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CHAPTERS ON THE MODERN GEOMETRY OF THE POINT, LINE, AND CIRCLE.

VOL. I.

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CHAPTERS

ON THE

MODERN GEOMETRY

OF THE

POINT, LINE, AND CIRCLE;

BEING THE SUBSTANCE OF

LECTURES DELIVERED IN THE UNIVERSITY OF DUBLIN TO THE CANDIDATES FOR HONORS OF THE FIRST YEAR IN ARTS.

BY THE

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VOL. I.

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PREFACE.

The work now offered to the public contains, as its title indicates, the substance of lectures delivered for some years in the University of Dublin to the candidates for mathematical honors of the first year in arts; and supposes, accordingly, a previous acquaintance only with the first six books of the Elements of Euclid, and with just that amount of the principles of Elementary Algebra essential to an intelligent conception of the nature of signs, and of the meaning and use of the ordinary symbols of operation and quantity.

The acknowledged want of a systematic treatise on Modern Elementary Geometry, adapted to the requirements of students unacquainted with the higher processes of Algebraic Analysis, which of late years have been applied so successfully to the extension of geometrical knowledge, has induced the author to come forward with the present attempt to supply the deficiency. The only existing work of the same nature in the English language with which he is acquainted, the "Principles of Modern Geometry," of the late lamented Dr. Mulcahy, published in the year 1852, being now confessedly behind the present state of the subject; and the only other work of the same nature in any language with which he is acquainted, the elaborate and masterly "Traité de Géométrie Supérieure" of the justly celebrated M. Chasles, published in the same year, having become so scarce as to be now hardly attainable at any price.

Though designed mainly for the instruction of students of the comparatively limited mathematical knowledge generally possessed at the transition from school to university life, and arranged with special reference to the existing course of mathematical instruction in the University of Dublin, the author has spared no pains to render the work as generally interesting and instructive as the extent of his subject admitted. The order adopted, though framed on the basis of an existing arrangement, appeared as natural as any other he could have substituted for it; the principles established have been considered in all the generality, and stated with all the freedom from ambiguity, of which they appeared to him susceptible; and the demonstrations submitted, which are to a considerable extent original, have been presented as directly referred to ultimate principles, and as completely disencumbered of unessential details, as he was capable of rendering them.

To the second of the works above referred to, the "Traité de Géométrie Supérieure" of M. Chasles, the author is indebted for many important suggestions in the advanced chapters of the work; in those especially on the Theories of Anharmonic Section, Homographic Division, Involution, &c., of which its illustrious author was virtually the originator as well as the nomenclator, it will be at once seen that he has profited largely by the results so ably developed in the corresponding chapters of that elaborate work, while at the same time he can in no sense be regarded as the mere copyist of any of its contents.

To the Board of Trinity College the best thanks of the author are due for the liberal assistance they have given towards defraying the expenses of the work.

TRINITY COLLEGE, DUBLIN,
October, 1863.

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* In the statement of the above property given in the text an omission occurs which the reader is requested to supply as follows:—for the words "*of the corresponding segments*" substitute "*of the sines of the corresponding segments.*" As the property is very frequently referred to in the sequel the omission should be supplied before passing on to the subsequent Articles.

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THE MODERN GEOMETRY OF THE POINT, LINE, AND CIRCLE.

CHAPTER I.

ON THE DOUBLE ACCEPTATION OF GEOMETRICAL TERMS.

1. GEOMETRICAL propositions refer either to the comparative *magnitudes* of geometrical *quantities*, as in the propositions: "Rectangle under sum and difference = difference of squares," "Square of sum + square of difference = twice sum of squares," &c., or to the relative *positions* of geometrical *figures*, as in the propositions: "All points equidistant from the same point lie on the same circle," "All lines equidistant from the same point touch the same circle," &c. Hence the modern division of the science of Geometry into the two departments of *Geometry of magnitude or quantity* and *Geometry of position or figure* respectively.

2. The ordinary terms of Geometry are, with few exceptions, employed in double acceptations with reference to these two departments, and denote sometimes *magnitudes* and sometimes *figures*; the familiar term "line," for instance, denoting sometimes the indefinite figure so denominated extending to infinity in both directions and sometimes the distance from one point to another; the equally familiar term "angle," again denoting sometimes the complete figure formed by two indefinite lines extending to infinity in both directions and sometimes the inclination of one line to another. The ambiguity

arising from this duality of application rarely, if ever, causes any inconvenience or confusion, as the sense in which geometrical terms are employed is generally apparent from the context in which they occur, as for instance in the expressions: "points of bisection of a line," "lines of bisection of an angle," &c.

3. The literal symbols also by which geometrical figures of all kinds are wont to be represented, are employed occasionally in a similar duality of application with reference to the two departments of geometry; thus if A and B represent two points, AB represents indifferently the indefinite line passing through both and the linear interval between them. If A and B represent two lines, AB represents indifferently the unique point common to both and the angular interval between them. If A and B represent one a point and the other a line, AB represents indifferently the indefinite line passing through the former at right angles to the latter, and the perpendicular interval between them; for the reason already stated the ambiguity arising from this duality of application rarely, if ever, causes any inconvenience or confusion in practice.

4. Of the two different ways in which linear and angular magnitudes are alike ordinarily represented, viz. by the two letters which represent their extreme points or lines, or by a single letter denoting the number of linear or angular units they contain; the latter or *uniliteral* notation is generally the more convenient when *magnitude* only need be attended to, as in the familiar instance of the triangle in which the three sides are ordinarily represented by the three small letters a, b, c, and the three respectively opposite angles by the three corresponding capitals A, B, C, a notation than which nothing could be more convenient; but the former or *biliteral* notation is, on the contrary, the more convenient, when, as is often the case, *direction* as well as magnitude has to be taken into account, which under the biliteral notation may be indicated, in a manner at once simple and expressive, merely by the *order* in which the two letters are written, AB naturally representing the segment, or the angle, or the perpendicular

intercepted between the two points, or the two lines, or the point and line, *A* and *B* considered as measured in the direction from *A* to *B*, and *BA* the same segment, or angle, or perpendicular considered as measured in the opposite direction from *B* to *A*; this mode of distinction we shall have frequent occasion to employ in the sequel.

5. When a geometrical magnitude of any kind is represented or said to be represented, as it often is, by a *number*, or by a letter regarded as the representative of a number, it is always to be remembered that what is meant by such number or representative letter is the ratio the magnitude bears to some other magnitude of the same kind, given or assumed arbitrarily, but not either evanescent or infinite, to which it is implicitly, if not expressly, referred as a standard, and which is called the *unit* of that particular kind of magnitude, because that when the compared and standard magnitudes are equal, the number representing the former is then unity. The given or assumed unit of any particular kind of magnitude may have theoretically *any* finite value, as, whatever it is, it always disappears whenever different magnitudes of the same kind are compared with each other, their relative magnitudes, or ratios to each other, being of course independent of the arbitrary standard by which their absolute values may happen to be estimated; it is thus, and thus only, that magnitudes other than abstract numbers become subjects of *calculation*, the proper and only subjects of which are *numbers* and numbers alone.

6. With respect to the three species of geometrical magnitude, *length*, *area*, and *volume*, it is to be observed that as the magnitudes themselves are not all independent of each other, but on the contrary vary simultaneously according to known laws, their three units consequently are never *all* arbitrary *together*, but are always made to correspond to each other according to the same laws of simultaneous variation in a manner at once obvious and natural; areas and volumes varying, cæteris paribus, as the squares and as the cubes respectively of the lengths on which they depend, the unit of area

accordingly is always the square and the unit of volume the cube of the unit of length; the latter, however, or more generally some one of the three, being arbitrary; it is for this reason that we are justified in asserting the area of a parallelogram and the volume of a parallelopiped to be equal in abstract numbers to the products of their two and of their three dimensions respectively, and similarly of other areas and volumes as having the same necessary and known connection with the lengths on which they depend.

7. With respect to the only remaining species of geometrical magnitude, viz. *inclination*, as no connection exists between it and any of the other three, its unit is therefore at once arbitrary and independent of any of theirs; any finite angle, consequently, may be given or assumed at pleasure, considered as the angular unit, and all other angles estimated by the numbers, integer or fractional, of such units contained in them; and this accordingly is what is done in Astronomy, Geography, Navigation, Geodesy, &c., and in other practical applications of Geometry where angles are ordinarily estimated by the numbers of *degrees*, *minutes*, and *seconds*, &c. which they contain.

Theoretically considered, the most convenient unit of angular measure as well in Geometry as in the science which treats more especially of angles and their relations, is *the angle which from the centre of a circle subtends an arc = the radius*, and which, as all circles are similar figures, is consequently unique, because in reference to it as unit the numerical value of any angle is simply *the ratio of the subtending arc to the radius* in any circle described round the vertex as centre, a value simpler than for any other unit. Practically considered, however, this unit has the twofold disadvantage; firstly, of being so large that angles of ordinary magnitude, if referred to it, must be expressed as fractions; and, secondly, of not being a sub-multiple of, or even commensurable with, four right angles, the exact divisions and sub-divisions of which are of such importance in all practical subjects.

8. As in Arithmetic the third proportional to any number and unity is termed the *reciprocal* of the number, so in Geo-

metry the third proportional to any magnitude and the unit, whatever it be, to which it is referred, is termed the *reciprocal* of the magnitude.

By taking the reciprocal of the reciprocal, as thus defined, either of a magnitude or number, we evidently get back again the original magnitude or number. Hence the reason why magnitudes or numbers so related are termed *reciprocals* to each other, the process by which either produces the other always reciprocally reproducing itself from the other.

The product of the extremes being equal to the square of the mean in every proportion of three terms, the product of every pair of numbers reciprocals to each other = 1, and that of every pair of magnitudes of any kind reciprocals to each other = the square of the common unit, whatever it be, to which they are referred; and, conversely, if the product of two numbers = 1, or the product of two magnitudes of any kind = the square of the unit to which they are referred, such numbers or magnitudes are reciprocals to each other.

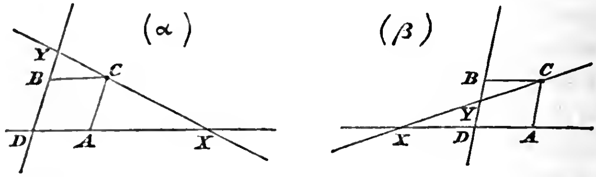
When two ratios $a : b$ and $c : d$ are reciprocals to each other, the four component magnitudes a, b, c, d, whatever be their nature, are evidently "reciprocally proportional" in Euclid's meaning of the phrase. (Euc. VI. 14, 15, 16).

9. As in Arithmetic the numbers *nothing* and *infinity* are reciprocals to each other, each being evidently the third proportional to the other and *any finite number*, so in Geometry *evanescent* and *infinite* values of any kind of magnitude are always reciprocals to each other, *whatever be the absolute value of the unit to which they are referred*, each being evidently the third proportional to the other and *any finite value* of the same kind of magnitude.

The rectangle under two linear magnitudes, reciprocals to each other, being constant and = the square of the unit, whatever it be, to which they are referred. The reader, familiar with the Second Book of Euclid, may take as exercises in its principles the four following problems: "*Given the sum, difference, sum of squares, or, difference of squares, of two linear magnitudes reciprocals to each other to a given unit, to determine the magnitudes.*"

10. There are several constructions by which pairs of reciprocals in linear magnitudes may be simultaneously determined, of which the following is perhaps the simplest:

Round any one of the four corners C of any equilateral parallelogram $ABCD$ the common length of whose four sides



= the linear unit, let an indefinite line XY be conceived to revolve intersecting the two sides AD and BD opposite to C in two variable points X and Y; the intercepts AX and BY between the two points of meeting and the two corners A and B adjacent to C are always reciprocals to each other.

For, by similar triangles XAC and CBY,

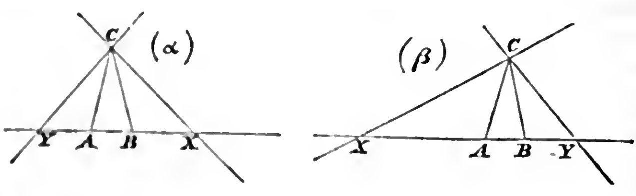
$$AX : AC = BC : BY \text{ or } AX.BY = AC.BC$$

in every position of the revolving line, and therefore, &c.

The parallelogram in the above need not be equilateral; any parallelogram, the rectangle under whose adjacent sides $CA.CB=$ the square of the linear unit, would obviously do as well.

11. In case it should be desirable to have the simultaneous reciprocals AX and BY measured on the same in place of on different lines, the following modification of the above may be employed for the purpose:

Round the vertex C of any isosceles triangle ACB, the



common length of whose sides = the linear unit, let two indefinite lines CX and CY inclined to each other at a constant angle equal to either base angle of the triangle be conceived to revolve intersecting the base AB in two variable points X and Y; the intercepts AX and BY between the two points of meeting and the two extremities of the base AB, for which the three angles CAX, CBY, and XCY are of the same affection; that is, all three acute (fig. α) or all three obtuse (fig. β) are always reciprocals to each other.

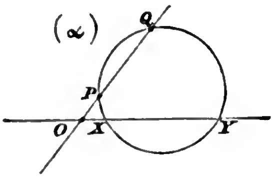
This is obviously identical with the preceding construction modified by turning the unit line CB round C, bringing with it the two indefinite lines BY and CY until the former coincides with AX, and the same demonstration, word for word, and letter for letter, applies indifferently to either.

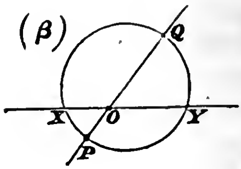
Since during the revolution of the constant angle XCY its acute and obtuse regions alternately comprehend the intercept XY, should any doubt exist in any particular position as to how the two points X and Y correspond to the two A and B in measuring the reciprocals AX and BY, it will be at once settled by remembering, as above stated, (see the figures of the original as well as of the modified construction which have been drawn to correspond) that the angles CAX and CBY must be always of the same affection with XCY.

If the vertical angle of the isosceles triangle ACB were nothing, its unit sides CA and CB would coincide and be perpendicular to XY; the constant revolving angle XCY would be right in every position; the two reciprocals AX and BY would be measured from a common origin, and the ambiguity adverted to above would not exist: in the corresponding case of the original construction the lozenge $ABCD$ would evidently be a square.

12. The following, however, is the most convenient construction for the simultaneous determination of pairs of conterminous reciprocals upon any given indefinite line MN, inasmuch as by it they may be determined at pleasure either in similar or in opposite directions from any given common origin O.

Drawing arbitrarily from the common origin O in any similar or opposite directions, according as the directions of the reciprocals are to be similar or opposite, any two lengths OP





and OQ, the rectangle under which $OP.OQ=$ the square of the linear unit. Every circle passing through their two extremities P and Q intersects the given line MN in two points X and Y whose distances from O are always reciprocals to each other.

For, Euc. III. 35, 36, $OX.OY=OP.OQ$, whatever be the circle, and therefore, &c.

The above three methods have all the common advantage of allowing to both reciprocals every range of magnitude from nothing to infinity, and of shewing very clearly how the passage of either through nothing or infinity is accompanied by the simultaneous passage of the other through infinity or nothing, whatever, in any case, be the absolute value of the unit to which they are referred, provided only it be finite.

13. Geometrical magnitudes of every kind, when compared with others of the same kind, present in their evanescent and infinite states some anomalous peculiarities, to which, as constantly occurring in geometrical investigation, we proceed to call early attention.

The product of an evanescent or of an infinite with any finite magnitude and the ratio of an evanescent or of an infinite to any finite magnitude being necessarily evanescent or infinite, when therefore two geometrical magnitudes of any kind have *any finite* product or ratio, one necessarily becomes infinite as the other vanishes, and conversely, in the former case, and both vanish or become infinite together in the latter case; hence, as in abstract numbers the product of 0 with ∞ or of ∞ with 0, and the ratio of 0 to 0 or of ∞ to ∞ is plainly *indeterminate*, so in geometrical magnitudes of every kind, the product of an evanescent with an infinite or of an infinite with an evanescent magnitude, and the ratio of an evanescent to an evanescent or of an infinite to an infinite magnitude, *considered in the abstract*, is also indeterminate; though in every particular instance in which either product or ratio actually arises it has generally some particular definite value determinable and to be determined from consideration of the particular circumstances under which it arises; as, for instance, if the product or ratio were *constant* in the general and therefore in every particular state of the magnitudes.

14. The ratio of two magnitudes of any kind, considered in the abstract, being thus indeterminate when the magnitudes are both either evanescent or infinite, it follows therefore that the two *criteria of equality* between two magnitudes of the same kind when compared with each other, viz. that 1°. their ratio = 1, and 2°. their difference = 0, each of which necessarily involves the other so long as the magnitudes are finite, do not involve each other when the magnitudes are either evanescent or infinite, for while the difference between two evanescent magnitudes is always = 0, their ratio, as above shewn, may have any value = or not = 1, and while the ratio of two infinite magnitudes may be and often is = 1, their difference, as may be easily shewn, may have any value = or not = 0.

15. The following useful example may be taken as an illustration of the preceding observation:

The ratio of the distances of a point P at infinity from any two points A and B not at infinity is always equal to unity, though their difference may (Euc. I. 20) *have any value from nothing to the interval AB.*

For, whatever be the position of P, whether at or not at infinity, or on or not on the line AB, since (Euc. I. 20) PA differs from PB by a quantity not exceeding AB, therefore $PA : PB$ differs from $PB : PB$, or 1, by a quantity not exceeding $AB : PB$, which quantity = 0, whatever be the length of AB whether evanescent or finite, when $PB = \infty$, that is, when P is any where at infinity whether on or not on the line AB.

In the particular case when the two points A and B coincide, then for every position of P, whether at infinity or not, the two criteria of equality $PA : PB = 1$, and $PA \sim PB = 0$ evidently hold, except only for the point $A = B$ itself, for which the ratio assumes the indeterminate form $0 : 0$, and, therefore, (13) may have any value as well as 1. This particular case often occurs in geometrical investigations, and whenever it does its peculiarity must always be attended to.

In the general case when A and B do not coincide, for every point P on the indefinite line bisecting internally at right angles the interval AB, whether at infinity or not, both criteria of equality $PA : PB = 1$, and $PA \sim PB = 0$, hold without any exception, while for a point P not on that line $PA \sim PB$ is never $= 0$, and $PA : PB$ is therefore $= 1$ only when P is at infinity.

In the general case again, for every point P on the indefinite line AB itself, whether at infinity or not, $PA \sim PB$, except only for the finite interval between A and B, has (Euc. I. 20) the greatest possible value AB, and therefore for points external to that interval $PA : PB = 1$ only when P is at infinity, in which position it is consequently termed *the point of external bisection* of the segment AB. Hence we see that—

The point of external bisection of any finite segment of a line is the point at which the line intersects infinity, and conversely, the point at which a line intersects infinity is the point of external bisection of any finite segment of the line.

In the particular case when the segment is evanescent, then, as already stated, every point on the line, except only that at which the extremities coincide, is indifferently a point of external bisection of the segment.

16. Admitting that any number of lines passing through a common point divide similarly (Euc. VI. 10) any two parallel lines in the ratio of their distances from the point, and that, conversely, any number of lines dividing any two parallel lines similarly in any ratio pass through a common point whose distances from the parallels are in that ratio; the following very important, but at first sight somewhat paradoxical, con-

clusion respecting points at infinity, results immediately from the general property of the preceding article, viz.—

Every system of lines passing through a common point at infinity is a system of parallel lines; and conversely, every system of parallel lines is a system of lines passing through a common point at infinity.

For, conceiving any two parallel lines L and L' drawn arbitrarily intersecting the entire system of lines, in either case, in two systems of points A, B, C, D, &c. and A', B', C', D', &c., then since, in the former case, the several lines AA', BB', CC', DD', &c. pass, by hypothesis, through a common point O, therefore, (Euc. VI. 4)

$$AB : A'B' = AC : A'C' = AD : A'D', \text{\&c.} = AO : A'O, = 1,$$

since, by hypothesis, O is at infinity (15); therefore $AB = A'B'$, $AC = A'C'$, $AD = A'D'$, &c.; and therefore (Euc. I. 33) BB', CC', DD', &c. are all parallel to AA' and to each other; and since, in the latter case, the several lines AA', BB', CC', DD', &c. are, by hypothesis, parallel; therefore (Euc. I. 34) $AB = A'B'$, $AC = A'C'$, $AD = A'D'$, &c.; and, therefore, as $AB : A'B' = AC : A'C' = AD : A'D'$, &c., the several lines BB', CC', DD', &c. all intersect AA' at the same point O (Euc. VI. 4); and as the common ratio $= 1$ that point O is at infinity (15).

The above is but one of a multitude of arguments for the truth of a conclusion long placed beyond all question by the simplest considerations of projection and perspective.

By a very slight modification Euclid's excellent definition of parallel lines, those, viz., "which lying in the same plane never meet though indefinitely produced," might be made to express the preceding most important and indeed fundamental property of such lines without failing to convey at the same time the notion intended by the original. The simple substitution of the two words *until infinitely* in place of the two *though indefinitely* would manifestly effect this.

It is evident from the above that the *position* of a point at infinity both determines and is determined by the *direction* of any line passing through it.

17. If a variable line be conceived to revolve continuously in one direction round a fixed point and to intersect in every position a fixed line not passing through the point; the point of intersection evidently traverses continuously in one direction the entire fixed line in the course of each complete semi-revolution of the variable line; approaches to infinity in the direction of its motion as the latter approaches to a position of parallelism with the former; reaches infinity as that position is attained; and emerges again from infinity from the opposite direction when that position is passed; from this and from many other considerations geometers have long satisfied themselves that

The two opposite directions of every line, not itself at infinity, are to be regarded, not as reaching infinity at two different and opposite points, but as running into each other and meeting at a single point at infinity.

Hence the propriety of the expression "point of external bisection" of any finite segment of a line (15).

Paradoxical as the above conclusion may appear when first stated, the grounds confirmatory of it are so numerous and varied that any early hesitation in admitting its legitimacy is generally very rapidly got over.

18. If the centre of a variable circle touching a fixed line at a fixed point be conceived to traverse continuously in one direction the entire circuit (17) of the orthogonal line passing through the point, starting from and returning to the point through infinity (Euc. III. 19). The circle itself evidently commences from evanescence with the commencement of the motion; expands continuously at the side of the line corresponding to its direction during the first half of the circuit; opens out into the line itself as infinity is reached; contracts continuously at the opposite side of the line during the second half of the circuit; and, terminates in evanesence with the completion of the motion. Hence, and from many other considerations, it appears that—

Every point not at infinity may be regarded as a circle of evanescent radius whose centre is the point; and every line not at infinity as a circle of infinite radius whose centre is the point at infinity in the direction orthogonal to the line (16).

In the geometry of the point, line, and circle, therefore the point and line are the limiting forms of the circle in the extreme cases of its radius being evanescent and infinite.

19. If a variable line be considered to revolve continuously in one direction round a fixed point, and to intersect in every position a fixed circle passing through the point, the variable point of intersection evidently traverses continuously in one direction the entire circumference of the circle in the course of each complete semi-revolution of the line; and, on its way every time, approaches to, reaches, and passes through the fixed point as the line approaches to, reaches, and passes through the particular position *in which it is a tangent to the circle at that point.* Hence, and from innumerable other considerations, it appears that—

When the two points of intersection of a line and circle coincide, the line and circle touch at the point of coincidence.

And generally that—

When two points of intersection of any two figures coincide, the figures themselves touch at the point of coincidence.

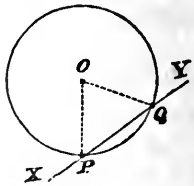
This, indeed, as fundamentally correct in conception and invariably simple in application, might be made the formal *criterion of contact* in elementary, as it is in advanced, geometry; and from it the several known properties respecting the contact of circles with lines and with each other, established in the Third Book of Euclid and elsewhere, might be easily shewn to be mere corollaries from more general properties respecting their intersection, deduced by simply introducing into the latter the particular supposition of coincidence between their two, in general separate, points of intersection. A few examples will shew this more clearly.

Ex. 1°. *A line and circle or two circles having contact at any point can never meet again either by contact or intersection.* (Euc. III. 13 and 16.)

For they can never under any circumstances meet at all in *more* than *two* points (Euc. III. 2 and 10); which property being true *in general*, whatever be the interval between the points, is *therefore* true *in the particular case* where the interval $=0$; that is, when the figures touch.

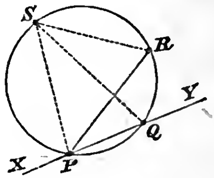
Ex. 2°. *At every point on a circle the tangent is perpendicular to the radius.* (Euc. III. 18.)

Let P be the point, XY *any* line passing through it, Q the other point in which XY meets the circle, and OP and OQ the radii to P and Q; then, *whatever be the interval* PQ, since the triangle POQ is always isosceles, the two external angles OPX and OQY are *always* equal (Euc. I. 5); they are *therefore* equal *in the particular case* when Q coincides with P, and therefore OQ with OP, and therefore the angle OQY with the angle OPY, in *that case*, therefore, the angles OPX and OPY are equal; and, therefore, (Euc. I. def. 11) the radius OP is perpendicular to the *tangent* XY.

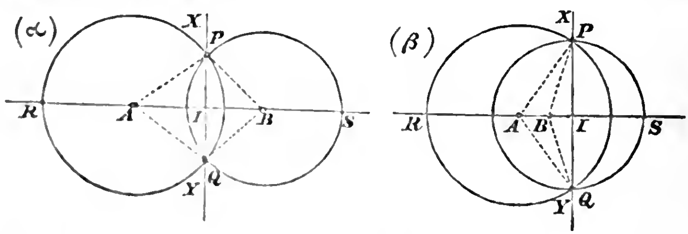


Ex. 3°. *At every point on a circle the angles made by any chord with the tangent are equal to the angles in the alternate segments*, (Euc. III. 32).

Let P be the point, PR the chord, XY *any* line passing through P, Q the other point in which XY meets the circle, S any arbitrary point on the circle, and SP, SQ, SR the lines connecting it with P, Q, R, then, *whatever be the interval* PQ, the angles RPY and RSQ being in the same segment are *always* equal (Euc. III. 21); they are *therefore* equal *in the particular case* when Q coincides with P, and therefore SQ with SP, and therefore the angle RSQ with the angle RSP, in *that* case therefore the angles RPY and RSP are equal, that is, the angle the chord PR makes with the *tangent* XY is equal to the angle in the alternate segment PSR.



Ex. 4°. *When two circles touch, externally or internally, the line joining their centres passes through the point of contact and is perpendicular to the line touching both at that point*, (Euc. III. 11 and 12).



Let PQR and PQS be *any* two *intersecting* circles, P and Q their two points of intersection, A and B their two centres, and XY the indefinite line passing through P and Q; then, on account of the two isosceles triangles PAQ and PBQ formed by connecting A and B with P and Q, the line AB connecting their vertices A and B always both bisects and is perpendicular to their common base PQ (Euc. III. 3); and this being *always* true in general, whatever be the length of PQ, is *therefore* true *in the particular case* when that length $= 0$, that is, when the two points P and Q coincide, but when they do, their middle point I coincides with both, the two circles touch, externally or internally, at the point of coincidence, and the indefinite line XY touches both at that point.

In every application of the above method, one precaution, observed it will be perceived in each of the above illustrative examples, is invariably to be attended to. The supposition of coincidence between the two points of intersection P and Q, in which the contact of the figures consists, is *never* to be introduced *until* the more general property, independent of the distance between them, has *first* been established.

20. As in the compound figure consisting of a line and circle variable in relative position with respect to each other, the two points common to both pass evidently from separation, through coincidence, to simultaneous disappearance, or conversely, as the distance of the line from the centre passes from being $<$, through being $=$, to being $>$ the radius of the circle, or conversely; so in the compound figure consisting of a point and circle variable in relative position with respect to each other, the two tangents common to both pass similarly from separation, through coincidence, to simultaneous disappearance, or conversely, as the distance of the point from the centre passes from being $>$, through being $=$, to being $<$ the radius of the circle, or conversely. Hence, as in many ways otherwise, it appears that—

As every tangent to a circle or any other figure is the connector of two coincident points on the circle or figure, and conversely, so every point on a circle or any other figure is the intersection of two coincident tangents to the circle or figure, and conversely.

In the applications of this, as of the preceding principle, of which it is the correlative, the same precaution again is invariably to be observed; in investigating any property of a point on a circle or any other figure regarded as the intersection of two coincident tangents to the circle or figure, the supposition of coincidence between the two tangents is never to be introduced until the more general property of the point of intersection of *any* two tangents, in which it is involved, has first been established.

21. In the language of modern geometry every two points, lines, or other similar elements of, or connected with, any compound figure, which with change of relative position among the constituents of the figure pass or are liable to pass, as above described, from *separation*, through *coincidence*, to *simultaneous disappearance*, or conversely, are termed *contingent* as distinguished from *permanent* elements of the figure, and are said to be *real* or *imaginary* according as they *happen to be* apparent or non-apparent to sense or conception. Geometers of course have not, nor do they profess to have, any conception of the nature of contingent elements in their imaginary state, but they find it preferable, on the grounds both of convenience and accuracy, to regard and speak of them as imaginary rather than as non-existent in that state: in the transition from the real to the imaginary state, and conversely, contingent elements pass invariably through coincidence, through which, as above described, they always change state together.

In the geometry of the point, line, and circle, it is only in figures involving, directly or indirectly, the latter in its finite form, that contingent elements from their nature could occur; in figures, however complicated, consisting of points and lines only all elements not depending on the circle in its finite form are invariably permanent.

22. When a line and figure of any kind intersect, the angles between the line and the tangents to the figure at the several points of intersection are termed the angles of intersection of the line and figure at the points; when two figures of any kind intersect, the angles between the tangents to them at the several points of intersection are termed the angles of intersection of the figures at the points; in the cases of a line and circle and of two circles the angles of intersection at the two points of intersection being evidently equal, each separately is called *the angle of intersection* of the figures.

With respect to the angle of intersection of a line and circle it is evident that:

1°. Every line passing through the centre of a circle intersects the circle at right angles; and conversely, every line intersecting a circle at right angles passes through the centre of the circle, (Euc. III. 18, 19).

2°. Every line dividing a circle into segments containing any angle, intersects the circle at the angle in the segments; and conversely, every line intersecting a circle at any angle divides the circle into segments containing the angle, (Euc. III. 32).

3°. A variable line whose distance from a fixed point is constant intersects at a constant angle every circle of which the point is the centre.

And with respect to the angle of intersection of two circles that:

1′. Every circle touching at either extremity any diameter of another circle intersects the other at right angles; and conversely, every circle intersecting another at right angles touches at each point of intersection a diameter of the other.

2′. Every circle touching at either extremity any chord of another circle intersects the other at the angle in the segments determined by the chord; and conversely, every circle intersecting another at any angle touches at each point of intersection a chord dividing the other into segments containing the angle.

3′. A variable circle of constant radius the distance of whose centre from a fixed point is constant intersects at a constant angle every circle of which the point is the centre.

A line and circle, two circles, or any other two figures, intersecting at right angles, are said *to cut orthogonally*, or, as it is sometimes termed, *to be orthotomic*.

23. In order to avoid the ambiguity as to which of the two supplemental angles, regarded as magnitudes, between the two tangents at either point of intersection of two circles is to be regarded as *the* angle of intersection of the circles,

in cases in which it is necessary, as it often is, to distinguish between them, the following convention has been agreed to by geometers.

The radius being perpendicular to the tangent at every point of a circle, and the two supplemental angles between any two lines being equal to those between any two perpendiculars to them, if from either point of intersection P or Q (fig., Ex. 4°, Art. 19) of the two circles, the two radii, PA and PB, or, QA and QB, be drawn, one of the two supplemental angles between the two tangents is equal to the *internal* and the other to the *external* angle between the two radii; the *former*, APB or AQB, is that which is considered as the angle of intersection of the circles; this is obviously tantamount to regarding that angle as measured either from the convex circumference of one circle to the concave circumference of the other; or, *vice versâ*, from the concave of one to the convex of the other; but not either from the concave of one to the concave of the other, or from the convex of one to the convex of the other.

In accordance with this convention the angle of intersection of two circles is to be regarded as acute, right, or obtuse, according as the square of the distance between their centres A and B is less than, equal to, or greater than the sum of the squares of their radii AP and BP, or AQ and BQ (Euc. II. 12, 13); in the extreme case of the former when $AB =$ the difference of the radii, that is, when the circles touch at the same side of their common tangent, the angle of intersection is to be regarded as $= 0$; and in the extreme case of the latter when $AB =$ the sum of the radii, that is, when the circles touch at opposite sides of their common tangent, the angle of intersection is to be regarded as = two right angles; and, for the same reason, generally, when any two figures touch, their angle of intersection at the point of contact is to be regarded as $= 0$, or = two right angles, according as they lie at the same side or at opposite sides of their common tangent at the point.

24. In *every* case of the *comparison* of two or more angles regarded as magnitudes (2) it is to be remembered: 1°. That

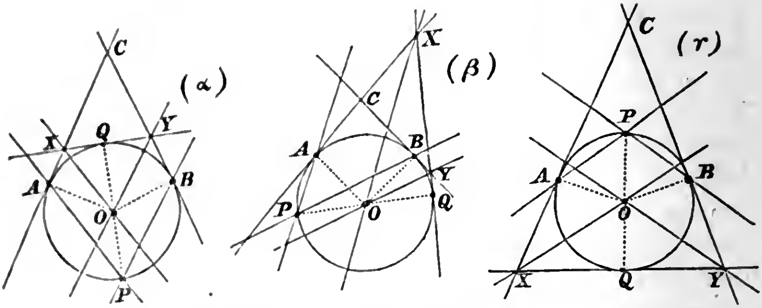
every two finite conterminous lines determine *two* different angular intervals of separation from each other, one exceeding by as much as the other falls short of two right angles, and having in the abstract equal claims to be regarded as *the* angle between the lines; and, 2°. That every two intersecting indefinite lines determine *two* pairs of opposite equiangular regions, one exceeding, in angular interval of separation between the determining lines, by as much as the other falls short of a right angle, and having in the abstract equal claims to be regarded as *the* angle between the lines. The twofold source of ambiguity thus arising must always be attended to in comparing angular magnitudes, as, whatever be the nature of two compared angles, the greater interval for one corresponds often to the lesser for the other in the former case, and the obtuse region for one corresponds often to the acute for the other in the latter case; and that even for angles *similar* as *figures*, that is, whose sides, whether finite or indefinite, are capable of simultaneous coincidence. Whenever, therefore, two angles different in position but similar in form, are said, as they often are, to be *equal*, and when an angle variable in position but invariable in form is said, as it often is, to be *constant*, the terms so employed, though applicable properly to magnitudes only, are to be regarded as indicating the aforesaid similarity or invariability of *form*, rather than absolute equality or constancy of *value*, in such cases generally.

25. The two following examples, of repeated occurrence in the modern geometry of the circle, are important illustrations of the preceding observations.

1°. *A variable point on the circumference of a fixed circle subtends a constant angle at any two fixed points on the circle.*

2°. *The segment of a variable tangent intercepted between any two fixed tangents to a circle subtends a constant angle at the centre of the circle.*

To prove 1°. Let O be the centre of the circle, A and B the two fixed points and P the variable point; the angle APB is, according to the position of P, equal to half the less or greater



angular interval AOB, and therefore constant in the sense above explained.

For, joining OA, OB, OP, and producing the latter through O to meet the circle again at Q; then, as in Euc. III. 21, 22, the angles APO and BPO being the halves of the angles AOQ and BOQ, the sum, or difference as the case may be, of the former, that is the angle APB, = half the sum, or difference, of the latter, that is, half the (less or greater) angle AOB; and therefore &c.

To prove 2°. Let AC and BC be the two fixed tangents, XY the segment of the variable tangent intercepted between them, Q its point of contact, and O, as before, the centre of the circle; the angle XOY is, according to the position of XY, equal to half the less or greater angular interval AOB, and therefore constant in the sense above explained.

For, joining OA, OB, OQ; then, Euc. III. 17, the angles XOQ and YOQ being the halves of the angles AOQ and BOQ, the sum, or difference as the case may be, of the former, that is, the angle XOY = half the sum, or difference, of the latter, that is half the (less or greater) angle AOB; and therefore &c.

Now it is evident that it is as figures and not as magnitudes (2) the two angles APB and XOY are strictly speaking invariable; for as the two points P and Q, on which their positions depend, traverse the entire circumference of the circle, their magnitudes in the positions indicated in fig. (γ), in which they are halves of the greater angular interval AOB, are evidently *the supplements* of their magnitudes in the positions indicated in the figures (α) and (β), in which they are halves of the lesser angular interval AOB; and so universally in all cases of the same nature, two finite conterminous lines presenting indifferently their greater and lesser angular intervals of separation, and two indefinite intersecting lines their obtuse and acute regions of figure, when revolving through four right angles.

In the particular cases when either the two fixed points A and B or the two variable points P and Q are diametrically opposite points of the circle, the two constant angles APB and XOY are always not only similar as figures but equal as magnitudes; for in the former case, whatever be the positions of P and Q, the two pairs of lines PA and PB, OX and OY intersect evidently at right angles, and therefore &c., and in the latter case (that represented in the figures), whatever be the positions of A and B, the two pairs of lines PA and OX, PB and OY are evidently parallels, and therefore &c.

CHAPTER II.

ON THE DOUBLE GENERATION OF GEOMETRICAL FIGURES.

26. When a variable point moving according to some law lies in every position on a figure of any form, such figure is termed the *locus* of the point. When a variable line moving according to some law touches in every position a figure of any form, such figure is termed the *envelope* of the line. As every simple figure, whatever be its form, may be conceived to be generated, either, if not itself a point, by the continued motion of a point, or, if not itself a line, by the continued motion of a line; with those two exceptions therefore every simple figure in geometry, whether existing alone or in combination with other figures, may be regarded *either as the locus of a variable point or as the envelope of a variable line.*

27. The law directing the movement of the generating point or line being given, the nature of the figure described or enveloped is implicitly given with it, though its actual determination presents of course very different degrees of difficulty in different cases; thus, for instance, the locus of a variable point, or the envelope of a variable line, moving so as to preserve a constant distance from a fixed point, is evidently a circle of which the fixed point and constant distance are the centre and radius.

28. But the law directing the movement of the generating point or line, by which a figure, the nature of which is given, may be described or enveloped, need not necessarily be that expressing the primary or fundamental property by which such figure may have been defined, but on the contrary may be one resulting from *any* of its secondary or derived properties in-

stead: thus, though a circle may, as above, be regarded either as the locus of a variable point, or as the envelope of a variable line, the distance or the square of the distance of which from a fixed point is constant; it may also, as will hereafter appear, be regarded either as the locus of a variable point the sum of the squares of whose distances, or as the envelope of a variable line the sum of whose distances, from any number of fixed points is constant.

29. A single geometrical condition governing the movement of a variable point or line is sufficient in all cases to restrict the point or line to some locus or envelope; thus, for instance, the single condition that a variable point subtend, or that a variable line intersect, a fixed circle at a constant angle, is sufficient to restrict the point or line to a concentric circle as its locus or envelope, of this the reason is evident, for while no condition on the one hand leaves the point or line free to occupy *any* position, two conditions on the other hand suffice when independent to fix it altogether.

30. The locus of a variable point or the envelope of a variable line may be, and often is, a compound figure whose component simple figures satisfy separately the condition governing the movement of the point or line; thus, for instance, the locus of a variable point whose distances from two fixed lines are equal consists evidently of the two lines of bisection external and internal of the angle determined by the lines, and the envelope of a variable line whose distances from two fixed points are equal consists evidently of the two points of bisection external and internal of the segment determined by the points; and similarly for any other constant ratio as well as that of equality. In such cases the compound figure consisting of the two or more simple figures is sometimes termed the *complete* locus or envelope of the point or line.

31. With respect to *particular cases* of loci and envelopes it is to be observed in general that—

1°. A locus or envelope, or any part of either if a compound figure, which, under the general circumstances of the conditions under which it arises, is a circle in its finite form, may, and often does, under particular circumstances of the conditions, assume the evanescent or infinite form of point or line (18): thus, for instance, the locus of a variable point or the envelope of a variable line whose distance from a fixed point is constant, which in general is the circle whose centre and radius are the point and constant, becomes of course evanescent or infinite when the constant $= 0$ or ∞.

2°. A locus or envelope, which, under the general circumstances of the conditions under which it arises, is a single figure of any form, often breaks up under particular circumstances of the conditions into two or more figures of simpler forms; thus, for instance, the locus of a variable point, the product of whose distances from any number of fixed lines, or the envelope of a variable line, the product of whose distances from any number of fixed points, is constant, which in general is a single figure of form depending on the number and disposition of the points or lines, breaks up into the entire system of lines or points when the constant $= 0$.

3°. A locus or envelope, which, under the general circumstances of the conditions under which it arises, is a definite determinate figure, simple or compound, becomes often *indeterminate* under particular circumstances of the conditions; thus, for instance, the locus of a variable point whose distances form two fixed lines, or the envelope of a variable line whose distances form two fixed points, are equal, which in general consists of the two lines or points of bisection of the angle or segment determined by the lines or points, becomes indeterminate when the lines or points coincide; *every* point in the former case, or line in the latter, then evidently satisfying the conditions of the locus or envelope.

As particular examples of loci and envelopes will appear in numbers in the course of the following pages, we shall not delay to give any here, but shall devote instead the remainder of the present chapter to the theory and properties of *similar figures* considered under their double aspect as loci of points and as envelopes of lines.

82. Two geometrical figures of any kind F and F', whether regarded as loci or envelopes, whose generating points or enveloping lines A, B, C, D, &c. and A', B', C', D', &c. correspond in pairs A to A', B to B', C to C', D to D', &c. are said to be *similar* when two points O and O', whether belonging to the figures or not, exist, such that for *every* two pairs of corresponding distances or perpendiculars OA and $O'A'$, OB and $O'B'$, the two angles AOB and $A'O'B'$ and the two ratios $OA : OB$ and $O'A' : O'B'$ are equal; and so also are two figures composed of systems of any common number of isolated points or lines, or mixed points and lines, A, B, C, D, &c. and A', B', C', D', &c. whose constituent elements correspond in pairs fulfilling the same conditions.

Two figures thus related to each other are said, like two hands or two feet, to be *both right or left* or *one right and the other left* according as the directions of rotation of the several pairs of corresponding angles AOB and $A'O'B'$, BOC and $B'O'C'$, COD and $C'O'D'$, &c. are similar or opposite.

As two angles, two ratios, or two magnitudes of any kind when equal to a third are equal to each other, it is evident from the conditions of similitude as above stated, that *two figures of any kind when similar to a third are similar to each other.*

33. Since, for two figures fulfilling the conditions of similarity, the ratios of the several pairs of corresponding distances or perpendiculars OA and $O'A'$, OB and $O'B'$, OC and $O'C'$, OD and $O'D'$, &c., by the second condition, are all equal, the constant value common to them all is termed the *ratio of similitude* of the figures; in the particular case when the ratio of similitude $= 1$, that is, when the several pairs of corresponding distances or perpendiculars are all equal, the figures themselves also are said to be equal.

Since again, for two figures fulfilling the conditions of similarity, the angles between the several pairs of corresponding distances or perpendiculars OA and $O'A'$, OB and $O'B'$, OC and $O'C'$, OD and $O'D'$, &c., by the first condition, are all equal when the figures are both right or left, and all bisected by the same two rectangular directions when they are one right and the other left, the constant value common to them all in the former case is termed the *angle of inclination*, and the fixed directions of bisection common to them all in the latter the *directions of symmetry* of the figures; when in the former case the angle of inclination $=0$ or $=$ two right angles, that is, when the directions of the several pairs of corresponding distances or perpendiculars (in both cases of course parallel) are all similar or opposite, the figures (in both cases said also to be parallel) are said to be *similarly* or *oppositely* placed.

34. From the preceding it is evident, conversely, that—

When two lines OA and $O'A'$, variable in length according to any law, turn in similar or opposite directions round two fixed extremities O and O', revolving simultaneously through equal angles and preserving as they revolve a constant ratio to each other, their two variable extremities A and A' describe, and the two perpendiculars to them at their variable extremities A and A' envelope, similar figures, whose ratio of similitude and angle of inclination or directions of symmetry are those of the lines.

For if A and A', B and B' be any two pairs of corresponding positions of the variable extremities, it follows at once from the conditions of revolution that the two angles AOB and $A'O'B'$ and the two ratios $OA : OB$ and $O'A' : O'B'$ are equal, and the two conditions of similarity of the figures described or enveloped being thus satisfied, the other circumstances respecting their ratio of similitude and angle of inclination or directions of symmetry are in fact stated in the conditions of revolution.

When the two fixed extremities O and O' coincide, and the two variable lines OA and $O'A'$ revolve in the same direction round the common extremity O, the species of the variable triangle AOA' is evidently constant, hence—

If one vertex of a triangle variable in magnitude and position but invariable in figure be fixed, the two variable vertices describe, and the two perpendiculars through them to the conterminous sides envelope similar figures, whose common ratio of similitude and angle of inclination are those of the variable sides containing the fixed vertex.

35. Two similar figures may be of such a form that a correspondence between their points or lines, in pairs satisfying the conditions of similarity, may exist *in more ways than one*, in the case of two *regular polygons* of any common order n, for instance, it may exist in n ways, and in the case of two *circles* in *an infinite number of ways*, and that whether the two figures be regarded as both right or left or one right and the other left. For, if O and O' be the centres of the two figures in either case, *any* pair or vertices or sides of the polygons, and *any* pair of points or tangents of the circles may be regarded as corresponding, and the correspondence between one pair of points or lines of the figures A and A' once established, that of all the remaining pairs B and B', C and C', D and D', &c. is of course fixed by the conditions that the several pairs of corresponding angles AOB and $A'O'B'$, AOC and $A'O'C'$, AOD and $A'O'D'$, &c., measured all either in similar or opposite directions of rotation, are equal. Such cases of similar figures are of course exceptional, but whenever they occur, as they necessarily do frequently in the geometry of the circle, their peculiarity in this respect leads sometimes to consequences not existing in the general case when the correspondence between the points or lines of the figures is unique.

36. In the particular cases when the radii of two circles regarded as similar figures are either evanescent or infinite; that is, when the two circles are either points or lines, *their ratio of similitude, being in all cases that of their radii, is indeterminate.* This peculiarity, which is evident on the general principles explained in (13), may easily be shewn, *a priori*, for both species of figures separately. For if I and I' be any two lines regarded as loci of points, or any two points regarded as envelopes of lines, O and O' in either case *any* two points taken arbitrarily, and A and A', B and B', C and C', &c. any number of pairs of points on the lines or of lines through the points, for which the several pairs of angles IOA and $I'O'A'$, IOB and $I'O'B'$, IOC and $I'O'C'$, &c. measured all in similar or opposite distances of rotation round O and O' are equal; since then in either case the several ratios $OA : O'A'$, $OB : O'B'$,

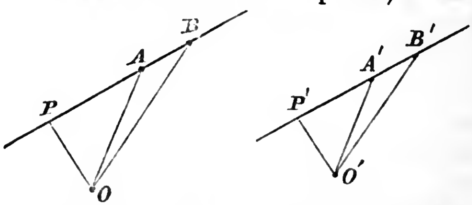
$OC : O'C'$, &c. are equal, the two figures are similar, and since in either case their common value $= OI : O'I'$, their ratio of similitude, the two points O and O' on which it depends being arbitrary, is indeterminate.

The preceding peculiarities of circles in general, and of points and lines in particular, regarded as similar figures, must always be carefully attended to in every application of the general theory of similar figures to their particular cases.

37. *For every pair of corresponding points of two similar figures F and F' regarded as loci, the two lines of connection with O and O' make equal angles and ratios with the two perpendiculars on their tangents from O and O'.*

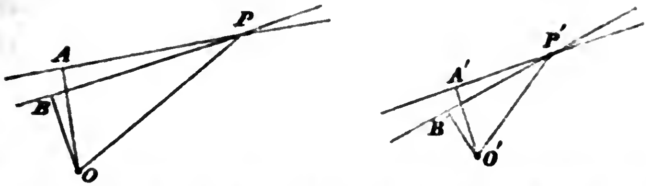
For every pair of corresponding tangents to two similar figures F and F' regarded as envelopes, the two perpendiculars from O and O' make equal angles and ratios with the two lines connecting their points of contact with O and O'.

To prove the first. If A and A' be the two points, B and B' any other pair of corresponding points, OP and $O'P'$ the two perpendiculars from O and O' upon the two indefinite lines



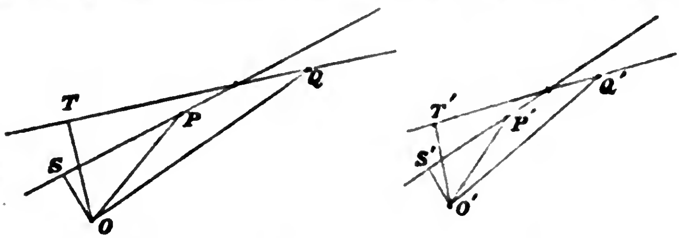
AB and $A'B'$, then since, whatever be the positions of the two pairs of corresponding points A and A', B and B', the two triangles AOB and $A'O'B'$ are by hypothesis similar (32), therefore the two triangles AOP and $A'O'P'$ are also similar, and therefore the two angles AOP and $A'O'P'$ and the two ratios $OA : OP$ and $O'A' : O'P'$ are equal; and this being true *in general*, whatever be the common magnitude of the two equiangular intervals AOB and $A'O'B'$, is *therefore* true *in the particular case* when that interval is evanescent, that is, when (19) the two lines AB and $A'B'$ are the two tangents to the two figures at the two points A and A'.

To prove the second. If A and A' be the two tangents, B and B' any other pair of corresponding tangents, OA and $O'A'$, OB and $O'B'$, the two pairs of perpendiculars upon them from O and O', and P and P' the two points of intersection of



A and B, and of A' and B'; then since, whatever be the positions of the two pairs of corresponding tangents A and A', B and B', the two triangles AOB and A'O'B' are by hypothesis similar (32); therefore the two triangles AOP and A'O'P' are also similar, and therefore the two angles AOP and A'O'P', and the two ratios OA : OP and O'A' : O'P' are equal, and this being true *in general*, whatever be the common magnitude of the two equiangular intervals AOB and A'O'B', is *therefore* true *in the particular case* when that interval is evanescent, that is when (20) the two points P and P' are the two points of contact with the two figures of the two tangents A and A'.

38. *When two figures regarded as loci of points are similar, they are also similar regarded as envelopes of lines, and conversely.*



For if P and P', Q and Q', be any two pairs of corresponding points, S and S', T and T', the two accompanying pairs of corresponding tangents; then since, by the preceding, the two pairs of angles POS and P'O'S', QOT and Q'O'T', and the two pairs of ratios OP : OS and O'P' : O'S', OQ : OT and O'Q' : O'T', are equal, when the figures whether regarded as loci or envelopes are similar; therefore the equality of the two angles POQ and P'O'Q', and of the two ratios OP : OQ and O'P' : O'Q' involves that of the two angles SOT and S'O'T', and of the two ratios OS : OT and O'S' : O'T', and conversely, and therefore &c.

39. When two figures F and F' are similar, every two points or lines X and X', whether belonging to the figures or not, which are such that for *any one* pair of points or lines of the figures A and A', the two angles AOX and $A'O'X'$ and the two ratios $OX : OA$ and $O'X' : O'A'$ are equal, are evidently, from the conditions of similarity (32), such that for *every other* pair B and B', the two angles BOX and $B'O'X'$ and the two ratios $OX : OB$ and $O'X' : O'B'$ are also equal. Every two such points or lines, whether belonging to the figures or not, are said to be *similarly situated*, and are termed *homologous points or lines*, with respect to the figures; all pairs, of corresponding points or lines A and A', of tangents T and T' at pairs of corresponding points P and P', and of points of contact P and P' of pairs of corresponding tangents T and T', of the figures, are evidently homologous.

From the nature of homologous points and lines as thus defined, it is evident for similar figures in general that—

1°. *If X and X' be any pair of homologous points or lines with respect to two similar figures F and F', the two distances or perpendiculars OX and OX' have the constant ratio of the similitude of the figures.*

For if A and A' be any pair of corresponding points or lines of the figures, since then, by hypothesis, $OX : OA = O'X' : O'A'$, therefore, by alternation, $OX : O'X' = OA : O'A'$, and therefore &c.

2°. *If X and X' be any pair of homologous points or lines with respect to two similar figures F and F', the two distances or perpendiculars OX and $O'X'$ have the same angle of inclination or directions of symmetry as the figures.*

For, if A and A' be any pair of corresponding points or lines of the figures, since then, by hypothesis, the two angles AOX and $A'O'X'$ are equal; therefore, according as their directions of rotation are similar or opposite, the two distances or perpendiculars OX and $O'X'$ have the same angle of inclination or directions of symmetry as the two OA and $O'A'$, and therefore &c.

3°. *If P and P', Q and Q' be any two pairs of homologous* *points with respect to two similar figures F and F', the two connectors PQ and $P'Q'$ have the constant ratio of the similitude of the figures.*

For, if A and A' be any pair of corresponding points or lines of the figures, since then, by hypothesis, the two pairs of angles AOP and $A'O'P'$, AOQ and $A'O'Q'$, and the two pairs of ratios $OP : OA$ and $O'P' : O'A'$, $OQ : OA$ and $O'Q' : O'A'$ are equal; therefore the two angles POQ and $P'O'Q'$ and the two ratios $OP : OQ$ and $O'P' : O'Q'$ are equal; and therefore, by similar triangles (Euc. VI. 4),

$$PQ : P'Q' = OP : O'P' = OQ : O'Q' = OA : O'A',$$

and therefore &c.

4°. *If P and P', Q and Q' be any two pairs of homologous points with respect to two similar figures F and F', the two connectors PQ and $P'Q'$ have the same angle of inclination or directions of symmetry as the figures.*

For, if A and A' be any pair of corresponding points or lines of the figures, since then, by hypothesis, the two pairs of angles AOP and $A'O'P'$, AOQ and $A'O'Q'$, and the two pairs of ratios $OP : OA$ and $O'P' : O'A'$, $OQ : OA$ and $O'Q' : O'A'$ are equal; therefore the two angles of inclination of PQ to OA and of $P'Q'$ to $O'A'$ are equal; and, therefore, according as their directions of rotation are similar or opposite, the two connectors PQ and $P'Q'$ have the same angle of inclination or directions of symmetry as the two distances or perpendiculars OA and $O'A'$, and therefore &c.

5°. *If P and P' be any pair of homologous points and L and L' any pair of homologous lines with respect to two similar figures F and F', the two perpendiculars PL and $P'L'$ have the ratio of similitude and the angle of inclination or directions of symmetry of the figures.*

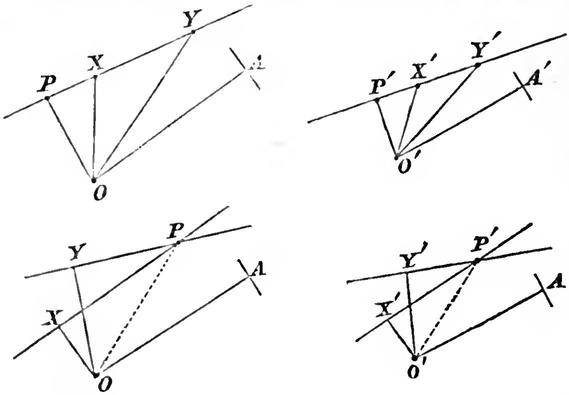
For, if A and A' be any pair of corresponding points or lines of the figures, since then, by hypothesis, the two pairs of angles AOP and $A'O'P'$, AOL and $A'O'L'$, and the two pairs of ratios $OP : OA$ and $O'P' : O'A'$, $OL : OA$ and $O'L' : O'A'$ are equal; therefore the two angles POL and

$P'O'L'$ and the two ratios $OP : OL$ and $O'P' : O'L'$ are equal; and therefore by pairs of similar right-angled triangles

$$PL : P'L' = OP : O'P' = OL : O'L' = OA : O'A',$$

and therefore &c; the second part being evident from the parallelism of PL and OL and of $P'L'$ and $O'L'$.

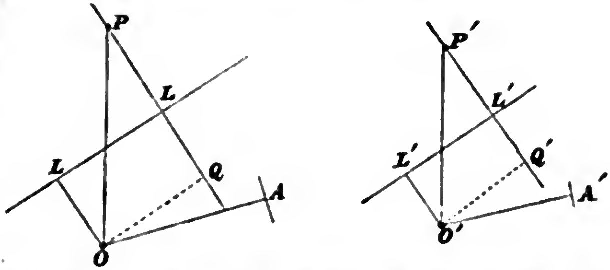
6°. *If X and X', Y and Y' be any two pairs of homologous points or lines with respect to two similar figures F and F', the two lines of connection or points of intersection XY and $X'Y'$ are homologous lines or points with respect to the figures.*



For, drawing the two perpendiculars or connectors OP and $O'P'$ from O and O' to XY and $X'Y'$. Since then for every pair of corresponding points or lines A and A' of the two figures F and F', the two pairs of angles AOX and $A'O'X'$, AOY and $A'O'Y'$, and the two pairs of ratios $OX : OA$ and $O'X' : O'A'$, $OY : OA$ and $O'Y' : O'A'$ are by hypothesis equal; therefore the two angles AOP and $A'O'P'$ and the two ratios $OP : OA$ and $O'P' : O'A'$ are equal, and therefore &c.

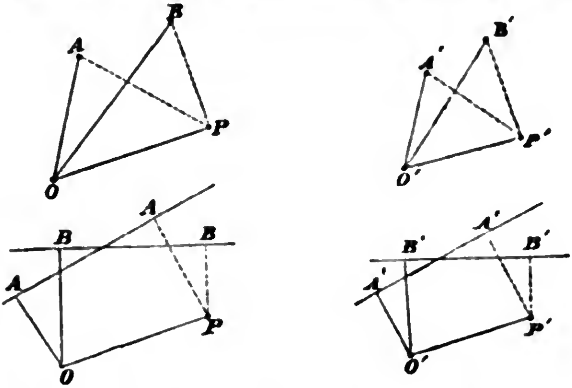
7°. *If P and P' be any pair of homologous points and L and L' any pair of homologous lines with respect to two similar figures F and F', the two perpendiculars PL and $P'L'$ are homologous lines and their two intersections with L and L' are homologous points with respect to the figures.*

For, drawing from O and O' the two perpendiculars OQ and $O'Q'$ to PL and $P'L'$. Since then for every pair of



corresponding points or lines A and A' of the two figures F and F', the two pairs of angles AOP and $A'O'P'$, AOL and $A'O'L'$, and the two pairs of ratios $OP : OA$ and $O'P' : O'A'$, $OL : OA$ and $O'L' : O'A'$, are by hypothesis equal, therefore the two angles AOQ and $A'O'Q'$ and the two ratios $OQ : OA$ and $O'Q' : O'A'$ are equal, and therefore &c.; the second part following from the first by the second part of 6°.

8°. *Any two homologous points P and P' with respect to two similar figures F and F' may be substituted for the two O and O' without violating the conditions of similitude of the figures.*



For, if A and A', B and B' be any two pairs of corresponding points or lines of the figures; then since by hypothesis the two pairs of angles AOP and $A'O'P'$, BOP and $B'O'P'$, and the two pairs of ratios $OA : OP$ and $O'A' : O'P'$, $OB : OP$ and $O'B' : O'P'$ are equal; therefore, by pairs of similar triangles, the two angles APB and $A'P'B'$ and the two ratios $PA : PB$ and $P'A' : P'B'$ are equal, and therefore &c., (32).

9°. *For every two similar figures F and F′ if any number of points connected with either F lie on a line L, the homologous points with respect to the other F′ lie on the homologous line L′, and, if any number of lines connected with either F pass through a point P, the homologous lines with respect to the other F′ pass through the homologous point P′.*

For, since by 5°, for every pair of homologous points P and P', and for every pair of homologous lines L and L', of the figures, $PL : P'L' =$ the constant ratio of similitude of F and F', therefore if either of them $= 0$ so also is the other, that is, if the point P lie on the line L the point P' lies on the line L', and if the line L pass through the point P the line L' passes through the point P', and therefore &c.

10°. *For every two similar figures F and F′, if any number of points or lines connected with either F lie on or touch a circle C, the homologous points or lines with respect to the other F′ lie on or touch a circle C′, the centres of the two circles being homologous points and their radii having the ratio of similitude of the figures.*

For, since by 3° and 5° or by 8°, for every pair of homologous points P and P', and for any number of pairs of homologous points or lines X and X', Y and Y', Z and Z', &c. of the figures $PX : P'X' = PY : P'Y' = PZ : P'Z'$, &c. = the constant ratio of similitude of F to F', therefore if $PX = PY = PZ$, &c., that is if X, Y, Z, &c. lie on or touch a circle of which P is the centre and their common distance from it the radius, then $P'X' = P'Y' = P'Z'$, &c., that is X', Y', Z', &c. lie on or touch a circle of which P' is the centre and their common distance from it the radius, and therefore &c.

11°. *If a pair of homologous points or lines X and X′ with respect to two similar figures F and F′ vary simultaneously according to any law, the two figures G and G′ they describe or envelope are similar and have the same ratio of similitude and the same angle of inclination or directions of symmetry as the original figures.*

For, if A and A' be any pair of corresponding points or lines of F and F', then since in every position of the two variable homologues X and X', the two angles AOX and

$A'O'X'$ and the two ratios $OX : O'X'$ and $OA : O'A'$ are equal, therefore &c., (32). This general property, here established on general principles, includes of course the particular cases 9° and 10° established above by particular considerations.

Every two figures G and G' described or enveloped as above are said to be *homologous figures* with respect to the originals F and F', which again reciprocally are evidently homologous figures with respect to G and G', and every pair of points P and P', of lines L and L', of circles C and C', and generally of figures of any kind E and E', which are homologous with respect to either pair F and F' are evidently also homologous with respect to the other pair G and G', and conversely.

40. *If a figure of any invariable form revolve round any point invariably connected with it as a fixed centre, varying in magnitude as it revolves according to any law, all points invariably connected with it describe, and all lines invariably connected with it envelope, similar figures, all right or left, whose ratios of similitude and angles of inclination two and two are those of the distances of the describing points or enveloping lines from the fixed centre.*

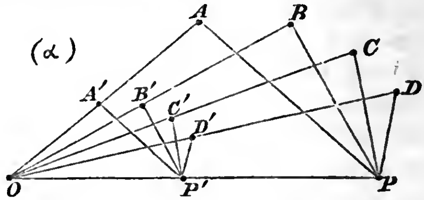
For, if O be the fixed point, and X, Y, Z, &c. any number of variable points or lines all invariably connected with the variable figure; then since the form of the figure, whatever be the law of its variation in magnitude while revolving round O, is by hypothesis invariable, therefore, by the preceding (39), the several angles XOY, YOZ, &c., and the several ratios $OX : OY$, $OY : OZ$, &c. are all constant, and therefore &c., (32).

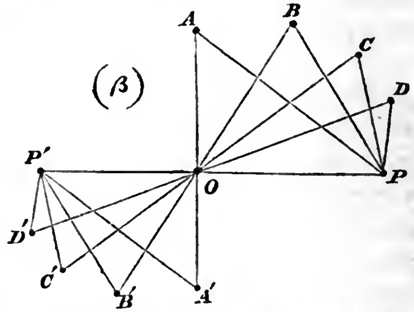
For points and lines of the revolving figure not evanescently or infinitely distant from O, it is easy to verify by particular considerations as in 9° and 10° of the preceding article, that in particular, *if any one point P describe a line or circle all points P, Q, R, &c. describe lines or circles, and if any one line L envelope a point or circle all lines L, M, N, &c. envelope points or circles;* this verification, there gone through in detail, need not of course be repeated here.

41. *When two similar figures of any kind, both right or left, are similarly or oppositely placed* (33), *all lines AA', BB',*

CC', DD', &c. connecting pairs of corresponding points pass through a common point O, and are there cut, externally or internally, in the ratio of the similitude of the figures.

For if O be the point in which any one of them AA' intersects the line PP', connecting any pair of homologous points P and P' with respect to the figures; since then, by hypothesis (33), the two directions PA and $P'A'$, whether similar (fig. α) or opposite (fig. β), are parallel; therefore, by similar triangles, the two ratios $OP : OP'$ and $OA : OA'$ are each = the ratio $PA : P'A'$ = the ratio of similitude of the figures; therefore all connectors AA', BB', CC', DD', &c. cut and are cut by the same line PP' at the same point O, and in the same ratio $OP : OP'$, and therefore &c.





Conversely, *if the several lines connecting any arbitrary point O with all the points A, B, C, D, &c. of a figure of any kind, be increased or diminished in similar or opposite directions in any common ratio, the several extremities A', B', C', D', &c. of the increased or diminished distances determine a second figure similar to the original, and similarly or oppositely placed with it according as the directions of the original and altered distances are similar or opposite.*

For, every pair of corresponding angles AOB and $A'OB'$ and every pair of corresponding ratios $OA : OB$ and $OA' : OB'$ being equal, the figures are similar; and every pair of corresponding directions OA and OA', OB and OB', OC and OC', &c. being similar or opposite, the figures are similarly or oppositely placed (33), and therefore &c.

Since, by pairs of similar triangles AOB and $A'OB'$, the two lines AB and $A'B'$ connecting any two points A and B

of either figure, and the two corresponding points A' and B' of the other are always parallel, whatever be the angle between the two lines AA' and BB' passing through O, they are therefore so in the particular case where that angle $= 0$, that is, when AA' and BB' coincide and when therefore (19) AB and $A'B'$ are the two tangents to the figures at A and A'. Hence, *when two similar right or left figures are similarly or oppositely placed, all pairs of tangents at pairs of corresponding points, like all other pairs of homologous lines of the figures, are parallel.*

42. The point O related as above to two similar right or left figures, when similarly or oppositely placed, is termed their *centre of similitude*, and is said to be *external* or *internal*, with respect to them, according as the section by it of all lines connecting pairs of homologous points in the common ratio of their similitude is external or internal, that is, according as they are similarly or oppositely placed; when the two figures in either case are given in absolute position, their centre of similitude O is evidently given by the intersection of any two lines PP' and QQ' connecting pairs of homologous points on or in any way situated with respect to them.

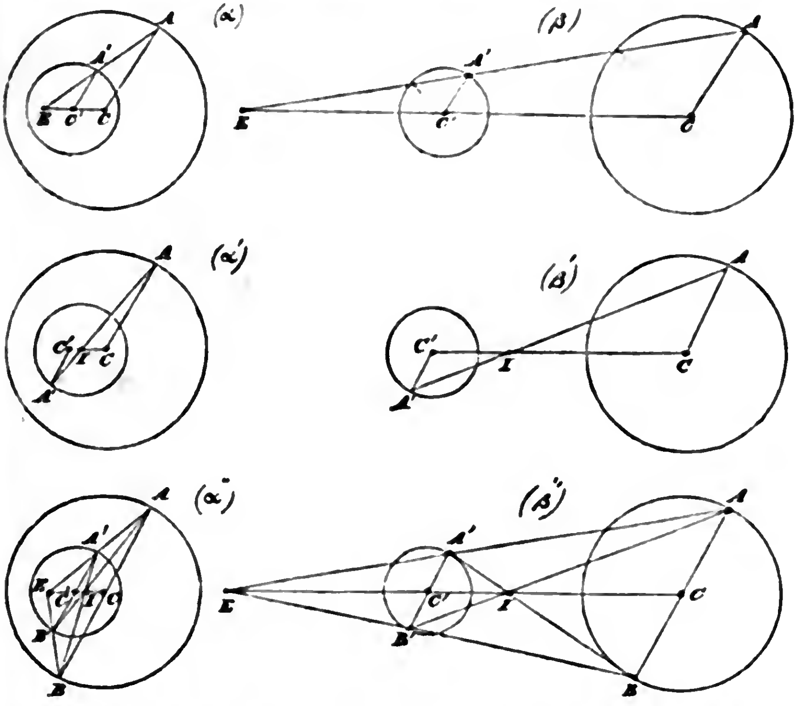
As all lines connecting pairs of homologous points P and P', Q and Q', R and R', S and S', &c., situated in any manner with respect to the figures, pass through O, and are there cut in the ratio of their similitude, externally or internally, according as their positions are similar or opposite; so, conversely, all pairs of points P and P', Q and Q', R and R', S and S', &c., which connect by lines passing through O, and there cut in their ratio of similitude, externally or internally according as their positions are similar or opposite, are evidently homologous pairs with respect to the figures; and the two similar and similarly or oppositely placed figures $PQRS$ &c. and $P'Q'R'S'$ &c., determined by any number of such pairs, are evidently similarly situated with respect to, and have the same centre and ratio of similitude with, the original figures $ABCD$ &c. and $A'B'C'D'$ &c.

Every line passing through O being evidently its own homologue with respect to both figures and intersecting them, if it meet them at all, at pairs of corresponding points A and A', B and B', C and C', &c., of which the number depends, of course, on the nature of the figures; at which the several pairs of corresponding tangents, by (41), are parallel; and for which the several pairs of ratios $OA : OA'$, $OB : OB'$, $OC : OC'$, &c., by the same, are equal to the ratio of similitude of the figures, so that if $OA = OB$, or any two of the points of meeting for either figure coincide, then also $OA' = OB'$, or the two corresponding points of meeting for the other also coincide. Hence, *when two similar right or left figures are similarly or oppositely placed, every line passing through their centre of similitude, like every pair of homologous lines in general, divides and is divided by them similarly into pairs of corresponding segments in the linear ratio of their similitude, intersects them at equal angles at every pair of corresponding points of meeting, and if it touch either at any point of meeting touches the other also at the corresponding point of meeting.*

43. Two similar figures of such a form, that a correspondence between their points and lines in pairs satisfying the conditions of similarity, exists in more ways than one (35), may be, moreover, of such a form that when similarly placed for one mode of correspondence, they are at the same time oppositely placed for another, or conversely; as for instance, *two similar parallelograms*, or, more generally, *two similar polygons of any even degree whose several pairs of opposite sides are equal and parallel*; every two such figures when thus at once similarly and oppositely placed have of course *two different centres of similitude*, one external corresponding to their similar, and the other internal corresponding to their opposite, parallelism, each determined, as in the general case, by the intersection of any two lines connecting pairs of homologous points for the relative positions corresponding to itself, and each possessing all the properties of the unique centre of similitude of the same kind with itself in the general case.

44. Of figures coming under the above head *two circles*, however circumstanced as to magnitude or position, absolute

or relative, provided only they be in the same plane, possess, for the reason explained in (35), the property, confined to them exclusively, of being *always* at once similarly and oppositely placed, and of having therefore in *every* position two



different centres of similitude, one E external as similarly placed, figs. (α) and (β), and the other I internal as oppositely placed, figs. (α′) and (β′); both situated on the line CC' connecting their centres C and C' and dividing that line, the former externally and the latter internally, in the ratio of their radii; both determined by the intersections with that line of the lines AA' connecting the extremities of any two parallel radii CA and $C'A'$ drawn in similar directions, figs. (α) and (β), for the former, and in opposite directions, figs. (α′) and (β′), for the latter—or, which comes to the same thing, by the intersections with each other of the pairs of lines AA' and BB', AB' and BA', connecting the extremities, adjacent for the former and non-adjacent for the latter, of any two parallel diameters AB and $A'B'$, figs. (α'') and (β''); and each possessing, as for every two figures coming under the same head, all the properties of the unique centre of similitude of its kind for any two similar figures similarly or oppositely placed, (41) and (42).

As every line touching two circles in the same plane connects the extremities of the two parallel radii to which it is perpendicular (Euc. III. 18); and consequently, by the above, passes through either the external or the internal centre of similitude of the circles, according as the directions of the radii are similar or opposite; hence two circles in the same plane, however circumstanced as to magnitude and position, admit, in general, of two, and of but two, pairs of common tangents, real or imaginary, both symmetrically situated with respect to, and intersecting upon, their line of centres; one, termed in consequence the external pair, intersecting at their external centre of similitude, and the other, termed in consequence the internal pair, intersecting at their internal centre of similitude; and, evidently, both real, both imaginary, or, one real and one imaginary, according as the distance between their centres is greater than the sum, less than the difference, or, intermediate between the sum and difference, of their radii.

The two centres of similitude, external and internal, of two given circles, determined as above, or by any other method, give, consequently, in two conjugate pairs (Euc. III. 17), the four solutions, real or imaginary, of the problem "To draw a common tangent to the two circles."

X

CHAPTER III.

THEORY OF MAXIMA AND MINIMA.

45. When a geometrical magnitude of any kind, which varies continuously according to any law, passes in the course of its variation through a value greater than either its preceding or succeeding values, it is said to be *a maximum*, even though at some other stage of its variation it may pass through a value absolutely greater; and, on the other hand, when it passes in the course of its variation through a value less than either its preceding or succeeding values it is said to be *a minimum*, even though at some other stage of its variation it may pass through a value absolutely less; the terms "maximum" and "minimum," as employed in geometry, are therefore relative, not absolute.

46. As, to a traveller on a road which is not a dead level, the top of *every* hill is *a* position of maximum, and the bottom of *every* hollow *a* position of minimum, elevation above the sea or any other standard level; so, for geometrical figures of the higher orders, the different variable magnitudes connected with them, may pass in the course of their variation through several maxima and several minima values, of course necessarily alternating with each other in the order of their occurrence; as, for instance, the linear distance from any fixed point, or the perpendicular distance from any fixed line, of a variable point, traversing the entire figure or any part of it; for the point, line, and circle, however the variable magnitudes most commonly considered in connection with them and their combinations, rarely pass during their variations through more than a single maximum and a single minimum value; as, for instance, the distance of a variable point on a circle from any fixed point or line situated in any manner with respect to it, which, in either case (Euc. III. 7, 8, 19), is a maximum for and only for the distance which passes through the centre, and a minimum for and only for the distance which if produced would pass through it; in all such cases the single maxima and minima values are not only relatively but also absolutely the greatest and least values through which the variable magnitude passes in the course of its variation.

47. As every increase or diminution of a magnitude of any kind is necessarily accompanied by the simultaneous diminution or increase of its reciprocal (8); it follows, of course, that when a variable magnitude passes under any circumstances through a maximum or minimum value, its reciprocal to any unit, passes simultaneously through a minimum or maximum value.

48. The following are a few simple but fundamental examples of maxima and minima, to which many others are reducible:—

Ex. 1°. *When two sides of a triangle are given in magnitude the area is a maximum (in this case the maximum) when they contain a right angle.*

For (Euc. I. 41), whatever be their angle of intersection, acute, right, or obtuse, the area = half the product of either into the perpendicular on its direction from the remote extremity of the other, which perpendicular is evidently equal to the other for the right and less than the other for any position at either side of the right angle; and, in the same way generally, when one side of a triangle is constant the area varies as, and therefore passes through, its maxima and minima values with the perpendicular upon its direction from the opposite vertex.

Ex. 2°. *For the point of internal bisection of any segment of a line the product of the distances from the extremities is a maximum, and the sum of their squares a minimum.*

For (Euc. II. 5 and 9, 10), the product for that point exceeds the product for any other point of internal section on either side by the square, and the sum of the squares for that point falls short of the sum of the squares for any other point of section, external or internal, on either side, by twice the square, of the distance between that and the other point of section; and, in the same way generally, for any two magnitudes expressed in numbers, as product = square of half sum − square of half difference, and as sum of squares = twice square of half sum + twice square of half difference; if the sum be constant, the product is a maximum and the sum of the squares a minimum; and if the product or the sum of the squares be constant, the sum is a minimum in the former case and a maximum in the latter, when the magnitudes are equal.

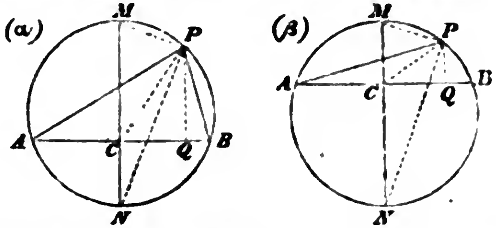
Ex. 3°. *For any two magnitudes expressed in numbers whose sum is constant, the sum, product, sum of squares, and product of squares, of the reciprocals are all minima when the magnitudes are equal.*

For, the product of the reciprocals being = the reciprocal of the product, and the product of the squares of the reciprocals being = the reciprocal of the square of the product are both minima when the product is a maximum, that is, Ex. 2°, when the magnitudes are equal; and again, the sum of the reciprocals being = the sum divided by the product, and the sum of the squares of the reciprocals being = the sum of the squares divided by the product of the squares, are both minima also, when the product is a maximum, the sum being constant by hypothesis, and the sum of the squares being then a minimum, Ex. 2°.

Ex. 4°. *For the point of internal bisection of any side of a triangle the area of the inscribed parallelogram formed by drawing parallels to the other two sides is a maximum.*

For, whatever be the position of the point of section, the angle of the parallelogram being constant, its area (Euc. VI. 23) varies as the product of the parallels; that is, as the product of the segments of the divided side determined by the point of section, the former being to the latter product in the constant ratio of the rectangle under the other two sides to the square of that side (Euc. VI. 23); but the latter product being a maximum, by Ex. 2°, for the point of bisection of the side, so therefore is the former, and therefore the area of the parallelogram; and, in the same manner exactly, it appears that, for the point of internal bisection of any side of the triangle the product of the perpendiculars on the other two sides, or more generally of the two lines drawn in any two given directions to meet them, is a maximum.

Ex. 5°. *For the point of internal bisection of any arc of a circle, the sum of the squares of the linear distances from the extremities is a maximum or a minimum, and for the point of external bisection a minimum or a maximum, according as the arc is greater or less than a semicircle.*



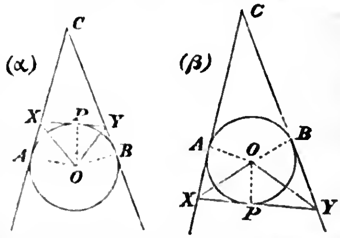
For, if AB be the arc, C the middle point of its chord, M and N its two points of bisection, internal and external, P any other point on the circle, and PQ the perpendicular from P on AB; then since (Euc. II. 12, 13), whatever be the position of P, $PA^2 = PC^2 + CA^2 \pm 2CA.CQ$, and $PB^2 = PC^2 + CB^2 \mp 2CB.CQ$, therefore $PA^2 + PB^2 = CA^2 + CB^2 + 2.CP^2$, which is a maximum or a minimum when CP is a maximum or a minimum; that is (Euc. III. 7), when P is at M or N in the former case, and at N or M in the latter; and, in the same manner, it appears generally that the sum of the squares of the linear distances of a variable point P, on any geometrical figure from any two fixed points A and B, situated in any manner with respect to the figure, increases and diminishes and passes through its maxima and minima values, with the distance PC of the variable point P from the middle point C of the line AB connecting the two fixed points A and B.

Ex. 6°. *For each point of bisection, internal and external, of any arc of a circle, the sum and product of the linear distances from the extremities, and the area of the triangle they determine with the chord, are all maxima.*

For, since whatever be the position of P, (same figures as in last), $PA.PB = MN.PQ$ (Euc. VI. 16), and area $APB = \frac{1}{2}AB.PQ$ (Euc. I. 41); the property is evident as regards the product and area, and it remains only to prove it for the sum $PA + PB$, which is easily done as follows: since for every position of P at the same side of the chord with M (as in the figures), by Ptolemy's Theorem (Euc. VI. 16, Cor.), $PA.NB + PB.NA = PN.AB$, and since, by hypothesis, $NA = NB$, therefore $PA + PB : PN :: AB : AN$ or BN, that is, in a constant ratio, and therefore $PA + PB$ is a maximum when PN is a maximum, that is, when P is at M; and in the same way it may be shewn (by simply substituting M for N in the above) that for positions of P at the same side of AB with N, $PA + PB$ varies as PM, and is therefore a maximum when P is at N.

Ex. 7°. *For each point of bisection, internal and external, of any arc of a circle, the segment of the tangent intercepted between the tangents at the extremities, and the area of the triangle it subtends at the centre of the circle, are both minima.*

For, if AB be the arc, AC and BC the tangents at its extremities, XY the segment intercepted between them of the tangent at any other point P, and O the centre of the circle; then, since whatever be the position of P, the lines OX and OY bisect the angles AOP and BOP (Euc. III. 17), the angle between them, XOY is equal to half the angle AOB subtended at O by the arc APB, therefore



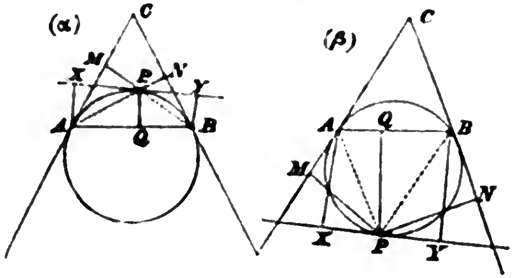
in the triangle whose vertex is O and base XY, the altitude OP and vertical angle XOY are both constant; and it is evident from the preceding, or independently, that when the vertical angle of a triangle is constant, the altitude and area are both maxima for a given base, and the base and area both minima for a given altitude, when the triangle is isosceles, that is, for the triangle XOY when P is a point of bisection, internal or external, of the arc AB.

Ex. 8°. *For the point of internal bisection of any arc of a circle, the area of the triangle formed by the tangent with the tangents at the extremities is a maximum or a minimum, and for the point of external bisection a minimum or a maximum, according as the arc is less or greater than a semicircle.*

For, since in either case (same figures as in last), the pentagonal area $XAOBY$, being double the triangular area XOY, is a minimum, by the preceding, for each point of bisection of AB; and the quadrilateral area $AOBC$ being of course constant, whatever be the position of XY, therefore the triangular area XCY, being = the quadrilateral − the pentagon in one case (fig. α), and = the quadrilateral + the pentagon in the other case (fig. β), is a maximum in the former case and a minimum in the latter.

Ex. 9°. *For each point of bisection, internal and external, of any arc of a circle, the product of the perpendiculars upon the tangents at the extremities, and the product of the perpendiculars from the extremities upon the tangent, are both maxima.*

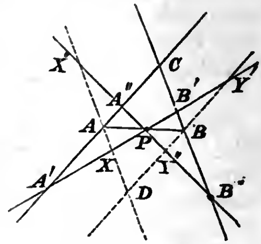
For, if AB be the arc, P any point upon it, external or internal, PM and PN the perpendiculars from P upon the tangents at A and B, AX and BY the perpendiculars from A and B upon the tangent at P, and PQ the perpendicular from P upon the chord AB; then, joining P with A and B, by pairs of equal triangles APM and PAX, BPN and PBY, we have $PM = AX$ and $PN = BY$, and therefore $PM.PN = AX.BY$, and by pairs of similar triangles APM (or PAX) and BPQ, BPN (or PBY) and APQ (Euc. III. 32), we have PM or $AX : PQ :: PQ : PN$ or BY, both being $= PA : PB$, therefore $PM.PN$ and $AX.BY$ both $= PQ^2$, and therefore &c.



Ex. 10°. *Of all lines passing through a fixed point that which determines with two fixed lines the triangle of minimum area is that whose segment intercepted between the lines is bisected at the point.*

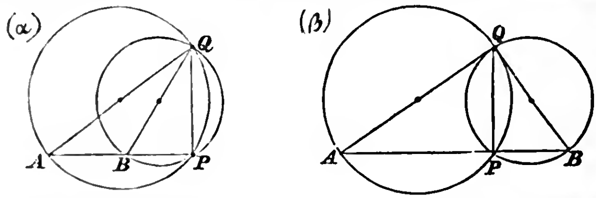
For, if P be the point, AC and BC the lines, AB the intercept bisected at P, and $A'B'$ or $A''B''$ any other intercept; then through A and B drawing AD and BD parallels to BC and AC, meeting $A'B'$

or $A''B''$ at X' and Y' or X'' and Y''. As the two triangles APX' and BPB', or the two triangles BPY'' and APA'', are evidently equal (Euc. I. 4); therefore the triangle ACB is less than the triangle $A'CB'$ or $A''CB''$, and therefore &c.



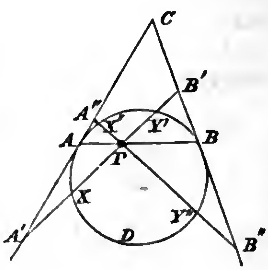
The point and lines being given, to draw AB so as to be bisected at P, is, of course, but a particular case of the more general problem to draw it so as to be cut in any given ratio, of which the preceding construction suggests the following obvious solution: drawing from P any line PA' or PA'' to either line CA, and producing it through P to Y' or Y'' so that $PA' : PY'$ or $PA'' : PY''$ = the given ratio, the parallel $Y'B$ or $Y''B$ to CA through Y' or Y'' evidently intersects the other line CB in the extremity B of the required line AB.

Ex. 11°. *Of all lines passing through either point of intersection of two circles, that whose segment intercepted between the circles is of maximum length, and subtends at the other point of intersection, the triangle of maximum area is that which is perpendicular to the chord of intersection.*



For, if PQA and PQB be the circles, P and Q their points of intersection, and AB any line passing through either of them P and meeting the circles at A and B; then since, joining A and B with the other intersection Q, the angles PAQ and PBQ are both constant (Euc. III. 21), the triangle AQB is constant in species, whatever be the position of AB, and therefore its base AB, area AQB, and sides QA and QB are all maxima together; but the sides QA and QB are maxima when they are diameters of their respective circles, that is (Euc. III. 31) when AB is perpendicular to PQ.

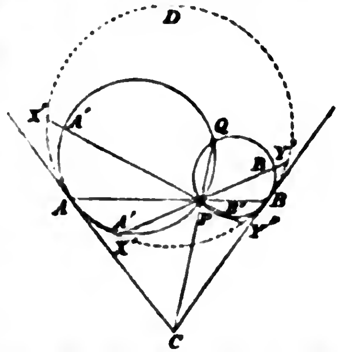
Ex. 12°. *Of all lines passing through a fixed point that whose segments intercepted in opposite directions between the point and two fixed lines contain the rectangle of minimum area is that which makes equal angles with the lines.*



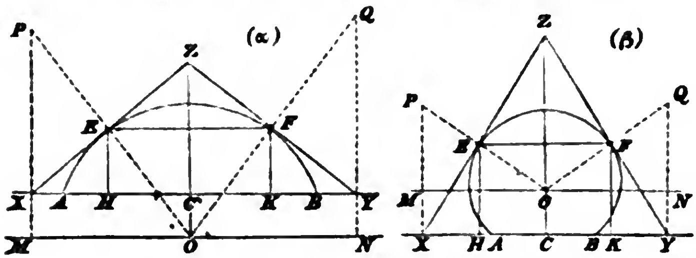
For, if P be the point, AC and BC the lines, AB the line through P making equal angles with AC and BC, and $A'B'$ or $A''B''$ any other line through P; then, as evidently the circle ADB touching AC and BC at A and B intersects $A'B'$ or $A''B''$ at points X' and Y' or X'' and Y'' internal to A' and B' or A'' and B'', the rectangle $PA \cdot PB$ which is equal to the rectangle $PX' \cdot PY'$ or $PX'' \cdot PY''$ (Euc. III. 35), is therefore less than the rectangle $PA' \cdot PB'$ or $PA'' \cdot PB''$, and therefore &c.

Ex. 13°. *Of all lines passing through either point of intersection of two circles that whose segments intercepted in opposite directions between the point and circles contain the rectangle of maximum area is that which makes equal angles with the circles* (22).

For, if PAQ and PBQ be the circles, P and Q their two points of intersection, AB the line passing through either of them P making equal angles with the circles, that is (22) with the tangents to them AC and BC at its extremities A and B, and $A'B'$ or $A''B''$ any other line through P; then, as evidently the circle ADB touching AC and BC at A and B intersects $A'B'$ or $A''B''$ at points X' and Y' or X'' and Y'' external to A' and B' or A'' and B''; the rectangle $PA \cdot PB$ which is equal to the rectangle $PX' \cdot PY'$ or $PX'' \cdot PY''$ (Euc. III. 35) is therefore greater than the rectangle $PA' \cdot PB'$ or $PA'' \cdot PB''$, and therefore &c.



Ex. 14°. *The rectangle of maximum area inscribed in any segment of a circle, or of any other convex figure, is that whose side parallel to the base of the segment bisects the sides of the triangle formed with the base by the lines touching at its extremities the circle or figure.*



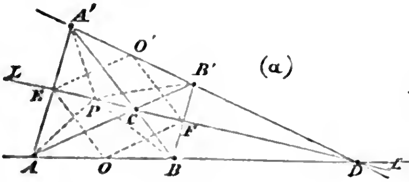
For, if $AEFB$ be the segment, EF the chord parallel to its base AB, which bisects the sides XZ and YZ of the triangle XZY formed with AB by the tangents at E and F; then, by Ex. 4°, the rectangle (or parallelogram) $EFKH$ is the maximum that could be inscribed in the triangle XZY, and therefore, *a fortiori*, in the segment $AEFB$ to which the triangle is external.

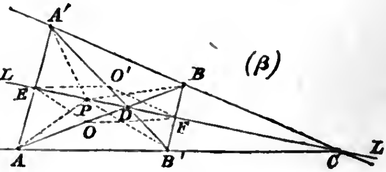
To draw EF so as to bisect the tangents ZX and ZY is, of course, a particular case of the more general problem, to draw it so as to cut them in any given ratio, which for the circle may be done as follows: through the centre O drawing OC and MN perpendicular and parallel to AB (the former of course passing through Z), and through X and Y, supposed found, drawing XM and YN parallel to OC to meet the radii OE and OF, supposed found, at P and Q respectively; then by pairs of similar triangles PEX and OEZ, QFY and OFZ, the two ratios $PE : EO$ and $QF : FO$ each = the given ratio of the tangents, and therefore as EO and FO are given and equal, PE and QF, PO and QO, and the rectangles $PE \cdot PO$ and $QF \cdot QO$, are given and equal; but by other pairs of similar triangles PEX and PMO, QFY and QNO, $PM \cdot PX = PE \cdot PO$, and $QN \cdot QY = QF \cdot QO$, therefore the rectangles $PM \cdot PX$ and $QN \cdot QY$ are given and equal; but MX and NY, being each = CO, are also given and equal; therefore (Euc. II. 6) PM and QN, PX and QY, and the angles POM and QON are given and equal, and therefore E and F are known.

49. The next example we give separately as the basis of some useful properties of the triangle.

a. The lines connecting a variable point on a fixed line with two fixed points at the same side of the line have the maximum difference when they coincide in direction, and the minimum sum when the angle between them is bisected (of course externally) by the line.

b. The lines connecting a variable point on a fixed line with two fixed points at opposite sides of the line have the minimum sum when they coincide in direction, and the maximum difference when the angle between them is bisected (of course internally) by the line.





Let LL, figs. α and β, be the fixed line, A and B the two fixed points, AE and BF the two perpendiculars from them on LL, A' and B' the two points on the perpendiculars for which $AE = EA'$ and $BF = FB'$, then the distances of any point P on LL from A and A', or from B and B', being equal

(Euc. I. 4), if D be the point on it at which AB or $A'B'$ intersects it, that is the point on it for which PA and PB coincide in direction, and if C be the point on it at which AB' or $A'B$ intersects it, that is the point on it for which the angle APB is bisected (externally fig. α, or internally fig. β) by it; it is to be shewn that, in fig. α, $DA \sim DB > PA \sim PB$, and $CA + CB < PA + PB$, and that, in fig. β, $DA + DB < PA + PB$, and $CA \sim CB > PA \sim PB$, which are evident, the first for each figure from the triangle APB or $A'PB'$, and the second for each figure from the triangle APB' or $A'PB$, any side of a triangle (Euc. I. 20) being greater than the difference and less than the sum of the other two.

The maximum difference in a (fig. α), or minimum sum in b (fig. β), is of course the distance AB between the two points A and B; the minimum sum in a (fig. α), or maximum difference in b (fig. β), may be expressed in terms of the distances of the points from the line and from each other as follows:

In both cases the four points $ABA'B'$ lie evidently in a circle, and the two pairs of opposite connectors AB and $A'B'$, AB' and $A'B$ are evidently equal; therefore, by Ptolemy's Theorem (Euc. VI. 16, Cor.), $AA'.BB' = AB'.A'B - AB.A'B'$ in fig. α, and $= AB.A'B' - AB'.A'B$ in fig. β; but $AA' = 2.AE$, $BB' = 2.BF$, and $AB' = A'B = AC + BC$ in fig. α, and $= AC \sim BC$ in fig. β; therefore

$$(AC + BC)^2 = AB^2 + 4.AE.BF \text{.......in fig. } \alpha,$$

and

$$(AC \sim BC)^2 = AB^2 - 4.AE.BF \text{.......in fig. } \beta,$$

which are the formulæ by which to calculate in numbers the minimum sum or maximum difference when the distances of the points from the line and from each other are given.

The line LL being in fig. α the external and in fig. β the internal bisector of the vertical angle C of the triangle ACB, we see from the above formulæ that—

If from the extremities of the base of a triangle perpendiculars be let fall upon the external or internal bisector of the vertical angle, their rectangle = square of half sum of sides − square of half base in the former case, and = square of half base − square of half difference of sides in the latter case.

If the interval AB between the two points A and B be bisected or conceived to be bisected at O, and the point of bisection O connected or conceived to be connected with the feet of the two perpendiculars E and F; then, evidently, $OE = \frac{1}{2}BA'$ and $OF = \frac{1}{2}AB'$, therefore $OE = OF = \frac{1}{2}(AC + BC)$ in fig. α, and $= \frac{1}{2}(AC \sim BC)$ in fig. β. Hence—

If from the extremities of the base of a triangle perpendiculars be let fall upon the external or internal bisector of the vertical angle, their feet are equidistant from the middle point of the base by an interval = half the sum of the sides in the former case, and = half the difference of the sides in the latter case.

From these last two properties combined we see that, when the base of a triangle is fixed and the sum or difference of the sides constant, if perpendiculars be let fall from the extremities of the base upon the external or internal bisector of the vertical angle—

a. Their feet are equidistant from the middle point of the base by a constant interval = half the sum or difference of the sides.

b. Their rectangle is constant and = square of half sum or difference of sides ~ square of half base.

The interval EF between the feet of the perpendiculars being a chord of the circle round O as centre, whose radius $= \frac{1}{2}(AC + BC)$ in fig. α, and $= \frac{1}{2}(AC \sim BC)$ in fig. β, and the square of the semi-interval AB between the two points A and B being = the square of the radius of the circle $\mp$ the rectangle $AE.BF$, we see that—

The two perpendiculars erected at the extremities of any chord of a circle meet any diameter of the circle at two points equidistant from the centre and contain a rectangle = the square of the radius of the circle ~ the square of the semi-interval they intercept on the diameter.

A useful property of the circle which the reader may very easily prove, *à priori*, for himself.

50. If from any point A a perpendicular be let fall upon any line L, and produced, as in the preceding, through the line to a second point A' equidistant from L with A, the new point A' is termed *the reflexion* of the original point A with respect to the line L; and, generally, if from all the points A, B, C, D, &c. of any geometrical figure perpendiculars be let

fall upon any line L and produced through L, in the same manner, to their reflexions at the opposite side, the new figure A', B', C', D', &c. is termed *the reflexion* of the original with respect to the line.—A convenient term introduced into Geometry from the science of Optics.

The relation between any figure and its reflexion with respect to any line is evidently *reciprocal* (8); that is, if one figure F' be the reflexion of another F with respect to a line L, the latter F is reciprocally the reflexion of the former F' with respect to the same line L; it is evident also, that every two figures F and F' reflexions of each other with respect to any line L are *right and left figures* (32), *similar* in *form*, *equal* in *magnitude*, and *symmetrically situated*, like two hands or two feet, with respect to the line and to each other.

Thus, the reflexion of a line is a line, of a circle a circle, of the line passing through two points the line passing through the reflexions of the points, of the circle passing through three points or touching three lines the circle passing through the reflexions of the points or touching the reflexions of the lines, &c.; and, generally, of any figure intersecting or touching another a similar and equal figure touching or intersecting the reflexion of the other at the reflexions of the point or points of intersection or contact of the original figures.

All points common to two figures reflexions of each other lie of course on the line (or *axis* as it is sometimes termed) of reflexion, which evidently bisects at once all the angles finite or evanescent at which they intersect or touch each other.

Every circle having its centre on the axis of reflexion of any two figures reflexions of each other evidently intersects or touches both, when it meets them at all, at pairs of points reflexions of each other with respect to the axis; this peculiarity of the circle arises from the evident circumstance that every diameter of the figure divides it into two halves reflexions of each other with respect to itself.

If the plane of any figure be turned round any line in itself through an angle of 180°, the figure in the new is evidently the reflexion of itself in the old position with respect to the line.

From properties *a* and *b* of the preceding article it appears that—

When the lines connecting a variable point on a fixed line with two fixed points not on the line are reflexions of each other with respect to the line, their sum is a minimum or their difference a maximum according as the points lie at the same or at opposite sides of the line.

51. The next example again we give separately as the basis of some important properties of the circle.

The lines connecting a variable point on a fixed line, circle, or any other geometrical figure with two fixed points situated in any manner with respect to the figure, contain a maximum or minimum angle for every point at which a variable circle passing through the points touches the figure.

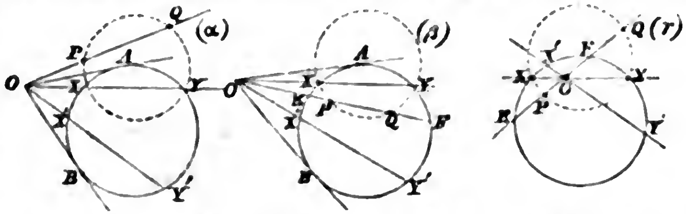
For, (Euc. III. 21 and I. 16), every chord of a circle subtends at any point on the circle an angle greater than at any point outside and less than at any point inside the circle, or conversely, according as the lesser or greater angular interval between the containing lines (24) is the subject of comparison for each angle; and, when a circle touches a line, circle, or any other figure, while the point of contact is common to the circle and figure, those at both sides of it on the figure are either both outside or both inside the circle according as the contact of the former with the latter is external or internal, and therefore &c.

The problem "*to find the points on a given line, circle, or any other geometrical figure which subtend maxima or minima angles at two given points*" is reduced, therefore, to the problem "*to describe a circle passing through the two given points and touching the given line, circle, or other figure;*" the solutions of which for the line and circle are respectively as follows:

For the line. If P and Q be the points and MN the line (figs. α and β, Art. 12); describing any circle $PQXY$ passing through P and Q and intersecting or not intersecting MN, and drawing to it a tangent OT from the point O in which the line PQ intersects MN, the circle round O as centre whose radius $= OT$ intersects MN in the points of contact A and B of the two circles required.

For, from the described circle $PQXY$ (Euc. III. 36), $OT^2 = OP.OQ$, and, by construction, OA^2 and OB^2 each $= OT^2$, therefore OA^2 and OB^2 each $= OP.OQ$, and therefore (Euc. III. 37) the circles PQA and PQB touch respectively at A and B the given line MN.

For the circle. If P and Q be the points and MN the circle; describing any circle $PQXY$ passing through P and Q



and intersecting MN in two points X and Y, and from the point O in which the chord of intersection XY meets the line PQ drawing the two tangents OA and OB to MN, their points of contact A and B are those of the two circles required.

For, from the given circle, OA^2 and OB^2 each $= OX.OY$ (Euc. III. 36), and from the described circle $OX.OY = OP.OQ$, therefore OA^2 and OB^2 each $= OP.OQ$, and therefore (Euc. III. 37) the circles PQA and PQB touch respectively at A and B the given circle MN.

If either of the points P or Q were on the line or circle MN, the other not being on it, the two points A and B would evidently coincide with it and with each other; and if P and Q were at opposite sides of the line or circumference MN, A and B would evidently be both impossible as no circle passing through P and Q could then possibly touch MN.

Hence, for the line or circle alike, the two solutions of the problem would be *distinct* if P and Q were at the same side of MN, *coincident* if either P or Q were upon MN, and *impossible* if P and Q were at opposite sides of MN.

52. With respect to the point O, determined as above in the solution for the circle, the following property is important—

The extremities X' and Y' of every chord of MN whose direction passes through O lie in the same circle with P and Q, and conversely, the chord of intersection $X'Y'$ of every circle passing through P and Q and meeting MN passes through O.

For, in the first case, the rectangles $OX'.OY'$ and $OP.OQ$ being each equal to the rectangle $OX.OY$ are equal to each other, and therefore &c.; and, in the second case, conceiving O connected with either point of intersection X' of the two circles MNX' and PQX', and supposing the connecting line OX' to meet them again if possible at two different points Y' and Y'', we would have the two different rectangles $OX'.OY'$ and $OX'.OY''$ equal to the same rectangle $OP.OQ$ which could not be, and therefore &c.

Hence the general property that—

If a variable circle pass through two fixed points P and Q and intersect a fixed circle MN, the variable chord of intersection XY passes through a fixed point O on the line PQ, that, viz., for which the constant rectangle $OX.OY$ = the fixed rectangle $OP.OQ$.

The circle MN being given, if P and Q be both given, O is of course implicitly given with them, being, as above, the point in which XY (the chord of intersection with MN of any circle through P and Q) meets PQ; but, if on the other hand, O only be given, P and Q may be (as in 12) on any line passing through O, and at any two distances from O (measured in similar or opposite directions according as O is external or internal to MN) for which $OP.OQ$ = the given rectangle $OX.OY$.

53. The problem to describe a circle passing through two given points P and Q and touching a given line or circle MN, is evidently a particular case of the problem.

To describe a circle passing through two given points P and Q and intercepting on a given line or circle MN a segment or chord of given length XY.

To solve which, as the direction of XY passes, by the preceding, in either case through O, we have $OX.OY = OP.OQ$ and $OX \mp OY = XY$ according as P and Q are at similar or opposite sides of MN, therefore, by Euc. II. 6 or 5, we have OX and OY and therefore X and Y themselves.

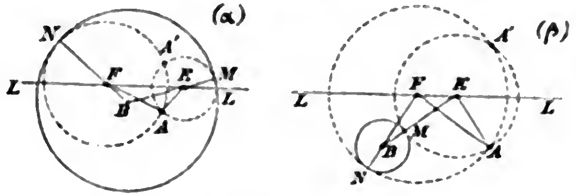
When P and Q are at the same side of MN, any length of segment or chord XY (less of course than the diameter in the case of the circle) might be intercepted by a circle through

P and Q, but when P and Q are at opposite sides of MN, figs. β Art. 12 and γ Art. 51, since the rectangle under the segments of a line cut internally can never (Euc. II. 5) exceed the square of half the line, no length less than twice the side of the square = the rectangle $OP.OQ$ could be intercepted; in that case, therefore, the two solutions of the problem are, distinct for any greater length, coincident for that particular length, and impossible for any lesser length.

COR. Since a circle passing through a fixed point and having its centre on a fixed line passes necessarily through a second fixed point the reflexion of the first with respect to the line (50), the four following problems are reduced immediately to the preceding.

To describe a circle passing through a given point, having its centre on a given line, and touching, or intercepting a given segment or chord of, a given line or circle.

54. If A be any point, A' its reflexion with respect to any line L, and E and F the centres of the two circles passing through A and A' and touching any circle MN, figs. α and β,



then, if B be the centre of MN, it is evident that $AE+BE$ and $AF+BF$ in fig. α, and $AE \sim BE$ and $AF \sim BF$ in fig. β = the radius BM or BN of MN. Hence the following solutions of the two useful problems—

On a given line L to determine the two points E and F, the sum or difference of whose distances from two given points A and B shall be given.

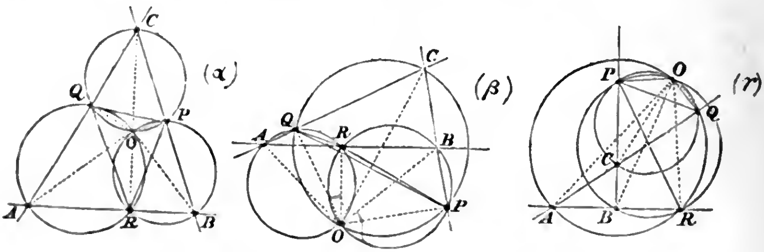
With either of the two given points B as centre and with a radius BM or BN = the given sum (fig. α) or difference (fig. β) describe a circle MN, the centres E and F of the two circles passing through the other given point A and its reflexion A' with respect to the given line L and touching that circle, are the two points required.

Should MN happen to pass through either A or A' the two points of contact M and N would evidently coincide at whichever of them it passed through; therefore the two centres E and F would also coincide, and the construction *then at the extreme limit of possibility or impossibility* would become that already given in (49) for the *minimum* sum and *maximum* difference of the distances in question.

55. The next example, again, we give separately as leading naturally to an important property of similar figures.

a. Of all triangles of any constant species, whose sides pass through three fixed points, the maximum is that the perpendiculars to whose sides at the points intersect at a common point.

b. Of all triangles of any constant species, whose vertices lie on three fixed lines, the minimum is that the perpendiculars to the lines at whose vertices intersect at a common point.



For if ABC and PQR be any two triangles such that the sides of ABC pass through the vertices of PQR, or the vertices of PQR lie on the sides of ABC; the three circles QAR, RBP, PCQ pass evidently in all cases (Euc. III. 21, 22) through a common point O, for which the three angles QOR, ROP, POQ are equal or supplemental to the three angles BAC, CBA, ABC respectively, and the three angles BOC, COA, AOB to the sums or differences of the three pairs of angles BAC and QPR, CBA and RQP, ACB and PRQ respectively (see 24), and which, when either of the two triangles ABC or PQR is fixed and the species of the other constant, is therefore fixed, and determines with the three sides of the variable triangle, whichever it be, three variable triangles *BOC*, *COA*, *AOB*, or *QOR*, *ROP*, *POQ* of constant species revolving round it as a common vertex. Hence, *O* being fixed in both cases, when, as in (*a*), *PQR* is fixed and *ABC* variable, *BC*, *CA*, *AB* are maxima with *OA*, *OB*, *OC*, that is, when the latter are diameters of the three fixed circles *QOR*, *ROP*, *POQ* respectively, and therefore &c.; and when, as in (*b*), *ABC* is fixed and *PQR* variable, *QR*, *RP*, *PQ* are minima with *OP*, *OQ*, *OR*, that is, when the latter are perpendiculars to the three fixed lines *BC*, *CA*, *AB* respectively, and therefore &c.

Hence, *to construct the triangle of given species and maximum area ABC whose sides shall pass through three given points PQR, or the triangle of given species and minimum area PQR whose vertices shall lie on three given lines BC, CA, AB.* The three angles *QOR*, *ROP*, *POQ* in the former case, and the three *BOC*, *COA*, *AOB* in the latter, being given by the above relations, the point *O* therefore in either case is given immediately by the common intersection of three given circles (Euc. III. 33), and therefore the three perpendiculars *BC*, *CA*, *AB* to *OP*, *OQ*, *OR* in the former case, and the three *OP*, *OQ*, *OR* to *BC*, *CA*, *AB* in the latter, are given, and therefore &c.

Cor. 1°. By aid of the point *O*, determined as above, the two problems: *to construct a triangle ABC or PQR of given magnitude and species, whose three sides BC, CA, AB shall pass through three given points P, Q, R, or whose three vertices P, Q, R shall lie on three given lines BC, CA, AB*, of which the two above are the extreme cases, may be solved with equal readiness; for, the species of the six triangles *BOC*, *COA*, *AOB* and *QOR*, *ROP*, *POQ* being given in both cases, when, as in the former case, the three lengths *BC*, *CA*, *AB* are given, so therefore are the three *OA*, *OB*, *OC*, and therefore the three points *A*, *B*, *C* on the three given circles *QOR*, *ROP*, *POQ*, and when, as in the latter case, the three lengths *QR*, *RP*, *PQ* are given, so therefore are the three *OP*, *OQ*, *OR*, and therefore the three points *P*, *Q*, *R* on the three given lines *BC*, *CA*, *AB*.

Hence, again, as in the problems, Arts. 51 and 53, the two solutions of the problem are distinct, coincident, or impossible according as the given magnitude of the triangle to be constructed ABC or PQR, is less than, equal to, or greater than its maximum value in the former case, or greater than, equal to, or less than its minimum value in the latter.

COR. 2°. By aid of the same again the two problems: *to construct a quadrilateral of given species, whose four sides A, B, C, D shall pass through four given points P, Q, R, S, or whose four vertices P, Q, R, S shall lie on four given lines* A, B, C, D may be readily solved. For, in the former case, to find any vertex AB of the required quadrilateral $ABCD$. As the two triangles PRQ and PSQ, through whose common vertices P and Q the two sides A and B corresponding to that vertex pass, are given, and as the two triangles ACB and ADB, which they determine with the other two sides C and D, are of given species; therefore by the above the circle passing through P and Q and though the required vertex AB passes through two given points M and N, whose distances from AB have a given ratio and which therefore determine AB. And, in the latter case, to find any side PQ of the required quadrilateral $PQRS$, as the two triangles ACB and ADB, on whose common sides A and B the two vertices P and Q corresponding to that side lie, are given, and as the two triangles PRQ and PSQ which they determine with the other two vertices R and S are of given species; therefore, by the above, the circle passing through the intersection of A and B and though the extremities of the required side PQ passes through two given points M and N, which consequently determine that circle and with it therefore the two points P and Q at which it intersects the two given lines A and B.

The same problem, the solutions of which, differing from those of Cor. 1°, are always in both cases unique and possible, may also, in the former case, to which the latter is evidently reducible, be solved otherwise thus as follows: Since the diagonal connecting any pair of opposite vertices AB and CD of the required quadrilateral $ABCD$ divides the two corresponding angles AB and CD each into segments of given magnitude; it therefore intersects the two given circles through P and Q and through R and S, on which AB and CD lie, at two given points I and J which consequently determine that diagonal and therefore the quadrilateral.

N.B. If the two points M and N in the former or the two I and J in the latter of the constructions just given *happened to coincide*, the construction otherwise determinate would be evidently indeterminate, and consequently *an infinite number of quadrilaterals could be constructed satisfying the conditions of the problem.* The circumstances under which such cases arise in general will be considered further on.

Cor. 3°. In the particular case of the above when, as is nearly the case in fig. β, one angle of the triangle PQR = two right angles, and when therefore the other two each = 0, it is evident from the values of the three angles BOC, COA, AOB, as given above, that the point O lies on the circle circumscribing the triangle ABC. Hence we see that—

a. If three points P, Q, R, taken arbitrarily on the three sides BC, CA, AB of any triangle ABC, lie in a right line; the three circles QAR, RBP, PCQ intersect at a common point O on the circle ABC.

b. If while the triangle is fixed the three points P, Q, R vary so as to preserve the constancy of the three ratios $QR : RP : PQ$, the point of intersection O is a fixed point, and conversely.

The four lines BPC, CQA, ARB and PQR, in the above, being entirely arbitrary, it follows at once from property a, as the reader may very easily prove *à priori* for himself, that—

The four circles circumscribing the four triangles determined by any four arbitrary lines taken three and three intersect at a common point.

By Cors. 1° and 2° applied to the same particular case we obtain ready solutions of the two following problems, viz.

1°. *To draw a line intersecting three given lines so that its segment intercepted between any two of them shall be cut in given lengths by the third.*

2°. *To draw a line intersecting four given lines so that its segment intercepted between any two of them shall be cut in given ratios by the other two.*

56. From the nature of similar figures and of their homologous points and lines, it appeared (40) that if one point O of or connected with a figure F of any nature variable in magnitude and position but invariable in form be fixed, all points

P, Q, R, S, &c. of or connected with it describe, and all lines A, B, C, D, &c. of or connected with it envelope similar figures, so that in such a case if one point P move on a line or describe a circle, all points Q, R, S, &c. move on lines or describe circles, and if one line A turn round a point or envelope a circle, all lines B, C, D, &c. turn round points or envelope circles. Hence from the preceding it follows that—

For a figure F of any nature variable in magnitude and position but invariable in form, if three points P, Q, R connected with it in any manner move on fixed lines A, B, C, all points S, T, &c. connected with it move on fixed lines D, E, &c., and if three lines A, B, C connected with it in any manner turn round fixed points P, Q, R, all lines D, E, &c. connected with it turn round fixed points S, T, &c.

For, in the former case, the variable triangle P, Q, R, whose vertices move on the three fixed lines A, B, C, and in the latter case the variable triangle A, B, C, whose sides pass through the three fixed points P, Q, R, being invariable in form; therefore by the preceding the point O, connected as above with the variable triangle and therefore with the figure, in both cases is a fixed point, and therefore &c.

Cor. 1°. The above general properties supply obvious solutions of the four following general problems, viz.

To construct a figure of given form, 1°. four of whose points shall lie on given lines; 2°. four of whose lines shall pass through given points; 3°. three of whose points shall lie on given lines, and one of whose lines shall pass through a given point; 4°. three of whose lines shall pass through given points, and one of whose points shall lie on a given line.

Of these four general problems 1°. and 2°. admit always of possible and generally of unique solutions, depending on the unique point of intersection of two lines in 1°., and on the unique line of connection of two points in 2°., which may however by the possible coincidence of the two lines in 1°., or of the two points in 2°. become in certain cases *indeterminate* (55, Cor. 2°.); 3°. and 4°. on the other hand admit in all cases of two solutions, distinct, coincident, or impossible according to circumstances.

The circumstances under which the solutions of 1°. and 2°.

may in certain cases become indeterminate, appear at once from the two general properties of the present article; the four points of the figure P, Q, R, S, in 1°. may so correspond to the four given lines A, B, C, D, or the four lines of the figure A, B, C, D in 2°. to the four given points P, Q, R, S, that when in 1°. three of the points P, Q, R lie on three of the lines A, B, C, the fourth point S must lie on the fourth line D, or when in 2°. three of the lines A, B, C pass through three of the points P, Q, R, the fourth line D must pass through the fourth point S; in either case the problem would evidently admit of an infinite number of solutions and consequently be indeterminate.

Cor. 2°. The same again by aid of the principles established in the preceding article supply obvious solutions of the four following additional problems, viz.—

To construct a figure of given form and of minimum or given magnitude, three of whose points shall lie on given lines.

To construct a figure of given form and of maximum or given magnitude, three of whose lines shall pass through given points.

Of which the two for the cases of given magnitude admit each, as in Cor. 1°. of the preceding, of two solutions, distinct, coincident, or impossible, according as the given magnitude happens to be within, upon, or beyond the limiting value of which it is susceptible under the circumstances of the case.

57. There are many cases in which a variable magnitude is shewn to be a maximum (or a minimum) in some particular relative position of the elements of the figure with which it is connected, by its being shewn that for any other relative position it could be increased (or diminished), and that every change which would increase (or diminish) it would tend to bring it to the particular configuration in question, of this the four following instructive examples may be taken as illustrations:

Ex. 1. The sum of the distances of a variable point on a fixed line from two fixed points at the same side of the line is a minimum when they make equal angles with the line (Ex. *a*, 49); from this it follows that—

Of all polygons of any order whose vertices in any assigned order lie on fixed lines, that of minimum perimeter is that whose several angles are all bisected externally by the lines on which their vertices lie.

For, supposing any angle of the polygon not to be so bisected, the removal of its vertex to the point at which it would be so bisected, would, *without affecting in any manner the remaining sides of the polygon*, diminish the sum of the containing sides, and therefore the entire perimeter of which that sum is a part.

Ex. 2. For the middle point of any arc of a circle: 1°. The sum of the chords of the segments and the area of the triangle they form with the chord of the arc, are both maxima (Ex. 6°, 48); 2°. The perimeter and area of the quadrilateral formed by the tangent with the chord of the arc and the tangents at its extremities are both minima (Ex. 7°, 48); from these it follows that—

1°. *Of all polygons of the same order inscribed in the same circle, that of maximum perimeter and area is the regular.*

2°. *Of all polygons of the same order circumscribed about the same circle, that of minimum perimeter and area is the regular.*

For, supposing any vertex of the polygon in 1°. not to bisect the arc of the circle intercepted between the adjacent two, its removal to the middle point would, *without affecting in any way the remainder of the polygon*, increase both the perimeter and area of the triangle it determines with the chord of the arc, and therefore of the entire figure of which that triangle is a part; and, supposing the point of contact of any side of the polygon in 2°. not to bisect the arc of the circle intercepted between those of the adjacent two, its removal to the middle point would, *without affecting in any way the remainder of the polygon*, diminish both the perimeter and area of the quadrilateral determined by that side with the chord of the arc and the tangents at its extremities, and therefore of the entire figure of which that quadrilateral is a part.

Ex. 3. When a line of any length is cut into two equal parts, the product of the parts is greater, and the sum of their squares less, than if it were cut into any two unequal parts (Ex. 2°, 48); from this it follows that—

1°. *When a line of any length is cut into any number of equal parts, the continued product of all the parts is greater, and the sum of their squares less, than if it were cut in any way into the same number of unequal parts.*

2°. *When a line of any length is cut into any number of parts a, b, c, d, &c. in the ratios of any set of integer numbers $\alpha, \beta, \gamma, \delta$, &c., the product $a^\alpha . b^\beta . c^\gamma . d^\delta$, &c. is greater, and the sum $\frac{a^2}{\alpha} + \frac{b^2}{\beta} + \frac{c^2}{\gamma} + \frac{d^2}{\delta}$ + &c. is less, than if it were cut in any other way into the same number of parts.*

To prove 1°. Supposing any two of the parts not to be equal, the equable division of their sum would, *without affecting any of the remaining parts*, increase the product and diminish the sum of the squares of those two, and therefore increase the product and diminish the sum of the squares of the entire set.

To prove 2°. Conceiving a subdivided into α equal parts, b into β equal parts, c into γ equal parts, d into δ equal parts, &c., then since

$a^\alpha = \alpha^\alpha$ times the continued product of the α subdivisions of a, $b^\beta = \beta^\beta$ times the continued product of the β subdivisions of b, $c^\gamma = \gamma^\gamma$ times the continued product of the γ subdivisions of c, $d^\delta = \delta^\delta$ times the continued product of the δ subdivisions of d, &c.; and since $a^2 = \alpha$ times the sum of the squares of the α subdivisions of a, $b^2 = \beta$ times the sum of the squares of the β subdivisions of b, $c^2 = \gamma$ times the sum of the squares of the γ subdivisions of c, $d^2 = \delta$ times the sum of the squares of the δ subdivisions of d, &c., therefore $a^\alpha . b^\beta . c^\gamma . d^\delta$. &c. $= \alpha^\alpha . \beta^\beta . \gamma^\gamma . \delta^\delta$. &c. times the continued product of the whole $\alpha + \beta + \gamma + \delta$ + &c. subdivisions, of the entire line, and $\frac{a^2}{\alpha} + \frac{b^2}{\beta} + \frac{c^2}{\gamma} + \frac{d^2}{\delta}$ + &c. = the sum of the squares of the same $\alpha + \beta + \gamma + \delta$ + &c. subdivisions; and therefore, by 1°., the former is a maximum and the latter a minimum when the several subdivisions are *all* equal, that is, as a contains α of them, b, β of them, c, γ of them d, δ of them, &c., when $a : b : c : d$, &c. :: $\alpha : \beta : \gamma : \delta$, &c. Q.E.D.

Ex. 4. When two conterminous lines of any lengths are placed at a right angle, the area of the triangle they determine is greater than if they were placed at any other angle obtuse or acute (Ex. 1°., 48) from this it follows that—

1°. *When all the sides but one of a polygon are given in length and order, the area of the figure is the maximum when the semicircle described on the closing side as diameter passes through all its vertices.*

2°. *When all the sides of a polygon are given in length and order, the area of the figure is the maximum when all its vertices lie in a circle.*

3°. *When the extremities of a bent line of given length are connected by a straight line, the area of the enclosed figure is the maximum when its form is a semicircle.*

4°. *When the perimeter of a closed figure is given, its area is the maximum when its form is a circle.*

To prove 1° and 3°. Supposing any single vertex P of the polygon in the former case, or any single point P of the bent line in the latter case, not to lie upon the semicircle described on the closing side or connecting line AB, then the two conterminous lines AP and BP not being at a right angle (Euc. III. 31). The putting of them at a right angle would, *without affecting in any way, except in position, the remaining portions of the figure which might be regarded as attached to and moveable with them,* increase the area of the triangle APB and therefore of the entire figure of which it is a part.

To prove 4°. Supposing the perimeter to form a circle, then any diameter AB would divide the whole figure into two semicircles, one or both of which would necessarily be altered in form and therefore diminished in area (3°.) by any change whatever from the circular form of the entire.

To prove 2°. Supposing the several vertices of the polygon to lie in a circle, then conceiving the circle described through them, any change whatever in the figure of the polygon would *without affecting in any way except in position the circular segments on the several sides which might be* *regarded as attached to and moveable with them* alter the form and therefore diminish the area of the entire circular figure (4°.), and consequently of the polygon itself the only part of the whole undergoing change of area.

Otherwise thus, from 1°. without the aid of 4°. the polygon and circle being supposed described as before; then, firstly, if any two vertices of the former A and B happened to determine a diameter of the latter, that diameter would divide the polygon into two whose areas, by 1°., would both be diminished by any change of figure they could receive; and, secondly, if no two vertices happened to determine a diameter, then drawing any diameter AB, and connecting its extremities A and B each with the two adjacent vertices of the polygon M and N, P and Q, between which it lies, that diameter would divide the entire figure consisting of the variable original polygon and the two invariable appended triangles MAN and PBQ into two parts, whose areas, by 1°., would both be diminished by any change of figure they could receive; therefore in either case any change of figure in the original polygon, as necessarily producing a change of figure in one or both of the partial polygons, would diminish the area of one or both, and therefore of the whole.

The former demonstration, though perhaps less elementary, will probably be regarded by the reader as simpler than the latter.

58. In the Theory of Maxima and Minima it happens very often, so often as to require special notice at the very outset of the subject, that a variable magnitude which in a certain relative position of the elements of the figure with which it is connected has a maximum and a minimum value each for the proper position corresponding to itself, appears in another relative position of the very same elements to have two maxima or two minima values for the same positions alternating with two minima values each $= 0$, or two maxima values each $= \infty$, at certain intermediate positions, as, for example, the distance of a variable point on a fixed circle from a fixed line, which when the circle and line do not intersect, is a maximum for the farther and a minimum for the nearer extremity of the diameter perpendicular to the line, but which when they do intersect has apparently maxima values at both those extremities alternating with apparently minima values each $= 0$ at the two points of intersection.

In the preceding, and in all similar cases, however—as will more fully appear when we come to the subject of the *Signs* of geometrical magnitudes—*a change of sign takes place at each passage of the variable through* 0 *or* ∞, after which a negative increase is of course a positive decrease, and conversely, and a negative maximum consequently a positive minimum, and conversely; and the two values $= 0$ or ∞ are not real *minima* or *maxima* values at all (45), but merely *the particular values through which the variable magnitude in continuous decrease or increase passes at the moment of changing sign.* Of course if absolute values of magnitudes only were taken into account, as in arithmetic and in the geometry of ancient times, the particular values 0 and ∞ would be the least of all minima and the greatest of all maxima for magnitudes of every kind; but in the geometry of the present day, in which magnitudes of certain kinds are regarded as having not only absolute value but also sign, they are looked on as in no way differing from any other particular values through which variable magnitudes in continuous decrease or increase may happen to pass. In the case of magnitudes incapable of change of sign, the values 0 and ∞ are of course the extreme minima and maxima values in modern as in ancient geometry, and it might at first sight appear questionable whether it would not be better to regard them as such for magnitudes of all kinds as well. The advantages, however, resulting from the convention of signs in modern geometry are so numerous and considerable, that in the present state of the science it could scarcely be regarded as optional to forego them or not.

59. The extreme maxima and minima values of variable magnitudes, in whichever light regarded, give evidently in all cases the extreme *limits of possibility and impossibility* in the solutions of all problems involving the magnitudes; it being of course impossible to construct a magnitude of any kind greater than the extreme maximum or less than the extreme minimum of which it is susceptible under the circumstances of its data.

Should the extreme maxima and minima values of a magnitude variable in position *happen to be equal*, of course all intermediate values would be also equal, and the magnitude would be *constant;* in every such case the problem to construct the magnitude so as to have *a given* value would of course be *impossible* for *any other* than the constant value, while for that value it would evidently *admit of an infinite number of solutions* or be *indeterminate* as it is termed.

When on the other hand, as is of course the case generally, the extreme maxima and minima values of a magnitude variable in position are not equal, the problem to construct the magnitude so as to have any intermediate value, admits always of at least *two* distinct and definite solutions, more or less separated from each other, which approach to coincidence as the value continuing within the limits of possibility approaches either limit, which actually coincide for each limiting value, and which become impossible together once the limits are passed; and the same is the case generally for all problems admitting of two solutions and therefore for all in which, directly or indirectly, the circle is involved, *the two solutions in general distinct become coincident at the limits of possibility and impossibility, and so pass together through coincidence from possibility to impossibility, and conversely*, (See 21).

As an example of the preceding principles: suppose it were required to draw from a given point to a given circle a line of given length. For the centre of the circle the solutions of the problem would manifestly be impossible for any value of the given length different from the radius and indeterminate for that value; while for every point different from the centre it would admit of two, and but two, determinate solutions which would be distinct, coincident, or both impossible, according as the given length happened to lie between, upon, or beyond the extreme limits for the point.

The above principles are all general and deserving of particular attention; for, 1°.—No problem in geometry admitting in its general form of but a single solution ever becomes impossible, however in certain cases it may appear to do so; 2°.—Whenever a problem admitting in its general form of two solutions becomes impossible, the two solutions always become impossible together, and pass invariably through coincidence in their transition from possible to impossible, and conversely; and 3°.—There is no problem in geometry that does not become indeterminate under certain circumstances of its data.

CHAPTER IV.

ON THE TRIGONOMETRICAL FUNCTIONS OF ANGLES.

60. If from any point P taken arbitrarily on either M of two indefinite lines M and N intersecting at a point O and constituting an angle of any form MN a perpendicular PQ be let fall upon the other line N, the perpendicular determines with the two lines a right-angled triangle PQO whose form it is evident depends only on that of the angle, and every two of whose sides determine two reciprocal ratios which are implicitly given with, and which, reciprocally, implicitly give the form of the angle. The six ratios thus determined from their importance in the science of Trigonometry are termed the *trigonometrical functions* of the angle, and are designated in that science by appropriate names as follows:

1°. The ratio of the perpendicular PQ to the interval PO between its head and the vertex of the angle is termed the *sine* of the angle.

2°. The ratio of the perpendicular PQ to the interval QO between its foot and the vertex of the angle is termed the *tangent* of the angle.

3°. The ratio of the former interval PO to the latter interval QO is termed the *secant* of the angle.

4°. The ratio of the interval OQ to the distance OP is termed the *co-sine* of the angle.

5°. The ratio of the interval OQ to the perpendicular PQ is termed the *co-tangent* of the angle.

6°. The ratio of the distance OP to the perpendicular PQ is termed the *co-secant* of the angle.

Upon the question as to the origin and appropriateness of the names 'sine,' 'tangent,' and 'secant,' we need not enter here; the three simple ratios so designated are of such frequent occurrence, the first of them especially, in geometrical researches, as absolutely to require some distinguishing appellations; and the old and familiar names by which they have always been known in another science, are at least as convenient as any others that might be proposed for the purpose; the remaining three, termed respectively co-sine, co-tangent, and co-secant, have been so named as being to *the complement of the angle* what the sine, tangent, and secant, are to the angle itself.

If the angle determined by the two lines be conceived to change figure and to pass continuously through every variety of form, from the extreme of two parallel to the opposite extreme of two rectangular lines; the whole six ratios will pass evidently in the course of the variation, the sine, tangent, and secant in continuous increase, and the co-sine, co-tangent, and co-secant in continuous decrease, through every variety of value of which they are severally susceptible; the sine from 0 up to 1 and the co-sine from 1 down to 0, the secant from 1 up to ∞ and the co-secant from ∞ down to 1, the tangent from 0 up to ∞ and the co-tangent from ∞ down to 0; the whole six being of course implicitly given for each particular form of angle, and any one of them reciprocally determining the corresponding form of the angle and with it of course the remaining five.

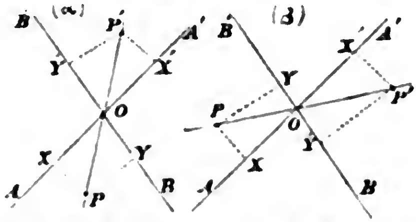
Of all the trigonometrical functions of the angle the *sine* is that which enters most largely into the investigations of modern geometry, and we shall accordingly devote the present chapter to the consideration of a few simple but very important properties involving the sines of angles.

61. *The ratio of the sines of the segments into which an angle is divided by any line passing through its vertex is the same as that of the perpendiculars on its sides from any point on the line; and conversely, the ratio of the perpendiculars from any point on the sides of an angle is the same as that of the sines of the segments into which the angle is divided by the line connecting its vertex with the point.*

For if AA' and BB', or M and N, be the sides of the angle; PP', or L, the line passing through its vertex O; P, or P', the point, and PX and PY, or $P'X'$ and $P'Y'$, the perpendiculars. Then since by definition $PX : PO$ or $P'X' : P'O = \sin LM$

and $PY : PO$ or $P'Y' : P'O = \sin LN$, therefore $PX : PY$ or $P'X' : P'Y' = \sin LM : \sin LN$, and therefore &c.

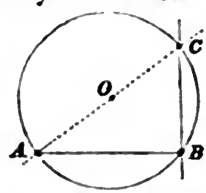
Cor. 1°. If the two perpendiculars PX and PY, or $P'X'$ and $P'Y'$, were turned round P, or P', through any common angle so as to become, more generally, isoclinals inclined at any equal angles to the sides of the given angle M and N, the same property would obviously be true of the isoclinals as of the perpendiculars, as the ratio of the former would evidently be constant and equal to that of the latter through whatever angles they were turned.



Cor. 2°. A very obvious solution of a very useful problem "*to divide a given angle internally or externally into two parts whose sines shall have a given ratio*" might evidently be based on the above, but another and in many respects more convenient method of effecting the same division will be given further on.

62. *In a circle the ratio of any chord to the diameter is the sine of the constant angle subtended by the chord at every point on the circumference of the circle* (25).

If AB be the chord; through either extremity of it A drawing the diameter AC and joining CB, then since the angle subtended by the chord at any point on the circle is independent as to form of the position of the point (25), if the theorem be true for any one point on the circle it is true for every point, but it is true for the point C, for the angle ABC being in a semicircle and consequently a right angle, therefore by (60) the ratio of $AB : AC$ is the sine of the angle ACB, and therefore &c.



Cor. 1°. Hence *two or any number of chords of the same circle are to each other as the sines of the angles they severally subtend at the circumference of the circle;* for each chord, by the above, being equal to the diameter of the circle multiplied by the sine corresponding to itself, and the diameter being the same for them all, therefore &c.

COR. 2°. The angle any chord of a circle makes with the tangent at either of its extremities being similar in form to that in the two segments into which it divides the circle (22). Hence from Cor. 1°.—

Two or any number of chords of the same circle are to each other as the sines of the angles they make with the tangents at their several extremities.

COR. 3°. The several chords may be conterminous, in which case it appears at once from Cor. 2°., that—

Two or any number of chords diverging from the same point on the circumference of a circle are to each other as the sines of the angles they severally make with the tangent at the point.

COR. 4°. Any two adjacent sides of any polygon inscribed in a circle being conterminous chords of the circle; therefore from Cor. 3°.—

The tangents at the several vertices of any polygon inscribed in a circle divide the several angles of the polygon externally into parts whose sines are in the ratios of the adjacent sides of the polygon.

COR. 5°. The three sides of every triangle being chords of the same circle, that viz. which passes through its three vertices, and the three angles being those subtended by their opposite sides at the circumference of the circle; hence at once, from the above, the important property of the triangle, that—

The sine of any angle of a triangle is equal to the opposite side divided by the diameter of the circle circumscribing the triangle; and conversely, the diameter of the circle circumscribing any triangle is equal to any side of the triangle divided by the sine of the opposite angle.

COR. 6°. Denoting in any triangle by a, b, c the three sides, and by A, B, C the three respectively opposite angles, then always—

$$a \div \sin A = b \div \sin B = c \div \sin C,$$

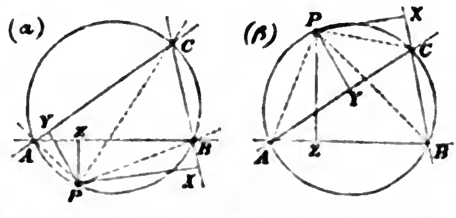
for each by the above is equal to the diameter of the circle circumscribing the triangle.

COR. 7°. Denoting by d the diameter of the circumscribing circle, and by p, q, r the three perpendiculars from the three vertices A, B, C upon the respectively opposite sides a, b, c; then since (60), $p = c.\sin B$ or $b.\sin C$, $q = a.\sin C$ or $c.\sin A$, $r = b.\sin A$ or $a.\sin B$, and since, Cor. 6°., $d = a \div \sin A = b \div \sin B = c \div \sin C$, therefore $pd = bc$, $qd = ca$, $rd = ab$, and therefore generally—

In every triangle the product of any two sides is equal to the product of the diameter of the circumscribing circle into the perpendicular on the third side from the opposite vertex.

This property supplies an obvious method of solving the problem: "*given of a triangle one side, the opposite angle, and the product of the other two sides to construct it.*"

COR. 8°. If P be any point on the circumscribing circle, and PA, PB, PC the three lines connecting it with the three vertices A, B, C, then since, whatever be the position of P, any two of the connecting chords



PA and PB, divided each by the diameter of the circle d, are the sines of the two segments PCA and PCB into which the third PC divides, internally or externally, the angle ACB through whose vertex it passes; it follows that—

The two general problems: "to divide a given angle internally or externally into two parts whose sines shall have any given relation to each other," and "to divide a given arc of a circle, internally or externally into two parts whose chords shall have the same relation to each other," are identical.

COR. 9°. If PX, PY, PZ be the three perpendiculars from P on the three sides BC, CA, AB of the triangle ABC, then since, Cor. 7°.,

$$PB.PC = d.PX, \quad PC.PA = d.PY, \quad PA.PB = d.PZ;$$

therefore

$$\sin PAB.\sin PAC = PX \div d,$$
$$\sin PBC.\sin PBA = PY \div d,$$
$$\sin PCA.\sin PCB = PZ \div d;$$

and therefore generally—

The product of the sines of the segments into which any angle of a triangle is divided by a variable line passing through its vertex, varies as the perpendicular to the opposite side from the point in which the line meets the circumscribing circle.

This property gives a very definite conception of the law according to which the product of the sines of the segments of a fixed angle by a variable line of section varies with the position of the dividing line, and supplies moreover an obvious solution of the useful problem—

To divide a given angle internally or externally into two parts whose sines shall have a given product.

Cor. 10°. If A, B, C, D be any four points on a circle, PX and PX', PY and PY', PZ and PZ' the three pairs of perpendiculars from any fifth point P on the circle upon the three pairs of opposite chords BC and AD, CA and BD, AB and CD they determine, and d the diameter of the circle; then since by Cor. 7°.,

$$PX = PB.PC \div d \text{ and } PX' = PA.PD \div d,$$
$$PY = PC.PA \div d \text{ and } PY' = PB.PD \div d,$$
$$PZ = PA.PB \div d \text{ and } PZ' = PC.PD \div d;$$

therefore

$$PX.PX' = PY.PY' = PZ.PZ' = PA.PB.PC.PD \div d^2;$$

and therefore—

The products of the three pairs of perpendiculars from any point on a circle upon the three pairs of opposite chords connecting any four points on the circle are equal; and their common value is equal to the product of the distances of the one point from the four divided by the square of the diameter of the circle.

If the six perpendiculars were turned round the point P through any common angle, so as to become, more generally, isoclinals inclined at any equal angles to the six chords; the products of the three pairs of isoclinals for opposite pairs of chords would still continue equal, each isoclinal being equal to the corresponding perpendicular multiplied by the secant of the angle of rotation.

63. *In every triangle the ratio of the sines of any two of the angles is the same as that of the sides opposite to them.*

For if A, B, C be the three angles, a, b, c the three opposite sides, and p, q, r the three perpendiculars on the latter from the opposite vertices; since then by definition $\sin B = p : c$ and $\sin C = p : b$, $\sin C = q : a$ and $\sin A = q : c$, $\sin A = r : b$ and $\sin B = r : a$, therefore at once

$$\sin B : \sin C = b : c, \ \sin C : \sin A = c : a, \ \sin A : \sin B = a : b,$$

and therefore generally for all three,

$$\sin A : \sin B : \sin C = a : b : c,$$

or, *in every triangle the sines of the angles are as the opposite sides.*

Otherwise thus: conceiving a circle circumscribed round the triangle, then since, by the preceding (62), each side divided by the diameter of the circle is the sine of the opposite angle, therefore &c.

This latter though less direct has the advantage over the former and more ordinary method of proving this important theorem, that besides establishing the proposition it gives at the same time the common value of the three equivalent quotients $a \div \sin A$, $b \div \sin B$, $c \div \sin C$, viz. *the diameter of the circle circumscribing the triangle.*

Cor. 1°. The angle between any two lines being similar in form to that between parallels to them through any point, it follows at once from the above, that—

Every three lines drawn from a point parallel and equal to the three sides of a triangle are to each other each as the sine of the angle between the other two.

That is, if O be the point, and OA, OB, OC the three lines, then $OA : OB : OC = \sin BOC : \sin COA : \sin AOB$.

Cor. 2°. In every parallelogram any two adjacent sides and the conterminous diagonal being equal and parallel to the three sides of either triangle into which the parallelogram is divided by the diagonal. Hence from Cor. 1°—

Each side of every parallelogram is divided by the diagonal which passes through it into parts whose sines are in the inverse ratio of the adjacent sides of the parallelogram.

That is, if OA and OB be the sides about the angle, and OD the diagonal, then $\sin AOD : \sin BOD = OB : OA$.

Cor. 3°. The above supplies obvious and rapid solutions of the two following problems:

1°. *To divide two or four right angles into three parts whose sines shall be as three given numbers.*

2°. *To determine two angles whose sines shall be to the sine of their sum or difference as two given numbers to a third.*

For, in the case of 1°., constructing any triangle whose three sides are as the three numbers, its three internal angles furnish obviously the solution for two and its three external for four right angles, (Euc. I. 32); and in the case of 2°., constructing any triangle two of whose sides are to the third as the two given numbers to the third, its two internal angles opposite to the two sides furnish obviously the solution for the case of the sum, and either of them with the external adjacent to the other for the case of the difference (Euc. I. 32).

If the three given numbers were such that three lines representing them were incapable of forming a triangle, that is, (Euc. I. 20), if one of them were greater than the sum or less than the difference of the other two, the above constructions would of course fail; thus showing that in such cases the required division or determination would be impossible.

Cor. 4°. The three internal angles BOC, COA, AOB, subtended by the three sides BC, CA, AB of any triangle ABC at any arbitrary point O, being either together equal to four right angles, or each separately equal to the sum or difference of the other two, according as the point O is within or without the triangle; the above leads again, as in Cor. 3°, to the four solutions of the following problem, viz.—

To determine the point O for which the sines of the three angles subtended by the three sides of one given triangle ABC shall be as the three sides of another given triangle $A'B'C'$.

For the three angles subtended at O by the three sides of ABC being, according to the position of O, as just observed, either the three external or one of the external and two of the internal corresponding angles of $A'B'C'$; therefore describing on the three sides of ABC as chords the three pairs of equal circles which intersect them internally and externally at the three corresponding angles, internal and external, of $A'B'C'$ (22); of the six circles thus described, the three which intersect the sides of ABC internally at the three internal and externally at the three external corresponding angles of $A'B'C'$ intersect with each other at a common point O, which is one of those required, and intersect with the remaining three, each with the two not corresponding to itself, at three other common points P, Q, R, which are the remaining three of those required.

In the particular case when the three pairs of corresponding angles of the two triangles ABC and $A'B'C'$ are equal, and when the triangles themselves are therefore similar; while the three circles determining the point O intersect at the point of concurrence of the three perpendiculars from the vertices on the opposite sides of ABC, that being the point for which the three internal angles BOC, COA, AOB, are the supplements of the three internal angles BAC, CBA, ACB, of the triangle; the remaining three evidently coincide with each other and with the circle circumscribing ABC, and the three points P, Q, R are consequently indeterminate. This is also evident *a priori* from (62); every point on the circle circumscribing any triangle ABC subtending, as there shewn, its three sides at angles whose sines are proportional to their lengths.

Cor. 5°. Denoting by P, Q, R the radii of the three equal pairs of conjugate circles in the preceding corresponding to the three sides BC, CA, AB, respectively of the triangle ABC; since then three of those circles for different sides pass through the point O, and consequently circumscribe the three partial triangles BOC, COA, AOB, therefore by (62).

$$2P = BC \div \sin BOC,\ 2Q = CA \div \sin COA,\ 2R = AB \div \sin AOB,$$

and therefore

$$P : Q : R = BC \div \sin BOC : CA \div \sin COA : AB \div \sin AOB.$$

Hence again, by Cor. 4°., the four solutions of the following additional problem, viz.—

To determine the point O such that for three given points A, B, C the radii P, Q, R of the three circles BOC, COA, AOB, shall be as three given numbers.

For since from the propositions just stated

$$\sin BOC : \sin COA : \sin AOB = BC \div P : CA \div Q : AB \div R,$$

the problem is therefore reduced at once to that of Cor. 4°., the two groups of three ratios $BC : CA : AB$ and $P : Q : R$ being both given, and therefore with them the group to which the three sines are proportional.

In the particular case, when $P=Q=R$, since then

$$\sin BOC : \sin COA : \sin AOB = BC : CA : AB,$$

the point O as already noticed in Cor 4°., is either the unique point of concurrence of the three perpendiculars from the vertices on the opposite sides, or any point indifferently on the circumscribing circle, of the triangle ABC; the three equal circles BOC, COA, AOB, in the former case being equal to, and in the latter case coinciding with, the circle ABC.

64. *In every triangle the ratio of double the area to the rectangle under any two of the sides is the sine of the angle contained by those sides.*

For if, as in the preceding, A, B, C be the three angles, a, b, c the three opposite sides, and p, q, r the three perpendiculars on the latter from the opposite vertices; since then (Euc. I. 42.) $2 \text{ area} = ap = bq = cr$, and since (60) $p = b \,.\, \sin C$ or $c \sin B$, $q = c \sin A$ or $a \sin C$, $r = a \sin B$ or $b \sin A$, therefore

$$2 \text{ area} = bc \,.\, \sin A = ca \,.\, \sin B = ab \,.\, \sin C,$$

and therefore &c.

Cor. 1°. Since from the above

$$\text{area} = \tfrac{1}{2}bc \,.\, \sin A = \tfrac{1}{2}ca \,.\, \sin B = \tfrac{1}{2}ab \,.\, \sin C,$$

therefore—

In every triangle the area is equal to half the product of any two of the sides multiplied into the sine of the included angle.

Hence if two sides of a triangle be given, the area varies as the sine of the included angle, has equal values for every pair of supplemental angles, and is the maximum for a right angle.

Cor. 2°. Denoting by R the radius of the circle circumscribing the triangle, then since by (62),

$$\sin A = a \div 2R, \sin B = b \div 2R, \sin C = c \div 2R,$$

and since by the above, $\sin A = 2 \text{ area} \div bc$, $\sin B = 2 \text{ area} \div ca$, $\sin C = 2 \text{ area} \div ab$, therefore $R = abc \div 4 \text{ area}$, or—

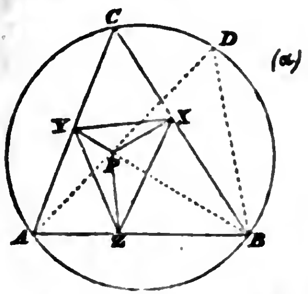
In every triangle the radius of the circumscribing circle is equal to the product of the three sides divided by four times the area.

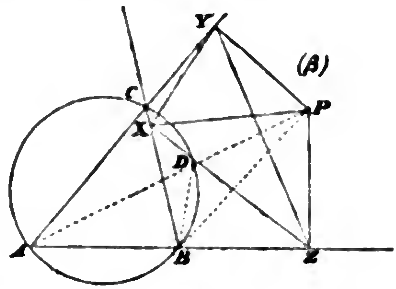
Which is the well-known formula by which to calculate in numbers the value of R, when those of a, b, c are given.

COR. 3°. *If from any point P perpendiculars PX, PY, PZ, be let fall upon the sides BC, CA, AB, of any triangle ABC, then*

$$2 \text{ area of triangle } XYZ = (OR^2 \sim OP^2) \,.\, \sin A \,.\, \sin B \,.\, \sin C$$

where O and OR are the centre and radius of the circle circumscribing the triangle ABC.





For, connecting P with any two of the vertices A and B of the triangle ABC, and the point D where the connector for either A intersects the circumscribing circle with the other B; then by the above, 2 area $XYZ = ZX \,.\, ZY \,.\, \sin XZY$; but the two groups of four points Y, P, Z, A and X, P, Z, B being evidently concyclic, and PA and PB being the diameters of the two circles (Euc. III. 31.); therefore (62), $ZY = PA \,.\, \sin A$, $ZX = PB \,.\, \sin B$, and (Euc. III. 21. 22.) angle XZY = angle PBD, the two angles PZX and PZY being equal to the two PBX and PAY or CBD respectively; therefore

$$2 \text{ area } XYZ = PA \,.\, PB \,.\, \sin A \,.\, \sin B \,.\, \sin PBD,$$

but (63) $PB \,.\, \sin PBD = PD \,.\, \sin PDB = PD \,.\, \sin C$, (Euc. III. 21.) therefore

$$2 \text{ area } XYZ = PA \,.\, PD \,.\, \sin A \,.\, \sin B \,.\, \sin C,$$

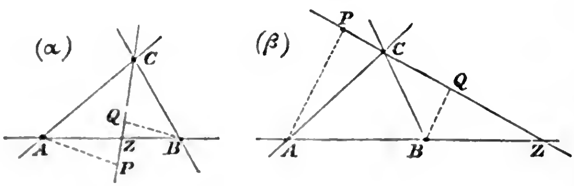
and therefore &c; since (Euc. III. 35. 36.) $PA \,.\, PD = (OR^2 - OP^2)$ or $(OP^2 - OR^2)$ according as P is within (fig α) or without (fig β) the circle circumscribing ABC.

If in the above the three perpendiculars PX, PY, PZ, were turned round P in the same direction of rotation through any common angle, so as to become, more generally, isoclinals PX', PY', PZ' inclined at the complement of the angle to the sides; the same value multiplied by the square of the secant of the angle of rotation, or, which is the same thing, divided by the square of the sine of the angle of inclination to the sides (60), would evidently (Euc. VI. 19.) be the value of the area of the triangle $X'Y'Z'$.

Cor. 4°. It follows of course from the preceding, Cor. 3°., that, whether for perpendiculars or isoclinals, *the area of the triangle XYZ is*—1°. *constant, when P is on any circle concentric with O;* 2°, *evanescent when P is on the circle circumscribing ABC;* 3°, *a maximum in absolute value* (58) *when P is at O, or at infinity;* from the second of which it appears that, *the feet of the perpendiculars upon the sides of a triangle from any point on its circumscribing circle, or more generally of any isoclinals inclined to the perpendiculars at the same angles and in the same directions of rotation, lie in a line;* a property the reader may easily prove directly for himself. See figs. Cor. 9°. Art. 62.

Cor. 5°. The preceding properties, Cor. 4°., supply obvious solutions of the three following problems:—"*On a given line or circle to determine the point or points from which if perpendiculars be let fall upon three given lines the area of the triangle determined by their feet shall be a minimum, a maximum, or given;*" or more generally of the three corresponding problems in which the perpendiculars are replaced by isoclinals inclined to them in either direction at any given angle of rotation.

65. *Every line passing through any vertex of a triangle divides the opposite side into segments in the ratio compounded of that of the conterminous sides and of that of the corresponding segments into which it divides the angle at the vertex.*



For, if ABC be the triangle, C the vertex, and CZ the line; letting fall upon CZ from the other two vertices A and B, the two perpendiculars AP and BQ, then since by similar triangles $AZ : BZ = AP : BQ$, and since by (60),

$$AP = AC.\sin ACP = AC.\sin ACZ,$$

and $BQ = BC.\sin BCQ = BC.\sin BCZ$;

therefore $AZ : BZ = AC.\sin ACZ : BC.\sin BCZ$,

that is, Euc. VI. (23) = the ratio compounded of the two ratios $AC : BC$ and $\sin ACZ : \sin BCZ$, and therefore &c.

Otherwise thus, since by triangles having the same altitude, $AZ : BZ = \text{area } ACZ : \text{area } BCZ$, and since by (64)

$$\text{area } ACZ = \tfrac{1}{2} AC.CZ.\sin ACZ,$$

and $$\text{area } BCZ = \tfrac{1}{2} BC.CZ.\sin BCZ,$$

therefore as before,

$$AZ : BZ = AC.\sin ACZ : BC.\sin BCZ,$$

and therefore &c.

Cor. 1°. If the sides AC and BC about the vertex be equal, then $AZ : BZ = \sin ACZ : \sin BCZ$, or—

Every line passing through the vertex of an isosceles triangle divides the base into segments whose ratio is the same as that of the sines of the segments into which it divides the vertical angle.

Cor. 2°. If CZ bisect the angle through whose vertex it passes either internally or externally, then, as in either case $\sin ACZ = \sin BCZ$, therefore $AZ : BZ = AC : BC$, or (Euc. VI. 3)—

The line bisecting internally or externally any angle of a triangle divides the opposite side internally or externally into segments in the ratio of the conterminous sides.

Cor. 3°. If CZ divide the angle through whose vertex it passes into segments whose sines are in the inverse ratio of the adjacent sides, that is, so that $\sin ACZ : \sin BCZ = BC : AC$, then $AZ : BZ = 1$, or—

The line dividing internally or externally any angle of a triangle into segments whose sines are in the inverse ratio of the adjacent sides bisects internally or externally the opposite side.

Cor. 4°. If CZ divide the angle through whose vertex it passes into segments whose sines are in the direct ratio of the adjacent sides, that is, so that $\sin ACZ : \sin BCD = AC : BC$, then $AZ : BZ = AC^2 : BC^2$, or—

The line dividing internally or externally any angle of a triangle into segments whose sines are in the direct ratio of the adjacent sides divides internally or externally the opposite side into segments in the duplicate ratio of the conterminous sides.

COR. 5°. As each angle of a triangle is divided externally into segments similar in form to the other two angles both by the parallel through its vertex to the opposite side (Euc. I. 32), and by the tangent at its vertex to the circumscribing circle (Euc. III. 32), the sines of the segments are therefore by (63), inversely in the former case and directly in the latter, in the ratio of the adjacent sides, and therefore, by Cors. 3° and 4° above—

Each side of a triangle is bisected externally by the parallel to it through the opposite vertex, and divided externally into segments in the duplicate ratio of the conterminous sides by the tangent to the circumscribing circle at the opposite vertex.

COR. 6°. Of the many methods of effecting the very useful division "*to divide a given angle internally or externally into two parts whose sines shall have a given ratio*," the following based on the above is perhaps on the whole the most convenient.

Connecting any two points A and B taken arbitrarily one on each side of the given angle ACB, (see figures) and cutting the connecting line AB (Euc. VI. 10), internally or externally as the case may be, in the ratio compounded of the known ratio of $AC : BC$ and of the given ratio of the required segments, the line CZ connecting the point of section Z with the vertex of the angle C divides by the above the angle as required.

The two points A and B being both arbitrary, they might be taken so that $AC = BC$, in which case Z would be simply the point of section, internal or external, of AB in the given ratio of the sine ACZ : sin BCZ (Cor. 1° above), or they might be taken so that $AC : BC$ in the inverse of the given ratio of sin ACZ : sin BCZ, in which case Z would be simply the point of bisection, internal or external, of AB (Cor. 3° above).

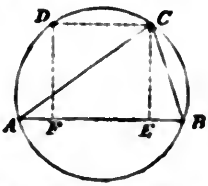
66. *The difference of the squares of the sines of any two angles is equal to the product of the sines of the sum and of the difference of the angles.*

The product of the sines of any two angles is equal to the *difference of the squares of the sines of half the sum and of half the difference of the angles.*

The reader familiar with the Second Book of Euclid will at once perceive that these are not two different propositions, but only two different modes of stating the same general property respecting the equal and unequal divisions of an angle; nor can he fail to observe at the same time the complete analogy between the common property they express, and the general property respecting the equal and unequal divisions of a line contained in propositions 5 and 6 of that Book.

On account of their importance, however, we shall give separate and independent demonstrations of each.

To prove the first. Constructing a triangle ABC, two of whose angles A and B are equal to the two angles, and through the third vertex C drawing the chord CD of its circumscribing circle parallel to the opposite side AB; then since $AD = BC$ (Euc. III. 30) and therefore $AC \sim BC = CD$ the four chords



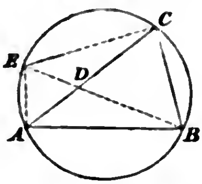
AC, BC, AB, CD divided each by the diameter of the circle are respectively (62) the sines of the four angles B, A, $B + A$, $B \sim A$, and to prove the theorem it remains only to shew that $AC^2 \sim BC^2 = AB \, . \, CD$.

From C and D letting fall CE and DF perpendiculars on AB, then (Euc. I. 47),

$$AC^2 \sim BC^2 = AE^2 \sim BE^2 = (AE + BE) \, . \, (AE \sim BE)$$
$$= AB \, . \, EF = AB \, . \, CD,$$

and therefore &c.

To prove the second. Constructing as before a triangle ABC, two of whose angles A and B are equal to the two angles, measuring from its third vertex C on either of the opposite sides CA a length CD equal to the other CB, joining BD meeting the circumscribing circle of the triangle at E, and drawing AE and CE; then,



the angles CBE and ABE being respectively half the sum and half the difference of the angles B and A, the four chords AC, BC, CE, AE divided each by the diameter of the circle are respectively (62) the sines of the four angles B, A, $\frac{1}{2}(B+A)$, $\frac{1}{2}(B \sim A)$, and to prove the theorem it remains only to shew that $AC \cdot BC = CE^2 \sim AE^2$.

The triangle BCD being isosceles by construction, so is the triangle AED which is similar to it (Euc. III. 21), therefore, (Euc. II. 5, 6, Cor.), $EC^2 \sim EA^2 = CA \cdot CD = CA \cdot CB$.

Cor. 1°. The preceding furnish obvious solutions of the two following problems:

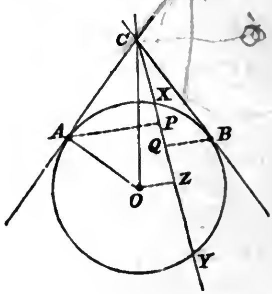
1°. *To divide a given angle, internally or externally, so that the difference of the squares of the sines of the segments shall be given.*

2°. *To divide a given angle, internally or externally, so that the product of the sines of the segments shall be given.*

Cor. 2°. The following deduction from the above furnishes a convenient mode of representation, as well as a very definite conception, of the law according to which the product of the sines of the segments of an angle varies with the change of position of its line of section.

If a circle of any radius be inscribed in an angle the product of the sines of the segments into which the angle is divided by a variable line passing through its vertex varies as the square of the segment of the line intercepted by the circle.

Let O be the centre of the circle, A and B its points of contact with the sides of the angle, and XY the line passing through C; then letting fall OZ perpendicular from O on XY, we have by the above



$$\sin ACZ \cdot \sin BCZ = \sin^2 OCA \sim \sin^2 OCZ$$
$$= (OA^2 \sim OZ^2) \div OC^2$$
$$= (OX^2 \sim OZ^2) \div OC^2$$
$$= XZ^2 \div OC^2 = XY^2 \div 4OC^2:$$

therefore $\propto XZ^2$ or XY^2. Q.E.D.

When the variable line of section in the course of its revolution round C enters the supplemental region of the angle ACB, the circle AOB is of course no longer available for the above representation; but then it may be replaced by another $A'O'B'$ inscribed in the supplemental region, and the new circle will continue to represent the law of the variation on the same scale as before, provided only the distance CO' of its centre from the vertex of the angle is equal to the distance CO of the centre of the original circle from the same.

Cor. 3°. Letting fall AP and BQ perpendiculars from A and B on XY; then, since $AP.BQ \div AC.BC = \sin ACZ.\sin BCZ$, therefore, from Cor. 2°., $AP.BQ \div XZ^2 = AC.BC \div OC^2$, or

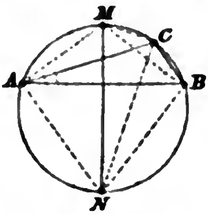
$$4\,AP.BQ \div XY^2 = AC.BC \div OC^2,$$

a property of the circle which may be easily proved directly.

67. *The sum of the sines of any two angles is equal to twice the product of the sines of half the sum and of the complement of half the difference of the angles.*

The difference of the sines of any two angles is equal to twice the product of the sines of half the difference and of the complement of half the sum of the angles.

Constructing, as in the properties of the preceding article, a triangle ABC, two of whose angles A and B are the two angles, bisecting internally or externally the arc ACB of the circumscribing circle at M and N respectively, and connecting both points of bisection with A, B, and C; then the angles MNA, or MNB, and MNC being respectively half the sum and half the difference of the angles CNA and CNB, that is, of the angles B and A, the four chords CA, CB, MA, or MB, and MC divided each by the diameter of the circle are respectively the sines of the four angles B, A, $\frac{1}{2}(A+B)$ and $\frac{1}{2}(A \sim B)$, and the two chords NA, or NB, and NC divided each by the diameter are the sines of the complements of $\frac{1}{2}(A+B)$ and $\frac{1}{2}(A \sim B)$; and to prove the theorems it remains only to shew



that, $\quad (CA + CB) : CN :: (MA + MB) : MN,$

and that $\quad (CA \sim CB) : CM :: (NA + NB) : NM.$

From the two inscribed quadrilaterals $MNCA$ and $MNCB$, since by Ptolemy's theorem,

$$CA.MN = CN.MA \pm CM.NA$$

and

$$CB.MN = CN.MB \mp CM.NB,$$

therefore by addition and subtraction

$$(CA + CB) . MN = CN . (MA + MB)$$

and $$(CA \sim CB) . MN = CM . (NA + NB),$$

and therefore &c.

Or, more directly, from the two inscribed quadrilaterals $ABCN$ and $ABCM$, since by the same theorem

$$CA . NB + CB . NA = CN . AB,$$

and $$CA . MB \sim CB . MA = CM . AB,$$

therefore at once

$$(CA + CB) : CN = AB : (AN \text{ or } BN) = (MA + MB) : MN,$$

and $$(CA \sim CB) : CM = AB : (AM \text{ or } BM) = NA + NB : NM,$$

and therefore &c.

Cor. The preceding supply evident solutions of the four problems:

To divide a given angle, internally or externally, into two parts whose sines shall have a given sum or difference.

And the proportions on which they depend of the two problems.

Given of a triangle (ACB) the base, the vertical angle, and the sum or the difference of the sides, to construct it.

CHAPTER V.

ON THE CONVENTION OF POSITIVE AND NEGATIVE IN GEOMETRY.

68. The most striking characteristic of modern as contrasted with ancient geometry is comprehensiveness of language and demonstration. General enunciations on the one hand, and general demonstrations on the other, comprehending in the geometry of the present day all the different cases of the various properties considered, arising from variations in number, position, or magnitude, among the elements of the figures involved, which in the geometry of former days would have been regarded as so many distinct propositions, requiring each a separate statement and independent proof of its own. All such enunciations and demonstrations, moreover, unencumbered, in consequence of this very character of comprehensiveness and generality, with the accidental peculiarities and unessential details of particular cases, and involving accordingly the essential elements of abstract principles only, being thus the more readily apprehended, easily remembered, and instructively suggestive, in proportion as they are comprehensive and general. These important and characteristic advantages are mainly due to the employment, now universally recognised by geometers, of the algebraic signs + and − to indicate the directions in which the various magnitudes coming under their consideration are measured, with regard to which they have laid down the following general rule of convention.

In every case of the comparison of magnitudes susceptible of measurement in either of two opposite directions the signs + and − are employed to distinguish between the directions.

Segments measured on the same line, arcs measured on the same circle, angles measured round the same vertex, triangles or parallelograms described on the same base, perpendiculars or any other isoclinals erected to or let fall upon the same line, are obvious examples of different kinds of geometrical magnitudes coming under the above head; every two of each of which, when considered together in any number, are therefore to be regarded as having similar or opposite signs according as the directions in which they are measured are similar or opposite.

69. In every application of the above principle of convention it is optional which of the two opposites is to be regarded as the positive and which the negative direction, but the selection once made, and either sign given to either direction in any case, the same sign must be given to the same direction and the opposite sign to the opposite direction throughout the entire case. In the comparison of magnitudes whose directions of measurement are not either similar or opposite, such as segments on different lines, triangles or parallelograms on different bases, perpendiculars or isoclinals to different lines, not parallel to each other, the selection for each separate direction and its opposite is also optional; but once made for each in any case must invariably be adhered to throughout the entire case.

It is this distinctive principle of modern as contrasted with ancient geometry, this recognition of magnitudes as having not only absolute or numerical value but also sign determined by application of the above general rule of convention, which has mainly tended to render the language and demonstrations of the former independent of all accidental variations among the component elements of the figures to which they refer.

70. In accordance with the preceding principle the familiar terms "sum" and "difference" are employed in the geometry of the present day with an important modification of their accustomed significations as employed in the geometry of former times, and to the present day in arithmetic, which must be carefully attended to in order to an accurate, and in many cases even an intelligible conception of the true meaning intended to be conveyed by their use, which is as follows:

The term "sum" as employed in arithmetic is used to denote the result of adding together the numerical values of any number of magnitudes taken absolutely without any regard to their signs, so that there it is always a positive quantity; in geometry, on the contrary, it is applied to the same result with this difference that the signs of the several magnitudes are taken into account in the addition; so that the geometric sum of any number of magnitudes really means the arithmetic sum of all that are positive among them minus the arithmetic sum of all that are negative, and this is what is uniformly meant by the term "sum" as now invariably employed in geometry unless the contrary be expressly stated.

It thus appears that the sum of any number of geometrical magnitudes is to be regarded as positive, negative, or nothing, according as the aggregate of the positive individuals or terms composing it happens to exceed, fall short of, or equal, that of the negative.

All that has been said in the above remarks applies equally to the term "difference" as employed in the geometry of the present day in reference to two magnitudes. It denotes in arithmetic the result of subtracting one from the other attending only to their absolute values, and in geometry the same result taking into account also their signs; thus the geometrical difference of two magnitudes may be their arithmetic sum, and conversely.

71. Similar remarks apply to the terms "product" and "quotient" as employed in the geometry of the present day, compared with their known significations as employed in arithmetic; in the latter, as in the cases of "sum" and "difference," the absolute values of the magnitudes only being taken into account, while in the former their signs also are attended to. Hence, since in the multiplication or division of any two quantities like signs produce always a positive and unlike signs a negative result, the product or quotient of any two geometrical magnitudes is to be regarded as positive or negative according as they have similar or opposite signs; and so, more generally, is the product of any number of magnitudes according as there happens to be an even or an odd number of negative signs amongst them.

The rectangle under any two lines being the same as their product, and the ratio of any two lines the same as their quotient: it follows from the above that the rectangle and the ratio of any two lines have always the same sign, and are positive or negative together according as the lines themselves have similar or opposite signs. The square of every real line for the same reason is always positive, whether the line itself be positive or negative.

72. The terms "Arithmetic Mean" and "Geometric Mean," as employed in the geometry of the present day in reference to any number of magnitudes, ought for uniformity sake to bear the same relation to their "Arithmetic Sum" and "Geometric Sum" respectively. Such however is not the case, those terms having been employed to denote two entirely different things long before the consideration of signs had been forced on the attention of geometers, the former to denote *the* n^{th} *part of the sum*, and the latter to denote *the* n^{th} *root of the product* of any n magnitudes. In the same acceptations they are still employed, only with this difference, that in estimating the sum or product the signs as well as the absolute values of the several magnitudes are taken into account.

In geometry, therefore, the terms "Arithmetic Mean" and "Geometric Mean," in reference to any number of magnitudes, denote respectively *the* n^{th} *part of their geometric sum* and *the* n^{th} *root of their geometric product*, n being the number of the magnitudes. Hence, n being necessarily a positive integer, the former is positive or negative with the sum in every case, and the latter positive or negative with the product when n is odd, but real or imaginary and of either sign indifferently according as the product is positive or negative when n is even.

N.B. The term "Arithmetic Mean" is employed in geometry in the same sense as the term "mean" or "average" is employed in ordinary language.

73. Since by the evident *law of continuity*, as it is termed in geometry, a magnitude of any kind which varies *continuously* according to any law cannot possibly pass either in increase or decrease from any one value to any other *without passing through every intermediate value on the way*. It might appear at first sight as if a variable magnitude at the point of transition from positive to negative, or conversely, should necessarily pass *always* through the particular value 0. Such however

is not the case. Magnitudes susceptible of indefinite increase, as for instance the distance of a variable from a fixed point on a line, passing as often through ∞ as through 0 in changing sign.

To see this, if indeed it be not evident of itself from the example adduced, we have but to conceive two *reciprocal* magnitudes of any kind (8) to vary continuously, and either of them to change sign by passing through 0; for since the product of two such magnitudes is, from the nature of their connection, invariable both in magnitude and sign, every change of sign in either is necessarily accompanied by a simultaneous change of sign in the other, and every passage of either through 0 or ∞ by the simultaneous passage of the other through ∞ or 0, and therefore &c.

On the other hand, however, magnitudes unsusceptible of indefinite increase, and oscillating therefore as they vary between their extreme maxima and minima values (59), if they change sign at all, do so only by passing through 0 at each point of transition; thus for instance, the sine of an angle regarded as a magnitude, whose absolute value can never exceed 1 (60), changes sign only by passing through 0, its value whenever the angle itself in continuous increase or decrease $= \pm 2n$ right angles, n being any integer of the natural series 0, 1, 2, 3, 4, 5, 6, &c. to infinity.

74. In every application of the principle of signs, some method of notation which would indicate the directions, as well as represent the magnitudes, of the quantities considered would be of manifest convenience, and should as far as possible be systematically adhered to; the biliteral notation (4) which represents a magnitude of any kind by means of the two letters representing its extremities, whenever otherwise convenient, effects this purpose in as simple and expressive a manner as could be desired, by merely the order (4) in which the two letters are written.

Thus, a geometrical magnitude of any kind whose extremities are A and B is to be considered as measured, if represented by AB in the direction from A to B, and if by BA in the opposite direction from B to A. So that in accordance with the convention of signs, AB is always to be regarded as $= - BA$, or which is the same thing $AB + BA = 0$, whatever be the nature of A and B and of the magnitude intercepted between them (3).

This premised, we proceed now to illustrate the convenience of the convention of signs by a few applications of very general utility in almost every department of pure and applied geometry.

75. *If A and B be any two points on a line, and P any third point taken arbitrarily on the same line, then whatever be the position of P with respect to A and B,*

$$AP - BP = AB,$$

regard being had to the signs as well as the magnitudes of the three intervals involved.

For, if 1°., P be external to AB at the side of B, then as AP, BP, and AB have all the same direction, and therefore the same sign, the relation is evident; if 2°. P be external to AB at the side of A, then, as by case 1°., $BP - AP = BA$, and as by the convention of signs $BA = -AB$, therefore &c. And if 3°., P be internal to AB, then as evidently $AP + PB = AB$, and as by the convention of signs $PB = -BP$, therefore &c.

A point P thus taken arbitrarily upon a line AB is said to divide that line, externally or internally according to its position, into two segments AP and BP, which, whether both measured from the extremities of the line to the point of section or from the point of section to the extremities of the line, have evidently similar or opposite directions, and therefore similar or opposite signs, according as the point of section is external or internal to the line. Hence the above relation expresses the general property that, *when a line AB is cut, externally or internally, at any point P, the geometrical difference* (70) *of the segments into which it is divided is constant and equal to the length of the line.*

The segments of a line AB divided at any point P having similar or opposite directions, and therefore similar or opposite signs, according as the point of section is external or internal to the line, *their rectangle and ratio are therefore both positive in the former case and negative in the latter.*

Hence, the problems "to divide a given line into segments, having a given rectangle or ratio," which would be ambiguous were the absolute magnitude of the rectangle or ratio alone given, becomes completely determinate when the sign also is given with it.

Cor. 1°. *If a line AB be cut, externally or internally, at any point P, then whatever be the position of P with respect to A and B,*

$$AP^2 + BP^2 = AB^2 + 2AP.BP;$$

regard being had to the signs as well as the magnitudes of the two segments AP and BP.

For, since by the above $AP - BP = AB$, whatever be the position of P, therefore $AP^2 + BP^2 - 2AP.BP = AB^2$, and therefore &c.

This relation being true for every position of P includes therefore the two properties (Euc. II. 7 and 4), the rectangle $AP.BP$ being positive or negative according as P is external or internal to AB.

Cor. 2°. *If from any point P a perpendicular PQ be let fall upon a line AB, then whatever be the position of P with respect to A and B,*

$$AP^2 - BP^2 = AB^2 + 2AB.BQ,$$

regard being had to the signs as well as the magnitudes of AB and BQ.

For by (Euc. I. 47, Cor.) $AP^2 - BP^2 = AQ^2 - BQ^2$, and by the preceding Cor. 1°., $AQ^2 = AB^2 + BQ^2 + 2AB.BQ$, therefore &c.

This relation being true for every position of P includes therefore the two properties (Euc. II. 12 and 13), the rectangle $AB.BQ$ being positive or negative according as the angle PBA is obtuse or acute.

76. *If A and B be any two points on a line, C the point on the line for which $AC + BC = 0$, and P any other point on the line, then whatever be the position of P,*

$$AP + BP = 2.CP \qquad (1),$$

$$AP.BP = AC.BC + CP^2 \qquad (2),$$

$$AP^2 + BP^2 = AC^2 + BC^2 + 2CP^2 \qquad (3),$$

$$AP^2 - BP^2 = 2AB.CP \qquad (4),$$

regard being had to the signs as well as the magnitudes of the several segments involved.

For, taking the sum, product, sum of squares, and difference of squares of the relations,

$$AP = AC + CP, \text{ and } BP = BC + CP,$$

which by the preceding (75) are true, whatever be the position of C, and remembering that by hypothesis $AC + BC = 0$, and that always $AC - BC = AB$, the above relations are the immediate results.

The point C on the line AB for which as above $AC + BC = 0$ being evidently the point of internal bisection of the line; the second of the above relations includes therefore the two properties (Euc. II. 5 and 6), and the third the two (Euc. II. 9 and 10), both being independent of the position of P; the first expresses that whatever be the position of P *the distance CP is the arithmetic mean of the distances AP and BP*; and the fourth, that whatever be the position of P *the difference of the squares of the distances AP and BP varies as the distance CP.* The four combined also supply obvious solutions of the four general problems: "To cut a line of given length, so that the sum, difference, sum of squares, or difference of squares of the segments, shall have a given magnitude and sign."

COR. *If AB and $A'B'$ be any two segments on the same line, C and C' their two middle points, then always*

$$CC' = \frac{AA' + BB'}{2} \text{ or } \frac{AB' + BA'}{2},$$

regard being had to the signs as well as the magnitudes of the several segments involved.

For, since for any arbitrary point P on the line, by the first of the above relations

$$2.CP = AP + BP \text{ and } 2.C'P = A'P + B'P,$$

therefore by subtraction

$$2(CP - C'P) = (AP - A'P) + (BP - B'P) \text{ or } (AP - B'P) + (BP - A'P),$$

and therefore as above (see 75)

$$2CC' = AA' + BB' \text{ or } AB' + BA'. \quad \text{Q.E.D.}$$

77. *If A and B be any two points on a line, a and b any two numbers positive or negative whose sum is not* $=0$, *O the point on the line for which* $a.AO+b.BO=0$, *and P any other point on the line, then, whatever be the position of P,*

$$a.AP+b.BP=(a+b).OP \quad \text{......................(1)},$$

$$a.AP^2+b.BP^2=a.AO^2+b.BO^2+(a+b).OP^2 \quad \text{...(2)},$$

regard being had to the signs as well as the magnitudes of the several quantities involved.

For, since by (75), $AP=AO+OP$ and $BP=BO+OP$, whatever be the position of O, multiplying the first by a and the second by b and adding, then multiplying the square of the first by a and the square of the second by b and adding, remembering in both cases that by hypothesis $a.AO+b.BO=0$, the above relations are the immediate result.

Given the two points A and B and the two multiples a and b, to determine the point O, for which as above $a.AO+b.BO=0$, and which is evidently internal or external to AB according as a and b have similar or opposite signs. Assuming arbitrarily any point P on the line AB, and measuring from it a length PO equal in magnitude and sign to the sum $\dfrac{a.PA+b.PB}{a+b}$, the point O by the first of the above relations is that required, and by aid of it the two relations supply obvious solutions of the two following problems: "*on a given line AB to determine the point P for which the sum* $a.AP+b.BP$ *or the sum of the squares* $a.AP^2+b.BP^2$ *shall have a given magnitude and sign.*"

In the particular case when $a+b=0$, the value of PO, on which the position of O determined as above depends, being then infinite, the point O is therefore at an infinite distance, and the above relations both fail in consequence of their right-hand members becoming both indeterminate (13). Since, however, in that case $b=-a$, the sum

$$a.AP+b.BP=a.(AP-BP)=a.AB, \ (75),$$

and therefore is *constant*; and the sum of the squares

$$a.AP^2+b.BP^2=a(AP^2-BP^2)=2a.AB.CP, \ (76),$$

C being the middle point of AB, and therefore *varies as CP*;

relations simpler than those for the general case where $a+b$ is not $=0$.

COR. *If from the three points A, B, and O, perpendiculars or any other isoclinals AL, BL, and OL be let fall upon any arbitrary line L, then, whatever be the position of L,*

$$a.AL+b.BL=(a+b).OL,$$

regard being had to the signs as well as the magnitudes of the several quantities involved.

For, in the particular case when L is parallel to AB, since then $AL=BL=OL$ the relation is evident; and in any other case if P be the point in which L intersects AB, since by similar triangles $AL:BL:OL=AP:BP:OP$, and since by the first of the above relations $a.AP+b.BP=(a+b)\,OP$, therefore &c.

78. *If A, B, C, D, E, F, &c. be any number of points on a line, situated in any manner with respect to each other, then, whatever be their order and disposition—*

For every three of them A, B, C,

$$AB+BC+CA=0.$$

For every four of them A, B, C, D,

$$AB+BC+CD+DA=0.$$

For every five of them A, B, C, D, E,

$$AB+BC+CD+DE+EA=0;$$

and so on for any number, the last being always connected with the first in completing the circuit, and the signs as well as the magnitudes of the several intercepted segments being always taken in account in the summation.

For, since by (75),

$$AB+BC=AC,\quad AC+CD=AD,$$

$$AD+DE=AE,\quad AE+EF=AF, \text{ \&c.,}$$

therefore,

$$AB+BC+CA=AC+CA=0,$$

$$AB+BC+CD+DA=AD+DA=0,$$

$$AB+BC+CD+DE+EA=AE+EA=0, \text{ \&c.,}$$

and therefore &c. Q.E.D.

79. *If A, B, C, D, &c. be any number (n) of points on a line, disposed in any manner, O the point on the line for which* $AO + BO + CO + DO + \&c. = 0$, *and P any other point on the line, then, whatever be the position of P,*

$$AP + BP + CP + DP + \&c. = n . OP \quad\text{.......................(1)},$$

$$AP^2 + BP^2 + CP^2 + DP^2 + \&c. = AO^2 + BO^2 + CO^2 + DO^2 + \&c. + n . OP^2 \quad\text{......................(2)},$$

the signs as well as the magnitudes of the several segments being taken into account in the first.

For, taking the sum and the sum of the squares of the several relations $AP = AO + OP$, $BP = BO + OP$, $CP = CO + OP$, $DP = DO + OP$, &c., which by (75) are true whatever be the position of O, and remembering that, by hypothesis,

$$AO + BO + CO + DO + \&c. = 0,$$

the above relations are the immediate result.

The point O on the line for which, as above,

$$AO + BO + CO + DO + \&c. = 0,$$

or, as it may be more concisely written, $\Sigma(AO) = 0$, being such by relation 1, that for every other point P on the line *the distance OP is the arithmetic mean of the several distances AP, BP, CP, DP, &c.*, is termed, in consequence, *the mean centre of the system of points A, B, C, D, &c.;* and to determine its position when the latter are given, we have but to assume arbitrarily any point P on the line, and to measure from it a distance PO equal in magnitude and sign to the n^{th} part of the sum of the distances PA, PB, PC, PD, &c., or, as it may be more concisely written, $= \frac{\Sigma(PA)}{n}$; the point O, by relation 1, is that required, and by its aid the two relations 1 and 2 supply obvious solutions of the two general problems: "*Given any number of points A, B, C, D, &c. on a line, to determine the point P on the line for which the sum* $\Sigma(AP)$ *or the sum of the squares* $\Sigma(AP^2)$ *shall be given.*"

COR. 1°. *If at the mean centre O a perpendicular OS be erected to the line whose square* $OS^2 =$ *the* n^{th} *part of the sum of the squares* $\Sigma(AO^2)$, *then for any point P on the line the sum of the squares* $\Sigma(AP^2) = n . SP^2$.

For, since by relation 2, $\Sigma(AP^2) = \Sigma(AO^2) + n.OP^2$, and since by construction, $\Sigma(AO^2) = n.OS^2$, therefore

$$\Sigma(AP^2) = n(OS^2 + OP^2) = n.SP^2.$$

Hence the variable sum $\Sigma(AP^2)$ has equal values for every two points on the line equidistant from O, and the minimum value for the point O itself.

Cor. 2°. Since when, as in relation 1, $\Sigma(AP) = n.OP$, then, as in relation 2,

$$\Sigma(AP^2) = n.OP^2 + \Sigma(AO^2) = n.OP^2 + \Sigma(AP - OP)^2,$$

it follows consequently that—

When the same sum, $\Sigma(AP)$, is cut into any number n of unequal parts AP, BP, CP, DP, &c., and also into the same number n of equal parts OP, OP, OP, OP, &c., the sum of the squares of the n unequal parts $\Sigma(AP^2)$ is equal to the sum of the squares of the n equal parts $n.OP^2$ + the sum of the squares of the n differences $\Sigma(AP - OP)^2$.

Cor. 3°. *If A, B, C, D, &c. and A', B', C', D', &c. be two systems of any common number of points on the same line, O and O' their mean centres, and n their common number of points, then*

$$OO' = \frac{AA' + BB' + CC' + DD' + \&c.}{n}$$

any mode of correspondence between the points of the systems in pairs being adopted in the summation.

For since, for any arbitrary point P on the line, by relation 1,

$$n.OP = AP + BP + CP + DP + \&c.,$$

and $$n.O'P = A'P + B'P + C'P + D'P + \&c.$$

therefore

$$n.(OP - O'P) = (AP - A'P) + (BP - B'P) + (CP - C'P) + (DP - D'P) + \&c.,$$

or (75) $$n.OO' = AA' + BB' + CC' + DD' + \&c.,$$

and therefore &c.

Cor. 4°. *If A, B, C, D, &c. and A', B', C', D', &c. be any two systems of points on the same line, O and O' their mean centres, and n and n' their numbers of points, then*

$$OO' = \frac{\Sigma(AA')}{nn'},$$

every point of one system being combined in the summation with every point of the other.

For, adding together the several relations,

$$AA' + BA' + CA' + DA' + \&c. = n.OA',$$
$$AB' + BB' + CB' + DB' + \&c. = n.OB',$$
$$AC' + BC' + CC' + DC' + \&c. = n.OC',$$
$$AD' + BD' + CD' + DD' + \&c. = n.OD', \&c.$$

there results at once the relation

$$\Sigma(AA') = n.(OA' + OB' + OC' + OD' + \&c.) = n.\Sigma(OA') = nn'.OO',$$

and therefore &c.

80. *If A, B, C, D, &c. be any system of points on a line, disposed in any manner, a, b, c, d, &c. any system of corresponding multiples, positive or negative, whose sum is not* = 0, *O the point on the line for which*

$$a.AO + b.BO + c.CO + d.DO + \&c. = 0,$$

and P any other point on the line, then, whatever be the position of P,

$$a.AP + b.BP + c.CP + d.DP + \&c. = (a + b + c + d + \&c.).OP \ldots (1),$$

$$a.AP^2 + b.BP^2 + c.CP^2 + d.DP^2 + \&c.$$
$$= a.AO^2 + b.BO^2 + c.CO^2 + d.DO^2 + \&c.$$
$$+ (a + b + c + d + \&c.).OP^2 \ldots\ldots\ldots\ldots (2),$$

regard being had to the signs as well as the magnitudes of the several quantities involved.

For, multiplying the several relations $AP = AO + OP$, $BP = BO + OP$, $CP = CO + OP$, $DP = DO + OP$, &c., which, by (75), are true whatever be the position of O, and also their squares, by the several corresponding multiples a, b, c, d, &c., and adding, remembering in both cases that by hypothesis $\Sigma(a.AO) = 0$, the above relations are the immediate result.

The point O on the line, for which as above $\Sigma(a.AO) = 0$, is termed, in virtue of relation 1, *the mean centre of the system of points A, B, C, D, &c. for the system of multiples a, b, c, d, &c.;* and to determine its position when the several points and multiples are given, we have but to assume arbitrarily any point P on the line, and to measure from it a length PO equal in magnitude and sign to $\frac{\Sigma(a.PA)}{\Sigma(a)}$, the point O, by relation 1, is that required, and by its aid the two relations 1 and 2 supply obvious solutions of the two general problems: "*Given any number of points on a line A, B, C, D, &c., and the same number of corresponding multiples a, b, c, d, &c. whose sum is not* $=0$. *To determine the point P on the line for which the sum* $\Sigma(a.AP)$, *or the sum of the squares* $\Sigma(a.AP^2)$ *shall be given.*"

In the particular case when $\Sigma(a)=0$, the value of PO, as given by the above formula, being then infinite, the point O is therefore at an infinite distance, and the relations 1 and 2 both fail in consequence of their right-hand members becoming both indeterminate (13). This case, the laws of which, though simpler, differ altogether from those of the general case when $\Sigma(a)$ is not $=0$, will be considered separately in the next section.

COR. 1°. *If round the mean centre O as centre and with a radius OP whose square equal to the absolute value of* $\frac{\Sigma(a.AO^2)}{\Sigma(a)}$, *disregarding its sign, a circle be described intersecting the line at the points M and N, and the perpendicular to it through O in either direction at the point S, then for any point P on the line the sum of squares* $\Sigma(a.AP^2)=\Sigma(a).SP^2$ *or* $\Sigma(a).MP.NP$, *according as* $\Sigma(a)$ *and* $\Sigma(a.AO^2)$ *have similar or opposite signs.*

For since, by relation 2, $\Sigma(a.AP^2)=\Sigma(a).OP^2+\Sigma(a.AO^2)$, and since by construction $\Sigma(a.AO^2)=\pm\Sigma(a).OR^2$, therefore

$$\Sigma(a.AP^2)=\Sigma(a).(OP^2\pm OR^2)=\Sigma(a).SP^2 \text{ or } \Sigma(a).MP.NP.$$

Hence, in both cases, the variable sum $\Sigma(a.AP^2)$ has equal values for every two points on the line equidistant from O, and the minimum value for the point O itself; it being remembered however that as it vanishes in the second case at the two points M and N, and increases negatively from each up to O, the term *minimum* is to be understood in the sense of *negative maximum* in that case, see (58).

COR. 2°. *If a system of any number of points on a line A, B, C, D, &c., and their mean centre O for any system of multiples a, b, c, d, &c., be projected by perpendiculars or any other parallels AA′, BB′, CC′, DD′, &c., and OO′ upon any arbitrary line L, then, whatever be the position of L.*

a. The projection O' of the mean centre is the mean centre of the projections A', B', C', D', &c. of the points for the same system of multiples.

b. The projector OO' of the mean centre, is the mean of the projectors AA', BB', CC', DD', &c. of the points for the same system of multiples.

For, as in Cor. 1 (77), for the case of two points. If L be parallel to the line of the points, both properties are evident; and in any other position, if P be the point in which the two lines intersect, since by similar triangles,

$$AP : BP : CP : DP, \&c. : OP = A'P : B'P : C'P : D'P, \&c. : O'P$$
$$= AA' : BB' : CC' : DD', \&c. : OO',$$

and since by relation 1,

$$\Sigma(a.AP) = \Sigma(a).OP; \text{ therefore } \Sigma(a.A'P) = \Sigma(a).O'P,$$

and $\Sigma(a.AA') = \Sigma(a).OO'$, and therefore &c.

Cor. 3°. *If A, B, C, D, &c. and A', B', C', D', &c. be two systems of any common number of points on the same line, O and O' their mean centres for any common system of multiples a, b, c, d, &c. then*

$$OO' = \frac{a.AA' + b.BB' + c.CC' + d.DD' + \&c.}{a + b + c + d + \&c.},$$

pairs of points having common multiples being combined in the summation.

For, since for any arbitrary point P on the line, by relation 1,

$$\Sigma(a).OP = a.AP + b.BP + c.CP + d.DP + \&c.,$$

and $\Sigma(a).O'P = a.A'P + b.B'P + c.C'P + d.D'P + \&c.$,

therefore

$$\Sigma(a).(OP - O'P) = a.(AP - A'P) + b.(BP - B'P)$$
$$+ c.(CP - C'P) + a.(DP - D'P) + \&c.$$

and therefore as above

$$\Sigma(a).OO' = a.AA' + b.BB' + c.CC' + d.DD' + \&c.$$

Cor. 4°. *If A, B, C, D, &c. and A', B', C', D', &c. be any two systems of points on the same line, O and O' their mean centres for any two systems of multiples a, b, c, d, &c., and a', b', c', d', &c., then*

$$OO' = \frac{\Sigma(aa'.AA')}{\Sigma(a).\Sigma(a')},$$

every point of one system being combined in the summation with every point of the other.

For, adding together the several relations

$$a.AA' + b.BA' + c.CA' + d.DA' + \&c. = \Sigma(a).OA',$$
$$a.AB' + b.BB' + c.CB' + d.DB' + \&c. = \Sigma(a).OB',$$
$$a.AC' + b.BC' + c.CC' + d.DC' + \&c. = \Sigma(a).OC',$$
$$a.AD' + b.BD' + c.CD' + d.DD' + \&c. = \Sigma(a).OD', \&c.$$

multiplied respectively by a', b', c', d', &c. there results immediately the relation

$$\Sigma(aa'.AA') = \Sigma(a).\Sigma(a'.OA') = \Sigma(a).\Sigma(a').OO',$$

and therefore &c.

81. *If A, B, C, D, &c. be any system of points on a line disposed in any manner, a, b, c, d, &c. any system of corresponding multiples, some positive and some negative, whose sum $= 0$, then for every point P on the line not at infinity the sum $\Sigma(a.AP)$ has the same constant value.*

In the same case, if I be the point on the line for which the sum $\Sigma(a.AI^2) = 0$, then for every other point P on the line the sum $\Sigma(a.AP^2) = 2k.IP$, k being the constant value of the sum $\Sigma(a.AP)$ for every point on the line.

To prove the first,—since for any two points P and Q on the line by (75),

$$AP - AQ = QP,\ BP - BQ = QP,\ CP - CQ = QP,\ DP - DQ = QP, \&c.$$

therefore, multiplying by a, b, c, d, &c. and adding,

$$\Sigma(a.AP) - \Sigma(a.AQ) = \Sigma(a).QP = 0,$$

when $\Sigma(a) = 0$, whatever be the positions of P and Q, provided neither of them be at infinity, and therefore &c.

To prove the second,—since for any two points P and Q on the line by (76 (4)),

$$AP^2 - AQ^2 = 2AR.QP, \quad BP^2 - BQ^2 = 2BR.QP,$$
$$CP^2 - CQ^2 = 2CR.PQ, \quad DP^2 - DQ^2 = 2DR.PQ, \&c.$$

R being the middle point of PQ; therefore, multiplying by a, b, c, d, &c. and adding,

$$\Sigma(a.AP^2) - \Sigma(a.AQ^2) = 2\Sigma(a.AR).QP = 2k.QP,$$

when $\Sigma(a) = 0$, whatever be the positions of P and Q; and therefore when either of them Q is the particular point I for which $\Sigma(a.AI^2) = 0$, then for the other P, whatever be its position, $\Sigma(a.AP^2) = 2k.IP$, as above stated.

From the above relations it appears that, while the sum $\Sigma(a.AP)$ is invariable, the sum $\Sigma(a.AP^2)$ follows a very simple law of variation when $\Sigma(a) = 0$, being simply proportional to the distance of P from a certain point I on the line, admitting therefore of no maximum or minimum value, but susceptible of every value positive and negative from 0 to ∞, passing through infinity as P passes through infinity, and through nothing as P passes through I, and changing from positive to negative, and conversely, at the passage through each.

To determine the point I, when the several points A, B, C, D, &c. and the several multiples a, b, c, d, &c. are given; assuming arbitrarily any point P on the line, and measuring from it a length PI equal in magnitude and sign to

$$\frac{\Sigma(a.PA^2)}{-2k} = \frac{\Sigma(a.PA^2)}{2.\Sigma(a.PA)},$$

the point I, by relation 2, is that required, and by its aid the same relation supplies an obvious solution of the more general problem, "*to determine the point P on the line for which the $\Sigma(a.PA^2)$ shall have any given magnitude and sign.*"

In the particular case where the constant $k = 0$, the value of PI, as given by the above formula, being then infinite, the point I is therefore at an infinite distance, and the relation $\Sigma(a.AP^2) = 2k.IP$ fails in consequence of its right-hand member becoming indeterminate (13). In that case however it is easy to see that, as it ought, *the sum $\Sigma(a.AP^2)$ has the same constant value for every point on the line not at infinity.*

For since for every two points P and Q on the line, as above shown, $\Sigma(a.AP^2) - \Sigma(a.AQ^2) = 2k.QP$, whatever be the value of k, therefore when $k = 0$, $\Sigma(a.AP^2) = \Sigma(a.AQ^2)$, whatever be the positions of P and Q, provided neither of them be at infinity, and therefore &c.

Among the various ways in which the constant k may be represented in the form of a single quantity, when the several points A, B, C, D, &c. and the several multiples a, b, c, d, &c. are given, the following is perhaps the most convenient.

Conceiving the entire system of points $\Sigma(A)$ *divided into two distinct groups, one* $\Sigma(A_+)$ *corresponding to the positive, and the other* $\Sigma(A_-)$ *to the negative multiples. If* O_+ *and* O_- *be the mean centres of the two groups for their respective systems of multiples* $\Sigma(a_+)$ *and* $\Sigma(a_-)$, *the constant sum*

$$\Sigma(a.AP) = \Sigma(a_+).O_+O_-, \text{ or } = \Sigma(a_-).O_-O_+.$$

For, $\Sigma(a.AP) = \Sigma(a_+.A_+P) + \Sigma(a_-.A_-P) = \Sigma(a_+).O_+P$
$+ \Sigma(a_-).O_-P$, by (80),

but $\Sigma(a_+) + \Sigma(a_-) = \Sigma(a) = 0$, by hypothesis,

therefore $\Sigma(a.AP) = \Sigma(a_+).(O_+P - O_-P)$,

$$\text{or} = \Sigma(a_-).(O_-P - O_+P) = \Sigma(a_+).O_+O_-, \text{ or } = \Sigma(a_-).O_-O_+.$$

Hence when the two points O_+ *and* O_- *coincide, the constant* $k=0$ *at all points of the line.*

In every case where the constant $k=0$, *the position of the mean centre* O *of the entire system of points* $\Sigma(A)$ *for the entire system of multiples* $\Sigma(a)$ *is indeterminate.* The relation $\Sigma(a.AO)=0$, by which that point, in general unique, is characterized (80), being then satisfied indifferently by *every* point on the line. An example of this for the case of three points will be given in the next number.

Hence, generally, the position of the mean centre O of any system of points A, B, C, D, &c. on a line for any system of multiples a, b, c, d, &c. whose sum $=0$, is either indeterminate or impossible *at any finite distance*, indeterminate if the value of the constant $k=0$, impossible if not.

82. *If* A, B, C, D *be four points on a line disposed in any manner, then always, none of the four being at infinity,*

$$BC.AD + CA.BD + AB.CD = 0,$$

regard being had to the signs as well as the magnitudes of the six segments involved.

For since whatever be the positions of the four points (75),

$$AD - CD = AC, \text{ and } BD - CD = BC,$$

therefore, multiplying the first by BC and the second by AC, and subtracting

$$BC.AD + CA.BD + (AC - BC)\,CD = 0,$$

the same as above, $AC - BC$ being always $=AB$ (75).

Hence, (see preceding article), *the mean centre O of three points A, B, C on a line for three multiples a, b, c, proportional in magnitude and sign to the three intervals BC, CA, AB is indeterminate.* Every point P on the line in virtue of the above relation, satisfying indifferently the characteristic condition,

$$a.AP + b.BP + c.CP = 0.$$

As four points on a line A, B, C, D, however disposed, determine in every case six different segments corresponding to each other two and two in three different sets of opposite pairs BC and AD, CA and BD, AB and CD, the above is the general relation connecting those six segments in all cases, regard being had to their signs as well as their magnitudes, and interpreted absolutely, disregarding signs, it expresses evidently the general property that—

Whatever be the disposition of four points on a line the rectangle under one pair of opposites of the six segments they determine is numerically equal to the sum of the rectangles under the other two pairs.

If the four points in the order of their disposition be denoted by 1, 2, 3, 4 respectively, it is easy to see that in all cases the rectangle $\overline{13}.\overline{24}$ is the one that is equal to the sum of the other two $\overline{12}.\overline{34}$ and $\overline{23}.\overline{14}$; for denoting by x, y, z the absolute intervals from 1 to 2, 2 to 3, 3 to 4, respectively, disregarding their signs, the relation

$$(x+y)(y+z) = xz + y(x+y+z),$$

is evidently in all cases identically true.

Cor. 1°. *If A, B, C, D be four points on a line disposed in any manner, and O any point whatever not at infinity, then always*

$$\text{area } BOC.\text{area } AOD + \text{area } COA.\text{area } BOD + \text{area } AOB.\text{area } COD = 0,$$

regard being had to their signs as well as their magnitudes.

For in the relation $BC.AD + CA.BD + AB.CD = 0$, multiplying each segment by half the length of the perpendicular from O on the line, the relation just given is the immediate result.

Cor. 2°. *More generally if A, B, C, D be any four points* *and O any fifth point, none of the five being at infinity, then always*

$$\text{area } BOC \,.\, \text{area } AOD + \text{area } COA \,.\, \text{area } BOD + \text{area } AOB \,.\, \text{area } COD = 0,$$

regard being had to their signs as well as their magnitudes.

For conceiving the four lines AO, BO, CO, DO, met by any fifth line L not parallel to one of themselves in the four points A', B', C', D', since then (64)

$$\text{area } BOC : \text{area } B'OC' = OB \,.\, OC : OB' \,.\, OC',$$

and $$\text{area } AOD : \text{area } A'OD' = OA \,.\, OD : OA' \,.\, OD',$$

both pairs of triangles having the same angles at O; therefore

$$\text{area } BOC \,.\, \text{area } AOD : \text{area } B'OC' \,.\, \text{area } A'OD' = OA \,.\, OB \,.\, OC \,.\, OD : OA' \,.\, OB' \,.\, OC' \,.\, OD',$$

and (both remaining pairs of corresponding products having for the same reason the same ratio) therefore

$$\text{area } BOC . \text{area } AOD : \text{area } COA . \text{area } BOD : \text{area } AOB . \text{area } COD$$
$$= \text{area } B'OC' \,.\, \text{area } A'OD' : \text{area } C'OA' \,.\, \text{area } B'OD' : \text{area } A'OB' \,.\, \text{area } C'OD';$$

but by Cor. 1°. the sum of the three consequents $= 0$, therefore, also the sum of the three antecedents $= 0$, and therefore &c.

Cor. 3°. *If OA, OB, OC, OD be four lines passing through a point, then in all cases whatever be their directions,*

$$\sin BOC . \sin AOD + \sin COA . \sin BOD + \sin AOB . \sin COD = 0,$$

regard being had to the signs as well as the magnitudes of the six angles involved.

For, if A, B, C, D be the four points in which any line not passing through O intersects the four lines; since then by (64)

$$OB . OC . \sin BOC = 2 \text{area} BOC \text{ and } OA . OD . \sin AOD = 2 \text{area} AOD,$$

therefore

$$OA \,.\, OB . OC . OD . \sin BOC . \sin AOD = 4 \text{ area} . BOC \,.\, \text{area } AOD,$$

and, similar relations for the same reason existing for the other two pairs, therefore,

$$\sin BOC \,.\, \sin AOD : \sin COA \,.\, \sin BOD : \sin AOB \,.\, \sin COD$$
$$= \text{area} BOC . \text{area} AOD : \text{area} COA . \text{area} BOD : \text{area} AOB . \text{area} COD,$$

but by Cor. 1°. the sum of the three antecedents $= 0$, therefore also the sum of the three consequents $= 0$, and therefore &c.

Otherwise thus, if A, B, C, D, be the four points in which any circle passing through O intersects the four lines, then since (62) diameter of circle . $\sin BOC =$ chord BC, and diameter of circle . $\sin AOD =$ chord AD; therefore diameter² of circle . $\sin BOC . \sin AOD =$ chord BC . chord AD, and (similar relations for the same reason existing for the other two pairs) therefore

$$\sin BOC . \sin AOD : \sin COA . \sin BOD : \sin AOB . \sin COD$$
$$= \text{chord } BC . \text{chord } AD : \text{chord } CA . \text{chord } BD : \text{chord } AB . \text{chord } CD;$$

but by Ptolemy's theorem (Euc. VI. 16, Cor.) one of the three consequents is always numerically equal to the sum of the other two, therefore, disregarding signs, the same is true also of the three antecedents, and therefore &c.

Cor. 4°. *If A, B, C be any three points in a line, and AL, BL, CL their three distances perpendicular or in any common direction from any line L not at infinity, then always*

$$BC . AL + CA . BL + AB . CL = 0,$$

regard being had to the signs as well as the magnitudes of the several quantities involved.

For, if L be parallel to the line containing the points, then since $AL = BL = CL$, and since by (78) $BC + CA + AB = 0$, therefore &c., and if not, then if P be the intersection of the two lines, since $AL : BL : CL :: AP : BP : CP$, and since by the above $BC . AP + CA . BP + AB . CP = 0$, therefore &c.

Cor. 5°. *If L, M, N be any three parallel lines and PL, PM, PN their three distances perpendicular or in any common direction from any point P not at infinity, then always*

$$MN . PL + NL . PM + LM . PN = 0,$$

regard being had to the signs as well as the magnitudes of the several quantities involved.

For if A, B, C be the three points in which any line through P not parallel to their common direction intersects L, M, N, then since $MN : NL : LM : PL : PM : PN :: BC : CA : AB : PA : PB : PC$, and since by the above $BC . PA + CA . PB + AB . PC = 0$, therefore &c.

Cor. 6°. *If L, M, N be any three lines passing through a point O, and PL, PM, PN the three perpendiculars or any other isoclinals upon them from any point P not at infinity, then always*

$$\sin MN \, . \, PL + \sin NL \, . \, PM + \sin LM \, . \, PN = 0,$$

regard being had to the signs as well as the magnitudes of the several quantities involved.

For, dividing by PO the distance of P from O, or more generally by the diameter of the circle passing through P and O and through the feet of the three perpendiculars or isoclinals, the relation becomes evidently identical with that of Cor. 3°. for the four lines OL, OM, ON, OP, and therefore &c.

The three sides of every triangle being as the three sines of the opposite angles (63), the three sines in the preceding formula may therefore be replaced by the three sides of any triangle formed by parallels to the three lines.

Cor. 7°. *If α, β, γ be the three angles of any triangle, and α', β', γ' those at which the three opposite sides a, b, c intersect any line d, then always*

$$\sin \alpha \, . \sin \alpha' \, . + \sin \beta \, . \sin \beta' + \sin \gamma \, . \sin \gamma' = 0,$$

regard being had to the signs as well as the magnitudes of the six angles involved.

For, drawing through any arbitrary point O four lines OA, OB, OC, OD parallel to a, b, c, d, then since by parallels $BOC = \alpha$, $COA = \beta$, $AOB = \gamma$, and $AOD = \alpha'$, $BOD = \beta'$, $COD = \gamma'$, the relation is evident from that of Cor. 3°.

83. *If A, B, C be three points on a line disposed in any manner, and AP, BP, CP the three lines connecting them with any point P not at infinity, then always*

$$BC.AP^2 + CA.BP^2 + AB.CP^2 = - BC.CA.AB,$$

regard being had to the signs as well as the magnitudes of the three segments involved.

For letting fall from P the perpendicular PQ on the line, then since (75 Cor. 2°.)

$$AP^2 - CP^2 = AC^2 + 2\,AC.CQ,$$

and $$BP^2 - CP^2 = BC^2 + 2\,BC.CQ,$$

therefore, multiplying the first by BC and the second by AC, and subtracting,

$$BC.AP^2 + CA.BP^2 + (AC - BC).CP^2 = BC.AC^2 - AC.BC^2$$
$$= - BC.CA.(AC - BC)$$

the same as the above, $AC - BC$ being always $= AB$ (75).

From the above which is *the general relation connecting any three lines drawn from a point to a line, and the three segments they intercept on the line*; it is evident that *when A, B, and C are fixed, the sum $BC.AP^2 + CA.BP^2 + AB.CP^2$ is independent of the position of P and therefore constant for all points at a finite distance*; an example of the general property established in (81), that when, as in the present instance (see preceding article), the sum $\Sigma(a.AQ)$ is *nothing* for every point on a line, then the sum $\Sigma(a.AQ^2)$ is *constant* for every point on the line, and therefore for every point whatever not at infinity, the quantity $\Sigma(a)PQ^2$ by which the sums for the two points P and Q differ, Euc. I. 47, vanishing with $\Sigma(a)$ for every position of P for which PQ is not infinite.

Dividing both sides of the above relation by its right-hand member $-BC.CA.AB$, it assumes the not less symmetrical but more compact form

$$\frac{AP^2}{AB.AC} + \frac{BP^2}{BC.BA} + \frac{CP^2}{CA.CB} = 1,$$

regard being had of course to the signs as well as the magnitudes of the three rectangles $AB.AC$, $BC.BA$, $CA.CB$ in the addition.

Cor. 1°. *If A, B, C be three points on a line disposed in any manner, and AR, BS, CT the three tangents from them to any circle, not either at infinity or infinite in radius, then always*

$$BC.AR^2 + CA.BS^2 + AB.CT^2 = - BC.CA.AB,$$

regard being had to the signs as well as the magnitudes of all the quantities involved.

For, if P be the centre of the circle, then since

$$AR^2 = AP^2 - PR^2,\quad BS^2 = BP^2 - PS^2,\quad CT^2 = CP^2 - PT^2,$$

and since $PR = PS = PT =$ radius of circle, therefore

$$BC.AR^2 + CA.BS^2 + AB.CT^2 = BC.AP^2 + CA.BP^2$$
$$+ AB.CP^2 - (BC + CA + AB).\text{radius}^2 \text{ of circle},$$

the first part of which by the above $= - BC.CA.AB$, and the second point of which by (78) $= 0$, and therefore &c.

Dividing, as in the original, both sides of this latter relation by its right-hand member $-BC.CA.AB$, it too assumes the more compact and not less symmetrical form

$$\frac{AR^2}{AB.AC}+\frac{BS^2}{BC.BA}+\frac{CT^2}{CA.CB}=1,$$

regard again of course being had to the signs as well as the magnitudes of all the quantities involved.

COR. 2°. *If CZ be any line drawn from the vertex C to the base AB of any triangle ACB, then always*

$$AZ.CB^2 - BZ.CA^2 = AB.(CZ^2 - AZ.BZ),$$

regard being had to the signs as well as the magnitudes of the three intercepts AZ, BZ, and AB.

This relation is obviously the same as the above, only stated in the form in which it most naturally presents itself in the process by which it was established above.

The following particular cases are deserving of notice:

1°. If Z bisect AB, then $AZ=\frac{1}{2}AB$ and $BZ=-\frac{1}{2}AB$, and the relation becomes

$$CZ^2 - AZ.BZ = \tfrac{1}{2}(CA^2 + CB^2),$$

the known relation connecting the base, bisector of base, and sides of a triangle, (Euc. II. 12, 13, Cor.).

2°. If CZ bisect ACB externally or internally, then as $AZ:BZ=\pm AC:BC$, (Euc. VI. 3), therefore $AZ.CB=\pm BZ.CA$ according as the bisection is external or internal, and the relation, remembering that in either case $AZ-BZ=AB$ (75), becomes

$$CZ^2 - AZ.BZ = \mp CA.CB,$$

the known relation connecting the sides of a triangle, either bisector external or internal of the vertical angle, and the segments into which it divides the base.

3°. If the triangle be isosceles, then $CA = CB$, and the relation, remembering as before that always $AZ-BZ=AB$, becomes $CZ^2 - AZ.BZ = CA^2$ or CB^2 or $CA.CB$,

the known relation connecting either side of an isosceles triangle, any line drawn from the vertex to the base, and the rectangle under the segments into which it divides the base (Euc. II. 5, 6, Cor.).

4°. If the triangle be right-angled, then $CA^2 + CB^2 = AB^2$, and the relation, multiplying its two sides, the first by $AZ - BZ$, and the second by its equivalent AB, which causes the rectangle $AZ.BZ$ to disappear in virtue of the property of the triangle, becomes

$$BC^2.AZ^2 + AC^2.BZ^2 = AB^2.CZ^2,$$

the general relation connecting the sides and the distances of any point on the hypotenuse from the vertices of a right-angled triangle.

COR. 3°. *If A, B, C, D be any four points on a circle taken in the order of their disposition, and P any fifth point without, within, or upon the circle, but not at infinity, then always*

$$\text{area } BCD.AP^2 - \text{area } CDA.BP^2 + \text{area } DAB.CP^2 - \text{area } ABC.DP^2 = 0,$$

regard being had only to the absolute magnitudes of the several areas which from their disposition are incapable of being compared in sign.

For, joining P with the intersection O of the two chords AC and BD, which from their positions necessarily intersect internally; then from the relation, Cor. 1°., applied successively to the two triangles APC and BPD, disregarding all signs in each, and attending only to absolute values throughout,

$$CO.AP^2 + AO.CP^2 = AC.(PO^2 + AO.CO),$$

$$DO.CP^2 + BO.DP^2 = BD.(PO^2 + BO.DO),$$

from which, as $AO.CO = BO.DO$, (Euc. III. 35), therefore immediately

$$BD.CO.AP^2 + BD.AO.CP^2 = AC.DO.BP^2 + AC.BO.DP^2,$$

which is evidently identical with the other, the four rectangles $BD.CO$, &c. multiplied each by the sine of the angle of intersection of the two chords AC and BD being respectively the double areas of the four triangles BCD, &c.

This theorem is due to Dr. Salmon, who has given it in his *Conic Sections* as the geometrical interpretation of the analytical condition that four points A, B, C, D should lie on a circle.

COR. 4°. *If A, B, C, D be any four points on a circle taken* *in the order of their disposition, and AQ, BR, CS, DT the four tangents from them to another circle not either infinitely distant or infinite in radius, then always*

$$\text{area } BCD.AQ^2 - \text{area } CDA.BR^2 + \text{area } DAB.CS^2 - \text{area } ABC.DT^2 = 0,$$

regard being had, as in Cor. 3°., only to the absolute values of the several areas.

For, if P be the centre of the latter circle, then since

$$PQ = PR = PS = PT = \text{radius of that circle},$$

and since

$$\text{area } BCD + \text{area } DAB = \text{area } CDA + \text{area } ABC = \text{area of quadrilateral } ABCD,$$

therefore

$$\text{area } BCD.PQ^2 - \text{area } CDA.PR^2 + \text{area } DAB.PS^2 - \text{area } ABC.PT^2 = 0,$$

which relation, subtracted from that of Cor. 3°., leaves immediately that just stated, and therefore &c.

If in this relation, as in that of Cor. 1°., any of the points A, B, C, D be within the second circle, the squares of the corresponding tangents are of course negative.

Cor. 5°. *If OA, OB, OC, OD be four lines passing through a point, then in all cases, whatever be their directions,*

$$\frac{\sin BOD.\sin COD}{\sin BOA.\sin COA} + \frac{\sin COD.\sin AOD}{\sin COB.\sin AOB} + \frac{\sin AOD.\sin BOD}{\sin AOC.\sin BOC} = 1,$$

regard being had to the signs as well as the magnitudes of the six angles involved.

For, drawing any line L parallel to OD, meeting OA, OB, OC in A, B, C, then since (63)

$$\frac{\sin BOD}{\sin BOA} = \frac{AO}{AB} \text{ and } \frac{\sin COD}{\sin COA} = \frac{AO}{AC};$$

therefore
$$\frac{\sin BOD.\sin COD}{\sin BOA.\sin COA} = \frac{AO^2}{AB.AC},$$

and, similarly,

$$\frac{\sin COD.\sin AOD}{\sin COB.\sin AOB} = \frac{BO^2}{BC.BA} \text{ and } \frac{\sin AOD.\sin BOD}{\sin AOC.\sin BOC} = \frac{CO^2}{CA.CA};$$

but, by the original relation of the present article, the sum of the three right-hand members = 1, therefore also the sum of the left-hand members = 1, and therefore &c.

Cor. 6°. *If OA, OB, OC be three lines passing through a point, and PA, PB, PC the three perpendiculars upon them from any point P not at infinity, then always whatever be their directions*

$$PB.PC.\sin BOC + PC.PA.\sin COA + PA.PB.\sin AOB$$
$$= -PO^2.\sin BOC.\sin COA.\sin AOB,$$

regard being had to the signs as well as the magnitudes of the several quantities involved.

For, dividing both sides of the relation by its right-hand member $-PO^2.\sin BOC.\sin COA.\sin AOB$, the relation of Cor. 5°., for the four lines OA, OB, OC, and OP, is the immediate result, and therefore &c.

Cor. 7°. *If α, β, γ be the three angles of any triangle, and α', β', γ' those at which the three opposite sides a, b, c intersect any line d, then always*

$$\frac{\sin\beta'.\sin\gamma'}{\sin\beta.\sin\gamma} + \frac{\sin\gamma'.\sin\alpha'}{\sin\gamma.\sin\alpha} + \frac{\sin\alpha'.\sin\beta'}{\sin\alpha.\sin\beta} = 1,$$

regard being had to the signs as well as the magnitudes of the six angles involved.

For, as in Cor. 5°. of the preceding article, drawing through any arbitrary point O, four lines OA, OB, OC, OD parallel to a, b, c, d; then since $BOC = \alpha$, $COA = \beta$, $AOB = \gamma$, and $AOD = \alpha'$, $BOD = \beta'$, $COD = \gamma'$, the relation is evident from that of Cor. 5°.

Cor. 8°. *If A, B, C be the three vertices of any triangle, and AX, BY, CZ three parallels drawn from them in any direction to meet the three opposite sides BC, CA, AB, then always*

$$\frac{BX.CX}{AX^2} + \frac{CY.AY}{BY^2} + \frac{AZ.BZ}{CZ^2} = 1,$$

regard being had to the signs as well as the magnitudes of the three rectangles involved.

For, if α, β, γ be the three angles of the triangle at A, B, C, and α', β', γ' those at which the three opposite sides intersect any line parallel to the common direction of the three parallels, since then

$$\frac{BX}{AX}=\frac{\sin\gamma'}{\sin\beta} \text{ and } \frac{CX}{AX}=\frac{\sin\beta'}{\sin\gamma};$$

therefore
$$\frac{BX.CX}{AX^2}=\frac{\sin\beta'.\sin\gamma'}{\sin\beta.\sin\gamma},$$

and similarly,

$$\frac{CY.AY}{BY^2}=\frac{\sin\gamma'.\sin\alpha'}{\sin\gamma.\sin\alpha} \text{ and } \frac{AZ.BZ}{CZ^2}=\frac{\sin\alpha'.\sin\beta'}{\sin\alpha.\sin\beta},$$

and the relation consequently is evident from that of Cor. 7°.

84. We shall conclude the present chapter with one or two applications of a very simple problem of very frequent occurrence in Pure and Applied Geometry.

Given in magnitude and sign the ratio $m : n$ *of the segments* AP *and* BP *into which a given line* AB *is cut at a point* P, *to determine the segments in magnitude and sign.*

Since, by hypothesis, $AP : BP = m : n$, therefore

$$AP : AP - BP : m : m - n, \text{ and } BP : BP - AP = n : n - m,$$

and since in all cases $AP - BP = AB$, and $BP - AP = BA$, therefore

$$AP = \frac{m}{m-n}.AB = \frac{m}{n-m}.BA,$$

$$BP = \frac{n}{m-n}.AB = \frac{n}{n-m}.BA,$$

which are the general formulæ by which to calculate in numbers the segments of a line of given length cut in any given ratio.

Cor. 1°. As an application of the preceding let it be required *to determine for any triangle* ABC *the lengths of the bisectors, external and internal, of the three angles, and the segments they intercept on the opposite sides.*

If AX, BY, CZ be the three external, and AX', BY', CZ' the three internal bisectors, then since (Euc. VI. 3)

$$\frac{BX}{CX}=+\frac{BA}{CA} \text{ and } \frac{BX'}{CX'}=-\frac{BA}{CA};$$

therefore, by the above,

$$BX = \frac{BA}{BA - CA}.BC, \quad BX' = \frac{BA}{BA + CA}.BC,$$

$$CX = \frac{CA}{CA - BA}.CB, \quad CX' = \frac{CA}{CA + BA}.CB,$$

and therefore at once, by subtraction, remembering that similar formulæ for the same reason hold for the other two sides,

$$X'X = \frac{2BA.CA}{BA^2 - CA^2}.BC, \quad Y'Y = \frac{2CB.AB}{CB^2 - AB^2}.CA,$$

$$Z'Z = \frac{2AC.BC}{AC^2 - BC^2}.AB,$$

which are the general formulæ by which to calculate in numbers the lengths of the three intercepts $X'X$, $Y'Y$, $Z'Z$, when the sides of the triangle are given.

Since again, at once, by multiplication,

$$BX.CX = \frac{BA.CA}{(BA - CA)^2}.BC^2, \text{ and } BX'.CX' = -\frac{BA.CA}{(BA + CA)^2}.BC^2,$$

with corresponding values for the other two sides, therefore, by Cor. 2°. (83),

$$AX^2 = BA.CA\left\{\left(\frac{BC}{BA - CA}\right)^2 - 1\right\},$$

and
$$AX'^2 = BA.CA\left\{1 - \left(\frac{BC}{BA + CA}\right)^2\right\},$$

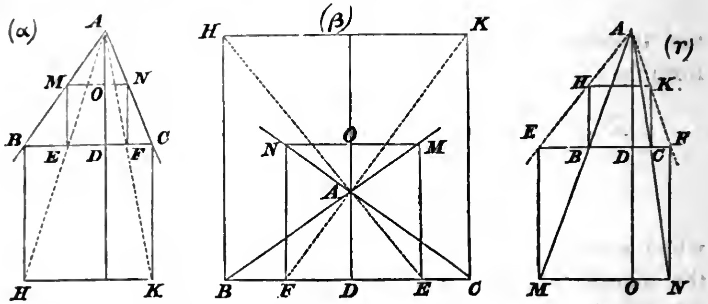
which with similar values for the other two sides are the formulæ by which to calculate in numbers the lengths of the six bisectors AX and AX', BY and BY', CZ and CZ' when the sides of the triangle are given.

From the above values for $X'X$, $Y'Y$, $Z'Z$, it is evident that their reciprocals are connected in all cases by the two following relations:

$$\frac{1}{XX'} + \frac{1}{YY'} + \frac{1}{ZZ'} = 0, \text{ and } \frac{BC^2}{XX'} + \frac{CA^2}{YY'} + \frac{AB^2}{ZZ'} = 0,$$

from which, regarding them as positive or negative according as they are similar or opposite in direction with the sides of the triangle measured from B to C, from C to A, and from A to B respectively, it is evident that one of them must have in all cases the sign opposite to that of the other two.

COR. 2°. As a second application of the same, let it be required *to determine for any triangle ABC the sides of the squares exscribed and inscribed to the three sides, and the segments they intercept on the perpendiculars from the opposite vertices.*



Let $EFMN$ be, fig. α, the inscribed, or, figs. β and γ, the exscribed square corresponding to the side BC of the triangle BAC; then drawing AD the perpendicular on that side from the opposite vertex A, intersecting MN in O, by similar triangles MAN and BAC, we have $MN : AO = BC : AD$, but, on account of the square, $MN = OD$, therefore, disregarding signs for a moment, $DO : AO = BC : AD$; that is the perpendicular AD is cut at the point O, internally, fig. α, in the case of the inscribed, and externally, figs. β and γ, in the case of the exscribed square in the ratio of $BC : AD$; and therefore, by the above

$$OD = \frac{BC}{BC \pm AD} . AD = \frac{BC.AD}{BC \pm AD},$$

the upper sign corresponding to the inscribed and the lower to the exscribed square.

Similar formulæ holding of course for the other two sides CA and AB; if a, b, c be the three sides of the triangle, p, q, r the three perpendiculars upon them from the opposite vertices, x, y, z the sides of the three inscribed, and x', y', z' those of the three exscribed squares; then, by the above,

$$x = \frac{ap}{a+p}, \quad y = \frac{bq}{b+q}, \quad z = \frac{cr}{c+r},$$

$$x' = \frac{ap}{a-p}, \quad y' = \frac{bq}{b-q}, \quad z' = \frac{cr}{c-r},$$

which are the general formulæ by which to calculate in numbers the sides of the six squares when the sides of the triangle are given.

It is evident from these formulæ, or directly, that while the inscribed square corresponding to any side of a triangle lies always on the same side of that side with the triangle itself, (fig. α); the exscribed square on the contrary lies on the same or on the opposite side, figs. β and γ, according as the side of the triangle to which it corresponds is greater or less than the perpendicular upon it from the opposite vertex; in the particular case when a side of a triangle is equal to the perpendicular upon it from the opposite vertex, the exscribed square corresponding to such side is infinite, and may therefore be regarded as lying indifferently in either direction.

Combining the above formulæ in corresponding pairs, by addition and subtraction, we have immediately

$$x'+x=\frac{2ap}{a^2-p^2}.a,\quad y'+y=\frac{2bq}{b^2-q^2}.b,\quad z'+z=\frac{2cr}{c^2-r^2}.c,$$

$$x'-x=\frac{2ap}{a^2-p^2}.p,\quad y'-y=\frac{2bq}{b^2-q^2}.q,\quad z'-z=\frac{2cr}{c^2-r^2}.r,$$

which latter, regard being had to their signs as well as their magnitudes, are the formulæ for the lengths of the segments intercepted on the three perpendiculars of the triangle by the three pairs of squares.

Taking again the reciprocals of the above formulæ, viz.

$$\frac{1}{x}=\frac{1}{p}+\frac{1}{a},\quad \frac{1}{y}=\frac{1}{q}+\frac{1}{b},\quad \frac{1}{z}=\frac{1}{r}+\frac{1}{c},$$

$$\frac{1}{x'}=\frac{1}{p}-\frac{1}{a},\quad \frac{1}{y'}=\frac{1}{q}-\frac{1}{b},\quad \frac{1}{z'}=\frac{1}{r}-\frac{1}{c},$$

and combining them also in corresponding pairs, by addition and subtraction, we get

$$\frac{1}{x}+\frac{1}{x'}=\frac{2}{p},\quad \frac{1}{y}+\frac{1}{y'}=\frac{2}{q},\quad \frac{1}{z}+\frac{1}{z'}=\frac{2}{r},$$

$$\frac{1}{x}-\frac{1}{x'}=\frac{2}{a},\quad \frac{1}{y}-\frac{1}{y'}=\frac{2}{b},\quad \frac{1}{z}-\frac{1}{z'}=\frac{2}{c},$$

which are the formulæ by which to calculate in numbers a side and perpendicular of a triangle when their inscribed or exscribed squares are given.

From the several preceding formulæ it is evident that any two of the four corresponding magnitudes, viz., a side of a triangle, the perpendicular upon it, the inscribed and exscribed squares resting upon it, determine the other two.

The sides of the squares inscribed and exscribed to any side BC of a triangle ABC, being given by the above formulæ, the squares themselves can of course be immediately constructed; if however it were required only to construct them without having also to calculate their sides, of the several methods of doing so the following is perhaps the most convenient.

On the side BC of the triangle upon which the squares are to be constructed, describe the square $BCHK$, and connect its two opposite vertices H and K with the opposite vertex A of the triangle; the two connecting lines HA and KA will intercept on BC the base EF of the required inscribed or exscribed square $EFMN$—of the inscribed if HK and A lie at opposite sides of BC (fig. α)—of the exscribed if they lie at the same side of it (figs. β and γ).

For, drawing EM and FN perpendiculars to BC and joining MN; as the three lines AH, AK, and AB pass through a point A, and as EM and EF are parallels to HB and HK, therefore $EM : EF = HB : HK$, and similarly $FN : FE = KC : KH$, but by construction $HB = KC = HK$, therefore $EM = FN = EF$, and therefore &c.

A method exactly similar might obviously be employed to solve the more general problem: "*On any side BC of a given triangle ABC to inscribe or exscribe a parallelogram of any given form.*"

CHAPTER VI.

THEORY GENERAL OF THE MEAN CENTRE OF ANY SYSTEM OF POINTS FOR ANY SYSTEM OF MULTIPLES.

85. The main features of this theory for the particular case of a system of points disposed along a line having been already given in sections 79, 80, 81 of the preceding, its extension to a system of points disposed in any manner will form the chief subject of the present chapter; the following fundamental theorem may be regarded as the basis of this extension.

If A, B, C, D, &c. be any system of points, disposed in any manner, but none infinitely distant, a, b, c, d, &c. any system of corresponding multiples, positive or negative, but none infinitely great, and O a point such that for two lines M and N passing through it $\Sigma(a.AM) = 0$, and $\Sigma(a.AN) = 0$; then for every line L passing through O $\Sigma(a.AL) = 0$, regard being had in all the sums to the signs as well as the magnitudes of the several quantities involved.

For, if O be at an infinite distance, then for the several points by Cor. 5°. (Art. 82) of the preceding chapter,

$$MN.AL + NL.AM + LM.AN = 0,$$
$$MN.BL + NL.BM + LM.BN = 0,$$
$$MN.CL + NL.CM + LM.CN = 0,$$
$$MN.DL + NL.DM + LM.DN = 0, \text{ \&c.}$$

And, if O be at a finite distance, then for the several points by Cor. 6°. (Art. 82) of the same,

$$\sin MN.AL + \sin NL.AM + \sin LM.AN = 0,$$
$$\sin MN.BL + \sin NL.BM + \sin LM.BN = 0,$$
$$\sin MN.CL + \sin NL.CM + \sin LM.CN = 0,$$
$$\sin MN.DL + \sin NL.DM + \sin LM.DN = 0, \text{ \&c.}$$

which multiplied in either case by a, b, c, d, &c. and added, give at once, in the former case the relation

$$MN.\Sigma(a.AL) + NL.\Sigma(a.AM) + LM.\Sigma(a.AN) = 0,$$

and in the latter case the relation

$$\sin MN.\Sigma(a.AL) + \sin NL.\Sigma(a.AM) + \sin LM.\Sigma(a.AN) = 0,$$

from which it follows immediately in either case that if any two of the three sums $\Sigma(a.AL)$, $\Sigma(a.AM)$, $\Sigma(a.AN) = 0$, the third also $= 0$, and therefore &c.

The case of O at an infinite distance corresponds, as may be easily shewn, to that of $\Sigma(a) = 0$, a case requiring, as we shall see, special treatment in almost every point connected with the present subject; for, since $\Sigma(a.AN) - \Sigma(a.AM) = \Sigma(a).MN$ for every two parallel lines M and N whatever be their interval of separation MN; therefore if, as above, $\Sigma(a.AN) = \Sigma(a.AM)$ for any two parallel lines M and N not coinciding with each other, then $\Sigma(a)=0$, and if conversely $\Sigma(a)=0$, then $\Sigma(a.AN)=\Sigma(a.AM)$ for every two parallel lines M and N not infinitely distant from each other, and therefore &c.

86. The point O related as above to a system of points A, B, C, D, &c. that for every line L passing through it the sum

$$a.AL + b.BL + c.CL + d.DL + \&c. = 0,$$

is termed *the centre of mean position*, or more shortly *the mean centre* of the system of points for the system of multiples a, b, c, d, &c. and is in general a unique point depending upon and varying with the positions of the points and the values of the multiples; the propriety of the name depending on the properties of the point will appear in the sequel.

In the science of Mechanics, if A, B, C, D, &c. be the positions, and a, b, c, d, &c. the masses of any system of material particles situated in the same plane, then is the point O, as above defined, *the centre of gravity* of the system; in that science, therefore, all propositions relating to this subject are of considerable importance.

87. For every system of points A, B, C, D, &c. there exists a particular system of multiples a, b, c, d, &c. indeterminate of

course in absolute but fixed and unique in relative values, such that for *every* line L not actually at infinity, the sum $\Sigma(a.AL) = 0$, and for which therefore *the mean centre O of the system is indeterminate;* in all such cases it is easy to see, 1°. *that* $\Sigma(a) = 0$, and 2°. *that each point of the system is the mean centre of the others for their respective multiples;* for, the values of $\Sigma(a.AL)$ being by hypothesis $= 0$ for two different lines passing through a point at infinity, therefore by the preceding $\Sigma(a) = 0$, and being again by hypothesis $= 0$ for two different lines passing through any point of the system, therefore by the same that point is the mean centre of the others for their respective multiples; instances of such cases are of course exceptional, but whenever they present themselves, as they occasionally do, their exceptional peculiarities must always be attended to.

88. From the fundamental property of the preceding article, it is easy to see that if a system of multiples a, b, c, d, &c. corresponding to a system of points A, B, C, D, &c. be such that for any *three* lines L, M, N *not passing through a common point* $\Sigma(a.AL) = 0$, $\Sigma(a.AM) = 0$, $\Sigma(a.AN) = 0$, then for *every* line I not actually at infinity $\Sigma(a.AI) = 0$. For, if L', M', N' be any three lines passing respectively through the three points MN, NL, LM, and intersecting on I, then since by (85),

$$\Sigma(a.AL') = 0, \quad \Sigma(a.AM') = 0, \quad \Sigma(a.AN') = 0,$$

therefore by the same $\Sigma(a.AI) = 0$, and therefore &c.

89. From the same again it appears, that if for a system of multiples a, b, c, d, &c. a system of points A, B, C, D, &c. have *two* different mean centres O_1 and O_2, then is *every* point O indifferently a mean centre of the same system of points for the same system of multiples; for, whatever be the position of O, since for the two lines L_1 and L_2 connecting it with O_1 and O_2, the two sums $\Sigma(a.AL_1)$ and $\Sigma(a.AL_2)$ are both $= 0$, therefore for every line L passing through O the sum $\Sigma(a.AL) = 0$, and therefore &c. *Hence, whatever be the positions of the points A, B, C, D, &c. and whatever be the values of the multiples a, b, c, d, &c. the mean centre O is always either indeterminate or unique.*

90. *If A, B, C, D, &c. be the several vertices of a regular polygon of any order, and O the geometric centre of the figure, then is O the mean centre of the several points A, B, C, D, &c. for the particular system of multiples each = unity.*

For, if the polygon be of an even order, since for every line passing through O the several pairs of perpendiculars from pairs of opposite vertices are equal and opposite, therefore for every line passing through O the sum of the perpendiculars from all the vertices = 0, and therefore &c.; and, if the polygon be of an odd order, since for every line passing through O and through a vertex of the figure the several pairs of perpendiculars from pairs of vertices equidistant from that through which the line passes are equal and opposite, and the one from that vertex itself = 0, therefore for every line passing through O and through a vertex of the figure, and therefore by the preceding for every line passing through O, the sum of the perpendiculars from all the vertices = 0, and therefore &c.

In consequence of the above, all properties true in general of the mean centre of any system of points A, B, C, D, &c. for any system of multiples a, b, c, d, &c. whose sum is not = 0, are true in particular of the geometric centre of any regular polygon regarded as the mean centre of its several vertices for the particular system of multiples each = unity.

91. *If A, B, C be the three vertices of any triangle, and O their mean centre for any three multiples a, b, c, then always—*

1°. *The three lines AO, BO, CO intersect with the three opposite sides BC, CA, AB at three points X, Y, Z such that*

$$b.BX + c.CX = 0, \quad c.CY + a.AY = 0, \quad a.AZ + b.BZ = 0.$$

2°. *The three triangles BOC, COA, AOB are connected with the three multiples a, b, c by the proportions*

$$\text{area}\,BOC : \text{area}\,COA : \text{area}\,AOB = a : b : c,$$

regard being had to the signs as well as the magnitudes of the several quantities involved in each.

To prove 1°. Since for every three lines L, M, N passing through O, (86)

$$a.AL + b.BL + c.CL = 0, \quad a.AM + b.BM + c.CM = 0,$$
$$a.AN + b.BN + c.CN = 0,$$

if L pass through A, then

$$AL = 0 \text{ and } BL : CL = BX : CX,$$

and therefore $b.BX + c.CX = 0$; if M pass through B, then

$$BM = 0 \text{ and } CM : AM = CY : AY,$$

and therefore $c.CY + a.AY = 0$; and if N pass through C, then

$$CN = 0 \text{ and } AN : BN = AZ : BZ,$$

and therefore $a.AZ + b.BZ = 0$.

To prove 2°. Since the two triangles AOB and AOC have a common base AO, therefore

$$\text{area} AOB : \text{area} AOC = BL : CL = BX : CX,$$

since the two BOC and BOA have a common base BO, therefore

$$\text{area} BOC : \text{area } BOA = BM : CM = BY : CY,$$

and since the two COA and COB have a common base CO, therefore

$$\text{area } COA : \text{area} COB = AN : BN = AZ : BZ;$$

and the proportions 2°. follow therefore immediately from the relation 1°.

Cor. The above relations supply each an obvious method of determining the mean centre O of any three points A, B, C forming a triangle, for any three multiples a, b, c given in magnitude and sign; the two following particular cases are deserving of attention:

1°. If in absolute magnitude $a = b = c$, then AX, BY, CZ bisect the three sides of the triangle, all internally or two externally and one internally according as the signs of a, b, c are all similar or two opposite to the third; O in either case is the intersection of the three bisectors; and the three areas BOC, COA, AOB are equal in absolute magnitude and have signs in accordance with those of a, b, c.

2°. If in absolute magnitude $a : b : c = BC : CA : AB$, then AX, BY, CZ bisect the three angles of the triangle, all internally or two externally and one internally according as the signs of a, b, c are all similar or two opposite to the third; O in either case is the intersection of the three bisectors, and therefore the centre of the inscribed or of one of the three exscribed circles of the triangle; and the three areas BOC, COA, AOB are proportional in absolute magnitude to the three sides BC, CA, AB, and have signs in accordance with those of a, b, c.

92. *If A, B, C, D, &c. be any system of points, O their mean centre for any system of multiples a, b, c, d, &c. whose sum is not $= 0$, and L any arbitrary line, then always whatever be the position of L*

$$\Sigma (a.AL) = \Sigma (a).OL,$$

regard being had to the signs as well as the magnitudes of the several quantities involved.

For, drawing through O the line M parallel to L, then since for any two parallel lines L and M whatever be their common direction or distance asunder $\Sigma (a.AL) - \Sigma (a.AM) = \Sigma (a).ML$, if, as in the present case, one of them M passes through O, since for it $\Sigma (a.AM) = 0$ (86) therefore for the other L whatever be its position $\Sigma (a.AL) = \Sigma (a).OL$, and therefore &c.

Cor. 1°. This is the property which gives to the point O its designation of "Mean Centre" of the system of points A, B, C, D, &c. for the system of multiples a, b, c, d, &c., and by its aid when the latter are both given the former may be determined in all cases by the following general construction:

Drawing arbitrarily any two lines L and L' not parallel to each other, the two parallels to them M and M' distant from them by the intervals LM and $L'M'$ equal in magnitude and sign to the quantities $\frac{\Sigma (a.LA)}{\Sigma (a)}$ and $\frac{\Sigma (a.L'A)}{\Sigma (a)}$ pass, by the above, through, and therefore intersect at, the mean centre O; in the particular case where $\Sigma (a) = 0$, the position of O thus given is at infinity (85), *unless also* $\Sigma (a.LA)$ *and* $\Sigma (a.L'A)$ *both* $=0$, *in which exceptional case it is indeterminate* (87).

Cor. 2°. The mean centre O of any given system of points A, B, C, D, &c. for any given system of multiples a, b, c, d, &c. may also be determined by the following in general less rapid, but in many cases not less convenient process, based like that just given on the above, viz.:

Connect any two points A and B of the system, and take on the connecting line AB the point P for which $a.AP + b.BP = 0$ (77). Connect then the point P with any third point C of the system, and take on the connecting line PC the point Q for which $(a+b).PQ + c.CQ = 0$. Connect then the point Q with any fourth point D of the system, and take on the connecting line QD the point R for which $(a+b+c).QR + d.DR = 0$. Connect then the point R with any fifth point E of the system, and take on the connecting line RE the point S for which $(a+b+c+d).RS+e.ES=0$, and so on, until all the points of the system are exhausted, the last point O thus determined is the mean centre required.

For since for every arbitrary line L, by (77) Cor.

$$a.AL+b.BL=(a+b).PL,$$
$$(a+b).PL+c.CL=(a+b+c).QL,$$
$$(a+b+c).QL+d.DL=(a+b+c+d).RL,$$
$$(a+b+c+d).RL+e.EL=(a+b+c+d+e).SL, \text{ \&c.}$$

therefore for the last point O, by addition

$$a.AL+b.BL+c.CL+d.DL+\text{\&c.}=(a+b+c+d+\text{\&c.}).OL,$$

which, by the above, is the characteristic property of the mean centre.

In the particular case when $\Sigma(a)=0$, the point O thus determined being the point of external bisection of the last connecting line in the above process is therefore at infinity, *unless when the length of that connecting line* $=0$ *in which exceptional case it is indeterminate.*

COR. 3°. Stating the above general relation in the equivalent form $\Sigma(a.AL)-\Sigma(a).OL=0$, it appears that, if to any system of points A, B, C, D, &c. be added their mean centre O for any system of multiples a, b, c, d, &c., then is the system of points A, B, C, D, &c. and O, for the system of multiples a, b, c, d, &c. and $-\Sigma(a)$, of the exceptional character mentioned in (87), for which for *every* line L not at infinity the sum $\Sigma(a.AL)=0$, and for which therefore the mean centre is *indeterminate.* Hence the original system of points A, B, C, D, &c. and of multiples a, b, c, d, &c. being entirely arbitrary, it appears that—

For a system of the exceptional character whose mean centre is indeterminate, all but one of the points may have any positions whatever, and their corresponding multiples any values whatever, provided only the remaining point be the mean centre of the others for their system of multiples, and the remaining multiple corresponding to it be equal in magnitude and opposite in sign to the sum of the others.

COR. 4°. Since for every line L tangent to any circle round O as centre the distance OL is constant and equal the radius of the circle, and since, by the above, the sum $\Sigma(a.AL)$ is constant when the radius OL is constant, and conversely, therefore—

If A, B, C, D, &c. be any system of points, and O their mean centre for any system of multiples a, b, c, d, &c. whose sum is not $= 0$, then for every line L tangent to any circle round O as centre the sum $\Sigma(a.AL)$ is constant and = the radius of the circle multiplied by $\Sigma(a)$, and, conversely, every line L for which the sum $\Sigma(a.AL)$ is constant touches the circle round O as centre whose radius = the constant sum divided by $\Sigma(a)$.

This property supplies obvious solutions of the following general problems, viz.: "Given any system of points A, B, C, D, &c., and any system of corresponding multiples a, b, c, d, &c. whose sum is not $= 0$, to draw a line L parallel to a given line, or passing through a given point, or touching a given circle, so that the sum $\Sigma(a.AL)$ shall $= 0$, or be a maximum, or have any given value."

Cor. 5°. *For every line L tangent to the circle inscribed in any triangle ABC the sum of the three rectangles*

$$BC.AL + CA.BL + AB.CL$$

is constant and equal to double the area of the triangle.

For, by (91), the centre O of that circle being the mean centre of the three points A, B, C for the three multiples BC, CA, AB, therefore, by the above,

$$BC.AL + CA.BL + AB.CL = (BC + CA + AB).OL;$$

but

$$BC.OL = 2\,\text{area}\,BOC,\ CA.OL = 2\,\text{area}\,COA,\ AB.OL = 2\,\text{area}\,AOB;$$

therefore their sum $= 2$ area ABC, and therefore &c.

A relation exactly similar holds of course for each of the three exscribed circles of the triangle, the sign of the side to which the circle is exscribed being merely changed in the above, see 91, Cor.

Cor. 6°. *For every line L tangent to any circle concentric with a regular polygon of any order n the sum of the perpendiculars from the several vertices is constant and $= n$ times the radius of the circle.*

For, by (90), the centre O of the polygon being the mean centre of the several vertices A, B, C, D, &c. for the particular system of multiples each $= 1$, therefore, by the above, $\Sigma(AL) = n.OL$, and therefore &c.

For a regular polygon of any order n the sum of the perpendiculars from any point P upon the several sides is also constant and = n times the radius of the circle inscribed in the figure.

For the sums of the perpendiculars from the centre O and from any other point P upon the several sides multiplied each by the common length of all the sides = double the area of the figure, and therefore &c.

93. *If any system of points $\Sigma(A)$ and of corresponding multiples $\Sigma(a)$ be divided into any number of groups $\Sigma(A_1)$, $\Sigma(A_2)$, $\Sigma(A_3)$, $\Sigma(A_4)$, &c., and $\Sigma(a_1)$, $\Sigma(a_2)$, $\Sigma(a_3)$, $\Sigma(a_4)$, &c., none of the latter being $=0$; then, if O_1, O_2, O_3, O_4, &c. be the several mean centres of the several groups of points for the several groups of corresponding multiples, the mean centre O of the system of points O_1, O_2, O_3, O_4, &c. for the systems of multiples $\Sigma(a_1)$, $\Sigma(a_2)$, $\Sigma(a_3)$, $\Sigma(a_4)$, &c. is the same as that of the system of points A, B, C, D, &c. for the system of multiples a, b, c, d, &c.*

For, since for every arbitrary line L, by the preceding,

$$\Sigma(a_1.A_1L) = \Sigma(a_1).O_1L, \quad \Sigma(a_2.A_2L) = \Sigma(a_2).O_2L,$$

$$\Sigma(a_3.A_3L) = \Sigma(a_3).O_3L, \quad \Sigma(a_4.A_4L) = \Sigma(a_4).O_4L, \text{ \&c.};$$

therefore the sum of all the first members = the sum of all the second members; but, by hypothesis, the sum of all the first members $= \Sigma(a.AL)$, and, by the preceding, the sum of all the second members

$$= \{\Sigma(a_1) + \Sigma(a_2) + \Sigma(a_3) + \Sigma(a_4) + \text{\&c.}\}.OL;$$

from which, since by hypothesis

$$\Sigma(a_1) + \Sigma(a_2) + \Sigma(a_3) + \Sigma(a_4) + \text{\&c.} = \Sigma(a);$$

therefore $\Sigma(a.AL) = \Sigma(a).OL$, and therefore &c.

Cor. In the particular case when $\Sigma(a) = 0$, if $\Sigma(A)$ be divided into any two groups $\Sigma(A_1)$ and $\Sigma(A_2)$ for which $\Sigma(a_1)$ and $\Sigma(a_2)$ are not separately $=0$; then since, by hypothesis, $\Sigma(a_1) + \Sigma(a_2) = 0$, if O_1 and O_2 be the mean centres of the two groups for their respective shares of the multiples, that of the entire system for all the multiples being, by the above, the point of external bisection of the line O_1O_2 is therefore the unique point in which that line intersects infinity (15), *except only when the two partial mean centres O_1 and O_2 coincide in* *which exceptional case it is indeterminate* (87). The division of $\Sigma(A)$ may, if we please, be into the two groups $\Sigma(A_+)$ and $\Sigma(A_-)$ corresponding to the division of $\Sigma(a)$ into its positive and negative constituents $\Sigma(a_+)$ and $\Sigma(a_-)$ respectively; or one group may, if we please, consist of but a single point and the other of all the rest.

94. *If A, B, C, D, &c. be any system of points, M any line parallel to the direction of their infinitely distant mean centre for any system of multiples a, b, c, d, &c. whose sum $=0$, and L any other line, then, whatever be the position of L,*

$$\Sigma(a.AL) = k.\sin ML,$$

k being a constant depending only on the disposition of the points and the values of the multiples.

For, if N be a third line passing through the intersection P of L and M, and perpendicular to the latter, then as in (85) the three lines LMN passing through a common point P,

$$\sin MN.\Sigma(a.AL) + \sin NL.\Sigma(a.AM) + \sin LM.\Sigma(a.AN) = 0,$$

from which as $\Sigma(a.AM) = 0$ from the property of the mean centre (86), and as $\sin MN = 1$ from the right angle MN (60), therefore

$$\Sigma(a.AL) = \Sigma(a.AN).\sin ML,$$

which proves the proposition, the two sums $\Sigma(a.AL)$ and $\Sigma(a.AN)$ depending when $\Sigma(a) = 0$ (85) only on the directions and not on the absolute positions of L and N.

Otherwise thus, as a corollary from the general case, when $\Sigma(a)$ is not $=0$; conceiving the entire system of points $\Sigma(A)$ divided into any two groups $\Sigma(A_1)$ and $\Sigma(A_2)$ for which the sums $\Sigma(a_1)$ and $\Sigma(a_2)$ of the corresponding groups of multiples are not separately $=0$; then, by the general relation of the preceding article, if O_1 and O_2 be the mean centres of the two partial groups for their respective systems of multiples, and L any line intersecting O_1O_2 at any point P and at any angle α,

$$\Sigma(a_1.A_1L) = \Sigma(a_1).O_1L, \text{ and } \Sigma(a_2.A_2L) = \Sigma(a_2).O_2L,$$

and therefore, by addition,

$$\Sigma(a.AL) = \Sigma(a_1).O_1L + \Sigma(a_2).O_2L;$$

but $O_1L = O_1P.\sin\alpha$, $O_2L = O_2P.\sin\alpha$, and $\Sigma(a_1) + \Sigma(a_2) = 0$;

therefore

$$\Sigma(a.AL) = \Sigma(a_1).O_1O_2.\sin\alpha, \text{ or } \Sigma(a_2).O_2O_1.\sin\alpha,$$

which proves the proposition, and gives at the same time in its most convenient form the value of the constant k or $\Sigma(a.AN)$ viz. $\Sigma(a_1).O_1O_2$ or $\Sigma(a_2).O_2O_1$. See (81).

The law of the variation of the sum $\Sigma(a.AL)$ for different positions of L is therefore very simple when $\Sigma(a) = 0$; depending only on the direction and not on the absolute position of L; vanishing for the direction of the infinitely distant mean centre of the system; being a maximum for the rectangular direction; and varying as the sine of the angle of inclination to the central for every intermediate direction; *in the exceptional case where the two partial mean centres O_1 and O_2 coincide, and when* (93, Cor.) *the position of O is consequently indeterminate, the sum $\Sigma(a.AL)$ undergoes no variation and is absolutely* $= 0$ *for every position of L not actually at infinity*, see (87).

Cor. 1°. By means of the above relation the direction of the infinitely distant mean centre of a given system of points A, B, C, D, &c. for a given system of multiples a, b, c, d, &c. whose sum $= 0$, if not previously known may be readily determined. For drawing arbitrarily any two lines L and L' not parallel to each other, the line M dividing the angle between them LL' so that in magnitude and sign

$$\sin ML : \sin ML' = \Sigma(a.AL) : \Sigma(a.AL')$$

gives, by the above, the required direction; *in the exceptional case when the two sums $\Sigma(a.AL)$ and $\Sigma(a.AL')$ are both* $=0$, *the mean centre of the system is indeterminate*, see (87).

Cor. 2°. The above relation also supplies obvious solutions of the six following problems, viz.: "given any system of points A, B, C, D, &c., and any system of corresponding multiples a, b, c, d, &c. whose sum $= 0$, to draw a line L passing through a given point or touching a given circle so that the sum $\Sigma(a.AL)$ shall be nothing, or a maximum, or have any given value."

95. *If any system of points A, B, C, D, &c. and their mean centre O for any system of multiples a, b, c, d, &c. be projected in any common direction upon any line L, the projection O' of the*

mean centre is always the mean centre of the projections A', B', C', D', &c. of the several points for the same system of multiples.

For, from the several points A, B, C, D, &c. conceiving lines AA_1, BB_1, CC_1, DD_1, &c. drawn parallel to the line L to meet the line OO'; then since, Euc. I. 34, $AA_1 = A'O'$, $BB_1 = B'O'$, $CC_1 = C'O'$, $DD_1 = D'O'$, &c., and since, by the fundamental property of the mean centre (86), $\Sigma(a.AA_1) = 0$, therefore $\Sigma(a.A'O') = 0$, and therefore O' is the mean centre of the system of points A', B', C', D', &c. for the system of multiples a, b, c, d, &c.; when L passes through O then $\Sigma(a.A'O) = 0$ and O itself is the mean centre of the projected as well as of the original system for the same system of multiples.

In the particular case when O is at an infinity, and when therefore $\Sigma(a) = 0$, its projection O' upon every base L is of course also at infinity, *except only when the direction of projection is parallel to that of O itself in which case it is indeterminate.*

In the exceptional case when O itself is indeterminate, and when therefore again $\Sigma(a) = 0$, its projection O' upon every base and for every direction of projection is of course also indeterminate.

96. *If A, B, C, D, &c. be any system of points, O their mean centre for any system of multiples a, b, c, d, &c. whose sum is not $= 0$, and L and M any two parallel lines, then always*

$$\Sigma(a.AL^2) - \Sigma(a.AM^2) = \Sigma(a).(OL^2 - OM^2),$$

whatever be the common direction and distance asunder of L and M.

For, identically,

$$\Sigma(a.AL^2) - \Sigma(a.AM^2) = \Sigma(a).\{(AL + AM).(AL - AM)\},$$

from which since $(AL - AM) =$ the constant interval between L and $M = (OL - OM)$, and since, by (92), $\Sigma(a.AL) = \Sigma(a).OL$, and $\Sigma(a.AM) = \Sigma(a).OM$, therefore at once

$$\Sigma(a.AL^2) - \Sigma(a.AM^2) = \Sigma(a).(OL + OM).(OL - OM)$$
$$= \Sigma(a).(OL^2 - OM^2). \quad \text{Q.E.D.}$$

Cor. 1°. When one of the lines M passes through O, then for the other L,

$$\Sigma(a.AL^2) = \Sigma(a.AM^2) + \Sigma(a).OL^2,$$

from which it appears that for a given direction of L the sum

$\Sigma(a.AL^2)$ is a minimum when L passes through O, and has equal values for every two positions equidistant in opposite directions from O; it appears also from the same that if the sum $\Sigma(a.AM^2)$ is constant for all lines passing through O the sum $\Sigma(a.AL^2)$ is constant for all lines touching a circle of any radius described round O as centre.

COR. 2°. The same relation also supplies an obvious solution of the general problem: "Given any system of points A, B, C, D, &c., and any system of corresponding multiples a, b, c, d, &c., whose sum is not $=0$; to draw a line L in a given direction so that the sum $\Sigma(a.AL^2)$ shall be given."

97. *If A, B, C, D, &c. be any system of points, a, b, c, d, &c. any system of corresponding multiples whose sum $=0$, and L and M any two parallel lines, then always*

$$\Sigma(a.AL^2) - \Sigma(a.AM^2) = 2.k.\sin\alpha.ML$$

k and α having the same signification as in (94).

For, as in the preceding, identically

$$\Sigma(a.AL^2) - \Sigma(a.AM^2) = \Sigma(a).\{(AL+AM).(AL-AM)\}$$

from which since $(AL-AM) = ML$ and since (94)

$$\Sigma(a.AL) = \Sigma(a.AM) = k.\sin\alpha,$$

therefore at once, as above,

$$\Sigma(a.AL^2) - \Sigma(a.AM^2) = 2k.\sin\alpha.ML.$$

COR. 1°. When one of the lines M is the particular line for its direction for which the sum $\Sigma(a.AM^2) = 0$, then for the other L,

$$\Sigma(a.AL^2) = 2k.\sin\alpha.ML,$$

from which it appears that for a given direction of L the sum $\Sigma(a.AL)$ follows a very simple law of variation when $\Sigma(a)=0$; being simply proportional in sign as well as in magnitude to the distance of L from a certain line M in that direction; admitting therefore of no minimum or maximum value; passing through 0 and ∞ with the distance ML; and changing sign at the passage through each. *For the particular direction for which $\alpha=0$ whatever be the value of k, and for the exceptional case for which $k=0$ whatever be the value of α, the sum $\Sigma(a.AL^2)$ undergoes no variation with the movement of L, but* *preserves in magnitude and sign the same constant value for every position of L in the same constant direction.*

Cor 2°. To find the line M corresponding to a given direction of L, for which in the general case the sum $\Sigma(a.AM^2)=0$; drawing arbitrarily any line L in the given direction, the parallel M distant from it by the interval $ML =$ in magnitude and sign to the quantity $\frac{\Sigma(a.AL^2)}{2k.\sin\alpha} = \frac{\Sigma(a.AL^2)}{2\Sigma(a.AL)}$, by the relation of Cor. 1°, is that required. *For the particular direction for which $\alpha=0$ whatever be the value of k, and for the exception case for which $k=0$ whatever be the value of α, the sum $\Sigma(a.AL)$ being $=0$, the position of M given by the above is at infinity, unless at the same time the sum $\Sigma(a.AL^2)$ also $=0$ in which case it is indeterminate.*

Cor. 3°. The above supplies an obvious solution of the following general problem: "Given any system of points A, B, C, D, &c., and any system of corresponding multiples a, b, c, d, &c. whose sum $=0$, to draw a line L in any given direction so that the sum $\Sigma(a.AL^2)$ shall have a given magnitude and sign."

98. *If A, B, C, D, &c. be any system of points, O their mean centre for any system of multiples a, b, c, d, &c. whose sum is not $=0$, and P any arbitrary point, then always, whatever be the position of P,*

$$\Sigma(a.AP^2) = \Sigma(a.AO^2) + \Sigma(a).OP^2$$

the same relation as for a system of points on a line and leading to the same consequences. See (80).

For, from the several points A, B, C, D, &c. conceiving perpendiculars AA', BB', CC', DD' &c., let fall upon the line OP, then since (75, Cor. 2°.)

$$AP^2 = AO^2 + OP^2 + 2A'O.OP$$
$$BP^2 = BO^2 + OP^2 + 2B'O.OP$$
$$CP^2 = CO^2 + OP^2 + 2C'O.OP$$
$$DP^2 = DO^2 + OP^2 + 2D'O.OP, \text{ \&c.}$$

therefore multiplying by a, b, c, d, &c. and adding

$$\Sigma(a.AP^2) = \Sigma(a.AO^2) + \Sigma(a).OP^2 + 2.\Sigma(a.A'O).OP,$$

from which, since by (95), $\Sigma(a.A'O)=0$, therefore &c.

COR. 1°. *If round O as centre and with a radius OR whose square = the absolute value of* $\frac{\Sigma(a.AO^2)}{\Sigma(a)}$, *disregarding its sign, a circle be described intersecting the line OP at the two points M and N, and the perpendicular to it through O in either direction at the point S, then, whatever be the position of P, the sum* $\Sigma(a.AP^2) = \Sigma(a).SP^2$ *or* $= \Sigma(a).MP.NP$ *according as* $\Sigma(a)$ *and* $\Sigma(a.AO^2)$ *have similar or opposite signs.*

For, since, by the above relation,

$$\Sigma(a.AP^2) = \Sigma(a).OP^2 + \Sigma(a.AO^2),$$

and since, by construction, $\Sigma(a.AO^2) = \pm\Sigma(a).OR^2$, therefore

$$\Sigma(a.AP^2) = \Sigma(a).(OP^2 \pm OR^2) = \Sigma(a).SP^2 \text{ or } \Sigma(a).MP.NP.$$

Hence, in both cases, the variable sum $\Sigma(a.AP^2)$ has the same value for all positions of P equidistant from O, and the minimum value for the point O itself; it being remembered however that as it vanishes in the second case for all points on the circle OR, and increases negatively from the circumference in to the centre, the term minimum is to be understood in the sense of negative maximum in that case. See (80), Cor. 1°.

COR. 2°. *For every point P on any circle round O as centre, the sum* $\Sigma(a.AP^2)$ *is constant and* $= \Sigma(a.OA^2) + \Sigma(a).\text{radius}^2$ *of circle; and, conversely, every point P for which the sum* $\Sigma(a.AP^2)$ *is constant lies on the circle round O as centre the square of whose radius* $= \frac{\Sigma(a.AP^2) - \Sigma(a.AO^2)}{\Sigma(a)}$.

These are both evident from the above, the first from the general relation $\Sigma(a.AP^2) = \Sigma(a.AO^2) + \Sigma(a).OP^2$, and the other from its equivalent $\Sigma(a.AP^2) - \Sigma(a.AO^2) = \Sigma(a).OP^2$; and they supply obvious solutions of the six general problems, viz.: "Given any system of points A, B, C, D, &c. and any system of corresponding multiples a, b, c, d, &c. whose sum is not $= 0$, to determine on a given line or circle the point P for which the sum $\Sigma(a.AP^2)$ shall be a maximum, a minimum, or given."

From the general property of this corollary, combined with that of Cor. 4°. (92), it follows evidently that every circle round O as centre is at once the locus of a variable point P for which the sum $\Sigma(a.AP^2)$ is constant, and the envelope of a variable line L for which the sum $\Sigma(a.AL)$ is constant.

COR. 3°. *For every point P on the circle inscribed in any triangle ABC, the sum* $BC.AP^2 + CA.BP^2 + AB.CP^2$ *is constant, and exceeds the corresponding sum for the centre O by double the area of the triangle multiplied by the radius of the circle.*

For, by (91), the centre O of that circle being the mean centre of the three points A, B, C for the three multiples BC, CA, AB, therefore, by the above,

$$BC.AP^2 + CA.BP^2 + AB.CP^2 = BC.AO^2 + CA.BO^2 + AB.CO^2 + (BC + CA + AB).OP^2;$$

but, as in Cor. 5°., (92),

$$(BC + CA + AB).OP = 2 \text{ area of triangle};$$

and therefore &c.

A relation exactly similar holds of course for each of the three exscribed circles of the triangle, the sign of the side to which the circle is exscribed being merely changed in the above. See (91), Cor.

COR. 4°. *If O be the centre and OR the radius of the circle which passes through the several vertices A, B, C, D, &c. of a regular polygon of any order n, then for every point P without, within, or upon the circle* $\Sigma(AP^2) = n.(OR^2 + OP^2)$.

For O being the mean centre of the system of points A, B, C, D, &c. for the system of multiples each $= 1$ (90); therefore, by the above,

$$\Sigma(AP^2) = \Sigma(AO^2) + n.OP^2;$$

but $OA = OB = OC = OD, \&c. = OR,$

therefore $\Sigma(AP^2) = n.OR^2 + n.OP^2.$

In the particular case when P is on the circle, since then $OP = OR$, therefore $\Sigma(AP^2) = 2n.OR^2$.

99. *If A, B, C, D, &c. be any system of points, and O their mean centre for any system of multiples a, b, c, d, &c. whose sum is not* $= 0$, *then always*

$$\Sigma(a).\Sigma(a.AO^2) = \Sigma(ab.AB^2),$$

every binary combination of the points of the system being included in the latter summation.

For, in the general relation of the preceding article,

$$\Sigma(a.AP^2) = \Sigma(a).PO^2 + \Sigma(a.AO^2),$$

conceiving the arbitrary point P to coincide successively with the several points A, B, C, D, &c. of the system, then

$$a.AA^2+b.AB^2+c.AC^2+d.AD^2+\&c.=\Sigma(a).AO^2+\Sigma(a.AO^2),$$
$$a.BA^2+b.BB^2+c.BC^2+d.BD^2+\&c.=\Sigma(a).BO^2+\Sigma(a.AO^2),$$
$$a.CA^2+b.CB^2+c.CC^2+d.CD^2+\&c.=\Sigma(a).CO^2+\Sigma(a.AO^2),$$
$$a.DA^2+b.DB^2+c.DC^2+d.DD^2+\&c.=\Sigma(a).DO^2+\Sigma(a.AO^2),\&c.;$$

which multiplied by a, b, c, d, &c. and added give, as $AA=0$, $BB=0$, $CC=0$, $DD=0$, &c., the relation

$$\Sigma(ab.AB^2+ba.BA^2)=\Sigma(a).\Sigma(a.AO^2)+\Sigma(a).\Sigma(a.AO^2);$$

or, which is the same thing, the relation

$$2\Sigma(ab.AB^2)=2\Sigma(a).\Sigma(a.AO^2),$$

the same as the above multiplied by 2.

The relation just proved, as furnishing for any given system of points and multiples the value of the indispensable constant $\Sigma(a.AO^2)$ without requiring the previous determination of the point O, is, consequently, of considerable importance in every numerical application of the formulæ of the preceding article.

Cor. 1°. *If O be the centre of the circle inscribed in any triangle ABC, then*

$$BC.AO^2+CA.BO^2+AB.CO^2=BC.CA.AB,$$

with similar relations for the centres of the three exscribed circles, the sign of the side corresponding to each being simply changed in the above.

For, by (91), O being the mean centre of the three vertices A, B, C for the three multiples BC, CA, AB, therefore, by the above,

$$(BC.AO^2+CA.BO^2+AB.CO^2).(BC+CA+AB)$$
$$=(BA.CA.BC^2+CB.AB.CA^2+AC.BC.AB^2)$$
$$=(BC.CA.AB).(BC+CA+AB);$$

which is the same as the above relation multiplied by

$$BC+CA+AB.$$

Cor. 2°. *If O be the centre and OR the radius of the circle which passes through the several vertices A, B, C, D, &c. of a regular polygon of any order n, then always $\Sigma(AB^2)=n^2.OR^2$.*

For, by (90), O being the mean centre of the system of points A, B, C, D, &c. for the system of multiples each $=1$; therefore, by the above, $\Sigma(AB^2)=n.\Sigma(AO^2)$, but

$$OA=OB=OC=OD, \text{ \&c.} = OR;$$

therefore $\Sigma(OA^2)=n.OR^2$, and therefore &c.

100. *If A, B, C, D, &c. be any system of points, M and N any two lines perpendicular to the direction of their infinitely distant mean centre O for any system of multiples a, b, c, d, &c. whose sum $=0$, and P and Q any two points on M and N not either of them at infinity, then always*

$$\Sigma(a.AP^2)-\Sigma(a.AQ^2)=2k.NM,$$

k having the same signification as in (94).

For, drawing through P and Q two other lines M_0 and N_0 parallel to the direction of O, and therefore at right angles to M and N, then (Euc. I. 47)

$$\Sigma(a.AP^2)=\Sigma(a.AM^2)+\Sigma(a.AM_0^2),$$
$$\Sigma(a.AQ^2)=\Sigma(a.AN^2)+\Sigma(a.AN_0^2);$$

from which, by subtraction, remembering (97) that

$$\Sigma(a.AM^2)-\Sigma(a.AN^2)=2k.NM,$$

and that $$\Sigma(a.AM_0^2)-\Sigma(a.AN_0^2)=0,$$

the relation above stated is the immediate result.

Cor. 1°. From the relation just proved it follows that the two sums $\Sigma(a.AP^2)$ and $\Sigma(a.AQ^2)$ are both constant as long as the two points P and Q continue on the same two lines M and N perpendicular to the direction of O. If one of them N be the particular line in that perpendicular direction for every point Q of which the sum $\Sigma(a.AQ^2)=0$, then for every point P on the other M not at infinity

$$\Sigma(a.AP^2)=2k.NM=2k.NP;$$

from which it appears that the sum $\Sigma(a.AP^2)$ follows, for different positions of P, a very simple law of variation when $\Sigma(a)=0$; being simply proportional in sign as well as magnitude to the distance NP of the variable point P from a constant fixed line N perpendicular to the direction of O; admitting therefore of no minimum or maximum value; passing through nothing and infinity with the distance NP; and changing sign at the passage through each. *In the exceptional case when* $k=0$, *and when therefore* (94) *the position of* O *is indeterminate, the sum* $\Sigma(a.AP^2)$ *undergoes no variation with the variation of* P, *but preserves in magnitude and sign the same constant value for all positions of* P *not actually at infinity;* an instance of which we have met with in (83), where for three points A, B, C on a line, we have seen that for the three multiples BC, CA, AB, the sum

$$BC.AP^2 + CA.BP^2 + AB.CP^2$$

is constant, whatever be the position of P provided only it be not at infinity.

COR. 2°. To find the particular line N perpendicular to the direction of O for every point of which in the general case the sum $\Sigma(a.AQ^2)=0$; drawing arbitrarily any line M perpendicular to the direction of O, the parallel to it N distant from it by the interval NM = in magnitude and sign to the quantity

$$\frac{\Sigma(a.AM^2)}{2k} = \frac{\Sigma(a.AM^2)}{2\Sigma(a.AM)},$$

by the above is that required. *In the exceptional case when* $k=0$, *and when the direction of* L *is therefore indeterminate with that of* O, *the position of* N *given by the above is at infinity, unless at the same time* $\Sigma(a.AM^2)$ *also* $=0$ *in which case it is indeterminate.*

COR. 3°. The above supplies an obvious solution of the following general problem: "Given any system of points A, B, C, D, &c. and any system of corresponding multiples a, b, c, d, &c. whose sum $=0$, to determine on a given line or circle or any other figure the point or points P for which the sum $\Sigma(a.AP^2)$ shall have a given magnitude and sign."

101. The law, determined directly in the preceding, of the variation of $\Sigma(a.PA^2)$ for the particular case of $\Sigma(a)=0$, may also be inferred as a corollary from that of the same for the general case of $\Sigma(a)$ not $=0$, given in (98); for, as in (81) and (94), conceiving the entire system of points $\Sigma(A)$ divided into any two groups $\Sigma(A_1)$ and $\Sigma(A_2)$ for which the sums $\Sigma(a_1)$ and $\Sigma(a_2)$ of the corresponding groups of multiples are not

separately $=0$; then, by the general relation of that article (98), if O_1 and O_2 be the mean centres of the two partial groups for their respective systems of multiples, and P any arbitrary point not at infinity, as

$$\Sigma(a_1.A_1P^2) = \Sigma(a_1.A_1O_1^2) + \Sigma(a_1).O_1P^2,$$

and $$\Sigma(a_2.A_2P^2) = \Sigma(a_2.A_2O_2^2) + \Sigma(a_2).O_2P^2;$$

therefore, by addition, remembering that $\Sigma(a_1.A_1O_1^2)$ and $\Sigma(a_2.A_2O_2)^2$ are both constant, and that $\Sigma(a_1)+\Sigma(a_2)$ by hypothesis $=0$, it appears that the sum $\Sigma(a.AP^2)$ depends on the quantity $\Sigma(a_1).(O_1P^2 - O_2P^2)$ or its equivalent $\Sigma(a_2).(O_2P^2 - O_1P^2)$, that is, on the difference of the squares of O_1P and O_2P, and is therefore constant (Euc. I. 47, Cor.) when P is any where on the same line perpendicular to O_1O_2, and therefore &c. *In the exceptional case when O_1 and O_2 coincide, and when therefore O is indeterminate, as $O_1P^2 - O_2P^2 = 0$ for every position of P not at infinity, the sum $\Sigma(a.AP^2)$ undergoes therefore no variation, but preserves in magnitude and sign the same constant value (which may $=0$) for all positions of P not at infinity.*

COR. If I be the line bisecting at right angles the interval O_1O_2; since then (76), $O_1P^2 - O_2P^2 = 2.O_1O_2.IP$, therefore $\Sigma(a_1).(O_1P^2 - O_2P^2)$, or its equivalent $\Sigma(a_2).(O_2P^2 - O_1P^2)$, $=2\Sigma(a_1).O_1O_2.IP$, or its equivalent $2\Sigma(a_2).O_2O_1.IP$, $=2k.IP$, (94); and therefore if P and Q be any two points on any two lines M and N parallel to I, that is, perpendicular to O_1O_2 the direction of O, then, by the above,

$$\Sigma(a.AP^2) - \Sigma(a.AQ^2) = 2k.(IP - IQ) = 2k.NM;$$

the same formula exactly as that found directly in the preceding and leading of course to the same consequences there given.

102. *If O be the centre of the circle inscribed in any triangle ABC, O', O'', O''' those of the three exscribed to the three sides a, b, c, and s the semi-perimeter, then*

1°. *For every arbitrary line L not at infinity,*

$$(s-a).O'L + (s-b).O''L + (s-c).O'''L - s.OL = 0 \ldots (1).$$

2°. *For every arbitrary point P not at infinity,*

$$(s-a).O'P^2 + (s-b).O''P^2 + (s-c).O'''P^2 - s.OP^2 = 2abc \ldots (2).$$

To prove 1°. From the general relation $\Sigma(a.AL) = \Sigma(a).OL$, (92) applied successively to the four points O, O', O'', O''' regarded (91) as the four mean centres of the three points A, B, C for the four varieties of signs of the three multiples a, b, c,

$$\left.\begin{aligned} a.AL+b.BL+c.CL &= (a+b+c).OL = 2s.OL \\ b.BL+c.CL-a.AL &= (b+c-a).O'L = 2(s-a).O'L \\ c.CL+a.AL-b.BL &= (c+a-b).O''L = 2(s-b).O''L \\ a.AL+b.BL-c.CL &= (a+b-c).O'''L = 2(s-c).O'''L \end{aligned}\right\}\text{..(3)},$$

and it is evident, from mere inspection of their right-hand numbers, that, as above stated, the first is = the sum of the other three.

To prove 2°. From the general relation

$$\Sigma(a.AP^2) - \Sigma(a.AO^2) = \Sigma(a).OP^2, \text{ (98)}$$

applied successively to the four points O, O', O'', O''' regarded as before, and remembering that by Cor. 1°. (99), $\Sigma(a.AO^2) = abc$, and that by the same $\Sigma(a.AO'^2) = \Sigma(a.AO''^2) = \Sigma(a.AO'''^2) = -abc$,

$$\left.\begin{aligned} a.AP^2+b.BP^2+c.CP^2-abc &= (a+b+c).OP^2 = 2s.OP^2 \\ b.BP^2+c.CP^2-a.AP^2+abc &= (b+c-a).O'P^2 = 2(s-a).O'P^2 \\ c.CP^2+a.AP^2-b.BP^2+abc &= (c+a-b).O''P^2 = 2(s-b).O''P^2 \\ a.AP^2+b.BP^2-c.CP^2+abc &= (a+b-c).O'''P^2 = 2(s-c).O'''P^2 \end{aligned}\right\}$$

.... (4),

the first of which subtracted from the sum of the other three, gives evidently the above relation multiplied by 2.

Cor. 1°. Since for every line L passing through any one of the four points O, O', O'', O''' the perpendicular from that point $= 0$, therefore, by relation 1, *each of the four points O, O', O'', O''' is the mean centre of the remaining three for the corresponding three of the four multiples* $-s$, $s-a$, $s-b$, $s-c$; a property the reader may easily prove directly for himself.

Cor. 2°. Denoting by r, r', r'', r''' the radii of the four circles, and by R that of the circle circumscribing the triangle, it may be shown at once—

1°. From relation 1, that

$$\frac{O'L}{r'} + \frac{O''L}{r''} + \frac{O'''L}{r'''} - \frac{OL}{r} = 0 \text{ .... (5).}$$

2°. And from relation 2, that

$$\frac{O'P^2}{r'} + \frac{O''P^2}{r''} + \frac{O'''P^2}{r'''} - \frac{OP^2}{r} = 8R \text{ .... } (6),$$

for since by (92, Cor. 5°.) $(s-a)\,r' = (s-b)\,r'' = (s-c)\,r''' = sr =$ area of triangle $= \Delta$, and since by (64, Cor. 2°.) $abc = 4R\Delta$; therefore dividing 1 and 2 by Δ they assume at once the forms 5 and 6; from the first of which again, as in Cor. 1°. it follows that *each of the four points O, O', O'', O''' is the mean centre of the remaining three for the corresponding three of the four multiples* $-\frac{1}{r}$, $\frac{1}{r'}$, $\frac{1}{r''}$, $\frac{1}{r'''}$, the reciprocals of $-s$, $(s-a)$, $(s-b)$, $(s-c)$ *to the unit whose square* $= \Delta$; a property again as easily proved directly for the reciprocal as for the original multiples.

Cor. 3°. Conceiving in the four relations (4), the arbitrary point P to coincide with the centre of the circle circumscribing the triangle, and denoting in that case by D, D', D'', D''' the four distances OP, $O'P$, $O''P$, $O'''P$, then, as $AP = BP = CP = R$, the four relations become

$$\left.\begin{aligned}(a+b+c).R^2 - abc &= (a+b+c).D^2\\ (b+c-a).R^2 + abc &= (b+c-a).D'^2\\ (c+a-b).R^2 + abc &= (c+a-b).D''^2\\ (a+b-c).R^2 + abc &= (a+b-c).D'''^2\end{aligned}\right\} \text{ .... } (7),$$

which are the formulæ by which to calculate in numbers the four distances D, D', D'', D''' when the sides of the triangle are given; and from which again, as for any other position of P, it follows that

$$(s-a)\,D'^2 + (s-b)\,D''^2 + (s-c)\,D'''^2 - sD^2 = 2abc \text{ ... } (8).$$

Cor. 4°. Substituting in the four relations (7), for abc its value $4R\Delta$ (64, Cor. 2°.), and for s, $(s-a)$, $(s-b)$, $(s-c)$ their values $\frac{\Delta}{r}$, $\frac{\Delta}{r'}$, $\frac{\Delta}{r''}$, $\frac{\Delta}{r'''}$, we get at once the values of the four distances in the well known forms*

$$D^2 = R^2 - 2Rr, \quad D'^2 = R^2 + 2Rr', \quad D''^2 = R^2 + 2Rr'',$$
$$D'''^2 = R^2 + 2Rr''' \text{ .... } (9),$$

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* See Galbraith and Haughton's *Manual of Euclid*, Book IV., Appendix.

from which it appears that *the radii of two circles and the distance between their centres must fulfil a certain relation of condition, in order to the possibility of a triangle being at once circumscribed to one of them and inscribed or exscribed to the other*; a particular case of a more general property which will be given in another chapter.

Cor. 5°. If OT, $O'T$, $O''T'$, $O'''T''$ be the four tangents from the four points O, O', O'', O''' to the circle circumscribing the triangle; since then

$$OT^2 = D^2 - R^2, \quad O'T^2 = D'^2 - R^2, \quad O''T'^2 = D''^2 - R^2,$$
$$O'''T''^2 = D'''^2 - R^2,$$

therefore, by relations 7,

$$OT^2 = -\frac{abc}{a+b+c}, \quad O'T^2 = \frac{abc}{b+c-a}, \quad O''T'^2 = \frac{abc}{c+a-b},$$
$$O'''T''^2 = \frac{abc}{a+b-c} \quad \ldots\ldots\ldots (10),$$

which are the formulæ by which to calculate in numbers the lengths of the four tangents OT, $O'T$, $O''T'$, $O'''T''$ when the sides of the triangle are given; and from which, as is otherwise evident, it appears that the first of the four OT is always imaginary and the remaining three $O'T$, $O''T'$, $O'''T''$ always real.

Cor. 6°. Taking the values of $D^2 - R^2$, $D'^2 - R^2$, $D''^2 - R^2$, $D'''^2 - R^2$ from relation 9, we see again that

$$OT^2 = -2Rr, \quad O'T^2 = 2Rr', \quad O''T'^2 = 2Rr'',$$
$$O'''T''^2 = 2Rr''' \quad \ldots\ldots\ldots (11),$$

which are the formulæ by which to calculate in numbers the length of any one of the four tangents OT, $O'T$, $O''T'$, $O'''T''$ when the radii of the circumscribed and of the corresponding inscribed or exscribed circles are given; and from which it follows at once

that $O'T^2 : O''T'^2 : O'''T''^2 : OT^2 = r' : r'' : r''' : -r$ ... (12),

that $\dfrac{1}{O'T^2} + \dfrac{1}{O''T'^2} + \dfrac{1}{O'''T''^2} + \dfrac{1}{OT^2} = 0$ .... (13),

that $O'T^2 + O''T'^2 + O'''T''^2 + OT^2 = 8R^2$ .... (14), and that each of the four points O, O', O'', O''' is the mean centre of the remaining three for the corresponding three of the four multiples $\frac{1}{OT^2}$, $\frac{1}{O'T'^2}$, $\frac{1}{O''T''^2}$, $\frac{1}{O'''T'''^2}$.

COR. 7°. Regarding the point O as the mean centre of the three O', O'', O''' for the three multiples $\frac{1}{r'}$, $\frac{1}{r''}$, $\frac{1}{r'''}$, (Cor. 2°.) it follows at once from the general relation

$$\Sigma(ab.AB^2) = \Sigma(a).\Sigma(a.AO^2)\ (99),$$

that

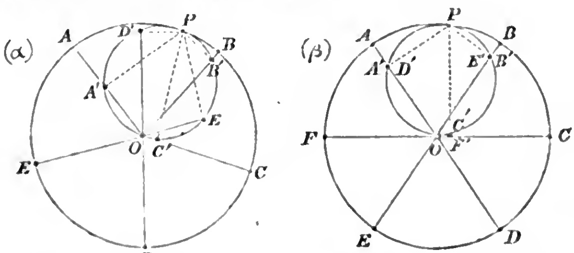
$$\frac{O''O'''^2}{r''r'''} + \frac{O'''O'^2}{r'''r'} + \frac{O'O''^2}{r'r''} = \frac{OO'^2}{rr'} + \frac{OO''^2}{rr''} + \frac{OO'''^2}{rr'''} = 8\frac{R}{r}\ \ldots\ (15),$$

which would also follow at once, as in the general relation referred to, by conceiving the arbitrary point P in relation 6, Cor. 2°., to coincide successively with each of the four points O, O', O'', O'''.

103. *If P be any point on the circle passing through the several vertices A, B, C, D, &c. of a regular polygon of any order n, and L any line passing through the centre O of the figure, then—*

1°. *The sum of the squares of the perpendiculars from P upon the several radii OA, OB, OC, OD, &c. is constant, and $= \frac{1}{2}n$ times the square of the radius of the circle.*

2°. *The sum of the squares of the perpendiculars upon L from the several vertices A, B, C, D, &c. is also constant, and $= \frac{1}{2}n$ times the square of the radius of the circle.*



To prove 1°. On the radius OP as diameter conceiving another circle described intersecting the several radii OA, OB, OC, OD, &c. in the feet A', B', C', D', &c. of the several perpendiculars upon them from P; then the several angles $A'OB'$,

$B'OC'$, $C'OD'$, $D'OE'$, &c. being equal, and each = the n^{th} part of four right angles, the several points A', B', C', D', &c. form therefore on the auxiliary circle, if n be odd (fig. α) the n vertices of a regular polygon of the order n, and if n be even (fig. β) the $2\,\frac{n}{2}$ vertices of two coincident regular polygons of the order $\frac{n}{2}$ (since in that case they evidently coincide two and two in opposite pairs), and therefore in either case $\Sigma(PA'^2)$ (98), Cor. 4°. = $2n$ times the square of the radius of the auxiliary, that is = $\frac{1}{2}n$ times the square of the radius of the original circle, and therefore &c.

N.B. In the same way exactly, it appears that the sum of the squares of the several intercepts OA', OB', OC', OD', &c. between the centre of the circle and the feet of the several perpendiculars from P upon the n radii OA, OB, OC, OD, &c. is constant, and = $\frac{1}{2}n$ times the square of the radius of the circle.

To prove 2°. Since for any two points on a circle, the perpendicular from either upon the diameter passing through the other = the perpendicular from the latter upon the diameter passing through the former, therefore the several perpendiculars from the n points A, B, C, D, &c. upon the one diameter passing through any other point P on the circle = the several perpendiculars from the one point P upon the several diameters passing through the n points A, B, C, D, &c.; but by 1°. the sum of the squares of the latter is constant, and = $\frac{1}{2}n$ times the square of the radius of the circle, therefore so is also the sum of the squares of the former, and therefore &c.

N.B. In the same way exactly it appears, from the note to 1°., that the sum of the squares of the several intercepts between the centre of the circle and the feet of the several perpendiculars from the n vertices A, B, C, D, &c. upon any diameter L is constant, and = $\frac{1}{2}n$ times the square of the radius of the circle.

Cor. 1°. From the above 1°. and 2°. combined with the two properties of regular polygons given in (92, Cor. 6°.) it follows that—

If O be the centre of a regular polygon of any order n, OQ

and OR the radii of its inscribed and circumscribed circles, P any arbitrary point, and L any arbitrary line, then—

1°. *The sum of the squares of the perpendiculars from P upon the several sides is constant and variable with OP, and*

$$= n\,(OQ^2 + \tfrac{1}{2}\,OP^2).$$

2°. *The sum of the squares of the perpendiculars upon L from the several vertices is constant and variable with OL, and*

$$= n\,(OL^2 + \tfrac{1}{2}\,OR^2).$$

To prove 1°. Since by (92, Cor. 6°.), the sum of the perpendiculars from P on the several sides = n times OQ, therefore by (79, Cor. 2°.) the sum of their squares = n times OQ^2 + the sum of the squares of the n differences between each of themselves and OQ; but the circle on OP as diameter intersecting the several perpendiculars from P in the feet of the several perpendiculars upon them from O, and intercepting therefore upon them the several differences in question, therefore by the above 1°. the sum of the squares of the n differences = $\frac{1}{2}n$ times the square of OP, and therefore &c.

To prove 2°. Since by (92, Cor. 6°.) the sum of the perpendiculars upon L from the several vertices = n times OL, therefore by (79, Cor. 2°.) the sum of their squares = n term OL^2 + the sum of the squares of the n differences between each of themselves and OL; but the parallel to L passing through O cutting off from the several perpendiculars the n differences in question, therefore by the above 2°. the sum of the squares of the n differences = $\frac{1}{2}n$ times the square of OP, and therefore &c.; and the same thing is also evident from the general property (96, Cor. 1°.) of which this is evidently a particular case.

N.B. It is evident from the above 1°. and 2°. that every circle concentric with a regular polygon of any order, is at once the locus of a variable point the sum of the squares of whose distances from the several sides is constant, and the envelope of a variable line the sum of the squares of whose distances from the several vertices is constant.

Cor. 2°. Conceiving, in the above, the arbitrary point P to be on the circle OQ, and the arbitrary line L to touch the circle OR, it follows at once that—

1°. *If from any point on the circle inscribed in a regular polygon of any order n perpendiculars be let fall on the several sides, the sum of their squares is constant, and* = $\frac{3}{2}n$*.radius*^{2} *of circle.*

2°. *If upon any line L touching the circle circumscribed to a regular polygon of any order n perpendiculars be let fall from the several vertices, the sum of their squares is constant, and* = $\frac{3}{2}n$*.radius*^{2} *of circle.*

For, when in 1°. $OP = OQ$, then $OQ^2 + \frac{1}{2}OP^2 = \frac{3}{2}OQ^2$, and therefore &c.; and when in 2°. $OL = OR$, then $OL^2 + \frac{1}{2}OR^2 = \frac{3}{2}OR^2$, and therefore &c.

Cor. 3°. Comparing with each other the values of the two sums of squares in the particular cases just given, it follows also that—

1°. *If two regular polygons of any common order n be constructed one circumscribed and the other inscribed to the same circle, the constant sum of the squares of the perpendiculars from any point on the circle upon all the sides of the former = the constant sum of the squares of the perpendiculars upon any tangent to the circle from all the vertices of the latter.*

2°. *If two circles be described one circumscribed and the other inscribed to the same regular polygon of any order n, the constant sum of the squares of the perpendiculars upon all the sides of the polygon from any point on the former = the constant sum of the squares of the perpendiculars from all the vertices of the polygon upon any tangent to the latter.*

For, by the above 1°. and 2°. both constant sums, in the former case = $\frac{3}{2}n$.radius^2 of common circle, and in the latter case = n.radius^2 of inscribed circle + $\frac{1}{2}n$.radius^2 of circumscribed circle, and therefore &c.

CHAPTER VII.

ON COMPLETE AND INCOMPLETE FIGURES OF POINTS AND LINES.

104. EVERY system of points or lines, whatever be their number and position, in which the several lines of connection or points of intersection of each point or line with all the others are taken into account without exception, is said to form a *complete* figure, which in the absence of any as yet generally recognized nomenclature may be termed a *polystigm* in the former case and a *polygram* in the latter. A system of points or lines, on the other hand, in which any of the lines of connection or points of intersection of each point or line with all the others are omitted, is said to form an *incomplete* figure, whose degree of incompleteness depends of course on the number of the omitted points or lines. In the extreme case of the latter, when the lines of connection or points of intersection of each point or line with but two others are taken into account, the figure evidently is simply a *polygon*, of which the several points or lines of the system are the several vertices or sides, and of which the shape and character depend, of course, on the *order of sequence* in which the several points or lines of the system are taken in the several connections or intersections of each with the two regarded as adjacent to it.

105. The several points or lines constituting the vertices or sides of a polygon of any order being always taken in some definite order of sequence, it is therefore an intelligible mode of expression to speak, as is often done, of "opposite vertices" and of "opposite sides" in one of an even order, or, of "the vertex opposite to a side" and of "the side opposite to a vertex" in one of an odd order; but to speak similarly of the constituent points or lines determining a complete figure of any

order would be meaningless and consequently inadmissible; each point or line standing by itself absolutely, and having no relation of the nature expressed by such terms as "adjacent," "opposite," &c. to any other.

106. But though *the determining points or lines* in complete figures have no relation amongst each other as regards order of sequence, *certain other elements* of the figures may be, and often are, with propriety and convenience, said to be *opposites* to each other; thus, for instance, in a tetrastigm or tetragram every line of connection of two points or point of intersection of two lines is said to be the opposite of that of the remaining two; in a hexastigm or hexagram every triangle determined by three points or lines is said to be the opposite of that determined by the remaining three; and, generally in a polystigm or polygram of any even order, every two polystigms or polygrams of inferior orders determined by half the points or lines and by the remaining half are said to be opposites to each other, &c.

107. In the complete figure determined by any number of points, every two points are said to determine *a line of connection*, and every two lines of connection to determine *an angle of connection* of the figure. In the complete figure determined by any number of lines, every two lines are said to determine *a point of intersection*, and every two points of intersection to determine *a chord of intersection* of the figure; for the same obvious reason as for the extreme case of incomplete figures, the several chords of intersection in the latter case are sometimes termed also *diagonals* of the figure.

108. If n be the number of the points or lines determining a complete figure of either species, it may be easily shown that, generally:

1°. *The entire number of lines of connection or of points of intersection of the figure* $= \frac{n(n-1)}{2}$.

2°. *The entire number of angles of connection or of chords of intersection of the figure* $= \frac{n(n-1)(n-2)(n-3)}{8}$.

3°. *The entire number of polygons of which the determining points or lines are the vertices or sides* $= \frac{(n-1)(n-2)(n-3)\ldots 1}{2}$.

For, in the case of 1°., the n points or lines connecting or intersecting each with the remaining $(n-1)$ produce $n(n-1)$ lines of connection or points of intersection *coinciding in pairs*, and therefore &c.; in the case of 2°., the $\frac{n(n-1)}{2}$ lines of connection or points of intersection of two points or lines intersecting or connecting with the $\frac{(n-2)(n-3)}{2}$ for the remaining $(n-2)$ produce $\frac{n(n-1)(n-2)(n-3)}{4}$ angles of connection or chords of intersection *coinciding in pairs*, and therefore &c.; and, in the case of 3°., any one of the n points or lines, taken arbitrarily as *first* vertex or side of all the polygons, may be followed in order of sequence by any one of the remaining $(n-1)$ as *second* vertex or side, each of which $(n-1)$ second vertices or sides may be followed in order of sequence by any one of the remaining $(n-2)$ as *third* vertex or side, each of which $(n-1)(n-2)$ third vertices or sides may be followed in order of sequence by any one of the remaining $(n-3)$ as *fourth* vertex or side, and so on to the last, thus producing $(n-1).(n-2).(n-3).(n-4)$, &c. 2.3.1 *last* vertices or sides, and therefore the same number of polygons *coinciding in pairs*, every order of sequence giving evidently the same polygon as the reverse order, and therefore &c.

109. A polygon of any order greater than three is said to be *convex*, *reentrant*, or *intersecting*, according as every two of its non-adjacent sides intersect externally, as any two of them intersect one externally and one internally, or as any two of them intersect internally; thus the quadrilateral *ABCD* in

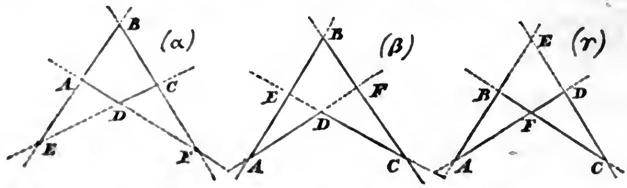
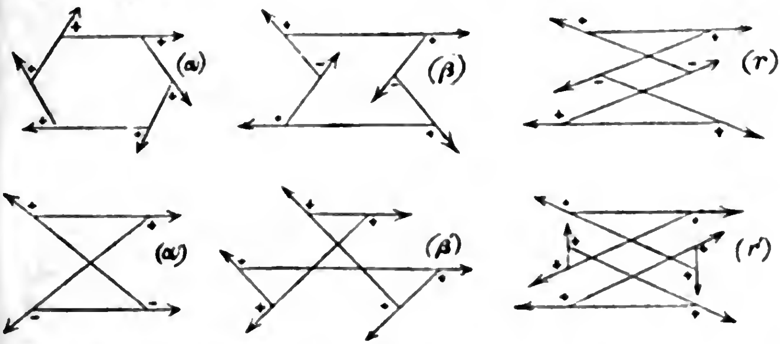


fig. α is convex, in fig. β is reentrant, and in fig. γ is intersecting, and it is evident from any of the three figures that of the three different quadrilaterals $ABCD$, $AECF$, $BEDF$, determined by the same four lines (108), one is always convex, one always reentrant, and one always intersecting.

110. *The sum of the external angles of a polygon of any order, convex, reentrant, or intersecting, regard being had to their signs as well as their magnitudes,* $= \pm 4m$ *right angles,* m *being some integer of the natural series* 0, 1, 2, 3, &c. *less than half the order of the polygon.*

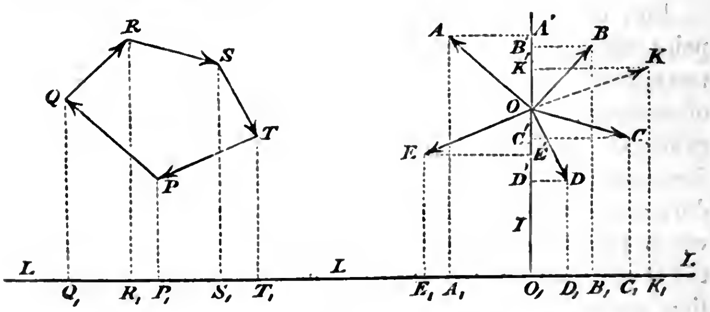


For, conceiving the polygon described by the motion of a point setting out from any position on one of its sides, and traversing its entire perimeter, returning again to the point of starting; the several *external* angles of the polygon are then evidently the several *deviations to the right or left*, in the direction of its motion, made by the describing point in passing during the circuit from the several sides to their successors, which for convex polygons universally (fig. α), and for others too occasionally (figs. β' and γ'), take place all in the same direction, and have therefore all the same sign, but which for reentrant and intersecting polygons generally (figs. β, γ, and α') take place some in one and others in the opposite direction, and have therefore some one and others the opposite sign; but since, on the completion of the entire circuit, the original direction of the motion is always finally regained, therefore *the total amount of deviation* however made up, that is the sum with their proper signs of the external angles of the polygon, $=0$, or $=4$ right angles, or $=4m$ right angles, m however being always less than half the order of the polygon, the deviation at each angle being necessarily limited to two right angles.

In the three polygons represented in figs. α, β, and γ, and in all convex polygons universally, $m=1$; in the three represented in figs. α', β', and γ', $m=0$, $=2$, and $=3$ respectively; and in all six alike the sides are supposed to be described in the directions indicated by the arrow heads in the figures, and the deviations are supposed to be positive or negative according as they take place to the right or to the left, as marked in the figures.

Any two sides of a polygon are said *to be measured cyclically in similar or opposite directions*, according as a moving point, going round as above the entire perimeter continuously in the same cyclic direction, would describe both in directions similar or opposite to those of their measurement or describe one in the similar and the other in the opposite direction.

111. *If the several sides of any polygon measured cyclically in the same direction be projected in any direction upon any line, the sum of the projections, regard being had to their signs as well as to their magnitudes*, $=0$.



For, if P, Q, R, S, T, &c. be the several vertices of the polygon, and P_1, Q_1, R_1, S_1, T_1, &c. their several projections upon any arbitrary line L, then the several sides, measured cyclically in the common direction indicated by the arrow heads in the figure, being PQ, QR, RS, ST, &c. returning back again to P, their several projections on L are respectively P_1Q_1, Q_1R_1, R_1S_1, S_1T_1, &c. returning back again to P_1, and the sum of the latter (by 78) being always $=0$, therefore &c.

The above useful property may obviously be stated otherwise thus, as follows—

If the several sides of any polygon be projected in any direction upon any line, the projection of any one side measured cyclically in either direction, or more generally the sum of the projections of any number of the sides measured cyclically in either direction, is equal to the sum of the projections of the remaining sides measured cyclically in the opposite direction.

112. Assuming the evident property that, if two finite parallel lines, however circumstanced as to absolute position, be equal and co-directional, their projections in any direction upon any line are equal and co-directional; the following consequences result immediately from the very useful property of the preceding article, viz.—

1°. *A system of any number of finite lines given in length and direction but not in absolute position would form a polygon if placed end to end in any order of sequence, provided that for two different directions of projection the sum of their projections upon any line* = 0.

For, if when placed end to end in any one of the different orders of sequence in which they could be disposed, the last extremity of the last side failed to coincide with the first extremity of the first side, then, though the sum of their projections would = 0 upon every line *for the particular direction of projection parallel to the line connecting those two extremities*, such obviously would not be the case upon any line for *any other* direction of projection, and therefore &c.

2°. *If a system of any number of finite lines given in length and direction but not in absolute position would form a polygon if placed end to end in any one order of sequence, they would do so equally for every order of sequence in which they could be disposed.*

For, if for any one order of sequence they formed a polygon, then since, by (111), the sum of their projections in every direction upon every line = 0, therefore, by 1°., they would form a polygon for every order of sequence, and therefore &c.

3°. *If a system of any number of finite lines, however circumstanced as to direction, length, and position, be such that for two* *different directions of projection the sum of their projections upon any line* $=0$, *then for every direction of projection the sum of their projections upon any line* $=0$.

For, if without alteration of length or direction they were, if not already in such a position, placed end to end in any order of sequence, then, since by hypothesis the sum of their projections for two different directions $=0$, therefore, by 1°., they would form a polygon, and therefore, by (111), the sum of their projections for every direction $=0$.

113. If from any point O as common origin (fig., Art. 111) a system of finite lines OA, OB, OC, OD, &c. be drawn parallel, equal, and co-directional to the several sides PQ, QR, RS, ST, &c. of any polygon $PQRST$ &c. measured cyclically in the same direction, it is easy to see from the same property that—

1°. *The sum of their projections in any direction upon any line* $=0$.

2°. *The sum of the perpendiculars, or any other isoclinals, from their extremities upon any line passing through* $O=0$.

3°. *The sum of the areas of the triangles they subtend at any point not at infinity* $=0$.

4°. *The sum of the rectangles under them and the perpendiculars upon them from any point not at infinity* $=0$.

To prove 1°. and 2°. If O_1, A_1, B_1, C_1, D_1, &c. be the several projections in any direction OO_1 of the several points O, A, B, C, D, &c. upon any line L; P_1, Q_1, R_1, S_1, T_1, &c. those of P, Q, R, S, T, &c. in the same direction on the same line, and AA', BB', CC', DD', &c. the several isoclinals from A, B, C, D, &c. to OO_1 in the direction parallel to L; then since, by hypothesis, Art. 112, and Euc. I. 34, $P_1Q_1=O_1A_1=A'A$, $Q_1R_1=O_1B_1=B'B$, $R_1S_1=O_1C_1=C'C$, $S_1T_1=O_1D_1=D'D$, &c., and since, by (111),

$$P_1Q_1+Q_1R_1+R_1S_1+S_1T_1+\&c.=0,$$

therefore $$O_1A_1+O_1B_1+O_1C_1+O_1D_1+\&c.=0,$$

and $$AA'+BB'+CC'+DD'+\&c.=0,$$

and therefore &c., the directions of L and of OO_1 being entirely arbitrary.

To prove 3°. If I be the point and AA', BB', CC', DD', &c. the several perpendiculars from A, B, C, D, &c. upon the line OI passing through the two points O and I, then since, by 2°.,

$$AA' + BB' + CC' + DD' + \&c. = 0,$$

and since by hypothesis OI is not infinite, therefore

$$OI.AA' + OI.BB' + OI.CC' + OI.DD' + \&c. = 0,$$

and therefore &c., each rectangle being double the area of the triangle subtended by its base at the point I.

To prove 4°. If I, as before, be the point and IX, IY, IZ, &c. the several perpendiculars from it upon OA, OB, OC, &c., then since $OA.IX = 2$ area OAI, $OB.IY = 2$ area OBI, $OC.IZ = 2$ area OCI, &c., and since, by 3°.,

$$2 \text{ area } OAI + 2 \text{ area } OBI + 2 \text{ area } OCI + \&c. = 0,$$

therefore $\quad OA.IX + OB.IY + OC.IZ + \&c. = 0,$

and therefore &c.

Of the above properties, 2°. shews evidently (86) that the point O is the mean centre of the system of points A, B, C, D, &c. for any system of multiples having a common magnitude and sign; and 4°. expresses obviously for any number of lines OA, OB, OC, OD, &c. passing through a common point O, what the property, Cor. 6°., Art. 82, established on other considerations in Chapter V., expresses for three.

114. When any number of lines OA, OB, OC, OD, &c. diverging from a common origin O, are, as in the preceding article, parallel, equal, and co-directional to the several sides of a polygon $PQRST$ &c. measured cyclically in the same direction, any one of them OE turned without change of length round the common origin O into the opposite direction OK is termed *the resultant* of the others OA, OB, OC, OD, &c., a name borrowed from the Science of Mechanics, in which the properties of the preceding and of some of the following articles are of considerable importance.

As all the sides but one of a polygon of any order are of course perfectly arbitrary in length and direction, the length and direction of the last however being implicitly given with those of the others, therefore the several lines OA, OB, OC, OD, &c. composing the system of which OK is the resultant as above defined are equally arbitrary in length and direction, but their lengths and directions once given their resultant in length and direction is implicitly given with them; two very rapid constructions for its determination in all cases will be presently given.

In the particular case of but two components OA and OB, the resultant OK in length and direction is evidently the conterminous diagonal of the parallelogram of which OA and OB in length and direction are adjacent sides. *All properties therefore which are true in general of any system of coinitial lines and their resultant are true in particular of two adjacent sides and the conterminous diagonal of any parallelogram.*

115. Since, in accordance with the foregoing definition, the several pairs of magnitudes OE and OK, OE' and OK', OEI and OKI, O_1E_1 and O_1K_{11}, EE' and KK', &c., in the figure of Art. 111, are equal and opposite, it follows at once from the several properties of Article 113 that the resultant OK of any system of lines OA, OB, OC, OD, &c. diverging from a common origin O possesses the following properties with respect to the component lines of the system—

1°. *The sum of the projections of the components in any direction upon any line is equal in magnitude and sign to the projection of the resultant in the same direction upon the same line.*

2°. *The sum of the perpendiculars or other isoclinals from the extremities of the components upon any line passing through the common origin O is equal in magnitude and sign to the perpendicular or isoclinal from the extremity of the resultant on the same line.*

3°. *The sum of the areas of the triangles subtended by the components at any point not at infinity is equal in magnitude and sign to the area of the triangle subtended by the resultant at the same point.*

4°. *The sum of the rectangles under the components and the perpendiculars upon them from any point not at infinity is equal in magnitude and sign to the rectangle under the resultant and the perpendicular upon it from the same point.*

These properties require no proof, they result immediately, as above enumerated from those similarly mentioned in Art. 113, from the obvious consideration that when the sum of a number of magnitudes of any kind = 0 then any one of them changed in sign = the sum of all the others; and it follows at once from any or all of them, as is also evident from the fundamental definition of the preceding article, that for a system of components parallel, equal, and co-directional to the several sides of any polygon measured cyclically in the same direction, the resultant is in magnitude evanescent and in direction indeterminate.

116. *Given in magnitude and direction any number of lines OA, OB, OC, OD, &c. diverging from a common origin O, to determine their resultant OK in magnitude and direction.*

First method. From any arbitrarily assumed point P (fig., Art. 111), drawing a line PQ parallel, equal, and co-directional to any one of the components OA; from its opposite extremity Q a second QR parallel, equal, and co-directional to a second of them OB; from its opposite extremity R a third RS parallel, equal, and co-directional to a third of them OC; from its opposite extremity S a fourth ST parallel, equal, and co-directional to a fourth of them OD; and so on until all the components are exhausted. The line OK from O parallel, equal, and co-directional to the line PT connecting the first extremity P of the first parallel PQ with the last extremity T of the last parallel ST is (114) the resultant required.

Should the last point T, determined by this construction, coincide with the first point P, assumed originally, that is, should the given lines OA, OB, OC, OD, &c. form a system parallel, equal, and co-directional to the several sides of any polygon measured cyclically in the same direction; their resultant OK, thus determined would, as it ought (115), be evanescent in magnitude and indeterminate in direction.

Second method. Projecting all the components OA, OB, OC, OD, &c. in any direction upon any line OO_1 (same figure) passing through their common origin O, and measuring from O on OO_1 a length OK' equal in magnitude and sign to the sum of the several projections OA', OB', OC', OD', &c., the length OK' thus determined is (115) the corresponding projection of the required resultant OK. Repeating the same process with a different direction of projection on the same or another line passing through O, the new length similarly determined is a second projection of the required resultant OK, and therefore &c.

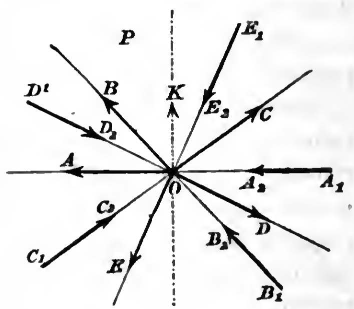
Should the two different lengths, determined as above, be both $=0$, that is, (112, 1°.) should the given lines OA, OB, OC, OD, &c. form a system parallel, equal, and co-directional to the several sides of any polygon measured cyclically in the same direction, their resultant OK thus determined would again, as it ought, be evanescent in magnitude and indeterminate in direction.

Of the above two general constructions the second, though less obvious and simple, is better adapted to numerical computation than the first.

117. The principles established in the preceding articles supply a ready solution of the very general problem—

Required the locus of a variable point P for which the sum of the areas of the system of triangles A_1PA_2, B_1PB_2, C_1PC_2, D_1PD_2, &c., subtended by any number of fixed bases A_1A_2, B_1B_2, C_1C_2, D_1D_2, &c. is constant, the length and line of direction with the positive and negative sides of each base being given.

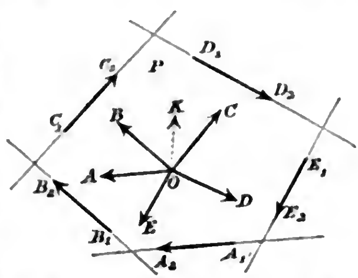
Case 1°. When the several lines of direction of the several bases A_1A_2, B_1B_2, C_1C_2, D_1D_2, &c. pass through a common point O; from the common point O measuring on the several lines of direction lengths OA, OB, OC, OD, &c., equal to the several lengths A_1A_2, B_1B_2, C_1C_2, D_1D_2, &c., and in directions, indicated by the arrow heads in the figure, such that the positive and nega-



tive sides of the several bases correspond to the right and left sides of the several directions, and taking, by (116), in length and direction the resultant OK of the several coinitial lines OA, OB, OC, OD, &c. thus obtained; then for every arbitrary point P not at infinity, since, by Euc. I. 38, the sum of the system of triangles $\Sigma(A_1PA_2)$ = the sum of the system of triangles $\Sigma(OPA)$, and since, by (115, 3°.), the sum of the latter system of triangles = the single triangle OPK, therefore for every position of P not at infinity the sum of the system of triangles $\Sigma(A_1PA_2)$ = the single triangle OPK; but the base OK of the latter being fixed its area is positive, negative, or nothing, according as its vertex P lies on the right or left side of or upon the line of direction of OK, and if its area is constant the locus of its vertex P is a line parallel to OK and at a distance from it equal in magnitude and sign to twice the constant value divided by OK.

COR. In the particular case when $OK=0$, that is, when the several bases A_1A_2, B_1B_2, C_1C_2, D_1D_2, &c. are parallel, equal, and co-directional with the several sides of a polygon measured cyclically in the same direction; then, as is evident from the above, *the sum of the areas of the system of triangles* $\Sigma(A_1PA_2)=0$ *for every position of P not at infinity.*

Case 2°. When the several lines of direction of the several bases A_1A_2, B_1B_2, C_1C_2, D_1D_2, &c. do not pass through a common point; assuming arbitrarily any fixed point O not at infinity, drawing from it a system of lines OA, OB, OC, OD, &c., parallel and equal to the several bases A_1A_2, B_1B_2, C_1C_2, D_1D_2, &c., and in directions, indicated



by the arrow heads in the figure, such that the positive and negative sides of the several bases correspond as before to the right and left sides of the several directions, and taking as before in magnitude and direction the resultant OK of the system of coinitial lines OA, OB, OC, OD, &c. thus obtained; then for every arbitrary point P not at infinity, since, by (75), the sum of the system of triangles $\Sigma(A_1PA_2)$ = the sum of the system of triangles $\Sigma(A_1OA_2)$ + the sum of the system of triangles $\Sigma(OPA)$, and since, by (115, 3°.), the sum of the latter system of triangles = the single triangle OPK, therefore for every position of P not at infinity, the sum of the system of triangles $\Sigma(A_1PA_2)$ = the sum of the system of triangles $\Sigma(A_1OA_2)$ + the single triangle OPK; but the sum of the system of triangles $\Sigma(A_1OA_2)$ being fixed with the point O, and the base OK of the single triangle OPK being also fixed with the same, if the sum of the system of triangles $\Sigma(A_1PA_2)$ be constant, the locus of P is a line parallel to OK and distant from it by an interval equal in magnitude and sign to the constant sum $\Sigma(A_1PA_2)$ − the fixed sum $\Sigma(A_1OA_2)$ divided by half the length of OK.

COR. In the particular case when $OK=0$, that is, when the several bases A_1A_2, B_1B_2, C_1C_2, D_1D_2, &c. are parallel, equal, and co-directional with the several sides of a polygon measured cyclically in the same direction, then, as is evident from the above, *the sum of the areas of the system of triangles* $\Sigma(A_1PA_2)$ *is constant for every position of* P *not at infinity.*

118. As the several fixed bases A_1A_2, B_1B_2, C_1C_2, D_1D_2, &c., in the general case of the preceding, may be in length and direction the several sides of any one of the different polygons determined by their several lines of direction (108, 3°.) measured cyclically in the same direction, and as then, by the corollary to that case, the sum of the areas of the several triangles A_1PA_2, B_1PB_2, C_1PC_2, D_1PD_2, &c. is constant for every position of P not at infinity; hence the important property that—

For a polygon of any form, convex, reentrant, or intersecting, the sum of the several triangular areas subtended by the several sides at any point not at infinity is constant, any two of the triangles being regarded as having similar or opposite signs according as they lie at similar or opposite sides of their respective bases measured cyclically in either common direction.

This property is important as supplying a formal *definition* of the *area* of a *polygon*, which is applicable without exception to every variety of form whether convex, reentrant, or intersecting, viz., "The constant sum of the areas of the several triangles subtended by the several sides at any arbitrary point not at infinity and regarded as positive or negative according as they lie at the positive or negative sides of their several bases measured cyclically in either common direction."

If an intersecting polygon were of such a form that the sum of the triangular elements constituting its area as thus defined $=0$ for any one point not at infinity, they would of course by virtue of the above $=0$ for every point not at infinity, and the area of the polygon would consequently $=0$; an intersecting quadrilateral in which the two opposite sides that do not intersect internally are equal and parallel, (as in fig. a', Art. 110), furnishes the simplest example of a polygon of this nature.

119. The linear locus in the general case of Art. 117 supplies obvious solutions of the four following very general problems—

Given in magnitude, position, and direction any number of fixed bases A_1A_2, B_1B_2, C_1C_2, D_1D_2, *&c. to determine—*

1°. *On a given line the point* P *for which the sum of the several triangular areas* $\Sigma(A_1PA_2)$ *shall have any given value, positive, negative, evanescent, or infinite.*

2°. *On a given circle the point* P *for which the sum of the several triangular areas* $\Sigma(A_1PA_2)$ *shall have the maximum, the minimum, or any intermediate given value.*

In the particular case when the several bases A_1A_2, B_1B_2, C_1C_2, D_1D_2, &c. are parallel, equal, and co-directional with the several sides of a polygon measured cyclically in the same direction, the sum $\Sigma(A_1PA_2)$ being then constant for every position of P not at infinity, these several problems are in consequence all indeterminate.

120. Denoting by A, B, C, D, &c. the several indefinite lines of direction, and by a, b, c, d, &c. the several numerical values to any unit of linear measure of the several fixed bases A_1A_2, B_1B_2, C_1C_2, D_1D_2, &c. in the linear locus of Art. 117; it follows immediately from the general property there established that—

If A, B, C, D, *&c. be any system of lines disposed in any manner, but none infinitely distant, and* a, b, c, d, *&c. any system of corresponding multiples positive or negative, but none infinitely great, the locus of a variable point* P *for which the sum*

$$a.PA + b.PB + c.PC + d.PD + \&c.,$$

or more shortly $\Sigma(a.PA)$, *has any constant value, positive, negative, or nothing, is a line whose direction depends on the directions of the lines and the ratios of the multiples, and whose position depends on the value of the constant.*

The positions and sides, positive and negative, of the several lines A, B, C, D, &c., and the magnitudes and signs, positive or negative, of the several multiples a, b, c, d, &c. being given, to determine the common direction of the several loci for all values of the constant, the particular position of the locus for any particular value of the constant, and the law governing the variation of the locus for the variation of the constant; on the several lines A, B, C, D, &c. from any arbitrarily assumed points A_1, B_1, C_1, D_1, &c., taking any system of lengths A_1A_2, B_1B_2, C_1C_2, D_1D_2, &c., proportional to the numerical values of the several multiples a, b, c, d, &c., and in directions, indicated by the arrow heads in the figures, such that the positive and negative signs of the several products $a.PA$, $b.PB$, $c.PC$, $d.PD$, &c. shall correspond to the right and left sides of the several directions; then since for every position of P not at infinity $A_1A_2.PA = 2$ area A_1PA_2, $B_1B_2.PB = 2$ area B_1PB_2, $C_1C_2.PC = 2$ area C_1PC_2, $D_1D_2.PD = 2$ area D_1PD_2, &c., and since therefore $\Sigma(A_1A_2.PA) = 2\Sigma(A_1PA_2)$, therefore, by (117), the locus of P for which $\Sigma(A_1A_2.PA)$ has any constant value, positive, negative, or nothing, is a line L parallel to the resultant OK of any coinitial system of lines OA, OB, OC, OD, &c., parallel, equal, and co-directional with A_1A_2, B_1B_2, C_1C_2, D_1D_2, &c., and distant from it by an interval equal in magnitude and sign to the quantity $\dfrac{\Sigma(A_1A_2.PA) - \Sigma(A_1A_2.OA)}{OK}$, or to its equivalent $\dfrac{\Sigma(a.PA) - \Sigma(a.OA)}{k}$, where k is the numerical value of OK to the same unit that a, b, c, d, &c. are those of OA, OB, OC, OD, &c.

If I be the particular line of the system parallel to OK for which the value of the constant $= 0$, it is easy to see that for any other line L of the system its value $= k.LI$; for, since for any two points P and Q on any two lines L and M parallel to OK, by (117),

$$\Sigma(a.PA) = \Sigma(a.OA) + k.LO \text{ and } \Sigma(a.QA) = \Sigma(a.OA) + k.MO,$$

therefore at once, by subtraction,

$$\Sigma(a.PA) - \Sigma(a.QA) = k.LM,$$

and therefore if M be the particular line I of the system for every point Q of which $\Sigma(a.QA) = 0$, then for every point P of any other line L of the system $\Sigma(a.PA) = k.LI$, as above stated.

Given the particulars of the system of lines A, B, C, D, &c. and of the system of multiples a, b, c, d, &c. to determine the line I. Assuming arbitrarily any point O, and drawing from it in magnitude and direction the resultant OK of the coinitial system of lines parallel, equal, and co-directional with the several segments A_1A_2, B_1B_2, C_1C_2, D_1D_2, &c. determined as above, the line I parallel to OK, distant from it by the interval $\Sigma(a.OA) \div k$, and at the positive or negative side of its direction according as the sign of $\Sigma(a.OA)$ is negative or positive, is, by the above, that required.

The line I, for every point Q of which the constant sum $\Sigma(a.QA) = 0$, is termed *the central axis* of the system of lines A, B, C, D, &c. for the system of multiples a, b, c, d, &c., and, by aid of it, determined as above or otherwise, the position of the parallel line L for every point P of which the constant sum $\Sigma(a.PA)$ shall have any given value, positive or negative, is given at once by the above; for it is distant from I by the interval $\Sigma(a.PA) \div k$, and it lies at its positive or negative side according as the sign of $\Sigma(a.PA)$ is positive or negative.

In the particular case when $k = 0$, that is (116), when the several segments A_1A_2, B_1B_2, C_1C_2, D_1D_2, &c., determined as above, are parallel, equal, and co-directional with the several sides of a polygon measured cyclically in the same direction, *the central axis I is at infinity, except only when the value of $\Sigma(a.PA)$, which* (117, Cor.) *is then constant for every position of P not at infinity, $= 0$, in which exceptional case it is indeterminate.* And for the same reason generally the several parallel loci of the present article are all at infinity, except only the particular one corresponding to the constant value of $\Sigma(a.PA)$, which one is indeterminate.

121. *If A, B, C be any three lines, I their central axis for any three multiples a, b, c, and P, Q, R the three points at which A, B, C intersect with I, then always* (see 91, 1°.)

$$b.PB + c.PC = 0, \quad c.QC + a.QA = 0, \quad a.RA + b.RB = 0.$$

For, since for every three points P, Q, R on I, by the preceding, $a.PA + b.PB + c.PC = 0$, $a.QA + b.QB + c.QC = 0$, $a.RA + b.RB + c.RC = 0$; if P lie on A, then $b.PB + c.PC = 0$;

if Q lie on B, then $c.QC+a.QA=0$; if R lie on C, then $a.RA+b.RB=0$; and therefore &c.

Of the above, which supplies an obvious and very rapid method of determining the central axis I of any three lines A, B, C for any three multiples a, b, c, the two following particular cases are deserving of attention. See (91, Cor.).

1°. If in absolute magnitude $a=b=c$ the three lines connecting P, Q, R with the three opposite vertices bisect (61) the three opposite angles BC, CA, AB of the triangle ABC, all externally, or one externally and two internally, according as the signs of a, b, c are all similar, or that of one opposite to those of the other two.

2°. If in absolute magnitude $a:b:c$ as the lengths of the three corresponding sides of the triangle ABC, the three points P, Q, R bisect (65, Cor. 3°.) the three sides on which they lie, all externally, or one externally and two internally, according as the signs of a, b, c are all similar, or that of one opposite to those of the other two.

In the first case of 2°., the three points of external bisection of the three sides of the triangle ABC being at infinity, so therefore is the central axis I which contains them; this is in exact accordance with the closing observation of the preceding article, the three segments A_1A_2, B_1B_2, C_1C_2, determined as there directed on the three lines A, B, C, being then parallel, equal, and co-directional with the three sides of the triangle ABC measured cyclically in the same direction.

122. The linear loci of Art. 120, determinable as there explained for all given values of the constant $\Sigma(a.PA)$, supply obvious solutions of the four following very general problems analogous to those of Art. 119—

Given the positions and sides, positive and negative, of any system of lines A, B, C, D, &c., and the magnitudes and signs, positive or negative, of any corresponding system of multiples a, b, c, d, &c., to determine—

1°. *On a given line the point P for which the sum $\Sigma(a.PA)$ shall have any given value, positive, negative, evanescent, or infinite.*

2°. *On a given circle the point P for which the sum $\Sigma(a.PA)$*

shall have the maximum, the minimum, or any intermediate given value.

In the particular case, when, as explained in the closing paragraph of that article (120), the particulars of the lines and multiples are such that the sum $\Sigma(a.PA)$ has the same value for every position of P not at infinity, then such problems are of course indeterminate for that particular value, and impossible at a finite distance for every other value of the sum.

123. Since in the particular case when the several segments A_1A_2, B_1B_2, C_1C_2, D_1D_2, &c., determined as in (120), on the several lines A, B, C, D, &c. are parallel, equal, and co-directional with the several sides of a polygon measured cyclically in the same direction, then, by (117), the sum $\Sigma(a.PA)$ has the same constant value for every position of P not at infinity, which value $= 0$ when the lines pass through a common point. Hence—

When a number of fixed lines A, B, C, D, &c. are parallel to the several sides a, b, c, d, &c. of any polygon, and that their positive and negative sides correspond to those of the sides of the polygon measured cyclically in either common direction, then for every point P not at infinity the sum $\Sigma(a.PA)$ is constant, and $= 0$ when the lines pass through a common point (see 113, 4°.).

When the polygon is equilateral, since then $a = b = c = d$, &c., therefore $\Sigma(a.PA) = a.\Sigma(PA)$, and therefore the sum $\Sigma(PA)$ is constant for every point not at infinity. Hence—

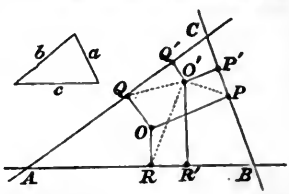
When a number of fixed lines A, B, C, D, &c. are parallel to the several sides of any equilateral polygon, and that their positive and negative sides correspond to those of the sides of the polygon measured cyclically in either common direction, then for every point P not at infinity the sum $\Sigma(PA)$ is constant, and $= 0$ when the lines pass through a common point.

Of all equilateral polygons of any order, one, the regular, being also equiangular, the term "equilateral" may therefore be replaced by "equiangular" in the statement of the latter property, the altered however being but a particular case of the original property, and no new principle of any kind being involved or expressed in the change.

124. The general property of the preceding article supplies ready solutions of the two following problems—

Given three points, or two points and a line, P, Q, R, to determine the point O for which the sum $a.OP+b.OQ+c.OR$ *shall be the minimum; a, b, c being any three positive multiples no one of which is greater than the sum or less than the difference of the other two.*

For, if O be the point for which the three perpendiculars at P, Q, R to OP, OQ, OR in the former case, or the two perpendiculars at P and Q to OP and OQ with the line R in the latter case, determine a triangle ABC similar to that determined by the three multiples a, b, c, and including O within its area; then if O' be any other point, and $O'P'$, $O'Q'$, $O'R'$ the three perpendiculars from it upon the three sides of ABC, since, by the preceding,



$$a.OP+b.OQ+c.OR=a.O'P'+b.O'Q'+c.O'R',$$

therefore $a.OA+b.OB+c.OC<a.O'P+b.O'Q+c.O'R$ in the former case, and $<a.O'P+b.O'Q+c.O'R'$ in the latter case, and therefore O in either case is the point required; but O is the common intersection of the three known circles QOR, ROP, POQ in the former case, and the intersection of the two known directions PO and QO in the latter case, and therefore &c.

When the three given multiples a, b, c are incapable of forming a triangle, the above method of determining O of course fails in both cases, but it is easily seen, at once without any construction, that if any of the three multiples a, b, c in the former case, or either of the two a and b corresponding to the two points P and Q in the latter case be $=$ or $>$ the sum of the other two, the point itself corresponding to that multiple is that for which the sum $a.OP+b.OQ+c.OR$ is the minimum.

But if the multiple c corresponding to the line R in the latter case be $=$ or $>$ the sum of the other two a and b corresponding to the two points P and Q, then through the required point O as before is easily seen without any construction to be on the line R, to determine its position on that line, that is, the position of the point O on R for which the sum $a.OP+b.OQ$ is the minimum, is a problem incapable of solution by the geometry of the point, line, and circle.

125. In the linear loci of Art. 120 the several distances PA, PB, PC, PD, &c. need not be measured perpendicularly to the several lines A, B, C, D, &c.; they might be measured in directions inclined to them at any constant angles α, β, γ, δ, &c., and the several conclusions there established, with some slight and obvious modifications, would be true for the oblique as well as for the perpendicular distances.

For, PA, PB, PC, PD, &c. being the several oblique distances, and a, b, c, d, &c. as before the several corresponding multiples, if $PA_{\prime}$, $PB_{\prime}$, $PC_{\prime}$, $PD_{\prime}$, &c. be the several perpendicular distances, and $a_{\prime}$, $b_{\prime}$, $c_{\prime}$, $d_{\prime}$, &c. a system of multiples corresponding to them, having to the original multiples a, b, c, d, &c. the constant ratios of the several oblique to the corresponding perpendicular distances; then, since for every position of P not at infinity $a.PA=a_{\prime}.PA_{\prime}$, $b.PB=b_{\prime}.PB_{\prime}$, $c.PC=c_{\prime}.PC_{\prime}$, $d.PD=d_{\prime}.PD_{\prime}$, &c., therefore $\Sigma(a.PA)=\Sigma(a_{\prime}.PA_{\prime})$, and therefore &c., the multiples for the perpendiculars being simply those for the oblique distances divided by the sines of the constant angles of inclination.

By virtue of the above the four general problems of Art. 122 may be still further generalised, by the substitution for perpendicular of oblique distances measured in any given directions from the required point P to the given lines A, B, C, D, &c.

126. If the several oblique distances PA, PB, PC, PD, &c. in the preceding, were all measured in the same absolute direction, their several points of meeting A, B, C, D, &c. with the several fixed lines would then lie in a line L passing through P parallel to the direction, and the sum $\Sigma(a.PA)$ would (Art. 80) $=\Sigma(a).PO$, when O is the mean centre on the line L of the system of points A, B, C, D, &c. for the system of multiples a, b, c, d, &c. Hence, by the preceding—

For a variable line L moving parallel to itself in any constant direction, and intersecting the several fixed lines of a polygram of any form in a system of variable points A, B, C, D, &c.

1°. *The locus of the mean centre O of the system of points A, B, C, D, &c. for any system of multiples a, b, c, d, &c. is a line M whose position depends on the direction of L.*

2°. *The locus, more generally, of the point P on L for which the sum $\Sigma (a.PA)$ has any constant value, positive or negative, is a line N parallel to M and distant from it in the direction of L by the interval $NM = \Sigma (a.PA) \div \Sigma (a)$.*

In the particular case when $a=b=c=d$, &c., the several lines M, loci of O for different directions of L, are termed, from the analogy of the circle, *diameters* of the polygram. The latter being given, the position of the particular diameter M corresponding to any given direction of L is determined by drawing any two lines L_1 and L_2 parallel to the given direction, taking the two mean centres O_1 and O_2 of the two systems of points A_1, B_1, C_1, D_1, &c. and A_2, B_2, C_2, D_2, &c., in which they intersect the several lines of the figure, and drawing the indefinite line O_1O_2. In the particular case where all the lines of the figure pass through a common point O, as every diameter M, corresponding to every direction of L, passes evidently through it, a single other point O_1 is therefore sufficient to determine each particular diameter in that case. Remarks precisely similar apply, of course, when the several multiples a, b, c, d, &c. have any values whatever.

In an equilateral triangle the several diameters of the figure envelope the inscribed circle; this very particular case of a much more general property, to be given in a future Chapter, is left for the present as an exercise to the reader.

127. The general property of the preceding article, combined with that of Art. 80, Cor. 1°., may be employed for the solution of the following very general problem—

Given any system of lines A, B, C, D, &c. and any corresponding system of multiples a, b, c, d, &c. to determine the point P for which the sum $\Sigma (a.PA^2)$ is the minimum.

Drawing arbitrarily any two parallel lines L_1 and L_2 intersecting the entire system of lines A, B, C, D, &c. at the system of angles α, β, γ, δ, &c.; taking the two mean centres O_1 and O_2 of the two systems of intersections A_1, B_1, C_1, D_1, &c. and A_2, B_2, C_2, D_2, &c. for the system of multiples $a \div \sin^2\alpha$,

$b \div \sin^2\beta$, $c \div \sin^2\gamma$, $d \div \sin^2\delta$, &c.; drawing then the indefinite line O_1O_2 intersecting the entire system of lines A, B, C, D, &c. at the system of angles $\alpha', \beta', \gamma', \delta'$, &c.; and taking finally the mean centre O' of the system of intersections A', B', C', D', &c. for the system of multiples $a \div \sin^2\alpha'$, $b \div \sin^2\beta'$, $c \div \sin^2\gamma'$, $d \div \sin^2\delta'$, &c.; the point O' thus determined is that required.

For, by (80, Cor. 1°.), O' is the point on the line O_1O_2 for which the sum $\Sigma(a.PA^2)$ is the minimum for points confined to that line; and supposing, if possible, a point I not on that line were that for which it were absolutely the minimum, the line L passing through I parallel to L_1 and L_2 would intersect the line O_1O_2 at a point O, which, by the preceding, would be the mean centre for the system of multiples $a \div \sin^2\alpha$, $b \div \sin^2\beta$, $c \div \sin^2\gamma$, $d \div \sin^2\delta$, &c. of the system of points in which it would intersect the system of lines A, B, C, D, &c., and for which therefore, by (80, Cor. 1°.), the sum $\Sigma(a.PA^2)$ would be the minimum for points confined to the line L, and consequently less than for the point I, which therefore could not, as supposed, be off the line O_1O_2; and therefore &c.

It is easy to see from the more general property (98, Cor. 1°.), that the point P, however determined, for which the sum $\Sigma(a.PA^2)$ is the minimum, is the mean centre of the feet of the several perpendiculars PA, PB, PC, PD, &c. for the system of multiples a, b, c, d, &c.

128. We shall conclude the present Chapter with a direct demonstration of the general property of Art. 120, not based like that there given upon any property of polygons, but resulting immediately from the nature of independent lines; the following general theorem, analogous to that established in Art. 85 for any system of points, may be regarded as the basis of the direct demonstration—

If A, B, C, D, &c. be any system of lines, disposed in any manner, but none infinitely distant, and a, b, c, d, &c. any system of corresponding multiples, positive or negative, but none infinitely great, then for every three points P, Q, R in a line, supposed all at a finite distance,

$$QR.\Sigma(a.PA) + RP.\Sigma(a.QA) + PQ.\Sigma(a.RA) = 0,$$

the distances under the symbols of summation being measured in *directions inclined at any constant angles* α, β, γ, δ, *&c. to the several lines* A, B, C, D, *&c.*

For, the three points P, Q, R being by hypothesis in a line, therefore, for the several lines A, B, C, D, &c., by (82, Cor. 4°.),

$$QR.PA + RP.QA + PQ.RA = 0,$$
$$QR.PB + RP.QB + PQ.RB = 0,$$
$$QR.PC + RP.QC + PQ.RC = 0,$$
$$QR.PD + RP.QD + PQ.RD = 0, \text{ \&c.},$$

from which, multiplying horizontally by a, b, c, d, &c. and then adding vertically, the above relation at once results, and from it the following consequences may be immediately inferred—

1°. *When two of the three sums* $\Sigma(a.QA)$ *and* $\Sigma(a.RA)=0$, *the third* $\Sigma(a.PA)$ *also* $=0$; this is evident, as the three coefficients QR, RP, PQ are by hypothesis all finite. *Hence the locus of a variable point P for which the sum* $\Sigma(a.PA)=0$ *is a line*, the central axis I of the system for the particulars of the case.

2°. *When two of the three sums* $\Sigma(a.QA)$ *and* $\Sigma(a.RA)$ *have equal values, the third* $\Sigma(a.PA)$ *has the same value;* this is evident, as the sum of the three coefficients QR, RP, PQ is always $=0$ (Art. 78). *Hence the locus of a variable point P for which the sum* $\Sigma(a.PA)$ *has any constant value, positive or negative, is a line L parallel to the central line I;* for if it met the latter at any finite distance, the sum $\Sigma(a.PA)$ for the point of intersection would have at once the two different values corresponding to the two lines.

3°. *When one of the three sums* $\Sigma(a.RA)=0$, *then for the other two* $\Sigma(a.PA):\Sigma(a.QA)=PR:QR=PI:QI$; this is evident, as R, by 1°., is then on the central axis I. *Hence for every point P on any line L parallel to I, the constant sum* $\Sigma(a.PA)$ *is proportional in magnitude and sign to the distance LI;* these are the principal results for the general case as otherwise established in Art. 120.

4°. *In the particular case when the central axis I of the system is at infinity, the sum* $\Sigma(a.PA)$ *has the same value for every position of P at a finite distance;* for since, by 3°., for every two points P and Q at a finite distance $\Sigma(a.PA):\Sigma(a.QA)=PI:QI$, whatever be the position of I, therefore for every two points

P and Q at a finite distance $\Sigma(a.PA) : \Sigma(a.QA) = 1$ when I is at infinity, and therefore &c.

5°. *When the sum* $\Sigma(a.PA)$ *has the same value, finite or evanescent, for three points* P, Q, R *not in the same line, it has the same value for every point* O *at a finite distance;* for having the same value for the three points P, Q, R, it has it, by 2°., for the three points X, Y, Z, in which the three lines OP, OQ, OR intersect with the three QR, RP, PQ, and having it for each pair of points P and X, Q and Y, R and Z, it has it, by the same, for the point O which is in the same line with each pair; and therefore &c.

6°. *The particulars of the system being all given, the position of the central axis* I *may be determined rapidly as follows:* assuming arbitrarily any three points P, Q, R not in the same line, and dividing the three distances QR, RP, PQ at X, Y, Z respectively, so that in magnitude and sign

$$QX : RX = \Sigma(a.QA) : \Sigma(a.RA),$$
$$RY : PY = \Sigma(a.RA) : \Sigma(a.PA),$$
$$PZ : QZ = \Sigma(a.PA) : \Sigma(a.QA);$$

the three points X, Y, Z thus determined lie, by 3°., on the central axis I of the system, and therefore &c.; when the three sums $\Sigma(a.PA)$, $\Sigma(a.QA)$, $\Sigma(a.RA)$ have the same value, the three points X, Y, Z being then at infinity or indeterminate, according as the common value is finite or evanescent, so also is the central axis.

CHAPTER VIII.

ON COLLINEAR AND CONCURRENT SYSTEMS OF POINTS AND LINES.

129. SYSTEMS of points ranged on lines, and of lines passing through points, enter largely into the investigations of modern geometry, and are distinguished by appropriate names, as follows:

A system of points ranged along a line is termed a *collinear* system, the figure they constitute a *row* of points, and the line on which they lie the *base* or *axis* of the row. A system of lines passing through a point is termed a *concurrent* system, the figure they constitute a *pencil* of lines, or *rays* as they are sometimes called, and the point through which they pass the *vertex* or *centre* or *focus* of the pencil. The terms "Ray," "Pencil," and "Focus," have been introduced into geometry from the science of Optics.

The axis of a row of points, or the centre of a pencil of lines, might be at infinity; in the former case the points of the row would, of course, be all at infinity, and in the latter case the lines of the pencil would (16) be all parallel; but in no other respects is there any difference between these particular and the general cases, when the axis of the row is any line whatever, and the centre of the pencil any point whatever.

Two points of a row or rays of a pencil determine, of course, the axis or vertex of the row or pencil to which they belong.

130. Two rows of any common number of points on different axes, or pencils of any common number of rays through different centres, A, B, C, D, &c., and A', B', C', D', &c., whose constituents correspond in pairs A to A', B to B', C to C', D to D', &c., are said to be *in perspective*, in the former case when the several lines of connexion AA', BB' CC' DD', &c., of pairs of

corresponding points are concurrent, and in the latter case when the several points of intersection AA', BB', CC', DD', &c., of pairs of corresponding lines are collinear. In the former case the centre of the pencil determined by the several concurrent connectors is termed *the centre of perspective* of the rows, and in the latter case the axis of the row determined by the several collinear intersections is termed *the axis of perspective* of the pencils. Every two rows of points determined on different axes by the same pencil of rays, and every two pencils of rays determined at different centres by the same row of points, are evidently in perspective; the centre of the determining pencil being the centre of perspective of the rows in the former case, and the axis of the determining row being the axis of perspective of the pencils in the latter case.

The centre of perspective of two rows in perspective, or the axis of perspective of two pencils in perspective, might be at infinity; in the former case the several connectors AA', BB', CC', DD', &c. being all parallel, the two rows of points would be similar (32), and in the latter case the several pairs of corresponding rays A and A', B and B', C and C', D and D', &c., being two and two parallel, the two pencils would be similar, and at once similarly and oppositely placed (33). In these particular cases of perspective the two rows or pencils $ABCD$ &c., and $A'B'C'D'$ &c., are said also to be *projections* of each other; though both terms "perspective" and "projection" are often applied indifferently as well to the general as to the particular case, and, as we shall see in the sequel, to other figures as well as to rows and pencils.

131. Every two rows or pencils of but *two* points or rays each having different axes or vertices being, of course, necessarily in perspective, however circumstanced as to position, absolute or relative, or whichever way regarded as corresponding two and two. Hence for two segments or angles AB and $A'B'$ having different axes or vertices, the two points of intersection, or lines of connection, of AB with $A'B'$, and, of AB' with $A'B$, are termed respectively *the two centres of perspective of the segments*, or, *the two axes of perspective of the angles*—names, at once convenient and expressive, by which to designate a pair

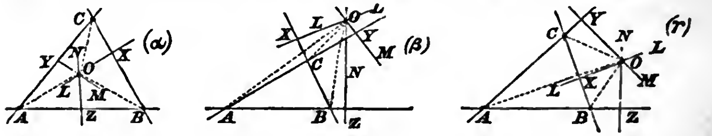
of points or lines of very frequent occurrence in geometrical research.

Reserving for future chapters the remarkable developments of modern Geometry as regards collinear and concurrent systems in general, we shall devote the present chapter to the consideration of some of their most important properties as regards the sides and angles of rectilinear figures in general, and of triangles in particular.

132. *When three lines LX, MY, NZ intersecting at right angles the three sides BC, CA, AB of any triangle ABC are concurrent, they divide them at the three parts of meeting X, Y, Z so as to fulfil the relation*

$$(BX^2 - CX^2) + (CY^2 - AY^2) + (AZ^2 - BZ^2) = 0$$

and, conversely, when they divide them at the three points of meeting so as to fulfil the above relation, they are concurrent.



To prove the first or direct part; if O be the point of concurrence of the three lines LX, MY, NZ, then joining OA, OB, OC, since (Euclid I, 47, Cor.) $(BX^2 - CX^2) = (BO^2 - CO^2)$, $(CY^2 - AY^2) = (CO^2 - AO^2)$, $(AZ^2 - BZ^2) = (AO^2 - BO^2)$, therefore

$$(BX^2 - CX^2) + (CY^2 - AY^2) + (AZ^2 - BZ^2)$$
$$= (BO^2 - CO^2) + (CO^2 - AO^2) + (AO^2 - BO^2) = 0$$

as above stated. And to prove the second or converse part; if O be the point of intersection of any two of them LX and MY, and Z' the point at which the parallel through O to the third NZ intersects the line AB to which the third is perpendicular; then since by the first part

$$(BX^2 - CX^2) + (CY^2 - AY^2) + (AZ'^2 - BZ'^2) = 0,$$

and since by hypothesis

$$(BX^2 - CX^2) + (CY^2 - AY^2) + (AZ^2 - BZ^2) = 0;$$

therefore $(AZ'^2 - BZ'^2) = (AZ^2 - BZ^2)$, and therefore $Z' = Z$, which, of course, could not be the case unless, as above stated, NZ passed through O.

A relation of exactly the same form, and proved in precisely the same manner as the above, connects the several pairs of segments into which the several sides of any polygon are divided by every concurrent system of perpendiculars to them. But the converse property which establishes the relation as *a criterion of concurrence* of the several perpendiculars *is true only for the triangle.* The reasoning by which it was inferred as above for that case from the direct property evidently proving only for the general case of any order (n), that when $(n-1)$ of them pass through a common point the n^{th} passes through the same point.

The relation itself may evidently be written also in the following form, viz.—

$$BX^2 + CY^2 + AZ^2 = CX^2 + AY^2 + BZ^2,$$

which in cases of numerical calculation is sometimes more convenient than the original.

133. The following are a few examples of the application of the preceding relation as a criterion of the concurrence of three lines perpendiculars at three points X, Y, Z to the three sides of a triangle ABC.

Ex. 1°. *The three perpendiculars at the middle points of the sides of a triangle are concurrent.*

For, since here by hypothesis, $BX = CX$, $CY = AY$, $AZ = BZ$, therefore the criterion relation $(BX^2 - CX^2) + (CY^2 - AY^2) + (AZ^2 - BZ^2) = 0$ is satisfied identically in the simplest manner of which it is susceptible, and therefore &c.

Ex. 2°. *The three perpendiculars through the vertices to the opposite sides of a triangle are concurrent.*

For, since here, Euc. I. 47,

$$(BX^2 - CX^2) = (BA^2 - CA^2),\ (CY^2 - AY^2) = (CB^2 - AB^2),$$
$$(AZ^2 - BZ^2) = (AC^2 - BC^2);$$

therefore the criterion relation again is satisfied, and therefore &c.

Ex. 3°. *The three perpendiculars to the sides of a triangle at the internal points of contact of the three excribed circles are concurrent.*

For, if a, b, c be the three sides and s the semi-perimeter of the triangle, then since, Euc. IV. Appendix,

$$BZ = CY = (s - a),\ CX = AZ = (s - b),\ AY = BX = (s - c);$$

therefore, as in Ex. 1°., the criterion relation is identically satisfied, and therefore &c.

Ex. 4°. *Every two perpendiculars to sides of a triangle at points of contact of exscribed circles external to the same vertex are concurrent with the perpendicular to the opposite side at the point of contact of the inscribed circle.*

For, if A be the vertex to which the two contacts are external; then since, Euc. IV., Appendix,

$$AY = BZ = (s - b),\quad AZ = CX = (s - c),\quad BZ = CY = s;$$

therefore, here again, as in the preceding example, the criterion relation is identically satisfied, and therefore &c.

Ex. 5°. *When three circles touch two and two, the three tangents at the three points of contact are concurrent.*

For, if A, B, C be the centres of the three circles, a, b, c the three radii, and X, Y, Z the three opposite points of contact, then since

$$AY = AZ = a,\quad BZ = BX = b,\quad CX = CY = c;$$

therefore, as in each of the preceding examples, the criterion relation is again identically satisfied, and therefore &c.

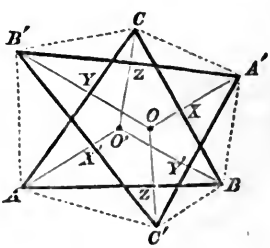
Ex. 6°. *When three perpendiculars to the sides of a triangle are concurrent, the other three equidistant from the middle points of the sides are also concurrent.*

For, if LX, MY, NZ and $L'X'$, $M'Y'$, $N'Z'$ be the two sets of perpendiculars, then since by hypothesis $BX = CX'$ and $CX = BX'$, $CY = AY'$ and $AY = CY'$, $AZ = BZ'$ and $BZ = AZ'$; therefore

$$(BX^2 - CX^2) + (CY^2 - AY^2) + (AZ^2 - BZ^2)$$
$$= (CX'^2 - BX'^2) + (AY'^2 - CY'^2) + (BZ'^2 - AZ'^2),$$

and therefore when either equivalent = 0, so is the other; that is, when either set of perpendiculars is concurrent, so is the other.

Ex. 7°. *When the three perpendiculars from the vertices of one triangle upon the sides of another are concurrent, the three corresponding perpendiculars from the vertices of the latter upon the sides of the former are also concurrent.*



Let ABC and $A'B'C'$ be the two triangles. If $A'X$, $B'Y$, $C'Z$ pass through a common point O, then AX', BY', CZ' pass also through a common point O', and conversely. For, joining A with B' and C', or A' with B and C; B with C' and A', or B' with C and A; C with A' and B', or C' with A and B; that is, each vertex of either triangle with the two of the other it does not correspond to, then

$$(BX^2 - CX^2) + (CY^2 - AY^2) + (AZ^2 - BZ^2)$$
$$= (BA'^2 - CA'^2) + (CB'^2 - AB'^2) + (AC'^2 - BC'^2)$$
$$= (C'A^2 - B'A^2) + (A'B^2 - C'B^2) + (B'C^2 - A'C^2)$$
$$= (C'X'^2 - B'X'^2) + (A'Y'^2 - C'Y'^2) + (B'Z'^2 - A'Z'^2),$$

but of these four equivalents the first = 0 is the condition for $A'X$, $B'Y$, $C'Z$ to pass through a common point O, and the last = 0 is the condition for AX', BY', CZ' to pass through a common point O', and therefore &c.

134. *When three points X, Y, Z lying on the three sides BC, CA, AB of any triangle ABC are collinear* (figs. α, β, γ).

a. They divide the three sides so as to fulfil the relation

$$\frac{BX}{CX}\cdot\frac{CY}{AY}\cdot\frac{AZ}{BZ}=+1.$$

b. They connect with the opposite vertices so as to fulfil the relation

$$\frac{\sin BAX}{\sin CAX}\cdot\frac{\sin CBY}{\sin ABY}\cdot\frac{\sin ACZ}{\sin BCZ}=+1;$$

and conversely, when they either divide the three sides so as to fulfil the former relation, or connect with the opposite vertices so as to fulfil the latter relation, they are collinear.

When three lines AX, BY, CZ passing through the three vertices A, B, C of any triangle ABC are concurrent (figs. α', β', γ').

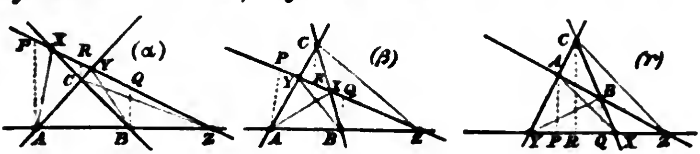
a'. They divide the three angles so as to fulfil the relation

$$\frac{\sin BAX}{\sin CAX}\cdot\frac{\sin CBY}{\sin ABY}\cdot\frac{\sin ACZ}{\sin BCZ}=-1.$$

b'. They intersect the opposite sides so as to fulfil the relation

$$\frac{BX}{CX}\cdot\frac{CY}{AY}\cdot\frac{AZ}{BZ}=-1;$$

and conversely, when they either divide the three angles so as to fulfil the former relation, or intersect the opposite sides so as to fulfil the latter relation, they are concurrent.



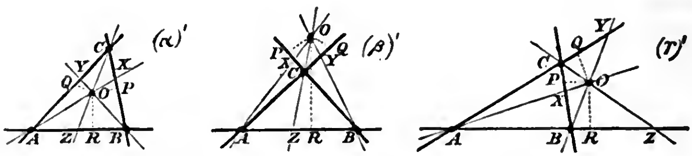
To prove the first or direct part of *a*.—From the three vertices of the triangle ABC drawing three perpendiculars, or parallels in any arbitrary direction, AP, BQ, CR to meet the line containing, by hypothesis, the three points X, Y, Z, then since (Euc. VI. 4) $BX : CX = BQ : CR$, $CY : AY = CR : AP$, $AZ : BZ = AP : BQ$; therefore, as above stated, the compound of the three antecedents $= +1$; the reason of the positive sign appearing also from the obvious consideration that three collinear points on the sides of a triangle necessarily divide an *even* number of them *internally* (75). And to prove the second or converse part of the same.—If Z' be the point at which the line containing any two of the points X and Y meets the side AB of the triangle containing the third Z; since then, by the first part, X, Y, and Z' being collinear,

$$(BX : CX).(CY : AY).(AZ' : BZ') = +1,$$

and since also, by hypothesis,

$$(BX : CX).(CY : AY).(AZ : BZ) = +1,$$

therefore $AZ' : BZ' = AZ : BZ$ in magnitude and sign, and therefore (75) $Z' = Z$, so that, as above stated, Z is collinear with X and Y.



To prove the first or direct part of a'.—From the point O through which, by hypothesis, the three lines AX, BY, CZ concur, letting fall three perpendiculars, or isoclinals at any arbitrary inclination, OP, OQ, OR upon the three sides BC, CA, AB of the triangle ABC; then since (61)

$$\sin BAX : \sin CAX = -OR : OQ,\quad \sin CBY : \sin ABY = -OP : OR,$$
$$\sin ACZ : \sin BCZ = -OQ : OP,$$

therefore, as above stated, the compound of the three antecedents $= -1$; the reason of the negative sign appearing also from the obvious consideration that three concurrent lines through the vertices of a triangle necessarily divide an *odd* number of the angles *internally* (75). And to prove the second or converse part of the same.—If CZ' be the line by which the point O, common to any two of the lines AX and BY, connects with the vertex C of the triangle through which the third CZ passes; since then, by the first part, AX, BY, and CZ' being concurrent,

$$(\sin BAX : \sin CAX).(\sin CBY : \sin ABY).(\sin ACZ' : \sin BCZ') = -1,$$

and since also, by hypothesis,

$(\sin BAX : \sin CAX).(\sin CBY : \sin ABY).(\sin ACZ : \sin BCZ) = -1,$

therefore $\sin ACZ' : \sin BCZ' = \sin ACZ : \sin BCZ$ in magnitude and sign, and therefore (75) $CZ' = CZ$, so that, as above stated, CZ is concurrent with AX and BY.

Relations of exactly the same form, and proved in precisely the same manner as the above (a and a'), connect the several pairs of segments into which the several sides of any polygon are divided by every collinear system of points lying upon them, and into which the several angles of any polygon are divided by every concurrent system of lines passing through them; the only modification being that while, in the former case, the sign of the compound is, as above, always positive, in the latter case it is negative only when, as above, the order of the polygon is odd, but positive when it is even. The converse properties however, which establish the relations a and a' as *criteria of collinearity and concurrence* of the several points and lines *are true only for the triangle;* the reasoning by which they have been inferred, as above, for that case from the direct properties proving only for the general case of any order (n), that when $(n-1)$ of the points in the former case are collinear the n^{th} is collinear with them, and that when $(n-1)$ of the lines in the latter case are concurrent the n^{th} is concurrent with them.

To prove b and b'.—Since, by (65), whatever be the positions of X, Y, Z in the former case, or the directions of AX, BY, CZ in the latter case,

$$\frac{BX}{CX} = \frac{BA}{CA} \cdot \frac{\sin BAX}{\sin CAX}, \quad \frac{CY}{AY} = \frac{CB}{AB} \cdot \frac{\sin CBY}{\sin ABY}, \quad \frac{AZ}{BZ} = \frac{AC}{BC} \cdot \frac{\sin ACZ}{\sin BCZ};$$

therefore, the two compounds

$$\frac{BX}{CX} \cdot \frac{CY}{AY} \cdot \frac{AZ}{BZ} \text{ and } \frac{\sin BAX}{\sin CAX} \cdot \frac{\sin CBY}{\sin ABY} \cdot \frac{\sin ACZ}{\sin BCZ}$$

are always equal in magnitude and similar in sign; whenever, therefore, either $= \pm 1$, so is also the other, and therefore &c.

Relations of exactly the same form with these last (b and b'), and very easily proved directly, as they too may be, connect the several pairs of segments into which, for any polygon of an *odd* order (105), the several angles are divided by their connectors with collinear systems of points on the opposite sides, and the several sides at their intersections with concurrent systems of lines through the opposite vertices; but, as in the cases of a and a', the converse properties which establish the relations as *criteria of collinearity and concurrence* of the several points and lines, are, for the same reason as in their cases, true only for the triangle.

The criteria (a and b') for three points X, Y, Z on the sides of a triangle to be collinear and to connect with the opposite vertices by three concurrent lines AX, BY, CZ, and the criteria (b and a') for three lines AX, BY, CZ through the vertices of a triangle to be concurrent and to intersect with the opposite sides at three collinear points X, Y, Z, being in both cases identical, if signs be disregarded or unknown; should any ambiguity arise in consequence, as to which of the two relations in either case is indicated by the fulfilment of the criterion in any particular instance, in which the signs of the compound ratios are not explicitly given or known; the obvious consideration, on which the difference of sign in each case depends, that an *odd* number of the points or lines must be *external* to their respective sides or angles for *collinearity*, and *internal* to them for *concurrence*, is sufficient always to remove it.

135. The following is an obvious corollary from, or rather indeed a different manner of, stating the two general properties a' and b of the preceding article, viz.,

When three points P, Q, R, however situated, connect with the three vertices A, B, C of a triangle ABC by three lines AP, BQ, CR which are either concurrent or collinearly intersectant with the opposite sides, the three pairs of perpendiculars PP' and PP'', QQ' and QQ'', RR' and RR'' from them upon the three pairs of sides containing the respective vertices are connected by the relation

$$\frac{PP'}{PP''} \cdot \frac{QQ'}{QQ''} \cdot \frac{RR'}{RR''} = \pm 1,$$

and conversely, when the three pairs of perpendiculars from them upon the sides of the three angles of the triangle are connected by the above relation, the three lines connecting them with the *corresponding vertices are either concurrent or collinearly intersectant with the opposite sides.*

For, whatever be the positions of P, Q, R, since (61)

$$PP' : PP'' = -\sin PAB : \sin PAC,$$
$$QQ' : QQ'' = -\sin QBC : \sin QBA,$$
$$RR' : RR'' = -\sin RCA : \sin RCB,$$

therefore the two compounds

$$\frac{PP'}{PP''} \cdot \frac{QQ'}{QQ''} \cdot \frac{RR'}{RR''} \text{ and } \frac{\sin BAP}{\sin CAP} \cdot \frac{\sin CBQ}{\sin ABQ} \cdot \frac{\sin ACR}{\sin BCR}$$

are always equal in magnitude and opposite in sign, and therefore when either $= \pm 1$ the other then $= \mp 1$, and therefore &c.

136. Two very important conclusions, one respecting points at infinity, the other respecting parallel lines, result immediately from the general relations a or b, and a' or b' of Art. 134, regarded as criteria of the collinearity of three points X, Y, Z on the sides, and of the concurrence of these lines AX, BY, CZ through the vertices, of a triangle ABC—

1°. Every three points X, Y, Z at infinity evidently divide the three sides BC, CA, AB of every triangle ABC whose directions pass through them, so as to fulfil identically the relation

$$(BX : CX).(CY : AY).(AZ : BZ) = +1,$$

and as evidently connect with the opposite vertices, so as to fulfil identically the relation

$$(\sin BAX : \sin CAX).(\sin CBY : \sin ABY).(\sin ACZ : \sin BCZ) = +1,$$

therefore, by relation a or b, they are collinear, and therefore—

Every three, and therefore all, points at infinity are collinear.

2°. Every three parallel lines AX, BY, CZ evidently divide the three angles BAC, CBA, ACB of every triangle ABC whose vertices lie on them, so as to fulfil identically the relation

$$(\sin BAX : \sin CAX).(\sin CBY : \sin ABY).(\sin ACZ : \sin BCZ) = -1,$$

and as evidently intersect with the opposite sides, so as to fulfil the relation

$$(BX : CX).(CY : AY).(AZ : BZ) = -1,$$

therefore, by relation a' or b', they are concurrent, and therefore—

Every three, and therefore all, parallel lines are concurrent.

Paradoxical as these conclusions always appear when first stated, all doubt of their legitimacy has been long set at rest by the number and variety of the considerations tending to verify and confirm them.

137. In the following examples of the application of the preceding relations, as criteria of the collinearity of three points X, Y, Z on three lines, and of the concurrence of three lines AX, BY, CZ through three points, one only of the two relations equally establishing the circumstance being proved in each case, the verification *à priori* of the other may be taken as an exercise by the reader.

Ex. 1°. *Every three points of bisection of different sides of a triangle are collinear, or connect concurrently with the opposite vertices, according as an odd number of them is external or internal.*

For, since by hypothesis, $BX : CX = \pm 1$, $CY : AY = \pm 1$, $AZ : BZ = \pm 1$, according as each section is external or internal, therefore the criterion relation (a or b') for collinearity or concurrence, viz.

$$(BX : CX).(CY : AY).(AZ : BZ) = \pm 1,$$

according as an odd number of them is external or internal, is satisfied in the simplest manner of which it is susceptible, and therefore &c.

Ex. 2°. *Every three lines of bisection of different angles of a triangle are concurrent, or intersect collinearly with the opposite sides, according as an odd number of them is internal or external.*

For, since by hypothesis,

$$\sin BAX : \sin CAX = \pm 1,\ \sin CBY : \sin ABY = \pm 1,\ \sin ACZ : \sin BCZ = \pm 1,$$

according as each section is external or internal, therefore the criterion relation (a' or b) for concurrence or collinearity, viz.

$$(\sin BAX : \sin CAX).(\sin CBY : \sin ABY).(\sin ACZ : \sin BCZ) = \mp 1,$$

according as an odd number of them is internal or external, is satisfied in the simplest manner of which it is susceptible, and therefore &c.

Ex. 3°. *In every triangle circumscribed to a circle the three points of contact of the sides connect concurrently with the opposite vertices.*

For, if X, Y, Z be the three points of contact, then, since, by pairs of equal tangents from ABC to the circle, $AY = AZ$, $BZ = BX$, $CX = CY$, therefore, as in Ex. 1°., the criterion relation (b') for the concurrence of AX, BY, CZ is identically satisfied; it being evident, from the nature of the case, that the three points X, Y, Z must, according to circumstances, be either all internal or one internal and two external to their respective sides, and therefore &c.

Ex. 4°. *In every triangle inscribed in a circle the three tangents at the vertices intersect collinearly with the opposite sides.*

For if AX, BY, CZ be the three tangents, then, since, by pairs of equal angles between BC, CA, AB and the circle, $\sin CBY = \sin BCZ$, $\sin ACZ = \sin CAX$, $\sin BAX = \sin ABY$, therefore, as in example 2°., the criterion relation (*b*) for the collinearity of XYZ is identically satisfied; it being evident, from the nature of the case, that the three lines AX, BY, CZ must, under all circumstances, be external to their respective angles, and therefore &c.

Ex. 5°. *In every triangle the three perpendiculars through the vertices to the opposite sides are concurrent* (See Ex. 2°, 133).

For, if AX, BY, CZ be the three perpendiculars, then, since, by pairs of similar right-angled triangles about A, B, C as common vertices,

$$\sin ABY = \sin ACZ, \quad \sin BCZ = \sin BAX, \quad \sin CAX = \sin CBY,$$

therefore the criterion relation (a') for the concurrence of AX, BY, CZ is identically satisfied; it being evident, from the nature of the case, that, according as the triangle is acute or obtuse angled, they are either all internal or one internal and two external to their respective angles, and therefore &c.

Ex. 6°. *In every triangle the three perpendiculars through any point to the three lines connecting them with the vertices intersect collinearly with the opposite sides.*

For, if O be the point, and OX, OY, OZ the three perpendiculars through it to OA, OB, OC respectively, then, since, by (65),

$$BX : CX = (BO : CO) . (\sin BOX : \sin COX),$$

$$CY : AY = (CO : AO) . (\sin COY : \sin AOY),$$

$$AZ : BZ = (AO : BO) . (\sin AOZ : \sin BOZ);$$

and since, by pairs of perpendiculars,

$$\sin COY = \sin BOZ, \quad \sin AOZ = \sin COX, \quad \sin BOX = \sin AOY,$$

therefore the criterion relation (*a*) for the collinearity of XYZ is satisfied; it being evident, from the nature of the case, that they must be, according to circumstances, either all external or one external and two internal to their respective sides, and therefore &c.

Ex. 7°. *If the three sides of a triangle be reflected with respect to any line* (50), *the three lines through the vertices parallel to the reflexions of the opposite sides are concurrent.*

For, if AX, BY, CZ be the three parallels, then, since, by differences of pairs of equal angles (50),

$$\sin ABY = \sin ACZ, \quad \sin BCZ = \sin BAX, \quad \sin CAX = \sin CBY,$$

therefore the criterion relation (a') for the concurrence of AX, BY, CZ is identically satisfied; it being evident, from the nature of the case, that, according as the axis of reflexion is or is not parallel to a bisector of an angle of the triangle, either two of them coincide with the sides of that angle or two are external and one internal to their respective angles, and therefore &c.

Ex. 8°. *If the three vertices of a triangle be reflected with respect to any line* (50), *the three lines connecting the reflexions with any point on the line intersect collinearly with the opposite sides.*

For, if A', B', C' be the three reflexions, O the point on the line, and X, Y, Z the three intersections of OA', OB', OC', with BC, CA, AB respectively, then, since, by (65),

$$BX : CX = (BO : CO) . (\sin BOX : \sin COX),$$
$$CY : AY = (CO : AO) . (\sin COY : \sin AOY),$$
$$AZ : BZ = (AO : BO) . (\sin AOZ : \sin BOZ);$$

and since, by differences of pairs of equal angles (50),

$$\sin COY = \sin BOZ, \quad \sin AOZ = \sin COX, \quad \sin BOX = \sin AOY,$$

therefore the criterion relation a, for the collinearity of X, Y, Z, is satisfied exactly as in Ex. 6°; it being evident, from the nature of the case, that, here as well as there, they must, according to circumstances, be either all external or one external and two internal to their respective sides, and therefore &c.

Ex. 9°. *When three of the six intersections of a circle with the three sides of a triangle connect concurrently with the opposite vertices, the remaining three also connect concurrently with the opposite vertices.*

For, if X, Y, Z and X', Y', Z' be the two sets of intersections, then, since, by Euc. III. 35, 36,

$$AY . AY' = AZ . AZ', \quad BZ . BZ' = BX . BX', \quad CX . CX' = CY . CY',$$

therefore

$$(AY : AZ) . (BZ : BX) . (CX : CY) = (AZ' : AY') . (BX' : BZ') . (CY' : CX'),$$

and therefore when either equivalent $= -1$ so is the other; that is, when either set of connectors AX, BY, CZ, or AX', BY', CZ' is concurrent so is the other. As no three points on a circle could be collinear, neither equivalent could $= +1$ in this case.

Ex. 10°. *When three of the six tangents to a circle from the three vertices of a triangle intersect collinearly with the opposite sides, the remaining three also intersect collinearly with the opposite sides.*

For, if AX, BY, CZ and AX', BY', CZ' be the two sets of tangents, and a, b, c the lengths of the three chords intercepted by the circle on the three sides of the triangle, since then, by (66, Cor. 2°),

$$\sin BAX . \sin BAX' : \sin CAX . \sin CAX' = c^2 : b^2,$$
$$\sin CBY . \sin CBY' : \sin ABY . \sin ABY' = a^2 : c^2,$$
$$\sin ACZ . \sin ACZ' : \sin BCZ . \sin BCZ' = b^2 : a^2,$$

therefore

$(\sin BAX : \sin CAX).(\sin CBY : \sin ABY).(\sin ACZ : \sin BCZ)$

$= (\sin CAX' : \sin BAX').(\sin ABY' : \sin CBY').(\sin BCZ' : \sin BCZ'),$

and therefore when either equivalent $= +1$ so is the other; that is, when either set of intersections X, Y, Z or X', Y', Z' is collinear so is the other. As no three tangents to a circle could be concurrent, neither equivalent could $= -1$ in this case.

Ex. 11°. *When three points on the sides of a triangle are either collinear or concurrently connectant with the opposite vertices, the other three equally distant from the bisections of the sides are also either collinear or concurrently connectant with the opposite vertices.*

For, if X, Y, Z and X', Y', Z' be the two sets of points, then, since, by hypothesis, $BX = CX'$ and $CX = BX'$, $CY = AY'$ and $AY = CY'$, $AZ = BZ'$ and $BZ = AZ'$, therefore

$(BX:CX).(CY:AY).(AZ:BZ) = (CX':BX').(AY':CY').(BZ':AZ'),$

and therefore when either equivalent $= \pm 1$ so is also the other; that is, when either set of points X, Y, Z or X', Y', Z' is collinear, or, when either set of lines AX, BY, CZ or AX', BY', CZ' is concurrent, so is also the other, and therefore &c.

Ex. 12°. *When three lines through the vertices of a triangle are either concurrent or collinearly intersectant with the opposite sides, the other three equally inclined to the bisectors of the angles are also either concurrent or collinearly intersectant with the opposite sides.*

For, if AX, BY, CZ and AX', BY', CZ' be the two sets of lines, then since by hypothesis $BAX = CAX'$ and $CAX = BAX'$, $CBY = ABY'$ and $ABY = CBY'$, $ACZ = BCZ'$ and $BCZ = ACZ'$, therefore

$(\sin BAX : \sin CAX).(\sin CBY : \sin ABY).(\sin ACZ : \sin BCZ)$

$= (\sin CAX' : \sin BAX').(\sin ABY' : \sin CBY').(\sin BCZ' : \sin ACZ'),$

and therefore when either equivalent $= \mp 1$ so is also the other, that is when either set of lines AX, BY, CZ or AX', BY', CZ' is concurrent, or when either set of points X, Y, Z or X', Y', Z' is collinear, so is also the other, and therefore &c.

Ex. 13°. *When three lines through the vertices of a triangle are concurrent, the six bisectors of the three angles they determine intersect with the corresponding sides of the triangle at six points, every three of which on different sides are either collinear or concurrently connectant with the opposite vertices, according as an odd number of them is external or internal.*

For, if O be the point of concurrence of the lines, and X, Y, Z the intersections with the sides of the triangle of any three of the six bisectors of the three angles BOC, COB, AOB, then, since Euc. VI. 3,

$BX : CX = \pm BO : CO, \quad CY : AY = \pm CO : AO, \quad AZ : BZ = \pm AO : BO,$

according as each bisector is external or internal, therefore

$$(BX : CX).(CY : AY).(AZ : BZ) = \pm 1,$$

according as an odd number of them is external or internal, and therefore &c.

Ex. 14°. *When three points on the sides of a triangle are collinear, the six bisections of the three segments they determine connect with the corresponding vertices of the triangle by six lines, every three of which through different vertices are either concurrent or collinearly intersectant with the opposite sides, according as an odd number of them is internal or external.*

For, if P, Q, R be the three collinear points, and AX, BY, CZ any three of the six lines through A, B, C bisecting externally and internally the three intercepted segments QR, RP, PQ, since then, by (65, Cor 3°.),

$$\sin BAX : \sin CAX = \pm AQ : AR, \quad \sin CBY : \sin ABY = \pm BR : BP,$$

$$\sin ACZ : \sin BCZ = \pm CP : CQ,$$

according as each bisector divides its angle of the triangle externally or internally, and since, by (a),

$$(BP : CP).(CQ : AQ) : (AR : BR) = +1,$$

the three points P, Q, R being by hypothesis collinear, therefore

$$(\sin BAX : \sin CAX).(\sin CBY : \sin ABY).(\sin ACZ : \sin BCZ) = \mp 1,$$

according as an odd number of the bisectors is internal or external, and therefore &c.

N.B. With respect to this last example and all others of the same kind, it is to be observed that, since, of the three segments intercepted on any line by the three angles of any triangle, *two* are always comprehended in the *internal* and *one* always in the *external* regions of the intercepted angle, (see figs. α, β, γ, Art. 134), therefore an odd number of sections of either kind for the segments corresponds always to an odd number of sections of the other kind for the angles, and conversely.

138. The two last Examples, 13° and 14°, of the preceding Article are particular cases of the two following general properties—

1°. *When three points X, Y, Z on the sides of a triangle ABC are collinear or connect concurrently with the opposite vertices, their connectors OX, OY, OZ with any arbitrary point O divide the three angles BOC, COA, AOB subtended at that point by the sides of the triangle, so as to fulfil the relation*

$$\frac{\sin BOX}{\sin COX} \cdot \frac{\sin COY}{\sin AOY} \cdot \frac{\sin AOZ}{\sin BOZ} = \pm 1,$$

and conversely, when they connect with any point O so as to fulfil the above relation they are collinear or connect concurrently with the opposite vertices.

2°. *When three lines AX, BY, CZ through the vertices of a triangle ABC are concurrent or intersect collinearly with the opposite sides, their intersections X, Y, Z with any arbitrary line L divide the three segments QR, RP, PQ intercepted on that line by the angles of the triangle, so as to fulfil the relation*

$$\frac{QX}{RX} \cdot \frac{RY}{PY} \cdot \frac{PZ}{QZ} = \pm 1,$$

and conversely, when they intersect with any line L so as to fulfil the above relation they are concurrent or intersect collinearly with the opposite sides.

For, whatever be the positions of X, Y, Z in 1°, since, by (65), disregarding signs for a moment,

$$\sin BOX : \sin COX = (BX : CX).(CO : BO),$$
$$\sin COY : \sin AOY = (CY : AY).(AO : CO),$$
$$\sin AOZ : \sin BOZ = (AZ : BZ).(BO : AO);$$

and since, evidently, the internal and external sections of BC, CA, AB and of BOC, COA, AOB always correspond, therefore the two compounds,

$$(\sin BOX : \sin COX).(\sin COY : \sin AOY).(\sin AOZ : \sin BOZ)$$

and $$(BX : CX).(CY : AY).(AZ : BZ)$$

are always equal in magnitude and similar in sign, and therefore when either of them $= \pm 1$ so is the other also, which, by relations a and b' of the preceding, proves both parts of 1°. And whatever be the positions of AX, BY, CZ in 2°, since, by the same, again disregarding signs for a moment,

$$QX : RX = (QA : RA).(\sin CAX : \sin BAX),$$
$$RY : PY = (RB : PB).(\sin ABY : \sin CBY),$$
$$PZ : QZ = (PC : QC).(\sin BCZ : \sin ACZ);$$

and since, by (a), the three points P, Q, R being collinear,

$$(QA : RA).(RB : PB).(PC : QC) = +1;$$

therefore, remembering (see note to the preceding Article) that the odd number of sections of either kind for QR, RP, PQ

corresponds always to the odd number of sections of the other kind for BAC, CBA, ACB, and conversely, the two compounds

$$(QX : RX).(RY : PY).(PZ : QZ)$$

and

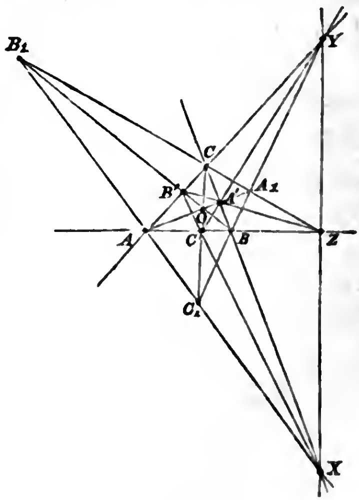
$$(\sin CAX : \sin BAX).(\sin ABY : \sin CBY).(\sin BCZ : \sin ACZ),$$

are always equal in magnitude and opposite in sign, and therefore when either of them $= \pm 1$ the other then $= \mp 1$, which, by relations a' and b of the preceding, prove both parts of 2°, and therefore &c.

139. The next example is given separately from the utility of the double property in the modern geometry of the triangle.

a. When three lines through the vertices of a triangle are concurrent, their three points of intersection with the opposite sides determine an inscribed triangle whose sides intersect collinearly with those of the original to which they correspond.

b. When three points on the sides of a triangle are collinear, their three lines of connection with the opposite vertices determine an exscribed triangle whose vertices connect concurrently with those of the original to which they correspond.



To prove *a.*—If ABC be the original triangle, $A'B'C'$ *any* inscribed triangle, and X, Y, Z the three intersections of their three pairs of corresponding sides BC and $B'C'$, CA and $C'A'$, AB and $A'B'$, then, whatever be the positions of $A'B'C'$, since, by (134, *a.*),

$$BX : CX = (BC' : AC').(AB' : CB'),$$

$$CY : AY = (CA' : BA').(BC' : AC'),$$

$$AZ : BZ = (AB' : CB').(CA' : BA'),$$

therefore, in all cases,

$$(BX : CX).(CY : AY).(AZ : BZ) = (CA' : BA')^2.(AB' : CB')^2.(BC' : AC')^2,$$

and therefore, as above stated, when AA', BB', CC' are concurrent X, Y, Z are collinear, and conversely, both equivalents being then $=+1$.

To prove *b*.—If ABC be the original triangle, $A_1B_1C_1$ *any* exscribed triangle, and X, Y, Z the three intersections of their three pairs of corresponding sides BC and B_1C_1, CA and C_1A_1, AB and A_1B_1, then, whatever be the directions of AX, BY, CZ, since, by (134, *a'*.),

$$\sin BAA_1 : \sin CAA_1 = -(\sin BCZ : \sin ACZ).(\sin ABY : \sin CBY),$$
$$\sin CBB_1 : \sin ABB_1 = -(\sin CAX : \sin BAX).(\sin BCZ : \sin ACZ),$$
$$\sin ACC_1 : \sin BCC_1 = -(\sin ABY : \sin CBY).(\sin CAX : \sin BAX),$$

therefore, in all cases,

$$(\sin BAA_1 : \sin CAA_1).(\sin CBB_1 : \sin ABB_1).(\sin ACC_1 : \sin BCC_1)$$
$$= -(\sin CAX : \sin BAX)^2.(\sin ABY : \sin CBY)^2.(\sin BCZ : \sin ACZ)^2,$$

and therefore, as above stated, when X, Y, Z are collinear AA_1, BB_1, CC_1 are concurrent, and conversely, both equivalents being then $=-1$.

COR. 1°. When, as in *a*, the three lines AA', BB', CC' are concurrent, and the three points X, Y, Z therefore collinear, or conversely, it is easy to see that then always

$$\frac{BX}{CX} = -\frac{BA'}{CA'}, \quad \frac{CY}{AY} = -\frac{CB'}{AB'}, \quad \frac{AZ}{BZ} = -\frac{AC'}{BC'},$$

relations which give at once, numerically, the positions of the three points X, Y, Z when those of the three A', B', C' are known, and conversely.

For, by (134) *a* and *b'*, the common values of the three pairs of equivalents are expressed alike by the three compounds,

$$(BC' : AC').(AB' : CB'), \quad (CA' : BA').(BC' : AC'),$$
$$(AB' : CB').(CA' : BA'),$$

respectively, and therefore &c.

COR. 2°. When, as in *b*, the three points X, Y, Z are collinear, and the three lines AA_1, BB_1, CC_1 therefore concurrent, or conversely, it is easy to see that then always

$$\frac{\sin BAX}{\sin CAX} = -\frac{\sin BAA_1}{\sin CAA_1}, \quad \frac{\sin CBY}{\sin ABY} = -\frac{\sin CBB_1}{\sin ABB_1},$$
$$\frac{\sin ACZ}{\sin BCZ} = -\frac{\sin ACC_1}{\sin BCC_1},$$

relations which give at once, numerically, the directions of the three lines AA_{1}, BB_{1}, CC_{1}, when those of the three AX, BY, CZ are known, and conversely.

For, by (134) b and a', the common values of the three pairs of equivalents are expressed alike by the three compounds

$$(\sin BCZ : \sin ACZ).(\sin ABY : \sin CBY),$$
$$(\sin CAX : \sin BAX).(\sin BCZ : \sin ACZ),$$
$$(\sin ABY : \sin CBY).(\sin CAX : \sin BAX),$$

respectively, and therefore &c.

COR. 3°. From the preceding relations it may be easily shown, that, for the same triangle ABC, the same line XYZ corresponds always to the same point O, and the same point O to the same line XYZ, in the two properties a and b.

For, if XYZ be given, then since, by the relations of Cor. 1°, the three sets of lines BY, CZ, and AA', CZ, AX, and BB', AX, BY, and CC' in (a) are concurrent, and since, by hypothesis, the three sets BY, CZ, and AA_{1}, CZ, AX, and BB_{1}, AX, BY, and CC_{1} in (b) are concurrent, therefore three pairs of lines AA' and AA_{1}, BB' and BB_{1}, CC' and CC_{1} coincide, and therefore &c. And, if O be given, then since, by the relations of Cor. 2°, the three sets of lines BO, CO, and $B_{1}C_{1}$, CO, AO, and $C_{1}A_{1}$, AO, BO, and $A_{1}B_{1}$ in (b) intersect collinearly with the opposite sides of ABC, and since, by hypothesis, the three sets BO, CO, and $B'C'$, CO, AO, and $C'A'$, AO, BO, and $A'B'$ in (a) do the same, therefore the three points X, Y, Z are the same for both, and therefore &c.

COR. 4°. Given, with the triangle ABC, either the point O or the line I containing the three points X, Y, Z, which in the modern geometry of the triangle are intimately connected with each other, and distinguished by correlative names expressive of their relation to each other and the triangle, the other may be immediately determined by mere linear constructions based on the above properties a and b, as follows—

For, the triangle ABC being given, the point O gives the three lines OA, OB, OC, they the three points A', B', C', they the three lines $B'C'$, $C'A'$, $A'B'$, they the three points X, Y, Z, and they finally the line I, by property (a); and conversely, the triangle ABC being given, the line I gives the three points X, Y, Z, they the three lines AX, BY, CZ, they the three points $A_{,}$, $B_{,}$, $C_{,}$, they the three lines $AA_{,}$, $BB_{,}$, $CC_{,}$, and they finally the point O, by property (b).

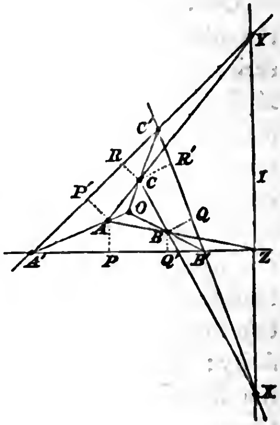
COR. 5°. The point O, or line I—and with either of course the other—being given, if from the original triangle ABC two series of triangles $A'B'C'$, $A''B''C''$, $A'''B'''C'''$, &c., and $A_{,}B_{,}C_{,}$, $A_{,,}B_{,,}C_{,,}$, $A_{,,,}B_{,,,}C_{,,,}$, &c. be derived by the continued repetition of the two inverse constructions indicated in the statements of the properties a and b; applied first to the original triangle itself ABC, as in the figure, producing the two first derivatives $A'B'C'$ and $A_{,}B_{,}C_{,}$, then to each of them, in the same manner, producing the two second derivatives $A''B''C''$ and $A_{,,}B_{,,}C_{,,}$, then to each of them again, in the same manner, producing the two third derivations $A'''B'''C'''$ and $A_{,,,}B_{,,,}C_{,,,}$, and so on to infinity; the two series of triangles thus derived from ABC, by the directing agency of O and I, would form evidently, through the connecting link of the original, one continuous, and in both directions unlimited, system of connected triangles, each inscribed to one and exscribed to the other of the two between which it lies; their three systems of corresponding sides passing in different directions through the same three points X, Y, Z on the line I; their three systems of corresponding vertices lying in different positions on the same three lines OA, OB, OC through the point O; and the point and line O and I having to each and all of them, individually and collectively, the same relations as to the original ABC.

In the particular case of the line I being at infinity, the triangles constituting the system would evidently be all similar, alternately similarly and oppositely placed, and having all the point O for common centre of similitude, (42).

140. The next Example, again, is given separately from the importance of the property as the basis of the theory of perspective, or homology, as it is termed by the French writers, in the geometry of plane figures.

For two triangles of any nature whose vertices and sides correspond in pairs, when the three pairs of corresponding vertices connect concurrently the three pairs of corresponding sides intersect *collinearly, and conversely, when the three pairs of corresponding sides intersect collinearly the three pairs of corresponding vertices connect concurrently.*

For, if ABC and $A'B'C'$ be *any* two triangles whose vertices and sides correspond in pairs, AA', BB', CC', the three connectors of corresponding pairs of vertices, and X, Y, Z the three intersections of corresponding pairs of sides; from the vertices ABC of either letting fall pairs of perpendiculars AP and AP', BQ and BQ', CR and CR' upon the pairs of sides about the corresponding vertices A', B', C' of the other, since then, in all cases,



$$BQ : CR' = BX : CX,$$

$$CR : AP' = CY : AY,$$

$$AP : BQ' = AZ : BZ;$$

therefore, in all cases, the two compounds

$$(BQ : CR').(CR : AP').(AP : BQ'),$$

or $$(AP : AP').(BQ : BQ').(CR : CR'),$$

and $$(BX : CX).(CY : AY).(AZ : BZ),$$

are equal in magnitude and similar in sign, and therefore when either $= +1$ so is the other also; but when the former $= +1$ the three lines AA', BB', CC' through the vertices of A', B', C' are concurrent, and conversely, (135), and when the latter $= +1$ the three points X, Y, Z on the sides of ABC are collinear, and conversely, (134, *a*.), and therefore &c. Of course when either equivalent $= -1$ so too is the other also, but the general property resulting from the circumstance, though equally obvious, is not equally important in that case.

As both parts, *a* and *b*, of the property of the preceding Article are evidently included in the above as particular cases, the former, therefore, though independently established in the text, are not really independent, but are merely converse properties; which is evident also from the obvious consideration, adverted to in Cor. 5°, that the two derived triangles $A'B'C'$ and $A_1B_1C_1$, see figure to the preceding Article, are related each to the original ABC as the original to the other.

141. From the above the following important extension of itself may be readily inferred, viz.—

For two geometrical figures of any kind, F and F', which are of such a nature that, to every point of one corresponds a point of the other, to every line of one a line of the other, to every point of intersection of two lines of one the point of intersection of the two corresponding lines of the other, and to every line of connection of two points of one the line of connection of the two corresponding points of the other; when the several pairs of corresponding points connect concurrently the several pairs of corresponding lines intersect collinearly, and conversely, when the several pairs of corresponding lines intersect collinearly the several pairs of corresponding points connect concurrently.

For, if, in the former case, L and L', M and M', N and N' be any three pairs of corresponding lines, and therefore, by the assumed connections, MN and $M'N'$, NL and $N'L'$, LM and $L'M'$ three pairs of corresponding points, of the figures; since then, by hypothesis, the three latter connect concurrently, therefore, by the above, the three former intersect collinearly; and the property being thus true for every three is therefore true for all pairs of corresponding lines, and therefore &c.; and if, in the latter case, P and P', Q and Q', R and R' be any three pairs of corresponding points, and therefore, by the assumed connections, QR and $Q'R'$, RP and $R'P'$, PQ and $P'Q'$ three pairs of corresponding lines, of the figures; since then, by hypothesis, the three latter intersect collinearly, therefore, by the above, the three former connect concurrently; and the property being thus true for every three is therefore true for all pairs of corresponding points, and therefore &c.

Every two triangles, or figures of any nature related as above to each other, when so relatively situated that their several pairs of corresponding points connect concurrently and their several pairs of corresponding lines intersect collinearly, are said to be *in perspective*, or, as the French writers term it, *in homology* with each other; and, in the same case, the point of concurrence O of the several concurrent connectors, and the line of collinearity I of the several pairs of collinear intersections, either or both of which may be at infinity, are termed respectively *the centre* and *the axis* of perspective or homology; the meaning and origin of the terms are obvious.

142. Two *similar* figures F and F', both right or left, whatever be their nature, when placed either in similar or in opposite positions with respect to each other (41), furnish the most obvious as well as the simplest example of figures in perspective; for, their several pairs of homologous points P and P' connect concurrently through their *centre of similitude* (42), which therefore in their case is the centre of perspective; and, their several pairs of homologous lines L and L', being parallels, intersect collinearly on *the line at infinity* (136, 1°.), which therefore in their case is the axis of perspective.

Conversely, when the axis of perspective of two figures F and F' in perspective is at infinity, the figures themselves, whatever be their nature, are similar, both right or left, and either similarly or oppositely placed; for, as their several pairs of corresponding lines L and L' intersect at infinity, they are parallel, and, as their several pairs of corresponding points P and P' connect through the centre of perspective, that point satisfies for the figures the conditions of similitude (32), and therefore &c. When, in addition, the centre of perspective also is at infinity, the ratio of similitude being then $= + 1$, the figures are not only similar in form, and similarly placed in position, but also equal in magnitude.

143. Two figures F and F' composed of pairs of corresponding points P and P', Q and Q', R and R', &c., connecting by parallel lines all cut in the same ratio by the same line I, furnish another obvious example of figures in perspective, the line of section being the axis, and the point at infinity in the direction of the parallels the centre, of perspective; for perpendicular section generally, every two such figures are said also to be *refractions*, and in the particular case when the ratio of section $= - 1$, as already mentioned in (50), to be *reflections* of each other, with respect to the line or axis of section; the general, like the particular, name having been introduced for convenience into Geometry from the science of Optics.

Conversely, when the centre of perspective of two figures F and F' in perspective is at infinity, the figures themselves, whatever be their nature, are connected with each other by the preceding relation; for, as every two connectors PP' and QQ' of their pairs of corresponding points intersect at infinity, they are parallel, and, as the two corresponding lines PQ and $P'Q'$ connecting their extremities intersect on the axis of perspective, they are divided by that line in the same ratio, and therefore &c. When, in addition, the axis of perspective also is at infinity, the ratio of section being then $= +1$, the figures, which for that ratio would necessarily coincide were the axis not at infinity, are, as already noticed in the preceding article, exact duplicates in form, magnitude, and direction, and merely separated from each other in absolute position by an interval of finite magnitude.

144. Two figures F and F' may be of such a nature as to form, and so circumstanced as to position, that a correspondence between their points and lines in pairs, satisfying the conditions of perspective, may exist in more ways than one. Two similar figures, for instance, of such a form as to be susceptible simultaneously of similar and opposite positions by different ways of correspondence (35), are of such a character, and are accordingly not only in perspective but *doubly* in perspective when in any positions of similitude or opposition, the two centres of similitude, external and internal, being the centres of the two perspectives, and the line at infinity the common axis of both.

Two *circles* being similar figures which, however situated, are *always* at once in similar and opposite positions with respect to each other, are therefore always in perspective for each centre of similitude; but, as we shall see in another chapter, they possess moreover the additional property of being not only in perspective but *doubly* in perspective for *each* centre of similitude, the line at infinity being the common axis for two of the perspectives, and another line at a finite distance the common axis for the other two.

145. In the perspective of two rows of points on different axes or of pencils of lines through different vertices, already

alluded to in (130), an exceptional peculiarity presents itself, which, if not attended to, might cause embarrassment in the applications of the general theory to their particular cases; while the centre of perspective in the case of the rows, and the axis of perspective in the case of the pencils, is unique and determinate (130), the axis in the former case, and the centre in the latter, is indeterminate; every line concurrent with the axes of the rows in the former case, and every point collinear with the centres of the pencils in the latter case, being indifferently an axis of perspective in the one case, and a centre of perspective in the other. All such cases however are exceptional, figures in perspective having in general but a single centre and a single axis of perspective, both generally at a finite distance, but either or both of which may be at infinity.

146. The following are a few consequences from the fundamental theorem of the preceding article (140) respecting triangles in perspective—

a. When three pairs of points P *and* P', Q *and* Q', R *and* R' *connect concurrently, the six centres of perspective* X *and* X', Y *and* Y', Z *and* Z' *of the three pairs of segments* QQ' *and* RR', RR' *and* PP', PP' *and* QQ' *they determine* (131), *lie in four groups of three* XYZ, $Y'Z'X$, $Z'X'Y$, $X'Y'Z$ *on four lines; each pair of corresponding centres thus constituting a pair of opposite intersections of the same tetragram* (106).

a'. When three pairs of lines L *and* L', M *and* M', N *and* N' *intersect collinearly, the six axes of perspective* U *and* U', V *and* V', W *and* W' *of the three pairs of angles* MM' *and* NN', NN' *and* LL', LL' *and* MM' *they determine* (131), *pass in four groups of three* UVW, $V'W'U$, $W'U'V$, $U'V'W$ *through four points; each pair of corresponding axes thus constituting a pair of opposite connections of the same tetrastigm* (106).

For, in the former case, the directions of the three segments PP', QQ', RR' being by hypothesis concurrent, the three pairs of points P and P', Q and Q', R and R' determine therefore four pairs of triangles PQR and $P'Q'R'$, QRP' and $Q'R'P$, RPQ' and $R'P'Q$, PQR' and $P'Q'R$, whose pairs of corresponding sides by (140) intersect collinearly at the six centers of perspective of the three segments, viz. QR and $Q'R'$ at X,

RP and $R'P'$ at Y, PQ and $P'Q'$ at Z, QR' and $Q'R$ at X', RP' and $R'P$ at Y', PQ' and $P'Q$ at Z', and therefore &c.; and, in the latter case, the vertices of the three angles LL', MM', NN' being by hypothesis collinear, the three pairs of lines L and L', M and M', N and N' determine therefore four pairs of triangles LMN and $L'M'N'$, MNL' and $M'N'L$, NLM' and $N'L'M$, LMN' and $L'M'N$, whose pairs of corresponding vertices, by (140), connect concurrently by the six axes of perspective of the three angles, viz., MN and $M'N'$ by U, NL and $N'L'$ by V, LM and $L'M'$ by W, MN' and $M'N$ by U', NL' and $N'L$ by V', LM' and $L'M$ by W', and therefore &c.

b. When three triads of points P, Q, R; P', Q', R'; P'', Q'', R'' determine three triangles whose sides pass concurrently through three collinear points, the three conjugate triads P, P', P''; Q, Q', Q''; R, R', R'' also determine three triangles whose sides pass concurrently through three collinear points.

b'. When three triads of lines L, M, N; L', M', N'; L'', M'', N'' determine three triangles whose vertices lie collinearly on three concurrent lines, the three conjugate triads L, L', L''; M, M', M''; N, N', N'' also determine three triangles whose vertices lie collinearly on three concurrent lines.

For, in the former case, if L, M, N; L', M', N'; L'', M'', N'' be the three triads of sides of the three original, and U, U', U''; V, V', V''; W, W', W'' those of the three conjugate triangles; then, since by hypothesis the three triads of points $L'L''$, $M'M''$, $N'N''$; $L''L$, $M''M$, $N''N$; LL', MM', NN' are collinear, therefore by (140) the three triads of lines U, V, W; U', V', W'; U'', V'', W'' are concurrent; and again, since by hypothesis the three triads of lines L, L', L''; M, M', M''; N, N', N'' are concurrent, therefore, by (140), the three triads of points VW, $V'W'$, $V''W''$; WU, $W'U'$, $W''U''$; UV, $U'V'$, $U''V''$ are collinear, and therefore &c. And in the latter case, if P, Q, R; P', Q', R'; P'', Q'', R'' be the three triads of vertices of the three original, and X, X', X''; Y, Y', Y''; Z, Z', Z'' those of the three conjugate triangles; then, since by hypothesis the three triads of lines $P'P''$, $Q'Q''$, $R'R''$; $P''P$, $Q''Q$, $R''R$; PP', QQ', RR' are concurrent, therefore, by (140), the three triads of points X, Y, Z; X', Y', Z'; X'', Y'', Z'' are collinear; and again, since by hypothesis the three triads of points P, P', P''; Q, Q', Q'';

R, R', R'' are collinear, therefore, by (140), the three triads of lines YZ, $Y'Z'$, $Y''Z''$; ZX, $Z'X'$, $Z''X''$; XY, $X'Y'$, $X''Y''$ are concurrent, and therefore &c.

c. When three figures of any kind F, F', F'', in prespective two and two, have a common axis of perspective, the three centers of perspective of the three pairs they determine are collinear.

c'. When three figures of any kind F, F', F'', in perspective two and two, have a common centre of perspective, the three axes of perspective of the three pairs they determine are concurrent.

For, in the former case, if P, Q, R; P', Q', R'; P'', Q'', R'' be any three triads of corresponding points of the three figures; then, since by hypothesis the three triads of lines QR, $Q'R'$, $Q''R''$; RP, $R'P'$, $R''P''$; PQ, $P'Q'$, $P''Q''$ pass concurrently through three collinear points, therefore, by the preceding (b), the three triads of lines $P'P''$, $Q'Q''$, $R'R''$; $P''P$, $Q''Q$, $R''R$; PP', QQ', RR' also pass concurrently through three collinear points, and therefore &c. And, in the latter case, if L, M, N; L', M', N'; L'', M'', N'' be any three triads of corresponding lines of the three figures; then, since by hypothesis the three triads of points MN, $M'N'$, $M''N''$; NL, $N'L'$, $N''L''$; LM, $L'M'$, $L''M''$ lie collinearly on three concurrent lines, therefore, by the preceding (b'), the three triads of points $L'L''$, $M'M''$, $N'N''$; $L''L$, $M''M$, $N''N$; LL', MM', NN' also lie collinearly on three concurrent lines, and therefore &c.

These two latter properties the reader may easily verify, *a priori*, for the particular cases when the common axis in the former case and the common centre in the latter case is at infinity.

147. *When two triangles ABC and $A'B'C'$, whose vertices and sides correspond in pairs, are in perspective.*

a. The sides of each intersect with the non-corresponding pairs of sides of the other so as to fulfil (see fig.)

for ABC the general relation

$$\frac{BY.BZ'}{CY.CZ'} \cdot \frac{CZ.CX'}{AZ.AX'} \cdot \frac{AX.AY'}{BX.BY'} = +1,$$

for $A'B'C'$ the corresponding relation

$$\frac{B'Y'.B'Z}{C'Y'.C'Z} \cdot \frac{C'Z'.C'X}{A'Z'.A'X} \cdot \frac{A'X'.A'Y}{B'X'.B'Y} = +1.$$

b. The vertices of each connect with the non-corresponding pairs of vertices of the other so as to fulfil (see fig.) *for ABC the general relation*

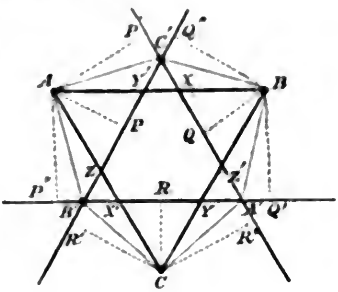
$$\frac{\sin BAB'.\sin BAC'}{\sin CAB'.\sin CAC'}\cdot\frac{\sin CBC'.\sin CBA'}{\sin ABC'.\sin ABA'}\cdot\frac{\sin ACA'.\sin ACB'}{\sin BCA'.\sin BCB'}=+1,$$

for A'B'C' the corresponding relation

$$\frac{\sin B'A'B.\sin B'A'C}{\sin C'A'B.\sin C'A'C}\cdot\frac{\sin C'B'C.\sin C'B'A}{\sin A'B'C.\sin A'B'A}\cdot\frac{\sin A'C'A.\sin A'C'B}{\sin B'C'A.\sin B'C'B}$$
$$=+1,$$

and conversely, when two triangles ABC and A'B'C', whose vertices and sides correspond in pairs, are such that the sides of one intersect with the non-corresponding pairs of sides of the other so as to fulfil either relation (*a*), *or that the vertices of one connect with the non-corresponding pairs of vertices of the other so as to fulfil either relation* (*b*), *they are in perspective.*

For, from the three vertices of either triangle ABC, letting fall the three triads of perpendiculars AP, AP', AP''; BQ, BQ', BQ''; CR, CR', CR'' upon the three sides, corresponding and non-corresponding, of the other $A'B'C'$; then, since, in the case of (*a*), by pairs of similar right-angled triangles,



$$BY : CY = BQ' : CR \text{ and } BZ' : CZ' = BQ : CR'',$$
$$CZ : AZ = CR' : AP \text{ and } CX' : AX' = CR : AP'',$$
$$AX : BX = AP' : BQ \text{ and } AY' : BY' = AP : BQ'';$$

and since, in the case of (*b*), by (61), directly

$$\sin B'A'B : \sin C'A'B = BQ' : BQ$$
and $$\sin B'A'C : \sin C'A'C = CR : CR'',$$
$$\sin C'B'C : \sin A'B'C = CR' : CR$$
and $$\sin C'B'A : \sin A'B'A = AP : AP'',$$
$$\sin A'C'A : \sin B'C'A = AP' : AP$$
and $$\sin A'C'B : \sin B'C'B = BQ : BQ'',$$

therefore the left-hand numbers of the first relation (*a*) and of the second relation (*b*) are always equal in magnitude and sign to the compound

$$(BQ' : CR'').(CR' : AP'').(AP' : BQ''),$$

or which is the same thing to the compound

$$(AP' : AP'').(BQ' : BQ'').(CR' : CR''),$$

which, by (135), $= +1$ when the triangles are in perspective, and conversely, and therefore &c.

By simply interchanging the two triangles in the preceding construction and demonstrations, the second relation (*a*), which is for $A'B'C'$ what the first is for ABC, and the first relation (*b*), which is for ABC what the second is for $A'B'C'$, result of course in the same manner.

148. With an important example of the application of each of the preceding criteria of perspective between triangles, whose vertices and sides correspond in pairs, we shall conclude the present chapter.

Example of criterion (*a*).—*In every hexagon inscribed in a circle the two triangles determined by the two sets of alternate sides are in perspective, opposite sides in the hexagon being corresponding sides in the perspective.*

For, supposing in the figure of the preceding article the six vertices X and X', Y and Y', Z and Z' of the hexagon $YZ'XY'ZX'$ determined by the six sides of the two triangles ABC and $A'B'C'$ to be six points on a circle; then, since Euc. III. 35, 36,

$$AX.AY' = AZ.AX',\ BY.BZ' = BX.BY',\ CZ.CX' = CY.CZ',$$

therefore relation (*a*) for the triangle ABC is satisfied in the simplest manner of which it is susceptible, and therefore &c.

This is the celebrated *Theorem of Pascal* respecting a hexagon inscribed in a circle, and accordingly the centre and axis of the perspective in this case are often spoken of as *the Pascal point and line of the hexagon.*

Example of criterion (*b*).—*In every hexagon circumscribed to a circle the two triangles determined by the two sets of alternate vertices are in perspective, opposite vertices in the hexagon being corresponding vertices in the perspective.*

For, supposing in the same figure the six sides BC' and $B'C'$, CA' and $C'A$, AB' and $A'B$ of the hexagon $BC'AB'CA$, determined by the six vertices of the two triangles ABC and $A'B'C'$ to be six tangents to a circle; then, if a, b, c be the lengths of the three chords intercepted by the circle on the three sides BC, CA, AB of either triangle ABC, since, by (66, Cor. 2°),

$$\sin BAB'.\sin BAC' : \sin CAB'.\sin CAC' = c^2 : b^2,$$

$$\sin CBC'.\sin CBA' : \sin ABC'.\sin ABA' = a^2 : c^2,$$

$$\sin ACA'.\sin ACB' : \sin BCA'.\sin BCB' = b^2 : a^2,$$

therefore relation (b) is satisfied for the triangle ABC, and therefore &c.

This is the celebrated *Theorem of Brianchon* respecting a hexagon circumscribed to a circle, and accordingly the centre and axis of the perspective in this case are often spoken of as *the Brianchon point and line of the hexagon.*

If of the two triangles ABC and $A'B'C'$ one be either inscribed or exscribed to the other, and the latter therefore either exscribed or inscribed to the former, the circle in either of the above properties would manifestly pass through the three vertices of the inscribed and there touch the three sides of the exscribed triangle, and the two properties of triangles circumscribed and inscribed to a circle, given in examples 3° and 4°, Art. 137, would follow at once as particular cases from either of the above.

CHAPTER IX.

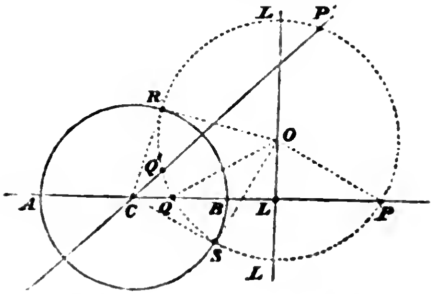
THEORY OF INVERSE POINTS WITH RESPECT TO A CIRCLE.

149. Every two points P and Q on any diameter of a circle, the rectangle $CP.CQ$ under whose distances from the centre C is equal in magnitude and sign to the square of the radius CR, are said to be *inverse points* with respect to the circle.

From the mere definition of inverse points it is evident that: 1°. When the radius of the circle is real they always lie at the same side of the centre and at opposite sides of the circumference, and coincide on the latter when their common distance from the former is equal to the radius; 2°. When the radius is imaginary they always lie at opposite sides of the centre, never coincide, and are at their least distance asunder when equidistant from the centre; 3°. Whether the radius be real or imaginary, as one recedes from, the other approaches to the centre, and conversely, and when one is at infinity in any direction the other is at the centre, and conversely; 4°. In the extreme case when the radius is evanescent, and the circle therefore a point, one is always at the point and the other any where indifferently; 5°. In the other extreme case when the radius is infinite, and the part of the circle not at infinity therefore a line, they are simply reflexions of each other with respect to the line (50). Of these particulars the last, less evident than the others, will appear more fully from the following general property of inverse points.

150. *If P and Q be any pair of inverse points with respect to a circle of any nature, A and B the two extremities, real or imaginary, of the diameter on which they lie, and C the centre of the circle, then always*

$$AP^2 : AQ^2 = BP^2 : BQ^2 = CP : CQ.$$



For, since, by hypothesis, $CP.CQ = CR^2$, therefore

$$CP : CR = CR : CQ = CP \pm CR : CR \pm CQ,$$

and therefore $(CP \pm CR)^2 : (CR \pm CQ)^2 = CP : CQ$, but

$$CP + CR = AP,\ CQ + CR = AQ,\ CP - CR = BP,\ CQ - CR = BQ;$$

and therefore &c.

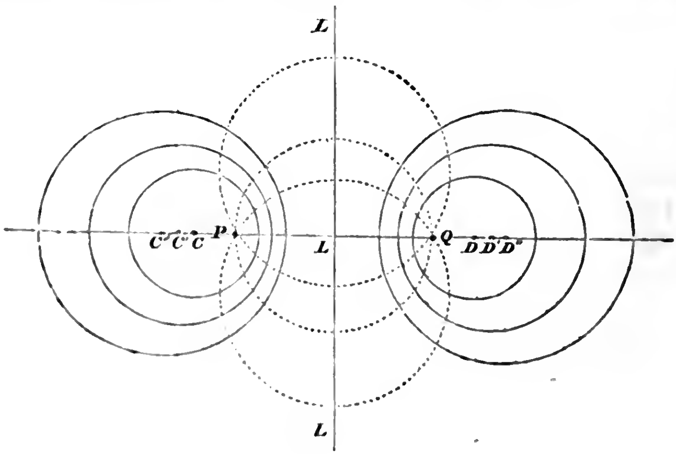
Hence, in the particular case when C is at infinity, that is, when the part of the circle not at infinity with it is a line at a finite distance; since then $CP : CQ = 1$, therefore, by the above, $AP^2 : AQ^2 = 1$ and $BP^2 : BQ^2 = 1$, or the two points A and B are the two points of bisection, external and internal, of the segment PQ, and therefore, as stated in the preceding Article, *the two points P and Q are in that case reflexions of each other with respect to the line into which the part of the circle not at infinity then opens out.*

In the Geometry of the Circle, upon which we are now formally entering, the reader will find, as he proceeds, that universally, as above, when the centre of a circle goes off to infinity without carrying the entire circle with it, the line at a finite distance, into which the figure in its limiting form for the extreme magnitude of its radius $= \infty$ then opens out (18), is in reality but *part* of the entire circle; *the line at infinity* (136) being invariably the remaining part, and possessing, in combination with the line not at infinity, all the properties of the complete circle in the general case; instances confirmatory of this will appear in numbers in the sequel, and though to avoid circumlocution we shall continue generally to speak, as we have hitherto done, of a circle becoming a line when its centre goes to infinity leaving itself behind, the circumstance that the line at infinity is always to be associated with the line not at infinity as part of the entire circle must never, in such cases, be lost sight of whenever it may be necessary, as it often is, to take it into account.

151. Whatever be the nature of the circle, the inverse Q of every point P, not the centre C, is evidently unique and determinate, being on the line CP connecting two known points C and P, and at a distance CQ from one of them C of known magnitude and sign; of the centre itself, however, the inverse, being on the line connecting two coincident points, is indeterminate, *any point at infinity* when the radius is finite, and *any point whatever* in the particular case when it is evanescent, satisfying evidently the conditions that determine it.

When two points P and Q are such that one P is the inverse of the other Q with respect to any circle, the latter Q is, of course, reciprocally, the inverse of the former P with respect to the same circle.

152. As every circle has an infinite number of pairs of inverse points P and Q, P' and Q', P'' and Q'', &c., whose lines of connection all pass through its centre C, and for which the several rectangles $CP.CQ$, $CP'.CQ'$, $CP''.CQ''$, &c. are all equal in magnitude and sign to the square of its radius CR; so conversely, every two points P and Q have an infinite number of circles to which they are inverse, whose centres C, C', C'', &c. all lie on their



line of connection PQ, and the squares of whose radii CR, $C'R'$, $C''R''$, &c. are severally equal in magnitude and sign to the corresponding rectangles $CP.CQ$, $C'P.C'Q$, $C''P.C''Q$, &c.; every such circle is said, for a reason that will appear in another chapter, to be *coaxal* with the two points P and Q, and its radius CR is evidently real or imaginary according as its centre C is external or internal to the segment PQ, evanescent when C coincides with either point P or Q, and infinite when C is at infinity, in which case the line into which the part of the circle not at infinity then opens out is (150) the axis of reflexion L of the two points P and Q.

Every two circles belonging to such a system being evidently equal in magnitude when their centres C and D, C' and D', C'' and D'', &c. are equidistant in opposite directions from the middle point of PQ, the entire system consists therefore of two similar and opposite groups, symmetrically disposed in equal and opposite pairs, reflexions of each other with respect to the axis of reflexion of P and Q, by and through which, in combination with the line at infinity, the circles of one group are separated from and pass into those of the other; each circle of each group enclosing all within and being enclosed by all without itself; and each point P and Q being the nucleus round which the circles of its own group are eccentrically disposed, and the evanescent limit through which they pass from real to imaginary, and conversely.

In the particular case when the two points P and Q coincide, the circles of the system are all real, the range of centres PQ for which they are imaginary in the general case being then evanescent. In this, the only case in which any two circles of the entire system have a common point or any two of the same group a common tangent, they evidently all pass through the point of coincidence $P=Q$, and all touch at that point the line L passing through it perpendicular to their line of centres; and all the other particulars respecting their distribution, as above stated for the general case, are obvious, and have been already stated in Art. 18.

153. In connection with the subject of the preceding Article the following problem not unfrequently presents itself:

Given two pairs of points P and Q, P' and Q' on the same line, to determine the centre C and radius CR of the circle coaxal with both.

To solve which, since, by the preceding,

$$CP.CQ = CP'.CQ' = CR^2,$$

therefore, assuming arbitrarily any point M not on the line, describing through it the two circles PMQ and $P'MQ'$, and drawing their chord of intersection MN intersecting the given line at the point C; the circle round C as centre, the square of whose radius CR is equal in magnitude and sign to the rectangle $CM.CN$, is evidently that required. For (Euc. III. 35, 36)

$$CP.CQ = CP'.CQ' = CM.CN = CR^2,$$

and therefore &c.

The circle thus determined, though its centre C is always real, is itself imaginary when the two points P and Q alternate with the two P' and Q' in the order of their occurrence on their common axis; this is evident from the obvious circumstance that the rectangle $CM.CN$ is then necessarily negative; in every other case however it is positive, and the circle is therefore real.

In the particular case when the two intercepted segments PQ and $P'Q'$ have a common middle point, the centre C, determined as above, being then at infinity, the part of the circle itself not at infinity opens out, as it ought, into the common axis of reflexion of the two pairs of points P and Q, P' and Q', see (150).

154. Any two segments of the same diameter of a circle, which are such that the extremities of one are the inverses of the extremities of the other with respect to the circle, are termed *inverse segments* with respect to the circle; thus, if PP' be the segment intercepted between any two points P and P' on the same diameter of a circle, and QQ' that intercepted between the two inverse points Q and Q', the two segments PP' and QQ' are inverse segments with respect to the circle.

Since, from the interchangeability of inverse points (151), every two pairs of inverse points P and Q, P' and Q' on the same diameter of a circle, determine evidently two different pairs of inverse segments PP' and QQ', PQ' and QP', hence connected with every pair of inverse segments PP' and QQ' with respect to any circle, there exists always a conjugate pair PQ' and QP' with respect to the same circle.

Again, as every two segments PQ and $P'Q'$ of the same line thus determine two different pairs of segments PP' and QQ', PQ' and QP' inverse to the unique circle coaxal with themselves (153), so conversely, they determine two different circles with respect to which they are themselves inverse segments, one that coaxal with the two PP' and QQ', and the other that coaxal with the two PQ' and QP' (153).

Hence the useful problem, *given two segments PQ and $P'Q'$ of the same line, to determine the two circles with respect to which they are inverse segments*, is reduced to that of the preceding Article (153), viz. to determine the two circles which are coaxal, one with the two segments PP' and QQ', and the other with the two PQ' and QP', and which, from the construction given in that Article, are easily seen to be both real in the case when the extremities of the two given segments PQ and $P'Q'$ alternate with each other in the order of their occurrence on their common axis, and to be one real and one imaginary in either of the two cases when they do not.

155. *Every two points and their two inverses with respect to the same circle lie in a circle.*

For, if (fig., Art. 150) P and P' be the two points, Q and Q' their two inverses, and C the centre of the circle; then since, by the definition of inverse points, $CP.CQ = CP'.CQ'$, each being = the square of the radius of the circle, therefore &c.

Conversely, *every circle passing through a pair of inverse points with respect to another circle determines a pair of inverse points on every diameter of the other.*

For, if P and Q, P' and Q' (same figure) be the two pairs of points in which any circle intersects any two diameters of any other circle, and C the centre of the latter; then, since $CP.CQ = CP'.CQ'$, if either rectangle = the square of either radius, so is the other.

Cor. 1°. It is evident from the above that if the same circle pass through a pair of inverse points with respect to one circle, and also through a pair of inverse points with respect to another circle, it cuts the diameter common to both in a pair of inverse points common to both.

Cor. 2°. The preceding furnishes an obvious solution of the problem, "*to determine on the common diameter of two given circles the two points, real or imaginary, inverse to both.*" For, assuming arbitrarily any point P, and describing the circle passing through it and through its two inverses Q and R with respect to the two circles; the circle PQR thus described intersects, by the preceding, the common diameter in the two points required.

The two points E and F thus determined are imaginary when the two circles intersect, real when they do not, and coincident at the point of contact when they touch. See Art. 152.

156. *Every circle passing through a pair of inverse points with respect to another circle is orthogonal to the other.* (22).

For, if C (fig., Art. 150) be the centre of any circle, P and Q any pair of inverse points with respect to it, and R either point in which any circle through P and Q intersects it; since then by hypothesis $CP.CQ=CR^2$, therefore CR, a radius of one circle, is a tangent to the other, and therefore &c. (22).

Conversely, *every circle orthogonal to another determines pairs of inverse points on all diameters of the other.*

For, if C (same fig.) be the centre of either circle, P and Q the two points in which any line through it meets the other, and R either point of intersection of the two; then since the radius CR of the former is, by hypothesis, a tangent to the latter, therefore $CP.CQ=CR^2$, and therefore &c. (22).

Cor. 1°. It is evident from the above that *every circle passing through the common pair of inverse points with respect to two others* (155, Cor. 2°) *is orthogonal to both*, and conversely that, *every circle orthogonal to two others passes through their common pair of inverse points.*

Cor. 2°. It is also evident from the same that *all the circles of a system having a common pair of inverse points* (see the undotted circles of fig., Art. 152) *are cut orthogonally by every circle passing through the points*, and, conversely, that *all the circles of a system passing through a pair of common points* (see the dotted circles of same figure) *are cut orthogonally by every circle coaxal with the points.*

Cor. 3°. It follows also from the above and from Cor. 1°. that *if a variable circle pass through a fixed point and cut a fixed circle at right angles, or, more generally, if it cut two fixed circles at right angles, the locus of its centre is a line*; for passing through the point and its inverse with respect to the circle in the former case, and through the common pair of inverse points with respect to the two circles in the latter case, its centre in either case describes therefore the axis of reflexion of the two points through which it passes; a more general proof for the second case will be given in another chapter.

Cor. 4°. The preceding supply obvious solutions of the three following problems: "*To describe a circle*, 1°. *passing through two given points and cutting a given circle at right angles*; 2°. *passing through a given point and cutting two given circles at right angles*; 3°. *cutting three given circles at right angles.*" For the circle passing through the two points and through the inverse of either with respect to the circle, in the first case; that passing through the point and its two inverses with respect to the two circles, in the second case; and that orthogonal to any one of the three circles, and passing through the common pair of inverse points with respect to the other two, in the third case; is evidently that required; a more general construction for the third case will be given in another chapter.

157. *The two tangents to a circle from any point in the axis of reflexion of any pair of inverse points are equal to the two distances of the point from the inverse points.*

For, if P and Q (fig., Art. 150) be the inverse points, O any point in their axis of reflexion L, and OR and OS the two tangents from O to the circle; since then, by the preceding, the circle round O as centre which passes through P and Q cuts the original circle at right angles, it passes through R and S, and therefore &c.

Conversely, *the locus of a variable point, not at infinity* (15), *the tangents from which to a fixed circle are equal to its distance from a fixed point is a line, the axis of reflexion of the point and its inverse with respect to the circle.*

For, if P (same fig.) be the fixed point, and O any point for which the two tangents OR and OS to the fixed circle are each equal to the distance OP; since then the circle round O as centre which passes through P passes through R and S, it cuts the fixed circle at right angles, and therefore passes also through Q, the inverse of P with respect to the fixed circle, and therefore &c.

Cor. 1°. It is evident from the first part of the above that when (152) any number of circles have a common pair of inverse points P and Q, tangents to them all from any point in the axis of reflexion L of the two points are equal.

Cor. 2°. The second part of the above supplies of itself obvious solutions of the two following problems:

1°. *To determine the point on a given line or circle, the tangents from which to a given circle shall be equal to its distance from a given point.*

2°. *To determine the point, the tangents from which to two given circles shall be equal to its distances from two given points.*

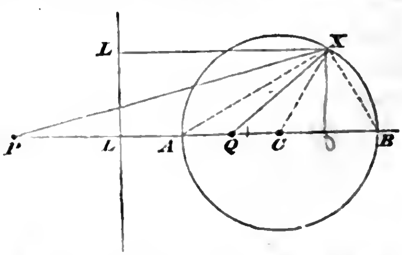
And, by aid of Cor. 2°., Art. (155) of the two following:

1'. *To determine the point on a given line or circle, the tangents from which to two given circles shall be equal.*

2'. *To determine the point, the tangents from which to three given circles shall be equal.*

158. *The squares of the distances of a variable point on a fixed circle from any fixed pair of inverse points have the constant ratio of the distances of the centre from the inverse points.*

For, if C be the centre of the circle, P and Q the fixed pair of inverse points, and X the variable point on the circle; since then, by hypothesis, $CP.CQ = CX^2$, or, which is the same thing, $CP : CX :: CX : CQ$; therefore the triangles PCX and XCQ are similar, and



therefore $PX^2 : QX^2 :: PC^2 : CX^2 :: CX^2 : QC^2 :: PC : QC$. The property of Art. (150) is evidently a particular case of this.

Conversely, *the locus of a variable point the distances of which from two fixed points have any constant ratio, is the circle coaxal with the fixed points* (152) *whose centre divides the distance between them in the duplicate of the constant ratio.*

For, if P and Q be the two fixed points, X any position of the variable point, and C the point on PQ for which $CP.CQ = CX^2$; then since, as above, the triangles PCX and XCQ are similar, therefore, as above, $PC : QC :: PX^2 : QX^2$, which being by hypothesis constant, therefore &c.

If while the two points P and Q remain fixed, the constant ratio $PX : QX$ be conceived to vary and pass continuously through all values from 0 to ∞, the locus circle will pass evidently through all the phases of coaxality with P and Q described in (152); see fig. of that article. Commencing with the point P as the nascent limit for the extreme value 0; opening out into the axis of reflexion L of P and Q as the part of the locus not at infinity (150) for the mean value 1; and ending with the point Q as the evanescent limit for the extreme value ∞.

Since for *every* point X *at infinity* the ratio $PX : QX = 1$ (15), the *complete* locus, which for every value of the ratio not $= 1$ is by the above a single unbroken circle in its general form, consists therefore for the particular value of the ratio $= 1$ of *two* lines, viz. the axis of reflexion of P and Q, and the line at infinity (136); this is an instance confirmatory of the general statement made at the close of Art. (150), that when the centre of a circle of infinite radius is at infinity the circle itself breaks up into two lines, one at a finite distance, and the other at infinity.

Cor. 1°. Since from the similarity of the two triangles PCX and QCX in the first part of the above, the two pairs of angles XPC and QXC, XQC and PXC are always similar (24), it follows consequently that—

Of the two lines connecting any point on a circle with any pair of inverse points, the angle determined by either with the radius at the point is similar to that determined by the other with the diameter containing the inverse points.

Cor. 2°. The second part of the above supplies obvious solutions of the two following problems:

1°. *To determine the point on a given line or circle, the ratio of whose distances from two given points shall be given.*

2°. *To determine the point, the ratios of whose distances from three given points shall be given.*

159. *The square of the distance of a variable point on a fixed circle from any fixed point varies as its distance from the axis of reflexion of the point and its inverse with respect to the circle.*

For, if C (figure of last Article) be the centre of the circle, P and Q the fixed point and its inverse, X any position of the variable point on the circle, and XL the perpendicular from X on the axis of reflexion L of P and Q; since then, Euc. II., 5, 6,

$$PX^2 - QX^2 = 2PQ.LX = 2(PC - QC)LX,$$

and since, by the preceding, $PX^2 : QX^2 :: PC : QC$, therefore $PX^2 = 2PC.LX$ and $QX^2 = 2QC.LX$, and therefore &c.

Conversely, *the locus of a variable point the square of whose distance from a fixed point varies as its distance from a fixed line is a circle coaxal with the point and its reflexion with respect to the line* (152).

For, if P (same fig.) be the fixed point, L the fixed line, Q the reflexion of P with respect to L, X any position of the variable point, XL its distance from the fixed line, and C the point on PQ for which $PX^2 = 2PC.LX$; since then, as above,

$$PX^2 - QX^2 = 2PQ.LX = 2(PC - QC)LX,$$

therefore $QX^2 = 2QC.LX$, and therefore $PX^2 : QX^2 :: PC : QC$, from which, since by hypothesis PC is constant, and therefore C fixed, it follows from the preceding that the locus of X is the circle coaxal with P and Q whose centre is C.

If while the point and line P and L remain fixed, the base PC of the variable rectangle $PC.LX$ be conceived to vary and take successively in the direction opposite to that of PQ all values from 0 to ∞, the locus circle will pass evidently through half the system of phases of coaxality with P and Q described in (152); commencing with P as the nascent limit for the extreme value 0, and ending with L as the part not at infinity of the infinite limit for the extreme value ∞. And if then after passing through infinity PC be conceived to change direction and take successively all values from ∞ to PQ, the locus circle will pass evidently through the remaining half of the same series of phases; commencing with L as the part not at infinity of the infinite limit for the extreme value ∞, and ending with Q as the evanescent limit for the extreme value PQ; after which, changing its nature, it will evidently become and continue imaginary for all lesser values from PQ down to 0.

COR. The second part of the above supplies obvious solutions of the two following problems:

1°. *To determine the point on a given line or circle, the square of whose distance from a given point shall be equal to the rectangle under a given base and its distance from a given line.*

2°. *To determine the point, the squares of whose distances from two given points shall be equal to the rectangles under two given bases and its distances from two given lines.*

160. *The angle connecting any point on a circle with any pair of inverse points is bisected, internally and externally, by the lines connecting the point with the extremities of the diameter containing the inverse points.*

For, if (same figure as in Art. 158) C be the centre of the circle, P and Q the pair of inverse points, A and B the extremities of the diameter on which they lie, and X any point on the circle; since then, by the first part of (158),

$$PA^2 : QA^2 = PB^2 : QB^2 = PX^2 : QX^2 = PC : QC,$$

therefore (Euc. VI. 3) the angle PXQ is bisected internally and externally by the two lines PA and PB, and therefore &c.

Conversely, *the locus of a variable point the angle connecting which with two of three fixed collinear points is bisected, internally or externally, by the line connecting it with the third, is the circle coaxal with the two which passes through the third.*

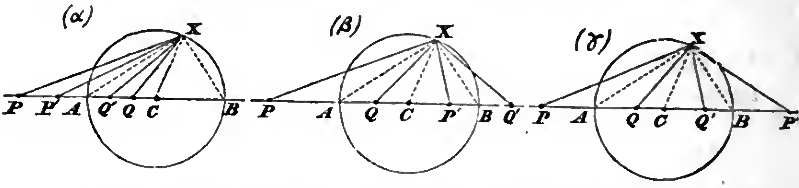
For, if P and Q (same figure) be the first and second of the fixed points, A or B the third, C the point on their common line for which $PC : QC = PA^2 : QA^2$ or $PB^2 : QB^2$, and X any position of the variable point; since then, by hypothesis, the angle PXQ is bisected by the line PA or PB, therefore (Euc. VI. 3) $PX^2 : QX^2 = PA^2 : QA^2$ or $PB^2 : QB^2 = PC : QC$, consequently, by the second part of (158), $CX^2 = CP.CQ = CA^2$ or CB^2, and therefore &c.

COR. The second part of the above supplies obvious solutions of the two following problems:

1°. *To determine the point on a given line or circle, the angle connecting which with two of three given points in a line shall be bisected by the line connecting it with the third.*

2°. *To determine the point, the angles connecting which with the extremities of two given lines shall be bisected by the lines connecting it with two given points on the lines.*

161. *Every two inverse segments of any diameter of a circle* (154) *subtend similar angles* (24) *at every point on the circle.*



For, if PP' and QQ' be the two inverse segments, P and Q, P' and Q' their two pairs of inverse extremities, A and B the extremities of the diameter to which they belong, C the middle point of AB, and X any point on the circle; then since, by the first part of the preceding (160), the two angles PXQ and $P'XQ'$ have the same bisectors XA and XB, therefore the two angles PXP' and QXQ' (and also the two PXQ' and QXP' (154)) are similar, and therefore &c.

Conversely, *the locus of a variable point the angles subtended at which by two fixed coaxal segments are similar, consists of the two circles* (154) *with respect to which the two segments are inverse.*

For, if PP' and QQ' (same figures) be the two segments, and X any position of the variable point; then since, by hypothesis, the two angles PXP' and QXQ' are similar, therefore either the two angles PXQ and $P'XQ'$, or the two PXQ' and QXP', have the same bisectors; in the former case (that of the figures), if C be the middle point of the segment AB intercepted on the axis of the segments by the common bisectors XA and XB, then since, as in the second part of the preceding (160), $CX^2 = CP.CQ = CP'.CQ'$, therefore C and CX are the centre and radius of the circle coaxal with PQ and $P'Q'$ (153), and therefore &c.; and in the latter case (not that of the figures), if C' be the middle point of the segment $A'B'$ intercepted on the axis of the segments by the common bisectors XA' and XB', then since, for the same reason as before, $C'X^2 = C'P.C'Q' = C'Q.C'P'$, therefore C' and $C'X$ are the centre and radius of the circle coaxal with PQ' and $P'Q$ (153), and therefore &c.

Of the two different circles comprising the above locus, though the first is real for all the three possible modes (82) in which the two segments PP' and QQ' could be disposed on their common axis, as represented in the three figures (α), (β), (γ), the second is real only for the disposition, represented in fig. (β), in which the extremities of one segment alternate with those of the other in the order of their occurrence on their common axis (see 153).

Cor. 1°. From the similarity of the two pairs of angles PXP' and QXQ', PXQ' and QXP' in the first part of the above, it follows immediately from (65), combined with (158), that

$$\frac{PP'.PQ'}{QP'.QQ'} = \frac{PX^2}{QX^2} = \frac{PA'^2}{QA'^2} = \frac{PB'^2}{QB'^2} = \frac{PC}{QC},$$

and, of course, for the same reason that

$$\frac{P'P.P'Q}{Q'P.Q'Q} = \frac{P'X^2}{Q'X^2} = \frac{P'A'^2}{Q'A'^2} = \frac{P'B'^2}{Q'B'^2} = \frac{P'C}{Q'C},$$

and therefore, generally, that—

The rectangles under the distances of any pair of inverse points from any other pair on the same diameter are as the squares of their distances from each extremity of the diameter, and as their distances from the centre of the circle.

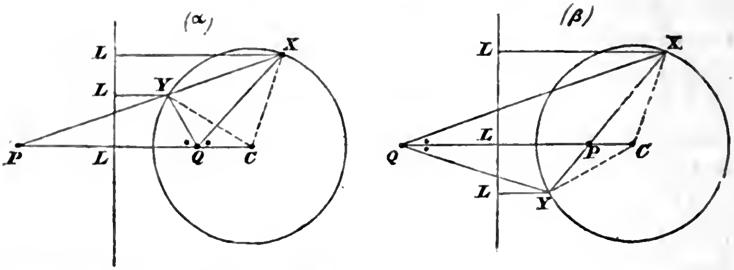
Cor. 2°. The second part of the above supplies obvious solutions of the two following problems:

1°. *To determine the point on a given line or circle, the angles subtended at which by two given coaxal segments shall be similar.*

2°. *To determine the point, the angles subtended at which by three given coaxal segments shall be similar.*

162. *The extremities of any chord of a circle, the centre, and the inverse of any point on the chord, lie in a circle.*

For, if C be the centre of the circle, X and Y the ex-



tremities of any chord, P any point, external or internal, on XY, and Q the inverse of P with respect to the circle; since then CX^2 or $CY^2 = CP^2 - PX.PY$, by the isosceles triangle XCY, and $= CP.CQ = CP^2 - PC.PQ$, by the inverse points P and Q, therefore $PX.PY = PC.PQ$, and therefore &c.

Conversely, *every circle passing through the centre of another circle passes through the inverse of every point on its chord of intersection with the other.*

For, if C be the centre of any circle, X and Y its points of intersection with any circle passing through C, P any point, external or internal, on XY, and Q the point in which the circle XCY intersects the line CP; since then $PC.PQ = PX.PY$, therefore $PC^2 - PC.PQ = PC^2 - PX.PY$, that is, $CP.CQ = CX^2$ or CY^2, and therefore &c.

Cor. 1°. From the above, supposing the two points P and Q to remain fixed, and the line and circle XY and XCY to vary simultaneously, it appears that—

If a variable line pass through a fixed point and intersect a fixed circle, the circle passing through the points of intersection and though the centre of the latter passes through a second fixed point, the inverse of the first with respect to the fixed circle.

And conversely, that—

If a variable circle pass through a fixed point and through the centre of a fixed circle, its chord of intersection with the latter passes through a second fixed point, the inverse of the first with respect to the fixed circle.

Cor. 2°. From the same, supposing, conversely, the line XY and circle XCY to remain fixed, and the two points P and Q to vary simultaneously, it appears again that—

If a variable point describe a fixed line, its inverse with respect *to any circle describes the circle determined by the centre of the latter and by its intersections with the fixed line.*

And conversely, that—

If a variable point describe a fixed circle, its inverse with respect to any cirele through whose centre it passes describes the line determined by the points of intersection of the two circles.

Cor. 3°. In the particular case when P is the middle point of the chord XY, since then CQ is evidently a diameter of the circle XCY, therefore the two angles CXQ and CYQ are both right, and therefore, from the above—

The middle point of any chord of a circle and the intersection of the two tangents at its extremities, and conversely, the intersection of any two tangents to a circle and the middle point of their chord of contact, are inverse points with respect to the circle.

163. *The diameter containing any pair of inverse points with respect to a circle bisects, externally or internally, the angle subtended at either point by any chord of the circle whose direction passes through the other.*

For, if P and Q (figures of last article) be the two points, X and Y the extremities of any chord passing through either of them P, and C the centre of the circle; then, since by (158), $PX^2 : QX^2 = PY^2 : QY^2$, each being $= PC : QC$, therefore, by alternation, $PX^2 : PY^2 = QX^2 : QY^2$, and therefore, Euc. VI. 3, the angle XQY is bisected, externally or internally, by QP; or, since, by (162), the circle XCY passes through Q, as the arc XY is bisected, externally or internally, at C, so is the angle XQY by QC.

Conversely, *if two points on the same diameter of a circle be such that the angle subtended at one of them by any chord of the circle, not perpendicular to the diameter, whose direction passes through the other is bisected by the diameter, they are inverse points with respect to the circle.*

For, if P and Q (same figures as before) be the two points, and XY the chord whose direction passes through one of them P; then, since by hypothesis, the angle XQY is bisected, externally or internally, by QP, therefore, Euc. VI. 3, $PX : QX = PY : QY$, and therefore (158) X and Y are two points on the same circle coaxal with P and Q, which, as its centre lies on the line PQ, unless in the particular case when XY is perpendicular to PQ, coincides therefore necessarily with the original circle, and therefore &c.; or, if C be the point in which the circle XQY intersects the line PQ, since by hypothesis the angles XQC and YQC are equal or supplemental, therefore the lines CX and CY are equal, and therefore either XY is perpendicular to PQ, or C is the centre of the original circle, in which case (162) $CP.CQ = CX^2$ or CY^2, and therefore &c.

Cor. 1°. It is evident from the above, that *when any number of circles have a common pair of inverse points* (152), *all pairs of opposite segments, intercepted by pairs of them on any line passing through either, subtend similar angles at the other.* For, if XY and $X'Y'$ be the two chords intercepted by any two of them on any line passing through either point P, the two angles XQY and $X'QY'$, subtended by them at the other Q, have the same bisector PQ, and therefore the two pairs of angles XQX' and YQY', XQY' and YQX' are similar.

Cor. 2°. It is also evident from the converse, that *the two centres of perspective of any two parallel chords of a circle are inverse points with respect to the circle.* For, when two chords XY and $X'Y'$ are parallel, the two pairs of opposite lines XX' and YY', XY' and YX' connecting their extremities, two and two, intersect evidently upon, and make equal angles with, the same diameter, and therefore &c.

164. *If a variable chord of a fixed circle turn round a fixed point, the rectangles under the distances of its extremities from the inverse of the point and from the axis of reflexion of the point and its inverse are both constant.*

For, if C (same figures as before) be the centre of the circle, P the fixed point, Q its inverse with respect to the circle, L the axis of reflexion of P and Q, and XY any position of the variable chord turning round P; then, to prove the first, since, by (158),

$$QX^2 : PX^2 = QY^2 : PY^2 = QC : PC,$$

therefore $$QX.QY : PX.PY = QC : PC,$$

and since (Euc. III. 35, 36)

$$PX.PY = PC.PQ = 2PC.PL,$$

therefore $$QX.QY = QC.QP = 2QC.QL,$$

and therefore &c.; and, to prove the second, since, by (159),

$$PX^2 = 2PC.LX \text{ and } PY^2 = 2PC.LY,$$

therefore $$LX.LY = PX^2.PY^2 \div 4PC^2,$$

and since (Euc. III. 35, 36)

$$PX^2.PY^2 = PC^2.PQ^2 = 4PC^2.PL^2,$$

therefore $LX.LY = LP^2$, and therefore &c.

Conversely, *if a variable chord of a fixed circle, turning round one of two fixed points on the same diameter of a circle, be such that the rectangle under the distances of its extremities either from the other or from the axis of reflexion of the two is constant, the two points are inverse points with respect to the circle.*

These are both evident from the direct properties, by taking the two extreme positions of the variable chord, those, viz. in which it coincides with the diameter containing the points, and in which it either intersects that diameter at right angles or touches the circle according as the point round which it turns is external or internal to the latter.

Cor. It follows at once from the above, that *for a system of circles having a common pair of inverse points* (152), *the several rectangles under the distances of the extremities of all chords passing through either from the other are constant, and from the axis of reflexion of both are constant and equal to the square of the semi-segment intercepted between them.*

CHAPTER X.

THEORY OF POLES AND POLARS WITH RESPECT TO A CIRCLE.

165. THE line passing through the inverse of any point with respect to a circle, and intersecting at right angles the diameter containing the point, is termed the *polar* of the point with respect to the circle; and, conversely, the inverse of the foot of the perpendicular from the centre of a circle upon any line is termed the *pole* of the line with respect to the circle.

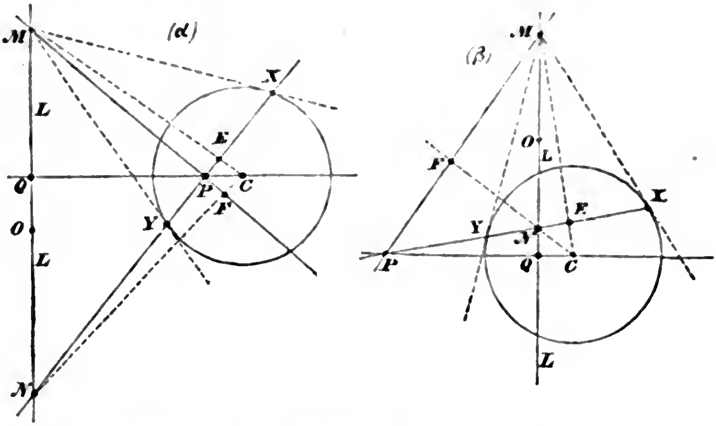
From the mere definition of a point and line, pole and polar to each other with respect to a circle, it is evident that—In the general case when the radius of the circle is finite, 1°. They lie at the same side or at opposite sides of the centre, according as the circle is real or imaginary; 2°. In either case, as one approaches to or recedes from, the other, conversely, recedes from or approaches to, the centre; 3°. The polar of the centre is the line at infinity, and conversely, the pole of the line at infinity is the centre; 4°. The polar of any point on the circle is the tangent at the point, and conversely, the pole of any tangent to the circle is the point of contact; 5°. The polar of any point at infinity is the diameter perpendicular to the direction of the point, and conversely, the pole of any diameter is the point at infinity in the direction perpendicular to the diameter; 6°. The point of intersection and chord of contact of any two tangents to the circle are pole and polar to each other with respect to the circle (162, Cor. 3°.). In the extreme case when the radius of the circle is evanescent, 1°. Every line, however situated, is a polar of the centre; 2°. Every line, not passing through the centre, is a polar of the centre only; 3°. Every line passing through the centre is a polar, not only of the centre, but of every point indifferently on the orthogonal line passing through the centre. And in the extreme case when the radius is infinite, the polar of every point, however situated, is parallel to the line into which the part of the circle not at infinity then opens out, and distant from it at the opposite side by an interval equal to that of the point.

Every two angles being similar whose sides are mutually perpendicular, it is evident also that, whatever be the nature of the circle, the angle subtended at the centre by any two points is similar to that determined by the polars of the points, and conversely, the angle determined by any two lines is similar to that subtended at the centre by the poles of the lines.

In the theory of poles and polars with respect to a circle, the diameter passing through any point is termed *the polar axis of the point*, and the projection of the centre on any line *the polar centre of the line.*

166. Of the various properties of points and lines, pole and polar to each other with respect to a circle, the two following, converse to each other, lead to the greatest number of consequences, and may be regarded as fundamental.

When a line passes through a point, its pole with respect to any circle lies on the polar of the point with respect to the circle; and conversely, *when a point lies on a line, its polar with respect to any circle passes through the pole of the line with respect to the circle.*



To prove which, P and L being the point and line, pole and polar to each other, and C the centre of the circle; if, in the first case, XY be any line through P, CE the perpendicular from C on XY, and M the point in which CE intersects L; then since, by similar right-angled triangles CEP and CQM, the rectangles $CE.CM$ and $CP.CQ$ are equal, and since, by hypothesis, the latter rectangle $CP.CQ =$ the square of the radius of the circle, therefore the former rectangle $CE.CM$ is also = the square of the radius, and therefore the point M is the pole of the line XY with respect to the circle. And if, in the second case, M be any point on L, MC the line connecting it with C, and PE the line through P perpendicular to MC; then, as before, $CE.CM = CP.CQ =$ square of radius of circle, and therefore the line PE is the polar of the point M with respect to the circle.

Cor. 1°. Since, by the above, the pole of every line passing through P lies on L, and, conversely, the pole of every point lying on L passes through P, it follows consequently that—

If any number of lines of any geometrical figure pass through a point, their poles with respect to any circle lie on a line, the polar of the point with respect to the circle; and conversely, *if any number of points of any geometrical figure lie on a line, their polars with respect to any circle pass through a point, the pole of the line with respect to the circle.*

Cor. 2°. If, in the above, one pole and polar P and L be conceived to remain fixed with the circle, and the other M and XY to vary, it appears again that—

If a variable line turn round a fixed point, its pole with respect to any fixed circle describes a fixed line, the polar of the point with respect to the circle; and conversely, *if a variable point describe a fixed line, its polar with respect to any fixed circle turns round a fixed point, the pole of the line with respect to the circle.*

Cor. 3°. Since when, as in the figures, the points X and Y are real, tangents at them intersect at M, and conversely (162, Cor. 3°), it follows of course, as included in the preceding, that—

If a variable chord of a fixed circle pass through a fixed point, the two tangents at its extremities intersect on a fixed line, the polar of the point; and conversely, *if two variable tangents to a fixed circle intersect on a fixed line, their chord of contact passes through a fixed point, the pole of the line.*

COR. 4°. It being evident, from the right angle PEC, that as the point M describes the line L its inverse E with respect to the circle describes the circle on PC as diameter, and conversely. Hence, as shewn otherwise for a particular case in (162, Cor. 2°)—

If a point describe a line, its inverse with respect to any circle describes the circle passing oppositely through the centre of the circle and the pole of the line; and conversely, *if a point describe a circle, its inverse with respect to any circle through whose centre it passes describes the line polar with respect to the latter of the point of the former opposite to its centre.*

The above properties, suitably modified, are of course all true in the particular cases when either of the two points P or Q is at infinity, and the other therefore at the centre of the circle.

167. From the fundamental property of the preceding article, it is evident with respect to any circle, that—

The line of connection of any two points is the polar of the point of intersection of the polars of the points; and, reciprocally, the point of intersection of any two lines is the pole of the line of connection of the poles of the lines.

For, by that property, when a line passes through two points its pole lies on the polars of both, and reciprocally, when a point lies on two lines its polar passes through the poles of both, and therefore &c.

The point of intersection and the chord of contact of any two tangents to a circle being pole and polar to each other with respect to the circle (162, Cor. 3°), it follows, of course, as included in the preceding, that *the point of intersection of the two chords of contact and the line of connection of the two points of intersection of any two pairs of tangents to the same circle, are pole and polar to each other with respect to the circle.*

Of the many consequences from the above, which in the modern geometry of the circle are numerous and remarkable, the six next articles contain a few of the most important.

168. *When a triangle is such that two of its vertices and their opposite sides are pole and polar to each other with respect to a circle, the third vertex and its opposite side are pole and polar to each other with respect to the same circle.*

For, since for three points PMN, (see figures of article 166), when P is the pole of MN, and M the pole of PN, then, by the preceding, N is the pole of PM, and therefore &c.

Every triangle MPN thus related to a circle, that its three vertices and their opposite sides are pole and polar to each other, is said (for a reason that will presently appear) to be *self-reciprocal* with respect to the circle; and it is evident from the definition of pole and polar, in Art. 165, that *in every self-reciprocal triangle with respect to a circle, the three perpendiculars from the vertices upon the opposite sides intersect at the centre, and are there divided so that the rectangle under the segments of each = the square of the radius of the circle.*

Since in every triangle ABC the three perpendiculars AX, BY, CZ from the vertices upon the opposite sides intersect at a common point O for which the three rectangles $OA.OX$, $OB.OY$, $OC.OZ$ are equal in magnitude and sign; therefore, by the above, *every triangle ABC is self-reciprocal with respect to the circle whose centre is the intersection O of the three perpendiculars AX, BY, CZ from its vertices on its opposite sides and the square of whose radius is the common value of the three equal rectangles $OA.OX$, $OB.OY$, $OC.OZ$, and which is therefore real or imaginary according as that common value is positive or negative, that is, according as the triangle is obtuse or acute angled.*

In the particular case of a right-angled triangle of any finite magnitude, the point O being the vertex of the right angle, and the common value of the three rectangles $OA.OX$, $OB.OY$, $OC.OZ$ being $= 0$; hence, from the above, *every right-angled triangle of finite magnitude is self-reciprocal with respect to the circle of evanescent radius whose centre is the vertex of the right angle.*

If, while the vertex of the right angle remains at a finite distance, the opposite side be conceived to recede to infinity; since, then, the common value of the three rectangles $OA.OX$, $OB.OY$, $OC.OZ$ is indeterminate (13), hence, again, from the above, *every right-angled triangle whose hypotenuse is at infinity is self-reciprocal with respect to every circle of finite radius whose centre is the vertex of the right angle.*

For any triangle ABC, whatever be its magnitude and form, if A, B, C be its three angles, and d the diameter of its circumscribing circle; the square of the radius OR of the circle to which it is self-reciprocal is given in all cases by the formula

$$OR^2 = -d^2.\cos A.\cos B.\cos C,$$

which, as the cosine of a right angle is evanescent, includes evidently with all others the two particular cases just noticed.

For, since for its centre O, which, in virtue of the property of the present article, is termed *the polar centre*, as the circle itself is, for the same reason, *the polar circle* of the triangle, the three circles BOC, COA, AOB are all equal to the circle ABC, therefore, by (62, Cor. 7°) and by (62), disregarding signs,

$$OX = OB.OC \div d,\quad OY = OC.OA \div d,\quad OZ = OA.OB \div d,$$

and $\quad OA = d.\cos A,\quad OB = d.\cos B,\quad OC = d.\cos C,$

therefore

$$OR^2 = OX.OA = OY.OB = OZ.OC$$
$$= OA.OB.OC \div d = d^2.\cos A.\cos B.\cos C,$$

and as the two magnitudes thus shewn to be always equal in absolute value are evidently always opposite in sign, therefore &c.

If a, b, c be the three sides of the triangle, and d, as before, the diameter of its circumscribing circle, it is easy to see from the above, or directly, that also

$$OR^2 = \tfrac{1}{2}(a^2 + b^2 + c^2) - d^2,$$

which is the formula for the square of the radius of the polar circle in terms of the three sides of the triangle.

In every triangle the polar circle, real or imaginary, intersects at right angles the three circles, of which the three sides are diameters. For the extremities of each perpendicular of the triangle being inverse points with respect to the polar circle (149), and the circle on each side as diameter passing through the four extremities of the two perpendiculars to the other two sides (Euc. III. 31), therefore &c. (156).

169. *When two triangles, whose vertices and sides correspond in pairs, are such with respect to a circle, that each vertex of one is the pole of the corresponding side of the other, or conversely,*

then, reciprocally, each vertex of the latter is the pole of the corresponding side of the former, or conversely.

For if P, Q, R be the three vertices of either triangle, and L', M', N' the three corresponding sides of the other; then, since, by hypothesis, P is the pole of L', Q of M', R of N', thereore, by (167), QR is the polar of $M'N'$, RP of $N'L'$, PQ of $L'M'$, and therefore &c.

More generally, *when two polygons of any order are such with respect to a circle, that every vertex of one is the pole of a corresponding side of the other, or conversely, then, reciprocally, every vertex of the latter is the pole of a corresponding side of the former, or conversely.*

For, if P, Q, R, S, &c. be the several vertices of either polygon, and L', M', N', O', &c. the several corresponding sides of the other; then, since, by hypothesis, P is the pole of L', Q of M', R of N', S of O', &c., therefore, by (167), PQ is the polar of $L'M'$, QR of $M'N'$, RS of $N'O'$, &c., and therefore &c.

More generally still, *when two figures of any nature are such with respect to a circle, that every point of one is the pole of a corresponding tangent to the other, or conversely, then, reciprocally, every point of the latter is the pole of a corresponding tangent to the former, or conversely.*

For if P and Q be any two points of either figure F, and L' and M' the two corresponding tangents to the other F', then, since, by hypothesis, P is the pole of L' and Q of M', therefore by (167), PQ is the polar of $L'M'$; and this being true in all cases, whatever be the separation of Q from P or of M' from L', is therefore true in the particular case when Q coincides with P, and consequently M' with L'; that is, when (19) PQ is the tangent L at the point P to the figure F, and when (20) $L'M'$ is the point of contact P' of the tangent L' with the figure F', and therefore &c.

170. Every two triangles, polygons, or figures of any kind F or F' then reciprocally related to each other, that the several points of either and the corresponding lines of the other are pole and polar to each other with respect to a circle, are said, each to be *the polar* of the other, and both together to be *reciprocal polars* to each other, with respect to the circle; the reciprocality

between them consisting in the circumstance, above established, that when either is the polar of the other with respect to a circle, then, reciprocally, the latter is the polar of the former with respect to the same circle.

Two polygons of any order, one inscribed and the other circumscribed to a circle at the same system of points on its circumference, furnish an obvious example of a pair of polygons reciprocal polars to each other with respect to the circle; the vertices and sides of the former being respectively the points of contact of the sides and the chords of contact of the angles of the latter. Two concentric circles again furnish another obvious example of a pair of figures, reciprocal polars to each other with respect to the concentric circle the square of whose radius equals the rectangle under their radii; either being indifferently the locus of the poles of all the tangents to the other, or the envelope of the polars of all the points of the other, with respect to that circle.

A figure of any nature F is said to be *self-reciprocal* with respect to a circle, when its several points and lines correspond in pairs pole and polar to each other with respect to the circle; thus, as stated in (168), every triangle ABC is self-reciprocal with respect to the particular circle, real or imaginary, to which its vertices and opposite sides are pole and polar to each other.

If either of two figures of any nature, reciprocal polars with respect to any circle, be turned round the centre of the circle into the opposite position, the two figures will then evidently be reciprocal polars with respect to the concentric circle the square of whose radius is equal in magnitude and opposite in sign to that of the original circle; of the two circles, for the two opposite positions, one therefore is always real and the other always imaginary.

171. Every two figures F and F', reciprocal polars to each other with respect to a circle, possess evidently (165 and 166) the following reciprocal properties:

1°. Every line L of either is perpendicular to that connecting the corresponding point P' of the other with the centre of the circle; and conversely.

2°. The angle determined by any two lines L and M of either is similar to that subtended by the two corresponding points P' and Q' of the other at the centre of the circle; and conversely.

3°. When of three lines L, M, N of either, two make equal angles with the third, then of the lines connecting the three corresponding points P', Q', R' of the other with the centre of the circle, the corresponding two make equal angles with the third; and conversely.

4°. The rectangle under the distances of any point P of either and of the corresponding line L' of the other from the centre of the circle is constant; and conversely.

5°. When two points P and Q of either are equidistant from the centre of the circle, the two corresponding lines L' and M' of the other are equidistant from the centre of the circle; and conversely.

6°. When three points P, Q, R of either are collinear, the three corresponding lines L', M', N' of the other are concurrent; and conversely.

172. Any figure F being given or taken arbitrarily, its polar F' with respect to any circle can always be derived from it, by the simple construction of taking either the polars of its several points or the poles of its several lines with respect to the circle; and the repetition of the same process to the new figure F', thus determined by either construction, always (169) reproduces the original figure F; thus, every figure F, whatever be its nature, has its polar figure F' with respect to every circle, and every two figures F and F', reciprocal polars to each other with respect to any circle, always produce and reproduce each other alternately by continued repetition of either process by which one may be derived from the other.

The process of transformation, just described, by which all the points of a figure of any nature are changed into their polars, and all the lines of the figure into their poles, with respect to an arbitrary circle, is sometimes termed *polarization*, the circle by aid of which it is performed *the polarizing circle*, and the centre and radius of the circle *the centre and radius of polarization*; but from the reciprocality, as above explained, existing between the original and derived figures, the process of transformation is more generally known as *reciprocation*, the circle by aid of which it is performed as *the reciprocating circle*, and the centre and radius of the circle as *the centre and radius of reciprocation.*

In the process of reciprocation, the reciprocating circle, provided only it be of a finite radius and at a finite distance, being otherwise entirely arbitrary as to magnitude and position, should of course, when necessary, be selected so as to accord most conveniently with the circumstances of the case; as, for instance, if it were required to obtain the reciprocal of any property of a single circle as far only as another property of a single circle is concerned, the circle itself, or at least one concentric with it, should be made the reciprocating circle, as one not concentric with it would transform it by reciprocation into a figure of more general form than a circle; or, if it were convenient for any reason to have any point or line of the reciprocal figure at infinity, the centre of the reciprocating circle should be placed on the corresponding line or at the corresponding point of the original figure, as any other position of its centre would leave the point or line in question at a finite distance (165, 3°, 5°); thus, a tetrastigm in its general form reciprocates into a tetragram in its general form, into a trapezium, or into a parallelogram, according as the centre of reciprocation is arbitrary, on any one of its six lines of connection, or at the vertex of any one of its three angles of connection (107); a circle, as above stated, reciprocates into a figure of more general form or into a circle, according as the centre of reciprocation is arbitrary or at its centre; and similarly, for figures of all kinds, the reciprocals of whose properties adapted to reciprocation are often much simplified by a convenient selection of the reciprocating circle.

173. As figures consisting of combinations of points and lines give by reciprocation to every circle figures consisting of combinations of lines and points, *all properties of such figures adapted to reciprocation are accordingly double, and from either of two reciprocal properties established for such a figure the other may always be inferred without further demonstration;* thus, from the Theorem of Pascal (148, *a*), that "in every hexagon inscribed in a circle the three pairs of opposite sides intersect

collinearly," may be, and in fact originally was, derived, by reciprocation to the circle, the Theorem of Brianchon (148, *b*), that "in every hexagon circumscribed to a circle the three pairs of opposite vertices connect concurrently," or conversely, (see 171, 6°)—*Hence one very important use of the reciprocating process as enabling us at once to double our previous knowledge of all properties adapted to reciprocation in the geometry of the point and line.*

Again, as circles give by reciprocation to circles not concentric with themselves figures of more general forms than circles, *all properties of circles obtained by reciprocation are consequently true of the more general figures derived from them by reciprocation, and from either of two reciprocal properties established for a circle, the other may always be inferred without further demonstration for the more general figures into which the circle reciprocates for different positions of the centre of reciprocation;* thus, from either of the two aforesaid reciprocal properties of Pascal and Brianchon established for the circle, the other may be inferred without further demonstration for every variety of figure into which the circle reciprocates—*Hence another and still more important use of the reciprocating process, as enabling us to evolve from the familiar and comparatively simple properties of the circle adapted to reciprocation, all the reciprocal properties for the more general figures into which the circle becomes transformed by reciprocation.*

In a treatise confined like the present to the geometry of the point, line, and circle, any examples of the reciprocating process in its second and higher use cannot of course be given, nor would they be intelligible to the reader without some previous knowledge of the Theory of Conic Sections; in its other use, however, examples of reciprocal properties of elementary figures, grouped in reciprocal pairs, marked by corresponding numbers or letters, but independently established, will be found in considerable numbers all through the advanced chapters of the work; the process of connecting the several pairs by reciprocation as they occur, thus furnishing a continued and very valuable exercise to the reader.

174. The two fundamental properties of Art. 167, from which the important consequences of the several succeeding

Articles have been inferred, may obviously be stated otherwise thus, as follows—

When two points are such that one lies on the polar of the other with respect to a circle, then, reciprocally, the latter lies on the polar of the former with respect to the circle; and, conversely, when two lines are such that one passes through the pole of the other with respect to a circle, then, reciprocally, the latter passes through the pole of the former with respect to the circle.

For, as there proved, see figures of that Article, when M lies on L then P lies on XY, and, conversely, when XY passes through P then L passes through M, and therefore &c.

Every two points thus related to each other, that each lies on the polar of the other with respect to a circle, are termed *conjugate points* with respect to the circle; and every two lines thus related to each other, that each passes through the pole of the other with respect to a circle, are termed *conjugate lines* with respect to the circle; in the figures of Art. 166 the two points M and N are evidently conjugate points, and the two lines PM and PN are evidently conjugate lines with respect to the circles.

From 5°, Art. 165, it is evident that—*Every two points at infinity in directions at right angles to each other are conjugate points with respect to every circle, and every two lines at right angles to each other are conjugate lines with respect to every circle whose centre is the intersection of the lines.*

175. Conjugate points and lines with respect to a circle possess evidently, see figures of Art. 166, the following general properties—

1°. Every point has an infinite number of conjugates, viz. all points lying on its polar; and, every line has an infinite number of conjugates, viz. all lines passing through its pole.

2°. When two points are conjugate so are their polars; and, conversely, when two lines are conjugate so are their poles.

3°. The common conjugate to any two points is the pole of their line of connection; and, conversely, the common conjugate to any two lines is the polar of their points of intersection.

4°. The lines by which two conjugate points connect with the pole of their line of connection are the polars of the points;

and, conversely, the points at which two conjugate lines intersect with the polar of their point of intersection are the poles of the lines.

5°. Every two conjugate points connect with the pole of their line of connection by a pair of conjugate lines; and, conversely, every two conjugate lines intersect with the polar of their point of intersection at a pair of conjugate points.

6°. Every two conjugate points determine with the pole of their line of connection a self-reciprocal triangle (168); and, conversely, every two conjugate lines determine with the polar of their point of intersection a self-reciprocal triangle (168). Hence, every self-reciprocal triangle with respect to a circle is said also to be *self-conjugate* with respect to the circle.

176. For every pair of conjugate points with respect to a circle the following metric relations exist, each of which reciprocally determines a pair of conjugate points with respect to the circle.

1°. *The square of the distance between them is equal to the sum of the squares of the tangents from them to the circle.*

2°. *The semi-distance between them is equal to the length of the tangent from its middle point to the circle.*

3°. *The rectangle under their distances from the polar centre of their line of connection is equal in magnitude and opposite in sign to the square of the tangent from that point to the circle.*

For, if M and N (figures, Art. 166) be any two points, O and Q the middle point and polar centre of their line of connection, C the centre of the circle, and P the intersection of the three perpendiculars MF, NE, and CQ of the triangle MCN, then—

To prove 1° and its converse. Since, by Euc. II. 12, 13, $MN^2 = CM^2 + CN^2 - 2\,CM.CE$ or $-2\,CN.CF$; when M and N are conjugate points, and when therefore $CM.CE$, or its equivalent $CN.CF$, = rad² of circle, then $MN^2 = CM^2 + CN^2 - 2$ rad² of circle $= (CM^2 - \text{rad}^2) + (CN^2 - \text{rad}^2) = \tan^2$ from $M + \tan^2$ from N; and, conversely, when the latter relation holds, then $CM.CE$, or its equivalent $CN.CF$, = rad² of circle, and therefore M and N are conjugate points with respect to the circle.

To prove 2° and its converse. Since, by 98, or Euc. II. 12, 13, Cor.,
$$CM^2 + CN^2 = OM^2 + ON^2 + 2\,OC^2,$$

and consequently

$$CM^2 + CN^2 - 2 \text{ rad}^2 \text{ of circle} = OM^2 + ON^2 + 2 \tan^2$$

from O to circle; when M and N are conjugate points, and when therefore, by 1°,

$$CM^2 + CN^2 - 2 \text{ rad}^2 \text{ of circle} = MN^2 = 2\,(OM^2 + ON^2),$$

then $OM^2 + ON^2 = 2 \tan^2$ from O to circle, and therefore $OM^2 = ON^2 = \tan^2$ from O to circle; and, conversely, when the latter relation exists, then $CM^2 + CN^2 - 2 \text{ rad}^2$ of circle $= MN^2$, and therefore, by 1°, M and N are conjugate points with respect to the circle.

To prove 3° and its converse. Since, by either pair of similar right-angled triangles MQP and CQN, or NQP and CQM, the two ratios $QM : QP$ and $QC : QN$, and therefore the two rectangles $QM.QN$ and $QP.QC$, are equal in magnitude and opposite in sign; when M and N are conjugate points, and when therefore (174) P is the pole of MN, then the latter rectangle (165) is equal in magnitude and sign to the square of the tangent from Q to the circle; and, conversely, when the latter rectangle is equal in magnitude and sign to the square of that tangent, then (165) P is the pole of MN, and therefore (174) M and N are conjugate points with respect to the circle.

In the particular case when the radius of the circle is evanescent, the above properties all follow immediately from the obvious consideration (168) that every two conjugate points with respect to an evanescent circle subtend a right angle at the centre of the circle, and that, conversely, every two points which subtend a right angle at the centre of an evanescent circle are conjugate points with respect to the circle.

177. *Every circle having for diameter the interval between two conjugate points with respect to another circle is orthogonal to the other.*

For, the circle on MN as diameter (figures, Art. 166) passes evidently through the two points E and F, which are the inverses of M and N when the latter are conjugates with respect to the circle C, and therefore &c. (156).

Conversely, *When two circles intersect at right angles, the extremities of every diameter of either are conjugate points with respect to the other.*

For, MN (same figures) being any diameter of either, C the centre of the other, and E and F the two points in which the former intersects the two diameters CM and CN of the latter; since then (156) E and F are the two inverses of M and N with respect to the latter, therefore (165) EN and FM are the two polars of M and N with respect to the same, and therefore &c. (174).

COR. 1°. The above property is evidently identical with 2° of the preceding Article, and from either it obviously follows immediately that—

1°. *The line connecting any two conjugate points with respect to a circle may be turned round its middle point through any angle without its extremities ceasing to be conjugate points with respect to the circle.*

2°. *When the distance between two conjugate points with respect to a circle of given radius is given, the distance of their middle point from the centre of the circle is also given, and conversely.*

3°. *If the same circle be orthogonal to a number of others, the extremities of every diameter of it are conjugate points with respect to all the others.*

4°. *The locus of points having a common conjugate with respect to three circles is the circle intersecting the three at right angles.*

COR. 2°. Since, when two points are conjugates with respect to a number of circles, the polars of either with respect to them all pass through the other (174); hence, from 3° and 4°, Cor. 1°—

1°. *If the same circle be orthogonal to a number of others, the polars of every point on it with respect to them all pass through the diametrically opposite point.*

2°. *The locus of points whose polars with respect to three circles are concurrent is the circle intersecting the three at right angles.*

COR. 3°. By aid of 156, Cor. 3°, the above supply obvious solutions of the four following problems—

1°. *On a given line or circle to determine two points separated by a given interval which shall be conjugates with respect to a given circle.*

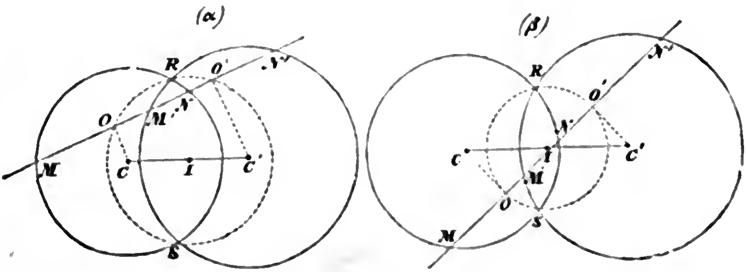
2°. *On a given line or circle to determine two points which shall be at once conjugates with respect to two given circles.*

178. *When a line intersecting two circles meets either in a pair of conjugate points with respect to the other.*

1°. *Then reciprocally it meets the latter in a pair of conjugate points with respect to the former.*

2°. *Its two segments intercepted by them are bisected by the circle passing through their points of intersection whose centre bisects the distance between their centres.*

3°. *The rectangle under its distances from their centres is equal in magnitude and sign to half the sum of the squares of their radii − half the square of the distance between their centres.*



For, if C and C' be the centres of the two circles, R and S their two points of intersection, MN and $M'N'$ the two segments they intercept on the line, O and O' the two middle points of the segments, and I the middle point of CC'; then—

To prove 1°. The relation $OM'.ON' = (\frac{1}{2}MN)^2$, or the equivalent relation $O'M.O'N = (\frac{1}{2}M'N')^2$, (Euc. II. 5, 6), being at once the condition (176, 2° and 3°) that M and N should be conjugate points with respect to the circle C', and that M' and N' should be conjugate points with respect to the circle C, therefore &c.

To prove 2°. Since

$$OM'.ON' = (\tfrac{1}{2}MN)^2 \text{ and } O'M.O'N = (\tfrac{1}{2}M'N')^2,$$

therefore $$C'O^2 - C'R^2 = CR^2 - CO^2,$$

and $$CO'^2 - CR^2 = C'R^2 - C'O'^2;$$

therefore $$CO^2 + C'O^2 = CO'^2 + C'O'^2 = CR^2 + C'R^2,$$

from which it follows, by (98, Cor. 2°.), that O, O', and R lie on the same circle having I for centre, and therefore &c.

To prove 3°. Since OC and $O'C'$ are perpendiculars at the extremities of the chord OO' of the circle ORO', meeting the diameter CC' at the points C and C' equidistant from the centre I; therefore (49)

$$CO.C'O' = IR^2 - (\tfrac{1}{2}CC')^2 = \tfrac{1}{2}(CR^2 + C'R^2 - CC'^2), \text{(83, Cor. 2°.)};$$

and therefore &c.

Cor. 1°. In the particular case when the two circles intersect at right angles, since then (23) $CR^2 + C'R^2 = CC'^2$, therefore, from the above (3°), $CO.C'O' = 0$; and therefore, as proved otherwise in the preceding Article—

When two circles intersect at right angles every line intersecting either in a pair of conjugate points with respect to the other passes through one of their centres.

Cor. 2°. The above (2° and 3°) supply obvious solutions of the two following problems—

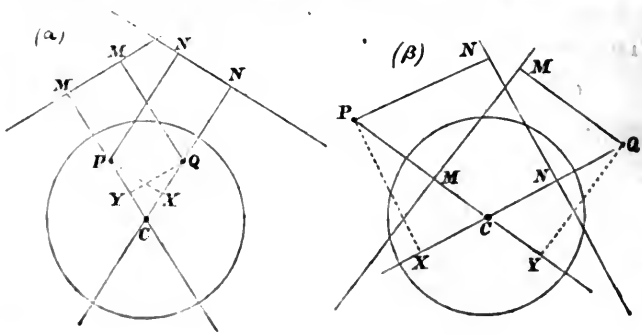
1°. *Through a given point to draw a line intersecting one of two given circles in a pair of conjugate points with respect to the other.*

2°. *To draw a line intersecting two of three given circles in pairs of conjugate points with respect to the third.*

179. In connection with the subject of poles and polars with respect to the circle, the following useful theorem is due to Dr. Salmon.

The distances of any two points from the centre of a circle have the same ratio as their distances each from the polar of the other with respect to the circle.

If P and Q be the two points, M and N their two polars, and C the centre of the circle, then $PC : QC = PN : QM$; for, letting



fall from P and Q the perpendiculars PX and QY upon the diameters CQ and CP, then since (165)

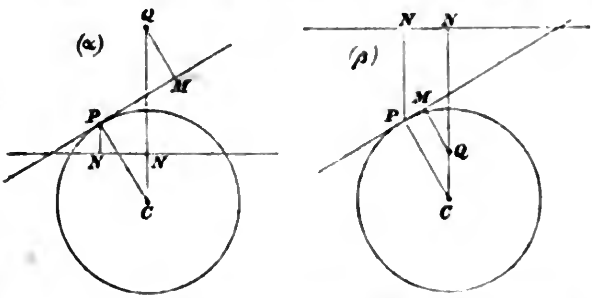
$$CP.CM = CQ.CN = \text{rad}^2 \text{ of circle},$$

therefore $CP : CQ = CN : CM = CX + PN : CY + QM$,

but, by similar right-angled triangles, $CP : CQ = CX : CY$, therefore $CP : CQ = PN : QM$, and therefore &c.

In the particular case when $CP = CQ$, it is evident without proof that then $PN = QM$, or, in general, that—*Every two points equidistant from the centre of a circle are equidistant each from the polar of the other;* and, in particular, that—*Every two points on the circumference of a circle are equidistant each from the tangent at the other.*

Cor. 1°. If one of the two points Q, in the above, with its polar N, be supposed fixed and arbitrary, and the other P, with its polar M, variable and confined to the circumference of the circle; since then the ratio $CP : CQ$ is constant, therefore,



by the above, its equivalent $PN : QM$ is also constant, and therefore, the polar of any point on a circle being the tangent at the point,

The distance of a variable point on a fixed circle from any fixed line is to the distance of the tangent at the point from the pole of the line in the constant ratio of the radius of the circle to the distance of the pole from its centre.

Cor. 2°. The following among many consequences follow immediately from Cor. 1°—

1°. The product of any number of constant ratios being of course constant, therefore—

The rectangle under the distances of a variable point on a *fixed circle from any two fixed lines is to the rectangle under the distances of the tangent at the point from the poles of the lines in the constant ratio of the square of the radius of the circle to the rectangle under the distances of the poles from its centre.*

2°. Every two polygons reciprocal polars to each other with respect to a circle (170) being such that the vertices of either and the corresponding sides of the other are pole and polar to each other with respect to the circle, therefore—

For every two polygons, reciprocal polars to each other with respect to a circle, the product of the distances of any point on the circle from the n sides of either is to the product of the distances of the tangent at the point from the n vertices of the other in the constant ratio of the n^{th} power of the radius of the circle to the product of the distances of the n vertices from its centre.

3°. For every two polygons, one inscribed and the other circumscribed to a circle at the same system of points on its circumference (polygons which evidently come under the preceding head) the products of the distances of the two sets of sides from any point on the circle being equal (48, Ex. 9°.), therefore—

For every two polygons, one inscribed and the other circumscribed to a circle at the same system of points on its circumference, the products of the distances of the two sets of vertices from any tangent to the circle have the constant ratio of the products of their distances from the centre of the circle.

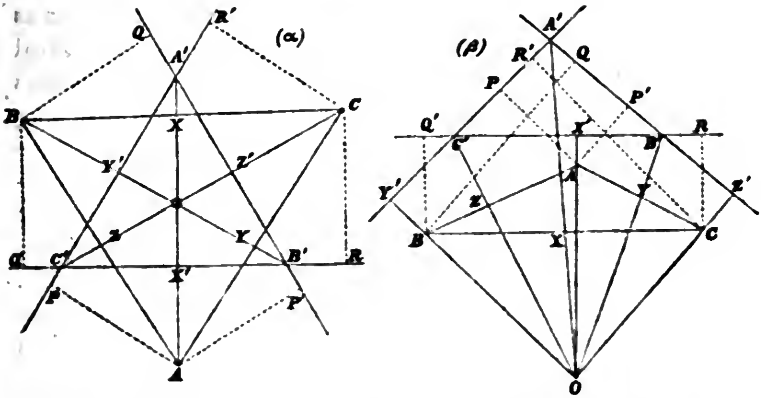
4°. In every tetrastigm inscribed in a circle, the rectangles under the distances of the three pairs of opposite connectors from any point on the circle being equal (62, Cor. 10°.), therefore—

In every tetragram circumscribed to a circle, the rectangles under the distances of the three pairs of opposite intersections from any tangent to the circle have the constant ratios of the rectangles under their distances from the centre of the circle.

180. With the three following properties of two triangles, reciprocal polars to each other with respect to a circle, we shall close the present chapter.

1°. *Every two triangles reciprocal polars to each other with respect to a circle are in perspective* (140).

For, if ABC and $A'B'C'$ be the two triangles, and O the



centre of the circle, from the three vertices A, B, C of either triangle, letting fall the three pairs of perpendiculars AP and AP', BQ and BQ', CR and CR' upon the three pairs of sides about the corresponding vertices of the other $A'B'C'$; then, since by Dr. Salmon's Theorem (179),

$$\frac{BQ}{CR'}=\frac{OB}{OC}, \quad \frac{CR}{AP'}=\frac{OC}{OA}, \quad \frac{AP}{BQ'}=\frac{OA}{OB},$$

therefore, at once, by composition

$$\frac{AP}{AP'}\cdot\frac{BQ}{BQ'}\cdot\frac{CR}{CR'}=1,$$

and therefore &c. (140).

In the particular case of two triangles, one inscribed and the other circumscribed to a circle at the same three points on its circumference, this general property obviously gives at once the two reciprocal properties established on other principles in Examples 3° and 4°, Art. 137. See also (148), where the same properties have been already inferred as particular cases from the general theorems of Pascal and Brianchon, respecting any hexagons inscribed and circumscribed to a circle.

2°. *Any two triangles, however circumstanced as to magnitude and form, may be placed relatively to each other, so as for any assigned correspondence of vertices and sides to be reciprocal polars with respect to a circle; and that in one or in three pairs of opposite positions* (170), *according as the two sets of corresponding vertices are disposed in similar or opposite directions of rotation round the two triangles.*

For, that the two triangles ABC and $A'B'C'$ (same figures as before) should be reciprocal polars with respect to a circle, real or imaginary, it is sufficient that the three perpendiculars AX', BY', CZ' from the vertices of ABC upon the corresponding sides of $A'B'C'$ pass through a common point O, and that the three $A'X$, $B'Y$, $C'Z$ from the vertices of $A'B'C'$ upon the corresponding sides of ABC pass through the same point O (133, Ex. 7°.); those conditions securing (by pairs of similar triangles, see figures), that the six rectangles $OA.OX'$, $OB.OY'$, $OC.OZ'$, $OA'.OX$, $OB'.OY$, $OC'.OZ$ shall be equal in magnitude and sign; taking therefore, according as the corresponding vertices of the two triangles are disposed in similar or opposite directions of rotation, as in figs. α and β respectively, for the triangle ABC, the internal or one of the three external points O for which the three angles BOC, COA, AOB are similar to the three $B'A'C'$, $C'B'A'$, $A'C'B'$, and for the triangle $A'B'C'$, the internal or corresponding external point O' for which the three angles $B'O'C'$, $C'O'A'$, $A'O'B'$ are similar to the three BAC, CBA, ACB (63, Cor. 4°.); and then placing the two triangles, so that the two points O and O' shall coincide, and that the six connectors OA, OB, OC, OA', OB', OC' shall be similar or opposite in direction with the six perpendiculars OX', OY', OZ', OX, OY, OZ, the required position is obtained; the circle, to which the triangles are polars, being real in the former case and imaginary in the latter (170).

In the particular case when the two triangles are similar, and when the correspondence is between their homologous vertices and sides, the two points O and O', evidently homologous points with respect to the triangles (39), are, for similar directions of rotation, fig. α, the two points of concurrence of their two sets of perpendiculars (63, Cor. 5°.), and for opposite directions of rotation, fig. β, any two homologous points on their circumscribing circles (63, Cor. 5°.); hence, as is also evident directly—*Every triangle reciprocates into a similar triangle to every circle whose centre is either the unique point of concurrence of its three perpendiculars or any point indifferently on its circumscribing circle; the two similar triangles being both right or left in the former case, and one right and one left in the latter* (32); in the former case also their homologous sides being evidently parallel, they are consequently similarly or oppositely placed (33), thus verifying for their particular case the general property 1°, see (142).

3°. *If ABC be any triangle, A'B'C' its polar triangle with respect to any circle, O the centre and OR the radius of the circle, then*

$$(A'B'C') = \frac{OR^4}{4} \cdot \frac{(ABC)^2}{(BOC).(COA).(AOB)},$$

and similarly for (ABC) in terms of (A'B'C'); the quantities within the parentheses signifying the areas of the several triangles they respectively represent.

For, since, by pairs of similar angles (64), (same figures as before)

$$\frac{(B'OC')}{(ABC)} = \frac{OB'.OC'}{AB.AC}, \quad \frac{(C'OA')}{(ABC)} = \frac{OC'.OA'}{BC.BA}, \quad \frac{(A'OB')}{(ABC)} = \frac{OA'.OB'}{CA.CB},$$

and, since, by (165),

$$OA'.OX = OB'.OY = OC'.OZ = OR^2,$$

therefore

$$(B'OC') = \frac{OR^4.(ABC)}{(AB.OZ).(AC.OY)} = \frac{OR^4}{4} \cdot \frac{(ABC)}{(AOB).(AOC)},$$

$$(C'OA') = \frac{OR^4.(ABC)}{(BC.OX).(BA.OZ)} = \frac{OR^4}{4} \cdot \frac{(ABC)}{(BOC).(BOA)},$$

$$(A'OB') = \frac{OR^4.(ABC)}{(CA.OY).(CB.OX)} = \frac{OR^4}{4} \cdot \frac{(ABC)}{(COA).(COB)};$$

and therefore, by addition, remembering whatever be the position of O (118), that

$$(B'OC') + (C'OA') + (A'OB') = (A'B'C'),$$

and that $$(BOC) + (COA) + (AOB) = (ABC),$$

the above relation is the evident result.

It is evident from the above, that for a given triangle ABC, and for a circle of given radius, but variable centre O, the area of the polar triangle $A'B'C'$ varies inversely as the product of the three areas (BOC), (COA), (AOB), and is therefore a minimum when that product is a maximum, that is (57, Ex. 3°.), when its three factors, their sum being constant (118), are equal, or when (91, Cor.) O is the mean centre of the three points A, B, C for multiples all $= 1$.

It may also be readily shewn from the same that—*In every triangle the polars of the middle points of the sides with respect to the inscribed circle determine a triangle equal in area to the original;* for a, b, c being the three sides, α, β, γ the three perpendiculars, s the semi-perimeter, and r the radius of the inscribed circle of any triangle; if A, B, C be the middle points of its sides, and O the centre of its inscribed circle, the three areas (BOC), (COA), (AOB), in the above, are easily seen, on drawing a figure, to be equal to the three products

$$\left(\frac{\alpha}{2}-r\right)\frac{a}{4}, \quad \left(\frac{\beta}{2}-r\right)\frac{b}{4}, \quad \left(\frac{\gamma}{2}-r\right)\frac{c}{4},$$

from which, since

$$a\alpha = b\beta = c\gamma = 2sr = 2 \text{ area of } abc = 8\,(ABC),$$

it follows, without difficulty, from the above, that

$$(A'B'C') = \frac{s^3r^3}{s(s-a)(s-b)(s-c)} = \frac{\text{area}^3 \text{ of } abc}{\text{area}^2 \text{ of } abc} = \text{area of } abc,$$

and therefore &c.

CHAPTER XI.

ON THE RADICAL AXES OF CIRCLES CONSIDERED IN PAIRS.

181. THE line intersecting at right angles the common diameter of two circles, and dividing the internal AB between their centres A and B at the point I for which the difference of the squares of the segments $AI^2 - BI^2$ is equal in magnitude and sign to the difference of the squares of the conterminous radii $AR^2 - BS^2$, is termed *the radical axis* of the circles.

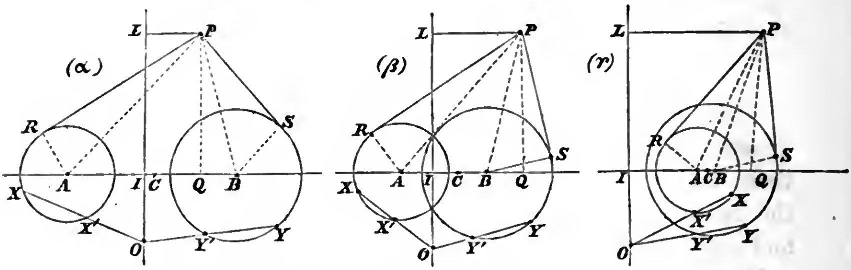
From the mere definition of the radical axis of two circles, it is evident that: 1°. when the circles intersect, it passes through the two points of intersection (Euc. I. 47); 2°. when they touch, it touches both at the point of contact (Euc. III. 16); 3°. when they are equal and not concentric, it coincides with their axis of reflexion (50); 4°. when they are concentric and not equal, it coincides with the line at infinity (136, 1°); 5°. when they are at once equal and concentric, it is indeterminate (13); 6°. when one is a line and the other not, it coincides with the line (150); 7°. when one is a point and the other not, it coincides with the axis of reflexion of the point and its inverse with respect to the other (157); and 8°. when they are both points or lines, the case comes under the head of 3°. or of 5°. Of these particulars, some, less evident than the others, will appear more fully from the general properties of the radical axis of any two circles, which will form the main subject of the present chapter.

When two circles, whatever be their nature, are given in magnitude and position, their radical axis, when not indeterminate, is of course implicitly given with them; the relation $AI^2 - BI^2 = AR^2 - BS^2$ fixing, evidently, the position, when determinate, of the point I at which it intersects at right angles their line of centres AB.

182. Of all the properties of the radical axis of two circles, the following leads to the greatest number of consequences, and may be regarded as fundamental.

The difference of the squares of the tangents from any point to two circles = twice the rectangle under the distance between their centres and the distance of the point from their radical axis.

For, if A and B be the centres of the two circles, AR and



BS their radii, IL their radical axis, P the point, PR and PS the tangents from it to the circles, PL and PQ the perpendiculars from it on IL and AB, and C the middle point of AB; then, since, (Euc. I. 47),

$$PR^2 = AP^2 - AR^2 \text{ and } PS^2 = BP^2 - BS^2,$$

therefore

$$(PR^2 - PS^2) = (AP^2 - BP^2) - (AR^2 - BS^2), \text{ but, (Euc. I. 47),}$$

$$AP^2 - BP^2 = AQ^2 - BQ^2 = 2AB.CQ, \text{ (Euc. II. 5, 6),}$$

and, by the definition of the radical axis (181),

$$AR^2 - BS^2 = AI^2 - BI^2 = 2AB.CI, \text{ (Euc. II. 5, 6),}$$

therefore

$$(PR^2 - PS^2) = 2AB.(CQ - CI) = 2AB.IQ = 2AB.LP,$$

and therefore &c.

Cor. 1°. If $PL = 0$, then $PR^2 - PS^2 = 0$; and, conversely, if $PR^2 - PS^2 = 0$, then $PL = 0$. Hence—*Tangents to two circles from any point on their radical axis are equal; and, conversely, when tangents to two circles from a point not at infinity are equal, the point is on the radical axis of the circles.*

It is this property, of which that of (157) is evidently a particular case, which has given the name "Radical axis" to the line in question, the tangents to two circles from any point being expressed by *radicals*, and the locus of points for which they are equal being a *line*.

The two tangents to the same circle from any point being equal, it follows of course from the second part of the above, that—

The tangents to two circles at their points of contact with any circle touching both intersect on their radical axis.

COR. 2°. If $PS = 0$, then $PR^2 = 2AB.LP$, and conversely, if $PR^2 = 2AB.LP$, then $PS = 0$. Hence—

The square of the tangent to either of two circles from any point on the other varies as the distance of the point from their radical axis; and, conversely, when the square of the tangent from a point to a circle varies as the distance of the point from a line, the point lies on another circle, of which and the original the line is the radical axis.

Of this property, that of (159) is evidently a particular case.

COR. 3°. If O be the intersection of any two chords XX' and YY' of the circles. Since, when their four extremities are concyclic, then $OX.OX' = OY.OY'$, and conversely, (Euc. III. 35, 36); and since, by Cor. 1°., the same is the condition that the point O should be on the radical axis of the circles, and conversely. Hence—

Every two chords of two circles whose four extremities are concyclic intersect on their radical axis; and, conversely, when two chords of two circles intersect on their radical axis, their four extremities are concyclic.

This property will be stated more generally in the next article.

COR. 4°. The point O, as before, being on the radical axis, if $OX = OY$, that is, if X and Y be two of the four intersections with the two circles of any third circle having its centre on their radical axis; then, since, by Cor. 3°., $XX' = YY'$, and since, by (62), $XX' = 2AX.\cos AXO$, and $YY' = 2BY.\cos BYO$, therefore $\cos AXO : \cos BYO = BY : AX$, and both angles having evidently the same affection (11). Hence—

Every circle having its centre on the radical axis of two others intersects them at angles, of the same affection, whose cosines are inversely as their radii; and, conversely, every circle intersecting *two others at angles, of the same affection, whose cosines are inversely as their radii, has its centre on their radical axis.*

The general property, of which this is a particular case, will be given further on.

Cor. 5°. In the same case, since when $XX' = 0$, then $YY' = 0$, and conversely, therefore as a particular case of Cor. 4°., or as is also evident directly from Cor. 1°.—

Every circle having its centre on the radical axis of two others, and intersecting either at right angles, intersect the other at right angles; and, conversely, every circle intersecting two others at right angles has its centre on their radical axis.

This last is the more general proof of the latter property to which allusion was made in Art. 156, Cor. 3°. of chap. IX.

Cor. 6°. Whatever be the position of the point O, whether on the radical axis or not, since, by the fundamental property above,

$$OX.OX' - OY.OY' = 2AB.LO,$$

where LO is the distance of O from the radical axis, if $OX = OY$, that is, if X and Y be two of the four intersections with the two circles of any third circle having its centre at O, then

$$OX.(XX' - YY') = 2AB.OL,$$

and therefore

$$OX : OL = 2AB : XX' - YY' = AB : AX.\cos AXO - BY.\cos BYO,$$

a ratio which is constant when the two angles of intersection AXO and BYO, whatever be their affections, are constant. Hence—

If a variable circle intersect two fixed circles at two constant angles, its radius is to the distance of its centre from their radical axis in a constant ratio; and, conversely, if a variable circle, whose radius is to the distance of its centre from the radical axis of two fixed circles in a constant ratio, intersect either circle at a constant angle, it intersects the other also at a constant angle.

Cor. 7°. As either angle of intersection may $= 0$, or $=$ two right angles. Hence, by (23), as a particular case of the preceding—

If a variable circle touch two fixed circles, the nature of its contact with each being invariable, its radius is to the distance of its centre from their radical axis in a constant ratio; and, con-

versely, if a variable circle, whose radius is to the distance of its centre from the radical axis of two fixed circles in a constant ratio, touch in every position either circle with contact of the same species, it intersects the other at a constant angle, which may = 0 *or two right angles.*

COR. 8°. The ratio $OL : OX$ being (22) the cosine of the angle, real or imaginary, at which the variable circle in Cors. 6°. and 7°. intersects the radical axis of the two fixed circles. Hence, in general, from Cor. 6°—

A variable circle intersecting two fixed circles at constant angles intersects their radical axis at a constant angle; and, conversely, a variable circle intersecting either of two fixed circles and their radical axis at constant angles intersects the other at a constant angle.

And, in particular, from Cor. 7°—

A variable circle touching two fixed circles, the nature of the contact with each being invariable, intersects their radical axis at a constant angle; and, conversely, a variable circle intersecting the radical axis of two fixed circles at a constant angle, and touching either circle with contact of invariable species, intersects the other at a constant angle, which may = 0 *or two right angles.*

The general property established in this corollary is but a particular case of another still more general, which will be given in a subsequent article of the present chapter.

COR. 9°. It is immediately evident from Cor. 1°. that—

The radical axis of two circles bisects the four segments of their four common tangents, real or imaginary, intercepted between their points of contact with the circles; and, conversely, the line joining the middle points of the intercepted segments of any two of the four common tangents to two circles, or, more generally, any two points the tangents from which to two circles are equal, is the radical axis of the circles.

And, from the first part of this latter property, that—

The two chords of contact of two circles with each pair of their common tangents, external and internal, are equidistant in opposite directions from their radical axis; and so, therefore, are the two chords for both pairs in the two circles from each other.

COR. 10°. Since when two circles intersect at right angles, their chord of intersection is the polar of the centre of each with respect to the other (165, 6°.). Hence from Cors. 5°. and 3°. see (166)—

The chords of intersection with two circles of every circle orthogonal to both pass through the poles of their radical axis.

The polars with respect to two circles of any point on their radical axis intersect on their radical axis.

This latter property is evidently true also of the line at infinity, a line which we shall see, in the sequel, possesses with respect to two circles nearly all the properties of their radical axis.

183. The following general property of any three circles includes evidently the first part of that established in Cor. 3°. of the preceding article as a particular case, viz.—

The three radical axes of any three circles, taken two and two, intersect at a common point, termed the radical centre of the circles.

For, if A, B, C be the three centres of the circles, AR, BS, CT their three radii, L, M, N the three radical axes of their three groups of two, and X, Y, Z, the three points in which L, M, N intersect at right angles the three sides BC, CA, AB of the triangle ABC; then since, by definition (181),

$$(BX^2 - CX^2) = (BS^2 - CT^2),$$
$$(CY^2 - AY^2) = (CT^2 - AR^2),$$
$$(AZ^2 - BZ^2) = (AR^2 - BS^2),$$

therefore

$$(BX^2 - CX^2) + (CY^2 - AY^2) + (AZ^2 - BZ^2) = 0,$$

and therefore (132) the three perpendiculars L, M, N intersect at a common point O.

Cor. 1°. It is evident from Cors. 1°. 4°. and 5°. of the preceding, that—

1°. *The six tangents, real or imaginary, to three circles from their radical centre are equal; and, conversely, when the six tangents, real or imaginary, to three circles from a point, not at infinity, are equal, the point is their radical centre.*

2°. *Every circle having its centre at the radical centre of three others intersects them at angles, of the same affection, whose cosines are inversely as their radii; and, conversely, every circle intersecting three others at angles, of the same affection, whose cosines* *are inversely as their radii, has its centre at their radical centre.*

3°. *The circle having its centre at the radical centre of three others, and intersecting one of them at right angles, intersects the other two at right angles; and, conversely, the circle intersecting three others at right angles has its centre at their radical centre.*

The obvious solution of the problem "to describe the circle which intersects three given circles at right angles," furnished by this latter property, is that to which allusion was made in Art. 156, Cor. 4°, of Chap. IX.

Since for every three chords XX', YY', ZZ' of any three circles A, B, C, which concur to their radical centre O, the three products $OX.OX'$, $OY.OY'$, $OZ.OZ'$ are always equal in magnitude and sign; their common value is sometimes termed *the radical product* of the three circles, and is, of course, always equal in magnitude and sign to the square of the radius of their orthogonal circle, which circle, consequently, is real or imaginary according as the sign of the radical product is positive or negative.

COR. 2°. *The point of concurrence O of the three perpendiculars AP, BQ, CR of any triangle ABC is the radical centre of the three circles of which the three sides BC, CA, AB are diameters.*

For, as the three circles on BC, CA, AB as diameters pass respectively through the three pairs of points Q and R, R and P, P and Q, (Euc. III. 31), therefore AP, BQ, CR are the three common chords of those circles, taken two and two, and therefore &c. (181, 1°.).

COR. 3°. *More generally, the point of concurrence O of the three perpendiculars AP, BQ, CR of any triangle ABC is the radical centre of the three circles, of which any three lines AX, BY, CZ drawn from the vertices to the opposite sides BC, CA, AB are diameters.*

For, whatever be the positions of the three diameters AX, BY, CZ, the three perpendiculars AP, BQ, CR are three chords of the three circles concurring to a point O for which the three products $OA.OP$, $OB.OQ$, $OC.OR$ are equal in magnitude and sign (168), and therefore &c. (Cor. 1°, 1°.)

Cor. 4°. *For any system of three combined with any system of two circles, both systems being arbitrary.*

a. The six radical axes of the six combinations of one of the three with one of the two determine two triangles in perspective (140).

b. The radical centre of the three and the radical axis of the two are the centre and axis of the perspective (141).

For, if A, B, C be the system of three, E and F the system of two, U, V, W and X, Y, Z the two sets of three radical axes of A, B, C combined each with E and F respectively, L, M, N the three radical axes of B and C, C and A, A and B respectively, which by the above intersect at the radical centre O of A, B, C, and I the radical axis of E and F; then, by the above, the three points UX, VY, WZ lie on I, and the three pairs of points VW and YZ, WU and ZX, UV and XY lie on L, M, N respectively, and therefore &c.

The radical axis of two circles which intersect being their chord of intersection (181, 1°.), the properties just proved are consequently true, in particular, of the two triangles determined by the six chords of intersection of any two with any three circles with which they intersect, both systems in all other respects being arbitrary.

Cor. 5°. *If A, B, C be the three centres, and AR, BS, CT the three radii, of any three circles, L, M, N the three radical axes of their three groups of two, O their radical centre, P and PQ the centre and radius of any fourth circle which intersects them, and α, β, γ the three angles of intersection; then—*

$$PL : PQ = BS.\cos\beta - CT.\cos\gamma : BC,$$

$$PM : PQ = CT.\cos\gamma - AR.\cos\alpha : CA,$$

$$PN : PQ = AR.\cos\alpha - BS.\cos\beta : AB.$$

For, if X, Y, Z be three of their six points of intersection with the fourth circle, and X', Y', Z' their three second points of intersection with its three radii PX, PY, PZ; then, since, by the fundamental property of the preceding article (182),

$$PY.PY' - PZ.PZ' = 2.BC.PL,$$

$$PZ.PZ' - PX.PX' = 2.CA.PM,$$

$$PX.PX' - PY.PY' = 2.AB.PN,$$

therefore, as in Cor. 6°. of the same,

$$PL : PQ = YY' - ZZ' : 2BC,$$
$$PM : PQ = ZZ' - XX' : 2CA,$$
$$PN : PQ = XX' - YY' : 2AB,$$

and since (62),

$$XX' = 2AR.\cos\alpha,$$
$$YY' = 2BS.\cos\beta,$$
$$ZZ' = 2CT.\cos\gamma,$$

therefore &c.

Hence, for the three circles whose centres are A, B, C and radii AR, BS, CT, the centre P of the circle which intersects them at the three angles α, β, γ lies on the line passing through their radical centre O which makes with the three radical axes L, M, N angles whose sines are proportional (61) to the three quantities

$$\frac{BS.\cos\beta - CT.\cos\gamma}{BC}, \quad \frac{CT.\cos\gamma - AR.\cos\alpha}{CA}, \quad \frac{AR.\cos\alpha - BS.\cos\beta}{AB}$$

and which therefore is given when the three circles and the three angles of intersection are given.

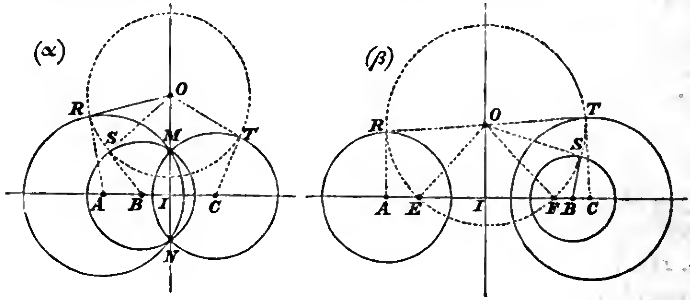
COR. 6°. It appears, from the preceding, (Cor. 5°.), that the solution of the general problem "*to describe the circle which intersects three given circles at three given angles*," is reduced to that of the problem "*to describe the circle having its centre on a given line and intersecting two given circles at given angles*," which in the particular case of *contacts* of assigned species with the two (23), (to which, as we shall see in the sequel, every other case may be reduced), can always be solved by (54); the sum or difference, according to the nature of the contacts, of the distances of its centre from those of the two circles being evidently given. Of the celebrated problem "*to describe the circle having contacts of given species with three given circles*," which is of course a particular case of the above, a more general and instructive solution will be given in the next chapter.

184. Any number of circles whose centers A, B, C, &c. are collinear, and whose radii AR, BS, CT, &c. are such that

$$AR^2 - AI^2 = BS^2 - BI^2 = CT^2 - CI^2 = \&c. = \pm k^2,$$

I being any point on the line of centres, are said to be *coaxal*, every two of them having evidently the same radical axis, viz. the perpendicular to the line of centres at the point I (181).

Of coaxal systems of circles there are two species, the sign of the constant difference or *modulus*, as it is termed, of the system $\pm k^2$ being positive for one and negative for the other; in the



former case, if M and N (fig. α) be the two points on the radical axis, for which

$$IM^2 = IN^2 = AR^2 - AI^2 = BS^2 - BI^2 = CT^2 - CI^2 = \&c. = k^2,$$

all the circles of the system (Euc. I. 47) pass evidently through M and N, and the system accordingly is said to be of *the common points species;* the two points M and N being common to all the circles, which, in that case, are all real, whatever be the positions of their centres A, B, C &c. upon the line on which they all lie; and, in the latter case, if E and F (fig. β) be the two points on the line of centers, for which

$$IE^2 = IF^2 = AI^2 - AR^2 = BI^2 - BS^2 = CI^2 - CT^2 \ \&c. = k^2,$$

all the circles of the system (Euc. II. 5, 6) have evidently E and F for a common pair of inverse points (149), and the system accordingly is said to be of *the limiting points species;* the two points E and F being evanescent limits (152) to the circles, which, in that case, are real or imaginary, according as their centres A, B, C, &c. are external or internal to the intercepted segment EF of the line on which they all lie. In both cases alike the radical axis itself is evidently the part not at infinity of the particular circle of the system corresponding to the particular position of the centre at an infinite distance, and is the common axis of reflexion (50) of the several pairs of evidently equal circles of the system whose centres are equidistant in opposite directions from the central point I; for which particular point as centre the square of the radius of the corresponding circle of the system is evidently the absolute minimum in the former case, and the negative maximum in the latter.

In the particular case when the constant $k=0$, the system may be regarded as belonging indifferently to either of the above species, or, more properly, as being at once in the limiting state of each, and at the transition phase from one to the other; the two common points M and N of the former species, or the two limiting points E and F of the latter, then evidently coinciding at the point I, the circles of the system all passing through that point, and the system itself being of the comparatively simple kind considered in (18).

It is evident from the above that two circles, given in magnitude and position, determine in all cases the coaxal system, whatever be its species, to which they belong; for, by the preceding article (181), they determine the position of the central point I, and with it the value of the modulus $\pm k^2$, of the system, and therefore &c.

185. Connected with every coaxal system of either species, as explained in the preceding article, there exists a conjugate system of the other species; the two common or limiting points of one being the two limiting or common points of the other; the radical and central axes of one being the central and radical axes of the other; the constant difference of squares or modulus for one being equal in magnitude and opposite in sign to the constant difference of squares or modulus for the other; and every circle of one system intersecting at right angles every circle of the other system; which latter property is evident from the consideration, that every circle coaxal (152) with the two common points M and N of a common points system, or passing through the two limiting points E and F of a limiting points system (see the dotted circles in figs. α and β of the preceding article) intersects at right angles, by (156), every circle of the system.

Between the original and its conjugate or orthogonal system, as in consequence of the preceding property it is also termed, the relations, as above stated, are evidently reciprocal (8); either being transformable into the other by the simple interchange of the elements peculiar to its character, and every property true of either in relation to the other being, consequently, true also of the latter in relation to the former.

186. *Given two circles of a coaxal system of either species, to determine the circle of the system which,* 1°. *passes through a given point;* 2°. *cuts orthogonally a given line or circle;* 3°. *touches a given line or circle.*

These three problems, to which many others in the theory of coaxal circles are reducible, require different solutions according as the system to which the given circles belong is of the common or of the limiting points species; in the former case, the two points common to both on their radical axis are the common points of the system, and in the latter case, the two points inverse to both on their central axis (155, Cor. 2°.) are the limiting points of the system; and the common or limiting points, as the case may be, being thus given, the solutions, based in the latter case on the general property of the preceding article, are respectively as follows:

To solve 1°.; in the former case, the circle passing through the given point and through the two common points is that required; and in the latter case, the tangent at the given point to the circle passing through it and through the two limiting points intersects the central axis at the centre of the required circle (152). To solve 2°.; in the former case, the circle passing through the two common points and through the inverse of either with respect to the line or circle is that required (156); and in the latter case, the two circles passing through the two limiting points and touching the line or circle (51) determine on the latter its two points of intersection with the required circle. And to solve 3°.; in the former case, the two circles passing through the two common points and touching the line or circle (51) are those required; and in the latter case, the circle passing through the two limiting points and through the inverse of either with respect to the line or circle determines on the latter its points of contact with the two circles required.

187. For coaxal systems in general, whatever be their species, it is evident, from Cors. 1°, 3°, 4°, 5°, Art. 182, that—

1°. The tangents, real or imaginary, to all the circles of a coaxal system from any point on their radical axis are equal; and, conversely, when three or more circles are such that for two points, not at infinity, the tangents to them, real or imaginary, are equal, they are coaxal, and the line containing the two points is their radical axis.

2°. The chords of intersection, real or imaginary, of all the circles of a coaxal system with any arbitrary circle concur to a point on their radical axis; and, conversely, when three or more circles are such that their chords of intersection, real or imaginary, with two others are concurrent, they are coaxal, and the line containing the two points of concurrence is their radical axis.

3°. Every circle having its centre on the radical axis intersects all the circles of a coaxal system at angles, of the same affection, whose cosines are inversely as their radii; and, conversely, when three or more circles are intersected by two others at angles, of the same affection, whose cosines are inversely as their radii, they are coaxal, and the line of centres of the two is their radical axis.

4°. Every circle having its centre on the radical axis and intersecting any circle of a coaxal system at right angles intersects every circle of the system at right angles; and, conversely, when three or more circles intersect two others at right angles, they are coaxal, and the line of centres of the two is their radical axis.

It is evident also from (177), combined with the preceding property 4°, that—

5°. Every point has a common conjugate with respect to all the circles of a coaxal system, viz. the diametrically opposite point of the circle of the orthogonal system which passes through it; and, conversely, when three or more circles have two pairs of common conjugate points, whose distances are not at once equal and concentric, they are coaxal, as intersecting two different circles at right angles (4°).

188. For systems of the limiting points species in particular, it is also evident, from the properties referred to, that—

1°. The two limiting points are inverse points with respect to every circle of the system; and, conversely, when two circles have a common pair of inverse points, those points are the limiting points of the coaxal system they determine (152).

2°. Each limiting point and the perpendicular to the line of centres passing through the other are pole and polar with respect to every circle of the system; and, conversely, when two circles have a common pole and polar, the pole and polar centre are the limiting points of the coaxal system they determine (165).

3°. The tangents to every circle of the system from each limiting point are bisected by the radical axis; and, conversely, when the tangents to two circles from a point on their line of centers are bisected by their radical axis, that point is a limiting point of the coaxal system they determine (157).

4°. The tangents to every circle of the system from any point in the radical axis are equal to the distances of the point from the two limiting points; and, conversely, when the tangents to two circles from any point in their radical axis are equal to the distances of the point from two points on their line of centres, the latter are the limiting points of the coaxal system they determine (157).

5°. Every circle passing through the two limiting points is orthogonal to every circle of the system; and, conversely, when two circles which do not intersect are orthogonal to two circles which do, the common points of the coaxal system determined by the intersecting pair are the limiting points of the coaxal system determined by the non-intersecting pair (156).

6°. For every line touching two circles of the system, the segment intercepted between the points of contact subtends a right angle at each limiting point; and, conversely, when for a line touching two circles the segment intercepted between the points of contact subtends right angles at two points on their line of centres, those points are the limiting points of the coaxal system they determine. (22, 1′, and Euc. III. 31.)

189. *If X, Y, Z be any three collinear points on the three sides BC, CA, AB of any triangle ABC.*

1°. *The three circles on the three connectors AX, BY, CZ, as diameters, are coaxal.*

2°. *The four polar centres of the four triangles YAZ, ZBX, XCY, and ABC are on their radical axis.*

3°. *The four polar circles of the four triangles YAZ, ZBX, XCY, and ABC are coaxal.*

4°. *The three middle points of the three connectors AX, BY, CZ are on their radical axis.*

For, the three connectors AX, BY, CZ being three lines from the vertices to the opposite sides of each of the four triangles YAZ, ZBX, XCY, and ABC; therefore, by Cors. 3°. and 1°. Art. 183, the four polar circles of the four triangles (168) intersect at right angles the three circles of which the three connectors are diameters, and, as consequently the circles of the two groups are conjugately coaxal (185), therefore &c.

Cor. 1°. The four lines BXC, CYA, AZB, and XYZ in the above being entirely arbitrary, the four properties consequently may be stated, otherwise thus, as follows—

1°. *The three circles of which the three chords of intersection of any four lines are diameters are coaxal.*

2°. *The four polar centres of the four triangles determined by the four lines are on their radical axis.*

3°. *The four polar circles of the four triangles determined by the four lines are coaxal.*

4°. *The three middle points of the three chords of intersection of the four lines are on their radical axis.*

Cor. 2°. The centres of all circles of a coaxal system being collinear (184), and the two lines of centres of two conjugate systems being orthogonal (185), it follows, consequently, from Cor. 1°., that, for every system of four arbitrary lines—

1°. *The three middle points of the three chords of intersection they determine are collinear.*

2°. *The four polar centres of the four triangles they determine are collinear.*

3°. *The two lines of collinearity for the middle points of the chords and for the polar centres of the triangles are orthogonal.*

The preceding properties may be established by many other considerations, but by none more simply or elegantly than the above, which are due to Mr. W. F. Walker.

190. *If D, E, F be three circles connected with three others*

A, B, C by the relations that D is coaxal with B and C, E with C and A, and F with A and B, then—

1°. *They have always the same radical centre and product with A, B, C.*

2°. *When they pass through a common point P they pass through a second common point P'.*

3°. *When their centres are collinear they are themselves coaxal.*

For, if RR', SS', TT' be any three chords of A, B, C concurring to their radical centre O, and UU', VV', WW' any three of D, E, F concurring to the same point O; then, to prove 1°., since by hypothesis, the three groups of circles B, C, and D; C, A, and E; A, B, and F are coaxal, therefore (187, 1°) the three groups of rectangles $OS.OS'$, $OT.OT'$, and $OU.OU'$; $OT.OT'$, $OR.OR'$, and $OV.OV'$; $OR.OR'$, $OS.OS'$, and $OW.OW'$ are equal in magnitude and sign, and therefore the two groups of circles D, E, F and A, B, C have the same radical centre and product (183, Cor. 1°.); to prove 2°., when D, E, F pass through a common point P they pass also through a second common point P', that viz. on the line OP for which the product $OP.OP'$ is equal in magnitude and sign to their radical product, and of course to that of A, B, C (1°.); and to prove 3°., when the centres of D, E, F are collinear, if X, Y, Z be their three centres, XU, YV, ZW their three radii, and I the foot of the perpendicular from O on the line XYZ; since then, by 1°,

$$XO^2 - XU^2 = YO^2 - YV^2 = ZO^2 - ZW^2,$$

therefore (Euc. I. 47),

$$XI^2 - XU^2 = YI^2 - YU^2 = ZI^2 - ZW^2,$$

and therefore the three circles D, E, F are coaxal (184).

Otherwise thus: the circle G orthogonal to the three A, B, C being, by (182, Cor. 5°.), orthogonal also to the three D, E, F, its centre O and the square of its radius OG^2 are, by (183, Cor. 1°.), the radical centre and product of both triads A, B, C and D, E, F which proves 1°.; when D, E, F pass through a common point P, they pass also, by (156), through its inverse P' with respect to the circle G, which proves 2°.; and when the centres of D, E, F are collinear, they are at once orthogonal to the circle G and to the line of their centres (22, 1°.), and therefore coaxal (187, 4°.), which proves 3°.

COR. In the general case, if P and P', Q and Q', R and R' be the three pairs of intersections of the three pairs of circles E and F, F and D, D and E, and if X, Y, Z be the three intersections of the three pairs of lines QR and $Q'R'$, RP and $R'P'$, PQ and $P'Q'$; then since, by 1°, the three lines PP', QQ', RR' are concurrent, and the three rectangles $OP.OP'$, $OQ.OQ'$, $OR.OR'$ are equal in magnitude and sign, therefore (140) the three points X, Y, Z are collinear, and, (Euc. III. 35, 36), the three pairs of rectangles $XQ.XR$ and $XQ'.XR'$, $YR.YP$ and $YR'.YP'$, $ZP.ZQ$ and $ZP'.ZQ'$ are equal in magnitude and sign; or, in other words, the two triangles PQR and $P'Q'R'$ are in perspective, and the centre and axis of their perspective are the radical centre of the three circles A, B, C and the radical axis of the two PQR and $P'Q'R'$.

191. *If A, B, C be the three centres, and AR, BS, CT the three radii, of any three coaxal circles, the relation*

$$\frac{AR^2}{AB.AC}+\frac{BS^2}{BC.BA}+\frac{CT^2}{CA.CB}=1$$

is true in all cases, whatever be the species of the systems to which they belong.

For since, by hypothesis (184), I being the central point of the system, $AR^2-AI^2=BS^2-BI^2=CT^2-CI^2=\pm k^2=$ the modulus of the system; therefore,

$$BC.AR^2+CA.BS^2+AB.CT^2$$
$$=BC.AI^2+CA.BI^2+AB.CI^2\pm(BC+CA+AB).k^2;$$

but, by (78), $BC+CA+AB=0$, and, by (83),

$$BC.AI^2+CA.BI^2+AB.CI^2=-BC.CA.AB,$$

therefore

$$BC.AR^2+CA.BS^2+AB.CT^2=-BC.CA.AB,$$

and therefore &c., the latter relation being evidently equivalent to the above.

The above general relation, which when the circles belong to a system of the common points species is evident from (83), may be regarded as *the criterion of coaxality* between three circles whose centres are collinear, and by aid of it the radius corresponding to a given centre, of any circle coaxal with two others, is given at once without requiring the previous determination of the central point I and of the modulus $\pm k^2$ of the system; it is evident also from it that when two of three coaxal circles are concentric and unequal, the third, as it ought (181, 4°.), is concentric with both.

Cor. 1.° If $CT=0$, that is, if, in a system of the limiting points species, C be one of the two limiting points; then, whatever be the positions of A and B and the magnitudes of AR and BS,

$$\frac{AR^2}{AC} - \frac{BS^2}{BC} = AB,$$

which accordingly is the relation by which to calculate in numbers the positions, real or imaginary, of the two limiting points, when two circles of the system are given in magnitude and position.

Cor. 2°. If $AR=0$ and $BS=0$, that is, if, in a system of the limiting points species, A and B are the two limiting points; then, for every position of C, whatever be the interval AB,

$$CT^2 = CA.CB,$$

from which it appears, as stated in (184), that, in a system of the limiting points species, the two limiting points are inverse points with respect to every circle of the system (152).

192. *If A, B, C be the centres of three coaxal circles, AR, BS, CT their three radii, and PR, PS, PT the three tangents to them from any point P not at infinity, the relation*

$$BC.PR^2 + CA.PS^2 + AB.PT^2 = 0$$

is true in all cases, whatever be the species of the system to which they belong.

For, since, by the general relation of Art. 83,

$$BC.AP^2 + CA.BP^2 + AB.CP^2 = -BC.CA.AB;$$

and since, by the general relation of the preceding article,

$$BC.AR^2 + CA.BS^2 + AB.CT^2 = -BC.CA.AB;$$

therefore, at once, by subtraction,

$$BC.(AP^2 - AR^2) + CA.(BP^2 - BS^2) + AB.(CP^2 - CT^2) = 0,$$

which is manifestly the same as the above.

Otherwise thus: if D be the centre of the circle of the system which passes through P, then since, by Cor. 2°, Art. 182,

$$PR^2 = 2.AD.LP, \quad PS^2 = 2.BD.LP, \quad PT^2 = 2.CD.LP;$$

therefore, multiplying by BC, CA, AB, and adding

$$BC.PR^2 + CA.PS^2 + AB.PT^2$$
$$= 2.LP.(BC.AD + CA.BD + AB.CD) = 0,$$

since LP by hypothesis is not $= \infty$, and therefore &c.

Cor. 1°. If $PT = 0$, that is, if P be on the circle C, then

$$BC.PR^2 + CA.PS^2 = 0, \text{ or } PR^2 : PS^2 = AC : BC,$$

and, conversely, if the latter relation exist, then $PT = 0$, or P is on the circle C. Hence—

When three circles are coaxal, the squares of the tangents to two of them from any point on the third have the constant ratio of the distances of their centres from the centre of the third; and, conversely, the locus of a variable point the squares of the tangents from which to two fixed circles have any constant ratio, positive or negative, is the coaxal circle whose centre divides the distance between their centres in the magnitude and sign of the ratio.

By aid of Cor. 2°, Art. 182, this important property, which obviously includes those of Art. 158, and of Cor. 1°, Art. 182, as particular cases, may be proved, otherwise thus, as follows: since, by the corollary in question, when P lies on the circle C, then $PR^2 = 2.AC.LP$, $PS^2 = 2.BC.LP$, and conversely, therefore, at once, by division, $PR^2 : PS^2 = AC : BC$, and therefore &c.

Cor. 2°. If $PR = 0$, and $PS = 0$, that is, if P be on two of the circles A and B at once, then $PT = 0$, or P is on the third circle also. Hence, as already stated in (184), when two circles intersect, every third circle coaxal with them passes through their points of intersection.

Cor. 3°. *If M and N be the two points of contact with any line of the two circles of any coaxal system which touch it, P and Q its two points of intersection with any third circle of the system, and O its point of intersection with the radical axis, then always*

$$PM^2 : QM^2 = PN^2 : QN^2 = PO : QO.$$

For, if A and B be the centres of the two circles touching the line at M and N, C that of the circle intersecting it at P and Q, and L the radical axis of the system; then since, as above, by Cor. 2°, Art. 182,

$$PM^2 = 2.AC.PL, \quad QM^2 = 2.AC.QL,$$
$$PN^2 = 2.BC.PL, \quad QN^2 = 2.BC.QL;$$

therefore, at once, by division,

$$PM^2 : QM^2 = PN^2 : QN^2 = PL : QL,$$

and since, by similar triangles, $PL : QL = PO : QO$, therefore &c.

Cor. 4°. *In the same case, for a system of the limiting points species, if E and F be the two limiting points, the two angles MEN and MFN are right angles, and their sides bisect externally and internally the two angles PEQ and PFQ respectively:* see 186, 6°.

For, since by Cor. 1°,

$$PM^2 : PN^2 : PE^2 : PF^2 = QM^2 : QN^2 : QE^2 : QF^2,$$

therefore at once, by alternation,

$$PM^2 : QM^2 = PN^2 : QN^2 = PE^2 : QE^2 = PF^2 : QF^2,$$

and therefore &c. (Euc. VI. 3.)

Cor. 5°. *If P, Q, R be the three vertices of any triangle inscribed in any circle of a coaxal system, X, Y, Z the three external, and X', Y', Z' the three internal, points of contact with its sides QR, RP, PQ of the six circles of the system which touch them in pairs* (186, 3°), *then always—*

a. The four groups of three points Y', Z', X; Z', X', Y; X', Y', Z; and X, Y, Z are collinear.

b. The four groups of three lines QY, RZ, PX'; RZ, PX, QY'; PX, QY, RZ'; and PX', QY', RZ' are concurrent.

For, if A and A', B and B', C and C' be the centers of the six circles touching QR, RP, PQ at X and X', Y and Y', Z and Z' respectively, and D the centre of the circle containing P, Q, R, then since, by Cor. 1°,

$$\frac{PY^2}{PZ^2} = \frac{BD}{CD}, \quad \frac{QZ^2}{QX^2} = \frac{CD}{AD}, \quad \frac{RX^2}{RY^2} = \frac{AD}{BD},$$

with seven other groups of the same form, one for the accented, and six for the mixed accented and unaccented letters; therefore, at once, by composition of ratios,

$$\frac{PY^2}{PZ^2} \cdot \frac{QZ^2}{QX^2} \cdot \frac{RX^2}{RY^2} = 1,$$

and similarly for each of the seven remaining groups, and therefore &c.

It is evident also, from Cor. 3°, that the three intercepts XX', YY', ZZ' between the points of contact of the three pairs of circles touching the three sides of the triangle, are cut internally and externally in common ratios by every circle of the system, and are bisected internally by its radical axis.

Cor. 6°. The general relation of the present article may obviously be stated in the equivalent form

$$BC.PX.PX' + CA.PY.PY' + AB.PZ.PZ' = 0,$$

X and X', Y and Y', Z and Z', being the pairs of intersections with the three circles of any three lines passing through P. This form has the comparative advantage, that the three rectangles it involves, whatever be their signs, are always real, whereas the three tangents, whose squares are involved in the original form, may be, and often are, some or all, imaginary.

193. *If A, B, C be the centres of three coaxal circles, AR, BS, CT their three radii, and α, β, γ their three angles of intersection with any arbitrary circle whose centre is not at infinity, the relation*

$$BC.AR.\cos\alpha + CA.BS.\cos\beta + AB.CT.\cos\gamma = 0$$

is true in all cases, whatever be the species of the system to which they belong.

For, if P be the centre of the arbitrary circle, PQ its radius, X, Y, Z three of its six points of intersection with the three coaxal circles, and X', Y', Z' their three second points of intersection with its three radii PX, PY, PZ; then since, by the general property of the preceding article,

$$BC.PX.PX' + CA.PY.PY' + AB.PZ.PZ' = 0;$$

and since in the present case $PX = PY = PZ = PQ$, therefore

$$(BC + CA + AB).PQ^2 + (BC.XX' + CA.YY' + AB.ZZ').PQ = 0,$$

from which as $BC+CA+AB=0$, and as PQ not $=\infty$, therefore

$$BC.XX'+CA.YY'+AB.ZZ'=0,$$

which, as

$$XX'=2.AR.\cos\alpha,\ \ YY'=2.BS.\cos\beta,\ \ ZZ'=2.CT.\cos\gamma,$$

is therefore equivalent to the above.

Otherwise thus: by Cor. 6°, Art. 182, see also Cor. 5°, Art. 183,

$$\frac{BS.\cos\beta-CT.\cos\gamma}{BC}=\frac{CT.\cos\gamma-AR.\cos\alpha}{CA}=\frac{AR.\cos\alpha-BS.\cos\beta}{AB},$$

each being $=PL:PQ=$ the cosine of the angle, real or imaginary, Cor. 8°, Art. 182, at which the arbitrary circle intersects the radical axis L of the three coaxal circles A, B, C, and from either of these equalities the above manifestly results immediately.

This latter method has the advantage over the former, of not only establishing the general relation connecting the cosines of the three angles of intersection, real or imaginary, of any arbitrary circle with three coaxal circles, but of connecting with them at the same time the cosine of its angle of intersection, real or imaginary, with their radical axis.

Cor. 1°. When C is such, that

$$BC.AR.\cos\alpha=AC.BS.\cos\beta,$$

or, which is the same thing, that

$$AC:BC=AR.\cos\alpha:BS.\cos\beta,$$

then $CT.\cos\gamma=0$, and therefore $\cos\gamma=0$; except only when $CT=0$, in which case it is indeterminate. Hence—

Every circle intersecting two circles A and B at two angles α and β intersects at right angles the coaxal circle C whose centre is given by the preceding proportion; except only when that circle is a point, in which case it passes through it, and intersects it at an indeterminate angle.

Cor. 2°. When in Cor. 1°, $\cos\alpha:\cos\beta=\pm 1$, that is, when α and β are equal or supplemental, then $AC:BC=\pm AR:BS$, and therefore (44) C is a centre of similitude, external in the former case and internal in the latter case, of the circles A and B. Hence—

Every circle intersecting two circles A and B at equal or supplemental angles, intersects at right angles the coaxal circle, real or imaginary, whose centre is the external or internal centre of similitude of A and B; except only when that circle is a point, in which case it passes through it, and intersects it at an indeterminate angle.

COR. 3°. When in Cor. 1°, $AR.\cos\alpha : BS.\cos\beta = \pm 1$, that is, when $\cos\alpha : \cos\beta = \pm BS : AR$, then $AC : BC = \pm 1$, and therefore C is a point of bisection, external in the former case and internal in the latter case, of the interval AB. Hence—

Every circle intersecting two others A and B at angles, of similar or opposite affections, whose cosines are inversely as their radii, intersects at right angles the coaxal circle whose centre bisects externally or internally the interval between the centres of A and B; except only, in the latter case, when that circle is a point, in which case it passes through it, and intersects it at an indeterminate angle.

COR. 4°. When $AR.\cos\alpha = 0$, that is, when either $AR = 0$ or $\cos\alpha = 0$, then $AC.BS.\cos\beta = AB.CT.\cos\gamma$, or, as before (Cor. 1°.), $BS.\cos\beta : CT.\cos\gamma = BA : CA$. Hence—

Every circle either passing through a point or cutting orthogonally a line or circle A, and intersecting another line or circle B at any other constant angle β, intersects every third line or circle C coaxal with A and B at a third constant angle γ, whose cosine is given by the preceding relation.

COR. 5°. When $AR.\cos\alpha = 0$, and $BS.\cos\beta = 0$, that is, when either $AR = 0$ or $\cos\alpha = 0$, and either $BS = 0$ or $\cos\beta = 0$, then $CT.\cos\gamma = 0$, whatever be the position of C, and therefore $\cos\gamma = 0$; except only when $CT = 0$, in which case it is indeterminate. Hence, see 156 and 185—

Every circle passing through two points, or cutting orthogonally two circles, or passing through a point and cutting orthogonally a circle, cuts orthogonally every circle coaxal with the two; except only when that circle is a point, in which case it passes through it, and intersects it, like every other evanescent circle, at an indeterminate angle.

COR. 6°. When C is such, that

$$BC.AR.\cos\alpha - AC.BS.\cos\beta = \pm AB.CT,$$

then $\cos\gamma = \mp 1$, and therefore γ either = two right angles or = 0. Hence—

Every circle intersecting two circles A and B at two angles α *and* β *touches, one externally and one internally, the two coaxal circles whose centres are given by the preceding relation.*

Cor. 7°. When $\cos\alpha = \pm 1$, and $\cos\beta = \pm 1$, that is, when α either = 0 or = two right angles, and β either = 0 or = two right angles, then $AB.CT.\cos\gamma = \mp BC.AR \pm AC.BS$. Hence—

Every circle touching, with definite contacts, two circles A and B intersects any coaxal circle C at the angle γ *whose cosine is given by the preceding relation.*

Cor. 8°. In general, when two of the circles A and B and the two corresponding angles of intersection α and β are given, then, in virtue of the general relation, the third circle C determines the third angle γ, and conversely. Hence, generally—

Every circle intersecting two circles A and B at the same two angles α *and* β, *intersects every third circle C coaxal with them at the same third angle* γ *determined by the general relation, cuts orthogonally the particular circle D determined by the relation* Cor. 1°, *and touches, one internally and one externally, the two particular circles E and F determined by the relation* Cor. 6°.

The two circles A and B and the two angles α and β being given, to determine the two circles E and F coaxal with A and B which are touched by the intersecting circle in every position; describing *any* circle K intersecting A and B at the given angles α and β, the two circles E and F coaxal with A and B which touch the circle K (186, 3°), by the above, are those required.

Of the many circles K which could be described intersecting A and B at the given angles α and β, one of *given* radius is that most easily constructed; for when a circle of given radius intersects two given circles at given angles, its centre lies evidently on two concentric circles of given radii, and is therefore given.

It is evident that when one of the two intersected circles A is evanescent, then one of the two enveloped circles E coincides with it; and, that when the two intersected circles A and B are evanescent, then the two enveloped circles E and F coincide with them.

COR. 9°. By aid of the general property Cor. 8°—the general problem "*to describe a circle intersecting three given circles A, B, C at three given angles α, β, γ*" may be readily reduced to the particular case of itself: "*to describe a circle having contacts of assigned species with three given circles.*" For the required circle to intersect the three circles A, B, C at the three angles α, β, γ must touch with opposite contacts three pairs of circles D' and D'', E' and E'', F' and F'', coaxal with B and C, C and A, A and B respectively, and determinable by Cor. 8°; any three of these six, for different pairs of the given circles, being constructed by Cor. 6°, the circle touching them with the species of contact to which they correspond is that required.

By supposing first one and then two of the three given circles A, B, C to become evanescent, the two problems "*to describe a circle passing through a given point and intersecting two given circles at given angles,*" and "*to describe a circle passing through two given points and intersecting a given circle at a given angle,*" are obviously included in the above as particular cases.

COR. 10°. As, by the same general property Cor. 8°, the circle intersecting three given circles A, B, C at three given angles α, β, γ cuts orthogonally three circles D, E, F coaxal with B and C, C and A, A and B respectively, and determinable very readily by Cor. 1°; and as the problem to describe the circle orthogonal to three others is one that presents no difficulty (183, Cor. 1°); it might at first sight appear as if an easier solution of the general problem Cor. 9°, would be obtained by substituting the three auxiliary circles D, E, F in place of the three employed in the construction actually given; such, however, would not be the case, the three circles D, E, F being, as may be easily shewn, coaxal, and consequently admitting of an infinite number of orthogonal circles (185).

For, A, B, C being the three centres, and AR, BS, CT the three radii of the three given circles; if X, Y, Z be the three centres of the three circles D, E, F, then since, by Cor. 1°,

$$\frac{BX}{CX}=\frac{BS.\cos\beta}{CT.\cos\gamma},\quad \frac{CY}{AY}=\frac{CT.\cos\gamma}{AR.\cos\alpha},\quad \frac{AZ}{BZ}=\frac{AR.\cos\alpha}{BS.\cos\beta},$$

therefore, at once, by composition,

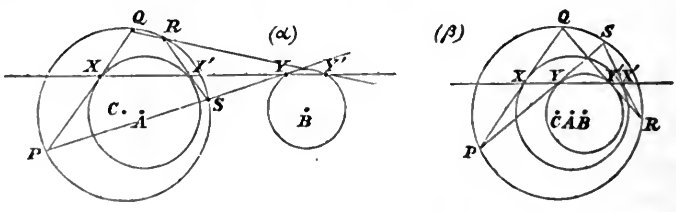
$$\frac{BX}{CX}\cdot\frac{CY}{AY}\cdot\frac{AZ}{BZ}=1,$$

consequently (134, a) the three centres X, Y, Z are collinear, and therefore (190, 3°) the three circles D, E, F are coaxal.

The general problem itself, "to describe the circle intersecting three given circles at three given angles," is of course, for the same reason, indeterminate or impossible when the circles are coaxal; indeterminate where the circles and angles are such as to fulfil the general relation of the present article, impossible when they are not.

194. With the two following converse properties of coaxal circles, and a few of the consequences to which they lead, we shall close the present Chapter.

When four points on two circles are collinear, the four vertices of the quadrilateral of which the tangents at the two on each are opposite sides lie on a third circle coaxal with both; and, conversely, when of a quadrilateral inscribed in a circle two opposite sides touch a second circle, the remaining two touch a third circle coaxal with the first and second, and the four points of contact with the two circles touched are collinear.



To prove the first part: If X and X', Y and Y' (figs. α and β) be the two pairs of points, A and B the centres of the two circles, α and β their two angles of intersection with the line of the points, and P, Q, R, S the four vertices of the quadrilateral, that is, the four points in which the two tangents at X and X' to one circle intersect the two at Y and Y' to the other; then since, in the four triangles XPY, XQY', $X'RY'$, $X'SY$, the four ratios $PX : PY$, $QX : QY'$, $RX' : RY'$, $SX' : SY$

are equal to the same ratio $\sin\beta : \sin\alpha$ (63), they are equal to each other, and therefore (192, Cor. 1°) the four points P, Q, R, S lie on the circle coaxal with A and B whose centre C is given by the relation $AC : BC = \sin^2\beta : \sin^2\alpha$; and therefore &c.

To prove the second part: If P, Q, R, S (same figures) be the four vertices of the quadrilateral, C the centre of the circle on which they lie, X and X' the points of contact of its pair of opposite sides PQ and RS with the second circle, A the centre of that circle, Y and Y' the points of intersection of the line XX' with the other pair of opposite sides PS and RQ of the quadrilateral, and B the centre of the circle which (Euc. III. 21, 22) touches those sides at Y and Y'; then since, by the first part, the circle C is coaxal with the two A and B, therefore, reciprocally, either of the latter is coaxal with the other and C, and therefore &c.

As to the possibility of a circle touching PS and RQ at Y and Y', it is evident, from Euc. III. 21, 22, that for the three pairs of angles of intersection α and α', β and β', γ and γ' of any line with the three pairs of opposite connectors L and L', M and M', N and N' of any four points P, Q, R, S on a circle, if $\alpha = \alpha'$ then $\beta = \beta'$ and $\gamma = \gamma'$, and therefore &c.

Cor. 1°. In both the above properties, while the three circles A, B, C remain fixed, the line and quadrilateral may vary simultaneously, provided only the ratio $\sin\alpha : \sin\beta$, of which, by the above, the ratio $BC : AC$ is the duplicate, be constant; or, which is the same thing, provided the ratio $XX' : YY'$ be constant, since (62) $XX' = 2.AX.\sin\alpha$, and $YY' = 2.BY.\sin\beta$. Hence—

If a variable line intersect two fixed circles at angles whose sines have any constant ratio, or, which is the same thing, intercept in them chords having any constant ratio, the four vertices of the quadrilateral, of which the tangents at the points of intersection with each are opposite sides, lie on the fixed circle coaxal with both whose centre divides the distance between their centres in the inverse duplicate of the constant ratio of the sines.

And, conversely—

If from a variable point on one of three fixed coaxal circles pairs of tangents be drawn to the other two, the four lines containing a point of contact with one and a point of contact with *the other intersect them at angles the squares of whose sines have the constant inverse ratio of the distances of their centres from the centre of the first, and therefore intercept in them chords whose squares divided by the squares of their radii have the same constant ratio.*

In the particular case, when $\sin\alpha : \sin\beta = 1$, or (62) when $XX' : YY' = AX : BY$, or (44) when the line of intersection passes through a centre of similitude, external or internal, of the circles intersected, then $AC : BC = 1$, and therefore, of the four vertices of the quadrilateral $PQRS$, two opposites lie on the line at infinity, and the remaining two lie on the radical axis of the circles A and B, the two lines into which the coaxal circle C then breaks up (184); and the same is evident from the consideration that when $\alpha = \beta$ the pairs of tangents at two pairs of intersections X and Y, X' and Y' are parallel, and intersect consequently at infinity, and the pairs of tangents at the remaining pairs of intersections X and Y', X' and Y form isosceles triangles with the line of intersection and intersect consequently on the radical axis of A and B. (182, Cor. 1°.)

Cor. 2°. *If L and L', M and M', N and N' be the three pairs of opposite lines connecting any four points P, Q, R, S on a circle, X and X', Y and Y', Z and Z' their three pairs of intersections with any line making equal angles $\alpha = \alpha'$ with one pair of them LL', and therefore* (Euc. III. 21, 22) *pairs of equal angles $\beta = \beta'$ and $\gamma = \gamma'$ with the remaining two pairs MM' and NN'; the three circles touching L and L', M and M', N and N' at X and X', Y and Y', Z and Z' are coaxal with each other and with the circle $PQRS$.*

For, by the first part of the above, the latter circle is coaxal with every two of them, and therefore &c.

If the intersecting line pass, as it or a parallel to it in every case may, through one of the three points LL', MM', NN', the corresponding circle of contact being then evanescent, that point is consequently a limiting point of the coaxal system to which the remaining two and the circle $PQRS$ belong; and if it pass through two of them at once, which, in compliance with the condition restricting it to one or other of two rectangular directions, it only could do when one of the three is at infinity, the two corresponding circles of contact being then evanescent, these points are consequently the two limiting points of the coaxal system to which the third and the circle $PQRS$ belong. Hence, see Cor. 2°, Art. 163, *the two centres of perspective of any two parallel chords of a circle are at once inverse points with respect to the circle itself and to that which touches the two chords at their middle points;* a property the reader may easily verify, *a priori,* for himself.

Cor. 3°. *If X and X', Y and Y', Z and Z' be the three pairs of intersections of an arbitrary line with any three circles, L and L', M and M', N and N' the three pairs of tangents at them to the circles; the three circles containing the vertices of the three quadrilaterals, of which MM' and NN', NN' and LL', LL' and MM' are pairs of opposite sides, are coaxal.*

For, if A, B, C be the centres of the three original circles, α, β, γ their three angles of intersection with the line, and A', B', C' the centres of the three circles containing the vertices of the three quadrilaterals, which, by the above, are coaxal with the pairs of the originals whose centres are B and C, C and A, A and B respectively; then since, by the above,

$$\frac{BA'}{CA'} = \frac{\sin^2\gamma}{\sin^2\beta}, \quad \frac{CB'}{AB'} = \frac{\sin^2\alpha}{\sin^2\gamma}, \quad \frac{AC'}{BC'} = \frac{\sin^2\beta}{\sin^2\alpha};$$

therefore, at once, by composition of ratios,

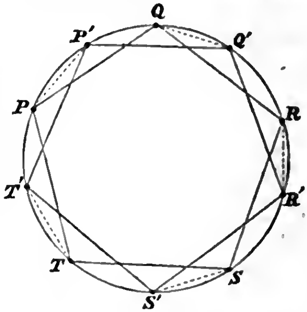
$$\frac{BA'}{CA'} \cdot \frac{CB'}{AB'} \cdot \frac{AC'}{BC'} = 1,$$

consequently (134, a) the three centres A', B', C' are collinear, and therefore &c., (190, 3°.).

In the particular case when the centres of the three original circles A, B, C are collinear, those of the three derived circles A', B', C' are of course necessarily collinear with them; but the preceding relation, proved exactly as above, exists in the particular as in the general case, and equally in both establishes the coaxality of the derived system; the same remark applies to the similar property proved, in a similar manner, in Cor. 10° of the preceding Article.

Cor. 4°. *For a variable polygon of any order inscribed to a fixed circle of any coaxal system, if all the sides but one touch in every position fixed circles of the system, that one also touches in every position a fixed circle of the system.*

Let P, Q, R, S, &c. T, and P', Q', R', S', &c. T' be any two positions of the vertices of the polygon on the circle of the system round which they move. If in the two positions the several pairs of sides PQ and $P'Q'$, QR and $Q'R'$, RS and $R'S'$, &c. up to, but not including, the last, touch the same circles of the system, the last pair TP and $T'P'$ also touch the same circle of the system.



For, joining the extremities PP', QQ', RR', SS', &c. TT' of the several pairs of sides touching the same circles in the two positions of the polygon; then since, by hypothesis, PQ and $P'Q'$ touch a common circle of the system, therefore, by the second part of the above, PP' and QQ' touch a common circle of the system; since again, by hypothesis, QR and $Q'R'$ touch a common circle of the system, therefore again, by the same, QQ' and RR' touch a common circle of the system; since again, by hypothesis, RS and $R'S'$ touch a common circle of the system, therefore again, by the same, RR' and SS' touch a common circle of the system; and so on to the last pair of sides but one; from which it follows that the first and last connectors PP' and TT' touch a common circle of the system, and therefore, by the same as before, that the last pair of sides TP and $T'P'$ touch a common circle of the system.

This simple and elegant demonstration of the above celebrated Theorem of Poncelet is due to Dr. Hart, who published an extension of it in *The Quarterly Journal of Pure and Applied Mathematics*, Vol. II., page 143; a proof nearly identical was arrived at independently about the same time by Mr. Casey.

Cor. 5°. The principle of the above demonstration depending on the circumstance that the several chords of connection PP', QQ', RR', SS', &c. TT' for any two positions of the polygon all touch a common circle of the system, and that again depending only on the circumstance that every circle of the system touched by a side of the polygon in one position is touched also by a side of the polygon in the other position, irrespectively altogether of the circumstance as to whether the contacts of the several sides of the polygon with the circles they touch take place in the same order of sequence in the two positions or not; hence, *in Poncelet's Theorem the order of sequence in which the several circles enveloped are touched by the several successive sides of the variable polygon in its different positions is entirely arbitrary, provided only no circle touched in one position be omitted in another;* a circumstance noticed by Poncelet himself, and established by him, in connection with the Theorem, on principles instructive and suggestive but involving conceptions beyond the limits of mere elementary geometry.

To see this clearly, the figure and notation employed above, for facility of conception in the first instance, being adapted only to the case when the order of the contacts with the several circles touched is the same in the two positions of the polygon; denoting by P_1P_2 and $P_1'P_2'$, Q_1Q_2 and $Q_1'Q_2'$, R_1R_2 and $R_1'R_2'$, S_1S_2 and $S_1'S_2'$, &c., the several pairs of sides of the two polygons corresponding to the two positions which touch the same circles A, B, C, D, &c. of the system, measured cyclically in the same direction for each polygon (110), but in similar or opposite directions for both, and independently altogether of the order of sequence in either; then since, by the above, the several pairs of connecting chords P_1P_1' and P_2P_2', Q_1Q_1' and Q_2Q_2', R_1R_1' and R_2R_2', S_1S_1' and S_2S_2', &c. touch the same circles of the system A', B', C', D', &c.; and since, from the nature of the case (every side of a polygon being conterminous with the two adjacent), every connector of the system P_1P_1', Q_1Q_1', R_1R_1', S_1S_1', &c. coincides necessarily with some connector of the opposite system P_2P_2', Q_2Q_2', R_2R_2', S_2S_2', &c., and conversely; therefore the several circles A', B', C', D', &c. touched by the several pairs of connectors all coincide, and therefore &c.

Cor. 6°. It appears at once from the above, Cors. 4° and 5°, that the general problem, "*to inscribe in a given circle of a coaxal system a polygon of any degree whose several sides in any order of sequence shall touch given circles of the system,*" is indeterminate when the circles are such that for every polygon inscribed to the first, all whose sides but one touch in any order of sequence all the others but one, the last side touches the last circle; when this is not the case the four common tangents, real or imaginary, to the last circle and to that touched in every position by the last side (Cor. 4°), give the last sides of the four polygons that solve the problem, and with them therefore the polygons themselves.

Since, when two circles intersect, two of their four common tangents, those passing through their external centre of similitude, are always real, and the other two, those passing through their internal centre of similitude, are always imaginary; hence when, in the above problem, the coaxal system to which the circles belong is of the common points species, two of the four polygons that solve it are always real and the other two always imaginary; when, however, the system is of the limiting points species, all four may be real or all four imaginary according to circumstances.

Cor. 7°. As all the circles touched by the several sides of the variable polygon in every position may coincide, thus reducing the several circles in the general case to two, it appears therefore, from the same, that the modified problem, "*to construct a polygon of any order all whose vertices shall lie on one given circle and all whose sides shall touch another given circle*," is indeterminate when the two circles are such that for every polygon of the required order all whose vertices lie on the first, and all whose sides but one touch the second, the last side also touches the second. When this is not the case the four common tangents, real or imaginary, to the second circle, and to the third circle, coaxal with the first and second, which is touched in every position by the last side (Cor. 4°), give, as in Cor. 6°, the last sides of the four polygons that solve the problem; which polygons for all *odd* orders, by taking the two symmetrical positions for which the last side is perpendicular to the line of centres of the three circles, are easily seen to be all real, all imaginary, or, two real and two imaginary, according as the distance between the centres of the two given circles is greater than the sum, less than the difference, or, intermediate between the sum and difference, of their radii.

In the particular case when the polygon is a triangle, the condition for indeterminateness, as regards the centres and radii of the two given circles, is given immediately by the known relation (102, Cor. 4°) that for every triangle, having no exceptional peculiarity of form, the square of the distance between the centres of the circle passing through its three vertices and of any of the four touching its three sides = the square of the radius of the former ± twice the rectangle under the radii of both; when, therefore, for two circles given in magnitude and position, the centres and radii fulfil either condition expressed in that relation, the problem to construct a triangle having its three vertices points on one and its three sides tangents to the other is indeterminate; and when they do not, though four or two real solutions still exist under the circumstances stated above, the resulting triangles, as may be easily seen on drawing the figures corresponding to the two cases, have each a pair of coincident sides, and therefore, besides their ordinary inscribed and exscribed circles, which for them as for every other triangle fulfil the relation, have each an indefinite number of other circles touching its three sides, which, including the given circle touched by the three, do not fulfil either relation. The fact as well as the explanation of the existence of real solutions in the latter case has hitherto been very generally overlooked by geometers.

CHAPTER XII.

ON THE CENTRES AND AXES OF PERSPECTIVE OF CIRCLES CONSIDERED IN PAIRS.

195. The two points on the common diameter of two circles which divide the interval between their centres, externally and internally, in the ratio of the conterminous radii, are termed indifferently (44) the two *centres of similitude*, external and internal, and also (144) the two *centres of perspective*, external and internal, of the circles; that they possess a double right to the latter appellation will appear in the sequel.

From the mere definition of the centres of similitude or perspective of two circles, it is evident that: 1°. When the circles intersect, they connect with each point of intersection by the two bisectors, external and internal, of the angle between the radii, and therefore (23) of the angle between the circles at the point (Euc. vi. 3); 2°. When the circles touch, one of them, the external or the internal according to circumstances, coincides with the point of contact; 3°. When the circles are equal and not concentric, they bisect, externally and internally, the interval between the two centres; 4°. When the circles are concentric and not equal, they both coincide with the common centre; 5°. When the circles are at once concentric and equal, one, the internal, coincides with the common centre, while the other, the external, is entirely indeterminate (15); 6°. When one circle is a point and the other not, they both coincide with the point; 7°. When one circle is a line and the other not, they coincide with the extremities of the diameter of the latter whose direction is perpendicular to the former; 8°. When both circles are points, with the exception of dividing, externally and internally, in a common ratio the interval between the points, they are otherwise both indeterminate (13);

and 9°. When both circles are lines, they connect from infinity, as in 1°, with the point of intersection by the two bisectors, external and internal, of the angle determined by the lines. Of these particulars, some, less evident than the others, will appear more fully from the general properties of the centres of similitude or perspective of any two circles, which will form the main subject of the present chapter.

When two circles, whatever be their nature, are given in magnitude and position, their two centres of perspective, external and internal, being in fact the two centres of perspective, external and internal, of any pair of their parallel diameters (131), are of course implicitly given with them; and, as already stated in (44), possess with respect to the circles, considered as similar figures at once similarly and oppositely placed, all the properties of the corresponding centres of similitude of similar figures of any form similarly or oppositely placed; all lines passing through either intersecting them at equal angles, dividing them into pairs of similar segments, determining on them pairs of homologous points at which the radii and tangents are parallel, and intercepting in them pairs of homologous chords in the constant ratio of the radii; and the two particular lines, real or imaginary, which are tangents to either circle being tangents to the other also (42).

196. The circle on the interval between the centres of similitude of two circles as diameter, which when the circles intersect passes evidently through the two points of intersection (195, 1°), is sometimes called *the circle of similitude* of the circles, and may be easily shown to be always coaxal with them, and to be such that from every point of it they subtend equal angles, real or imaginary.

For, the distances of every point on it from their centres having, by (158), the constant ratio of their radii, therefore, by pairs of similar right-angled triangles, the tangents to them from every point on it have the same constant ratio; but because the ratio of the tangents to them from every point of it is constant, it is coaxal with them (192, Cor. 1°); and because the constant ratio is that of their radii, the pairs of tangents to them from every point of it contain equal angles, real or imaginary, and therefore &c.

Or, more briefly, thus: the tangents, real or imaginary, to two circles from each of their centres of similitude having the ratio of their radii (44), therefore, by (192, Cor. 1°), so have the tangents, real or imaginary, to them from every point of the circle of which the interval between the two centres of similitude is diameter, and therefore &c.

We shall see, in the next article, that the three circles of similitude of the three groups of two determined by any system of three arbitrary circles, besides being thus coaxal each with the two original circles of its own group, are also coaxal with each other.

197. *For any three circles, whose centres are A, B, C, and radii AR, BS, CT, if X and X', Y and Y', Z and Z' be the three pairs of centres of similitude, external and internal, of the three groups of two whose centres are B and C, C and A, A and B, respectively, then—*

1°. *The six points X and X', Y and Y', Z and Z' lie three and three on four lines.*

2°. *The six lines AX and AX', BY and BY', CZ and CZ' pass three and three through four points.*

3°. *The three middle points of the three segments XX', YY', ZZ' are collinear.*

4°. *The three circles of which the three segments XX', YY', ZZ' are diameters are coaxal.*

Of these properties, the two first follow at once from the general criteria a and b' of Art. 134, by virtue of the relations (195) which determine the three pairs of points X and X', Y and Y', Z and Z' on the three sides BC, CA, AB of the triangle ABC, viz.:

$$\frac{BX}{CX}=+\frac{BS}{CT},\quad \frac{CY}{AY}=+\frac{CT}{AR},\quad \frac{AZ}{BZ}=+\frac{AR}{BS},$$

$$\frac{BX'}{CX'}=-\frac{BS}{CT},\quad \frac{CY'}{AY'}=-\frac{CT}{AR},\quad \frac{AZ'}{BZ'}=-\frac{AR}{BS};$$

and the two last follow immediately from the first, by virtue of the two general properties 1° and 4° of Cor. 1°, Art. 189, of which they furnish obvious examples; or they may be established independently as follows.

If U, V, W be the three middle points of the three segments XX', YY', ZZ', then since, by (150),

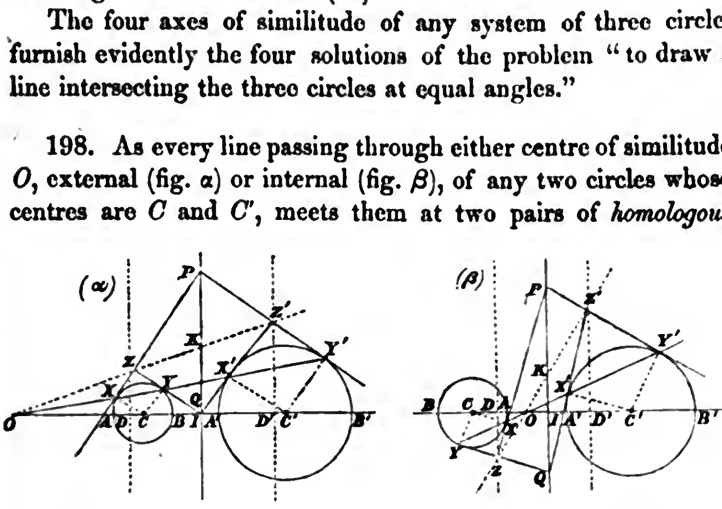
$$\frac{BU}{CU}=\frac{BS^2}{CT^2},\quad \frac{CV}{AV}=\frac{CT^2}{AR^2},\quad \frac{AW}{BW}=\frac{AR^2}{BS^2},$$

therefore, by (134, a), the three points U, V, W are collinear; and because the centres of the three circles of which the three segments XX', YY', ZZ' are diameters are collinear, the three circles themselves, being, by the preceding (196), coaxal each with the corresponding pair of the original circles to which it is the circle of similitude, are therefore, by (190, 3°), coaxal with each other.

The four lines $Y'Z'X$, $Z'X'Y$, $X'Y'Z$, and XYZ, on which the six points X and X', Y and Y', Z and Z', by property 1°, are grouped three and three, are termed, from their origin, the four *axes of similitude* of the three original circles, and occur very frequently in Modern Geometry in enquiries connected with systems of three circles. As passing each through a centre of similitude of every two of the three, they each, if they meet the three circles at all, intersect them at three equal angles; determine on them two systems of three points at which the radii and tangents are parallel; intercept in them three chords in the ratios of their radii; and, if happening to touch one of the three, touching the other two also (42).

The four axes of similitude of any system of three circles furnish evidently the four solutions of the problem "to draw a line intersecting the three circles at equal angles."

198. As every line passing through either centre of similitude O, external (fig. α) or internal (fig. β), of any two circles whose centres are C and C', meets them at two pairs of *homologous*



points (39) X and X', Y and Y', at which the two pairs of corresponding radii CX and $C'X'$, CY and $C'Y'$, and of corresponding tangents ZX and $Z'X'$, ZY and $Z'Y'$ are parallel (42); so it meets them at two pairs of *anti-homologous* points, as they are termed, X and Y', Y and X', at which the two pairs of corresponding radii CX and $C'Y'$, CY and $C'X'$, and of corresponding tangents ZX and $Z'Y'$, ZY and $Z'X'$, though not parallel, make equal angles and determine isosceles triangles with the line. Hence any two circles C and C' may be conceived to be divided by a variable line revolving round either of their centres of similitude O, and simultaneously exhausting them both, either into pairs of homologous points X and X' or Y and Y', or into pairs of anti-homologous points X and Y' or Y and X'; the distances of every two of the former from the centre of similitude to which they correspond having, as already shown in (42), a constant *ratio* termed that of *the similitude* of the figures, and the distances of every two of the latter from the same having, as may be easily shewn, a constant *product* termed that of *the anti-similitude* of the figures.

For, since, by (Euc. III. 35, 36), the two rectangles $OX.OY$ and $OX'.OY'$ are both constant, and since, by (42), the two ratios $OX:OX'$ and $OY:OY'$ are both constant and equal, therefore the two rectangles $OX.OY'$ and $OY.OX'$ are both constant and equal, and therefore &c.

The constant ratio $OX:OX'$ or $OY:OY'$ being positive or negative according as O is the external or the internal centre of similitude, and the two constant rectangles $OX.OY$ and $OX'.OY'$ being both positive and both negative together according as O is external or internal to the circles; hence, as regards the two centres of similitude of any two real circles, the constant rectangle of anti-similitude $OX.OY'$ or $OY.OX'$ is positive for the external and negative for the internal, positive for the internal and negative for the external, or, positive for both, according as the distance between the centres of the circles CC' is greater than the sum, less than the difference, or, intermediate between the sum and difference, of their radii.

199. From the properties of the preceding article, it follows evidently, conversely, that—

If on a variable line, revolving round a fixed point O and intersecting a fixed circle C in two variable points X and Y, two variable points Y′ and X′ be taken, such that $OX.OY' = OY.OX'$ *= any constant magnitude, positive or negative; the locus of the two points Y′ and X′ is another circle C′, with respect to which and the original the point O is a centre of similitude, the external or the internal according as the two constant rectangles* $OX.OY'$, *or* $OY.OX'$, *and* $OX.OY$ *have similar or opposite signs.*

For, if on the diameter AB of the original circle which passes through O (figs. of last article) the two points B' and A' be taken for which $OA.OB' = OB.OA'$ = the given rectangle, the circle on $A'B'$ as diameter fulfils evidently, by the preceding, the conditions of the required locus; but since, as regards it and the original, if C' be its centre, as $OA.OB' = OB.OA'$, therefore

$$OA : OA' = OB : OB' = OC : OC' = CA : CA' = CB : C'B',$$

and therefore &c.

If D and D' (same figures) be the two inverses of the point O with respect to the two circles; since then, by (164),

$$OX.OY = OC.OD \text{ and } OX'.OY' = OC'.OD',$$

therefore the constant product of anti-similitude for the point O, viz.,

$$OX.OY' \text{ or } OY.OX' = OC.OD' \text{ or } OC'.OD;$$

a value found very useful in the modern Theory of Inversion.

200. The two products of anti-similitude, external and internal, for any two circles may be expressed, in terms of their radii and of the distance between their centres, as follows:

If (same figures as before) C and C' be their two centres, CR and $C'R'$ their two radii, and O either centre of similitude, external (fig. α) or internal (fig. β), then since (Euc. III. 35, 36),

$$OX.OY = OC^2 - CR^2 \text{ and } OX'.OY' = OC'^2 - C'R'^2,$$

and since (42)

$$OX : OX' = OY : OY' = OC : OC' = \pm (CR : C'R'),$$

according as O is external or internal, therefore $OX.OY'$, or its equivalent $OY.OX'$, $= OC.OC' \mp CR.C'R'$; but, by (84),

$$OC = \frac{CR}{CR \mp C'R'}.C'C \text{ and } OC' = \frac{C'R'}{C'R' \mp CR}.CC',$$

therefore, denoting by r and r' the two radii and by d the distance between the two centres, the two products of anti-similitude, external and internal, have respectively for values

$$\frac{rr'}{(r-r')^2}\cdot\{d^2-(r-r')^2\} \text{ and } \frac{rr'}{(r+r')^2}\cdot\{(r+r')^2-d^2\};$$

which are the formulæ by which to calculate them in numbers when the centres and the radii of the circles are given, and which for real circles, it will be observed, give them signs in exact accordance with the particulars already stated in Art. 198.

201. The two circles round the two centres of similitude of any two circles as centres, the squares of whose radii are equal in magnitude and sign to the corresponding rectangles of anti-similitude, are termed the two *circles of anti-similitude*, external and internal, of the original circles. When the latter intersect, they evidently (198) pass through their two points, and bisect, externally and internally, their two angles, of intersection, and are therefore in that case coaxal with them and with their circle of similitude (196); that they are so in all cases may easily be shewn as follows:

Since for each centre of similitude O (same figures as before)

$$OX : OX' = OY : OY' = OC : OC',$$

therefore

$$OC'.OX.OY - OC.OX'.OY'$$
$$= (OC - OC').(OX.OY' \text{ or } OY.OX'),$$

but $OC - OC' = C'C$, and $OX.OY'$ or its equivalent $OY.OX'$ = the square of the radius of the circle of anti-similitude round O, $= - OX''.OY''$, if X'' and Y'' be any two diametrically opposite points of that circle; therefore for the three circles whose collinear centres are C, C' and O,

$$OC'.OX.OY - OC.OX'.OY' = CC'.OX''.OY'',$$

and therefore by (192, Cor. 6°) those three circles are coaxal.

As every two anti-homologous points with respect to either centre of similitude of two circles are evidently inverse points (149) with respect to the circle of anti-similitude corresponding to that centre, it follows therefore, from (156), that *every circle passing through any pair of anti-homologous points with respect to* *either centre of similitude of two circles intersect at right angles the circle of anti-similitude corresponding to that centre.*

Again, as every circle orthogonal to two others is orthogonal to every circle coaxal with the two (187, 4°), it follows, of course, from the relations of coaxality, established above and in (196), between any two circles, their circle of similitude, and their two circles of anti-similitude, that *every circle orthogonal to two others is orthogonal at once to their circle of similitude and also to their two circles of anti-similitude.*

202. *For any three circles A, B, C, if D and D', E and E', F and F' be the three pairs of circles of anti-similitude, external and internal, of B and C, C and A, A and B respectively; then—*

1°. *The four groups of three circles E', F', D; F', D', E; D', E', F; and D, E, F are coaxal.*

2°. *Their four radical axes pass through the radical centre of the original group A, B, C.*

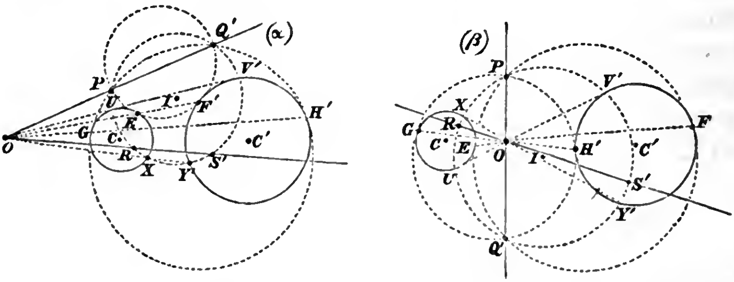
3°. *Their four pairs of common points, real or imaginary, are inverse pairs with respect to the four axes of similitude, and to the orthogonal circle, of the group A, B, C.*

4°. *Their four pairs of limiting points, imaginary or real, are the intersections of the four axes of similitude with the orthogonal circle of the group A, B, C.*

These several properties follow immediately from, or rather are all examples of, the general properties of Art. 190; the three pairs of circles D and D', E and E', F and F' being coaxal with the three pairs B and C, C and A, A and B (201); the four groups of centres of the four groups of circles E', F', D; F', D', E; D', E', F; and D, E, F being collinear (197, 1°); their four lines of centres Y', Z', X; Z', X', Y; X', Y', Z; and X, Y, Z being the four axes of similitude of the group A, B, C (197); and the whole six circles themselves being all cut orthogonally by the common circle, real or imaginary, orthogonal to the three A, B, C (183, Cor. 1°).

203. As for any two circles, regarded as similar figures, every two points P and P', or Q and Q', whether on the circles or not, which connect through either centre of similitude O, and which are such that the ratio of their distances from it $OP : OP'$, or $OQ : OQ'$, is equal in magnitude and sign to the constant ratio

of similitude for it, are termed homologous points with respect to it (42); so for any two circles, regarded as anti-similar figures, every two points P and Q', or R and S', whether on the



circles or not, which connect through either centre of anti-similitude O, external (fig. α) or internal (fig. β), and which are such that the product of their distances from it $OP.OQ'$, or $OR.OS'$, is equal in magnitude and sign to the constant product of anti-similitude for it, are termed anti-homologous points with respect to it. And again, as in the former case, every two connectors PQ and $P'Q'$ of two points P and Q, and of their two homologues P' and Q' with respect to either centre of similitude O, are termed homologous lines with respect to that centre (39, 6°); so, in the latter case, every two connectors PR and $Q'S'$ of two points P and R, and of their two anti-homologues Q' and S' with respect to either centre of anti-similitude O, external (fig. α) or internal (fig. β), are termed anti-homologous lines with respect to that centre.

It is evident (Euc. III. 35, 36) that every two pairs of anti-homologous points P and Q', R and S' with respect to either centre of similitude O of two circles, whether on the circles or not, lie, when not collinear, on a circle, the square of the tangent to which from that centre is equal in magnitude and sign to the corresponding product of anti-similitude of the circles; and, conversely, that every circle passing through any pair of anti-homologous points P and Q' with respect to either centre of similitude O of two circles, whether on the circles or not, determines pairs of anti-homologous points R and S', real or imaginary, with respect to that centre on all lines passing through it; intersects the circles themselves in two pairs of anti-homologous points U and V', X and Y', real or imaginary, with respect to the same; and, when, by the coincidence of the two points of intersection at E or G, touching either circle, then, by the simultaneous coincidence of the two anti-homologous points of intersection at F' or H', touching the other also (19).

It is evident also that, in their more general as in their more restricted acceptation (198), all pairs of anti-homologous points P and Q', R and S', &c. with respect to either centre of similitude O of two circles are inverse pairs with respect to the circle of anti-similitude corresponding to that centre (201); and that, consequently, all circles passing, as in the above figures, through any pair of them P and Q', with respect to either centre O, intersect at right angles the circle of anti-similitude corresponding to that centre (156).

204. *All pairs of homologous tangents with respect to either centre of similitude of two circles intersect on the line at infinity.*

All pairs of anti-homologous tangents with respect to either centre of similitude of two circles intersect on their radical axis.

For, if X and X' or Y and Y' (figures of Art. 198) be any pair of homologous points on the circles, X and Y' or Y and X' any pair of anti-homologous points on the same, and O the centre of similitude, external or internal, to which they correspond; then the two pairs of tangents at the former being parallel (41) intersect therefore on the line at infinity (16); and the two pairs at the latter determining isosceles triangles XPY' and YQX' with the line of the points (198) intersect therefore on the radical axis (182, Cor. 1°).

Conversely, *if from any point either on the line at infinity or on the radical axis of two circles four tangents be drawn to the circles, their four chords of contact with different circles intersect two and two at the two centres of similitude, external and internal, of the circles.*

For, the four tangents being parallel in the case of the line at infinity (16) and equal in the case of the radical axis (182, Cor. 1°), their four chords of contact with different circles in either case make equal angles with the circles, and therefore &c. (42).

Cor. Since, in the converse property, the two chords of contact of the two pairs of tangents to the same circles intersect, in either case, on the line containing the point, and pass, in either case, through its two poles with respect to the two circles, see Art. 182, Cor. 10°; it follows consequently, from that property, that—

The two centres of perspective, external and internal, of any two chords of two circles which pass through the two poles with respect to the circles either of the line at infinity or of their radical axis, and which intersect on the line whichever it be, are the two centres of similitude, external and internal, of the circles.

205. *The interval between the polars of either centre of similitude of two circles, of course bisected externally by the line at infinity, is bisected internally by the radical axis of the circles.*

For, if X and Y, X' and Y' (same figures as before) be the four intersections with the circles of any line passing through either centre of similitude O; then since, of the four vertices of the parallelogram $PZQZ'$ determined by their four tangents (41), the two opposites Z and Z', at which the pairs of tangents to the same circles intersect, lie on the two polars of O with respect to the two circles (166, Cor. 3°), and the two opposites P and Q, at which the pairs of tangents to different circles intersect, lie on the radical axis of the circles (204); and since in every parallelogram the two diagonals mutually bisect internally (Euc. I. 34), therefore &c.

Conversely, *if the interval between two homologous points with respect to either centre of similitude of two circles, of course bisected externally by the line at infinity, be bisected internally by the radical axis of the circles, the two points lie on the two polars of that centre of similitude with respect to the circles.*

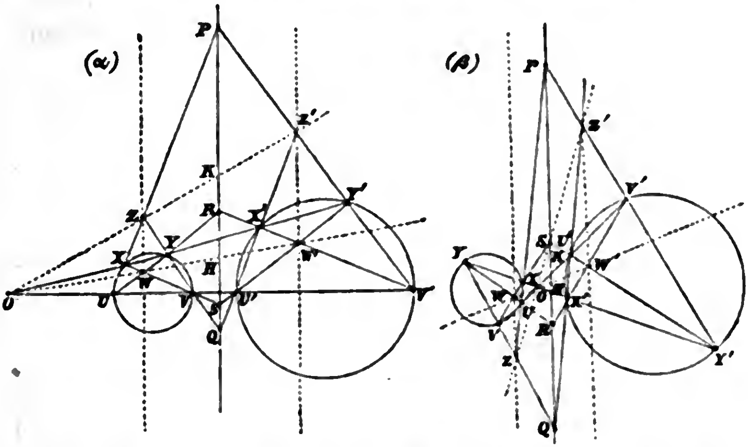
For, if connected with another pair of homologous points on the two polars in question, the interval between which, by the above, is also bisected by the radical axis, the two homologous connectors, being parallel to each other (41), would be parallel to the radical axis (Euc. VI. 2), and therefore &c.

Cor. It follows evidently from the above, as proved before for a particular case in (182, Cor. 9), that *for any two circles, however circumstanced as to magnitude and position, the two polars of the two centres of similitude with respect to each are equidistant in the two.*

206. The two properties of Art. 204 are evidently particular cases of the two following, viz.—

All pairs of homologous chords with respect to either centre of similitude of two circles intersect on the line at infinity.

All pairs of anti-homologous chords with respect to either centre of similitude of two circles intersect on their radical axis.



For, if UX and $U'X'$, or UY and $U'Y'$, or VX and $V'X'$, or VY and $V'Y'$ be any pair of homologous chords of the circles, UX and $V'Y'$, or UY and $V'X'$, or VX and $U'Y'$, or VY and $U'X'$ any pair of anti-homologous chords of the same, and O the centre of similitude, external (fig. α) or internal (fig. β), to which they correspond; then since, by the similitude of the figures,

$$OU : OU' = OV : OV' = OX : OX' = OY : OY',$$

therefore the directions of the several pairs of homologous chords are parallel, and therefore the several pairs themselves intersect on the line at infinity (16); and since, by the anti-similitude of the figures,

$$OU.OV' = OV.OU' = OX.OY' = OY.OX',$$

therefore the extremities of the several pairs of anti-homologous chords are concyclic, and therefore the several pairs themselves intersect on the radical axis of the circles (182, Cor. 3°).

In the two parallelograms $PZQZ'$ and $RWSW'$ (see figs.) formed by the four pairs of homologous and of anti-homologous chords determined by any two lines passing through O and intersecting the circles, the two diagonals ZZ' and WW', which connect the homologous intersections of pairs of chords of the same circles, being both bisected by the radical axis, their extremities lie consequently, as in the preceding (205), on the two polars of the point O with respect to the two circles; lines which with respect to that point possess evidently the property peculiar to themselves of being at once homologous and anti-homologous chords of the figures.

In the application of the above properties to any system of two circles, it is evident, from Art. 204, Cor., that—

All pairs of chords passing through the two poles of and intersecting upon the line at infinity are homologous pairs with respect to both centres of similitude.

All pairs of chords passing through the two poles of and intersecting upon the radical axis are anti-homologous pairs with respect to both centres of similitude.

207. The two general properties of the preceding article establish, as stated in (144), the quadruple relation of perspective existing between every two circles in the same plane, however circumstanced as to position and magnitude; the first their double relation of perspective as similar figures at once similarly and oppositely placed, and the second their double relation of perspective as anti-similar figures at once similarly and oppositely placed; the line at infinity and their radical axis being the axes of their double perspective in the two cases respectively, and the two centres of similitude or of anti-similitude, external and internal, being the centres of their double perspective in both cases alike.

As every two figures in perspective, whatever be their nature (141), evidently intersect their axis of perspective, whatever be its position, (or each axis of perspective if like two circles they have more than one), at the same system of points, real or imaginary, whose number depends, of course, on the nature of the figures; it follows, consequently, from the above, that *for every two circles in the same plane, however circumstanced as to magnitude and position, the radical axis and the line at infinity, being both axes of perspective, are both chords of intersection; the corresponding points of intersection, real or imaginary, according* *to circumstances in the case of the former, being of course from the nature of the figures always imaginary in the case of the latter.* This remarkable conclusion, as regards the line at infinity in relation to every two circles, the reader will find abundantly verified by various other considerations in the course of the sequel.

As again, every two figures in perspective, whatever be their nature, subtend, as stated in (41), their centre of perspective, whatever be its position, (or each centre of perspective if like two circles they have more than one), in the same system of tangents, real or imaginary, whose number depends, as before, on the nature of the figures. Hence, and from the above, the following pair of analogous properties respecting the two centres and the two axes of perspective of every two circles in the same plane, viz.—

Every two circles in the same plane, however circumstanced as to magnitude and position, subtend the same two angles, real or imaginary, at their two centres of perspective.

Every two circles in the same plane, however circumstanced as to magnitude and position, intercept the same two segments, real or imaginary, on their two axes of perspective.

208. The following pairs of polar relations, common respectively to both centres and to both axes of perspective of two circles, supply additional illustrations of the analogy noticed at the close of the preceding article, viz.—

a. The two poles of every line through either centre of perspective of two circles connect through the same centre of perspective.

a′. The two polars of every point on either axis of perspective of two circles intersect on the same axis of perspective.

For, in the former case, the two polars of the line are evidently homologous points with respect to the centre, whichever it be, and therefore &c. (41); and, in the latter case, the property is evidently that already noticed in Art. 204, Cor., and therefore &c.

b. Every two lines through either centre of perspective of two circles which are conjugates with respect to either circle are conjugates with respect to the other also.

b′. Every two points on either axis of perspective of two circles *which are conjugates with respect to either circle are conjugates with respect to the other also.*

For, in the former case, the two lines, passing each through the pole of the other with respect to one of the circles (174), pass, therefore, by (a), each through the pole of the other with respect to the other circle also, and therefore &c.; and, in the latter case, the two points, lying each on the polar of the other with respect to one of the circles (174), lie, therefore, by (a'), each on the polar of the other with respect to the other circle also, and therefore &c.

c. In every two circles the two centres of perspective are those of every two inscribed chords whose poles coincide on either axis of perspective.

c'. In every two circles the two axes of perspective are those of every two circumscribed angles whose polars coincide through either centre of perspective.

For, in the former case, the four extremities of the two chords determine, according to the axis, two pairs either of homologous or of anti-homologous points with respect to both centres of perspective (204), and therefore &c.; and, in the latter case, the four sides of the two angles determine, whichever be the centre, two pairs of homologous and two pairs of anti-homologous tangents with respect to the centre (204), and therefore &c.

d. In every two circles the two centres of perspective divide, externally and internally, in common ratios the intervals between the two poles of each axis of perspective.

d'. In every two circles the two axes of perspective divide, externally and internally, in common ratios the intervals between the two polars of each centre of perspective.

For, in the latter case, the two axes of perspective, as already shewn in (205), bisect, externally and internally, the intervals between the polars of each centre of perspective, and therefore &c.; and, in the former case, the two centres of perspective being, by (204, Cor.), those of every pair of chords of the circles which pass through the poles of and intersect on either axis of perspective, are therefore those of the particular pair perpendicular to the line of centres, the interval between which they consequently divide, externally and internally, in the ratio of their lengths, and therefore &c.

e. When two circles intersect at right angles, the polar of either centre of perspective with respect to either circle is the polar of the other centre of perspective with respect to the other circle.

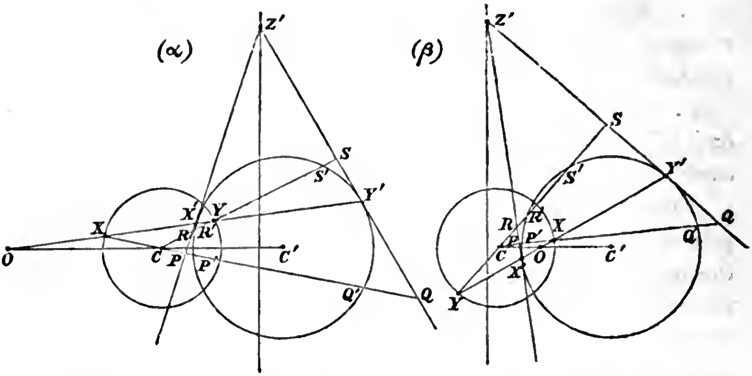
e′. When two circles intersect at right angles, the pole of either axis of perspective with respect to either circle is the pole of the other axis of perspective with respect to the other circle.

For, in the latter case, the centre of each circle being the pole of the line at infinity with respect to itself (165, 3°), and the pole of the radical axis with respect to the other (165, 6°), therefore &c.; and, in the former case, as the two lines connecting either point of intersection of the circles with their two centres of perspective make each half a right angle with each of the two radii at the point of intersection (195, 1°), therefore the two lines from either point of intersection which make each half a right angle with the line of centres of the circles intersect that line at two points, each of which, by (158, Cor. 1°), is the inverse of one centre of perspective with respect to one circle, and of the other centre of perspective with respect to the other circle, and therefore &c. (165).

f. When two circles intersect at right angles, every two tangents to either which intersect on a polar of either centre of perspective are conjugate lines with respect to the other.

f′. When two circles intersect at right angles, every two points of either which connect through a pole of either axis of perspective are conjugate points with respect to the other.

For, in the latter case, the centres of the two circles being the poles of both their axes of perspective (165, 3°, 6°), and the extremities of all diameters of either being conjugate points with respect to the other (177), therefore &c.; and, in the former case, if C and C' be the centres of the two circles, O either of their centres of perspective, the external (fig. α) or the internal (fig. β), X and Y, X' and Y' their two pairs of intersections with any line passing through O, $X'Z'$ and $Y'Z'$ the two tangents to either C' at its pair of intersections X' and Y', which, by (166, Cor. 3°), intersect on the polar of O with respect to itself, P and Q, R and S their two pairs of intersections with the two homologous radii CX and CY of the other C, to which, by (198, and Euc. III. 18), they are respectively at right angles, and, P' and Q', R' and S' the two pairs of intersections, real or



imaginary, of their circle with the same radii; then, since, by the isosceles triangle $X'Z'Y'$, the two angles at X' and Y' are equal, therefore, by (63), or by (134, a),

$$PX'^2 : QY'^2 = PX^2 : QX^2, \text{ and } RX'^2 : SY'^2 = RY^2 : SY^2,$$

and therefore, by Euc. III. 35, 36,

$$PP'.PQ' : QP'.QQ' = PX^2 : QX^2,$$

and $$RR'.RS' : SR'.SS' = RY^2 : SY^2;$$

but, since the circles, by hypothesis, intersect at right angles, therefore, by (156),

$$CP'.CQ' = CX^2, \text{ and } CR'.CS' = CY^2,$$

and therefore, by (161, Cor. 1°),

$$CP.CQ = CX^2, \text{ and } CR.CS = CY^2,$$

from which, since, by (198), the two tangents $Z'X'$ and $Z'Y'$ are perpendiculars to the two radii CX and CY, it follows, consequently, from (165), that they are the polars of the two points Q and R with respect to the circle C, and therefore &c. (174).

209. *Every circle having contacts of similar species with two others touches them at a pair of anti-homologous points with respect to their external centre of perspective.*

Every circle having contacts of opposite species with two others touches them at a pair of anti-homologous points with respect to their internal centre of perspective.

For, if C and C' (figures, Art. 198) be the centres of the two touched circles, and X and Y', or Y and X', their two points of contact with the touching circle; then since the chord of contact XY', or YX', makes equal angles with the radii of the latter, it does so with those of the former at its extremities, and therefore (42) passes through a centre at perspective of the former, the external (fig. α) or the internal (fig. β), according as their radii CX and $C'Y'$, or CY and $C'X'$, at its extremities are at similar or opposite sides of it (44); that is, according as the contacts of the touching with the touched circles are of similar or opposite species, and therefore &c.

Conversely, *Every circle passing through a pair of anti-homologous points with respect to either centre of perspective of two others, and touching either with contact of either species, touches the other with contact of similar or opposite species, according as the centre of perspective is external or internal.*

For the line XY', or YX', (same figures as before), passing through a pair of anti-homologous points X and Y', or Y and X', with respect to a centre of perspective O of the two circles whose centres are C and C', makes equal angles with their radii at the points, and also with those of every circle passing through the points; consequently, if the latter circle have contact of either species with either of the former, it has contact of similar or opposite species with the other, according as their radii at the points, (those of itself lying necessarily at the same side), lie at similar or opposite sides of the line; that is, according as the centre of perspective is external (fig. α) or internal (fig. β), and therefore &c. See also Art. 203.

Cor. 1°. Every two anti-homologous points with respect to either centre of perspective of two circles being inverse points with respect to the corresponding circle of anti-similitude (203), it follows at once (156) from the first part of the above, that—

Every circle having contacts of similar species with two others intersects at right angles their external circle of anti-similitude.

Every circle having contacts of opposite species with two others intersects at right angles their internal circle of anti-similitude.

Properties which, as the two circles of anti-similitude are coaxal with the original circles, coincide consequently with those already established on other principles in (193, Cor. 2°), viz. that—

Every circle having contacts of similar species with two others intersects at right angles the coaxal circle whose centre is their external centre of perspective.

Every circle having contacts of opposite species with two others intersects at right angles the coaxal circle whose centre is their internal centre of perspective.

COR. 2°. Since when a number of circles are orthogonal to the same circle, the radical axis of every two of them passes through, and the radical centre of every three of them coincides with, its centre; it follows consequently, from Cor. 1°, or indeed again directly from the first part of the above, that—

For every two circles having contacts of similar species with two others, the radical axis passes through their external centre of perspective.

For every two circles having contacts of opposite species with two others, the radical axis passes through their internal centre of perspective.

For every three circles having contacts of similar species with two others, the radical centre coincides with their external centre of perspective.

For every three circles having contacts of opposite species with two others, the radical centre coincides with their internal centre of perspective.

COR. 3. Since when three circles are orthogonal to three others, both systems are coaxal and conjugate to each other (185), it follows also from Cor. 1°, or again, directly from the first part of the above, that—

The circle orthogonal to three others, and the two circles touching the three with contacts of similar species, are coaxal, and have for radical axis the line passing through the three external centres of perspective of the three groups of two contained in the three (197, 1°).

The circle orthogonal to three others, and the two circles touching the same two of them with contacts of similar species and the third with contact of the opposite species, are coaxal, and have for radical axis the line passing through the external centre of perspective of the two and the two internal centres of perspective of the two combined each with the third (197, 1°).

COR. 4°. The second part of the above supplies obvious and rapid solutions of two following problems, viz.—

To describe a circle passing through a given point and having contacts of similar or opposite species with two given circles.

For by it, (see figures Art. 203), the two circles, real or imaginary, passing through the given point P and its anti-homologue Q' with respect to either centre of perspective O of the given circles, and touching either circle, touch the other with contact of similar or opposite species, according as O is (fig. α) the external or (fig. β) the internal centre of perspective of the two, and therefore &c.

Of the four circles supplied in pairs by the two cases of the above, each evidently is the unique solution of some one of the four different cases of the more definite problem: "*To describe a circle passing through a given point and having contacts of assigned species with two given circles.*"

COR. 5°. Since if a circle O have contacts of definite species with three given circles A, B, C, a concentric circle O' passing through the centre of any one of them C evidently touches with contacts of definite species two circles A' and B' concentric with the other two A and B, whose radii are equal to the sums or differences, according to circumstances, of the radii of A and C and of B and C, and which are therefore given with the latter; hence the unique solution of the definite problem: "*To describe a circle having contacts of given species with three given circles,*" is reduced at once to that of the definite problem just stated: "*To describe a circle passing through a given point and having contacts of given species with two given circles;*" and, consequently, the eight different solutions of the celebrated problem: "*To describe a circle touching three given circles,*" corresponding to the eight different combinations of contacts of both kinds with the three, may be regarded as all given in detail by so many applications of the definite construction of Cor. 4°, which, though indirect, is perhaps on the whole the simplest of which they are susceptible, see 183, Cor. 6° and 186, 3°.

Of all the constructions ever given for the direct determination of the eight circles of contact of three given circles, that of M. Gergonne, who regarded them as divided into four conjugate pairs having contacts of opposite species with the three given circles, and who determined simultaneously the six points of contact of each pair of conjugates, is decidedly the most elegant. The principles on which it depends are contained also in the above, and form the subject of the next article.

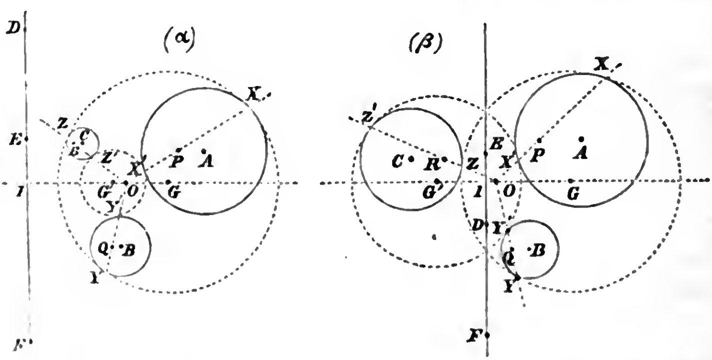
210. *When two circles have contacts of opposite species with each of three others.*

a. If they have each contacts of similar species with the three, their radical axis passes through the three external centres of perspective of the three groups of two contained in the three.

b. If they have each contacts of similar species with the same two of the three and contact of the opposite species with the third, their radical axis passes through the external centre of perspective of the two and through the two internal centres of perspective of the two combined each with the third.

c. Their three chords of contact with the three pass, in either case, through the radical centre of the three and through the three poles of their radical axis with respect to the three.

For, if A, B, C be the three centres of the touched circles,



X, Y, Z and X', Y', Z' their six points of contact with the two touching each with contacts of opposite species and having themselves each either contacts of similar species with all three (fig. α), or contacts of similar species with two of them A and B and contact of the opposite species with the third C (fig. β), D, E, F the three centres of perspective, external or internal, of B and C, C and A, A and B respectively, through which the three pairs of connectors YZ and $Y'Z'$, ZX and $Z'X'$, XY and $X'Y'$ by (209) pass, and O the internal centre of perspective of XYZ and $X'Y'Z'$, through which the three connectors XX', YY', ZZ' by the same pass; then, since, by (198),

$$DY.DZ = DY'.DZ',\ EZ.EX = EZ'.EX',\ FX.FY = FX'.FY',$$

therefore, by (182, Cor. 1°), the line DEF (197, 1°) is the radical axis of the two circles XYZ and $X'Y'Z'$, which proves a and b; since, by (198), $OX.OX' = OY.OY' = OZ.OZ'$, therefore, by (183, Cor. 1°), the point O is the radical centre of the three circles A, B, C, which proves the first part of c; and, since, by (182, Cor. 1°), the three pairs of tangents at X and X', Y and Y', Z and Z' to the two circles XYZ and $X'Y'Z'$ intersect on their radical axis DEF, therefore, by (166, Cor. 3°), their three chords of contact XX', YY', ZZ' pass through the three poles P, Q, R of that line with respect to the three circles A, B, C, which proves the second part of c; and therefore &c.

COR. 1°. Hence the following elegant construction of M. Gergonne for determining directly the six points of contact X, Y, Z and X', Y', Z' of any particular conjugate pair of the eight circles of contact of three given circles A, B, C.

Take the axis of similitude DEF of the three given circles (197) which, by the above (a or b), is the radical axis of the conjugate pair whose points of contact are required, and connect its three poles P, Q, R with respect to the given circles with their radical centre O; the three connecting lines OP, OQ, OR intersect the three circles in three pairs of points, real or imaginary, X and X', Y and Y', Z and Z', which, by the above (c), are the six points required.

The unique solution of the definite problem, "to describe a circle having contacts of given species with three given circles," is of course involved in this construction, which with its three points of contact gives evidently those of its conjugate at the same time.

COR. 2°. If G and G' be the centres of the two circles XYZ and $X'Y'Z'$, since, by (199) and (181), the line GG' passes through the point O and intersects at right angles the line DEF. Hence—

For the eight circles of contact of any system of three circles, the eight centres lie two and two in conjugate pairs on the four perpendiculars to the four axes of similitude through the radical centre of the three.

Cor. 3°. The circle round O as centre, the square of whose radius is equal in magnitude and sign to the common value of the three equal rectangles $OX.OX'$, $OY.OY'$, $OZ.OZ'$, being, by (201), the internal circle of anti-similitude of the two XYZ and $X'Y'Z'$, and, by (183, Cor. 1°), the orthogonal circle of the three A, B, C. Hence—

Of the eight circles of contact of any system of three circles, the four conjugate pairs have a common internal centre and a common internal circle of anti-similitude, viz. the radical centre and the orthogonal circle of the three.

Cor. 4°. Each circle of anti-similitude, external and internal, of any two circles being coaxal with the two (201). Hence by Cor. 3°.—

Of the eight circles of contact of any system of three circles, the four conjugate pairs belong to the four coaxal systems determined by the four axes of similitude with the orthogonal circle of the three. See Cor. 3°, Art. 209.

211. The two properties of Art. 209 are evidently particular cases of the two following, viz.—

For any system of two circles, every circle passing through any pair of anti-homologous points with respect to their external centre of perspective intersects them at equal angles, and every circle passing through any pair of anti-homologous points with respect to their internal centre of perspective intersects them at supplemental angles.

For, if C and C' (figs. Art. 203) be the centres of the two circles, P and Q' any pair of anti-homologous points with respect to either of their centres of perspective, the external (fig. α) or the internal (fig. β), UX and $V'Y'$ their pair of anti-homologous chords of intersection (203) with any circle passing through P and Q', and I the centre of that circle; then, since (Euc. I. 5) the two pairs of angles IUV' and $IV'U$, IXY' and $IY'X$ are equal, and since (198) the two pairs of angles CUV' and $C'V'U$, CXY' and $C'Y'X$ are equal (fig. α) or supplemental (fig. β), therefore the two pairs of angles IUC and $IV'C'$, IXC and $IY'C'$ are equal (fig. α) or supplemental (fig. β), and therefore &c. (23).

Conversely, *For any system of two circles, every circle intersecting them at equal angles intersects them in a pair of anti-homologous chords with respect to their external centre of perspective, and every circle intersecting them at supplemental angles intersects them in a pair of anti-homologous chords with respect to their internal centre of perspective.*

For, if C and C' (same figures as before) be the centres of the two circles, UX and $V'Y'$ their two chords of intersection with any circle intersecting them at equal angles (fig. α) or at supplemental angles (fig. β), and I the centre of that circle; then, since (Euc. I. 5) the two pairs of angles IUV' and $IV'U$, IXY' and $IY'X$ are equal, and since, by hypothesis, the two pairs of angles IUC and $IV'C'$, IXC and $IY'C'$ are equal (fig. α) or supplemental (fig. β), therefore the two pairs of angles CUV' and $C'V'U$, CXY' and $C'Y'X$ are equal (fig. α) or supplemental (fig. β), and therefore &c. (198.)

Cor. 1°. Every two anti-homologous points with respect to either centre of perspective of two circles being inverse points with respect to the corresponding circle of anti-similitude (201), it follows at once from the second part of the above, precisely as in Cor. 1°, Art. 209, that—

Every circle intersecting two others at equal angles intersects at right angles their external circle of anti-similitude, and every circle intersecting two others at supplemental angles intersects at right angles their internal circle of anti-similitude.

Properties which, as both circles of anti-similitude are coaxal with the original circles, coincide evidently, as in the corollary referred to, with those already established on other principles in (193, Cor. 2°), viz. that—

Every circle intersecting two others at equal angles intersects at right angles the coaxal circle whose centre is their external centre of perspective, and every circle intersecting two others at supplemental angles intersects at right angles the coaxal circle whose centre is their internal centre of perspective.

Cor. 2°. Again, as in Cor. 2°, Art. 209, since, when a number of circles are orthogonal to the same circle, the radical axis of every two of them passes through, and the radical centre of every three of them coincides with, its centre; it follows therefore, from Cor. 1°, as in the corollary referred to, that—

When two circles intersect two others at equal angles their radical axis passes through the external centre of perspective of the two, and when two circles intersect two others at supplemental angles their radical axis passes through the internal centre of perspective of the two.

When three circles intersect two others at equal angles their radical centre coincides with the external centre of perspective of the two, and when three circles intersect two others at supplemental angles their radical centre coincides with the internal centre of perspective of the two.

Cor. 3°. Again, as in Cor. 3°, Art. 209, since when three circles are orthogonal to three others, both systems are coaxal and conjugate to each other (185); it follows also from Cor. 1°, as in the corollary referred to, that—

Every three circles intersecting three others at equal angles are coaxal, and have for radical axis the line passing through the three external centres of perspective of the three groups of two contained in the three (197, 1°).

Every three circles intersecting the same two of three others at equal angles and the third at the supplemental angle are coaxal, and have for radical axis the line passing through the external centre of perspective of the two and the two internal centres of perspective of the two combined each with the third (197, 1°).

Cor. 4°. As the unique circle, real or imaginary, orthogonal to three others intersects the three at equal angles, and, at the same time, every two of the three at equal angles and the third at the supplemental angle, it follows immediately as a particular case of Cor. 3°, that—

The unique circle orthogonal to three others is coaxal with every two circles intersecting the three at equal angles, and also with every two intersecting the same two of them at equal angles and the third at the supplemental angle; the axis of similitude of the three external to them all in the former case, and that *external to the two and internal to the third in the latter case, being the corresponding radical axis of the system.*

COR. 5°. In the particular case where one of the two intersecting circles has one combination of the angle of intersection and its supplement, and the other the opposite combination of the same angle of intersection and its supplement, with the three; then, by the second part of Cor. 1°, for the same reason as in (210, Cor. 3°), the radical centre and orthogonal circle of the three are the internal centre and circle of anti-similitude of the two. Hence the following extension of the property Cor. 3°, of the preceding article, viz.—

The unique circle orthogonal to three others is the common internal circle of anti-similitude of every pair of conjugate circles intersecting the three at any opposite combinations of the same angle and its supplement.

Cor. 6°. The following properties of a variable circle intersecting a system of two or three fixed circles at equal or supplemental angles are evident, from Cors. 1°, 3° and 4° of the above, viz.—

a. A variable circle passing through a fixed point and intersecting two fixed circles at equal or at supplemental angles passes through a second fixed point, the anti-homologue of the first with respect to the corresponding centre of perspective of the circles.

b. A variable circle intersecting three fixed circles at equal or at any invariable combination of equal and supplemental angles describes the coaxal system determined by the corresponding axis of similitude with the orthogonal circle of the three.

Properties, the converses of which supply obvious solutions of the several problems of the three following groups, viz.—

To describe a circle (a) passing through two given points and intersecting two given circles at equal or at supplemental angles, (b) passing through a given point and intersecting three given circles at equal or at any assigned combinations of equal and supplemental angles, (c) intersecting four given circles at equal or at any assigned combinations of equal and supplemental angles.

212. With the two following properties of a system of three arbitrary circles, we shall conclude the present chapter and volume.

1°. *For any system of three circles, the three pairs of points, at which they are touched by the three pairs of circles tangent to one and orthogonal to the other two, lie on three circles, coaxal each with the two of the original three to which it does not correspond, and coaxal with each other.*

2°. *For any system of three circles, the three pairs of points, at which they are touched by any conjugate pair of their eight circles of contact, lie on three circles, coaxal each with the two of the original three to which it does not correspond, and coaxal with each other.*

For, if A_0, B_0, C_0 be the three circles, A, B, C their three centres, O their radical centre, P and P', Q and Q', R and R' the three pairs of points of contact in either case, and X_0, Y_0, Z_0 the three circles passing through them and having their three centres X, Y, Z on the three lines BC, CA, AB respectively; then, since P and P', Q and Q', R and R', in the case of 1°, by 186, 2°, are the intersections with A_0, B_0, C_0 of the three circles orthogonal to themselves and coaxal with B_0 and C_0, C_0 and A_0, A_0 and B_0 respectively, and in the case of 2°, by 210, *c.*, are collinearly distant from O by intervals such that the three rectangles $OP.OP'$, $OQ.OQ'$, $OR.OR'$ are equal in magnitude and sign, the first parts of both properties are evident; and it remains only to shew that in both cases the three points X, Y, Z on the three lines BC, CA, AB are collinear, in order to shew that in both cases the three circles X_0, Y_0, Z_0, of which they are the centres, are coaxal. See 190, 3°.

In the case of 1°, if α, β, γ be the three angles of intersection, real or imaginary, of the three pairs of original circles B_0 and C_0, C_0 and A_0, A_0 and B_0 respectively; then, since by the above, the three circles X_0, Y_0, Z_0 are coaxal with B_0 and C_0, C_0 and A_0, A_0 and B_0, and orthogonal to A_0, B_0, C_0 respectively, therefore, by 193, Cor. 1°,

$$\frac{BX}{CX}=\frac{BQ.\cos\gamma}{CR.\cos\beta},\quad \frac{CY}{AY}=\frac{CR.\cos\alpha}{AP.\cos\gamma},\quad \frac{AZ}{BZ}=\frac{AP.\cos\beta}{BQ.\cos\alpha},$$

and therefore &c. (134, *a.*). See also 193, Cor. 10°, where it was shewn, in a manner exactly similar, that three circles orthogonal to the same circle and coaxal each with a different pair of three others, all four being arbitrary, are coaxal with each other.

In the case of 2°, if PB_0 and PC_0, QC_0 and QA_0, RA_0 and RB_0 be the three pairs of tangents, real or imaginary, from the three points P, Q, R to the three pairs of original circles B_0 and C_0, C_0 and A_0, A_0 and B_0 respectively, and D, E, F the three centres of perspective of the latter at which the three lines QR, RP, PQ intersect collinearly with the three BC, CA, AB respectively (210, *a* and *b*); then, since by 134, *a*.,

$$\frac{QD}{RD}\cdot\frac{RE}{PE}\cdot\frac{PF}{QF}=1, \text{ and } \frac{BD}{CD}\cdot\frac{CE}{AE}\cdot\frac{AF}{BF}=1,$$

therefore, from the first, immediately,

$$\frac{QD.QR}{RD.RQ}\cdot\frac{RE.RP}{PE.PR}\cdot\frac{PF.PQ}{QF.QP}=-1;$$

but since from the three constant ratios of similitude of the three pairs of circles B_0 and C_0, C_0 and A_0, A_0 and B_0 respectively, by Euc. III. 35, 36,

$$\frac{QD.QR}{RD.RQ}=-\frac{BD}{CD}\cdot\frac{QC_0^2}{RB_0^2},$$

$$\frac{RE.RP}{PE.PR}=-\frac{CE}{AE}\cdot\frac{RA_0^2}{PC_0^2},$$

$$\frac{PF.PQ}{QF.QP}=-\frac{AF}{BF}\cdot\frac{PB_0^2}{QA_0^2},$$

therefore, from the second, by composition,

$$\frac{QC_0^2}{RB_0^2}\cdot\frac{RA_0^2}{PC_0^2}\cdot\frac{PB_0^2}{QA_0^2}=1;$$

from which, since by 192, Cor. 1°,

$$\frac{PB_0^2}{PC_0^2}=\frac{BX}{CX},\quad \frac{QC_0^2}{QA_0^2}=\frac{CY}{AY},\quad \frac{RA_0^2}{RB_0^2}=\frac{AZ}{BZ},$$

therefore &c. (134, *a*.)

Of the eight circles of contact of any system of three arbitrary circles, Dr. Hart has shewn, by a process, of the first part of which he has given an abstract in the *Quarterly Journal of Pure and Applied Mathematics*, Vol. IV. page 260, that they may always be divided, in four different ways, into two groups of four and their four conjugates, having each a fourth common circle of contact in addition to the original three; and Dr. Salmon, by an analysis of remarkable elegance, which he has given in Vol. VI. page 67 of the same periodical, has verified Dr. Hart's results and extended them to the more general figures of which circles are particular cases. The methods employed by both geometers, however, involve principles beyond the limits of the present work; and a demonstration of the property by Elementary Geometry, within the domain of which it manifestly lies, has not, so far as the Author is aware, been yet given.

END OF VOL. I.

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CHAPTERS

ON THE

MODERN GEOMETRY

OF THE

POINT, LINE, AND CIRCLE;

BEING THE SUBSTANCE OF

LECTURES DELIVERED IN THE UNIVERSITY OF DUBLIN TO THE CANDIDATES FOR HONORS OF THE FIRST YEAR IN ARTS.

BY THE

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THE MODERN GEOMETRY OF THE POINT, LINE, AND CIRCLE.

CHAPTER XIII.

THEORY OF HARMONIC SECTION.

213. A line AB cut at two points X and Y (fig. 1), or an angle AB cut by two lines X and Y (fig. 2), is said *to be cut harmonically* when the ratios ($AX : BX$ and $AY : BY$) of the two pairs of segments into which it is divided in the former case, or ($\sin AX : \sin BX$ and $\sin AY : \sin BY$) of the sines of the two

Fig (1)

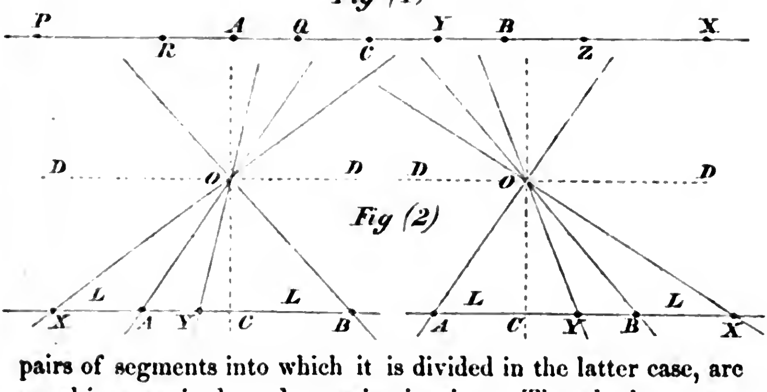


Fig (2)

pairs of segments into which it is divided in the latter case, are equal in magnitude and opposite in sign. The absolute magnitude common to the two ratios (or to their reciprocals according to the extremities of the line or angle from which the antecedents and consequents are respectively measured) is called the *ratio*, and sometimes the *modulus*, of the harmonic section; the two points or lines of section X and Y are termed *harmonic conjugates* to each other with respect to the extremities of the line or angle A and B; and the four points or lines A and B, X and Y, taken together, are said to form *an harmonic system.*

As the ratio of the harmonic section of a line or angle may have any value, real or imaginary, a line or angle may be cut harmonically in an infinite number of ways; but the ratio of the harmonic section, or the position of one of the two conjugates, is of course sufficient to determine the particular harmonic section in the case of a given line or angle.

214. The relation characteristic of the harmonic section of a line or angle AB by a pair of conjugates X and Y, viz. $AX : BX = -AY : BY$, or $\sin AX : \sin BX = -\sin AY : \sin BY$, may obviously be stated in the more symmetrical form $AX : BX + AY : BY = 0$, or $\sin AX : \sin BX + \sin AY : \sin BY = 0$, which is that most generally employed, and which is called *the equation of harmonicism* of the row or pencil of four points or rays A, B, X, Y.

When three points or rays of an harmonic row or pencil are given, the fourth evidently is implicitly given with them; provided, of course, it be known to which one of the given three it is to be conjugate.

215. In the theory of harmonic section either pair of conjugates A and B, or X and Y, may be imaginary; and as cases of each, of the section of a real line or angle by an imaginary pair of conjugates, and of an imaginary line or angle by a real pair of conjugates, are of familiar and necessary occurrence in every application of the theory to the geometry of the circle, the reader must be prepared from the outset to encounter and not be embarrassed by them.

When a line or angle and its ratio of harmonic section are both real, the two points or lines of harmonic section are of course real also, and necessarily one external and the other internal to the line or angle; the former corresponding to the positive, and the latter to the negative sign of the ratio.

216. Conceiving the ratio of harmonic section of a real line or angle AB to take successively all real values from 0 to ∞, the following particulars respecting the simultaneous positions and changes of position of the two conjugates X and Y are evident from the mere definition of harmonic section (213), viz.

1°. In the extreme case when the ratio = 0, the two antecedents AX and AY, or $\sin AX$ and $\sin AY$, in the two ratios of section simultaneously vanish; and, therefore, the two conjugates X and Y coincide at the extremity A of the line or angle from which the antecedents are measured.

2°. In the extreme case when the ratio = ∞, the two consequents BX and BY, or $\sin BX$ and $\sin BY$, in the two ratios of section simultaneously vanish; and, therefore, the two conjugates X and Y coincide at the extremity B of the line or angle from which the consequents are measured.

Hence, *for the two extreme values* 0 *and* ∞ *of the ratio of harmonic section of a real line or angle, the two points or lines of harmonic section coincide with each other and with an extremity of the line or angle.*

3°. In the particular case when the ratio = 1, the two conjugates X and Y are the two points or lines of bisection, external and internal, of the line or angle; and are, therefore, at their greatest distance asunder; being infinitely distant from each other in the case of the line, and at right angles to each other in the case of the angle.

Hence, *for the mean value*, 1, *of the ratio of harmonic section of a real line or angle, the two points or lines of harmonic section are in their position of greatest separation from each other; being infinitely distant from each other in the former case, and at right angles to each other in the latter case.*

4°. For all values of the ratio < 1, the two antecedents AX and AY, or $\sin AX$ and $\sin AY$, in the two ratios of section, are less than the two consequents BX and BY, or $\sin BX$ and $\sin BY$, and diminish or increase simultaneously with the diminution or increase of the ratio; therefore the two conjugates X and Y lie in the same segment or angle, intercepted between the two points or lines of bisection of the line or angle, with the extremity A from which the antecedents are measured, and simultaneously approach to or recede from that extremity and each other as the ratio approaches to or recedes from 0.

5°. For all values of the ratio > 1, the two consequents BX and BY or $\sin BX$ and $\sin BY$ in the two ratios of section are less than the two antecedents AX and AY, or $\sin AX$ and $\sin AY$, and diminish or increase simultaneously with the increase or diminution of the ratio; therefore the two conjugates X and Y lie in the same segment or angle, intercepted between the two points or lines of bisection of the line or angle, with the extremity B from which the consequents are measured, and simultaneously approach to or recede from that extremity and each other as the ratio approaches to or recedes from ∞.

Hence, *for all values of the ratio of harmonic section of a real line or angle different from* 1, *the two points or lines of harmonic section lie in the same segment or angle intercepted between the two points or lines of bisection of the line or angle; and move or revolve in opposite directions with the change of the ratio; approaching to or receding from each other and the extremity of the line or angle at the side of which they lie as the ratio recedes from or approaches to* 1.

These several particulars undergo, as will appear in the sequel, considerable modifications when the extremities A and B of the line or angle are, as they often are, imaginary.

217. Every two lines whose intersections with the axis of a segment cut the segment harmonically are termed *conjugate lines with respect to the segment*; and every two points whose connectors with the vertex of an angle cut the angle harmonically are termed *conjugate points with respect to the angle*.

It is evident, from the definition of harmonic section, that every line has an infinite number of conjugates with respect to every segment, all passing through the point on the axis of the segment which with the intersection of the line and axis cuts the segment harmonically, and which is termed *the pole of the line with respect to the segment*; and that every point has an infinite number of conjugates with respect to every angle, all lying on the line through the vertex of the angle which with the connector of the point and vertex cuts the angle harmonically, and which is termed *the polar of the point with respect to the angle*;

the origin and appropriateness of these several names, based as they have been on the analogy of the circle (165), will appear in a subsequent chapter.

For every two lines or points M and N, conjugates to each other with respect to a segment or angle AB, it is evident, from the equation of harmonic section (213), that in either case $AM : BM = -AN : BN$ (Euc. VI. 4, and Art. 61), or more symmetrically $$AM : BM + AN : BN = 0 \quad (214);$$ a relation which, conversely, may be regarded as a criterion of two lines or points M and N being conjugates to each other with respect to a segment or angle AB.

218. *When a line or angle AB is cut harmonically by two points or lines X and Y, then, reciprocally, the line or angle XY is cut harmonically by the two points or lines A and B.*

For, the relation,

$$AX : BX + AY : BY = 0, \text{ or } \sin AX : \sin BX + \sin AY : \sin BY = 0,$$

which (214) expresses the harmonic section of AB by X and Y, gives at once, by simple alternation, the relation

$$XA : YA + XB : YB = 0, \text{ or } \sin XA : \sin YA + \sin XB : \sin YB = 0,$$

which expresses the harmonic section of XY by A and B (214).

Hence, *When four points on a common axis, or rays through a common vertex A, B, X, Y form an harmonic system* (213)*; the two segments or angles AB and XY, intercepted between the two pairs of conjugate points or rays, cut each other harmonically; and the equation of harmonicism of the system* (214) *is the expression of the fact of their mutual harmonic section.*

In exactly the same manner it may be shewn from the closing relation of the preceding article (217), that *When two lines or points M and N are conjugates to each other with respect to a segment or angle AB, then, reciprocally, the two points or lines A and B are conjugates to each other with respect to the angle or segment MN;* a very important property of harmonic section which will be presently considered under another form.

It follows, of course, from the above, that every property of harmonic section which is true of X and Y in relation to A and B, is true reciprocally of A and B in relation to X and Y, and conversely.

219. *When four collinear points or concurrent lines, in conjugate pairs A and B, X and Y, form an harmonic system; the three pairs of opposite segments or angles they determine* (82) *are connected two and two.*

a. In the former case by the three following relations

$$AX.BY + AY.BX = 0 \ldots\ldots\ldots\ldots (1),$$

$$AB.XY + 2AY.BX = 0 \ldots\ldots\ldots\ldots (2),$$

$$AB.YX + 2AX.BY = 0 \ldots\ldots\ldots\ldots (3).$$

a'. In the latter case by the three corresponding relations

$$\sin AX.\sin BY + \sin AY.\sin BX = 0 \ldots\ldots (1'),$$

$$\sin AB.\sin XY + 2\sin AY.\sin BX = 0 \ldots\ldots (2'),$$

$$\sin AB.\sin YX + 2\sin AX.\sin BY = 0 \ldots\ldots (3'),$$

the signs as well as the magnitudes of the several segments or angles being regarded in all.

For, the first relation of each group is manifestly equivalent to the equation of harmonicism of the system (214), which it expresses in perhaps its most convenient form; and the second and third of each follow immediately from the first, in virtue of the general relation (82) connecting the six segments or angles determined by any four points on a common axis (82) or rays through a common vertex (82, Cor. 3°).

Since, in virtue of the general relation in question, any one of the three relations in each group involves the other two; each, therefore, by itself singly, may be regarded as characteristic of an harmonic system, and sufficient to determine it.

220. *When four collinear points or concurrent lines, in conjugate pairs A and B, X and Y, form an harmonic system; the three pairs of segments or angles determined by any one of them A, and by any arbitrary fifth collinear point or concurrent line K, with the remaining three X, Y, and B, are connected.*

a. In the former case by the following relation

$$KX : AX + KY : AY = 2.KB : AB.$$

a'. In the latter case by the corresponding relation

$$\sin KX : \sin AX + \sin KY : \sin AY = 2.\sin KB : \sin AB;$$

the signs as well as the magnitudes of the several segments or angles involved being regarded in each.

For, whatever be the position or direction of K, since, in the former case, by the general relation of Art. 82,

$$BY.KX - BX.KY = XY.KB;$$

and, in the latter case, by the corresponding relation, Cor. 3°, of the same article,

$$\sin BY.\sin KX - \sin BX.\sin KY = \sin XY.\sin KB;$$

and, again, from the harmonicism of the system A, B, X, Y, since, in the former case, by relations (a) of the preceding article,

$$BY.AX = - BX.AY = \tfrac{1}{2}.XY.AB;$$

and, in the latter case, by the corresponding relations (a') of the same article,

$$\sin BY.\sin AX = - \sin BX.\sin AY = \tfrac{1}{2}.\sin XY.\sin AB;$$

therefore, in the former case,

$$KX : AX + KY : AY,$$

or its equivalent

$$KX.BY : AX.BY + KY.BX : AY.BX,$$

$$= XY.KB : \tfrac{1}{2}.XY.AB = 2.KB : AB;$$

and, in the latter case,

$$\sin KX : \sin AX + \sin KY : \sin AY,$$

or its equivalent

$$\sin KX.\sin BY : \sin AX.\sin BY + \sin KY.\sin BX : \sin AY.\sin BX,$$

$$= \sin XY.\sin KB : \tfrac{1}{2}.\sin XY.\sin AB = 2.\sin KB : \sin AB;$$

and therefore &c.

By taking the arbitrary fifth point or ray K to coincide successively with the three B, X, and Y of the four A, B, X, Y constituting the harmonic system, the above useful relations become obviously those of the preceding article in the order of their enumeration; which accordingly they include as particular cases, and equally with which they may be regarded as characteristic of the relation of harmonicism between four points or rays, and sufficient to determine it.

221. *Every harmonic pencil of rays determines an harmonic row of points on every axis; and, conversely, every harmonic row of points determines an harmonic pencil of rays at every vertex.*

For, if, in either case, O (fig. 2, Art. 213) be the vertex of the pencil, and A, B, X, Y the four points of the row; then since, by (65),

$$\frac{AX}{BX} = \frac{AO}{BO} \cdot \frac{\sin AOX}{\sin BOX}, \text{ and } \frac{AY}{BY} = \frac{AO}{BO} \cdot \frac{\sin AOY}{\sin BOY};$$

therefore at once, by division of ratios,

$$\frac{AX}{BX} : \frac{AY}{BY} = \frac{\sin AOX}{\sin BOX} : \frac{\sin AOY}{\sin BOY};$$

consequently, if either equivalent $= -1$ so is the other also, that is, if either the row or the pencil be harmonic (213) so is the other also; and therefore &c.

There is one case, and one only, in which the above demonstration fails, that, viz. when the vertex O of the pencil is at an infinite distance; but in that case, the four rays of the pencil being parallel (16), the property is evident without any demonstration (Euc. VI. 10).

Of all properties of harmonic section the above, *from which it appears that the relation of harmonicism of a row of points or pencil of rays is preserved in perspective* (130), is by far the most important. As an abstract proposition it was known to the Ancients, but it was only in modern and comparatively recent times that its importance was perceived. It is to it indeed mainly that the theory of harmonic section owes its utility and power as an instrument of investigation in modern geometry.

222. Among the many consequences deducible from the general property of the preceding article, the following are of repeated occurrence in the applications of the theory of harmonic section.

1°. *When a pencil of four rays determines an harmonic row of points on any axis, it does so on every axis; and, reciprocally, when a row of four points determines an harmonic pencil of rays at any vertex, it does so at every vertex.*

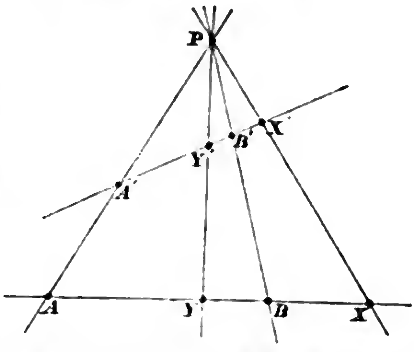
For, the pencil, in the former case, as determining an harmonic row of points on an axis, is itself harmonic; and the row, in the latter case, as determining an harmonic pencil of rays at a vertex, is itself harmonic; and therefore &c.

2°. *When a row of four points or pencil of four rays is harmonic, the perspective of either to any centre and axis is also harmonic.*

For, the row and its perspective, in the former case, connect with the centre of perspective by the same pencil of rays; and the pencil and its perspective, in the latter case, intersect with the axis of perspective at the same row of points; and therefore &c.

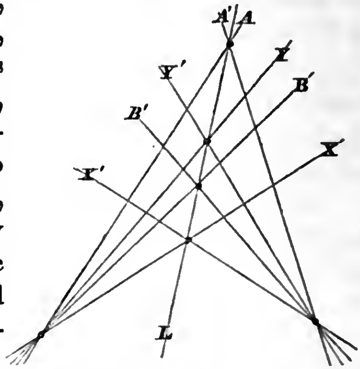
3°. *When two harmonic rows of points on different axes A, B, X, Y and A', B', X', Y' are such that any pair of their points A and A', the conjugate pair B and B', and either of the remaining pairs X and X' connect by lines AA', BB', XX' passing through a common point P, the fourth pair Y and Y' connect also by a line YY' passing through the same point P.*

For, the two rows of four points A, B, X, Y and A', B', X', Y' being, by hypothesis, harmonic, so therefore, by the preceding, are the two pencils of four rays $P(A, B, X, Y)$ and $P(A', B', X', Y')$; but three pairs of corresponding rays of those two harmonic pencils PA and PA', PB and PB', PX and PX', by hypothesis, coincide; therefore (214) the fourth pair PY and PY' coincide also; and therefore &c.



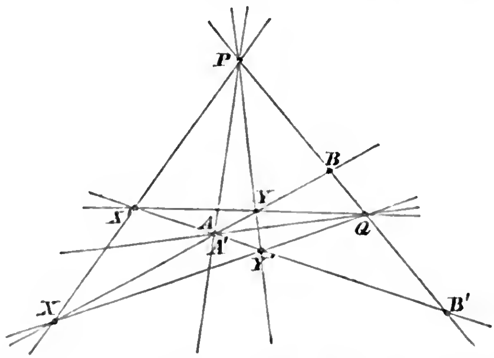
4°. *When two harmonic pencils of rays through different vertices A, B, X, Y and A', B', X', Y' are such that any pair of their rays A and A', the conjugate pair B and B', and either of the remaining pairs X and X' intersect at points AA', BB', XX' lying on a common line L, the fourth pair Y and Y' intersect also at a point YY' lying on the same line L.*

For, the two pencils of four rays A, B, X, Y and A', B', X', Y' being, by hypothesis, harmonic, so therefore, by the preceding, are the two rows of four points $L(A, B, X, Y)$, and $L(A', B', X', Y')$; but three pairs of corresponding points of those two harmonic rows LA and LA', LB and LB', LX and LX', by hypothesis, coincide, therefore (214) the fourth pair LY and LY' coincide also; and therefore &c.



5°. *When of two harmonic rows of points on different axes A, B, X, Y and A', B', X', Y', any pair of points A and A' coincide at the intersection of the axes, the conjugate pair B and B' are collinear with the two centres of perspective P and Q of the two segments XY and $X'Y'$ determined by the remaining two pairs X and X', Y and Y'.*

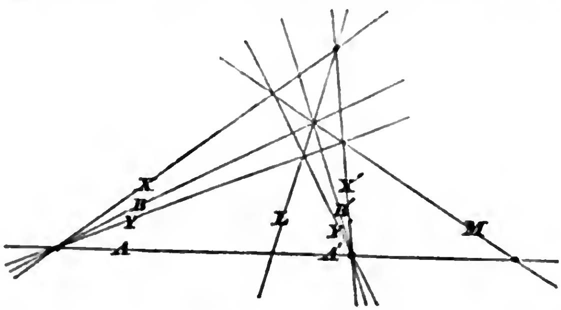
For, as in 3°, of which this is evidently a particular case,



the two rows of four points A, B, X, Y and A', B', X', Y' being, by hypothesis, harmonic, so therefore, by the preceding, are the four pencils of four rays $P(A, B, X, Y)$ and $P(A', B', X', Y')$, $Q(A, B, X, Y)$ and $Q(A', B', X', Y')$; but, for both pairs of harmonic pencils, three pairs of corresponding rays, viz. PX and PX', PY and PY', PA and PA' for the first pair, and, QX and QY', QY and QX', QA and QA' for the second pair, by hypothesis, coincide, therefore the fourth pairs for both, viz. *PB* and *PB'* for the first, and, *QB* and *QB'* for the second, coincide also; and therefore &c.

6°. *When, of two harmonic pencils of rays through different vertices A, B, X, Y and A', B', X', Y', any pair of rays A and A' coincide along the connector of the vertices, the conjugate pair B and B' are concurrent with the two axes of perspective L and M of the two angles XY and X'Y' determined by the remaining two pairs X and X', Y and Y'.*

For, as in 4°, of which this is evidently a particular case,



the two pencils of four rays *A*, *B*, *X*, *Y* and *A'*, *B'*, *X'*, *Y'* being, by hypothesis, harmonic, so therefore, by the preceding, are the four rows of four points *L*(*A*, *B*, *X*, *Y*) and *L*(*A'*, *B'*, *X'*, *Y'*), *M*(*A*, *B*, *X*, *Y*) and *M*(*A'*, *B'*, *X'*, *Y'*); but, for both pairs of harmonic rows, three pairs of corresponding points, viz. *LX* and *LX'*, *LY* and *LY'*, *LA* and *LA'* for the first pair, and, *MX* and *MY'*, *MY* and *MX'*, *MA* and *MA'* for the second pair, by hypothesis, coincide, therefore the fourth pairs for both, viz. *LB* and *LB'* for the first, and, *MB* and *MB'* for the second, coincide also; and therefore &c.

223. *When four points form an harmonic row, their four polars with respect to any circle form an harmonic pencil; and, conversely, when four lines form an harmonic pencil, their four poles with respect to any circle form an harmonic row* (166, Cor. 1°).

For, in either case, the pencil determined by the four rays being similar to that subtended by the four points at the centre of the circle (171, 2°), the harmonicism of either, consequently, involves and is involved in that of the other (213); but, by virtue of the general property of Art. 221, the harmonicism of the latter pencil involves and is involved in that of the row determined by the four points, and therefore &c.

In the applications of the theory of harmonic section, the above property, *from which it appears that the relation of harmonicism of a row of points or pencil of rays is preserved in reciprocation* (172), ranks next in importance to that of Art. 221, from which, as above demonstrated, it is indeed an inference. *By virtue of it all harmonic properties of geometrical figures are in fact double, every harmonic property of any figure being accompanied by a corresponding harmonic property of its reciprocal figure to any circle* (172), *the establishment of either of which involves that of the other without the necessity of any further demonstration* (173). The principal harmonic properties of figures consisting only of points and lines, which will form the subject of the next chapter, will be found arranged throughout in reciprocal pairs, placed in immediate connection with each other, and marked by corresponding letters, accented and unaccented, so as to keep the circumstance of this remarkable duality continually present before the reader, and supply him at the same time with numerous examples by which to keep up the valuable exercise of inferring one from the other by the reciprocating process described in Art. 172. The principal harmonic properties of figures involving circles, which will form the subject of the following chapter, will also, when their reciprocals are properties involving no higher figures (173), be arranged as far as possible on a similar plan.

224. *When two angles having a common vertex cut each other harmonically, every chord of either parallel to a side of the other is bisected internally by the second side of the other.*

For, by the general property of Art. 221, every chord of either, whatever be its direction, is cut harmonically by the sides of the other; but for the particular direction in question, one point of harmonic section is at an infinite distance (16), and therefore the other is the middle point of the chord (216, 3°).

Conversely, *When two angles having a common vertex are* *such that a side of one bisects while its second side is parallel to any chord of the other, they cut each other harmonically.*

For, the extremities of the chord with its points of internal and external bisection form an harmonic row (216, 3°); and therefore, by the same general property (221), subtend an harmonic pencil at every vertex.

COR. The preceding furnishes a rapid method of constructing the fourth ray of an harmonic pencil conjugate to any assigned one of three given rays; for, drawing any transversal parallel to the assigned conjugate, and bisecting its segment intercepted between the other two rays, the line connecting the point of bisection with the vertex of the pencil is the fourth ray required.

225. *When two segments having a common axis cut each other harmonically, the rectangle under the distances of the extremities of either from the middle point of the other, is equal in magnitude and sign to the square of half the other.*

Let AB and XY (fig. 1, Art. 213) be the segments, C and Z their middle points; then since, by hypothesis,

$$AX : BX + AY : BY = 0,$$

therefore

$$(AX + BX) : (AX - BX) :: (AY - BY) : (AY + BY),$$

but (76, (1), and 75)

$$AX + BX = 2CX, \quad AY + BY = 2CY, \quad AX - BX = AY - BY = AB,$$

therefore $$2CX : AB :: AB : 2CY,$$

and therefore $4CX.CY = AB^2$, or $CX.CY = (\frac{1}{2}AB)^2$;

and in the same manner exactly it may be proved, that

$$4ZA.ZB = XY^2, \text{ or } ZA.ZB = (\tfrac{1}{2}XY)^2,$$

and therefore &c.

Conversely, *When two segments having a common axis are such, that the rectangle under the distances of the extremities of one from the middle point of the other is equal in magnitude and sign to the square of half the other, they cut each other harmonically.*

For, since, by hypothesis,

$$CX.CY = (\tfrac{1}{2}AB)^2, \text{ or } 4CX.CY = AB^2,$$

therefore $2CX : AB :: AB : 2CY$,

but (76, (1), and 75)

$2CX = AX + BX$, $2CY = AY + BY$, $AB = AX - BX = AY - BY$,

therefore $AX + BX : AX - BX :: AY - BY : AY + BY$,

and therefore $AX : BX + AY : BY = 0$;

and similarly, if it had been given that

$$ZA.ZB = (\tfrac{1}{2}XY^2), \text{ or } 4ZA.ZB = XY^2,$$

and therefore &c.

Of all properties of the harmonic section of lines, the above leads to the greatest variety of consequences, and, as a criterion of the relation between two segments having a common axis, is generally found the most readily applicable, especially in questions relating to the circle. An analogous criterion of harmonic section between two angles, having a common vertex might be established, in precisely the same manner, with or without the aid of Trigonometry, but the general property (221) renders this unnecessary, and reduces at once all questions respecting the harmonic section of angles to the corresponding questions respecting the harmonic section of lines.

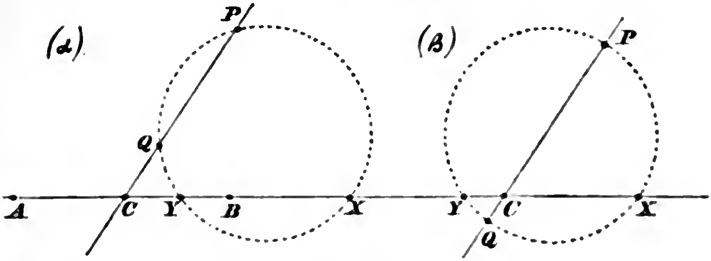
COR. 1°. Since $CA.CB = -(\tfrac{1}{2}AB)^2$, and $ZX.ZY = -(\tfrac{1}{2}XY)^2$, the preceding relations may obviously be stated in the forms

$$CX.CY + CA.CB = 0, \text{ and, } ZA.ZB + ZX.ZY = 0;$$

which, therefore, equally with their equivalents, express each the mutual harmonic section of the two coaxal segments XY and AB.

COR. 2°. A convenient and rapid construction, for determining in any number segments of a given axis cutting a given segment AB harmonically, is supplied immediately by the above; drawing a line, in any direction different from that of the given axis, through the middle point C of the given segment AB, and taking upon it any segment PQ for which the rectangle $CP.CQ$ is equal in magnitude and sign to the square of half the given segment AB; every circle passing through P and Q will intercept on the given axis a segment XY cutting harmonically the given segment AB.

For, Euclid III. 35, 36, $CX.CY = CP.CQ$, which by construction $= (\tfrac{1}{2}AB)^2$, and therefore &c.



From this construction it appears at once, as observed in (215), that when A and B are real, and therefore $(\frac{1}{2}AB)^2$ positive (fig. α), X and Y may be, as they often are, imaginary; and that when A and B are, as they may be and often are, imaginary, and therefore $(\frac{1}{2}AB)^2$ negative (fig. β), X and Y are always real.

Cor. 3°. Since for a fixed segment AB, real or imaginary, cut harmonically by a variable pair of conjugates X and Y, the rectangle $CX.CY$, as appears from the above, is constant, and equal to the square, positive or negative, of half the fixed segment; the following particulars respecting the simultaneous positions and fluctuations of X and Y may be immediately inferred:

1°. When A and B are real, and $(\frac{1}{2}AB)^2$ therefore positive; they lie at the same side of the point C, move in opposite directions on the axis AB, and coincide with each other at each of the points A and B (fig. α).

2°. When A and B are imaginary, and $(\frac{1}{2}AB)^2$ therefore negative; they lie at opposite sides of the point C, move in the same direction on the axis AB, and are at their least distance asunder when equidistant from C (fig. β).

3°. Whether A and B be real or imaginary; when either of them is at or passes through C, the other is at or passes through infinity; and, conversely, when either of them is at or passes through infinity, the other is at or passes through C (figs. α and β).

Cor. 4°. Again, for a fixed angle AB, real or imaginary, cut harmonically by a variable pair of conjugates X and Y, if O

be its vertex and C and D its two lines of bisection (fig. 2, Art. 213); the following analogous particulars, respecting the simultaneous positions and movements of X and Y, follow immediately from the preceding by virtue of the general property of Art. 221, viz.:

1°. When A and B are real; they lie in the same region of the angle CD, revolve in opposite directions round the vertex O, and coincide with each other at each of the lines A and B.

2°. When A and B are imaginary; they lie in different regions of the angle CD, revolve in the same direction round the vertex O, and are at their least separation asunder when equally inclined to C or D.

3°. Whether A and B be real or imaginary; when either of them is upon or passes over either bisector C or D, the other is upon or passes over the other bisector D or C.

226. *If a variable segment XY of a fixed axis cut a fixed segment AB of the axis harmonically—*

1°. *The circle on the variable segment XY as diameter determines a coaxal system* (184), *whose limiting points* (184) *are the extremities, real or imaginary, of the fixed segment AB.*

2°. *The circle on the variable segment XY as chord which passes through any fixed point P, not on the axis, passes also through a second fixed point Q, on the line connecting the first, real or imaginary, with the middle point C of the fixed segment AB.*

Both these properties follow at once from the preceding. The first from the consideration that for the variable circle of which XY is diameter, and therefore Z centre, $CZ^2 - (\frac{1}{2}XY)^2$, which (Euc. II. 5, 6) $= CX.CY$, is constant and $= (\frac{1}{2}AB)^2$ (184); and the second from the consideration that for the variable circle PXY, if Q be the second point in which it intersects the line PC (figs. α and β, Cor. 2°, Art. 225), $CP.CQ$, which (Euc. III. 35, 36) $= CX.CY$, is constant and $= (\frac{1}{2}AB)^2$.

Conversely, *Every circle of a coaxal system cuts harmonically the segment, real or imaginary—*

1°. *Of the line of centres intercepted between the two limiting points of the system.*

2°. *Of any line intercepted between its two points of contact with circles of the system.*

For, AB as before being the segment of the line, XY the diameter or chord of the circle, and C and Z the middle points of AB and XY; then since in the case of 1°, by (184),

$$CZ^2 - (\tfrac{1}{2}XY)^2 = CA^2 = CB^2 = (\tfrac{1}{2}AB)^2,$$

therefore $CX.CY = (\tfrac{1}{2}AB)^2$, and therefore &c. (225), and since in the case of 2°, by (182, Cor. 9°), C is on the radical axis of the system, therefore $CX.CY = CA^2 = CB^2 = (\tfrac{1}{2}AB)^2$, and therefore &c. (225).

227. *When two segments having a common axis cut each other harmonically, the square of the distance between their middle points is equal to the sum of the squares of their semi-lengths.*

Let, as before, AB and XY be the segments, C and Z their middle points; then, since (Euc. II. 5, 6) $CZ^2 = CX.CY + (\tfrac{1}{2}XY)^2$ or $= ZA.ZB + (\tfrac{1}{2}AB)^2$, and since (225) $CX.CY = (\tfrac{1}{2}AB)^2$ and $ZA.ZB = (\tfrac{1}{2}XY)^2$, therefore $CZ^2 = (\tfrac{1}{2}AB)^2 + (\tfrac{1}{2}XY)^2$, and therefore &c.

Conversely, *when two segments having a common axis are such that the square of the distance between their middle points is equal to the sum of the squares of their semi-lengths, they cut each other harmonically.*

For, since, by hypothesis, $CZ^2 = (\tfrac{1}{2}AB)^2 + (\tfrac{1}{2}XY)^2$, therefore $CZ^2 - (\tfrac{1}{2}XY)^2$, or $CX.CY$, $= (\tfrac{1}{2}AB)^2$, and $CZ^2 - (\tfrac{1}{2}AB)^2$, or $ZA.ZB$, $= (\tfrac{1}{2}XY)^2$, and therefore &c. (225).

Cor. 1°. Since, in a right-angled triangle, the square of the side subtending the right angle is equal to the sum of the squares of the sides containing the right angle, and, conversely, (Euc. I. 47, 48), it appears immediately, from the above, that—

If two coaxal segments which cut each other harmonically be turned round their middle points and made conterminous in position, they will form a right angle; and, conversely, If two conterminous segments which form a right angle be turned round their middle points and made coincident in direction, they will cut each other harmonically.

Cor. 2°. Since, when two circles intersect at right angles, the square of the distance between their centres is equal to the sum of the squares of their radii, and conversely (23), it appears again, from the above, that—

When two coaxal segments cut each other harmonically, the two circles of which they are diameters intersect at right angles; and, conversely, when two circles intersect at right angles, their two diameters which coincide in direction cut each other harmonically.

Cor. 3°. The above, also, supplies obvious solutions of the three following problems:

1°. *Given one segment AB of a line and the length XY of another cutting it harmonically, to determine the middle point Z of the other.*

2°. *Given one segment AB of a line and the middle point Z of another cutting it harmonically, to determine the length XY of the other.*

3°. *Given two segments AB and $A'B'$ of a line, to determine the middle point Z and the length XY of the segment which cuts both harmonically.*

228. *When two segments having a common axis cut each other harmonically, every circle passing through the extremities of either cuts orthogonally the circle of which the other is a diameter.*

Let, as before, AB and XY be the segments, C and Z their middle points; then since (225) $CX.CY = (\frac{1}{2}AB)^2$, therefore (Euc. III. 35, 36) square of tangent from C to any circle passing through X and $Y =$ square of radius of circle of which AB is diameter; and since $ZA.ZB = (\frac{1}{2}XY)^2$, therefore square of tangent from Z to any circle passing through A and $B =$ square of radius of circle of which XY is diameter; and therefore &c. (23).

Conversely, *when two circles of any radii cut each other orthogonally, every diameter of either is cut harmonically by the other.*

Let AB be any diameter of either, C its middle point, and X and Y the two points, real or imaginary, at which it intersects the other; then since (Euc. III. 35, 36) $CX.CY =$ square of tangent from C to the latter, that is, as the circles cut orthogonally, $=$ square of radius of former, $= (\frac{1}{2}AB)^2$, therefore &c. (225).

COR. 1°. Since a variable circle passing through a fixed point, and cutting a fixed circle orthogonally, passes through a second fixed point, the inverse of the first with respect to the fixed circle (149), it appears at once from the above, as already noticed in (226, 2°), that—

A variable circle passing through a fixed point, and cutting a fixed segment of a fixed axis harmonically, passes also through a second fixed point, on the line connecting the first with the centre of the fixed segment.

COR. 2°. Again, since a variable circle cutting two fixed circles orthogonally determines a coaxal system, whose radical axis is the line of centres, whose line of centres is the radical axis, and whose limiting points are the intersections, real or imaginary, of the fixed circles (185); it appears also, from the above, that—

A variable circle cutting two fixed segments of two fixed axes harmonically determines a coaxal system, whose radical axis is the line of centres, whose line of centres is the radical axis, and whose limiting points are the intersections, real or imaginary, of the circles of which the fixed segments are diameters.

COR. 3°. Since (156, Cor. 4°) a circle may be described, 1° passing through two given points and cutting a given circle orthogonally; 2° passing through a given point and cutting two given circles orthogonally; 3° cutting three given circles orthogonally; the radical centre of the given group and its tangential distance from each circle of the group, evanescent or finite, being the centre and radius of the cutting circle in each case; the above furnishes solutions at once simple and obvious of the three following problems, viz.

To describe a circle, 1° *passing through two given points and cutting a given segment of a given axis harmonically;* 2° *passing through a given point and cutting two given segments of two given axes harmonically;* 3° *cutting three given segments of three given axes harmonically.*

COR. 4°. As three segments of three axes may be the three sides of the triangle determined by the axes, the problems of the preceding corollary (3°) consequently include as particular cases the three following, respectively, viz.—

To describe a circle, 1° passing through two given points and cutting a side of a given triangle harmonically; 2° passing through a given point and cutting two sides of a given triangle harmonically; 3° cutting the three sides of a given triangle harmonically.

COR. 5°. Since (168) the three circles of which the sides of any triangle are diameters are cut orthogonally by the polar circle, real or imaginary, of the triangle; that is, by the circle round the intersection of its three perpendiculars as centre, the square of whose radius is equal, in magnitude and sign, to the common value of the three equal rectangles under the segments into which they mutually divide each other; hence again, from the above, it appears that—

In every triangle the polar circle, real or imaginary, cuts the three sides harmonically.

COR. 6°. Since (189, 1°, Cor. 1°) the three circles of which the three chords of intersection of any tetragram are diameters are coaxal, and since consequently (185) every circle cutting two of them orthogonally cuts the third also orthogonally; hence also, from the above, it appears that—

Every circle cutting two of the three chords of intersection of any tetragram harmonically cuts the third also harmonically.

229. *If a line AB be cut harmonically by two pairs of conjugates X and Y, X' and Y', both pairs being arbitrary.*

a. The three circles on XX', YY', and AB (and also the three on XY', YX', and AB) as diameters are coaxal.

b. The three circles on XX', YY', and AB (and also the three on XY', YX', and AB) as chords, which pass through any common point P not on the line, pass also through a second common point Q not on the line.

To prove (*a*). Since by hypothesis

$$AX : AY = -BX : BY \text{ and } AX' : AY' = -BX' : BY',$$

therefore, by composition of ratios,

$$AX.AX' : AY.AY' :: BX.BX' : BY.BY',$$

and therefore &c. (192, Cor. 1°).

To prove (*b*). Since, by (*a*), there exists a point O on AB for which $OX.OX' = OY.OY' = OA.OB$, therefore, if Q be the point on OP for which each $= OP.OQ$, the three circles XPX', YPY', and APB all pass through Q, and therefore &c.

The point O on AB for which $OX.OX' = OY.OY'$, and each therefore $= OA.OB$, is evidently that determined by the relation $OZ.X'Y' + OZ'.XY = 0$, Z and Z' being the middle points of XY and $X'Y'$; for, since when $OX.\ OX' = OY.OY'$ then $OX : OY = OY' : OX'$, therefore

$$OX + OY : OX - OY = OY' + OX' : OY' - OX',$$

or $2.OZ : YX = 2.OZ' : X'Y'$, and therefore &c.

230. In the applications of the theory of harmonic section to the geometry of the circle, the solutions of a variety of problems are reduced to those of the following:

Given two segments or angles AB and $A'B'$ having a common axis or vertex, to determine the segment or angle XY which cuts both harmonically.

By virtue of the general relation of Art. 221, the case of the angle is of course reduced at once to that of the segment, which is given immediately by any of the three following constructions, all based on the property of Art. 225, viz.:

1°. Describing the two circles of which AB and $A'B'$, bisected at C and C' respectively, are diameters; any circle cutting them both orthogonally will intercept on the given axis the required segment XY.

For (228) $CX.CY = (\frac{1}{2}AB)^2$, and $C'X.C'Y = (\frac{1}{2}A'B')^2$, and therefore &c. (225).

2°. Taking arbitrarily any point P not on the given axis, and describing the two circles PAB and $PA'B'$; their chord of intersection PQ will intersect the given axis at the middle point Z of the required segment XY; and the circle round Z as centre, the square of whose radius is equal to the rectangle $ZP.ZQ$, will intercept on the given axis the required segment itself.

For, (Euc. III. 35, 36) $ZA.ZB = ZP.ZQ = (\frac{1}{2}XY)^2$, and $ZA'.ZB' = ZP.ZQ = (\frac{1}{2}XY)^2$, and therefore &c. (225).

3°. Taking arbitrarily any point P not on the given axis, connecting it with the middle points C and C' of AB and $A'B'$, and taking on the connecting lines PC and PC' the two points Q and Q', for which $CP.CQ = (\frac{1}{2}AB)^2$, and $C'P.C'Q' = (\frac{1}{2}A'B')^2$;

the circle QPQ' will intercept on the given line the required segment XY.

For, (Euc. III. 35, 36) $CX.CY = CP.CQ = (\frac{1}{2}AB)^2$, and $C'X.C'Y = C'P.C'Q' = (\frac{1}{2}A'B')^2$, and therefore &c. (225).

If either or both of the given segments AB and $A'B'$ be imaginary, the last alone of the preceding constructions is applicable; and the problem, as solved by it, is obviously in its most general form equivalent to the following, viz.:

On a given line to determine the two points X and Y the rectangles under whose distances from each of two given points on the line C and C' are given in magnitude and sign.

When the two given segments or angles AB and $A'B'$ are such that A and B alternate with A' and B' in order of succession, the segment or angle XY which cuts them both harmonically is of course necessarily imaginary; its two points or lines of bisection are however in all cases real (225, Cor. 2°).

231. The harmonic relation of a system of four points on a common axis A, B, X, Y may be expressed in terms of the three distances of any three of them from the fourth as follows:

If A be the point from which the distances of the remaining three are measured; substituting for BX and BY their equivalents $AX - AB$ and $AY - AB$, the fundamental proportion of harmonic section (213) becomes

$$AX : AY :: AX - AB : AB - AY \quad\text{......... (1).}$$

If B be the point; substituting for AX and AY their equivalents $BX - BA$ and $BY - BA$, it becomes

$$BX : BY :: BX - BA : BA - BY \quad\text{......... (2).}$$

If X be the point; substituting for YA and YB their equivalents $XA - XY$ and $XB - XY$, it becomes

$$XA : XB :: XA - XY : XY - XB \quad\text{......... (3).}$$

And if Y be the point; substituting for XA and XB their equivalents $YA - YX$ and $YB - YX$, it becomes

$$YA : YB :: YA - YX : YX - YB \quad\text{......... (4),}$$

in each of which the relation is expressed in terms of the distances of three of the points from the fourth, in a form which is precisely the same from whichever of the four the three distances are measured.

COR. 1°. The four preceding relations give at once the equalities

$$\left.\begin{array}{l} 2.AX.AY=(AX+AY).AB \\ 2.BX.BY=(BX+BY).BA \\ 2.XA.XB=(XA+XB).XY \\ 2.YA.YB=(YA+YB).YX \end{array}\right\} \text{.......... (I)},$$

from which it follows immediately, that

$$\left.\begin{array}{l} AB=\dfrac{2.AX.AY}{AX+AY}=\dfrac{AX.AY}{\frac{1}{2}(AX+AY)} \\ BA=\dfrac{2.BX.BY}{BX+BY}=\dfrac{BX.BY}{\frac{1}{2}(BX+BY)} \\ XY=\dfrac{2.XA.XB}{XA+XB}=\dfrac{XA.XB}{\frac{1}{2}(XA+XB)} \\ YX=\dfrac{2.YA.YB}{YA+YB}=\dfrac{YA.YB}{\frac{1}{2}(YA+YB)} \end{array}\right\} \text{...... (II)},$$

relations which express the distance of any point of an harmonic system from its conjugate, in terms of its distances from the remaining two points of the system.

COR. 2°. The reciprocals of the four latter relations (II), give again immediately

$$\left\{\begin{array}{l} \dfrac{1}{AX}+\dfrac{1}{AY}=\dfrac{2}{AB} \\ \dfrac{1}{BX}+\dfrac{1}{BY}=\dfrac{2}{BA} \\ \dfrac{1}{XA}+\dfrac{1}{XB}=\dfrac{2}{XY} \\ \dfrac{1}{YA}+\dfrac{1}{YB}=\dfrac{2}{YX} \end{array}\right\} \text{ or } \left\{\begin{array}{l} \dfrac{1}{AB}=\dfrac{1}{2}\left(\dfrac{1}{AX}+\dfrac{1}{AY}\right) \\ \dfrac{1}{BA}=\dfrac{1}{2}\left(\dfrac{1}{BX}+\dfrac{1}{BY}\right) \\ \dfrac{1}{XY}=\dfrac{1}{2}\left(\dfrac{1}{XA}+\dfrac{1}{XB}\right) \\ \dfrac{1}{YX}=\dfrac{1}{2}\left(\dfrac{1}{YA}+\dfrac{1}{YB}\right) \end{array}\right\} \text{... (III)},$$

which express, in a remarkably simple manner, the harmonic relation of four points, in terms of the reciprocals of the distances of any three of them from the fourth.

COR. 3°. If C and Z (fig. 1, Art. 213) be the middle points of the two conjugate segments AB and XY respectively; then since (76, 1)

$$AX+AY=2.AZ, \quad BX+BY=2.BZ,$$
$$XA+XB=2.XC, \quad YA+YB=2.YC,$$

the four general relations (I) of Cor. 1°, are obviously equivalent to the following:

$$\left.\begin{cases} AX.AY = AZ.AB \\ BX.BY = BZ.BA \\ XA.XB = XC.XY \\ YA.YB = YC.YX \end{cases}\right\} \quad \ldots\ldots\ldots\ldots\ldots \text{(IV)},$$

relations of considerable utility, each of which, like any of the preceding, is characteristic of an harmonic system, and sufficient to determine it.

232. When four points on a common axis A and B, X and Y form an harmonic system, the three distances from any one of them to the remaining three, regard being had to their signs as well as to their magnitudes, are said *to be in harmonic progression*, and the distance from each to its conjugate is termed *the harmonic mean* of the distances from it to the other two.

Thus, A and B, X and Y being the two pairs of conjugates, the four sets of their magnitudes AX, AB, and AY; BX, BA, and BY; XA, XY, and XB; YA, YX, and YB, taken all with the proper signs due to their several directions, are each in harmonic progression, AB, BA, XY, and YX being the harmonic means in the four cases respectively.

From the invariable order of the four points of an harmonic system when all real, it is evident, from the above definition, that the harmonic mean of two magnitudes has the sign common to both when their signs are similar, and that of the numerically lesser of the two when their signs are opposite.

According to the analogy of arithmetic and geometric progression, any number of magnitudes are said to be in harmonic progression when every consecutive three of them are in such progression.

233. The several groups of relations of Art. 231 and its corollaries, interpreted in accordance with the above definitions, express all the ordinary properties of three or more magnitudes in harmonic progression, regard being had to their signs as well as to their absolute values in every case.

The group of proportions (1), (2), (3), (4) express immediately that—

1°. *When three magnitudes are in harmonic progression, the first : the third :: the first – the second : the second – the third.*

The group of equalities (II) that—

2°. *When three magnitudes are in harmonic progression, the mean = twice the product of the extremes divided by their sum, or = the product of the extremes divided by half their sum.*

The group of equalities (III) that—

3°. *When three magnitudes are in harmonic progression, the sum of the reciprocals of the extremes = twice the reciprocal of the mean; or, the reciprocal of the mean = half the sum of the reciprocals of the extremes.*

As half the sum of two magnitudes = their arithmetic mean, and the product of two magnitudes = the square of their geometric mean; the group of equalities (I) or (IV) shew that—

4°. *The product of the arithmetic and harmonic means of two magnitudes = the square of their geometric mean.*

As three magnitudes are in geometric progression when the product of the first and third = the square of the second; it appears, from 4°, that—

5°. *The arithmetic, geometric, and harmonic means of two magnitudes are in geometric progression.*

As three magnitudes are in arithmetic progression when the sum of the first and third = twice the second, or, the second = half the sum of the first and third; it appears, from 3°, that—

6°. *When three or any number of magnitudes are in harmonic progression, their reciprocals are in arithmetic progression.*

Between the three kinds of progression, arithmetic, geometric, and harmonic, the following relation appears from 1°—

7°. *For every three consecutive terms a, b, c, the difference $(a - b)$: the difference $(b - c)$, in arithmetic progression :: a : a, in geometric progression :: a : b, and in harmonic progression :: a : c.*

An extension of the term *harmonic mean* from two to any number of magnitudes, by the same kind of analogy by which the terms arithmetic mean and geometric mean have been similarly extended, has been suggested by 3°.

8°. As, for two magnitudes a, b, we say that—

Arithmetic mean = half of $(a+b)$,

Geometric mean = square root of $(a \times b)$,

Harmonic mean = reciprocal of $\frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}\right)$.

So, by analogy, for n magnitudes a, b, c, d, &c., we say that—

Arithmetic mean = n^{th} part of $(a+b+c+d+\&c.)$,

Geometric mean = n^{th} root of $(a \times b \times c \times d \times \&c.)$,

Harmonic mean = reciprocal of $\frac{1}{n}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+\&c.\right)$.

The harmonic mean of any number of magnitudes thus signifying the magnitude whose reciprocal = the arithmetic mean of the reciprocals of the magnitudes.

234. More generally (231) the harmonic relation of a system of four points on a common axis A, B, X, Y, may be expressed in terms of their four distances from any arbitrary point P on the axis of the system, as follows:—

In the fundamental proportion of harmonic section (213) substituting for AX, AY, BX, and BY their equivalents $PX-PA$, $PY-PA$, $PX-PB$, and $PY-PB$, the result

$$(PX-PA):(PX-PB)+(PY-PA):(PY-PB)=0 \ldots (1),$$

or, which is the same thing,

$$(PX-PA).(PY-PB)+(PY-PA).(PX-PB)=0 \ldots (1'),$$

expresses the relation in terms of the four distances in question, and may, like any of the preceding, be regarded as characteristic of an harmonic system, and sufficient to determine it.

Dividing both terms of the proportion (1) by the ratio $PA:PB$, or of the equality (1′) by the product $PA.PB.PX.PY$, the resulting proportion

$$\left(\frac{1}{PX}-\frac{1}{PA}\right):\left(\frac{1}{PX}-\frac{1}{PB}\right)+\left(\frac{1}{PY}-\frac{1}{PA}\right):\left(\frac{1}{PY}-\frac{1}{PB}\right)=0 \ldots (2),$$

or the resulting equality

$$\left(\frac{1}{PX}-\frac{1}{PA}\right)\cdot\left(\frac{1}{PY}-\frac{1}{PB}\right)+\left(\frac{1}{PY}-\frac{1}{PA}\right)\cdot\left(\frac{1}{PX}-\frac{1}{PB}\right)=0 \ldots (2'),$$

expresses again the relation in terms of the reciprocals of the four distances, in precisely the same form as in terms of the distances themselves.

Cor. 1°. The first of the preceding proportions (1), or its equivalent (1'), gives at once the equality

$$2.PX.PY + 2.PA.PB = (PA + PB).(PX + PY) \ldots (3),$$

and the second (2), or its equivalent (2'), the corresponding equality

$$\frac{2}{PX.PY} + \frac{2}{PA.PB} = \left(\frac{1}{PA} + \frac{1}{PB}\right) \cdot \left(\frac{1}{PX} + \frac{1}{PY}\right) \ldots (4),$$

in which the forms again, as they ought to be, are identical.

Cor. 2°. If C and Z be the middle points of AB and XY respectively, then, as $PA + PB = 2.PC$ and $PX + PY = 2.PZ$, the first of these latter equalities (3) becomes

$$PX.PY + PA.PB = 2.PC.PZ \ldots\ldots\ldots\ldots (5),$$

a relation of considerable utility in the applications of the theory of harmonic section.

Cor. 3°. If Q and Q' be the harmonic conjugates of P with respect to AB and XY respectively, then, as

$$\frac{1}{PA} + \frac{1}{PB} = \frac{2}{PQ}, \text{ and } \frac{1}{PX} + \frac{1}{PY} = \frac{2}{PQ'}, \text{ (231, III.)},$$

the second (4) becomes

$$\frac{1}{PX.PY} + \frac{1}{PA.PB} = \frac{2}{PQ.PQ'} \ldots\ldots\ldots\ldots (6);$$

a relation again identical in form, and, as may be easily seen from (233, 6°), in meaning too, with that for the direct distances (5) to which it corresponds.

235. *If a line AB cut harmonically at two points X and Y be again cut harmonically at two other points P and Q, both pairs of conjugates being arbitrary, then*

$$PX.PY = 2PC.RZ \ldots\ldots\ldots\ldots\ldots\ldots (\alpha),$$

$$QX.QY = 2QC.RZ \ldots\ldots\ldots\ldots\ldots\ldots (\beta),$$

C, R, and Z being the middle points of the three segments AB, PQ, and XY respectively. (See fig. 1, Art. 213).

For, by (234, Cor. 2°),

$$PX.PY + PA.PB = 2PC.PZ,$$

and, by (231, Cor. 3°),

$$PA.PB = PC.PQ = 2.PC.PR,$$

therefore $PX.PY = 2.PC.(PZ - PR) = 2.PC.RZ$;

and, similarly,

$$QX.QY = 2.QC.(QZ - QR) = 2.QC.RZ,$$

and therefore &c.

Otherwise thus, by (Euc. II. 5, 6),

$$PX.PY - CX.CY = PZ^2 - CZ^2 = (PZ - CZ).(PZ + CZ)$$
$$= PC.(PZ + QZ - QC) = 2.PC.RZ - PC.QC,$$

and, by (225), $CX.CY = CP.CQ$, therefore $PX.PY = 2.PC.RZ$; and, similarly, $QX.QY = 2.QC.RZ$; and therefore &c.

The latter proof, depending only on the single consideration that the two rectangles $CX.CY$ and $CP.CQ$ are equal in magnitude and sign, shews that the relations themselves, (α) and (β), depend on that circumstance alone, and are therefore independent of the accident as to whether the two points A and B are real or imaginary.

Cor. 1°. Taking successively the sum, difference, product, and quotient of the above equalities (α) and (β), we get at once the four following relations:—

1°. Adding, remembering that $PC + QC = 2.RC$, we get

$$PX.PY + QX.QY = 4.RC.RZ \text{ .... (1).}$$

2°. Subtracting, remembering that $PC - QC = PQ$, we get

$$PX.PY - QX.QY = 2.PQ.RZ \text{ .... (2).}$$

3°. Multiplying, remembering that $4.CP.CQ = AB^2$ (225), we get

$$PX.PY \times QX.QY = AB^2.RZ^2 \text{ .... (3).}$$

4°. Dividing, we get at once, without any reduction,

$$PX.PY : QX.QY :: PC : QC \text{ ....(4),}$$

relations which, like those from which they are derived, are perfectly general, and independent alike of the position of either pair of conjugates X and Y, or P and Q, and of the accident of A and B being real or imaginary.

COR. 2°. From (α) and (β), and from (3°, Cor. 1°), we get at once the equalities

$$PC = \frac{PX.PY}{2.RZ}, \quad QC = \frac{QX.QY}{2.RZ} \ldots\ldots\ldots\ldots(5),$$

$$AB^2 = \frac{PX.PY.QX.QY}{RZ^2} \ldots\ldots\ldots\ldots\ldots\ldots(6),$$

which are the simplest formulæ by which to calculate in numbers the position and length of AB when those of PQ and XY are given; a problem for which, it will be remembered, various constructions were given in Art. 230.

COR. 3°. If, while P and Q, and therefore R, are supposed to remain fixed, X and Y, and therefore Z, be conceived to vary, and in the course of their variation to coincide all three first at A and then at B; we see, from (4°, Cor. 1°), that $PX.PY : QX.QY$ = a constant ratio, and also that

$$\frac{PX.PY}{QX.QY} = \frac{PA^2}{QA^2} = \frac{PB^2}{QB^2} = \frac{PC}{QC} \ldots\ldots\ldots\ldots(7),$$

relations which, for the particular positions of A and B, may be easily verified from the fundamental conception of harmonic section. See Arts. 150, and 161, Cor. 1°.

COR. 4°. If XY, $X'Y'$, $X''Y''$, &c. be any number of segments cutting the same segment AB harmonically, Z, Z', Z'', &c. their several middle points, P and Q as before any arbitrary pair of conjugates, and R their middle point; then since, from (α) and (β),

$$PX.PY : PX'.PY' : PX''.PY'', \text{\&c.}$$

$$= QX.QY : QX'.QY' : QX''.QY'', \text{\&c.} = RZ : RZ' : RZ'', \text{\&c.},$$

if the several distances RZ, RZ', RZ'', &c. form an arithmetic, geometric, or harmonic series, so do the two sets of rectangles $PX.PY$, $PX'.PY'$, $PX''.PY''$, &c., and $QX.QY$, $QX'.QY'$, $QX''.QY''$, &c., whatever be the positions of P and Q.

COR. 5°. If XY, $X'Y'$, $X''Y''$ be any three segments cutting the same segment AB harmonically, Z, Z', Z'' their three middle points, and P any arbitrary point on the axis of the segments; then

$$PX.PY.Z'Z'' + PX'.PY'.Z''Z + PX''.PY''.ZZ' = 0 \ldots(8),$$

a theorem due to Chasles, and made much use of by him in the theory of involution.

For, Q being the harmonic conjugate of P with respect to AB, and R the middle point of PQ, therefore by (α),

$$PX.PY=2PC.RZ,\ PX'.PY'=2PC.RZ',\ PX''.PY''=2PC.RZ'',$$

and R, Z, Z', Z'' being four points on a common axis, therefore, by (82),

$$RZ.Z'Z''+RZ'.Z''Z+RZ''.ZZ'=0;$$

and therefore &c.

This proof, it will be observed, is independent of the circumstance as to whether A and B are real or imaginary.

Cor. 6°. If POQ and XOY be two angles cutting harmonically the same angle AOB, then, all three being otherwise entirely arbitrary,

$$\frac{\sin POX.\sin POY}{\sin QOX.\sin QOY}=\frac{\sin^2 POA}{\sin^2 QOA}=\frac{\sin^2 POB}{\sin^2 QOB}\ \dots\dots (9),$$

which are the formulæ by which to calculate in numbers the positions of the sides of the angle AOB when those of the angles POQ and XOY are given.

For, if PQ, XY, and AB be the three segments intercepted by the three angles on any arbitrary line not passing through their common vertex O, then since, by (65),

$$\frac{\sin POX}{\sin QOX}=\frac{PX}{QX}:\frac{PO}{QO},\ \text{and}\ \frac{\sin POY}{\sin QOY}=\frac{PY}{QY}:\frac{PO}{QO},$$

therefore, at once, by composition of ratios,

$$\frac{\sin POX.\sin POY}{\sin QOX.\sin QOY}=\frac{PX.PY}{QX.QY}:\frac{PO^2}{QO^2};$$

and since, by the same again directly,

$$\frac{\sin^2 POA}{\sin^2 QOA}=\frac{PA^2}{QA^2}:\frac{PO^2}{QO^2},\ \text{and}\ \frac{\sin^2 POB}{\sin^2 QOB}=\frac{PB^2}{QB^2}:\frac{PO^2}{QO^2},$$

therefore &c.; the rest being evident from relation (7), Cor. 3°.

Cor. 7°. In the particular case when the angle POQ is right, that is, when the two conjugates OP and OQ are the two bisectors, internal and external, of the angle AOB (216, 3°), the sines of the several angles measured from OQ may be replaced by the cosines of the corresponding angles measured from OP, or conversely, and the above relation (9) becomes for the harmonic section of an angle what that of Art. 225 is for that of a line, viz.

$$\tan COX . \tan COY = \tan^2 COA = \tan^2 COB \ldots\ldots (10),$$

OC being either bisector, internal or external, of the angle AOB. This latter relation, however, appears more immediately from that of the article referred to, by drawing the arbitrary line in the general proof of (9) perpendicular to the direction of OC, and then dividing the relation of that article, $CX . CY = CA^2 = CB^2$, by the square of OC (60).

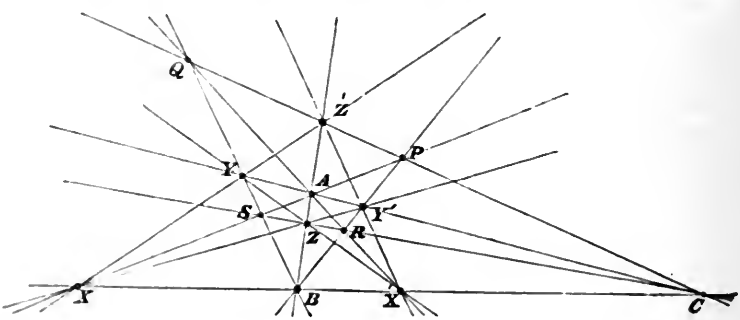
CHAPTER XIV.

HARMONIC PROPERTIES OF THE POINT AND LINE.

236. Of the various harmonic properties of figures of points and lines, the two following, reciprocals of each other (173), lead to the greatest number of consequences, and may be regarded as fundamental.

a. In every tetragram the three pairs of opposite intersections (106) *divide harmonically the three sides of the triangle determined by their three lines of connection.*

a′. In every tetrastigm the three pairs of opposite connectors (106) *divide harmonically the three angles of the triangle determined by their three points of intersection.*



To prove *a*. If X and X', Y and Y', Z and Z' be the three pairs of opposite intersections of the tetragram determined by the four lines in the figure on which they lie, three and three, and ABC the triangle determined by their three lines of connection; the three segments XX', YY', ZZ' cut harmonically and are cut harmonically by the three BC, CA, AB.

For, in the triangle XYX', having any one of the three former XX' for a side, and either extremity Y' of either of the remaining two YY' for the opposite vertex; the axis of the third ZZ' intersecting with the three sides at three collinear points Z, Z', B, and the other extremity Y' of the second connecting with the three vertices by three concurrent lines XZ, $X'Z'$, YC, therefore, by relations a and b', Art. 134,

$$\frac{XB}{X'B} = + \frac{XZ'}{YZ'} \cdot \frac{YZ}{X'Z}, \text{ and } \frac{XC}{X'C} = - \frac{XZ'}{YZ'} \cdot \frac{YZ}{X'Z},$$

which evidently (213) prove the property for the pair of segments XX' and BC; and, as it may be proved exactly similarly for the remaining two pairs YY' and CA, ZZ' and AB, therefore &c.

To prove a'. If QR and PS, RP and QS, PQ and RS be the three pairs of opposite connectors of the tetrastigm determined by the four points P, Q, R, S in the same figure, and ABC the triangle determined by their three points of intersection; the three angles PAQ, QBR, RCP cut harmonically and are cut harmonically by the three BAC, CBA, ACB.

For, in the triangle PAQ, having any one of the three former PAQ for an angle, and either side PQ of either of the remaining two RCP for the opposite side; the vertex of the third QBR connecting with the three vertices by three concurrent lines PR, QS, AB, and the other side RS of the second intersecting with the three sides at three collinear points R, S, C, therefore, by relations a' and b, Art. 134,

$$\frac{\sin PAB}{\sin QAB} = - \frac{\sin PQS}{\sin AQS} \cdot \frac{\sin APR}{\sin QPR},$$

and

$$\frac{\sin PAC}{\sin QAC} = + \frac{\sin PQS}{\sin AQS} \cdot \frac{\sin APR}{\sin QPR},$$

which evidently (213) prove the property for the pair of angles PAQ and BAC; and, as it may be proved exactly similarly for the remaining two pairs QBR and CBA, RCP and ACB, therefore &c.

237. That the two properties just established are reciprocals of each other, in the sense explained in Art. 173, may readily be shewn, in general terms, as follows:—

If L, M, N, O be any four lines, and P, Q, R, S their four poles with respect to any circle, either system being arbitrary; U and U', V and V', W and W' the three pairs of opposite intersections MN and LO, NL and MO, LM and NO of the four lines; X and X', Y and Y', Z and Z' the three pairs of opposite connectors QR and PS, RP and QS, PQ and RS of the four points; A, B, C the three vertices of the triangle determined by the three connectors UU', VV', WW'; and D, E, F the three sides of the triangle determined by the three intersections XX', YY', ZZ'; then, since, by the fundamental property of poles and polars (167), the several pairs of points and lines U and X, V and Y, W and Z; U' and X', V' and Y', W' and Z'; A and D, B and E, C and F are pole and polar to each other with respect to the circle, therefore, by the general property of Art. 223, the harmonicism of the three rows of four collinear points B, C, U, U'; C, A, V, V'; A, B, W, W' involves and is involved in that of the three pencils of four concurrent lines E, F, X, X'; F, D, Y, Y'; D, E, Z, Z'; and therefore &c.

The reader understanding the spirit of the above mode of reasoning is recommended to apply it for himself to the several other examples of pairs of reciprocal properties which will be given him in abundance in the course of the sequel. He will in general find the transformation of a property into its reciprocal to be a process almost purely mechanical, consisting ordinarily of little more than merely changing in its statement all points into lines and lines into points, all connectors of points into intersections of lines and intersections of lines into connectors of points, all points on a circle into tangents to the circle and tangents to a circle into points on the circle, &c. In cases presenting any exceptional peculiarity, or involving the necessity of any intermediate considerations, the reciprocality of the properties will occasionally be proved for him, but in all ordinary cases, like the above, the process of tracing it will be left as an exercise to himself; especially when, as in the preceding article, the demonstrations actually given of the reciprocal properties are themselves also reciprocal; a circumstance which in that article would have been rendered more apparent by the employment, as above, of corresponding notation applied to the reciprocal parts of separate figures for both properties, had not, for other reasons which will appear in the sequel, the figures for the two been combined in their case instead.

238. It is easy to see, from the general property of Art. 221, that the harmonicism of any one of the three rows of four points X, X', B, C; Y, Y', C, A; Z, Z', A, B in property *a*, or of any one of the three pencils of four rays QR, PS, AB, AC; RP, QS, BC, BA; PQ, RS, CA, CB in property *a'* of Art. 236, (see figure of that article), involves that of the other two; for, in the former case, the two rows for every two of the three connectors XX', YY', ZZ' being in perspective at both extremities of the third, viz. Y, Y', C, A and Z, Z', A, B at X and X'; Z, Z', A, B and X, X', B, C at Y and Y'; X, X', B, C and Y, Y', C, A at Z and Z' (see fig.); and, in the latter case, the two pencils for every two of the three intersections A, B, C being in perspective on both lines determining the third, viz. RP, QS, BC, BA and PQ, RS, CA, CB on QR and PS; PQ, RS, CA, CB and QR, PS, AB, AC on RP and QS; QR, PS, AB, AC and RP, QS, BC, BA on PQ and RS (see fig.); therefore, by property 2°, Art. 222, if any one of the three rows in the former case, or of the three pencils in the latter case, be harmonic, so are the other two; and therefore &c.

It is again easy to see, from the same, that of the two reciprocal properties themselves, *a* and *a'*, either involves the other directly without the aid of the reciprocating process explained in Art. 173, and applied in Art. 237. For, in the triangle ABC (see fig.), if the three sides BC, CA, AB are cut harmonically by the three pairs of conjugates X and X', Y and Y', Z and Z', then, by the general relation of Art. 221, the three opposite angles BAC, CBA, ACB are cut harmonically by the three pairs of conjugates QR and PS, RP and QS, PQ and RS; and, conversely, if the three angles BAC, CBA, ACB are cut harmonically by the three pairs of conjugates QR and PS, RP and QS, PQ and RS, then, by the same, the three opposite sides BC, CA, AB are cut harmonically by the three pairs of conjugates X and X', Y and Y', Z and Z'; and therefore &c.

A tetragram and tetrastigm, related as in the figure to the same central triangle ABC, possess many interesting harmonic properties in connexion with each other and the triangle, some of which will be noticed in the course of the sequel.

239. In the particular cases, when, in property *a* of Art. 236, one of the four lines $X'Y'Z'$ constituting the tetragram in the general case is the line at infinity (131), and when, in property a' of the same article, one of the four points S constituting the tetrastigm in the general case is the polar centre of the triangle PQR determined by the remaining three (168); since, in the former case, the three pairs of harmonic conjugates X and X', Y and Y', Z and Z' connect by infinite intervals, they bisect, internally and externally, the three sides BC, CA, AB of the triangle ABC determined by their three lines of connection (216, 3°); and since, in the latter case, the three pairs of harmonic conjugates QR and PS, RP and QS, PQ and RS intersect at right angles, they bisect, externally and internally, the three angles BAC, CBA, ACB of the triangle ABC determined by their three points of intersection (216, 3°); hence, the two reciprocal properties themselves, *a* and a', shew for these particular cases, as is otherwise evident, that—

a. In every triangle the three vertices bisect the three sides of the triangle determined by the directions of the three parallels through them to the opposite sides.

a'. In every triangle the three sides bisect the three angles of the triangle determined by the intersections of the three perpendiculars to them through the opposite vertices.

These latter properties are not reciprocals in the same sense as those from which they have been inferred; each, to an arbitrary circle, reciprocating, not into the other, but into the more general property of which the other is a particular case. In reciprocating the first, the line at infinity (136), on which the three parallels through the vertices intersect with the opposite sides of the triangle, must be taken into account, with the latter, in order to complete the tetragram of the general property, under which, as above shewn, it comes as a particular case.

In the particular case when the tetragram in property *a* of Art. 236 is a parallelogram; since then one chord of intersection, XX' suppose (see figure of that article), of the figure, and with it, of course, the side BC of the triangle ABC, is at infinity; therefore, by virtue of that property, the other two chords of intersection YY' and ZZ' mutually bisect each other at the opposite vertex A of the triangle ABC. Hence the familiar property that *in every parallelogram the two diagonals mutually bisect each other*, comes as another particular case under the same general property a; and, to an arbitrary circle, reciprocates (like the above a) into the general property a' reciprocal to a.

240. From the two fundamental properties of Art. 236, the following general consequences, in pairs reciprocals of each other, may be immediately inferred, viz.—

a. The two centres of perspective of any two segments (131) *divide harmonically the segment intercepted on their line of connexion by the axes of the segments.*

a'. The two axes of perspective of any two angles (131) *divide harmonically the angle subtended at their point of intersection by the vertices of the angles.*

For, if, in the figure of that article, any two of the three pairs of opposite intersections X and X', Y and Y' of the tetragram be regarded as the extremities of the two segments in a; then are the remaining pair Z and Z' the two centres of perspective (131) of those segments, and, by property a of the article in question, they divide harmonically the segment AB intercepted on their line of connection by the axes of the segments; and therefore &c. And, if, in the same figure, any two of the three pairs of opposite connectors QR and PS, RP and QS of the tetrastigm be regarded as the sides of the two angles in a'; then are the remaining pair PQ and RS the two axes of perspective (131) of those angles, and, by property a' of the same article, they divide harmonically the angle ACB subtended at their point of intersection by the vertices of the angles; and therefore &c.

b. The two centres of perspective of any two segments connect harmonically with the vertex of the angle determined by the axes of the segments.

b'. The two axes of perspective of any two angles intersect harmonically with the axis of the segment determined by the vertices of the angles.

For, if, in the figure of the same article, any pair of opposite connectors QR and PS of the tetrastigm be regarded as the two segments in b; then are the two intersections B and C of the other two pairs RP and QS, PQ and RS the two centres of perspective of those segments, and, by property a' of the article in question, they connect harmonically with the vertex A of the angle determined by the axes of the segments; and therefore &c. And, if, in the same figure, the two pairs of lines determining any pair of opposite intersections X and X' of the tetragram be regarded as the two angles in b'; then are the two connectors YY' and ZZ' of the other two pairs Y and Y', Z and Z' the two axes of perspective of those angles, and, by property a of the same article, they intersect harmonically with the axis BC of the segment XX' determined by the vertices of the angles; and therefore &c.

c. *The two centres of perspective of any two segments are conjugate points* (217) *with respect to the angle determined by the axes of the segments.*

c'. *The two axes of perspective of any two angles are conjugate lines* (217) *with respect to the segment determined by the vertices of the angles.*

These, by Art. 217, are obviously but another mode of stating the two general properties b and b'; which, though proved independently above by reciprocal demonstrations, follow at once, it may be observed, from the two a and a', by virtue of the general property of Art. 221. For, since, by a, the three rows of four points B, C, X, X'; C, A, Y, Y'; A, B, Z, Z' (see fig.) are harmonic, therefore, by the general property in question, the three pencils of four rays ZZ', YY', AX, AX'; XX', ZZ', BY, BY'; YY', XX', CZ, CZ' are harmonic, and therefore &c. And, since, by a', the three pencils of four rays AB, AC, QR, PS; BC, BA, RP, QS; AB, AC, PQ, RS (see fig.) are harmonic, therefore, by the same general property, the three rows of four points B, C, X', X; C, A, Y', Y; A, B, Z', Z are harmonic, and therefore &c.

241. The two fundamental properties of Art. 236 supply also obvious solutions, by linear constructions only without the aid of the circle, of the two following reciprocal problems, viz.—

a. Given three points of an harmonic row, to determine the fourth conjugate to any assigned one of the given three.

a′. Given three rays of an harmonic pencil, to determine the fourth conjugate to any assigned one of the given three.

Thus, in the figure of that article, of the harmonic row B, C, X, X' given the three points X, X', C to determine the fourth B conjugate to C; and, in the same figure, of the harmonic pencil AB, AC, AX, AX' given the three rays AX, AX', AB to determine the fourth AC conjugate to AB.

To solve the first; on any line CA, drawn arbitrarily through the point C whose conjugate is to be determined, taking arbitrarily any two points Y and Y'; their connectors with the other two points X and X' determine the two centres of perspective Z and Z' of the two segments XX' and YY', whose line of connection ZZ', by *a*, Art. 236, intersects with the axis of the given points at the required conjugate B. And, to solve the second, through any point B, taken arbitrarily on the ray AB whose conjugate is to be determined, drawing arbitrarily any two lines BY and BY'; their intersections with the other two rays AX and AX' determine the two axes of perspective PQ and RS of the two angles XAX' and YBY', whose point of intersection C, by *a′*, Art. 236, connects with the vertex of the given rays by the required conjugate AC.

COR. Every point of an harmonic row being the pole of every line through its conjugate with respect to the segment determined by the remaining two points, and every ray of an harmonic pencil being the polar of every point on its conjugate with respect to the angle determined by the remaining two rays (217); the above reciprocal constructions give, consequently, solutions of the two following reciprocal problems, as well as of those for which they have been given, viz.—

To determine by linear constructions only without the aid of the circle: a. the pole of a given line with respect to a given segment; a′. the polar of a given point with respect to a given angle.

242. The two fundamental properties themselves, of Art. 236, may obviously be stated in the following equivalent forms, in which they express two reciprocal harmonic properties of triangles, viz.—

a. Every three collinear points on the sides of a triangle determine with the opposite vertices three segments dividing harmonically the sides of the triangle determined by their axes.

a'. Every three concurrent lines through the vertices of a triangle determine with the opposite sides three angles dividing harmonically the angles of the triangle determined by their vertices.

For, if (fig. of Art. 236) XYZ be any triangle, and X', Y', Z' any three collinear points on its three sides; then, since, in the tetragram determined by the line of collinearity with the three sides of the triangle, the three segments XX', YY', ZZ', by property *a* of that article, are intersected harmonically each by the axes of the other two, therefore &c. And if (same fig.) PQR be any triangle, and PS, QS, RS any three concurrent lines through its three vertices; then, since, in the tetrastigm determined by the point of concurrence with the three vertices of the triangle, the three angles XAX', YBY', ZCZ', by property *a'* of the same article, are subtended harmonically each by the vertices of the other two, therefore &c.

Cor. 1°. As every three lines through the vertices of a triangle which intersect collinearly with the opposite sides determine an exscribed triangle in perspective with it, and as every three points on the sides of a triangle which connect concurrently with the opposite vertices, determine an inscribed triangle in perspective with it (141); it appears consequently, from the above reciprocal properties, or from those of Cors. 1° and 2°, Art. 139, with which they are evidently identical, that—

a. When a triangle exscribed to another is in perspective with it, its sides are cut harmonically by the corresponding vertices and sides of the other.

a'. When a triangle inscribed to another is in perspective with it, its angles are cut harmonically by the corresponding sides and vertices of the other.

Cor. 2°. Since, for every two triangles in perspective, the three pairs of corresponding vertices connect through the centre of perspective, and the three pairs of corresponding sides intersect on the axis of perspective (140); it follows consequently, from the two reciprocal properties of the preceding corollary, that—

a. When a triangle exscribed to another is in perspective with *it, its sides are the polars of the centre of perspective with respect to the corresponding angles of the other.*

a'. When a triangle inscribed to another is in perspective with it, its vertices are the poles of the axis of perspective with respect to the corresponding sides of the other.

243. Of the various other harmonic properties of triangles, the following, in pairs reciprocals of each other, result immediately from the four general relations of Art. 134.

a. When three points on the three sides of a triangle are collinear, their three harmonic conjugates with respect to the sides connect concurrently with the opposite vertices; and conversely.

a'. When three lines through the three vertices of a triangle are concurrent, their three harmonic conjugates with respect to the angles intersect collinearly with the opposite sides; and conversely.

For, in the case of *a*, if A, B, C be the three vertices of the triangle; X, Y, Z *any* three points on its three opposite sides; and X', Y', Z' their three harmonic conjugates with respect to the three segments BC, CA, AB respectively; then, since, by the definition of harmonic section (213),

$$\frac{BX}{CX}+\frac{BX'}{CX'}=0,\quad \frac{CY}{AY}+\frac{CY'}{AY'}=0,\quad \frac{AZ}{BZ}+\frac{AZ'}{BZ'}=0,$$

therefore at once, by composition of ratios,

$$\frac{BX}{CX}\cdot\frac{CY}{AY}\cdot\frac{AZ}{BZ}+\frac{BX'}{CX'}\cdot\frac{CY'}{AY'}\cdot\frac{AZ'}{BZ'}=0,$$

consequently, when either compound $=\pm 1$, the other then $=\mp 1$, and therefore &c. (134). And, in the case of a', if A, B, C be the three sides of the triangle; X, Y, Z *any* three lines through its three opposite vertices; and X', Y', Z' their three harmonic conjugates with respect to the three angles BC, CA, AB respectively; then, since, by the definition of harmonic section (213),

$$\frac{\sin BX}{\sin CX}+\frac{\sin BX'}{\sin CX'}=0,\quad \frac{\sin CY}{\sin AY}+\frac{\sin CY'}{\sin AY'}=0,$$

$$\frac{\sin AZ}{\sin BZ}+\frac{\sin AZ'}{\sin BZ'}=0,$$

therefore at once, by composition of ratios,

$$\frac{\sin BX}{\sin CX}\cdot\frac{\sin CY}{\sin AY}\cdot\frac{\sin AZ}{\sin BZ}+\frac{\sin BX'}{\sin CX'}\cdot\frac{\sin CY'}{\sin AY'}\cdot\frac{\sin AZ'}{\sin BZ'}=0,$$

consequently, when either compound $=\mp 1$, the other then $=\pm 1$, and therefore &c. (134).

b. When three points on the three sides of a triangle are collinear, their three harmonic conjugates with respect to the sides determine with them the three pairs of opposite intersections of a tetragram (106).

b'. When three lines through the three vertices of a triangle are concurrent, their three harmonic conjugates with respect to the angles determine with them the three pairs of opposite connectors of a tetrastigm (106).

For, in the case of b, employing the same notation as in the proof of a, the four compounds

$$\frac{BX}{CX}\cdot\frac{CY}{AY}\cdot\frac{AZ}{BZ},\quad \frac{CY'}{AY'}\cdot\frac{AZ'}{BZ'}\cdot\frac{BX}{CX},$$

$$\frac{AZ'}{BZ'}\cdot\frac{BX'}{CX'}\cdot\frac{CY}{AY},\quad \frac{BX'}{CX'}\cdot\frac{CY'}{AY'}\cdot\frac{AZ}{BZ},$$

being always equal in magnitude and sign, when any one of the four $=+1$, the remaining three each $=+1$, that is (134, a) when any one of the four groups of three points X, Y, Z; Y', Z', X; Z', X', Y; X', Y', Z is collinear, the remaining three are also collinear, and, the four lines of collinearity consequently determining a tetragram of which X and X', Y and Y', Z and Z' are the three pairs of opposite intersections (106), therefore &c. And, in the case of b', employing the same notation as in the proof of a', the four compounds

$$\frac{\sin BX}{\sin CX}\cdot\frac{\sin CY}{\sin AY}\cdot\frac{\sin AZ}{\sin BZ},\quad \frac{\sin CY'}{\sin AY'}\cdot\frac{\sin AZ'}{\sin BZ'}\cdot\frac{\sin BX}{\sin CX},$$

$$\frac{\sin AZ'}{\sin BZ'}\cdot\frac{\sin BX'}{\sin CX'}\cdot\frac{\sin CY}{\sin AY},\quad \frac{\sin BX'}{\sin CX'}\cdot\frac{\sin CY'}{\sin AY'}\cdot\frac{\sin AZ}{\sin BZ},$$

being always equal in magnitude and sign, when any one of the four $=-1$, the remaining three each $=-1$, that is (134, a') when any one of the four groups of three lines X, Y, Z; Y', Z', X; Z', X', Y; X', Y', Z is concurrent, the remaining three are also concurrent, and the four points of concurrence consequently determining a tetrastigm of which X and X', Y and Y', Z and Z' are the three pairs of opposite connectors (106), therefore &c.

c. When three points on the three sides of a triangle are collinear, their three polars (217) *with respect to the three opposite angles are concurrent.*

c'. When three lines through the three vertices of a triangle are concurrent, their three poles (217) *with respect to the three opposite sides are collinear.*

These two reciprocal properties follow at once from the two *a* and *a'*, by virtue of the general property of Art. 221; the harmonic conjugate of each point in *c* with respect to its own side connecting with the opposite angle by the polar of the point with respect to that angle; and, the harmonic conjugate of each line in *c'* with respect to its own angle intersecting with the opposite side at the pole of the line with respect to that side.

d. The three poles of any line with respect to the three sides of a triangle connect concurrently with the opposite vertices.

d'. The three polars of any point with respect to the three angles of a triangle intersect collinearly with the opposite sides.

These two reciprocal properties are obviously identical with the two *a* and *a'*; the three poles of the line with respect to the three sides in *d* being the three harmonic conjugates of its three points of intersection with the sides; and the three polars of the point with respect to the three angles in *d'* being the three harmonic conjugates of its three lines of connection with the vertices (217).

In the particular cases when the line in *d* is the line at infinity, and the point in *d'* any point at infinity; since, in the former case, the three poles of the line at infinity with respect to the three sides are the three middle points of the sides (216, 3°); and since, in the latter case, the three polars of the point at infinity with respect to the three angles bisect internally the three segments intercepted by the angles on any line passing through the direction of the point (224); the two reciprocal properties *d* and *d'* become, consequently, those already established on other principles in examples (1° and 13°, Art. 137), viz.—

In every triangle, α. the three middle points of the sides connect concurrently with the opposite vertices; α′. the three lines connecting the vertices with the middle points of the segments intercepted by the corresponding angles on any line, intersect collinearly with the opposite sides.

Of the several pairs of reciprocal properties established in this article, it may be observed that either reciprocal would follow directly from the other by virtue of the general property of Art. 121; from which it follows, evidently, for a triangle, that every two points harmonic conjugates with respect to any side connect harmonically with the opposite angle, and, that every two lines harmonic conjugates with respect to any angle intersect harmonically with the opposite side.

244. From the fundamental properties of Art. 236, combined with the two a and a' of the preceding article, the two following reciprocal properties of the tetragram and tetrastigm may be readily inferred, viz.—

a. In every tetragram, the three pairs of opposite intersections connect with the opposite vertices of the triangle determined by their three lines of connection by six lines passing three and three through four points, and thus determining the three pairs of opposite connectors of a tetrastigm.

a′. In every tetrastigm, the three pairs of opposite connectors intersect with the opposite sides of the triangle determined by their three points of intersection at six points lying three and three on four lines, and thus determining the three pairs of opposite intersections of a tetragram.

For, in the case of the tetragram, the three pairs of opposite intersections X and X', Y and Y', Z and Z' (fig. Art. 236), dividing harmonically, by (236, a), the three sides of the triangle ABC determined by their three lines of connection, and lying, by hypothesis, three and three on four lines YZX', ZXY', XYZ', $X'Y'Z'$; therefore, by (243, a), they connect with the opposite vertices A, B, C by three pairs of lines AX and AX', BY and BY', CZ and CZ' passing three and three through four points P, Q, R, S; and therefore &c. And, in the case of the tetrastigm, the three pairs of opposite connectors *QR* and *PS*, *RP* and *QS*, *PQ* and *RS* (same fig.) dividing harmonically, by (236, *a'*), the three angles of the triangle *ABC* determined by their three points of intersection, and passing, by hypothesis, three and three through four points *P*, *Q*, *R*, *S*; therefore, by (243, *a'*), they intersect with the opposite sides *BC*, *CA*, *AB* at three pairs of points *X* and *X'*, *Y* and *Y'*, *Z* and *Z'* lying three and three on four lines *YZX'*, *ZXY'*, *XYZ'*, *X'Y'Z'*; and therefore &c.

It will be seen in the sequel that the four lines *YZX'*, *ZXY'*, *XYZ'*, *X'Y'Z'* and the four points *P*, *Q*, *R*, *S*, related as above to each other, possess also several other reciprocal harmonic relations in connection with the triangle *ABC*.

245. From the same fundamental relations, combined with the two (7) and (9) of Art. 235, two other important reciprocal properties of the tetragram and tetrastigm may again be readily inferred, viz.—

a. In the triangle determined in a tetragram by the axes of the three chords of intersection of the figure (107), *when three points on the sides are either collinear or concurrently connectant with the opposite vertices, their three harmonic conjugates with respect to the three chords of intersection are also either collinear or concurrently connectant with the opposite vertices.*

a'. In the triangle determined in a tetrastigm by the vertices of the three angles of connection of the figure (107), *when three lines through the vertices are either concurrent or collinearly intersectant with the opposite sides, their three harmonic conjugates with respect to the three angles of connection are also either concurrent or collinearly intersectant with the opposite sides.*

To prove *a*. If, as in the figure of Art. 236, *X* and *X'*, *Y* and *Y'*, *Z* and *Z'* be the three pairs of opposite intersections of the tetragram; *ABC* the triangle determined by their three lines of connection; *U*, *V*, *W* *any* three points on its three sides *BC*, *CA*, *AB*; and *U'*, *V'*, *W'* their three harmonic conjugates with respect to the three chords of intersection *XX'*, *YY'*, *ZZ'* of the figure; then since, by hypothesis and (236, *a*), the three latter segments are cut harmonically at once by the three *UU'*,

VV', WW', and also by the three BC, CA, AB, therefore by (7) Art. 235,

$$\frac{BU.BU'}{CU.CU'} = \frac{BX^2}{CX^2} = \frac{BX'^2}{CX'^2},$$

$$\frac{CV.CV'}{AV.AV'} = \frac{CY^2}{AY^2} = \frac{CY'^2}{AY'^2},$$

$$\frac{AW.AW'}{BW.BW'} = \frac{AZ^2}{BZ^2} = \frac{AZ'^2}{BZ'^2},$$

and since, by a and b', Art. 134, the two compounds

$$\frac{BX^2}{CX^2}\cdot\frac{CY^2}{AY^2}\cdot\frac{AZ^2}{BZ^2} \text{ and } \frac{BX'^2}{CX'^2}\cdot\frac{CY'^2}{AY'^2}\cdot\frac{AZ'^2}{BZ'^2}$$

both $= +1$, therefore the compound

$$\frac{BU.BU'}{CU.CU'}\cdot\frac{CV.CV'}{AV.AV'}\cdot\frac{AW.AW'}{BW.BW'} = +1 \text{ .... } (\alpha),$$

consequently, when either of the two compounds

$$\frac{BU}{CU}\cdot\frac{CV}{AV}\cdot\frac{AW}{BW} \text{ or } \frac{BU'}{CU'}\cdot\frac{CV'}{AV'}\cdot\frac{AW'}{BW'}$$

$= \pm 1$, the other also $= \pm 1$, and therefore &c. (134, a and b').

To prove a'. If, as in the same figure, QR and PS, RP and QS, PQ and RS be the three pairs of opposite connectors of the tetrastigm; ABC the triangle determined by their three points of intersection; AU, BV, CW *any* three lines through its three vertices A, B, C; and AU', BV', CW' their three harmonic conjugates with respect to the three angles of connection XAX', YBY', ZCZ' of the figure; then since, by hypothesis and (236, a'), the three latter angles are cut harmonically at once by the three UAU', VBV', WCW', and also by the three BAC, CBA, ACB, therefore by (9) Art. 235,

$$\frac{\sin BAU.\sin BAU'}{\sin CAU.\sin CAU'} = \frac{\sin^2 BAX}{\sin^2 CAX} = \frac{\sin^2 BAX'}{\sin^2 CAX'},$$

$$\frac{\sin CBV.\sin CBV'}{\sin ABV.\sin ABV'} = \frac{\sin^2 CBY}{\sin^2 ABY} = \frac{\sin^2 CBY'}{\sin^2 ABY'},$$

$$\frac{\sin ACW.\sin ACW'}{\sin BCW.\sin BCW'} = \frac{\sin^2 ACZ}{\sin^2 BCZ} = \frac{\sin^2 ACZ'}{\sin^2 BCZ'},$$

and since, by b and a', Art. 134, the two compounds

$$\frac{\sin^2 BAX}{\sin^2 CAX} \cdot \frac{\sin^2 CBY}{\sin^2 ABY} \cdot \frac{\sin^2 ACZ}{\sin^2 BCZ}$$

and

$$\frac{\sin^2 BAX'}{\sin^2 CAX'} \cdot \frac{\sin^2 CBY'}{\sin^2 ABY'} \cdot \frac{\sin^2 ACZ'}{\sin^2 BCZ'}$$

both $= +1$, therefore the compound

$$\frac{\sin BAU . \sin BAU'}{\sin CAU . \sin CAU'} \cdot \frac{\sin CBV . \sin CBV'}{\sin ABV . \sin ABV'} \cdot \frac{\sin ACW . \sin ACW'}{\sin BCW . \sin BCW'}$$
$$= +1 \ldots\ldots\ldots\ldots (\alpha'),$$

consequently, when either of the two compounds

$$\frac{\sin BAU}{\sin CAU} \cdot \frac{\sin CBV}{\sin ABV} \cdot \frac{\sin ACW}{\sin BCW}$$

or

$$\frac{\sin BAU'}{\sin CAU'} \cdot \frac{\sin CBV'}{\sin ABV'} \cdot \frac{\sin ACW'}{\sin BCW'}$$

$= \mp 1$, the other also $= \mp 1$, and therefore &c. (134, b and a').

COR. 1°. Since, by (217), every two lines which intersect with the axes of any number of segments harmonically are conjugate lines with respect to all the segments; and, every two points which connect with the vertices of any number of angles harmonically are conjugate points with respect to all the angles; it follows, consequently, from the first (and more important) parts of the above properties a and a', that—

b. Every two lines conjugates to each other with respect to two of the three chords of intersection of a tetragram are conjugates to each other with respect to the third also.

b'. Every two points conjugates to each other with respect to two of the three angles of connection of a tetrastigm are conjugates to each other with respect to the third also.

Every two lines I and I', thus conjugates to each other with respect to the three chords of intersection of a tetragram, are said to be *conjugate lines with respect to the tetragram*; and, every two points O and O', thus conjugates to each other with respect to the three angles of connection of a tetrastigm, are said to be *conjugate points with respect to the tetrastigm.* Every two conjugates in both cases are evidently interchangeable.

COR. 2°. Since, by the general property of Art. 221, every two lines I and I' conjugates to each other with respect to a tetragram (Cor. 1°, *b*) divide harmonically the three angles subtended at their point of intersection II' by the three chords of intersection XX', YY', ZZ' of the figure; and, every two points O and O' conjugates to each other with respect to a tetrastigm (Cor. 1°, *b'*) divide harmonically the three segments intercepted on their line of connection OO' by the three angles of connection XAX', YBY', ZCZ' of the figure; hence, from the same, the reciprocal properties that—

c. Every two conjugate lines with respect to a tetragram divide harmonically the three angles subtended at their point of intersection by the three chords of intersection of the figure.

c'. Every two conjugate points with respect to a tetrastigm divide harmonically the three segments intercepted on their line of connection by the three angles of connection of the figure.

Cor. 3°. Since again, by the first parts of *a* and *a'*, combined with the general property of Art. 221, the two lines, real or imaginary (230), which divide harmonically two of the three angles subtended at any arbitrary point O by the three chords of intersection XX', YY', ZZ' of a tetragram (107) divide harmonically the third also; and, the two points, real or imaginary (230), which divide harmonically two of the three segments intercepted on any arbitrary line I by the three angles of connection XAX', YBY', ZCZ' of a tetrastigm (107) divide harmonically the third also; hence, from the same again, the reciprocal properties that—

d. The three angles subtended at any point by the three chords of intersection of a tetragram have a common angle of harmonic section, real or imaginary.

d'. The three segments intercepted on any line by the three angles of connection of a tetrastigm have a common segment of harmonic section, real or imaginary.

Cor. 4°. Every harmonic pencil of rays, whatever be its vertex, determining an harmonic row of points on every axis, and every harmonic row of points, whatever be its axis, determining an harmonic pencil of rays at every vertex (221); it appears consequently, from the two reciprocal properties of the preceding corollary (3°) applied to the particular cases when the point in *d* and the line in *d'* are at infinity, that—

e. The three segments determined on any axis, by the three pairs of perpendiculars, or any other isoclinals, through the three pairs of opposite intersections of any tetragram, have a common segment of harmonic section, real or imaginary.

e'. The three angles determined at any vertex, by the three pairs of parallels, or any other isoclinals, to the three pairs of opposite connectors of any tetrastigm, have a common angle of harmonic section, real or imaginary.

COR. 5°. Every two points harmonic conjugates to each other with respect to any segment being each the pole of every line through the other with respect to the segment (217), and, every two lines harmonic conjugates to each other with respect to any angle being each the polar of every point on the other with respect to the angle (217); the first parts of the original properties *a* and *a'* may consequently be stated otherwise thus as follows—

f. In every tetragram, the three poles of any line with respect to the three chords of intersection are collinear.

f'. In every tetrastigm, the three polars of any point with respect to the three angles of connection are concurrent.

COR. 6°. In the particular cases where the line in *f* (Cor. 5°) is the line at infinity, and the point in *f'* (same Cor.) any point at infinity; since, in the former case, the three poles of the line at infinity with respect to the three chords are the three middle points of the chords (216, 3°), and since, in the latter case, the three polars of the point at infinity with respect to the three angles bisect internally the three segments intercepted by the angles on any line passing through the direction of the point (224); from the properties themselves (*f* and *f'*, Cor. 5°) applied to those cases, it appears, consequently, that—

g. In every tetragram, the three middle points of the three chords of intersection are collinear.

g'. In every tetrastigm, the three lines connecting the vertices of the three angles of connection with the middle points of the three segments they intercept on any arbitrary line are concurrent.

Of these properties the first (*g*), it will be observed, is identical with that already established on other principles in (189, Cor. 2°).

COR. 7°. In the particular cases, when, in the original property a, one of the four lines $X'Y'Z'$ constituting the tetragram is the line at infinity (136), and, in the original property a', one of the four points S constituting the tetrastigm is the polar centre of the triangle determined by the remaining three P, Q, R (168); since, in the former case, the three pairs of harmonic conjugates X and X', Y and Y', Z and Z', connecting by infinite intervals, bisect, internally and externally, at once the three segments UU', VV', WW' and the three BC, CA, AB (216, 3°); and since, in the latter case, the three pairs of harmonic conjugates QR and PS, RP and QS, PQ and RS, intersecting at right angles, bisect, internally and externally, at once the three angles UAU', VBV', WCW' and the three BAC, CBA, ACB (216, 3°); hence the two reciprocal properties themselves (a and a') shew, for these particular cases, that—

h. When three points on the sides of a triangle are either collinear or concurrently connectant with the opposite vertices, the conjugate three equally distant from the bisections of the sides are also either collinear or concurrently connectant with the opposite vertices.

h'. When three lines through the vertices of a triangle are either concurrent or collinearly intersectant with the opposite sides, the conjugate three equally inclined to the bisectors of the angles are also either concurrent or collinearly intersectant with the opposite sides.

Properties which, it will be remembered, have been already established, on other principles, in Examples 11° and 12°, Art. 137.

246. The two following reciprocal properties are evident from the fundamental relation of harmonic section (214), combined with the general property of Art. 221, viz.—

a. If on a variable line L, turning round a fixed point O, and intersecting with two fixed lines A and B at two variable points X and Y, a variable point P be taken so as to satisfy in every position the relation

$$\frac{PX}{OX} + \frac{PY}{OY} = 0;$$

the point P moves on a fixed line I, passing through the intersection of A and B; the polar, viz., of the point O with respect to the angle AB (217).

a′. If through a variable point P, moving on a fixed line I, and connecting with two fixed points A and B by two variable lines U and V, a variable line L be drawn so as to satisfy in every position the relation

$$\frac{\sin LU}{\sin IU} + \frac{\sin LV}{\sin IV} = 0;$$

the line L turns round a fixed point O, lying on the connector of A and B; the pole, viz. of the line I with respect to the segment AB (217).

For, in the case of *a*, the two points *O* and *P*, being harmonic conjugates (214) with respect to the two *X* and *Y*, connect harmonically (221) with the vertex of the angle *AB*; and in the case of *a′*, the two lines *I* and *L*, being harmonic conjugates (214) with respect to the two *U* and *V*, intersect harmonically (221) with the axis of the segment *AB*; and therefore &c.

247. The two reciprocal properties of the preceding article are evidently particular cases of the two following, viz.—

a. If on a variable line L, turning round a fixed point O, and intersecting with two fixed lines A and B at two variable points X and Y, a variable point P be taken so as to satisfy in every position the relation

$$a.\frac{PX}{OX} + b.\frac{PY}{OY} = 0,$$

a and b being any two finite multiples, positive or negative; the point P moves on a fixed line I, passing through the intersection of A and B, and termed the polar of the point O with respect to the two lines A and B for the two multiples a and b.

a′. If through a variable point P, moving on a fixed line I, and connecting with two fixed points A and B by two variable lines U and V, a variable line L be drawn so as to satisfy in every position the relation

$$a.\frac{\sin LU}{\sin IU} + b.\frac{\sin LV}{\sin IV} = 0,$$

a and b being any two finite multiples, positive or negative; the line L turns round a fixed point O, lying on the connector of A and B, and termed the pole of the line I with respect to the two points A and B for the two multiples a and b.

To prove which. In the case of a, if OA and OB, PA and PB be the four perpendiculars from O and P upon A and B; then since, by (Euc. VI. 4),

$$\frac{PX}{OX}=\frac{PA}{OA}, \text{ and } \frac{PY}{OY}=\frac{PB}{OB},$$

therefore, by the relation determining the position of P on L,

$$a.\frac{PA}{OA}+b.\frac{PB}{OB}=0,$$

from which, the ratio of PA to PB being constant, it follows, consequently, that P lies on the line I which divides the angle AB into segments whose sines are in the constant ratio (61); and therefore &c. And, in the case of a', if AI and BI, AL and BL be the four perpendiculars from A and B upon I and L; then since, by (61),

$$\frac{\sin LU}{\sin IU}=\frac{AL}{AI}, \text{ and } \frac{\sin LV}{\sin IV}=\frac{BL}{BI},$$

therefore, by the relation determining the direction of L through P,

$$a.\frac{AL}{AI}+b.\frac{BL}{BI}=0,$$

from which, the ratio of AL to BL being constant, it follows, consequently, that L passes through the point O which divides the interval AB into segments in the constant ratio (Euc. VI. 4); and therefore &c.

COR. 1°. From the relations in properties a and a' above, it is evident, by mere inversion of ratios, that—

b. When two points P and Q are such that one of them P lies on the polar of the other Q with respect to two lines A and B for two multiples a and b; then the latter Q lies on the polar of the former P with respect to the two lines for the reciprocals of the two multiples.

b'. When two lines L and M are such that one of them L passes through the pole of the other M with respect to two points A and B for two multiples a and b; then the latter M passes through the pole of the former L with respect to the two points for the reciprocals of the two multiples.

For, the relations of condition that the first parts be true, viz.—

$$a.\frac{PA}{QA}+b.\frac{PB}{QB}=0, \text{ and } a.\frac{AL}{AM}+b.\frac{BL}{BM}=0,$$

give immediately, by inversion of the two ratios in each, the relations

$$\frac{1}{a}\cdot\frac{QA}{PA}+\frac{1}{b}\cdot\frac{QB}{PB}=0, \text{ and } \frac{1}{a}\cdot\frac{AM}{AL}+\frac{1}{b}\cdot\frac{BM}{BL}=0,$$

which are the relations of condition that the second parts be true; and therefore &c.

COR. 2°. From the same relations again, it is evident, by mere alternation of proportions, that, for two points P and Q, two lines L and M, and two multiples a and b—

c. When P lies on the polar of Q with respect to L and M for a and b; then L passes through the pole of M with respect to P and Q for a and b.

c′. When L passes through the pole of M with respect to P and Q for a and b; then P lies on the polar of Q with respect to L and M for a and b.

For, the relations of condition that both parts of each be true, viz.—

$$a\cdot\frac{PL}{QL}+b\cdot\frac{PM}{QM}=0, \text{ and } a\cdot\frac{PL}{PM}+b\cdot\frac{QL}{QM}=0,$$

are evidently identical, by mere alternation of either; and therefore &c.

COR. 3°. In the same case, if X and Y be the two points of intersection with L and M of the line PQ, and if U and V be the two lines of connection with P and Q of the point LM; then again, by mere alternation of proportions, it is evident that—

d. When P lies on the polar of Q with respect to L and M for a and b; then X lies on the polar of Y with respect to U and V for a and b.

d′. When L passes through the pole of M with respect to P and Q for a and b; then U passes through the pole of V with respect to X and Y for a and b.

For, the relations of condition that the first parts of each be true, viz.

$$a\cdot\frac{PX}{QX}+b\cdot\frac{PY}{QY}=0, \text{ and } a\cdot\frac{\sin LU}{\sin MU}+b\cdot\frac{\sin LV}{\sin MV}=0,$$

give immediately, by alternation of the proportions in each,

$$a\cdot\frac{XP}{YP}+b\cdot\frac{XQ}{YQ}=0, \text{ and } a\cdot\frac{\sin UL}{\sin VL}+b\cdot\frac{\sin UM}{\sin VM}=0,$$

which are the relations of condition that the second parts of each be true; and therefore &c.

COR. 4°. In connection with the subject of the present article, it may be readily shewn, that generally—

When four collinear points P, Q, X, Y lie on four concurrent lines L, M, U, V, or, when four concurrent lines L, M, U, V pass through four collinear points P, Q, X, Y; then, for every two finite multiples a and b, the two relations

$$a.\frac{PX}{QX}+b.\frac{PY}{QY}=0, \text{ and } a.\frac{\sin LU}{\sin MU}+b.\frac{\sin LV}{\sin MV}=0,$$

with the two equivalent relations derived from them by alternation,

$$a.\frac{XP}{YP}+b.\frac{XQ}{YQ}=0, \textit{ and } a.\frac{\sin UL}{\sin VL}+b.\frac{\sin UM}{\sin VM}=0,$$

mutually involve and are involved in each other.

For, evidently, of the two additional relations

$$a.\frac{PU}{QU}+b.\frac{PV}{QV}=0, \text{ and } a.\frac{XL}{YL}+b.\frac{XM}{YM}=0,$$

the first is equivalent to each of the first two, and the second to each of the second two, of the above; and therefore &c.

248. The two reciprocal properties of the preceding article are again evidently particular cases of the two following; which follow readily, the first from the general property of Art. 120, respecting the central axis of any system of lines for any system of multiples, and the second from the general property of Art. 86, respecting the mean centre of any system of points for any system of multiples; viz.—

a. If on a variable line L, turning round a fixed point O, and intersecting with any system of fixed lines A, B, C, &c. at a system of variable points X, Y, Z, &c., a variable point P be taken so as to satisfy in every position the relation

$$a.\frac{PX}{OX}+b.\frac{PY}{OY}+c.\frac{PZ}{OZ}+\&c.=0,$$

a, b, c, &c. being any system of finite multiples, positive or negative; the point P moves on a fixed line I, termed the polar of the *point O with respect to the system of lines A, B, C, &c. for the system of multiples a, b, c, &c.*

a'. If through a variable point P, moving on a fixed line I, and connecting with any system of fixed points A, B, C, &c. by a system of variable lines U, V, W, &c., a variable line L be drawn so as to satisfy in every position the relation

$$a.\frac{\sin LU}{\sin IU}+b.\frac{\sin LV}{\sin IV}+c.\frac{\sin LW}{\sin IW}+\&c.=0,$$

a, b, c, &c. being any system of finite multiples, positive or negative; the line L turns round a fixed point O, termed the pole of the line I with respect to the system of points A, B, C, &c. for the system of multiples a, b, c, &c.

To prove *a*. From the two points O and P conceiving the two systems of perpendiculars OA, OB, OC, OD, &c. and PA, PB, PC, PD, &c. let fall upon the system of lines A, B, C, D, &c.; then since, by (Euc. VI. 4),

$$\frac{PX}{OX}=\frac{PA}{OA},\quad \frac{PY}{OY}=\frac{PB}{OB},\quad \frac{PZ}{OZ}=\frac{PC}{OC},\ \&c.$$

therefore, by the relation determining the position of P on L,

$$a.\frac{PA}{OA}+b.\frac{PB}{OB}+c.\frac{PC}{OC}+d.\frac{PD}{OD}+\&c.=0,$$

from which, as it follows, by (120), that *the point P lies on the central axis I of the system of lines A, B, C, D, &c. for the system of multiples* $a \div OA$, $b \div OB$, $c \div OC$, $d \div OD$, &c., therefore &c.

To prove *a'*. From the system of points A, B, C, D, &c. conceiving the two systems of perpendiculars AI, BI, CI, DI, &c. and AL, BL, CL, DL, &c. let fall upon the two lines I and L; then since, by (61),

$$\frac{\sin LU}{\sin IU}=\frac{AL}{AI},\quad \frac{\sin LV}{\sin IV}=\frac{BL}{BI},\quad \frac{\sin LW}{\sin IW}=\frac{CL}{CI},\ \&c.,$$

therefore, by the relation determining the direction of L through P,

$$a.\frac{AL}{AI}+b.\frac{BL}{BI}+c.\frac{CL}{CI}+d.\frac{DL}{DI}+\&c.=0,$$

from which, as it follows, by (86), that *the line L passes through* *the mean centre O of the system of points A, B, C, D, &c. for the system of multiples* $a \div AI$, $b \div BI$, $c \div CI$, $d \div DI$, *&c.*, therefore &c.

COR. 1°. It having been shewn in the demonstrations just given, that—

b. The polar of a point O with respect to any system of lines A, B, C, D, &c. for any system of multiples a, b, c, d, &c. is the central axis (120) *of the system of lines for the system of multiples* $a \div OA$, $b \div OB$, $c \div OC$, $d \div OD$, *&c.*

b′. The pole of a line I with respect to any system of points A, B, C, D, &c. for any system of multiples a, b, c, d, &c. is the mean centre (86) *of the system of points for the system of multiples* $a \div AI$, $b \div BI$, $c \div CI$, $d \div DI$, *&c.*

It follows, consequently, that the two general problems: "To determine" *a.* "the polar of a given point with respect to a given system of lines for a given system of multiples;" *a′.* "the pole of a given line with respect to a given system of points for a given system of multiples;" are reduced at once to the two: "To determine" *b.* "the central axis of a given system of lines for a given system of multiples;" *b′.* "the mean centre of a given system of points for a given system of multiples;" constructions for which in their most general forms have been already given in articles (120) and (92).

COR. 2°. In the particular case when the fixed point *O*, in property *a*, is at infinity in any direction; since then, whatever be the position of the variable line *L* passing through it, the several ratios $OX : OY : OZ : \&c. = 1$; therefore, for the variable point *P*, by the relation determining its position on *L*,

$$a.PX + b.PY + c.PZ + \&c. = 0.$$

And, in the particular case, when the fixed line *I*, in property *a′*, is the line at infinity; since then, whatever be the position of the variable point *P* lying upon it, the several ratios $AI : BI : CI : \&c.$ all $= 1$; therefore, for the variable line *L*, by the relation determining its direction through *P*,

$$a.AL + b.BL + c.CL + \&c. = 0.$$

Hence by (126) and (86) it appears that—

c. The polar with respect to any system of lines, for any *system of multiples, of a point at infinity, is the diameter of the system of lines, for the system of multiples, corresponding to the direction of the point.*

c'. The pole with respect to any system of points, for any system of multiples, of the line at infinity, is the mean centre of the system of points, for the system of multiples.

Cor. 3°. In the particular case when the several fixed lines A, B, C, D, &c., in property a, pass through a common point P; since then, for the particular line L passing through the two points O and P, the several segments PX, PY, PZ, &c. all $=0$; therefore the point P, on that line L passing through O, satisfies, for every system of finite multiples a, b, c, d, &c., the relation

$$a.\frac{PX}{OX}+b.\frac{PY}{OY}+c.\frac{PZ}{OZ}+\text{\&c.}=0,$$

and consequently lies on the polar of the point O with respect to the system of lines A, B, C, D, &c. for the system of multiples a, b, c, d, &c. And, in the particular case, when the several fixed points A, B, C, D, &c., in property a', lie on a common line L; since then, for the particular point P lying on the two lines I and L, the sines of the several angles LU, LV, LW, &c. all $=0$; therefore the line L, through that point P lying on I, satisfies, for every system of finite multiples a, b, c, d, &c., the relation

$$a.\frac{\sin LU}{\sin IU}+b.\frac{\sin LV}{\sin IV}+c.\frac{\sin LW}{\sin IW}+\text{\&c.}=0,$$

and consequently passes through the pole of the line I with respect to the system of points A, B, C, D, &c. for the system of multiples a, b, c, d, &c. Hence it appears that—

d. For a concurrent system of lines, the polar of every point, for every system of finite multiples, passes through the point of concurrence.

d'. For a collinear system of points, the pole of every line, for every system of finite multiples, lies on the line of collinearity.

Cor. 4°. In the particular case when the several multiples a, b, c, d, &c. each $=1$; it may be easily shewn that in every position of the variable line L, in property a, *the distance OP is the harmonic mean of the several distances* OX, OY, OZ, &c. (233, 8°.).

For, in the relation determining the position of P on L, substituting for the several distances PX, PY, PZ, &c. their equivalents $OX - OP$, $OY - OP$, $OZ - OP$, &c. (75), and dividing by the interval OP, there results immediately the relation

$$\frac{a}{OX} + \frac{b}{OY} + \frac{c}{OZ} + \&c. = \frac{a+b+c+\&c.}{OP},$$

which, when $a = b = c = d$, &c. $= 1$, becomes

$$\frac{1}{OX} + \frac{1}{OY} + \frac{1}{OZ} + \&c. = \frac{n}{OP},$$

n being the number of multiples; and therefore &c. (233, 8°.).

N.B. It is this latter case (that, viz. in which the several multiples a, b, c, d, &c. all $= 1$) that is always implicitly intended whenever the terms "polar of a point with respect to a system of lines" and "pole of a line with respect to a system of points" are employed, as they often are, absolutely, without specifying the system of multiples to which they correspond.

249. That the two general properties a and a' of the preceding article (248) are reciprocals of each other in the sense explained in Art. 173, being less obvious than for any of the other pairs of properties stated in the preceding articles of the present chapter, may be readily shewn as follows:

If P, Q, X, Y, Z, &c. be any number of collinear points; L, M, U, V, W, &c. their several polars with respect to any circle; or conversely; and O the centre of the circle; then, the several angles subtended at O by the several pairs of the former being similar, by (171, 2°.), to those determined by the several corresponding pairs of the latter; therefore, by the general property, Art. 65,

$$\frac{PX}{QX} = \frac{PO}{QO} \cdot \frac{\sin POX}{\sin QOX} = \frac{PO}{QO} \cdot \frac{\sin LU}{\sin MU},$$

$$\frac{PY}{QY} = \frac{PO}{QO} \cdot \frac{\sin POY}{\sin QOY} = \frac{PO}{QO} \cdot \frac{\sin LV}{\sin MV},$$

$$\frac{PZ}{QZ} = \frac{PO}{QO} \cdot \frac{\sin POZ}{\sin QOZ} = \frac{PO}{QO} \cdot \frac{\sin LW}{\sin MW}, \&c.,$$

which, multiplied by any system of finite multiples a, b, c, &c., and added, give at once the relation

$$\Sigma\left(a.\frac{PX}{QX}\right) = \frac{PO}{QO}.\Sigma\left(a.\frac{\sin LU}{\sin MU}\right),$$

from which it follows that, under all circumstances of the several points and lines, the two relations

$$\Sigma\left(a.\frac{PX}{QX}\right) = 0, \text{ and } \Sigma\left(a.\frac{\sin LU}{\sin MU}\right) = 0,$$

mutually involve each other, and therefore that, under all circumstances, the two general properties they express are reciprocals of each other in the sense explained in that article (173); but, by property a of the preceding article, the first expresses that when Q is fixed, and X, Y, Z, &c. move on fixed lines, then P moves on a fixed line; and, by property a' of the same article, the second expresses that when M is fixed, and U, V, W, &c. turn round fixed points, then L turns round a fixed point; those properties are therefore reciprocals of each other, and either might be inferred from the other, without independent demonstration, by the reciprocating process described in Art. 173 and applied as above, in virtue of the general property, Cor. 2°., Art. 166, that *when a variable point moves on a fixed line, its polar with respect to any circle turns round a fixed point, the pole of the line with respect to the circle;* and, conversely, that *when a variable line turns round a fixed point, its pole with respect to any circle moves on a fixed line, the polar of the point with respect to the circle.*

That the two general properties, to which they have been reduced in the independent demonstrations given of them in the preceding article, are also reciprocals in the same sense, being again less obvious than for any of the more ordinary pairs of properties previously stated as such, may as readily be shewn as follows:

If P, A, B, C, &c. be any number of points disposed in any manner; L, E, F, G, &c. their several polars with respect to any circle; or conversely; and O the centre of the circle; then since, by Dr. Salmon's property, Art. 179,

$$\frac{PE}{PO} = \frac{AL}{AO}, \quad \frac{PF}{PO} = \frac{BL}{BO}, \quad \frac{PG}{PO} = \frac{CL}{CO}, \text{ \&c.};$$

therefore, multiplying by any system of finite multiples **a, b, c,** &c., and adding,

$$\Sigma\left(a.\frac{PE}{PO}\right)=\Sigma\left(a.\frac{AL}{AO}\right), \quad \text{or} \quad \Sigma\left(\frac{a}{PO}.PE\right)=\Sigma\left(\frac{a}{AO}.AL\right),$$

from which it follows that, under all circumstances of the several points and lines, the two relations

$$\Sigma(a.PE)=0, \text{ and } \Sigma(a'.AL)=0,$$

where $a'=a\div AO$, $b'=b\div BO$, $c'=c\div CO$, &c., mutually involve each other, and therefore that, under all circumstances, the two general properties they express are reciprocals of each other; but, by the general property of Art. 120, the first expresses that when E, F, G, &c. are fixed, and P variable, then P moves on a fixed line; and, by the general property of Art. 86, the second expresses that when A, B, C, &c. are fixed, and L variable, then L turns round a fixed point; these properties therefore are reciprocals of each other, and either might be inferred, as above, from the other, without independent demonstration, by the same reciprocating process, and in virtue of the same general property of poles and polars, as the more general two reduced to them in the preceding article.

250. With a few polar properties respecting triangles we shall now conclude the present chapter.

a. If A, B, C be any three lines, a, b, c any three corresponding multiples, O any arbitrary point, I the polar of O with respect to the three lines for the three multiples, and P, Q, R the three intersections of I with A, B, C respectively; then always

$$b.\frac{PB}{OB}+c.\frac{PC}{OC}=0, \quad c.\frac{QC}{OC}+a.\frac{QA}{OA}=0, \quad a.\frac{RA}{OA}+b.\frac{RB}{OB}=0.$$

a'. If A, B, C be any three points, a, b, c any three corresponding multiples, I any arbitrary line, O the pole of I with respect to the three points for the three multiples, and L, M, N the three connectors of O with A, B, C respectively; then always

$$b.\frac{BL}{BI}+c.\frac{CL}{CI}=0, \quad c.\frac{CM}{CI}+a.\frac{AM}{AI}=0, \quad a.\frac{AN}{AI}+b.\frac{BN}{BI}=0.$$

For, in the case of a, since for every three points P, Q, R on the line I, by property a of the preceding article (248),

$$a.\frac{PA}{OA}+b.\frac{PB}{OB}+c.\frac{PC}{OC}=0,$$

$$a.\frac{QA}{OA}+b.\frac{QB}{OB}+c.\frac{QC}{OC}=0,$$

$$a.\frac{RA}{OA}+b.\frac{RB}{OB}+c.\frac{RC}{OC}=0,$$

therefore for the three particular points P, Q, R on it for which respectively $PA=0$, $QB=0$, $RC=0$ the above relations are true; and therefore &c. And, in the case of a', since for every three lines L, M, N through the point O, by property a' of the preceding article (248),

$$a.\frac{AL}{AI}+b.\frac{BL}{BI}+c.\frac{CL}{CI}=0,$$

$$a.\frac{AM}{AI}+b.\frac{BM}{BI}+c.\frac{CM}{CI}=0,$$

$$a.\frac{AN}{AI}+b.\frac{BN}{BI}+c.\frac{CN}{CI}=0,$$

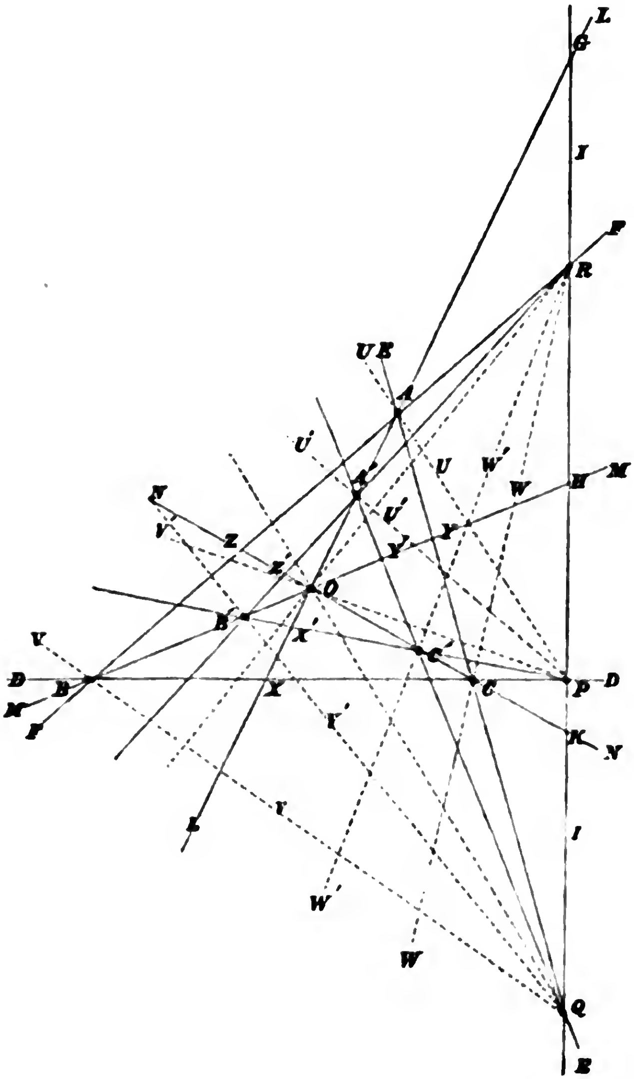
therefore for the three particular lines L, M, N through it for which respectively $AL=0$, $BM=0$, $CN=0$ the above relations are true; and therefore &c.

COR. 1°. It is evident, from the above relations, that, in a, *the three points P, Q, R connect with the three BC, CA, AB by the three polars of the point O with respect to the three pairs of lines B and C, C and A, A and B for the three pairs of multiples b and c, c and a, a and b respectively* (247, a); and, that, in a', *the three lines L, M, N intersect with the three BC, CA, AB at the three poles of the line I with respect to the three pairs of points B and C, C and A, A and B for the three pairs of multiples b and c, c and a, a and b respectively* (247, a'). Properties which, for any given system of three lines or points A, B, C, supply obvious and rapid constructions for determining, for any given system of corresponding multiples a, b, c, the polar I of any given point O with respect to the former, or the pole O of any given line I with respect to the latter.

COR. 2°. In the particular case when $a=b=c=1$, it is evident, from Cor. 1°, that, in a, *the three points* P, Q, R *on the three lines* A, B, C *are conjugates to the point* O *with respect to the three opposite angles* BC, CA, AB *of the triangle determined by the lines* (217); and that, in a', *the three lines* L, M, N *through the three points* A, B, C *are conjugates to the line* I *with respect to the three opposite sides* BC, CA, AB *of the triangle determined by the points* (217). It is evident again, from the same, that, the line I, in the former case, is *the line of collinearity of the three points* P, Q, R *at which the three polars of the point* O *with respect to the three angles intersect with the opposite sides of the triangle determined by the lines* (243, d'); and that, the point O, in the latter case, is *the point of concurrence of the three lines* L, M, N *by which the three poles of the line* I *with respect to the three sides connect with the opposite angles of the triangle determined by the points* (243, d). And it is evident also, from the same, that *when, for any triangle* ABC, *a line* I *is the polar of a point* O *with respect to the three sides, then, reciprocally, the point* O *is the pole of the line* I *with respect to the three vertices, and conversely* (139, Cor. 3°); a property which we shall presently see is true generally, not only for the particular system of multiples each $=1$, but for any system of finite multiples as well.

In the figure of Art. 236, the four points P, Q, R, and S are connected with the four lines YZX', ZXY', XYZ', and $X'Y'Z'$ respectively by the above relation of being pole and polar to each other with respect to the vertices and sides of the central triangle ABC; and, in the figure of Art. 139, the point O is connected with the line XYZ by the same relation of being pole and polar to each other with respect to the vertices and sides, not only of the original triangle ABC, but of the several derivatives of both species $A'B'C'$ and $A_{,}B_{,}C_{,}$, $A''B''C''$ and $A_{,,}B_{,,}C_{,,}$, $A'''B'''C'''$ and $A_{,,,}B_{,,,}C_{,,,}$, &c. obtained from it, through their directing agency, by the continued application of the two inverse processes of construction described in Cors. 4° and 5° of that article.

251. If A, B, C be the three vertices, and D, E, F the three opposite sides of any triangle; O and I any arbitrary point and



line; *L*, *M*, *N* the three connectors of *O* with the vertices; *P*, *Q*, *R* the three intersections of *I* with the sides; *U*, *V*, *W* the three connectors of *P*, *Q*, *R* with the vertices; and *X*, *Y*, *Z* the three intersections of *L*, *M*, *N* with the sides; then—

a. The point O is the pole of the line I with respect to the three points A, B, C for any three multiples a, b, c such that

$$b.\frac{BX}{BP}+c.\frac{CX}{CP}=0,\quad c.\frac{CY}{CQ}+a.\frac{AY}{AQ}=0,\quad a.\frac{AZ}{AR}+b.\frac{BZ}{BR}=0.$$

a'. The line I is the polar of the point O with respect to the three lines D, E, F for any three multiples a, b, c such that

$$b.\frac{\sin EU}{\sin EL}+c.\frac{\sin FU}{\sin FL}=0,\quad c.\frac{\sin FV}{\sin FM}+a.\frac{\sin DV}{\sin DM}=0,$$

$$a.\frac{\sin DW}{\sin DN}+b.\frac{\sin EW}{\sin EN}=0.$$

For, the three relations, in the case of a, being evidently equivalent to the three

$$b.\frac{BL}{BI}+c.\frac{CL}{CI}=0,\quad c.\frac{CM}{CI}+a.\frac{AM}{AI}=0,\quad a.\frac{AN}{AI}+b.\frac{BN}{BI}=0;$$

and the three, in the case of a', to the three

$$b.\frac{PE}{OE}+c.\frac{PF}{OF}=0,\quad c.\frac{QF}{OF}+a.\frac{QD}{OD}=0,\quad a.\frac{RD}{OD}+b.\frac{RE}{OE}=0,$$

which being identical with those in a' and a of the preceding article (250), therefore &c.

COR. 1°. Since, for the same triangle, by Cor. 4°, Art. 247, the two groups of three relations in a and a' of the above mutually involve each other for the same system of multiples a, b, c; hence, generally, as noticed in Cor. 2° of the preceding article for the particular system of multiples each = 1.—

When, for any triangle ABC, a line I is the polar of a point O with respect to the three sides for any system of multiples a, b, c; then, reciprocally, the point O is the pole of the line I with respect to the three vertices for the same system of multiples a, b, c.

This property, it may be observed, would have followed also from those of Cor. 1° of the preceding article (250), combined with those of Cor. 3° of Art. 247.

COR. 2°. Since, for two triangles in perspective (140), the three lines of connection L, M, N of the three pairs of corresponding vertices A and A', B and B', C and C' pass through a common point O, their centre of perspective, and the three points of intersection P, Q, R of the three pairs of corresponding sides D and D', E and E', F and F' lie on a common line I, their axis of perspective (see figure); hence again, from the above relations a and a' combined with the general property of Cor. 4°, Art. 247, the two following polar properties of two triangles in perspective—

b. For every two triangles in perspective, the centre is the pole of the axis of perspective with respect to the vertices of both triangles for the same system of multiples.

b'. For every two triangles in perspective, the axis is the polar of the centre of perspective with respect to the sides of both triangles for the same system of multiples.

For, the four lines I, L, M, N and the four points O, P, Q, R, in the relations a and a', being the same for both triangles (see figure), if U', V', W' and X', Y', Z' be for the triangle $A'B'C'$ what U, V, W and X, Y, Z as above stated are for the triangle ABC; then since the three pairs of corresponding systems of four collinear points B, C, X, P and B', C', X', P; C, A, Y, Q and C', A', Y', Q; A, B, Z, R and A', B', Z', R are in perspective at the point O (130), therefore, by the general property, Cor. 4°, Art. 247, the three pairs of corresponding relations

$$b.\frac{BX}{BP}+c.\frac{CX}{CP}=0, \text{ and } b.\frac{B'X'}{B'P}+c.\frac{C'X'}{C'P}=0,$$

$$c.\frac{CY}{CQ}+a.\frac{AY}{AQ}=0, \text{ and } c.\frac{C'Y'}{C'Q}+a.\frac{A'Y'}{A'Q}=0,$$

$$a.\frac{AZ}{AR}+b.\frac{BZ}{BR}=0, \text{ and } a.\frac{A'Z'}{A'R}+b.\frac{B'Z'}{B'R}=0,$$

(see property a) mutually involve each other for the same system of multiples a, b, c, and therefore &c. as regards b; and since the three pairs of corresponding systems of four concurrent lines E, F, U, L and E', F', U', L; F, D, V, M and F', D', V', M; D, E, W, N and D', E', W', N are in perspective on the line I (130), therefore, by the same general property, Cor. 4°, Art. 247, the three pairs of corresponding relations

$$b.\frac{\sin EU}{\sin EL}+c.\frac{\sin FU}{\sin FL}=0, \text{ and } b.\frac{\sin E'U'}{\sin E'L}+c.\frac{\sin F'U'}{\sin F'L}=0,$$

$$c.\frac{\sin FV}{\sin FM}+a.\frac{\sin DV}{\sin DM}=0, \text{ and } b.\frac{\sin F'V'}{\sin F'M}+c.\frac{\sin D'V'}{\sin D'M}=0,$$

$$a.\frac{\sin DW}{\sin DN}+b.\frac{\sin EW}{\sin EN}=0, \text{ and } a.\frac{\sin D'W'}{\sin D'N}+b.\frac{\sin E'W'}{\sin E'N}=0,$$

(see property a') mutually involve each other for the same system of multiples a, b, c, and therefore &c. as regards b'.

N.B. These two reciprocal properties, though thus established independently, evidently involve each other by virtue of Cor. 1°.

Cor. 3°. It follows, of course, from the two reciprocal properties of the preceding corollary, that *when any number of triangles ABC, $A'B'C'$, $A''B''C''$, &c., in perspective two and two, have a common centre and axis of perspective O and I; the centre is the pole of the axis with respect to the vertices, and the axis the polar of the centre with respect to the sides, of all of them alike, for the same system of multiples a, b, c;* for, the three lines L, M, N and the three points P, Q, R (see figure) being then necessarily the same for the entire system, and the property, as above shewn, being consequently true for every two of the component triangles, therefore &c. Of this general property that stated in the closing paragraph of Cor. 2° of the preceding article (250) is evidently a particular case.

CHAPTER XV.

HARMONIC PROPERTIES OF THE CIRCLE.

252. Of the various harmonic properties of the circle, the two following, reciprocals of each other (173), are those to which the appellation is most commonly given, and which, on the whole, lead, perhaps, to the greatest number of consequences—

a. When four points on a circle determine an harmonic pencil of rays at any fifth point on the circle, they do so at every point on the circle.

a'. When four tangents to a circle determine an harmonic row of points on any fifth tangent to the circle, they do so on every tangent to the circle.

For, in the former case, if A, B, X, Y be the four points, P the fifth point, and Q any sixth point on the circle; then since, by property 1°, Art. 25, the two pencils of four rays PA, PB, PX, PY and QA, QB, QX, QY, or, as they may be more concisely denoted, $P.ABXY$ and $Q.ABXY$, are similar, therefore (213) the harmonicism of either involves that of the other, and therefore &c. And, in the latter case, if C, D, U, V be the four tangents, L the fifth tangent, and M any sixth tangent to the circle; then since, by property 2°, Art. 25, the two rows of four points LC, LD, LU, LV and MC, MD, MU, MV, or, as they may be more concisely denoted, $L.CDUV$ and $M.CDUV$, subtend similar pencils at the centre O of the circle, therefore, by the general property of Art. 221, the harmonicism of either involves that of the other, and therefore &c.

253. These two properties, thus established independently, are evidently reciprocals of each other to the circle to which the points or tangents belong (see Art. 172); for, if A, B, X, Y

be any four points on a circle, and C, D, U, V the four tangents at them to the same circle, or conversely; P any fifth point on the circle, and L the tangent at it to the circle, or conversely; then, since, by (165, 6°), the four collinear points $L.CDUV$ are the four poles with respect to the circle of the four concurrent lines $P.ABXY$, and conversely, therefore, by the general property of Art. 223, the harmonicism of either system involves that of the other, and therefore &c.

From this last, or from properties 1° and 2°, Art. 25, it is evident that *when four points A, B, X, Y on a circle determine (as in a) an harmonic pencil of rays $P.ABXY$ at every fifth point P on the circle, the four tangents at them, C, D, U, V, determine (as in a') an harmonic row of points $L.CDUV$ on every fifth tangent L to the circle; and conversely.*

254. Every four points A, B, X, Y on a circle which determine, as above (a), an harmonic pencil of rays at every fifth point P on the circle are said to form *an harmonic system of points on the circle*, whose two pairs of conjugates A and B, X and Y correspond, of course, to those of the pencil $P.ABXY$ they determine at P; and every four tangents C, D, U, V to a circle which determine, as above (a'), an harmonic row of points on every fifth tangent L to the circle are said to form *an harmonic system of tangents to the circle*, whose two pairs of conjugates C and D, U and V correspond, of course, to those of the row $L.CDUV$ they determine on L. In either case the two arcs of the circle intercepted between the two pairs of conjugates are said *to cut each other harmonically;* and, of two arcs of a circle thus cutting each other harmonically, either may be, and in fact is, as often imaginary as real (215).

When four points A, B, X, Y on a circle form an harmonic system, the two lines of connection AB and XY of the two pairs of conjugates A and B, X and Y are termed *conjugate lines with respect to the circle;* and when four tangents C, D, U, V to a circle form an harmonic system, the two points of intersection CD and UV of the two pairs of conjugates C and D, U and V are termed *conjugate points with respect to the circle.* It will appear in the sequel that every two lines or points in this sense conjugates to each other with respect to a circle are also conjugates to each other with respect to the circle in the more general sense in which the same term was employed in Art. 174.

By virtue of the general relation of Art. 221, it is evident that *when four points on a circle form an harmonic system, their four lines of connection with any fifth point on the circle determine an harmonic row of points on every axis;* and that *when four tangents to a circle form an harmonic system, their four points of intersection with any fifth tangent to the circle determine an harmonic pencil of rays at every vertex.*

255. From the fundamental reciprocal properties of Art. 252 the two following, also reciprocals to each other, result at once by virtue of the two general principles explained in articles (19) and (20); viz.—

a. When four points on a circle form an harmonic system, the tangent to the circle at each forms an harmonic pencil with its three lines of connection with the remaining three.

a'. When four tangents to a circle form an harmonic system, the point of contact with the circle of each forms an harmonic row with its three points of intersection with the remaining three.

For, if A, B, X, Y be the four points, and C, D, U, V the four tangents; then, since, in the former case, for *every* point P on the circle, by property a, Art. 252, the pencil of four rays $P.ABXY$ is harmonic, therefore, for the four points A, B, X, Y, the four pencils of four rays $A.ABXY$, $B.ABXY$, $X.ABXY$, $Y.ABXY$ are harmonic; but of these four pencils the four rays AA, BB, XX, YY, by (19), are the four tangents to the circle at the four points A, B, X, Y; and therefore &c. And, since, in the latter case, for *every* tangent L to the circle, by property a', Art. 252, the row of four points $L.CDUV$ is harmonic, therefore, for the four tangents C, D, U, V, the four rows of four points $C.CDUV$, $D.CDUV$, $U.CDUV$, $V.CDUV$ are harmonic; but of these four rows the four points CC, DD, UU, VV, by (20), are the four points of contact with the circle of the four tangents C, D, U, V; and therefore &c.

It is, of course, evident conversely, as in Art. 252, that the harmonicism of any one of the four pencils of rays $A.ABXY$, $B.ABXY$, $X.ABXY$, $Y.ABXY$ in the former case, or of any one of the four rows of points $C.CDUV$, $D.CDUV$, $U.CDUV$, $V.CDUV$ in the latter case, involves that of the remaining three; for it involves, in the former case, that of the system of four points A, B, X, Y on the circle, and in the latter case, that of the system of four tangents C, D, U, V to the circle; and therefore &c.

256. *When two arcs of a circle cut each other harmonically, the two pairs of chords connecting the extremities of either with those of the other have equal ratios; and, conversely, when two arcs of a circle are such that the two pairs of chords connecting the extremities of either with those of the other have equal ratios, they cut each other harmonically.*

For, if AB and XY be *any* two arcs of a circle, AX and BX, AY and BY the two pairs of chords connecting the extremities of either AB with those of the other XY, and P any arbitrary point on the circle; then since always, by (62, Cor. 1°), disregarding signs,

$$\frac{AX}{BX} : \frac{AY}{BY} = \frac{\sin APX}{\sin BPX} : \frac{\sin APY}{\sin BPY},$$

therefore when either equivalent in absolute value $= 1$ so is the other; but (252), when the arcs AB and XY cut each other harmonically, the latter equivalent in absolute value $= 1$; and conversely, when the latter equivalent in absolute value $= 1$, the arcs AB and XY cut each other harmonically; and therefore &c.

Since, by the above, immediately and by alternation,

$$\frac{AX}{BX} = \frac{AY}{BY}, \text{ and } \frac{AX}{AY} = \frac{BX}{BY},$$

therefore again, immediately and by alternation,

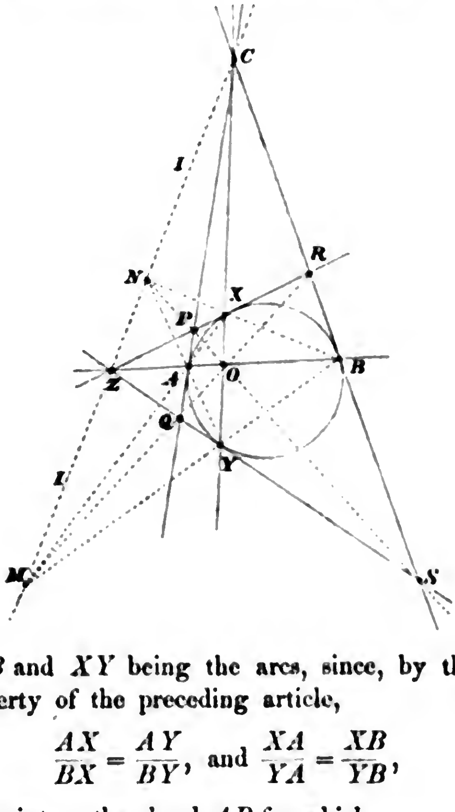
$$\frac{AX^2}{BX^2} = \frac{AX.AY}{BX.BY} = \frac{AY^2}{BY^2}, \text{ and } \frac{AX^2}{AY^2} = \frac{AX.BX}{AY.BY} = \frac{BX^2}{BY^2},$$

and again also, immediately or by alternation,

$$AX.BY = AY.BX = \tfrac{1}{2}.AB.XY, \text{ (see Art. 219, } a);$$

any of which, consequently, may be regarded as characteristic of the harmonic section of two arcs AB and XY of a circle, and sufficient to determine it.

257. *When two arcs of a circle cut each other harmonically; the tangents at the extremities of either intersect on the chord of the other; and, reciprocally, the chord of either passes through the intersection of the tangents at the extremities of the other.*



For, AB and XY being the arcs, since, by the first part of the property of the preceding article,

$$\frac{AX}{BX} = \frac{AY}{BY}, \text{ and } \frac{XA}{YA} = \frac{XB}{YB},$$

if Z be the point on the chord AB for which

$$\frac{AZ}{BZ} = \frac{AX^2}{BX^2} = \frac{AY^2}{BY^2} = \frac{AX.AY}{BX.BY},$$

and C the point on the chord XY for which

$$\frac{XC}{YC} = \frac{XA^2}{YA^2} = \frac{XB^2}{YB^2} = \frac{XA.XB}{YA.YB},$$

then, evidently, (Euc. III. 32, and VI. 4), the two tangents at X and Y pass both through Z, and the two at A and B pass both through C; and therefore &c.

Conversely, *Every two points on a circle which connect through the intersection of two tangents to the circle cut harmonically the arc of the circle intercepted between the tangents; and, reciprocally, every two tangents to a circle which intersect on the connector of two points on the circle cut harmonically the arc of the circle intercepted between the points.*

For, if A and B be the two points, AC and BC the two tangents, X and Y any two points connecting through C, and XZ and YZ any two tangents intersecting on AB; then since, for the pair of tangents XZ and YZ, and for the pair AC and BC, respectively, by Euc. III. 32, and VI. 4,

$$\frac{AX^2}{BX^2} = \frac{AY^2}{BY^2} = \frac{AZ}{BZ}, \text{ and } \frac{XA^2}{YA^2} = \frac{XB^2}{YB^2} = \frac{XC}{YC},$$

therefore, by the second part of the property of the preceding article, the two arcs AB and XY cut each other harmonically; and therefore &c.

Of all properties of harmonic systems, whether of points on, or of tangents to, a circle, the above gives, in either case, the most definite conception of the actual disposition of the two pairs of conjugates on or round the circle.

COR. 1°. The two points C and Z, in the above, being (165) the two poles with respect to the circle of the two lines AB and XY; it appears, consequently, from the first part of the above, as stated in other terms in Art. 254, that—

When four points on a circle form an harmonic system, the connectors of the two pairs of conjugates pass each through the pole of the other with respect to the circle; and, reciprocally, when four tangents to a circle form an harmonic system, the intersections of the two pairs of conjugates lie each on the polar of the other with respect to the circle. See (174).

And, from the second part of the same, conversely, that—

Every two lines passing each through the pole of the other with respect to a circle determine the two conjugate pairs of an harmonic system of points on the circle; and, reciprocally, every two points lying each on the polar of the other with respect to a circle determine the two conjugate pairs of an harmonic system of tangents to the circle.

These last, as thus stated, include evidently those cases in which one pair of the conjugates, whether points or tangents, is imaginary, as well as those in which both are real.

COR. 2°. The point of intersection O of the two lines AB and XY, and the line of connection I of the two points C and Z (see figure), being also, by (167), pole and polar to each other with respect to the circle; and the triangle determined by the three points C, Z, and O, or by the three lines AB, XY, and I, being, consequently, self-reciprocal with respect to the circle (168); it appears therefore, again, from both parts of the above, that—

Every triangle every two of whose sides determine harmonic systems of points on a circle, or every two of whose vertices determine harmonic systems of tangents to a circle, is self-reciprocal with respect to the circle: and, conversely, in every triangle self-reciprocal with respect to a circle, every two of the sides determine harmonic systems of points on the circle, and every two of the vertices determine harmonic systems of tangents to the circle.

Of the three pairs of points or tangents, thus determining two and two three harmonic systems, it is evident, from (168), or directly from the nature of harmonic section, that, for a real circle, while two of them are always real, the third is always imaginary.

COR. 3°. If, in the first part of the above, while the line AB and the point C are supposed to remain fixed, the line XY and the point Z be conceived to vary simultaneously; then since, by that part, as above shewn, XY passes in every position through C, and Z lies in every position on AB, therefore—

If a fixed arc of a fixed circle be cut harmonically by a variable pair of conjugates, either points or tangents; the latter intersect in every position on the fixed connector of its terminal points, and the former connect in every position through the fixed intersection of its terminal tangents.

COR. 4°. The two reciprocal properties of the second part of the above supply obvious and rapid solutions of the three following pairs of reciprocal problems, viz.—

1°. *To cut a given arc of a given circle harmonically, a. by two points connecting through a given point; a'. by two tangents intersecting on a given line.*

2°. *To cut two given arcs of a given circle harmonically, a. by the same two points on the circle; a'. by the same two tangents to the circle.*

3°. *To cut two given arcs of two given circles harmonically, a. by four collinear points on the circles; a'. by four concurrent tangents to the circles.*

As a circle of any magnitude may be described passing in any direction through the vertex of any angle, the solution of either problem (2°) respecting arcs gives obviously a direct solution (see Art. 230) of the corresponding problem respecting angles, viz.—

Given in magnitude and position two angles having a common vertex, to determine in magnitude and position the angle that cuts both harmonically.

Cor. 5°. Since, for a given circle, the length of an arc gives the length of its chord, and the points of bisection of an arc the direction of its chord; the same again supply obvious solutions of the two following problems:—

Given in magnitude and position one arc of a given circle, and the length or points of bisection of another cutting it harmonically, to determine the other.

For the same reason as in the preceding corollary, the solutions of these two problems respecting arcs give obviously direct solutions (see Art. 227, Cor. 3°) of the two corresponding problems respecting angles, viz.—

Given an angle in magnitude and position, and the magnitude or lines of bisection of another angle cutting it harmonically; to determine the latter.

Cor. 6°. The circle having C for centre which passes through A and B (see figure) being orthogonal to the original circle $ABXY$ (22, 1'); therefore, from both parts of the above, directly and conversely—

a. When two circles intersect at right angles; every two points of either which cut harmonically its arc intercepted by the other connect through the centre of the other; and, conversely, every two points of either which connect through the centre of the other cut harmonically its arc intercepted by the other.

b. When two circles intersect at right angles; every two *tangents to either which cut harmonically its arc intercepted by the other intersect on the common chord of both; and, conversely, every two tangents to either which intersect on the common chord of both cut harmonically its arc intercepted by the other.*

It follows, of course, from the second part of (*a*), that when the same circle is orthogonal to a number of others, every line passing through its centre cuts harmonically the several arcs it intercepts on them all.

COR. 7°. Every two points X and Y on the original circle (see figure) which connect through C, being inverse points (149) with respect to the circle having C for centre which passes through A and B, and every circle passing through them being consequently (156) orthogonal to that circle; therefore, again, from the above—

Every circle cutting an arc of another circle harmonically is orthogonal to the circle which passes orthogonally through the extremities of the arc; and, conversely, every circle intersecting two others orthogonally cuts harmonically the arcs of both intercepted between their points of intersection.

It follows, of course, from the second part of this, or of property (*a*) of the preceding corollary, (which it may be observed evidently involve each other), that every circle coaxal with the same two points (152) cuts harmonically the arcs intercepted by the latter on all circles passing through them.

COR. 8°. If, as in Cor. 3°, while the two points A and B with the original circle are supposed to remain fixed, the two points X and Y with the intersecting circle be conceived to vary simultaneously; then from the first part of the preceding, Cor. 7°, by (156)—

1°. *A variable circle, passing through a fixed point and cutting a fixed arc of a fixed circle harmonically, passes through a second fixed point, the inverse of the first with respect to the circle passing orthogonally through the extremities of the arc.*

2°. *A variable circle, cutting two fixed arcs of two fixed circles harmonically, passes through the two fixed points, real or imaginary, inverse to the two circles passing orthogonally through the extremities of the two arcs.*

Whether the two fixed points inverse to the two latter circles are imaginary or real, it follows of course, from (187, 4°), that the variable circle in 2° generates, in all cases, the coaxal system orthogonal to those circles.

Cor. 9°. As in Cor. 3°, Art. 228, the two properties of the preceding corollary (8°) reduce at once to those of Cor. 4°, Art. 156, the solutions of the three following problems, viz.—

To describe a circle, 1° passing through two given points and cutting a given arc of a given circle harmonically; 2° passing through a given point and cutting two given arcs of two given circles harmonically; 3° cutting three given arcs of three given circles harmonically.

258. *When four points on a circle from an harmonic system, the connector of either pair of conjugates and the tangent at either of its points intersect harmonically with the axis of the segment determined by the other pair; and, reciprocally, when four tangents to a circle form an harmonic system, the intersection of either pair of conjugates and the point of contact of either of its tangents connect harmonically with the vertex of the angle determined by the other pair.*

For, if, in the former case, the system of four concyclic points A and B, X and Y (figure of last article) be harmonic; then since, by (255, a), the two pencils of four rays $A.ABXY$ and $B.BAXY$ having the common ray AB, and the two $X.XYAB$ and $Y.YXAB$ having the common ray XY, are harmonic; therefore, by (222, 6°), they intersect collinearly, the former pair on the line XY at the harmonic row of four points $COXY$, and the latter pair on the line AB at the harmonic row of four points $ZOAB$; and therefore &c. And, if, in the latter case, the system of four concyclic tangents AC and BC, XZ and YZ (same figure) be harmonic; then since, by (255, a'), the two rows of four points $ACPQ$ and $BCRS$ having the common point C, and the two $XZPR$ and $YZQS$ having the common point Z, are harmonic; therefore, by (222, 5°), they connect concurrently, the former pair through the point Z by the harmonic pencil of four rays $Z.OCXY$, and the latter pair through the point C by the harmonic pencil of four rays $C.OZAB$; and therefore &c.

Conversely, since, in the former case, the harmonicism of any one of the four pencils of four rays $A.ABXY$, $B.BAXY$, $X.XYAB$, $Y.YXAB$ involves, by (252, *a*), that of the system of four points A, B, X, Y on the circle; and, since, in the latter case, the harmonicism of any one of the four rows of four points $ACPQ$, $BCRS$, $XZPR$, $YZQS$ involves, by (252, *a'*), that of the system of four tangents AC, BC, XZ, YZ to the circle; therefore, *the above reciprocal properties are criteria, the former of the harmonicism of four points on a circle, and the latter of the harmonicism of four tangents to a circle.*

The above demonstrations, as establishing directly the collinearity of the two triads of points X, Y, C and A, B, Z for an harmonic system of points A, B, X, Y on a circle, and the concurrence of the two triads of lines PR, QS, AB and PQ, RS, XY for an harmonic system of tangents AC, BC, XZ, YZ to a circle, and conversely, establish therefore, in a manner applicable to higher figures as well, the two reciprocal properties established in the preceding article by a method applicable to the circle alone.

COR. 1°. By virtue of the above, the two points C and Z (see figure) being the poles of the two lines AB and XY with respect to the two segments XY and AB (217); and the two lines AB and XY being the polars of the two points C and Z with respect to the two angles XZY and ACB (217); it appears, consequently, from it, that—

The intersection of the two terminal tangents, and the connector of the two terminal points, of any arc of a circle, are pole and polar to each other with respect, at once to the segment determined by every two points on the circle which connect through the former, and to the angle determined by every two tangents to the circle which intersect on the latter.

COR. 2°. Again, the point O and the line I (see figure) being, by the above, pole and polar to each other with respect at once to the two segments XY and AB, and to the two angles XZY and ACB; and, the three points C, Z, and O, and the three lines AB, XY, and I, being, by (167), pole and polar to each other with respect to the circle itself; hence, again, from the above, for every point and line pole and polar to each other with respect to a circle, it appears that—

Every point and line, pole and polar to each other with respect to a circle, are also pole and polar to each other with respect, at once to the segment determined by every two points on the circle which connect through the former, and to the angle determined by every two tangents to the circle which intersect on the latter.

COR. 3°. The triangle determined by the three points C, Z, and O, or by the three lines AB, XY, and I, (see figure) being self-reciprocal with respect to the circle (168), each vertex and its opposite side being pole and polar to each other with respect to the circle; hence, also, from the above, see Cor. 2° of the preceding article.

In every triangle self-reciprocal with respect to a circle, the circle divides harmonically the three sides, and subtends harmonically the three vertices; and, conversely, the circle which divides harmonically the three sides or subtends harmonically the three vertices of a triangle is the polar circle of the triangle (168).

259. Of the various reciprocal properties of points and lines, pole and polar to each other with respect to a circle, the two following, termed their harmonic properties, and obviously tantamount to those just stated in Cor. 2° of the preceding article, are second only in importance to those of Art. 166, and lead, next to them, to the greatest number and variety of remarkable consequences in the modern geometry of the circle:—

a. Every two conjugate points with respect to a circle are harmonic conjugates with respect to the two collinear points on the circle; and, conversely, every two points harmonic conjugates with respect to the two collinear points on a circle are conjugate points with respect to the circle (174).

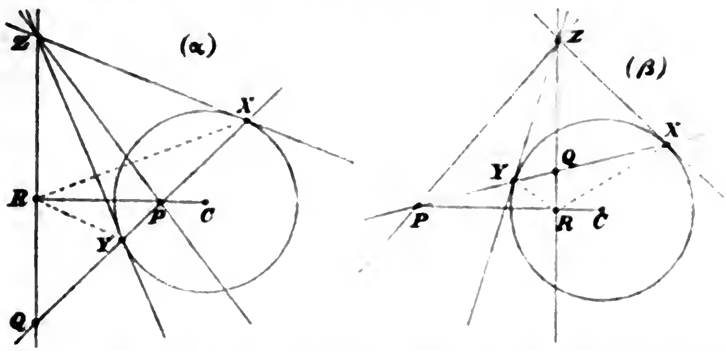
a′. Every two conjugate lines with respect to a circle are harmonic conjugates with respect to the two concurrent tangents to the circle; and, conversely, every two lines harmonic conjugates with respect to the two concurrent tangents to a circle are conjugate lines with respect to the circle (174).

These properties follow immediately, indirectly, from those of the preceding article; the two points O and C (see figure of that article) being at once conjugate points with respect to the circle, and harmonic conjugates with respect to the two collinear points X and Y on the circle, and the two lines ZO and ZC

being at once conjugate lines with respect to the circle and harmonic conjugates with respect to the two concurrent tangents ZX and ZY to the circle; the two points O and Z being at once conjugate points with respect to the circle, and harmonic conjugates with respect to the two collinear points A and B on the circle, and the two lines CO and CZ being at once conjugate lines with respect to the circle and harmonic conjugates with respect to the two concurrent tangents CA and CB to the circle; the two points C and Z being at once conjugate points with respect to the circle and harmonic conjugates with respect to the two imaginary collinear points on the circle, and the two lines OC and OZ being at once conjugate lines with respect to the circle, and harmonic conjugates with respect to the two imaginary concurrent tangents to the circle; and therefore &c.

From their importance, however, we subjoin the ordinary direct demonstrations of them, based on the fundamental definition of poles and polars with respect to the circle given in Art. 165.

If, as regards (a), P and Q be any two points, X and Y the



two collinear points on any circle C, and R the inverse of either of them P with respect to the circle; then since, by (221), the harmonicism of the row of four points $PQXY$ involves and is involved in that of the pencil of four rays $R.PQXY$, and since, by (216, 3°), for the harmonicism of the pencil, the ray RP being always equally inclined to the two rays RX and RY (163), it is necessary and sufficient that the conjugate ray RQ be at right angles to the ray RP, that is, that the point Q be on the polar of the point P with respect to the circle (165), therefore &c. And, if, as regards (a'), ZP and ZQ be any two lines, ZX and ZY the two concurrent tangents to the circle, and P and Q the two points at which their chord of contact XY intersects with ZP and ZQ; then since, by (221), the harmonicism of the pencil of four lines $Z.PQXY$ involves and is involved in that of the row of four points $PQXY$, and since, by (175, 5°), P and Q are conjugate points when ZP and ZQ are conjugate lines with respect to the circle, and conversely, therefore &c., the rest being evident from (a).

From these properties, thus, or in any other manner, independently established, those of the preceding article, with all the consequences to which they lead, follow of course indirectly; both pairs of reciprocal properties, as above shewn, being, in fact, virtually identical.

Cor. 1°. From the first parts of the above, by virtue of the properties (225) and (235, Cor. 7°), it is evident that (see Art. 176)—

a. Every two conjugate points with respect to a circle determine with the polar centre of their line of connection two segments, whose product is constant and equal in magnitude and sign to the square of the semi-chord intercepted by the circle on the line.

a'. Every two conjugate lines with respect to a circle determine with the polar axis of their point of intersection two angles, the product of whose tangents is constant and equal in magnitude and sign to the square of that of the semi-angle subtended by the circle at the point.

Cor. 2°. By virtue of the general property of Art. 218, it is evident, also, from the same, that (see Art. 178)—

a. When a line intersects one of two circles at a pair of conjugate points with respect to the other, then, reciprocally, it intersects the latter at a pair of conjugate points with respect to the former.

a'. When a point subtends one of two circles by a pair of conjugate lines with respect to the other, then, reciprocally, it subtends the latter by a pair of conjugate lines with respect to the former.

COR. 3°. For the particular case when the circles, in the preceding corollary, intersect at right angles, from the same again, by virtue of properties f' and f, Art. 208, it appears that—

a. Every line intersecting two orthogonal circles in an harmonic system of points passes through one or other common pole of one axis of perspective with respect to one circle and of the other axis of perspective with respect to the other circle (208, e'.)

a'. Every point subtending two orthogonal circles in an harmonic system of tangents lies on one or other common polar of one centre of perspective with respect to one circle and of the other centre of perspective with respect to the other circle (208, e.)

COR. 4°. By aid of the solutions (227, Cor. 3°) and (257, Cor. 5°), the second parts of the above supply obvious solutions of the four following problems, viz.—

a'. On a given line to determine two points conjugates to a given circle and either separated by a given interval or having a given middle point.

a'. At a given point to determine two lines conjugates to a given circle and either separated by a given interval or having a given middle line.

And by aid of the solutions (230) and (257, Cor. 4°), the same, again, supply obvious solutions of the two following problems—

b. On a given line to determine the pair of points conjugates at once to two given circles.

b'. At a given point to determine the pair of lines conjugates at once to two given circles.

260. The line at infinity being the polar of any point with respect to any circle having its centre at the point (165), and the points of intersection of any circle with any line being the points of contact of the tangents to the circle from the pole of the line (165); the following remarkable consequences result from the reciprocal properties of the preceding article, applied to the particular cases of conjugate points at infinity, and of conjugate lines through the centres of circles:

1°. Every two points at infinity in directions at right angles to each other being conjugate points with respect to every circle (174), and every two lines through any point at right angles to each other being conjugate lines with respect to every circle having its centre at the point; hence, from properties a and a' of the preceding article, respectively—

a. Every two points at infinity in directions at right angles to each other are harmonic conjugates with respect to the two imaginary points at which any circle, however situated, intersects with the line at infinity.

a'. Every two lines through any point at right angles to each other are harmonic conjugates with respect to the two imaginary tangents from the point to any circle having its centre at the point.

2°. If O, O', O'', &c. be the several centres of any number of circles situated in any manner; X and Y, X' and Y', X'' and Y'', &c., the several pairs of imaginary points at which they intersect with the line at infinity; and, P and Q, R and S any two pairs of points at infinity in directions at right angles to each other; then since, by the same, the several segments XY, $X'Y'$, $X''Y''$, &c. divide harmonically the same two segments PQ and RS, therefore (230) they coincide with each other; and since, by the same again, the several angles XOY, $X'O'Y'$, $X''O''Y''$, &c. divide harmonically the several pairs of parallel angles POQ and ROS, $PO'Q$ and $RO'S$, $PO''Q$ and $RO''S$, &c. therefore (230) they are parallel to each other; consequently—

b. All circles, however situated, intersect with the line at infinity at the same pair of imaginary points, termed the two circular points at infinity.

b'. All circles, however situated, subtend at their several centres pairs of imaginary tangents parallel to the directions of the two circular points at infinity.

3°. In the particular case where the several points O, O', O'', &c. coincide, that is, when the several circles are concentric; since then not only the several segments XY, $X'Y'$, $X''Y''$, &c. but also the several angles XOY, $X'OY'$, $X''OY''$, &c. coincide, and since, consequently, the several circles have not only a common pair of imaginary points X and Y at infinity, but also a common pair of imaginary tangents OX and OY at those points, therefore—

All concentric circles not only intersect but also touch at the two circular points at infinity.

4°. The following property of rectangular lines, which is one of considerable importance in the higher branches of geometry, is evident from the preceding properties 1° and 2° combined, viz.—

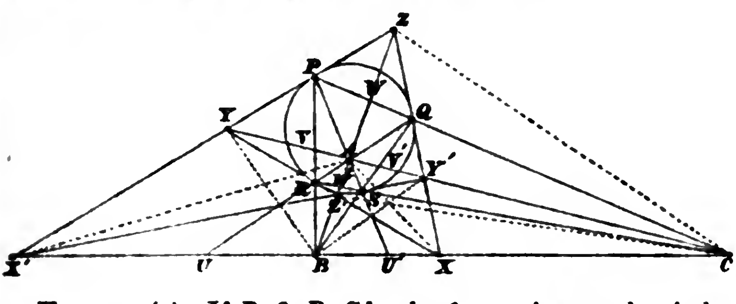
Every two lines intersecting at right angles are conjugate lines with respect to the two circular points at infinity; and, conversely, the two circular points at infinity are conjugate points with respect to every two lines intersecting at right angles (217).

Paradoxical as the above conclusions 2° and 3°, like those of Art. 136, always appear when first stated, all doubt of their legitimacy soon vanishes on consideration of their meaning; every system of figures, in perspective two and two, which, like circles however situated, have a common axis of perspective, intersecting (141) at the same system of points, real or imaginary, on their axis of perspective, and touching (20) at that system of points, if, like concentric circles, they have also a second common axis of perspective coinciding with the first (181, 4°. and 207).

261. The two following reciprocal properties, one of every tetrastigm determined by four points on a circle, and the other of every tetragram determined by four tangents to a circle, result also immediately from the two reciprocal properties of Art. 259, viz.—

a. In every tetrastigm determined by four points on a circle, the intersections of the three pairs of opposite connectors determine a self reciprocal triangle with respect to the circle (170).

a'. In every tetragram determined by four tangents to a circle, the connectors of the three pairs of opposite intersections determine a self reciprocal triangle with respect to the circle (170).



To prove (a). If P, Q, R, S be the four points on the circle; A, B, C the three intersections of their three pairs of opposite connectors QR and PS, RP and QS, PQ and RS; and, U and U', V and V', W and W' the three pairs of intersections of the same pairs of connectors with the three opposite sides BC, CA, AB of the triangle ABC; then since, by the fundamental property (a') of Art. 236, the three pencils of four rays $A.BCUU'$, $B.CAVV'$, $C.ABWW'$ are harmonic, and since, consequently, by (221), the six rows of four points $QRAU$ and $PSAU'$, $RPBV$ and $QSBV'$, $PQCW$ and $RSCW'$ are harmonic, therefore, by the property (a) of Art. 259, the three pairs of points U and U', V and V', W and W' lie on the three polars of the three points A, B, C with respect to the circle; and therefore &c. (170).

To prove a'. If X and X', Y and Y', Z and Z' be the three pairs of opposite intersections of the tetragram determined by the four tangents at the four points P, Q, R, S on the circle; and A, B, C the three vertices of the triangle determined by their three lines of connection XX', YY', ZZ'; then since, by the fundamental property (a) of Art. 236, the three rows of four points $BCXX'$, $CAYY'$, $ABZZ'$ are harmonic, and since, consequently, by (221), the six pencils of four rays $X.QRAX'$ and $X'.PSAX$, $Y.RPBY'$ and $Y'.QSBY$, $Z.PQCZ'$ and $Z'.RSCZ$ are harmonic, therefore, by the property (a') of Art. 259, the three pairs of lines XA and $X'A$, YB and $Y'B$, ZC and $Z'C$ pass through the three poles of the three lines XX', YY', ZZ' with respect to the circle; and therefore &c. (170).

The reader will perceive immediately, that not only are the above properties reciprocals to their common circle in the figure, but that the demonstrations above given of them are reciprocals to it also.

Cor. 1°. Since, in the former case, by (166), the four tangents to the circle at the four points P, Q, R, S intersect two and two in opposite pairs on the polars of the three points A, B, C with respect to the circle; and since, in the latter case, by (166), the four points of contact with the circle P, Q, R, S of the four tangents connect two and two in opposite pairs through the poles of the three lines BC, CA, AB with respect to the circle; hence, from the above properties a and a' combined, it appears that—

In the tetrastigm determined by any four points on a circle, and in the tetragram determined by the four corresponding tangents *to the circle, or conversely; the two self-reciprocal triangles determined by the vertices of the three angles of connection, in the former case, and by the axes of the three chords of intersection, in the latter case, are identical.*

COR. 2°. Again, from the harmonicism of the three pencils of four rays $A.BCUU'$, $B.CAVV'$, $C.ABWW'$, with that of the several rows they determine on all axes, in the former case, and of that of the three rows of four points $BCXX'$, $CAYY'$, $ABZZ'$, with that of the several pencils they determine at all vertices, in the latter case; it appears from the same that—

In the tetrastigm determined by any four points on a circle, and in the tetragram determined by the four corresponding tangents to the circle, or conversely—

a. The three pairs of opposite connectors of the former divide harmonically the three angles of the triangle determined by the axes of the three chords of intersection of the latter.

a'. The three pairs of opposite intersections of the latter divide harmonically the three sides of the triangle determined by the vertices of the three angles of connection of the former.

COR. 3°. Again, the concurrence of the four triads of lines PA, PB, PC; QA, QB, QC; RA, RB, RC; SA, SB, SC involving, by (243, a'), the collinearity of the four triads of points U, V', W'; V, W', U'; W, U', V'; U, V, W in the former case; and the collinearity of the four triads of points Y, Z, X'; Z, X, Y'; X, Y, Z'; X', Y', Z' involving, by (243, a), the concurrence of the four triads of lines AX, BY', CZ'; BY, CZ', AX'; CZ, AX', BY'; AX, BY, CZ in the latter case; it appears from the same that—

In the tetrastigm determined by any four points on a circle, and in the tetragram determined by the four corresponding tangents to the circle, or conversely—

a. The three pairs of opposite connectors of the former intersect with the axes of the three chords of intersection of the latter at six points lying three and three on four lines.

a'. The three pairs of opposite intersections of the latter connect with the vertices of the three angles of connection of the former by six lines passing three and three through four points.

COR. 4°. Again, as the four points P, Q, R, S on the circle, taken in different orders, determine the three different inscribed quadrilaterals whose pairs of opposite vertices connect by the three pairs of lines QR and PS, RP and QS, PQ and RS; and as the four tangents at them to the circle, taken in different orders, determine the three corresponding exscribed quadrilaterals whose pairs of opposite sides intersect at the three pairs of points X and X', Y and Y', Z and Z'; it appears also from the same that—

In each pair of corresponding quadrilaterals determined by any four points on a circle taken in any order and by the four corresponding tangents to the circle taken in the same order—

a. The two pairs of intersections of opposite sides are collinear and harmonic.

a'. The two pairs of connectors of opposite vertices are concurrent and harmonic.

Cor. 5°. Again, the three pairs of points B and C, C and A, A and B being the three pairs of centres of perspective of the three pairs of opposite segments QR and PS, RP and QS, PQ and RS, in the former case; and the three pairs of lines YY' and ZZ', ZZ' and XX', XX' and YY' being the three pairs of axes of perspective of the three pairs of opposite angles QXR and $PX'S$, RYP and $QY'S$, PZQ and $RZ'S$, in the latter case; it appears also from the same that—

a. The two centres of perspective of any two chords inscribed to a circle are conjugate points with respect to the circle, and connect harmonically with the intersection of the axes of the chords by a pair of conjugate lines with respect to the circle.

a'. The two axes of perspective of any two angles exscribed to a circle are conjugate lines with respect to the circle, and intersect harmonically with the connector of the vertices of the angles at a pair of conjugate points with respect to the circle.

From these latter properties it may be easily shewn conversely that—

b. When the directions of any two chords inscribed to a circle divide harmonically the angle determined by any two conjugate lines with respect to the circle, the two centres of perspective of the chords are the two poles of the lines.

b'. When the vertices of any two angles exscribed to a circle divide harmonically the segment determined by any two conjugate *points with respect to the circle, the two axes of perspective of the angles are the two polars of the points.*

For, in the former case, if QR and PS be the two chords; AB and AC the two conjugate lines; B the pole of either of them AC with respect to the circle; BQ and BR its connectors with the extremities of either chord QR; and $P'S'$ the connector of the two second intersections of BR and BQ with the circle; then since, by the above (a), the two lines QR and $P'S'$ pass through the point A and divide harmonically the angle BAC, and since, by hypothesis, the two lines QR and PS do the same, therefore the two lines PS and $P'S'$ coincide; and therefore &c. And, in the latter case, if QXR and $PX'S$ be the two angles; B and C the two conjugate points; AC the polar of either of them B with respect to the circle; Y and Y' its intersections with the sides of either angle QXR; and X'' the intersection of the two second tangents from Y and Y' to the circle; then since, by the above (a'), the two points X and X'' lie on the line BC and divide harmonically the segment BC, and since, by hypothesis, the two points X and X' do the same, therefore the two points X' and X'' coincide; and therefore &c.

It is evident, from these latter properties, that *for every triangle self-reciprocal with respect to a circle, an infinite number of tetrastigms could be inscribed to the circle whose pairs of opposite points would connect through the vertices of the triangle, and an infinite number of tetragrams could be exscribed to the circle whose pairs of opposite lines would intersect on the sides of the triangle.* For, by those properties, every pair of lines dividing harmonically any angle of the triangle would determine four points on the circle fulfilling the former condition, and every pair of points dividing harmonically any side of the triangle would determine four tangents to the circle fulfilling the latter condition; and therefore &c.

Cor. 6°. Again, the three pairs of points X and X', Y and Y', Z and Z', and the three pairs of lines QR and PS, RP and QS, PQ and RS being pole and polar to each other with respect to the circle; therefore, from the harmonicism of the three rows of four points $BCXX'$, $CAYY'$, $ABZZ'$, and of the three pencils of four rays $A.BCUU'$, $B.CAVV'$, $C.ABWW'$, it appears from the same, as in Cors. 2° and 4°, that—

a. For every two chords inscribed to a circle, the two poles of their directions are collinear with, and harmonic conjugates with respect to, their two centres of perspective.

a′. For every two angles exscribed to a circle, the two polars of their vertices are concurrent with, and harmonic conjugates with respect to, their two axes of perspective.

COR. 7°. Again, since, by the second part of Art. 257, the three lines BC, CA, AB determine the three pairs of points on the circle, and the three points A, B, C determine the three pairs of tangents to the circle, which divide harmonically the three pairs of arcs QR and PS, RP and QS, PQ and RS; it appears also, from the same, that—

a. The two centres of perspective of any two chords inscribed to a circle are collinear with the two points on the circle which divide harmonically the two arcs intercepted by the chords.

a′. The two axes of perspective of any two angles exscribed to a circle are concurrent with the two tangents to the circle which divide harmonically the two arcs intercepted by the angles.

Of these latter properties the first (a) supplies an obvious and very rapid method of determining by linear constructions only, without the aid of a circle, the two points on a given circle which divide two given arcs of it harmonically. See Arts. 230 and 257, Cor. 4°.

COR. 8°. Again, every three of the four points P, Q, R, S on the circle, in the former case, determining an inscribed triangle whose three sides pass through the three points A, B, C, every two of which are conjugates to each other and to the third with respect to the circle; and every three of the four tangents at the four points P, Q, R, S to the circle, in the latter case, determining an exscribed triangle whose three vertices lie on the three lines BC, CA, AB, every two of which are conjugates to each other and to the third with respect to the circle; it appears also, from the same, that—

a. In every triangle inscribed to a circle, every two of the three sides intersect with every line conjugate to the third at a pair of conjugate points with respect to the circle.

a′. In every triangle exscribed to a circle, every two of the three vertices connect with every point conjugate to the third by a pair of conjugate lines with respect to the circle.

From these latter properties it may be easily shewn that conversely—

b. When, of a triangle inscribed to a circle, two of the three sides pass through a pair of conjugate points with respect to the circle, the third passes through the pole of their line of connection.

b'. When, of a triangle exscribed to a circle, two of the three vertices lie on a pair of conjugate lines with respect to the circle, the third lies on the polar of their point of intersection.

For, in the former case, if PQR be the inscribed triangle whose two sides PR and PQ pass through the two conjugate points B and C with respect to the circle; A the pole of the line BC; and V and W the two points at which the two lines AC and AB, which, by (175, 5°), are the polars of the two points B and C, intersect with the aforesaid sides PR and PQ of the triangle; then, the two rows of four points $PRBV$ and $PQCW$, having the common point P, being harmonic (259, *a*), therefore, by (222, 5°), the three lines BW, CV, and QR are concurrent; and therefore &c. And, in the latter case, if XYZ be the exscribed triangle whose two vertices Y and Z lie on the two conjugate lines AC and AB with respect to the circle; BC the polar of the point A; and BY and CZ the two lines by which the two points B and C, which, by (175, 5°), are the polars of the two lines AC and AB, connect with the aforesaid vertices Y and Z of the triangle; then, the two pencils of four rays $Y.ZXBC$ and $Z.YXBC$, having the common ray YZ, being harmonic (259, *a'*), therefore, by (222, 6°), the three points B, C, and X are collinear; and therefore &c.

Cor. 9°. If, while the two triangles PQR and XYZ determined by any three of the four points on the circle, in the former case, and by the corresponding three of the four tangents to the circle, in the latter case, with the circle to which they are respectively inscribed and exscribed, are supposed to remain fixed; the triangle ABC, connected with them as above, be conceived to vary, in consequence of the simultaneous variation of the fourth point S and of the corresponding tangent $X'Y'Z'$, on which, in that case, it of course depends; then since, by the above, the triangle ABC in every position is self-reciprocal with respect to the circle; it appears, consequently, that—

For every two triangles determined by any three points on a *circle and by the three corresponding tangents to the circle, or conversely, an infinite number of triangles could be constructed, at once inscribed to the former and exscribed to the latter, and all self-reciprocal with respect to the circle.*

It is evident, from this latter, that the solutions of the two reciprocal problems, "for a given circle to determine a self-reciprocal triangle either inscribed to any triangle inscribed to itself or exscribed to any triangle exscribed to itself," are both indeterminate.

COR. 10°. If, on the other hand, while the triangle ABC, with the circle to which it is self-reciprocal, are supposed to remain fixed; the two triangles PQR and XYZ, connected with them as above, be conceived to vary simultaneously, in consequence of the simultaneous variation of the point S and of the tangent $X'Y'Z'$, on which, in that case, they of course depend; then since, by the above, the two triangles PQR and XYZ respectively inscribed and exscribed to the circle are respectively exscribed and inscribed to the triangle ABC; it appears, consequently, that—

For every triangle self-reciprocal with respect to a circle, an infinite number of triangles could be constructed at once inscribed to the circle and exscribed to the triangle; and, also, an infinite number of corresponding triangles at once exscribed to the circle and inscribed to the triangle.

It is evident, from this latter, that the solutions of the two reciprocal problems, "for a given circle to determine either an inscribed triangle exscribed to, or an exscribed triangle inscribed to, any self-reciprocal triangle with respect to itself," are both indeterminate.

COR. 11°. Of the three triangles PQR, XYZ, and ABC, thus constituting in every position a cycle in which each triangle is inscribed to one and exscribed to the other of the remaining two; the two first being in perspective, with the third by virtue of their relations of connexion with it, and with each other by virtue of the general property 1° of Art. 180; it appears consequently that—

In every cycle of three triangles determined by any arbitrary triangle, any exscribed triangle inscribed to its polar circle, and *the corresponding inscribed triangle exscribed to its polar circle, every two of the three are in perspective* (140).

It will be shewn, in another chapter, that for every cycle of three triangles, however originating, in which, as above, each triangle is inscribed to one and exscribed to the other of the remaining two, when any two of the three are in perspective every two of the three are in perspective.

COR. 12°. The centre of perspective S of the two triangles PQR and ABC being a point on the circle, and the axis of perspective $X'Y'Z'$ of the two XYZ and ABC being a tangent to the circle; it appears consequently also that—

a. The centre of perspective of any triangle, with any exscribed triangle inscribed to its polar circle, is a point on the circle.

a'. The axis of perspective of any triangle, with any inscribed triangle exscribed to its polar circle, is a tangent to the circle.

It is evident that when, for the same original triangle, the two derived triangles in those properties correspond, the point on and tangent to the polar circle correspond also.

COR. 13°. If O be the point of concurrence of the three lines of connection PX, QY, RZ of the three pairs of corresponding vertices P and X, Q and Y, R and Z, and I the line of collinearity of the three points of intersection P', Q', R' of the three pairs of corresponding sides QR and YZ, RP and ZX, PQ and XY, of the two triangles PQR and XYZ; then, from the harmonicism of the three pencils of four rays $P.QRXX'$, $Q.RPYY'$, $R.PQZZ'$ (236, a), and consequently (221) of the two rows of four points determined by any two of them on the two non-corresponding sides of the triangle ABC through whose intersection their two vertices connect, which two rows have that intersection for a common point; therefore, by (222, 5°), the point O, that is, the centre of perspective of the two triangles PQR and XYZ, is collinear with the three points U, V, W, that is, with the axis of perspective of the two triangles PQR and ABC; and, from the harmonicism of the three rows of four points $YZPP'$, $ZXQQ'$, $XYRR'$ (236, a'), and consequently (221) of the two pencils of four rays determined by any two of them at the two non-corresponding vertices of the triangle ABC on whose connector their two axes intersect, which two pencils have that connector for a common ray; therefore, by (222, 6°), the line I, that is, the axis of perspective of the two triangles PQR and XYZ, is concurrent with the three lines AX, BY, CZ, that is, with the centre of perspective of the two triangles XYZ and ABC; hence it appears that—

In every cycle of three triangles determined by any arbitrary triangle, any exscribed triangle inscribed to its polar circle, and the corresponding inscribed triangle exscribed to its polar circle.

a. The centre of perspective of the second and third lies on the axis of perspective of the first and second.

a′ The axis of perspective of the second and third passes through the centre of perspective of the first and third.

The centre of perspective S of the two triangles PQR and ABC lying also, evidently, on the axis of perspective $X'Y'Z'$ of the two triangles XYZ and ABC; these properties for the whole three triangles may consequently be stated more symmetrically as follows:—

In every cycle of three triangles determined by any arbitrary triangle, any exscribed triangle inscribed to its polar circle, and the corresponding inscribed triangle exscribed to its polar circle; the centre of perspective of each with that to which it is inscribed lies on its axis of perspective with that to which it is exscribed.

It will be seen, in another chapter, that this latter property is true generally of every cycle of three triangles, each inscribed to one and exscribed to the other of the remaining two and in perspective with either and consequently with both.

Cor. 14°. Since, from the harmonicism of the three rows of four points $QRAU$, $RPBV$, $PQCW$, the line of collinearity of the three points U, V, W is the polar of the point of concurrence of the three lines AP, BQ, CR with respect to the three sides of the triangle PQR (250, Cor. 2°); and since, from the harmonicism of the three pencils of four rays $X.YZAX'$, $Y.ZXBY'$, $Z.XYCZ'$, the point of concurrence of the three lines AX, BY, CZ is the pole of the line of collinearity of the three points X', Y', Z' with respect to the three vertices of the triangle XYZ (250, Cor. 2°); hence, conceiving the point S and the line $X'Y'Z'$ to vary while the two triangles PQR and XYZ remain fixed, it follows from the two reciprocal properties a and a' of the preceding corollary (12°), that—

a. If a variable point describe a fixed circle, its polar with respect to the three sides of any inscribed triangle turns round a fixed point, the centre of perspective of the inscribed with the corresponding exscribed triangle.

a'. If a variable line envelope a fixed circle, its pole with respect to the three vertices of any exscribed triangle moves upon a fixed line, the axis of perspective of the exscribed with the corresponding inscribed triangle.

In the particular case when both triangles are equilateral, their centre and axis of perspective, in all cases evidently pole and polar to each other with respect to the circle, being then the centre of the circle and the line at infinity (142), it follows from the converses of the preceding properties a and a', that—

b. If a variable line turn round a fixed point, its pole with respect to the three vertices of any equilateral triangle concentric with the point describes the circle circumscribed to the triangle.

b'. If a variable point describe the line at infinity, its polar with respect to the three sides of any equilateral triangle envelopes the circle inscribed to the triangle.

The polar of a point at infinity, with respect to any system of lines, being the diameter, corresponding to its direction, of the polygram determined by the lines (248, c); this latter property b' is therefore identical with that stated in the concluding paragraph of Art. 126, viz., that *in every equilateral triangle the several diameters of the figure envelope its inscribed circle.*

COR. 15°. Again, since from the harmonicism of the three rows of four points $BCUU'$, $CAVV'$, $ABWW'$, the line of collinearity of the three points U, V, W is the polar of the point of concurrence of the three lines AU', BV', CW' with respect to the three sides of the triangle ABC; and since, from the harmonicism of the three rows of four points $BCXX'$, $CAYY'$, $ABZZ'$, the point of concurrence of the three lines AX, BY, CZ is the pole of the line of collinearity of the three points X', Y', Z' with respect to the three vertices of the triangle ABC; hence, from the same, again, it appears also, that—

a. The pole, with respect to the three vertices of any triangle, of its axis of perspective with any exscribed triangle inscribed to its polar circle, is a point on the circle.

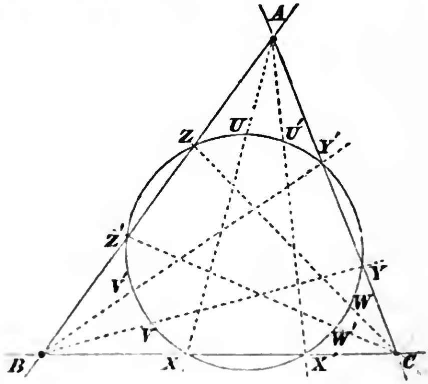
a'. The polar, with respect to the three sides of any triangle, of its centre of perspective with any inscribed triangle exscribed to its polar circle, is a tangent to the circle.

It is evident, as in Cor. 12°, that when, for the same original triangle, the two derived triangles in those properties correspond, the point on and tangent to the polar circle correspond also.

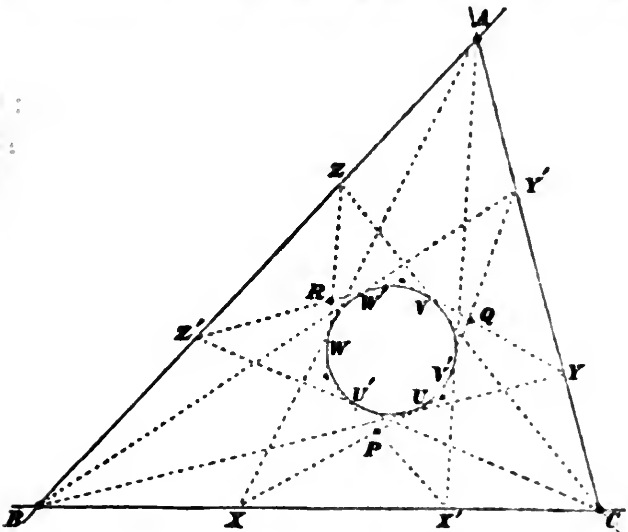
262. From the two reciprocal properties of the preceding article, combined with the general property (180, 1°) that every two triangles reciprocal polars to each other with respect to a circle are in perspective, the two following reciprocal properties of a triangle with respect to an arbitrary circle can be readily inferred, viz.—

a. The three angles, subtended at the vertices of a triangle by the three pairs of intersections of its opposite sides with an arbitrary circle, determine three second pairs of intersections with the circle whose connectors intersect collinearly with the corresponding sides of the triangle.

a'. The three segments, intercepted on the sides of a triangle by the three pairs of tangents from its opposite vertices to an arbitrary circle, determine three second pairs of tangents to the circle whose intersections connect concurrently with the corresponding vertices of the triangle.



For, in the case of *a*, if *A*, *B*, *C* be the three vertices of the triangle; *X* and *X'*, *Y* and *Y'*, *Z* and *Z'* the three pairs of intersections of its opposite sides with the circle; *U* and *U'*, *V* and *V'*, *W* and *W'* the three second pairs of intersections of the three angles *XAX'*, *YBY'*, *ZCZ'* with the circle; and *P*, *Q*, *R* the three intersections of the three pairs of lines *UU'* and *XX'*, *VV'* and *YY'*, *WW'* and *ZZ'*; then since, by the general property (*a*) of the preceding article, the three points *P*, *Q*, *R* lie on the polars of the three points *A*, *B*, *C* with respect to the circle; therefore, by the general property (180, 1°), they lie on the axis of perspective of the triangle *ABC* and its polar triangle *A'B'C'* with respect to the circle; and therefore &c. And, in the case of *a'*, if *A*, *B*, *C*, as before, be the three vertices of the triangle; *XX'*, *YY'*, *ZZ'* the three segments intercepted on its opposite sides by the three pairs of



tangents from them to the circle; and *P*, *Q*, *R* the three intersections of the three second pairs of tangents *XU* and *X'U'*, *YV* and *Y'V'*, *ZW* and *Z'W'* from the three pairs of points *X* and *X'*, *Y* and *Y'*, *Z* and *Z'* to the circle; then since, by the general property (*a'*) of the preceding article, the three lines *PA*, *QB*, *RC* pass through the poles of the three lines *BC*, *CA*, *AB* with respect to the circle; therefore, by the general property

(180, 1°), they pass through the centre of perspective of the triangle ABC and its polar triangle $A'B'C'$ with respect to the circle; and therefore &c.

Cor. In the particular cases when the circle either passes through the three vertices or touches the three sides of the triangle; since then, in either case, the three lines UU', VV', WW', in a, are the polars of the three vertices, and the three points P, Q, R, in a', are the poles of the three sides, of the triangle ABC, with respect to the circle; the above properties give consequently, in those cases, the two reciprocal properties, established originally on other principles in Examples 3° and 4°, Art. 137, and inferred subsequently, as particular cases, firstly, from the two reciprocal theorems of Pascal and Brianchon (148, a and b) respecting any hexagon inscribed and exscribed to a circle, and afterwards, from the general property of (180, 1°) respecting any two triangles reciprocal polars to each other with respect to a circle.

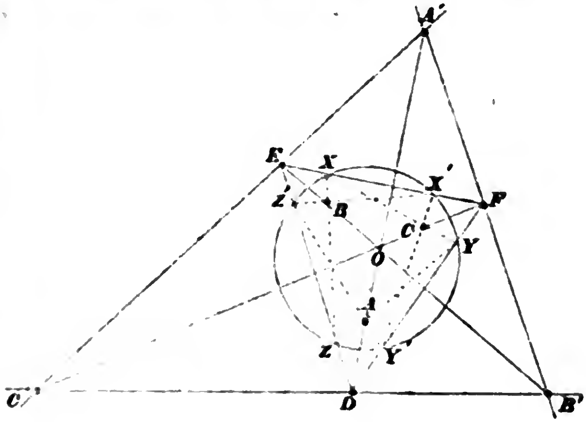
263. By virtue of the same general property (180, 1°) respecting the perspective of every two triangles reciprocal polars to each other with respect to a circle, the two reciprocal properties, b and b', Cor. 5°, of the same article (261), supply the following very elegant reciprocal solutions of the two reciprocal problems—

a. To construct a triangle at once exscribed to a given triangle and inscribed to a given circle.

a'. To construct a triangle at once inscribed to a given triangle and exscribed to a given circle.

In a, if ABC be the given triangle; $A'B'C'$ its polar triangle with respect to the given circle; and D, E, F the three points of intersection of the three concurrent connectors AA', BB', CC' with the three corresponding sides $B'C'$, $C'A'$, $A'B'$ of the latter triangle; then, of the six intersections of the three lines EF, FD, DE with the circle, one set of three for different lines determine one XYZ, and the other set of three the other $X'Y'Z'$, of the two triangles required.

For, the three pairs of lines $B'C'$ and AD, $C'A'$ and BE, $A'B'$ and CF being conjugate pairs with respect to the circle (174), and the three angles they determine being cut har-

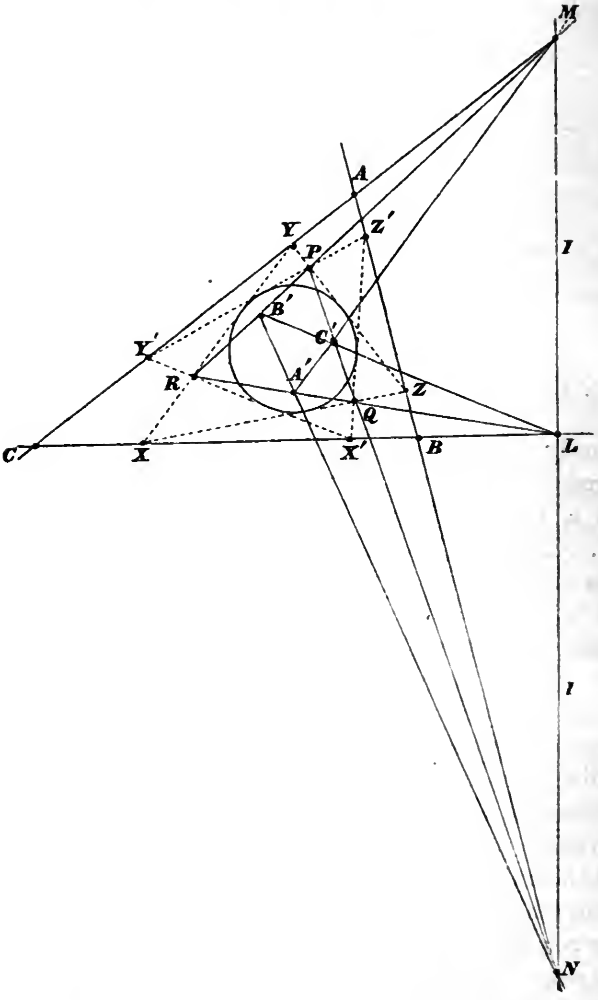


monically by the corresponding angles of the triangle DEF determined by their three vertices (242, a'); therefore, by property b, Cor. 5° of Art. 261, the three pairs of lines YZ and $Y'Z'$, ZX and $Z'X'$, XY and $X'Y'$ pass through the three points A, B, C respectively; and therefore &c.

In a', if ABC (fig., page 98) be the given triangle; $A'B'C'$ its polar triangle with respect to the given circle; L, M, N the three collinear intersections of BC and $B'C'$, CA and $C'A'$, AB and $A'B'$; and P, Q, R the three vertices of the triangle determined by the three connectors of L, M, N with the corresponding vertices A', B', C' of the polar triangle; then, of the six tangents from the three points P, Q, R to the circle, one set of three for different points determine one XYZ, and the other set of three the other $X'Y'Z'$, of the two triangles required.

For, the three pairs of points A' and L, B' and M, C' and N being conjugate pairs with respect to the circle (174), and the three segments they determine being cut harmonically by the corresponding sides of the triangle PQR determined by their three axes (242, a); therefore, by property b', Cor. 5° of Art. 261, the three pairs of points X and X', Y and Y', Z and Z' lie on the three lines BC, CA, AB respectively; and therefore &c.

Cor. In the particular case when the given triangle is self-reciprocal with respect to the given circle (170), the two triangles ABC and $A'B'C'$ then coincide, and the two preceding



constructions are consequently indeterminate; hence, as shewn already in Cor. 10°, Art. 261, the solutions of the two reciprocal problems *to construct a triangle either exscribed to a given triangle and inscribed to its polar circle or inscribed to a given triangle and exscribed to its polar circle* are both indeterminate.

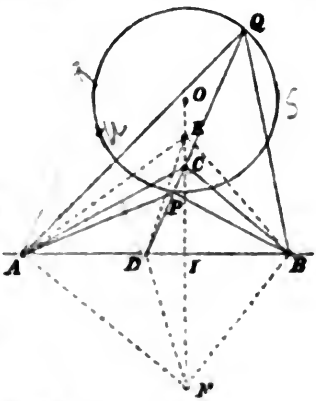
264. For every triangle self-reciprocal with respect to a circle (170), the following metric relation results readily from the general property *a* of Art. 259, combined with the property of every right-angled triangle given in 4°, Cor. 2°, Art. 83, viz.—

For any triangle self-reciprocal with respect to a circle, if A, B, C be the three vertices, AR, BS, CT the three tangents from them to the circle, and P any arbitrary point on the latter, then always—

$$AR^2.(BPC)^2 + BS^2.(CPA)^2 + CT^2.(APB)^2 = 0;$$

the quantities within the parentheses signifying the areas of the three triangles they respectively represent.

For, if C be the vertex of the triangle internal to the circle, Q the second intersection of the line CP with the circle, D its intersection with the side AB of the triangle opposite to C, and DU the tangent from D to the circle; then since, from the harmonicism of the system of four points $CDPQ$ (259, *a*),



$$\frac{CP^2}{DP^2} = -\frac{CP.CQ}{DP.DQ} = -\frac{CT^2}{DU^2},$$

(Euc. III. 35, 36), therefore, at once, multiplying by the square of AB,

$$CP^2.AB^2.DU^2 + DP^2.AB^2.CT^2 = 0,$$

from which, assuming for a moment that

$$BD^2.AR^2 + AD^2.BS^2 = AB^2.DU^2,$$

it follows of course, at once, that

$$CP^2.BD^2.AR^2 + CP^2.AD^2.BS^2 + DP^2.AB^2.CT^2 = 0,$$

which is manifestly equivalent to the above, the three parallelograms $CP.BD$, $CP.AD$, $DP.AB$ (see figure) being the doubles of the three triangles BPC, CPA, APB respectively; and therefore &c.

To prove the relation assumed in the above; if E and F (see figure) be the two points inverse at once to the circle and to the line AB (149), which, by (156) and (177), lie both on the circle on AB as diameter; then since, by (157), $AR = AE = AF$, $BS = BE = BF$, $DU = DE = DF$, and since by the relation 4°, Cor. 2°, Art. 83, the two angles AEB and AFB being both right (Euc. III. 31),

$$BD^2.AE^2 + AD^2.BE^2 = AB^2.DE^2,$$

and $$BD^2.AF^2 + AD^2.BF^2 = AB^2.DF^2,$$

therefore $$BD^2.AR^2 + AD^2.BS^2 = AB^2.DU^2;$$

and therefore &c.

265. The two following reciprocal properties, again of every tetrastigm inscribed and of every tetragram exscribed to a circle, result readily from those of Cor. 10°, Art. 62, and of 4°, Cor. 2°, Art. 179, combined with those of Cors. 3° and 6°, Art. 235, viz.—

a. The four segments intercepted on any arbitrary line, by any circle and by the three angles of connection of any inscribed tetrastigm, have a common segment of harmonic section, real or imaginary.

a'. The four angles subtended at any arbitrary point, by any circle and by the three chords of intersection of any exscribed tetragram, have a common angle of harmonic section, real or imaginary.

To prove a. If L and L', M and M', N and N' be the three pairs of opposite connectors of the tetrastigm; X and X', Y and Y', Z and Z' their three pairs of intersections with the line; and P and Q the two intersections of the latter with the circle; then since, by the property, Cor. 10°, Art. 62,

$$PL.PL' = PM.PM' = PN.PN',$$

$$QL.QL' = QM.QM' = QN.QN';$$

and, since evidently, by pairs of similar triangles,

$$\frac{PL}{QL} = \frac{PX}{QX}, \quad \frac{PM}{QM} = \frac{PY}{QY}, \quad \frac{PN}{QN} = \frac{PZ}{QZ},$$

$$\frac{PL'}{QL'} = \frac{PX'}{QX'}, \quad \frac{PM'}{QM'} = \frac{PY'}{QY'}, \quad \frac{PN'}{QN'} = \frac{PZ'}{QZ'},$$

therefore, at once, by division,

$$\frac{PX.PX'}{QX.QX'} = \frac{PY.PY'}{QY.QY'} = \frac{PZ.PZ'}{QZ.QZ'},$$

and therefore &c.; the three segments XX', YY', ZZ' having consequently, by (235, Cor. 3°), a common segment of harmonic section, real or imaginary, with the segment PQ.

To prove *a'*. If P and P', Q and Q', R and R' be the three pairs of opposite intersections of the tetragram; U and U', V and V', W and W' their three pairs of connectors with the point; and L and M the two tangents from the latter to the circle; then since, by the property 4°, Cor. 2°, Art. 179,

$$\frac{PL.P'L}{PM.P'M}=\frac{QL.Q'L}{QM.Q'M}=\frac{RL.R'L}{RM.R'M};$$

and since, evidently, by (61),

$$\frac{PL}{PM}=\frac{\sin LU}{\sin MU}, \quad \frac{QL}{QM}=\frac{\sin LV}{\sin MV}, \quad \frac{RL}{RM}=\frac{\sin LW}{\sin MW},$$

$$\frac{P'L}{P'M}=\frac{\sin LU'}{\sin MU'}, \quad \frac{Q'L}{Q'M}=\frac{\sin LV'}{\sin MV'}, \quad \frac{R'L}{R'M}=\frac{\sin LW'}{\sin MW'},$$

therefore, at once, by substitution,

$$\frac{\sin LU.\sin LU'}{\sin MU.\sin MU'}=\frac{\sin LV.\sin LV'}{\sin MV.\sin MV'}=\frac{\sin LW.\sin LW'}{\sin MW.\sin MW'},$$

and therefore &c.; the three angles UU', VV', WW' having consequently, by (235, Cor, 6°), a common angle of harmonic section, real or imaginary, with the angle LM.

The above reciprocal properties evidently verify, for the particular cases of tetrastigms inscribed and of tetragrams exscribed to circles, the general properties established for all tetrastigms and tetragrams in Cor. 3°, Art. 245.

COR. 1°. The extremities of the common segment of harmonic section, in the former case, being conjugate points, by (245, Cor. 1°, *b'*), with respect to the tetrastigm, and, by (259, *a*), with respect to the circle; and the sides of the common angle of harmonic section, in the latter case, being conjugate lines, by (245, Cor. 1°, *b*), with respect to the tetragram, and, by (259, *a'*), with respect to the circle; hence, from the above, it appears that—

a. Every two conjugate points with respect to any tetrastigm inscribed to a circle are conjugate points with respect to the circle.

a'. Every two conjugate lines with respect to any tetragram exscribed to a circle are conjugate lines with respect to the circle.

COR. 2°. Since, for every two points conjugates at once with respect to a circle and to any inscribed tetrastigm, the four polars of either, with respect to the circle and to the three angles of connection of the tetrastigm, pass through the other (174, and 245, Cor. 5°, f'); and since, for every two lines conjugates at once with respect to a circle and to any exscribed tetragram, the four poles of either, with respect to the circle and to the three chords of intersection of the tetragram, lie on the other (174, and 245, Cor. 5°, f); hence also, from the above, it appears that—

a. The four polars of any point, with respect to any circle and to the three angles of connection of any inscribed tetrastigm, are concurrent.

a'. The four poles of any line, with respect to any circle and to the three chords of intersection of any exscribed tetragram, are collinear.

Cor. 3°. The centre of any circle and the line at infinity being pole and polar to each other with respect to the circle; it appears, from the latter properties, for the particular cases when the point in a is the centre of the circle and when the line in a' is the line at infinity, that—

a. The three polars of the centre of a circle, with respect to the three angles of connection of any inscribed tetrastigm, are concurrent with the line at infinity.

a'. The three poles of the line at infinity, with respect to the three chords of intersection of any tetragram exscribed to a circle, are collinear with the centre of the circle.

The pole of any segment with respect to the line at infinity being the middle point of the segment (216, 3°); the latter property a' may be stated, otherwise thus, as follows—

In every tetrastigm determined by four tangents to a circle, the three middle points of the three chords of intersection of the figure are collinear with the centre of the circle.

This property the reader may very easily verify, *à priori*, for himself.

Cor. 4°. Again, every point at infinity and the diameter of any circle perpendicular to its direction being pole and polar to each other with respect to the circle; it appears also, from the same properties, for the particular cases where the point in a is at infinity and where the line in a' passes through the centre of the circle, that—

a. The three polars of any point at infinity, with respect to the three angles of connection of any tetrastigm inscribed to a circle, are concurrent with the diameter of the circle perpendicular to the direction of the point.

a'. The three poles of any diameter of a circle, with respect to the three chords of intersection of any tetragram exscribed to the circle, are collinear with the point at infinity in the direction perpendicular to the diameter.

COR. 5°. Again, every point on a circle and the tangent at it to the circle being pole and polar to each other with respect to the circle; it appears also, from the same properties, for the particular cases when the point in *a* is on the circle and when the line in *a'* touches the circle, that—

a. The three polars of any point on a circle, with respect to the three angles of connection of any inscribed tetrastigm, are concurrent with the tangent at the point.

a'. The three poles of any tangent to a circle, with respect to the three chords of intersection of any exscribed tetragram, are collinear with the point of contact of the tangent.

COR. 6°. In the particular cases, of the original properties of the present article, when the arbitrary line in *a* is the line at infinity and when the arbitrary point in *a'* is the centre of the circle; since, by (260, 1°, *a'*), every two points harmonic conjugates with respect to the two circular points at infinity subtend right angles at all points not at infinity, and since, by (216, 3°), the sides of all right angles are the bisectors of all angles they cut harmonically; it appears, consequently, from them, for those particular cases, that—

a. In every tetrastigm inscribed to a circle, the three segments intercepted by the three angles of connection on the line at infinity subtend at every point three angles having a common pair of bisectors.

a'. In every tetragram exscribed to a circle, the three angles subtended by the three chords of intersection at the centre of the circle have a common pair of bisectors.

These properties, like those of Cor. 3°, the reader may easily verify, *à priori*, for himself.

266. The two following reciprocal properties, analogous to those of Art. 246, are evident from those of Art. 259, viz.—

a. If on a variable line L, turning round a fixed point O, and intersecting a fixed circle A_0 at two variable points X_1 and X_2, a variable point P be taken so as to satisfy the relation

$$\frac{PX_1}{OX_1} + \frac{PX_2}{OX_2} = 0;$$

the point P moves on a fixed line I; the polar, viz., of the point O with respect to the circle.

a'. If through a variable point P, moving on a fixed line I, and subtending a fixed circle A_0 by two variable tangents U_1 and U_2, a variable line be drawn so as to satisfy the relation

$$\frac{\sin LU_1}{\sin IU_1} + \frac{\sin LU_2}{\sin IU_2} = 0;$$

the line L turns round a fixed point O; the pole, viz., of the line I with respect to the circle.

For, in the case of a, the two points O and P, being harmonic conjugates with respect to the two X_1 and X_2 (214), are conjugate points with respect to the circle (259), and therefore &c. (175, 1°); and, in the case of a', the two lines I and L, being harmonic conjugates with respect to the two U_1 and U_2 (214), are conjugate lines with respect to the circle (259), and therefore &c. (175, 1°).

267. The two reciprocal properties of the preceding article, respecting a single circle, are evidently particular cases of the two following, respecting a system of any number of circles; which are analogous to those of Art. 248 respecting a system of any number of lines, and with the establishment of which we shall conclude the present long Chapter.

a. If on a variable line L, turning round a fixed point O, and intersecting any system of fixed circles A_0, B_0, C_0, &c. at a system of pairs of variable points X_1 and X_2, Y_1 and Y_2, Z_1 and Z_2, &c., a variable point P be taken so as to satisfy the relation

$$a.\left(\frac{PX_1}{OX_1} + \frac{PX_2}{OX_2}\right) + b.\left(\frac{PY_1}{OY_1} + \frac{PY_2}{OY_2}\right) + c.\left(\frac{PZ_1}{OZ_1} + \frac{PZ_2}{OZ_2}\right) + \&c. = 0,$$

a, b, c, &c. being any system of finite multiples, positive or negative; the point P moves on a fixed line I, termed the polar of the point O with respect to the system of circles A_0, B_0, C_0, &c. for the system of multiples a, b, c, &c.

a′. If through a variable point P, moving on a fixed line I, and subtending any system of fixed circles A_0, B_0, C_0, &c. by a system of pairs of variable tangents U_1 and U_2, V_1 and V_2, W_1 and W_2, &c., a variable line L be drawn so as to satisfy the relation

$$a.\left(\frac{\sin LU_1}{\sin IU_1}+\frac{\sin LU_2}{\sin IU_2}\right)+b.\left(\frac{\sin LV_1}{\sin IV_1}+\frac{\sin LV_2}{\sin IV_2}\right)$$
$$+c.\left(\frac{\sin LW_1}{\sin IW_1}+\frac{\sin LW_2}{\sin IW_2}\right)+\&c.=0,$$

a, b, c, &c. being any system of finite multiples, positive or negative; the line L turns round a fixed point O, termed the pole of the line I with respect to the system of circles A_0, B_0, C_0, &c. for the system of multiples a, b, c, &c.

To prove *a*. If X, Y, Z, &c. be the several points of intersection of the variable line L with the several polars A, B, C, &c. of the fixed point O with respect to the several fixed circles A_0, B_0, C_0, &c.; then since, from the harmonicism of the several systems of four points OXX_1X_2, OYY_1Y_2, OZZ_1Z_2, &c., whatever be the position of P on their common axis, by (220, *a*),

$$\frac{PX_1}{OX_1}+\frac{PX_2}{OX_2}=2.\frac{PX}{OX},$$

$$\frac{PY_1}{OY_1}+\frac{PY_2}{OY_2}=2.\frac{PY}{OY},$$

$$\frac{PZ_1}{OZ_1}+\frac{PZ_2}{OZ_2}=2.\frac{PZ}{OZ}, \ \&c.,$$

which, multiplied horizontally by *a*, *b*, *c*, &c., and added vertically, give the equality

$$\Sigma\left\{a.\left(\frac{PX_1}{OX_1}+\frac{PX_2}{OX_2}\right)\right\}=2.\Sigma\left(a.\frac{PX}{OX}\right),$$

from which it follows that when either equivalent = 0, so is the other also; but when the latter equivalent = 0, then, by (248, *a*), *the point P lies on the polar of the point O with respect to the system of lines A, B, C, &c. for the system of multiples a, b, c, &c.;* and therefore &c.

To prove *a′*. If U, V, W, &c. be the several lines of connection of the variable point P with the several poles A, B, C, &c. of the fixed line I with respect to the several fixed circles

A_0, B_0, C_0, &c.; then since, from the harmonicism of the several systems of four lines IUU_1U_2, IVV_1V_2, IWW_1W_2, &c., whatever be the direction of L through their common vertex, by (220, a'),

$$\frac{\sin LU_1}{\sin IU_1} + \frac{\sin LU_2}{\sin IU_2} = 2 . \frac{\sin LU}{\sin IU},$$

$$\frac{\sin LV_1}{\sin IV_1} + \frac{\sin LV_2}{\sin IV_2} = 2 . \frac{\sin LV}{\sin IV},$$

$$\frac{\sin LW_1}{\sin IW_1} + \frac{\sin LW_2}{\sin IW_2} = 2 . \frac{\sin LW}{\sin IW}, \text{ \&c.}$$

which, multiplied horizontally by a, b, c, &c., and added vertically, give the equality

$$\Sigma \left\{a . \left(\frac{\sin LU_1}{\sin IU_1} + \frac{\sin LU_2}{\sin IU_2}\right)\right\} = 2 . \Sigma \left(a . \frac{\sin LU}{\sin IU}\right),$$

from which it follows that when either equivalent = 0, so is the other also; but when the latter equivalent = 0, then, by (248, a'), *the line L passes through the pole of the line I with respect to the system of points A, B, C, &c. for the system of multiples a, b, c, &c.;* and therefore &c.

The above very general properties are not reciprocals in the same sense as those of Art. 248, to which they are analogous; each, to an arbitrary circle, reciprocating, not into the other, but into the corresponding property of the figures of more general forms into which circles reciprocate for all positions of the centre of reciprocation not coinciding with their own (172).

COR. It being evident from the above demonstrations, that—

a. The polar of any point, with respect to any system of circles, for any system of multiples, is the same as if the several circles were all removed, and replaced by the several polars of the point with respect to themselves.

a'. The pole of any line, with respect to any system of circles, for any system of multiples, is the same as if the several circles were all removed, and replaced by the several poles of the line with respect to themselves.

All questions concerning the polars of points or the poles of lines, with respect to systems of circles, for systems of multiples, may therefore, in all cases, be regarded as reduced to the corresponding questions in which the several circles are replaced by lines in the former case and by points in the latter case, the main points connected with which have been already very fully discussed in Arts. 246 to 251 at the close of the preceding Chapter.

CHAPTER XVI.

THEORY OF ANHARMONIC SECTION.

268. WHEN a line, or angle, AB, is cut at two points, or by two lines, C and D, each ratio of ratios of the two pairs of segments, or of the sines of the two pairs of segments, into which it divided, is termed *an anharmonic ratio* of the section of the line, or angle, by the two points, or lines.

Thus, if $AC : BC$, or $\sin AC : \sin BC, = m$, and $AD : BD$, or $\sin AD : \sin BD, = n$, the two ratios $m : n$ and $n : m$ are what are termed anharmonic ratios of the section of the line, or angle, AB, by the two points, or lines, C and D; *for any line or angle, AB, every two points or lines of section, C and D, determine therefore two different anharmonic ratios, reciprocals of each other.*

The name "anharmonic" was given to this simple function of the section of a line or angle by Chasles, who was the first to perceive its utility and to apply it extensively in geometry; because that in the particular case when m and n are equal in magnitude and opposite in sign, the section of the line or angle becomes what from ancient times had been familiarly known as "harmonic," and which from its special importance has been treated of separately in Chapter XIII.

269. The two anharmonic ratios of the section of a line or angle AB by any two points or lines of section C and D, like every other pair of magnitudes reciprocals to each other, have of course always the same sign, positive if C and D be both external or both internal, and negative if one be external and the other internal to AB; but the positions, absolute or relative, of C and D, being quite arbitrary, either may have any absolute magnitude from 0 to ∞, and the other the reciprocal of the same from ∞ to 0.

When either anharmonic ratio $=0$, the other $=\infty$; and, conversely, when either $=\infty$, the other $=0$; in both those extreme cases it is evident that one or other of the two points or lines of section, C and D, coincides with one or other of the two extremities of the line or angle, A and B.

When either anharmonic ratio $=\pm 1$, the other also $=\pm 1$; these are the only two cases in which the two anharmonic ratios of the section of a line or angle are equal, $+1$ and -1 being the only two numbers which are equal to their reciprocals; in the latter case the section of the line or angle is, as already noticed, harmonic; and in the former case it is evident that either the two points or lines of section, C and D, or the two extremities of the line or angle, A and B, coincide with each other.

For the three particular values of either anharmonic ratio of the section of a line or angle AB by two points or lines C and D, 0, ∞, *and* $+1$, *some two of the four points or lines A, B, C, D, therefore, coincide.* For every other value of either, however, they are all four distinct from each other.

270. When for one of the two points or lines of section, D suppose, the two simple ratios for the single section each $=1$, that is, when D is the point or line of external bisection of the segment or angle AB; then, whatever be the position of the other point or line of section C, *the two anharmonic ratios for the double section by C and D combined, become in that case the two simple ratios for the single section by C alone.* This particular case is deserving of special attention, not only on account of its comparative simplicity, but because, as we shall presently see, *every other case of anharmonic section of a line or angle, whatever be the positions of the two points or lines of section, may be reduced to it.*

271. When two segments, or angles, or a segment and an angle, AB and $A'B'$, are cut in equal anharmonic ratios by two pairs of sectors, C and D, C' and D', they are said to be cut *equianharmonically;* and so, also, is the same segment, or angle, AB, when cut in equal anharmonic ratios by two different pairs of sectors, C and D, C' and D'. A more general definition of the relation of equianharmonicism will be given further on.

Two pairs of lines, L and M, L' and M', are said sometimes to intersect, and sometimes to divide, two segments, AB and $A'B'$, equianharmonically, when their pairs of intersections with their axes divide them equianharmonically; and, two pairs of points, P and Q, P' and Q', are said sometimes to subtend, and sometimes to divide, two angles, AB and $A'B'$, equianharmonically, when their pairs of connectors with their vertices divide them equianharmonically. These modes of expression are frequently employed for shortness in the applications of the theory of anharmonic section.

272. *When a segment or angle AB is cut equianharmonically by the two pairs of sectors C and D, C' and D', it is also cut equianharmonically by the two pairs C and C', D and D'; and conversely.*

For since, by hypothesis,

$$\frac{AC}{BC} : \frac{AD}{BD} = \frac{AC'}{BC'} : \frac{AD'}{BD'}, \text{ or, } \frac{\sin AC}{\sin BC} : \frac{\sin AD}{\sin BD} = \frac{\sin AC'}{\sin BC'} : \frac{\sin AD'}{\sin BD'};$$

Therefore, at once, by alternation,

$$\frac{AC}{BC} : \frac{AC'}{BC'} = \frac{AD}{BD} : \frac{AD'}{BD'}, \text{ or, } \frac{\sin AC}{\sin BC} : \frac{\sin AC'}{\sin BC'} = \frac{\sin AD}{\sin BD} : \frac{\sin AD'}{\sin BD'};$$

and therefore &c.

In exactly the same manner it may be shown, in accordance with the mode of expression noticed at the close of the preceding article, that *when a segment AB is divided equianharmonically by the two pairs of lines L and M, L' and M', it is also divided equianharmonically by the two pairs L and L', M and M';* and that *when an angle AB is subtended equianharmonically by the two pairs of points P and Q, P' and Q', it is also subtended equianharmonically by the two pairs P and P', Q and Q'.* For since, in the two cases, respectively,

$$\frac{AL}{BL} : \frac{AM}{BM} = \frac{AL'}{BL'} : \frac{AM'}{BM'}, \text{ and, } \frac{AP}{BP} : \frac{AQ}{BQ} = \frac{AP'}{BP'} : \frac{AQ'}{BQ'};$$

therefore, at once, by alternation, in the two, respectively,

$$\frac{AL}{BL} : \frac{AL'}{BL'} = \frac{AM}{BM} : \frac{AM'}{BM'}, \text{ and, } \frac{AP}{BP} : \frac{AP'}{BP'} = \frac{AQ}{BQ} : \frac{AQ'}{BQ'};$$

and therefore &c.

The following is an obvious corollary from the above:

When a segment or angle AB is cut harmonically by the two segments or angles XY and X'Y', it is cut equianharmonically by the two XX' and YY'. And also *every two lines L and M and their two poles P and Q with respect to any segment AB, or any two points P and Q and their two polars L and M with respect to any angle AB, divide the segment or angle equianharmonically.*

273. *The two anharmonic ratios of the section of any segment or angle AB by any two points or lines C and D are the same in magnitude and sign as those of the section of the segment or angle CD by the two points or lines A and B.*

For, by simple alternation,

$$\frac{AC}{BC}:\frac{AD}{BD}=\frac{CA}{DA}:\frac{CB}{DB}, \text{ or, } \frac{\sin AC}{\sin BC}:\frac{\sin AD}{\sin BD}=\frac{\sin CA}{\sin DA}:\frac{\sin CB}{\sin DB},$$

and

$$\frac{AD}{BD}:\frac{AC}{BC}=\frac{CB}{DB}:\frac{CA}{DA}, \text{ or, } \frac{\sin AD}{\sin BD}:\frac{\sin AC}{\sin BC}=\frac{\sin CB}{\sin DB}:\frac{\sin CA}{\sin DA},$$

and therefore &c.

Of this general property of anharmonic section, from which it appears that *every two segments having a common axis, or angles having a common vertex, AB and CD, cut each other equianharmonically*, the property of harmonic section proved in Art. 218 is obviously a particular case.

In the same manner exactly it may be shown, in accordance with the mode of expression noticed at the close of Art. 271, that *the two anharmonic ratios of the section of any segment or angle AB by any two lines or points C and D are the same in magnitude and sign as those of the angle or segment CD by the two points or lines A and B.*

For, since in either case, by simple alternation,

$$\frac{AC}{BC}:\frac{AD}{BD}=\frac{CA}{DA}:\frac{CB}{DB}, \text{ and, } \frac{AD}{BD}:\frac{AC}{BC}=\frac{CB}{DB}:\frac{CA}{DA},$$

therefore &c. This important property of anharmonic section, from which it appears that *every segment and angle, AB and CD, however circumstanced as to magnitude or position, divide* *each other equianharmonically*, will be presently considered under another form.

The sign common to the two reciprocal equianharmonic ratios determined by the mutual section of two segments or angles, or of a segment and angle, AB and CD, depends of course on the relative positions of their respective extremities, A and B, C and D; being obviously negative when those of one alternate with those of the other in the order of their occurrence, and positive when they do not.

274. As four points on a common axis or rays through a common vertex A, B, C, D determine, whatever be their order and disposition, six different segments or angles corresponding to each other two and two in three sets of opposite pairs BC and AD, CA and BD, AB and CD; and as, by the preceding, the two segments or angles constituting each pair of opposites cut each other in the same two anharmonic ratios, reciprocals of each other; it follows, therefore, that *four points on a common axis or rays through a common vertex determine in general six different anharmonic ratios, in pairs reciprocals of each other.*

These three pairs of reciprocal ratios corresponding to the three pairs of opposite segments or angles BC and AD, CA and BD, AB and CD, are respectively as follows:

For four points on a common axis,

$$\frac{BA}{CA}:\frac{BD}{CD} \text{ and } \frac{CA}{BA}:\frac{CD}{BD}, \text{ or, } \frac{BA.CD}{CA.BD} \text{ and } \frac{CA.BD}{BA.CD}, \ldots (1).$$

$$\frac{CB}{AB}:\frac{CD}{AD} \text{ and } \frac{AB}{CB}:\frac{AD}{CD}, \text{ or, } \frac{CB.AD}{AB.CD} \text{ and } \frac{AB.CD}{CB.AD}, \ldots (2).$$

$$\frac{AC}{BC}:\frac{AD}{BD} \text{ and } \frac{BC}{AC}:\frac{BD}{AD}, \text{ or, } \frac{AC.BD}{BC.AD} \text{ and } \frac{BC.AD}{AC.BD}, \ldots (3).$$

For four rays through a common vertex,

$$\frac{\sin BA}{\sin CA}:\frac{\sin BD}{\sin CD} \text{ and } \frac{\sin CA}{\sin BA}:\frac{\sin CD}{\sin BD},$$

$$\text{or, } \frac{\sin BA.\sin CD}{\sin CA.\sin BD} \text{ and } \frac{\sin CA.\sin BD}{\sin BA.\sin CD}, \ldots\ldots\ldots (1').$$

$$\frac{\sin CB}{\sin AB}:\frac{\sin CD}{\sin AD} \text{ and } \frac{\sin AB}{\sin CB}:\frac{\sin AD}{\sin CD},$$

or, $\dfrac{\sin CB.\sin AD}{\sin AB.\sin CD}$ and $\dfrac{\sin AB.\sin CD}{\sin CB.\sin AD}$, ... (2').

$\dfrac{\sin AC}{\sin BC}:\dfrac{\sin AD}{\sin BD}$ and $\dfrac{\sin BC}{\sin AC}:\dfrac{\sin BD}{\sin AD}$,

or, $\dfrac{\sin AC.\sin BD}{\sin BC.\sin AD}$ and $\dfrac{\sin BC.\sin AD}{\sin AC.\sin BD}$; .... (3').

which may, for convenience, be represented in either case by the abridged notation P and $\frac{1}{P}$, Q and $\frac{1}{Q}$, R and $\frac{1}{R}$ respectively; and which, as is evident from mere inspection of their values, are connected in either case by the relations

$$P.Q.R = -1 \text{ and } \frac{1}{P}\cdot\frac{1}{Q}\cdot\frac{1}{R} = -1.$$

275. In the case of points on a common axis, if any one of the four, D suppose, be at infinity, the six simple ratios $BD:CD$ and $CD:BD$, $CD:AD$ and $AD:CD$, $AD:BD$ and $BD:AD$, into which that point enters, are all $= 1$; and the six anharmonic ratios for the entire four A, B, C, D become consequently the six simple ratios for the remaining three A, B, C; which, in the order above given in the general case, viz., for BC cut at A, for CA cut at B, and for AB cut at C, are respectively as follows,

$$\frac{BA}{CA} \text{ and } \frac{CA}{BA},\quad \frac{CB}{AB} \text{ and } \frac{AB}{CB},\quad \frac{AC}{BC} \text{ and } \frac{BC}{AC};$$

and which are evidently connected by the same relations as in the general case.

To this comparatively simple case it will appear in the sequel that every other case of anharmonic ratio, whether of points or rays, whatever be the order or disposition of either, may be reduced.

276. Whatever be the order and disposition of four points or rays constituting a row or pencil, A, B, C, D, it is evident that *of the three pairs of reciprocal anharmonic ratios they determine two are always positive and the third always negative;* the negative corresponding to the pair of opposite segments or angles they determine whose extremities alternate with each

other in the order of their succession, and the two positive to the two pairs whose extremities do not so alternate. Hence, as seen above, for three of them P, Q, R, and for their three reciprocals, the product of any three of them of different pairs is always negative.

When three points or rays of an anharmonic system, any one of its six anharmonic ratios, and the order in which the four constituents enter in the formation of the ratio are given, the fourth point or ray is of course implicitly given also; its determination depending only on the section of a given segment or angle into two parts whose lengths or sines shall have a given magnitude and sign.

277. *The six anharmonic ratios P and $\frac{1}{P}$, Q and $\frac{1}{Q}$, R and $\frac{1}{R}$ determined by the same row of four points or pencil of four rays A, B, C, D are connected two and two by the three relations*

$$R+\frac{1}{Q}=1, \quad P+\frac{1}{R}=1, \quad Q+\frac{1}{P}=1,$$

whatever be the order and disposition of the constituents of either.

For, since for every system of four points A, B, C, D on a common axis, whatever be their order and disposition (82),

$$BC.AD+CA.BD+AB.CD=0 \,\ldots\ldots\ldots\ldots\, (a);$$

and, since for every system of four rays A, B, C, D through a common vertex, whatever be their order and disposition (82, Cor. 3°),

$$\sin BC.\sin AD+\sin CA.\sin BD+\sin AB.\sin CD=0 \,\ldots\, (a');$$

therefore, dividing each successively by each of its three components, the three relations above given result at once in each case.

In the particular case of points on a common axis, when one of the four D is at infinity, since then $P=BA:CA$, $Q=CB:AB$, $R=AC:BC$, the above relations are evident to mere inspection; and to this comparatively simple case, as already stated, all others may be reduced.

From the above relations, combined with those already given, it appears that the six anharmonic ratios of the same row or pencil of four points or rays, though in general all different, are never independent of each other, but that, on the contrary, whatever be the order and disposition of the constituent points or rays, *they are always so connected with each other that any one of the entire six determines the remaining five*; so that if any one of them be given or known, all the others may be regarded as implicitly given or known with it.

Thus, supposing P known, then, from the above, $Q = \frac{P-1}{P}$ and $R = \frac{1}{1-P}$; and the three P, Q, R thus known, so of course are their three reciprocals, which are the remaining three ratios.

As an example, let $P = -1$, that is, let the row or pencil form an harmonic system (213); then $Q = 2$ and $R = \frac{1}{2}$. Hence, when four points or rays form an harmonic row or pencil, and when therefore one pair of their reciprocal anharmonic ratios $= -1$, the other two pairs are 2 and $\frac{1}{2}$, and $\frac{1}{2}$ and 2 respectively; the same results obtained in a different manner in Art. 219.

278. *When two rows of four points or pencils of four rays, or a row of four points and a pencil of four rays, A, B, C, D and A', B', C', D', are such that a single anharmonic ratio is the same for both systems, the entire six anharmonic ratios are the same for both systems.*

For, denoting by P and $\frac{1}{P}$, Q and $\frac{1}{Q}$, R and $\frac{1}{R}$ the six for the system A, B, C, D, and by P' and $\frac{1}{P'}$, Q' and $\frac{1}{Q'}$, R' and $\frac{1}{R'}$ the six for the system A', B', C', D'; since, by the preceding article,

$$Q = \frac{P-1}{P}, \quad Q' = \frac{P'-1}{P'}, \quad R = \frac{1}{1-P}, \quad R' = \frac{1}{1-P'},$$

when $P = P'$, then $Q = Q'$ and $R = R'$, and therefore &c. the reciprocals of equal magnitudes being of course equal.

Two rows of four points or pencils of four rays, or a row of four points and a pencil of four rays, thus related to each other that the six anharmonic ratios are the same for both systems, are said *to be equianharmonic* (271); and the pairs of constituents, A and A', B' and B', C and C', D and D', which enter similarly into the several pairs of equal ratios, are said to be *corresponding* or *homologous* pairs.

In every case of equianharmonicism between two systems of four constituents, A, B, C, D and A', B', C', D', which correspond two and two in pairs, A and A', B and B', C and C', D and D', when three pairs of corresponding constituents and one constituent of the fourth pair are given, the second constituent of the fourth pair is of course implicitly given also. Various constructions for determining it will be given further on.

COR. 1°. Since for every two equianharmonic systems of four constituents A, B, C, D and A', B', C', D', by the above, $P=P'$, $Q=Q'$, $R=R'$, therefore for the same, by (277),

$$R+\frac{1}{Q'}=1,\quad P+\frac{1}{R'}=1,\quad Q+\frac{1}{P'}=1,$$

$$R'+\frac{1}{Q}=1,\quad P'+\frac{1}{R}=1,\quad Q'+\frac{1}{P}=1.$$

relations often of much use in establishing the circumstance of equianharmonicism between two systems of four constituents when the simpler relations $P=P'$, $Q=Q'$, $R=R'$ are not as readily applicable.

COR. 2°. Since two or more magnitudes of any kind when equal to the same magnitude are equal to each other, it follows evidently, from the nature of equianharmonicism, as above explained, that *when two or more systems of four constituents A', B', C', D'; A'', B'', C'', D''; A''', B''', C''', D'''; &c. are equianharmonic with the same system A, B, C, D, they are equianharmonic with each other.*

279. Dr. Salmon has employed the following very convenient notation for expressing the equianharmonicism of two or more systems of four constituents A, B, C, D; A', B', C', D'; A'', B'', C'', D''; &c. viz.:

$$\{ABCD\}=\{A'B'C'D'\}=\{A''B''C''D''\}=\&c.,$$

where the symbol $\{ABCD\}$ is regarded as the general representative of the entire six anharmonic ratios for the system A, B, C, D; $\{A'B'C'D'\}$ as that of the entire six for the system A', B', C', D'; $\{A''B''C''D''\}$ as that of the entire six for the system A'', B'', C'', D''; &c. and where the letters representing *corresponding* constituents are invariably written *in the same* *order* in all; so that the *corresponding* groups of ratios, which alone are *equal* in the several systems, may be evident to inspection, without the trouble of seeking for, or the danger of mistaking them. We shall employ the same notation generally in the comparison of equianharmonic systems.

And in the same manner as the *notation* $\{ABCD\}$ is to be regarded as the general *symbolical* representative of the entire six anharmonic ratios of the system of four constituents A, B, C, D, the precaution respecting similarity of order being invariably observed in all cases of comparison with other systems, so the *expression* "anharmonic ratio of four points or rays" when used, as it constantly is, in the singular number, is to be regarded as the general *nominal* representative of the entire six for the system; the same precaution respecting similarity of order being invariably attended to in all cases of comparison between two or more systems.

As the symbol $\{ABCD\}$ is employed to denote, in the sense above explained, the anharmonic ratio of the system of four points A, B, C, D when collinear, or of the system of four lines A, B, C, D when concurrent; so, for shortness, the symbol $\{O.ABCD\}$ is employed to represent, in the same sense, the anharmonic ratio of the pencil of four lines by which the system of four points A, B, C, D, whether collinear or not, connects with the vertex O, or of the row of four points at which the system of four lines A, B, C, D, whether concurrent or not, intersects with the axis O.

280. *Any order of the four constituents of an anharmonic system of points or rays, A, B, C, D suppose, may be altered in three different ways, viz. into B, A, D, C, or C, D, A, B, or D, C, B, A, without affecting, either in magnitude or sign, any of the six anharmonic ratios of the system corresponding to that order.*

To prove this, or, which is the same thing, to shew that always

$$\{ABCD\} = \{BADC\} = \{CDAB\} = \{DCBA\},$$

it will only be necessary (278) to establish its truth for any one of the six ratios for the first order, compared with the three that correspond to it in the other three.

Taking then arbitrarily any one for the first, $\frac{BA.CD}{CA.BD}$ suppose, for the case of points, or its analogue $\frac{\sin BA.\sin CD}{\sin CA.\sin BD}$ for the case of rays, and placing beside it its three correspondents in the other three; we have, for the whole four, the system

$$\frac{BA.CD}{CA.BD},\quad \frac{AB.DC}{DB.AC},\quad \frac{DC.AB}{AC.DB},\quad \frac{CD.BA}{BD.CA},$$

in the case of points, or the analogous system

$$\frac{\sin BA.\sin CD}{\sin CA.\sin BD},\quad \frac{\sin AB.\sin DC}{\sin DB.\sin AC},\quad \frac{\sin DC.\sin AB}{\sin AC.\sin DB},\quad \frac{\sin CD.\sin BA}{\sin BD.\sin CA}$$

in the case of rays; which on mere inspection are seen, in either case, to be equal both in magnitude and sign.

In comparing together the preceding, or any other four equivalent orders; it appears that, to go from any order to an equivalent order, *any two of the four constituents may be interchanged provided the remaining two be interchanged also.* Hence the following simple rule for the formation from any given order of its three equivalents, viz. *every interchange of two constituents is to be accompanied by the interchange of the other two;* this is Chasles' rule, the reason of which is evident from the obvious signification of the double interchange as regards the three pairs of opposite segments or angles determined by the four points or rays of the system.

Cor. Since, from the above, for any row of four points or pencil of four rays A, B, C, D,

$$\{ABCD\} = \{BADC\} = \{CDAB\} = \{DCBA\} \ldots. \ (a);$$

and since again, for any other row of four points or pencil of four rays A', B', C', D',

$$\{A'B'C'D'\} = \{B'A'D'C'\} = \{C'D'A'B'\} = \{D'C'B'A'\} \ldots (a');$$

it follows therefore that the equality of *any* one of the four equivalents of group a to *any* one of the four of group a', whether the two compared correspond to each other or not, is sufficient to establish the equianharmonicism of the two systems, if so related to each other. This is an important consideration of which frequent use is made in the applications of the theory of anharmonic section.

281. *There is one, and but one, case in which one pair of constituents of an anharmonic row or pencil, when all four distinct from each other, may be interchanged, without requiring the simultaneous interchange of the other pair in order to preserve the anharmonic equivalence of the changed to the original order; viz. when the system is harmonic, and when the interchanged constituents are conjugates.*

For, if a system A, B, C, D be such that

$$\{ABCD\} = \{ABDC\},$$

then, according as it consists of points or rays,

$$\frac{AC}{BC} : \frac{AD}{BD} = \frac{AD}{BD} : \frac{AC}{BC}, \text{ or, } \frac{\sin AC}{\sin BC} : \frac{\sin AD}{\sin BD} = \frac{\sin AD}{\sin BD} : \frac{\sin AC}{\sin BC},$$

and therefore, as the case may be,

$$\frac{AC}{BC} : \frac{AD}{BD}, \text{ or } \frac{\sin AC}{\sin BC} : \frac{\sin AD}{\sin BD}, = \pm 1;$$

but, for the positive sign the two points or rays C and D coincide (269), and, for the negative sign they are harmonic conjugates with respect to the two A and B (213); and therefore &c.

Of all criteria of the mutual harmonic section of two segments or angles AB and CD (218), the above relation $\{ABCD\} = \{ABDC\}$ is probably the most universally applicable, especially in the higher departments of geometry. Several examples of its application will be given in the sequel; and it will appear that, in some of the cases of harmonic section established in the two preceding chapters, it might, under a different order of treatment, have replaced with advantage the criteria employed.

282. *When two systems of any common number of constituents, both of points, or both of rays, or one of points and one of rays, A, B, C, D, E, F, G, &c. and A', B', C', D', E', F', G', &c. which correspond in pairs, A and A', B and B', C and C', D and D', E and E', F and F', G and G', &c. are such that any three pairs of corresponding constituents, A and A', B and B', C and C', form equianharmonic systems with every other pair, D and D', E and E', F and F', G and G', &c. then also 1°, every two of the former and every two of the latter; 2°, every one*

of the former and every three of the latter; 3°, *every four of the latter, form equianharmonic systems.*

To prove 1°, or to shew that, when $\{ABCD\} = \{A'B'C'D'\}$ and $\{ABCE\} = \{A'B'C'E'\}$, then $\{BCDE\} = \{B'C'D'E'\}$ and $\{CADE\} = \{C'A'D'E'\}$ and $\{ABDE\} = \{A'B'D'E'\}$.

Since, by hypothesis, $\{ABCD\} = \{A'B'C'D'\}$, therefore

$$\frac{BA}{CA} : \frac{BD}{CD} \text{ or } \frac{\sin BA}{\sin CA} : \frac{\sin BD}{\sin CD} = \frac{B'A'}{C'A'} : \frac{B'D'}{C'D'} \text{ or } \frac{\sin B'A'}{\sin C'A'} : \frac{\sin B'D'}{\sin C'D'},$$

and since, by hypothesis, $\{ABCE\} = \{A'B'C'E'\}$, therefore

$$\frac{BA}{CA} : \frac{BE}{CE} \text{ or } \frac{\sin BA}{\sin CA} : \frac{\sin BE}{\sin CE} = \frac{B'A'}{C'A'} : \frac{B'E'}{C'E'} \text{ or } \frac{\sin B'A'}{\sin C'A'} : \frac{\sin B'E'}{\sin C'E'},$$

therefore, at once, by division of ratios,

$$\frac{BD}{CD} : \frac{BE}{CE} \text{ or } \frac{\sin BD}{\sin CD} : \frac{\sin BE}{\sin CE} = \frac{B'D'}{C'D'} : \frac{B'E'}{C'E'} \text{ or } \frac{\sin B'D'}{\sin C'D'} : \frac{\sin B'E'}{\sin C'E'},$$

and therefore $\{BCDE\} = \{B'C'D'E'\}$; and similarly

$$\{CADE\} = \{C'A'D'E'\} \text{ and } \{ABDE\} = \{A'B'D'E'\};$$

and therefore &c.

To prove 2°, or to shew that,

when $\{ABCD\} = \{A'B'C'D'\}$ and $\{ABCE\} = \{A'B'C'E'\}$
and $\{ABCF\} = \{A'B'C'F'\}$,

then $\{ADEF\} = \{A'D'E'F'\}$ and $\{BDEF\} = \{B'D'E'F'\}$
and $\{CDEF\} = \{C'D'E'F'\}$.

Since, by hypothesis,

$$\{ABCD\} = \{A'B'C'D'\} \text{ and } \{ABCE\} = \{A'B'C'E'\}$$

and $\{ABCF\} = \{A'B'C'F'\}$,

therefore by 1°,

$$\{ABDE\} = \{A'B'D'E'\} \text{ and } \{ABDF\} = \{A'B'D'F'\},$$

and therefore, by the same,

$$\{ADEF\} = \{A'D'E'F'\};$$

and similarly

$$\{BDEF\} = \{B'D'E'F'\} \text{ and } \{CDEF\} = \{C'D'E'F'\};$$

and therefore &c.

To prove 3°, or to shew that,

when $\{ABCD\} = \{A'B'C'D'\}$ and $\{ABCE\} = \{A'B'C'E'\}$
and $\{ABCF\} = \{A'B'C'F'\}$ and $\{ABCG\} = \{A'B'C'G'\}$,
then $\{DEFG\} = \{D'E'F'G'\}$.

Since, by hypothesis,

$\{ABCD\} = \{A'B'C'D'\}$ and $\{ABCE\} = \{A'B'C'E'\}$
and $\{ABCF\} = \{A'B'C'F'\}$ and $\{ABCG\} = \{A'B'C'G'\}$,
therefore, by 2°,

$\{ADEF\} = \{A'D'E'F'\}$ and $\{ADEG\} = \{A'D'E'G'\}$,
and therefore, by 1°,

$$\{DEFG\} = \{D'E'F'G'\};$$

and therefore &c.

Two systems of any common number of constituents, A, B, C, D, E, F, G, &c. and A', B', C', D', E', F', G', &c., thus corresponding in pairs, A and A', B and B', C and C', &c., and thus related to each other that every four pairs of corresponding constituents form equianharmonic systems, have been termed by Chasles *homographic*, and will be treated of at length under that denomination in another chapter. They occur very frequently in the applications of the theory of anharmonic section, and the relation between them may, when necessary, be represented by the obvious extension of Dr. Salmon's notation for simple equianharmonicism between two systems of four (279), viz.:

$$\{ABCDEFG, \&c.\} = \{A'B'C'D'E'F'G', \&c.\};$$

the same precaution respecting *order* among the representatives of corresponding constituents, so essential in the simpler, being, of course, not less indispensable in the more general case. See Art. 279.

Cor. 1°. If, while three pairs of corresponding constituents, A and A', B and B', C and C', of two equianharmonic systems, both of points, or both of rays, or one of points and one of rays, A, B, C, D and A', B', C', D', are supposed to remain fixed, the fourth pair, D and D', be conceived to vary, preserving always, however, the equianharmonicism of the two systems; then, every two positions of the variable pair may be conceived to take the places of D and D', E and E', every three positions those of D and D', E and E', F and F', and every four positions those of D and D', E and E', F and F', G and G', in the preceding; hence, from the above, it appears that—

When a variable pair of constituents, points, or rays, or a point and ray, D and D', form in every position equianharmonic systems with three fixed pairs, A and A', B and B', C and C', then—

1°. *Every two positions of D form, with BC, with CA, and with AB, systems equianharmonic with those formed by the two corresponding positions of D', with $B'C'$, with $C'A'$, and with $A'B'$.*

2°. *Every three positions of D form, with A, with B, and with C, systems equianharmonic with those formed by the three corresponding positions of D', with A', with B', and with C'.*

3°. *Every four positions of D form a system equianharmonic with the four corresponding positions of D'.*

Cor. 2°. Since, during the variation of D and D', in the above, the relation $\{ABCD\} = \{A'B'C'D'\}$ is constantly preserved with A and A', B and B', C and C', therefore throughout the entire variation

$$\frac{AD}{BD} : \frac{AC}{BC} \text{ or } \frac{\sin AD}{\sin BD} : \frac{\sin AC}{\sin BC} = \frac{A'D'}{B'D'} : \frac{A'C'}{B'C'} \text{ or } \frac{\sin A'D'}{\sin B'D'} : \frac{\sin A'C'}{\sin B'C'},$$

and therefore, by alternation,

$$\frac{AD}{BD} \text{ or } \frac{\sin AD}{\sin BD} : \frac{A'D'}{B'D'} \text{ or } \frac{\sin A'D'}{\sin B'D'} = \frac{AC}{BC} \text{ or } \frac{\sin AC}{\sin BC} : \frac{A'C'}{B'C'} \text{ or } \frac{\sin A'C'}{\sin B'C'}$$

= a constant ratio, A, B, C and A', B', C' being fixed; hence it appears that—

When two fixed segments or angles, or a fixed segment and a fixed angle, AB and $A'B'$, are cut by two variable sectors, D and D', so that, throughout their variation, $\frac{AD}{BD}$ or $\frac{\sin AD}{\sin BD} : \frac{A'D'}{B'D'}$ or $\frac{\sin A'B'}{\sin A'D'}$ in any constant ratio, then—

1°. *Every two positions of D form with A and B a system equianharmonic with that formed by the two corresponding positions of D' with A' and B'.*

2°. *Every three positions of D form with A and with B systems equianharmonic with those formed by the three corresponding positions of D' with A' and with B'.*

3°. *Every four positions of D form a system equianharmonic with the four corresponding positions of D'.*

COR. 3°. In the particular case, of these latter properties, when the two segments or angles AB and $A'B'$ coincide, so as to form but a single segment or angle AB, the constant ratio $\frac{AD}{BD}:\frac{A'D'}{B'D'}$ or $\frac{\sin AD}{\sin BD}:\frac{\sin A'D'}{\sin B'D'}$ becomes then (since $A=A'$ and $B=B'$) the anharmonic ratio of the section of the segment or angle AB by the two points or lines of section D and D' (268); hence it appears that—

If a fixed segment or angle AB be cut in any constant anharmonic ratio by a variable pair of sectors D and D'; then—

1°. *Every two positions of D and the two corresponding positions of D' form equianharmonic systems with A and B.*

2°. *Every three positions of D and the three corresponding positions of D' form equianharmonic systems with A and with B.*

3°. *Every four positions of D and the four corresponding positions of D' form equianharmonic systems.*

COR. 4°. In the particular case, of these latter properties, when the constant anharmonic ratio of the section $= -1$, that is, when the section of the fixed segment or angle AB by the variable pair of sectors D and D' is constantly harmonic (213); since then, and then only, $\{ABDD'\} = \{ABD'D\}$ in every position of D and D' (281), that is, since then, and then only, the two points or lines of section are interchangeable in every position without violating the constant anharmonic ratio of their section of the fixed segment or angle AB; hence it appears that—

When a fixed segment or angle AB is cut harmonically by a variable pair of conjugates D and D', then—

1°. *Every two positions of D and the two corresponding positions of D' determine four constituents, every two of which and their two conjugates form equianharmonic systems with A and B.*

2°. *Every three positions of D and the three corresponding positions of D' determine six constituents, every three of which and their three conjugates form equianharmonic systems with A and with B.*

3°. *Every four positions of D and the four corresponding* *positions of D' determine eight constituents, every four of which and their four conjugates form equianharmonic systems.*

COR. 5°. When, in these latter properties, the two conjugates D and D' coincide, in one of their positions, with the two D' and D, in another of their positions; the properties themselves become evidently modified as follows—

1°. *Every position of D and the corresponding position of D' determine two constituents, which taken in both orders form equianharmonic systems with A and B.*

2°. *Every two positions of D and the two corresponding positions of D' determine four constituents, every three of which and their three conjugates form equianharmonic systems with A and with B.*

3°. *Every three positions of D and the three corresponding positions of D' determine six constituents, every four of which and their four conjugates form equianharmonic systems.*

N.B. To the principles established in this article the important modern theories, of Homographic Division, of Double Points and Rays in Homographic Division, and of Involution, may all be referred; as will appear in the sequel in the chapters in which they are severally discussed.

283. *When two triads of points on a common axis or rays through a common vertex, A, B, C and A', B', C', which correspond in pairs, A and A', B and B', C and C', are such that any two systems determined by four of the six constituents and their four correspondents are equianharmonic, then every two systems determined by four of them and their four correspondents are equianharmonic.*

For, in either case, the relation

$$\{BCAA'\} = \{B'C'A'A\}, \dots\dots\dots\dots\dots\dots (1°),$$

gives at once, by (272) and (280), the two equivalent relations

$$\{B'CAA'\} = \{BC'A'A\} \text{ and } \{BC'AA'\} = \{B'CA'A\}; \dots (1');$$

the first of which, combined with the original, gives, by virtue of the general property 1° of the preceding article, the relation

$$\{CABB'\} = \{C'A'B'B\}, \dots\dots\dots\dots\dots\dots (2°),$$

and with it, by (272) and (280), the two equivalent relations

$$\{C'ABB'\} = \{CA'B'B\} \text{ and } \{CA'BB'\} = \{C'AB'B\}; \ldots(2');$$

and the second of which, combined with the original, gives, by virtue of the same general property, the relation

$$\{ABCC'\} = \{A'B'C'C\}, \ldots\ldots\ldots\ldots\ldots (3^{\circ}),$$

and with it, by (272) and (280), the two equivalent relations

$$\{A'BCC'\} = \{AB'C'C\} \text{ and } \{AB'CC'\} = \{A'BC'C\}; \ldots(3');$$

and, each of the six cases of anharmonic equivalence (1°), (2°), (3°) and (1′), (2′), (3′) thus involving the remaining five, therefore &c.

Cor. 1°. That, *for every two triads related as above to each other, the three segments or angles, AA', BB', CC', determined by the three pairs of corresponding constituents, A and A', B and B', C and C', have a common segment or angle MN of harmonic section, real or imaginary*, (see 3°, Cor. 5°, of the preceding article), may be easily shewn as follows: If MN be the common segment or angle of harmonic section, real or imaginary, of any two of them, AA' and BB' suppose, then since, by 2°, Cor. 5°, of the preceding article,

$$\{MBAA'\} = \{MB'A'A\}, \text{ or, } \{MABB'\} = \{MA'B'B\},$$
$$\{NBAA'\} = \{NB'A'A\}, \text{ or, } \{NABB'\} = \{NA'B'B\};$$

and since, by relations (1°) and (1′), or (2°) and (2′), of the above,

$$\{CBAA'\} = \{C'B'A'A\}, \text{ or, } \{CABB'\} = \{C'A'B'B\},$$
$$\{C'BAA'\} = \{CB'A'A\}, \text{ or, } \{C'ABB'\} = \{CA'B'B\};$$

therefore, at once, in either case, by virtue of the general property (3°) of the preceding article,

$$\{MNCC'\} = \{MNC'C\},$$

and therefore &c.; MN thus cutting CC' also harmonically (281).

Cor. 2°. That *every three lines through a point determine with the three perpendiculars to them through the point a system of six rays related as above to each other*, is evident from the circumstance that every two pencils determined by four of the six constituent rays and their four perpendiculars are similar, and therefore equianharmonic (278). And the general property of Cor. 1° is obviously verified for this particular case by that established on other principles in Art. 260, viz., that *all right angles, having a common vertex, have a common imaginary angle of harmonic section, viz., that subtended at their common vertex by the two cyclic points at infinity.*

N.B. The property of the present article has been made by Chasles the basis of the modern theory of Involution, and, as such, has been discussed by him at considerable length in his Chapter on that subject.

284. *When two equianharmonic systems of points on a common axis or rays through a common vertex, A, B, C, D and A', B', C', D', are such that any two of their corresponding constituents may be interchanged without violating their relation of equianharmonicism, then every two of their corresponding constituents may be interchanged without violating their relation of equianharmonicism.*

For, in either case, the two relations

$$\{ABCD\} = \{A'B'C'D'\} \text{ and } \{A'BCD\} = \{AB'C'D'\}$$

give, by virtue of the general property 1° of Art. 282, the three

$$\{AA'CD\} = \{A'AC'D'\}, \quad \{AA'DB\} = \{A'AD'B'\},$$
$$\{AA'BC\} = \{A'AB'C'\},$$

and with them consequently, by (272) and (280), the equivalent three

$$\{AA'CD'\} = \{A'AC'D\}, \quad \{AA'DB'\} = \{A'AD'B\},$$
$$\{AA'BC'\} = \{A'AB'C\},$$

of which, the first terms of the second, third, and first, combined respectively with the second terms of the third, first, and second, give, by virtue of the same general property 1° of Art. 282, the three

$$\{AB'CD\} = \{A'BC'D'\}, \quad \{ABC'D\} = \{A'B'CD'\},$$
$$\{ABCD'\} = \{A'B'C'D\},$$

in each of which, since again, for the same reason, the original pair of constituents A and A' may be interchanged, thus giving the three

$$\{A'B'CD\} = \{ABC'D'\}, \quad \{A'BC'D\} = \{AB'CD'\},$$
$$\{A'BCD'\} = \{AB'C'D\},$$

and so exhausting the entire number of different combinations of four and their four correspondents that could be formed from the four pairs of corresponding constituents A and A', B and B', C and C', D and D'; therefore &c.

COR. 1°. That, *for every system of eight collinear or concurrent constituents, corresponding in pairs, which are thus related to each other that every two systems determined by four of them and their four correspondents are equianharmonic, the four segments or angles, AA', BB', CC', DD', determined by the four pairs of corresponding constituents, A and A', B and B', C and C', D and D', have a common segment or angle MN of harmonic section, real or imaginary*, (see 3°, Cor. 4°, Art. 282), may be shewn in precisely the same manner as for the particular case established in the corollary of the preceding article. If MN be the common segment or angle of harmonic section, real or imaginary, of any two of them, AA' and BB', then since, by 2°, Cor. 5°, Art. 282,

$$\{MBAA'\} = \{MB'A'B\}, \text{ or, } \{MABB'\} = \{MA'B'B\},$$
$$\{NBAA'\} = \{NB'A'B\}, \text{ or, } \{NABB'\} = \{NA'B'B\};$$

and since, as shewn above,

$$\{CBAA'\} = \{C'B'A'B\}, \text{ or, } \{CABB'\} = \{C'A'B'B\},$$
$$\{C'BAA'\} = \{CB'A'B\}, \text{ or, } \{C'ABB'\} = \{CA'B'B\};$$

with relations exactly similar in which C and C' are replaced by D and D'; therefore at once, in either case, by virtue of the general property 3° of Art. 282,

$$\{MNCC'\} = \{MNC'C\}, \text{ and, } \{MNDD'\} = \{MND'D\},$$

and therefore &c.; MN thus cutting CC' and DD' also harmonically (281).

COR. 2°. That *every four lines through a point determine with the four perpendiculars to them through the point a system of eight rays related as above to each other*, is evident (as in Cor. 2° of the preceding article) from the consideration that every two pencils determined by four of the eight constituent rays and their four perpendiculars are similar, and therefore equianharmonic (278). And (as in that same corollary) the general property of Cor. 1° is obviously verified for their particular case by that established on other principles in Art. 260, respecting the harmonic section of every right angle by the two imaginary lines connecting its vertex with the two cyclic points at infinity.

Cor. 3°. That, as above stated, *the property of the preceding is a particular case of that of the present article*, appears at once by supposing the fourth pair of corresponding constituents, D and D', in the above, to coincide, successively, with the first, second, and third pairs, A and A', B and B', C and C'; its unaccented taking the places of their accented constituents, and conversely; as the three groups of relations 1° and 1', 2° and 2', 3° and 3' of the preceding would then result evidently from those of the present article, therefore &c.

N.B. The property of this article is also of considerable importance in the Theory of Involution, under which head it will again be referred to in a subsequent chapter.

285. *Every pencil of four rays determines a row of four points equianharmonic with itself on every axis; and, conversely, every row of four points determines a pencil of four rays equianharmonic with itself at every vertex.*

Let, in either case, O be the vertex of the pencil, and A, B, C, D the four points of the row; then since, by (65),

$$\frac{BA}{CA} = \frac{BO}{CO} \cdot \frac{\sin BOA}{\sin COA}, \text{ and, } \frac{BD}{CD} = \frac{BO}{CO} \cdot \frac{\sin BOD}{\sin COD},$$

therefore, at once, by division of ratios,

$$\frac{BA}{CA} : \frac{BD}{CD} = \frac{\sin BOA}{\sin COA} : \frac{\sin BOD}{\sin COD} \ldots\ldots\ldots\ldots (1),$$

and similarly $$\frac{CB}{AB} : \frac{CD}{AD} = \frac{\sin COB}{\sin AOB} : \frac{\sin COD}{\sin AOD} \ldots\ldots\ldots\ldots (2),$$

and finally $$\frac{AC}{BC} : \frac{AD}{BD} = \frac{\sin AOC}{\sin BOC} : \frac{\sin AOD}{\sin BOD} \ldots\ldots\ldots\ldots (3),$$

and therefore &c. (268).

As for the corresponding property of harmonic section, proved in Art. 221, which is manifestly a particular case of the above, there is one case, and one only, in which the above demonstration fails, viz., when the vertex O of the pencil is at an infinite distance; but in that case, as noticed before in the article referred to, the four rays of the pencil being parallel (16), the property is evident without any demonstration (Euc. VI. 10).

Of all properties of anharmonic section, the above, *which shews that all anharmonic ratios whether of rows of points or pencils of rays are preserved unchanged in perspective* (130), is much the most important; as an abstract proposition, like its particular case already referred to, it was known to the Ancients, but it was only in modern and comparatively recent times that its importance was perceived; it is to it indeed mainly that the theory of anharmonic section owes its utility and power as an instrument of investigation and proof in modern geometry. See Art. 221.

COR. 1°. When one of the four points of the row, D suppose, is at infinity, that is, when the axis of the row is parallel to the corresponding ray OD of the pencil, or conversely (16); since then the three ratios $BD : CD$, $CD : AD$, $AD : BD$ and their three reciprocals are all $= 1$, and since, therefore, the six anharmonic ratios of the row are the three simple ratios $BA : CA$, $CB : AB$, $AC : BC$ and their three reciprocals (275); hence, from the above—

The six anharmonic ratios of any pencil of four rays are equal to the six simple ratios of the three segments, taken in pairs, intercepted by any three of them on any axis parallel to the fourth.

COR. 2°. As every row of four points determines six segments, and every pencil of four rays determines six angles, corresponding two and two in opposite pairs (274), the general property itself, as in fact it was proved above, may be stated otherwise thus, as follows—

Every two angles having a common vertex and the two segments they intercept on any axis, and conversely, every two segments having a common axis and the two angles they subtend at any vertex, cut each other equianharmonically.

Cor. 3°. When, in Cor. 2°, the common axis of the segments is parallel to a side of one of the angles, it follows evidently, from Cor. 1°, that—

For every two angles having a common vertex, every chord of either parallel to a side of the other is cut by the second side of the latter in the two anharmonic ratios of their mutual section.

N.B. Of this latter property that of Art. 224 is evidently a particular case.

286. Among the immediate consequences from the general property of the preceding article may be noticed the following:

1°. *The same pencil of four rays determines equianharmonic rows of four points on all axes; and, the same row of four points determines equianharmonic pencils of four rays at all vertices.*

For, the several rows, in the former case, are all equianharmonic with the pencil, and therefore with each other; and the several pencils, in the latter case, are all equianharmonic with the row, and therefore with each other.

2°. *Every two rows of four points or pencils of four rays in perspective with each other are equianharmonic.*

For, the two rows, in the former case, subtend the centre of perspective by the same pencil of four rays; and, the two pencils, in the latter case, intersect the axis of perspective at the same row of four points; and therefore &c.

3°. *Every two rows of four points or pencils of four rays in perspective with the same row or pencil are equianharmonic.*

For, they are both equianharmonic with the row or pencil with which they are both in perspective, and therefore with each other.

4°. *Every two rows of three points in perspective with each other form equianharmonic systems with the intersection of their axes; and, every two pencils of three rays in perspective with each other form equianharmonic systems with the connector of their vertices.*

For, the two rows, in the former case, combined each with the intersection of their axes, subtend the centre of perspective by the same pencil of four rays; and, the two pencils, in the latter case, combined each with the connector of their vertices, intersect the axis of perspective at the same row of four points; and therefore &c.

5°. *A fixed pencil of four rays determines on a variable line, moving according to any law, a variable row of four points having a constant anharmonic ratio; and, a fixed row of four points determines at a variable point, moving according to any law, a variable pencil of four rays having a constant anharmonic ratio.*

For, the variable row, in the former case, is equianharmonic in every position with the fixed determining pencil; and, the variable pencil, in the latter case, is equianharmonic in every position with the fixed determining row; and therefore &c.

6°. *A fixed pencil of three rays determines, on a variable line turning round a fixed point, a variable row of three points forming with the fixed point a system having a constant anharmonic ratio; and, a fixed row of three points determines, at a variable point moving on a fixed line, a variable pencil of three lines forming with the fixed line a system having a constant anharmonic ratio.*

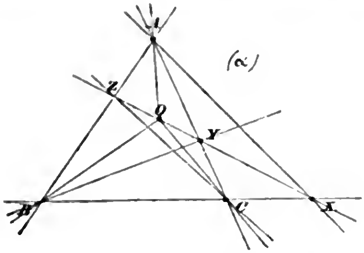
For, the variable row, in the former case, combined with the fixed point, is equianharmonic in every position with the fixed pencil, combined with the line common to its vertex and the fixed point; and, the variable pencil, in the latter case, combined with the fixed line, is equianharmonic in every position with the fixed row, combined with the point common to its axis and the fixed line; and therefore &c.

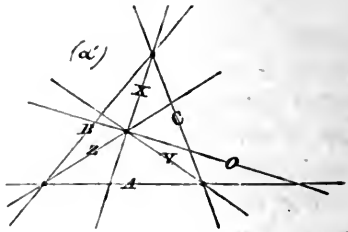
N.B. Of the above inferences, 4° is evidently a particular case of 2°, and 6° is evidently a particular case of 5°.

287. The same general property supplies obvious solutions of the two following reciprocal problems, to which, as will appear in the sequel, several others in the theory of anharmonic section may be reduced, viz.—

a. Through a given point to draw the line whose intersections with three given lines, which are not concurrent, shall determine, with the given point, a system of four points having, in a given assigned order, a given anharmonic ratio.

a'. On a given line to find the point whose connectors with three given points, which are not collinear, shall determine, with the given line, a system of four rays having, in a given assigned order, a given anharmonic ratio.





For, if, in the former case, O (fig. α) be the given point; X, Y, Z the three intersections of the required with the three given lines; and A, B, C the three vertices of the triangle determined by the latter; then since, by the general property of Art. 285, the three pencils of four rays $A.XYZO$, $B.XYZO$, $C.XYZO$, which have each three rays given, are all equianharmonic with the row of four points $XYZO$; and since, by hypothesis, the anharmonic ratio of the latter is given; therefore the anharmonic ratios of the former, and with them their fourth rays AX, BY, CZ are given; and therefore &c. And, if, in the latter case, O (fig. α') be the given line; X, Y, Z the three connectors of the required with the three given points; and A, B, C the three sides of the triangle determined by the latter; then since, by the general property of Art. 285, the three rows of four points $A.XYZO$, $B.XYZO$, $C.XYZO$, which have each three points given, are all equianharmonic with the pencil of four rays $XYZO$; and since, by hypothesis, the anharmonic ratio of the latter is given; therefore the anharmonic ratios of the former, and with them their fourth points AX, BY, CZ, are given; and therefore &c.

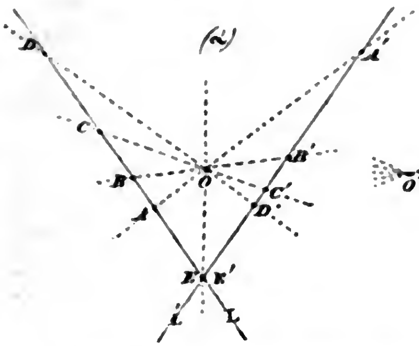
These two reciprocal solutions may be briefly summed up in one as follows: Since, in both cases alike, by the general property of Art. 285, the three systems $A.XYZO$, $B.XYZO$, $C.XYZO$, which have each three constituents given, are equianharmonic with the system $XYZO$, whose anharmonic ratio is, by hypothesis, given; therefore their fourth constituents AX, BY, CZ are implicitly given; and therefore &c.

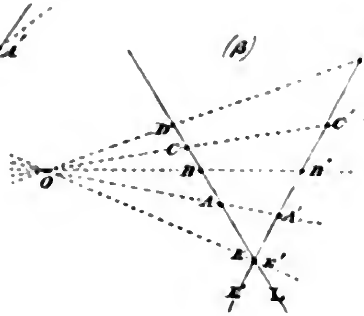
N.B. The preceding problems are manifestly indeterminate or impossible, the former when the three given lines are concurrent, and the latter when the three given points are collinear; for, by 6° of the preceding article, the anharmonic ratio of the system $XYZO$ is then *constant*, whatever be the direction of its axis in the former case, or the position of its vertex in the latter case; and therefore &c. (59).

288. The two following reciprocal inferences from the same general property are, evidently, the converses of the two embodied, in a single statement, in 2°, Art. 286, viz.—

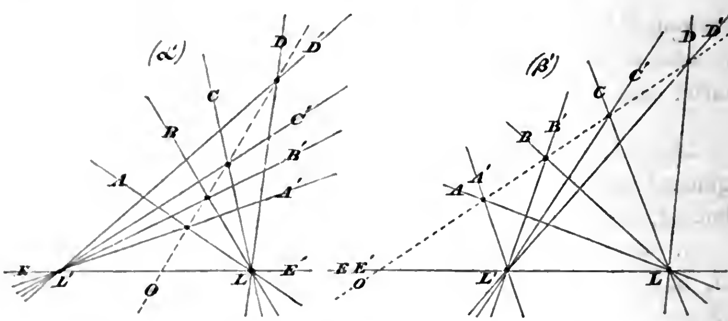
a. When two equianharmonic rows of points on different axes A, B, C, D and A', B', C', D' are such that three of their pairs of corresponding points A and A', B and B', C and C' connect by lines AA', BB', CC' passing through a common point O, the fourth pair D and D' connect also by a line DD' passing through the same point O.

a'. When two equianharmonic pencils of rays through different vertices A, B, C, D and A', B', C', D' are such that three of their pairs of corresponding rays A and A', B and B', C and C' intersect at points AA', BB', CC' lying on a common line O, the fourth pair D and D' intersect also at a point DD' lying on the same line O.





For, in the former case (figs. α and β), the two rows of points A, B, C, D and A', B', C', D' being, by hypothesis, equianharmonic, so therefore (285) are the two pencils of rays OA, OB, OC, OD, and OA', OB', OC', OD'; but three pairs of corresponding rays of those two equianharmonic pencils OA and OA', OB and OB', OC and OC' coincide; therefore the fourth pair OD and OD' coincide also; and therefore &c. And, in the latter case (figs. α' and β'), the two pencils of rays A, B, C, D and A', B', C', D' being, by hypothesis, equian-



harmonic, so therefore (285) are the two rows of points OA, OB, OC, OD, and OA', OB', OC', OD'; but three pairs of corresponding points of those two equianharmonic rows OA and OA', OB and OB', OC and OC' coincide; therefore the fourth pair OD and OD' coincide also; and therefore &c.

The above reciprocal properties may be briefly summed up in one as follows—

When, of two equianharmonic rows of four points or pencils of four rays A, B, C, D and A', B', C', D', three pairs of corresponding constituents A and A', B and B', C and C' are in perspective (130), *the fourth pair D and D' are in perspective with them.*

And so also may the reciprocal demonstrations above given of them, as follows—

Since, in both cases, by hypothesis, $\{ABCD\} = \{A'B'C'D'\}$; therefore, in both cases, by (285), $\{O.ABCD\} = \{O.A'B'C'D'\}$, O being the centre (or axis) of perspective of A and A', B and B', C and C'; but, in both cases, by hypothesis, $OA = OA'$, $OB = OB'$, $OC = OC'$; therefore, in both cases, $OD = OD'$; and therefore &c.

289. The two following, again, are very important particular cases of those of the preceding article; and are, also, evidently, the converses of the two combined, in a single statement, in 4°, Art. 286, viz.—

a. When two equianharmonic rows of points on different axes B, C, D, E and B', C', D', E' are such that a pair of their corresponding points E and E' coincide at the intersection of the axes, the remaining three pairs B and B', C and C', D and D'

connect by three lines BB', CC', DD' passing through a common point O.

a'. When two equianharmonic pencils of rays through different vertices B, C, D, E and B', C', D', E' are such that a pair of their corresponding rays E and E' coincide along the connector of the vertices, the remaining three pairs B and B', C and C', D and D' intersect at three points BB', CC', DD' lying on a common line O.

For, if, in the former case, (figs. α and β of the preceding article), O be the intersection of any two of them BB' and CC'; then, the two rows of points B, C, D, E and B', C', D', E' being, by hypothesis, equianharmonic, so therefore (285) are the two pencils of rays OB, OC, OD, OE and OB', OC', OD', OE'; but three pairs of their corresponding rays OB and OB', OC and OC', OE and OE' coincide; therefore the fourth pair OD and OD' coincide also; and therefore &c. And if, in the latter case, (figs. α' and β' of the preceding article), O be the connector of any two of them BB' and CC'; then, the two pencils of rays B, C, D, E and B', C', D', E' being, by hypothesis, equianharmonic, so therefore (285) are the two rows of points OB, OC, OD, OE and OB', OC', OD', OE'; but three pairs of their corresponding points OB and OB', OC and OC', OE and OE' coincide; therefore the fourth pair OD and OD' coincide also; and therefore &c.

Like those of the preceding article, of which they are important particular cases, the above reciprocal properties may be briefly summed up in one, as follows—

When, of two equianharmonic rows of four points or pencils of four rays B, C, D, E and B', C', D', E', one pair of corresponding constituents E and E' coincide, the remaining three pairs B and B', C and C', D and D' are in perspective.

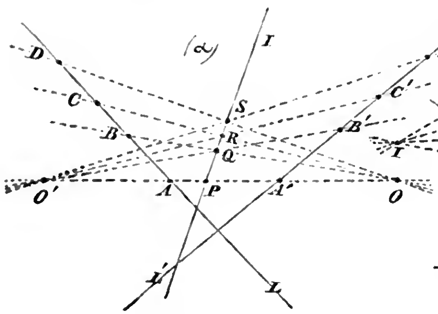
And so, like those of the same, may the reciprocal demonstrations above given of them, as follows—

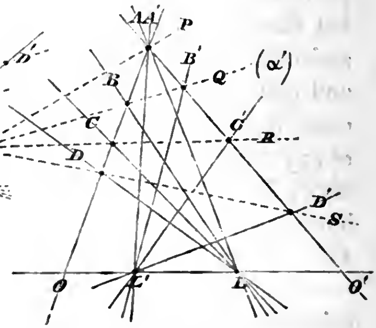
Since, in both cases, by hypothesis, $\{BCDE\} = \{B'C'D'E'\}$; therefore, in both cases, by (285), $\{O.BCDE\} = \{O.B'C'D'E'\}$, O being the point (or line) common to the two lines (or points) BB' and CC'; but, in both cases, by hypothesis, $OB = OB'$, $OC = OC'$, $OE = OE'$; therefore, in both cases, $OD = OD'$; and therefore &c.

290. The two following, again, are useful inferences from the important reciprocal properties of the preceding article, viz.—

a. Every two equianharmonic rows of points on different axes A, B, C, D and A′, B′, C′, D′ determine two pencils of rays in perspective at any two vertices O and O′ lying on the line of connection AA′ of any of their four pairs of corresponding points A and A′.

a′. Every two equianharmonic pencils of rays through different vertices A, B, C, D and A′, B′, C′, D′ determine two rows of points in perspective on any two axes O and O′ passing through the point of intersection AA′ of any of their four pairs of corresponding rays A and A′.





For, in the former case, the two rows of points *A, B, C, D* and *A′, B′, C′, D′* (fig. α) being, by hypothesis, equianharmonic, so therefore (285) are the two pencils of rays *OA, OB, OC, OD* and *O′A′, O′B′, O′C′, O′D′*; but, for the two vertices *O* and *O′*, the pair of corresponding rays *OA* and *O′A′*, by hypothesis, coincide; therefore (289, *a′*) the three remaining pairs *OB* and *O′B′*, *OC* and *O′C′*, *OD* and *O′D′*, intersect at three points *Q, R, S*, lying on a common line *I*; and therefore &c. And, in the latter case, the two pencils of rays *A, B, C, D* and *A′, B′, C′, D′* (fig. α′) being, by hypothesis, equianharmonic, so therefore (285) are the two rows of points *OA, OB, OC, OD* and *O′A′, O′B′, O′C′, O′D′*; but, for the two axes *O* and *O′*, the pair of corresponding points *OA* and *O′A′*, by hypothesis, coincide; therefore (289, *a*) the three remaining pairs *OB* and *O′B′*, *OC* and *O′C′*, *OD* and *O′D′* connect by three lines *Q, R, S* passing through a common point *I*; and therefore &c.

As, in the two preceding articles, these two reciprocal demonstrations may be briefly summed up in one as follows—

Since, in both cases, $\{ABCD\} = \{A'B'C'D'\}$; therefore, in both, $\{O.ABCD\} = \{O'.A'B'C'D'\}$; but, in both cases, OA and $O'A'$ coincide; therefore, in both, the two systems OB, OC, OD and $O'B'$, $O'C'$, $O'D'$ are in perspective; and therefore &c.

N.B. In the above reciprocal properties, the two points (or lines) O and O' might, of course, coincide respectively with the two A' and A; the properties themselves as above stated, and their demonstrations as above given, would remain unchanged; but the axis (or centre) of perspective I of the two pencils (or rows) $O.ABCD$ and $O'.A'B'C'D'$ would then have certain important relations with respect to the two determining rows (or pencils) A, B, C, D and A', B', C', D', which will be considered at length in another chapter.

291. The two reciprocal properties of the preceding article supply ready solutions, by linear constructions only, without the aid of the circle, of the two following reciprocal problems, viz.—

a. Given three pairs of corresponding constituents A and A', B and B', C and C' of two equianharmonic systems of points A, B, C, D and A', B', C', D' on different axes, and the fourth point D of either system; to determine the fourth point D' of the other system.

a'. Given three pairs of corresponding constituents A and A', B and B', C and C' of two equianharmonic systems of rays A, B, C, D and A', B', C', D' through different vertices, and the fourth ray D of either system; to determine the fourth ray D' of the other system.

For, in the former case, from any two points O and O', taken arbitrarily on the line of connection AA' of any one of the three given pairs of corresponding points A and A' (fig. a of preceding Art.), drawing the two pairs of lines OB and $O'B'$, OC and $O'C'$ intersecting at the two points Q and R; and from the point of intersection S of the two lines QR and OD drawing the line SO'; the latter line, by property (a) of the preceding article, intersects with the axis L' of the system A', B', C', D' at the required point D'. And, in the latter case, on any two lines O and O', drawn arbitrarily through the point of intersection AA' of any one of the three given pairs of corresponding rays A and A' (fig. α' of preceding Art.), taking the two pairs of points OB and $O'B'$, OC and $O'C'$ connecting by the two lines Q and R; and on the line of connection S of the two points QR and OD taking the point SO'; the latter point, by property (a') of the preceding article, connects with the vertex L' of the system A', B', C', D' by the required ray D'.

These two reciprocal constructions may be briefly summed up in one as follows: The two given points or rays A and A' give the line or point AA'; on or through which are taken or drawn arbitrarily the two points or lines O and O'; which, with the two given pairs of points or lines B and B', C and C', give the two pairs of lines or points OB and $O'B'$, OC and $O'C'$; which give the two points or lines Q and R; which give the line or point QR; which, with the line or point OD, gives the point or line S; which, with the point or line O', gives the line SO'; which, with the axis or vertex L' of the system A', B', C', D', gives the required fourth point or ray D'.

N.B. As noticed at the close of the preceding article, the two assumed points or lines O and O', in the two preceding reciprocal constructions, might be taken to coincide with the given two A' and A respectively; but no simplification worth mentioning would be obtained by so taking them.

In the particular case when the three given pairs of corresponding constituents A and A', B and B', C and C' are in perspective (130), either constituent of the remaining pair D and D' is given immediately with the other, being, by (288), in perspective with it to the same centre or axis.

292. *Every row of four points is equianharmonic with the pencil of four rays determined by their four polars with respect to any circle; and, conversely, every pencil of four rays is equianharmonic with the row of four points determined by their four poles with respect to any circle* (166, Cor. 1°).

For, in either case, the pencil determined by the four rays being similar to that subtended by the four points at the centre of the circle (171, 2°); and the latter pencil, by virtue of the general property of Art. 285, being equianharmonic with the row determined by the four points; therefore &c. The property of harmonic section established in Art. 223 is evidently a particular case of this.

In the applications of the theory of anharmonic section, the above property, *from which it appears that all anharmonic ratios, whether of rows of points or pencils of rays, are preserved unchanged in reciprocation* (172), ranks next in importance to that of Art. 285, from which, as above demonstrated, like its particular case already referred to, it is indeed an inference. *By virtue of it all anharmonic properties of geometrical figures are in fact double, every anharmonic property of any figure being accompanied by a corresponding anharmonic property of its reciprocal figure to any circle* (172), *the establishment of either of which involves that of the other without the necessity of any further demonstration* (173). As, in the applications of the theory of harmonic section given in Chapters XIV. and XV., the principal anharmonic properties of figures consisting only of points and lines, and also of figures involving circles so far as their reciprocals are properties involving no higher figures (173), will be given in the next and following chapters, arranged for the most part in reciprocal pairs, placed in immediate connection with each other, and marked by corresponding letters, accented and unaccented, so as to keep the circumstance of this duality, which forms such a remarkable feature in modern geometry, continually present before the reader, and furnish him at the same time with numerous additional examples by which to exercise and perfect himself in the reciprocating process described in Art. 172, and already exemplified at some length in the chapters referred to. The five articles immediately preceding the present furnish obvious examples of this mode of arrangement; and, until the closing chapter, where it would be inadmissible for the reason mentioned in Art. 173, the same will be adhered to as systematically as possible throughout the remainder of the work.

CHAPTER XVII.

ANHARMONIC PROPERTIES OF THE POINT AND LINE.

293. In the applications of the theory of anharmonic section to the geometry of the Point and Line, the two following properties, reciprocals of each other, present themselves so frequently that we shall commence this chapter on the subject with their statement and proof.

If A, B, C be any three points lying on a line (or lines passing through a point) O, and A', B', C' any three others lying on another line (or passing through another point) O'; the three intersections (or connectors) A'', B'', C'' of the three pairs of connectors (or intersections) BC' and $B'C$, CA' and $C'A$, AB' and $A'B$ lie on a third line (or pass through a third point) O''; which determines with O and O' a triangle $OO'O''$, whose opposite vertices (or sides) I, I', I'' are connected with the three collinear (or concurrent) triads A, B, C; A', B', C'; A'', B'', C'' by the three groups of equianharmonic relations

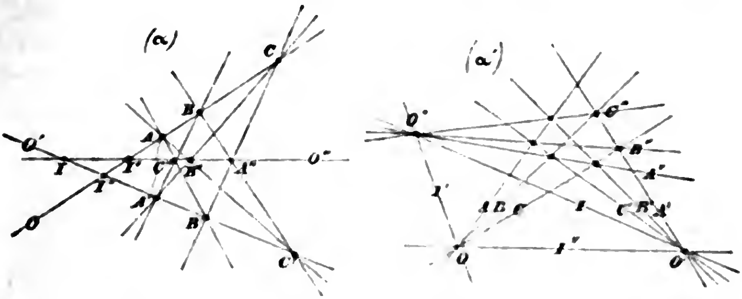
$$\left.\begin{array}{l}\{BCI'I''\} = \{B'C'I''I\} = \{B''C''II'\}\\ \{CAI'I''\} = \{C'A'I''I\} = \{C''A''II'\}\\ \{ABI'I''\} = \{A'B'I''I\} = \{A''B''II'\}\end{array}\right\} \ldots\ldots\ldots (1),$$

and, as a consequence from them, also by the two

$$\left.\begin{array}{l}\{ABCI'\} = \{A'B'C'I''\} = \{A''B''C''I\}\\ \{ABCI''\} = \{A'B'C'I\} = \{A''B''C''I'\}\end{array}\right\} \ldots\ldots (2).$$

For, if O'' be the line of connection (fig. α) (or the point of intersection (fig. α')) of some two, A'' and B'' suppose, of the three points (or lines) A'', B'', C''; and I, I', I'' the three opposite vertices (or sides) of the triangle determined by the three lines (or points) O, O', O''; then, since the two triads of points (or lines) B, C, I' and C', B', I are in perspective, therefore, by (286, 4°)

$$\{BCI'I''\} = \{C'B'II''\} = \{B'C'I''I\} \text{ (280)};$$



and, since the two triads of points (or lines) C, A, I' and A', C', I are in perspective, therefore, by the same,

$$\{CAI'I''\} = \{A'C'II''\} = \{C'A'I''I\} \text{ (280)};$$

therefore, by the general property (1°), Art. (282),

$$\{ABI'I''\} = \{A'B'I''I\} = \{B'A'II''\} \text{ (280)};$$

and therefore, by (289), the two triads of points (or lines) A, B, I' and B', A', I are in perspective; which proves the first parts of both properties, in which, as is evident from the figures and mode of establishment, any two of the three collinear (or concurrent) triads A, B, C; A', B', C'; A'', B'', C'' may be regarded as the original pair, and the third as that derived from them by the construction involved in the corresponding statement, whichever it be.

To prove the second parts of both properties; since, in either case, as shown above for the two original triads A, B, C and A', B', C',

$$\{BCI'I''\} = \{B'C'I''I\}, \quad \{CAI'I''\} = \{C'A'I''I\},$$
$$\{ABI'I''\} = \{A'B'I''I\};$$

therefore, by cyclic interchange between each and the derived triad A'', B'', C'',

$$\{BCI'I''\} = \{B'C'I''I\} = \{B''C''II'\},$$
$$\{CAI'I''\} = \{C'A'I''I\} = \{C''A''II'\},$$
$$\{ABI'I''\} = \{A'B'I''I\} = \{A''B''II'\};$$

which are the relations (1) as above stated; and, as from them the relations (2) follow immediately in virtue of the general property (2°), Art. (282), therefore &c.

In the particular case when any two of the three triads, A, B, C and A', B', C' suppose, are in perspective; it is evident, from (240), that the line of collinearity (or point of concurrence) O'' of the third A'', B'', C'' is the polar of their centre (or the pole of their axis) of perspective, with respect to the angle (or segment) OO' determined by their two lines of collinearity (or points of concurrence) O and O' (217). In that case the three lines (or points) O, O', O'' being concurrent (or collinear), and the three points (or lines) I, I', I'' consequently coincident, therefore, by relations (2) above,

$$\{ABCI\} = \{A'B'C'I'\} = \{A''B''C''I''\},$$

and therefore, by (289), the three triads A, B, C; A', B', C'; A'', B'', C'' are two and two in perspective. Hence—

When, of three collinear (or concurrent) triads A, B, C; A', B', C'; A'', B'', C'' connected cyclically as above, any two are in perspective, then every two are in perspective; and the axis (or vertex) of each is the polar of the centre (or the pole of the axis) of perspective of the other two, with respect to the angle (or segment) determined by their axes (or vertices).

294. Among the numerous inferences from the two reciprocal properties of the preceding article, the following, in pairs reciprocals of each other, are deserving of attention.

1°. The three pairs of points (or lines) A and A', B and B', C and C' may be regarded as determining three segments (or angles) AA', BB', CC', which, taken in pairs, have a common centre (or axis) of perspective I'', and of which, taken in pairs, the three points (or lines) A'', B'', C'' are the remaining three centres (or axes) of perspective; and similarly for the three pairs A' and A'', B' and B'', C' and C'', and for the three A'' and A, B'' and B, C'' and C; hence, generally, from the first parts of the two reciprocal properties in question—

a. When, of three segments taken in pairs, three of the six centres of perspective coincide, the remaining three are collinear.

a'. When, of three angles taken in pairs, three of the six axes of perspective coincide, the remaining three are concurrent.

2°. The two triads of points (or lines) A, B, C and A', B' C' may be regarded as the two sets of three alternate vertices (or sides) of a hexagon $AB'CA'BC'$, of which A and A',

B and B', C and C' are the three pairs of opposite vertices (or sides), and A'', B'', C'' the three intersections (or connectors) of the three pairs of opposite sides (or vertices) BC' and $B'C$, CA' and $C'A$, AB' and $A'B$; and similarly for the two triads A', B', C' and A'', B'', C'', and for the two A'', B'', C'' and A, B, C; hence, generally, from the same again—

a. When, of a hexagon, both triads of alternate vertices are collinear, the three intersections of opposite sides are collinear.

a'. When, of a hexagon, both triads of alternate sides are concurrent, the three connectors of opposite vertices are concurrent.

3°. The three triads of points (or lines) A, B, C; A', B', C'; A'', B'', C'' may be regarded as determining six different cycles of three triangles, each inscribed to one and exscribed to the other of the remaining two, viz.—

$$BA'C,\ B'A''C',\ B''AC'' \text{ and } BA''C,\ B'AC',\ B''A'C'',$$

$$CB'A,\ C'B''A',\ C''BA'' \text{ and } CB''A,\ C'BA',\ C''B'A'',$$

$$ACB',\ A'C''B',\ A''CB'' \text{ and } AC''B,\ A'CB',\ A''C'B'',$$

for each of which the three points (or lines) I, I', I'' are the points of intersection (or lines of connection) of the three pairs of corresponding sides (or vertices) O' and O'', O'' and O, O and O' respectively; and similarly for the remaining pairs of corresponding sides (or vertices); hence, generally, from relations (1) and (2) of the preceding article, respectively—

In every cycle of three triangles each inscribed to one and exscribed to the other of the remaining two.

a. The sides and opposite vertices of each divide equianharmonically the corresponding sides of that to which it is inscribed.

a'. The vertices and opposite sides of each divide equianharmonically the corresponding angles of that to which it is exscribed.

b. The pairs of corresponding sides of every two intersect equianharmonically the corresponding sides of the third.

b'. The pairs of corresponding vertices of every two subtend equianharmonically the corresponding angles of the third.

4°. When, for the system of three triangles constituting any one of the six cycles, $BA'C$, $B'A''C'$, $B''AC''$ suppose, in the preceding, the equianharmonic section in properties (a) and (a') is harmonic; since then, by the two reciprocal properties (a) and (a') of Art. (243), the three triangles constituting the cycle are two and two in perspective, and conversely; and since always, by the first parts of the two reciprocal properties of the preceding article, the three triads of points of intersection (or lines of connexion) of

$$BA' \text{ and } B'A'',\quad CA' \text{ and } C'A'',\quad B'B'' \text{ and } C'C'',$$
$$B'A'' \text{ and } B''A,\quad C'A'' \text{ and } C''A,\quad B''B \text{ and } C''C,$$
$$B''A \text{ and } BA',\quad C''A \text{ and } CA',\quad BB' \text{ and } CC',$$

are collinear (or concurrent); hence generally, as noticed in Cors. (11°) and (13°), Art. (261)—

In a cycle of three triangles each inscribed to one and exscribed to the other of the remaining two.

a. When any two of the three are in perspective, every two of the three are in perspective.

b. For each triangle, its centre of perspective with that to which it is inscribed lies on its axis of perspective with that to which it is exscribed, and reciprocally, its axis of perspective with that to which it is exscribed passes through its centre of perspective with that to which it is inscribed.

5°. In the two triads of points (or lines) A, B, C and A', B', C', if, while the three constituents A, B, C of either and any two A' and B' of the other are supposed to remain fixed, the third constituent C' of the latter be conceived to vary, causing of course the simultaneous variation of the two constituents A'' and B'' of the third triad A'', B'', C''; since then, in every position of the variable triangle $A''C'B''$, the three vertices (or sides) lie on the three fixed lines (or pass through the three fixed points) $B'C$, $A'B'$, CA', while the three sides (or vertices) pass through the three fixed points (or lie on the three fixed lines) A, C'', B; hence again, generally, from the first parts of the reciprocal properties of the preceding article—

a. When, of a variable triangle whose three vertices move on fixed lines, two of the sides turn round the corresponding vertices of any fixed triangle exscribed to that determined by the lines, the third turns round its third vertex.

a′. When, of a variable triangle whose three sides turn round fixed points, two of the vertices move on the corresponding sides of any fixed triangle inscribed to that determined by the points, the third moves on its third side.

6°. When the two fixed triangles, in the two reciprocal properties a and a' of the preceding (5°), are in perspective; since then, by property a of (4°), the variable triangle, in every position, is in perspective with both; and since also, by property b of the same, its centre of perspective with that to which it is exscribed lies, in every position, on the axis of perspective of the two, while its axis of perspective with that to which it is inscribed passes, in every position, through the centre of perspective of the two; hence, generally, from those properties combined, it appears that—

In a cycle of three triangles, each inscribed to one and exscribed to the other of the remaining two and in perspective with both; if, while two of the three are supposed to remain fixed, the third be conceived to vary, then—

a. The centre of perspective, of the variable with the fixed triangle to which it is exscribed, moves on the axis of perspective of the two fixed triangles.

a'. The axis of perspective, of the variable with the fixed triangle to which it is inscribed, turns round the centre of perspective of the two fixed triangles.

7°. Since, in every cycle of three triangles each inscribed to one and exscribed to the other of the remaining two; by virtue of the two reciprocal properties a and a' of (3°), or of either of them combined with the general property of Art. 285, the opposite vertices and sides of each divide in the same anharmonic ratios, the corresponding sides of that to which it is inscribed, and the corresponding angles of that to which it is exscribed; which three sets of equal anharmonic ratios are of course fixed when two of the three triangles of the cycle are fixed, however the third may vary; hence, generally, from those properties combined with those of (5°), it appears that—

In any cycle of three triangles, each inscribed to one and exscribed to the other of the remaining two; if, while two of the three are supposed to remain fixed, the third be conceived to vary, then of the latter with respect to the two former—

a. The opposite vertices and sides divide in the same constant anharmonic ratios, the corresponding sides of that to which it is inscribed, and the corresponding angles of that to which it is exscribed.

b. The opposite sides and angles are divided in the same constant anharmonic ratios, the former by the corresponding vertices and sides of that to which it is exscribed, and the latter by the corresponding sides and vertices of that to which it is inscribed.

8°. Since again, by virtue of the same two reciprocal properties, or of either of them combined with that of Art. 285, the opposite vertices and sides of every two triangles, either inscribed to a third and exscribed to a fourth exscribed to the third, or exscribed to a third and inscribed to a fourth inscribed to the third, divide, in equal anharmonic ratios, the corresponding sides of that to which they are inscribed, and the corresponding angles of that to which they are exscribed; while their sides and angles are divided in the same equal anharmonic ratios, the former by the corresponding vertices and sides of that to which they are exscribed, and the latter by the corresponding sides and vertices of that to which they are inscribed; hence, again, from the same, conversely, as may also be easily shewn directly, it appears that—

a. When, of two triangles inscribed to a third, the opposite vertices and sides divide in any equal anharmonic ratios the corresponding sides of the third; the intersections of their pairs of corresponding sides determine a fourth triangle, inscribed to each of themselves and exscribed to the third, whose opposite sides and vertices divide in the same equal anharmonic ratios their pairs of corresponding sides and the corresponding angles of the third.

a'. When of two triangles exscribed to a third, the opposite sides and vertices divide in any equal anharmonic ratios the corresponding angles of the third; the connectors of their pairs of corresponding vertices determine a fourth triangle, exscribed to each of themselves and inscribed to the third, whose opposite vertices and sides divide in the same equal anharmonic ratios their pairs of corresponding angles and the corresponding sides of the third.

9°. In the particular case when, in the two reciprocal properties *a* and *a'* of the preceding (8°), the three sets of equal anharmonic ratios are all harmonic; since then, by (243), the several pairs of corresponding triangles are in perspective, and conversely; hence from those properties combined with those of the latter article, it appears that—

a. When two triangles are each inscribed to a third and in perspective with it; the intersections of their pairs of corresponding sides determine a fourth triangle, inscribed to each of themselves, exscribed to the third, and in perspective with all three.

a'. When two triangles are each exscribed to a third and in perspective with it; the connectors of their pairs of corresponding vertices determine a fourth triangle, exscribed to each of themselves, inscribed to the third, and in perspective with all three.

10°. Since, when two triangles are each inscribed or exscribed to a third, the sides of the third are the connectors of their pairs of corresponding vertices in the former case, and the vertices of the third the intersections of their pairs of corresponding sides in the latter case; it appears, consequently, from the two separate parts of either of the two reciprocal, and also converse, properties of the preceding (9°), that—

*a. Of two triangles whose vertices and sides correspond in pairs; when the connectors of their pairs of corresponding vertices determine a common exscribed triangle in perspective with both, the intersections of their pairs of corresponding sides determine a common inscribed triangle in perspective with both; and conversely.**

---

* The above is evidently a particular case of the following:

Of two triangles whose vertices and sides correspond in pairs; when the connectors of their pairs of corresponding vertices determine a common exscribed triangle in perspective with either, the intersections of their pairs of corresponding sides determine a common inscribed triangle in perspective with the other: and conversely.

Which may be proved readily, from the general relations *a* and *b'* of Art. 134, as follows:

If X, Y, Z be the three vertices of one of the original triangles; X', Y', Z' those of the other; A, B, C those of the common exscribed triangle determined by the three connectors; and A', B', C' those of the common inscribed triangle determined by the three intersections; since then *always*, in virtue of relation *a* of the article in question,

$$\frac{BX}{CX}\cdot\frac{CY}{AY}\cdot\frac{AZ}{BZ}=\frac{Y'A'}{Z'A'}\cdot\frac{Z'B'}{X'B'}\cdot\frac{X'C'}{Y'C'}\quad\text{..........}\quad(1),$$

and

$$\frac{BX'}{CX'}\cdot\frac{CY'}{AY'}\cdot\frac{AZ'}{BZ'}=\frac{XA'}{ZA'}\cdot\frac{ZB'}{XB'}\cdot\frac{XC'}{YC'}\quad\text{..........}\quad(2),$$

b. Of the two derived thus each in perspective with both the original triangles; that exscribed to the original two is inscribed to the other of themselves, that inscribed to the original two is exscribed to the other of themselves, and they are also in perspective with each other.

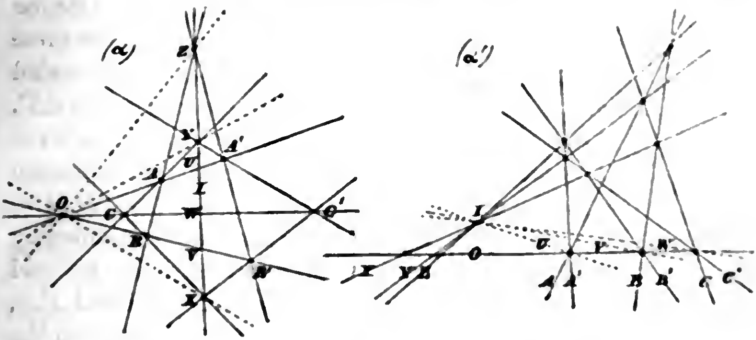
295. The two following properties, reciprocals of each other, form, as explained in Art. 140, the basis of the theory of perspective in modern geometry; and establish, at the same time, the equianharmonic relations connecting the several pairs of corresponding points and lines of every two figures in perspective with their centre and axis of perspective (141).

If A, B, C be any three points on three concurrent lines (or lines through three collinear points); A', B', C' any three other points on the same lines (or lines through the same points); and X, Y, Z the three intersections (or connectors) of the three pairs of connectors (or intersections) BC and $B'C'$, CA and $C'A'$, AB and $A'B'$; the three points (or lines) X, Y, Z are collinear (or concurrent); and their line of collinearity (or point of concurrence) I intersects with the three lines (or connects with the three points) AA', BB', CC' at three points (or by three lines) U, V, W which, with their point of concurrence (or line of collinearity) O, determine the group of equianharmonic relations

$$\{AA'OU\} = \{BB'OV\} = \{CC'OW\}.$$

For, if I be the line of connection (fig. α) (or the point of intersection (fig. α')) of any two, X and Y suppose, of the three points (or lines) X, Y, Z; and U, V, W its three points of intersection (or lines of connection) with the three lines (or points) AA', BB', CC'; then, since the two triads of points (or lines) B, B', V and C, C', W are in perspective, therefore, by (286, 4°), $\{BB'OV\} = \{CC'OW\}$; and, since the two triads of points (or lines) C, C', W and A, A', U are in perspective, therefore, by the same, $\{CC'OW\} = \{AA'OU\}$; from which, since imme-

consequently, when either equivalent of either relation $= -1$, the other also $= -1$; that is, in virtue of relation b' of the same, when XYZ and ABC are in perspective, then $X'Y'Z'$ and $A'B'C'$ are in perspective, and conversely, by (1); and when $X'Y'Z'$ and ABC are in perspective, then XYZ and $A'B'C'$ are in perspective, and conversely, by (2); and therefore &c.



diately, $\{AA'OU\} = \{BB'OV\}$, therefore, by (289), the two triads of points (or lines) A, A', U and B, B', V are in perspective; and therefore &c., the second parts of both properties having been established in the demonstrations of the first.

COR. 1°. If X', Y', Z' be the three intersections (or connectors) of the three pairs of connectors (or intersections) BC' and $B'C$, CA' and $C'A$, AB' and $A'B$; it may, of course, be shewn, in precisely the same manner, that the three triads of points (or lines) Y', Z', X; Z', X', Y; X', Y', Z are also collinear (or concurrent); their three lines of collinearity (or points of concurrence) determining, with that of the triad X, Y, Z, a tetragram (or tetrastigm), of which the three pairs of corresponding points (or lines) X and X', Y and Y', Z and Z' are the three pairs of opposite intersections (or connectors); and each line (or point) determining, by its intersections (or connectors) with the original three, a group of equianharmonic relations similar to the above, and differing only in the interchange of the constituents of the corresponding reversed pair of the three A and A', B and B', C and C' in the equivalent which contains them.

COR. 2°. In the particular case when the three equianharmonic systems of points (or rays) A, A', O, U; B, B', O, V; C, C', O, W are harmonic, that is, when the three segments (or angles) AA', BB', CC' are cut harmonically by the three OU, OV, OW; since then (281),

$$\{AA'OU\} = \{A'AOU\},\ \{BB'OV\} = \{B'BOV\},$$
$$\{CC'OW\} = \{C'COW\},$$

therefore the three points (or lines) U, V, W are the same for the three lines of collinearity (or points of concurrence) of the three triads Y', Z', X; Z', X', Y; X', Y', Z' as for that of the triad X, Y, Z; and therefore the whole six points (or lines) X and X', Y and Y', Z and Z' lie on the same line (or pass through the same point) I; which, in that case, is consequently, the common polar of the point (or the common pole of the line) O with respect at once to the six angles (or segments) determined by the six pairs of opposite connectors (or intersections) BC and $B'C'$, BC' and $B'C$; CA and $C'A'$, CA' and $C'A$; AB and $A'B'$, AB' and $A'B$ (217); a property which is also evident from the two reciprocal properties 5° and 6° of Art. 222.

The three pairs of points (or tangents) A and A', B and B', C and C' determined by any circle on any three concurrent lines (or at any three collinear points) OU, OV, OW furnish an obvious and important example of the particular case in question. For, the three pairs of points (or lines) X and X', Y and Y', Z and Z' lying on the polar (or passing through the pole) I of the point of concurrence (or line of collinearity) O with respect to the circle (261); and the three A and A', B and B', C and C' being consequently pairs of harmonic conjugates with respect to the three O and U, O and V, O and W respectively (259); therefore &c.

Cor. 3°. In the same case it is evident also, from the general property of Art. 285, that, of the six points (or lines) A and A', B and B', C and C', every two conjugates connect (or intersect) with the remaining four equianharmonically; for since, by the general property in question,

$$\{A.BB'CC'\} = \{ZZ'YY'\} = \{A'.BB'CC'\} \ldots\ldots (1°),$$

$$\{B.CC'AA'\} = \{XX'ZZ'\} = \{B'.CC'AA'\} \ldots\ldots (2°),$$

$$\{C.AA'BB'\} = \{YY'XX'\} = \{C'.AA'BB'\} \ldots\ldots (3°),$$

therefore &c. That, in the same case, the same property is true of every two of the six points (or lines) A and A', B and B', C and C', whether conjugates or not, might be shewn without difficulty; but a more general property, which will include it as a particular case, will form the subject of a subsequent article of the present chapter.

Cor. 4°. That the triad of points (or rays) U, V, W determines with the triad X, Y, Z in the general case, and therefore also with each of the three triads Y', Z', X; Z', X', Y; X', Y', Z in the particular case considered in the two preceding corollaries, a system of six constituents corresponding two and two in opposite pairs, every four of which are equianharmonic with their four opposites (283), may be easily shewn as follows: The three pencils of four rays (or rows of four points)

$$O.UVWX,\quad O.UVWY,\quad O.UVWZ,$$

(see figures) being in perspective with the three

$$A.UZYX,\quad B.ZVXY,\quad C.YXWZ,$$

on the three lines (or at the three points) BC, CA, AB respectively, or with the three

$$A'.UZYX,\quad B'.ZVXY,\quad C'.YXWZ,$$

on the three lines (or at the three points) $B'C'$, $C'A'$, $A'B'$ respectively; therefore, from either set of perspectives, by the general property of Art. 285,

$$\{UVWX\} = \{UZYX\} = \{XYZU\}\ (280) \ldots\ldots (1),$$

$$\{UVWY\} = \{ZVXY\} = \{XYZV\}\ (280) \ldots\ldots (2),$$

$$\{UVWZ\} = \{YXWZ\} = \{XYZW\}\ (280) \ldots\ldots (3),$$

and therefore &c. (283). That the two triads X, Y, Z and X', Y', Z' when collinear (or concurrent), Cors. 2° and 3°, are connected by the same relation, might be shewn, without difficulty, either from the above or independently; but another more general property, under which it will come as a particular case, will form the subject of another subsequent article of the present chapter.

Cor. 5°. If E and E', F and F', G and G' be the three pairs of intersections (or connectors), with the three pairs of connectors (or intersections) BC and $B'C'$, CA and $C'A'$, AB and $A'B'$ respectively, of any line passing through (or point lying on) O; and K its intersection (or connector) with I; it may be easily shown, from the general property of Art. 285, that the three quartets of points or lines E, E', O, K; F, F', O, K; G, G', O, K are equianharmonic with each other and with the three A, A', O, U; B, B', O, V; C, C', O, W. For since in either case, by the property in question,

$$\{EE'OK\} = \{BB'OV\} = \{CC'OW\} \text{ ....} (1),$$

$$\{FF'OK\} = \{CC'OW\} = \{AA'OU\} \text{ ....} (2),$$

$$\{GG'OK\} = \{AA'OU\} = \{BB'OV\} \text{ ....} (3),$$

therefore &c. Three similar triads of equianharmonic quartets, result, of course, on the successive interchanges (as in Cor. 1°) of A and A', B and B', C and C' in the constructions determining E and E', F and F', G and G' as above given; all of which, in common with the original triad, are evidently alike harmonic in the particular case considered in Cor. 2°. The three pairs of points (or tangents) A and A', B and B', C and C', determined by any circle on any three concurrent lines (or at any three collinear points) OU, OV, OW, furnish, as observed at the close of that corollary, an important example of a case in which they are all thus harmonic.

Cor. 6°. In the same case it follows immediately, from the general property (2°) of Art. 282, that the two triads of points (or lines) E, F, G and E', F', G' of the preceding corollary (5°) determine equianharmonic systems, both with the point (or line) O, and with the point (or line) K. For since, by the property in question, the two equianharmonic relations of the preceding corollary, viz.:

$$\{EE'OK\} = \{FF'OK\} = \{GG'OK\} \text{..............} (1),$$

give immediately the two

$$\{EFGO\} = \{E'F'G'O\} \text{ and } \{EFGK\} = \{E'F'G'K\} \text{ ...} (2),$$

therefore &c. When, as for the three pairs of points (or tangents) A and A', B and B', C and C' determined by any circle on any three concurrent lines (or at any three collinear points) OU, OV, OW, the three equianharmonic systems in (1) are all harmonic; it follows also, from (3°, Cor. 5°) of the same article (282), that the same two triads E, F, G and E', F', G' determine in three opposite pairs E and E', F and F', G and G' a system of six collinear (or concurrent) constituents, every four of which are equianharmonic with their four opposites (283).

296. Among the various inferences from the two reciprocal properties of the preceding article, the following, in pairs reciprocals of each other, are deserving of attention.

1°. The three pairs of points (or lines) A and A', B and B', C and C', may be regarded as determining three segments (or angles) AA', BB', CC'; whose axes (or vertices) are concurrent (or collinear), and of which, taken in pairs, the three points (or lines) X, Y, Z are three of the six centres (or axes) of perspective, every two of which become changed into their opposites by the interchange of extremities of one of the two determining segments (or angles), those of the other remaining unchanged; hence, generally, from the two properties in question, as shewn in part on other principles in Art. 146, it appears that—

a. When the axes of three segments are concurrent, the six centres of perspective of the three pairs they determine lie, three and three, on four lines; each of which, with the point of concurrence of the axes, divides the three segments equianharmonically.

a'. When the vertices of three angles are collinear, the six axes of perspective of the three pairs they determine pass, three and three, through four points; each of which, with the line of collinearity of the vertices, divides the three angles equianharmonically.

2°. The two triads of points (or lines) A, B, C and A', B', C' may be regarded as determining two triangles ABC and $A'B'C'$; the connectors (or intersections) AA', BB', CC' of whose pairs of corresponding vertices (or sides) A and A', B and B', C and C' are concurrent (or collinear); and so may also the three pairs of triads B, C, A' and B', C', A; C, A, B' and C', A', B; A, B, C' and A', B', C resulting from them by the three different interchanges of corresponding constituents. Hence again from the same, and from the obvious inference from them contained in Cor. 5° of the preceding article, it appears generally, as shewn in part on other principles in Art. 140, that—

For two triangles whose vertices and sides correspond in pairs—

a. When the connectors of the three pairs of corresponding vertices are concurrent, the intersections of the three pairs of corresponding sides are collinear; and, reciprocally, when the intersections of the three pairs of corresponding sides are collinear, *the connectors of the three pairs of corresponding vertices are concurrent.**

b. When thus related to each other, the line of collinearity and the point of concurrence divide equianharmonically, at once the three segments determined by the three pairs of corresponding vertices, and the three angles determined by the three pairs of corresponding sides.

c. In the same case, more generally, the same point and line divide equianharmonically, at once all segments determined by their pairs of corresponding sides on lines passing through the former, and all angles determined by their pairs of corresponding vertices at points lying on the latter.

When two triangles, related as above to each other, are either both inscribed or both exscribed to the same circle; they furnish, see Cor. 2° of the preceding article, an important example of the particular case in which the several equianharmonic sections in the two latter properties *b* and *c* are all harmonic. Hence, since, by 3°, Cor. 5°, Art. 282, every three pairs of points or lines, harmonic conjugates with respect to the same pair, determine a system of six collinear or concurrent constituents every four of which are equianharmonic with their four conjugates, it follows from the above (*c*), that—

When two triangles either both inscribed or both exscribed to the same circle are in perspective.

a. Every line passing through their centre of perspective intersects with their three pairs of corresponding sides at six points,

* This important property, which, as stated in Art. 140, is the basis of the theory of perspective in the geometry of plane figures, may be established, even more readily than either by the above or by the method employed in that article, as follows:

If ABC and $A'B'C'$ (fig. *a*, Art. 295) be the two triangles; then in the quadrilateral $CC'XY$ determined by any two pairs of their corresponding sides BC and $B'C'$, CA and $C'A'$, to which that determined by the two pairs of opposite vertices A and A', B and B' is inscribed, the relation

$$AC.BX.B'C'.A'Y = A'C'.B'X.BC.AY,$$

by virtue of the general relation *a* of Art. 134, being at once the criterion, that the two lines AA' and BB' should intersect at the same point O on the diagonal CC', and that the two AB and $A'B'$ should intersect at the same point Z on the diagonal XY, therefore &c.

in opposite pairs, every four of which are equianharmonic with their four opposites.

d'. Every point lying on their axis of perspective connects with their three pairs of corresponding vertices by six lines, in opposite pairs, every four of which are equianharmonic with their four opposites.

3°. If, in the preceding (2°), while the point (or line) O, one of the two triangles ABC, and two X and Y of the three points (or lines) X, Y, Z, are supposed to remain fixed, the other triangle $A'B'C'$ be conceived to vary consistently with the restriction of the fixity of X and Y; then, in every position of the variable triangle, since, by the first parts of the same properties, the third point (or line) Z is also fixed, being the point of intersection (or line of connection) of the two fixed lines (or points) AB and XY, and since, by the two reciprocal properties Cor. 4° of the same article, the three points (or lines) X, Y, Z determine with the three U, V, W a system of six collinear (or concurrent) constituents, corresponding two and two in opposite pairs U and X, V and Y, W and Z, every four of which are equianharmonic with their four opposites; hence, generally,—

a. When, of a variable triangle whose three vertices move on three fixed concurrent lines, two of the sides turned round two fixed points, the third turns round a third fixed point collinear with the other two.

a'. When, of a variable triangle whose three sides turn round three fixed collinear points, two of the vertices move on two fixed lines, the third moves on a third fixed line concurrent with the other two.

b. The three fixed points, and the three intersections of their line of collinearity with the three fixed lines, in the former case, determine, in three opposite pairs, a row of six points every four of which are equianharmonic with their four opposites.

b'. The three fixed lines, and the three connectors of their point of concurrence with the three fixed points, in the latter case, determine, in three opposite pairs, a pencil of six rays every four of which are equianharmonic with their four opposites.

4°. Any two pairs of corresponding vertices (or sides) A and A', B and B' of the two triangles ABC and $A'B'C'$, in the same (2°), may be regarded as the four points of a tetrastigm (or the four lines of a tetragram); and the two pairs of opposite sides (or vertices) BC and $B'C'$, CA and $C'A'$ as the four lines of an exscribed tetragram (or the four points of an inscribed tetrastigm); hence, again, from the first parts of the same properties, it appears (see note to 2°, *a*, of the present article) that—

a. For a tetrastigm inscribed to a tetragram, when the intersection of a pair of opposite connectors of the former is collinear with a pair of opposite intersections of the latter, the intersections of the remaining two pairs of opposite connectors of the former are collinear each with one of the two remaining pairs of opposite intersections of the latter.

a'. For a tetragram exscribed to a tetrastigm, when the connector of a pair of opposite intersections of the former is concurrent with a pair of opposite connectors of the latter, the connectors of the remaining two pairs of opposite intersections of the former are concurrent each with one of the two remaining pairs of opposite connectors of the latter.

297. As, from property *a* of inference 2° of the preceding article, it was shewn without difficulty in Art. 141 that the same property is true generally of every two figures in perspective with each other; so, from properties *b* and *c* of the same, it follows immediately, that, generally—

For every two figures in perspective with each other, the centre and axis of perspective divide equianharmonically, at once all segments determined by pairs of corresponding points, and all angles determined by pairs of corresponding lines.

And the same appears at once, *à priori*, from the consideration that the property being, by virtue of the general property of Art. 285, evidently true for *every two* is *therefore* true for *all* such segments or angles.

This constant anharmonic ratio of section is termed, with respect to the figures, *their anharmonic ratio of perspective;* and from the circumstance of its constancy in all cases of perspective, it follows that, for figures whose centre or axis of perspective is at infinity, all segments determined by pairs of corresponding points are cut in the same ratio by the one not at infinity (270).

A property already, it will be remembered, established on other principles in Arts. 142 and 143.

In the particular case, when, for two figures in perspective, the anharmonic ratio of perspective $= -1$; all segments determined by pairs of corresponding points, and all angles determined by pairs of corresponding lines, are cut harmonically, by the centre and axis of perspective; the figures themselves are said to be in harmonic perspective; and when either their centre or axis of perspective is at infinity, all segments determined by pairs of corresponding points are bisected internally by the one not at infinity.

Every two figures inscribed, or exscribed, to the same circle furnish, when in perspective, an obvious and important example of two figures whose anharmonic ratio of perspective $= -1$. For, their centre and axis of perspective being then (167) pole and polar to each other with respect to the circle to which they are both inscribed or exscribed, and consequently (259) dividing harmonically at once all segments determined by their pairs of corresponding points, and all angles determined by their pairs of corresponding lines, therefore &c. When, in their case, either the centre or axis of perspective is at infinity, the internal bisection of all segments determined by their pairs of corresponding points by the one not at infinity is evident from 3° and 5°, Art. 165.

COR. As two circles, however circumstanced as to magnitude and position, are always doubly in perspective with respect to each centre of perspective (207); the line at infinity and their radical axis being the axes of their two perspectives for both (206); it appears consequently, from the above, that—

For two circles, however circumstanced as to magnitude and position—

a. The line at infinity and each centre of perspective divide in the same constant anharmonic ratio, at once all segments determined by pairs of homologous points, and all angles determined by pairs of homologous lines, with respect to that centre of perspective.

b. The radical axis and each centre of perspective divide in the same constant anharmonic ratio, at once all segments *determined by pairs of antihomologous points, and all angles determined by pairs of antihomologous lines, with respect to that centre of perspective.*

Since, for any two circles, the line at infinity and the radical axis bisect externally and internally the two segments, real or imaginary, intercepted between the two pairs of at once homologous and antihomologous points determined by the two common tangents through each centre of perspective (182); while the centre of perspective itself divides them in the positive or negative ratio of the similitude of the circles according as it is external or internal (198); it follows, consequently (268), that the two constant anharmonic ratios of perspective, both in *a* and *b* separately for the two centres of perspective, and also in *a* and *b* combined for each centre of perspective, are equal in magnitude and opposite in sign; the absolute value common to the whole four being the constant ratio of similitude of the circles.

298. Since, for every two figures reciprocal polars to each other with respect to a circle (170), there correspond: 1°. To every point or line of either, a line or point of the other (170). 2°. To every connector of two points or intersection of two lines of either, the intersection of the two corresponding lines or the connector of the two corresponding points of the other (167). 3°. To every collinear system of points or concurrent system of lines of either, the concurrent system of corresponding lines or the collinear system of corresponding points of the other (166, Cor. 1°). 4°. To every anharmonic row of four points or pencil of four rays of either, the equianharmonic pencil of four corresponding rays or row of four corresponding points of the other (292). Hence, from the general property of the preceding article combined with those of Art. 141, the following important properties of figures in perspective, as regards reciprocation to an arbitrary circle (172)—

a. Every two figures in perspective with each other reciprocate to any circle into two figures in perspective with each other.

b. The centre of perspective of either pair of figures and the axis of perspective of the other pair are pole and polar to each other with respect to the circle.

c. The anharmonic ratios of perspective of the two pairs, original and reciprocal, are equal.

These properties are evident from the considerations, respectively: *a.* That, as in the original figures all pairs of corresponding points connect concurrently and all pairs of corresponding lines intersect collinearly, so in the reciprocal figures all pairs of corresponding lines intersect collinearly and all pairs of corresponding points connect concurrently; *b.* That the point of concurrence and the line of collinearity for the original figures are the pole and polar respectively of the line of collinearity and the point of concurrence for the reciprocal figures; and, *c.* That all anharmonic ratios whether of rows or pencils are preserved unchanged in reciprocation to any arbitrary circle.

COR. Since, with respect to any circle, the polar of its centre is the line at infinity, and the pole of any line passing through its centre is the point at infinity in the direction perpendicular to the line (165, 3°, 5°); it follows, consequently, from the above, for the particular cases when the centre of the reciprocating circle is (1°) at the centre, and (2°) on the axis, of perspective of the original figures, that—

1°. *Every two figures in perspective with each other reciprocate, to any circle whose centre is at their centre of perspective, into two similar and similarly (or oppositely) placed figures, whose ratio of similitude is equal to their anharmonic ratio of perspective* (see Art. 142).

2°. *Every two figures in perspective with each other reciprocate, to any circle whose centre is on their axis of perspective, into two figures consisting of pairs of points connecting by parallel lines all cut by the same line in their anharmonic ratio of perspective* (see Art. 143).

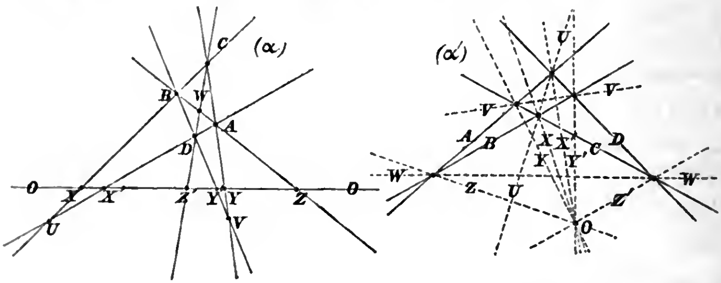
In the particular case when the anharmonic ratio of perspective of the original figures = − 1; the lines connecting the several pairs of corresponding points of the two reciprocal figures are all bisected internally, by their centre of perspective in the case of 1°, and by their axis of perspective in the case of 2°.

As all anharmonic ratios whether of rows or pencils are preserved unchanged in reciprocation (293); it follows, consequently, from these latter properties 1° and 2°, that all anharmonic properties of pairs of figures in either of the two particular cases of perspective in which the axis or the centre of perspective is at infinity are true generally of all pairs of figures in perspective with each other.

299. The two following properties, reciprocals of each other, are in the theory of anharmonic what those of Art. 236 are in that of harmonic section.

a. In every tetrastigm, the three pairs of opposite connectors intersect with every line at three pairs of opposite points, every four of which are equianharmonic with their four opposites.

a'. In every tetragram, the three pairs of opposite intersections connect with every paint by three pairs of opposite rays, every four of which are equianharmonic with their four opposites.



For, if A, B, C, D be the four points constituting the tetrastigm (fig. α), or the four lines constituting the tetragram (fig. α'); BC and AD, CA and BD, AB and CD the three pairs of opposite connectors (or intersections) of the figure; X and X', Y and Y', Z and Z' their three pairs of opposite intersections (or connectors) with any arbitrary line (or point) O; and U, V, W the three points of intersection (or chords of connection) of the figure; then—

1°. The two rows of four points (or pencils of four rays) Y, Z, X, X' and Z', Y', X, X' being in perspective, at the points (or on the lines) A and D, with the row (or pencil) C, B, X, U; and the two Y', Z, X', X and Z', Y, X', X being so, at the points (or on the lines) B and C, with the row (or pencil) D, A, X', U; therefore, by (286, 3°),

$$\left.\begin{aligned} &\{YZXX'\} = \{Z'Y'XX'\} = \{Y'Z'X'X\} \ (280), \\ \text{and} \quad &\{Y'ZX'X\} = \{Z'YX'X\} = \{YZ'XX'\} \ (280), \end{aligned}\right\} \ldots (a).$$

2°. The two rows of four points (or pencils of four rays) Z, X, Y, Y' and X', Z', Y, Y' being in perspective, at the points (or on the lines) B and D, with the row (or pencil) A, C, Y, V; and the two Z', X, Y', Y and X', Z, Y', Y being so, at the points (or on the lines) C and A, with the row (or pencil) D, B, Y', V; therefore, by the same,

$$\left.\begin{aligned} &\{ZXYY'\} = \{X'Z'YY'\} = \{Z'X'Y'Y\} \ (280), \\ \text{and} \quad &\{Z'XY'Y\} = \{X'ZY'Y\} = \{ZX'YY'\} \ (280), \end{aligned}\right\} \ldots (b).$$

3°. The two rows of four points (or pencils of four rays) X, Y, Z, Z' and Y', X', Z, Z' being in perspective, at the points (or on the lines) C and D, with the row (or pencil) B, A, Z, W; and the two X', Y, Z', Z and Y', X, Z', Z being so, at the points (or on the lines) A and B, with the row (or pencil) D, C, Z', W; therefore, by the same,

$$\left.\begin{aligned} &\{XYZZ'\} = \{Y'X'ZZ'\} = \{X'Y'Z'Z\} \ (280), \\ \text{and} \quad &\{X'YZ'Z\} = \{Y'XZ'Z\} = \{XY'ZZ'\} \ (280), \end{aligned}\right\} \ldots\ldots (c),$$

and therefore &c. (283).

Cor. 1°. In the particular case when the line (or point) O passes through (or lies on) any two of the three points of intersection (or lines of connection) U, V, W of the figure; since then, $X = X' = U$ if it pass through (or lie on) U; $Y = Y' = V$ if it pass through (or lie on) V; $Z = Z' = W$ if it pass through (or lie on) W; therefore, from 1°, 2°, 3°, respectively, of the above—

1°. If it pass through (or lie on) V and W, then

$$\{VWXX'\} = \{VWX'X\} \ldots\ldots\ldots\ldots\ldots (1°).$$

2°. If it pass through (or lie on) W and U, then

$$\{WUYY'\} = \{WUY'Y\} \ldots\ldots\ldots\ldots\ldots (2°).$$

3°. If it pass through (or lie on) U and V, then

$$\{UVZZ'\} = \{UVZ'Z\} \ldots\ldots\ldots\ldots\ldots (3°),$$

and therefore (281), as established before on other principles in (236)—

a. In every tetrastigm, the three pairs of opposite connectors divide harmonically, each the segment determined by the intersections of the remaining two (107).

a'. In every tetragram, the three pairs of opposite intersections divide harmonically, each the angle determined by the connectors of the remaining two (107).

COR. 2°. In the particular cases when the line (or point) O is at infinity; the six intersections, in the former case, being the six points at infinity in the directions of the six connectors of the tetrastigm, and the six connectors, in the latter case, being the six parallels in the direction of O through the six intersections of the tetragram; while, in every case, every four of the former are equianharmonic with the pencil they determine at any point, and every four of the latter with the row they determine on any line. Hence, from the general properties applied to those cases, it appears that—

a. The six parallels through any point, to the six connectors of any tetrastigm, determine at the point, in three opposite pairs, a pencil of six rays every four of which are equianharmonic with their four opposites.

a'. The six perpendiculars to any line, through the six intersections of any tetragram, determine on the line, in three opposite pairs, a row of six points every four of which are equianharmonic with their four opposites.

N.B. The six parallels, in the former case, and the six perpendiculars, in the latter case, might evidently be turned in the same direction of rotation through any common angle, without affecting in either case the above relations between them.

COR. 3°. As the three segments (or angles) determined by three pairs of opposite constituents every four of which are equianharmonic with their four opposites, have in all cases a common segment (or angle) of harmonic section, real or imaginary (283, Cor. 1°); it follows consequently from the above, as shewn already on other principles in (245, Cor. 3°), that—

a. The three segments intercepted on any line, by the three pairs of opposite connectors of any tetrastigm, have a common segment of harmonic section, real or imaginary.

a'. The three angles subtended at any point, by the three pairs of opposite intersections of any tetragram, have a common angle of harmonic section, real or imaginary.

COR. 4°. As any three A, B, C of the four points (or lines) A, B, C, D constituting the tetrastigm (or tetragram) may be regarded as the three vertices (or sides) of a triangle ABC, and the fourth D as the point of concurrence (or line of collinearity) of any three concurrent lines through its three vertices (or collinear points on its three sides); the two reciprocal properties of the present article, respecting the tetrastigm and tetragram, may consequently be regarded as anharmonic properties of the triangle, and stated accordingly as follows—

a. The three sides of any triangle, and any three concurrent lines through the three vertices, intersect with every line at six points, corresponding two and two in opposite pairs, every four of which are equianharmonic with their four opposites.

a′. The three vertices of any triangle, and any three collinear points on the three sides, connect with every point by six lines, corresponding two and two in opposite pairs, every four of which are equianharmonic with their four opposites.

COR. 5°. In the particular cases of the latter properties, when the fourth point (or line) D is at infinity; then, since, in the former case, the three lines of connection AD, BD, CD are parallel, and since, in the latter case, the three points of intersection AD, BD, CD are the three at infinity in the directions of the three sides of the triangle; hence, from those properties applied to these particular cases, it appears that—

a. The three intersections with any line of the three sides of any triangle determine, with the three projections on the line of the three vertices of the triangle, a system of six points, corresponding two and two in opposite pairs, every four of which are equianharmonic with their four opposites.

a′. The three connectors with any point of the three vertices of any triangle determine, with the three parallels through the point to the three sides of the triangle, a system of six rays, corresponding two and two in opposite pairs, every four of which are equianharmonic with their four opposites.

COR. 6°. In the particular case when the arbitrary point in property a' of the preceding corollary (5°) is the polar centre of the triangle (168); since, then, each connector and the corresponding parallel are perpendiculars to each other, it follows consequently, from that property applied to this particular case, as is *à priori* evident (283, Cor. 2°), that—

Every three lines through a point determine, with the three perpendiculars to them through the point, a system of six rays, corresponding two and two in opposite pairs, every four of which are equianharmonic with their four opposites.

Cor. 7°. Since from the original property a' of the present article, combined with that of the preceding corollary (6°), it follows that—*when, of the three angles subtended at a point by the three chords of intersection of any tetragram, two are right, the third is right also;* hence, from the familiar property (Euc. III. 31), that the vertices of all right angles subtending a common segment lie on the circle of which the segment is a diameter, it appears, as proved already more generally on other principles in Art. 189, Cor. 1°, that—

The three circles on the three chords of intersection of any tetragram as diameters pass each through the two points of intersection of the other two; and have, therefore, all three, a common pair of points, real or imaginary.

300. The two reciprocal properties of the preceding article supply obvious solutions, by linear constructions only without the aid of the circle, of the two following reciprocal problems, viz.—

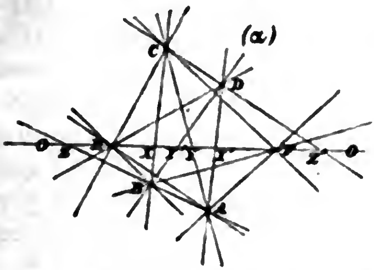
Of two triads of collinear points (or concurrent lines), which correspond two and two in opposite pairs, and every four of which are equianharmonic with their four opposites; given either triad and any two constituents of the other, to determine the third constituent of the latter.

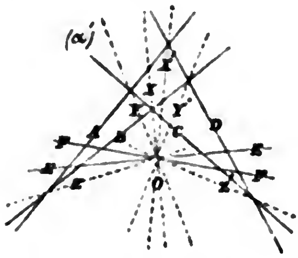
Thus, in the figures (α) and (α') of that article, given the triad of points (or rays) X, Y, Z, and any two constituents X' and Y' of the other X', Y', Z', to determine the third constituent Z' of the latter. In the former case, through the three given points X, Y, Z (fig. α) drawing arbitrarily any three non-concurrent lines; the three opposite vertices A, B, C of the triangle they determine, by property a of the preceding article, connect with their three opposites X', Y', Z' by three concurrent lines AX', BY', CZ', two of which AX' and BY' being given determine the point of concurrence D, and therefore the third CZ', which intersects with the line O at the required point Z'. And, in the latter case, on the three given lines X, Y, Z (fig. α') taking arbitrarily any three non-collinear points; the three opposite sides A, B, C of the triangle they determine, by property a' of the preceding article, intersect with their three opposites X', Y', Z' at three collinear points AX', BY', CZ', two of which AX' and BY' being given determine the line of collinearity D, and therefore the third CZ', which connects with the point O by the required line Z'.

301. Of all anharmonic properties of figures of points and lines, the two following, reciprocals of each other, lead to the greatest number of consequences in the theory of conic sections, viz.—

a. When, of six points, any four connect equianharmonically with the remaining two, then every four connect equianharmonically with the remaining two.

a'. When, of six lines, any four intersect equianharmonically with the remaining two, then every four intersect equianharmonically with the remaining two.





Let A, B, C, D, E, F be the six points (fig. α) or the six lines (fig. α'); when any four of them C, D, E, F connect (or intersect) equianharmonically with the remaining two A and B, then any other four of them A, B, E, F connect (or intersect) equianharmonically with the remaining two C and D. For, O being the line of connection (or the point of intersection) of the two points (or lines) E and F which are common to the two systems of four A, B, E, F and C, D, E, F for which the relation is given and to be proved respectively, if X and X', Y and Y', Z and Z' be the three pairs of opposite intersections (or connectors) of O with the three pairs of opposite connectors (or intersections) BC and AD, CA and BD, AB and CD of the tetrastigm (or tetragram) determined by the four A, B, C, D which are not common to the two systems; then since, by hypothesis, $\{A.CDEF\} = \{B.CDEF\}$, and since, consequently, by (285), $\{YX'EF\} = \{XY'EF\}$, therefore, by (272), $\{YXEF\} = \{X'Y'EF\}$, and consequently, by (285), $\{C.ABEF\} = \{D.ABEF\}$; and therefore &c.

The above demonstration, though apparently establishing the property only for the six cases in which the quartet for which the relation is given has but *two* constituents in common with that for which it is to be proved, in reality establishes it for the eight cases in which the two quartets have *three* constituents in common as well; for establishing it, as shewn above, for every quartet having but two constituents in common with that for which it is given, it consequently establishes it at the same time for every quartet having but two constituents in common with each of the latter; and therefore &c.

Two collinear triads of points on different axes (or concurrent triads of lines through different vertices) (293) furnish an obvious, but very particular, example of a system of six points (or lines) every four of which connect (or intersect) equianharmonically with the remaining two. Every system of six points on (or tangents to) the same circle, as will be shewn at the opening of the next chapter, also comes under the same head, and possesses, in consequence, every property of the more general system depending only on the existence of the aforesaid equianharmonic relations between its constituent points (or lines).

Cor. 1°. As the three pairs of points (or lines) A and B, C and D, E and F (or any other three pairs into which the six may be resolved) may be regarded as determining three segments (or angles) AB, CD, EF, the extremities of some, and therefore of every, two of which connect (or intersect) with those of the third equianharmonically; the above reciprocal properties may, consequently, be stated (as indeed they were proved) in the following equivalent, but less general, forms, viz.—

a. When, of three segments, the extremities of any two connect equianharmonically with those of the third, then the extremities of every two connect equianharmonically with those of the third.

a′. When, of three angles, the sides of any two intersect equianharmonically with those of the third, then the sides of every two intersect equianharmonically with those of the third.

COR. 2°. As, in the tetrastigm (or tetragram) determined by the four points (or lines) A, B, C, D (or by any other four of the six), the three pairs of equianharmonic relations, for the three pairs of opposite segments (or angles) BC and AD, CA and BD, AB and CD, with each other, and with the segment (or angle) EF determined by the remaining two E and F, viz.—

$\{A.CBEF\}=\{D.CBEF\}$ and $\{B.DAEF\}=\{C.DAEF\}$...(1°),
$\{B.ACEF\}=\{D.ACEF\}$ and $\{C.DBEF\}=\{A.DBEF\}$...(2°),
$\{C.BAEF\}=\{D.BAEF\}$ and $\{A.DCEF\}=\{B.DCEF\}$...(3°),

and the three corresponding pairs for the three pairs of segments (or angles) XY and $X'Y'$, YZ and $Y'Z'$, ZX and $Z'X'$ they determine on the line (or at the point) EF, viz.—

$\{YZEF\}=\{Y'Z'FE\}$ and $\{Y'ZEF\}=\{YZ'FE\}$...... (1′),
$\{ZXEF\}=\{Z'X'FE\}$ and $\{Z'XEF\}=\{ZX'FE\}$...... (2′),
$\{XYEF\}=\{X'Y'FE\}$ and $\{X'YEF\}=\{XY'FE\}$...... (3′),

by virtue of (285) and (280), mutually involve each other; hence, see (299) and (284), it appears generally that—

a. Every two points, which connect equianharmonically with the four points of any tetrastigm, form, with the six determined on their line of connection by the six connectors of the tetrastigm, a system of eight points, in four opposite pairs, every four of which are equianharmonic with their four opposites; and, conversely, every two points, which form such a system with the six determined on their line of connection by the six connectors of any tetrastigm, connect equianharmonically with the four points of the tetrastigm.

a′. Every two lines, which intersect equianharmonically with the four lines of any tetragram, form, with the six determined at their point of intersection by the six intersections of the tetragram, a system of eight rays, in four opposite pairs, every four of which are equianharmonic with their four opposites; and, conversely, *every two lines, which form such a system with the six determined at their point of intersection by the six intersections of any tetragram, intersect equianharmonically with the four lines of the tetragram.*

Cor. 3°. As an example of the criterion of the above relation of equianharmonicism for a system of six points or lines supplied by the second parts of the two reciprocal properties a and a' of the preceding corollary (Cor. 3°); suppose the two triangles ABC and DEF employed in its establishment were both self-reciprocal with respect to the same circle (168); since then evidently, by (167), the several pairs of points (or lines) X and X', Y and Y', Z and Z', E and F would be pairs of conjugates with respect to the circle (174), and consequently pairs of harmonic conjugates with respect to the two points (or tangents), real or imaginary, determined with the circle by the line or point EF (259); therefore, by (3°, Cor. 4°, Art. 282), they would satisfy the criterion expressed in the two properties; and therefore—

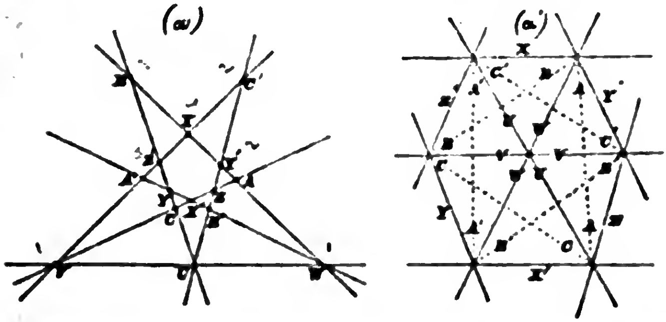
For every two self reciprocal triangles with respect to the same circle, every four of the six vertices connect, and every four of the six sides intersect, equianharmonically with the remaining two.

302. Reserving for the next chapter the principal consequences resulting, in the geometry of the circle, from the circumstance of every six concyclic points or tangents being connected by the equianharmonic relations of the preceding article; we shall conclude the present with the two following reciprocal properties of such systems in general, and with a few of the many consequences to which they lead in the geometry of the point and line.

a. In a hexagon, when the intersections of the three pairs of opposite sides are collinear, every four of the six vertices connect equianharmonically with the remaining two; and, conversely, when any four of the six vertices connect equianharmonically with the remaining two, the intersections of the three pairs of opposite sides are collinear.

a'. In a hexagon, when the connectors of the three pairs of opposite vertices are concurrent, every four of the six sides intersect equianharmonically with the remaining two; and, conversely, when any four of the six sides intersect equianharmonically with *the remaining two, the connectors of the three pairs of opposite vertices are concurrent.*

For, if X and X', Y and Y', Z and Z' be the three pairs of opposite vertices (fig. α), or sides (fig. α'), of the hexagon; A and A', B and B', C and C' the three pairs of corresponding vertices (or sides) of the two triangles determined by its two triads of alternate sides (or vertices); and U, V, W the three points of intersection (or lines of connection) of its three pairs



of opposite sides (or vertices); then since, by the general property of Art. (285), the three pairs of equianharmonic relations

$\{Y.X'Y'Z'X\} = \{Z.X'Y'Z'X\}$ and $\{Y'.XYZX'\} = \{Z'.XYZX'\}$ .... (1°),

$\{Z.Y'Z'X'Y\} = \{X.Y'Z'X'Y\}$ and $\{Z'.YZXY'\} = \{X'.YZXY'\}$ .... (2°),

$\{X.Z'X'Y'Z\} = \{Y.Z'X'Y'Z\}$ and $\{X'.ZXYZ'\} = \{Y'.ZXYZ'\}$ .... (3°),

and the three corresponding pairs

$\{WY'BX\} = \{VC'Z'X\}$ and $\{WYB'X'\} = \{VCZX'\}$ ... (1'),

$\{UZ'CY\} = \{WA'X'Y\}$ and $\{UZC'Y'\} = \{WAXY'\}$ ... (2'),

$\{VX'AZ\} = \{UB'Y'Z\}$ and $\{VXA'Z'\} = \{UBYZ'\}$ ... (3'),

mutually involve each other; and since, by (286, 4°), the latter are all involved in, while, by (289), any one of them involves, the collinearity (or concurrence) of the three points (or lines) U, V, W; therefore &c.

The hexagons originally considered in the celebrated theorems of Pascal and Brianchon established on other principles in

Art. 148, coming under the second parts of the above reciprocal properties *a* and *a′* respectively; the names "Pascal hexagon" and "Brianchon hexagon" are in consequence applied generally, the former to all hexagons whose pairs of opposite sides intersect collinearly, and the latter to all whose pairs of opposite vertices connect concurrently; the line of collinearity in the former case, and the point of concurrence in the latter case being termed respectively the "Pascal line" and "Brianchon point" of the hexagon. For the same reason the names "Pascal hexastigm" and "Brianchon hexagram" are applied generally, the former to all systems of six points, every four of which connect equianharmonically with the remaining two, and the latter to all systems of six lines, every four of which intersect equianharmonically with the remaining two; all hexagons determined by such systems being, by virtue of the same properties, Pascal and Brianchon hexagons in the two cases respectively.

Since for every two triangles in perspective, the three pairs of corresponding sides intersect collinearly on the axis of perspective, and the three pairs of corresponding vertices connect concurrently through the centre of perspective (140); it follows consequently, from the above, that *for every two triangles in perspective; the four hexagons of which their pairs of corresponding are pairs of opposite sides are Pascal hexagons, of which their axis of perspective is the common Pascal line; and the four of which their pairs of corresponding are pairs of opposite vertices are Brianchon hexagons, of which their centre of perspective is the common Brianchon point.*

The same hexagon might be at once a Pascal and a Brianchon hexagon, and when such would of course in its double capacity combine the properties of both; every hexagon at once inscribed to one circle and exscribed to another circle furnishes an example of a hexagon of this nature.

303. From the two reciprocal properties of the preceding article, combined with the fundamental two of Art. 140 respecting triangles in perspective, the following consequences, in pairs reciprocals of each other, may be readily inferred, viz.—

a. The intersections of the six pairs of alternate sides of a

Pascal hexagon, taken in consecutive order, determine a Brianchon hexagon.

a'. The connectors of the six pairs of alternate vertices of a Brianchon hexagon, taken in consecutive order, determine a Pascal hexagon.

For, if (figures of last article) X and X', Y and Y', Z and Z' be the three pairs of opposite vertices (or sides) of the original hexagon, and A and A', B and B', C and C' those of the derived hexagon; then since, by hypothesis, the three pairs of corresponding sides (or vertices) BC and $B'C'$, CA and $C'A'$, AB and $A'B'$ of the two triangles ABC and $A'B'C'$ intersect collinearly (or connect concurrently), therefore, by (140), their three pairs of corresponding vertices (or sides) A and A', B and B', C and C' connect concurrently (or intersect collinearly); and therefore &c.

b. When the vertices of three angles are collinear, the twelve remaining intersections of their six determining lines may be divided, in four different ways, into two groups of six, determining one a Pascal and the other a Brianchon hexagon.

b'. When the axes of three segments are concurrent, the twelve remaining connectors of their six determining points may be divided, in four different ways, into two groups of six, determining one a Brianchon and the other a Pascal hexagon.

For, the four hexagons, of which the three pairs of lines (or points) determining the three angles (or segments) are the three pairs of opposite sides (or vertices), being, by the preceding article, Pascal (or Brianchon) hexagons; and, the four hexagons determined by the intersections (or connectors) of their pairs of alternate sides (or vertices) being, by the two properties *a* and *a'* just proved, Brianchon (or Pascal) hexagons; therefore &c.

In the figures of the preceding article, the three collinear points (or concurrent lines) U, V, W being the three vertices (or axes) of the three determining angles (or segments), the four sets of two complementary groups of six points (or lines)

$X, B', C;\ X', B, C'$ and $A, Y', Z;\ A', Y, Z'$ .... (1),

$Y, C', A;\ Y', C, A'$ and $B, Z', X;\ B', Z, X'$ .... (2),

$Z, A', B;\ Z', A, B'$ and $C, X', Y;\ C', X, Y'$ .... (3),

$X, Y, Z;\ X', Y', Z'$ and $A, B, C;\ A', B', C'$ .... (4), are those determining the four Pascal and Brianchon (or Brianchon and Pascal) hexagons in the four cases respectively.

c. Of the sixty hexagons determined by the same Pascal hexastigm, the sixty Pascal lines pass three and three through twenty points.

c′. Of the sixty hexagons determined by the same Brianchon hexagram, the sixty Brianchon points lie three and three on twenty lines.

For, of the four hexagons, of which the two triads of points (or lines) X, Y, Z and X', Y', Z', in the figures of the last article, are the two triads of alternate vertices (or sides), viz.:

$$XY'YX'ZZ',\ YZ'ZY'XX',\ ZX'XZ'YY',\ XZ'YX'ZY',$$

while the Pascal line (or Brianchon point) of the fourth is the line (or point) UVW, those of the three first are the three lines (or points) AA', BB' CC' respectively, which, by the preceding a and a', or by the general property of triangles in perspective (140), are concurrent (or collinear); and, the same being of course true for every other similarly circumstanced three of the entire sixty, therefore, &c.

The above theorem c (and with it of course its reciprocal c') is due to M. Steiner, who was the first to direct the attention of geometers to the complete figure determined by a system of six points (or lines), every four of which connect (or intersect) equianharmonically with the remaining two. The subject has since, from time to time, engaged the attention of different eminent geometers, including M. Plücker, Dr. Salmon, Professor Cayley, and Mr. Kirkman, by whom several other properties of the same nature have been discovered; of the principal of which, an abstract will be found in Dr. Salmon's Conic Sections, Ed. 4, note 1, page 357, and further details in Mr. Kirkman's published paper, Cambridge and Dublin Mathematical Journal, Vol. v., p. 185.

d. For each of the fifteen triads of non-conterminous segments determined by the same Pascal hexastigm, the six centres of perspective of the three pairs they determine lie three and three on four lines.

d′. For each of the fifteen triads of non-conterminous angles determined by the same Pascal hexagram, the six axes of perspec-

tive of the three pairs they determine pass three and three through four points.

For if XX', YY', ZZ' (same figures) be any triad of non-conterminous segments (or angles) determined by the six points (or lines) of the hexastigm (or hexagram); U, V, W the three intersections (or connectors) of the three pairs of connectors (or intersections) YZ' and $Y'Z$, ZX' and $Z'X$, XY' and $X'Y$; and U', V', W' those of the three pairs YZ and $Y'Z'$, ZX and $Z'X'$, XY and $X'Y'$; then, in the four hexagons $YXZY'X'Z'$, $ZYXZ'Y'X'$, $XZYX'Z'Y'$, $YZ'XY'ZX'$, the four triads of points (or lines) V' W' U; W' U', V; U', V', W; U, V, W, being the four triads of intersections (or connectors) of pairs of opposite sides (or vertices), are collinear (or concurrent); and the same being of course true for each of the remaining fourteen triads of non-conterminous segments (or angles) determined by the hexastigm (or hexagram), therefore, &c.

304. The two following reciprocal criteria that six points lying in pairs on the three sides of a triangle should determine a Pascal hexastigm, and that six lines passing in pairs through the three vertices of a triangle should determine a Brianchon hexagram, result immediately from the two of Art. 147 for the perspective of two triangles; viz.—

When three pairs of points (or lines) X and X', Y and Y', Z and Z', lying on the three sides (or passing through the three vertices) BC, CA, AB of a triangle whose three opposite vertices (or sides) are A, B, C, determine a Pascal hexastigm (or Brianchon hexagram), they satisfy—

a. In the former case the general relation

$$\frac{BX.BX'}{CX.CX'}\cdot\frac{CY.CY'}{AY.AY'}\cdot\frac{AZ.AZ'}{BZ.BZ'}=+1,$$

a'. In the latter case the reciprocal relation

$$\frac{\sin BX.\sin BX'}{\sin CX.\sin CX'}\cdot\frac{\sin CY.\sin CY'}{\sin AY.\sin AY'}\cdot\frac{\sin AZ.\sin AZ'}{\sin BZ.\sin BZ'}=+1;$$

and, conversely, when of the above two reciprocal relations they satisfy the one corresponding to their case, they determine a Pascal hexastigm (or Brianchon hexagram).

For, of those two reciprocal relations, that corresponding to the case being, by (147), the criterion that the triangle determined by the three lines (or points) XX', YY', ZZ', or the triangle ABC, should be in perspective with each of the eight triangles determined by the eight triads of points (or lines) YZ', ZX', XY'; $YZ', ZX, X'Y'$; $YZ, Z'X, X'Y'$; $YZ, Z'X', XY'$; $Y'Z, Z'X, X'Y$; $Y'Z, Z'X', XY$; $Y'Z', ZX', XY$; $Y'Z', ZX, X'Y$; and, conversely, being, by the same, fulfilled when it is in perspective with any one of them; therefore, &c. (302).

COR. 1°. Since, for the same triangle, by (65); every three pairs of points on the three sides which satisfy relation (a) connect with the opposite vertices by three pairs of lines satisfying relation (a'); while, conversely, every three pairs of lines through the three vertices which satisfy relation (a') intersect with the opposite sides at three pairs of points satisfying relation (a); hence, from the above, it appears, generally, that—

When three pairs of points on the three sides of a triangle determine a Pascal hexastigm, their three pairs of connectors with the opposite vertices determine a Brianchon hexagram; and, conversely, when three pairs of lines through the three vertices of a triangle determine a Brianchon hexagram, their three pairs of intersections with the opposite sides determine a Pascal hexastigm.

COR. 2°. Since, for any triangle, by (134); every two triads of points X, Y, Z, and X', Y', Z' on the three sides BC, CA, AB, which are both either collinear or concurrently connectant with the opposite vertices, satisfy relation (a); while, reciprocally, every two triads of lines X, Y, Z and X', Y', Z' through the three vertices BC, CA, AB, which are both either concurrent or collinearly intersectant with the opposite sides, satisfy relation (a'); hence, again, from the above, it appears that—

When two triads of points on the three sides of a triangle are both either collinear or concurrently connectant with the opposite vertices they determine a Pascal hexastigm; and, reciprocally, when two triads of lines through the three vertices of a triangle are both either concurrent or collinearly intersectant with the opposite sides they determine a Brianchon hexagram.

COR. 3°. Since again, conversely, for any triangle, by the same, if, of two triads of points X, Y, Z and X', Y', Z' on the three sides BC, CA, AB which satisfy relation (a), one be either collinear or concurrently connectant with the opposite vertices, so is the other also; while, reciprocally, if, of two triads of lines X, Y, Z and X', Y', Z' through the three vertices BC, CA, AB which satisfy relation (a'), one be either concurrent or collinearly intersectant with the opposite sides, so is the other also; hence also, from the above, it appears, conversely, that—

Of two triads of points on the three sides of a triangle which determine a Pascal hexastigm, if one be either collinear or concurrently connectant with the opposite vertices, so is the other also; and, reciprocally, of two triads of lines through the three vertices of a triangle, which determine a Brianchon hexagram, if one be either concurrent or collinearly intersectant with the opposite sides, so is the other also.

N.B.—Of the reciprocal properties of this corollary, those established, on other considerations, in Examples 9°, 10°, 11°, 12°, Art 137, are evidently particular cases.

Cor. 4°. Since, by the two reciprocal relations (a) and (a') of Art. 245, every three pairs of points (or lines) X and X', Y and Y', Z and Z', harmonic conjugates with respect to the three chords of intersection (or angles of connection) of any tetragram (or tetrastigm), divide the three sides (or angles) BC, CA, AB of the triangle determined by the axes (or vertices) of the three chords (or angles) so as to satisfy the above relations (a) and (a') respectively; hence, again, from the above, it appears generally that—

a. Every three pairs of points, harmonic conjugates with respect to the three chords of intersection of a tetragram, determine a Pascal hexastigm.

a'. Every three pairs of lines, harmonic conjugates with respect to the three angles of connection of a tetrastigm, determine a Brianchon hexagram.

N.B.—Of the two reciprocal properties of this corollary, the two (a) and (b) of the article referred to in their proof (245) are evidently particular cases.

Cor. 5°. Assuming, as will be shown in the next chapter, that when five of the six points (or lines) determining a Pascal hexastigm (or Brianchon hexagram) are points on (or tangents to) a common circle, the sixth also is a point on (or tangent to) the same circle; it follows evidently, from the two reciprocal properties a and a' of the preceding corollary (4°), that—

a. Every circle, dividing two of the three chords of intersection of a tetragram harmonically, divides the third also harmonically.

a'. Every circle, subtending two of the three angles of connection of a tetrastigm harmonically, subtends the third also harmonically.

Properties, the first of which, it will be remembered, was proved before, on other principles, in Art. 228, Cor. 6°.

CHAPTER XVIII.

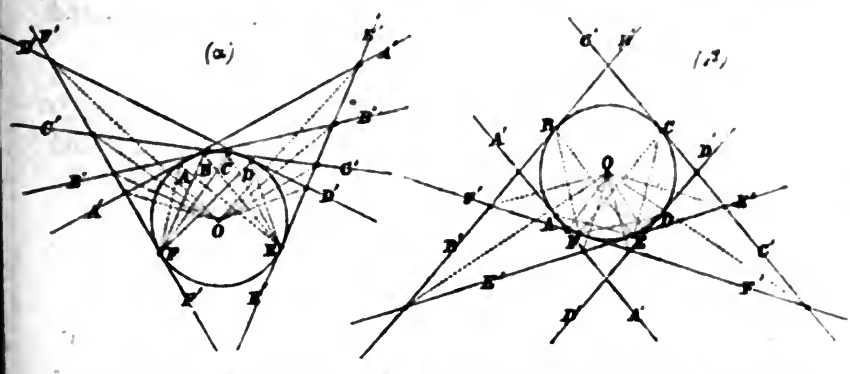
ANHARMONIC PROPERTIES OF THE CIRCLE.

305. Among the various anharmonic properties of the circle, the two following, reciprocals of each other, are those to which the designation is most commonly applied; and they obviously include, as particular cases, the two already given in Art. 252 at the commencement of Chapter XV., viz.:

a. Every system of four points on a circle determines equianharmonic pencils of rays at every two, and therefore at all, points on the circle.

a'. Every system of four tangents to a circle determines equianharmonic rows of points on every two, and therefore on all, tangents to the circle.

For, in the former case, if A, B, C, D be any four points on



a circle; then since, for every two points E and F on the circle, the two pencils of four rays $E.ABCD$ and $F.ABCD$ are similar (25, 1°), therefore &c. And, in the latter case, if A', B', C', D' be any four tangents to a circle; then since, for every two tangents E' and F' to the circle, the two rows of four points

$E'.A'B'C'D'$ and $F'.A'B'C'D'$ determine similar pencils at the centre O of the circle (25, 2°), therefore &c. (285).

Since, by virtue of the above reciprocal properties, every six points A, B, C, D, E, F on a circle form a system of six points, every four of which connect equianharmonically with the remaining two, and every six tangents A', B', C', D', E', F' to a circle form a system of six lines, every four of which intersect equianharmonically with the remaining two; all anharmonic properties, consequently, which are true, in general, of any system of six points or lines thus related to each other (301), are true, in particular, of every system of six points on or tangents to the same circle. See Arts. 301 to 304.

Again, since, under the process of reciprocation to an arbitrary circle (172), all systems of points and tangents of the original become transformed into systems of tangents and points of the reciprocal figure (159), and all anharmonic rows and pencils of the original into equianharmonic pencils and rows of the reciprocal figure (292); it follows, consequently, that the above reciprocal properties, with all the consequences to which they lead in the geometry of the circle, are true, more generally, not only of the circle, but also of every figure into which the circle can become transformed by reciprocation; either in the original involving the other in the reciprocal figure, and conversely. See Art. 173.

306. If, in the preceding, while the four points A, B, C, D and one of the two E, F, or the four tangents A', B', C', D' and one of the two E', F', are supposed to remain fixed, the remaining point, or tangent, be conceived to vary, and, in the course of its variation, to go round the entire circle; since then, in every position of the variable point, or tangent,

$$\{E.ABCD\} = \{F.ABCD\}, \text{ or, } \{E'.A'B'C'D'\} = \{F'.A'B'C'D'\},$$

the two reciprocal properties of the preceding article may consequently be stated as follows:

a. Every system of four fixed points on a circle determines, at a variable fifth point on the circle, a variable pencil of four rays having a constant anharmonic ratio.

a'. Every system of four fixed tangents to a circle determines, *on a variable fifth tangent to the circle, a variable row of four points having a constant anharmonic ratio.*

When the variable point, or tangent, in the course of its variation, coincides with one of the four fixed points, or tangents; the corresponding ray of the variable pencil, or point of the variable row, becomes then the tangent at the fixed point (19), or the point of contact of the fixed tangent (20); but the entire pencil, or row, has still the same constant anharmonic ratio as for every other position of the variable point, or tangent. See Art. 255.

This constant anharmonic ratio is commonly termed that of the four fixed points on the circle, in the former case, and that of the four fixed tangents to the circle, in the latter case; it being, of course, always implicitly understood to mean, as above explained, that of the pencil determined by the four at any fifth point on the circle, in the former case, and that of the row determined by the four on any fifth tangent to the circle, in the latter case. See Art. 279.

For the same reasons stated in the concluding paragraph of the preceding article, the above reciprocal properties are true generally, not only of the circle, but also of every figure into which the circle can become transformed by reciprocation; either in the original involving the other in the reciprocal figure, and conversely.

307. *The pencil of four rays determined by any system of four points on a circle at any fifth point on the circle, and the row of four points determined by the corresponding system of four tangents to the circle on any fifth tangent to the circle, are equianharmonic.*

For, if A, B, C, D be any four points on a circle, and A', B', C', D' the four corresponding tangents to the circle; E any fifth point on the circle, and E' any fifth tangent to the circle; then since, for all positions of E and E', by (25, 1° and 2°), the pencil of four rays $E.ABCD$ is similar to that determined by the row of four points $E'.A'B'C'D'$ at the centre of the circle, therefore &c. (285).

In the particular case when the fifth tangent E' is that corresponding to the fifth point E; the four points $E'.A'B'C'D'$

are the four poles, with respect to the circle, of the four points $E.ABCD$ (165, 6°); and their equianharmonicism follows, as a particular case, from the general property of Art. 292.

The above anharmonic equivalence is generally represented, for shortness, by Dr. Salmon's Notation, Art. 279, viz.—

$$\{ABCD\} = \{A'B'C'D'\};$$

it being, of course, always understood that the two equivalents refer, respectively, to the pencil determined by the four points A, B, C, D at any fifth point on the circle, and to the row determined by the four corresponding tangents $A'B'C'D'$ on any fifth tangent to the circle.

Again, for the reasons stated in the concluding paragraph of Art. 305, the above property is true generally, not only of the circle, but also of every figure into which the circle can become transformed by reciprocation.

308. *The six anharmonic ratios* P *and* $\frac{1}{P}$, Q *and* $\frac{1}{Q}$, R *and* $\frac{1}{R}$ *of the pencil determined by any four points* A, B, C, D *on a circle at any fifth point* E *on the circle, or by the four corresponding tangents* A', B', C', D' *on any fifth tangent* E' *to the circle* (307), *may be expressed in terms of the six chords connecting the four points two and two, exactly as for four points on a line* (274), *viz.*—

$$\frac{BA}{CA} : \frac{BD}{CD} \text{ and } \frac{CA}{BA} : \frac{CD}{BD}, \text{ or, } \frac{BA.CD}{CA.BD} \text{ and } \frac{CA.BD}{BA.CD} \ldots\ldots (1),$$

$$\frac{CB}{AB} : \frac{CD}{AD} \text{ and } \frac{AB}{CB} : \frac{AD}{CD}, \text{ or, } \frac{CB.AD}{AB.CD} \text{ and } \frac{AB.CD}{CB.AD} \ldots\ldots (2),$$

$$\frac{AC}{BC} : \frac{AD}{BD} \text{ and } \frac{BC}{AC} : \frac{BD}{AD}, \text{ or, } \frac{AC.BD}{BC.AD} \text{ and } \frac{BC.AD}{AC.BD} \ldots\ldots (3).$$

For, in the three pairs of reciprocal ratios (1), (2), (3), dividing each chord involved by the diameter of the circle, and substituting for the resulting quotient the sine of the angle subtended by that chord at any point E on the circle (62); the three pairs of corresponding anharmonic ratios of the pencil $E.ABCD$ determined by the four points A, B, C, D at the point E, viz.—

$$\frac{\sin BEA}{\sin CEA} : \frac{\sin BED}{\sin CED} \text{ and } \frac{\sin CEA}{\sin BEA} : \frac{\sin CED}{\sin BED} \ldots\ldots (1'),$$

$$\frac{\sin CEB}{\sin AEB} : \frac{\sin CED}{\sin AED} \text{ and } \frac{\sin AEB}{\sin CEB} : \frac{\sin AED}{\sin CED} \ldots\ldots (2'),$$

$$\frac{\sin AEC}{\sin BEC} : \frac{\sin AED}{\sin BED} \text{ and } \frac{\sin BEC}{\sin AEC} : \frac{\sin BED}{\sin AED} \ldots\ldots (3'),$$

or their three corresponding equivalents (see (1'), (2'), (3') (Art. 274), are the immediate result; and therefore &c.

309. Two different systems of four points on, or tangents to, the same circle, or two different circles, A, B, C, D and A', B', C', D', which correspond in pairs A to A', B to B', C to C', D to D', are said to be equianharmonic, when the pencils of four rays, or the rows of four points, they determine at all points on, or on all tangents to, their circle, or circles, are equianharmonic. With the same understanding, as to meaning, as in the particular case considered in Art. 307, all such cases of anharmonic equivalence may in general be represented, for shortness, by Dr. Salmon's Notation (279), viz.—

$$\{ABCD\} = \{A'B'C'D'\};$$

the several pairs of corresponding constituents being, of course, invariably written in the same order in the two equivalents, in every case of its employment.

Two *similar* systems of four points on the same circle, or on two different circles, as determining similar pencils of four rays at all points on their circle, or circles, furnish the most obvious as well as the simplest example of two equianharmonic systems of concyclic points; and the two systems of corresponding tangents to the circle, or circles, furnish the most obvious as well as the simplest example of two equianharmonic systems of concyclic tangents, in the sense above defined. Thus, for two circles, every two systems determined by four points or tangents of either, and by the four homologous points or tangents of the other with respect to either centre of similitude of the two (198), as being evidently similar, are equianharmonic in that sense. For the same circle, it is evident, from 2°, Art. 286, that every two systems of four points which determine pencils in perspective at any two points on the circle, and every two systems of four tangents which determine rows in perspective on any two tangents to the circle, are equianharmonic in the same sense.

310. As, in Art. 290, for two equianharmonic rows of four points on different axes, or for two equianharmonic pencils of four rays through different vertices; so, for two concyclic systems of four points, or tangents, equianharmonic in the sense of the preceding article, it is evident, from the two reciprocal properties (a) and (a') of Art. 289, that—

a. Every two equianharmonic systems of four points on the same circle determine two pencils of rays in perspective, either at any point of the other, and the latter at the corresponding point of the former.

a'. Every two equianharmonic systems of four tangents to the same circle determine two rows of points in perspective, either on any tangent of the other, and the latter on the corresponding tangent of the former.

For, if A, B, C, D and A', B', C', D' be the two systems of points, or tangents; then, each pair of pencils or rows

$$A.A'B'C'D' \text{ and } A'.ABCD,\quad B.A'B'C'D' \text{ and } B'.ABCD,$$

$$C.A'B'C'D' \text{ and } C'.ABCD,\quad D.A'B'C'D' \text{ and } D'.ABCD,$$

being equianharmonic, and having a common ray or point, therefore &c. (289).

Cor. As, in Art. 291, for two equianharmonic rows of four points on different axes, or pencils of four rays through different vertices; so, for two concyclic systems of four points, or tangents, equianharmonic in the sense in question, the above reciprocal properties supply ready solutions, by linear constructions only, without the aid of a second circle, of the two following reciprocal problems, viz.—

a. Given three pairs of corresponding constituents of two equianharmonic systems of points on the same circle, and the fourth point of either system, to determine the fourth point of the other system.

a'. Given three pairs of corresponding constituents of two equianharmonic systems of tangents to the same circle, and the fourth tangent of either system, to determine the fourth tangent of the other system.

The two reciprocal constructions given in the article referred to (291), modified in the manner mentioned in the note at its close, apply, word for word and letter for letter, to these problems also.

311. The two following reciprocal cases of anharmonic equivalence, between concyclic systems of points and tangents, result immediately from the fundamental properties of Art. 305, and lead to several remarkable consequences in the modern geometry of the circle, viz.—

If A, B, C, D be any four points on (or tangents to) the same circle, and X, Y, Z the three points of intersection (or lines of connection) of the three pairs of connectors (or intersections) BC and AD, CA and BD, AB and CD (figs. α and α'); *then* (see Art. 272).

1°. *For every pair of concyclic points (or tangents) E and F which connect through (or intersect on) X,*

$$\{ABEF\} = \{CDEF\} \text{ and } \{ACEF\} = \{BDEF\} \;.....(a);$$

2°. *For every pair G and H which connect through (or intersect on) Y,*

$$\{BCGH\} = \{ADGH\} \text{ and } \{BAGH\} = \{CDGH\} \;...(b);$$

3°. *For every pair K and L which connect through (or intersect on) Z,*

$$\{CAKL\} = \{BDKL\} \text{ and } \{CBKL\} = \{ADKL\} \;.... (c);$$

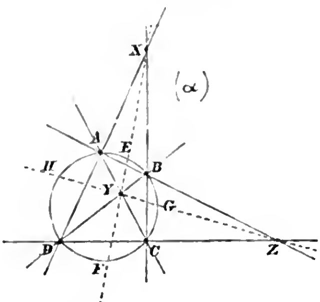
and, conversely, all pairs of concyclic points (or tangents) which fulfil either, and therefore the other (272), *of any of the three preceding pairs of equianharmonic relations, connect through (or intersect on) the corresponding one of the three points (or lines) X, Y, Z.*

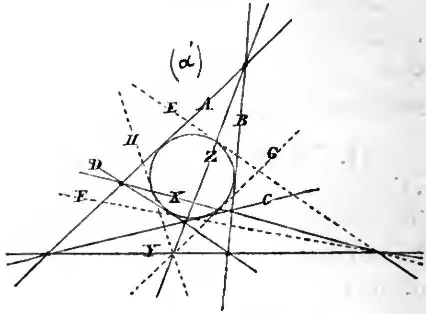
For, by the two converse pairs of reciprocal properties (286, 4°) and (289), the collinearity (or concurrence) of the three points (or lines) E, F, and X, in the first case, involves, and is involved in either of, the two equianharmonic relations

$$\{B.ACEF\} = \{A.BDEF\} \text{ and } \{C.ABEF\} = \{A.CDEF\};$$

that of the three G, H, and Y, in the second case, involves, and is involved in either of, the two

$$\{C.BAGH\} = \{B.CDGH\} \text{ and } \{A.BCGH\} = \{B.ADGH\};$$





and that of the three K, L, and Z, in the third case, involves, and is involved in either of, the two

$$\{A.CBKL\} = \{C.ADKL\} \text{ and } \{B.CAKL\} = \{C.BDKL\};$$

and therefore &c. (305).

Cor. 1°. The three points (or lines) X, Y, Z in the above, taken in pairs Y and Z, Z and X, X and Y, being the three pairs of centres (or axes) of perspective of the three pairs of inscribed chords (or exscribed angles) BC and AD, CA and BD, AB and CD determined by the four concyclic points (or tangents) A, B, C, D; it follows consequently, from the three pairs of equianharmonic relations (a), (b), (c), that—

a. Every two points on a circle, which connect through either centre of perspective of any two inscribed chords, divide equianharmonically the two arcs of the circle intercepted by the chords; and, conversely, every two points on a circle, which divide equianharmonically any two arcs of the circle, connect through one or other of the two centres of perspective of the two inscribed chords determined by the arcs.

a′. Every two tangents to a circle, which intersect on either axis of perspective of any two exscribed angles, divide equianharmonically the two arcs of the circle intercepted by the angles; and, conversely, every two tangents to a circle, which divide equianharmonically any two arcs of the circle, intersect on one or other of the two axes of perspective of the two exscribed angles determined by the arcs.

Cor. 2°. In the particular case where the two arcs of the circle, intercepted between two of the four points (or tangents),

B and C suppose, and between the remaining two A and D, are equal; since then, evidently, one of the two centres of perspective Z (fig. a) of the two inscribed chords they determine is at an infinite distance, while one of the two axes of perspective Z (fig. a') of the two exscribed angles they determine passes through the centre of the circle; and since, consequently, the two circular points at infinity (260) connect through the former, while the two tangents from the centre of the circle intersect on the latter; hence, from the first parts of the above, it appears that—

a. *Every two, and therefore all, equal arcs of the same circle are cut equianharmonically by the two circular points at infinity.*

a'. *Every two, and therefore all, equal arcs of the same circle are cut equianharmonically by the two central tangents to the circle.*

COR. 3°. In the particular case when, of the four points (or tangents) A, B, C, D, any two, B and C suppose, and also the remaining two, A and D, coincide; and when, consequently (19 and 20), the point (or line) X is the intersection of the terminal tangents (or the connector of the terminal points) of the arc of the circle AB intercepted between the two pairs of coincident points (or tangents) $B = C$ and $A = D$; since then, for every pair of concyclic points (or tangents) E and F which connect through (or intersect on) X, by either relation (a) of the above, $\{ABEF\} = \{BAEF\}$; and since, by (281), every pair of concyclic points or tangents E and F which fulfil the latter relation are harmonic conjugates to each other with respect to the two A and B; hence also, from the above, as already established on other principles in Art. 257, it appears that—

a. *Every two points on a circle, which connect through the intersection of any two tangents to the circle, divide harmonically the arc of the circle intercepted by the tangents; and, conversely, every two points on a circle, which divide any arc of the circle harmonically, connect through the intersection of the terminal tangents of the arc.*

a'. *Every two tangents to a circle, which intersect on the connector of any two points on the circle, divide harmonically the arc of the circle intercepted by the points; and, conversely,* *every two tangents to a circle, which divide any arc of the circle harmonically, intersect on the connector of the terminal points of the arc.*

COR. 4°. When, in the general case, the two points (or tangents) E and F connect through (or intersect on) not only X but also Y, or the two G and H connect through (or intersect on) not only Y but also Z, or the two K and L connect through (or intersect on) not only Z but also X; since then, by the first and second of the general relations (a) and (b), (b) and (c), (c) and (a), respectively, combined—

$$\{ABEF\} = \{BAEF\} \text{ and } \{CDEF\} = \{DCEF\},$$
$$\{BCGH\} = \{CBGH\} \text{ and } \{ADGH\} = \{DAGH\},$$
$$\{CAKL\} = \{ACKL\} \text{ and } \{BDKL\} = \{DBKL\};$$

and since, consequently, by (281), the several systems of four constituents are all harmonic; hence again, from the above, as already shewn on other principles in Art. 261, it appears that—

a. The two points on a circle, which are collinear with the two centres of perspective of any two inscribed chords, divide harmonically the two arcs of the circle intercepted by the chords; and, conversely, the two points on a circle, which divide harmonically the two arcs of the circle intercepted by any two inscribed chords, are collinear with the two centres of perspective of the chords.

a′. The two tangents to a circle, which are concurrent with the two axes of perspective of any two exscribed angles, divide harmonically the two arcs of the circle intercepted by the angles; and, conversely, the two tangents to a circle, which divide harmonically the two arcs of the circle intercepted by any two exscribed angles, are concurrent with the two axes of perspective of the angles.

COR. 5°. Since, in the same case (see figures), by virtue of the general property of Art. 285—

$$\{EFXY\} = \{D.EFAB\} = \{C.EFBA\} = \{B.EFCD\} = \{A.EFDC\},$$
$$\{EFYX\} = \{C.EFAB\} = \{D.EFBA\} = \{A.EFCD\} = \{B.EFDC\},$$

with similar groups of relations for the system G, H, Y, Z, and for the system K, L, Z, X, (which, it will be observed, prove directly the harmonic relations of the preceding corollary); it follows consequently, by (306), that—

$$\{EFXY\}=\{EFYX\}, \{GHYZ\}=\{GHZY\}, \{KLZX\}=\{KLXZ\};$$

and therefore, by (281), as established already on other considerations in Art. 261, that—

a. The two centres of perspective of any two chords inscribed to a circle divide harmonically the segment, real or imaginary, intercepted between the two collinear points on the circle.

a'. The two axes of perspective of any two angles exscribed to a circle divide harmonically the angle, real or imaginary, intercepted between the two concurrent tangents to the circle.

COR. 6°. In the particular case when the four points (or tangents) A, B, C, D are in pairs, A and B, C and D suppose, diametrically opposite to each other; since then, evidently, the two centres of perspective Z and X (fig. α) of the two inscribed chords AB and CD, and with them of course all collinear points, are at infinity, while the two axes of perspective Z and X (fig. α') of the two exscribed angles AB and CD, and with them of course all concurrent lines, pass through the centre of the circle; hence, from Cor. 4°, as established already on other considerations in Art. 260, it appears that—

a. Every two, and therefore all, semicircular arcs of the same circle are cut harmonically by the two circular points at infinity.

a'. Every two, and therefore all, semicircular arcs of the same circle are cut harmonically by the two central tangents to the circle.

COR. 7°. The two reciprocal properties of Cor. 1° supply obvious solutions of the three following pairs of reciprocal problems, viz.—

a. To draw a line, 1°. *passing through a given point and determining two points on a given circle dividing two given arcs of the circle equianharmonically;* 2°. *touching one given circle and determining two points on another given circle dividing two given arcs of the latter circle equianharmonically;* 3°. *determining two points on each of two given circles dividing two given arcs of each circle equianharmonically.*

a'. To find a point, 1°. *lying on a given line and determining two tangents to a given circle dividing two given arcs of the circle* *equianharmonically; 2°. lying on one given circle and determining two tangents to another given circle dividing two given arcs of the latter circle equianharmonically; 3°. determining two tangents to each of two given circles dividing two given arcs of each circle equianharmonically.*

Since, by the corollary in question (Cor. 1°), every two points on (or tangents to) a circle, which connect through either centre of perspective of the two inscribed chords (or intersect on either axis of perspective of the two exscribed angles) determined by any two arcs of the circle, divide those arcs equianharmonically; it follows, consequently, that, of the above pairs of reciprocal problems, the first of each group admits of two, and the second and third of each admit of four, different solutions, the two points (or tangents) corresponding to any or all of which may, according to circumstances, be as often imaginary as real.

Cor. 8°. Since (156) every circle, whose chord of intersection with either of two orthogonal circles passes through the centre of the other, is orthogonal to the latter; while, conversely, every circle orthogonal to one of two orthogonal circles determines a chord of the other passing through the centre of the former; it follows consequently, from the first and second parts of property (*a*) of the same corollary (Cor. 1°) respectively, that—

Every circle orthogonal to either of two orthogonal circles cuts equianharmonically every two arcs of the other intercepted between two diameters of the former; and, conversely, every circle cutting any two arcs of another circle equianharmonically is orthogonal to one or other of the two circles orthogonal to the latter, and to each other, 176, 1°, *whose centres are the two centres of perspective of the chords of the arcs.*

Cor. 9°. Since (187, 2°) a variable circle, whose chords of intersection, real or imaginary, with two fixed circles pass through two fixed points, describes the coaxal system orthogonal to the pair of circles concentric with the points and orthogonal to the circles; it follows evidently, from the second part of the general property of the preceding corollary, that—

1°. *A variable circle, passing through a fixed point and* *cutting two fixed arcs of a fixed circle equianharmonically, passes also through the inverse of the point with respect to one or other of the two circles orthogonal to the fixed circle and concentric with the two centres of perspective of the chords of its arcs.*

2°. *A variable circle, cutting two fixed arcs of each of two fixed circles equianharmonically, describes one or other of the four coaxal systems orthogonal to a circle of each pair orthogonal to one of the fixed circles and concentric with the two centres of perspective of the chords of its arcs.*

COR. 10°. Since (156, Cor. 4°) a circle may be described, 1°. passing through two given points and cutting a given circle orthogonally; 2°. passing through a given point and cutting two given circles orthogonally; 3°. cutting three given circles orthogonally; the first part of the same general property (that of Cor. 8°) supplies obviously the two, four, and eight solutions, respectively, of the three following problems, viz.—

To describe a circle, 1°. passing through two given points and cutting two given arcs of a given circle equianharmonically; 2°. passing through a given point and cutting two given arcs of each of two given circles equianharmonically; 3°. cutting two given arcs of each of three given circles equianharmonically.

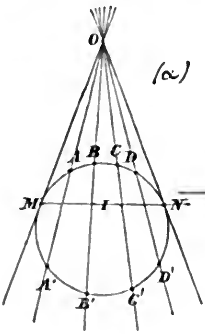
312. Since every two similar angles, however circumstanced as to position, absolute or relative, intercept equal arcs on every circle passing through their two vertices, and since every circle, whatever be its magnitude or position, passes through the two circular points at infinity (260); it follows consequently, from relations (*a*) of Cors. 2° and 6° of the preceding article, that—

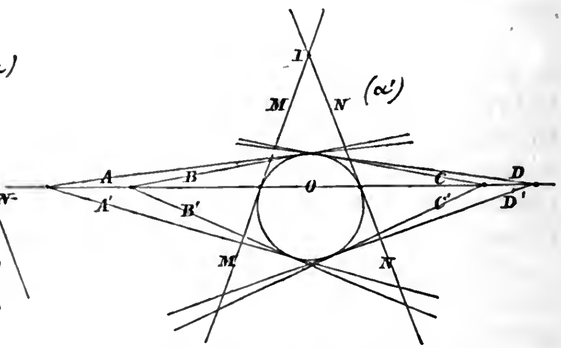
Every two, and therefore all, similar angles, however circumstanced as to position, absolute or relative, are cut equianharmonically, and if right angles harmonically, by the lines connecting their vertices with the two circular points at infinity; the value of the common anharmonic ratio of their section depending on their common form, and being = − 1 when that form is rectangular.

This remarkable result, which for the particular case of right angles has, it will be remembered, been already established on other principles in Art. 260, is of considerable importance in the higher departments of modern geometry; as bringing at once under the operation of all processes of geometrical transformation, such as reciprocation, projection, &c., under which anharmonic ratios remain unchanged, all properties of geometrical figures involving similar angles; and shewing, in general, what such properties become by transformation when the angles themselves, as they generally do, lose by change of form their character of similarity under the process of transformation.

313. From the two reciprocal properties of Art. 311 the two following, also reciprocal, properties respecting concyclic triads of points and tangents in perspective may be immediately inferred, viz.—

When two systems of three points on (or tangents to) the same circle A, B, C and A′, B′, C′, which correspond two and two in opposite pairs A and A′, B and B′, C and C′, are in perspective, every pair of systems determined by four of the six constituents and their four opposites are equianharmonic; and, conversely, when they are such that any pair of systems determined by four of the six constituents and their four opposites are equianharmonic, they are in perspective.





For, if O be the point of concurrence (fig. α), or the line of collinearity (fig. α'), of two of the three lines of connection (or points of intersection) AA', BB', CC' of the three pairs of opposite constituents A and A', B and B', C and C'; then since, by the two reciprocal properties of the article in question (311), the concurrence (or collinearity) with them of the third involves both, and is involved in either of, the two equianharmonic relations, 1°. If BB' and CC' be the two, and AA' the third,

$$\left.\begin{aligned} &\{BCAA'\} = \{C'B'AA'\} = \{B'C'A'A\}, \ (280) \\ \text{and} \quad &\{BC'AA'\} = \{CB'AA'\} = \{B'CA'A\}, \ (280) \end{aligned}\right\} \ldots.(a);$$

2°. If CC' and AA' be the two, and BB' the third,

$$\left.\begin{aligned} &\{CABB'\} = \{A'C'BB'\} = \{C'A'B'B\}, \ (280) \\ \text{and} \quad &\{CA'BB'\} = \{AC'BB'\} = \{C'AB'B\}, \ (280) \end{aligned}\right\} \ldots.(b);$$

3°. If AA' and BB' be the two, and CC' the third,

$$\left.\begin{aligned} &\{ABCC'\} = \{B'A'CC'\} = \{A'B'C'C\}, \ (280) \\ \text{and} \quad &\{AB'CC'\} = \{BA'CC'\} = \{A'BC'C\}, \ (280) \end{aligned}\right\} \ldots\ldots(c);$$

therefore &c. (283).

Otherwise thus: Since, when the two triads are in perspective, their three pairs of opposite constituents A and A', B and B', C and C' divide harmonically the arc MN, real or imaginary, intercepted between the two tangents to the circle from the centre of perspective (or the two intersections with the circle of the axis of perspective) O (257), therefore, by 3°, Cor. 5, Art. 282, every four of the six constituents and their four opposites form equianharmonic systems. And since, conversely, when any four of the six constituents and their four opposites form equianharmonic systems, the three pairs of opposite constituents divide harmonically a common arc MN, real or imaginary (Cor. 1°, Art. 283), therefore, (by 257), the two triads are in perspective. This latter demonstration, though perhaps on the whole simpler, yet, as involving the contingent elements M and N, is consequently less general than the former in which all the elements involved are permanent (21).

In the particular cases when, in the first parts of the above reciprocal properties, the centre of perspective O, in the former case, is the centre of the circle, and the axis of perspective O, in the latter case, is the line at infinity; the three pairs of corresponding constituents A and A', B and B', C and C' being then diametrically opposite pairs with respect to the circle, the two properties are evident, *à priori*, from the obvious similarity and consequent equianharmonicism of every two systems determined by four of them and their four opposites. See Cor. 2°, Art. 283, from which also, Euc. III. 31, the properties evidently follow in the same cases.

COR. 1°. It follows, indirectly, from both parts of the above properties combined, that *when a system of six points on (or tangents to) the same circle, which correspond two and two in opposite pairs A and A′, B and B′, C and C′, is such that any two systems determined by four of the six constituents and their four opposites are equianharmonic, then every two systems determined by four of them and their four opposites are equianharmonic.* For, by the second parts of the above properties, the equianharmonicism of any one of the six pairs of conjugate groups of four into which the system may be divided involves the perspective of the two conjugate triads A, B, C and A', B', C' of which it consists, and consequently, by the first parts of the same properties, the equianharmonicism of the remaining five. This property, it will be remembered, was proved directly for collinear points and concurrent lines, and with them, implicitly, for concyclic points and tangents, in Art. 283, and the above indirect verification of it for the latter may of course be regarded as extending to the former also.

COR. 2°. The first parts of the above reciprocal properties supply obvious solutions of the two reciprocal problems, *of two triads of concyclic points (or tangents) A, B, C and A′, B′, C′, which correspond two and two in opposite pairs, and every four of which are equianharmonic with their four opposites; given any two pairs of corresponding constituents A and A′, B and B′, and either constituent C of the third pair C and C′, to determine the second constituent C′ of that pair.* For, the two lines of connection (or points of intersection) AA' and BB' of the two given pairs A and A', B and B' determine, by their point of intersection (or line of connection), the centre (or axis) of perspective O of the two triads, and with it, consequently, the conjugate C' to the given constituent C of the third pair C and C'.

314. From the two reciprocal properties of the preceding article, respecting concyclic triads of points or tangents in perspective, it may be readily inferred, for concyclic quartets of points or tangents in perspective, that, more generally—

When two systems of four points on (or tangents to) the same circle A, B, C, D and A′, B′, C′, D′, which correspond two and *two in opposite pairs A and A′, B and B′, C and C′, D and D′, are in perspective, every pair of systems determined by four of the eight constituents and their four opposites are equianharmonic; and, conversely, when two equianharmonic systems of four points on (or tangents to) the same circle A, B, C, D and A′, B′, C′, D′ are such that a pair of their corresponding constituents may be interchanged without violating their relation of equianharmonicism, they are in perspective.*

For, from the general property (1°) Art. 282, see figures of preceding article.

1°. Any two of the three equianharmonic relations

$$\{CDAA'\} = \{C'D'A'A\},\ \{DBAA'\} = \{D'B'A'A\},$$
$$\{BCAA'\} = \{B'C'A'A\},$$

involve and are involved in the two

$$\{BCDA\} = \{B'C'D'A'\} \text{ and } \{BCDA'\} = \{B'C'D'A\} \ldots (a);$$

2°. Any two of the three equianharmonic relations

$$\{DABB'\} = \{D'A'B'B\},\ \{ACBB'\} = \{A'C'B'B\},$$
$$\{CDBB'\} = \{C'D'B'B\},$$

involve and are involved in the two

$$\{CDAB\} = \{C'D'A'B'\} \text{ and } \{CDAB'\} = \{C'D'A'B\} \ldots (b);$$

3°. Any two of the three equianharmonic relations

$$\{ABCC'\} = \{A'B'C'C\},\ \{BDCC'\} = \{B'D'C'C\},$$
$$\{DACC'\} = \{D'A'C'C\},$$

involve and are involved in the two

$$\{DABC\} = \{D'A'B'C'\} \text{ and } \{DABC'\} = \{D'A'B'C\} \ldots (c);$$

4°. Any two of the three equianharmonic relations

$$\{BCDD'\} = \{B'C'D'D\},\ \{CADD'\} = \{C'A'D'D\},$$
$$\{ABDD'\} = \{A'B'D'D\},$$

involve and are involved in the two

$$\{ABCD\} = \{A'B'C'D'\} \text{ and } \{ABCD'\} = \{A'B'C'D\} \ldots (d);$$

and, as three similar quartets of equianharmonic relations result evidently from the interchange; firstly, of B and B' in 1°, of C and C' in 2°, of D and D' in 3°, and of A and A' in 4°; secondly, of C and C' in 1°, of D and D' in 2°, of A and A' in

3°, and of B and B' in 4°; and thirdly, of D and D' in 1°, of A and A' in 2°, of B and B' in 3°, and of C and C' in 4°; therefore &c.

Otherwise thus, as for the properties of the preceding article, which are included in the above as particular cases. Since (same figures) when the two quartets are in perspective, their four pairs of opposite constituents A and A', B and B', C and C', D and D' divide harmonically the arc MN, real or imaginary, intercepted between the two tangents to the circle from the centre of perspective (or the two intersections with the circle of the axis of perspective) O (257); therefore, by 3°, Cor. 4°, Art. 282, every four of the eight constituents and their four opposites form equianharmonic systems; and since, conversely, when the two quartets are equianharmonic, and preserve their equianharmonicism on the interchanges of a pair of their corresponding constituents, the four pairs of corresponding constituents divide harmonically a common arc MN, real or imaginary, (Cor. 1°, Art. 284); therefore, by (257), the two quartets are in perspective. This latter demonstration has the same advantages and disadvantages, compared with the former, as for the properties of the preceding article.

In the particular cases when, in the first parts of the above reciprocal properties, the centre of perspective O, in the former case, is the centre of the circle, and the axis of perspective O, in the latter case, is the line at infinity; the four pairs of corresponding constituents A and A', B and B', C and C', D and D' being then diametrically opposite pairs with respect to the circle, the two properties are evident, *à priori*, from the obvious similarity, and consequent equianharmonicism, of every two systems determined by four of them and their four opposites. See Cor. 2°, Art. 284, from which also, Euc. III. 31, the properties evidently follow in the same cases.

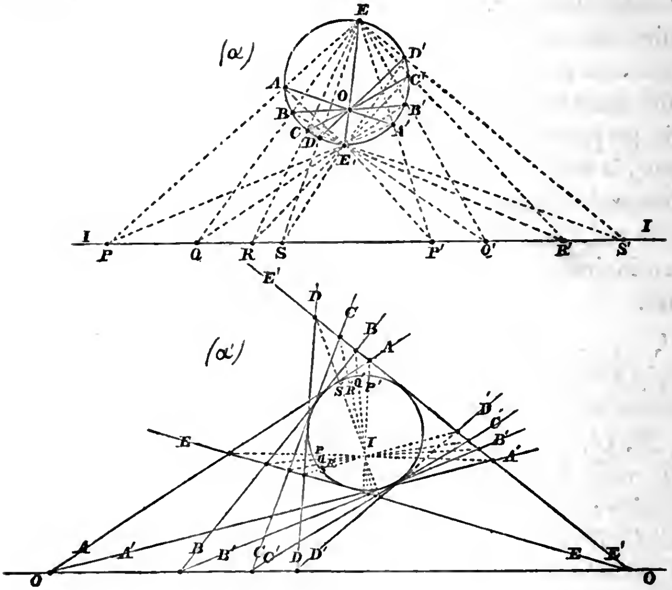
Cor. 1°. It follows, indirectly, from both parts of the above properties combined, that *when two equianharmonic systems of four points on (or tangents to) the same circle A, B, C, D and A', B', C', D' are such that a pair of their corresponding constituents may be interchanged without violating their relation of equianharmonicism, then every pair of their corresponding constituents may be interchanged without violating their relation of* *equianharmonicism.* For, by the second parts of the above properties, the possibility of a single such interchange involves the perspective of the systems, and consequently, by the first parts of the same properties, the possibility of every such interchange. This property also, like that of the corollary of the preceding article, it will be remembered, was proved directly for collinear points and concurrent lines, and therefore implicitly for concyclic points and tangents, in Art. 284, and the above may be regarded as an indirect verification of it for the former as well as for the latter.

COR. 2°. The first parts of the above reciprocal properties supply obvious solutions of the two reciprocal problems; *of two quartets of concyclic points (or tangents) A, B, C, D and A', B', C', D', which correspond two and two in opposite pairs, and every four of which are equianharmonic with their four opposites; given any two pairs of corresponding constituents A and A', B and B', and any two non-corresponding constituents of the remaining two pairs, to determine the remaining two constituents of those pairs.* For, the two lines of connection (or points of intersection) AA' and BB' of the two given pairs of corresponding constituents A and A', B and B' determine, by their point of intersection (or line of connection), the centre (or axis) of perspective O of the two quartets, and with it, consequently, the two correspondents to the two given non-corresponding constituents of the remaining two pairs.

315. The two reciprocal properties of the preceding article are sometimes ennunciated as follows:

When three pairs of points on (or tangents to) the same circle A and A', B and B', C and C' determine two triads in perspective A, B, C and A', B', C'; every fourth pair D and D' determines with them two quartets A, B, C, D and A', B', C', D' which if in perspective are equianharmonic, and which if equianharmonic are in perspective.

And the following reciprocal demonstrations, based on the two reciprocal properties a and a' of Art. 261, are sometimes given of them. If O be the centre (fig. a), or the axis (fig. a'), of perspective of the two triads A, B, C and A', B', C'; I its polar (fig. a), or its pole (fig. a'), with respect to the circle;



E and E' any fifth pair of points (or tangents) connecting through (or intersecting on) O; P, Q, R, S the four points of intersection (or lines of connection) of the four pairs of lines of connection (or points of intersection) EA and $E'A'$, EB and $E'B'$, EC and $E'C'$, ED and $E'D'$; and P', Q', R', S' the four for the four pairs EA' and $E'A$, EB' and $E'B$, EC' and $E'C$, ED' and $E'D$; then since, by the properties in question, (a and a', Art. 261), the two triads of points (or lines) P, Q, R and P', Q', R' lie on the line (or pass through the point) I; and since on the collinearity (or concurrence) of the two S and S' with them depends, at once, the circumstance of the two points (or tangents) D and D' connecting through (or intersecting on) O (a and a', Art. 261), and the circumstance of the two pairs of pencils (or rows) $E.ABCD$ and $E'.A'B'C'D'$, $E.A'B'C'D'$ and $E'.ABCD$, that is, of the two quartets of points (or tangents) A, B, C, D and A', B', C', D', being equianharmonic (a and a', Art. 288); therefore &c.

Cor. 1°. Since, when two systems of any common number of points on (or tangents to) the same circle A, B, C, D, &c. and A', B', C', D', &c. which correspond in pairs A and A',

B and B', C and C', D and D', &c. are in perspective, any pair, or any number of pairs, of their corresponding constituents may evidently be interchanged without violating their relation of perspective; therefore, in the equianharmonic relation $\{ABCD\} = \{A'B'C'D'\}$, which connects, as above shewn, every two quartets A, B, C, D and A', B', C', D' in perspective, the accented and unaccented constituents may be interchanged at pleasure without violating the relation of equianharmonicism; thus, for the eight different combinations of four and their four correspondents that could be formed from the four pairs of corresponding constituents A and A', B and B', C and C', D and D' (314), giving rise (see figures), as observed in Arts. 284 and 314, to the eight following different cases of anharmonic equivalence, viz.—

$$\{ABCD\} = \{A'B'C'D'\} = \{PQRS\} = \{P'Q'R'S'\},$$
$$\{A'BCD\} = \{AB'C'D'\} = \{P'QRS\} = \{PQ'R'S'\},$$
$$\{AB'CD\} = \{A'BC'D'\} = \{PQ'RS\} = \{P'QR'S'\},$$
$$\{ABC'D\} = \{A'B'CD'\} = \{PQR'S\} = \{P'Q'RS'\},$$
$$\{ABCD'\} = \{A'B'C'D\} = \{PQRS'\} = \{P'Q'R'S\},$$
$$\{A'B'CD\} = \{ABC'D'\} = \{P'Q'RS\} = \{PQR'S'\},$$
$$\{A'BC'D\} = \{AB'CD'\} = \{P'QR'S\} = \{PQ'RS'\},$$
$$\{A'BCD'\} = \{AB'C'D\} = \{P'QRS'\} = \{PQ'R'S\};$$

for none of which, however, is it to be supposed, as is sometimes erroneously done by beginners, that the anharmonicism is that of the pencil (or row) determined, at the centre (or on the axis) of perspective O, by the four lines of connection (or points of intersection) AA', BB', CC', DD', whose concurrence (or collinearity) constitutes the common perspective of them all.

COR. 2°. If, while, of two concyclic quartets of points or tangents in perspective A, B, C, D and A', B', C', D', three pairs of corresponding constituents A and A', B and B', C and C' are supposed to remain fixed, the fourth pair D and D' be conceived to vary; and, in the course of their variation, to coincide successively; firstly, D with A' and D' with A; secondly, D with B' and D' with B; thirdly, D with C' and D' with C; since then, for every position of D and D', by the above, $\{ABCD\} = \{A'B'C'D'\}$, therefore for the particular positions in question (see figures)—

$$\{BCAA'\} = \{B'C'A'A\} = \{QRPP'\} = \{Q'R'P'P\} \ldots (1°),$$
$$\{CABB'\} = \{C'A'B'B\} = \{RPQQ'\} = \{R'P'Q'Q\} \ldots (2°),$$
$$\{ABCC'\} = \{A'B'C'C\} = \{PQRR'\} = \{P'Q'R'R\} \ldots (3°);$$

and from them, by the interchange, as explained in Cor. 1°, of B and B' in (1°), of C and C' in (2°), and of A and A' in (3°),

$$\{B'CAA'\} = \{BC'A'A\} = \{Q'RPP'\} = \{QR'P'P\} \ldots\ldots (1'),$$
$$\{C'ABB'\} = \{CA'B'B\} = \{R'PQQ'\} = \{RP'Q'Q\} \ldots\ldots (2'),$$
$$\{A'BCC'\} = \{AB'C'C\} = \{P'QRR'\} = \{PQ'R'R\} \ldots\ldots (3');$$

and, conversely, if, for two concyclic triads of points or tangents A, B, C and A', B', C', any of the six preceding relations (1°) or (1'), (2°) or (2'), (3°) or (3') exist, since then the necessary perspective of the repeated with each unrepeated pair of corresponding constituents involves, by the above, the perspective of the triads; therefore, see Cor. 3°, Art. 284, the two reciprocal properties respecting concyclic triads in perspective, established on other considerations in Art. 313, are particular cases of those respecting concyclic quartets in perspective, established by their aid in the subsequent article.

316. From the general property of the preceding article, that every two quartets of points or tangents of the same circle in perspective are equianharmonic, combined with the circumstance of the evident equianharmonicism of every two similar quartets of points or tangents, either of the same or of different circles (309); it follows immediately that—

For any two circles, every two systems determined by four points or tangents of either, and by the four antihomologous points or tangents of the other with respect to either centre of perspective of the two (198), *are equianharmonic.*

For, the system for either circle being similar to the homologous system for the other with respect to either centre of perspective of the two, and the latter being in perspective with the corresponding antihomologous system for the same circle (198), therefore &c.

The above property may obviously be stated otherwise (206), as follows—

For any two circles, every two pairs of antihomologous arcs, with respect to either centre of perspective, divide each other equianharmonically.

COR. 1°. As every circle, intersecting two others at any equal or supplemental angles, intercepts on them a pair of antihomologous arcs with respect to their external or internal centre of perspective, according as the angles of intersection are equal or supplemental (211); hence, from the above, it appears that—

a. Every circle intersecting two others at any equal, or supplemental, angles divides equianharmonically all pairs of their antihomologous arcs with respect to their external, or internal, centre of perspective.

b. When two circles each intersect two others at any equal or supplemental angles, the pairs of arcs they intercept on them divide each other equianharmonically.

c. When two circles intersect two others, one at any angles and the other at the same or the supplemental angles, their pairs of arcs intercepted by them divide each other equianharmonically.

d. When two circles intersect two others, both at the same equal or supplemental angles, the pairs of arcs they intercept on them, and their pairs of arcs intercepted by them, both divide each other equianharmonically.

COR. 2°. Since every circle orthogonal to two others intersects the two at once at equal and at supplemental angles, and since, of the entire system of circles orthogonal to the same two, one, viz. their common diameter, is a line; it follows, consequently, from *a* and *d* of the preceding (Cor. 1°), that—

a. Every circle orthogonal to two others divides equianharmonically all pairs of their antihomologous arcs with respect to either of their centres of perspective.

b. Every two circles determine on every circle orthogonal to them both a system equianharmonic with that they determine on their common diameter.

c. When two circles are orthogonal to two others, both pairs *determine equianharmonic systems, each on the circles of the other and on their own common diameter.*

COR. 3°. If, in property *c* of Cor. 1°, one of the two intersecting circles be conceived to vary while the other and the two intersected circles remain fixed; since then, by virtue of the general property (193, Cor. 8°), the variable circle intersecting two fixed circles at two constant angles intersects at a third constant angle every third fixed circle coaxal with them, it follows consequently, from that property, that—

When a variable circle intersects any two fixed circles at any two constant angles; a. its arc intercepted by either is cut in a constant anharmonic ratio by the other; b. its arcs intercepted by both are cut in constant anharmonic ratios by all fixed circles coaxal with both.

317. The two following reciprocal properties, respecting any two concyclic triads of points and tangents, are in the modern geometry of the circle what those of Art. (293) are in that of the point and line, and lead to as many and important consequences in the applications of the theory of anharmonic section.

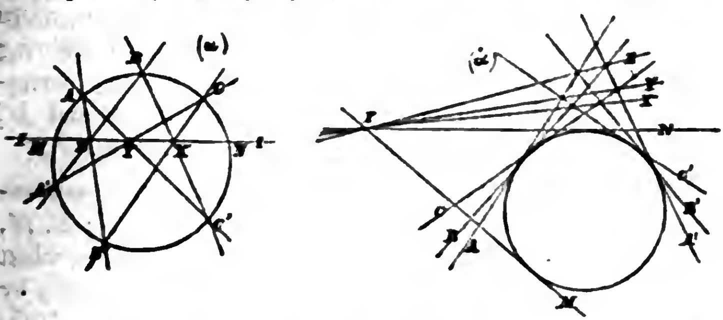
If A, B, C be any three points on (or tangents to) a circle, A', B', C' any other three points on (or tangents to) the same circle, and X, Y, Z the three intersections (or connectors) of the three pairs of connectors (or intersections) BC' and $B'C$, CA' and $C'A$, AB' and $A'B$; the three points (or lines) X, Y, Z are collinear (or concurrent); and their line of collinearity (or point of concurrence) determines with the circle two points (or tangents) M and N connected with the two original triads A, B, C and A', B', C' by the three groups of equianharmonic relations

$$\left.\begin{array}{l}\{BCMN\} = \{B'C'MN\}\\ \{CAMN\} = \{C'A'MN\}\\ \{ABMN\} = \{A'B'MN\}\end{array}\right\} \text{ .... (1)},$$

$$\{AA'MN\} = \{BB'MN\} = \{CC'MN\} \text{ .... (2)},$$

$$\left.\begin{array}{l}\{ABCM\} = \{A'B'C'M\}\\ \{ABCN\} = \{A'B'C'N\}\end{array}\right\} \text{ .... (3)}.$$

For, if M and N be the two points, fig. (α), (or tangents, fig. (α')), determined with the circle by the line of connection (or point of intersection) I of any two, X and Y suppose, of the three points (or lines) X, Y, Z; then since the two triads of



concyclic points (or tangents) B, C, M and C', B', N are in perspective, therefore, by the first parts of the two reciprocal properties of Art. 311,

$$\{BCMN\} = \{B'C'MN\} \text{ and } \{BB'MN\} = \{CC'MN\} \ldots (a);$$

and since the two triads C, A, M and A', C', N are in perspective, therefore, by the same,

$$\{CAMN\} = \{C'A'MN\} \text{ and } \{CC'MN\} = \{AA'MN\} \ldots (b);$$

therefore, by the general property (1°) Art. 282, or directly as regards the second equivalents,

$$\{ABMN\} = \{A'B'MN\} \text{ and } \{AA'MN\} = \{BB'MN\} \ldots (c);$$

and therefore, by the second parts of the two reciprocal properties of Art. 311, the two triads A, B, M and B', A', N are in perspective; which proves the first parts of the above reciprocal properties, and with them the two groups of equianharmonic relations (1) and (2), from either of which the group (3) follows immediately by virtue of the general property 2°, Cor. 3°, Art. 282.

Cor. 1°. If X', Y', Z' be the three intersections (or connectors) of the three pairs of connectors (or intersections) BC and $B'C'$, CA and $C'A'$, AB and $A'B'$; it may, of course, be shewn, in precisely the same manner, that the three triads of points (or lines) Y', Z', X; Z', X', Y; X', Y', Z are also collinear (or concurrent); their three lines of collinearity (or points of concurrence) determining, with that of the triad X, Y, Z, a tetragram (or tetrastigm), of which the three pairs of corresponding points (or lines) X and X', Y and Y', Z and Z'

are the three pairs of opposite intersections (or connectors); and each line (or point) determining two points on (or tangents to) the circle connected with the original six by three groups of equianharmonic relations similar to the above, and differing only in the interchange of the constituents of the two corresponding reversed pairs of the three A and A', B and B', C and C' in the several equivalents which contain them.

Cor. 2°. In the particular case when the three equianharmonic systems of points (or tangents) A, A', M, N; B, B', M, N; C, C', M, N of group (2) are harmonic, that is, when the three intercepted arcs AA', BB', CC' are cut harmonically by the intercepted arc MN; since then (281) $\{AA'MN\} = \{A'AMN\}$, $\{BB'MN\} = \{B'BMN\}$, $\{CC'MN\} = \{C'CMN\}$, therefore the two points (or tangents) M and N are the same for the three lines of collinearity (or points of concurrence) of the three triads Y', Z', X; Z', X', Y; X', Y', Z, as for that of the triad X, Y, Z; and therefore the whole six points (or lines) X and X', Y and Y', Z and Z' lie on the same line (or pass through the same point) I. In this case it is evident, from (257), that the three lines of connection (or points of intersection) AA', BB', CC' of the three pairs of corresponding points (or tangents) A and A', B and B', C and C' are concurrent (or collinear), and that the line (or point) I is the polar of their point of concurrence (or the pole of their line of collinearity) O with respect to the circle; a property, the converse of which, for two concyclic triads of points (or tangents) in perspective, is evident from Art. 261.

Cor. 3°. In the same case it is easily seen that, as the three pairs of concyclic points (or tangents) A and A', B and B', C and C' divide harmonically the arc of the circle MN intercepted between the two points (or tangents) M and N, so the three pairs of collinear points (or concurrent lines) X and X', Y and Y', Z and Z' divide harmonically the segment (or angle) MN intercepted between them. For, since, by the general property of Art. 285,

$$\{MNXX'\} = \{C'.MNBB'\} \text{ or } \{B'.MNCC'\},$$
$$\{MNYY'\} = \{A'.MNCC'\} \text{ or } \{C'.MNAA'\},$$
$$\{MNZZ'\} = \{B'.MNAA'\} \text{ or } \{A'.MNBB'\};$$

and since, by hypothesis, the three concyclic systems of points (or tangents) M, N, A, A'; M, N, B, B'; M, N, C, C' are harmonic; therefore the three collinear (or concurrent) systems of points (or lines) M, N, X, X'; M, N, Y, Y'; M, N, Z, Z' are harmonic, and therefore &c. The converse of this property, for two concyclic triads of points (or tangents) in perspective, is evident from Cor. 5°, Art. 282.

Cor. 4°. That, in the same case, the three pairs of collinear points (or concurrent lines) X and X', Y and Y', Z and Z' constitute a system of six constituents, corresponding two and two in opposite pairs, every four of which are equianharmonic with their four opposites, follows also immediately from the preceding Cor. 3°. For, the three intercepted segments (or angles) XX', YY', ZZ' having a common segment (or angle) of harmonic section, real or imaginary, MN, therefore &c. The converse of this property also, for two concyclic triads of points (or tangents) in perspective, is, like the preceding, evident from Cor. 5°, Art. 282.

318. From the two reciprocal properties of the preceding article, the following inferences, in pairs reciprocals of each other, may be shown in precisely the same manner as the corresponding inferences of Art. (294) from those of its preceding article (293).

1°. The three pairs of concyclic points (or tangents) A and A', B and B', C and C' may be regarded as determining three chords (or angles) AA', BB', CC' inscribed (or exscribed) to the circle to which they belong, of which, taken in pairs, the three points (or lines) X, Y, Z are three of the six centres (or axes) of perspective; every two of which evidently become changed into their two opposites by the interchange of extremities of one of the two determining chords (or angles), those of the other remaining unchanged; hence, generally, from the first parts, and from the equianharmonic relations (1) of the second parts, of the two reciprocal properties in question.

a. For every three chords inscribed to the same circle, taken in pairs, the six centres of perspective lie three and three on four lines; each of which determines two points on the circle which divide equianharmonically the three arcs intercepted by the chords.

a'. For every three angles exscribed to the same circle, taken in pairs, the six axes of perspective pass three and three through four points; each of which determines two tangents to the circle which divide equianharmonically the three arcs intercepted by the angles.

In the particular case when the directions of the three chords (or the vertices of the three angles) AA', BB', CC' are concurrent (or collinear); the six centres (or axes) of perspective of the three pairs they determine, being then, by Art. 261, all collinear with the polar of their point of concurrence (or concurrent with the pole of their line of collinearity) with respect to the circle, the four lines (or points) of the general case then coincide; and the two points or tangents they determine with the circle, by Art. 257, divide harmonically the three arcs intercepted by the chords (or angles).

2°. The two concyclic triads of points (or tangents) A, B, C and A', B', C' may be regarded as the two triads of alternate vertices (or sides) of a hexagon $AB'CA'BC'$ inscribed (or exscribed) to the circle to which they belong, of which A and A', B and B', C and C' are the three pairs of opposite vertices (or sides), and X, Y, Z the three intersections (or connectors) of the three pairs of opposite sides (or vertices) BC' and $B'C$, CA' and $C'A$, AB' and $A'B$; hence, generally, from the first parts, and from the equianharmonic relations (2) and (3) of the second parts of the same, respectively—

a. In every hexagon inscribed to a circle, the three intersections of opposite sides are collinear; and their line of collinearity determines two points on the circle which form equianharmonic systems, separately with the two triads of alternate, and conjointly with the three pairs of opposite, vertices of the hexagon.

a'. In every hexagon exscribed to a circle, the three connectors of opposite vertices are concurrent; and their point of concurrence determines two tangents to the circle which form equianharmonic systems, separately with the two triads of alternate, and conjointly with the three pairs of opposite, sides of the hexagon.

By virtue of the fundamental property of triangles in perspective (140), the first parts of these latter properties are evidently identical with the celebrated theorems of Pascal and Brianchon, established already, on other considerations, in Art. 148, and generalized subsequently, on principles independent of the circumstance as to whether the two points (or tangents) M and N are imaginary or real, in Art. 302.

3°. In the two concyclic triads of points (or tangents) A, B, C and A', B', C', if, while the three constituents A, B, C of either and any two A' and B' of the other are supposed to remain fixed, the third constituent C' of the latter be conceived to vary, causing of course the simultaneous variation of the two constituents X and Y of the collinear (or concurrent) triad X, Y, Z; since then, of the variable triangle XYC', the three sides (or vertices) turn round the three fixed points (or move on the three fixed lines) A, B, Z, and the two vertices (or sides) X and Y move on the two fixed lines (or turn round the two fixed points) CB' and CA', while the third vertex (or side) C' describes (or envelopes) the circle to which the concyclic points (or tangents) belong; hence, conversely—

a. When, of a variable triangle whose sides turn round fixed points, two of the vertices move on fixed lines whose intersections with each other, and with the corresponding sides of the fixed triangle determined by the points, form with the opposite vertices of that triangle a concyclic system of points; the third vertex describes the circle determined by the five points.

a'. When, of a variable triangle whose vertices move on fixed lines, two of the sides turn round fixed points whose connectors with each other, and with the corresponding vertices of the fixed triangle determined by the lines, form with the opposite sides of that triangle a concyclic system of tangents; the third side envelopes the circle determined by the five tangents.

The locus and envelope of these latter properties, as well as those of 5°, Art. 294, are evidently particular cases of the more general "locus of the third vertex of a variable triangle whose remaining vertices move on fixed lines while its three sides turn round fixed points" and "envelope of the third side of a variable triangle whose remaining sides turn round fixed points while its three vertices move on fixed lines;" which, in general, by reciprocation of the above to an arbitrary circle, are easily seen to be the more general figures into which the circle becomes transformed by reciprocation (173).

4°. In the two concyclic triads of points (or tangents) A, B, C and A', B', C', if, while two pairs of corresponding constituents A and A', B and B' are supposed to remain fixed, the third pair be conceived to vary, causing of course the simultaneous variation of the two non-corresponding constituents X and Y of the collinear (or concurrent) triad X, Y, Z; since then, in every position of the variable tetragram (or tetrastigm) determined by the four lines (or points) AC' and $A'C$, BC' and $B'C$ turning round the four fixed points (or moving on the four fixed tangents) A and A', B and B', the pair of opposite intersections (or connectors) C and C' lie on (or touch) the circle to which the concyclic triads belong, while the remaining two pairs connect through (or intersect on) the two centres (or axes) of perspective of the two inscribed chords (or exscribed angles) AB and $A'B'$; hence, generally—

a. When, of a variable tetragram whose four lines turn round four fixed concyclic points, a pair of opposite intersections describe the circle determined by the points, the two remaining pairs connect through the intersections of the two corresponding pairs of opposite connectors of the points.

b. When, of a variable tetrastigm whose four points move on four fixed concyclic tangents, a pair of opposite connectors envelope the circle determined by the tangents, the two remaining pairs intersect on the connectors of the two corresponding pairs of opposite intersections of the tangents.

319. The two groups of equianharmonic relations (a) and (a') of the same article (317) supply obvious and rapid solutions of the two following pairs of reciprocal problems, than which, as will appear in the sequel, none, perhaps, are of more importance in the applications of the theory of anharmonic section, viz.—

Given two concyclic triads of points (or tangents) A, B, C and A', B', C' whose constituents correspond in pairs A and A', B and B', C and C'; to determine the two concyclic points (or tangents) M and N which form equianharmonic systems; 1°, separately with the two triads; 2°, conjointly with their three pairs of corresponding constituents.

For, constructing the hexagon $AB'CA'BC'$ (see figures of Art. 317) of which the two given triads of points (or tangents) A, B, C and A', B', C' are the two triads of alternate vertices (or sides), and their three pairs of corresponding constituents

A and A', B and B', C and C' the three pairs of opposite vertices (or sides); that is, the hexagon determined by the two lines of connection (or points of intersection) of each constituent A, B, C of either triad with the two non-corresponding constituents B' and C', C' and A', A' and B' of the other triad; then, by the relations in question, the line of collinearity (or point of concurrence) of the three intersections of its opposite sides (or the three connectors of its opposite vertices) X, Y, Z determines with the circle the two points (or tangents) M and N, real or imaginary, which (see 2°, of the preceding article) solve at once the two problems.

In the particular case when the three lines of connection (or points of intersection) AA', BB', CC' of the three pairs of corresponding constituents A and A', B and B', C and C' are concurrent (or collinear), that is, when the two given triads of points (or tangents) are in perspective; then, as already noticed in Cor. 2°, of Art. 317, the polar of their point of concurrence (or the pole of their line of collinearity) with respect to the circle is the line (or point) which determines with the latter the two points (or tangents) M and N, real or imaginary, which solve at once the two problems.

320. The two following reciprocal properties, respecting the two triangles determined by any two concyclic triads of points or tangents in perspective, follow also from the same, or from the reciprocal theorems of Pascal and Brianchon, Arts. (148) and (302), with which, as shewn in the preceding 318, 2°, their first parts are virtually identical, viz.—

a. When two triangles inscribed to the same circle are in perspective, the three lines of connection of the vertices of either with any point on the circle intersect with the corresponding sides of the other at three points collinear with each other and with the centre of perspective.

a'. When two triangles exscribed to the same circle are in perspective, the three points of intersection of the sides of either with any tangent to the circle connect with the corresponding vertices of the other by three lines concurrent with each other and with the axis of perspective.

For, if A, B, C and A', B', C' be the two triads of vertices (or sides) of the two triangles, O their centre (or axis) of perspective, D any arbitrary point on (or tangent to) the circle, and X, Y, Z the three points of intersection (or lines of connection) of the three lines (or points) DA', DB', DC' with the three BC, CA, AB respectively; then since, in the three Pascal (or Brianchon) hexagons whose vertices (or sides) in consecutive order are respectively $DB'BACC'$, $DC'CBAA'$, $DA'ACBB'$, the three triads of points (or lines) YOZ, ZOX, XOY are those determining their three Pascal lines (or Brianchon points) respectively, therefore &c.—

N.B. In the particular case when, in the first of the above pair of reciprocal properties (a), the centre of perspective O of the two triangles ABC and $A'B'C'$ is the centre of the circle; the three lines DA', DB', DC' being then perpendiculars to the three DA, DB, DC (Euc. III. 31), the property consequently becomes that established on other principles in Ex. 6°, Art. 137.

If D' be the point (or tangent) corresponding to D in the same perspective with the two inscribed (or exscribed) triangles ABC and $A'B'C'$; and X', Y', Z' the three points of intersection (or lines of connection) of the three lines (or points) $D'A$, $D'B$, $D'C$ with the three $B'C'$, $C'A'$, $A'B'$ respectively; it is easy to shew, in the same manner precisely as above, that *the three points (or lines) X', Y', Z', which by the above are collinear (or concurrent) with each other and with the point (or line) O, are also collinear (or concurrent) with the three X, Y, Z, with which they consequently* (313) *determine, in three opposite pairs X and X', Y and Y', Z and Z', a system of six constituents, every four of which are equianharmonic with their four opposites.* For, in the three Pascal (or Brianchon) hexagons whose vertices (or sides) in consecutive order are respectively $B'DD'CAA'$, $C'DD'ABB'$, $A'DD'BCC'$, the three triads of points (or lines) YOZ', ZOX', XOY' being those determining their three Pascal lines (or Brianchon points) respectively, therefore &c.

In the same case, it is easy to shew also that *the two triads of collinear points (or concurrent lines) X, Y, Z and X', Y', Z' determine equianharmonic systems with the centre (or axis) of perspective O.* For, since, by (285), $\{XYZO\} = \{D.A'B'C'D'\}$ and $\{X'Y'Z'O\} = \{D'.ABCD\}$, and since, by (314), $\{A'B'C'D'\} = \{ABCD\}$, therefore $\{XYZO\} = \{X'Y'Z'O\}$; and therefore &c.

This property is evidently a particular case of that established on other principles for any two triangles in perspective in Art. 295, Cor. 6°.

By reciprocation to an arbitrary circle, the above, as well as all the other pairs of reciprocal properties established in this chapter, with all the consequences to which they lead in the geometry of the circle, are seen at once to be true, not only of circles, but generally of all figures into which circles become transformed by reciprocation; all such, as noticed in the opening article (305), possessing alike the two fundamental anharmonic properties a and a' of that article, from which, as has been seen, all the others established in the chapter have been successively inferred.

CHAPTER XIX.

THEORY OF HOMOGRAPHIC DIVISION.

321. Two rows of points or pencils of rays, or a row of points and a pencil of rays, A, B, C, D, E, F, &c. and A', B', C', D', E', F', &c. whose constituents correspond in pairs A and A', B and B', C and C', D and D', E and E', F and F', &c. are said to be *homographic* (282) when every four constituents of one and the four corresponding constituents of the other are equianharmonic (278). Every two similar rows of points or pencils of rays (268); every row of points and pencil of rays determined by it, or pencil of rays and row of points determined by it (285); every row of points and pencil of rays reciprocal to each other with respect to any circle (292); are evidently thus related to each other.

In accordance with the above definition of homography between two rows of points or pencils of rays, or a row of points and a pencil of rays, whose constituents correspond in pairs; two variable points or lines, or a variable point and line, dividing two fixed segments or angles, or a fixed segment and angle, so that every four positions of one and the four corresponding positions of the other are equianharmonic, are said *to divide homographically* the two segments or angles, or the segment and angle; the two systems of constituents determined by their several pairs of corresponding positions being, as above defined, homographic. Hence the meaning and origin of the name *homographic division* as applied, by Chasles, to this the process by which homographic systems are most frequently generated in modern geometry.

322. Two systems of points on or tangents to, or a system of points on and a system of tangents to, the same circle, or two

different circles, A, B, C, D, E, F, &c. and A', B', C', D', E', F', &c. whose constituents correspond in pairs A and A', B and B', C and C', D and D', E and E', F and F', &c. are also said to be homographic under the same circumstances as rows of points and pencils of rays; viz., when every four constituents of one and the four corresponding constituents of the other are equianharmonic (309). Every two similar systems of points on or tangents to the same circle or two different circles (305); every system of points on and the corresponding system of tangents to the same circle (307); every two systems of points on or tangents to the same circle in perspective with each other to any centre or axis (315); are evidently thus related to each other.

It will appear in the sequel that homographic systems of points on, or of tangents to, the same circle possess not unfrequently comparative facilities of management in the general case when the radius of the circle is finite, which are altogether lost in the two extreme cases when it is either evanescent or infinite; and when, consequently, the two systems of points are collinear in the one case, and the two systems of tangents concurrent in the other.

323. As two or more magnitudes of any kind when equal to a common magnitude are equal to each other; it is evident, from the conditions of homography as stated in the two preceding articles (see Cor. 2°, Art. 278), that *when two or more systems of any species are homographic with a common system, they are homographic with each other; and their several pairs or groups of constituents which correspond to the same constituent of the common system correspond to each other.* All rows of points or pencils of rays in perspective with the same row or pencil (285); all rows or points or pencils of rays reciprocals to the same pencil or row with respect to different circles (292); all pencils of rays determined by the same system of points on a circle at different points on the circle, and all rows of points determined by the same system of tangents to a circle on different tangents to the circle (305); all systems of points determined by the same pencil of rays on different circles passing through its vertex, and all systems of tangents determined by the same row of points

to different circles touching its axis (309); are thus homographic with each other.

And as, again, two or more magnitudes of any kind when equal, not all as above to a common magnitude, but each instead to a different one of as many equal magnitudes, are also equal to each other; it follows consequently, as evidently, from the same conditions, that *when two or more systems of any species are homographic each with a different one of as many homographic systems, they are homographic with each other; and their several pairs or groups of constituents which correspond to corresponding pairs or groups of the homographic systems correspond to each other.* All rows of points or pencils of rays determined by homographic pencils of rays or rows of points (285); all rows of points or pencils of rays in perspective with homographic rows or pencils (285); all rows of points or pencils of rays reciprocals to homographic pencils of rays or rows of points with respect to circles (292); all systems of points determined by homographic pencils of rays on circles passing through their vertices, or systems of tangents determined by homographic rows of points to circles touching their axes (305); all systems of points on or tangents to common circles in perspective with homographic systems of points on or tangents to the same circles (315); are thus homographic with each other.

324. The relation of homography between two or more systems of any species, whose constituents correspond in pairs or groups A, A', A'', &c.; B, B', B'', &c.; C, C', C'', &c.; D, D', D'', &c.; E, E', E'', &c.; F, F', F'', &c.; &c., may be always symbolically represented, as observed in Art. 282, by the obvious extension of Dr. Salmon's very convenient notation for the equianharmonicism of any groups of their corresponding quartets, viz.—

$$\{ABCDEF\text{ \&c.}\} = \{A'B'C'D'E'F'\text{ \&c.}\} = \{A''B''C''D''E''F''\text{ \&c.}\}$$

= &c. The essential precaution, respecting uniformity of order among the corresponding constituents in the several groups, being of course invariably attended to in every case of its employment (see 279.)

325. The following are some fundamental examples of cases of homographic division, grouped in reciprocal pairs, in all of which the relation of homography between the generated systems appears from the nature of the law connecting the several pairs, or groups, of corresponding constituents, which is given in each.

Ex. *a*. *Two variable points on a fixed line or circle, either separated by a constant interval, or having a fixed middle point, determine two homographic systems of points on the line or circle.*

Ex. *a'*. *Two variable tangents to a fixed point or circle, either inclined at a constant angle, or having a fixed middle tangent, determine two homographic systems of tangents to the point or circle.*

For, in each of the eight cases alike, the two generated systems are evidently similar, and therefore homographic by the simplest criterion of the relation (321).

Ex. *b*. *A variable line, intersecting a fixed circle at any constant angle, determines two homographic systems of points on every concentric circle.*

Ex. *b'*. *A variable point, subtending a fixed circle at any constant angle, determines two homographic systems of tangents to every concentric circle.*

Here again, in both cases alike, the two generated systems are evidently similar, and therefore, as in the preceding examples, homographic by the simplest criterion of the relation (321).

Ex. *c*. *A variable line, enveloping a fixed circle, determines homographic systems of points on all fixed tangents to the circle.*

Ex. *c'*. *A variable point, describing a fixed circle, determines homographic systems of rays at all fixed points on the circle.*

For, all the pencils in the latter case being similar (25, 1°), and all the rows in the former case determining similar pencils at the centre of the circle (25, 2°); therefore &c. (285).

Ex. *d*. *A variable line, turning round a fixed point, determines homographic systems of points, on all fixed lines, and on all fixed circles passing through the point.*

Ex. *d'*. *A variable point, moving on a fixed line, determines homographic systems of tangents, to all fixed points, and to all fixed circles touching the line.*

For, all the generated systems being, in each case, homographic with the determining pencil or row (285 and 306); therefore &c. (323).

Ex. *e*. *Two variable lines, turning round a fixed point, and either inclined at a constant angle or having a fixed middle line, determine homographic systems of points, on all fixed lines, and on all fixed circles passing through the point.*

Ex. *e'*. *Two variable points, moving on a fixed line, and either separated by a constant interval or having a fixed middle point, determine homographic systems of tangents, to all fixed points, and to all fixed circles touching the line.*

For, in both cases of each, as in the preceding examples, all the generated systems being homographic with their determining pencils or rows (285 and 306); and the latter, by examples *a* and *a′*, being homographic with each other; therefore &c. (323).

Ex. *f*. *A variable line, turning round a fixed point, determines two homographic systems of points on any fixed circle.*

Ex. *f′*. *A variable point, moving on a fixed line, determines two homographic systems of tangents to any fixed circle.*

For, in both cases alike, every two quartets of corresponding constituents of the two generated systems are equianharmonic (315, *a* and *a′*); and therefore &c. (321).

Ex. *g*. *A variable line, turning round either centre of perspective of two fixed circles* (207), *determines four homographic systems of points on the two circles.*

Ex. *g′*. *A variable point, moving on either axis of perspective of two fixed circles* (207), *determines four homographic systems of tangents to the two circles.*

For, in both cases, each system for either circle being homologous or antihomologous (198 and 204), and therefore homographic (316), with one of the two for the other circle; and the two for the same circle being homographic, by the preceding examples *f* and *f′*; therefore &c. (323).

Ex. *h*. *Two variable points on a fixed line or circle, dividing harmonically a fixed segment of the line or arc of the circle, determine two homographic systems of points on the line or circle.*

Ex. *h′*. *Two variable tangents to a fixed point or circle, dividing harmonically a fixed angle at the point or arc of the circle, determine two homographic systems of tangents to the point or circle.*

For, in the latter cases of both, the two systems, being in perspective (257), are consequently, as in examples *f* and *f′*, homographic by (315, *a* and *a′*); and they evidently involve the former (309); which however follow at once directly from (3°, Cor. 4°, Art. 282).

Ex. *i*. *Two variable points on a fixed line or circle, dividing equianharmonically two fixed segments of the line or arcs of the circle, determine two homographic systems of points on the line or circle.*

Ex. *i′*. *Two variable tangents to a fixed point or circle, dividing equianharmonically two fixed angles at the point or arcs of the circle, determine two homographic systems of tangents to the point or circle.*

For, in the latter cases of both again, the two systems, being in perspective (313), are consequently, as in the preceding examples, homographic by (315, *a* and *a′*); and they also evidently involve the former (309); which however are reduced at once to those of the preceding examples by (283, Cor. 1°), from which it appears that any two sectors C and C', which cut AB' and BA' equianharmonically, are harmonic conjugates with respect to the two M and N which cut AA' and BB' harmonically; and therefore &c.

Ex. *j*. *Two variable lines, dividing a fixed angle harmonically, or two fixed angles having a common vertex equianharmonically, determine homo-*

graphic systems of points, on all fixed lines, and on all fixed circles passing through the vertex of the angle or angles.

Ex. *j'*. *Two variable points, dividing a fixed segment harmonically, or two fixed segments having a common axis equianharmonically, determine homographic systems of tangents, to all fixed points, and to all fixed circles touching the axis of the segment or segments.*

For, in both cases of each, as in examples *e* and *e'*, all the generated systems are homographic with their determining pencils or rows (285 and 306); and the latter, by examples *h* and *h'*, or *i* and *i'*, are homographic with each other; and therefore &c. (323).

Ex. *k*. *When, of a variable polygon of any order inscribed to a fixed circle, all the sides but one turn round fixed points, or envelope fixed circles concentric with the original; the several vertices determine so many homographic systems of points on the circle.*

Ex. *k'*. *When, of a variable polygon of any order exscribed to a fixed circle, all the vertices but one move on fixed lines, or describe fixed circles concentric with the original; the several sides determine so many homographic systems of tangents to the circle.*

These follow immediately as corollaries from examples *f* and *f'*, or *b* and *b'*; the two extremities of every restricted (and therefore (323) of the single unrestricted) side, in the former case, and the two sides of every restricted (and therefore (323) of the single unrestricted) angle, in the latter case, determining homographic systems; and therefore &c.

Ex. *l*. *When, of a variable polygon of any order all whose vertices move on fixed lines, all the sides but one turn round fixed points, or envelope fixed circles touching the pairs of lines on which their extremities move; the several vertices determine so many homographic systems of points on the several lines.*

Ex. *l'*. *When, of a variable polygon of any order all whose sides turn round fixed points, all the angles but one move on fixed lines, or describe fixed circles passing through the pairs of points round which their extremities turn; the several sides determine so many homographic systems of rays at the several points.*

These follow immediately as corollaries from examples *d* and *d'*, or *c* and *c'*; by virtue of which the demonstrations just given for the two preceding examples *k* and *k'*, without modification of any kind, apply word for word to them also. The first parts of both are evidently included in those of the two following, under which they come respectively as particular cases; viz.—

Ex. *m*. *When, of a variable polygon of any order all whose vertices move on fixed lines, all the sides but one subtend at fixed points angles, of constant magnitudes, or having fixed middle lines, or dividing harmonically fixed angles, or dividing equianharmonically pairs of fixed angles, at the points: the several vertices determine so many homographic systems of points on the several lines.*

Ex. *m'*. *When, of a variable polygon of any order all whose sides turn round fixed points, all the angles but one intercept on fixed lines segments,* *of constant magnitudes, or having fixed middle points, or dividing harmonically fixed segments, or dividing equianharmonically pairs of fixed segments, on the lines; the several sides determine so many homographic systems of rays at the several points.*

These follow immediately as corollaries from examples *e* and *e'*, or *j* and *j'*; by virtue of which the same demonstrations again, without modification of any kind, apply to them also. That all four cases of both properties are included in a single reciprocal pair, under which they come alike as particular cases, will appear further on in the present chapter.

Ex. *n*. *When, of a figure of any nature, variable in magnitude and position but invariable in form, three points fixed relatively to it move on fixed lines; all points fixed relatively to it move on fixed lines, and determine homographic systems of points on the several lines.*

Ex. *n'*. *When, of a figure of any nature, variable in magnitude and position but invariable in form, three lines fixed relatively to it turn round fixed points; all lines fixed relatively to it turn round fixed points, and determine homographic systems of rays at the several points.*

The first parts of these properties have been already established in Art. 56; from which as it appears also, from the invariability of one point fixed relatively to the figure in either case, that the several points determine similar rows on their several lines in the former case, and that (as is otherwise evident from their necessarily revolving simultaneously through equal angles) the several lines determine similar pencils at their several points in the latter case; therefore &c.

Ex. *o*. *If the three vertices of a variable triangle of constant species move on fixed lines, their mean centres, for all triads of constant multiples* (86), *move on fixed lines, and determine homographic systems of points on the several lines.*

Ex. *o'*. *If the three sides of a variable triangle of constant species turn round fixed points, their central axes, for all triads of constant multiples* (120), *turn round fixed points, and determine homographic systems of rays at the several points.*

These properties are evidently particular cases of the preceding examples; the mean centres of the three vertices, and the central axes of the three sides, of a triangle of given species, being evidently fixed relatively to the figure for all systems of fixed multiples; and therefore &c. They have been given merely with prospective reference to the more general properties into which they become transformed by a process to be explained in another chapter.

326. The several examples of the preceding article have been given, as stated at its commencement, grouped in pairs, one concerning systems of points and the other concerning systems of lines, and each reciprocating to an arbitrary circle

either into the other or into some more general property involving the other. Those of the present article again, though all concerning systems of points only, and those determined by variable circles, are also given grouped in pairs connected by a different and not less interesting law, which will form the subject of another chapter.

Ex. *a*. *A variable circle, passing through two fixed points, determines two homographic systems of points on any fixed line* (*or circle*.)

For, in the former case, the pencil determined by either system at either point is similar to that determined by the other system at the other point (Euc. III. 21, 22); and therefore &c. by Art. 323. And, in the latter case the two systems are in perspective at a centre on the line containing the points (52); and therefore &c. by Ex. *f* of the preceding article.

Ex. *b*. *A variable circle, coaxal with two fixed points, determines two homographic systems of points on any fixed line* (*or circle*).

For, in the former case, the pencils determined by the two systems at either point are similar (192, Cor. 4°); and therefore &c. by Art. 323. And, in the latter case, as in the preceding example, the two systems are in perspective at a centre on the axis of reflexion of the points (187, 2°); and therefore &c., by Ex. *f* of the preceding article. This example and the preceding combined express evidently a common property of a variable circle of any coaxal system (184).

Ex. *c*. *A variable circle, passing through two fixed points, determines two homographic systems of points on any two fixed lines* (*or circles*) *passing each through one or both through either of the points.*

For, in either case of the former, the two systems are evidently similar to that determined by the centre of the variable circle on the fixed line it describes; and therefore &c. (323). And, in either case of the latter, the pencils determined by the two systems at the fixed point or points through which their containing circles pass, are evidently orthogonal, and therefore similar, to those determined at the centers of those circles by the centre of the variable circle; and both systems being consequently, as in the former case, homographic with the row determined by the centre of the common generating circle, therefore &c. (323).

Ex. *d*. *Two variable circles, passing through two fixed points, and either intersecting at a constant angle or making equal* (*or supplemental*) *angles with a fixed circle passing through the points, determine two homographic systems of points on any fixed line* (*or circle*) *passing through either point.*

For, in either case of the former, the two systems being evidently similar to those determined by the respective centers of their generating circles on the common axis they describe; and the latter being homographic, by Ex. *e* of the preceding article; therefore &c. (323). And, in either case of the latter, the pencils determined by the two systems at the fixed point through which their containing circle passes, being evidently orthogonal, and therefore similar, to those determined at the centre of that circle by the respective centres of their generating circles; and the two rows determined by the latter on the common axis they describe being, as in the former case, homographic; therefore &c. (323).

Ex. *e*. *A variable circle, passing through a fixed point, and intersecting a fixed line* (*or circle*) *at right angles, determines two homographic systems of points on the line* (*or circle*).

For, in either case, the variable circle passing also in every position through a second fixed point, the inverse of the original with respect to the fixed line or circle (156), the property, in either case, is consequently evident from Ex. *a*. of the present article.

Ex. *f*. *A variable circle, passing through a fixed point, and intersecting a fixed line* (*or circle*) *at any constant angle, determines two homographic systems of points on the line* (*or circle*).

For, in the former case, the two variable lines, determined by the two variable points of intersection with the fixed point, intersect at the constant angle of intersection (22, 2°); and therefore &c. by Ex. *e*. of the preceding article. And, in either case, the two variable circles, determined by the two variable points of intersection with the original fixed point and with any second arbitrarily assumed on the fixed line or circle, intersect at the constant angle of intersection; and therefore &c., by Ex. *d*. of the present article. It is assumed in the latter proof that, *of the four circles which pass each through a different triad of the same four points, the angle of intersection of any two is equal or supplemental to that of the remaining two;* a property the reader may very easily prove for himself.

Ex. *g*. *A variable circle, intersecting two fixed lines* (*or circles*) *at right angles, determines four homographic systems of points on the two lines* (*or circles*).

For, in the former case, the centre of the variable circle being evidently fixed at the intersection of the lines, the four generated systems are consequently similar and equal; and therefore &c. And, in the latter case, its centre describing the radical axis of the circles (182, Cor. 5°), its four radii to its four points of intersection with them, determine, by Ex. *g'* of the preceding article, four homographic systems of tangents to them; and therefore &c. (322).

Ex. *h*. *A variable circle, intersecting two fixed lines* (*or circles*) *at any two constant angles, determines four homographic systems of points on the two lines* (*or circles*).

For, in the former case, the four systems are obviously similar to that determined by the centre of the generating circle on the line on which it evidently moves; and therefore &c. (321). And, in either case, if U and V be the two variable points of intersection with one line or circle, X and Y those with the other, and P any fixed point arbitrarily assumed on either; then, the two angles of intersection, and therefore their difference and their sum, being constant, the two variable circles UPX and VPY pass through a second fixed point Q, and the two UPY and VPX through a second fixed point R, both on the other; therefore, by Ex. *e* of the present article, the two systems determined by the two variable points U and V are homographic, respectively, with the two determined by the two X and Y in consequence of the two fixed points P and Q, and with the two determined by the two Y and X in consequence of the two P and R; and therefore &c. It is assumed in the latter proof that *a variable circle, passing through two fixed points on two fixed lines or circles, intersects the latter at angles whose difference or sum is constant*, and that when the lines or circles, with the sum or difference of the angles are given, the point on either determines that on the other. Of these, however, the former can present no difficulty to the reader, and the latter is but an obvious inference from it.

Ex. *i.* *A variable circle, intersecting two fixed lines* (*or circles*) *at constant angles, determines 2n homographic systems of points on any n lines passing through the point* (*or circles passing through the two points*) *of intersection of the two.*

For, a variable circle, intersecting two rays of a pencil, or circles of a coaxal system, at constant angles, intersects, evidently in the former case, and by (193, Cor. 8°) in the latter case, all rays of the pencil, or circles of the system, at constant angles; and therefore &c. by the preceding Ex. *h*. In the particular cases when the original two, or any two, of the angles of intersection are right angles; then, evidently in the former case, and by (193, Cor. 5°) in the latter case, all the angles of intersection are right angles; and the property is consequently evident from the comparatively simple case of Ex. *g*.

Ex. *j.* *A variable circle, passing through a fixed point, and intersecting two fixed lines* (*or circles*) *at equal or supplemental angles, determines four homographic systems of points on the two lines* (*or circles*).

For, the variable circle passing, evidently in the former case, and by (211, Cor. 6°, *a*) in the latter case, through a second fixed point, the reflexion of the original (50) with respect to the corresponding bisector, external or internal, of the angle determined by the lines, or the antihomologue of the original (203) with respect to the corresponding centre of perspective, external or internal, of the circles, determines consequently, by Ex. *a*. of the present article, two homographic systems on each line or circle separately; and the systems on different lines or circles being in pairs, evidently similar and equal in the former case, and, by (203), antihomologous with respect to the corresponding centre of perspective in the latter case; therefore &c., by (321), and Ex. *g*. of the preceding article.

Ex. *k.* *A variable circle, intersecting three fixed lines* (*or circles*) *at equal or at any invariable combination of equal and supplemental angles, determines six homographic systems of points on the three lines* (*or circles*).

For, the variable circle determining, evidently in the former case, and by (211, Cor. 6°, *b*) in the latter case, a system concentric with the corresponding circle of the four that touch the three lines, or coaxal with the corresponding pair of conjugates of the eight that touch the four circles, determines consequently, evidently in the former case, and by Ex. *b*. of the present article in the latter case, six similar and equal systems on the three lines, or two homographic systems on each circle separately; and as also, in the latter case by (211), it determines on each pair of circles two systems of antihomologous points with respect to their external or internal centre of perspective according as its angles of intersection with them are equal or supplemental, the two systems it determines on each circle are consequently, by Ex. *g* of the preceding article, homographic each with one of the two it determines on each of the remaining two; and therefore &c.

327. *When a variable pair of corresponding constituents of any common or different species, D and D', are connected in every position with three fixed pairs, A and A', B and B', C and C', by the constant relation* $\{ABCD\} = \{A'B'C'D'\}$; *they determine two homographic systems, of which A and A', B and B', C and C' are three pairs of corresponding constituents.*

For, as shewn in Art. 282, Cor. 1°, every four positions of D and the four corresponding positions of D' are equianharmonic; and when, in the course of their variation, D coincides with any of the three A, B, C, then D' coincides with the corresponding one of the three A', B', C'; and therefore &c. (321).

It follows immediately from the above that, *as regards two homographic systems of any common or different species, any three pairs of corresponding constituents A and A', B and B', C and C' may be entirely arbitrary; but that once given, or taken, or known in any manner, they determine completely the systems, and with them, of course, all particulars directly or indirectly connected with them.* For, the relation $\{ABCD\} = \{A'B'C'D'\}$, necessary to the homography of the systems (321), determines, when A and A', B and B', C and C' are known, the constituent D' or D of either system corresponding to any given or assumed fourth constituent D or D' of the other; and, by the above, the two systems determined by the simultaneous variation of D and D', in accordance with that relation, are homographic, and have A and A', B and B', C and C' for three of their pairs of corresponding constituents.

Since, conversely, for any two homographic systems of any common or different species, if A and A', B and B', C and C' be any three pairs of corresponding constituents, then for every other pair D and D', by (321), $\{ABCD\} = \{A'B'C'D'\}$; the criterion of homography furnished by the above is consequently perfectly general, and applicable to every case, without exception, of the generation of two homographic systems of any common or different species by the simultaneous variation of a pair of connected constituents D and D'.

328. *When a variable pair of corresponding constituents of any two collinear or concurrent systems, or of a collinear and a concurrent system, C and C', are connected in every position with two fixed pairs, A and A', B and B', by the constant relation*

$$\left(\frac{AC}{BC} \text{ or } \frac{\sin AC}{\sin BC}\right) : \left(\frac{A'C'}{B'C'} \text{ or } \frac{\sin A'C'}{\sin B'C'}\right)$$

in any constant ratio, positive or negative; they determine two homographic systems, of which A and A', B and B' are two pairs of corresponding constituents.

For, as shewn in Art. 282, Cor. 2°, if any one position C and C' of the variable pair be regarded as fixed; then, since for every other position D and D', by division of ratios,

$$\left(\frac{AC}{BC} : \frac{AD}{BD}\right) \text{ or } \left(\frac{\sin AC}{\sin BC} : \frac{\sin AD}{\sin BD}\right) = \left(\frac{A'C'}{B'C'} : \frac{A'D'}{B'D'}\right)$$

or $$\left(\frac{\sin A'C'}{\sin B'C'} : \frac{\sin A'D'}{\sin B'D'}\right),$$

therefore, as in the preceding article, $\{ABCD\} = \{A'B'C'D'\}$; and therefore &c.

Conversely, *For any two homographic systems, both collinear or concurrent, or one collinear and one concurrent; if A and A', B and B' be any two pairs of corresponding constituents, then, for every other pair C and C', the ratio*

$$\left(\frac{AC}{BC} \text{ or } \frac{\sin AC}{\sin BC}\right) : \left(\frac{A'C'}{B'C'} \text{ or } \frac{\sin A'C'}{\sin B'C'}\right)$$

is constant, both in magnitude and sign.

For, since for every other two pairs C and C', D and D', by the homography of the systems, $\{ABCD\} = \{A'B'C'D'\}$, and since, consequently,

$$\left(\frac{AC}{BC} : \frac{AD}{BD}\right) \text{ or } \left(\frac{\sin AC}{\sin BC} : \frac{\sin AD}{\sin BD}\right) = \left(\frac{A'C'}{B'C'} : \frac{A'D'}{B'D'}\right)$$

or $$\left(\frac{\sin A'C'}{\sin B'C'} : \frac{\sin A'D'}{\sin B'D'}\right);$$

therefore at once, by alternation,

$$\left(\frac{AC}{BC} \text{ or } \frac{\sin AC}{\sin BC}\right) : \left(\frac{A'C'}{B'C'} \text{ or } \frac{\sin A'C'}{\sin B'C'}\right)$$

$$= \left(\frac{AD}{BD} \text{ or } \frac{\sin AD}{\sin BD}\right) : \left(\frac{A'D'}{B'D'} \text{ or } \frac{\sin A'D'}{\sin B'D'}\right);$$

and therefore &c.

It follows evidently, from this latter property, that the criterion of homography furnished by the above is, as regards collinear and concurrent systems, as general as that of the preceding article; and applies, equally with it, to every case, without exception, of the generation of two homographic systems of either species by the simultaneous variation of a pair of connected constituents C and C'.

COR. From the first part of the above general property, it follows, immediately, that—

When two variable lines or points, or a variable line and point, I and I', are connected in every position with two fixed pairs of points or lines, or with a fixed pair of points and a fixed pair of lines, A and B, A' and B', by the constant relation

$$\left(\frac{AI}{BI} : \frac{A'I'}{B'I'}\right) \textit{ or its equivalent } \left(\frac{AI}{A'I'} : \frac{BI}{B'I'}\right)$$

in any constant ratio, positive or negative; they divide the two segments or angles, or the segment and angle, AB and $A'B'$, homographically; and the two pairs of corresponding constituents in the two ratios, A and A', B and B', are two pairs of corresponding constituents in the two divisions.

For, if C and C' be the two points of intersection or lines of connection, or the point of intersection and line of connection, of I and I' with AB and $A'B'$; then, since, according to the case, evidently,

$$\left(\frac{AC}{BC} \text{ or } \frac{\sin AC}{\sin BC}\right) = \frac{AI}{BI} \text{ and } \left(\frac{A'C'}{B'C'} \text{ or } \frac{\sin A'C'}{\sin B'C'}\right) = \frac{A'I'}{B'I'};$$

therefore &c. These properties are useful in the modern theories of homographic and of correlative transformation, as will appear in the sequel in the chapters in which they are respectively discussed.

329. *Two variable sectors, C and C', dividing a fixed segment or angle, AB, in any constant anharmonic ratio, positive or*

negative, determine two homographic systems of points or rays; of which the two extremities, A and B, of the fixed segment or angle constitute each a pair of corresponding constituents coinciding with each other.

For, as in the more general property of the preceding article (under the first part of which, as observed in Art. 282, Cor. 3°, this manifestly comes as a particular case), if any one position C and C' of the variable pair be regarded as fixed; then since for every other position D and D', by hypothesis, $\{ABCC'\} = \{ABDD'\}$, therefore, by (272), $\{ABCD\} = \{ABC'D'\}$, and therefore, by (327), D and D' determine two homographic systems, of which A ($= A'$) and B ($= B'$) constitute each a pair of corresponding constituents coinciding with each other; as it is evident *à priori* they ought, two variable magnitudes of any kind having a constant ratio to each other (268), whatever be its magnitude or sign, provided only it be finite, necessarily vanishing, becoming infinite, and changing sign together.

Conversely, *For any two homographic rows of points or pencils of rays having a common axis or vertex, if $A = A'$ and $B = B'$ be two pairs of corresponding constituents which coincide with each other; then, for every other pair C and C' of their corresponding constituents, the anharmonic ratio of section of the intercepted segment or angle AB is constant both in magnitude and sign.*

For, since, for every other two pairs C and C', D and D', by the homography of the systems and the hypothesis that A' and B' coincide with A and B respectively, $\{ABCD\} = \{ABC'D'\}$; therefore, at once, by (272), $\{ABCC'\} = \{ABDD'\}$; and therefore &c.

It will be shewn in the next chapter that, for every two homographic rows of points or pencils of rays having a common axis or vertex, there exist always two pairs of corresponding points or rays, real or imaginary, which thus coincide with each other, and which have been termed in consequence, by Chasles, *the double points or rays of the systems.* Their properties and uses are among the most interesting and important in the whole theory of homographic division, and will form the entire subject of the chapter.

330. *Two variable points B and B′ on two fixed lines, the ratio of whose distances from two fixed points A and A′ on the*

lines is constant both in magnitude and sign, determine two homographic systems, of which A and A', and the two points at infinity on the lines, are two pairs of corresponding constituents.

For, the systems being similar are therefore homographic; and, whatever be the magnitude and sign of the ratio, provided only it be finite, the two variable distances AB and $A'B'$ vanish and become infinite together; and therefore &c.

Conversely, *when two homographic rows of points are such that the two points at infinity on their axes,* ∞ *and* ∞', *are corresponding constituents of the systems, they are similar.*

For, if A and A', B and B', C and C' be any other three pairs of their corresponding constituents, then since, by hypothesis, $\{ABC\infty\} = \{A'B'C'\infty'\}$, therefore (275)

$$BC : CA : AB = B'C' : C'A' : A'B';$$

and therefore &c.

The *criterion of similitude* between two homographic rows of points on any axes, supplied by the second part of the above, viz. *the correspondence of the point at infinity of one to the point at infinity of the other*, is in all cases very readily applicable; depending, as there shewn, on the circumstance of similitude only, and being independent, as shewn in the first part, of the magnitude and sign of the ratio of similitude, provided only it be finite.

COR. In the particular case when the two points A and A' coincide at the intersection I of the lines; the species of the variable triangle BIB' being then evidently constant, it follows consequently from the first part of the above, (as is otherwise evident) that *a variable line, determining with two fixed lines a triangle of constant species, divides the lines homographically; the common point and the point at infinity corresponding in each division to the common point and the point at infinity in the other.*

331. *Two variable points C and C' on two fixed lines, the rectangle under whose distances form two fixed points A and B' on the lines is constant both in magnitude and sign, determine two homographic systems; of which A and B' correspond to the points at infinity on the lines.*

For, if any one position C and C' of the variable pair be regarded as fixed, then since, for every other position D and D', by hypothesis, $AC.B'C' = AD.B'D'$, and since consequently $CA : DA = D'B' : C'B'$, therefore, by (275),

$$\{CDA\infty\} = \{D'C'B'\infty'\} = \{C'D'\infty'B'\} \text{ (280)},$$

and therefore, by (327), D and D' determine two homographic systems, of which A and ∞', ∞ and B' are two pairs of corresponding constituents; as it is evident *à priori* they ought, one side of a variable rectangle of any constant area, whatever be its magnitude or sign, provided only it be finite, necessarily vanishing as the other becomes infinite, and conversely.

Conversely, *for any two homographic rows of points on any axes, if A and B' be their points corresponding to those at infinity on the axes, then, for every pair C and C' of their corresponding constituents, the rectangle $AC.B'C'$ is constant both in magnitude and sign.*

For, since, for every two pairs C and C', D and D', by the homography of the systems and the hypothesis as to A and B',

$$\{CDA\infty\} = \{C'D'\infty'B'\} = \{D'C'B'\infty'\} \text{ (280)},$$

therefore, by (275), $CA : DA = D'B' : C'B'$; from which, since of course immediately $AC.B'C' = AD.B'D'$, therefore &c.

In the exceptional case of two similar rows, the two points A and B' being then at infinity (330), the constant rectangle $AC.B'C'$ becomes accordingly infinite, and the second part of the above consequently useless; in every other case however it is finite, and the property itself is one of the most useful in the entire theory of homographic division.

COR. In the particular case when the two points A and B' coincide at the intersection I of the lines; the area of the variable triangle CIC' being then evidently constant, it follows consequently from the first part of the above that *a variable line, determining with two fixed lines a triangle of constant area, divides the lines homographically; the common point corresponding in each division to the point at infinity in the other.*

332. The general property of the preceding article, applied to the particular case of homographic division considered in the first part of Art. 329, gives immediately the general property of a fixed segment AB cut in any constant anharmonic ratio by a

variable pair of sectors, corresponding to that given in Art. 225 for the particular case of harmonic section; viz.—

For a fixed segment AB, cut in any constant anharmonic ratio by a variable pair of sectors C and C'; if P and Q be its two points of section in the ratio and its reciprocal, the rectangle PC.QC' is constant both in magnitude and sign, and equal to the rectangle PA.QA, or to the rectangle PB.QB.

For, the two variable sectors determining two homographic systems, of which each extremity of the fixed segment constitutes a pair of corresponding constituents coinciding with each other (329); and of which the two points corresponding to that at infinity on the common axis (regarded as belonging first to one and then to the other system) divide the segment, one in the constant ratio and the other in its reciprocal (275); therefore &c. by the general property of the preceding article (331).

In the particular case when the constant anharmonic ratio of the section $= -1$, that is, when the section is constantly harmonic (268); the two points P and Q coincide at the middle point of the segment (216, 3°), and the property, as observed above, becomes that established on other principles in Art. 225.

333. To the same particular, but important, case of homographic division considered in the first part of Art. 329, several others are reducible. The following are a few examples, grouped in reciprocal pairs, of cases coming under this head:

Ex. *a*. *A variable point moving on a fixed line, and its polar with respect to the sides of any fixed angle, determine homographic systems for all pairs of constant multiples* (247, *a*).

Ex. *a'*. *A variable line turning round a fixed point, and its pole with respect to the extremities of any fixed segment, determine homographic systems for all pairs of constant multiples* (247, *a'*).

For, by (247, Cor. 4°), whatever be the values of the two multiples, the variable point and line divide, in every position, the fixed angle in the former case, and the fixed segment in the latter case, in two constant anharmonic ratios, equal in magnitude and opposite in sign to the two reciprocal ratios of the multiples (268); and therefore &c. (323).

Ex. *b*. *A variable point describing a fixed circle, and its polar with respect to the sides of any fixed angle whose vertex lies on the circle, determine homographic systems for all pairs of constant multiples.*

Ex. *b'*. *A variable line enveloping a fixed circle, and its pole with respect* *to the extremities of any fixed segment whose axis touches the circle, determine homographic systems for all pairs of constant multiples.*

These follow precisely as in the two preceding examples; the variable point and line dividing, in every position, the fixed angle in the former case, and the fixed segment in the latter case, in two constant anharmonic ratios, equal in magnitude and opposite in sign to the two reciprocal ratios of the multiples; and therefore &c. (323).

Ex. c. *If a variable point move on a fixed line, its polar, with respect to the sides of any fixed triangle, divides the sides homographically for all triads of constant multiples* (248, a).

Ex. c'. *If a variable line turn round a fixed point, its pole with respect to the vertices of any fixed triangle, divides the angles homographically for all triads of constant multiples* (248, a').

For, whatever be the values of the three multiples; the variable point, in the former case, and the intersection of its polar with any side of the triangle, divide the opposite angle (250, a); and the variable line, in the latter case, and the connector of its pole with any vertex of the triangle, divide the opposite side (250, a'); in two constant anharmonic ratios, equal in magnitude and opposite in sign to the two reciprocal ratios of the corresponding pair of multiples (268); therefore, in both cases, the three divisions, being homographic with that determined by the variable point or line, are homographic with each other (323); and therefore &c. The same evidently follows also from examples a and a', by virtue of the general property 250, Cor. 1°.

Ex. d. *If a variable point describe a fixed circle, its polar, with respect to the sides of any fixed triangle inscribed to the circle, divides the sides homographically for all triads of constant multiples.*

Ex. d'. *If a variable line envelope a fixed circle, its pole, with respect to the vertices of any fixed triangle exscribed to the circle, divides the angles homographically for all triads of constant multiples.*

These follow precisely as in the two preceding examples; the three divisions being, for the reason just given in their case, homographic with that determined by the variable point or line, and therefore with each other. It will be proved further on (as shewn in (261, Cor. 14°, a and a') for the particular case when the three multiples are all equal in magnitude and sign) that, for all triads of finite multiples, the polar in the former case turns round a fixed point, and the pole in the latter case moves on a fixed line; and that, consequently, the three divisions, in both cases, are not only homographic but in perspective to a common centre or axis.

Ex. e. *A variable line, intersecting with any four fixed lines at four points having any constant anharmonic ratio, determines four homographic systems of points on the four lines.*

Ex. e'. *A variable point, connecting with any four fixed points by four rays having any constant anharmonic ratio, determines four homographic systems of rays at the four points.*

For, the two intersections (or connectors) of the variable line (or point), with any two of the four fixed lines (or points), connect (or intersect) with the vertex (or axis) of the fixed angle (or segment) determined by the remaining two, by two variable lines (or at two variable points) which dividing, by hypothesis, that angle (or segment) in a constant anharmonic ratio (285), determine consequently two homographic systems of rays (or points); and therefore &c. (323). The general properties of homographic rows and pencils, converse to these reciprocal examples, will be given further on.

Ex. *f*. *A variable line, intersecting with any three fixed lines at three points the ratios of whose three intercepted segments are constant, determines three homographic systems of points on the three lines.*

Ex. *f′*. *A variable point, connecting with any three fixed points by three lines the ratios of whose three intercepts on any fixed line are constant, determines three homographic systems of rays at the three points.*

These are obviously particular cases of the two preceding examples; the variable line, in the former case, intersecting with the three fixed lines, and with the line at infinity, at a system of four points having a constant anharmonic ratio (275); and the variable point, in the latter case, connecting with the three fixed points, and with the point at infinity in the direction of the fixed line, by a system of four rays having a constant anharmonic ratio (285, Cor. 1°); and therefore &c. The first of these examples is obviously a particular case also of that established on other principles in Ex. *n*, Art. 325.

Ex. *g*. *A variable line, determining in every position an equianharmonic hexagram with any five fixed lines* (301), *determines five homographic systems of points on the five lines.*

Ex. *g′*. *A variable point determining in every position an equianharmonic hexastigm with any five fixed points* (301), *determines five homographic systems of rays at the five points.*

For, the variable line (or point), determining in every position with every four of the fixed lines (or points) a system of four points (or rays) equianharmonic with that determined with them by the fifth (301), determines with them, consequently, a system of four points (or rays) having a constant anharmonic ratio; and therefore &c. by examples *e* and *e′*. The general properties of homographic rows and pencils, converse to these reciprocal examples, will also be given further on.

Ex. *h*. *When, of a variable polygon of any order all whose vertices move on fixed lines, all the sides but one subtend, at fixed points, angles dividing fixed angles at the points in constant anharmonic ratios; the several vertices determine so many homographic systems of points on the several lines.*

Ex. *h′*. *When, of a variable polygon of any order all whose sides turn round fixed points, all the angles but one intercept, on fixed lines, segments dividing fixed segments on the lines in constant anharmonic ratios; the several sides determine so many homographic systems of rays at the several points*

These follow as immediate corollaries from the same, precisely as their particular cases in examples *m* and *m'* of Art. 325 from the properties on which they depend; the two extremities of every restricted (and therefore (323) of the single unrestricted) side, in the former case, and the two sides of every restricted (and therefore (323) of the single unrestricted) angle, in the latter case, determining, by the above, homographic systems; and therefore &c. These latter properties evidently include as particular cases all those of the examples referred to; and, as any or all of the fixed angles or segments may be imaginary as well as real, they are consequently themselves the most general of their class, and include a variety of other particular cases besides.

334. From the two pairs of reciprocal properties (288, *a* and *a'*) and (315, *a* and *a'*), the following reciprocal criteria of the relation of perspective between two homographic systems evidently result immediately, viz.—

a. When, of two homographic rows of points on different axes, or systems of points on the same circle, any two triads of corresponding constituents are in perspective, the systems themselves are in perspective.

a'. When, of two homographic pencils of rays through different vertices, or systems of tangents to the same circle, any two triads of corresponding constituents are in perspective, the systems themselves are in perspective.

For, in each of the four cases, all fourth pairs of corresponding constituents of the two systems, forming, by hypothesis, equianharmonic quartets with the two triads in perspective, are consequently, by the properties referred to, in perspective with them; and therefore &c.

From the nature of the relation of perspective between two systems of points or lines (130), it is of course evident, conversely, that when two homographic rows of points on different axes or systems of points on the same circle are in perspective, every line through their centre of perspective determines with the axes or circle a pair of corresponding points of the systems; and, that when two homographic pencils of rays through different vertices or systems of tangents to the same circle are in perspective, every point on their axis of perspective determines with the vertices or circle a pair of corresponding lines of the systems.

335. The two following reciprocal criteria of the relation of perspective between two homographic rows of points on different axes, or pencils of rays through different vertices, are generally much more readily applicable than those of the preceding article; of which however their second parts are but particular cases; viz.—

a. When two homographic rows of points on different axes are in perspective, the intersection of the axes constitutes a pair of corresponding constituents coinciding with each other; and, conversely, when, of two homographic rows of points on different axes, a pair of corresponding constituents coincide at the intersection of the axes, the systems are in perspective.

a'. When two homographic pencils of rays through different vertices are in perspective, the connector of the vertices constitutes a pair of corresponding constituents coinciding with each other; and, conversely, when, of two homographic pencils of rays through different vertices, a pair of corresponding constituents coincide along the connector of the vertices, the systems are in perspective.

For, as regards the first parts of both; as every line through the centre of perspective, in the former case, intersects with the axes at a pair of corresponding points of the rows, the particular line through the intersection of the axes does so like the rest; and, as every point on the axis of perspective, in the latter case, connects with the vertices by a pair of corresponding rays of the pencils, the particular point on the connector of the vertices does so like the rest: and therefore &c. And, as regards the second parts of both; as, in either case, the coincident pair determines, with any other two pairs of corresponding constituents of the systems, two triads of corresponding constituents in perspective; therefore, as in the preceding article, or by the two reciprocal properties (289, *a* and *a'*), the systems themselves are in perspective.

Cor. 1°. From the above reciprocal properties it is evident, that—

a. When two homographic rows of points on different axes are in perspective, either axis may be turned round its point of intersection with the other, without its row, supposed to be carried with but not to move along it, ceasing, in any position, to be in perspective with the other.

a'. When two homographic pencils of rays through different vertices are in perspective, either vertex may be moved along its line of connection with the other, without its pencil, supposed to be carried with but not to turn round it, ceasing, in any position, to be in perspective with the other.

For, the two systems being, by hypothesis, originally in perspective, their single common point or ray, by the first parts of the above, constitutes a pair of corresponding constituents coinciding with each other; and this coincidence not being affected by the supposed movement in either case, therefore, by the second parts of the same, they continue in perspective notwithstanding its taking place.

COR. 2°. From the same again it is also evident that—

a. Any two homographic rows of points, given in every particular, except position, may be placed, relatively to each other, in an infinite number of ways, so as to be in perspective with each other.

a'. Any two homographic pencils of rays, given in every particular, except position, may be placed, relatively to each other, in an infinite number of ways, so as to be in perspective with each other.

For, by bringing *any* pair of corresponding constituents to coincide in either case, the systems, whatever be the angle between their axes in the former case, or the interval between their vertices in the latter case, will, by the second parts of the above, be in perspective with each other; and therefore &c.

336. The two reciprocal properties of the preceding article supply, as there observed, very obvious, and in general very readily applicable, criteria for determining *whether two homographic rows of points on different axes or pencils of rays through different vertices, the law connecting whose several pairs of corresponding constituents is given, are in perspective or not.* For, the point or ray corresponding, in either system, to the intersection of the axes or the connector of the vertices, regarded as belonging to the other system, being determined by the given law of connection; the systems, by virtue of the criteria, are or are not in perspective according as the correspondent so determined does or does not coincide with the original point or ray.

The following are a few examples, grouped in reciprocal pairs, of the application of these criteria:

Ex. *a*. *Two vertices of a variable triangle A and B move on two fixed lines L and M, the two opposite sides E and F turn round two fixed points P and Q, and the third side G turns round a third fixed point R; required the condition that the third vertex C should move on a third fixed line N.*

Ex. *a'*. *Two sides of a variable triangle E and F turn round two fixed points P and Q, the two opposite vertices A and B move on two fixed lines L and M, and the third vertex C moves on a third fixed line N; required the condition that the third side G should turn round a third fixed point R.*

In the former case, the two rows determined by A and B on L and M being in all cases homographic (325, Ex. *d*), in order that the two homographic pencils (323) they determine at Q and P respectively should be in perspective, a pair of their corresponding positions should be collinear with those points, which could be the case only when the latter are collinear either with R or with the intersection of L and M. And, in the latter case, the two pencils determined by E and F at P and Q being in all cases homographic (325, Ex. *d'*), in order that the two homographic rows (323) they determine on M and L respectively should be in perspective, a pair of their corresponding positions should be concurrent with those lines; which could be the case only when the latter are concurrent either with N or with the connector of P and Q. See the pairs of reciprocal properties (296, 3°) and (294, 5°) where, on other principles, the perspectives were shewn to exist in those cases respectively.

Ex. *b*. *Two vertices of a variable triangle A and B move on two fixed lines L and M, the two opposite sides E and F turn round two fixed points P and Q, and the third side G envelopes a fixed circle touching L and M; required the condition that the third vertex C should move on a third fixed line N.*

Ex. *b'*. *Two sides of a variable triangle E and F turn round two fixed points P and Q, the two opposite vertices A and B move on two fixed lines L and M, and the third vertex C describes a fixed circle passing through P and Q: required the condition that the third side G should turn round a third fixed point R.*

In the former case, as in Ex. *a*, the two rows determined by A and B on L and M being in all cases homographic (325, Ex. *c*), in order that the two homographic pencils (323) they determine at Q and P respectively should be in perspective, a pair of their corresponding positions should be collinear with those points, which could be the case only when the latter connect by a tangent to the circle. And, in the latter case, as in Ex. *a'*, the two pencils determined by E and F at P and Q being in all cases homographic (325, Ex. *c'*), in order that the two homographic rows (323) they determine on M and L respectively should be in perspective, a pair of their corresponding positions should be concurrent with those lines; which could be the case only when the latter intersect at a point on the circle. That the perspectives exist in those cases, the reader can find no difficulty in shewing independently.

Ex. c. *If a variable point P move on a fixed line I, required the condition that its polar L, with respect to the three sides of a fixed triangle, for any triad of constant multiples, should turn round a fixed point.*

Ex. c'. *If a variable line L turn round a fixed point O, required the condition that its pole P, with respect to the three vertices of a fixed triangle, for any triad of constant multiples, should move on a fixed line.*

In the former case, if A, B, C be the three vertices of the triangle, and X, Y, Z the three intersections of L with its opposite sides E, F, G respectively; then, the anharmonic ratios of the three pencils $A.BCPX$, $B.CAPY$, $C.ABPZ$ being in all cases constant (250, *a*), and the three rows determined by X, Y, Z on E, F, G respectively being, consequently, in all cases homographic with that determined by P on I (329), and therefore with each other (323); in order that they should be in perspective, a pair of corresponding positions should coincide, of Y and Z at A, or of Z and X at B, or of X and Y at C; consequently, of BY and CZ with BA and CA, or of CZ and AX with CB and AB, or of AX and BY with AC and BC; and consequently (329) of BP and CP with BA and CA, or of CP and AP with CB and AB, or of AP and BP with AC and BC; which could be the case only when a position of P coincides with, and when consequently I passes through, one of the three points A, B, C. And, in the latter case, if E, F, G be the three sides of the triangle, and U, V, W the three connectors of P with its opposite vertices A, B, C respectively; then, the anharmonic ratios of the three rows $E.FGLU$, $F.GELV$, $G.EFLW$ being in all cases constant (250, *a'*), and the three pencils determined by U, V, W at A, B, C respectively being, consequently, in all cases homographic with that determined by L at O (329), and therefore with each other (323); in order that they should be in perspective, a pair of corresponding positions should coincide, of V and W along E, or of W and U along F, or of U and V along G; consequently, of FV and GW with FE and GE, or of GW and EU with GF and EF, or of EU and FV with EG and FG; and consequently (329), of FL and GL with FE and GE, or of GL and EL with GF and EF, or of EL and FL with EG and FG; which could be the case only when a position of L coincides with, and when consequently O lies on, one of the three lines E, F, G. That, in the former case, when I passes through any vertex A of the triangle, then L turns round a fixed point X on the opposite side BC, and that, in the latter case, when O lies on any side E of the triangle, then P moves on a fixed line U through the opposite vertex FG, is evident from the constancy of the anharmonic ratio, of the pencil $A.BCPX$ in the former case (250, *a*), and of the row $E.FGLU$ in the latter case (250, *a'*).

Ex. d. *If a variable point P describe a fixed circle, its polar L, with respect to the three sides of any fixed inscribed triangle, for any triad of constant multiples, turns round a fixed point O; the pole, viz., with respect to the three vertices of the triangle, of its axis of perspective I with the corresponding exscribed triangle, for the reciprocal triad of multiples.*

Ex. d'. If a variable line L envelope a fixed circle, its pole P, with respect to the three vertices of any fixed exscribed triangle, for any triad of constant multiples, moves on a fixed line I; the polar, viz., with respect to the three sides of the triangle, of its centre of perspective O with the corresponding inscribed triangle, for the reciprocal triad of multiples.

For, in the former case (employing the reasoning and notation of Ex. *c*) the three rows determined by *X, Y, Z* on *E, F, G* respectively, are homographic with the system determined by *P* on the circle, and therefore with each other; and they are always in perspective, because that, as *P* in the course of its revolution passes successively through every point on the circle, *Y* and *Z* coincide with it and with each other as it passes through *A*, *Z* and *X* with it and with each other as it passes through *B*, *X* and *Y* with it and with each other as it passes through *C*; and therefore &c. And, in the latter case (employing the reasoning and notation of Ex. *c'*) the three pencils determined by *U, V, W* at *A, B, C* respectively, are homographic with the system determined by *L* to the circle, and therefore with each other; and they are always in perspective, because that, as *L* in the course of its revolution passes successively through every tangent to the circle, *V* and *W* coincide with it and with each other as it passes through *E*, *W* and *U* with it and with each other as it passes through *F*, *U* and *V* with it and with each other as it passes through *G*; and therefore &c. That the common centre of perspective *O* in the former case, and the common axis of perspective *I* in the latter case, are the particular point and line stated in the above enunciations respectively; is evident from the general relations of Art. 251, by taking the three particular positions of *L* corresponding to the three passages of *P* through *A, B,* and *C* in the former case, and the three of *P* corresponding to the three of *L* through *E, F,* and *G* in the latter case. And from the properties themselves, thus or in any other manner obtained, inferences exactly analogous to those of Cor. 14°, Art. 261, for the particular cases there established on other principles, may of course be drawn in precisely the same manner.

337. From the same criteria of perspective between homographic rows and pencils, combined with the reciprocal properties of Arts. 293 and 317 respecting arbitrary pairs of corresponding triads, the four following general properties of homographic systems, in pairs reciprocals of each other, may be readily inferred; viz.—

a. For any two homographic rows of points on different axes, or systems of points on the same circle; all pairs of corresponding connectors of pairs of non-corresponding constituents (such as AB' and A'B, AC' and A'C, BC' and B'C, &c.) intersect on the same fixed line O; termed the directive axis of the systems.

d'. For any two homographic pencils of rays through different vertices, or systems of tangents to the same circle; all pairs of corresponding intersections of pairs of non-corresponding constituents (such as AB' and A'B, AC' and A'C, BC' and B'C, &c.) connect through the same fixed point O; termed the directive centre of the systems.

For, firstly, since, for each separate pair of corresponding constituents A and A', the two homographic pencils (or rows) $A.A'B'C'D'E'F'$ &c. and $A'.ABCDEF$ &c., they determine with the opposite systems, being, by the criteria of Art. 334, in perspective; therefore, for each separate pair A and A', the several points of intersection (or lines of connection) of AB' and $A'B$, AC' and $A'C$, AD' and $A'D$, &c., are collinear (or concurrent). And, secondly, since for every three pairs A and A', B and B', C and C', the three points of intersection (or lines of connexion) of BC' and $B'C$, of CA' and $C'A$, and of AB' and $A'B$, being, by the general properties of Arts. 293 and 317, collinear (or concurrent); therefore, for every three A and A', B and B', C and C', the three axes (or centres) of perspective of the three pairs of homographic pencils (or rows) $A.A'B'C'D'E'F'$ &c. and $A'.ABCDEF$ &c., $B.A'B'C'D'E'F'$ &c. and $B'ABCDEF$ &c., $C.A'B'C'D'E'F'$ &c. and $C'.ABCDEF$ &c., coincide. And, as their coincidence for any *two* arbitrary pairs involves evidently their coincidence for *all* pairs, therefore &c.

Given, in any of the four cases, three pairs of corresponding constituents A and A', B and B', C and C' of the two systems, to determine the line (or point) O. The three given pairs of corresponding constituents A and A', B and B', C and C' give at once the three pairs of corresponding connectors (or intersections) of pairs of non-corresponding constituents BC' and $B'C$, CA' and $C'A$, AB' and $A'B$; any two of whose three collinear points of intersection (or concurrent lines of connexion) determine, by the above, the required line (or point) O.

By aid of the line (or point) O, thus determined from three pairs of corresponding constituents A and A', B and B', C and C' of the systems, any number of other pairs D and D', E and E', F and F' may be obtained at pleasure. For, in the former case, two variable lines turning round A and A', or B and B', or C and C', and intersecting in every position on O, determine successively all other pairs D and D', E and E', F and F', &c.; and, in the latter case, two variable points, moving on A and A', or B and B', or C and C', and connecting in every position through O, determine successively all other pairs D and D', E and E', F and F', &c.; (see Arts. 291, Note, and 310, Cor.). Hence the name *directive* as applied, in either case, to O; the line or point so designated being the axis or centre that directs the moment of the two variable lines or points which, giving in every position a pair of corresponding constituents, thus by their variation generate the systems.

If M and N be the two points (or lines) of the systems which lie on (or pass through) O, and I the point (or ray) common to the axes (or vertices) in the case of the two rows (or pencils); then since, in their case, by relations (2) Art. 293,

$$\{ABCM\} = \{A'B'C'I\} \text{ and } \{A'B'C'N\} = \{ABCI\};$$

and since, in the case of the two concyclic systems of points (or tangents), by relations (3) Art. 317,

$$\{ABCM\} = \{A'B'C'M\} \text{ and } \{ABCN\} = \{A'B'C'N\};$$

the two points or lines M and N are, therefore, in the former case, the two constituents of the two systems corresponding to the point (or ray) I common to their axes (or vertices); and, in the latter case, the two *double* points (or lines), as they are termed, of the systems, that is, the two points on (or tangents to) their common circle at each of which a pair of their corresponding constituents coincide (see Art. 329).

When, in any of the four cases, the two systems are in perspective; their directive axis (or centre) O is then evidently (240 and 261) the polar of the centre (or the pole of the axis) of perspective, with respect to the angle (or segment) determined by the two axes (or vertices) in the case of the two rows (or pencils), or with respect to the common circle in the case of the two concyclic systems of points (or tangents). In the former case, therefore, the two points (or rays) M and N coincide, as they ought (334), with the point (or ray) I common to the two axes (or vertices).

338. Again, from the same criteria of perspective, combined with the particular case of homographic division considered in

Art. 329, and with the reciprocal properties of Arts. 240 and 261 respecting poles and polars to angles or segments and to circles, the four following general properties of homographic systems, in pairs reciprocals of each other, may as readily be inferred; viz.—

a. A variable line, determining two homographic rows of points on different axes, or systems of points on a common circle—

1°. Intersects with any four positions of itself at a quartet of points equianharmonic with the two corresponding quartets of the generated systems.

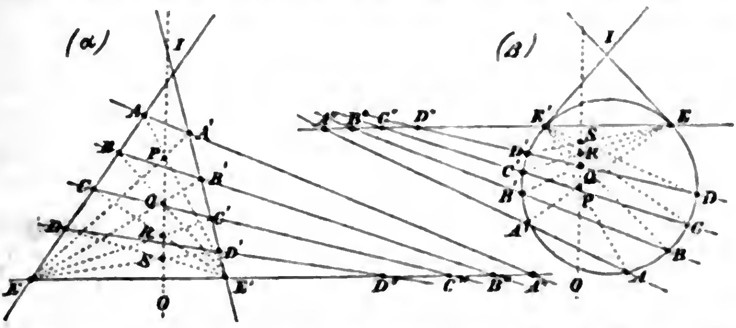
2°. Determines on all positions of itself rows of points homographic with each other and with the original systems.

a'. A variable point, determining two homographic pencils of rays at different vertices, or systems of tangents to a common circle—

1°. Connects with any four positions of itself by a quartet of rays equianharmonic with the two corresponding quartets of the generated systems.

2°. Determines at all positions of itself pencils of rays homographic with each other and with the original systems.

For, in the former case, if A and A', B and B', C and C', D and D' be the four pairs of corresponding constituents determined on the axes (fig. α) or circle (fig. β) by any four positions

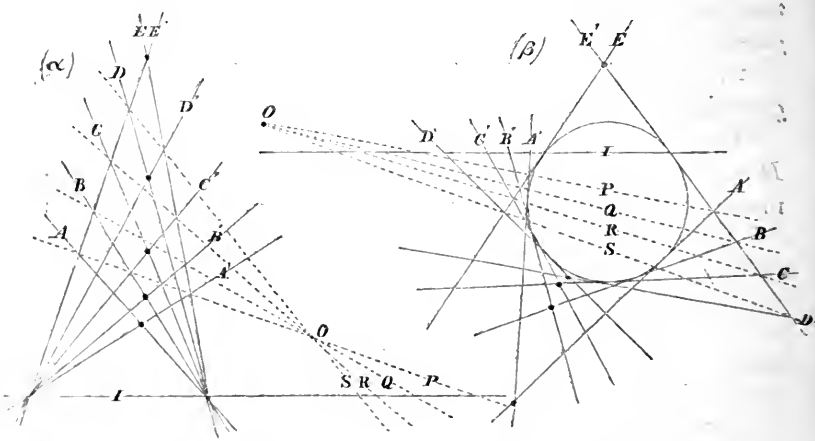


AA', BB', CC', DD' of the variable line; E and E' the fifth pair corresponding to any fifth positions EE'; A'', B'', C'', D'' the four intersections of the four positions with the fifth; I the intersection of the axes (fig. α) or of the tangents at E and E' to the circle (fig. β); and P, Q, R, S the four intersections of the four pairs of connectors EA' and $E'A$, EB' and $E'B$,

EC' and $E'C$, ED' and $E'D$, which, by the preceding article, are collinear, and lie on the directive axis O of the systems; then, since, by Arts. 240 and 261, the four lines IP, IQ, IR, IS are the four polars of the four points A'', B'', C'', D'' with respect to the axes (fig. α) or circle (fig. β); and since, consequently, in either case, the four pairs of lines IA'' and IP, IB'' and IQ, IC'' and IR, ID'' and IS are pairs of harmonic conjugates with respect to the two IE and IE'; therefore, in either case, by Art. 329, $\{I.A''B''C''D''\} = \{I.PQRS\}$; and therefore, in either, by Art. 285,

$$\{A''B''C''D''\} = \{PQRS\} = \{ABCD\} = \{A'B'C'D'\};$$

which being true for the intersections of every four with all fifth positions of the variable line, therefore &c. And, in the latter case, if A and A', B and B', C and C', D and D' be the



four pairs of corresponding constituents determined at the vertices (fig. α') or to the circle (fig. β') by any four positions AA', BB', CC', DD' of the variable point; E and E' the fifth pair corresponding to any fifth position EE'; A'', B'', C'', D'' the four connectors (not drawn in the figures) of the four positions with the fifth; I the connector of the vertices (fig. α') or of the points of contact of E and E' with the circle (fig. β'); and P, Q, R, S the four connectors of the four pairs of intersections EA' and $E'A$, EB' and $E'B$, EC' and $E'C$, ED' and $E'D$, which, by the preceding article, are concurrent, and pass through the directive centre O of the systems; then, since, by Arts. 240 and 261, the four points IP, IQ, IR, IS are the four poles of the four lines A'', B'', C'', D'' with respect to the vertices (fig. α') or circle (fig. β'); and since, consequently, in either case, the four pairs of points IA'' and IP, IB'' and IQ, IC'' and IR, ID'' and IS are pairs of harmonic conjugates with respect to the two IE and IE'; therefore, in either case, by Art. 329,

$$\{I.A''B''C''D''\} = \{I.PQRS\};$$

and therefore, in either, by Art. 285,

$$\{A''B''C''D''\} = \{PQRS\} = \{ABCD\} = \{A'B'C'D'\};$$

which being true for the connectors of every four with all fifth positions of the variable point, therefore &c.

The above reciprocal demonstrations (which it may be observed would be simplified for the concyclic systems in both cases by the general property of Art. 292) may be briefly summed up in one as follows. Since, in all four cases alike, by Arts. 240 and 261, IA'' and IP, IB'' and IQ, IC'' and IR, ID'' and IS are pairs of harmonic conjugates with respect to IE and IE'; therefore, in all four alike, by Art. 329,

$$\{I.A''B''C''D''\} = \{I.PQRS\};$$

and therefore, in all alike, by Art. 285,

$$\{A''B''C''D''\} = \{PQRS\} = \{ABCD\} = \{A'B'C'D'\};$$

which, for all alike, establishes at once the two properties in question.

Cor. 1°. It follows of course immediately, from the first parts of the above reciprocal properties, that *when a variable line, which does not turn round a fixed point, determines homographic rows on any two fixed lines, it coincides, once in the course of its entire variation, with each of the lines*; and, reciprocally, that *when a variable point, which does not move on a fixed line, determines homographic pencils at any two fixed points, it coincides, once in the course of its entire variation, with each of the points*. Which are also evident, *à priori*, from the consideration that, when the variable line (or point) passes, in the course of its variation, through the intersection of the fixed lines (or over the connector of the fixed points), if the corresponding constituents of the two homographic systems do not then coincide, as they do not when the systems are not in perspective (334), it must itself necessarily coincide with one or other fixed line (or point).

COR. 2°. It follows also immediately, from the same, that *a variable line, determining homographic rows of points on any two fixed lines, intersects with every four fixed positions of itself at a variable quartet of points having a constant anharmonic ratio, and determines with every five positions of itself an equianharmonic hexagram* (301); and, reciprocally, that *a variable point, determining homographic pencils of rays at any two fixed points, connects with every four fixed positions of itself by a variable quartet of rays having a constant anharmonic ratio, and determines with every five positions of itself an equianharmonic hexastigm* (301). These are the general properties of homographic rows and pencils, whose converses were given in examples e and e', g and g' of Art. 333.

339. As every two homographic pencils of rays through any vertices determine on every axis two homographic rows of points whose constituents at infinity correspond to those of the determining pencils to whose directions the axis is parallel (16); it follows, from the criterion of similitude between homographic rows given in Art. 330, that—

When two homographic pencils of rays through any vertices have a pair of corresponding constituents whose directions are parallel; they determine on every axis parallel to those directions two similar rows; whose ratio of similitude, evidently constant both in magnitude and sign when the two constituents coincide, varies when they do not with every position of the axis; changes sign, passing through 0, *as it passes in either direction through the vertex of the pencil of antecedents; again changes sign, passing through* ∞, *as it passes in either direction through the vertex of the pencil of consequents; and passes without change of sign through every intermediate absolute magnitude, in continuous increase from* 0 *to* ∞ *during its passage in either direction from the former to the latter, and in continuous decrease from* ∞ *to* 0 *during its passage in either direction from the latter to the former.*

Given the parallel pair of corresponding constituents, I and I', and any other two pairs, A and A', B and B', of the two pencils, the particular axis L parallel to I and I' for which the ratio of similitude shall have any given value, positive or negative, may be readily determined as follows: Denoting by O and O' the vertices of the two pencils, by X and X', Y and Y' the four intersections of L with A and A', B and B' respectively, and by Z its intersection with the line OO'; then, the ratios of OZ and $O'Z$ to XY and $X'Y'$ respectively being evidently given with the direction of L, when the ratio of XY to $X'Y'$, which is that of the similitude of the systems, is also given, that of $OZ : O'Z$, and with it of course the position of Z, is consequently given; and therefore &c. The particular cases when the given ratio has, as of course it may have, the particular values ± 1 (and when consequently the several segments intercepted on L by the several pairs of corresponding rays A and A', B and B', C and C', D and D', &c. of the two pencils, have a common magnitude or middle point) differ in no respect from the general case when it has any value positive or negative.

Since, for every two homographic pencils of rays through any vertices, there exist, as will be shewn in the next chapter, two pairs of corresponding constituents, real or imaginary, whose directions are parallel; it follows consequently, from the above, that—

For every two homographic pencils of rays through any vertices, there exist two directions, real or imaginary, on all lines parallel to either of which they determine similar rows of points; and, of the two systems of parallels determined by those directions, two particular lines on each of which their several pairs of corresponding rays intercept equal segments, and two others on each of which the several intercepted segments have a common middle point.

In the particular case when the two pencils are in perspective, it is evident, from Art. 334, or independently, that the two directions in question are parallel, one to the connector of their vertices and the other to their axis of perspective; on the former of which lines the several pairs of corresponding rays intercept evidently a common segment, and on the latter of which the several segments they intercept are of course all evanescent.

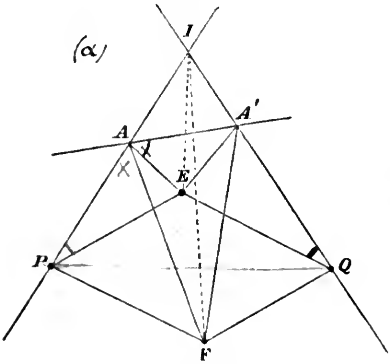
340. As, from the general property of Art. 330, it was shewn, at the close of the preceding article, that—

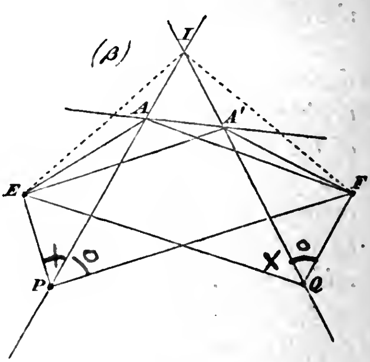
a. For every two homographic pencils of rays through different *vertices, there exist two lines, real or imaginary, on each of which the several pairs of corresponding rays intercept equal segments; and also two others, real or imaginary, for each of which the several intercepted segments have a common middle point.*

So, from the general property of Art. 331, it may be shewn that, reciprocally—

a'. For every two homographic rows of points on different axes, there exist two points, always real, at each of which the several pairs of corresponding points subtend equal angles; and also two others, sometimes imaginary, for each of which the several subtended angles have a common middle ray.

For, if I (figs. α and β) be the intersection of the axes; P and Q the points of their rows whose correspondents are at infinity; A and A' a variable pair of corresponding points of the systems; and E and F the two fixed points, real or imaginary, for which the two rectangles $PE.QE$ and $PF.QF$ are equal to the constant rectangle $PA.QA'$ (331), and for which the two pairs of angles IPE and IQE, IPF and IQF, measured in opposite directions of rotation (fig. α) and in the same direction of rotation (fig. β),





are equal; then, since, from the evident similarity in either case of the two pairs of triangles APE and EQA', APF and FQA' in every position of A and A', the two pairs of angles PAE and QEA', PAF and QFA' (or the two pairs PEA and $QA'E$, PFA and $QA'F$) are always equal; therefore, as A and A' vary, the two pairs of lines EA and EA', FA and FA' revolve always through equal angles, in the same direction of rotation (fig. α), and in opposite directions of rotation (fig. β), round the two fixed points E and F; and therefore &c.*

Of the two (evidently similar and equal) triangles PEQ and PFQ, whose two vertices E and F are the two points involved in the above properties, and which combined form evidently a parallelogram in the former case (fig. α); the common base PQ, the rectangle under the sides, and the difference (fig. α) or sum (fig. β) of the base angles, being known, the triangles are consequently completely determined; and, while evidently always real in the former case, are imaginary in the latter when the rectangle $PE.QE$ or $PF.QF$ is greater than for any point on the known circle PIQ, which in that case evidently passes always through E and F. In the particular case when the two rows are in perspective, like the analogous case in the preceding article, it is evident, from Art. 334, or independently, that, of the two points E and F in the former property, one coincides with the intersection of the axes and the other with the centre of perspective; the constant angle for the former being evidently the fixed angle between the axes, and for the latter being of course evanescent.

COR. 1°. The three pairs of lines EA and FA, EA' and FA', EI and FI, in (fig. α), being evidently equally inclined to the bisectors of the three corresponding angles of the triangle AIA'; and the three rectangles under the three pairs of perpendiculars from E and F upon the three sides of that triangle being consequently equal in magnitude and sign; hence, from the above, it appears that—

a. When a variable line intersects with two fixed lines homographically, the rectangle under its distances from the two fixed points, at which the several pairs of corresponding intersections subtend constant angles, is constant both in magnitude and sign.

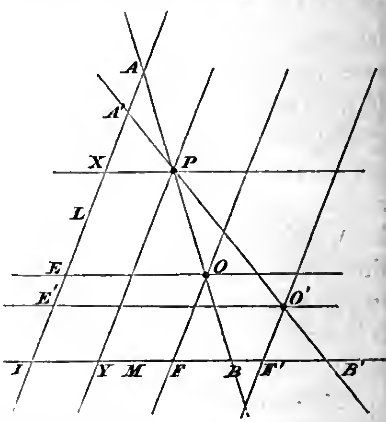
And from the analogous property of the preceding article, respecting homographic pencils of rays, it may be shown that reciprocally—

a'. When a variable point connects with two fixed points homographically, the rectangle under its distances from the two

* The above demonstration was communicated to the author by Mr. Casey.

fixed lines, on which the several pairs of corresponding connectors intercept constant segments, is constant both in magnitude and sign.

For, if O and O' be the two fixed points; P any position of the variable point; OE and $O'E'$, OF and $O'F'$ the two pairs of corresponding rays, of the two homographic pencils, whose directions are parallel; L and M the two parallels to them, intersecting at I, on each of which the several pairs of corresponding rays intercept constant segments, and on which consequently PO and PO' intercept segments AA' and BB' which are equal to EE' and FF' respectively; then, PX and PY being parallels through P to L and M, since, by pairs of similar triangles,



$$AX : PX = OF : BF \text{ and } A'X : PX = O'F' : B'F',$$

and since $AX - A'X = OF - O'F'$, and $BF = B'F'$, therefore $AX = OF$ and $A'X = O'F'$; from which, since $AP = OB$ and $A'P = O'B'$, therefore

$$PX.PY = OE.OF = O'E'.O'F', \text{ or, } PL.PM = OL.OM = O'L.O'M;$$

and therefore &c.

COR. 2°. Since, by (338, a and a', 2°), a variable line (or point), intersecting (or connecting) homographically with two fixed lines (or points), intersects (or connects) homographically with all positions of itself; it follows, consequently, from the two reciprocal properties a and a' of the preceding, Cor. 1°, that—

a. A variable line, the rectangle under whose distances from two fixed points is constant in magnitude and sign, intersects homographically with all positions of itself.

a'. A variable point, the rectangle under whose distances from two fixed lines is constant in magnitude and sign, connects homographically with all positions of itself.

COR. 3°. If X and Y be the feet of the perpendiculars from E and F on AA' in (fig. a), Z the intersection with AA' of the position BB' which intersects it at right angles, and O the middle point of EF; then since (49)

$$OX^2 = OY^2 = EX.FY + (\tfrac{1}{2}EF)^2,$$

and since (Euc. I. 48, and II. 5, 6),

$$OZ^2 = 2EX.FY + (\tfrac{1}{2}EF)^2;$$

it follows again, consequently, from a Cor. 1°, that—

When a variable line intersects homographically with two fixed lines—

a. In every position, the feet of the two perpendiculars on it, from the two fixed points for which their rectangle is constant, lie on a fixed circle; whose centre bisects the interval between the points, and the square of whose radius = the constant rectangle + the square of the semi-interval.

b. Every two of its positions which intersect at right angles intersect on another fixed circle; concentric with the former, the square of whose radius = twice the same constant rectangle + the square of the same semi-interval.

N.B. The several properties of homographic divisions contained in this and the two preceding articles are of very familiar occurrence in the Theory of Conic Sections, where alone indeed the subject has scope for the full and adequate developement due to its importance.

CHAPTER XX.

ON THE DOUBLE POINTS AND LINES OF HOMOGRAPHIC SYSTEMS.

341. WHEN the axes of two homographic rows of points or the vertices of two homographic pencils of rays coincide, there exist always two pairs of corresponding constituents, real or imaginary, whose positions coincide, and which accordingly have been termed by Chasles *the double points or rays of the systems* (329).

Assuming for the present the existence of such points or rays, the following properties are evident from their mere definition.

1°. *No more than two double points or rays could exist unless the systems altogether coincided.* For, since, for every four pairs of corresponding constituents A and A', B and B', C and C', D and D', the systems being homographic, $\{ABCD\}=\{A'B'C'D'\}$, if $A=A'$, and $B=B'$, and $C=C'$, then necessarily $D=D'$; and, as for the same reason $E=E'$, $F=F'$, $G=G'$, $H=H'$, &c., therefore &c.

2°. *Every three pairs of corresponding constituents A and A', B and B', C and C' are connected with the two double points or rays $M\,(=M')$ and $N\,(=N')$ taken separately by the relations $\{ABCM\}=\{A'B'C'M\}$ and $\{ABCN\}=\{A'B'C'N\}$; and every two pairs A and A', B and B' with both combined by the relation $\{MNAB\}=\{MNA'B'\}$.* These relations, which are evident from the homography of the systems and the hypothesis respecting M and N, are characteristic of the double points or rays, and sufficient in all cases to identify and distinguish them.

3°. Each double point or ray being of course equivalent to a pair of corresponding constituents, and three pairs of corresponding constituents being sufficient to determine any two homographic systems (327); one double point or ray with two pairs of corresponding constituents, or both double points or rays with a single pair of corresponding constituents, are therefore sufficient to determine two homographic rows or pencils whose axes or vertices coincide.

4°. For the same reason, three pairs of corresponding constituents, given, taken, or known in any manner, being sufficient in all cases to determine every thing connected with the two homographic systems to which they belong (327), are therefore sufficient to determine the two double points or rays of two homographic rows or pencils whose axes or vertices coincide. (See Art. 348).

5°. As two homographic rows of points on any axis determine two homographic pencils of rays at any vertex, and conversely, the two double points of one correspond always to the two double rays of the other, and conversely.

342. *Every two corresponding constituents of two homographic rows or pencils, whose axes or vertices coincide, divide in the same constant anharmonic ratio the segment or angle determined by the two double points or rays of the systems.*

For, since, for every two pairs of corresponding constituents A and A', B and B', by 2° of the preceding article,

$$\{MNAB\} = \{MNA'B'\},$$

therefore, by (272),

$$\{MNAA'\} = \{MNBB'\};$$

and, since, for the same reason,

$$\{MNBB'\} = \{MNCC'\}, \quad \{MNCC'\} = \{MNDD'\},$$

&c.; therefore &c.

The particular case when the constant anharmonic ratio of section $= -1$, that is, when the several pairs of corresponding constituents A and A', B and B', C and C', D and D', &c. divide harmonically the segment or angle MN, will be considered in connexion with the Theory of Involution in the next Chapter.

343. *Every two constituents of the two systems, corresponding to the same point on the common axis or ray through the common*

vertex, divide in the square of the above constant anharmonic ratio the segment or angle determined by the two double points or rays of the systems.

For, every point on the common axis or ray through the common vertex belonging of course indifferently to both systems, if P be the correspondent of any point or ray I regarded as belonging to one system, and Q the correspondent of the same point or ray I regarded as belonging to the other system, then since, by the preceding article,

$$\left(\frac{MP}{NP}:\frac{MI}{NI}\right) \text{ or } \left(\frac{\sin MP}{\sin NP}:\frac{\sin MI}{\sin NI}\right) = \text{const.} \ldots\ldots\ldots (1),$$

and

$$\left(\frac{MI}{NI}:\frac{MQ}{NQ}\right) \text{ or } \left(\frac{\sin MI}{\sin NI}:\frac{\sin MQ}{\sin NQ}\right) = \text{const.} \ldots\ldots\ldots (2),$$

therefore at once, by composition of ratios,

$$\left(\frac{MP}{NP}:\frac{MQ}{NQ}\right) \text{ or } \left(\frac{\sin MP}{\sin NP}:\frac{\sin MQ}{\sin NQ}\right) = \text{const}^2. \ldots\ldots (3),$$

and therefore &c. (268).

The particular case where this constant anharmonic ratio of section $= 1$, that is, when the two constituents of the two systems corresponding to the same point on their common axis or ray through their common vertex always coincide, will also be considered in reference to the Theory of Involution in the next Chapter.

Cor. 1°. When, in the above, the point or line I is either point or line of bisection, external or internal, of the segment or angle MN, then—

The segment or angle PQ has the same points or lines of bisection as the segment or angle MN.

For, since, in that case, $(MI : NI)$ or $(\sin MI : \sin NI) = \pm 1$, therefore, in the same case, by (1) and (2) above,

$(MP : NP).(MQ : NQ)$ or $(\sin MP : \sin NP).(\sin MQ : \sin NQ) = +1$;

and therefore &c.

Cor. 2°. The point of external bisection of every segment of a line being the point at infinity on the line, it follows immediately, from the preceding Cor. 1°, that—

In the case of two homographic rows of points on a common *axis, the segment MN intercepted between the two double points is concentric with the segment PQ intercepted between the two constituents of the systems corresponding to the point at infinity on the axis.*

A consequence which, by virtue of the general property of Art. 331, may also be proved otherwise as follows: Since, for every two pairs of corresponding constituents A and A', B and B' of the systems, by the property in question, $PA.QA' = PB.QB'$, when $A = A' = M$, and $B = B' = N$, then $PM.QM = PN.QN$; and therefore &c. (Euc. II. 5, 6).

COR. 8°. If, in the above, the point or line I be conceived to vary, causing of course the simultaneous variation of the two P and Q, then—

The two points or rays P and Q determine two homographic rows or pencils having the same double points or rays M and N with the original systems.

This follows immediately from relation (3) by virtue of the general property of Art. 329; and the same is evident also from the consideration that the systems determined by P and Q are both homographic with that determined by I, with which, combined separately, they constitute in fact the original systems.

N.B. From the general, or any derived, property of either this or the preceding article, it is evident that every two homographic rows of points or pencils of rays, whose axes or vertices coincide, are symmetrically disposed on opposite sides of each point or line of bisection, external and internal, of the segment or angle MN determined by the two double points or rays M and N of the systems; which two points or lines of bisection, always real, are therefore *the two points or lines of symmetry* of the systems.

344. *Of two homographic rows of points on a common axis; when one double point is at infinity, the rows are similar, and have the other double point for their centre of similitude; and, when both double points are at infinity, the rows are similar, similarly placed, and equal.*

For, since, by (342), whatever be the positions of the two double points, $\{MNAA'\} = \{MNBB'\} = \{MNCC'\} = \{MNDD'\} =$ &c.

= a constant; when one of them N is at infinity, then, for the other M, by (275),

$$MA : MA' = MB : MB' = MC : MC' = MD : MD' = \&c.$$

= a constant; and when the other of them M is also at infinity, then, by (15), the constant = + 1; and therefore &c. (41).

In the particular case when the constant = − 1, then

$$MA = -MA',\ MB = -MB',\ MC = -MC',\ MD = -MD',\ \&c.$$

and the systems, as in the preceding case, are similar and equal, but are oppositely, in place of similarity, placed on the axis (33). In this case, also, the several segments AA', BB', CC', DD', &c., intercepted between the several pairs of corresponding constituents of the rows, are evidently concentric, being all bisected internally by the double point M not at infinity.

The converse of the above, viz., that, *when two homographic rows of points on a common axis are similar, then, whatever be the magnitude and sign of their ratio of similitude, provided only it be finite, one double point is their centre of similitude and the other the point at infinity on their axis*, is evident from the constancy of the ratio $MA : MA'$, from which it follows at once that the two variable distances MA and MA' become necessarily evanescent and infinite together.

In the solitary and exceptional case of entire coincidence between two similar, similarly placed, and equal rows of points on a common axis, every point on the axis is of course indifferently a double point. (See 341, 1°).

345. From the general property of the preceding article, the two following results may be immediately inferred, viz.—

1°. *Any two homographic rows of points on a common axis, whose double points are real, may be regarded as the perspective, to any arbitrary centre, of two similar rows on a common axis depending in direction on the position of the centre.*

2°. *Any two homographic rows of points on a common axis, whose double points coincide, may be regarded as the perspective, to any arbitrary centre, of the two similar, similarly placed, and equal rows on a common axis depending in direction on the position of the centre.*

For, as the two homographic rows of points on the common axis determine in all cases two homographic pencils of rays at every vertex, whose double rays correspond to the double points on the axis (341, 5°); any axis parallel to the direction of either double ray if they be distinct, or to the common direction of the two if they coincide, would intersect the pencils in two homographic rows of points, having one double point at infinity in the former case, and both double points at infinity in the latter case; and therefore &c. (344).

346. The two following general properties of two homographic rows or pencils, whose double points or rays are imaginary, have been given by Chasles, viz.—

1°. *Any two homographic rows of points on a common axis, whose double points are imaginary, may be regarded as generated by the revolution of a variable angle of constant magnitude round one or other of two fixed vertices, reflexions of each other with respect to the axis.*

2°. *Any two homographic pencils of rays through a common vertex, whose double rays are imaginary, may be regarded as the perspective to any arbitrary axis of a pencil generated by the revolution of a variable angle of constant magnitude round a fixed vertex.*

To prove the first of these properties (which evidently involves the second), it is only necessary to shew that, under the circumstances of the case, a real point E (and with it, of course, its reflexion F with respect to the axis) can always be found, at which some three of the segments AA', BB', CC' intercepted between pairs of corresponding points shall subtend equal angles; for, if three of them subtend equal angles at any point, it follows necessarily, from the homography of the rows, that they must all subtend equal angles at the same point. And that two such points, reflexions of each other with respect to the axis, exist always in this case, is evident from Art. 161; for, the three circles, loci of points at which the three pairs of segments BB' and CC', CC' and AA', AA' and BB' subtend equal angles, are then (see the article in question) all real, and intersect at two real points E and F, which (the centres of the three circles being all on the axis) are of course reflexions of each other with respect to the axis.

The second property follows at once, as above observed, from the first; for, as the two homographic pencils, whose double rays are by hypothesis imaginary, intersect with every axis in two homographic rows whose double points are imaginary; and, as there always exist, by the above 1°, two real points E and F, with respect to each of which the latter may be regarded as generated by the revolution of a variable angle of constant magnitude revolving round it as a fixed vertex; therefore &c.

347. The entire preceding theory applies of course, in its main features, as well to two homographic systems of points on a common circle or of tangents to a common circle (322), as to two systems of points on a common axis or of rays through a common vertex; and every two such systems have accordingly, for every as well as for either limiting magnitude of the common circle, two pairs of corresponding constituents, real or imaginary, whose positions coincide, and which are therefore termed *the double points or tangents of the systems.*

That every two corresponding constituents A and A' of the systems divide in the same constant anharmonic ratio the arc of the common circle intercepted between the two double points or tangents M and N; that every two constituents P and Q of the systems corresponding to the same point or tangent I divide in the square of the same constant anharmonic ratio the same intercepted arc; that, as I varies, P and Q determine two homographic systems having the same double points or tangents with the original systems; that when I is equidistant from or equi-inclined to M and N then is it also equidistant from or equi-inclined to P and Q; and that the systems themselves are always symmetrically disposed on opposite sides of each of the two points or tangents equidistant from or equi-inclined to M and N, which two points or tangents are therefore *the two points or lines of symmetry of the systems*; appear all in precisely the same manner as for the two extreme states of the circle in Arts. 342 and 343.

As every two homographic systems of points on a common circle determine two homographic pencils of rays at any point on the circle, and conversely; and as every two homographic systems of tangents to a common circle determine two homographic rows of points on any tangent to the circle, and conversely; it is evident that the double points of one correspond always to the double rays of the other, and conversely, in the former case; and that the double tangents of one correspond always to the double points of the other, and conversely, in the latter case.

348. *Given three pairs of corresponding constituents, A and A′, B and B′, C and C′, of two homographic systems, of points on a common axis, or of rays through a common vertex, or of points on a common circle, or of tangents to a common circle; to construct the two double points or lines, M and N, of the systems.*

1°. In the case of points on a common circle. Drawing any two of the three pairs of corresponding connectors of pairs of non-corresponding constituents, *BC′* and *B′C*, *CA′* and *C′A*, *AB′* and *A′B*, (see fig. *a*, Art. 317); the line of connection *XY* of their two points of intersection *X* and *Y* (which by 337, *a*, is the directive axis of the systems) will pass through the intersection *Z* of the third pair (317), and will determine on the circle two points *M* and *N*, real or imaginary, which satisfy (317) the equianharmonic relations of 2°, Art. 341, and which are consequently the two double points of the systems.

2°. In the case of tangents to a common circle. Taking any two of the three pairs of corresponding intersections of pairs of non-corresponding constituents, *BC′* and *B′C*, *CA′* and *C′A*, *AB′* and *A′B*, (see fig. *a′*, Art. 317); the point of intersection *XY* of their two lines of connection *X* and *Y* (which, by 337, *a′*, is the directive centre of the systems) will lie on the connector *Z* of the third pair (317), and will determine to the circle two tangents *M* and *N*, real or imaginary, which satisfy (317) the equianharmonic relations of 2°, Art. 341, and which are consequently the two double lines of the systems.

3°. In the case of rays through a common vertex. Describing arbitrarily any circle passing through the common vertex, and taking on it its three pairs of second intersections with the three pairs of rays; the two double points, found by 1°, of the two homographic systems determined by the latter on the circle will connect with the common vertex by the required double rays (347).

4°. In the case of points on a common axis. Describing arbitrarily any circle touching the common axis, and drawing to it its three pairs of second tangents through the three pairs of points; the two double lines, found by 2°, of the two homographic systems determined by the latter to the circle will intersect with the common axis at the required double points (347).

These several constructions are all perfectly general, and applicable with equal facility to every variety of disposition of the three given pairs of corresponding constituents. Various direct constructions may also be given for the extreme cases of collinear and concurrent systems (4° and 3°) without reducing them, as above, to the general cases of concyclic systems (2° and 1°); but the above, though indirect, are on the whole the simplest of which they are susceptible.

349. The following construction for determining directly the two double points, in the case of collinear systems on a common axis, has been given by Chasles.

Assuming arbitrarily any point O not on the common axis, and describing through it any two of the three pairs of corresponding circles passing through pairs of non-corresponding constituents, BOC' and $B'OC$, COA' and $C'OA$, AOB' and $A'OB$, which intersect again respectively at three second points P, Q, R; the circle passing through O, and through the second intersections P and Q of the two described pairs, will pass through the second intersection R of the third pair, and will intersect with the common axis at the two double points M and N of the systems.

For, since the three circles BOC', $B'OC$, and MON pass through the two common points O and P, therefore, by similar pencils at O and P, $\{MNBB'\} = \{MNCC'\}$; and since the three COA', $C'OA$, and MON pass through the two common points O and Q, therefore, by similar pencils at O and Q, $\{MNCC'\} = \{MNAA'\}$; consequently at once $\{MNAA'\} = \{MNBB'\}$; and therefore, by similar pencils at O and R, the three circles AOB', $A'OB$, and MON pass through the two common points O and R; and since, as just shewn,

$$\{MNAA'\} = \{MNBB'\} = \{MNCC'\},$$

therefore (342) M and N are the two double points of the systems.

N.B. It will appear in the sequel that this construction is the transformation, by inversion from any point on the circle, of that given in 1° of the preceding article for concyclic systems on a common circle.

350. The following again, derived from the general property of Art. 332, is another construction for the direct determination of the two double points in the same case of collinear systems on a common axis.

Taking the two points P and Q corresponding in the two systems to the point at infinity on the common axis, and dividing their intercepted interval PQ at the two points M and N for which the two rectangles $PM.QM$ and $PN.QN$ are each equal in magnitude and sign to the common value of the three equal rectangles $PA.QA'$, $PB.QB'$, $PC.QC'$ (332); the two points of section M and N are evidently the required double points. See also Cor. 2°, Art. 343.

In any case of the construction of the double points or lines of two homographic systems by means of three pairs of corresponding constituents; if the two constituents A and A' of any pair happened to coincide, the point or line A $(= A')$ would itself be one of the required double points or lines, and the general construction for the other would be much simplified; and if, moreover, the two B and B' of either remaining pair happened also to coincide, the point or line B $(= B')$ would itself be the other, and all construction would be dispensed with. In this last case, the third pair C and C' would furnish the value of the constant anharmonic ratio $\{ABCC'\}$ distinctive of the particular pair of homographic systems determined by the three pairs of corresponding constituents (342).

351. The general constructions 1° and 2° of Art. 348, for the double points and lines in the cases of concyclic systems of points and tangents, lead each to a remarkable result when applied to the particular case of similar and similarly ranged systems separated from each other by an interval of any finite magnitude.

For, in that case, the several arcs AA', BB', CC', DD', &c., intercepted between the several pairs of corresponding constituents A and A', B and B', C and C', D and D', &c., being all equal and cyclically co-directional, the three points (or lines) X, Y, Z, and with them of course the two M and N, determined by the construction in question (1° or 2°, Art. 348), consequently (see Art. 312) lie on the line at infinity (or pass through the centre of the circle); which in that case is accordingly the directive axis (or centre) of the systems (337, a or a'). Hence the remarkable results, that—

a. Every two similar and similarly ranged systems of points on a common circle have the same two (imaginary) double points, whatever be their interval of separation from each other; viz. the fixed two lying on the line at infinity (260, 2°, b).

a'. Every two similar and similarly ranged systems of tangents to a common circle have the same two (imaginary) double lines, whatever be their interval of separation from each other; viz. the fixed two passing through the centre of the circle (260, 2°, b').

In the special case when the interval of separation between the systems is nothing, that is, when the systems altogether coincide; since then $A=A'$, $B=B'$, $C=C'$, $D=D'$, &c., the three points (or lines) X, Y, Z, and with them of course the two M and N, determined by the same constructions, become, as they ought, indeterminate; every point on (or tangent to) the common circle being then of course indifferently a double point (or line) of the systems.

352. As the two homographic systems of rays, generated by the sides of a variable angle of any invariable form revolving round a fixed vertex, determine two similar and similarly ranged systems of points on any circle passing through the fixed vertex, the double points of which correspond to the double rays of the determining pencils; hence, from property a of the preceding article, the remarkable result, that—

Every two homographic pencils of rays, determined by the sides of a variable angle of invariable form revolving round a fixed vertex, have the same (imaginary) double rays, whatever be the form of the angle; the connectors, viz. of the fixed vertex with the two fixed circular points at infinity.

A result from which it follows at once, by Art. 342, as shewn already on other principles in Art. 312, that the sides of a variable angle of invariable form revolving round a fixed vertex divide in a constant anharmonic ratio the angle subtended at the fixed vertex by the two fixed circular points at infinity; the value of the anharmonic ratio of section depending of course on the particular figure of the angle. A property which, established independently as in the article referred to or otherwise, involves evidently the above conversely, by virtue of Art. 329.

353. There is probably in the entire range of modern geometry no problem to some case or other of which a greater number and variety of others, admitting of two solutions, are reducible than that of the construction of the double points or lines of two homographic systems by means of three pairs of corresponding constituents; some connected directly with the subject of homographic division, but far the greater number having no apparent connexion with it. Of the former class, the following are a few, the applications of which are extremely numerous and varied.

Ex. 1°. *Given three pairs of corresponding constituents A and A', B and B', C and C' of two homographic systems of points on a common line or circle, or of tangents to a common point or circle; to determine the pair M and M' for which $AM = \pm A'M'$.*

Taking the three points or lines A'', B'', C'' connected with the given point or line A by the relations $AA'' = 0$, $AB'' = \pm A'B'$, $AC'' = \pm A'C'$; and constructing the second double point or line $M = M''$ of the two homographic systems determined by the three pairs of corresponding constituents A and A'', B and B'', C and C''; the point or line M is that which in the system A, B, C, &c. is connected with its correspondent M' in the system A', B', C', &c. by the required relation $AM = \pm A'M'$.

For, since, by construction, $A''B'' = \pm A'B'$ and $A''C'' = \pm A'C'$, the two homographic systems A'', B'', C'', &c. and A', B', C', &c. are similar, and their ratio of similitude $= \pm 1$; therefore $A''M''$ or $AM = \pm A'M'$; and therefore &c.

N.B. In the case of points on a common axis, the more general problem "Given three pairs of corresponding constituents A and A', B and B', C and C', to determine the pair M and M' for which the ratio $AM : A'M'$ shall have any given magnitude and sign," may evidently be solved in precisely the same manner.

Ex. 2°. *Given three pairs of corresponding constituents A and A', B and B', C and C' of two homographic systems of points on a common line or circle, or of tangents to a common point or circle; to determine the two pairs M and M', N and N' whose intercepted segments or angles MM', NN' shall have a given magnitude and sign.*

Taking the three points or lines A'', B'', C'' connected with the given three A, B, C by the common relation $AA'' = BB'' = CC''$ = the given segment or angle; constructing then the two double points or lines $M' = M''$ and $N' = N''$ of the two homographic systems determined by the three pairs of corresponding constituents A' and A'', B' and B'', C' and C''; and taking finally the two points or lines M and N connected with the two M' and N' by the common relation $MM' = NN'$ = the given segment or angle; the two pairs of constituents M and M', N and N', are those required.

For, since, by construction, $AA'' = BB'' = CC'' = MM'' = NN''$ = the given segment or angle, therefore $\{ABCMN\} = \{A''B''C''M''N''\}$; and since again, by construction, $M' = M''$ and $N' = N''$ are the two double points or lines of the two homographic systems A', B', C', &c. and A'', B'', C'', &c., therefore $\{A'B'C'M'N'\} = \{A''B''C''M''N''\}$; consequently therefore $\{ABCMN\} = \{A'B'C'M'N'\}$; or, the two pairs of constituents M and M', N and N', which by construction intercept the required segment or angle, are pairs of corresponding constituents of the two homographic systems A, B, C, &c. and A', B', C', &c.; and therefore &c.

N.B. The constructions in the present and preceding examples are both based on the obvious consideration that when two homographic systems of points, rays, or tangents have a common axis, vertex, or circle, a movement of either along the common axis, or round the common vertex or circle, the other remaining fixed, would alter (increase or diminish as the case might be) the distances between the several pairs of corresponding constituents by the amount of the movement; so that those correspondents which coincided before would be separated after by that amount, and conversely.

Ex. 3°. *Given three pairs of corresponding constituents A and A', B and B', C and C' of two homographic systems of points on a common line or circle, or of tangents to a common point or circle: to determine the two pairs M and M', N and N' whose intercepted segments or angles MM', NN' shall have a given middle point or line O.*

Taking the three points or lines A'', B'', C'' connected with the given three A, B, C by the common relation $OA'' = -OA$, $OB'' = -OB$, $OC'' = -OC$; constructing then the two double points or lines $M' = M''$ and $N' = N''$ of the two homographic systems determined by the three pairs of corresponding constituents A' and A'', B' and B'', C' and C''; and taking finally the two points or lines M and N connected with the two M' and N' by the common relation $OM = -OM'$, $ON = -ON'$; the two pairs of constituents M and M', N and N' are those required.

For, since, by construction, AA'', BB'', CC'', MM'', NN'' are all bisected by O, therefore $\{ABCMN\} = \{A''B''C''M''N''\}$; and since again, by construction, $M' = M''$ and $N' = N''$ are the two double points or lines of the two homographic systems A', B', C', &c. and A'', B'', CC'', &c., therefore $\{A'B'C'M'N'\} = \{A''B''C''M''N''\}$; consequently therefore $\{ABCMN\} = \{A'B'C'M'N'\}$; or, the two pairs of constituents M and M', N and N', which by construction have the required middle point or line O, are pairs of corresponding constituents of the two homographic systems A, B, C, &c. and A', B', C', &c.; and therefore &c.

N.B. The problems of the present and preceding examples are manifestly equivalent to the following, viz.: "Given three pairs of corresponding constituents A and A', B and B', C and C' of two homographic systems of points on a common line or circle, or of tangents to a common point or circle; to determine the two pairs M and M', N and N', for which the sum or difference $PM \pm PM'$, $PN \pm PN'$, P being a given point or tangent, shall have a given magnitude and sign."

Ex. 4°. *Given three pairs of corresponding constituents A and A', B and B', C and C' of two homographic systems of points on a common line or circle, or of tangents to a common point or circle: to determine the two pairs M and M', N and N' which shall form with two given points or tangents P and Q a system having a given anharmonic ratio.*

Taking the three points or lines A'', B'', C'' connected with the given three A, B, C by the common relation $\{PQAA''\} = \{PQBB''\} = \{PQCC''\}$ = the given anharmonic ratio; constructing then the two double points or lines $M' = M''$ and $N' = N''$ of the two homographic systems determined by the three pairs of corresponding constituents A' and A'', B' and B'', C' and C''; and taking finally the two points or lines M and N connected with the two M' and N' by the common relation $\{PQMM'\} = \{PQNN'\}$ = the given anharmonic ratio; the two pairs of constituents M and M', N and N' are those required.

For, since, by construction, $\{PQAA''\} = \{PQBB''\} = \{PQCC''\} = \{PQMM''\}$ $= \{PQNN''\}$ = the given anharmonic ratio, therefore (329) $\{ABCMN\}$ $= \{A''B''C''M''N''\}$; and, since again, by construction, $M' = M''$ and $N' = N''$ are the two double points or lines of the two homographic systems A', B', C', &c. and A'', B'', C'', &c. therefore $\{A'B'C'M'N'\}$ $= \{A''B''C''M''N''\}$; consequently therefore $\{ABCMN\} = \{A'B'C'M'N'\}$; or, the two pairs of constituents M and M', N and N', which by construction form with P and Q the given anharmonic ratio, are pairs of corresponding constituents of the two homographic systems A, B, C, &c. and A', B', C', &c.; and therefore &c.

Ex. 5°. *Given three pairs of corresponding constituents A and A', B and B', C and C' of one pair of homographic systems of points on a common line or circle, or of tangents to a common point or circle; and also three pairs P and P'', Q and Q'', R and R'' of another pair of homographic systems of points on the same line or circle, or of tangents to the same point* *or circle: to determine the two pairs M and M' or M'', N and N' or N'' common to both pairs of systems.*

Taking the three points or lines A'', B'', C'' connected with the given three A, B, C by the relations $\{PQRA\} = \{P''Q''R''A''\}$, $\{PQRB\} = \{P''Q''R''B''\}$, $\{PQRC\} = \{P''Q''R''C''\}$; constructing then the two double points or lines $M' = M''$ and $N' = N''$ of the two homographic systems determined by the three pairs of corresponding constituents A' and A'', B' and B'', C' and C''; and taking finally the two points or lines M and N connected with the two $M' = M''$ and $N' = N''$ by the relations $\{ABCM\} = \{A'B'C'M'\} = \{A''B''C''M''\}$ and $\{ABCN\} = \{A'B'C'N'\} = \{A''B''C''N''\}$; the two pairs of constituents M and M' or M'', N and N' or N'' are those required.

For, since, by virtue of the preceding relations, $\{ABCMN\} = \{A'B'C'M'N'\}$, and also (327) $\{PQRMN\} = \{P''Q''R''M''N''\}$; and since by construction $M' = M''$ and $N' = N''$; therefore M and M' or M'', N and N' or N'' are pairs of corresponding constituents of both pairs of homographic systems; and therefore &c.

N.B. This latter problem evidently comprehends the three preceding as particular cases; and with them a variety of others of the same nature corresponding to the variety of other ways in which homographic systems may be generated. See the various articles of the preceding chapter in which the principal of them are given.

Ex. 6°. *Given three pairs of corresponding constituents A and A', B and B', C and C' of two homographic systems of points on a common circle, or of tangents to a common circle; to determine the two pairs M and M', N and N'.*

a. Whose lines of connexion, in the former case, shall pass through a given point P.

a'. Whose points of intersection, in the latter case, shall lie on a given line L.

In the former case. Taking the three second intersections A'', B'', C'' with the circle of the three lines PA, PB, PC; constructing then the two double points $M' = M''$ and $N' = N''$ of the two homographic systems determined by the three pairs of corresponding constituents A' and A'', B' and B'', C' and C''; and taking finally the two second intersections M and N with the circle of the two lines PM' and PN'; the two pairs of points M and M', N and N' are those required.

In the latter case. Drawing the three second tangents A'', B'', C'' to the circle through the three points LA, LB, LC; constructing then the two double tangents $M' = M''$ and $N' = N''$ of the two homographic systems determined by the three pairs of corresponding constituents A' and A'', B' and B'', C' and C''; and drawing finally the two second tangents M and N to the circle through the two points LM' and LN'; the two pairs of tangents M and M', N and N' are those required.

For, in both cases, since, by construction, $\{A'B'C'M'N'\} = \{A''B''C''M''N''\}$, and since, by (315, *a* and *a'*), $\{A''B''C''M''N''\} = \{ABCMN\}$; therefore $\{ABCMN\} = \{A'B'C'M'N'\}$; and therefore &c.

Ex. 7°. *Given three pairs of corresponding constituents A and A', B and B', C and C' of two homographic systems of points on any two axes, or of rays through any two vertices; to determine the two pairs M and M', N and N'.*

a. Whose lines of connexion, in the former case, shall pass through a given point, or touch a given circle tangent to the two axes.

a'. Whose points of intersection, in the latter case, shall lie on a given line, or on a given circle passing through the two vertices.

Here, evidently, the three given pairs of corresponding constituents determine; in the former case, the corresponding three of two homographic systems of rays through the given point, or of tangents to the given circle, whose double lines intersect with the given axes at the required pairs of constituents; and, in the latter case, the corresponding three of two homographic systems of points on the given line, or circle, whose double points connect with the given vertices by the required pairs of constituents.

N.B. In the case of the latter property *a'*; the two rays, constituting each required pair of corresponding constituents, being parallel when the given line is at infinity (16), and intersecting at a given angle for every given circle passing through the two vertices (Euc. III. 21, 22); the two solutions, real or imaginary, (see Art. 339) of the problem "Given three pairs of corresponding constituents A and A', B and B', C and C' of two homographic systems of rays through different vertices, to determine the two pairs M and M', N and N' whose directions are parallel, or, more generally, intersect at any given angle," are consequently given by it for every form of the angle.

Ex. 8. *Given three pairs of corresponding constituents A and A', B and B', C and C' of two homographic systems of points on any two axes, or of rays through any two vertices: to determine the two pairs M and M', N and N'.*

a. Whose lines of connection with a given point P, in the former case, shall, 1°, coincide; 2°, contain a given angle; 3°, make equal angles with a given line through the point; 4°, divide in a given anharmonic ratio a given angle at the point.

a'. Whose points of intersection with a given line L, in the latter case, shall, 1°, coincide; 2°, intercept a given segment; 3°, make equal segments with a given point on the line: 4°, divide in a given anharmonic ratio a given segment of the line.

Here, in the former case; the required pairs of connectors PM and PM', PN and PN' are evidently, with respect to the two homographic pencils of rays determined at P by the three given pairs of connectors PA and PA', PB and PB', PC and PC'; in 1°, the double rays; in 2°, the pairs containing the given angle; in 3°, the pairs equally inclined to the given line; in 4°, the pairs dividing in the given anharmonic ratio the given angle. And, in the latter case; the required pairs of intersections LM and LM', LN and LN' are evidently, with respect to the two homographic rows of points determined on L by the three given pairs of intersections LA and LA', LB and LB', LC and LC'; in 1°, the double points; in 2°, the pairs intercepting the given segment; in 3°, the pairs equidistant from the given point; in 4°, the pairs dividing in the given anharmonic ratio the given segment. In both cases, consequently, while the solution of problem 1° is reduced at once to the corresponding case of Art. 348, those of problems 2°, 3°, 4° are reduced to those of examples 2°, 3°, 4° of the present article.

N.B. The above reciprocal solutions would all manifestly remain unchanged, if the point P were replaced by a circle touching the two axes, in the former case, and the line L by a circle containing the two vertices, in the latter case.

Ex. 9°. *Given two triads of corresponding constituents A, A', A'' and B, B', B'' of three homographic systems of points on a common line or circle, or of tangents to a common point or circle; to determine the three systems which shall have a pair of triple points or lines $M = M' = M''$ and $N = N' = N''$; with the positions of the two triple points or lines M and N.*

Constructing the two double points or lines M and N of the two homographic systems determined by the three pairs of corresponding constituents A and B, A' and B', A'' and B''; the two points or lines M and N are those required; and the position of either, as supplying a third triad of corresponding constituents in addition to the given two, determines, of course, the required systems (327).

For, since, by (342), $\{MNAB\} = \{MNA'B'\} = \{MNA''B''\}$; therefore &c.

Cor. 1°. The following property of two conjugate triads of homographic systems of points on a common line or circle, or of tangents to a common point or circle, follows immediately from the above.

If the three systems determined by the three triads of corresponding constituents A, A', A''; B, B', B''; C, C', C'' have a pair of triple points or lines M and N; the three determined by the three triads A, B, C; A', B', C'; A'', B'', C'' have also a pair of triple points or lines; and the triple points or lines are the same for both triads.

For, as the relations

$$\{MNABC\} = \{MNA'B'C'\} = \{MNA''B''C''\} \ldots\ldots\ldots\ldots (1),$$

involve reciprocally, by (272), the relations

$$\{MNAA'A''\} = \{MNBB'B''\} = \{MNCC'C''\} \ldots\ldots\ldots\ldots (2),$$

and conversely; therefore &c.

Cor. 2°. The comparison of both groups of relations (1) and (2) of the preceding (Cor. 1°) gives immediately (282) the three following groups of relations among the nine constituent points or lines themselves, viz.—

$$\{A'A''BC\} = \{A''AB'C'\} = \{AA'B''C''\} \quad \ldots\ldots\ldots (1),$$

$$\{B'B''CA\} = \{B''BC'A'\} = \{BB'C''A''\} \quad \ldots\ldots\ldots (2),$$

$$\{C'C''AB\} = \{C''CA'B'\} = \{CC'A''B''\} \quad \ldots\ldots\ldots (3),$$

which are therefore the conditions, necessary and sufficient, that either (and therefore the other) of the two triads of homographic systems, determined by the two conjugate triads of corresponding constituents, should have a pair of triple points or lines.

Ex. 10°, *a*. *Given two triads of corresponding constituents A, A', A'' and B, B', B'' of three homographic rows of points on any three axes, and one of the three lines L which intersect with the three axes at a triad of corresponding points P, P', P''; to determine the other two M and N which intersect with them also at triads of corresponding points Q, Q', Q'' and R, R', R''.*

Ex. 10°. *a'*. *Given two triads of corresponding constituents A, A', A'' and B, B', B'' of three homographic pencils of rays through any three vertices, and one of the three points P will connect with the three vertices by a triad of corresponding rays L, L', L''; to determine the other two Q and R which connect with them also by triads of corresponding rays M, M', M'' and N, N', N'';*

In the former case. Taking the two triads of points X, X', X'' and Y, Y', Y'' at which the sides of the two triangles $AA'A''$ and $BB'B''$ intersect with the given line L; and constructing the two double points E and F of the two homographic rows determined on L by the three pairs of corresponding constituents X and Y, X' and Y', X'' and Y''; the required lines M and N pass through E and F respectively, and may therefore be determined by the first case of Ex. 7°, *a*.

For, if E and F be the two points at which M and N intersect with L; then since, by the first case of (338, *a*),

$$\{EFXY\} = \{Q'R'A'B'\} = \{Q''R''A''B''\} \quad \ldots\ldots\ldots (1),$$

$$\{EFX'Y'\} = \{Q''R''A''B''\} = \{QRAB\} \quad \ldots\ldots\ldots (2),$$

$$\{EFX''Y''\} = \{QRAB\} = \{Q'R'A'B'\} \quad \ldots\ldots\ldots (3);$$

therefore at once $\{EFXY\} = \{EFX'Y'\} = \{EFX''Y''\}$; and therefore &c. (342).

In the latter case. Taking the two triads of lines X, X', X'' and Y, Y', Y'' by which the vertices of the two triangles A, A', A'' and B, B', B'' connect with the given point P; and constructing the two double rays E and F of the two homographic pencils determined at P by the three pairs of corresponding constituents X and Y, X' and Y', X'' and Y''; the required points Q and R lie on E and F respectively, and may therefore be determined by the first case of Ex. 7°, *a'*.

For, if E and F be the two lines by which Q and R connect with P; then since, by the first case of (338, a'),

$$\{EFXY\} = \{M'N'A'B'\} = \{M''N''A''B''\} \text{ .... } (1'),$$

$$\{EFX'Y'\} = \{M''N''A''B''\} = \{MNAB\} \text{ .... } (2'),$$

$$\{EFX''Y''\} = \{MNAB\} = \{M'N'A'B'\} \text{ .... } (3');$$

therefore at once $\{EFXY\} = \{EFX'Y'\} = \{EFX''Y''\}$; and therefore &c. (342).

Cor. 1°. If C, C', C'' be any third triad of corresponding constituents of the three systems, in either case; and Z, Z', Z'' the three points or lines at which the three sides or by which the three vertices of the triangle C, C', C'' intersect with the line L or connect with the point P; then since, in either case, for the same reason as above,

$$\{EFXYZ\} = \{EFX'Y'Z'\} = \{EFX''Y''Z''\},$$

and similarly for all triads; therefore the three rows or pencils X, Y, Z, &c.; X', Y', Z', &c.; X'', Y'', Z'', &c. are homographic, and have E and F for triple points or rays. Hence the following general properties of the three lines L, M, N in the former case, and of the three points P, Q, R in the latter case.

a. The sides of the system of triangles formed by the several triads of corresponding points determine on each line, in the former case, three homographic rows having a pair of triple points; and the triple points on each line are its intersections with the other two.

a'. The vertices of the systems of triangles formed by the several triads of corresponding rays determine at each point, in the latter case, three homographic pencils having a pair of triple rays; and the triple rays at each point are its connectors with the other two.

N.B. That, in both cases, the three triads of systems are homographic with each other and with the systems of the original triad, follows also immediately from the first parts of (338, b and b'); and that for each triad the points or lines in question are triple, is evident also from the obvious consideration, that, of a triangle, when the three vertices are collinear, the three sides intersect with every line at a triad of coincident points, and, when the three sides are concurrent, the three vertices connect with every point by a triad of coincident lines.

Cor. 2°. From the reciprocal properties of the preceding corollary it follows immediately, by virtue of (341, 1°), that—

For three homographic rows of points on different axes, or pencils of rays through different vertices, no more than three triads of corresponding constituents could be collinear in the former case, or concurrent in the latter case, unless all triads of corresponding constituents were collinear in the former case, or concurrent in the latter case.

For, if four collinear or concurrent triads existed, then, of the four lines of collinearity or points of concurrence, every three, by the properties in question, would intersect or connect with the fourth at three triple points or by three triple rays of the three homographic rows or pencils determined on or at it by the sides or vertices of the system of triangles formed by the several other triads; which three homographic rows or pencils should therefore (341, 1°) entirely coincide; and therefore &c.

N.B. Between the three lines L, M, N and the three axes of the rows in the former case, or between the three points P, Q, R, and the three vertices of the pencils in the latter case, no relation of connexion necessarily exists; both triads in either case may be given or taken arbitrarily; and give rise in all cases to two conjugate triads of homographic rows or pencils, determined; in the former case, by the three triads of corresponding constituents P, P', P''; Q, Q', Q''; R, R', R'' on the three axes, and by the three P, Q, R; P', Q', R'; P'', Q'', R'' on the three lines; and, in the latter case, by the three triads of corresponding constituents L, L', L''; M, M', M''; N, N', N'' at the three vertices, and by the three L, M, N; L', M', N'; L'', M'', N'' at the three points; between which there exist several interesting relations of connexion, though the two triads of lines or points which determine them are entirely arbitrary.

When, of six lines or points given or taken arbitrarily, any (and therefore every) four intersect or connect equianharmonically with the remaining two (301, a and a'); then (338), of the three homographic rows or pencils determined by any three of them on or at the remaining three, all triads of corresponding constituents are collinear or concurrent; and, conversely, when, of three homographic rows or pencils given or taken arbitrarily, any four (and therefore all) triads of corresponding constituents are collinear or concurrent (Cor. 2° above); then (338), of the three axes and any three lines of collinearity, or of the three vertices and any three points of concurrence, every four of the six intersect or connect equianharmonically with the remaining two.

Ex. 11°. a. *Given two triads of corresponding constituents A, A', A'' and B, B', B'' of three homographic systems, one A, B, C, &c. of points on an axis, and two A', B', C', &c. and A'', B'', C'', &c. of points on a circle; and one of three lines L which determine a collinear triad P, P', P''; to construct the other two M and N which determine collinear triads Q, Q', Q'' and R, R', R''.*

Ex. 11°. a'. *Given two triads of corresponding constituents A, A', A'' and B, B', B'' of three homographic systems, one A, B, C, &c. of rays through a vertex, and two A', B', C', &c. and A'', B'', C'', &c. of tangents to a circle; and one of the three points P which determine a concurrent triad L, L', L''; to construct the other two R and S which determine concurrent triads M, M', M'' and N, N', N''.*

Here, since in the former case, by the second case of (338, a), the several lines of connexion $A'A''$, $B'B''$, $C'C''$, &c. of the several pairs of corresponding points on the circle determine on any two of themselves $H'H''$ and $K'K''$ two collinear systems $A_{,}$, $B_{,}$, $C_{,}$, &c., and $A_{,,}$, $B_{,,}$, $C_{,,}$, &c., homographic with the two concyclic systems A', B', C', &c., and A'', B'', C'', &c. and therefore with the collinear system A, B, C, &c.; and, since in the latter case, by the second case of (338, a'), the several points of intersection $A'A''$, $B'B''$, $C'C''$, &c. of the several pairs of corresponding tangents to the circle determine at any two of themselves $H'H''$ and $K'K''$ two concurrent systems $A_{,}$, $B_{,}$, $C_{,}$, &c., and $A_{,,}$, $B_{,,}$, $C_{,,}$, &c., homographic with the two concyclic systems A', B', C', &c., and A'', B'', C'', &c., and therefore with the concurrent system A, B, C, &c.; the two reciprocal problems of the present are consequently reducible at once to those of the preceding example; and the various inferences there drawn are accordingly applicable here also.

354. In all the examples of the preceding article, the two homographic systems, whose double points or lines were the object of enquiry, direct or indirect, were supposed to have been given by means of three pairs of corresponding constituents A and A', B and B', C and C'; which, in all cases, as shewn in Art. 327, implicitly determine the systems, and all particulars connected with them. In the applications of the theory, however, it is the law connecting the several pairs of corresponding constituents, whatever it be, and not the actual triad of constituents themselves, which is generally given; and, should the law of connexion not be such as to furnish the required double points or lines directly by a simpler construction, a certain preliminary process is consequently necessary before the particular construction corresponding to the case, as already described, can be applied.

This preliminary process is however uniformly the same in all cases, and consists simply in taking arbitrarily any three constituents A, B, C of either system, and constructing their three correspondents A', B', C' of the other, in accordance with the given law of connexion, whatever it be. The three pairs of corresponding constituents A and A', B and B', C and C' necessary and sufficient to determine the two homographic systems, whose double points or lines give the two solutions of the proposed problem, are thus obtained; and the subsequent process is that already described and exemplified at some length in the preceding article.

355. If, in the performance of the preliminary process described in the preceding article, two of the three *coincidences* $A=A'$, $B=B'$, $C=C'$ should *happen* to result, the required double points or lines, and therefore the two solutions of the proposed problem, would of course be obtained without the necessity of any further construction. In the performance of the preliminary process, therefore, each arbitrary assumption of a point or line A of either system, from which to construct the corresponding point or line A' of the other system by application of the given law of connection between them, may be regarded as an *attempt* to solve the proposed problem by the *method of trial;* which would be *successful* if $A'=A$; but which of course results generally in a *failure*, of which AA' represents the amount of *error* both in magnitude and sign. And it is by a simple and uniform process, based on the data resulting from three such attempts and their failures, that, as in the *method of false position* in Arithmetic, the true solutions of the proposed problem are by this method eventually obtained.

356. With a few examples of problems solved by the above method of trial, and coming under the second class (353) of those reducible to the determination of the double points or lines of two homographic systems, we shall conclude the present chapter.

Ex. 1°. *To divide a given segment or angle EF in a given anharmonic ratio, by a segment or angle MM', or NN', of given magnitude, or having a given point or line of bisection.*

Assuming arbitrarily any three points on the axis of the segment or rays through the vertex of the angle A, B, C; and constructing the three A', B', C' for which $\{EFAA'\} = \{EFBB'\} = \{EFCC'\}$ = the given anharmonic ratio, and also the three A'', B'', C'' for which the three segments or angles AA'', BB'', CC'' have the given magnitude or bisector; if, having proceeded so far, two of the three coincidences $A' = A''$, $B' = B''$, $C' = C''$ happen to result, the problem is solved; if not, the two systems of points or rays A', B', C', &c. and A'', B'', C'', &c., being both homographic with the system A, B, C, &c. (329), and therefore with each other (323), the two double points or rays $M' = M''$ and $N' = N''$ of the two former, with their two correspondents M and N in the latter, give the two segments or angles MM' and NN' which satisfy its two conditions.

N.B. Of the above problems (which evidently include as particular cases those of 1° and 2°, Cor. 3°, Art. 227) the first may obviously be stated otherwise as follows: "To place two segments or angles of given magnitude so as to cut each other in a given anharmonic ratio."

Ex. 2°. *To divide two given segments or angles* **EF** *and* **GH**, *having a common axis or vertex, in two given anharmonic ratios, by a common segment or angle* MM', *or* NN'.

Assuming arbitrarily any three points on the axis or rays through the vertex A, B, C; and constructing the three A', B', C' for which $\{EFAA'\} = \{EFBB'\} = \{EFCC'\}$ = the given anharmonic ratio for EF, and also the three A'', B'', C'' for which $\{GHAA''\} = \{GHBB''\} = \{GHCC''\}$ = the given anharmonic ratio for GH; if, having proceeded so far, two of the three coincidences $A' = A''$, $B' = B''$, $C' = C''$ happen to result, the problem is solved; if not, the two systems of points or rays A', B', C', &c. and A'', B'', C'', &c. being homographic with the system A, B, C, &c. (329), and therefore with each other (323), the two double points or rays $M' = M''$ and $N' = N''$ of the two former, with their two correspondents M and N in the latter, give the two segments or angles MM' and NN' which satisfy its two conditions.

N.B. To the first of the above problems (which evidently include those of Art. 230 as particular cases) the following, by virtue of the general property of Art. 332, may obviously be reduced: "Given four points P, Q, R, S on a common axis, to determine the two M and M', or N and N', on the axis, for which the two rectangles $PM . QM'$ and $RM . SM'$, or $PN . QN'$ and $RN . SN'$, shall be given in magnitude and sign."

Ex. 3°. *Given two points on or tangents to a circle* **E** *and* **F**, *to divide their intercepted arc* **EF** *in a given anharmonic ratio by two others* **M** *and* M', *or* **N** *and* N';

a. Connecting, in the former case, through a given point **P**.

a'. Intersecting, in the latter case, on a given line **L**.

Assuming arbitrarily any three points on or tangents to the circle A, B, C; and constructing the three A', B', C' for which $\{EFAA'\} = \{EFBB'\} = \{EFCC'\}$ = the given anharmonic ratio, and also the three A'', B'', C'' which connect with A, B, C in the former case through the given point P, or intersect with A, B, C in the latter case on the given line L; if, having proceeded so far, two of the three coincidences $A' = A''$, $B' = B''$, $C' = C''$ happen to result, the problem is solved; if not, the two systems of points or tangents A', B', C', &c. and A'', B'', C'', &c. being both homographic with the system A, B, C, &c. (329 and 315), and therefore with each other (323), the two double points or tangents $M' = M''$ and $N' = N''$ of the two former, with their two correspondents M and N in the latter, give the two pairs of points or tangents M and M', N and N' which fulfil both required conditions.

N.B. To the above problems (which evidently include those of Cor. 4°, Art. 257, as particular cases) the following, by virtue of the general property of Art. 257, may obviously be reduced; viz. "To divide two given arcs of two given circles, one harmonically, and the other in any given anharmonic ratio, by four collinear points on, or by four concurrent tangents to, the circles."

Ex. 4°. *a*. *On two given lines L and L' to find two points M and M', or N and N', whose lines of connection with each of two given points P and P' shall*; 1°, *contain a given angle*; 2°, *make equal angles with a given line through the point*; 3°, *divide in a given anharmonic ratio a given angle at the point.*

Ex. 4°. *a'*. *Through two given points P and P' to draw two lines M and M', or N and N', whose points of intersection with each of two given lines L and L' shall*; 1°, *intercept a given segment*; 2°, *make equal segments with a given point on the line*; 3°, *divide in a given anharmonic ratio a given segment of the line.*

In the former case; taking arbitrarily, on either line L, any three points A, B, C; and constructing, on the other L', the three A', B', C' for which the three angles APA', BPB', CPC' fulfil the required condition for the point P, and also the three A'', B'', C'' for which the three $AP'A''$, $BP'B''$, $CP'C''$ fulfil that for the point P'. And, in the latter case; drawing arbitrarily, through either point P, any three rays A, B, C; and constructing, through the other P', the three A', B', C' for which the three segments ALA', BLB', CLC' fulfil the required condition for the line L, and also the three A'', B'', C'' for which the three $AL'A''$, $BL'B''$, $CL'C''$ fulfil that for the line L'. If, in either case, having proceeded so far, two of the three coincidences $A' = A''$, $B' = B''$, $C' = C''$ happen to result; the problem is solved; if not, the two systems of points or rays A', B', C', &c. and A'', B'', C'', &c. being homographic with the system A, B, C, &c. (see Ex. 8° of preceding Art.), and therefore with each other (323), the two double points or rays $M' = M''$ and $N' = N''$ of the two former, with their two correspondents M and N in the latter, are the two pairs of points or rays M and M', N and N' which fulfil the required conditions.

Ex. 5°. *a*. *Through a given point P to draw a line intersecting with four given lines L_1, L_2, L_3, L_4 at a system of four points M_1, M_2, M_3, M_4, or N_1, N_2, N_3, N_4, having a given anharmonic ratio.*

Ex. 5°. *a'*. *On a given line L to find a point connecting with four given points P_1, P_2, P_3, P_4 by a system of four rays M_1, M_2, M_3, M_4, or N_1, N_2, N_3, N_4, having a given anharmonic ratio.*

In the former case. Taking arbitrarily any three points A_1, B_1, C_1 on any one of the four given lines L_1; and drawing through them the three lines intersecting with the remaining three L_2, L_3, L_4 at the three triads of points A_2, A_3, A_4; B_2, B_3, B_4; C_2, C_3, C_4 determining with A_1, B_1, C_1 the given anharmonic ratio (287, *a*); if, having proceeded so far, two of the three lines so drawn happen to pass through the given point P, the problem is solved; if not, the four systems of points A_1, B_1, C_1, &c.; A_2, B_2, C_2, &c.; A_3, B_3, C_3, &c.; A_4, B_4, C_4, &c., on the four given lines L_1, L_2, L_3, L_4, being homographic (333, Ex. *e*), the two double rays of the two homographic pencils determined by any two of them at P are the two lines that solve it.

In the latter case. Drawing arbitrarily any three rays A_1, B_1, C_1 through any one of the four given points P_1; and taking on them the three points connecting with the remaining three P_2, P_3, P_4 by the three triads of rays A_2, A_3, A_4; B_2, B_3, B_4; C_2, C_3, C_4 determining with A_1, B_1, C_1 the given anharmonic ratio (287, *a'*); if, having proceeded so far, two of the three points so taken happen to lie on the given line L, the problem is solved; if not, the four systems of rays A_1, B_1, C_1, &c.; A_2, B_2, C_2, &c.; A_3, B_3, C_3, &c.; A_4, B_4, C_4, &c., through the four given points P_1, P_2, P_3, P_4, being homographic (333, Ex. *e'*), the two double points of the two homographic rows determined by any two of them on L are the two points that solve it.

Cor. 1°. Regarding the four lines in *a*, or the four points, in *a'*, as grouped in two pairs determining two angles in the former case, or two segments in the latter case; the above reciprocal problems may be stated otherwise as follows:

a. Through a given point to draw a line the segments intercepted on which by two given angles shall divide each other in a given anharmonic ratio.

a'. On a given line to find a point the angles subtended at which by two given segments shall divide each other in a given anharmonic ratio.

Cor. 2°. If any one of the four lines, in *a*, or of the four points, in *a'*, be at infinity; the problems for the remaining three (See Cor. 3°, Art. 285) become modified as follows:

a. Through a given point to draw a line intersecting with three given lines at three points the ratios of whose three intercepted segments shall be given.

a'. On a given line to find a point connecting with three given points by three lines the ratios of the three segments intercepted by which on a second given line shall be given.

N.B. Of these latter problems the first (*a*) is obviously a very particular case of that proposed, with others, for solution, on other principles, in 3°, Cor. 1°, Art. 56.

Ex. 6°. *a. To draw a line intersecting with six given lines L_1, L_2, L_3, L_4, L_5, L_6 (or five if any two coincide) so that four points of intersection M_1, M_2, M_3, M_4, or N_1, N_2, N_3, N_4, shall have one given anharmonic ratio, and four more M_1, M_2, M_5, M_6, or N_1, N_2, N_5, N_6, another given anharmonic ratio.*

Ex. 6°. *a'. To find a point connecting with six given points P_1, P_2, P_3, P_4, P_5, P_6 (or five if any two coincide) so that four rays of connexion M_1, M_2, M_3, M_4, or N_1, N_2, N_3, N_4, shall have one given anharmonic ratio, and four more M_1, M_2, M_5, M_6, or N_1, N_2, N_5, N_6, another given anharmonic ratio.*

In the former case. Taking arbitrarily any three points A_1, B_1, C_1 on L_1; and drawing through them the three lines intersecting with L_2, L_3, L_4 at the three triads of points A_2, A_3, A_4; B_2, B_3, B_4; C_2, C_3, C_4 determining with A_1, B_1, C_1 the first given anharmonic ratio, and also the three intersecting with L_2, L_5, L_6 at the three triads of points A'_2, A_5, A_6; B'_2, B_5, B_6; C'_2, C_5, C_6 determining with A_1, B_1, C_1 the second given anharmonic ratio (287, *a*); if, having proceeded so far, two of the three coincidences $A_2 = A'_2$, $B_2 = B_2'$, $C_2 = C_2'$ happen to result, the problem is solved; if not, the two systems of points A_2, B_2, C_2, &c. and A_2', B_2', C_2', &c. on L_2 being both homographic with the systems A_1, B_1, C_1, &c. on L_1 (333, Ex. *a*), and therefore with each other (323), the two double points $M_2 = M_2'$ and $N_2 = N_2'$ of the two former connect with their two correspondents M_1 and N_1 in the latter by the two lines which solve it.

In the latter case. Drawing arbitrarily any three rays A_1, B_1, C_1 through P_1; and taking on them the three points connecting with P_2, P_3, P_4 by the three triads of rays A_2, A_3, A_4; B_2, B_3, B_4; C_2, C_3, C_4 determining with A_1, B_1, C_1 the first given anharmonic ratio, and also the three connecting with P_2, P_5, P_6 by the three triads of rays A_2', A_5, A_6; B_2', B_5, B_6; C_2', C_5, C_6 determining with A_1, B_1, C_1 the second given anharmonic ratio (287, *a'*); if, having proceeded so far, two of the three coincidences, $A_2 = A_2'$, $B_2 = B_2'$, $C_2 = C_2'$ happen to result, the problem is solved; if not, the two systems of rays A_2, B_2, C_2, &c. and A_2', B_2', C_2', &c. through P_2 being both homographic with the system A_1, B_1, C_1, &c. through P_1 (333, Ex. *a'*), and therefore with each other (323), the two double rays $M_2 = M_2'$ and $N_2 = N_2'$ of the two former intersect with their two correspondents M_1 and N_1 in the latter at the two points which solve it.

Cor. 1°. Regarding the six lines, in *a*, or the six points, in *a'*, as grouped in three pairs determining three angles in the former case or three segments in the latter case; the above reciprocal problems, like those of the preceding example, may be stated otherwise as follows:

a. To draw a line the segments intercepted on which by three given angles shall divide each other two and two in given anharmonic ratios.

a'. To find a point the angles subtended at which by three given segments shall divide each other two and two in given anharmonic ratios.

Cor. 2°. If any two of the four lines, in *a*, or of the four points, in *a'*, which do not enter into both anharmonic ratios, coincide at infinity; the problems for the remaining four (see Cor. 3°, Art. 285) become modified as follows:—

a. To draw a line two of whose intersections with four given lines shall divide in given ratios its segment intercepted by the remaining two.

a'. To find a point two of whose connectors with four given points shall divide in given ratios the segment intercepted on a given line by the remaining two.

N.B. Of these latter problems, the first (*a*), it will be remembered, was already proposed for solution, on other principles, at the close of Art. 55.

Ex. 7°. *a*. *To a given circle to inscribe a polygon of any order, whose several sides shall pass through given points, or any or all of them touch instead given circles concentric with the original.*

Ex. 7°. *a'*. *To a given circle to exscribe a polygon of any order, whose several vertices shall lie on given lines, or any or all of them lie instead on given circles concentric with the original.*

In the former case. Taking arbitrarily any three points A_1, B_1, C_1 on the given circle; and constructing successively the several triads of inscribed chords A_1A_2, B_1B_2, C_1C_2; A_2A_3, B_2B_3, C_2C_3; A_3A_4, B_3B_4, C_3C_4; &c.; A_nA_{n+1}, B_nB_{n+1}, C_nC_{n+1} passing through (or touching) the several given points (or concentric circles) corresponding respectively to the several successive sides of the polygon; if, having proceeded so far, two of the three coincidences $A_{n+1} = A_1$, $B_{n+1} = B_1$, $C_{n+1} = C_1$ happen to result, the problem is solved; if not, the several systems of points A_1, B_1, C_1, &c.; A_2, B_2, C_2, &c.; A_3, B_3, C_3, &c., &c.; A_{n+1}, B_{n+1}, C_{n+1}, &c., being all homographic (315), the two double points $M_1 = M_{n+1}$ and $N_1 = N_{n+1}$ of the first and last give the first vertices of the two polygons that solve it.

In the latter case. Drawing arbitrarily any three tangents A_1, B_1, C_1 to the given circle; and constructing successively the several triads of exscribed angles A_1A_2, B_1B_2, C_1C_2; A_2A_3, B_2B_3, C_2C_3; A_3A_4, B_3B_4, C_3C_4; &c.; A_nA_{n+1}, B_nB_{n+1}, C_nC_{n+1} having their vertices on the several given lines (or concentric circles) corresponding respectively to the several successive vertices of the polygon; if, having proceeded so far, two of the three coincidences $A_{n+1} = A_1$, $B_{n+1} = B_1$, $C_{n+1} = C_1$ happen to result, the problem is solved; if not, the several systems of tangents A_1, B_1, C_1, &c.; A_2, B_2, C_2, &c.; A_3, B_3, C_3, &c., &c.; A_{n+1}, B_{n+1}, C_{n+1}, &c. being all homographic (315), the two double lines $M_1 = M_{n+1}$ and $N_1 = N_{n+1}$ of the first and last give the first sides of the two polygons that solve it.

N.B. Of the above reciprocal problems, those of Art. 263, solved there on other principles, are evidently particular cases.

Ex. 8°. *a*. *To a given circle to inscribe a polygon of any order, whose several sides shall divide in given anharmonic ratios given arcs of the circle.*

Ex. 8°. *a'*. *To a given circle to exscribe a polygon of any order, whose several angles shall divide in given anharmonic ratios given arcs of the circle.*

In the former case. Taking arbitrarily any three points A_1, B_1, C_1 on the given circle; and constructing successively the several triads of inscribed chords A_1A_2, B_1B_2, C_1C_2; A_2A_3, B_2B_3, C_2C_3; A_3A_4, B_3B_4, C_3C_4; &c. A_nA_{n+1}, B_nB_{n+1}, C_nC_{n+1} dividing in the several given anharmonic ratios the several given arcs of the circle corresponding respectively to the several successive sides of the polygon; if, having proceeded so far, two of the three coincidences $A_{n+1} = A_1$, $B_{n+1} = B_1$, $C_{n+1} = C_1$ happen to result, the problem is solved; if not, the several systems of points A_1, B_1, C_1, &c.; A_2, B_2, C_2, &c.; A_3, B_3, C_3, &c.; A_{n+1}, B_{n+1}, C_{n+1}, &c. being all homographic (329), the two double points $M_1 = M_{n+1}$ and $N_1 = N_{n+1}$ of the first and last give the first vertices of the two polygons that solve it.

In the latter case. Drawing arbitrarily any three tangents A_1, B_1, C_1 to the given circle; and constructing successively the several triads of exscribed angles A_1A_2, B_1B_2, C_1C_2; A_2A_3, B_2B_3, C_2C_3; A_3A_4, B_3B_4, C_3C_4 &c.; $A_nA_{n+1}, B_nB_{n+1}, C_nC_{n+1}$ dividing in the several given anharmonic ratios the several given arcs of the circle corresponding respectively to the several successive angles of the polygon; if, having proceeded so far, two of the three coincidences $A_{n+1} = A_1, B_{n+1} = B_1, C_{n+1} = C_1$ happen to result, the problem is solved; if not, the several systems of tangents A_1, B_1, C_1, &c.; A_2, B_2, C_2, &c.; A_3, B_3, C_3, &c.; $A_{n+1}, B_{n+1}, C_{n+1}$, &c. being all homographic (329), the two double lines $M_1 = M_{n+1}$ and $N_1 = N_{n+1}$ of the first and last give the first sides of the two polygons that solve it.

N.B. That the above reciprocal problems involve, as particular cases, those of the preceding example, is evident from Arts. 257 and 311, Cor. 3°.

Ex. 9°. *a*. *To construct a polygon of any order, whose several vertices shall lie on given lines, and whose several sides shall pass through given points, or any or all of them touch instead given circles tangent to the pairs of lines on which the adjacent vertices lie.*

Ex. 9°. *a'*. *To construct a polygon of any order, whose several sides shall pass through given points, and whose several vertices shall lie on given lines, or any or all of them lie instead on given circles passing through the pairs of points through which the adjacent sides pass.*

In the former case. On any one of the given lines L_1 taking arbitrarily any three points A_1, B_1, C_1; and on the several others L_2, L_3, L_4, &c. L_n taken in the order of the several successive vertices of the polygon, and finally on the original L_1 itself, constructing successively the several triads of points A_2, B_2, C_2; A_3, B_3, C_3; A_4, B_4, C_4; &c...A_n, B_n, C_n; and $A_{n+1}, B_{n+1}, C_{n+1}$ for which the several triads of connectors A_1A_2, B_1B_2, C_1C_2; A_2A_3, B_2B_3, C_2C_3; A_3A_4, B_3B_4, C_3C_4; &c.; $A_nA_{n+1}, B_nB_{n+1}, C_nC_{n+1}$ pass through (or touch) the several given points (or circles) corresponding respectively to the several successive sides of the polygon; if, having proceeded so far, two of the three coincidences $A_{n+1} = A_1, B_{n+1} = B_1, C_{n+1} = C_1$ happen to result, the problem is solved; if not, the several systems of points A_1, B_1, C_1, &c.; A_2, B_2, C_2, &c.; A_3, B_3, C_3, &c. &c.; $A_{n+1}, B_{n+1}, C_{n+1}$, &c. being all homographic (285 or 315), the two double points $M_1 = M_{n+1}$ and $N_1 = N_{n+1}$ of the first and last give the vertices on L_1 of the two polygons that solve it.

In the latter case. Through any one of the given points P_1 drawing arbitrarily any three lines A_1, B_1, C_1; and through the several others P_2, P_3, P_4, &c. P_n, taken in the order of the several successive sides of the polygon, and finally through the original P_1 itself, constructing successively the several triads of lines A_2, B_2, C_2; A_3, B_3, C_3; A_4, B_4, C_4, &c.; A_n, B_n, C_n; and $A_{n+1}, B_{n+1}, C_{n+1}$ for which the several triads of intersections A_1A_2, B_1B_2, C_1C_2; A_2A_3, B_2B_3, C_2C_3; A_3A_4, B_3B_4, C_3C_4, &c.; A_nA_{n+1}, B_nB_{n+1}, C_nC_{n+1} lie on the several given lines (or circles) corresponding respectively to the several successive vertices of the polygon; if, having proceeded so far, two of the three coincidences $A_{n+1} = A_1$, $B_{n+1} = B_1$, $C_{n+1} = C_1$ happen to result, the problem is solved; if not, the several systems of rays A_1, B_1, C_1, &c.; A_2, B_2, C_2, &c.; A_3, B_3, C_3, &c. &c.; A_{n+1}, B_{n+1}, C_{n+1}, &c. being all homographic (285 or 315), the two double lines $M_1 = M_{n+1}$ and $N_1 = N_{n+1}$ of the first and last give the sides through P_1 of the two polygons that solve it.

N.B. Of the above reciprocal problems, the first parts may evidently be stated in the common equivalent form; viz. "given two polygons of any common order, to construct a third at once inscribed to one and exscribed to the other of them."

Ex. 10°. *a. To construct a polygon of any order, whose several vertices shall lie on given lines, and whose several sides shall subtend angles at given points;* 1°, *of given magnitudes;* 2°, *having given lines of bisection through the points;* 3°, *dividing in given anharmonic ratios given angles at the points.*

Ex. 10°. *a′. To construct a polygon of any order, whose several sides shall pass through given points, and whose several angles shall intercept segments on given lines;* 1°, *of given magnitudes;* 2°, *having given points of bisection on the lines;* 3°, *dividing in given anharmonic ratios given segments of the lines.*

In the former case. On any one of the given lines L_1 taking arbitrarily any three points A_1, B_1, C_1; and on the several others L_2, L_3, L_4, &c. L_n taken in the order of the several successive vertices of the polygon, and finally on the original L_1 itself, constructing successively the several triads of points A_2, B_2, C_2; A_3, B_3, C_3; A_4, B_4, C_4, &c.; A_n, B_n, C_n; and A_{n+1}, B_{n+1}, C_{n+1} for which the several triads of segments A_1A_2, B_1B_2, C_1C_2; A_2A_3, B_2B_3, C_2C_3; A_3A_4, B_3B_4, C_3C_4, &c.; A_nA_{n+1}, B_nB_{n+1}, C_nC_{n+1} subtend at the several given points, corresponding respectively to the several successive sides of the polygon, angles fulfilling the required conditions 1° or 2° or 3°: if, having proceeded so far, two of the three coincidences $A_{n+1} = A_1$, $B_{n+1} = B_1$, $C_{n+1} = C_1$ happen to result, the problem is solved; if not, the several systems of points A_1, B_1, C_1, &c.; A_2, B_2, C_2, &c.; A_3, B_3, C_3, &c., &c.; A_{n+1}, B_{n+1}, C_{n+1}, &c. being all homographic (329), the two double points $M_1 = M_{n+1}$ and $N_1 = N_{n+1}$ of the first and last give the vertices on L_1 of the two polygons that solve it.

In the latter case. Through any one of the given points P_1 drawing arbitrarily any three lines A_1, B_1, C_1; and through the several others P_2, P_3, P_4, &c. P_n taken in the order of the several successive sides of the polygon, and finally through the original P_1 itself, constructing successively the several triads of lines A_2, B_2, C_2; A_3, B_3, C_3; A_4, B_4, C_4, &c.; A_n, B_n, C_n; and A_{n+1}, B_{n+1}, C_{n+1}, for which the several triads of angles A_1A_2, B_1B_2, C_1C_2; A_2A_3, B_2B_3, C_2C_3; A_3A_4, B_3B_4, C_3C_4, &c.; A_nA_{n+1}, B_nB_{n+1}, C_nC_{n+1}, intercept on the several given lines, corresponding respectively to the several successive angles of the polygon, segments fulfilling the required conditions 1° or 2° or 3°; if, having proceeded so far, two of the three coincidences

$A_{n+1} = A_1$, $B_{n+1} = B_1$, $C_{n+1} = C_1$ happen to result, the problem is solved; if not, the several systems of rays A_1, B_1, C_1, &c.; A_2, B_2, C_2, &c.; A_3, B_3, C_3, &c. &c.; $A_{n+1}, B_{n+1}, C_{n+1}$, &c. being all homographic (329), the two double lines $M_1 = M_{n+1}$ and $N_1 = N_{n+1}$ of the first and last give the sides through P_1 of the two polygons that solve it.

N.B. The above reciprocal constructions would evidently remain unaltered, if any line in the former case, or any point in the latter case, were replaced by a circle, containing in the former case the two points, or touching in the latter case the two lines, between which it lies in the order of the several successive vertices or sides of the polygon.

CHAPTER XXI.

ON THE RELATION OF INVOLUTION BETWEEN HOMOGRAPHIC SYSTEMS.

357. WHEN the axes of two homographic rows of points or the vertices of two homographic pencils of rays coincide, every point on the common axis or ray through the common vertex belongs of course indifferently to both systems, and has in general two different correspondents, one as belonging to one system, and the other as belonging to the other system; it sometimes happens, however, that these two correspondents always coincide, as appears from the following fundamental theorem:

When two homographic rows of points on a common axis, or pencils of rays through a common vertex, are such that any one point on the axis, or ray through the vertex, has the same correspondent to whichever system it be regarded as belonging, then every point on the axis, or ray through the vertex, possesses the same property.

Let A, B, C, D, E, F, &c. and A', B', C', D', E', F', &c. be the two systems; and let any one point or ray P, denoted by A or B' according to its system, have in both cases the same correspondent Q, denoted by A' or B according to its system; then every other point or ray R, denoted by C or D' according to its system, has in both cases the same correspondent S, denoted by C' or D according to its system.

For, the two systems being homographic, $\{ABCD\} = \{A'B'C'D'\}$; but, by hypothesis, $A = B' = P$, $B = A' = Q$, $C = D' = R$, therefore $\{PQRD\} = \{QPC'R\} = \{PQRC'\}$ (280); therefore, at once, $D = C' = S$; and therefore &c.

The same theorem may also be stated in the somewhat different, but obviously equivalent form, as follows:

For two homographic rows of points on a common axis, or pencils of rays through a common vertex, the interchangeability of *a single pair of corresponding constituents involves that of every pair.* (See Art. 284).

358. Two homographic rows of points on a common axis, or pencils of rays through a common vertex, related as above to each other, that every point on the common axis, or ray through the common vertex, has the same correspondent to whichever system it be regarded as belonging, are said *to be in involution with each other.* In the same case, their common axis or vertex is termed the axis or vertex of the involution; their two double points or rays (341) are termed the double points or rays of the involution; and their several pairs of corresponding constituents, from their property of interchangeability, are termed conjugate points or rays of the involution.

Every two conjugate groups of two homographic rows or pencils in involution are said also to be in involution with each other, provided they contain at least three points or rays each; that number of pairs of corresponding constituents of any two homographic systems being requisite (327) to determine the systems. Hence, two triads of corresponding points or rays, having a common axis or vertex, are said to be in involution, when the two homographic rows or pencils they determine are in involution with each other.

359. Two pairs of corresponding constituents are sufficient to determine two homographic rows of points on a common axis, or pencils of rays through a common vertex, when in involution with each other. For, the relation of involution between the two systems requiring (357) that every pair of corresponding constituents should be interchangeable, the interchange of the two constituents of either pair, when two are given or known, would supply the third pair necessary and sufficient to determine the systems (327).

From the nature of the relation of involution between two homographic rows or pencils (357), it is evident (285) that every two rows in involution on any axis determine two pencils in involution at every vertex, and, conversely, that every two pencils in involution at any vertex determine two rows in involution on every axis.

360. The fundamental theorem of Art. 357 applies, of course, as well to two homographic systems of points on a common circle, or of tangents to a common circle, as to two rows of points on a common axis, or pencils of rays through a common vertex; and two such systems accordingly, or any two conjugate groups of two such systems, containing at least three constituents each, are also said to be in involution with each other under the same circumstances exactly as if the circle were a line in the former case or a point in the latter case.

It is evident that systems of points in involution on any circle determine pencils of rays in involution at every point on the circle; and, conversely, that pencils of rays in involution at any vertex determine systems of points in involution on every circle passing through the vertex. Also, that systems of tangents in involution to any circle determine rows of points in involution on every tangent to the circle; and, conversely, that rows of points in involution on any axis determine systems of tangents in involution to every circle touching the axis.

361. The following are a few fundamental examples of two homographic systems in involution with each other; from which it will be seen that the relation, when existing between two systems otherwise known to be homographic, is generally apparent of itself when the law connecting the several pairs of corresponding constituents in the generation of the systems is given or known.

Ex. 1°. *A fixed segment or angle is cut harmonically by a variable pair of conjugates; the two homographic rows or pencils determined by the two points or lines of section* (329) *are in involution.*

For, each point or line of section has in every position the other for its correspondent to whichever system it be regarded as belonging; and therefore &c. (358).

Ex. 2°. *A variable segment or angle has a fixed pair of points or lines of bisection; the two similar and therefore homographic rows or pencils determined by its two bounding points or lines are in involution.*

For, each bounding point or line has in every position the other for its correspondent to whichever system it be regarded as belonging; and therefore &c. (358).

Ex. 3°. *Two variable points on a fixed line have a constant product of distances from a fixed point on the line; the two homographic rows they determine on the line* (331) *are in involution.*

For, each variable point has in every position the other for its correspondent to whichever system it be regarded as belonging; and therefore &c. (358).

Ex. 4°. *Two variable lines through a fixed point intersect constantly at right angles; the two similar and therefore homographic pencils they determine at the point are in involution.*

For, each variable line has in every position the other for its correspondent to whichever system it be regarded as belonging; and therefore &c. (358).

Ex. 5°. *Two variable points on a fixed circle connect constantly through a fixed point: the two homographic systems they determine on the circle* (315) *are in involution.*

For, each variable point has in every position the other for its correspondent to whichever system it be regarded as belonging; and therefore &c. (358).

Ex. 6°. *Two variable tangents to a fixed circle intersect constantly on a fixed line: the two homographic systems they determine to the circle* (315) *are in involution.*

For, each variable tangent has in every position the other for its correspondent to whichever system it be regarded as belonging; and therefore &c. (358).

N.B. To the first of the above examples, which the reader will readily perceive involves the remaining five, it will appear in the sequel that every case of involution between two homographic systems, of points on a common line or circle, or of tangents to a common point or circle, may be reduced.

362. The following additional examples of homographic systems in involution, all reducible to some or other of the preceding, and all of the same class with them, the law connecting the several pairs of corresponding constituents in their generation being given in all, are left as exercises to the reader.

Ex. 1°. A variable circle, passing through two fixed points, determines two systems of points in involution on any fixed line or circle.

Ex. 2°. A variable circle, coaxal with two fixed points, determines two systems of points in involution on any fixed line or circle.

Ex. 3°. A variable circle, of any coaxal system, determines two systems of points in involution on any fixed line or circle.

Ex. 4°. A variable circle, passing through a fixed point and intersecting a fixed line or circle at right angles, determines two systems of points in involution on the line or circle.

Ex. 5°. A variable circle, intersecting two fixed lines or circles (or a fixed line and circle) at right angles, determines two systems of points in involution on each line or circle.

Ex. 6°. A variable circle, passing through a fixed point and intersecting two fixed circles at equal (or supplemental) angles, determines two systems of points in involution on each circle.

Ex. 7°. A variable circle, intersecting three fixed circles at equal (or at any invariable combination of equal and supplemental) angles, determines two systems of points in involution on each circle.

Ex. 8°. A variable circle, having a fixed pole and polar, determines two systems; *a*. of points in involution on every line through the pole; *a'*. of rays in involution at every point on the polar.

Ex. 9°. Any number of circles, having a common pair of conjugate points or lines (174), determine two systems; *a*. in the former case, of points in involution on the connector of the points; *a'*. in the latter case, of rays in involution at the intersection of the lines.

Ex. 10°. Any number of circles, orthogonal to a common circle, determine two rows of points in involution on every diameter of the circle (156).

Ex. 11°. Any number of circles, intersecting two common circles at equal (or supplemental) angles, determine two rows of points in involution on every line passing through the external (or internal) centre of perspective of the circles (211, Cor. 1°.)

Ex. 12°. Any number of circles, intersecting two common circles at angles whose cosines have any constant ratio, determine two rows of points in involution on every line dividing the interval between their centres in the compound ratio of their radii and of the cosines of the angles. (193, Cor. 1°.)

N.B. In the first eight of the above examples, the relation of homography between the two systems has been already established in Art. 326, and in the remaining four it follows at once from examples 1° and 4° of the preceding article, by virtue of the properties referred to in their statements; the additional relation of involution between them appears in all from the same consideration (358) that, of every pair of constituents determined by the same circle, each has the other for its correspondent to whichever system it be regarded as belonging.

363. *Any two homographic systems of points on a common line or circle, or of tangents to a common point or circle, may, if not already in involution, be brought into the particular relative position constituting that relation, by the absolute movement of either, or both, on the common line or circle in the former case, or round the common point or circle in the latter case.*

For, taking arbitrarily any pair of corresponding constituents A and A' of the two systems, and determining, by Ex. 1°, Art. 353, the pair B and B' for which $AB = -A'B'$; a movement of either or both of the systems which would bring A to coincide with B' and B to coincide with A' would then, by the fundamental theorem of Art. 357, place them in involution with each other; and that without altering the relative directions of succession of the several constituents of one and of the corresponding constituents of the other.

Since, by the example in question (Ex. 1°, Art. 353), for each pair of corresponding constituents A and A' of the original systems, there exists, not only a pair B and B' for which $AB = -A'B'$, but also a pair C and C' for which $AC = +A'C'$; a movement of either or both of the systems which would bring A to coincide with C' and C to coincide with A' would also, by the same theorem of Art. 357, place them in involution with each other; but, of course, not without altering the direction of succession of the several constituents of one of them, that of the corresponding constituents of the other remaining unchanged.

Hence, *For every two homographic systems of points on a common line or circle, or of tangents to a common point or circle, there exist two different relative positions of involution with each other; the relative directions of succession of the several constituents of one and of the corresponding constituents of the other being opposites in the two positions.*

364. *For every two homographic systems in involution with each other, every three pairs of corresponding constituents determine a system of six points or lines, every four of which are equianharmonic with their four correspondents* (see Art. 283).

For, every point on the common line or circle, or tangent to the common point or circle, having the same correspondent to whichever system it be regarded as belonging (358), every two conjugate quartets determined by any three pairs of their corresponding constituents (283) are consequently conjugate quartets of the two systems, and as such are of course equianharmonic, the systems being homographic.

Conversely, *When, of two homographic systems of points on a common line or circle, or of tangents to a common point or circle, any two conjugate quartets determined by any three pairs of their corresponding constituents are equianharmonic, the two systems are in involution with each other.*

For, one of the three pairs of corresponding constituents being necessarily common to the two conjugate quartets (see

Art. 283), the equianharmonicism of the latter involves consequently the interchangeability of the former, and with it therefore, by the fundamental theorem of Art. 357, the involution of the systems.

Cor. 1°. As three pairs of corresponding constituents, of points on a common line or circle, or of tangents to a common point or circle, determine six different pairs of conjugate quartets (283), it follows, indirectly, from the above converse properties combined, *that the anharmonic equivalence of any one of the six pairs involves the anharmonic equivalence of each of the remaining five.* It was upon this property as basis (which it will be remembered was proved directly for collinear and concurrent systems in Art. 283, and otherwise indirectly for concyclic systems in Art. 313, Cor. 1°) that M. Chasles originally founded the whole theory of Involution; because that by means of it the relation is generally perceived to exist in cases (many of considerable interest which he was himself the first to investigate) where but three pairs of conjugates are given.

Cor. 2°. It is evident also from the same properties that *when any number of pairs of corresponding constituents, of points on a common line or circle, or of tangents to a common point or circle, form each an involution with the same two pairs, they form involutions three and three with each other;* or, to express the same thing differently, *when a variable pair of corresponding constituents, of points on a common line or circle, or of tangents to a common point or circle, form in every position an involution with two fixed pairs, they determine two homographic systems in involution with each other.*

365. The following are a few fundamental examples, grouped in reciprocal pairs, of cases of three pairs of corresponding constituents satisfying the criterion of the preceding article, and therefore in involution with each other. They were among the first originally given by Chasles, and have been shown to satisfy the criterion in the articles referred to with their statements respectively:

Ex. *a*. The three pairs of opposite connectors of every tetrastigm determine on every line a system of six points in involution (299, *a*).

Ex. a'. The three pairs of opposite intersections of every tetragram determine at every point a system of six rays in involution (299, a'),

Ex. b. The six parallels through any point to the three pairs of opposite connectors of any tetrastigm form a system of six rays in involution (299, Cor. 2°, a).

Ex. b'. The six projections on any line of the three pairs of opposite intersections of any tetragram form a system of six points in involution (299, Cor. 2°, a').

Ex. c. The three sides of any triangle, and any three concurrent lines through the three vertices, determine on every line a system of six points in involution (299, Cor. 4°, a).

Ex. c'. The three vertices of any triangle, and any three collinear points on the three sides, determine at every point a system of six rays in involution (299, Cor. 4°, a').

Ex. d. The three intersections with any line of the three sides of any triangle determine, with the three projections on the line of the three vertices of the triangle, a system of six points in involution (299, Cor. 5°, a).

Ex. d'. The three connectors with any point of the three vertices of any triangle determine, with the three parallels through the point to the three sides of the triangle, a system of six rays in involution (299, Cor. 5°, a'.).

Ex. e. Every circle, and any two of the three pairs of opposite connectors of any inscribed tetrastigm, determine on every line a system of six points in involution (301, Cor. 2°, a.).

Ex. e'. Every circle, and any two of the three pairs of opposite intersections of any exscribed tetragram, subtend at every point a system of six rays in involution (301, Cor. 2°, a'.).

Ex. f. Every three pairs of points on a circle which connect by concurrent lines form a system of six points in involution (313.).

Ex. f'. Every three pairs of tangents to a circle which intersect at collinear points form a system of six tangents in involution (313.).

Ex. g. Every three pairs of points on a line or circle, harmonic conjugates to each other with respect to the same two points on the line or circle, form a system of six points in involution (282, Cor. 5°, 3°.).

Ex. g'. Every three pairs of tangents to a point or circle, harmonic conjugates to each other with respect to the same two tangents to the point or circle, form a system of six tangents in involution (282, Cor. 5°, 3°.).

366. To the preceding fundamental cases of involution between three pairs of corresponding constituents, several others, involving like them but three pairs of conjugates, are reducible; the following are some examples, grouped in reciprocal pairs, the reductions of which are left as exercises to the reader.

Ex. a. When the directions of three segments are concurrent, the six centres of perspective of their three groups of two determine at every point a system of six rays in involution (295, Cor. 1°.).

Ex. *a'*. When the vertices of three angles are collinear, the six axes of perspective of their three groups of two determine on every line a system of six points in involution (295, Cor. 1°.).

Ex. *b*. When the extremities of three segments form an equianharmonic hexastigm, the six centres of perspective of their three groups of two determine at every point a system of six rays in involution (303, *d*.).

Ex. *b'*. When the sides of three angles form an equianharmonic hexagram, the six axes of perspective of their three groups of two determine on every line a system of six points in involution (303, *d'*.).

Ex. *c*. When three segments combine the characteristics of examples *a* and *b*, the six centres of perspective of their three groups of two are collinear and in involution (295, Cor. 4°.).

Ex. *c'*. When three angles combine the characteristics of examples *a'* and *b'*, the six axes of perspective of their three groups of two are concurrent and in involution (295, Cor. 4°.).

Ex. *d*. The six centres of perspective, of any three chords inscribed to a circle taken in pairs, determine at every point a system of six rays in involution (317, Cor. 1°.).

Ex. *d'*. The six axes of perspective, of any three angles exscribed to a circle taken in pairs, determine on every line a system of six points in involution (317, Cor. 1°.).

Ex. *e*. When the directions of three chords inscribed to a circle are concurrent, the six centers of perspective of their three groups of two are collinear and in involution (317, Cor. 4°.).

Ex. *e'*. When the vertices of three angles exscribed to a circle are collinear, the six axes of perspective of their three groups of two are concurrent and in involution (317, Cor. 4°.).

Ex. *f*. When two triangles either inscribed or exscribed to the same circle are in perspective, their three pairs of corresponding sides determine six points in involution on every line through the centre of perspective (320).

Ex. *f'*. When two triangles either exscribed or inscribed to the same circle are in perspective, their three pairs of corresponding vertices determine six rays in involution at every point on the axis of perspective (320).

Ex. *g*. When four circles pass through a common point, the six axes of perspective through the point of their six groups of two form a system of six rays in involution. (See Ex. *a'* of preceding Art.)

Ex. *g'*. When four circles touch a common line, the six centres of perspective on the line of their six groups of two form a system of six points in involution. (See Ex. *a* of preceding Art.)

Ex. *h*. Every two circles and their two centres of perspective subtend at every point a system of six rays in involution.

Ex. *h'*. Every two circles and their two axes of perspective determine on every line a system of six points in involution.

N.B. In the reduction of these two last examples to examples *e'* and *e* of the preceding article respectively, it is to be remembered, with respect to any two circles, that the two centres of perspective are a pair of opposite intersections of the tetragram exscribed to both determined by their four common tangents, and that the two axes of perspective are a pair of opposite connectors of the tetrastigm inscribed to both determined by their four common points. (See Art. 207.)

367. *When two homographic rows of points on a common axis, or pencils of rays through a common vertex, are in involution; every three pairs of corresponding constituents A and A', B and B', C and C' are connected.*

a. In the former case, by the symmetrical relation

$$\frac{BA'}{CA'} \cdot \frac{CB'}{AB'} \cdot \frac{AC'}{BC'} = 1,$$

or, which is the same thing, by the equivalent relation

$$BC'.CA'.AB' + B'C.C'A.A'B = 0;$$

a'. In the latter case, by the corresponding relation

$$\frac{\sin BA'}{\sin CA'} \cdot \frac{\sin CB'}{\sin AB'} \cdot \frac{\sin AC'}{\sin BC'} = 1,$$

or, which is the same thing, by the equivalent relation

$$\sin BC'.\sin CA'.\sin AB' + \sin B'C.\sin C'A.\sin A'B = 0;$$

every constituent being interchangeable with its conjugate in each (357). *And, conversely, when of two homographic rows of points on a common axis, or pencils of rays through a common vertex, any three pairs of corresponding constituents A and A', B and B', C and C' are connected by relation* (a) *in the former case, or by relation* (a') *in the latter case; the two systems are in involution.*

For, taking any four of the six points or rays, B, C, A', C' suppose, and equating any one of their six anharmonic ratios,

$$BA'.CC' \div CA'.BC' \text{ or } \sin BA'.\sin CC' \div \sin CA'.\sin BC'$$

suppose, to the corresponding anharmonic ratio

$$B'A.C'C \div C'A.B'C \text{ or } \sin B'A.\sin C'C \div \sin C'A.\sin B'C$$

of their four correspondents B', C', A, C (364), the relation a or a' immediately results; from which again, conversely, the anharmonic equivalence of the two conjugate quartets B, C, A', C' and B', C', A, C (or, from its symmetry, of each of the three pairs of conjugate quartets B, C, A', C' and B', C', A, C; C, A, B', A' and C', A', B, A; A, B, C', B' and A', B', C, B) reciprocally results; and therefore &c. (364).

The above relation, a or a' as the case may be, (or any of the three others of similar form resulting from it by the three interchanges (357) of the three involved pairs of conjugates) being characteristic of the involution of the three pairs of collinear points or concurrent rays A and A', B and B', C and C', is termed accordingly *their equation of involution;* and, geometrically interpreted, it expresses, in a form at once concise and symmetrical, the anharmonic equivalence (364) of every two conjugate quartets of their six constituent points or rays.

368. The following are a few examples of the application of the preceding relation as a criterion of involution between three pairs of collinear points or concurrent rays; in some of which the equianharmonic relations of Art. 364, previously established on other principles, may be regarded as thus verified at the same time:

Ex. 1°. *The three intersections with any line of the three sides of any triangle determine, with the three projections on the line of the three vertices of the triangle, a system of six points in involution* (Ex. *d*, Art. 365).

For, if P, Q, R be the three vertices of the triangle; A, B, C their three projections on the line; and A', B', C' the three intersections of the opposite sides with the same; then, since (Euc. VI. 4.)

$$\frac{BA'}{CA'}=\frac{BQ}{CR},\quad \frac{CB'}{AB'}=\frac{CR}{AP},\quad \frac{AC'}{BC'}=\frac{AP}{BQ},$$

therefore at once, by composition of ratios,

$$\frac{BA'}{CA'}\cdot\frac{CB'}{AB'}\cdot\frac{AC'}{BC'}=1;$$

and therefore &c., by relation (a).

Ex. 2°. *The three connectors with any point of the three vertices of any triangle determine, with the three parallels through the point to the three sides of the triangle, a system of six rays in involution* (Ex. *d'*, Art. 365.).

For, if A, B, C be the three vertices of the triangle; O the point; and OA', OB', OC' the three parallels through it to the opposite sides; then, since (63)

$$\frac{\sin BOA'}{\sin COA'}=\frac{OC}{OB},\quad \frac{\sin COB'}{\sin AOB'}=\frac{OA}{OC},\quad \frac{\sin AOC'}{\sin BOC'}=\frac{OB}{OA},$$

therefore at once, by composition of ratios,

$$\frac{\sin BOA'}{\sin COA'}\cdot\frac{\sin COB'}{\sin AOB'}\cdot\frac{\sin AOC'}{\sin BOC'}=1;$$

and therefore &c., by relation (a').

Ex. 3°. *The three sides of any triangle, and any three concurrent lines through the three vertices, determine on every line a system of six points in involution* (Ex. *c*, Art. 365).

For, if P, Q, R be the three vertices of the triangle; A, B, C their three perspectives on the line from any arbitrary point O; and A', B', C' the three intersections of the opposite sides with the same; then, since (134, *a*)

$$\frac{BA'}{CA'} = \frac{BQ}{CR} : \frac{OQ}{OR}, \quad \frac{CB'}{AB'} = \frac{CR}{AP} : \frac{OR}{OP}, \quad \frac{AC'}{BC'} = \frac{AP}{BQ} : \frac{OP}{OQ},$$

therefore at once, by composition of ratios,

$$\frac{BA'}{CA'} \cdot \frac{CB'}{AB'} \cdot \frac{AC'}{BC'} = 1;$$

and therefore &c., by relation (*a*).

Ex. 4°. *The three vertices of any triangle, and any three collinear points on the three sides, determine at every point a system of six rays in involution* (Ex. *c'*, Art. 365).

For, if A, B, C be the three vertices of the triangle; A', B', C' the three collinear points on the opposite sides; and O any arbitrary point; then, since (65)

$$\frac{\sin BOA'}{\sin COA'} = \frac{BA'}{CA'} : \frac{BO}{CO}, \quad \frac{\sin COB'}{\sin AOB'} = \frac{CB'}{AB'} : \frac{CO}{AO}, \quad \frac{\sin AOC'}{\sin BOC'} = \frac{AC'}{BC'} : \frac{AO}{BO},$$

therefore at once, by composition of ratios,

$$\frac{\sin BOA'}{\sin COA'} \cdot \frac{\sin COB'}{\sin AOB'} \cdot \frac{\sin AOC'}{\sin BOC'} = \frac{BA'}{CA'} \cdot \frac{CB'}{AB'} \cdot \frac{AC'}{BC'};$$

which latter being = 1 (134, *a*), therefore &c., by relation (*a'*).

Ex. 5°. *The six perpendiculars to any line through the three vertices, and through any three collinear points on the three sides, of any triangle determine on the line a system of six points in involution* (Ex. *b'*, Art. 365).

For, if P, Q, R be the three vertices of the triangle; A, B, C their three projections on the line; P', Q', R' the three collinear points on the opposite sides; and A', B', C' their three projections on the line; then, since (Euc. VI. 10.)

$$\frac{BA'}{CA'} = \frac{QP'}{RP'}, \quad \frac{CB'}{AB'} = \frac{RQ'}{PQ'}, \quad \frac{AC'}{BC'} = \frac{PR'}{QR'},$$

therefore at once, by composition of ratios,

$$\frac{BA'}{CA'} \cdot \frac{CB'}{AB'} \cdot \frac{AC'}{BC'} = \frac{QP'}{RP'} \cdot \frac{RQ'}{PQ'} \cdot \frac{PR'}{QR'};$$

which latter being = 1 (134, *a*), therefore &c., by relation (*a*).

Ex. 6°. *The six perpendiculars through any point to the three sides, and to any three concurrent lines through the three vertices, of any triangle determine at the point a system of six rays in involution* (Ex. *b*, Art. 365).

For, if P, Q, R be the three vertices of the triangle; I the point of concurrence of the three lines passing through them; O the point through which the six perpendiculars pass; OA, OB, OC the three of them to the three lines IP, IQ, IR; and OA', OB', OC' the three of them to the three opposite sides QR, RP, PQ; then since (63)

$$\frac{\sin BOA'}{\sin COA'} = \frac{IR}{IQ}, \quad \frac{\sin COB'}{\sin AOB'} = \frac{IP}{IR}, \quad \frac{\sin AOC'}{\sin BOC'} = \frac{IQ}{IP},$$

therefore at once, by composition of ratios,

$$\frac{\sin BOA'}{\sin COA'} \cdot \frac{\sin COB'}{\sin AOB'} \cdot \frac{\sin AOC'}{\sin BOC'} = 1;$$

and therefore &c., by relation (a').

Ex. 7°. *When three circles of a coaxal system touch the three sides of a triangle at three points which are either collinear or concurrently connectant with the opposite vertices; their three centres form, with those of the three circles of the system which pass through the three vertices of the triangle, a system of six points in involution.**

For, if P, Q, R be the three vertices of the triangle; P', Q', R' the three points of contact on the opposite sides; A, B, C the centres of the three circles passing through P, Q, R; and A', B', C' those of the three touching at P', Q', R'; then since (192, Cor. 1°.)

$$\frac{PQ'^2}{PR'^2} = \frac{AB'}{AC'}, \quad \frac{QR'^2}{QP'^2} = \frac{BC'}{BA'}, \quad \frac{RP'^2}{RQ'^2} = \frac{CA'}{CB'},$$

therefore at once, by composition of ratios,

$$\frac{PQ'^2}{PR'^2} \cdot \frac{QR'^2}{QP'^2} \cdot \frac{RP'^2}{RQ'^2} = \frac{AB'}{AC'} \cdot \frac{BC'}{BA'} \cdot \frac{CA'}{CB'};$$

the former of which being = 1 (134, a or b'), therefore &c., by relation (a).

369. The following additional examples of the application of the same relation, as a criterion of involution between three pairs of collinear points or concurrent rays, are left as exercises to the reader.

Ex. 1°. If a segment or angle AA' be cut harmonically by any two pairs of conjugates B and C, B' and C'; the three pairs of collinear points or concurrent rays A and A', B and B', C and C' are in involution.

Ex. 2°. The two pairs of conjugates A and A', B and B' of any harmonic system are in involution with the two harmonic conjugates C and C' of every collinear point or concurrent ray with respect to themselves.

Ex. 3°. If A, B, C be any three collinear points or concurrent rays, A' any fourth collinear point or concurrent ray, and B', C' the two har-

* This property was communicated to the Author by Mr. Casey.

monic conjugates of A' with respect to C and A, A and B respectively; the three pairs A and A', B and B', C and C' are in involution.

Ex. 4°. If A, B, C be any three collinear points or concurrent rays, and A', B', C' the three harmonic conjugates of any fourth collinear point or concurrent ray D with respect to B and C, C and A, A and B respectively; the three triads of pairs B and B', C and C', A and D; C and C', A and A', B and D; A and A', B and B', C and D are each in involution.

Ex. 5°. If A, B, C be any three collinear points or concurrent rays, A', B', C' any other three collinear or concurrent with them, and A'', B'', C'' the three harmonic conjugates of A', B', C' with respect to B and C, C and A, A and B respectively; the three triads A', B'', C''; A'', B', C''; A'', B'', C' are each in involution with the triad A, B, C.

370. *Every two conjugate points or lines of two homographic systems in involution are harmonic conjugates with respect to the two double points or lines, real or imaginary, of the systems.* (See Art. 342).

For, since, for any three pairs of conjugates A and A', B and B', C and C' in involution, $\{ABCC'\} = \{A'B'C'C\}$ (364); if $A = A' = M$ and $B = B' = N$, which is the characteristic of the two double points or lines (341), then $\{MNCC'\} = \{MNC'C\}$, whatever be the third pair C and C'; and therefore &c. (281).

This very simple law connecting the several pairs of corresponding constituents, in every case of involution between two homographic systems of any common species, would also have followed at once negatively from its converse shewn already (361, Ex. 1°) to result directly from the fundamental definition of involution (358). And, while confirming the statement in the note at the close of the same article (361), it evidently comprehends in a form at once simple and complete every other law connected with the subject.

371. The following are immediate consequences from the general property of the preceding article.

1°. *In every involution of points on, or tangents to, a common circle.*

a. The several pairs of conjugate points, in the former case, connect through a common point (257).

a'. The several pairs of conjugate lines, in the latter case, intersect on a common line (257).

b. The two double points, in the former case, lie on the polar of the common point with respect to the circle (165, 6°.).

b'. The two double lines, in the latter case, pass through the pole of the common line with respect to the circle (165, 6°.).

These properties, which shew in fact *that every two homographic systems of points on or tangents to a common circle in involution are in perspective, and that the two double points or lines lie on the polar of the centre or pass through the pole of the axis of perspective with respect to the circle,* would also have followed at once negatively from their converses shewn already (361, Ex. 5° and 6°) to result immediately from the fundamental definition of involution (358); or, they would have followed directly from the second part of the general property of Art. 313, by virtue of the equianharmonic relations of Art. 364.

2°. *In every involution of points on a common axis.*

a. The several circles passing through the several pairs of conjugates, and any common point not on the axis, pass all through a second common point not on the axis (226, 2°.).

b. The line connecting the two common points through which they all pass bisects the interval, real or imaginary, between the two double points of the systems (226, 2°.).

c. The rectangle under the distances of the several pairs of conjugates from the point of bisection is constant, and equal in magnitude and sign to the square of the semi-interval between the double points (225).

From the third of these properties (which, like that from which it results (370), would also have followed at once negatively from its converse shown already (361, Ex. 3°) to result from the fundamental definition of involution), it appears that—*For every two homographic rows of points in involution on a common axis, there exists a point (always real and evidently conjugate to that at infinity on the axis), the rectangle under whose distances from the several pairs of conjugates is constant, in magnitude and sign, and equal to the square of the semi-interval, real or imaginary, between their two double points.* The point possessing this property is termed *the centre of the involution;* and the involution itself is said *to be positive or negative* according as the sign of the constant rectangle is positive or negative, or, which is the same thing, according as the two double points of the systems are real or imaginary.

COR. 1°. The above properties 1° and 2° supply obvious and rapid solutions; the former of the following problems—

Given two pairs of conjugate points or tangents of two homographic systems in involution on a common circle; to determine, a. the centre or axis of perspective of the systems; b. the two double points or tangents, real or imaginary, of the systems; c. the conjugate to any third point or tangent of either system; d. the pair of conjugates having a given middle point or tangent; e. the two pairs of conjugates intercepting a chord or angle of given magnitude; f. the two pairs of conjugates determining with two given points or tangents a given anharmonic ratio.

Given two pairs of conjugate points or tangents of each of two different involutions on the same common circle; to determine the pair of conjugates common to both involutions.

And the latter of the corresponding problems—

Given two pairs of conjugates of two homographic rows of points in involution on a common axis; to determine, a. the centre of the involution; b. the two double points, real or imaginary, of the systems; c. the conjugate to any third point of either system; d. the pair of conjugates having a given middle point; e. the two pairs of conjugates intercepting a segment of given length; f. the two pairs of conjugates dividing a given segment in a given anharmonic ratio.

Given two pairs of conjugates of each of two different involutions of points on the same common axis; to determine the pair of conjugates common to both involutions.

The corresponding problems for homographic pencils of rays in involution through a common vertex are not included directly in any of the above; but they are evidently reducible immediately to those for the two homographic systems of points determined by the pencils on any circle passing through their common vertex (306), or on any line not passing through it (285).

COR. 2°. Since, for two homographic systems of points in involution on a common circle, there exists always one, and in general but one, pair of corresponding constituents diametrically opposite to each other, viz. those determined by the diameter of the circle which passes through their centre of perspective; and since when two then all pairs are diametrically opposite, the centre of the circle being in that case the centre of perspective of the systems. Hence, conceiving an arbitrary circle passing through the common vertex of any two homographic pencils of rays in involution, it follows at once (Euc. III. 31) that—

For two homographic pencils of rays in involution through a common vertex.

a. There exists always one, and in general but one, pair of conjugate rays which intersect at right angles.

b. When two pairs of conjugates intersect at right angles, then all pairs of conjugates intersect at right angles.

N.B. These latter properties, which admit also of easy direct demonstration, are often useful in the higher departments of geometry.

372. The property of the centre (371, 2°, *c*) in the case of two homographic rows of points in involution on a common axis, viz. that the rectangle under its distances from every pair of conjugates is constant in magnitude and sign, follows also immediately from the general property (331) of the two correspondents of the point at infinity of any two homographic rows of points on a common axis.

For, if P and Q be the two correspondents to the point at infinity on the common axis regarded as belonging first to one and then to the other of the two rows; since then always, for every pair of corresponding constituents A and A', by the property in question, the rectangle $PA.QA'$ is constant in magnitude and sign; therefore, when the rows are in involution, and when consequently (358) $P = Q = O$ (the point at infinity like every other point on the common axis having the same correspondent to whichever system it be regarded as belonging), the rectangle $OA.OA'$ is constant in magnitude and sign; and therefore &c.

This very simple property of involution might have been made the basis of the entire theory; but, as it belongs only to collinear systems of points on a common axis, that actually employed (358), being applicable alike to involutions of all species without exception, has been adopted in preference.

373. The property (363) that any two homographic rows of points on a common axis, may, if not already in involution,

be brought in two different ways into the particular relative position to be so, follows also as an easy consequence from the same general property (331).

For, if the two correspondents P and Q of the point at infinity on the common axis, regarded as belonging first to one and then to the other system, do not already coincide; they may first be brought together to a common point O by the absolute moment of one or both of the systems along the common axis, thus giving one position of involution (361, Ex. 3°); and then, when together, the axis may be turned round the point O as centre, carrying with it one system but not the other, and brought again to coincide with its original position in the opposite direction, thus giving another and opposite position of involution (361, Ex. 3°).

From this way of regarding the question, it appears that the constant rectangle $OA.OA'$ is the same in magnitude, but opposite in sign, in the two positions of involution of the two rows; hence, the value of that rectangle being of course the constant by which alone any one involution of points on a common axis differs from any other, all such involutions being evidently similar in figure and differing only in magnitude, it appears that—

In the two positions of involution of the same two homographic rows of points on a common axis, the two constants of involution are always equal in magnitude and opposite in sign.

374. The property of the centre supplies in many cases a very simple criterion of the relation of involution between three or more pairs of corresponding points on a common axis; as, for instance, in the three following examples:

Ex. 1°. *Every line passing through their radical centre intersects with any three circles at three pairs of points in involution.*

For, if A and A', B and B', C and C' be the three pairs of intersections, real or imaginary, and O the radical centre; then, since (183) $OA.OA' = OB.OB' = OC.OC'$, therefore &c.

Conversely, *Every line intersecting with any three circles at three pairs of points in involution passes through their radical centre.*

For, if A and A', B and B', C and C' be the three pairs of intersections, and O the centre of their involution; then, since $OA.OA' = OB.OB = OC.OC'$, therefore &c. (183).

Ex. 2°. *When a number of circles have a common radical centre, every line passing through it intersects with them at as many pairs of points in involution.*

For, if A and A', B and B', C and C', D and D', &c. be the several pairs of intersections, real or imaginary, and O the common radical centre; then since, as before, $OA \cdot OA' = OB \cdot OB' = OC.OC' = OD \cdot OD' =$ &c. therefore &c.

Conversely, *When a number of circles intersect with a line at as many pairs of points in involution, they have a common radical centre through which the line passes.*

For, if A and A', B and B', C and C', D and D', &c. be the several pairs of intersections, and O the centre of their involution; then, since $OA \cdot OA' = OB \cdot OB' = OC \cdot OC' = OD \cdot OD' =$ &c. therefore &c.

Ex. 3°. *When a number of circles have a common radical axis, every line intersects with them at as many pairs of points in involution.*

For, if A and A', B and B', C and C', D and D' be the several pairs of intersections, real or imaginary, with the several circles, and O the intersection with the radical axis; then, since (187, 1°) $OA \cdot OA' = OB \cdot OB' = OC \cdot OC' = OD \cdot OD' =$ &c. therefore &c.

Conversely, *When a number of circles intersect with three different lines, which are not concurrent, at as many pairs of points in involution, they have a common radical axis.*

For, at the centre of each involution they have a common radical centre; and as the three lines by hypothesis, are not concurrent, two at least of the three centres must necessarily be different; and therefore &c. (187, 1°).

375. For any two homographic systems of points on a common line or circle, or of tangents to a common point or circle, from the general properties of Arts. 364 and 370, it may be shown immediately that—

1°. *Every two corresponding pairs of non-corresponding constituents A and B', A' and B are in involution with the two double points or lines M and N, real or imaginary, of the systems.* (See Arts. 349 and 371, 2°, *a*).

For, since, for every four pairs of corresponding constituents A and A', B and B', C and C', D and D', by the homography of the systems, $\{ABCD\} = \{A'B'C'D'\}$; if $C = C' = M$ and $D = D' = N$, which is the characteristic of the two double points or lines (341), then $\{ABMN\} = \{A'B'MN\} = \{B'A'NM\}$ (280); and therefore &c. (364).

Or thus, for either case of the circle, to which the others are of course reducible. The three lines of connection (or points of intersection) AB', $A'B$, MN being concurrent (or collinear)

(337); therefore $\{ABMN\} = \{B'A'MN\}$ (313); and therefore &c. (364).

2°. *If P_1 and P_2 be the two correspondents of any constituent P, regarded as belonging first to one and then to the other system, and P_3 the harmonic conjugate of P with respect to P_1 and P_2; then, as P varies—*

a. The two systems determined by P_1 and P_2 are homographic, and have the same double points or lines with the original systems.

b. The two systems determined by P and P_3 are in involution, and have also the same double points or lines with the original systems.

c. In every position, P and P_3 are harmonic conjugates with respect to the two double points or lines of the original systems.

Of these properties; the first (a) is evident from the consideration that the two systems determined by P_1 and P_2 are homographic with that determined by P and therefore with each other (323), and that when P in the course of its variation coincides with either double point or line M or N of the original systems, its two correspondents P_1 and P_2 coincide with the same double point or line, and therefore with each other; and the second (b) follows immediately (370) from the third (c), which may be proved as follows:

The pair of points or lines P_1 and P_2, the pair M and N, and the coincident pair P and P, being in involution, by the preceding 1°, have therefore a common pair of harmonic conjugates (370); one of which being, of course, the double point or line P, its harmonic conjugate P_3 with respect to P_1 and P_2 is therefore its harmonic conjugate with respect to M and N also; and therefore &c.

Or thus, for either case of the circle, to which the others are of course reducible. The three lines of connexion (or points of intersection) P_1P_2, PP, MN being concurrent (or collinear) (337); and the three lines of connexion (or points of intersection) P_1P_2, PP, P_3P_3 being also concurrent (or collinear) (257); therefore the three lines of connexion (or points of intersection) PP, P_3P_3, MN are also concurrent (or collinear); and therefore &c. (257).

376. The general property of Art. 370 supplies obvious and rapid solutions of the two following pairs of reciprocal problems, viz.—

a. Through a given point to draw a line intersecting two given angles, or circles, so that the point shall be a double point of the involution determined by the two pairs of intersections.

a'. On a given line to find a point subtending two given segments, or circles, so that the line shall be a double line of the involution determined by the two pairs of subtenders.

For, since, by the property of that article (370), the two double points (or lines) of the involution in question are harmonic conjugates with respect to the two pairs of intersections (or subtenders), and therefore conjugate points (or lines) with respect to the two angles (or segments) (217), or to the two circles (259, *a* or *a'*); therefore, the two polars of the given point (or poles of the given line) with respect to the two given angles (or segments) (217), or to the two given circles (174), determine, by their point of intersection (or line of connexion), the second double point (or line) of that involution; and the two double points (or lines) being thus known, their line of connexion (or point of intersection) is of course the required line (or point).

N.B. When, in the former case, the given point has the same polar with respect to the two given angles or circles, and when, in the latter case, the given line has the same pole with respect to the two given segments or circles, the above reciprocal constructions become, as they ought, indeterminate; every line through the given point in the former case, and every point on the given line in the latter case, then evidently satisfying the conditions of the problem.

377. The equianharmonic relations of Art. 364, combined with the general property of Art. 327, reduce also the solutions of the two following reciprocal problems to those of the first parts of the two of Ex. 7°, Art. 353; viz.—

a. Through a given point to draw a line intersecting with five given lines, so that any two assigned pairs of the five intersections shall be in involution with the point and fifth.

a'. On a given line to find a point connecting with five given *points, so that any two assigned pairs of the five connectors shall be in involution with the line and fifth.*

For, denoting by C the given point (or line), by A and A', B and B' the two assigned pairs of intersections (or connectors) for *any* line drawn through (or point taken on) C, by C' the conjugate of C in the involution determined by A and A', B and B' on (or at) that line (or point), by O and O' the vertices of the two angles (or axes of the two segments) determined by the two pairs of the given lines (or points) corresponding to A and B, A' and B' respectively, and by I the fifth given line (or point) on (or through) which C' is to lie (or pass) in the required involution; then, whatever be the position of the line drawn through (or of the point taken on) C, since $\{ABCC'\} = \{B'A'CC'\}$ (364), therefore $\{O.ABCC'\} = \{O'.B'A'CC'\}$ (285); and since the three pairs of corresponding rays (or points) OA and $O'B'$, OB and $O'A'$, OC and $O'C$ are fixed, therefore the pair OC' and $O'C'$, which vary with the position of that line (or point), determine two homographic systems (327) whose two pairs of corresponding constituents intersecting on (or connecting through) I (353, Ex. 7°, a' or a) determine the two positions of C' whose connectors (or intersections) with C give the two solutions of the problem a (or a').

N.B. When the given line (or point) I coincides with one, and the given point (or line) C lies on (or passes through) the other, of the two axes (or centers) of perspective of the two angles (or segments) determined by the two pairs of given lines (or points) corresponding to the two pairs of conjugates A and A', B and B' of the involution; the two positions of C', given by the above, become, as they ought to be, indeterminate; every line passing through (or point lying on) C then evidently determining the required involution. (See examples a and a', Art. 365).

Cor. When, in the former problem, the line I is at infinity; then the point C, being the conjugate of the point at infinity on its axis, is consequently the centre of the involution determined on the required line by the two pairs of intersections A and A', B and B' (372); hence by the above are given the two solutions, real or imaginary, of the problem.

Through a given point to draw a line intersecting two given angles, so that the point shall be the centre of the involution determined by the two pairs of intersections.

When the two given angles are two opposite angles of a parallelogram, and when the given point is on the diagonal not passing through their vertices, this problem is indeterminate for the same reason as in the general case; the point being then evidently (Euc. VI. 16) the centre of the involution for every line passing through it.

378. From the two reciprocal properties of Art. 337, respecting the directive axis of two homographic rows of points on different axes, and the directive centre of two homographic pencils of rays through different vertices, the two following reciprocal properties of involution, with respect to such systems, may be immediately inferred; viz.—

a. Every line intersecting two homographic pencils of rays through different vertices in two homographic rows of points in involution passes through their directive centre; and, conversely, every line passing through the directive centre of two homographic pencils of rays through different vertices intersects them in two homographic rows of points in involution.

a′. Every point subtending two homographic rows of points on different axes by two homographic pencils of rays in involution lies on their directive axis; and, conversely, every point lying on the directive axis of two homographic rows of points on different axes subtends them by two homographic pencils of rays in involution.

For, if a line intersect (or a point connect) with the two systems of rays (or points) in two homographic rows (or by two homographic pencils) in involution, the two correspondents A' and B of every two rays (or points) A and B' which intersect on (or connect through) it, must also intersect on (or connect through) it (358), and therefore &c. (337); and, conversely, if a line pass through the directive centre (or a point lie on the directive axis) of the two systems of rays (or points), the two correspondents A' and B of every two rays (or points) A and B' which intersect on (or connect through) it, must also intersect on (or connect through) it (337), and therefore &c. (358).

By virtue of the fundamental theorem of Art. 357 the same results may be arrived at, without the aid of the reciprocal properties of Art. 337, from the consideration that when the point on the line at which it intersects with the ray common to the two pencils (or the line through the point by which it connects with the point common to the two rows) has the same correspondent to whichever of the two rows of intersection (or pencils of connection) it be regarded as belonging (357), the line (or point) itself passes through the intersection (or lies on the connector) of the two correspondents of the common ray (or point); and, conversely, when the line (or point) passes through that intersection (or lies on that connector), its two rows of intersection (or pencils of connection) with the two systems of rays (or points) have, in that intersection (or connector) and in its own intersection (or connector) with the ray (or point) common to the two systems, a pair of interchangeable correspondents; and therefore &c. (357).

COR. 1°. When, in the former case, the ray common to the two pencils is at infinity; that is, when the pencils consist each of parallel lines; their directive centre being then the conjugate to the point at infinity, and therefore the centre, of the involution they determine on every line passing through it (337), it appears consequently, from the above, that—

When two homographic pencils consist each of parallel lines, their directive centre is the centre of the involution they determine on every line passing through it.

COR. 2°. The above reciprocal properties supply, in the general case, obvious and rapid solutions of the following reciprocal problems; viz.—

a. To draw a tangent to a given point or circle, whose two triads of intersections with two given triads of concurrent lines through different vertices shall be in involution with any assigned correspondence of pairs of constituents.

a′. To find a point on a given line or circle, whose two triads of connectors with two given triads of collinear points on different axes shall be in involution with any assigned correspondence of pairs of constituents.

379. The two reciprocal properties of Art. 337, supply also solutions of the two following reciprocal problems; viz.—

a. Given three pairs of corresponding constituents of two homographic pencils of rays through different vertices; to describe a circle passing through their two vertices, and determining with them two concyclic triads of points in involution.

a'. Given three pairs of corresponding constituents of two homographic rows of points on different axes; to describe a circle touching their two axes, and determining with them two concyclic triads of tangents in involution.

For, as the two points (or tangents) P and Q, determined with the required circle by any pair of corresponding rays (or points) A and A' of the two given triads, must, by the properties in question, connect through the directive centre (or intersect on the directive axis) O of the two systems, which is given with the two triads (337); and as, in addition, the direction of their line of connection (or the sum or difference of the distances from A and A' of their point of intersection) PQ is given, being manifestly the same for *every* circle passing through the two vertices (or touching the two axes); the solution of the problem is therefore evident in the former case, and reducible to that of Art. 54 in the latter case; and therefore &c.

N.B. Since, for *every* pair of corresponding constituents A and A' of the two homographic pencils (or rows), the line (or point) PQ passes through (or lies on) their directive centre (or axis) O, when they determine with the circle two systems in involution (337); it follows, consequently, that the two determined systems, when in involution, are in perspective. A property which, it will be remembered, was proved for every two concyclic systems of points (or tangents) in involution, in 1°, Art. 371.

CHAPTER XXII.

METHODS OF GEOMETRICAL TRANSFORMATION. THEORY OF HOMOGRAPHIC FIGURES.

380. Two figures of any kind, F and F', in which correspond, to every point of either a point of the other, to every line of either a line of the other, to every connector of two points of either the connector of the two corresponding points of the other, and to every intersection of two lines of either the intersection of the two corresponding lines of the other, are said *to be homographic* when every two of their corresponding quartets whether of collinear points or of concurrent lines are equianharmonic. Every two figures in perspective with each other to any centre and axis (141) are evidently thus related to each other (286, 2°).

As two anharmonic quartets of any kind, when each equianharmonic with a common quartet, are equianharmonic with each other; it follows at once, from the above definition, that *when two figures of any kind F' and F'' are each homographic with a common figure F, they are homographic with each other.*

381. Every two figures F and F' satisfying the four preliminary conditions, whether homographic or not, possess evidently the following properties in relation to each other.

1°. *To every collinear system of points or concurrent system of lines of either, corresponds a collinear system of points or concurrent system of lines of the other.*

For, every connector of two points (or intersection of two lines) of either corresponding to the connector of the two corresponding points (or the intersection of the two corresponding lines) of the other, when, for any system of the points (or lines) of either, every two connect by a common line (or intersect at a common point), then, for the corresponding system of the points (or lines) of the other, every two connect by the corresponding line (or intersect at the corresponding point); and therefore, &c.

2°. *To every two collinear systems of points or concurrent systems of lines of either in perspective with each other, correspond two collinear systems of points or concurrent systems of lines of the other in perspective with each other.*

For, the concurrence (or collinearity) of the several lines of connection (or points of intersection) of the several pairs of corresponding constituents of the two systems, for either, involves, by 1°, a similar concurrence of connectors (or collinearity of intersections) of pairs of corresponding constituents of the two corresponding systems, for the other; and therefore &c. (130).

3°. *To every two figures of the points and lines of either in perspective with each other, correspond two figures of the points and lines of the other in perspective with each other.*

For, the concurrence of the several lines of connection of the several pairs of corresponding points, and the collinearity of the several points of intersection of the several pairs of corresponding lines, of the two figures, for either, involve, by 1°, a similar concurrence of connectors and collinearity of intersections of pairs of corresponding constituents of the two corresponding figures, for the other; and therefore &c. (141).

4°. *To a variable point moving on a fixed line or a variable line turning round a fixed point of either, corresponds a variable point moving on the corresponding fixed line or a variable line turning round the corresponding fixed point of the other.*

For, since every two positions of the variable point (or line) connect by the same fixed line (or intersect at the same fixed point) for the former; therefore by 1°, every two positions of the variable point (or line) connect by the corresponding fixed line (or intersect at the corresponding fixed point) for the latter; and therefore &c.

5°. *To a variable point or line of either the ratio of whose distances from two fixed lines or points is constant, corresponds a variable point or line of the other the ratio of whose distances from the two corresponding fixed lines or points is constant.*

For, since the variable point (or line) evidently moves on a line concurrent with the two fixed lines (or turns round a point collinear with the two fixed points) for the former; therefore, by the preceding property 4°, the variable point (or line) moves on a line concurrent with the two corresponding fixed lines (or turns round a point collinear with the two corresponding fixed points) for the latter; and therefore &c.

6°. *To a variable polygon of either all whose vertices move on fixed lines and all whose sides but one turn round fixed points, or conversely, corresponds a variable polygon of the other all whose vertices move on the corresponding fixed lines and all whose sides but one turn round the corresponding fixed points, or conversely.*

For, since, by 4°, to every variable point moving on a fixed line (or variable line turning round a fixed point) of either, corresponds a variable point moving on the corresponding fixed line (or a variable line turning round the corresponding fixed point) of the other; therefore &c.

7°. *To every harmonic row of four points or pencil of four rays of either, corresponds an harmonic row of four points or pencil of four rays of the other.*

For, as every harmonic row (or pencil) may be regarded as determined by two angles and their two axes of perspective on the connector of their vertices (or by two segments and their two centres of perspective at the intersection of their axes) (241); and, as to the vertices and axes of perspective of any two angles (or the axes and centres of perspective of any two segments) of either correspond the vertices or axes of perspective of the two corresponding angles (or the axes and centres of perspective of the two corresponding segments) of the other; therefore &c.

8°. *To every pair of lines or points conjugate to each other with respect to any segment or angle of either, correspond a pair of lines or points conjugate to each other with respect to the corresponding segment or angle of the other.*

For, as every two lines (or points) conjugate to each other with respect to any segment (or angle) intersect with the axis of the segment (or connect with the vertex of the angle) at two points (or by two lines) which divide the segment (or angle) harmonically (217); therefore &c. by the preceding property 7°.

9°. *To every point and line pole and polar to each other with respect to any triangle of either, correspond a point and line pole and polar to each other with respect to the corresponding triangle of the other.*

For, as every point, and the intersection of its polar with each side (or every line, and the connector of its pole with each vertex) of any triangle, are conjugate to each other with respect to the opposite angle (or side) of the triangle (250, Cor. 2°); therefore &c. by the preceding property 8°.

10°. *To a variable point or line of either determining with four fixed points or lines an harmonic pencil or row, corresponds a variable point or line of the other determining with the four corresponding fixed points or lines an harmonic pencil or row.*

For, the harmonicism of the quartet of variable rays (or points), in every position of the variable point (or line) for either, involving, by 7°, the harmonicism of the corresponding quartet of variable rays (or points), in every position of the variable point (or line) for the other; therefore &c.

11°. *To every two equianharmonic rows of four points or pencils of four rays of either, correspond two equianharmonic rows of four points or pencils of four rays of the other.*

For, as every two equianharmonic rows of four points (or pencils of four rays) may be regarded as determined, on their respective axes (or at their respective vertices), by two quartets of rays (or points) in perspective with each other (290); and, as to every two quartets of rays (or points) in perspective for either correspond two quartets of rays (or points) in perspective for the other (property 2° above); therefore &c.

12°. *To every equianharmonic hexastigm or hexagram of either, corresponds an equianharmonic hexastigm or hexagram of the other.*

For, the equianharmonicism of the two pencils of connection (or rows of intersection) of any two with the remaining four of the six points (or lines) for either hexastigm (or hexagram) involving, by the preceding property 11°, the equianharmonicism of the two corresponding pencils (or rows) for the other hexastigm (or hexagram); therefore &c. (301). The same result follows also from the reciprocal properties of Art. 302, by virtue of the preceding property 1°.

13°. *To a variable point or line of either determining with four fixed points or lines a pencil or row having a constant anharmonic ratio, corresponds a variable point or line of the other determining with the four corresponding fixed points or lines a pencil or row having a constant anharmonic ratio.*

For, the equianharmonicism of the two quartets of rays (or points), in every two positions of the variable point (or line) for either, involving, by property 11°, the equianharmonicism of the two corresponding quartets of rays (or points), in every two positions of the variable point (or line) for the other; therefore &c.

14°. *To every two homographic rows of points or pencils of rays of either, correspond two homographic rows of points or pencils of rays of the other.*

For, the equianharmonicism of every two quartets of corresponding constituents of the two rows (or pencils) for either involving, by property 11°, the equianharmonicism of every two quartets of corresponding constituents of the two corresponding rows (or pencils) for the other; therefore &c. (321).

15°. *To two homographic coaxal rows or concentric pencils of either in involution with each other, correspond two homographic coaxal rows or concentric pencils of the other in involution with each other.*

For, every interchange of corresponding constituents of the two rows (or pencils) for either involving evidently a corresponding interchange of corresponding constituents of the two corresponding rows (or pencils) for the other; the interchangeability of every pair of corresponding constituents for either involves consequently the interchangeability of every pair of corresponding constituents for the other; and therefore &c. (357). The same result follows also from the general property of Art. 370, by virtue of the preceding property 7°.

16°. *To the double points or rays of any two homographic coaxal rows or concentric pencils of either, correspond the double points or rays of the two corresponding coaxal rows or concentric pencils of the other.*

For, every coincidence of corresponding constituents of the two rows (or pencils) for either involving evidently a corresponding coincidence of corresponding constituents of the two corresponding rows (or pencils) for the other; the two coincidences, real or imaginary, of pairs of corresponding constituents, which constitute the two double points (or rays) for either, correspond consequently to the two coincidences, real or imaginary, of pairs of corresponding constituents, which constitute the two double points (or rays) for the other; and therefore &c. (341).

17°. *To a variable point or line of either connecting or intersecting with two fixed points or lines homographically, corresponds a variable point or line of the other connecting or intersecting with the two corresponding fixed points or lines homographically.*

For, the equianharmonicism of every two quartets of corresponding connectors (or intersections) of the variable with the two fixed points (or lines), for either, involving, by property 11°, the equianharmonicism of every two quartets of corresponding connectors (or intersections) of the variable with the two corresponding fixed points (or lines), for the other; therefore &c. (321).

18°. *To a variable point or line of either the rectangle under whose distances from two fixed lines or points is constant, corresponds a variable point or line of the other the rectangle under whose distances from two (not necessarily corresponding) fixed lines or points is constant.*

For, the variable line (or point) of the former intersecting (or connecting) with every two fixed positions of itself homographically (340, Cor. 2°); and the variable line (or point) of the latter consequently, by the preceding property 17°, intersecting (or connecting) with every two fixed positions of itself homographically; therefore &c. (340, Cor. 1°).

19°. *To a variable point or line of either whose angle of connection with two fixed points or chord of intersection with two fixed lines intercepts on a fixed line or subtends at a fixed point a segment or angle of constant magnitude, corresponds a variable point or line of the other whose angle of connection with the two corresponding fixed points or chord of intersection with the two corresponding fixed lines intercepts on a (not necessarily corresponding) fixed line or subtends at a (not necessarily corresponding) fixed point a segment or angle of constant magnitude.*

For, the variable line (or point) of the former intersecting (or connecting) with the two fixed lines (or points) homographically (325, *a* and *a'*); and the variable line (or point) of the latter consequently, by property 17°, intersecting (or connecting) with the two corresponding fixed lines (or points) homographically; therefore &c. (339 and 340).

20°. *For continuous figures, all pairs of corresponding points determine pairs of corresponding tangents, and all pairs of corresponding tangents determine pairs of corresponding points.*

For, every connector of two points of either corresponding to the connector of the two corresponding points of the other, and every intersection of two lines of either corresponding to the intersection of the two corresponding lines of the other; and the coincidence of any two points or lines of either involving the coincidence of the two corresponding points or lines of the other; therefore &c. (19 and 20).

382. From the fundamental definition of Art. 380, the following general property of homographic figures may be readily inferred; viz.—

If A and A', B and B' be any two fixed pairs of corresponding points (or lines) of any two homographic figures F and F', and I and I' any variable pair of corresponding lines (or points) of the figures; then, for every position of I and I', the ratio

$$\left(\frac{AI}{BI} : \frac{A'I'}{B'I'}\right) \text{ or its equivalent } \left(\frac{AI}{A'I'} : \frac{BI}{B'I'}\right)$$

is constant, both in magnitude and sign.

For, if Z and Z' be the two variable points of intersection (or lines of connection) of the two variable lines (or points) I and I' with the two fixed lines (or points) AB and $A'B'$ respectively; then, since, by hypothesis, Z and Z' determine two homographic rows (or pencils) of which A and A', B and B' are two pairs of corresponding constituents (380), therefore, by (328), the ratio

$$\left(\frac{AZ}{BZ} : \frac{A'Z'}{B'Z'}\right) \text{ or } \left(\frac{\sin AZ}{\sin BZ} : \frac{\sin A'Z'}{\sin B'Z'}\right),$$

to which, in the corresponding case, the above is manifestly equivalent, is constant both in magnitude and sign; and therefore &c.

Cor. 1°. If A and B' be the two lines of the two figures whose two correspondents A' and B coincide at infinity; since then, for every two pairs of corresponding points P and P',

Q and Q' of the figures, the two ratios $PB : QB$ and $P'A' : Q'A'$ each $= 1$ (15), and since, for all cases, by the above,

$$\left(\frac{PA}{PB} : \frac{P'A'}{P'B'}\right) = \left(\frac{QA}{QB} : \frac{Q'A'}{Q'B'}\right),$$

therefore, for the case in question, $PA.P'B' = QA.Q'B'$, and therefore—

For any two homographic figures F and F', if A and B' be the two lines whose two correspondents A' and B coincide at infinity, then, for every pair of corresponding points P and P' of the figures, the rectangle PA.P'B' is constant in magnitude and sign.

Cor. 2°. From the simple relation of the preceding corollary, the following properties of any two homographic figures F and F', with respect to their two lines A and B' whose correspondents A' and B coincide at infinity, may be immediately inferred; viz.—

1°. *Every two corresponding segments PQ and P'Q' of any two corresponding lines L and L' are cut in reciprocal ratios by the two lines A and B' respectively.*

For, since, by the relation, $PA.P'B' = QA.Q'B'$; therefore, at once, $PA : QA = Q'B' : P'B'$; and therefore &c. (Euc. VI. 4).

2°. *For a variable pair of corresponding points P and P' on any fixed pair of corresponding lines L and L', if H and K' be the intersections of the latter with A and B' respectively, the rectangle HP.K'P' is constant in magnitude and sign.*

For, the two ratios $PA : PH$ and $P'B' : P'K'$ being both constant, by hypothesis, and the rectangle $PA.P'B'$ being constant in magnitude and sign, by the relation; therefore &c.

3°. *For every pair of corresponding points P and P', if L and L' be any fixed pair of corresponding lines, the ratio* $PL^2 \div PA : P'L'^2 \div P'B'$ *is constant in magnitude and sign.*

For, since, by the general property of the present article, the two ratios $PL \div PA : P'L' \div P'A'$ and $PL \div PB : P'L' \div P'B'$ are constant in magnitude and sign, and since the ratio $PB : P'A' = 1$; therefore &c.

4°. *To every line L of F parallel to A, corresponds a line L' of F' parallel to B'; and conversely.*

For, since, for every two pairs of corresponding points P and P', Q and Q', on any pair of corresponding lines L and L', by

1°, $PA : QA = Q'B' : P'B'$; consequently when either equivalent $= +1$ so is the other also; and therefore &c. (15).

5°. *For every pair of corresponding lines L and L' parallel to A and B' respectively, the rectangle $AL.B'L'$ is constant in magnitude and sign.*

For, since, for any pair of corresponding points P and P' on L and L' respectively, the rectangle $PA.P'B'$ is constant in magnitude and sign, by the relation; therefore &c.

6°. *Every two corresponding lines L and L' parallel to A and B' respectively are divided similarly by the several pairs of corresponding points that lie on them.*

For, since, for any number of pairs of corresponding points P and P', Q and Q', R and R', &c. on L and L' respectively, if M and M' be any other pair of corresponding lines not parallel to A and B', by 3°, $PM' \div PA : QM' \div QA : RM' \div RA$, &c. $= P'M'' \div P'B' : Q'M'' \div Q'B' : R'M'' : R'B'$, &c.; and since, by hypothesis, $PA : QA : RA$, &c. $= P'B' : Q'B' : R'B'$, &c. $= 1$; therefore $PM' : QM' : RM'$, &c. $= P'M'' : Q'M'' : R'M''$, &c.; and therefore &c. (Euc. VI. 4).

N.B. Of these several results, the second, fourth, and sixth are also evident *à priori* from the fundamental definition of Art. 380; the fourth from (16), from the consideration that to every line L of F passing through the point AB, corresponds a line L' of F' passing through the corresponding point $A'B'$; the sixth form (330), from the consideration that to the point AB on L, corresponds the point $A'B'$ on L'; and the second from (331), from the consideration that to the two points H and K' at which L and L' intersect with A and B' respectively, correspond the two H' and K at which L' and L intersect with A' and B respectively.

Cor. 3°. When the figures are such that a pair of their corresponding lines A and A' coincide at infinity; then, since for every pair of corresponding points P and P' the ratio $PA : P'A' = 1$, therefore for every other fixed pair of corresponding lines B and B', by the above, the ratio $PB : P'B'$ is constant in magnitude and sign, however P and P' vary, and therefore—

When two homographic figures F and F' have a pair of corresponding lines A and A' coinciding at infinity, the distance *of a variable point P from any fixed line B, of either F, is to the distance of the corresponding variable point P' from the corresponding fixed line B', of the other F', in a ratio constant in magnitude and sign.*

COR. 4°. From the general property of the preceding corollary, the following consequences, respecting two homographic figures F and F' having a pair of corresponding lines A and A' coinciding at infinity, may be readily inferred.

1°. *To every two parallel lines L and M of either F, correspond two parallel lines L' and M' of the other F'.*

For, since, for every two pairs of corresponding points P and P', Q and Q' on either pair of corresponding lines L and L', by the preceding, $PM : P'M' = QM : Q'M'$; consequently, when always $PM = QM$, then always $P'M' = Q'M'$; and therefore &c.

2°. *For every two parallel lines L and M of either F having any fixed direction, and for the two corresponding parallel lines L' and M' of the other F', the ratio $LM : L'M'$ is constant in magnitude and sign.*

For, since, for a variable pair of corresponding points P and P' on either pair of corresponding lines L and L', by the same, the ratio $PM : P'M'$ is then constant in magnitude and sign; therefore &c.

3°. *Every two of their corresponding lines L and L' are divided similarly by their several pairs of corresponding points P and P', Q and Q', R and R', &c.*

For, if M and M' be any other pair of corresponding lines not parallel to L and L', and O and O' the two points of intersection LM and $L'M'$; then since, by the same, $PM : P'M' = QM : Q'M' = RM : R'M'$, &c. therefore, by Euc. VI. 4, $PO : P'O' = QO : Q'O' = RO : R'O'$, &c.; and therefore &c.

4°. *For every two points P and Q of either F whose line of connection is parallel to any fixed direction L, and for the two corresponding points P' and Q' of the other F', the ratio $PQ : P'Q'$ is constant in magnitude and sign.*

For, if M and M' be a pair of corresponding lines passing through either pair of corresponding points P and P' and parallel to any pair of corresponding fixed directions not coinciding with L and L'; then, since by the same, the ratio

$MQ : M'Q'$ is constant in magnitude and sign, therefore, by Euc. VI. 4, so is also the ratio $PQ : P'Q'$; and therefore &c.

5°. *For every three points P, Q, R of either F, and for the three corresponding points P', Q', R' of the other F', the area of the triangle PQR is to the area of the triangle $P'Q'R'$ in a constant ratio.*

For, if L and L' be a pair of corresponding lines passing through any pair P and P' of the corresponding points, and parallel to any fixed pair of corresponding directions of the figures; and S and S' their pair of intersections with the pair of opposite sides QR and $Q'R'$ of the triangles; then since, by the preceding properties 4° and 2°, the ratio $PS : P'S'$ is constant, and the two ratios $QL : Q'L'$ and $RL : R'L'$ are constant and equal, therefore, the difference of the two areas PQS and PRS, or the area PQR (75), is to the difference of the two areas $P'Q'S'$ and $P'R'S'$, or the area $P'Q'R'$ (75), in a constant ratio; and therefore &c.

6°. *For every system of points P, Q, R, S, T, &c. of either F, and for the corresponding system of points P', Q', R', S', T', &c. of the other F', the area of every polygon determined by the former* (108) *is to that of the corresponding polygon determined by the latter* (108) *in the same constant ratio.*

For, if $PQRST$, &c. and $P'Q'R'S'T'$, &c. be any pair of corresponding polygons determined by the two systems of points, and O and O' any independent pair of corresponding points of the figures; then since, by the preceding property 5°, the several triangular areas POQ, QOR, ROS, SOT &c. are to the several corresponding areas $P'O'Q'$, $Q'O'R'$, $R'O'S'$, $S'O'T'$ &c. in the same constant ratio, therefore the sum of the former, or the area of the polygon $PQRST$, &c. (118) is to the sum of the latter, or the area of the polygon $P'Q'R'S'T'$ &c. (118), in the same constant ratio; and therefore &c.

N.B. Of these several properties, the first and third are also evident *à priori* from (16) and (330), from the obvious consideration that when two homographic figures have a pair of corresponding lines coinciding at infinity, then to every point at infinity of either corresponds, by their fundamental definition, a point at infinity of the other.

383. From the same fundamental definition of Art. 380, it follows, precisely in the same manner as the general property of the preceding article, that—

If A and A', B and B', C and C' be any three fixed pairs of corresponding points (or lines) of any two homographic figures F and F', and I and I' any variable pair of corresponding lines (or points) of the figures; then, for every position of I and I', the three ratios

$$\left(\frac{BI}{CI} : \frac{B'I'}{C'I'}\right), \quad \left(\frac{CI}{AI} : \frac{C'I'}{A'I'}\right), \quad \left(\frac{AI}{BI} : \frac{A'I'}{B'I'}\right),$$

or their three equivalents

$$\left(\frac{BI}{B'I'} : \frac{CI}{C'I'}\right), \quad \left(\frac{CI}{C'I'} : \frac{AI}{A'I'}\right), \quad \left(\frac{AI}{A'I'} : \frac{BI}{B'I'}\right),$$

every two of which manifestly involve the third, are constant, both in magnitude and sign.

For, as in the preceding article, if X and X', Y and Y', Z and Z' be the three pairs of intersections (or connectors) of I and I' with BC and $B'C'$, CA and $C'A'$, AB and $A'B'$ respectively; then since, for the same reason as in the preceding article, the three ratios

$$\left(\frac{BX}{CX} : \frac{B'X'}{C'X'}\right) \text{ or } \left(\frac{\sin BX}{\sin CX} : \frac{\sin B'X'}{\sin C'X'}\right),$$

$$\left(\frac{CY}{AY} : \frac{C'Y'}{A'Y'}\right) \text{ or } \left(\frac{\sin CY}{\sin AY} : \frac{\sin C'Y'}{\sin A'Y'}\right),$$

$$\left(\frac{AZ}{BZ} : \frac{A'Z'}{B'Z'}\right) \text{ or } \left(\frac{\sin AZ}{\sin BZ} : \frac{\sin A'Z'}{\sin B'Z'}\right),$$

to which, in the corresponding cases, the above are manifestly equivalent, are constant both in magnitude and sign; therefore &c.

Cor. 1°. The above supplies obvious solutions of the two following problems: *Given, of two homographic figures F and F', three pairs of corresponding points (or lines) A and A', B and B', C and C', and a pair of corresponding lines (or points) D and D', to determine the line (or point) E of either of them F corresponding to any assumed line (or point) E' of the other F'.* For, since, by the above—

$$\frac{BE}{CE} : \frac{B'E'}{C'E'} = \frac{BD}{CD} : \frac{B'D'}{C'D'},$$

$$\frac{CE}{AE} : \frac{C'E'}{A'E'} = \frac{CD}{AD} : \frac{C'D'}{A'D'},$$

$$\frac{AE}{BE} : \frac{A'E'}{B'E'} = \frac{AD}{BD} : \frac{A'D'}{B'D'},$$

the three ratios $BE : CE$, $CE : AE$, $AE : BE$, which manifestly determine the position of the required line (or point) E, are consequently given; and therefore &c.

The particular cases where the given line (or point) E' is at infinity present no special peculiarity; the three ratios $B'E' : C'E'$, $C'E' : A'E'$, $A'E' : B'E'$ being simply all $= 1$ in the former case, and having for values $\sin B'L' : \sin C'L'$, $\sin C'L' : \sin A'L'$, $\sin A'L' : \sin B'L'$ respectively, where L' is any line parallel to the direction of E', in the latter case.

Cor. 2°. As three pairs of corresponding points (or lines) A and A', B and B', C and C' of two homographic figures F and F' determine (380) three pairs of corresponding lines (or points) BC and $B'C'$, CA and $C'A'$, AB and $A'B'$ of the figures; the solutions of the two problems: *Given, of two homographic figures F and F', four pairs of corresponding points (or lines) A and A', B and B', C and C', D and D'; to determine the point (or line) E of either of them F corresponding to any assumed point (or line) E' of the other F';* may consequently be regarded as included in those of the above; the particular cases where the given point (or line) E' is at infinity, presenting, as above observed, no exceptional or special peculiarity.

Cor. 3°. It appears also immediately from the above, that *when, for two homographic figures F and F', three pairs of corresponding points (or lines) A and A', B and B', C and C' coincide, the coincidence of any independent pair of corresponding lines (or points) D and D' involves the coincidence of every other pair E and E', and therefore of the figures themselves F and F'.* For, when, in the three relations of Cor. 1°, which as there shewn result immediately from it, $A = A'$, $B = B'$, $C = C'$, if, in addition, $D = D'$, then necessarily $E = E'$; and therefore &c.

Cor. 4°. For the same reason as in Cor. 2°, it follows of course from the preceding, Cor. 3°, that *when, for two homographic figures F and F', four independent pairs of corresponding points (or lines) A and A', B and B', C and C', D and D' coincide, then all pairs of corresponding points (or lines) E and E', and consequently the figures themselves F and F' coincide.* Which is also evident *à priori* from the fundamental characteristic of homographic figures (380) that, for every two corresponding quintets A, B, C, D, E and A', B', C', D', E' of their points (or lines), the five relations

$$\{A.BCDE\}=\{A'.B'C'D'E'\},\ \{B.CDEA\}=\{B'.C'D'E'A'\},$$
$$\{C.DEAB\}=\{C'.D'E'A'B'\},\ \{D.EABC\}=\{D'.E'A'B'C'\},$$
$$\{E.ABCD\}=\{E'.A'B'C'D'\}$$

must in all cases exist together; which, when $A=A'$, $B=B'$, $C=C'$, $D=D'$, would be manifestly impossible unless also $E=E'$; and therefore &c.

N.B. It will appear in the sequel that, for every pair of homographic figures F and F', there exists a unique triangle Δ, whose three elements of either species A, B, C, regarded as belonging to either figure, coincide, as supposed in the two latter corollaries 3° and 4°, with their three correspondents of the same species A', B', C' in the other figure. Of the triangle Δ, thus related to the two figures F and F', two pairs of opposite elements (vertices and sides) may be imaginary, but the third pair are always real.

384. On the converse of the property of the preceding Article, the following general construction for the double generation (26) of a pair of homographic figures, by the simultaneous variation of a pair of connected points, or lines, has been based by Chasles, the originator of the general theory.

If A and A', B and B', C and C' be the three pairs of corresponding sides (or vertices) of any two arbitrary fixed triangles ABC and $A'B'C'$, and I and I' a pair of variable points (or lines) so connected that, in every position, any two of the three ratios

$$\left(\frac{BI}{CI}:\frac{B'I'}{C'I'}\right),\ \left(\frac{CI}{AI}:\frac{C'I'}{A'I'}\right),\ \left(\frac{AI}{BI}:\frac{A'I'}{B'I'}\right),$$

or of their three equivalents

$$\left(\frac{BI}{B'I'}:\frac{CI}{C'I'}\right),\ \left(\frac{CI}{C'I'}:\frac{AI}{A'I'}\right),\ \left(\frac{AI}{A'I'}:\frac{BI}{B'I'}\right),$$

and with them of course the third, are constant in magnitude and sign; the two variable points (or lines) I and I' generate two homographic figures F and F', of which A and A', B and B', C and C' are three pairs of corresponding lines (or points).

That the two figures F and F' resulting from either mode of generation are thus homographic, follows of course conversely from the property of the preceding article; but it may be easily shewn directly that they fulfil all the conditions of connection of the fundamental definition of Art. 380; for—

1°. *To every point (or line) of either corresponds a point (or line) of the other.* This is evident from the law of their generation; every two points (or lines) I and I' connected by the above relations, whether generating pairs or not, thus corresponding with respect to them.

2°. *To every line (or point) of either corresponds a line (or point) of the other.* For, when a variable point (or line) I of the former is connected, in every position, with the three fixed lines (or points) A, B, C by a relation of the form

$$a.AI+b.BI+c.CI=0 \text{ .... } (\alpha),$$

where a, b, c are any three constant multiples; then, by virtue of the above relations, the corresponding point (or line) I' of the latter is connected with the three fixed lines (or points) A', B', C' by a corresponding relation of similar form

$$a'.A'I'+b'.B'I'+c'.C'I'=0 \text{ .... } (\alpha'),$$

where a', b', c' are three other constant multiples whose ratios to a, b, c respectively depend on and are given with those of the same relations; but, by the general property of Art. 120 (or 85), the former relation (α) is the condition that the variable point (or line) I should move on a fixed line (or turn round a fixed point) O, and the latter (α') is the condition that the corresponding point (or line) I' should move on a corresponding fixed line (or turn round a corresponding fixed point) O'; and therefore &c.

3°. *To the connector of any two points (or the intersection of* *any two lines) of either, corresponds the connector of the two corresponding points (or the intersection of the two corresponding lines) of the other.* For, since, to a line passing through any two points (or a point lying on any two lines) of either, corresponds, by the preceding property 2°, a line passing through the two corresponding points (or a point lying on the two corresponding lines) of the other; therefore &c.

4°. *To the intersection of any two lines (or the connector of any two points) of either, corresponds the intersection of the two corresponding lines (or the connector of the two corresponding points) of the other.* For, since, to two lines passing through any point (or two points lying on any line) of either, correspond, by the same property 2°, two lines passing through the corresponding point (or two points lying on the corresponding line) of the other; therefore &c.

5°. *Every two of their corresponding quartets of collinear points (or concurrent lines) are equianharmonic.* For, the four connectors (or intersections) of any quartet I_1, I_2, I_3, I_4 of the points (or lines) of the former, whether collinear (or concurrent) or not, with any vertex (or side) BC or CA or AB of the triangle ABC being (by Cor. Art. 328) equianharmonic with the four connectors (or intersections) of the corresponding quartet I_1', I_2', I_3', I_4' of the points (or lines) of the latter with the corresponding vertex (or side) $B'C'$ or $C'A'$ or $A'B'$ of the triangle $A'B'C'$; therefore &c. (285).

6°. *Every two of their corresponding quartets of concurrent lines (or collinear points) are equianharmonic.* For, the four intersections (or connectors) of any quartet O_1, O_2, O_3, O_4 of the lines (or points) of the former, whether concurrent (or collinear) or not, with any fifth line (or point) O_5 of the figure being (by the preceding properties 4° and 5°) equianharmonic with the four intersections (or connectors) of the corresponding quartet O_1', O_2', O_3', O_4' of the lines (or points) of the latter with the corresponding fifth line (or point) O_5' of the figure; therefore &c. (285).

That, for either mode of generation, the three pairs of corresponding vertices and sides of the two fixed triangles ABC and $A'B'C'$ are pairs of corresponding points and lines of the two resulting figures F and F', is evident from the relations of generation; from which it follows immediately, in either case, that the evanescence of any one or two of the three distances AI, BI, CI, for the former, involves necessarily the simultaneous evanescence of the corresponding one or two of the three corresponding distances $A'I'$, $B'I'$, $C'I'$, for the latter; and therefore &c.

N.B. When, of the two arbitrary triangles of construction ABC and $A'B'C'$ in either of the above modes of generation, the three pairs of corresponding elements A and A', B and B', C and C' coincide, the triangle ABC is then, with respect to the two resulting figures F and F', that to which allusion was made in the note at the close of the preceding article (383).

385. From the general constructions of the preceding article the following consequences respecting *the homographic transformation of figures* may be readily inferred, viz.—

1°. *Any figure F may be transformed homographically into another F', in which any four points (or lines), given or taken arbitrarily, shall correspond to any assigned four points (or lines) of the original figure.*

For, of the four given pairs of corresponding points or lines, any three determine the two fixed triangles of construction ABC and $A'B'C'$, and the fourth give the values of the three constant ratios of construction

$$\left(\frac{BI}{CI}:\frac{B'I'}{C'I'}\right),\ \left(\frac{CI}{AI}:\frac{C'I'}{A'I'}\right),\ \left(\frac{AI}{BI}:\frac{A'I'}{B'I'}\right);$$

and therefore &c. See Cors. 1° and 2°, Art. 283.

The obvious conditions, that when, for either of two homographic figures F and F', three points are collinear or three lines concurrent, then, for the other, the three corresponding points must also be collinear or the three corresponding lines concurrent, and that when, for either, four points by their collinearity or four lines by their concurrence form an anharmonic quartet, then, for the other, the four corresponding points by their collinearity or the four corresponding lines by their concurrence must form an equianharmonic quartet, are the only restrictions on the perfect generality of the above. The former condition may indeed be violated, but, when it is, it is easy to see, from the general process of construction, that the figure for which the three points are collinear or the three lines concurrent, when their three correspondents in the other are not, must (except for the fourth point or line of the other) have all its points collinear or all its lines concurrent with the three. For, if, in any position of I and I', any one, AI suppose, of the six distances AI and $A'I'$, BI and $B'I'$, CI and $C'I'$ be evanescent when its correspondent $A'I'$ is not, then, in every position of I and I', from the constancy of the three ratios of construction, either the same distance AI, or each of the two non-corresponding distances $B'I'$ and $C'I'$ is evanescent; and therefore &c. See the general remark 2° of Art. 31, an illustration of which is supplied by the above.

2°. *In the homographic transformation of any figure F into another F', the line (or any point) at infinity, regarded as belonging to either, may be made to correspond to any assigned line (or point), regarded as belonging to the other.*

This follows at once from the preceding property 1°; the three ratios of construction

$$\left(\frac{BI}{CI}:\frac{B'I'}{C'I'}\right),\quad \left(\frac{CI}{AI}:\frac{C'I'}{A'I'}\right),\quad \left(\frac{AI}{BI}:\frac{A'I'}{B'I'}\right)$$

being given as definitely (see Cors. 1° and 2°, Art. 383) when one of the two given points or lines I and I' is at infinity, as when both are at a finite distance; and therefore &c.

By virtue of the above general property 1°, combined with its particular case 2°, the tetrastigm or tetragram determined by any four points or lines of F may be transformed homographically into another of any simpler or more convenient form for F'; such, for instance, as the four vertices or sides of a parallelogram of any form, or, more generally, the three vertices or sides of a triangle of any form combined with any remarkable or convenient point or line connected with its figure. By this means, the demonstration of a property, or the solution of a problem, when such property or problem admits of homographic transformation, may frequently be much simplified; as, for instance, in the pairs of reciprocal properties a and a' of Art. 236, a and a' of Art. 245, a' and a of Art. 299, the demonstrations of which are comparatively easy (239) when, in the first case of each, one of the four lines of the tetragram is at infinity, and when, in the second case of each, one of the four points of the tetrastigm is the polar centre of the triangle determined by the remaining three; positions into which, if not originally in them, they may at once be thrown by homographic transformation, and so placed in the circumstances most favourable to their establishment.

3°. *In the homographic transformation of any figure F into another F', the correspondents to any assigned five points (or lines) of the original, no three of which are collinear (or concurrent), may be made to lie on (or touch) a circle, given or taken arbitrarily.*

To prove this, it is only necessary (380) to shew that, for every quintet of points (or lines) A, B, C, D, E, no three of which are collinear (or concurrent), a corresponding quintet of concyclic points (or tangents) A', B', C', D', E' may be found on (or to) any given circle, satisfying the five conditions

$$\{A'.B'C'D'E'\} = \{A.BCDE\},\ \{B'.C'D'E'A'\} = \{B.CDEA\},$$
$$\{C'.D'E'A'B'\} = \{C.DEAB\},\ \{D'.E'A'B'C'\} = \{D.EABC\},$$
$$\{E'.A'B'C'D'\} = \{E.ABCD\};$$

which will manifestly be the case if any collinear (or concurrent) quintet A'', B'', C'', D'', E'', can be found satisfying the five corresponding conditions

$$\{B''C''D''E''\} = \{A.BCDE\},\ \{C''D''E''A''\} = \{B.CDEA\},$$
$$\{D''E''A''B''\} = \{C.DEAB\},\ \{E''A''B''C''\} = \{D.EABC\},$$
$$\{A''B''C''D''\} = \{E.ABCD\};$$

inasmuch as their five connectors (or intersections) with any arbitrary point on (or tangent to) the circle, will of course determine, by their five second intersections with (or tangents to) the circle, a concyclic quintet of points (or tangents) A', B', C', D', E' satisfying the required conditions. And that every two D and E of the five points (or lines) A, B, C, D, E determine such a collinear (or concurrent) quintet with the three intersections (or connectors) of their line of connection (or point of intersection) DE with the three sides (or vertices) BC, CA, AB of the triangle determined by the remaining three A, B, C; may be readily shown as follows:

Denoting by X, Y, Z the three intersections (or connectors) of the three lines (or points) BC, CA, AB with the line (or point) DE; then since, immediately, by the general property of Art. 285, $\{ZYDE\} = \{A.BCDE\}$, $\{XZDE\} = \{B.CADE\}$, $\{YXDE\} = \{C.ABDE\}$; and since, from the perspective of $A.XYZD$ or $B.XYZD$ or $C.XYZD$ with $D.ABCE$, and of $A.XYZE$ or $B.XYZE$ or $C.XYZE$ with $E.ABCD$, by 4°, Art. 286, $\{XYZD\} = \{D.ABCE\}$ and $\{XYZE\} = \{E.ABCD\}$; therefore &c.

It follows immediately, from this latter property, that *every figure, locus of a variable point every six of whose positions form an equianharmonic hexastigm* (301, a), *or envelope of a variable line every six of whose positions form an equianharmonic hexagram* (301, a'), *may be transformed homographically into a circle;* for, if transformed, by the above, so that the correspondents to any five of its points (or tangents) shall lie on (or touch) a circle, the correspondent of every sixth point (or tangent) must, by virtue of its connection with the five, lie on (or touch) the same circle (305); and therefore &c. Thus: 1°. Every figure, locus of a variable point determining in every position an equianharmonic hexastigm with five fixed points, or envelope of a variable line determining in every position an equianharmonic hexagram with five fixed lines (301, a and a'); 2°. Every figure, locus of a variable point connecting with four fixed points by four lines, or envelope of a variable line intersecting with four fixed lines at four points, having any constant anharmonic ratio (333, e and e'); 3°. Every figure, locus of a variable point connecting homographically with two fixed points, or envelope of a variable line intersecting homographically with two fixed lines (338, Cor. 2°); 4°. Every figure, locus of a variable point the rectangle under whose distances form two fixed lines, or envelope of a variable line the rectangle under whose distances form two fixed points, is constant in magnitude and sign (340, Cor. 2°, a and a'); may be transformed homographically into a circle; and all their properties admitting of homographic transformation, such as their harmonic and anharmonic properties, consequently inferred from the comparatively simple and familiar properties of the circle. See Chapters XV. and XVIII.; all the properties of which, not involving the magnitudes of angles, are consequently true, not only of the circle, but of all the figures above enumerated also.

It follows also, from the same property, that *five points* (*or tangents*), *given or taken arbitrarily, completely determine any figure homographic to a circle;* for, if transformed, by the above, so that the correspondents of the five points (or tangents) shall lie on (or touch) a circle, all the other points (or tangents) of the figure are then implicitly given, as the correspondents to the several other points (or tangents) of the circle; and therefore &c.

Given five points (*or tangents*) A, B, C, D, E *of a figure homographic to a circle, the five corresponding tangents* (*or points*) AA, BB, CC, DD, EE *of the figure are given implicitly with them;* for, since, for the five corresponding points (or tangents) A', B', C', D', E' of the circle, by (306), $\{A'.A'B'C'D'E'\} = \{B'.A'B'C'D'E'\} = \{C'.A'B'C'D'E'\} = \{D'.A'B'C'D'E'\} = \{E'.A'B'C'D'E'\}$, therefore, for the five given points (or tangents) A, B, C, D, E of the figure, by (380) and (381, 20°), $\{A.ABCDE\} = \{B.ABCDE\} = \{C.ABCDE\} = \{D.ABCDE\} = \{E.ABCDE\}$; and, since, of each of these five latter homographic pencils (or rows), four rays (or points) are actually given, therefore, of each, the fifth ray (or point) is implicitly given; and therefore &c.

386. Of the numerous properties of the interesting and important class of figures into which the circle may be transformed homographically, the few following, derived on the preceding principles from those of the circle, may be taken as so many examples illustrative of the utility of the process of homographic transformation in modern geometry.

Ex. 1°. *No figure homographic to a circle could have either three collinear points or three concurrent tangents.*

For, if, of a figure homographic to a circle, three points were collinear, or three tangents concurrent, then, of the circle itself, by (381, 1°), the three corresponding points should be collinear, or the three corresponding tangents concurrent; and therefore &c.

N.B. The only exception to this fundamental property occurs in the cases noticed in connection with property 1° of the preceding article (385), when the figure is in one or other limiting state of its general form, and has either an infinite number of collinear points lying on one or other of two definite lines, or an infinite number of concurrent tangents passing through one or other of two definite points. See the general remark 2° of Art. 31; of which the above and all similar exceptional cases supply so many illustrations.

Ex. 2°. *No figure homographic to a circle could have either three points at infinity or three parallel tangents.*

This is manifestly a particular case of the general property of the preceding article; all points at infinity being collinear, and all parallel lines concurrent (136); and therefore &c.

N.B. As, in the process of homographic transformation of one figure into another, the correspondent to any line of the original may be thrown to infinity in the transformed figure, (See 2° of the preceding article), *a circle will consequently be transformed homographically into a figure having two distinct, coincident, or imaginary points at infinity, according as the line whose correspondent is thrown to infinity in the transformation intersects it at two distinct, coincident, or imaginary points* (21). Since, in the particular case of coincidence, the original and transformed lines are tangents to the original and transformed figures (19), *a circle may consequently be transformed homographically into a figure having a tangent at infinity, by merely throwing to infinity, in the transformation, the correspondent to any tangent to itself.* The transformed figure possesses in this latter case, as will be seen in the sequel, some special properties peculiar to the case.

Ex. 3°. *In every figure homographic to a circle, every three points (or tangents) and the three corresponding tangents (or points) determine two triangles in perspective* (140).

For, both properties, by examples 3° and 4° of Art. 137, being true of the circle itself, are consequently, by properties 1° and 20° of Art. 381, true of every figure homographic to it; and therefore &c.

N.B. The consequences resulting from this general property applied to the particular cases, when two of the three points are at infinity, the third being arbitrary, and when one of the three tangents is at infinity, the remaining two being arbitrary, are left as exercises to the reader.

Cor. The above, in the general case, supplies obvious solutions of the two following problems: *given, of a figure homographic to a circle, any three points (or tangents) and two of the three corresponding tangents (or points), to determine the third corresponding tangent (or point).*

Ex. 4°. *In every figure homographic to a circle, every six points (or tangents) determine an equianharmonic hexastigm (or hexagram)* (301).

For, both properties, by *a* and *a'* of Art. 305, being true of the circle itself, are consequently by 12° of Art. 381, true of every figure homographic to it; and therefore &c.

N.B. By virtue of this general property, every system of six points on (or tangents to) any figure homographic to a circle possesses all the properties of a system of six points (or lines) determining an equianharmonic hexastigm (or hexagram). See Arts. 301 to 304.

Cor. If, in the above, while any five of the six points (or tangents) are supposed to remain fixed, the sixth be conceived to vary, and in the course of its variation to coincide successively with each of the five that remain fixed; the theorems of Pascal and Brianchon (302, *a* and *a'*) applied to the five cases of coincidence, supply ready solutions by linear constructions only, without the aid of the circle, of the two following problems: *given of a figure homographic to a circle, any five points (or tangents), to determine the five corresponding tangents (or points).* For, of the three collinear intersections of pairs of opposite sides (or concurrent connectors of pairs of opposite vertices) of any one of the sixty hexagons determined by the six points (or tangents), in the general position of the variable point (or tangent), that corresponding to the opposite side (or vertex) of the pentagon they determine in any position of coincidence, gives at once the line of connection (or point of intersection) of the two coincident points (or lines), that is (19 and 20) the tangent (or point) corresponding to that position; and therefore &c. See also 3° of the preceding article.

Ex. 5°. *In every figure homographic to a circle, a variable point (or tangent) determines with every four fixed points (or tangents) a variable quartet of rays (or points) having a constant anharmonic ratio.*

For, both properties, by *a* and *a'* of Art. 306, being true of the circle itself, are consequently, by 13° of Art. 381, true of every figure homographic to it; and therefore &c.

Cor. 1°. From the first part of the above, applied to the particular case when two of the four fixed points are the two, real or imaginary, at which, the figure intersects infinity (see Ex. 2°, note); it appears at once, as shown already in Art. 312 for the particular case of the circle, that—

In every figure homographic to a circle, the angle connecting a variable with any two fixed points of the figure is cut in a constant anharmonic ratio by the angle connecting it with the two points, real or imaginary, at which the figure intersects infinity.

Cor. 2°. From the second part of the same, applied to the particular case when, for a figure having a tangent at infinity (see Ex. 2°, note), one of the four fixed tangents, whatever be the positions of the remaining three, is the tangent at infinity, it follows at once, by virtue of the general property of Art. 275, that—

When a figure homographic to a circle has a tangent at infinity, the segment of a variable intercepted by any two fixed tangents is cut in a constant ratio by every third fixed tangent to the figure.

Cor. 3°. Since, in the same case, by Art. 55, Cor. 3°, *b*, the variable circle circumscribing the triangle determined by the variable with any two of the three fixed tangents, in the preceding corollary, passes in every position through a fixed point on the circle circumscribing the triangle determined by the three fixed tangents; it follows consequently from the same, by virtue of the property referred to, that—

When a figure homographic to a circle has a tangent at infinity, the variable circle circumscribing the triangle determined by a variable with any two fixed tangents to the figure, passes through a fixed point.

Cor. 4°. From both parts of the above, applied to the case when, for any figure, the constant anharmonic ratio of the quartet of rays (or points) determined by the four fixed points (or tangents) with the variable fifth point (or tangent) = – 1, in which case the four former are said to form an harmonic system, it follows, precisely as shewn for the circle itself in Art. 311, Cor. 3°, a and a', that—

In every figure homographic to a circle, when four points (or tangents) form an harmonic system, the pair of tangents (or points) corresponding to either pair of conjugates are concurrent (or collinear) with the connector (or intersection) of the other pair.

Cor. 5°. From the converses of the two preceding properties, shewn with themselves for the circle in the place above referred to, and also on other principles in Art. 257, it follows immediately, as shewn for the circle itself in Art. 258, that—

a. In every figure homographic to a circle, the segment intercepted on a variable by any two fixed tangents is cut harmonically at the corresponding variable point and at its intersection with the connector of the two corresponding fixed points.

a'. In every figure homographic to a circle, the angle subtended at a variable by any two fixed points is cut harmonically by the corresponding variable tangent and by its connector with the intersection of the two corresponding fixed tangents.

Cor. 6°. From the first of the two preceding properties, applied to the particular case when the two fixed tangents are the two, real or imaginary, whose points of contact are at infinity (see Ex. 2°, note), since then the second point of harmonic section in every position of the variable tangent is at infinity, it follows consequently, by virtue of 3°, Art. 216, that—

In every figure homographic to a circle, the segment intercepted on a variable by the two fixed tangents, real or imaginary, whose points of contact are at infinity, is bisected at its point of contact with the figure.

N.B. With respect to the numerous inferences from the two very fertile properties of the present example, it may be observed generally that, by virtue of them, all the properties established for the circle in Chapters XV. and XVIII. not involving directly the magnitudes of angles, are true generally of all figures into which circles may be transformed homographically; the circumstance that such figures may have real points at infinity (Ex. 2°, note) giving rise sometimes, as in Cors. 2°, 3°, 6° above, to important modifications not occurring, from the absence of that circumstance, in the case of the circle itself.

Ex. 6°. *In every figure homographic to a circle, a variable point (or tangent) connects (or intersects) homographically with every two fixed points (or tangents).*

For, both properties, by examples c and c' of Art. 325, being true of the circle itself, are consequently, by property 14° of Art. 381, true of every figure homographic to it; and therefore &c.

COR. 1°. From the second part of the above, applied to the case of a figure having a tangent at infinity (Ex. 2°, note), since then, whatever be the positions of the two fixed tangents, their two points at infinity are corresponding points of the two homographic divisions determined on them by the variable tangent, it follows consequently, by virtue of the general property of Art. 330, that—

When a figure homographic to a circle has a tangent at infinity, a variable tangent divides every two, and therefore all, fixed tangents similarly.

COR. 2°. From the same, applied to the particular case when, for any figure, the two fixed tangents are the two, real or imaginary, whose points of contact are at infinity (Ex. 2°, note), since then their common intersection is the point on each corresponding to that at infinity on the other, it follows consequently, by virtue of the particular property of Art. 331, Cor. that—

In every figure homographic to a circle, a variable determines with the two fixed tangents, real or imaginary, whose points of contact are at infinity, a triangle of constant area.

COR. 3°. From the same again, applied to the case when the two fixed tangents, whatever be their common absolute direction, are parallel, since then the point on each corresponding to that at infinity on the other is its point of contact with the figure, it follows consequently, by virtue of the general property of Art. 331, that—

In every figure homographic to a circle, a variable intersects with every two fixed tangents, whose directions are parallel, at two variable points, the rectangle under whose distances from their two fixed points of contact is constant in magnitude and sign.

COR. 4°. Since, for any two fixed tangents, the point on each corresponding to that at infinity on the other, in their homographic division by a variable tangent, is that of its intersection with the second fixed tangent parallel to the other (Ex. 2°), it follows also from the same, by virtue of the same general property of Art. 331, that—

In every figure homographic to a circle, a variable tangent intersects with each pair of adjacent sides of any fixed parallelogram exscribed to the figure, at a pair of variable points, the rectangle under whose distances from their pair of non-conterminous extremities is constant in magnitude and sign.

COR. 5°. From the two parts of the above, by virtue of the two general properties *a* and *a'* of Art. 340, it appears that—

a. In every figure homographic to a circle, the angle subtended by a variable at any two fixed points of the figure intercepts segments of constant magnitude on each of two fixed lines, and also segments having fixed middle points on each of two other fixed lines.

a'. In every figure homographic to a circle, the segment intercepted on a variable by any two fixed tangents to the figure subtends angles of constant magnitude at each of two fixed points, and also angles having fixed middle lines at each of two other fixed points.

Cor. 6°. And, from both parts, again, by virtue of the two general properties *a* and *a'* of Cor. 1° of the same article (340), that—

a. For every figure homographic to a circle, there exist two points (always real) the rectangle under whose distances from a variable tangent to the figure is constant in magnitude and sign.

a'. For every figure homographic to a circle, there exist two lines (sometimes imaginary) the rectangle under whose distances from a variable point on the figure is constant in magnitude and sign.

From the first of these latter properties it follows, as shewn in Art. 340, Cor. 3°, *b*, that, *in every figure homographic to a circle, the locus of the intersections of all pairs of rectangular tangents is a circle, which of course opens out into a line in the particular case where the figure has a tangent at infinity.*

That the two lines in the second property are *the two tangents to the figure whose points of contact are at infinity* (Ex. 2°, note), is evident from Cor. 6° of the preceding combined with Cor. 2° of the present article; and the same may also be shewn directly in a variety of ways.

Ex. 7°. *When, of any figure homographic to a circle, two variable points (or tangents) connect through (or intersect on) a fixed point (or line), the two corresponding tangents (or points) intersect on (or connect through) a fixed line (or point).*

For, both properties, by Cor. 3° of Art. 166, being true of the circle itself, are consequently, by 1° and 20° of Art. 381, true of every figure homographic to it; and therefore &c.

N.B. Every point and line related to each other, with respect to any figure homographic to a circle, as in both parts of the above general property, are said, as in the case of the circle itself, *to be pole and polar to each other with respect to the figure;* and the entire nomenclature connected with the subject of poles and polars, as employed in Chapter X. with respect to the circle, being extended in the same manner to every figure possessing the corresponding properties, it follows evidently, for the same reason as above, from the nature of those properties as given in that chapter, and from the fundamental relations of homographic figures as stated in articles 380 and 381, that generally—

In every case of the transformation of a circle into any figure homographic to it, every point and line pole and polar to each other (165), *every two points or lines conjugate to each other* (174), *every two triangles or other figures reciprocal polars to each other* (170), *every triangle or other figure reciprocal to itself* (170), *&c. with respect to the circle, are transformed into correspondents of the same nature similarly related to each other with respect to the figure.*

Cor. From the second and first parts of the above, applied respectively to the particular cases when the fixed line, or polar, is at infinity, and when the fixed point, or pole, is at infinity in any direction, it follows at once that—

a. In every figure homographic to a circle, all pairs of points at which the corresponding tangents are parallel connect through a fixed point, the pole with respect to the figure of the line at infinity.

a'. In every figure homographic to a circle, all pairs of tangents whose chords of contact are parallel intersect on a fixed line, the polar with respect to the figure of the point at infinity in the direction of the chords.

N.B. The pole of the line at infinity and the several polars passing through it of the several points at infinity, possesses some remarkable properties with respect to the figure; the principal of which will appear from the general property of the next example, to which we now proceed.

Ex. 8°. *When, of any figure homographic to a circle, two variable points (or tangents) connect through (or intersect on) a fixed point (or line), the harmonic conjugate with respect to them of the fixed point (or line) moves on (or turns round) a fixed line (or point).*

For, both properties, by *a* and *a'* of Art. 259, being true of the circle itself, are consequently, by 10° and 20° of Art. 381, true of every figure homographic to it; and therefore &c.

Cor. From the first part of the above, applied to the particular cases when the fixed line, or polar, is at infinity, and when the fixed point, or pole, is at infinity in any direction, it follows immediately, by virtue of 3°, Art. 216, that—

a. In every figure homographic to a circle, every chord of the figure which passes through the pole of the line at infinity is bisected at that point.

a'. In every figure homographic to a circle, every chord of the figure whose direction passes through any point at infinity is bisected by the polar of that point.

By virtue of these two important properties of every figure homographic to a circle, the pole of the line at infinity, as bisecting every chord passing through it, is termed *the centre* of the figure, and the polar of every point at infinity, as bisecting every chord parallel to the direction of the point, is termed *a diameter* of the figure. That every diameter passes through the centre, and thus derives its name, is evident, either generally from the consideration that, as its pole lies on the line at infinity, it consequently passes itself through the pole of that line, or particularly, from the consideration that, as bisecting every chord whose direction passes through its pole, it consequently bisects the particular one which passes through the centre. In the particular case when the figure has a tangent at infinity (Ex. 2°, note), the pole of the line at infinity being then its point of contact with the figure (165, 4°), the centre consequently is at infinity, and every diameter consequently parallel to its direction.

In every figure not having a tangent at infinity, every two diameters whose directions pass each through the pole of the other (174) are termed *conjugate diameters* of the figure; they, evidently, bisect each all chords parallel to the other, pass each through the points of contact of the two tangents parallel to the other, and, like all other conjugate lines of the figure

(259), are harmonic conjugates to each other with respect to the two central tangents of the figure, that is (165, 6°) to the two tangents, real or imaginary, whose points of contact are at infinity. The two lines bisecting the two pairs of opposite sides of any inscribed parallelogram, as bisecting each a pair of chords parallel to the other, are consequently a pair of conjugate diameters of the figure; and so, for the same reason, by virtue of the property *a'* Cor. 1°, of the preceding example, are the two connecting the two pairs of opposite vertices of any exscribed parallelogram also. In the particular case when the figure is itself a circle, all pairs of diameters intersecting at right angles are evidently pairs of conjugate diameters, and conversely all pairs of conjugate diameters evidently intersect at right angles.

Of the different pairs of conjugate diameters, all of which, as just observed, divide harmonically the angle, real or imaginary, determined by the two central tangents to the figure, the particular pair which bisect that angle, externally and internally, and which are consequently at right angles to each other, are termed *the axes* of the figure; they evidently bisect each all chords perpendicular to itself, and consequently divide each the entire figure into two similar, equal, and symmetrical halves, reflexions of each other with respect to itself (50). For figures having a tangent at infinity (Ex. 2°, note) two axes also exist, but, for such figures, the centre being at infinity, one of the two axes is consequently also at infinity, and but one therefore at a finite distance. For the circle itself, every diameter is evidently an axis.

For a given figure homographic to a circle the centre and axes may be readily determined as follows: drawing any two pairs of parallel chords in different directions, the two connectors of their two pairs of middle points are two diameters of the figure which by their intersection consequently determine the centre; should the centre thus determined be at infinity, any two parallel chords perpendicular to its direction determine evidently, by their two middle points, the axis not at infinity; and should it not, any circle concentric with it, and intersecting the figure, determines evidently, by its four intersections (see 50), an inscribed rectangle whose two pairs of opposite sides determine, by their two pairs of middle points, the two axes. In the particular case when the figure is itself a circle, the directions of the axes determined by this latter part of the construction become, as they ought, indeterminate.

Ex. 9°. *Every two triangles reciprocal polars to each other with respect to any figure homographic to a circle are in perspective; and their centre and axis of perspective are pole and polar to each other with respect to the figure.*

For, the first part of the property, by 1°, Art. 180, being true of the circle itself, is consequently, by Ex. 7°, note, and 1°, Art. 381, true of every figure homographic to it; and the second part, by virtue of the general property of Art. 167, being evident alike for circle and figure, therefore &c. This general property evidently includes as particular cases those given in Ex. 3° of the present article.

Conversely, *every two triangles in perspective are, with their centre and* *axis of perspective, reciprocal polars to each other with respect to a unique figure homographic to a circle, which is implicitly given when the triangles themselves are given.* For, if A and A', B and B', C and C' (see figs. a and a', Art. 295) be the three pairs of corresponding vertices (fig. a), or sides (fig. a'), of the two triangles; O and I their centre and axis (or axis and centre) of perspective; U, V, W the three points of intersection (or lines of connection) of the three lines (or points) AA', BB', CC' with the line (or point) I; D and D', E and E', F and F' their three pairs of intersections (or connectors) with the three pairs of lines (or points) BC and $B'C'$, CA and $C'A'$, AB and $A'B'$; and, G and G', H and H', K and K' the three pairs of collinear points (or concurrent lines) which divide harmonically the three pairs of segments (or angles) AD' and $A'D$, BE' and $B'E$, CF' and $C'F$ (230); then, since, from the involution of the three triads of segments or angles AD', $A'D$, and OU; BE', $B'E$, and OV; CF', $C'F$, and OW (299, a and a'), and the consequent harmonicism of the three quartets of points (or rays) G, G', O, U; H, H', O, V; K, K', O, W (370), the six points (or lines) G and G', H and H', K and K' determine an equianharmonic hexastigm (or hexagram) (295, Cor. 3°), they are consequently six points on (or tangents to) the same figure homographic to a circle (Ex. 5°); which figure being determined by any five of them (3°, Art. 385), and being such that the nine pairs of points (or lines) A and D', B and E', C and F'; A' and D, B' and E, C' and F; O and U, O and V, O and W are pairs of conjugates with respect to it (Ex. 8°), therefore &c.

Ex. 10°. *Every figure homographic to a circle intersects (or subtends) harmonically the three sides (or angles) of every triangle self-reciprocal with respect to itself; and, conversely, every triangle whose three sides (or angles) are intersected (or subtended) harmonically by any figure homographic to a circle is self-reciprocal with respect to the figure.*

For, both properties, by a and a' of Art. 259, being true of the circle itself, are consequently, by 10° and 20° of Art. 381, true of every figure homographic to it; and therefore &c.

Given, of a figure homographic to a circle, a self reciprocal triangle ABC, and the position of the centre O, the directions of the two central tangents (real or imaginary) and of the two axes (always real) may be readily determined as follows: The three lines OA, OB, OC, connecting the centre with the three vertices, and the three parallels OA', OB', OC' through the centre to the three opposite sides, of the triangle, determining (by Ex. 8°) three pairs of conjugate diameters OA and OA', OB and OB', OC and OC' of the figure; the two double rays (real or imaginary) OM and ON (370), and the two rectangular rays (always real) OI and OI' (371, Cor. 2°), of the involution they determine (368, Ex. 2°), are respectively the two pairs of lines in question (Ex. 8°). In the particular case when O is the polar centre of the triangle ABC (168), the three pairs of lines OA and OA', OB and OB', OC and OC' being rectangular, so therefore are all pairs of conjugates of the involution they determine, and the directions of the axes are consequently indeterminate; as they ought, the figure being then a circle (168).

That, in the same case, the figure itself is implicitly given, may also be readily shewn as follows: If P and P', Q and Q', R and R' be its three pairs of intersections with the three diameters OA, OB, OC, and X, Y, Z the three intersections of the latter with the three sides BC, CA, AB of the triangle; then, the three central chords PP', QQ', RR' being all bisected at O, and cut harmonically at the three pairs of points A and X, B and Y, C and Z respectively (Ex. 8°), therefore (225) $OP^2 = OP'^2 = OA . OX$, $OQ^2 = OQ'^2 = OB . OY$, $OR^2 = OR'^2 = OC . OZ$; relations which give at once the six points P and P', Q and Q', R and R', and therefore the figure itself (385, 3°). In the particular case when O is the polar centre of the triangle ABC (168), the three rectangles $OA . OX$, $OB . OY$, $OC . OZ$ being equal in magnitude and sign; so therefore are the six semi-diameters OP and OP', OQ and OQ', OR and OR'; as they ought, the figure being then a circle (168).

More generally, *given, of a figure homographic to a circle, a self-reciprocal triangle, and a point and line pole and polar to each other; the two tangents, real or imaginary, to the figure through the former, and the two intersections, real or imaginary, of the figure with the latter, may be readily determined as follows:* If A, B, C be the three vertices (or sides) of the triangle, O and I the point and line (or line and point), and A', B', C' the three intersections (or connectors) of the latter with the three opposite sides (or vertices) BC, CA, AB of the triangle; then, the three pairs of lines (or points) OA and OA', OB and OB', OC and OC' being (174 and Ex. 7°) pairs of conjugates with respect to the figure, the two double rays (or points), real or imaginary, OM and ON of the involution they determine (299) are (370) the two tangents (or points) in question (Ex. 8°), and their two intersections (or connectors) with I, are the two corresponding points (or tangents) (Ex. 7°).

That, in the same case, the figure itself is implicitly given, may also be readily shewn as follows: If G and G', H and H', K and K' be its three pairs of intersections with (or tangents through) the three lines (or points) OA, OB, OC; X, Y, Z the three intersections (or connectors) of the latter with the three opposite sides BC, CA, AB of the triangle; and U, V, W their three intersections (or connectors) with the line (or point) I; then, the three pairs of points (or tangents) G and G', H and H', K and K', dividing, as they do, harmonically the three pairs of segments (or angles) AX and OU, BY and OV, CZ and OW respectively (Ex. 8°), are consequently given (230), and with them of course the figure itself (385, 3°).

Ex. 11°. *In every tetrastigm (or tetragram) determined by four points on (or tangents to) any figure homographic to a circle, the three intersections (or connectors) of the three pairs of opposite connectors (or intersections) determine a self-reciprocal triangle with respect to the figure.*

For, both properties, by a and a' of Art. 261, being true of the circle itself, are consequently, by Ex. 7°, note, true of every figure homographic to it; and therefore &c.

N.B. By virtue of the above, all the properties established for the circle, in the several corollaries of the article referred to in their proof (261), with the applications of them given in the two succeeding articles (262 and 263), are seen at once to hold, without modification of any kind, not only for the circle, but for every figure into which the circle may be transformed homographically also.

That, *for every figure homographic to a circle, the six vertices (or sides) of every two self reciprocal triangles determine an equianharmonic hexastigm (or hexagram)*; appears, in precisely the same manner as for the circle itself. See Art. 301, Cor. 3°.

Ex. 12°. *In every figure homographic to a circle, if A and A′, B and B′, C and C′ be the three pairs of opposite connectors (or intersections) of the tetrastigm (or tetragram) determined by any four fixed points on (or tangents to) the figure, and I a variable point (or tangent) of the figure; then, in every position of I, the three rectangles IA . IA′, IB . IB′, IC . IC′ are to each other, two and two, in constant ratios.*

For, since, by Ex. 6°, the variable point (or tangent) *I*, in the course of its variation, divides homographically the three pairs of angles (or segments) *BC′* and *CB′*, *CA′* and *AC′*, *AB′* and *BA′*, therefore, by Cor. Art. 328, the three ratios

$$\left(\frac{IB}{IC'} : \frac{IC}{IB'}\right),\ \left(\frac{IC}{IA'} : \frac{IA}{IC'}\right),\ \left(\frac{IA}{IB'} : \frac{IB}{IA'}\right)$$

are constant in magnitude and sign; and therefore &c.

N.B. That, for the circle itself, the three rectangles, in the former case are all equal, and in the latter case are proportional to the three *OA . OA′*, *OB . OB′*, *OC . OC′*, where *O* is the centre of the circle, has been shewn in Cor. 10°, Art. 62, and in 4°, Cor. 2°, Art. 179; from which, of course, the above would have followed also, by virtue of the general relations of construction given in Art. 384, but by a process on the whole less simple and instructive than that actually employed for their establishment.

Cor. From the above, applied to the particular case when two of the four fixed points (or tangents) coincide with the remaining two, since then, of the six lines of connection (or points of intersection) of the tetrastigm (or tetragram) they determine, four necessarily coincide, while the remaining two are the two tangents (or points) corresponding to the two coincident pairs of points (or tangents), it follows consequently that—

In every figure homographic to a circle, if A and B be any two fixed tangents (or points), C the connector (or intersection) of the two corresponding points (or tangents), and I a variable point (or tangent); then, in every position of I, the ratio $IA . IB : IC^2$ *is constant in magnitude and sign.*

N.B. That, for the circle itself, the constant ratio, in the former case = 1, and in the latter case = $OA . OB : OC^2$, where *O* is the centre of the circle, has been shewn in Art. 48, Ex. 9°, and in Art. 179, Cor. 1°; from which, as above observed for the general properties, those of the corollary itself would have followed by virtue of the general relations of construction given in Art. 384.

Ex. 13°. *In every figure homographic to a circle, if A, B, C be the three sides (or vertices) of any fixed triangle inscribed (or exscribed) to the figure, and I a variable point (or tangent) of the figure; then, in every position of I,*

$$\frac{a}{IA} + \frac{b}{IB} + \frac{c}{IC} = 0,$$

where a, b, c are three multiples, whose ratios to each other, two and two, are constant in magnitude and sign.

For, if A', B', C' be the three lines of connection (or points of intersection) of the three fixed vertices (or sides) of the triangle ABC with any fourth fixed point on (or tangent to) the figure; then, in every position of I, since, by the preceding example, the three rectangles $IA.IA'$, $IB.IB'$, $IC.IC'$ are to each other two and two in constant ratios, and since, by Cor. 6° (or 4°) of Art. 82, from the concurrence (or collinearity) of A', B', C',

$$a'.IA' + b'.IB' + c'.IC' = 0,$$

where a', b', c' are three multiples, whose ratios to each other two and two are constant in magnitude and sign; therefore &c.

N.B. That, for the circle itself, the three constant multiples are proportional, in the former case to the three sides a, b, c, and in the latter case to the three differences $s-a$, $s-b$, $s-c$ between the semi-perimeter and the three sides, of the triangle ABC; may be readily seen from Cor. 3°, Art. 64, and from Cor. 1°, Art. 179; or the reader may easily prove the same independently for himself.

Ex. 14°. *In every figure homographic to a circle, if A, B, C be the three sides (or vertices) of any fixed triangle exscribed (or inscribed) to the figure, and I a variable point (or tangent) of the figure; then, in every position of I,*

$$a.IA^{\frac{1}{2}} + b.IB^{\frac{1}{2}} + c.IC^{\frac{1}{2}} = 0,$$

where a, b, c are three multiples, whose ratios to each other, two and two, are constant in magnitude and sign.

For, if A', B', C' be the three sides (or vertices) of the corresponding fixed triangle inscribed (or exscribed) to the figure; then, in every position of I, since, by Ex. 10°, Cor. the three rectangles $IB.IC$, $IC.IA$, $IA.IB$ are to the three squares IA'^2, IB'^2, IC'^2 respectively in constant ratios, and since, by the preceding example,

$$\frac{a'}{IA'} + \frac{b'}{IB'} + \frac{c'}{IC'} = 0,$$

where a', b', c' are three multiples, whose ratios to each other two and two are constant in magnitude and sign; therefore &c.

N.B. That, for the circle itself, the three constant multiples are proportional, in the former case to the three sides a', b', c', and in the latter case to the three differences $s'-a'$, $s'-b'$, $s'-c'$ between the semi-perimeter and the three sides, of the triangle $A'B'C'$, appears at once, by virtue of the above demonstration, from the note to the preceding example; or, as there observed, the reader may easily prove the same independently for himself.

Ex. 15°. *In every figure homographic to a circle, if A, B, C be the three vertices (or sides) of any fixed triangle self reciprocal with respect to the figure, and I a variable tangent (or point) of the figure; then, in every position of I,*

$$a \,.\, IA^2 + b \,.\, IB^2 + c \,.\, IC^2 = 0$$

where a, b, c are three multiples, whose ratios to each other, two and two, are constant in magnitude and sign.

For, if X and Y be the two fixed points (or tangents) of the figure which are collinear (or concurrent) with any two, A and B, of the three vertices (or sides) of the fixed triangle ABC, and with which they consequently determine, by Ex. 10°, the harmonic row (or pencil) A, B, X, Y; then, since, by Ex. 7°, the two corresponding fixed tangents (or points) pass through (or lie on) the third vertex (or side) C, therefore, in every position of I, by Ex. 12°, Cor. the rectangle IX, IY is to the square of IC in a constant ratio; and it remains only to shew that, in every position of I, the same rectangle is connected with the squares of IA and IB by a relation of the above form.

The four points (or lines) A, B, X, Y being collinear (or concurrent) therefore, by Cors. 4° and 6° of Art. 82, whatever be the position of the line (or point) I,

$$(AB \text{ or } \sin AB)\, IX = -(BX \text{ or } \sin BX)\, IA + (AX \text{ or } \sin AX)\, IB,$$

$$(AB \text{ or } \sin AB)\, IY = -(BY \text{ or } \sin BY)\, IA + (AY \text{ or } \sin AY)\, IB,$$

from which, multiplying, remembering that, from the harmonicism of the system A, B, X, Y, by relations (1) and (1') Art. 219.

$$(AX \text{ or } \sin AX).(BY \text{ or } \sin BY) + (AY \text{ or } \sin AY).(BX \text{ or } \sin BX) = 0,$$

it follows, immediately, that

$$(AB \text{ or } \sin AB)^2 . IX . IY = (BX \text{ or } \sin BX) . (BY \text{ or } \sin BY) . IA^2$$
$$+ (AX \text{ or } \sin AX) . (AY \text{ or } \sin AY) . IB^2;$$

which being, in either case, of the form in question, therefore &c.

N.B. For the circle itself, the quantities, to which the three constant multiples a, b, c are proportional, are evidently given for the latter case in the relation of Art. 264, and can from the same, of course, be at once inferred for the former case by Dr. Salmon's Theorem, given in Art. 179. The reader however can have no difficulty in determining them directly in either case for himself.

387. With the four following general properties of any two homographic figures we shall conclude the present chapter.

1°. *For any two homographic figures F and F', the two* *correspondents* I_1 *and* I_2, *in the two figures, of a variable point or line* I, *moving according to any law, generate two homographic figures* G_1 *and* G_2, *in which all pairs of corresponding elements, whether points or lines, which coincide with each other are the same as in the original figures.*

For, the two figures G_1 and G_2, generated by the two variable points (or lines) I_1 and I_2, being each homographic with the figure G, generated by the variable point (or line) I (384), and therefore homographic with each other (380), therefore &c. as regards the first part; and since, when any two points (or lines) A_1 and A_2 of G_1 and G_2 coincide, then evidently the point (or line) A of G, to which they correspond in the original figures F and F', coincides with both, therefore &c. as regards the second part.

N.B. As, for two homographic rows of points or pencils of rays having a common axis or vertex (341), so, for two homographic figures of any kind F and F', every point or line at or along which a pair of corresponding elements coincide, is termed *a double point or line* of the figures. It was shewn in Cor. 4°, Art. 383, that, for two homographic figures of any kind, no more than three independent double points or lines could exist unless the figures altogether coincided; and it will be shewn in the next general property (2°) that, for every two homographic figures, three double elements of each species (two of which however may be and often are imaginary) do always exist, and constitute in fact the three elements of each species (vertices and sides) of the same triangle Δ.

2°. *For every two homographic figures* F *and* F', *however situated, there exists a triangle (unique or indeterminate)* Δ; *whose three elements of either species (sides or vertices) constitute each a pair of corresponding elements (lines or points) of the figures coinciding with each other.*

For, if X_1, Y_1, Z_1 and X_2, Y_2, Z_2 be the two triads of collinear points (or concurrent lines) corresponding in the two figures to the same arbitrary triad of collinear points (or concurrent lines) X, Y, Z regarded as belonging first to one and then to the other figure; and A, B, C the three lines (or points) which intersect with the three axes (or connect with the three vertices) of the three homographic rows (or pencils) determined by the three triads of corresponding points (or rays) X_1, X, X_2; Y_1, Y, Y_2; Z_1, Z, Z_2 at (or by) three triads of corresponding points (or rays) U_1, U, U_2; V_1, V, V_2; W_1, W, W_2 (Ex. 10°, Art. 353); the three lines (or points) A, B, C, as determining three pairs of corresponding lines (or points) of the figures U_1U and UU_2, V_1V and VV_2, W_1W and WW_2 (380) which coincide with each other, determine consequently a triangle Δ whose three sides (or vertices), and therefore whose three vertices (or sides) also (380), fulfil the conditions in question; and therefore &c.

N.B. The triangle Δ, when none of its elements are known, cannot in general be constructed by elementary geometry; but if one of its sides (or vertices) A be known, the remaining two B and C, which may be real or imaginary, are given implicitly with them, and may in all cases be determined immediately by the corresponding construction a or a' of the problem Ex. 10°, of Art. 353, applied to the arbitrary triad of points (or lines) X, Y, Z and the two X_1, Y_1, Z_1 and X_2, Y_2, Z_2 determinable from them by Cor. 1° of Art. 383.

3°. *When, for two homographic figures F and F', the three double lines (or points) A, B, C are concurrent (or collinear), the figures themselves are in perspective; and the point of concurrence (or line of collinearity) O is their centre (or axis) of perspective.*

For, if U and U', V and V', W and W' be any three pairs of corresponding points on (or lines through) the three double lines (or points) $A = A'$, $B = B'$, $C = C'$; and X and X', Y and Y', Z and Z', &c. any number of other pairs of corresponding points (or lines) of the figures; then since, by 380, O being evidently a double point (or line), $\{O.UVWXYZ$ &c.$\}$ = $\{O.U'V'W'X'Y'Z'$ &c.$\}$, and since, by hypothesis, $OU = OU'$, $OV = OV'$, $OW = OW'$, therefore (268) $OX = OX'$, $OY = OY'$, $OZ = OZ'$, &c.; and therefore &c. (141).

N.B. Since, for every two homographic figures F and F', every pair of corresponding lines L and L' parallel to the two particular lines A and B' whose two correspondents A' and B coincide at infinity are divided similarly by the several pairs of corresponding points P and P', Q and Q', R and R', S and S', &c. that lie on them (6°, Cor. 2°, Art. 382); and since, for every two figures F and F' in perspective with each other, the several lines of connection PP', QQ', RR', SS', &c. of all pairs of corresponding points P and P', Q and Q', R and R', S and S', &c. are concurrent (141); it follows consequently that when two homographic figures F and F' are in perspective, their two lines A and B' whose two correspondents A' and B coincide at infinity are parallel (Euc. VI. 2). In the particular case when the centre of perspective is at infinity, since then all pairs of corresponding points connect by parallel lines, and since consequently all pairs of corresponding lines however situated are divided similarly by the several pairs of corresponding points that lie on them, the figures themselves consequently have a pair of corresponding lines coinciding at infinity (3°, Cor. 4°, Art. 382).

4°. *Every two homographic figures F and F', which have not a double line at infinity, may be placed, in two pairs of different and opposite positions relatively to each other, so as to be in perspective with each other.*

For this the following (always possible and determinate) construction, based on the note to the preceding property 3°, has been given by Chasles. On any pair of corresponding lines L and L' of the figures parallel to the two A and B' whose two correspondents A' and B coincide at infinity (4°, Cor. 2°, Art. 382) taking arbitrarily any two pairs of corresponding points P and P', Q and Q'; drawing through them (by Ex. 1°, Art. 353) either of the two pairs (always real) of corresponding lines PO and $P'O'$, QO and $Q'O'$ for which the two pairs of angles OPQ and $O'P'Q'$, OQP and $O'Q'P'$ are equal in absolute magnitude, and for which consequently the pair of corresponding triangles POQ and $P'O'Q'$ are similar; and placing the figures, in either case, in either of the two opposite positions, relatively to each other, in which the pair of corresponding points O and O', and the two pairs of corresponding lines OP and $O'P'$, OQ and $O'Q'$ shall coincide; the four resulting positions thus obtained are positions of perspective.

For, since, in each, for all pairs of corresponding points P and P', Q and Q', R and R', S and S', &c. on the pair of (then parallel) corresponding lines L and L' (by 6°, Cor. 2°, Art. 382) $PQ : P'Q' = PR : P'R' = PS : P'S' =$ &c. therefore (by Euc. VI. 4) their several lines of connection PP', QQ', RR',

SS', &c. all concur to the double point $O = O'$; and therefore &c. See note to the preceding property 3°.

In the particular case when the two figures F and F' have a double line at infinity, the above construction of Chasles' becomes, as observed by himself, and as it ought, indeterminate; *the figures having then, in fact, not two (always real), but an infinite number (real or imaginary), of pairs of opposite positions of perspective with each other.* This is evident in the case of two *similar* figures, for which, when both right or left, all pairs of similar and opposite positions (33) are positions of perspective, the double line at infinity being for all alike the common axis of perspective (142); and, for any other two figures of the same class, it appears readily from the consideration that, if *any* two pairs of corresponding points P and P', Q and Q', not at infinity, can be found for which in absolute magnitude $PQ = P'Q'$, the placing of the figures in either of the two positions in which P shall coincide with P', and Q with Q', will place them in a position of perspective, by virtue of the general property 3° of the present article; the point at infinity on the double line PQ being then, by hypothesis, a third double point on that line. And that, for *every* pair of corresponding points P and P' of the figures, four different second pairs Q and Q', on two pairs of corresponding lines (real or imaginary) passing through P and P', can be found satisfying the required condition, may be readily shown as follows:

Drawing arbitrarily any two particular pairs of corresponding lines L and L', M and M' through any particular pair of corresponding points P and P'; then, since, for every other pair of corresponding points Q and Q', the two ratios $QL : Q'L'$ and $QM : Q'M'$ are given in magnitude and sign (Cor. 3°, Art. 382), if, in addition, the ratio $PQ : P'Q'$ be also given in absolute magnitude (as it is in the case in question), the two pairs of corresponding directions (real or imaginary) of the two lines PQ and $P'Q'$ (which when superposed, as above described, constitute the axis of perspective of the figures) are manifestly given with it; and therefore &c.

N.B. In every case when two homographic figures of any kind are brought by any means into any position of perspective with each other, it is evident, from the general property of

Art. 141, that, if either figure be turned through two right angles, either in its plane round the centre or with its plane round the axis of perspective, the other remaining unmoved, it will be in perspective, in its new as well as in its original position, with the other. From this, combined with the property, above established in the general case, that two homographic figures have in general but four different relative positions of perspective with each other, it follows indirectly that *if, in any position of perspective of any two homographic figures F and F′, either figure receive both the above movements in succession, the other the while remaining unmoved, its ultimate position as regards the other is independent of the order in which the movements take place.* A property of figures in perspective which the reader may easily verify directly for himself.

CHAPTER XXIII.

METHODS OF GEOMETRICAL TRANSFORMATION. THEORY OF CORRELATIVE FIGURES.

388. Two figures of any kind, F and F', in which correspond, to every point of either a line of the other, to every line of either a point of the other, to every connector of two points of either the intersection of the two corresponding lines of the other, and to every intersection of two lines of either the connector of the two corresponding points of the other, are said *to be correlative* when every quartet of collinear points or concurrent lines of either and the corresponding quartet of concurrent lines or collinear points of the other are equianharmonic. Every two figures reciprocal polars to each other with respect to any circle (170) are evidently thus related to each other (292).

As two anharmonic quartets of any kind, when each equianharmonic with a common quartet, are equianharmonic with each other; it follows at once, from the above definition, that *when two figures of any kind F' and F'' are each correlative with a common figure F, they are homographic with each other.* (See Art. 380).

389. Every two figures F and F' satisfying the four preliminary conditions, whether correlative or not, possess evidently the following properties in relation to each other.

1°. *To every collinear system of points or concurrent system of lines of either, corresponds a concurrent system of lines or collinear system of points of the other.*

For, every connector of two points (or intersection of two lines) of either corresponding to the intersection of the two corresponding lines (or the connector of the two corresponding points) of the other; when, for any system of the points (or lines) of either, every two connect by a common line (or intersect at a common point), then, for the corresponding system of the lines (or points) of the other, every two intersect at the corresponding point (or connect by the corresponding line); and therefore &c.

2°. *To every two collinear systems of points or concurrent systems of lines of either in perspective with each other, correspond two concurrent systems of lines or collinear systems of points of the other in perspective with each other.*

For, the concurrence (or collinearity) of the several lines of connection (or points of intersection) of the several pairs of corresponding constituents of the two systems, for either, involves, by 1°, the collinearity (or concurrence) of the several points of intersection (or lines of connection) of the several pairs of corresponding constituents of the two corresponding systems, for the other; and therefore &c. (130).

3°. *To every two figures of the points and lines of either in perspective with each other, correspond two figures of the lines and points of the other in perspective with each other.*

For, the concurrence of the several lines of connection of the several pairs of corresponding points, and the collinearity of the several points of intersection of the several pairs of corresponding lines, of the two figures, for either, involve, by 1°, the collinearity of the several points of intersection of the several pairs of corresponding lines, and the concurrence of the several lines of connection of the several pairs of corresponding points, of the two corresponding figures, for the other; and therefore &c. (141).

4°. *To a variable point moving on a fixed line or a variable line turning round a fixed point of either, corresponds a variable line turning round the corresponding fixed point or a variable point moving on the corresponding fixed line of the other.*

For, since every two positions of the variable point (or line) connect by the same fixed line (or intersect at the same fixed point) for the former; therefore, by 1°, every two positions of the variable line (or point) intersect at the corresponding fixed point (or connect by the corresponding fixed line) for the latter; and therefore &c.

5°. *To a variable point or line of either the ratio of whose distances from two fixed lines or points is constant, corresponds a* *variable line or point of the other the ratio of whose distances from the two corresponding fixed points or lines is constant.*

For, since the variable point (or line) evidently moves on a line concurrent with the two fixed lines (or turns round a point collinear with the two fixed points) for the former; therefore, by the preceding property 4°, the variable line (or point) turns round a point collinear with the two corresponding fixed points (or moves on a line concurrent with the two corresponding fixed lines) for the latter; and therefore &c.

6°. *To a variable polygon of either all whose vertices move on fixed lines and all whose sides but one turn round fixed points, or conversely, corresponds a variable polygon of the other all whose sides turn round the corresponding fixed points and all whose vertices but one move on the corresponding fixed lines, or conversely.*

For, since, by 4°, to every variable point moving on a fixed line (or variable line turning round a fixed point) of either, corresponds a variable line turning round the corresponding fixed point (or a variable point moving on the corresponding fixed line) of the other; therefore &c.

7°. *To every harmonic row of four points or pencil of four rays of either, corresponds an harmonic pencil of four rays or row of four points of the other.*

For, as every harmonic row (or pencil) may be regarded as determined by two angles and their two axes of perspective on the connector of their vertices (or by two segments and their two centres of perspective at the intersection of their axes) (241); and as, to the vertices and axes of perspective of any two angles (or the axes and centres of perspective of any two segments) of either, correspond the axes and centres of perspective of the two corresponding segments (or the vertices and axes of perspective of the two corresponding angles) of the other; therefore &c.

8°. *To every pair of lines or points conjugate to each other with respect to any segment or angle of either, correspond a pair of points or lines conjugate to each other with respect to the corresponding angle or segment of the other.*

For, as every two lines (or points) conjugate to each other with respect to any segment (or angle) intersect with the axis of the segment (or connect with the vertex of the angle) at two points (or by two lines) which divide the segment (or angle) harmonically (217); therefore &c. by the preceding property 7°.

9°. *To every point and line pole and polar to each other with respect to any triangle of either, correspond a line and point polar and pole to each other with respect to the corresponding triangle of the other.*

For, as every point and the intersection of its polar with each side (or every line and the connector of its pole with each vertex) of any triangle are conjugate to each other with respect to the opposite angle (or side) of the triangle (250, Cor. 2°); therefore &c. by the preceding property 8°.

10°. *To a variable point or line of either determining with four fixed points or lines an harmonic pencil or row, corresponds a variable line or point of the other determining with the four corresponding fixed lines or points an harmonic row or pencil.*

For, the harmonicism of the quartet of variable rays (or points) in every position of the variable point (or line), for either, involving, by 7°, the harmonicism of the corresponding quartet of variable points (or rays) in every position of the variable line (or point), for the other; therefore &c.

11°. *To every two equianharmonic rows of four points or pencils of four rays of either, correspond two equianharmonic pencils of four rays or rows of four points of the other.*

For, as every two equianharmonic rows of four points (or pencils of four rays) may be regarded as determined, on their respective axes (or at their respective vertices), by two quartets of rays (or points) in perspective with each other (290); and as, to every two quartets of rays (or points) in perspective, for either, correspond two quartets of points (or rays) in perspective, for the other (property 2° above); therefore &c.

12°. *To every equianharmonic hexastigm or hexagram of either, corresponds an equianharmonic hexagram or hexastigm of the other.*

For, the equianharmonicism of the two pencils of connection (or rows of intersection) of any two with the remaining four of the six points (or lines) of the hexastigm (or hexagram) for the former, involving, by the preceding property 11°, the equianharmonicism of the two corresponding rows (or pencils) of the corresponding hexagram (or hexastigm) for the latter; therefore &c. (301). The same result follows also from the reciprocal properties of Art. 302, by virtue of the preceding property 1°.

13°. *To a variable point or line of either determining with four fixed points or lines a pencil or row having a constant anharmonic ratio, corresponds a variable line or point of the other determining with the four corresponding fixed lines or points a row or pencil having a constant anharmonic ratio.*

For, the equianharmonicism of the two quartets of rays (or points), in every two positions of the variable point (or line), for the former, involving, by 11°, the equianharmonicism of the two corresponding quartets of points (or rays), in every two positions of the variable line (or point), for the latter; therefore &c.

14°. *To every two homographic rows of points or pencils of rays of either, correspond two homographic pencils of rays or rows of points of the other.*

For, the equianharmonicism of every two quartets of corresponding constituents of the two rows (or pencils), for the former, involving, by 11°, the equianharmonicism of every two quartets of corresponding constituents of the two corresponding pencils (or rows), for the latter; therefore &c. (321).

15°. *To two homographic coaxal rows or concentric pencils of either in involution with each other, correspond two homographic concentric pencils or coaxal rows of the other in involution with each other.*

For, every interchange of corresponding constituents of the two systems, for either, involving evidently a corresponding interchange of corresponding constituents of the two corresponding systems, for the other; the interchangeability of every pair of corresponding constituents, for either, involves consequently the interchangeability of every pair of corresponding constituents, for the other; and therefore &c. (357). The same result follows also from the general property of Art. 370, by virtue of the preceding property 7°.

16°. *To the double points or rays of any two homographic coaxal rows or concentric pencils of either, correspond the double rays or points of the two corresponding concentric pencils or coaxal rows of the other.*

For, every coincidence of corresponding constituents of the two systems, for either, involving, evidently, a corresponding coincidence of corresponding constituents of the two corresponding systems, for the other; the two coincidences, real or imaginary, of pairs of corresponding constituents, which constitute the two double points (or rays) for the former, correspond, consequently, to the two coincidences, real or imaginary, of pairs of corresponding constituents, which constitute the two double rays (or points) for the latter; and therefore &c. (341).

17°. *To a variable point or line of either connecting or intersecting with two fixed points or lines homographically, corresponds a variable line or point of the other intersecting or connecting with the two corresponding fixed lines or points homographically.*

For, the equianharmonicism of every two quartets of corresponding connectors (or intersections) of the variable with the two fixed points (or lines), for the former, involving, by 11°, the equianharmonicism of every two quartets of corresponding intersections (or connectors) of the variable with the two corresponding fixed lines (or points), for the latter; therefore &c. (321).

18°. *To a variable point or line of either the rectangle under whose distances from two fixed lines or points is constant, corresponds a variable line or point of the other the rectangle under whose distances from two (not necessarily corresponding) fixed points or lines is constant.*

For, the variable point (or line) of the former connecting (or intersecting) with every two fixed positions of itself homographically (340, Cor. 2°); and the variable line (or point) of the latter consequently, by the preceding property 17°, intersecting or connecting with every two fixed positions of itself homographically; therefore &c. (340, Cor. 1°).

19°. *To a variable point or line of either whose angle of connection with two fixed points or chord of intersection with two fixed lines intercepts on a fixed line or subtends at a fixed point a segment or angle of constant magnitude, corresponds a variable line or point of the other whose chord of intersection with the two corresponding fixed lines or angle of connection with the two corresponding fixed points subtends at a (not necessarily corresponding) fixed point or intercepts on a (not necessarily corresponding) fixed line an angle or segment of constant magnitude.*

For, the variable point (or line) of the former connecting (or intersecting) with the two fixed points (or lines) homographically (325, *a* and *a'*); and the variable line (or point) of the latter consequently, by 17°, intersecting (or connecting) with the two corresponding fixed lines (or points) homographically; therefore &c. (339 and 340).

20°. *For continuous figures, the tangent at any point of either corresponds to the point of contact of the corresponding tangent to the other, and the point of contact of any tangent to either corresponds to the tangent at the corresponding point of the other.*

For, every connector of two points of either corresponding to the intersection of the two corresponding lines of the other, and every intersection of two lines of either corresponding to the connector of the two corresponding points of the other; and the coincidence of any two points or lines of either involving the coincidence of the two corresponding lines or points of the other; therefore &c. (19 and 20).

390. From the fundamental definition of Art. 388, the following general property of any two correlative figures F and F' may be readily inferred; viz.—

If A and B be any two fixed points (or lines) of either figure, A' and B' the two corresponding lines (or points) of the other, I any variable line (or point) of the former, and I' the corresponding point (or line) of the latter; then, for every position of I and I', the ratio

$$\left(\frac{AI}{BI} : \frac{A'I'}{B'I'}\right) \text{ or its equivalent } \left(\frac{AI}{A'I'} : \frac{BI}{B'I'}\right)$$

is constant, both in magnitude and sign.

For, if Z be the variable point of intersection (or line of connection) of the variable line (or point) I with the fixed line (or point) AB, and Z' the variable line of connection (or point of intersection) of the variable point (or line) I' with the fixed point (or line) $A'B'$; then, since, by hypothesis, Z and Z' determine a homographic row and pencil (or pencil and row) of which A and A', B and B' are two pairs of corresponding constituents (388), therefore, by (328), the ratio

$$\left(\frac{AZ}{BZ} : \frac{\sin A'Z'}{\sin B'Z'}\right) \text{ or } \left(\frac{\sin AZ}{\sin BZ} : \frac{A'Z'}{B'Z'}\right),$$

to which, in the corresponding case, the above is manifestly equivalent, is constant both in magnitude and sign; and therefore &c.

391. From the same fundamental definition, it follows, precisely in the same manner as the general property of the preceding article, that, for any two correlative figures F and F',

If A, B, C be any three fixed points (or lines) of either figure, A', B', C' the three corresponding lines (or points) of the other, I any variable line (or point) of the former, and I' the corresponding point (or line) of the latter; then, for every position of I and I', the three ratios

$$\left(\frac{BI}{CI} : \frac{B'I'}{C'I'}\right), \quad \left(\frac{CI}{AI} : \frac{C'I'}{A'I'}\right), \quad \left(\frac{AI}{BI} : \frac{A'I'}{B'I'}\right),$$

or their three equivalents

$$\left(\frac{BI}{B'I'} : \frac{CI}{C'I'}\right), \quad \left(\frac{CI}{C'I'} : \frac{AI}{A'I'}\right), \quad \left(\frac{AI}{A'I'} : \frac{BI}{B'I'}\right),$$

any two of which manifestly involve the third, are constant, both in magnitude and sign.

For, as in the preceding article, if X, Y, Z be the three intersections (or connectors) of I with BC, CA, AB respectively, and X', Y', Z' the three connectors (or intersections) of I' with $B'C'$, $C'A'$, $A'B'$ respectively; then, since, for the same reason as in the preceding article, the three ratios

$$\left(\frac{BX}{CX} : \frac{\sin B'X'}{\sin C'X'}\right) \text{ or } \left(\frac{\sin BX}{\sin CX} : \frac{B'X'}{C'X'}\right),$$

$$\left(\frac{CY}{AY} : \frac{\sin C'Y'}{\sin A'Y'}\right) \text{ or } \left(\frac{\sin CY}{\sin AY} : \frac{C'Y'}{A'Y'}\right),$$

$$\left(\frac{AZ}{BZ} : \frac{\sin A'Z'}{\sin B'Z'}\right) \text{ or } \left(\frac{\sin AZ}{\sin BZ} : \frac{A'Z'}{B'Z'}\right),$$

to which, in the corresponding cases, the above are manifestly equivalent, are constant both in magnitude and sign; therefore &c.

Cor. 1°. The above supplies obvious solutions of the two following problems: *given, of either of two correlative figures F, three points (or lines) A, B, C and a line (or point) D, and, of the* *other* F', *the three corresponding lines* (*or points*) A', B', C' *and the corresponding point* (*or line*) D'; *to determine the line* (*or point*) E *of the former* F *corresponding to any assumed point* (*or line*) E' *of the latter* F'. For, since, by the above,

$$\frac{BE}{CE} : \frac{B'E'}{C'E'} = \frac{BD}{CD} : \frac{B'D'}{C'D'},$$

$$\frac{CE}{AE} : \frac{C'E'}{A'E'} = \frac{CD}{AD} : \frac{C'D'}{A'D'},$$

$$\frac{AE}{BE} : \frac{A'E'}{B'E'} = \frac{AD}{BD} : \frac{A'D'}{B'D'},$$

the three ratios $BE : CE$, $CE : AE$, $AE : BE$, which manifestly determine the position of the required line (or point) E, are consequently given; and therefore &c.

As already observed for homographic figures (Cor. 1°, Art. 383), the particular cases where the given point (or line) E' is at infinity present no special peculiarity; the three ratios $B'E' : C'E'$, $C'E' : A'E'$, $A'E' : B'E'$ having the values $\sin B'L' : \sin C'L'$, $\sin C'L' : \sin A'L'$, $\sin A'L' : \sin B'L'$ respectively, where L' is any line parallel to the direction of E', in the former case, and being simply all $= 1$, in the latter case.

COR. 2°. As three points (or lines) A, B, C of either of two correlative figures F, and the three corresponding lines (or points) A', B', C' of the other F', determine (388) three lines (or points) BC, CA, AB of the former, and the three corresponding points (or lines) $B'C'$, $C'A'$, $A'B'$ of the latter; the solutions of the two problems: *given, of either of two correlative figures* F, *four points* (*or lines*) A, B, C, D, *and, of the other* F', *the four corresponding lines* (*or points*) A', B', C', D'; *to determine the point* (*or line*) E *of the former* F *corresponding to any assumed line* (*or point*) E' *of the latter* F'; may consequently be regarded as included in those of the above; the particular cases where the given line (or point) E' is at infinity, presenting, as above observed, no exceptional or special peculiarity.

COR. 3°. It appears also immediately from the above, that *when, for two correlative figures* F *and* F', *three points* (*or lines*) A, B, C *of either* F *are interchangeable with the three corresponding lines* (*or points*) A', B', C' *of the other* F', *the interchange-*

ability of any independent line (or point) D of the former with the corresponding point (or line) D' of the latter involves the interchangeability of every other line (or point) E of the former with the corresponding point (or line) E' of the latter. For, when, in the three relations of Cor. 1°, which as there shewn result immediately from it, A and A', B and B', C and C' are interchangeable, if, in addition, D and D' are interchangeable, then necessarily, E and E' are interchangeable; and therefore &c.

COR. 4°. For the same reason as in Cor. 2°, it follows of course from the preceding, Cor. 3°, that *when, for two correlative figures F and F', four independent points (or lines) A, B, C, D of either F are interchangeable with the four corresponding lines (or points) A', B', C', D' of the other F', then every point (or line) E of the former is interchangeable with the corresponding line (or point) E' of the latter.* Which is also evident *à priori* from the fundamental characteristic of correlative figures (388) that, for every quintet A, B, C, D, E of the points (or lines) of either F and the corresponding quintet A', B', C', D', E' of the lines (or points) of the other F', the five relations $\{A.BCDE\} = \{A'.B'C'D'E'\}$, $\{B.CDEA\} = \{B'.C'D'E'A'\}$, $\{C.DEAB\} = \{C'.D'E'A'B'\}$, $\{D.EABC\} = \{D'.E'A'B'C'\}$, $\{E.ABCD\} = \{E'.A'B'C'D'\}$ must in all cases exist together; which, when A and A', B and B', C and C', D and D' are interchangeable, would be manifestly impossible unless also E and E' were interchangeable; and therefore &c.

N.B. It will appear in the sequel that, for every pair of correlative figures F and F', there exists a unique pair of corresponding triangles Δ and Δ', for which the three elements of either species A, B, C of either, regarded as belonging to either figure, correspond interchangeably, as supposed in the two latter corollaries 3° and 4°, to the three of the other species A', B', C' of the other, regarded as belonging to the other figure; and of which, as in the corresponding property of homographic figures (Note, Art. 383), though two pairs of corresponding elements may be imaginary, the third pair are always real. When the two triangles Δ and Δ', thus related to the two figures F and F', coincide, that is when the three pairs of interchangeable elements A and A', B and B', C and C' which determine them are the three pairs of opposite elements (vertices and sides) of the same triangle Δ; then, as will appear also in the sequel, all pairs of corresponding elements D and D', E and E', &c. of the figures are interchangeable as well.

392. On the converse of the property of the preceding article, the following general construction for the double generation (26) of a pair of correlative figures, by the simultaneous variation of a connected point and line, or line and point, has been based by Chasles, the originator of the general theory.

If A, B, C be the three sides (or vertices) and A', B', C' the three corresponding vertices (or sides) of any two arbitrary fixed triangles ABC and $A'B'C'$, and I and I' a variable point and line (or line and point) so connected that, in every position, any two of the three ratios

$$\left(\frac{BI}{CI} : \frac{B'I'}{C'I'}\right), \quad \left(\frac{CI}{AI} : \frac{C'I'}{A'I'}\right), \quad \left(\frac{AI}{BI} : \frac{A'I'}{B'I'}\right),$$

or of their three equivalents

$$\left(\frac{BI}{B'I'} : \frac{CI}{C'I'}\right), \quad \left(\frac{CI}{C'I'} : \frac{AI}{A'I'}\right), \quad \left(\frac{AI}{A'I'} : \frac{BI}{B'I'}\right),$$

and with them of course the third, are constant in magnitude and sign; the variable point and line (or line and point) I and I' generate two correlative figures F and F', in which A and A', B and B', C and C' correspond in pairs as line and point (or point and line).

That the two figures F and F' resulting from either mode of generation are thus correlative, follows of course conversely from the property of the preceding article; but, as in the corresponding case of homographic figures (Art. 384), it may be easily shewn directly that they fulfil all the conditions of connection of the fundamental definition of Art. 388; for—

1°. *To every point (or line) of the former corresponds a line (or point) of the latter.* This is evident from the law of their generation; every point and line (or line and point) I and I' connected by the above relations, whether generating pairs or not, thus corresponding with respect to them.

2°. *To every line (or point) of the former corresponds a point (or line) of the latter.* For, when a variable point (or line) I of the former is connected, in every position, with the three fixed lines (or points) A, B, C by a relation of the form

$$a.AI + b.BI + c.CI = 0 \;\ldots\ldots\ldots\ldots\ldots\; (\alpha),$$

where a, b, c are any three constant multiples, then, by virtue of the above relations, the corresponding line (or point) I' of the latter is connected, in every position, with the three fixed points (or lines) A', B', C' by a corresponding relation of similar form

$$a'.A'I' + b'.B'I' + c'.C'I' = 0 \;\ldots\ldots\ldots\ldots\; (\alpha'),$$

where a', b', c' are three other constant multiples whose ratios to a, b, c respectively depend on and are given with those of the same relations; but, by the general properties of Arts. 120 and 85, the former relation (a) is the condition that the variable point (or line) I should move on a fixed line (or turn round a fixed point) O, and the latter (α') is the condition that the corresponding line (or point) I' should turn round a corresponding fixed point (or move on a corresponding fixed line) O'; and therefore &c.

3°, *To the connector of any two points (or the intersection of any two lines) of the former corresponds the intersection of the two corresponding lines (or the connector of the two corresponding points) of the latter.* For, since, to a line passing through any two points (or a point lying on any two lines) of the former corresponds, by the preceding property 2°, a point lying on the two corresponding lines (or a line passing through the two corresponding points) of the latter; therefore &c.

4°. *To the intersection of any two lines (or the connector of any two points) of the former corresponds the connector of the two corresponding points (or the intersection of the two corresponding lines) of the latter.* For, since, to two lines passing through any point (or two points lying on any line) of the former, correspond, by the same property 2°, two points lying on the corresponding line (or two lines passing through the corresponding point) of the latter; therefore &c.

5°. *Every quartet of collinear points (or concurrent lines) of the former is equianharmonic with the corresponding quartet of concurrent lines (or collinear points) of the latter.* For, the four connectors (or intersections) of any quartet I_1, I_2, I_3, I_4 of the points (or lines) of the former, whether collinear (or concurrent) or not, with any vertex (or side) BC or CA or AB of the triangle ABC being (by Cor. Art. 328) equianharmonic with the four intersections (or connectors) of the corresponding quartet I_1', I_2', I_3', I_4' of the lines (or points) of the latter with the corresponding side (or vertex) $B'C'$ or $C'A'$ or $A'B'$ of the triangle $A'B'C'$; therefore &c. (285).

6°. *Every quartet of concurrent lines (or collinear points) of the former is equianharmonic with the corresponding quartet of collinear points (or concurrent lines) of the latter.* For, the four intersections (or connectors) of any quartet O_1, O_2, O_3, O_4 of the lines (or points) of the former, whether concurrent (or collinear) or not, with any fifth line (or point) O_5 of the figure, being (by the preceding properties 4° and 5°) equianharmonic with the four connectors (or intersections) of the corresponding quartet O_1', O_2', O_3', O_4' of the points (or lines) of the latter with the corresponding fifth point (or line) O_5' of the figure; therefore &c. (285).

That, for either mode of generation, the three vertices and sides of one correspond respectively to the three corresponding sides and vertices of the other of the two fixed triangles ABC and $A'B'C'$, as point and line and as line and point, in the two resulting figures F and F', is evident from the relations of generation; from which, as in the corresponding case of homographic figures (Art. 384), it follows immediately, in either case, that the evanescence of any one or two of the three distances AI, BI, CI, for the former, involves necessarily the simultaneous evanescence of the corresponding one or two of the three corresponding distances $A'I'$, $B'I'$, $C'I'$, for the latter; and therefore &c.

N.B. When, of the two arbitrary triangles of construction ABC and $A'B'C'$ in either of the above modes of generation, the three pairs of corresponding elements A and A', B and B', C and C' are the three pairs of opposite elements (vertices and sides) of a common triangle Δ; the triangle Δ is then, with respect to the two resulting figures F and F', that to which allusion was made in the note at the close of the preceding article (391).

393. From the general constructions of the preceding article the following consequences respecting *the correlative transformation of figures* may be immediately inferred, viz.—

1°. *Any figure F may be transformed correlatively into another F′, in which any four lines (or points), given or taken arbitrarily, shall correspond to any assigned four points (or lines) of the original figure.*

For, of the four given pairs of corresponding points and lines or lines and points, any three determine the two fixed triangles of construction ABC and $A'B'C'$, and the fourth gives the values of the three constant ratios of construction

$$\left(\frac{BI}{CI}:\frac{B'I'}{C'I'}\right),\quad \left(\frac{CI}{AI}:\frac{C'I'}{A'I'}\right),\quad \left(\frac{AI}{BI}:\frac{A'I'}{B'I'}\right);$$

and therefore &c. See Cors. 1° and 2°, Art. 391.

The obvious conditions, that when, for either of two correlative figures F and F', three points are collinear or three lines concurrent, then, for the other, the three corresponding lines must be concurrent or the three corresponding points collinear, and that when, for either, four points by their collinearity or four lines by their concurrence form an anharmonic quartet, then, for the other, the four corresponding lines by their concurrence or the four corresponding points by their collinearity must form an equianharmonic quartet, are the only restrictions on the perfect generality of the above. The former condition may indeed be violated, but, when it is, as in the corresponding case for homographic figures (385, 1°), it is easy to see, from the general process of construction, that the figure for which the three points are collinear, or the three lines concurrent, when their three correspondents in the other are not concurrent, or collinear, must (except for the fourth line or point of the other) have all its points collinear, or all its lines concurrent, with the three. For, if, in any position of I and I', any one, AI suppose, of the six distances AI and $A'I'$, BI and $B'I'$, CI and $C'I'$ be evanescent when its correspondent $A'I'$ is not, then in every position of I and I', from the constancy of the three ratios of construction, either the same distance AI, or each of the two non-corresponding distances $B'I'$ and $C'I'$, is evanescent; and therefore &c. See the general remark 2° of Art. 31, an illustration of which, as in the corresponding case of homographic figures above referred to, is supplied by the above.

2°. *In the correlative transformation of any figure F into another F', the line (or any point) at infinity, regarded as belonging to either, may be made to correspond to any assigned point (or line), regarded as belonging to the other.*

This follows at once from the preceding property 1°; the three ratios of construction

$$\left(\frac{BI}{CI}:\frac{B'I'}{C'I'}\right),\ \left(\frac{CI}{AI}:\frac{C'I'}{A'I'}\right),\ \left(\frac{AI}{BI}:\frac{A'I'}{B'I'}\right)$$

being given as definitely (see Cors. 1° and 2°, Art. 391) when, of the given point and line, or line and point, I and I', one is at infinity, as when both are at a finite distance; and therefore &c.

By virtue of the above general property 1°, combined with its particular case 2°, the tetrastigm or tetragram determined by any four points or lines of F may be transformed correlatively into a tetragram or tetrastigm of any arbitrary or convenient form for F'; such for instance (see 2°, Art. 385) as the four sides or vertices of a parallelogram of any form, or, more generally, the three sides or vertices of a triangle of any form, combined with any remarkable or convenient line or point connected with its figure. By this means, as in the corresponding case for homographic figures (2°, Art. 385), the demonstration of a property, or the solution of a problem, when such property or problem admits of correlative transformation, may frequently be much simplified; as, for instance, in the three pairs of reciprocal properties there referred to (2°, Art. 385), whose direct demonstrations are comparatively easy under the circumstances there stated, and which under any other circumstances may be transformed correlatively, each into the other, and brought by the transformation under the circumstances most favourable to their establishment.

3°. *In the correlative transformation of any figure F into another F', the correspondents to any assigned five points (or lines) of the original, no three of which are collinear (or concurrent), may be made to touch (or lie on) a circle, given or taken arbitrarily.*

From this property, which may be proved in precisely the same manner as the corresponding property of homographic figures given in 3°, Art. 385, it follows immediately that *every figure, locus of a variable point every six of whose positions form an equianharmonic hexastigm* (301, *a*), *or envelope of a variable line every six of whose positions form an equianharmonic hexagram* (301, *a'*), *may be transformed correlatively into a circle;* for, if transformed, by the above, so that the correspondents to any five of its points (or tangents) shall touch (or lie on) a circle, the correspondent to every sixth point (or tangent) must, by virtue of its connection with the five, touch (or lie on) the same circle (305); and therefore &c. Thus, the four classes of loci and envelopes enumerated in the article above referred to (385, 3°), may be transformed, not only homographically, as there shewn, but also correlatively, into circles, and all their properties admitting of correlative transformation, such as their harmonic and anharmonic properties, consequently inferred from the comparatively simple and familiar properties of the circle. See chapters XV. and XVIII.; all the properties of which, not involving the magnitudes of angles, are consequently true not only of circles, but of the several classes of figures there enumerated also.

It follows also from the same, as in the corresponding case for homographic figures (385, 3°), that *five points* (*or tangents*), *given or taken arbitrarily, completely determine any figure correlative to a circle;* for, if transformed, by the above, so that the correspondents of the five points (or tangents) shall touch (or lie on) a circle, all the other points (or tangents) of the figure are then implicitly given as the correspondents to the several other tangents (or points) of the circle; and therefore &c.

Given five points (*or tangents*) *A, B, C, D, E of a figure correlative to a circle, the five corresponding tangents* (*or points*), *AA, BB, CC, DD, EE of the figure are given implicitly with them;* for, since, for the five corresponding tangents (or points) A', B', C', D', E' of the circle, by (306,) $\{A'.A'B'C'D'E'\} = \{B'.A'B'C'D'E'\} = \{C'.A'B'C'D'E'\} = \{D'.A'B'C'D'E'\} = \{E'.A'B'C'D'E'\}$, therefore, for the five given points (or tangents) A, B, C, D, E of the figure, by (388) and (389, 20°), $\{A.ABCDE\} = \{B.ABCDE\} = \{C.ABCDE\} = \{D.ABCDE\} = \{E.ABCDE\}$; and since, of each of these five latter homographic pencils (or rows), four rays (or points) are actually given, therefore, of each, the fifth ray (or point) is implicitly given; and therefore &c.

394. Of the numerous properties of the interesting and important class of figures into which the circle may be transformed correlatively, the few following, derived on the preceding principles from those of the circle, and identical with those already derived from the same by homographic transformation in Art. 386, may be taken as so many examples illustrative of the utility of the process of correlative transformation in modern geometry.

Ex. 1°. *No figure correlative to a circle could have either three collinear points or three concurrent tangents.*

For, if, of a figure correlative to a circle, either three points were collinear or three tangents concurrent, then, of the circle itself, by (389, 1°) the three corresponding tangents should be concurrent, or the three corresponding points collinear; and therefore &c.

N.B. As in the corresponding property of figures homographic to the circle (386, Ex. 1°), the only exception to this fundamental property occurs in the cases noticed in connection with property 1° of the preceding article (393), where the figure is in one or other limiting state of its general form, and has either an infinite number of collinear points lying on one or other of two definite lines, or an infinite number of concurrent tangents passing through one or other of two definite points. See the general remark 2° of Art. 31; of which the above and all similar exceptional cases supply so many illustrations.

Ex. 2°. *No figure correlative to a circle could have either three points at infinity or three parallel tangents.*

This, as in the corresponding property of figures homographic to the circle (386, Ex. 2°), is manifestly a particular case of the general property of the preceding article; all points at infinity being collinear and all parallel lines concurrent (136); and therefore &c.

N.B. As, in the process of correlative transformation of one figure into another, the line corresponding to any point of the original may be thrown to infinity in the transformed figure (see 2°, of the preceding article), *a circle will consequently be transformed correlatively into a figure having two distinct, coincident, or imaginary points at infinity, according as the point whose correspondent is thrown to infinity in the transformation subtends it by two distinct, coincident, or imaginary tangents* (21). Since, in the particular case of coincidence, the original point and correspondent line are a point on and a tangent to the original and transformed figures respectively (20), *a circle may consequently be transformed correlatively into a figure having a tangent at infinity, by merely throwing to infinity in the transformation the correspondent to any point on itself.* As in the corresponding case of figures homographic to the circle, the transformed figure possesses in this latter case, as will be seen in the sequel, some special properties peculiar to the case.

Ex. 3°. *In every figure correlative to a circle, every three points (or tangents) and the three corresponding tangents (or points) determine two triangles in perspective* (140).

For, each property, by examples 3° and 4° of Art. 137, being true of the circle itself, the other consequently, by properties 1° and 20° of Art. 389, is true of every figure correlative to it; and therefore &c. (For consequences, see same Ex. Art. 386).

Ex. 4°. *In every figure correlative to a circle, every six points (or tangents) determine an equianharmonic hexastigm (or hexagram)* (301).

For, each property, by *a* and *a'* of Art. 305, being true of the circle itself, the other consequently, by 12° of Art. 389, is true of every figure correlative to it; and therefore &c. (For consequences, see same Ex. Art. 386.)

Ex. 5°. *In every figure correlative to a circle, a variable point (or tangent) determines with every four fixed points (or tangents) a variable quartet of rays (or points) having a constant anharmonic ratio.*

For, each property, by *a* and *a'* of Art. 306, being true of the circle itself, the other consequently, by 13° of Art. 389, is true of every figure correlative to it; and therefore &c. (For consequences, see same Ex. Art. 386).

Ex. 6°. *In every figure correlative to a circle, a variable point (or tangent) connects (or intersects) homographically with every two fixed points (or tangents).*

For, each property, by examples *c* and *c'* of Art. 325, being true of the circle itself, the other consequently, by property 14° of Art. 389, is true of every figure correlative to it; and therefore &c. (For consequences, see same Ex. Art. 386.)

Ex. 7°. *When, of any figure correlative to a circle, two variable points (or tangents) connect through (or intersect on) a fixed point (or line), the two corresponding tangents (or points) intersect on (or connect through) a fixed line (or point).*

For, each property, by Cor. 3° of Art. 166, being true of the circle itself, the other consequently, by 1° and 20° of Art. 389, is true of every figure correlative to it; and therefore &c. (For consequences and resulting definitions, see same Ex. Art. 386).

Ex. 8°. *When, of any figure correlative to a circle, two variable points (or tangents) connect through (or intersect on) a fixed point (or line), the harmonic conjugate with respect to them of the fixed point (or line) moves on (or turns round) a fixed line (or point).*

For, each property, by *a* and *a'* of Art. 259, being true of the circle itself, the other consequently, by 10° and 20° of Art. 389, is true of every figure correlative to it; and therefore &c. (For consequences and resulting definitions, see same Ex. Art. 386).

Ex. 9°. *Every two triangles reciprocal polars to each other with respect to any figure correlative to a circle are in perspective; and their centre and axis of perspective are pole and polar to each other with respect to the figure.*

For, the first part of the property, by 1° Art. 180, being true of the circle itself, is consequently, by 1° Art. 389, true of every figure correlative to it; and the second part, by virtue of the general property of Art. 167, being evident alike for circle and figure; therefore &c. (For consequences, see same Ex. Art. 286).

Ex. 10°. *Every figure correlative to a circle intersects (or subtends) harmonically the three sides (or angles) of every triangle self-reciprocal with respect to itself; and conversely, every triangle whose three sides (or angles) are intersected (or subtended) harmonically by any figure correlative to a circle is self-reciprocal with respect to the figure.*

For, each property, by a and a' of Art. 259, being true of the circle itself, the other consequently, by 10° and 20° of Art. 389, is true of every figure correlative to it; and therefore &c. (For consequences, see same Ex. Art. 386).

Ex. 11°. *In every tetrastigm (or tetragram) determined by four points on (or tangents to) any figure correlative to a circle, the three intersections (or connectors) of the three pairs of opposite connectors (or intersections) determine a self-reciprocal triangle with respect to the figure.*

For, each property, by a and a' of Art. 261, being true of the circle itself, the other consequently, by Ex. 7°, is true of every figure correlative to it; and therefore &c. (For consequences, see Arts. 261, 262, 263).

Ex. 12°. *In every figure correlative to a circle, if A and A', B and B', C and C' be the three pairs of opposite connectors (or intersections) of the tetrastigm (or tetragram) determined by any four fixed points on (or tangents to) the figure, and I a variable point (or tangent) of the figure; then, in every position of I, the three rectangles $IA \cdot IA'$, $IB \cdot IB'$, $IC \cdot IC'$ are to each other, two and two, in constant ratios.*

These properties follow from Ex. 6°, precisely in the same manner as for figures homographic to the circle in the corresponding example of Art. 386, and lead precisely to the same consequences; see note and corollary to that example in the article referred to.

Ex. 13°. *In every figure correlative to a circle, if A, B, C be the three sides (or vertices) of any fixed triangle inscribed (or exscribed) to the figure, and I a variable point (or tangent) of the figure; then, in every position of I,*

$$a \cdot IB \cdot IC + b \cdot IC \cdot IA + a \cdot IB \cdot IC = 0,$$

where a, b, c are three multiples, whose ratios to each other, two and two, are constant in magnitude and sign.

These properties follow from those of the preceding example (12°), precisely as for figures homographic to the circle in the corresponding example of Art. 386; and the three multiples a, b, c have for the circle itself the same values given for those figures in the note to that example.

Ex. 14°. *In every figure correlative to a circle, if A, B, C be the three sides (or vertices) of any fixed triangle exscribed (or inscribed) to the figure,* *and I a variable point (or tangent) of the figure: then, in every position of I,*

$$a \cdot IA^{\frac{1}{2}} + b \cdot IB^{\frac{1}{2}} + c \cdot IC^{\frac{1}{2}} = 0,$$

where a, b, c are three multiples, whose ratios to each other, two and two, are constant in magnitude and sign.

These properties follow from those of the preceding example (13°), precisely as for figures homographic to the circle in the corresponding example of Art. 386; and the three multiples a, b, c have for the circle itself the same values given for those figures in the note to that example.

Ex. 15°. *In every figure correlative to a circle, if A, B, C be the three vertices (or sides) of any fixed triangle self-reciprocal with respect to the figure, and I a variable tangent (or point) of the figure; then, in every position of I,*

$$a \cdot IA^2 + b \cdot IB^2 + c \cdot IC^2 = 0,$$

where a, b, c are three multiples, whose ratios to each other, two and two, are constant in magnitude and sign.

These properties follow from examples 10° and 12°, precisely in the same manner as for figures homographic to the circle in the corresponding example of Art. 386; and the three multiples a, b, c have for the circle itself the same values as for those figures. (See note to the example in question.)

N.B. From the properties of figures correlative to a circle given in the examples of the present article, compared with those of figures homographic to a circle given in the corresponding examples of Art. 386, the reader will at once perceive the complete identity existing between both classes of figures; and can consequently, in investigating their properties from those of the circle, employ in every case whichever mode of transformation appears best adapted to the case.

395. With the four following general properties of any two correlative figures we shall conclude the present chapter:

1°. *For any two correlative figures F and F', the two correspondents I_1 and I_2, in the two figures, of a variable line or point I, moving according to any law, generate two homographic figures G_1 and G_2, in which all pairs of corresponding elements, whether points or lines, which coincide with each other, have the same correspondents in the two original figures.*

For, the two figures G_1 and G_2 generated by the two variable points (or lines) I_1 and I_2 being each correlative with the figure G generated by the variable line (or point) I (392), and therefore homographic with each other (388), therefore &c. as regards the first part; and since, when any two points (or lines) A_1 and A_2 of G_1 and G_2 coincide, then evidently the line (or point) A of G, to which they correspond, corresponds in the two original figures F and F' to the point (or line) of coincidence, therefore &c. as regards the second part.

2°. *For every pair of correlative figures F and F', however situated, there exists a pair of corresponding triangles (unique or indeterminate) Δ and Δ', for which the three elements of either species (vertices or sides) of either are interchangeable, as regards the figures, with the three corresponding elements of the other species (sides or vertices) of the other.*

This follows from the preceding property 1°, by virtue of the general property 2° of Art. 387; for, since, by the preceding property 1°, the two correspondents I_1 and I_2 of a variable point (or line) I, moving according to any law, generate two homographic figures G_1 and G_2, of which every double element of either species (point or line) has the same correspondent in the two original figures F and F'; and, since, for the two generated figures G_1 and G_2, there exists, by the general property 2° of Art. 387, a (unique or indeterminate) triangle Δ', whose three elements of both species (vertices and sides) are double elements of those figures; therefore, for the two original figures F and F', there exist two corresponding triangles Δ and Δ', whose six pairs of corresponding elements of opposite species (of which two are always real though the remaining four may be imaginary) are pairs of interchangeable correspondents with respect to them; and therefore &c.

3°. *When, for two correlative figures F and F', the two corresponding triangles Δ and Δ' determined by the six pairs of interchangeable elements coincide, then all pairs of corresponding elements are alike interchangeable; and the figures themselves are reciprocal polars to each other with respect to the figure homographic to a circle whose centre is the correspondent in both to the line at infinity, and with respect to which the triangle of coincidence is self-reciprocal.*

To prove the first part. If A, B, C be the three vertices (or sides) of the triangle of coincidence, with which the three opposite sides (or vertices) BC, CA, AB are, by hypothesis, interchangeable as pairs of corresponding elements of the two figures; I any arbitrary line (or point); I_1 and I_2 the two points (or lines) corresponding to I in the two figures; X, Y, Z the three intersections (or connectors) of I with BC, CA, AB respectively; and U_1 and U_2, V_1 and V_2, W_1 and W_2 the three pairs of intersections (or connectors) of the three pairs of lines (or points) AI_1 and AI_2, BI_1 and BI_2, CI_1 and CI_2 with the same three lines (or points) BC, CA, AB respectively; then, since, by the fundamental definition of correlative figures (388),

$$\{BCXU_1\} = \{A.CBU_2X\} = \{BCXU_2\} \text{ (285 and 280)},$$
$$\{CAYV_1\} = \{B.ACV_2Y\} = \{CAYV_2\} \text{ (285 and 280)},$$
$$\{ABZW_1\} = \{C.BAW_2Z\} = \{ABZW_2\} \text{ (285 and 280)},$$

therefore, at once, the three pairs of points (or lines) U_1 and U_2, V_1 and V_2, W_1 and W_2, and with them consequently the pair I_1 and I_2, coincide; and therefore &c.

To prove the second part. If O be the point (or line) corresponding, by the first part, to the line (or point) I, in the two figures; U, V, W the three intersections (or connectors) of the three lines (or points) AO, BO, CO with the three BC, CA, AB respectively; and G and G', H and H', K and K' the three pairs of points (or lines) which divide harmonically the three pairs of segments (or angles) UX and BC, VY and CA, WZ and AB respectively, and which consequently, as I and O vary, are, by the above relations, the three pairs of double points (or rays) of the three involutions (357) determined by the three variable pairs of points (or rays) U and X, V and Y, W and Z on (or at) the three lines (or points) BC, CA, AB respectively; then, since, in every position of I and O, the three pairs of points (or lines) U and X, V and Y, W and Z are pairs of conjugate points (or lines) with respect to the figure homographic to a circle determined by the three pairs of fixed points (or tangents) G and G', H and H', K and K' (386, Exs. 4° and 10°), and since consequently the line and point (or point and line) I and O themselves are in every position polar and pole (or pole and polar) to each other with respect to that figure; therefore &c. the pole of the line at infinity with respect to any figure homographic to a circle being the centre of the figure (386, Ex. 8°, Cor.), and every triangle whose three sides (or vertices) are intersected (or subtended) harmonically by any such figure being self-reciprocal with respect to the figure (386, Ex. 10°).

4°. *Every two correlative figures F and F' may be placed, in two different positions relatively to each other, so that every pair of corresponding elements shall be interchangeable between them; and so as consequently (by the preceding property 3°) to be reciprocal polars to each other with respect to a figure homographic to a circle.*

For this the following (always possible and determinate) construction, based on the preceding property 3°, has been given by Chasles: Taking the two points O and P which correspond in F and F' respectively to the line at infinity I, and which it is evident must both be either at a finite or at an infinite distance (388); drawing through either of them O any three arbitrary lines U, V, W intersecting with I at three points X, Y, Z, and through the other P the three lines X', Y', Z' which correspond in F' to the three points X, Y, Z in F, and which intersect with I at three points U', V', W' corresponding in F' to the three lines U, V, W in F; and placing the figures in either of the two positions, relatively to each other, in which the two points O and P shall coincide, and in which the two homographic systems (388) determined by three pairs of corresponding rays U and X', V and Y', W and Z' shall be in involution (363); in each of the two resulting positions thus obtained the two figures F and F' are related to each other as required.

For, in each of the three triangles $U'OX$, $V'OY$, $W'OZ$, the three pairs of opposite elements U and U', X and X', I and O; V and V', Y and Y', I and O; W and W', Z and Z', I and O; respectively, are evidently interchangeable as pairs of corresponding elements of the figures (357); and therefore &c. by the preceding property 3°. This construction of Chasles is manifestly as readily applicable when the two points O and P are at infinity as when they are at a finite distance.

N.B. Since, as regards the two different positions of involution of the same two homographic pencils of rays, the two double rays of the systems are always real for one and imaginary for the other (370 and 373); and, since, when the point O is at a finite distance, [the two double rays of the involution determined by the three pairs of conjugate diameters U and X',

V and Y', W and Z' are the two central tangents of the figure with respect to which F and F' are reciprocal polars to each other (386, Ex. 8°, Cor.); it follows consequently, as regards the two different positions of F and F' given by the above construction, that, when the point O is not at infinity, the two central tangents, and therefore the two points at infinity, of that figure, are always real for one and imaginary for the other.

CHAPTER XXIV.

METHODS OF GEOMETRICAL TRANSFORMATION. THEORY OF INVERSE FIGURES.

396. EVERY two figures, F and F', which, however generated, are resolvable into pairs of corresponding points, P and P', Q and Q', R and R', S and S', &c. inverse to each other (149) with respect to a common circle, K, real or imaginary, are said *to be inverse to each other with respect to that circle.* Every two concentric circles, real or imaginary, are evidently thus related to each other with respect to the concentric circle, real or imaginary, the square of whose radius is equal in magnitude and sign to the rectangle under their radii (149).

397. The two figures F and F', thus inverse to each other with respect to a common circle K, might be two different parts of the same continuous figure. The two parts with which a line or circle, for instance, is divided by any circle intersecting it at right angles (22°, 1° and 1') are evidently thus related to each other with respect to that circle (149 and 156).

398. The same two figures F and F' might, by different modes of resolution into pairs of corresponding points, be inverse to each other with respect to more than one circle. Every two circles, for instance, being resolvable, however circumstanced as to magnitude and position, into pairs of antihomologous points (198) inverse to each other with respect to either of their two circles of antisimilitude (201), are consequently thus related to each other with respect to *each* of those circles.

399. Any figure, F, may be transformed into another, F', inverse to it with respect to any arbitrary circle K, real or imaginary, by simply changing all its points, P, Q, R, S, &c.

into their inverses, P', Q', R', S', &c. with respect to that circle (149). This simple (and, as regards the geometry of the circle, fertile) process of transformation is termed *inversion;* the circle, K, finite, evanescent, or infinite, with respect to which it is performed, is termed *the circle of inversion;* its centre, O, which is always real, *the centre of inversion;* and its radius, OR, which may be real or imaginary, *the radius of inversion.*

400. As regards the process of inversion generally, the three following particulars, though obvious, are deserving of attention:

1°. In the extreme case when the circle of inversion is a point; as then the inverse with respect to it of every point, not coinciding with itself, coincides evidently with it (149), it follows, consequently, that *the inverse of every figure with respect to a point not lying on itself is evanescent and coincides with the point.*

2°. In the other extreme case when the circle of inversion is a line; as then the inverse of every point with respect to it is the reflexion of the point with respect to it (150), it follows, consequently, that *the inverse of every figure with respect to a line is the reflexion of the figure with respect to the line* (50).

3°. In every case, as every point and its inverse with respect to any circle are reciprocally inverse to each other with respect to the circle (151), it follows, consequently, that *every figure and its inverse with respect to any circle are reciprocally inverse to each other with respect to that circle* (396).

401. Every figure F, and its inverse F' with respect to any circle K, possess evidently, in relation to each other and to the circle, the following general properties:

1°. *Every two of their corresponding points P and P' connect through the centre of inversion O.*

For, P and P' being, by hypothesis, inverse points with respect to the circle of inversion K (399); therefore &c. (149).

2°. *Every two pairs of their corresponding points P and P', Q and Q' are concyclic.*

For, P and P', Q and Q' being both, by hypothesis, inverse pairs with respect to the same circle K; therefore &c. (155).

3°. *At every two of their corresponding points P and P', the* *two tangents T and T' determine an isosceles triangle with the line of connection PP'.*

For, since, for every two pairs of corresponding points P and P', Q and Q', by the preceding property 2°, the two angles OPQ and $OQ'P'$ are always equal (Euc. III. 21, 22), therefore, when $P = Q$ and $P' = Q'$, and when consequently PQ and $P'Q'$ are the two tangents T and T' (19), those two angles are equal; and therefore &c.

4°. *Every line passing through the centre of inversion O which intersects either F at any point P intersects the other F' at the corresponding point P'.*

For, every line passing through the centre O of any circle K, and containing any point P, contains also the inverse point P' with respect to the circle (149); and therefore &c.

5°. *Every line passing through the centre of inversion O which touches either F at any point P touches the other F' at the corresponding point P'.*

For, since every line passing through O which intersects F at any two points P and Q intersects, by the preceding property 4°, F' at the two corresponding points P' and Q'; and since when the two points P and Q coincide their two inverses P' and Q' with respect to the circle K coincide also; therefore &c. (19).

6°. *Every line passing through the centre of inversion O intersects them at equal angles at each pair of corresponding intersections P and P'.*

For, since, by the preceding property 3°, the two tangents T and T' at every pair of their corresponding points P and P' determine an isosceles triangle with the line PP'; therefore &c. (22).

7°. *Every circle passing through any pair of inverse points P and P' with respect to the circle of inversion K which intersects either F at any point Q intersects the other F' at the corresponding point Q'.*

For, every two pairs of inverse points P and P', Q and Q' with respect to any circle K being in all cases concyclic (152); therefore &c.

8°. *Every circle passing through any pair of inverse points P and P' with respect to the circle of inversion K which touches* *either F at any point Q touches the other F' at the corresponding point Q'.*

For, since every circle passing through P and P' which intersects F at any two points Q and R intersects, by the preceding property 7°, F' at the two corresponding points Q' and R'; and since when the two points Q and R coincide their two inverses Q' and R' with respect to the circle K coincide also; therefore &c. (19).

9°. *Every circle passing through any pair of inverse points P and P' with respect to the circle of inversion K intersects them at equal angles at each pair of corresponding intersections Q and Q'.*

For, since, by the preceding property 3°, the two tangents T and T' to the figures at every pair of their corresponding points Q and Q' determine an isosceles triangle with the line QQ'; and since, evidently, the two tangents S and S' at Q and Q' to every circle passing through them do the same; therefore &c. (22).

10°. *Every point of intersection P of either F with the circle of inversion K is a corresponding point of intersection P' of the other F' with the same circle K.*

For, every two of their corresponding points P and P' being, by hypothesis, inverse points with respect to the circle K (399), when either P is on that circle the other P' necessarily coincides with it (149); and therefore &c.

11°. *Every point of contact P of either F with the circle of inversion K is a corresponding point of contact of the other F' with the same circle K.*

For, since every two points of intersection P and Q of either F with the circle K are, by the preceding property 10°, corresponding points of intersection P' and Q' of the other F' with the same circle K, the same is of course the case when P and Q coincide; and therefore &c. (19).

12°. *Every angle of intersection of either F with the circle of inversion K is equal to the corresponding angle of intersection of the other F' with the same circle K.*

For, since, by the preceding property 3°, the two tangents T and T' at their common point of intersection $P = P'$ with the circle K make equal angles with the corresponding radius of inversion OP, and since the tangent to the circle itself at the same point is perpendicular to the same radius; therefore &c. (22).

13°. *When either F has a point P at infinity, the other F' passes through the centre of inversion O, and conversely.*

For, since, for every pair of their corresponding points P and P', the rectangle $OP.OP'$ is constant in magnitude and sign, therefore when $OP=\infty$ then $OP'=0$, and conversely; and therefore &c.

14°. *When either F has a point P at infinity, the tangent T' to the other F' at the centre of inversion O passes through it, and conversely.*

For, since, as in the preceding, when $OP=\infty$ then $OP'=0$, and conversely, and since when $OP'=0$ the line OP' is the tangent T' to F' at O (19); therefore &c.

402. Every two figures E and F, and their two inverses E' and F' with respect to any common circle K, also possess evidently, in relation to each other and to the circle, the following general properties:

1°. *To every point of intersection P of either pair E and F corresponds an inverse point of intersection P' of the other pair E' and F'.*

For, the point P being, by hypothesis, common to the two figures E and F, its inverse P' with respect to any circle K is, consequently, common to their two inverses E' and F' with respect to the same circle K (399); and therefore &c.

2°. *To every point of contact P of either pair E and F corresponds an inverse point of contact P' of the other pair E' and F'.*

For, since, to every two points P and Q common to E and F correspond, by the preceding property 1°, two inverse points P' and Q' common to E' and F'; and since when P and Q coincide then P' and Q' coincide also (149); therefore &c. (19).

3°. *Every angle of intersection of either pair E and F is equal to the corresponding angle of intersection of the other pair E' and F'.*

For, since, at every pair of corresponding intersections P and P', of E and F, E' and F', respectively, the two pairs of corresponding tangents S and S', T and T', to E and E', F and

F', respectively, determine, by property 3° of the preceding article, two isosceles triangles with the common base PP'; therefore &c. (22).

4°. *When either pair E and F have a common point P at infinity, the other pair E' and F' have a common tangent T' at the centre of inversion O whose direction passes through P, and conversely.*

For, since, by hypothesis, $OP = \infty$ for both figures E and F, therefore, (see properties 13° and 14° of preceding Art.) $OP' = 0$ for both figures E' and F', which latter therefore both pass through the point O and there touch the line OP; and therefore &c.

N.B. Of all properties of inverse figures, the preceding 3°, which shews that *all angles preserve their magnitudes, or more properly speaking their forms* (24), *under the process of inversion*, is the most important connected with the subject of the present chapter.

403. The figures inverse to a line or circle with respect to any arbitrary circle of inversion, real or imaginary, are, under different circumstances of magnitude and relative position, respectively as follows:

1°. *The figure inverse to a line or circle with respect to any circle of evanescent radius, whose centre is not a point on itself, is evanescent and coincides with the centre of inversion* (400, 1°).

2°. *The figure inverse to a line or circle with respect to any circle of infinite radius, which is not itself at infinity, is the reflexion of the line or circle with respect to the line into which the part of the circle of inversion not at infinity then opens out* (400, 2°.)

3°. *The figure inverse to a line or circle with respect to any circle, finite, evanescent, or infinite, which it intersects at right angles, is the line or circle itself* (397).

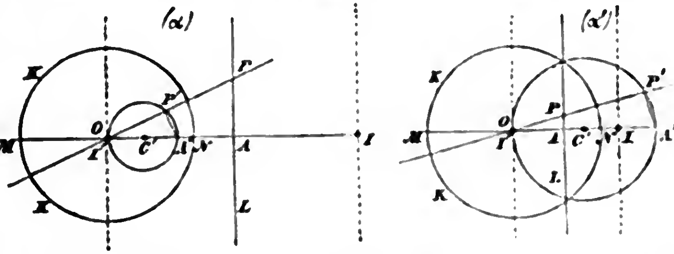
4°. *The figure inverse to a line with respect to any circle of finite radius, real or imaginary, whose centre is not a point on itself, is a circle passing through the centre of inversion and coaxal with itself and the circle of inversion* (184).

5°. *The figure inverse to a circle with respect to any circle of finite radius, real or imaginary, whose centre is a point on itself, is a line, the radical axis of itself and the circle of inversion* (181).

6°. *The figure inverse to a circle with respect to any circle of finite radius, real or imaginary, which it neither passes through the centre of, nor intersects at right angles, is another circle, in perspective with itself from the centre of inversion, and coaxal with itself and the circle of inversion* (199).

Of these several cases, thus stated together for convenience of reference, the three first have been already given in the previous articles referred to in their statements, and the last, which comprehends the preceding two (and indeed all the others) as particular cases, has been virtually established in its entire generality in Art. 199; from their importance however in the applications of the theory of inversion, we subjoin the ordinary independent proofs of the fourth and fifth, and repeat again that of the sixth with new figures adapted more particularly to the subject of the chapter.

To prove 4°. If O (figs. α and α') be the centre of inversion,



L the line, A the foot of the perpendicular upon it from O, and A' the inverse of A with respect to the circle of inversion K; the circle on OA' as diameter is the required inverse of L.

For, drawing any line through O intersecting L at P and the circle in question at P', and joining $A'P'$; then, since, by similar right-angled triangles AOP and $P'OA'$, the two rectangles $OP.OP'$ and $OA.OA'$ are always equal in magnitude and sign, therefore &c. (149).

The diameter MN of the circle of inversion perpendicular to L being bisected at O and cut harmonically at A and A' (225), therefore (231, Cor. 3°) the two rectangles $AA'.AO$ and $AM.AN$ are always equal in magnitude and sign, and therefore (181) the line L is always the radical axis of its inverse and the circle of inversion. A property which, when (as in fig. α') it intersects the latter at real points, is evident *à priori* from the general property 10° of Art. 401.

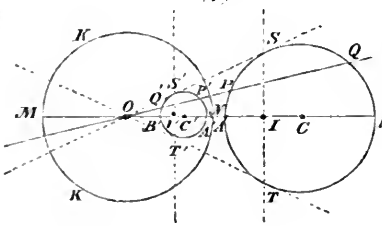
To prove 5°. If O (same figures) be the centre of inversion, A' the diametrically opposite point of the circle passing through it, and A the inverse of A' with respect to the circle of inversion K; the line L passing through A perpendicular to AA' is the inverse required.

For, since (as in the preceding case), for every line passing through O and intersecting the circle and L at P' and P respectively, the two rectangles $OP.OP'$ and $OA.OA'$ are always equal in magnitude and sign; therefore &c. (149).

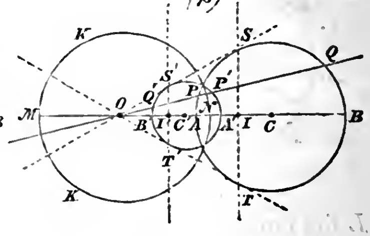
That, in all cases in which the original circle passes, as above, through the centre of inversion O, the line L inverse to it is the radical axis of itself and the circle of inversion K, appears of course in precisely the same manner as for the preceding property 4°.

To prove 6°. If O (figs. β and β', γ and γ') be the centre

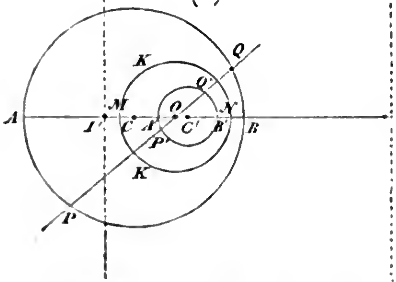
(β)



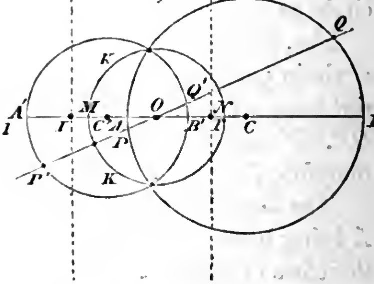
(β')



(γ)



(γ')



of inversion, C that of the circle whose inverse is required, A and B the extremities of its diameter passing through O, and A' and B' the two inverses of A and B with respect to the circle of inversion K; the circle on $A'B'$ as diameter is the inverse required.

For, O being, by virtue of the relation $OA.OA' = OB.OB'$ (149), one of the two centres of perspective (the external in figs. β and β', the internal in figs. γ and γ') of the two circles on AB and $A'B'$ as diameters (199), and every line passing through it consequently intersecting them at two pairs of antihomologous points P and P', Q and Q' with respect to it, for which the four rectangles $OP.OP'$, $OQ.OQ'$, $OA.OA'$, $OB.OB'$ are all equal in magnitude and sign (198); therefore &c. (149).

The diameter MN of the circle of inversion passing through the centres C and C' of the original and inverse circles being cut harmonically by each pair of inverse points A and A', B and B' (225), the three circles on AB, $A'B'$, and MN as diameters, that is the original and inverse circles and the circle of inversion, are consequently always coaxal (229). A property which, when (as in figs. β' and γ') the former and latter intersect at real points, is evident *à priori* from the general property 10° of Art. 401.

N.B. The several figures (α and α', β and β', γ and γ') given with the above proofs, though applicable as they stand only to the case in which the circle of inversion K is real, may be adapted immediately to that in which it is imaginary, by simply turning in each the inverse figure round the centre of inversion O through two right angles into the opposite position, the original figure remaining unchanged; the changed and original figures will then be inverse to each other with respect not to the real circle K but to the concentric imaginary circle K' the negative square of whose radius is equal in absolute magnitude to the positive square of the radius of K (149).

From a comparison of the two figures β and β' with the two γ and γ', both in their original and changed positions, it is evident that, while, for a real circle of inversion K, the centre of inversion O is the external or the internal centre of perspective of the original and inverse circles according as it is external or internal to both, the reverse exactly is the case for an imaginary circle of inversion K', the centre of inversion O being then their internal or external centre of perspective according as it is external or internal to both. The consideration of this difference is important whenever it is necessary, as it sometimes is, to compare as to magnitude, in accordance with the convention of Art. 23, the angles of intersection of two known circles and of their two inverses with respect to a known circle of inversion, real or imaginary.

404. The centre and radius of the circle inverse to a given line or circle with respect to a given circle of inversion K, real or imaginary, may be found immediately, on the principles just stated, as follows:

In the case of the line. If O (figs. α and α' of preceding article) be the centre and OR the radius of inversion, L the line, A the foot of the perpendicular OA on it from O, A' the inverse of A with respect to K, and C' the middle point of OA', which, by 4° of the preceding article, is the centre of the circle inverse to L; then, since $OA . OA' = OR^2$ (149), and since $OA' = 2OC'$, therefore $OC' = OR^2 \div 2OA = OR^2 \div 2OL$; which is consequently the formula by which to calculate in numbers the position of the centre C' and the magnitude of the radius OC' of the circle inverse to L with respect to K.

In the case of the circle. If, as before, O and OR (figs. β and β', γ and γ' of the preceding article) be the centre and radius of inversion, r and r' the radii of the original and inverse circles, d and d' the distances OC and OC' of their centres C and C' from O, and t and t' the lengths (real or imaginary) of the tangents OS and OS', or OT and OT', to them from O; then since, O being one of their two centres of similitude (199), $d' \div d = r' \div r = t' \div t = t't \div t^2 = OR^2 \div (d^2 - r^2)$, therefore

$$d' = \left(\frac{OR^2}{d^2 - r^2}\right) . d, \text{ and, } r' = \left(\frac{OR^2}{d^2 - r^2}\right) . r;$$

which are consequently the formulæ by which to calculate in numbers the position of the centre C' and the length of the radius r' of the inverse circle, when, with the circle of inversion K, the centre C and radius r of the original circle are given.

The same formulæ may also be obtained easily without the aid of the two tangents t and t' which are as often imaginary (figs. γ and γ') as real (figs. β and β'); for if P and P', Q and Q', be the two pairs of inverse points at which any line passing through O intersects the original and inverse circles; then since (199) $d' \div d = r' \div r = OP' \div OQ$ or $OQ' \div OP = (OP.OP'$ or $OQ.OQ') \div OP.OQ = OR^2 \div (d^2 - r^2)$, therefore &c.

405. The centre of the circle inverse to a given line or circle with respect to a given circle of inversion K, real or imaginary, is given also in every case by the general property, that—

The inverse of the centre of inversion with respect to any line or circle inverts into the centre of the circle inverse to the line or circle.

For, if O and OR (same figures and notation as before) be the centre and radius of inversion, I the inverse of O with respect to the original line or circle, and C' the centre of the inverse circle or line; then, since, in the case of the line (figs. α and α'), $OI = 2OA$ (150), therefore $OC'.OI = OA'.OA = OR^2$, and therefore &c. (149); and, since, in the case of the circle (figs. β and β', γ and γ'), $OC.OI = OP.OQ$ (149), therefore (199) $OC'.OI = OP.OP'$ or $OQ.OQ' = OR^2$; and therefore &c. (149).

Cor. 1°. If I' (same figures) be the inverse of O with respect to the inverse circle or line; then, since, for precisely the same reasons as above, $OC.OI' = OR^2$, therefore reciprocally—

The centre of any circle inverts into the inverse of the centre of inversion with respect to the inverse circle.

Cor. 2°. The two perpendiculars at the two points I and I' to the line II' (same figures) being (165) the two polars of the point O with respect to the original and inverse circles, and inverting with respect to K into the two circles on the two intervals OC' and OC as diameters (4°, Art. 403); it follows consequently that—

The polar with respect to any circle of the centre of inversion inverts into the circle whose diameter is the interval between that point and the centre of the inverse circle.

The circle on the interval between the centres of inversion and of any circle as diameter inverts into the polar of the centre of inversion with respect to the inverse circle.

N.B. A different, and more general, proof of these useful properties, on a principle common to the line and circle indifferently, will be presently given.

406. As regards the effect of inversion on the several anharmonic ratios of collinear and concyclic quartets of points (274 and 308), it may be easily shewn that in all cases—

Every four points on any line or circle and the four corresponding points on its inverse with respect to any circle of inversion, real or imaginary, are equianharmonic.

For, of the two inverse figures, when one is a line, and the other consequently a circle passing through the centre of inversion (4° and 5°, Art. 403), the two quartets are then equianharmonic with the common pencil they determine at that point (285 and 306), and therefore with each other; and when both are circles, and neither consequently passing through the centre of inversion, the two quartets are then antihomologous with respect to the centre of perspective of the circles which coincides with that point (403, 6°), and therefore &c. (316). In the particular case of a line or circle intersecting the circle of inversion at right angles, and consequently inverting into itself (403, 3°), all pairs of corresponding points are then harmonic conjugates with respect to the two points of intersection (149 and 257), and therefore &c. (282, Cor. 4°).

N.B. Next to the general property 3° of Art. 402, the above, from which it appears that *all anharmonic ratios of collinear and concyclic quartets of points are preserved unchanged in inversion*, is the most important connected with the subject of the present chapter.

407. As regards the general property of intersecting figures and their inverses, just adverted to in the note at the close of the preceding article, when applied to the case of intersecting circles and their inverses, it may be stated, in accordance with the convention of Art. 23, more definitely as follows:

When two circles intersect, their angle of intersection is equal or supplemental to that of their two inverses with respect to any circle of inversion, real or imaginary, according as the centre of inversion is external or internal to both, or external to one and internal to the other.

For, the centre of inversion being the external or the internal centre of perspective of both pairs of inverse circles in the former case, and the external of one pair and the internal of the other pair in the latter case (403, note); and every circle passing through either pair of inverse points of intersection (402, 1°) consequently intersecting both pairs at equal or at supplemental angles in the former case, and one pair at equal and the other pair at supplemental angles in the latter case (211); therefore &c.

N.B. It follows of course from the above that, though, in accordance with the general property 3° of Art. 402, the angle of intersection of two circles undergoes, as a figure, no change of form under the process of inversion, yet, in accordance with the convention of Art. 23, it may, and often does, as a magnitude, change into its supplement under that process (24). In the applications of the theory of inversion to the geometry of the circle, this circumstance must of course, when necessary, be always attended to. The two cases of *contact*, external and internal, come of course under it as particular cases (23); and in but one case alone, that of *orthogonal* intersection, which presents no ambiguity, can the precaution ever be entirely dispensed with.

408. Between the squares of the common tangents and the rectangles under the radii of any two intersecting circles and of their two inverses with respect to any circle of inversion, real or imaginary, the following metric relation results immediately from the general property of the preceding article, combined with that of Euc. II. 12, 13; viz.—

*When two circles intersect, the squares (disregarding signs) of their two common tangents are to the rectangle under their radii, as the squares (disregarding signs) of the two common tangents are to the rectangle under the radii of their two inverses with respect to any circle of inversion, real or imaginary; squares having similar or opposite signs corresponding in the two proportions, according as the angles of intersection of the two pairs of circles, original and inverse, are equal or supplemental.**

* The above, established generally for any two circles and their two inverses to any circle of inversion, has been applied with considerable success by Mr. Casey to the investigation of some interesting properties of circles, which, but for the length to which the present volume has extended, would have been noticed here. The same geometer has also obtained by inversion an indirect but general proof of the first part of Dr. Hart's

For if a, b, c be the three sides of the triangle determined by the two radii to either point of intersection and by the interval between the centres of the original pair of circles, and a', b', c' those of the corresponding triangle for the inverse pair; then, since, by the general property of the preceding article, the two angles of those triangles opposite to c and c' are either equal or supplemental, therefore, by Euc. II. 12 and 13, the two differences of squares $c^2-(a-b)^2$ and $(a+b)^2-c^2$, which (disregarding signs) are the squares of the two common tangents to the original pair, are to the two corresponding or non-corresponding differences of squares $c'^2-(a'-b')^2$ and $(a'+b')^2-c'^2$, which (disregarding signs) are the squares of the two common tangents to the inverse pair, as the rectangle ab, which is that under the radii of the original pair, is to the rectangle $a'b'$, which is that under the radii of the inverse pair; and therefore &c.

N.B. In the two extreme cases of real intersection, viz. contact, external and internal, the above ratios have in magnitude and sign the extreme values $+4$ and 0, 0 and -4 respectively; and in the mean case of real intersection, viz. orthogonal, they have in magnitude and sign the two intermediate mean values $+2$ and -2. The property however being true generally for every two circles and their two inverses to any circle of inversion, they may, for imaginary intersection, have in magnitude and sign, any value from 0 to $\pm\infty$, according as the distance between their centres is greater than the sum or less than the difference of their radii.

409. Intersecting circles possess also the following evident properties with respect to inversion:

1°. *When two circles intersect, every circle passing through their two points of intersection inverts into a circle passing through the two points of intersection of their two inverse circles.*

Theorem, respecting the eight circles of contact of three arbitrary circles, stated at the close of Art. 212. His paper on the latter subject in the *Quarterly Journal of Pure and Applied Mathematics* (Vol. v. page 318) had been in fact published when that article was written, but the author was unaware at the time that any demonstration, direct or indirect, had been obtained of it by Elementary Geometry.

For, since, to every circle of any figure corresponds a circle of the inverse figure (403, 6°), and since to every point of intersection of any two figures corresponds an inverse point of intersection of the two inverse figures (402, 1°); therefore &c.

2°. *When two circles intersect, the circle passing through the two points of intersection and through the centre of inversion inverts into the radical axis of the two inverse circles.*

For, that circle, passing through the centre of inversion, inverts into a line (403, 5°), and, passing through the two points of intersection of the two original circles, the line into which it inverts passes through the two points of intersection of the two inverse circles (402, 1°); and therefore &c.

3°. *Every two intersecting circles and their two circles of inversion* (398) *invert, to every circle, into two intersecting circles and their two circles of inversion* (398).

For, since, for the two original circles, the two circles of antisimilitude, or inversion (398), pass through the two points, and bisect, one externally and the other internally, the two angles, of intersection (201); therefore, for the two circles inverse to the two former, the two circles inverse to the two latter pass through the two points (402, 1°), and bisect, one externally and the other internally, the two angles (402, 3°), of intersection; and therefore &c. (201).

4°. *Every two intersecting circles invert into equal circles to every circle having its centre on either of their two circles of inversion, external or internal.*

For, that circle of inversion of the original circles then inverting into the radical axis of the two inverse circles (2° above), and two intersecting circles being evidently equal when their radical axis bisects their two angles of intersection (3° above); therefore &c.

5°. *Every two intersecting circles invert into circles whose radii have a constant ratio to every circle having its centre on the same circle passing through their two points of intersection.*

For, the latter circle inverting to every such circle into the radical axis of the two inverse circles (2° above), and the radii of any two intersecting circles being evidently in the inverse ratio of the sines of the segments into which their radical axis divides, externally or internally, their two angles of intersection; therefore &c. (402, 3°).

N.B. It will be presently shewn that the several properties above given are all general, and true, with obvious modifications, of any two circles whether intersecting or not.

410. If C be a circle of any radius, finite or infinitely great or small, and C' its inverse with respect to any arbitrary circle of inversion K, of which the centre is O; then always—

1°. *Every circle D orthogonal to C inverts into a circle D' orthogonal to C'.*

2°. *Every two points P and Q inverse to each other with respect to C invert into two points P' and Q' inverse to each other with respect to C'.*

3°. *Every diameter of C inverts into a circle through O orthogonal to C'; and, conversely, every circle through O orthogonal to C inverts into a diameter of C'.*

4°. *The centre of C inverts into the inverse of O with respect to C'; and, conversely, the inverse of O with respect to C inverts into the centre of C'.*

Of these very useful properties as regards inversion; the first follows immediately from the general property (407) that the angle of intersection of any two circles is equal or supplemental to that of their two inverses with respect to any circle of inversion K; the second from the first, from the consideration that as every circle D passing through P and Q intersects C at right angles (156), therefore, by 1°, every circle D' passing through P' and Q' intersects C' at right angles, and therefore &c. (156); the third also from the first, of which it is evidently a particular case, from the consideration that, by virtue of it, every line orthogonal to C inverts into a circle through O orthogonal to C' (403, 4°), and that, conversely, every circle through O orthogonal to C inverts into a line orthogonal to C' (403, 5°); and the fourth; either from the second, of which it is evidently a particular case, from the consideration that, by virtue of it, when P is at infinity then Q and P' are the centres of C and K respectively (149, 3°) and Q' consequently the inverse of the latter with respect to C', and when conversely P' is at infinity then Q' and P are the centers of C' and K respectively (149, 3°) and Q consequently the inverse of the latter with respect to C; or from the third, more readily perhaps, from the consideration that, by virtue of it, every line passing through the centre of C (22, 1°) inverts into a circle passing through O and through its inverse with respect to C' (156), and, conversely, every circle passing through O and through its inverse with respect to C (156) inverts into a line passing through the centre of C' (22, 1°); and therefore &c. (402, 1°). See also Art. 405, where this last property was established on other principles for the line and circle separately.

Cor. 1°. From properties 1° and 3° of the above, the following consequences are at once evident, viz.—

a. Every circle orthogonal to two circles inverts, to every circle, into a circle orthogonal to the two inverse circles.

b. The particular circle orthogonal to two circles, which passes through the centre of inversion, inverts into the line orthogonal, to the two inverse circles.

c. The line orthogonal to two circles inverts into the particular circle orthogonal to the two inverse circles which passes through the centre of inversion.

d. The circle orthogonal to three circles inverts, to every circle, into the circle orthogonal to the three inverse circles.

e. The circle orthogonal to three circles inverts, to every circle through whose centre it passes, into a line orthogonal to the three inverse circles.

f. Every system of circles having a common orthogonal circle inverts, to every circle, into a system having a common orthogonal circle.

g. And, to every circle having its centre on the common orthogonal circle, into a system having a common orthogonal line.

h. Every system of circles having a common pair of orthogonal circles inverts, to every circle, into a system having a common pair of orthogonal circles.

i. And, to every circle having its centre at either point of intersection of the common orthogonal pair, into a system having a common pair of orthogonal lines.

Cor. 2°. From properties 2° and 4° of the above, the following also are at once evident, viz.:

a. Every two non-intersecting circles and their common pair of inverse points invert, to every circle, into two non-intersecting circles and their common pair of inverse points.

b. Every two non-intersecting circles invert, to every circle having its centre at either of their common pair of inverse points, into two circles having a common centre, the inverse of the other.

c. Every two circles having a common centre invert, to every circle, into two non-intersecting circles whose common pair of inverse points are the centre of inversion and the inverse of the common centre.

d. Every system of circles having a common pair of inverse points inverts, to every circle, into a system having a common pair of inverse points, inverse to the original pair.

e. Every system of circles having a common pair of inverse points inverts, to every circle having its centre at either point, into a system having a common centre, the inverse of the other.

f. Every system of circles having a common centre inverts, to every circle, into a system having a common pair of inverse points, the centre of inversion and the inverse of the common centre.

411. Every two figures E and F, inverse to each other with respect to any line or circle C, being composed of pairs of points P and Q, R and S, &c. inverse to each other with respect to the line or circle C (396); it follows, consequently, from the general property 2° of the preceding article, that—

Every two figures E and F, inverse to each other with respect to any line or circle C, invert—

a. To every circle whose centre is not on the line or circle, into two figures inverse to each other with respect to the inverse circle.

b. To every circle whose centre is on the line or circle, into two figures reflexions of each other with respect to the inverse line.

Cor. Every two *circles*, however circumstanced as to magnitude and position, being figures inverse to each other with respect to *each* of their two circles of antisimilitude (398); it follows, consequently, from the above, as established already on other principles for the particular case of intersecting circles in 3° and 4° of Art. 409, that—

a. Every two circles and their two circles of inversion invert, to every circle, into two circles and their two circles of inversion.

b. Every two circles invert into equal circles to every circle having its centre on either of their two circles of inversion.

412. Coaxal circles (184) possess, with respect to inversion, the following among other important properties, viz.—

1°. *Every system of coaxal circles of the common points species inverts, from either common point as centre, into a system of concurrent lines, whose vertex is the inverse of the other common point.*

2°. *Every system of coaxal circles of the limiting points species inverts, from either limiting point as centre, into a system of concentric circles, whose centre is the inverse of the other limiting point.*

3°. *Every system of coaxal circles of either species, whose common or limiting points coincide, inverts, from the point of coincidence as centre, into a system of parallel lines, whose direction is that of their common tangent at the point.*

4°. *Every system of coaxal circles of either species inverts, from every centre, into a coaxal system of the same species, whose common or limiting points, distinct or coincident, are the inverses of those of the original system.*

Firstly, for a system having real common points M and N, distinct or coincident. Since, by hypothesis, the component figures of the system are all circles passing through M and N, therefore, by (403, 6° and 5°), those of the inverse system, for every centre O not coinciding with either M or N, are all circles passing through M' and N' the inverses, distinct or coincident, of M and N, and, for each centre O coinciding with either M or N, are all lines passing through the inverse N' or M' of the other N or M; and therefore &c. as regards 1° and the first part of 4°. And, since, when M and N coincide at O, then M' and N' coincide at infinity on the line MN (401, 14°); therefore &c. as regards the first part of 3°.

Secondly, for a system having real limiting points E and F, distinct or coincident. Since, by hypothesis, the component figures of the system are all circles intersecting at right angles every circle passing through E and F (188, 5°), therefore, by (403, 6° and 5°, and 407) those of the inverse system, for every centre O not coinciding with either E or F, are all circles intersecting at right angles every circle passing through E' and F' the inverses, distinct or coincident, of E and F, and, for each centre O coinciding with either E or F, are all circles intersecting at right angles every line passing through the inverse F' or E' of the other F or E; and therefore &c. as regards 2° and the second part of 4°. And, since, when E and F coincide at O, then E' and F' coincide at infinity on the line EF (401, 14°); therefore &c. as regards the second part of 3°.

In every case, the particular circles of the original and orthogonal systems (185) which pass through the centre of inversion O invert evidently into the radical and central axes of the inverse system. For, as passing both through the centre of inversion, they invert both into lines (403, 5°), of which one (the former) is a particular circle of the inverse system, and the other (the latter) intersects all its circles at right angles (407); and therefore &c.

And, conversely, for the same reason, the radical and central axes of the original system invert respectively, in all cases, into the particular circles of the inverse system and of its orthogonal system which pass through the centre of inversion.

N.B. That the two parts of property 3° above are in fact identical, and express a common property which appears at once from (403, 5°), is evident *à priori* from (184), the coaxal system consisting in both cases of circles having a common tangent at a common point. The consideration that a system of lines passing through a common point at infinity in any direction, and a system of circles having a common centre at infinity in the perpendicular direction, are in fact identical (16 and 18) explains *à posteriori* the reason why properties so different as 1° and 2° above lead to the common result they do in the particular case of 3°.

413. From properties 1° and 2° of the preceding article, including of course their common particular case 3°, it follows of course, conversely, that—

1°. *Every system of lines passing through a common point (whether at a finite distance or at infinity) inverts, from any centre, into a system of coaxal circles of the common points species; the centre of inversion and the inverse of the common point being the two common points of the inverse system.*

2°. *Every system of circles having a common centre (whether at a finite distance or at infinity) inverts, from any centre, into a* *system of coaxal circles of the limiting points species; the centre of inversion and the inverse of the common centre being the two limiting points of the inverse system.*

Both of which are also evident directly; the first from the consideration that every line of the system inverts into a circle passing through the centre of inversion and through the inverse of the common point (403, 4°); and the second from the consideration that, as every line passing through the common centre intersects every circle of the system at right angles, therefore every circle passing through the centre of inversion and through the inverse of the common centre (403, 4°) intersects every circle of the inverse system at right angles (407), and therefore &c. (188, 5°).

414. By virtue of the results established in the two preceding articles, every property of coaxal circles, involving only considerations of contact, or intersection at constant angles, with lines or circles, or anharmonic equivalence of quartets of points, or homographic division by lines or circles, which (406 and 407) undergo no change in the process of inversion, may be reduced, according to the nature of the system, to one or other of the comparatively simple cases of concurrent lines or concentric circles (including of course the common case of parallel lines), for which it is sometimes evident to mere perception. Thus for instance, the two general properties, established directly on other principles in (193, Cor. 8°) and in (326, Exs. *h* and *i*), that "*a variable circle intersecting any two fixed circles at any two constant angles intersects at constant angles all circles of the system coaxal with them, touches two particular circles of the system, and cuts a third at right angles;*" and that "*a variable circle intersecting any two fixed circles at any two constant angles determines four homographic systems of points on the circles themselves, and, generally,* 2*n* *homographic systems of points on every n circles of the system coaxal with them;*" are both evident to mere perception for the particular cases of a system of concurrent lines and of a system of concentric circles, to one or other of which every other case, by virtue of the preceding properties, may be reduced by inversion.

415. Every problem again connected with a system of coaxal circles, not involving other than similar considerations, may, by virtue of the same results, be reduced, according to the nature of the system, to the corresponding problem for a system of concurrent lines or of concentric circles; for which, as might naturally be expected, the solution, if not evident to mere inspection, is generally simpler than for the original system. Thus for instance the two following problems, "*To describe a circle of a coaxal system; 1°, intersecting a given circle at a given angle; 2°, dividing a given arc of a given circle in a given anharmonic ratio;*" are reducible to one or other pair, as the case may be, of the corresponding simpler problems: "*to draw a line passing through a given point, or to describe a circle having a given centre, and fulfilling the required condition* (1° *or* 2°) *for the inverse circle or arc;*" the solutions of which, for condition 1° are evident in either case, and for condition 2° have been given, actually in the former case and virtually in the latter case, in Ex. 3°, *a*, Art. 356. And the solution of every such problem, once obtained, after inversion, for the corresponding simplified form, gives of course, by inversion back again, the solution of the original in its general form.

416. Coaxal circles possess also, with respect to inversion, the following important property, particular cases of which have been already established on other principles in Art. 409, viz.—

Every two circles invert into two whose radii have a constant ratio from every point on any third circle coaxal with themselves.

For, if O and k be the centre and radius of inversion, A and B the centres of the two original circles, a and b their two radii, u and v the two tangents to them from O, a' and b' the radii of the two inverse circles, and Z the centre of the coaxal circle on which O lies; then, since (404) $a' = \frac{k^2}{u^2}\, a$ and $b' = \frac{k^2}{v^2}\, b$, therefore $\frac{a'}{b'} = \frac{a}{b} \cdot \frac{v^2}{u^2} = \frac{a}{b} \cdot \frac{BZ}{AZ}$ (192, Cor. 1°), which being of course constant when Z is fixed, whatever be the position of O on the coaxal circle of which it is the centre, therefore &c.

Cor. 1°. As the radius of a circle may have either sign indifferently, it appears from the above that, for every pair of original circles, two different coaxal circles loci of O correspond to each particular value of the constant ratio of the inverse radii; that, when the ratio is given, their two centres Z and Z' are given by the relations $\frac{AZ}{BZ} = \pm \frac{a}{b} \cdot \frac{b'}{a'}$ and $\frac{AZ'}{BZ'} = \mp \frac{a}{b} \cdot \frac{b'}{a'}$; and, as already shewn on other principles in (409, 5°), that, when the original circles intersect, they divide their angles of intersection, externally and internally, into parts whose sines are in the inverse of the ratio.

COR. 2°. In the particular case when $\frac{AZ}{BZ} = \pm \frac{a}{b}$ and $\frac{AZ'}{BZ'} = \mp \frac{a}{b}$, that is, when Z and Z' are the two centres of perspective of the two original circles (195); then $a' = b'$, and therefore, as already shewn on other principles in (411, Cor. b).

Every two circles invert into equal circles from every point on either of the two coaxal circles whose centres are their two centres of perspective.

COR. 3°. If A, B, C be any three given circles, and X and X', Y and Y', Z and Z' the three pairs of circles coaxal with B and C, C and A, A and B respectively for whose several points, as centres of inversion, the three pairs of radii b' and c', c' and a', a' and b' of the three corresponding pairs of inverse circles B' and C', C' and A', A' and B' have the three pairs of ratios determined by any three given lines taken two and two; since then (the compound of the three ratios being necessarily $= 1$) the six centres of the three pairs of circles X and X', Y and Y', Z and Z' (determinable, when the three ratios $a' : b' : c'$ are given, by the general relations of Cor. 1°) lie three and three on four lines (which when $a' = b' = c'$ are, by Cor. 2°, the four axes of similitude (197) of the three original circles A, B, C), the six circles themselves pass consequently three and three through four pairs of conjugate points P and P', Q and Q', R and R', S and S' reflexions of each other with respect to the four lines, and real or imaginary according to circumstances (190). Hence, from Cor. 1°, it appears that *there exist in general eight different points, corresponding to each other two and two in four conjugate pairs, real or imaginary according to circumstances, and determinable in every case by the intersections three and three of six determinable* *circles, for which three given circles invert into circles whose radii are proportional to three given lines.*

COR. 4°. By virtue of the preceding, many problems involving only contacts, or intersections at angles of prescribed forms, which undergo no change by inversion (407), may be transformed from one system of three circles to another, the relative magnitudes of whose radii may be more convenient for their solutions, and the solutions thus obtained then transformed back again by inversion to the original system. Thus, for instance, the two solutions, real or imaginary, of the particular problem, "*To describe a circle having contacts of the same species with three given circles, or, more generally, intersecting three given circles at combinations of the same affection of three given angles and their three supplements;*" are evident, respectively, for three equal circles, and for three whose radii are inversely as the cosines of the three corresponding angles, the radical centre of the three being then, in either case, the common centre of the required pair of circles (2, Cor. 1°, Art. 183). And, since to such a system every given system of three may be transformed by inversion from any one of eight different centres, corresponding two and two in four conjugate pairs, real or imaginary, according to circumstances (Cor. 3°), the four pairs of conjugate solutions, real or imaginary, of the general problem, "*To describe a circle touching three given circles, or, more generally, intersecting with three given circles at three angles of given forms;*" may consequently be obtained by four different inversions of the three given circles from any four, no two of which are conjugates, of the aforesaid eight centres, and by the four inversions back again of the four pairs of solutions thus obtained for their four corresponding triads of inverse circles.

417. In the applications of the theory of inversion, the distances, absolute and relative, between the inverses of pairs of points have occasionally to be considered. The following are the principal relations to be used in such cases:

If A, B, C, D, E, &c. be any number of points, and A', B', C', D', E', &c. their several inverses with respect to any centre and radius of inversion O and OR; then—

1°. *For every two points A, B, and their two inverses A', B',*

$$A'B' : AB = OR^2 : OA.OB.$$

2°. *For every three points A, B, C, and their three inverses A', B', C',*

$$B'C' : C'A' : A'B' = OA.BC : OB.CA : OC.AB.$$

3°. *For every four points A, B, C, D, and their four inverses A', B', C', D',*

$$B'C'.A'D' : C'A'.B'D' : A'B'.C'D' = BC.AD : CA.BD : AB.CD.$$

Of these relations, the second and third are both evident from the first, which may be easily proved as follows: Since $OA.OA' = OB.OB' = OR^2$, the two triangles AOB and $B'OA'$ are similar; therefore $A'B' : AB = OA' : OB = OA.OA' : OA.OB = R^2 : OA.OB$; and therefore &c.

Cor. 1°. It is evident, from the first of the preceding relations, that, for a given radius of inversion, the absolute distance $A'B'$ between the inverses of any two points A and B varies directly as the distance AB and inversely as the rectangle $OA.OB$; from the second, that, for any radius of inversion, the ratio $A'C' : B'C'$ of the distances of the inverses of any two points A and B from that of any third C varies directly as the ratio $AC : BC$ and inversely as the ratio $OA : OB$; and from the third, that, for any centre and radius of inversion, the three rectangles $B'C'.A'D'$, $C'A'.B'D'$, $A'B'.C'D'$ for the four inverses A', B', C', D' of any four points A, B, C, D are proportional to, and connected consequently by every linear relation connecting, the three corresponding rectangles $BC.AD$, $CA.BD$, $AB.CD$ for the four points themselves.

Cor. 2°. Any three points A, B, C being given, it is evident, from 2°. that, when the ratio $A'C' : B'C'$ is given, then O lies on a given circle coaxal with A and B (152), that, viz. for every point of which $OA : OB = AC \div BC : A'C' \div B'C'$ (158); and that, when the three ratios $B'C' : C'A' : A'B'$, and with them of course the species of the triangle $A'B'C'$, are given, then O is one or other of the two points common to the three given circles coaxal with B and C, C and A, A and B respectively (152), and with each other (190), for each point common to which

$$OA : OB : OC = B'C' \div BC : C'A' \div CA : A'B' \div AB \quad (158).$$

COR. 3°. When, for four points A, B, C, D, one of the three rectangles $BC.AD$, $CA.BD$, $AB.CD$ is equal in absolute value to the sum of the other two, it is evident, from 3°, that, for the four inverse points A', B', C', D', the corresponding rectangle is equal in absolute value to the sum of the other two; hence (82), *when four points are either collinear or concyclic, their four inverses with respect to any circle of inversion are also either concyclic or collinear.* A property verifying indirectly the results established in 4°, 5°, 6° of Art. 403.

418. From the principles of inversion established in the preceding articles, it is evident that every property in the geometry of the line and circle, involving only the equality or constancy of angles of intersection or of anharmonic quartets of points collinear or concyclic, may be transformed by inversion into another of the same character in which any circle will be changed into a line, or circle having any required centre or radius; any two circles into two lines, two concentric circles, two equal circles, or two circles whose radii shall have any required ratio to each other; or any three circles into three others, the distances of whose centres or the magnitudes of whose radii shall have any required ratios two and two to each other; and thus its establishment may be, and often is, rendered much simpler than in the original form. Thus, for instance, the properties (established in 326, Ex. *f*, and 193, Cor. 4°) that "*a variable circle passing through a fixed point and intersecting a fixed circle at a constant angle divides the latter homographically and envelopes another coaxal with it and the point*" become transformed by inversion, from the point as centre, into the self-evident properties that "*a variable line intersecting a fixed circle at a constant angle divides the circle homographically and envelopes a concentric circle;*" the properties (established in 326, Ex. *h*, and 143, Cor. 6°) that "*a variable circle intersecting two fixed circles at two constant angles divides both homographically and envelopes two others coaxal with both*" become transformed by inversion, from either point common to both circles when they intersect, or from either point inverse to both when they do not,

into the self-evident properties that "*a variable circle intersecting two fixed lines, or concentric circles, at two constant angles divides both homographically and envelopes two others concurrent, or concentric, with both;*" and the two properties (established in 326, Ex. k, and 211, Cor. 6°, b) that "*a variable circle intersecting three fixed circles at equal (or at any invariable combination of equal and supplemental) angles divides all three homographically and determines a coaxal system*" become transformed by inversion, from either of the corresponding pair of conjugates of the eight points into which the three circles invert into three of equal radii (416, Cor. 3°), into the self-evident properties that "*a variable circle intersecting three fixed circles of equal radii at equal angles divides all three homographically and determines a concentric system.*" And similarly for all properties of the same nature; any of which, when seen or proved to be true in their transformed, may always, by virtue of the general properties of Arts. 406 and 407, be regarded as established in their original forms.

419. Every problem too, in the geometry of the line and circle, involving only considerations of the same nature, may be transformed in the same manner by inversion into another of the same character, in which, while all other essential elements remain unchanged, any one, two, or three, of the circles involved shall be modified in any of the ways above enumerated; and its solution thus rendered, in many cases, much simpler than in its original form. Thus, for instance, the problem "*To describe a circle passing through two given points and intersecting a given circle at a given angle*" becomes transformed by inversion, from either point as centre, into the very simple problem "*to draw a line passing through a given point and intersecting a given circle at a given angle*"; the more general problem "*to describe a circle passing through a given point and intersecting two given circles at two given angles*" becomes transformed by inversion, from the point as centre, into the also very simple problem "*to draw a line intersecting two given circles at two given angles;*" and the still more general problem "*to describe a circle intersecting three given circles at three given angles,*" becomes transformed by inversion, from either point common to any two of the circles when

they intersect, or from either point inverse to any two of them when they do not, into one or other of the two comparatively simple problems: "*to describe a circle intersecting two given lines, or concentric circles, at two given angles, and a third given circle at a third given angle;*" (see also, as regards the latter problem, the different mode of solution by inversion given in Cor. 4°, Art. 416). And similarly for all problems of the same nature; their solutions, whether obvious or comparatively simple, in their modified forms, giving of course in all cases, by inversion back again, their solutions in their original forms.

420. Every property again of the same class, relating to the intersection or contact of figures, simple or compound, by lines or circles, gives evidently, by inversion from an arbitrary centre, a corresponding but more general property of the same character, in which while all the other essential features remain unaltered, all the lines, fixed or variable, are changed into circles passing through a common point, the centre of inversion (403, 4°). Of this use of inversion, as an instrument for the evolution of such more general properties from others well known and familiar, we shall terminate this chapter with some examples; in which the original and inverse properties are placed side by side in parallel columns, for convenience of comparison; and in all of which the reader is recommended to attempt the process of inversion himself, and to draw, if necessary, the requisite figures, before looking at the results.

Ex. 1°. A variable line turning round a fixed point determines two homographic divisions on every two fixed lines.	A variable circle passing through two fixed points determines two homographic divisions on every two fixed circles passing through either point.
Ex. 2°. A variable line turning round a fixed point determines two homographic divisions in involution on every fixed circle.	A variable circle passing through two fixed points determines two homographic divisions in involution on every fixed circle.
Ex. 3°. A variable line dividing a fixed arc of a fixed circle harmonically turns round a fixed point, the intersection of the two lines touching the circle at the extremities of the arc.	A variable circle passing through a fixed point and dividing a fixed arc of a fixed circle harmonically passes through another fixed point, the second intersection of the two circles through the point which touch the circle at the extremities of the arc.

Ex. 4°. A variable line dividing two fixed arcs of the same fixed circle equianharmonically turns round one or other of two fixed points, the intersections of the two pairs of lines connecting the extremities of one arc with those of the other.

Ex. 5°. A variable line intersecting a fixed circle at right angles passes in every position through a fixed point, the centre of the circle.

Ex. 6°. A variable line intersecting a fixed circle at any constant angle envelopes another fixed circle, concentric with the first.

Ex. 7°. A variable line intersecting a fixed circle at any constant angle intersects at constant angles all fixed circles concentric with the first.

Ex. 8°. A variable line intersecting a fixed circle at any constant angle determines two similar systems of points on the circle.

Ex. 9°. A variable line intersecting a fixed circle at any constant angle determines two similar systems of points on every fixed circle concentric with the first.

Ex. 10°. A variable circle intersecting two fixed lines at right angles determines by its variation a system concentric with the intersection of the lines.

Ex. 11°. A variable circle intersecting two fixed lines (or concentric circles) at two constant angles envelopes two other fixed lines (or circles) concurrent (or concentric) with the two first.

A variable circle passing through a fixed point and dividing two fixed arcs of the same fixed circle equianharmonically passes through one or other of two other fixed points, the second intersections of the two pairs of circles passing through the point and connecting the extremities of one arc with those of the other.

A variable circle passing through a fixed point and intersecting a fixed circle at right angles passes in every position through a second fixed point, the inverse of the first with respect to the circle.

A variable circle passing through a fixed point and intersecting a fixed circle at any constant angle envelopes another fixed circle, coaxal with the point and first.

A variable circle passing through a fixed point and intersecting a fixed circle at any constant angle intersects at constant angles all fixed circles coaxal with the point and first.

A variable circle passing through a fixed point and intersecting a fixed circle at any constant angle determines two homographic systems of points on the circle.

A variable circle passing through a fixed point and intersecting a fixed circle at any constant angle determines two homographic systems of points on every fixed circle coaxal with the point and first.

A variable circle intersecting two fixed intersecting circles at right angles determines by its variation a system coaxal with the two intersections of the circles.

A variable circle intersecting any two fixed intersecting (or non-intersecting) circles at any two constant angles envelopes two other fixed intersecting (or non-intersecting) circles coaxal, in either case, with the two first.

Ex. 12°. A variable circle intersecting two fixed lines (or concentric circles) at two constant angles intersects at constant angles all fixed lines (or circles) concurrent (or concentric) with the two.

A variable circle intersecting any two fixed intersecting (or non-intersecting) circles at any two constant angles intersects, in either case, at constant angles all fixed circles coaxal with the two.

Ex. 13°. A variable circle intersecting two fixed lines (or concentric circles) at two constant angles determines four similar systems of points on the two lines (or circles).

A variable circle intersecting any two fixed intersecting (or non-intersecting) circles at any two constant angles determines, in either case, four homographic systems of points on the two circles.

Ex. 14°. A variable circle intersecting two fixed lines (or concentric circles) at two constant angles divides similarly all lines (or circles) concurrent (or concentric) with the two.

A variable circle intersecting any two fixed intersecting (or non-intersecting) circles at any two constant angles divides, in either case, homographically all circles coaxal with the two.

Ex. 15°. When a variable circle intersects, in every position, two fixed lines (or concentric circles) at two constant angles, its centre describes a concurrent line (or concentric circle.)

When a variable circle intersects, in every position, any two fixed intersecting (or non-intersecting) circles at two constant angles, the inverse with respect to it of either point common (or inverse) to both describes, in either case, a coaxal circle.

Ex. 16°. A variable line intersecting any two fixed circles at equal (or supplemental) angles passes, in every position, through the centre of their external (or internal) circle of inversion.

A variable circle passing through a fixed point and intersecting any two fixed circles at equal (or supplemental) angles passes, in every position, through the inverse of the point with respect to their external (or internal) circle of inversion.

Ex. 17°. A variable line intersecting any two fixed circles at equal (or supplemental) angles determines four homographic divisions on the two circles.

A variable circle passing through a fixed point and intersecting any two fixed circles at equal (or supplemental) angles determines four homographic divisions on the two circles.

Ex. 18°. A variable line intersecting two fixed circles at two pairs of collinear points harmonically conjugate to each other divides every two positions of itself homographically.

A variable circle passing through a fixed point and intersecting two fixed circles at two pairs of concyclic points harmonically conjugate to each other divides every two positions of itself homographically.

Ex. 19°. In the particular case when the two fixed circles intersect at right angles, the variable line passes, in every position, through one or other of their two centres.

In the particular case when the two fixed circles intersect at right angles, the variable circle passes, in every position, through one or other of the two inverses of the fixed point with respect to them.

Ex. 20°. When a variable circle intersects in every position two fixed circles at right angles, its centre describes their radical axis.

When a variable circle intersects in every position two fixed circles at right angles, the inverse of any fixed point with respect to it describes their coaxal circle passing through the point.

Ex. 21°. In every system of three circles, the three radical axes of their three groups of two pass through a common point, the centre of their common orthogonal circle.

In every system of three circles, every three circles coaxal with their three groups of two which pass through a common point pass through a second common point, the inverse of the first with respect to their common orthogonal circle.

Ex. 22°. A variable line intersecting three fixed circles at three pairs of collinear points in involution turns round a fixed point, the centre of their common orthogonal circle.

A variable circle passing through a fixed point and intersecting three fixed circles at three pairs of concyclic points in involution passes through a second fixed point, the inverse of the first with respect to their common orthogonal circle.

Ex. 23°. In the particular case when the three fixed circles are coaxal, every line, however situated, intersects them at three pairs of collinear points in involution.

In the particular case when the three fixed circles are coaxal, every circle, however circumstanced as to magnitude and position, intersects them at three pairs of concyclic points in involution.

Ex. 24°. A variable line intersecting four fixed lines at four collinear points having a constant anharmonic ratio divides the four lines and all other positions of itself homographically.

A variable circle passing through a fixed point and intersecting four fixed circles passing through the point at four concyclic points having a constant anharmonic ratio divides the four circles and all other positions of itself homographically.

Ex. 25°. A variable line dividing any two fixed lines homographically divides all fixed positions of itself homographically, and determines with every four of them a collinear quartet of points having a constant anharmonic ratio.

A variable circle passing through a fixed point and dividing any two fixed circles passing through the point homographically divides all fixed positions of itself homographically, and determines with every four of them a concyclic quartet of points having a constant anharmonic ratio.

Ex. 26°. A variable line enveloping a fixed circle divides all fixed tangents to the circle homographically, and determines with every four of them a collinear quartet of points having a constant anharmonic ratio.

A variable circle passing through a fixed point and eveloping a fixed circle divides all fixed circles passing through the point and touching the circle homographically, and determines with every four of them a concyclic quartet of points having a constant anharmonic ratio.

Ex. 27°. Every two lines touching a circle make equal angles with the line passing through their point of intersection and through the centre of the touched circle.

Every two circles touching a circle make equal angles with the circle passing through their two points of intersection and through the inverse of either with respect to the touched circle.

Ex. 28°. Every two lines touching a circle make equal angles with the concentric circle passing through their point of intersection.

Every two circles touching a circle make equal angles with the circle passing through either of their two points of intersection and coaxal with the other and the circle.

Ex. 29°. Two variable lines touching a fixed circle and intersecting on a second concentric with the first intersect at a constant angle; and conversely.

Two variable circles passing through a fixed point, touching a fixed circle, and intersecting on a second fixed circle coaxal with the point and first, intersect at a constant angle; and conversely.

Ex. 30°. In the same case the line passing through their two points of contact envelopes a third fixed circle concentric with the other two.

In the same case the circle passing through the point and through their two points of contact envelopes a third fixed circle coaxal with the other two.

Ex. 31°. The angle between the lines connecting any two points on a circle with the centre is double the angle between the lines connecting the same points with any third point on the circle.

The angle between the circles connecting any two points on a circle with any pair of inverse points is double the angle between the circles connecting the same points with either inverse point and with any third point on the circle.

Ex. 32°. Two variable lines intersecting on a fixed circle and turning each round one of two fixed points on the circle intersect at a constant angle.

Two variable circles intersecting at a fixed point and on a fixed circle, and passing each through one of two fixed points on the circle, intersect at a constant angle.

Ex. 33°. In the particular case when the two fixed points are collinear with the centre of the fixed circle, the constant angle of intersection is a right angle.

In the particular case when the two latter fixed points are concyclic with the first and its inverse with respect to the fixed circle, the constant angle of intersection is a right angle.

Ex. 34°. When two variable lines intersecting on a fixed circle intersect at a constant angle, the line passing through their other two intersections with the circle envelopes a second fixed circle concentric with the first.

When two variable circles intersecting at a fixed point and on a fixed circle intersect at a constant angle, the circle passing through the point and through their other two intersections with the circle envelopes a second fixed circle coaxal with the point and first.

Ex. 35°. In the particular case when the constant angle of intersection is a right angle, the envelope is evanescent, and the enveloping line passes in every position through the centre of the fixed circle.

In the particular case when the constant angle of intersection is a right angle, the envelope is evanescent, and the enveloping circle passes in every position through the inverse of the fixed point with respect to the fixed circle.

Ex. 36°. The circle passing through the two points of contact and through the point of intersection of any two lines touching a circle passes through the centre of the touched circle.

The circle passing through the two points of contact and through either point of intersection of any two circles touching a circle passes through the inverse of the other point of intersection with respect to the touched circle.

Ex. 37°. The same circle intersects at right angles the line passing through the centre and through the point of intersection.

The same circle intersects at right angles the circle passing through the same inverse, and through the two points of intersection.

Ex. 38°. The two lines connecting the centre of a circle, with the two points at which a variable intersects with two fixed tangents to the circle, intersect at a constant angle, equal to half that determined by the fixed tangents.

The two circles connecting any two inverse points with respect to a circle, with the two points at which a variable intersects with two fixed tangent circles through either point to the circle, intersect at a constant angle, equal to half that determined by the fixed tangent circles.

Ex. 39°. In the particular case when the two fixed tangents are parallel, the constant angle of intersection is a right angle.

In the particular case when the two fixed tangent circles through the point touch at the point, the constant angle of intersection is a right angle.

Ex. 40°. When two variable points on a fixed circle connect in every position by a line passing through a fixed point, the two tangents at them intersect in every position on the fixed line passing through the points of contact of the two fixed tangents through the point.

When two variable points on a fixed circle connect in every position by a circle passing through two fixed points, the two tangent circles at them which pass through either point intersect in every position on the fixed circle passing through that point and through the points of contact of the two fixed tangent circles through both points.

Ex. 41°. In the same case, the point on the variable line of connection, harmonically conjugate to the fixed with respect to the two variable points, describes, in the course of its variation, the same fixed line on which the two variable tangents intersect in every position.

In the same case, the point on the variable circle of connection, harmonically conjugate to either of the two fixed with respect to the two variable points, describes, in the course of its variation, the same fixed circle on which the two variable tangent circles through the same fixed point intersect in every position.

Ex. 42°. In every system of two circles, the two pairs of conjugate lines touching both, intersect on, make equal angles with, and are reflexions of each other with respect to, the line cutting both circles at right angles.

In every system of two circles, the two pairs of conjugate circles touching both which pass through any common point, intersect on, make equal (or supplemental) angles with, and are inverses of each other with respect to, the circle through the point cutting both circles at right angles.

Ex. 43°. In the same case, the two intersections of the two pairs of conjugate lines divide harmonically the segment of the orthogonal line intercepted between the centres of the circles.

In the same case, the two second intersections of the two pairs of conjugate circles divide harmonically the arc of the orthogonal circle intercepted between the inverses of the point with respect to the original circles.

Ex. 44°. In every system of three circles, the six pairs of conjugate tangent lines to their three groups of two determine six points of intersection which lie three and three on four lines.

In every system of three circles, the six pairs of conjugate tangent circles to their three groups of two which pass through any common point determine six points of intersection which lie three and three on four circles passing through the point.

Ex. 45°. When four lines touch two circles at four points lying on a line, the two touching one circle intersect with the two touching the other circle at four points lying on a third circle coaxal with the other two.

When four circles passing through a common point touch two circles at four points lying on a circle passing through the point, the two touching one circle intersect with the two touching the other circle at four points lying on a third circle coaxal with the other two.

Ex. 46°. In every triangle formed by three lines, the three lines which pass through the vertices and intersect perpendicularly with the opposite sides pass through a common point.

In every triangle formed by three circles passing through a common point, the three circles through the point which pass through the vertices and intersect perpendicularly with the opposite sides pass through a second common point.

Ex. 47°. In every triangle found by three lines, the three pairs of lines which bisect exter-

In every triangle formed by three circles passing through a common point, the three pairs of circles through the point which nally and internally the three angles pass three and three through four points and intersect with the opposite sides at three pairs of points which lie three and three on four lines.

bisect externally and internally the three angles pass three and three through four points, and intersect with the opposite sides at three pairs of points which lie three and three on four circles passing through the common point.

Ex. 48°. In every triangle formed by three lines touching a common circle, the three lines which connect the points of contact with the opposite vertices are concurrent, and the three which connect them with each other two and two intersect with the third sides at three collinear points.

In every triangle formed by three circles passing through a common point and touching a common circle, the three circles through the point which connect the points of contact with the opposite vertices are coaxal, and the three through the point which connect them with each other two and two intersect with the third sides at three points concyclic with the point.

Ex. 49°. In every triangle formed by three lines intersecting two and two on a common circle, the three lines which touch the circle at the vertices intersect with the opposite sides at three collinear points, and with each other two and two at three points which connect with the third vertices by three concurrent lines.

In every triangle formed by three circles passing through a common point and intersecting two and two on a common circle, the three circles through the point which touch the circle at the vertices intersect with the opposite sides at three points concyclic with the point, and with each other two and two at three points which connect with the third vertices and with the point by three coaxal circles.

Ex. 50°. In every triangle formed by three lines, when three lines through the vertices intersect either with each other concurrently or with the opposite sides collinearily, their three reflexions with respect to the three lines which bisect, internally or externally, the corresponding angles fulfil the same condition.

In every triangle formed by three circles passing through a common point, when three circles through the point and vertices intersect either with each other concurrently or with the opposite sides concyclically with the point, their three inverses with respect to the three circles through the point which bisect, internally or externally, the corresponding angles fulfil the same condition.

Ex. 51°. In every triangle formed by three lines, when three lines through the vertices intersect either with each other concurrently or with the opposite sides collinearily, the three lines harmonically conjugate to them with respect to the corresponding angles fulfil the opposite condition.

In every triangle formed by three circles passing through a common point, when three circles through the point and vertices intersect either with each other concurrently or with the opposite sides concyclically with the point, the three circles through the point harmonically conjugate to them with respect to the corresponding angles fulfil the opposite condition.

Ex. 52°. In every triangle formed by three lines, when three points on the sides connect either with each other collinearly or with the opposite vertices concurrently, their three harmonic conjugates with respect to the corresponding sides fulfil the opposite condition.

In every triangle formed by three circles passing through a common point, when three points on the sides connect either with each other and the point concyclically or with the opposite vertices and the point concurrently, their three harmonic conjugates with respect to the corresponding sides fulfil the opposite condition.

Ex. 53°. In every triangle formed by three lines, the three sides and every three concurrent lines through the vertices intersect with every line at three triads of collinear points in involution; and, the three vertices and every three collinear points on the sides connect with every point by two triads of concurrent lines in involution.

In every triangle formed by three circles passing through a common point, the three sides and every three coaxal circles through the point and vertices intersect with every circle through the point at two triads of concyclic points in involution: and, the three vertices and every three points on the sides concyclic with the point connect through the latter with every point by two triads of coaxal circles in involution.

Ex. 54°. In every quadrilateral formed by four lines, the four circles circumscribing the four triangles determined by their four triads are concurrent.

In every quadrilateral formed by four circles passing through a common point, the four circles circumscribing the four triangles determined by their four triads are concurrent.

Ex. 55°. In every quadrilateral formed by four lines, the three circles passing through the three pairs of opposite vertices, and intersecting at right angles their three lines of connection, are coaxal.

In every quadrilateral formed by four circles passing through a common point, the three circles passing through the three pairs of opposite vertices, and intersecting at right angles their three circles of connection with the point, are coaxal.

Ex. 56°. In every quadrilateral formed by four lines, the three pairs of opposite vertices divide harmonically the three sides of the triangle determined by their three lines of connection.

In every quadrilateral formed by four circles passing through a common point, the three pairs of opposite vertices divide harmonically the three sides of the triangle determined by their three circles of connection with the point.

Ex. 57°. In every quadrilateral formed by four lines touching a common circle, the two lines connecting pairs of opposite vertices and the two connecting points of contact of

In every quadrilateral formed by four circles passing through a common point and touching a common circle, the two circles through the point which connect pairs of opposite vertices and the two which connect points of contact pairs of opposite sides are concurrent, and harmonically conjugate to each other.

Ex. 58°. In every quadrilateral formed by four lines intersecting two and two on a common circle, the two intersections of pairs of opposite sides, and the two of pairs of lines touching the circle at pairs of opposite vertices are collinear, and harmonically conjugate to each other.

Ex. 59°. In every hexagon formed by six lines touching a common circle, the three lines connecting the three pairs of opposite vertices are concurrent, and the two concurrent with them which touch the circle divide equianharmonically its three arcs intercepted by the three pairs of opposite sides.

Ex. 60°. In every hexagon formed by six lines intersecting two and two on a common circle, the three points of intersection of the three pairs of opposite sides are collinear, and the two collinear with them on the circle divide equianharmonically its three arcs intercepted by the three pairs of opposite vertices.

Ex. 61°. In a variable polygon of any order formed by any number of variable lines touching different fixed circles of a concentric system, if all the vertices but one describe fixed circles of the system the remaining one describes a fixed circle of the system; and all the vertices and sides alike divide their several circles similarly.

of pairs of opposite sides are coaxal, and harmonically conjugate to each other.

In every quadrilatera formed by four circles passing through a common point and intersecting two and two on a common circle, the two intersections of pairs of opposite sides and the two of pairs of circles through the point touching the circle at pairs of opposite vertices are concyclic with the point, and harmonically conjugate to each other.

In every hexagon formed by six circles passing through a common point and touching a common circle, the three circles connecting the three pairs of opposite vertices with the point are coaxal, and the two coaxal with them which touch the circle divide equianharmonically its three arcs intercepted by the three pairs of opposite sides.

In every hexagon formed by six circles passing through a common point and intersecting two and two on a common circle, the three points of intersection of the three pairs of opposite sides are concyclic with the point, and the two concyclic with them on the circle divide equianharmonically its three arcs intercepted by the three pairs of opposite vertices.

In a variable polygon of any order formed by any number of variable circles passing through a fixed point and touching different fixed circles of a system coaxal with the point, if all the vertices but one describe fixed circles of the system the remaining one describes a fixed circle of the system; and all the vertices and sides alike divide their several circles homographically.

Ex. 62°. In a variable polygon of any order formed by any number of variable lines intersecting two and two on different fixed circles of a concentric system, if all the sides but one envelope fixed circles of the system, the remaining one envelopes a fixed circle of the system; and all the sides and vertices alike divide their several circles similarly.

In a variable polygon of any order formed by any number of variable circles passing through a fixed point and intersecting two and two on different fixed circles of a system coaxal with the point, if all the sides but one envelope fixed circles of the system, the remaining one envelopes a fixed circle of the system; and all the sides and vertices alike divide their several circles homographically.

Ex. 63°. In a variable polygon of any order formed by any number of variable lines intersecting two and two on the same fixed circle of any coaxal system, if all the sides but one envelope fixed circles of the system, the remaining one envelopes a fixed circle of the system.

In a variable polygon of any order formed by any number of variable circles passing through a fixed point and intersecting two and two on the same fixed circle of any coaxal system, if all the sides but one envelope fixed circles of the system, the remaining one envelopes a fixed circle of the system.

The above examples might easily be multiplied to almost any extent, but they are abundantly sufficient, both in number and variety, to illustrate the use and fertility of the process of inversion on the modern geometry of the circle.

END OF VOL. II.

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