also, that one position of the variable line is taken for the axis of y. Then the other two fixed lines may be written as—

$$\left.\begin{array}{c} y = b_1 \\ l_1 x + n_1 z = 0 \end{array}\right\} \qquad \left.\begin{array}{c} y = b_2 \\ l_2 x + n_2 z = 0 \end{array}\right\}.$$

and the surface must evidently contain these. But these equations give

$$\begin{aligned} l_1 xy + n_1 b_1 z &= 0 \\ l_2 xy + n_2 b_2 z &= 0 \end{aligned}$$

---

* This only introduces one condition among the director lines, for the plane is arbitrary, and therefore may be taken parallel to any two given lines. It is, therefore, only the third director which is restricted.

and these will satisfy the equation $cz = xy$, if

$$c = -\frac{n_1 b_1}{l_1} = -\frac{n_2 b_2}{l_2}$$

conditions which we may always satisfy by a suitable choice of the axis of z. Moreover, it is clear that the equation $cz = xy$ will also be satisfied by the other system of lines

$$x = a$$

$$my + nz = 0.$$

$$\text{provided } c = -\frac{na}{m}.$$

Hence the equation of the surface may always be written as

$$cz = xy,$$

whatever point of the surface be chosen as origin.

There is always one point on the surface for which, when the equation is so written, z will be perpendicular to (xy); but when that is so, it will not be generally true that x and y are at right angles to one another.

If we leave the axis of z unchanged, and take, as new axes of x and y, the bisectors of the old axes, we shall obtain a rectangular equation of the form

$$\frac{z}{2k} = \frac{x^2}{l^2} - \frac{y^2}{m^2}$$

and the old axes will be given by

$$0 = \frac{x^2}{l^2} - \frac{y^2}{m^2}$$

which is the equation to the pair of asymptotic planes of the surface. When these asymptotic planes are at right angles to one another, $l=m$, and the hyperbolas, parallel to the principal section, are rectangular.

In that case the equation $cz=xy$ is referred to rectangular axes, and we have the following curious properties, the demonstration of which, whether by analysis or geometry, will be a useful exercise for the student. Those marked (1) and (4) hold, with slight modification, for the oblique case.

In the rectangular hyperbolic paraboloid $cz=xy$,

1. The areas of any two portions of surface which have similar and equal projections on the plane of (xy) at equal distances from the axis of z, are equal to one another.

2. The inclination of the tangent plane is constant at any given distance from the axis of z.
3. Any cylindric annulus of the surface, with the axis of z passing through its centre, may be measured by a parabolic arc.
4. If the projection of a portion of surface be any figure symmetrical to the projections of any two right line generators, the corresponding cylindrical volume shall be the product of the projected area into the ordinate at its middle point.
5. Gauss's measure of curvature is constant at a constant distance from the axis of z.

The Hyperboloid of One Sheet.

This in its most general form may be defined as the surface traced out by a line which meets any three fixed lines in space. It will be simplest to take the three lines as three edges (which do not meet) of a parallelepiped, to take axes parallel to them, and to take the centre of the parallelepiped as origin. Calling the lengths of these edges $2a$, $2b$, $2c$, their equations may be written as

$$y = b,\ z = -c$$
$$z = c,\ x = -a$$
$$x = a,\ y = -b$$

In order that an arbitrary line

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$$

may meet these three lines, we must have

$$\frac{b-\beta}{m} = -\frac{c+\gamma}{n}$$

$$\frac{c-\gamma}{n} = -\frac{a+\alpha}{l}$$

$$\frac{a-\alpha}{l} = -\frac{b+\beta}{n}$$

multiplying these together we get

$$(a-\alpha)(b-\beta)(c-\gamma) + (a+\alpha)(b+\beta)(c+\gamma) = 0,$$

or,

$$a\beta\gamma + b\gamma\alpha + c\alpha\beta + abc = 0.$$

And since (α, β, γ) is an arbitrary point on the line, the equation of the surface may be obtained by writing x, y, z, instead of α, β, γ, in this equation, that is

$$ayz + bzx + cxy + abc = 0.$$

The asymptotic cone is

$$ayz + bzx + cxy = 0,$$

which may also be written as

$$\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 0.$$

These are the forms of equation which arise most directly from the rectilinear generation of the surface. Another form of the equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1,$$

the asymptotic cone then becoming

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0.$$

Let us try to follow analytically the deformation indicated in model No. 16, starting, for simplicity, from the right cylinder.

Take for two of the axes the centre line of the cylinder and the line from its middle point to the middle point of its line of contact with the tangent plane. Instead of supposing that one ring only turns, it will be easier to suppose that we turn them both equally, one backwards and the other forwards, through an angle $=\theta$. Suppose, also, that for the tangent plane we take a string whose distance from the point of contact is $\tan\phi$.

Before deformation the equation of the cylinder will be $x^2 + y^2 = r^2$, and its tangent plane $y = r$.

The ends of the line of contact will be, at the top

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = c,$$

and at the bottom

$$x = r\cos\theta, \quad y = -r\sin\theta, \quad z = -c.$$

The equations of the line of contact will therefore be

$$x - r\cos\theta = 0, \qquad \frac{y}{r\sin\theta} = \frac{z}{c},$$

the square of the distance from the axis of any point at a height z is therefore

$$x^2 + y^2 = r^2 \cos^2 \theta + \frac{z^2}{c^2} r^2 \sin^2 \theta,$$

and this is the equation of the surface.

Now with reference to the string in the the tangent plane ; before deformation, its distance from the axis will be $r \sec \varphi$; after deformation, the position of its upper end will be

$$x = r \sec \varphi \cos (\varphi + \theta), \;\; y = r \sec \varphi \sin (\varphi + \theta), z = c,$$

and of its lower end

$$x = r \sec \varphi \cos (\varphi - \theta), \;\; y = r \sec \varphi \sin (\varphi - \theta), z = - c.$$

Hence its equations will be

$$\frac{x - r \sec \varphi \cos (\varphi + \theta)}{r \sec \varphi \{\cos (\varphi + \theta) - \cos (\varphi - \theta)\}}$$

$$= \frac{y \cos \varphi - r \sec \varphi \sin (\varphi + \theta)}{r \sec \varphi \{\sin (\phi + \theta) - \sin (\varphi - \theta)\}} = \frac{z - c}{2c}$$

or,

$$\frac{x \cos \varphi - r \cos (\varphi + \theta)}{r \sin \phi \sin \theta}$$

$$= \frac{y \cos \varphi - r \sin (\phi + \theta)}{r \cos \varphi \sin \theta} = \frac{z - c}{c}.$$

We easily find from these

$$\frac{y}{r} - \frac{z}{c} \sin \theta = \tan \varphi \cos \theta$$

$$\frac{x}{r} - \cos \theta = \frac{z - 2c}{c} \tan \varphi \sin \theta$$

and we can eliminate $\tan \varphi$ by simple division. This gives us the equation of the surface, since φ is a parameter which indicates only the position of the line in the surface, a hyperbolic paraboloid.

It will be an easy and useful exercise for the student to find the other system of generating lines of the paraboloid, and to show that they are horizontal.

The leading properties connected with rectilinear generation are as follows :—

A tangent plane to the hyperboloid cuts the surface in two right lines, which intersect at the point of contact.

A tangent plane to the asymptotic cone cuts the hyperboloid in a pair of parallel lines.

A plane parallel to a tangent plane of the asymptotic cone cuts the hyperboloid in a parabola.

Conoïds.

The simplest form of this surface is when a line moves parallel to the plane of (x, y) and always passes through the axis of z, passing also through a circle with its centre on the axis of x, and its plane at right angles to that axis. This is what Wallis named the cono-cuneus. Call the radius of the circle c, and let a be its distance from the origin. Then for any particular value of z, say $z = k$, the equations of the generating line will be $z = k$, and $\frac{y}{x} = \frac{\sqrt{c^2 - k^2}}{c}$.

The equation of the surface therefore will be

$$\frac{y}{x} = \frac{\sqrt{c^2 - z^2}}{c}, \text{ or}$$

$$c^2 (y^2 - x^2) + x^2 z^2 = 0.$$

The sections at right angles to the axis of x are therefore elliptic. Those parallel to the plane of x, y are evidently pairs of right lines. Other plane sections of the surface are in general curves of the fourth degree.

This particular case sufficiently exhibits the characteristic form of the conoïds. There is more complexity, but no real difficulty in obtaining the equations of the conoïds described under other conditions. They are chiefly met with in the case of splay arches.

Models Nos. 25, 39, and 44 are examples of this application of the surface.

Families of ruled Surfaces.

The general consideration of any class of ruled surfaces in which one or more of the director curves are left arbitrary. requires the use of the arbitrary functional symbol. This symbol can only be got rid of by partial differentiation.

Cylindrical Surfaces.

A right line moves always parallel to itself. Its equations will therefore be

$$\frac{x - \alpha}{l} = \frac{y - \beta}{m} = \frac{z - \gamma}{n}$$

where $l, m, n,$ are fixed quantities known, if the direction of the generating line be known; but α, β, and γ are variable quantities depending upon the director curve. We may write the equations as

$$\frac{x}{l} - \frac{y}{m} = \frac{\alpha}{l} - \frac{\beta}{m}$$

$$\frac{y}{m} - \frac{z}{n} = \frac{\beta}{m} - \frac{\gamma}{n}$$

Of the three quantities $\frac{\alpha}{l}\,\frac{\beta}{m}\,\frac{\gamma}{n}$, one is absolutely arbitrary, as we only have to do with their differences. We may therefore consider

$$\frac{\beta}{m} - \frac{\gamma}{n} \text{ to be a function of } \frac{\alpha}{l} - \frac{\beta}{m}$$

the particular form of the function depending upon the character of the director. We may therefore write the equation of cylindrical surfaces as

$$\frac{x}{l} - \frac{y}{m} = \mathrm{F}\left(\frac{y}{m} - \frac{z}{n}\right)$$

in which F is a functional symbol.

It might seem at first sight that the common equation of a right cylinder

$$x^2 + y^2 = a^2$$

is not of this form. But any change of variables such as will introduce the third variable z will at once reduce it to this form.

Conical Surfaces.

A right line always passes through a fixed point. Calling the point $\alpha\ \beta\ \gamma$, its equations are

$$\frac{x - \alpha}{l} = \frac{y - \beta}{m} = \frac{z - \gamma}{n}$$

whence

$$\frac{x - \alpha}{y - \beta} = \frac{l}{m}, \quad \frac{y - \beta}{z - \gamma} = \frac{m}{n}$$

and $\frac{l}{m}$ must be some function of $\frac{m}{n}$. This gives

$$\frac{x - \alpha}{y - \beta} = \mathrm{F}\left(\frac{y - \beta}{z - y}\right)$$

as the equation of conical surfaces. Its form expresses simply that $x - \alpha$, $y - \beta$, $z - \gamma$ are connected by a homogeneous equation.

Conoïds.

A right line is conditioned to pass through a fixed right line and remain parallel to a fixed plane. Take the fixed line for the axis of z, and the fixed plane for that of (x, y). Then the variable line may be written as

$$x + ky = 0 \qquad z = c,$$

where c and k are variable, and are evidently connected together by some equation depending upon a second director. We shall, therefore, have $k = \mathrm{F}(c) = \mathrm{F}(z)$, and the equation of the surface is

$$x + y\,\mathrm{F}(z) = 0.$$

It may also be written as $z = f\left(\dfrac{x}{y}\right)$.

If the director line and plane be taken arbitrarily, a change of coordinates will be necessary. But the effect of this will only be to put the equation in the form

$$a_1 x + b_1 y + c_1 z + e_1 = f\left(\frac{a_2 x + b_2 y + c_2 z + d_2}{a_3 x + b_3 y + c_3 z + d}\right).$$

The hyperbolic paraboloid is a particular case of the conoïd. So also is the common form of the skew helixoïd, or screw surface, not the one illustrated in this collection.

Ruled Surface with Director Plane.

This is more general than the conoïd, inasmuch as it is not restricted to pass through a right line director. The removal of this restriction introduces another arbitrary function. Taking the director plane through the origin and calling it

$$ax + by + cz = 0,$$

the equations of any line parallel to it may be written as

$$ax + by + cz = k, \; lx + my = 1,$$

where k, l, and m are arbitrary constants. We may regard l and m as functions of k, and we get

$$x\mathrm{F}(ax + by + cz) + yf(ax + by + cz) = 1.$$

Monge writes it in the form

$$z = x\,\mathrm{F}\,(ax + by + cz) + yf\,(ax + by + cz)$$

which comes to the same thing, as F and f are arbitrary functions. Monge's form may be obtained directly by taking the equations of the line as

$$ax + by + cz = k,\ lx + my = z.$$

If the plane of $x\,y$ be taken for the director plane, a and b vanish, and we may write the equation as

$$x\mathrm{F}z + yfz = 1 \text{ or } z.$$

Ruled Surface with right line Director.

This again requires two arbitrary functions. Taking the line for the axis of z, the projection of the variable line on the plane of (x, y) will be of the form

$$x - ky = 0,$$

while its projection on the other plane, say of (x, z), will be of the form

$$mx + n = z.$$

Now making m and n arbitrary functions of k or $\frac{x}{y}$ we have for the equation of the surface

$$z = x\,\mathrm{F}\left(\frac{y}{x}\right) + f\left(\frac{y}{x}\right).$$

To this family of surfaces belong the hyperboloid of one sheet, all conical surfaces, conoïds, skew helixoïds, the skew arch, known by the French name of the "*biais passé*" (Model No. 27 of this series), and the splay vaulting, known as the "*Arrière voussure de Marseille.*" Of the last, there are two specimens in the education museum at South Kensington, although it is not represented in this series.

The ruled surface with a director plane is the particular case of the ruled surface with a right line director, in which the right line has moved off to infinity. It follows that conoïds are a particular case of surfaces with two right line directors. These directors must not meet in a point, or the surface will reduce to a plane.

Ruled Surfaces with two right line Directors.

The most symmetrical form in which the two directors can be referred to rectangular axes is to take the shortest distance between them for one axis, and its middle point for origin, and to let the other axes bisect the angle between the two directors, whose equations are thus—

$$x = a, y = z \tan \theta$$
$$x = - a, y = - z \tan \theta.$$

Thus the line joining a point ($z = k$) on one to a point ($z = l$) on the other is

$$\frac{x - a}{2a} = \frac{y - k \tan \theta}{(k + l) \tan \theta} = \frac{z - k}{k - l}$$

Whence $$k = \frac{a}{\tan \theta} \cdot \frac{y + z \tan \theta}{x + a}$$

$$l = \frac{a}{\tan \theta} \cdot \frac{y - z \tan \theta}{x - a}$$

in which k and l are parameters, one of which may be assumed to be a function of the other. It follows that $\frac{k \tan \theta}{a}$ is a function of $\frac{l \tan \theta}{a}$. Hence the functional equation of the surface is

$$\frac{y + z \tan \theta}{x + a} = \mathrm{F}\left(\frac{y - z \tan \theta}{x - a}\right).$$

This is not the general form of the functional equation, but its simplest form, analogous to that which we obtained for a conical surface, by assuming the position of the vertex to be at the origin, or to that of the ruled surface with one right line director when we take that for an axis.

Another form, even more symmetrical, can be obtained by taking the origin and axis of x as before, and lines parallel to the directors as (oblique) axes. The equations of the directors are thus

$$x = a, \ z = 0$$
$$x = - a, y = 0$$

and the line joining a point $y = k$ on the first with a point $z = l$ on the second, is

$$\frac{x - a}{2a} = \frac{y - k}{k} = - \frac{z}{l}.$$

Whence $$\frac{l}{2a} = \frac{z}{a - x}, \ \frac{k}{2a} = \frac{y}{a + x}$$

and the functional equation of the surface is

$$\frac{z}{a - x} = \mathrm{F}\left(\frac{y}{a + x}\right).$$

Here, however, the coordinates are oblique.

The ruled surface with two director planes is cylindrical.

Ruled Surfaces generally.

It is not possible to express the general equation of ruled surfaces in a purely functional form, for the equations of a right line are only two in number, and they involve four arbitrary constants. We cannot clear all four by such suppositions as we have made in the previous cases without introducing some similar restriction. If, for instance, we take

$$ax = y + b$$
$$z = cx + e$$

and consider c and e functions of a we get

$$z = x\,\mathrm{F}\left(\frac{y + b}{x}\right) + f\left(\frac{y + b}{x}\right)$$

in which b remains perfectly arbitrary, and as it is not a function of x and y alone, we cannot get rid of b without bringing back one of the other arbitrary quantities or introducing a restriction. We cannot expunge them without partial differential equations.

It is also to be remarked that the functional equations which we have hitherto obtained are not quite general. In the case of cones, for instance, we have given the equation $\frac{x - a}{y - b} = \mathrm{F}\left(\frac{y - b}{z - c}\right)$ on the assumption that the vertex is at the known point, whose coordinates are (a, b, c). If the vertex is to be left arbitrary, so that the functional equation shall express all conical surfaces without reference to the position of the vertex, no such functional equation exists, nor can the condition in general be expressed by any one differential equation.

Developable Surfaces.

The characteristic of these surfaces is essentially differential and it cannot be expressed without partial differentials.

Surfaces of which the Directors are given, either explicitly or implicitly.

In these cases the arbitrary functions must be determined by the conditions in question. In the case of ruled surfaces with given director lines, through which the variable right line must pass, it will in general be sufficient, for the determination of the arbitrary function, to make the equations of the director satisfy the functional equation identically, for then the director will be a line traced upon the surface.

Differential Equations of Families of Surfaces.

In the restricted cases in which we have obtained functional equations, we may obtain the differential equations by the direct processes of partial differentiation. But they may also be obtained from the equation to the generating line, expressed in terms of the coefficients which we wish to retain; for our object is simply to obtain the relation between certain partial differential coefficients, and it is immaterial whether we obtain these by implicit differentiation from a line in which only one variable is independent, or by partial differentiation from a surface in which there are two independent variables.

The character of the restriction imposed on the motion of the line will determine what are the constants to be eliminated by the differentiation. For the motion of a point fixed while the line shifts, and the independent motion of the point along the line, each infinitesimal in amount, determine the tangent plane of the surface, and this tangent plane is one of the *complete* solutions of the partial differential equation, of which the functional equation of the surface is the *general* solution.

Any family of surfaces of which the actual surface is the envelope will be a complete solution of the differential equation of the surface, but the tangent plane is the only one in which we are certain that the right line generator must be contained.

Cylindrical Surfaces.

Starting from the right line

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$$

and observing that our object is to retain l, m, n, we obtain by implicit differentiation

$$\frac{dz}{dx}+\frac{dz}{dy}\frac{dy}{dx}=\frac{n}{l}\frac{dy}{dx}=\frac{m}{l}$$

therefore

$$l \frac{dz}{dx} + m \frac{dz}{dy} = n$$

which is the differential equation of all cylindrical surfaces whose director ratios are l, m, n.

If we take the functional equation

$$\frac{z}{n} - \frac{y}{m} = \mathrm{F}\left(\frac{y}{m} - \frac{x}{l}\right)$$

we obtain by partial differentiation

$$\frac{dz}{dx} = -\frac{n}{l}\mathrm{F}', \qquad \frac{dz}{dy} = \frac{n}{m} + \frac{n}{m}\mathrm{F}'$$

$$\therefore l \frac{dz}{dx} + m \frac{dz}{dy} = n \text{ as before.}$$

Conical Surfaces.

Our object here is to retain α, β, γ, and to get rid of l, m, n, by differentiation. For this purpose write

$$\frac{z-\gamma}{y-\beta} = \frac{n}{m} \qquad \frac{y-\beta}{x-\alpha} = \frac{m}{l}$$

$$(y-\beta)\left(\frac{dz}{dx} + \frac{dz}{dy}\frac{dy}{dx}\right) - (z-\gamma)\frac{dy}{dx} = 0.$$

$$(x-\alpha)\frac{dy}{dx} - (y-\beta) = 0.$$

$$\therefore (y-\beta)\left\{\frac{dz}{dx} + \frac{y-\beta}{x-\alpha}\frac{dz}{dy}\right\} = (z-\gamma)\frac{y-\beta}{x-\alpha}$$

or

$$(x-\alpha)\frac{dz}{dx} + (y-\beta)\frac{dz}{dy} = z - \gamma$$

is the differential equation of the surface.

Conoïds.

The generating line is, for the simplest form,

$$y = kx, \qquad z = c.$$

$$\therefore \frac{dz}{dx} + \frac{dz}{dy}\frac{dy}{dx} = 0, \qquad \frac{dy}{dx} = k = \frac{y}{x}$$

eliminating $\frac{dy}{dx}$, the partial differential equation of the surface is

$$x\frac{dz}{dx} + y\frac{dz}{dy} = 0.$$

Ruled Surface with director Plane.

Let the equations of the line be

$$Ax + By + Cz = k \qquad (1.)$$

$$ax + by = 1 \qquad (2.)$$

in which a, b, and k, are arbitrary constants. Differentiating implicitly

$$A + B\frac{dy}{dx} + C\left(\frac{dz}{dx} + \frac{dz}{dx}\frac{dy}{dx}\right) = 0 \qquad (3.)$$

$$a + b\frac{dy}{dx} = 0. \qquad (4.)$$

Proceeding to a second differentiation, in which, by virtue of the last equation, we may consider $\frac{dy}{dx}$ as constant, we obtain

$$\frac{d^2z}{dx^2} + z\frac{d^2z}{dxdy}\left(\frac{dy}{dx}\right) + \frac{d^2z}{dy^2}\left(\frac{dy}{dx}\right)^2 = 0. \qquad (5.)$$

Then, if we take (2) as the director plane, that is to say, if we make the director plane parallel to the axis of z, we must substitute in (5) the value of $\frac{dy}{dx}$ obtained from (4), and we get

$$b^2\frac{d^2z}{dx^2} - 2ab\frac{d^2z}{dxdy} + a^2\frac{d^2z}{dy^2} = 0.$$

or in the usual notation

$$b^2r - 2abs + a^2t = 0. \qquad (6.)$$

If, however, we take (1) as the director plane, we must determine $\frac{dy}{dx}$ from (3), which gives

$$\frac{dy}{dx} = -\frac{Cp + A}{Cq + B}$$

and the differential equation becomes

$$(Cq + B)^2r - 2(Cq + B)(Cp + A)s + (Cp + A)^2t = 0. \qquad (7.)$$

which reduces to the same form as (6) if $C = 0$.

Ruled Surface with right line Director.

The axis of z being taken as the director, the equations of the variable line may be written as

$$x = ky \qquad\qquad mx + n = z.$$

whence

$$\frac{dy}{dx} = \frac{1}{k} = \frac{y}{x} = \text{constant,}$$

and

$$m = \frac{dz}{dx} + \frac{dz}{dy}\frac{dy}{dx}$$

Differentiating again,

$$0 = \frac{d^2z}{dx^2} + 2\frac{d^2z}{dxdy}\frac{dy}{dx} + \frac{d^2z}{dy^2}\left(\frac{dy}{dx}\right)^2$$

Substituting $\frac{y}{x}$ for $\frac{dy}{dx}$ we obtain

$$x^2\frac{d^2z}{dx^2} + 2xy\frac{d^2z}{dxdy} + y^2\frac{d^2z}{dy^2} = 0. \qquad (8.)$$

which is the equation of the surface.

If we had taken any other right line than the axis of z for the director, we should have had the constants of its equations appearing in the differential equation (as in equation No. 7 of the previous section). Equation (8) is therefore not a general one, but a restricted form, in which the constants have received particular values which simplify the result.

Ruled Surfaces considered generally.

We have seen that we cannot express the equation of a ruled surface in a functional form, without differentiation. This happens because the simplest expression of a ruled surface is, that its tangent plane at any point shall meet it in a right line, or, what is the same thing, that one of its tangent lines at any point shall be wholly in the surface. Now the question of tangency is emphatically a question of differentiation.

In what follows it will be convenient to use the ordinary abridged notation of partial differential coefficients, namely,

$$p = \frac{dz}{dx} \qquad\qquad q = \frac{dz}{dy}$$

$$r = \frac{d^2z}{dx^2} \qquad s = \frac{d^2z}{dxdy} \qquad t = \frac{d^2z}{dy^2}$$

$$\alpha = \frac{d^3z}{dx^3} \qquad \beta = \frac{d^3z}{dx^2dy}$$

$$\gamma = \frac{d^3z}{dxdy^2} \qquad \delta = \frac{d^3z}{dy^3}\cdot$$

A ruled surface is expressed with complete generality by considering it to be traced out by the motion of the generating line

$$z = c_1x + c_3 \qquad y = c_2x + c_4$$

in which the four quantities c, which are constant so far as the equation of the line in any one position is concerned, but variable parameters, when considered with reference to the position of the line, are to be made to disappear. The obvious way of effecting this is to obtain, by means of implicit differentiation, a relation between the partial differential coefficients of z, which, when thus cleared of what is special to the particular generating line, will be the differential equation, in partial differentials, of the surface.

Operating implicitly upon the above equations we get

$$p + q\frac{dy}{dx} = c_1 \qquad \frac{dy}{dx} = c_2$$

whence

$$p + qc_2 = c_1 \qquad (1.)$$

Differentiating again upon the same suppositions we have

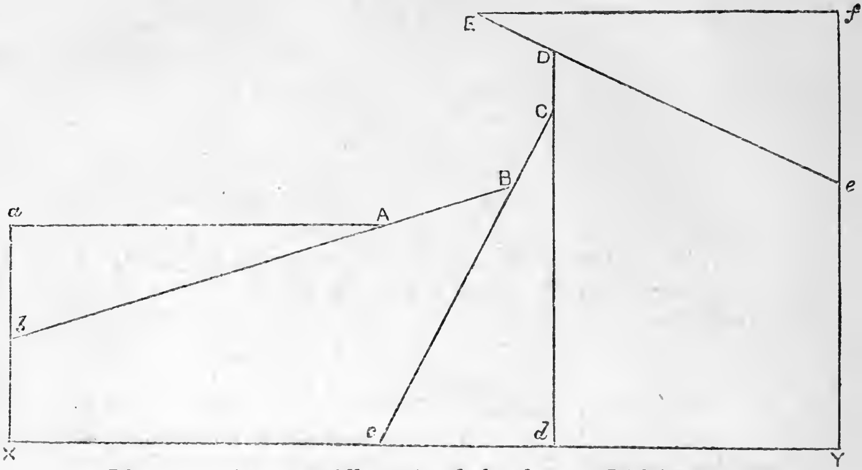
$$r + 2sc_2 + tc_2^2 = 0. \qquad (2.)$$

Differentiating a third time, we get

$$\alpha + 3\beta c_2 + 3\gamma c_2^2 + \delta c_2^3 = 0. \qquad (3).$$

If now, by ordinary algebra, we eliminate c_2 from equations (2) and (3), we get the differential equation of ruled surfaces. It is not worth while to write it down, as it is more conveniently used in the implicit form given above, than in its very cumbrous explicit form.

We have already noticed that the general form of a ruled surface cannot be expressed as a single functional equation. It follows that the differential equation has no *general* primitive.



If we consider a stiff card, of the form *a* X Y *f* E in the figure above, to be deeply scored along the right lines B A *b*, C B *c*, D C *d*, E D *e*, so that we can bend it along each of them, the broken line A B C D E will form a polygon, at first plane, but a skew polygon when we come to bend the the surface, which will then form a polyhedron, every edge of which will run into every successive edge, along the broken line A B C D E. It is evident that this condition is necessary to our being able to deform the surface. For if one of the scores (say *c* B) stopped short of the edge A B C, the card would not bend, and if A B C were not an actual edge, in that case it would not bend either.

Now if we consider a surface which can be formed by the gradual bending of a plane surface, the only departures from this type are,—

(1.) That the polyhedral surface is replaced by a curved surface.

(2.) That the polygon A B C D E is replaced by a curve.

(3.) That instead of a finite bending along a few lines A *a*, B *b*, C *c*, &c. we have an infinitesimal bending along an infinite number of such lines infinitely close together.

(4.) That all these lines are tangents to the curve A B C . . . which replaces the polygon.

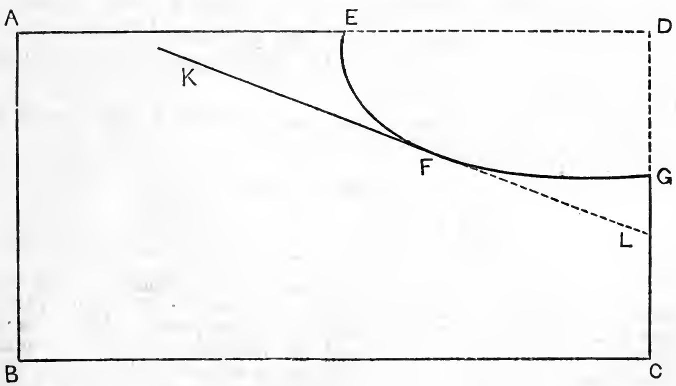
(5.) That the consecutive lines A *a*, B *b*, meet one another; that is to say, the shortest distance between them is an infinitesimal of a higher order than the distance between any other two points of them. This is implied in their being tangents to the same curve.

If the curve A B C D E degenerates into a single point, there will occur some singularities which will mask the general properties of these surfaces. This degeneration gives conical surfaces. When, further, the single point in question moves off to infinity, we get the cylindrical surfaces.

This is not all. If we take two cards, counterparts of one another, and glue them together along the polygon A B C D E, we can deform the double plane into a double polyhedron, the two sheets of which will be placed back to back. There will then be a finite angle between the corresponding gores in the two sheets. When we replace this polygon by a curve, this angle will be infinitesimal, and a section at right angles to A B C D E will be a cusped curve.

Each sheet will evidently be a ruled surface, the variable right lines of which will touch the curve which replaces the polygon A B C D E. It is evident that the right line generators must otherwise stop abruptly, for they could not get from one sheet to the other on any different conditions.

A good imitation of this can be made by taking two sheets of paper, fitted over one another, as A B C D, cutting out a curved piece (E F G) from both, and laying a smear of strong glue along E F G, so as to fasten them together



along that curved side only. Then take one sheet up by the corner A or B, and let the other sheet hang freely. If there be no crumpling or stretching, we shall thus obtain a developable surface of two sheets. A tangent to the curve E F G, such as K F L, will be partly on one sheet and partly on the other.

The curve E F G which connects the two sheets is called the *edge of regression*. It is not a very significant name, being a bad translation of the French *arète de rebroussement*. Still it is the usual name.

In the case of cones, this curve degenerates into a point It then ceases to be a necessity that the angle between the two sheets should be infinitesimal, and, accordingly, this angle is then generally finite.

The analytical criterion of a developable surface is derived from the consideration that its generating lines must be tangents to a curve in space. In analytical language, they must have an envelope. It must, therefore, be possible to get rid of the parameter which distinguishes one line from another by differentiating with regard to that parameter. Going back to equation (2), namely,

$$r + 2sc_2 + tc_2^2 = 0,$$

in which there is only one parameter, c_2, and differentiating with regard to that, we get

$$s + tc_2 = 0$$

Now eliminating c_2 by ordinary algebra, we obtain, as the differential equation of ruled surfaces,

$$rt - s^2 = 0.$$

The geometrical interpretation of this is that a section parallel to the tangent plane at any point, and infinitely near to it, is always a parabola. In other words, the curvature at every point is parabolic.

To resume then, the differential equation of ruled surfaces* is the result of eliminating c between

$$r + 2sc + tc^2 = 0$$

$$\text{and } \alpha + 3\beta c + 3\gamma c^2 + \delta c^3 = 0$$

* It seems at first sight rather singular that the elimination of a single constant should introduce four new partial differentials; but it must be recollected that the partial differentials are here only accidental. The work we are doing is implicit differentiation, and the variation of the whole equation simply amounts to the introduction of one new element. This consideration seems to dispose (affirmatively) of the question whether the resulting equation represents ruled surfaces only. The converse question, whether ruled surfaces are included in it, is evident at first sight.

while that of developable surfaces is

$$rt = s^2.$$

It will be observed that this is the condition that the first equation should give two equal values for c.

Cones and Cylinders.

The differential equations previously obtained are not those which distinguish these surfaces from other ruled or developing surfaces. On the contrary, they are restricted by conditions which settle the position of the vertex of the cone, or the direction of the generators of the cylinder. They are consequently expressed by differential equations of the first degree. If we desire to eliminate the constants of the vertex, or of direction, we must proceed to third differentials.

Writing the equation of the cone as

$$z - c = (x - a)\,\mathrm{F}\left(\frac{y - b}{x - a}\right)$$

and regarding x and y as independent variables, it is easy to verify that

$$\frac{\alpha\gamma - \beta^2}{r} = \frac{\alpha\delta - \beta\gamma}{2s} = \frac{\beta\delta - \gamma^2}{t};$$

but these equations are also derivable from the general equation of developable surfaces, $rt = s^2$, and do not distinguish conical surfaces from all others.

If again we write the equation of the cylinder as

$$ny - mz = \mathrm{F}(mx - ly)$$

we find that the numerators of the above functions vanish, or, as the result may be more simply stated

$$\frac{\alpha}{\beta} = \frac{\beta}{\gamma} = \frac{\gamma}{\delta}$$

and this appears to distinguish the cylindrical from other developable surfaces.

The theory of this part of the subject is recent, and far from complete. Meanwhile it is certain that what goes before is strictly true as stated; but the student must be cautious of drawing inferences which go beyond the text. For instance, Mr. Cayley has shown that the equations

$$\frac{\alpha}{\beta} = \frac{\beta}{\gamma} = \frac{\gamma}{\delta}$$

do not represent cylinders only, or even ruled surfaces only. It is when taken in connexion with the equation of developable surfaces that they represent cylinders.

The complexity of these results was to be expected. The general equation of cones involves an arbitrary function and three arbitrary constants, while that of cylinders involves a function and two constants. Now the ordinary practice in the formation of differential equations is to consider arbitrary functions on the one hand, as yielding what is called a *general* solution, or primitive; and arbitrary constants on the other hand, as yielding a *complete* solution or primitive. The simultaneous elimination of functions and constants would naturally be more complex, and the result at which we have arrived, namely, that it leads to the existence of simultaneous equations in differentials, is only reasonable.

The equations of the cones

$$\frac{\alpha\gamma - \beta^2}{r} = \frac{\alpha\delta - \beta\gamma}{2s} = \frac{\beta\delta - \gamma^2}{t}$$

in virtue of the relation $rt = s^2$, lead to the equation of the third degree

$$4(\alpha\gamma - \beta^2)(\beta\delta - \gamma^2) = (\alpha\delta - \beta\gamma)^2.$$

This equation is more general than that of conical surfaces. Its integration has not yet been effected, but it is known to be satisfied by surfaces traced by common parabolas intersecting consecutively; moreover, it is easily shown that it is also satisfied by the equation of the ruled surface with a director plane, a surface which is not generally developable. This can easily be verified by taking the director plane parallel to the axis of z, when it may be written as

$$z = x\,\mathrm{F}(ax + by) + yf(ax + by) + \varphi(ax + by).$$

The result, being invariant, will not be affected by a change of co-ordinates. The question has not yet, however, been fully studied.

The Screw, and Surfaces connected with it.

If a cylinder is put into the lathe and turned with a steady motion while a tool travels along in a direction parallel to the axis of the cylinder, also with a steady motion, the curve traced upon the cylinder is a *screw thread,* or *helix,* as it is sometimes called.

It is the only curve in which any portion can be superposed upon any other portion so as to fit exactly. The *only* curve, because the circle and straight line are simply extreme cases of it. If the tool is still as well as steady we get a circle. If the cylinder is still while the tool travels we get a right line.

The most convenient form of its equations is $x = a \cos \theta$, $y = a \sin \theta$, $z = c\,\theta$. Its tangent, of course, touches the cylinder $x^2 + y^2 = a^2$ and makes an angle with the axis whose trigonometrical tangent is a $\frac{a}{c}$. The radius of absolute curvature is evidently the same as that of the elliptic section of the cylinder passing through the tangent. The minor semi axis of this ellipse is a, and its major semi axis $\sqrt{a^2 + c^2}$. Hence the radius of absolute curvature is $\frac{a^2 + c^2}{a}$.

If we eliminate a and θ we get a surface which is independent of the diameter of the cylinder, or of the inclination of the particular helix got by varying the diameter of the cylinder at the same time that the pitch remains constant for all values of a. This is called the *skew helixoid*. It is the underside of a common screw staircase, or the driving side of a common screw propeller. A mooring screw and the twisted surfaces of a square-threaded screw are other examples. The equation is

$$\frac{y}{x} = \tan \frac{z}{c}.$$

Practical reasons made it inconvenient to include the common form of this surface in this series of models. Moreover, the examples of it are too frequently met with in practice to render its exhibition worth encountering an inconvenience. There is no difficulty in representing it by strings or threads; but there is a difficulty in deforming it with regularity.

The *skew helixoid* selected for representation (model No. 31) is one in which the generating line passes through the axis, and is inclined to it at a constant angle instead of being at right angles to it. It is the surface of what is called a screw with a triangular thread. If we call φ the angle which its generating line makes with the axis, this generating line will pass through the point of the helix ($a \cos \theta$, $a \sin \theta$, $c\,\theta$), and also the point of the axis $z = c\,\theta - a \cos \varphi$. Its equations are therefore

$$\frac{x}{\cos \theta} = \frac{y}{\sin \theta} = \frac{z - c\theta}{\cos \varphi} + a,$$

or eliminating θ

$$\cos\varphi \sqrt{x^2+y^2} = z - c\tan^{-1}\frac{y}{x} + a\cos\varphi.$$

If z is made constant, that is, if we take a section perpendicular to the axis, we get for the polar equation of the section,

$$r = k - l\theta,$$

which is the spiral of Archimedes.

Reverting to the helix, it may be remarked that its projections by lines parallel to any tangent are either cycloids or elliptic modifications of the cycloid, and that *all* its projections (by parallel lines) are either trochoids or elliptic modifications of the trochoids.

The tangent to the helix has for its equations

$$\frac{x - a\cos\theta}{-a\sin\theta} = \frac{y - a\sin\theta}{a\cos\theta} = \frac{z - c\theta}{c}$$

The first pair of terms gives $x\cos\theta + y\sin\theta = a$. Squaring all the terms, and adding the numerators and denominators of the first two we get

$$\frac{x^2 + y^2 + a^2 - 2a(x\cos\theta + y\sin\theta)}{a^2} = \left(\frac{z - c\theta}{c}\right)^2$$

whence

$$x^2 + y^2 - a^2 = \frac{a^2}{c^2}\left(z - c\theta\right)^2$$

this equation, with $x\cos\theta + y\sin\theta = a$, represents the tangent for any given value of θ. When $z = 0$ we get $r^2 - a^2 = a^2\theta^2$, which is the involute of the circle. This is geometrically evident, if we consider the helix as traced by winding a paper triangle on a cylinder.

The surface swept out by the tangent is the *developable helixoid* (model No. 30). Its equation is formed by eliminating θ from the above equations, which may easily be done, since $x\cos\theta + y\sin\theta = a$ gives

$$\sin\theta = \frac{ay + x\sqrt{x^2 + y^2 - a^2}}{x^2 + y^2}$$

The helix itself is the edge of regression of this surface

When the surface is actually developed or flattened, the helix becomes a circle whose radius is that of the absolute curvature of the helix, that is to say, $\frac{a^2 + c^2}{a}$.

THE FRENCH SKEW ARCH. (*Biais passé.*)

See model No. 27. Take the line joining the centres of the large semicircles as x, the middle point of it for origin, and the axis of z, vertical. Then if we call the radii of the small circles c and their distance from the axis of x, b, their equations may be written as

$$x = a, \ (y - b)^2 + z^2 = c^2$$
$$x = -a, \ (y + b)^2 + z^2 = c^2$$

A plane through the axis of x, inclined to the vertical at an angle φ, will be $y = z \tan \varphi$: substituting $y \cot \varphi$ for z in the above equations and solving for y, we find for the extremities of a generating line

$$x = a, \ y = b \sin^2 \varphi + \sin \varphi \sqrt{c^2 - b^2 \cos^2 \varphi}$$

$$x = -a, \ y = -b \sin^2 \varphi + \sin \varphi \sqrt{c^2 - b^2 \cos^2 \varphi}$$

The plan of the generating line is therefore

$$\frac{x}{a} = \frac{y - \sin \varphi \sqrt{c^2 - b^2 \sin^2 \varphi}}{b \sin^2 \varphi}$$

Now putting $\tan \varphi = \frac{z}{y}$ we obtain for the equation of the surface

$$\frac{bxy}{a} = z^2 + y^2 - \sqrt{c^2 (z^2 + y^2) - b^2 z^2}.$$

If we make $x = 0$ and $y = 0$ we get for the height at the middle point $z = \sqrt{c^2 - b^2}$, which shows that the arch droops in the middle, as already stated. To get the middle section, make $x = 0$, and we have

$$(z^2 + y^2)^2 = c^2 (z^2 + y^2) - b^2 z^2$$

or changing to polar co-ordinates

$$r^2 = c^2 - b^2 \sin^2 \theta$$

an equation which shows that the curve bears a close resemblance to the ellipse.

This curved surface is essentially unsymmetrical; it is evidently of the fourth degree.

The Splay Vault of Marseille.

This is a splay vault or pendentive used to connect a semicircular arch on one face of a wall with a flat or segmental arch on the other face. It is a ruled surface, of which the intrados of the two arches are two curvilinear directors, while the other director is a vertical right line. It is called the *Arrière-voussure de Marseille;* but when one of the arches is flat instead of segmental, it is sometimes called the *Arrière-voussure de Montpellier.* The latter form is selected for illustration here, as the investigation is somewhat simpler. The student will have no difficulty in extending the algebra to meet the more general case of the segmental arch.

Assuming, then, that the front arch is replaced by a horizontal line, whose height above the springing plane is h, and that the inner arch is a semicircle, radius c, let the distance between the two arches be b and let the vertical director be taken at a distance a behind the inner arch. Then if we take the vertical director for the axis of z, the equations of the other two directors will be

$$x = a, y^2 + z^2 = c^2$$

$$x = a + b, z = h$$

Take an auxiliary plane passing through the axis of z; its horizontal trace may be written as

$$y = x \tan \varphi$$

it will therefore meet the horizontal director in the points

$$x = (a + b), y = (a + b) \tan \varphi, z = h$$

and the semicircular director in the point

$$x = a, y = a \tan \varphi \quad z = \sqrt{c^2 - a^2 \tan^2 \varphi}.$$

The projection of the generator on the plane of (yz) will therefore be

$$\frac{y - (a + b) \tan \varphi}{b \tan \varphi} = \frac{z - h}{h - \sqrt{c^2 - a^2 \tan^2 \varphi}}$$

Eliminating tan φ by means of the horizontal trace of the generator, $y = x \tan \varphi$, we obtain after a few reductions

$$\left(\frac{ab + bz - hx}{a + b - x}\right)^2 = \frac{c^2 x^2 - a^2 y^2}{x^2}$$

It is easy to verify, by giving x the values 0, a, and $a + b$ successively, that this surface really contains the three directors.

This surface, like the last, is of the fourth degree, and in its ordinary construction is unsymmetrical. It may, however, be rendered symmetrical in a particular case by the assumption $h = 0$. If we then transfer the origin to the centre of the semicircle, the equation becomes

$$\left(\frac{bz}{b - x}\right)^2 + \left(\frac{ay}{a + x}\right)^2 = c^2.$$

In this case all sections perpendicular to x are ellipses.

In the more usual case which occurs in actual building construction, instead of a horizontal right line director there is in general a very flat segment of a circle. Whatever the curve may be, it will in general be possible both to construct the generating line at any point geometrically and to find the analytical equation of the surface, but the analytical expressions will become rather complicated. The expedient to be used in all constructions relating to ruled surfaces with a director line is to take auxiliary planes passing through that line.