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Faculty Working Papers

A CENTENNIAL: THE WALRAS VISION

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College of Commerce and Business Administration
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> Reste à savoir si c'est parce qu'on a payé 2 fr. de fermages, 2 fr. de salaires et 1 fr. d'intérêts que cette bouteille de vin se vend 5 fr., ou si ce ne serait pas plutôt parce que cette bouteille de vin se vend 5 fr. qu'on paie 2 fr. de fermages, 2 fr. de salaires et 1 fr. d'intérêts. [13], 176.[1]

1. INTRODUCTION

In 1874 Walras began publishing the first edition of his Eléments [12]. The purpose of the present paper is to observe the centennial by restating as simply and as succinctly as possible the core of the Walras vision: Let there be m outputs, n inputs, and s households. Industry demands inputs and supplies outputs; households demand outputs and supply inputs. Determine equilibrium demand, supply, and relative price in all resulting markets. In restating the core we shall use our own words, our own notation, and nothing more complicated than total differentials. Beyond simplicity and succinctness, we offer nothing original.

There was more to Walras than the core restated. As for scope,

there were insights into the nature of capital, income, and money. The first edition used fixed input-output coefficients. Once Barone had helped him understand marginal productivity, however, Walras became pleased with its symmetry and beauty and incorporated it into his fourth edition——but only towards its end so as not to disturb the core:

...la _théorie de la productivité marginale_...fournit ainsi le ressort de la demande des services et de l'offre des produits par les entrepreneurs, tout comme la _théorie de l'utilité finale_ fournit le ressort de la demande des produits et de l'offre des services par les propriétaires fonciers, travailleurs et capitalistes, mais que j'ai préféré ne pas introduire dans ma théorie générale de l'équilibre économique, déjà suffisamment compliquée, de peur que celle-ci ne devint trop difficile à saisir dans son ensemble. [13], 375-376.[2]

As for method, there were discussions of special cases (Lesson 7) in which the same number of equations and variables fails to produce a unique solution, and there was the messy process of groping, _tâtonnement_, by which Walras tried to trace the convergence of price towards equilibrium. Ignoring all this, let us restate the core, using the following notation:

Variables

C_k ≡ kth household's money expenditure on consumption

p_i ≡ price of ith input

P_j ≡ price of jth output

U_k ≡ utility to kth household

x_i ≡ ith input demanded by industry

x_{ki} ≡ ith input supplied by kth household

X_j ≡ jth output supplied by industry

X_{jk} ≡ jth output demanded by kth household

Y_k ≡ money income of kth household

Parameters

a_{ij} ≡ ith input demanded by industry per physical unit of jth output

q_{ki} ≡ endowment of kth household with ith input

2. HOUSEHOLD INCOME AND CONSUMPTION

The money value of consumption in the kth household is defined as

the sum of the money values of the outputs demanded by that household:

(1) $$C_k \equiv \sum_{j=1}^{m} (P_j X_{jk})$$

Of such equations we have s, one for each household. The money value of the income of the kth household is defined as the sum of the money values of the inputs supplied by that household:

(2) $$Y_k \equiv \sum_{i=1}^{n} (p_i x_{ki})$$

Of such equations we also have s, one for each household. The economy is a stationary one, hence there is no saving. Indeed, no household saves:

(3) $$C_k = Y_k$$

Not until the entire Walras system has unfolded before us will we be able to see that of such equations we have merely s - 1, to be demonstrated in Sec. 8.

3. HOUSEHOLD DEMAND FOR OUTPUTS AND SUPPLY OF INPUTS

Let the kth household possess the endowments $q_{k1}, \ldots, q_{kn}$ of potential inputs: An able-bodied man is endowed with potential man-years of labor, and a landowner is endowed with potential acre-years of land use. Some of the endowments may be supplied as inputs $x_{k1}, \ldots, x_{kn}$, but the remainder is withheld for the household itself to enjoy $q_{k1} - x_{k1}, \ldots, q_{kn} - x_{kn}$: Man-hours are withheld for leisure, without which no consumption can be enjoyed, and acre-years are withheld for garden use or hunting! Withholdings yield utility just like outputs demanded. The utility function of the kth household is, then, a function of outputs demanded as well as of withholding:

$$(4) \qquad U_k = U_k(X_{1k}, \ldots, X_{mk}, q_{k1} - x_{k1}, \ldots, q_{kn} - x_{kn})$$

where q_{ki} are parameters.[3] Let all marginal utilities $\partial U_k/\partial X_{jk}$ be positive but declining with rising X_{jk}. Let all marginal utilities

$\partial U_k/\partial x_{ki}$ be negative and declining——hence numerically rising——with rising x_{ki}.

Now assume the household to have succeeded in maximizing utility. In the immediate neighborhood of such a utility maximum let us consider the substitution between the demand for two outputs: Let the kth household change infinitesimally its X_{jk} by dX_{jk} and its $X_{(j+1)k}$ by $dX_{(j+1)k}$ subject to two constraints. First, the changes must not affect utility U_k which is already at its maximum. Second, the changes must not upset the equality (3) between consumption and income: If the household wants to demand more of one output, it must demand less of another. Thus

$$dU_k \equiv \frac{\partial U_k}{\partial X_{jk}} dX_{jk} + \frac{\partial U_k}{\partial X_{(j+1)k}} dX_{(j+1)k} = 0$$

$$dC_k \equiv \frac{\partial C_k}{\partial X_{jk}} dX_{jk} + \frac{\partial C_k}{\partial X_{(j+1)k}} dX_{(j+1)k} = 0$$

Assuming prices of outputs to be beyond the household's control use (1) to carry out the partial derivations and find

(5)
$$\frac{P_j}{\partial U_k/\partial X_{jk}} = \frac{P_{j+1}}{\partial U_k/\partial X_{(j+1)k}}$$

Second, consider the substitution between the supply of two inputs in the immediate neighborhood of the utility maximum: Let the kth household change infinitesimally its x_{ki} by dx_{ki} and its $x_{k(i+1)}$ by $dx_{k(i+1)}$ subject to two constraints. First, the changes must not affect utility U_k which is already at its maximum. Second, the changes must not upset the equality (3) between consumption and income: If the household wants to supply less of one input, it must supply more of another. Thus

$$dU_k \equiv \frac{\partial U_k}{\partial x_{ki}} dx_{ki} + \frac{\partial U_k}{\partial x_{k(i+1)}} dx_{k(i+1)} = 0$$

$$dY_k \equiv \frac{\partial Y_k}{\partial x_{ki}} dx_{ki} + \frac{\partial Y_k}{\partial x_{k(i+1)}} dx_{k(i+1)} = 0$$

Assuming prices of inputs, too, to be beyond the household's control use (2) to carry out the partial derivations and find

$$\frac{p_i}{\partial U_k/\partial x_{ki}} = \frac{p_{i+1}}{\partial U_k/\partial x_{k(i+1)}} \tag{6}$$

Third, consider the substitution between the demand for an output and the supply of an input: Let the kth household change infinitesimally its X_{jk} by dX_{jk} and its x_{ki} by dx_{ki} subject to two constraints. First, the changes must not affect utility U_k which is already at its maximum. Second, the changes must not upset the equality (3) between consumption and income: If the household wants to demand more of one output, it must supply more of some input. Thus

$$dU_k \equiv \frac{\partial U_k}{\partial X_{jk}} dX_{jk} + \frac{\partial U_k}{\partial x_{ki}} dx_{ki} = 0$$

$$dC_k - dY_k \equiv \frac{\partial C_k}{\partial X_{jk}} dX_{jk} - \frac{\partial Y_k}{\partial x_{ki}} dx_{ki} = 0$$

Assume prices of outputs and inputs to be beyond the household's control, use (1) and (2) to carry out the partial derivations, and find

$$\frac{P_j}{\partial U_k / \partial X_{jk}} = - \frac{p_i}{\partial U_k / \partial x_{ki}} \tag{7}$$

For each household the system (5) through (7) contains m + n - 1 equations, hence there are (m + n - 1)s such equations for the economy. The system (5) through (7) determines the kth household's demand for outputs and supply of inputs by requiring the ratio between price and marginal utility to be the same for all outputs demanded and inputs supplied. Walras held this principle to be the very foundation of the whole edifice of economics:

Tous les hommes au courant des choses savent que la théorie de l'échange qui proportionne le prix a l'intensité du dernier besoin

satisfait, au Final Degree of Utility, au Grenznutzen, théorie produite presque simultanément par Jevons, M. Menger et moi, et qui fournit le fondement de tout l'édifice, est une théorie acquise á la science en Angleterre, en Autriche, aux Etats-Unis et dans les autres pays ou l'économie pure est cultivée et enseignée. [13], xvi.[4]

4. SUPPLY OF OUTPUT

Under pure competition and freedom of entry and exit the jth industry will supply its output at a price equalling unit cost:

$$P_j = \sum_{i=1}^{n} (a_{ij} p_i) \tag{8}$$

where a_{ij} is the ith input demanded by industry per physical unit of jth output, a technologically given parameter called "coefficient de fabrication". Of such equations we have m, one for each output.

5. EQUILIBRIUM IN THE OUTPUT MARKET

The number of physical units supplied by the jth industry must

equal the sum of all physical units of its output demanded by households:

(9) $$X_j = \sum_{k=1}^{s} X_{jk}$$

Of such equations we have m, one for each output.

6. DEMAND FOR INPUT

By definition of a_{ij} the number of physical units of the ith input demanded by industry must equal the sum of all the m outputs, each multiplied by its input-output coefficient with respect to the ith input:

(10) $$x_i \equiv \sum_{j=1}^{m} (a_{ij}X_j)$$

Of such equations we have n, one for each input.

7. EQUILIBRIUM IN THE INPUT MARKET

The number of physical units of the ith input demanded by industry must equal the sum of all physical units of that input supplied by households:

$$(11) \qquad x_i = \sum_{k=1}^{s} x_{ki}$$

Of such equations we have n, one for each input.

8. WALRAS' LAW

We can now see why we have merely s - 1 equations of the type (3). Sum Equation (1) over all the s households. Upon the summation use (9), (8), and (10) in that order to find

$$\sum_{k=1}^{s} C_k = \sum_{i=1}^{n} (p_i x_i)$$

Upon this use (11) and (2) in that order to find

(12) $$\sum_{k=1}^{s} C_k = \sum_{k=1}^{s} Y_k$$

Without using (3), then, we have found that the aggregate money value of consumption equals the aggregate value of money income. Now let (3) be satisfied by each of the first s - 1 households, sum over those households, and find

(13) $$\sum_{k=1}^{s-1} C_k = \sum_{k=1}^{s-1} Y_k$$

Deduct (13) from (12) and find

(14) $$C_s = Y_s$$

Consequently, if (3) is satisfied by each of the first s - 1 households, it is also satisfied by the sth household. Eq. (14), then, may be derived from the entire system and is not an independent equation. Of independent equations of type (3) we have only s - 1.

This carries us to the counting of equations and variables.

9. COUNTING EQUATIONS AND VARIABLES

Of equations we have

Type	Number
(1)	s
(2)	s
(3)	$s - 1$
(4)	s
(5) through (7)	$(m + n - 1)s$
(8)	m
(9)	m
(10)	n
(11)	n

or a total of $(m + n)(s + 2) + 3s - 1$. Of variables we have

Variable	Number
C_k	s
p_i	n
P_j	m
U_k	s
x_i	n
x_{ki}	ns
X_j	m
X_{jk}	ms
Y_k	s

or a total of $(m + n)(s + 2) + 3s$. So the number of variables exceeds the number of equations by one. Finding himself one equation short, Walras suspected his system to be in some sense indeterminate. Which sense? Suppose his system to be satisfied by a set of prices p_i and P_j, money expenditures C_k, and money incomes Y_k. Now if in this set every variable were multiplied by the same arbitrary constant λ, Eqs.

(1) through (3) and (5) through (8) would still hold——and no other equations contain p_i, P_j, C_k, or Y_k. Consequently, if the system were satisfied by one set of prices, money expenditures, and money incomes, it would be satisfied by infinitely many. In this sense the system is indeterminate: It does not determine absolute prices, money expenditures, or money incomes.

But if we divide Eqs. (1) through (3) and (5) through (8) by the price p_i or P_j of any good, using that good for our *numéraire*, we eliminate the price of our *numéraire* as a variable. Prices of all other goods, money expenditures, and money incomes are then expressed relative to the price of the *numéraire*, and we are left with the same number of equations and variables, $(m + n)(s + 2) + 3s - 1$. This led Walras to believe that his system determined relative prices, money expenditures, and money incomes:

> ...les problèmes de l'échange, de la production..., ainsi posés, sont des problèmes déterminés, c'est-à-dire comportant des équations en nombre rigoureusement égal à celui des inconnues... [13], xv.[5]

Mathematicians know that it is neither a sufficient nor a necessary condition for the existence of a unique solution to have "des équations en nombre rigoureusement égal a celui des inconnues." If the nonmathematical reader needs to be convinced, let him not even worry about nonlinear systems with complex roots but merely consider a linear system of two equations in two variables x and y and three parameters a, b, and c:

$$y = a + cx \tag{15}$$

$$x = y + b \tag{16}$$

whose solutions are

$$x = (a + b)/(1 - c) \tag{17}$$

$$y = (a + cb)/(1 - c) \tag{18}$$

Now if $a + b \lessgtr 0$ and $c = 1$, (15) and (16) are inconsistent, and

no solution exists. If $a + b = 0$ and $c = 1$, (15) and (16) are equivalent, and infinitely many solutions exist. If $a + b = 0$ and $c \lessgtr 1$, the unique solution $x = 0$ and $y = a$ exists. If either $a + b > 0$ and $c > 1$ or $a + b < 0$ and $c < 1$, a unique but negative solution exists for x and y.

Like the man who had always been speaking prose, the reader may discover that he has always been using the system (15) and (16): If $a \equiv$ autonomous consumption, $b \equiv$ investment, $c \equiv$ marginal propensity to consume, $x \equiv$ output, and $y \equiv$ consumption, the system (15) and (16) is nothing but the simplest possible Keynesian model, usually illustrated by the familiar 45°-line diagram!

In his Lesson 7, Walras did discuss special cases, illustrated graphically, in which the same number of equations and variables failed to produce unique solutions. But in his remaining 35 lessons those examples never bothered him again.

Men like Wald [11], von Neumann [8], Arrow and Debreu [1], Gale [4], Nikaido [9], Debreu [3], McKenzie [6], and Negishi [7] were bothered enough to prove the existence of a general equilibrium

in a competitive economy. Hicks [5], Samuelson [10], and Arrow and Hurwicz [2] went further and helped prove the stability of such an equilibrium. But at the very least, their problems were first clearly seen and formulated by Walras.

F O O T N O T E S

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[1]"It still remains to be seen whether it was because 2 francs were paid out in rent, 2 francs in wages and 1 franc in interest that this bottle of wine sells for 5 francs, or whether it is because the bottle sells for 5 francs that 2 francs were paid out in rent, 2 francs in wages and 1 franc in interest." [14], 211. I wish to thank Professor Jaffé for permission to quote here and in the following the translation he prepared with loving care and superior skill.

[2] "...*the theory of marginal productivity*...shows the underlying motive of the demand for services and the offer of products by entrepreneurs, just as *the theory of final utility* shows the underlying motive of the demand for products and offer of services by landowners, workers and capitalists. I preferred, however, not

to introduce the theory of marginal productivity into my general theory of economic equilibrium, for fear that the general theory, which was already complicated enough, might then be too difficult to grasp in its entirety." [14], 385.

[3]Walras never wrote a utility function like this but merely its partial derivatives which he expressed as functions of merely the quantity with respect to which the partial derivative was taken. I consider my Eq. (4) to be merely a notational modernization of Walras.

[4]"Everyone competent in the field knows that the theory of exchange based on the proportionality of prices to _intensities of the last wants satisfied_ (i. e. to _Final Degrees of Utility_ or _Grenznutzen_), which was evolved almost simultaneously by Jevons, Menger and myself, and which constitutes the very foundation of the whole edifice of economics, has become an integral part of the science in England, Austria, the United States, and wherever pure economics is developed and taught." [14], 44.

[5] "...the aforementioned problems of exchange, production...are determinate problems, in the sense that the number of equations entailed is equal to the number of the unknowns..." [14], 43-44.

R E F E R E N C E S

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