UNIVERSITY OF ILLINOIS BULLETIN

Vol. X | OCTOBER 28, 1912 | No. 9

[Entered Feb. 14, 1902, at Urbana, Ill., as second-class matter under Act of Congress of July 16, 1894]

BULLETIN No. 61

CHARACTERISTICS AND LIMITATIONS OF THE SERIES TRANSFORMER

BY
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UNIVERSITY OF ILLINOIS
ENGINEERING EXPERIMENT STATION

URBANA, ILLINOIS
PUBLISHED BY THE UNIVERSITY

PRICE: TWENTY-FIVE CENTS
EUROPEAN AGENT
CHAPMAN AND HALL, LTD., LONDON

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CHARACTERISTICS AND LIMITATIONS OF THE SERIES TRANSFORMER

BY

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CHARACTERISTICS AND LIMITATIONS OF THE SERIES TRANSFORMER

I. Introduction

1. *Uses of the Series Transformer.*—High potential distribution and the large currents carried by feeders have made the use of series transformers, or so-called "current transformers," imperative. Where high voltages are used it would be a source of considerable danger to bring the potential of the distribution system to the switchboard and controlling apparatus. The use of the series transformer, in connection with the potential transformer, makes it possible to meter the power and control such system without handling voltages which are dangerous to life. Furthermore, if live potentials were brought to the switchboard, the proper insulation of instruments and controlling apparatus would be accomplished only with considerable difficulty and expense. Again in the present day system of distribution, even where the voltage may be so low as to overrule the above objection, the feeders leading from the station may carry such large currents as to make the use of series transformers a decided convenience. Obviously it is much cheaper and easier to run small wires to instruments requiring but a low value of current than to bring large feeders to a board on which are mounted cumbersome instruments of sufficient capacity to carry the full live current. The demands of convenience and safety, then, have placed the series transformer in its present position of importance.

In general, the function of the series transformer is to furnish a current in a circuit not metallically connected with the main circuit, which current shall bear a definite constant ratio to the main current, and shall be 180° out of phase with it. The uses to which this secondary current is put may be divided into two classes: (1) to give indications in metering instruments, and (2) to furnish power for the operation of regulators, time-limit relays, and like controlling apparatus. But with the series transformer the ideal is never completely realized, and consequently the above function is but imperfectly performed.

2. *Scope of Bulletin.*—It is the purpose of this bulletin to study the imperfections of the series transformer, to determine how and to what extent certain constants influence its operation, and to deduce certain general characteristics. The first part of the bulletin will be devoted to a discussion of the fundamental principles of the series transformer and the representation by vector diagrams of its operation. A deduction of current relations by the method of complex quantities, and

a discussion of conclusions that may be derived therefrom will follow. The derivation of current relations by the use of instantaneous current values will then be given. In this last named part particular stress will be laid upon the application of the current transformer for the purpose of recording transient phenomena. A comparison of the results obtained for stable condition by the two methods and a general summary will be given in conclusion.

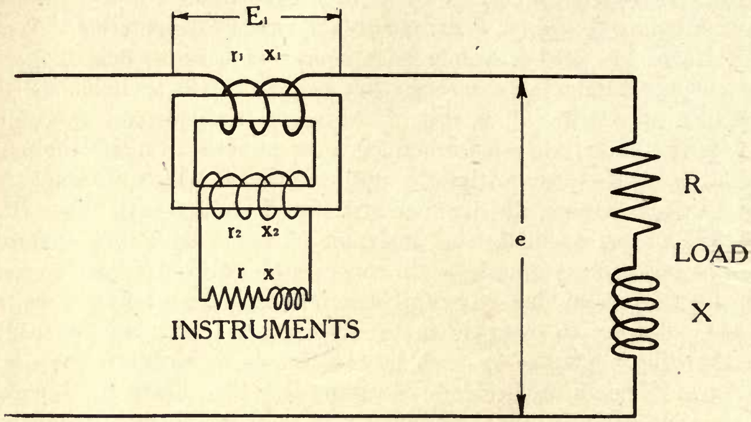

FIG. 1. SYMBOLS USED FOR CONSTANTS OF CIRCUIT

II. CHARACTERISTICS OF THE SERIES TRANSFORMER

3. *General Features of the Series Transformer.*—In construction the series transformer does not greatly differ from the ordinary power potential transformer, inasmuch as both consist essentially of two electrical circuits interlinked by a magnetic circuit. In the potential transformer for furnishing power there is, of course, a variation of primary current with secondary current; in fact, the primary current is a function of the secondary current. In the current transformer, however, the effect of the secondary current upon the primary current is so very small as to be negligible. The primary current is determined by the constants of the main circuit, and is taken to be independent of variations in secondary load and current. Herein then, lies the basic element of distinction between the power potential transformer and the current transformer. This dependence of the primary current upon the constants of the main circuit only, and its independence of variations in secondary current, is the starting point for the theory of the current transformer which follows. Naturally, this difference in operation between the two types of transformers makes a difference in their design, and just how and why some of the design constants differ will be pointed

out later after a development of the theory has made their bearing on the operation of the transformer clearer.

There are certain facts about the series transformer which are generally known among men who have anything at all to do with it. It is known that the series transformer consists of two coils metallically separate, wound on a laminated iron core; that the coil connected in the line generally has fewer turns than the secondary coil across which the instruments are connected, and that the ratio of the turns is about in inverse proportion to the ratio of the currents. It is known that it is dangerous to leave the secondary open-circuited because if a current flows in the primary, high voltages will be induced in the open-circuited secondary and there will be high heating of the iron core which may cause disaster. It is also known that slight variations in secondary load (or resistance and reactance) do not change the transformation ratio to a very great extent, and that the ratio is nearly constant for all values of primary current within a given range. It is also known that when used in connection with wattmeters annoyance is often caused by a certain angle of phase displacement from the ideal position. These very nearly constant factors, however, such as current ratio and phase angle, are very often subject to great variations, and in the following discussion it is proposed to investigate the relations between these factors and the constants of the transformer and load; to determine how and to what extent they are influenced by these constants.

4. *Vector Diagrams for the Series Transformer.*—At the outset it is thought advantageous to study the vector diagram representation of the current transformer in order to gain familiarity with its operation. Fig. 1 is given to indicate the constants represented by the different symbols. Fig. 2 is a simple vector diagram of a series transformer having no core loss. This would be the case with a transformer having an air core. The voltage e across the load is taken as the reference vector, and the primary current I' lags behind it an angle θ dependent upon the power-factor of the load. $\left(\tan\theta = \frac{X}{R}\right)$ In phase with I' is ϕ' the total primary flux, or the flux which could be set up in the existing magnetic circuit by the primary ampere-turns if no counteracting force were present. A part of this flux, ϕ_L', however, is set up in a path not interlinked by the secondary coil, that is, it does not thread the secondary coil, and hence has no effect on the secondary circuit.*

*The leakage flux of the primary, it is readily seen, acts only as a reactance in the primary circuit and does not affect the ratio of the currents in the transformer. Since, however, the primary leakage reactance is an inherent constant of the transformer, it is thought well to introduce it here for the completeness of the discussion.

This flux is, of course, in phase with the primary current. Since it is established in a path external to the secondary coil, it is very properly called leakage flux, and if subtracted from the total primary flux, gives as a resultant what might be called the useful primary flux, in phase with the primary current, and represented in the diagram by the vector ϕ'-ϕ_L'. Now in order that a current may flow in a secondary circuit having impedance, there must be induced in the secondary an e.m.f., and to accomplish this there must exist in the core a *real* flux leading the induced e.m.f. by 90°. This flux, indicated in the diagram by ϕ_M is the only flux *really existing* in the core and is the resultant of the useful primary flux and the useful secondary flux $\phi'' - \phi_L''$, as indicated in the diagram. ϕ'' is the total secondary flux, and ϕ_L'' is the secondary leakage flux, corresponding to ϕ' and ϕ_L', for the primary. I'' is, of course, in phase with ϕ''. The induced secondary voltage is indicated in the diagram by E''. The angle $\propto$ between E'' and I'' is dependent upon the resistance and reactance of the secondary circuit ($\tan \propto = \frac{x_2 + x}{r_2 + r}$),

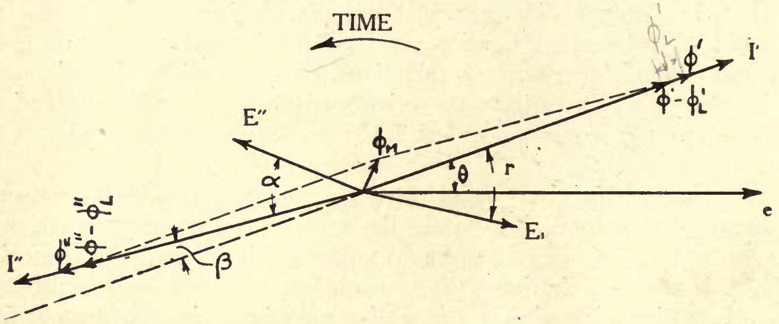

FIG. 2. VECTOR DIAGRAM FOR FLUX IN TRANSFORMER WITH ZERO CORE LOSS

where x_2 is the secondary *leakage* reactance.) β is known as the angle of phase displacement of the secondary current, and is the angle by which the secondary current leads its ideal position of 180° out of phase with the primary current. The function which gives β in terms of other constants of the transformer will be derived later. The variations in the positions and magnitudes of the vectors with the constants of the transformer are not readily seen from this diagram, but there are certain things which might be expected from an examination of the diagram. For instance, it might be expected that a decrease in secondary reactance or an increase in secondary resistance, either of which has the effect of decreasing $\propto$, would as a consequence increase β. It will be seen later that such a variation actually takes place.

It is oftentimes convenient to draw the vector diagram using currents in place of fluxes, as is done in Fig. 3. If the ratio of turns is other than 1 to 1, it will be necessary to consider the magnitude of the current vectors given in ampere turns, or what amounts to the same thing, if the primary current vector is drawn to scale the secondary current should be multiplied by the ratio of the number of secondary turns to primary turns to give the magnitude of the secondary current vector. The resulting current vectors will then be in terms of primary current directly, and it will not be necessary to divide by the number of turns as would be the case if the vectors were scaled off as ampere-turns.

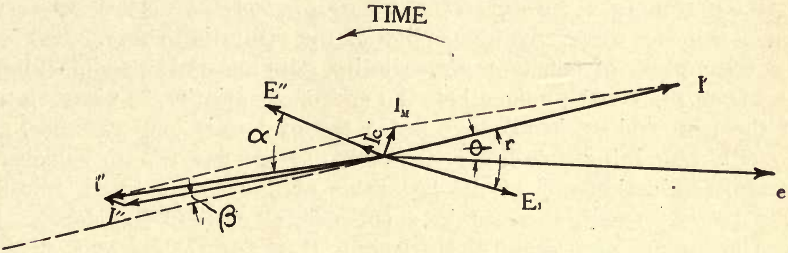

FIG. 3. VECTOR DIAGRAM FOR CURRENTS IN TRANSFORMER WITH ZERO CORE LOSS

Fig. 3 is a vector diagram of a current transformer with an iron core, having consequently some core loss. It is very similar to the diagram in Fig. 2 with the exception that the flux vectors are plotted as currents, and the core loss component I_c is taken into consideration. Core loss represents power dissipated in a secondary circuit, and requires in phase components of current and electromotive force. Since the dissipation of energy takes place in a secondary circuit and we have already drawn a vector for secondary e.m.f. the current vector I_c in phase with E'' will give the required representation of energy dissipation. This current I is consumed in the core, and hence is not available in the useful secondary circuit. Subtracting I_c from i'' gives the true secondary current I'' as represented in the diagram. The angles α and β correspond to the angles α and β of Fig. 2. In both Fig. 1 and Fig. 2, E_1 is the electromotive force across the primary and γ the angle that it makes with the primary current. From Fig. 3 the effect of core loss is readily seen. It decreases the secondary current I'', and with an inductive secondary circuit it decreases also the phase angle β. Since it is impossible to construct a transformer which has not at least the secondary reactance due to leakage flux, it must follow that in a commercial transformer core loss has both of the above effects. If these were its only effects it might be thought, ignoring efficiency, that core loss is desirable. But it will be seen later that hysteresis loss is accompanied by a variation in core reluctance that is extremely undesirable so far as regulation is concerned.

5. *Formulas for Current Ratio and Phase Angle—Use of Complex Quantities.*—With these vector diagrams in mind we may proceed with a mathematical discussion of the current transformer by the use of complex quantities. In the following derivation it will be considered for simplicity, that permeability of the iron is constant. The error thus introduced is not so great as might at first be expected, due to the fact that the iron in a current transformer is generally worked at a very low density on the straighter portion of the saturation curve. However, the assumption of constant permeability still makes it possible to discuss from the resulting equations the effects of variation in permeability. If the permeability is taken constant the hysteresis loop vanishes, and the only remaining core loss is the eddy current loss. With sufficiently careful lamination and at the low value of flux density used, this loss may be very small. Indeed in a commercial current transformer the core losses are very small. Neglecting then for the present all core losses, the following derivation will correspond to the vector diagram of Fig. 2 and will be rigid for a transformer with an air core. Let a sine wave of e.m.f. e instantaneous value $= e\sqrt{2} \sin \omega t$, be impressed upon the circuit as indicated in Fig. 1. Then since the effect of the secondary current upon the primary current is neglected

$$I' = \frac{e}{R - jX} = i + ji_1 \text{ * } \qquad \text{(1)}$$

where $R—jX$ is the load impedance.

where $i = \frac{eR}{R^2 + X^2}$ and $i_1 = \frac{eX}{R^2 + X^2}$

The flux set up by the primary current or the total primary flux is proportional to the primary current and in phase with it or

$$\Phi' = K_1 N_1 I' \times 10^8 \qquad \text{(2)}$$

where $K_1 = \frac{.4\pi}{\rho_1} \times 10^{-8}$; ($\rho_1$ = combined reluctance of iron circuit

*In all these following equations involving complex quantity, time is considered the independent variable and is represented by counter clock-wise rotation. The inductive impedance is $R - jX$ and thus there is a lagging current $I = i + ji_1$.

and leakage path), and N_1 is the number of primary turns. Of $\dot{\Phi}'$ a certain part known as leakage flux does not thread the secondary, but produces a reactive e. m. f. in the primary, which is given by the following expression:

$$\dot{e}_r' = j \frac{2\pi f N_1 \dot{\Phi}' L}{10^8} \qquad \text{.....................(3)}$$

It should perhaps be pointed out that the value given above for electromotive force, current, and flux, as well as those to follow, are all effective values.

Equation (3) gives an expression for reactive primary e.m.f. in terms of leakage flux. But the primary leakage reactance x_1 also gives the value of this e.m.f., as follows:

$$\dot{e}_r' = j \dot{I}' x_1 \qquad \text{.........................(4)}$$

Combining (3) and (4) we obtain

$$\frac{2 \pi f N_1 \dot{\Phi}_L'}{10^8} = \dot{I}' x_1$$

or

$$\Phi_L' = \frac{\dot{I}' x_1}{2 \pi f N_1} 10^8 = \frac{\dot{I}' x_1}{\omega N_1} 10^8 \qquad \text{..............(5)}$$

An expression for the magnetizing flux $\dot{\Phi}_M$ may be obtained by starting with the induced secondary e.m.f. This e.m.f. must be sufficient to overcome all of the impedance of the secondary circuit including the leakage x_2. Hence, we may write

$$\dot{E}'' = \dot{I}'' \dot{Z}'' \qquad \text{.......................(6)}$$

But the induced secondary e.m.f. is dependent upon the magnetizing flux in the following relation:

$$\dot{E}'' = j \frac{2 \pi f N_2 \dot{\Phi}_M}{10^8} \qquad \text{...................(7)}$$

Combining (6) and (7) an expression for magnetizing flux in terms of secondary current and impedance is obtained as follows:

$$j\,\frac{2\,\pi f\,N_2\,\Phi_M}{10^8} = I''\,Z''$$

Whence,

$$\Phi_M = -\,j\,\frac{I''\;Z''}{\omega N_2}\,10^8 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8)$$

The total secondary flux is proportional to the secondary current and in phase with it, or

$$\Phi'' = K_2\,N_2\,I'' \times 10^8 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9)$$

where $K_2 = \frac{4\pi}{\rho_2} \times 10^{-8}$; ($\rho_2$ = combined reluctance of iron circuit and leakage path).

In a manner similar to that for obtaining equation (5) for the primary leakage flux, the following expression for the secondary leakage flux is obtained,

$$\Phi_L'' = \frac{I''\,x_2}{\omega N_2}\,10^8 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(10)$$

If we subtract from the total primary flux the primary leakage flux we obtain what may be called the useful primary flux. Similarly the difference between the total secondary flux and the secondary leakage flux gives the useful secondary flux. The sum of these two useful fluxes then gives the magnetizing flux, or expressed in the form of an equation,

$$\Phi' - \Phi_L' + \Phi'' - \Phi_L'' = \Phi_M \quad\ldots\ldots\ldots\ldots\ldots\ldots(11)$$

which may be written

$$\Phi' - \Phi_L' + \Phi'' - \Phi_L'' - \Phi_M = 0 \quad\ldots\ldots\ldots\ldots\ldots(12)$$

Substituting in (12) the expressions for the various fluxes given by equations (2), (5), (9), (10), and (8), and dividing through by 10^8, we obtain

$$K_1\,N_1\,I' - \frac{I'\;x_1}{\omega N_1} + K_2\;N_2\;I'' - \frac{I''\;x_2}{\omega N_2} + j\,\frac{I''\;Z''}{\omega N_2} = 0 \;..(13)$$

or $$\left(K_1 N_1 - \frac{x_1}{\omega N_1}\right) I' + \left(K_2 N_2 - \frac{x_2}{\omega N_2} + j\frac{Z''}{\omega N_2}\right) I'' = 0 \quad \ldots (13')$$

Whence, $$I'' = -\frac{(K_1 N_1^2 \omega - x_1)\,\omega N_2}{\omega N_1 (K_2 N_2^2 \omega - x_2 + j Z'')} I' \quad \ldots\ldots\ldots (14)$$

But $$Z'' = (r_2 + r) - j(x_2 + x)$$

And $$j Z'' = (x_2 + x) + j(r_2 + r) \quad \ldots\ldots\ldots (15)$$

Substituting (15) in (14),

$$I'' = -\frac{(K_1 N_1^2 \omega - x_1) N_2}{N_1 [K_2 N_2^2 \omega + x + j(r_2 + r)]} I' \quad \ldots\ldots\ldots (16)$$

which might better be written

$$I'' = -\frac{N_2}{N_1} \frac{(K_1 N_1^2 \omega - x_1)}{(K_2 N_2^2 \omega + x)^2 + (r_2 + r)^2} [K_2 N_2^2 \omega + x - j(r_2 + r)] I' \quad (17)$$

The ratio of secondary current to primary current then becomes

$$\frac{I''}{I'} = -\frac{N_2}{N_1} \frac{(K_1 N_1^2 \omega - x_1)}{\sqrt{(K_2 N_2^2 \omega + x)^2 + (r_2 + r)^2}} \quad \ldots\ldots (18)$$

Equation (18) may be reduced as follows: K_1 and K_2 represent the primary and secondary magnetic conductances respectively. If $F = \frac{0.4\pi}{\rho} \times 10^{-8}$ represents the conductance of the iron path, and k_1 and k_2 represent the magnetic conductances of primary and secondary leakage paths respectively, then for K_1 and K_2 in equation (18) may be substituted:

$$K_1 = F + k_1$$

$$K_2 = F + k_2$$

$$\frac{I''}{I'} = -\frac{N_2}{N_1} \frac{(F N_1^2 \omega + k_1 N_1^2 \omega - x_1)}{\sqrt{(F N_2^2 \omega + k_2 N_2^2 \omega + x)^2 + (r_2 + r)^2}}$$

$$k_1 N_1^2 \omega = x_1$$

$$k_2 N_2^2 \omega = x_2$$

$$\frac{I''}{I'} = -\frac{N_2}{N} \frac{F N_1^2 \omega}{\sqrt{(F N_2^2 \omega + x_2 + x)^2 + (r_2 + r)^2}} \ldots . (19)$$

which shows the dependence of current ratio upon the primary leakage reactance. To show more clearly the relation between the currents and the number of turns equation (19) may be reduced to the form:

$$\frac{I''}{I'} = -\frac{N_1}{N_2} \frac{F \omega}{\sqrt{(F\omega + \frac{x_2 + x}{N_2^2})^2 + (\frac{r_2 + r}{N_2})^2}} \ldots . (20)$$

from which it is seen that with no secondary resistance or reactance the ratio of currents is inversely as the ratio of the number of turns. Furthermore, the lower the reluctance of the iron circuit, and consequently the greater the value of F, the higher the frequency, and the greater the number of secondary turns, the more nearly does the transformation ratio equal the ratio of the number of turns.

Equation (20) is useful in showing certain relations, but equation (18) may be reduced in another way which yields results of interest. From equation (2)

$$\Phi' = K_1 N_1 I' \times 10^8$$

and $N_1 \Phi' = K_1 N_1^2 I' \times 10^8$

But $N_1 \Phi' = L_1 I' \times 10^8$

Whence $K_1 N_1^2 = L_1$

and $K_1 N_1^2 \omega = L_1 \omega = X_1 \ldots\ldots\ldots\ldots\ldots . (21)$

Where X_1 is the total inductive reactance of the primary coil.

Similarly, $K_2 N_2^2 \omega = X_2 \ldots\ldots\ldots\ldots\ldots . (22)$

If the primary leakage reactance x_1 be subtracted from the total primary reactance X_1, and the difference multiplied by the ratio of the secondary turns to the primary turns the result will be the mutual inductive reactance X_M or

$$\frac{N_2}{N_1}(X_1 - x_1) = X_M \qquad \text{(23)}$$

Now substituting (21), (22) and (23), in (18), the following expression is obtained,

$$\frac{I''}{I'} = -\frac{X_M}{\sqrt{(X_2 + x)^2 + (r_2 + r)_2}} \qquad \text{(24)}$$

From (24) it is seen that with zero resistance of the secondary coil and load, the ratio of transformation is as the ratio of the mutual inductive reactance to the total reactance of the secondary circuit.

From equation (20) it is seen that the effect of increasing the secondary resistance or reactance is to decrease the secondary current; that the effect of increasing the frequency or increasing F, which means decreasing the magnetic reluctance, is to increase the secondary current. Equation (20) further shows that a large number of secondary turns makes variations in frequency, permeability, and secondary resistance and reactance less effective in influencing the transformation ratio, and hence points to the desirability of a large number of secondary turns.

Referring again to equation (17) it is seen that the phase angle between the secondary current and the projected primary current is

$$\beta = \text{arc tan}\,\frac{r_2 + r}{K_2 N_2^2\,\omega + x} = \text{arc tan}\,\frac{r_2 + r}{FN_2^2\,\omega + x_2 + x} \qquad \text{(25)}$$

$$\text{or} \quad \beta = \text{arc tan}\,\frac{r_2 + r}{X_2 + x} \qquad \text{(25')}$$

Stating equation (25′) in words the tangent of the phase angles is directly proportional to the total secondary resistance, and inversely proportional to the total secondary reactance.

If it is desired to study the current ratio from the design constants of a transformer, equation (18) will be found the more convenient; if it is desired to study this ratio from data taken experimentally, equation (24) may perhaps be used more conveniently. The equations for phase angles are so simple that there would be no trouble in using either one or the other.

As an example a transformer having the following constants may be considered:

$$r_1 = .0058 \text{ ohms.}$$

$$x_1 = .058 \text{ ohms at } 60\sim$$

$$K = .000015$$

$$F = .00001425$$

$$N_1 = 14$$

$$N_2 = 133$$

$$r_2 = 1 \text{ ohm}$$

$$x_1 = K N_1^2 \omega = 1.11 \text{ ohms at } 60\sim$$

$$x_2 = K N_2^2 \omega = 100 \text{ ohms at } 60\sim$$

$$\frac{N_2}{N_1} = 9.5$$

$$X_M = \frac{N_2}{N_1}(X_1 - x_1) = 10 \text{ ohms at } 60\sim$$

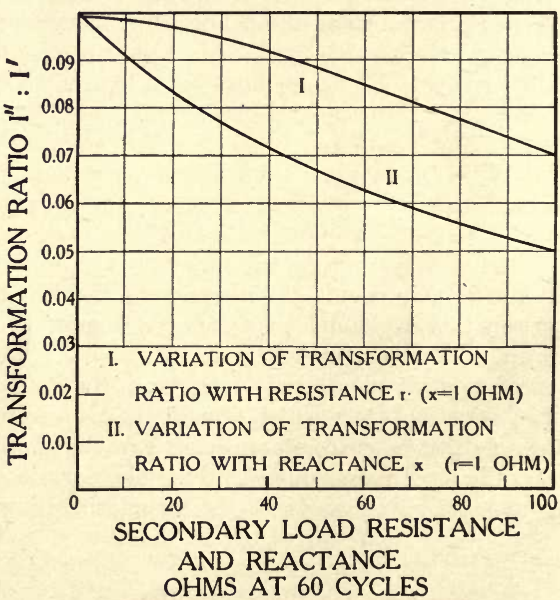

FIG. 4. EFFECT OF SECONDARY LOAD RESISTANCE AND REACTANCE ON TRANSFORMATION RATIO

Taking the secondary load reactance equal to 1 ohm, we may study the effect of secondary load resistance upon the transformation ratio. Fig. 4 Curve I. shows the relation between the transformation ratio and secondary load resistance. It is seen that for reasonable values of resistance (say up to 5 ohms) the ratio is but little affected. Beyond 10 ohms, however, an increase in the resistance results in quite an appreciable decrease in the value of secondary current, and the rate of decrease becomes greater with increased resistance. If the secondary load resistance be taken as 1 ohm, the effect of secondary load reactance upon the transformation ratio may be studied. Fig. 4 Curve II. shows the variation of the transformation ratio with secondary load reactance. It is seen that the ratio is very appreciably decreased by increasing the reactance and that the ratio of decrease is greatest for low values of reactance. Comparing then secondary resistance and reactance as regards their effect upon the transformation ratio, it may be said that for reasonable values the effect of increasing the former is to decrease but negligibly the ratio, whereas an increase in the latter results in a very considerable decrease of the ratio.

Taking the secondary load reactance equal to 1 ohm at 60 cycles, and choosing three values of secondary load resistance, viz., 1 ohm, 10 ohms, and 50 ohms, three curves were plotted, one for each value of resistance, showing the effect of varying the frequency upon the transformation ratio. These curves are given in Fig. 5. An inspection of these curves shows that for high values of frequency the effect of variation in frequency upon the transformation ratio becomes negligible, and also that the lower the secondary resistance, the lower the frequency at which the ratio becomes practically constant. With 1 ohm secondary load resistance it is seen that the ratio is practically constant above 25 cycles. With reasonably low secondary resistance then it may be concluded that the effect of frequency variation, within commercial limits, upon the transformation ratio is quite small. It should not be forgotten, however, that below a certain critical frequency for a given value of secondary resistance decreasing the frequency will result in a very rapid decrease of the transformation ratio.

The effect of secondary load resistance on the phase angle β will be considered next. Taking the secondary load reactance x equal to 1, the curve shown in Fig. 6, was plotted showing the variation of phase angle with the secondary load resistance. It is seen that the increase in phase angle is practically proportional to the increase in resistance for reasonable values of r. By taking the secondary load resistance equal to unity, the variation of phase angle with secondary load reactance may be studied. Fig. 7 shows the relation between these two quantities. It is noted that increasing the secondary reactance decreases the phase angle, the ratio of decrease being almost constant for reasonable values of x. The effect of x in decreasing the phase angle, however, is not nearly as great as the effect of r in increasing it. As shown by the curves, a given change in x changes the phase angle by less than 2% of that caused by an equivalent change in r.

By taking secondary load resistance equal to 1 ohm, and secondary load reactance equal to 1 ohm at 60 cycles, Fig. 8 was drawn showing the effect of frequency on the magnitude of the phase angle. This curve shows that the phase angle is very sensitive to changes in frequency within the range of commercial frequencies, the phase angle at 25 cycles being more than twice as great as at 60 cycles and over five times as great as at 133 cycles.

6. *Formulas for Current Ratio and Phase Angle of Series Transformers with Iron Cores.*—Throughout the above discussion zero core loss and constant magnetic reluctance have been assumed. In other words the above discussion applies rigidly to a transformer with an air core. The commercial series transformer, however, has an iron core. As was pointed out previously, due to the low flux density in the core and careful lamination, the core loss in such a transformer is very small; so small in fact, as to be of no consequence in affecting the validity of the foregoing discussion. In using an iron core, however, there is an effect more objectionable than the hysteretic core loss, viz., the accompanying variation in core reluctance.

An examination of equation (19) shows that with given constants and frequency, the ratio of secondary current to primary current is constant, independent of the value of primary current. This is true of an air core transformer where the permeability of the core, and consequently the factor F, are constant. But in a transformer with an iron core the permeability and consequently the factor F varies with the flux density. Since the flux density varies with the primary current, it must follow that the factor F is a function of the primary current, and consequently it is not to be expected that the ratio of secondary current to primary current will be constant for all values of primary current in a transformer with an iron core. In order to determine the nature of the variation in F and the consequent variation in the current ratio, it will be necessary to consider that portion of the saturation curve around which the transformer operates. In Fig. 9 Curve I. is given an assumed saturation curve. The scales of the co-ordinates are not given because they are not important for our purpose. The essential thing that concerns us is the shape of the saturation curve.

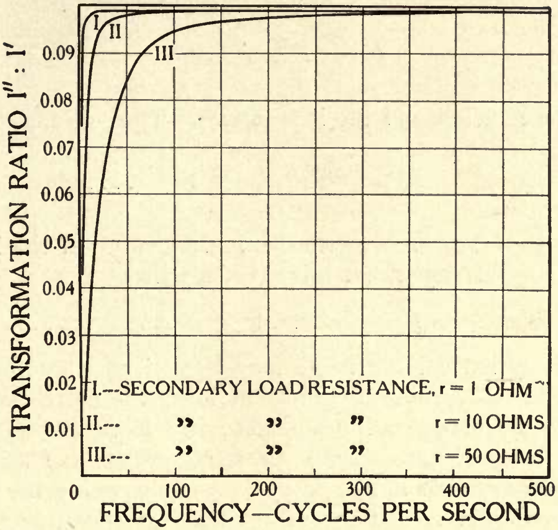

FIG. 5. EFFECT OF FREQUENCY ON TRANSFORMATION RATIO

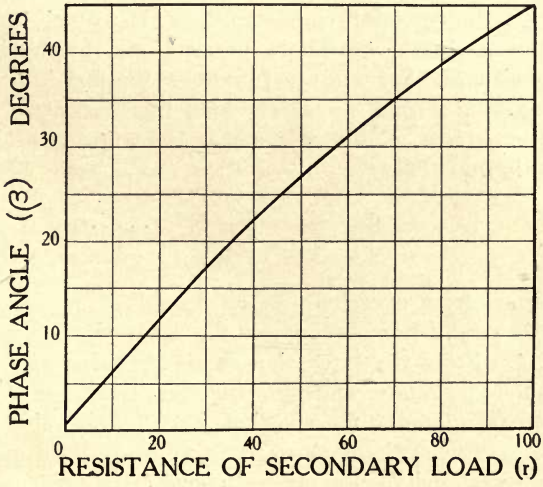

FIG. 6. EFFECT OF SECONDARY LOAD RESISTANCE OF PHASE ANGLE

It will be remembered that the factor F is given by the expression:

$$F = \frac{.4\pi}{\rho} \times 10^{-8}$$

where ρ is the reluctance of the iron circuit. Then we may write

$$F = \frac{.4\pi\, a\, \mu}{l} \times 10^{-8} \qquad \text{(26)}$$

where a is the average cross section and l the length of the iron circuit. Again for a given transformer, this may be written

$$F = \text{Constant} \times \mu \qquad \text{(27)}$$

From the saturation curve a curve may be derived whose ordinates are proportional to the permeability μ by dividing the ordinates of the saturation curve by corresponding abscissas, and plotting the quotients as ordinates of the new curve. From (27) the ordinates of this new curve will also be proportional to F, and by properly choosing the scale it will be possible to so plot it that the value of F corresponding to any point on the saturation curve may be directly read from it. It will be seen how this may be done.

The point of operation on the saturation curve depends upon the voltage that must be induced in the secondary to send the secondary current through the secondary impedance. This voltage is of course directly proportional to the secondary current, and to the impedance of the secondary circuit. For a given frequency the flux is proportional to the voltage. If the normal secondary load then is known and the full load secondary current, the flux for this condition may be computed from the relation.

$$\Phi_M = \frac{I'' Z''}{\omega N_2} 10^8 \qquad \text{(28)}$$

which is obtained from equation (8) by dropping the complex quantity symbols. This establishes a point on the saturation curve, as denoted by A in Fig. 4. From the saturation curve the value of permeability μ at this point may be found, and knowing the dimension of the iron circuit F may be calculated from equation (26). This value of F is then laid off to scale as an ordinate through point A, and the other ordinates are plotted proportionally, as explained above from equation (27). This gives Curve II of Fig. 9. As an example, let the normal secondary load for the transformer under consideration be $r = 1$ ohm and $x = 1$ ohm, and let the value given for F apply to the point of normal full load

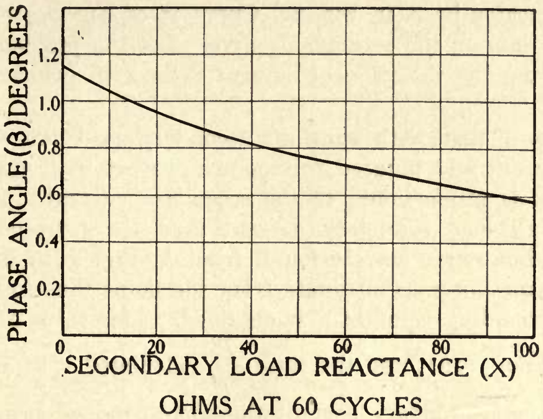

FIG. 7. EFFECT OF SECONDARY LOAD REACTANCE ON PHASE ANGLE

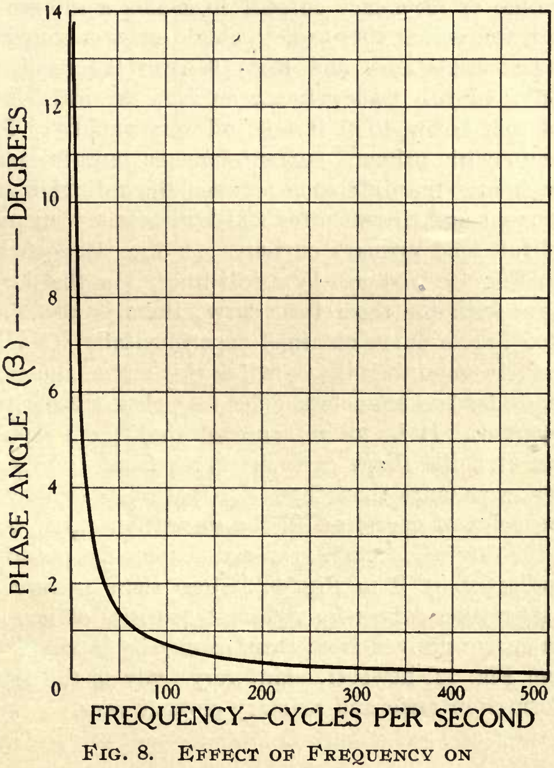

FIG. 8. EFFECT OF FREQUENCY ON PHASE ANGLE

operation indicated by A in Fig. 9. Curve II of Fig. 9 gives the values F at other points on the saturation curve. Let the problem of plotting a curve showing the variation of current ratio with primary current be considered.

It is known that with constant secondary load the variations in secondary current and induced voltage are proportional, and that therefore, the flux is proportional to the secondary current. For any percentage of full load secondary current then the corresponding point on the saturation curve may be found from A, Fig. 9 by direct proportion. Going up along the ordinate from the point thus found to Curve II, the corresponding value of F is obtained. This value of F is substituted in equation (19), and the ratio $\frac{I''}{I'}$ calculated. From this ratio and the value of secondary current the corresponding primary current in per cent of full load primary current may be determined, and thus a point on the required curve established. This is done for a sufficient number of values of secondary current to enable a smooth curve to be drawn through the points obtained. Table 1 gives a convenient tabulation for these computations, and Fig. 10 gives the result. It is seen that above 60% of full load primary current the ratio is very nearly constant, and that below 40% it falls off very rapidly. The variation in current ratio with primary current may be more clearly shown by plotting as ordinates the difference between the ratio for a given value of primary current and the ratio for full load primary current expressed in percent of full load primary current. In Fig. 11 such a curve, corresponding to Fig. 10, is shown by a full line. For the purpose of general comparison with the theoretical curve, there is also shown in Fig. 11, by a dotted line, a curve obtained experimentally. No unusual precision was used in obtaining the data for this curve, the currents being measured simply by two ammeters. The very close similarity of the two curves is apparent. It is to be expected that the magnitude of the ordinates, and even the shape of such curves from different transformers will differ even more than these do, inasmuch as they depend not only upon the point of operation on the saturation curve, but also upon the shape of the saturation curve, especially upon the prominence of the first bend, indicated by B in Fig. 9. Since there is very considerable variation in saturation curves for different qualities of iron, close agreement between curves derived from them cannot be expected. The curves in Fig. 10 and Fig. 11, however, point very truly to the general way in which transformation ratio and primary current vary.

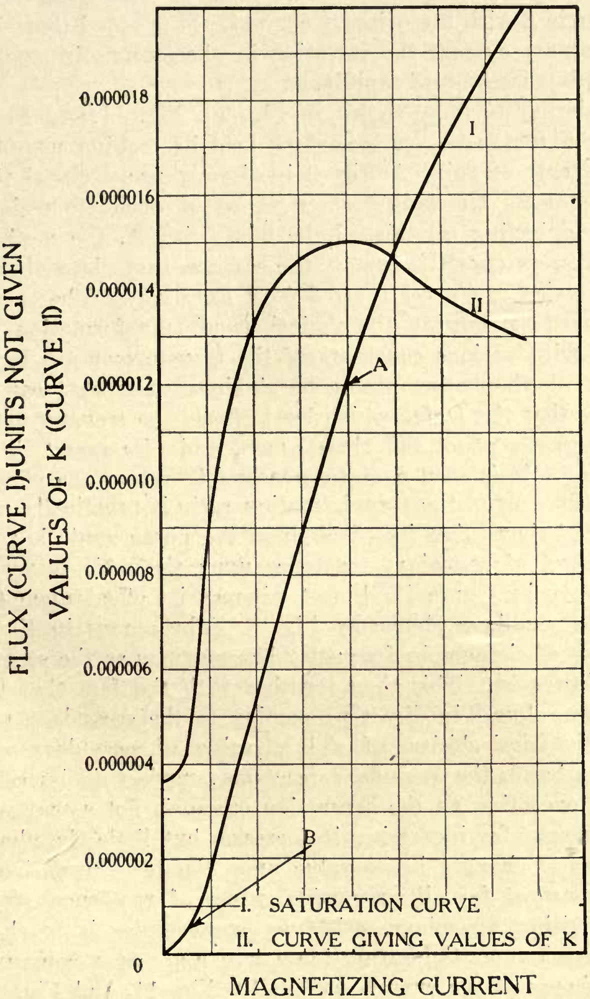

FIG. 9. SATURATION CURVE AND CURVE GIVING VALUES OF K

With values of F already determined for various values of primary current, it is an easy matter to substitute these in equation (25), and find the corresponding phase angles. Fig. 12 shows the variation of the phase angle β with the primary current. It is noted that above 60% full load primary current the variation in phase angle in small, but below 40% it increases quite rapidly.

Curves similar to those shown in Fig. 10, Fig. 11, and Fig. 12 may be determined for any other secondary load by making use of the fact that the ordinate of point A Fig. 9 is directly proportional to the secondary impedance. Knowing the new values of secondary resistance and reactance, and having established the new point A, the method of determining the new curves is exactly the same as that given above.

7. *Discussion of Curves Plotted from Formulas.*—The curves which have been plotted, showing the variations of transformation ratio and phase angle with various constants of the transformer are largely self-explanatory. Perhaps the most general observation that can be made from them is that the factor which least affects the transformation ratio is likely to greatly affect the phase angle, and *vice versa.* Thus from Curve I. Fig. 4 it is seen that for reasonable values the effect of secondary resistance upon the transformation ratio is practically negligible, whereas Fig. 6 shows that its effect upon the phase angle is very great. Again the effect of secondary reactance upon the ratio is very considerable as shown by Curve II. Fig. 4 whereas its effect upon the phase angle is quite small, as shown by Fig. 7. The curves in Fig. 5 show that the effect of changes in frequency becomes less as the secondary resistance is decreased. This then together with the fact that the phase angle increases almost in direct proportion to the secondary resistance points to the desirability of a low value of secondary resistance. On the other hand, the secondary reactance, neglecting its effect upon the point of operation on the saturation curve, is not a matter of such great consequence, for increasing it decreases but little the phase angle, and the *effect* of changes in its value upon the transformation ratio is practically *constant* for all reasonable values of reactance. The effect of changing either secondary resistance or reactance is to change the point of operation on the saturation curve and, as a consequence, to change the shapes of the curves in Fig. 10, Fig. 11, and Fig. 12. The desirability of using iron having as nearly constant permeability at low densities as possible is made evident by Fig. 10, Fig. 11, and Fig. 12. These figures also indicate that the reliable working range lies above 50 or 60 per cent full load current.

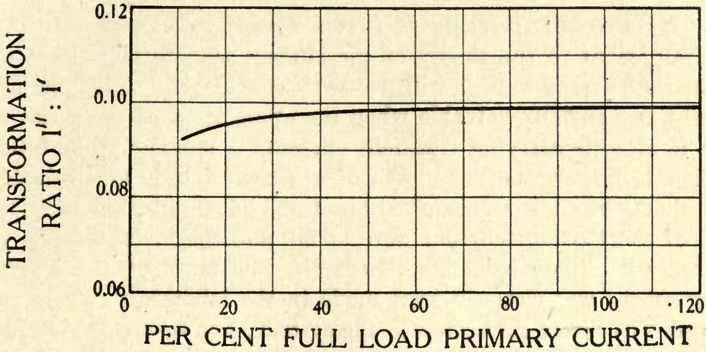

FIG. 10. VARIATION OF TRANSFORMATION RATIO WITH STRENGTH OF PRIMARY CURRENT

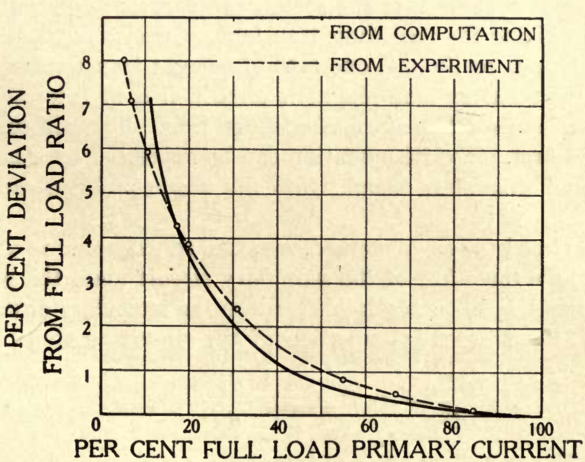

FIG. 11. DEVIATION FROM NORMAL TRANSFORMATION RATIO WITH STRENGTH OF PRIMARY CURRENT

III. THEORY OF THE SERIES TRANSFORMER BASED ON INSTANTANEOUS CURRENT VALUES

8. *Symmetrical Currents in Series Transformer with Air Core.*— The development of the theory of the current transformer based on instantaneous current values will be next considered. This method of treatment is especially valuable when the application of the series transformer to circuits carrying transient currents is studied. The following treatment is largely the outgrowth of a series of tests and calculations on the short-circuit currents of alternators. A number of oscillograph records of short-circuit current were taken, and the instantaneous values compared with those calculated from the constants of the alternator. The agreement was close in tests made in the laboratory where the oscillograph was direct connected to the system; but very considerable discrepancies were found in some tests made outside of the laboratory, where the oscillograph was connected in the secondary circuit of a series transformer. Since the theory of the short circuit currents of alternators would not explain the peculiarites in the latter curves, a study of the action of series transformers on unsymmetrical or transient currents was undertaken in an attempt to explain the phenomena. The results of this investigation show that the series transformer (especially with iron core) is unreliable for recording transient or unsymmetrical currents.

While there is really only one type of current transformer in practical use, which type has an iron magnetic circuit, it is of interest to consider the action of such a transformer without iron. The air core type will be discussed first, since its operation can be expressed by a mathematical equation, and thus clear insight into the whole problem can readily be given.

As has already been explained, a series transformer is a mutual inductance where the effect of the secondary circuit upon the primary current is so small as to be negligible. Since the secondary circuit is closed upon itself, the differential equation of the circuit becomes,

$$X_2 \frac{di''}{d\theta} + r_2 i'' + X_M \frac{di'}{d\theta} = e'' = 0 \qquad (29)$$

Where r_2, X_2, and i'' are the resistance, reactance, and current in the secondary circuit; X_M is the mutual inductive reactance between the primary and secondary circuits; and i' is the primary current. As stated above i' is not affected by i'' and therefore $\frac{di'}{d\theta}$ is not a function of i'', and the single differential equation is sufficient for the solution of the problem.

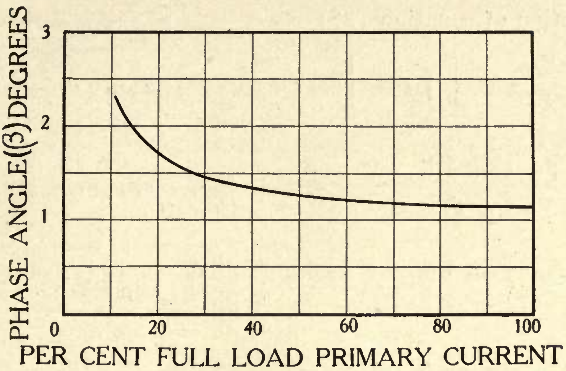

FIG. 12. VARIATION IN PHASE ANGLE WITH STRENGTH OF PRIMARY CURRENT

If the primary current i' contains a transient term, as the starting alternating current in an inductive circuit,

$$i' = A \left[\sin (\theta - s) - \sin (\theta_1 - s) \varepsilon^{-a\theta'} \right] \ldots\ldots\ldots (30)$$

Where A = the maximum value of voltage across the primary divided by the primary impedance $= \frac{E'}{Z'}$, $a = \frac{r_1}{x_1}$, $s = \text{arc tan} \frac{I}{a}$, θ_1 is the angular displacement of the voltage from its zero position when the circuit is closed, and θ' is the time in electrical radians counted from the time of closing the circuit, or $\theta' = (\theta - \theta_1)$. Differentiating equation (30)

$$\frac{di'}{d\theta} = A \left[\cos (\theta - s) + a \sin (\theta - s) \varepsilon^{-a\theta'} \right] \ldots\ldots (31)$$

Equation (31) substituted in equation (29) gives

$$X_2 \frac{di''}{d\theta} + r_2 i'' = - X_M A \left[\cos (\theta - s) + a \sin (\theta_1 - s) \varepsilon^{-a\theta'} \right] (32)$$

Equation (32) is easily written in the familiar form

$$\frac{di''}{d\theta} + bi'' = - \frac{X_M A}{X_Z} \left[\cos (\theta - s) + a \sin (\theta_1 - s) \varepsilon^{-a\theta'} \right] (33)$$

Where the factor $\frac{r_2}{X_2}$ is replaced by b

The solution of equation (33) is

$$i'' = -A\frac{X_M}{X_2}\varepsilon^{-b\theta}\left\{\int \varepsilon^{b\theta}\left[\cos(\theta - s) + a\sin(\theta_1 - s)\,\varepsilon^{-a\theta'}\right]d\theta + C'\right\}$$

$$= -A\frac{X_M}{X_2}\left\{\frac{\sin(\theta - s + \beta)}{\sqrt{b^2+1}} + \frac{a}{(b-a)}\sin(\theta_1 - s)\,\varepsilon^{-a\theta'} + C'\varepsilon^{-b\theta}\right\}$$

$$= -A\frac{X_M}{X_2}\left\{\frac{\sin(\theta - s + \beta)}{\sqrt{b^2+1}} + \frac{a}{(b-a)}\sin(\theta_1 - s)\,\varepsilon^{-a\theta'} + C\varepsilon^{-b\theta'}\right\} \quad (34)$$

Where $\beta = \text{arc tan } b$

When $\theta = \theta_1$, that is when $\theta' = 0$,

$i' = 0$ and $i'' = 0$

Therefore, solving equation (34) for C

$$C = -\left[\frac{\sin(\theta_1 - s + \beta)}{\sqrt{b^2+1}} + \frac{a}{(b-a)}\sin(\theta_1 - s)\right] \ldots (35)$$

$$= -\frac{b}{(b-a)}\sqrt{\frac{(1+a^2)}{(1+b^2)}}\sin(\theta_1 - s + \rho) \ldots\ldots\ldots (36)$$

Where $\rho = \text{arc tan}\dfrac{b-a}{1+ab}$ ....(36′

Knowing the instantaneous values of i' and i'' and the ratio of the secondary to primary turns $\dfrac{N_2}{N_1}$, the instantaneous value of magnetizing current i is given by the following equation,

$$i = i' + \frac{N_2}{N_1}i'' \ldots\ldots\ldots\ldots\ldots\ldots (37)$$

Substituting equation (34) and (30) in (37)

$$i = A\left\{\sin(\theta - s) - \sin(\theta_1 - s)\,\varepsilon^{-a\theta'} - \right.$$

$$\frac{X_M N_2}{X_2 N_1}\left[\frac{\sin(\theta - s + \beta)}{\sqrt{b^2+1}} + \frac{a}{b-a}\sin(\theta_1 - s)\,\varepsilon^{-a\theta'} + C\varepsilon^{-b\theta'}\right]\Bigg\} \quad (38)$$

Representing $\dfrac{X_M N_2}{X_2 N_1}$ by the constant K and replacing and combining the trigonometric functions into one, the equation becomes

$$i = A\left\{\sqrt{\left(1 - \frac{2k-k^2}{b^2+1}\right)}\sin(\theta - s - u) - \right.$$

$$\left.\left(1 + \frac{ka}{b-a}\right)\sin(\theta_1 - s)\,\varepsilon^{-a\theta'} - k\,C\,\varepsilon^{-b\theta'}\right\} \quad \ldots\ldots\ldots (39)$$

Where $u = \text{arc tan}\dfrac{kb}{(b^2+1-k)}$, which is the angle of lag of the stable magnetizing current behind the primary current.

Example 1. As an example of the application of this method the transformer for which data have already been given may be considered, with the additional data

$$A = 1$$

$$\theta_1 = 175^\circ$$

The following data, previously given, will be needed,

$$a = \frac{r_1}{x_1} = 0.1$$

$$b = \frac{r_2}{x_2} = 0.01$$

$$X_M = 10$$

$$X_2 = 100$$

$$\frac{N_2}{N_1} = 9.5$$

$$s = 85^\circ \text{ (calculated)}$$

Substituting these constants in equation (30)

$$i' = \sin(\theta - 85^\circ) - \varepsilon^{-.1\theta'} \quad \ldots\ldots\ldots\ldots\ldots (A')$$

ρ from equation (36′) is

$$\rho = \text{arc tan}\ \frac{b-a}{1+ab} = \text{arc tan}\ (-.09) = -5^\circ$$

The value of C from equation (36) becomes

$$C = .111$$

$$\beta = \text{arc tan}\ b = 35 \text{ minutes}$$

Hence, from equation (34)

$$i'' = -0.1\ [\sin(\theta - 84.4^\circ) - 1.11\ \varepsilon^{-.1\theta'} + .111\ \varepsilon^{-.01\theta'}] \ldots (B')$$

$$u = \text{arc tan}\ \frac{kb}{b^2 + 1 - k} = 10^\circ 45'$$

That is, the stable condition of the magnetizing current lags 10° 45′ behind the primary current.

i from equation (39) becomes

$$i = [.05 \sin(\theta - 95.75^\circ) + .055\ \varepsilon^{-.1\theta} - .1055\ \varepsilon^{-.01\theta}] \ldots (C')$$

In Fig. 13 equations (A′), (B′), and (C′) are plotted with θ as abscissas and i', i'', and i as ordinates. The ordinates derived from equation (B′) are multiplied by the transformation ratio $\frac{X_2}{X_M} = 10$ and turned 180° so as to be superimposed upon the primary current for better comparison. The curves thus plotted show clearly the error in secondary current due to the air core transformer with a 5% magnetizing current.

9. *Unsymmetrical Currents in Series Transformer with Air Core.*—As a second problem a series transformer traversed by an unsymmetrical primary current might be considered. If the current lies entirely above the zero line, the equation would be

$$i' = A\ [\sin\theta + 1 - C\ \varepsilon^{-a\theta'}] \ldots\ldots\ldots\ldots (40)$$

Where $C = \sin\theta_1 + 1$

Differentiating (40) and substituting in equation (29)

$$\frac{di''}{d\theta} + bi'' = -A\ \frac{X_M}{X_2}\ (\cos\theta + C\,a\ \varepsilon^{-a\theta'}) \ldots\ldots (41)$$

And the solution of equation (41) is

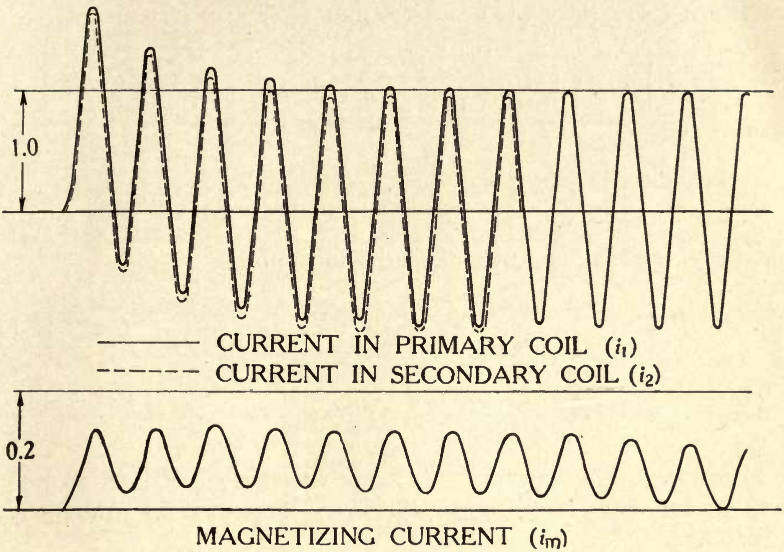

FIG. 13. CURRENT DIAGRAM FOR EXAMPLE 1

$$i'' = -A\frac{X_M}{X_2}\left[\frac{\sin(\theta+\beta)}{\sqrt{b^2+1}} + \frac{aC}{(b-a)}\varepsilon^{-a\theta'} + C'\varepsilon^{-b\theta'}\right] \quad (42)$$

Where β = arc tan b, and since when $\theta = \theta_1$, $\theta' = 0$ and $i'' = 0$

$$C' = -\left[\frac{\sin(\theta_1+\beta)}{\sqrt{b^2+1}} + \frac{aC}{(b-a)}\right]$$

If the time constant $\frac{1}{a}$ of the primary circuit is exceedingly small, then $\varepsilon^{-a\theta'}$ will practically be zero in a very short time; (a) will be large in comparison to (b) and equation (42) becomes, after an infinitesimal time.

$$i'' = -A\frac{X_M}{X_2}\left\{\frac{\sin(\theta+\beta)}{\sqrt{b^2+1}} - \left[\frac{\sin(\theta_1+\beta)}{\sqrt{b^2+1}} - \sin\theta_1 - 1\right]\varepsilon^{-b\theta'}\right\} (43)$$

And equation (40) becomes

$$i' = A[\sin\theta + 1] \; \ldots\ldots\ldots\ldots\ldots\ldots (44)$$

Substituting equations (44) and (43) in (37)

$$i = A\left[\sin\theta + 1 - \frac{k}{\sqrt{b^2+1}}\sin(\theta+\beta) + C\,\varepsilon^{-b\theta'}\right] \quad (45)$$

Where

$$C = k\left[\frac{\sin(\theta_1+\beta)}{\sqrt{b^2+1}} - \sin\theta_1 - 1\right] \quad (46)$$

Equation (45) may be further reduced to the form

$$i = A\left[\sqrt{1 - \frac{(2k-k^2)}{(b^2+1)}}\sin(\theta - s') + 1 + C\,\varepsilon^{-b\theta'}\right] \quad (47)$$

Where $s' = \arc\tan \dfrac{kb}{(b^2+1-k)}$

Example 2. Assume that the same transformer is used as in Example 1, but that the constant (a) is very large and the primary wave of the form,

$$i' = \sin\theta + 1 \quad (A'')$$

Then from equation (43)

$$i'' = -.1\,[\sin(\theta + 35') + \varepsilon^{-.01\theta'}] \quad (B'')$$

And from (47)

$$i = .05\sin(\theta - 10.75^\circ) - .95\,\varepsilon^{-.01\theta'} + 1 \quad (C'')$$

10. *Effect of Constants of Transformer on Instantaneous Current Values.*—The curves on Fig. 14, plotted from equations A″, B″, and C″ show clearly how the current in the secondary of the transformer gradually becomes symmetrical in reference to the zero line, while the magnetizing current creeps up to a line which is symmetrical in respect to the primary current.

The reactance of the secondary, X_2, as has previously been indicated, contains two factors, X'_2 and x_2, of which X'_2 represents that portion of the flux that interlinks with the primary coil, and x_2 represents that portion of the flux that does not interlink with the primary coil, and is known as leakage reactance. X'_2 and X_M hold a definite relation to each other, which ratio, is that of the secondary turns N_2 to the primary turns N_1. The transformation ratio of the transformer for zero secondary impedance is that ratio of the mutual inductive reactance X_M to the total secondary self inductive reactance X_2, which ratio can only be considered equal to the ratio of the respective turns in so far as x_2 can be neglected in comparison to X'_2. The resistance in the secondary cir-

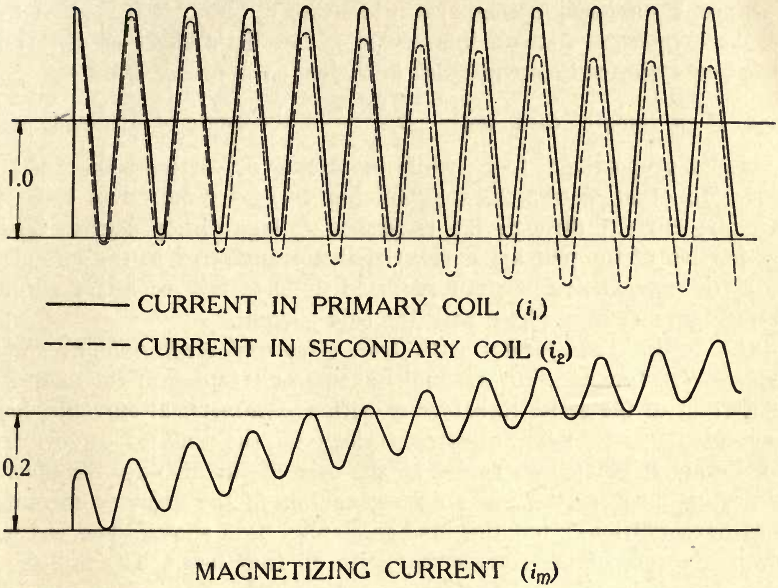

FIG. 14. CURRENT DIAGRAM FOR EXAMPLE 2

cuit has little effect on the transformation ratio, but does produce a displacement in phase in the stable condition, and introduces large errors in the transient term, since b is directly proportional to r_2. The smaller the factor b, the more closely the secondary current follows the primary current and therefore, for observing transient currents, the secondary leakage reactance improves the accuracy; while resistance, though small, introduces large errors. The value of x_2 is limited only in so far as its effect upon the primary current can be neglected. As increase of x_2 does increase the magnetizing current, but this increase is in phase with the primary current and therefore no displacement results, and the transformation ratio is only diminished. Therefore, a transformer designed to be used for transient phenomena should have an exceedingly large secondary reactance and a very small secondary resistance.

Under stable conditions, equation (34) becomes

$$i'' = -A \frac{X_M}{X_2} \frac{\sin(\theta - s + \beta)}{\sqrt{b^2 + 1}}$$

$$= -\frac{A X_M}{\sqrt{X_2^2 + r_2^2}} \sin(\theta - s + \beta) \qquad (48)$$

Since X'_2 is large in comparison to either r_2 *or* x_2, it is evident from the above equation that an increase of r_2 has little effect on the transformation ratio, while an increase in x_2 has an appreciable effect. Also, since β is small (arc tan $\dfrac{r_2}{X'_2 + x_2}$) the angle of displacement β is directly proportional to r_2, while increasing x_2 decreases the displacement. In other words, the magnetizing current required to force the secondary current through the resistance r_2 lags ninety degrees behind that portion of the primary current that is transferred to the secondary, while the magnetizing current required to force the secondary through the reactance x_2 is in phase with the said current.

11. *Series Transformer with Iron Core and Negligible Secondary Leakage Reactance.*—This gradual increase, or creeping of the magnetizing current of the series transformer with unsymmetrical currents, has a much greater and more disastrous effect in the case of an ordinary transformer with an iron core. In the case of the air core transformer the magnetizing current was a direct function of the primary current in all stable conditions; but this, as has already been shown, does not hold true in the case of the transformer with an iron core. The mutual inductive reactance X_M and the corresponding self-inductive reactance X'_2 depend, in exactly the same way, upon the reluctance of the core circuit, and consequently their ratio is a constant, with a value the same as that of the ratio of the number of primary and secondary turns. The leakage reactance x_2 is more or less independent of the permeability of the core, and consequently the transformation ratio $\dfrac{X_M}{X'_2 + x_2}$ is not a constant but depends upon the magnitude of the magnetizing current. Furthermore as X'_2 decreases it necessarily approaches the value of r_2 and the error due to the increasing factor b becomes very large.

It is unfortunate that the saturation curve of the iron cannot be represented by a mathematical equation and therefore the only solution of the problem is the tedious step by step method. Although this method obviously is not absolutely correct, it does give a close approximation. Consider first the problem where the secondary leakage reactance x_2 is negligible. From the terminal conditions and constants of the circuit, the primary current at any instant may be determined, as by equation (30). From equation (37),

$$i' - \frac{N_2}{N_1} i'' = i \qquad (49)$$

where the quantities are expressed arithmetically instead of algebraically.

The nature of what follows makes it more convenient to express the relation in this manner. Since the secondary leakage reactance is negligible, the change of flux need give only the e.m.f. necessary to force the secondary current through the resistance r_2, or

$$D \frac{d\phi}{d\theta} = i'' r_2 \quad \text{(50)}$$

where D is a proportionality constant. Equation (50) may be written

$$i = \frac{D}{r_2} \frac{d\phi}{d\theta} \quad \text{(51)}$$

Substituting (51) in (49)

$$i' - \frac{N_2}{N_1} \frac{D}{r_2} \frac{d\phi}{d\theta} = i \quad \text{(52)}$$

Changing from differential to difference, that is, replacing as approximation d by $\triangle$ gives

$$i' - \frac{N}{N_1} \frac{D}{r_2} \frac{\triangle\phi}{\triangle\theta} = i \quad \text{(53)}$$

Taking increments of 10° for $\triangle\ \theta$ gives

$$i' - \frac{N_2}{N_1} \frac{D}{r_2} 5.73 \triangle\phi = i$$

Whence $$\triangle\phi = \frac{N_1}{N_2} \frac{r_2}{D} .175 [i' - i] \quad \text{(54)}$$

If the remanent magnetism in the core of the transformer at the time the circuit is made be denoted by ϕ_0, the flux ϕ at any instant is given by the expression,

$$\phi = \phi_0 + \Sigma \triangle\phi \quad \text{(55)}$$

Substituting (54) in (55),

$$\phi = \phi_0 + \Sigma \frac{N_1}{N_2} \frac{r_2}{D} .175 [i' - i] \quad \text{(56)}$$

For convenience let r_2 have such a value that $.175 \frac{r_2}{D}$ becomes equal to unity. Then (56) becomes

$$\phi = \phi_0 + \Sigma \frac{N_1}{N_2} [i' - i] \quad \text{(57)}$$

As indicated above, the primary current may be determined from the conditions of the main circuit. Assuming a value for ϕ_0, we may take as a first approximation

$$\phi' = \phi_0 + \Sigma \frac{N_1}{N_2} i \ldots\ldots\ldots\ldots\ldots\ldots (58)$$

From the saturation curve a value of i corresponding to ϕ' is obtained, and substituted in equation (57). From (57) then a second approximate value of ϕ is obtained, which from the saturation curve gives a very close second approximate value of i. The following example may illustrate more fully the method of procedure.

Example 3. Assume the ratio of secondary to primary turns to be 10; the per cent of primary magnetizing current to be 1.2%; the primary current as given by equation (A′); a residual magnetism in the core of .5 units; and the saturation curve given in Fig. 15 to be that of the primary coil. The tabulation shown in Table 3 is a convenient form. Table 2 is a convenient tabulation for determining values of i'.

Column 2 of Table 3 is obtained from Table 2. The value of $\triangle \phi'$ in column 3 is equal to $\frac{N_1}{N_2}$ times the primary current i', and ϕ in column 4 is the sum of $\triangle \phi'$ in column 3 and the value of ϕ in column 8 of the preceding line. (i) in column 5 is found from the saturation curve for the value of flux corresponding to 4. From (i'—i) of column 6 the values of $\triangle \phi$ in column 7 are obtained and added to 8 of the preceding line. This gives the value of flux from which the final value of magnetizing current i is determined. From column 10, i'' is obtained. In Fig. 16 are plotted curves for the above case similar to those in Fig. 12. The much more disastrous effect of the magnetizing current in a transformer with an iron core is readily seen from a comparison of these two sets of curves.

12. *Series Transformer with Iron Core and Appreciable Secondary Leakage Reactance.*—If the reactance of the secondary must be considered the change of flux must give an e.m.f. sufficient to overcome the reactance drop as well as the resistance drop of the secondary, or

$$D \frac{d\phi}{d\theta} = r_2 i'' + x_2 \frac{di''}{d\theta} \ldots\ldots\ldots\ldots\ldots\ldots (59)$$

Example 4. Take the data of Example 3, but consider secondary leakage reactance. Changing from differentials to differences, and taking increments of $\theta = 10°$, we obtain

$$5\ 73\ D\triangle\phi = r_2 i'' + x_2\, 5.73\ \triangle i'' \ldots\ldots\ldots\ldots (60)$$

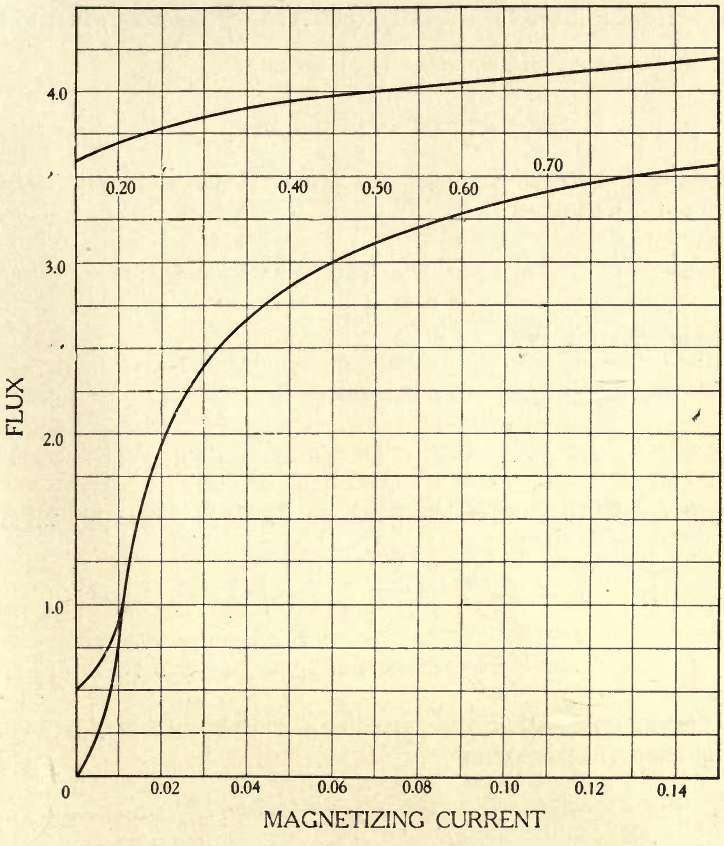

FIG. 15. SATURATION CURVE FOR EXAMPLE 3

And
$$i'' = \frac{5.73}{r_2}\,[D\triangle\phi - x_2\,\triangle i''] \ldots\ldots\ldots\ldots\ldots (61)$$

Substituting (61) in (49)

$$i' - \frac{N_2}{N_1}\,\frac{5.73}{r_2}\,[D\triangle\phi - x_2\,\triangle i''] = i$$

Which reduces to

$$\triangle\phi = \frac{.175 r_2}{D}\,\frac{N_1}{N_2}\,[i' - i] + \frac{x_2}{D}\,\triangle\, i'' \ldots\ldots\ldots (62)$$

Again, let r_2 have such a value that $.175 \frac{r_2}{D}$ becomes equal to unity. Then $D = .175\, r_2$, and equation (62) becomes

$$\triangle \phi = \frac{N_1}{N_2}[i' - i] + 5.73\, \frac{x_2}{r_2} \triangle i'' \quad \ldots\ldots\ldots\ldots (63)$$

Taking the secondary reactance equal to the secondary resistance, equation (63) becomes

$$\triangle \phi = \frac{N_1}{N_2}[i' - i] + 5.73 \triangle i'' \quad \ldots\ldots\ldots\ldots (64)$$

But from equation (49) $\triangle i'' = \frac{N_1}{N_2}[\triangle i' - \triangle i] \quad \ldots\ldots\ldots\ldots (65)$

Substituting (65) in (64) it becomes

$$\triangle \phi = \frac{N_1}{N_2}\left\{[i' - i] + 5.73\, [\triangle i' - \triangle i]\right\} \quad \ldots\ldots (66)$$

Now substituting equation (66) in equation (55), an expression for the flux is obtained

$$\phi = \phi_o + \Sigma \frac{N_1}{N_2}\left\{[i' - i] + 5.73\, [\triangle i' - \triangle i]\right\}$$

$$\phi = \phi_o + \Sigma \frac{N_1}{N_2}\left\{(i' + 5.73 \triangle i') - (i + 5.73 \triangle i)\right\} \ldots (67)$$

As a first approximation, the parenthesis containing i may be omitted, and equation (67) becomes

$$\phi = \phi_o + \Sigma \frac{N_1}{N_2}(i' + 5.73 \triangle i') \quad \ldots\ldots\ldots\ldots (68)$$

With this value of ϕ' an approximate value i is obtained from the saturation curve. By this value of i the parenthesis of equation (67) is then filled in, and a second approximate value of ϕ obtained, whence by means of the saturation curve, a second very close approximate value of i is obtained.

Table 4 is a convenient form for tabulation. Column 7 of Table 4 gives ϕ' as obtained by the first approximation, and column 8 gives the corresponding magnetizing current. Column 12 gives $\triangle\ \phi$ as obtained when using i, and column 13 gives the second approximation of ϕ. Column 14 gives the corresponding value of magnetizing current, and from 15 the value of secondary current may be obtained. ϕ' in column 7 is obtained by adding $\triangle\ \phi'$ of column 6 to ϕ of the previous line in column 13. In Fig. 17 are plotted curves from the above case similar to those shown in Fig. 13 and in Fig. 14.

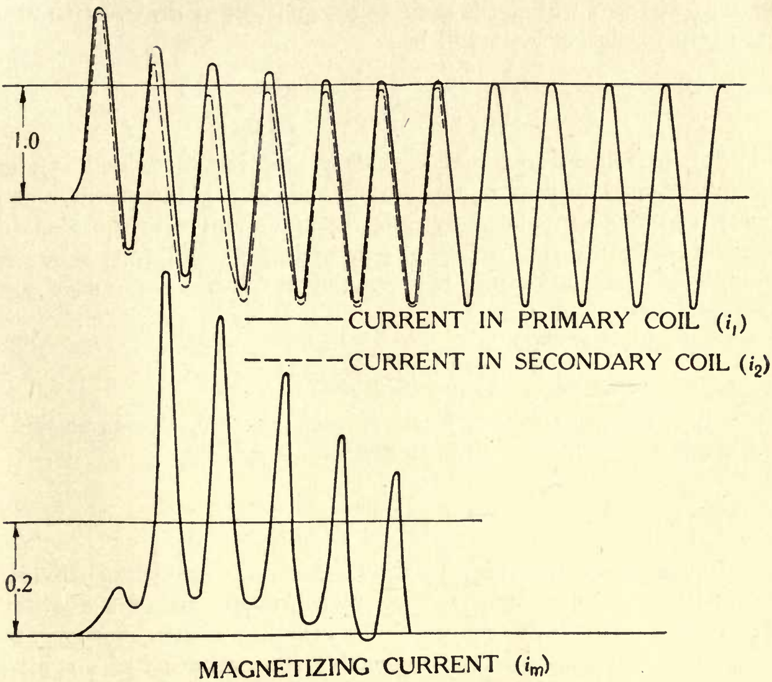

FIG. 16. CURRENT DIAGRAM FOR EXAMPLE 3

Fig. 16 and Fig. 17 (corresponding to examples 3 and 4), show clearly the errors introduced by the use of the series transformer with an iron core in recording transient phenomena. The "steadying" effect of secondary reactance is made evident by a comparison of the two curves.

The oscillograph record, shown in Fig. 18, is that of the current through the secondary of the series transformer together with the current through the primary, which is the starting current of an inductive circuit with an electrical time constant $(\frac{x}{r})$. of 10. These experimental curves are very similar to those as calculated from Examples 3 and 4 although the constants are somewhat different.

13. *Comparison of Methods of Computation for Series Transformers.*—The agreement between the two methods of solution, namely, the "complex quantity method" and the "differential equation method" may be pointed out briefly as follows. If transient terms be dropped, and the maximum value of equation (34) be divided by the maximum value of

equation (30) this will be the same as the quotient of the effective values, or the transformation ratio will be

$$\frac{i''}{i'} = -\frac{X_M}{X_2\sqrt{b^2+1}} = -\frac{X_M}{\sqrt{r_2^2+X_2^2}} \quad \ldots\ldots (69)$$

It will be remembered that secondary load resistance and reactance were taken equal to zero in the second method. If therefore, r and x in equation (24) be replaced by zero, the agreement with equation (69) is evident. Again a comparison of equations (34) and (30) shows that the angle of phase difference between the two currents i'' and i' under stable conditions is β and

$$\beta = \text{arc tan } b = \text{arc tan } \frac{r_2}{X_2}$$

If r and x in equation (25′) be replaced by zero, the agreement in phase angle as obtained by the two methods is evident.

IV. CONCLUSIONS

The conclusions that may be drawn from the preceding discussion in regard to the behavior of current transformers, are of a qualitative rather than a quantitative nature. The examples chosen, however, have been such as to give a fair and reasonable representation of magnitude, but the main purpose has been to derive and explain the general characteristics of the current transformer, and to point out some of its limitations.

To enumerate again in detail all the results of this investigation is thought unnecessary. An examination of the figures will show most of them. In conclusion a few of the most general and important results are given:

1. The transformation ratio and phase angle of a series transformer having a core of constant permeabilty (as air) are constant under given conditions for all values of primary current; but this is not so with a transformer having an iron core. With an iron-cored series transformer the form of variation depends upon the shape of the saturation curve, and upon the range over which the transformer operates. The range over which the *permeability* remains most nearly constant is the range over which the *ratio* remains most nearly constant. In a transformer of constant core reluctance the so-called magnetizing current is proportional to the primary current, and its phase position is constant.

2. The introduction of resistance in the secondary circuit of a series transformer has the effect of increasing the phase angle, and this increase in phase angle is practically proportional to the secondary resistance for

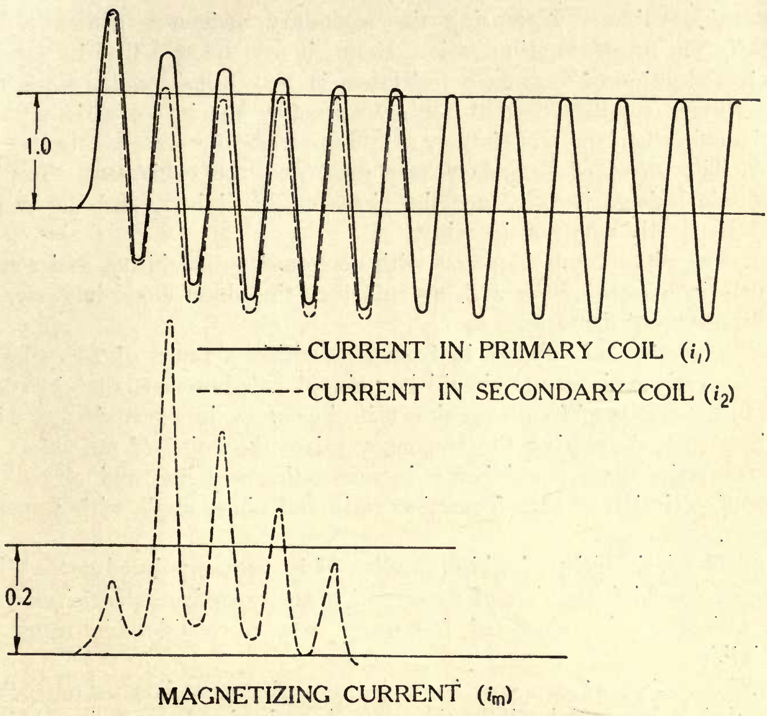

FIG. 17. CURRENT DIAGRAM FOR EXAMPLE 3—SECONDARY REACTANCE CONSIDERED.

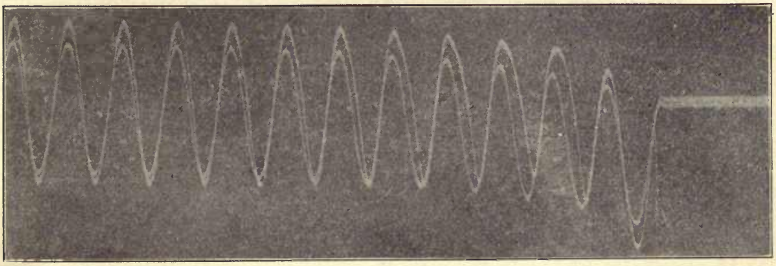

FIG. 18. OSCILLOGRAPH RECORD OF PRIMARY AND SECONDARY CURRENTS IN SERIES TRANSFORMER

reasonable values. Increasing the secondary resistance decreases but slightly the transformation ratio. Hence it may be said that in general the introduction of secondary resistance is very objectionable when the transformer supplies current for a wattmeter, but is not seriously objectionable when the transformer supplies current for an ammeter.

3. The effect of secondary reactance, and the equivalent effect of magnetic leakage is to reduce the phase angle slightly, and the transformation ratio very considerably.

4. The phase angle increases with decreased permeability, and consequently in a transformer with an iron core the phase angle increases as the line current decreases.

5. The effect of changes in frequency within a range of 10 cycles is not generally serious. It should be pointed out, however, that in addition to the effects of frequency shown in the curves, in a transformer with an iron core, decreasing the frequency raises the point of operation on the saturation curve, and hence increases the core loss and alters the form of variation of transformation ratio and phase angle with primary current.

6. The desirability of a high number of turns was pointed out. With a reasonably high number of turns and a not excessive value of secondary resistance the effect of frequency over a considerable range is negligible.

7. The effect of core loss is to decrease the secondary current, this effect being lessened by inductive secondary load. Increased core loss decreases the phase angles, and this effect is increased by inductive secondary load.

8. In an iron-core series transformer the value of flux density should be low (say $B = 2{,}000$ at full load). This means a *low* value of magnetizing current. To this end excessive secondary impedance should be avoided, as increased impedance requires an increase in flux in practically direct proportion to the impedance. Since the effect of magnetic leakage is equivalent to the effect of secondary reactance, the transformer should be designed with a view to minimum magnetic leakage. This requires a well closed iron circuit.

9. For recording instantaneous values of current in transient or unsymmetrical systems, the commercial series transformer with an iron core is quite inadequate, and cannot be relied upon.

10. If necessity demands the use of a series transformer in recording transient or unsymmetrical currents, an air-core transformer, designed to have a very small secondary resistance and a large secondary reactance will be found to give results nearer to those desired than can be obtained with an iron core transformer.

TABLE I.

NOMINAL AND ACTUAL CURRENT RATIOS

Secondary Current—per cent of Full Load	100	70	50	30	20	10
F	0.00001425	0.00001375	0.00001288	0.00001100	0.00000934	0.00000650
$F N_1^2 \omega$	1.052	1.017	0.951	0.812	0.690	0.480
$A = \frac{N_2}{N_1} F N_1^2 \omega$	10.00	9.67	9.04	7.71	6.55	4.56
$F N_2^2 \omega$	95.0	91.75	86.0	73.4	62.3	43.3
$F N_2^2 \omega + x_2 + x$	101.0	97.75	92.0	79.4	68.3	49.3
$(F N_2^2 \omega + x_2 + x)^2$	10200	9550	8460	6300	4660	2430
$(r_2 + r)^2$	4	4	4	4	4	4
$(F N_2^2 \omega + x_2 + x)^2 + (r_2 + r)^2$	10204	9554	8464	6304	4664	2434
$B = \sqrt{(F N_2^2 \omega + x_2 + x)^2 + (r_2 + r)^2}$	101.0	97.8	92.0	79.4	68.3	49.3
$\frac{I''}{I'} = -\frac{A}{B}$	0.0990	0.0988	0.0983	0.0971	0.0958	0.0925
Primary Current—per cent of Full Load	100	70.2	50.4	30.6	20.6	10.7
Deviation of $\frac{I''}{I'}$ *from Full Load Value*	0	0.0002	0.0007	0.0019	0.0032	0.0065
Error in Nominal Value of $\frac{I''}{I'}$ *— per cent*	0	0.202	0.707	1.920	3.230	6.570

TABLE 2

DETERMINATION OF VALUES OF i'

1	2	3	4	5	6	7	8	9	10
θ' degrees	θ' radians	(θ_1-s) degrees	$a\,\theta'$	$e^{-a\theta'}$	$\sin(\theta_1-s)\,e^{-a\theta'}$	θ	$(\theta-s)$	$\sin(\theta-s)$	i'
0	.0	90	0	1	1.000	175	90	1.000	0
10	.175	"	.0175	.983	.983	185	100	.985	—.002
20	.349	"	.0349	.965	.965	195	110	.940	—.025
30	.524	"	.0524	.950	.950	205	120	.866	—.084
40	.699	"	.0699	.935	.935	215	130	.766	—.169
50	.873	"	.0873	.92	.92	225	140	.643	—.277
60	1.048	"	.1048	.90	.90	235	150	.500	—.400
70	1.223	"	.1223	.885	.885	245	160	.342	—.543
80	1.398	"	.1398	.87	.87	255	170	.174	—.696
90	1.572	"	.1572	.855	.855	265	180	0	—.855
100	1.747	"	.1747	.84	.84	275	190	—.174	—1.014
110	1.922	"	.1922	.825	.825	285	200	—.342	—1.167
120	2.095	"	.2095	.81	.81	295	210	—.500	—1.310
130	2.27	"	.227	.795	.795	305	220	—.643	—1.438
140	2.44	"	.244	.785	.785	315	230	—.766	—1.551
150	2.62	"	.262	.77	.77	325	240	—.866	—1.636
160	2.79	"	.279	.76	.76	335	250	—.940	—1.700
170	2.97	"	.297	.745	.745	345	260	—.985	—1.730
180	3.14	"	.314	.73	.73	355	270	—1.000	—1.730
190	3.32	"	.332	.715	.715	365	280	—.985	—1.700
200	3.49	"	.349	.705	.705	375	290	—.940	—1.645
210	3.67	"	.367	.69	.69	385	300	—.866	—1.556
220	3.84	"	.384	.68	.68	395	310	—.766	—1.446
230	4.02	"	.402	.67	.67	405	320	—.643	—1.313
240	4.19	"	.419	.66	.66	415	330	—.500	—1.160
250	4.37	"	.437	.65	.65	425	340	—.342	—.992
260	4.54	"	.454	635	.635	435	350	—.174	—.809
270	4.72	"	.472	.625	.625	445	360	0	—.625
280	4.89	"	.489	.615	.615	455	370	.174	—.441
290	5.07	"	.507	.605	.605	465	380	.342	—.263
300	5.24	"	.524	.59	.59	475	390	.500	—.090
310	5.42	"	.542	.58	.58	485	400	.643	.063
320	5.59	"	.559	.57	.57	495	410	.766	.196
330	5.76	"	.576	.56	.56	505	420	.866	.306
340	5.94	"	.594	.55	.55	515	430	.940	.390
350	6.11	"	.611	.54	.54	525	440	.985	.445
360	6.28	"	.628	.535	.535	535	450	1.000	.465

TABLE 3

SOLUTION OF EXAMPLE 3

1	2	3	4	5	6	7	8	9	10
θ'	i'	$\Delta\phi'=\frac{N_1}{N_2}i'$	ϕ'	i	$(i'-i)$	$\Delta\phi=\frac{N_1}{N_2}(i'-i)$	ϕ	i	$(i'-i)=\frac{N_2}{N_1}i''$
0	0	0	.500	0	0	0	.500	0	0
10	0	0	.500	0	0	0	.500	0	0
20	.025	.002	.502	0	.025	.002	.502	0	.025
30	.084	.008	.510	.001	.083	.008	.510	.001	.083
40	.169	.017	.527	.002	.167	.017	.527	.002	.167
50	.277	.028	.555	.003	.274	.027	.554	.003	.274
60	.400	.040	.595	.004	.396	.040	.594	.004	.396
70	.543	.054	.649	.006	.537	.054	.648	.006	.537
80	.696	.070	.718	.007	.689	.069	.717	.007	.689
90	.855	.085	.803	.008	.847	.085	.802	.008	.847
100	1.014	.101	.903	.010	1.004	.100	.902	.010	1.004
110	1.167	.117	1.019	.011	1.156	.116	1.018	.011	1.156
120	1.310	.131	1.149	.012	1.298	.130	1.148	.012	1.298
130	1.438	.144	1.291	.013	1.425	.143	1.290	.013	1.425
140	1.551	.155	1.445	.014	1.537	.154	1.444	.014	1.537
150	1.636	.164	1.607	.016	1.620	.162	1.606	.016	1.620
160	1.700	.170	1.776	.018	1.682	.168	1.774	.018	1.682
170	1.730	.173	1.947	.020	1.710	.171	1.945	.020	1.710
180	1.730	.173	2.118	.023	1.707	.171	2.116	.023	1.707
190	1.700	.170	2.286	.026	1.674	.167	2.283	.026	1.674
200	1.645	.164	2.448	.031	1.614	.161	2.444	.031	1.614
210	1.556	.156	2.600	.036	1.520	.152	2.596	.036	1.520
220	1.446	.145	2.741	.043	1.403	.140	2.737	.043	1.403
230	1.313	.131	2.868	.050	1.263	.126	2.863	.050	1.263
240	1.160	.116	2.979	.059	1.101	.110	2.973	.058	1.102
250	.992	.099	3.072	.067	.925	.093	3.066	.066	.926
260	.809	.081	3.147	.074	.735	.073	3.139	.073	.736
270	.625	.062	3.202	.080	.545	.055	3.194	.079	.546
280	.441	.044	3.238	.085	.356	.036	3.229	.084	.357
290	.263	.026	3.256	.086	.177	.018	3.247	.085	.178
300	.090	.009	3.256	.086	.004	.000	3.247	.085	.005
310	—.063	—.006	3.241	.084	—.147	—.015	3.233	.083	—.146
320	—.196	—.020	3.213	.081	—.277	—.028	3.205	.080	—.276
330	—.306	—.031	3.174	.077	—.383	—.038	3.167	.076	—.382
340	—.390	—.039	3.128	.072	—.462	—.046	3.120	.071	—.461
350	—.445	—.044	3.076	.067	—.512	—.051	3.069	.066	—.511
360	—.465	—.046	3.023	.062	—.527	—.053	3.017	.061	—.526

TABLE 4

SOLUTION OF EXAMPLE 4

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
θ'	i'	$\Delta i'$	$5.73 \Delta i'$	$[i' + 5.73 \Delta i']$	$\Delta \phi'$	ϕ'	i	Δi	$5.73 \Delta i$	$[i + 5.73 \Delta i]$	$\Delta \phi$	ϕ	i	$(i'-i) = \frac{N_2}{N_1} i''$
0	0	0	0	0	0	.500	0	0	0	0	0	.500	0	0
10	0	0	0	0	0	.500	0	0	0	0	0	.500	0	0
20	.025	.025	.143	.168	.017	.517	.001	.001	.006	.007	.016	.516	.001	.024
30	.084	.059	.338	.422	.042	.558	.003	.002	.011	.014	.041	.557	.003	.081
40	.169	.085	.487	.656	.066	.622	.005	.002	.011	.016	.064	.621	.005	.164
50	.277	.108	.619	.896	.090	.710	.007	.002	.011	.018	.088	.709	.007	.270
60	.400	.123	.705	1.105	.110	.819	.009	.002	.011	.020	.108	.817	.009	.391
70	.543	.143	.820	1.363	.136	.953	.010	.001	.006	.016	.135	.952	.010	.533
80	.696	.153	.877	1.573	.157	1.109	.012	.002	.011	.023	.155	1.107	.012	.684
90	.855	.161	.923	1.778	.178	1.285	.013	.001	.006	.019	.176	1.283	.013	.842
100	1.014	.159	.911	1.925	.192	1.475	.015	.002	.011	.026	.190	1.473	.015	.999
110	1.167	.153	.877	2.044	.204	1.677	.017	.002	.011	.028	.202	1.674	.017	1.150
120	1.310	.143	.820	2.130	.213	1.887	.019	.002	.011	.030	.210	1.884	.019	1.291
130	1.438	.128	.733	2.171	.217	2.101	.022	.003	.017	.039	.213	2.097	.022	1.416
140	1.551	.113	.647	2.198	.220	2.317	.027	.005	.029	.056	.214	2.312	.027	1.524
150	1.636	.085	.487	2.123	.212	2.524	.033	.006	.034	.067	.206	2.517	.033	1.603
160	1.700	.064	.366	2.066	.207	2.724	.042	.009	.052	.094	.197	2.714	.042	1.658
170	1.730	.030	.172	1.902	.190	2.905	.053	.011	.063	.116	.179	2.893	.052	1.678
180	1.730	0	0	1.730	.173	3.066	.066	.013	.074	.140	.159	3.052	.065	1.665
190	1.700	—.030	—.172	1.528	.153	3.205	.080	.014	.080	.160	.137	3.189	.078	1.622
200	1.645	—.055	—.315	1.330	.133	3.322	.095	.015	.086	.181	.115	3.304	.093	1.552

TABLE 4.—*Continued*

SOLUTION OF EXAMPLE 4

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
θ'	i'	$\Delta\, i'$	$5.73\Delta\, i'$	$[i'+5.73\,\Delta\, i']$	$\Delta\,\phi'$	ϕ'	i	$\Delta\, i$	$5.73\Delta\, i$	$i\,+\,5.73\,\Delta\, i]$	$\Delta\,\phi$	ϕ	i	$(i'-i)=\frac{N_2}{N_1}i''$
210	1.556	—.089	—.510	1.046	.105	3.408	.110	.015	.086	.196	.085	3.389	.106	1.450
220	1.446	—.110	—.630	.816	.082	3.470	.122	.012	.069	.191	.062	3.451	.118	1.328
230	1.313	—.133	—.762	.551	.055	3.506	.129	.007	.040	.169	.038	3.489	.126	1.187
240	1.160	—.153	—.877	.283	.028	3.518	.132	.003	.017	.149	.013	3.503	.129	1.031
250	.992	—.168	—.963	.029	.003	3.506	.129	—.003	—.017	.112	—.008	3.495	.127	.865
260	.809	—.183	—1.048	—.239	—.024	3.471	.121	—.008	—.046	.075	—.031	3.463	.120	.689
270	.625	—.184	—1.054	—.429	—.043	3.420	.112	—.009	—.052	.060	—.049	3.414	.111	.514
280	.441	—.184	—1.054	—.613	—.061	3.353	.101	—.011	—.063	.038	—.065	3.349	.100	.341
290	.263	—.178	—1.020	—.757	—.076	3.273	.089	—.012	—.069	.020	—.078	3.271	.088	.175
300	.090	—.173	—.992	—.902	—.090	3.181	.077	—.012	—.069	.008	—.091	3.180	.077	.013
310	—.063	—.153	—.877	—.940	—.094	3.086	.068	—.009	—.052	.016	—.096	3.085	.068	—.131
320	—.196	—.133	—.762	—.958	—.096	2.989	.060	—.008	—.046	.014	—.097	2.988	.060	—.256
330	—.306	—.110	—.630	—.936	—.094	2.894	.052	—.008	—.046	.006	—.094	2.893	.052	—.358
340	—.390	—.084	—.481	—.871	—.087	2.806	.047	—.005	—.029	.018	—.089	2.805	.047	—.437
350	—.445	—.055	—.315	—.760	—.076	2.729	.042	—.005	—.029	.013	—.077	2.727	.042	—.487
360	—.465	—.020	—.114	—.579	—.058	2.669	.039	—.003	—.017	.022	—.060	2.667	.039	—.504

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