CHAUVENET'S

TREATISE ON

ELEMENTARY GEOMETRY

REVISED AND ABRIDGED

BY

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PREFACE.

In preparing this edition of Chauvenet's Geometry I have endeavored to compel the student to think and to reason for himself, and I have tried to emphasize the fact that he should not merely learn to understand and demonstrate a few set propositions, but that he should acquire the power of grasping and proving any simple geometrical truth that may be set before him; and this power, it must be remembered, can never be gained by memorizing demonstrations. Systematic practice in devising proofs of new propositions is indispensable.

On this account the demonstrations of the main propositions, which at first are full and complete, are gradually more and more condensed, until at last they are sometimes reduced to mere hints, by the aid of which the full proof is to be developed; and numerous additional theorems and problems are constantly given as exercises for practice in original work.

A syllabus, containing the axioms, the postulates, and the captions of the main theorems and corollaries, has been added to aid student and teacher in reviews and examinations, and to make the preparation of new proofs more easy and definite.

In the order of the propositions I have departed considerably from the larger Chauvenet's Geometry, with the double object of simplifying the demonstrations and of giving the student, as soon as possible, the few theorems which are the tools with which he must most frequently work in geometrical investigation.

Teachers are strongly advised to require as full and formal proofs of the corollaries and exercises as of the main propositions, and to lay much stress upon written demonstrations, which should be arranged as in the illustrations given at the end of Book I.

In preparing a written exercise, or in passing a written examination, the student should have the syllabus before him, and may then conveniently refer to the propositions by number. In oral recitation, however, he should quote the full captions of the theorems on which he bases his proof.

W. E. BYERLY.

CAMBRIDGE, MASS., 1887.

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ELEMENTS OF GEOMETRY.

INTRODUCTION.

1. EVERY person possesses a conception of *space* indefinitely extended in all directions. Material bodies occupy finite, or limited, portions of space. The portion of space which a body occupies can be conceived as abstracted from the matter of which the body is composed, and is called a *geometrical solid*. The material body filling the space is called a *physical solid*. A geometrical solid is, therefore, merely the *form*, or *figure*, of a physical solid. In this work, since only geometrical solids will be considered, we shall, for brevity, call them simply solids, and we shall define them formally, as follows:

Definition. A *solid* is a limited or bounded portion of space, and has *length*, *breadth*, and *thickness*.

2. The boundaries of a solid are *surfaces*.

Definition. A *surface* is that which has *length* and *breadth*, but no *thickness*.

If a surface is bounded, its boundaries are *lines*.

If two surfaces intersect, their intersection is a *line*.

Definition. A *line* is that which has *length*, but neither *breadth* nor *thickness*.

If a line is terminated, it is terminated by *points*.

If two lines intersect, they intersect in a *point*.

Definition. A *point* has *position*, but neither *length*, *breadth*, nor *thickness*.

3. If we suppose a *point* to move in space, its *path* will be a *line*, and it is often convenient to regard a line as the path, or *locus*, of a moving point.

If a point starts to move from a given position, it must move in some definite *direction;* if it continues to move in the same direction, its path is a *straight line.* If the direction in which the point moves is continually changing, the path is a *curved line.*

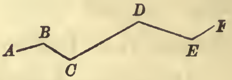
If a point moves along a line, it is said to *describe* the line.

By the *direction of a line* at any point we mean the direction in which a point describing the line is moving when it passes through the point in question.

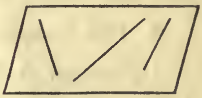
Definitions. A *straight line* is a line which has everywhere the same direction.

A *curved line* is one no portion of which, however short, is straight.

A *broken line* is a line composed of different successive straight lines.



4. *Definitions.* A *plane surface*, or simply a *plane*, is a surface such that, if any two points in it are joined by a straight line, the line will lie wholly in the surface.



A *curved surface* is a surface no portion of which, however small, is plane.

5. *Definitions.* A *geometrical figure* is any combination of points, lines, surfaces, or solids, formed under given conditions. Figures formed by points and lines in a plane are called *plane figures*. Those formed by straight lines alone are called *rectilinear*, or *right-lined*, figures; a straight line being often called a *right* line.

6. *Definitions. Geometry* may be defined as the science of extension and position. More specifically, it is the science which treats of the *construction* of figures under given conditions, of their *measurement* and of their *properties*.

Plane geometry treats of plane figures.

The consideration of all other figures belongs to the *geometry of space*, also called the *geometry of three dimensions*.

7. Some terms of frequent use in geometry are here defined.

A *theorem* is a truth requiring demonstration. A *lemma* is an auxiliary theorem employed in the demonstration of another theorem. A *problem* is a question proposed for solution. An *axiom* is a truth assumed as self-evident. A *postulate* (in geometry) assumes the possibility of the solution of some problem.

Theorems, problems, axioms, and postulates are all called *propositions*.

A *corollary* is an immediate consequence deduced from one or more propositions. A *scholium* is a remark upon one or more propositions, pointing out their use, their connection, their limitation, or their extension. An *hypothesis* is a supposition, made either in the enunciation of a proposition or in the course of a demonstration.

PLANE GEOMETRY.

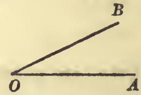
BOOK I.

RECTILINEAR FIGURES.

ANGLES.

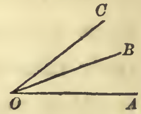
1. *Definition.* A *plane angle*, or simply an *angle*, is the *amount of divergence* of two lines which meet in a point or which would meet if *produced* (*i.e.*, prolonged).

The point is called the *vertex* of the angle, and the two lines the *sides* of the angle.



From the definition it is clear that the magnitude of an angle is independent of the length of its sides.

An isolated angle may be designated by the letter at its vertex, as "the angle O;" but when several angles are formed at the same point by different lines, as OA, OB, OC, we designate the angle intended by three letters; namely, by one letter on each of its sides, together with the one at its vertex, which must be written between the other two.



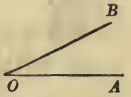
Thus, with these lines there are formed three different angles, which are distinguished as AOB, BOC, and AOC.

Two angles, such as AOB, BOC, which have the same vertex O and a common side OB between them, are called *adjacent*.

2. *Definitions.* Two angles are *equal* when one can be superposed upon the other, so that the vertices shall coincide and the sides of the first shall fall along the sides of the second.

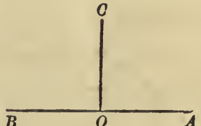
Two angles are *added* by placing them in the same plane with their vertices together and a side in common, care being taken that neither of the angles is superposed upon the other. The angle formed by the exterior sides of the two angles is their sum.

3. A clear notion of the magnitude of an angle will be obtained by supposing that one of its sides, as OB, was at first coincident with the other side OA, and that it has *revolved* about the point O (turning upon O as the leg of a pair of dividers turns upon its hinge) until it has arrived at the position OB. During this revolution the movable side makes with the fixed side a *varying* angle, which increases by insensible degrees, that is, *continuously;* and the revolving line is said to *describe*, or to *generate*, the angle AOB. By continuing the revolution, an angle of any magnitude may be generated.



4. *Definitions.* When one straight line meets another, so as to make two adjacent angles equal, each of these angles is called a *right angle;* and the first line is said to be *perpendicular* to the second.

Thus, if AOC and BOC are equal angles, each is a right angle, and the line CO is perpendicular to AB.



Intersecting lines not perpendicular are said to be *oblique* to each other.

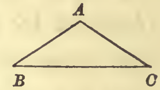
An *acute* angle is less than a right angle.

An *obtuse* angle is greater than a right angle.

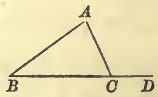
5. *Definition.* Two straight lines lying in the same plane and forming *no angle* with each other—that is, two straight lines in the same plane which will not meet, however far produced—are *parallel.*

TRIANGLES.

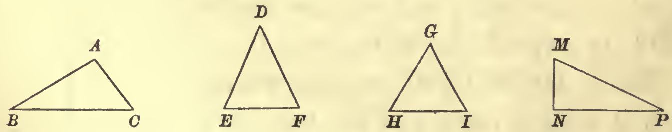
6. *Definitions.* A plane *triangle* is a portion of a plane bounded by three intersecting straight lines; as ABC. The *sides* of the triangle are the portions of the bounding lines included between the points of intersection; viz., AB, BC, CA. The angles of the triangle are the angles formed by the sides with each other; viz., CAB, ABC, BCA. The three angular points, A, B, C, which are the vertices of the angles, are also called the vertices of the triangle.



If a side of a triangle is produced, the angle which the prolongation makes with the adjacent side is called an *exterior angle;* as ACD.



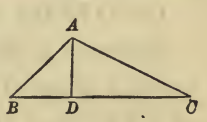
A triangle is called *scalene* (ABC) when no two of its sides are equal; *isosceles* (DEF) when two of its sides are equal; *equilateral* (GHI) when its three sides are equal.



A *right triangle* is one which has a right angle; as MNP, which is right-angled at N. The side MP, opposite to the right angle, is called the *hypotenuse.*

The *base* of a triangle is the side upon which it is supposed to stand. In general, any side may be assumed as the base; but in an isosceles triangle DEF, whose sides DE and DF are equal, the third side EF is always called the base.

When any side BC of a triangle has been adopted as the base, the angle BAC opposite to it is called the *vertical angle*, and its angular point A the vertex of the triangle. The perpendicular AD let fall from the vertex upon the base is then called the *altitude* of the triangle.



7. *Definition. Equal figures* are figures which can be made to coincide throughout if one is properly superposed upon the other.

Roughly speaking, *equal* figures are figures of the same size and of the same shape; *equivalent* figures are of the same size but not of the same shape; and *similar* figures are of the same shape but not of the same size.

POSTULATES AND AXIOMS.

8. *Postulate* I. Through any two given points one straight line, and only one, can be drawn.

Postulate II. Through a given point one straight line, and only one, can be drawn having any given direction.

9. *Axiom* I. A straight line is the shortest line that can be drawn between two points.

Axiom II. Parallel lines have the same direction.

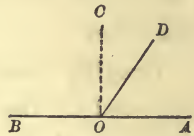
PROPOSITION I.—THEOREM.

10. *At a given point in a straight line one perpendicular to the line can be drawn, and but one.*

Let O be the given point in the line AB.

Suppose a line OD, constantly passing through O, to revolve about O, starting from the position OA and stopping at the position OB.

The angle which OD makes with OA will at first be less than the angle which it makes with OB, and will eventually become greater than the angle made with OB.

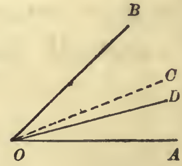


Since the angle DOA increases continuously (3), the line OD must pass through one position in which the angles DOA and DOB are equal. Let OC be this position. Then OC is perpendicular to AB by (4).*

There can be no other perpendicular to AB at O, for if OD is revolved from the position OC by the slightest amount in either direction, one of the adjacent angles will be increased at the expense of the other, and they will cease to be equal.

11. Corollary. *Through the vertex of any given angle one line can be drawn bisecting the angle, and but one.*

Suggestion. Suppose a line OD to revolve about O, as in the proof just given.



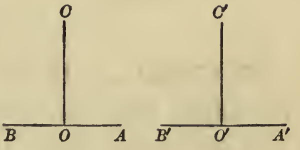
* An Arabic numeral alone refers to an article in the same Book; but in referring to articles in another Book, the number of the Book is also given.

PROPOSITION II.—THEOREM.

12. *All right angles are equal.*

Let AOC and $A'O'C'$ be any two right angles.

Superpose $A'O'C'$ upon AOC, placing the point O' upon the point O and making the line $O'A'$ fall along the line OA; then will $O'C'$ coincide with OC, for otherwise we should have two perpendiculars to the line AB at the same point O, which, by Proposition I., is impossible.



The angles $A'O'C'$ and AOC are then equal by definition (2).

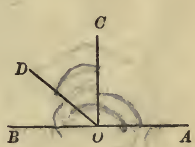
PROPOSITION III.—THEOREM.

13. *The two adjacent angles which one straight line makes with another are together equal to two right angles.*

If the two angles are equal, they are right angles by definition (4), and no proof is necessary.

If they are not equal, as AOD and BOD, still the sum of AOD and BOD is equal to two right angles.

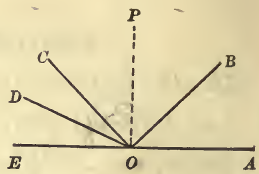
Let OC be drawn at O perpendicular to AB.



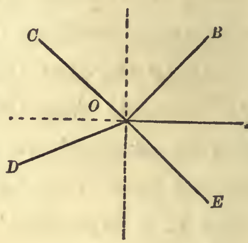
The angle AOD is the sum of the two angles AOC and COD (2). Adding the angle BOD, the sum of the two angles AOD and BOD is the sum of the three angles AOC, COD, and BOD.

The first of these three is a right angle, and the other two are together equal to the right angle BOC; hence the sum of the angles AOD and BOD is equal to two right angles.

14. Corollary I. *The sum of all angles having a common vertex, and formed on one side of a straight line, is two right angles.*



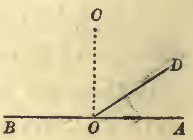
15. Corollary II. *The sum of all the angles that can be formed about a point in a plane is four right angles.*



EXERCISE.

Theorem.—If a line is perpendicular to a second line, then reciprocally the second line is perpendicular to the first.

16. *Definition.* When the sum of two angles is equal to a right angle, each is called the *complement* of the other. Thus, DOC is the complement of AOD, and AOD is the complement of DOC.



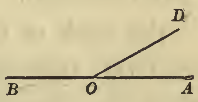
When the sum of two angles is equal to two right angles, each is called the *supplement* of the other. Thus, BOD is the supplement of AOD, and AOD is the supplement of BOD.

It is evident that the complements of equal angles are equal to each other; and also that the supplements of equal angles are equal to each other.

PROPOSITION IV.—THEOREM

17. *If the sum of two adjacent angles is equal to two right angles, their exterior sides are in the same straight line.*

Let the sum of the adjacent angles AOD, BOD, be equal to two right angles; then OA and OB are in the same straight line.



For BOD is the supplement of AOD, and is therefore identical with the angle which OD makes with the prolongation of AO (Proposition III.).

Therefore OB and the prolongation of AO are the same line.

18. Every proposition consists of an *hypothesis* and a *conclusion.* The *converse* of a proposition is a second proposition of which the hypothesis and conclusion are respectively the conclusion and hypothesis of the first. For example, Proposition III. may be enunciated thus:

Hypothesis—if two adjacent angles have their exterior sides in the same straight line, then—*Conclusion*—the sum of these adjacent angles is equal to two right angles.

And Proposition IV. may be enunciated thus:

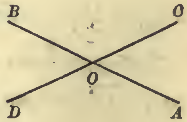
Hypothesis—if the sum of two adjacent angles is equal to two right angles, then—*Conclusion*—these adjacent angles have their exterior sides in the same straight line.

Each of these propositions is, therefore, the converse of the other.

A proposition and its converse are, however, not always both true.

PROPOSITION V.—THEOREM.

19. *If two straight lines intersect each other, the opposite (or vertical) angles are equal.*



Let AB and CD intersect in O; then will the opposite, or vertical, angles AOC and BOD be equal.

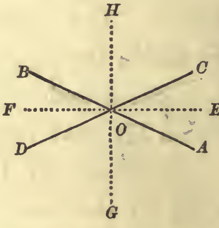
For each of these angles is, by Proposition III., the supplement of the same angle BOC, and hence they are equal (16).

In like manner it can be proved that the opposite angles AOD and BOC are equal.

EXERCISES.

1. *Theorem.—The line which bisects one of two vertical angles bisects the other.*

2. *Theorem.—The straight lines which bisect a pair of adjacent angles formed by two intersecting straight lines are perpendicular to each other.*

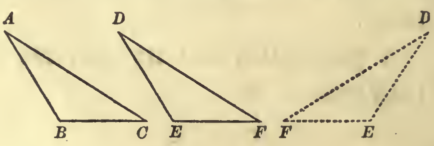


Suggestion. Prove $EOH = FOH$.

PROPOSITION VI.—THEOREM.

20. *Two triangles are equal when two sides and the included angle of the one are respectively equal to two sides and the included angle of the other.*

In the triangles ABC, DEF, let AB be equal to DE, BC to EF, and the included angle B equal to the included angle E; then the triangles are equal.



For, superpose the triangle ABC upon the triangle DEF, placing the point B

upon the point E, and making the side BC fall along the side EF. Then, since BC is equal to EF by hypothesis, the point C will fall upon the point F.

Since the angle B is equal to the angle E, and the side BC has been made to coincide with the side EF, BA must fall along ED, by definition (2); and, as BA is equal to ED, the point A will fall on the point D.

Since the point C has been proved to coincide with the point F, and the point A with the point D, the side CA must coincide with the side FD, by Postulate I. (8).

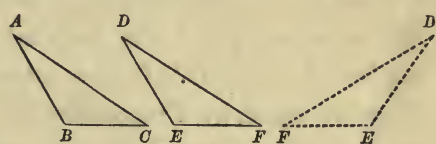
The two triangles have now been proved to coincide throughout, and are equal, by definition (7).

21. *Scholium.* When two triangles are equal, the equal angles are opposite to the equal sides.

PROPOSITION VII.—THEOREM.

22. *Two triangles are equal when a side and the two adjacent angles of the one are respectively equal to a side and the two adjacent angles of the other.*

In the triangles ABC, DEF, let BC be equal to EF, and let the angles B and C adjacent to BC be respectively equal to the angles E and F adjacent to EF; then the triangles are equal.



For, superpose the triangle ABC upon the triangle DEF, placing the point B upon the point E, and making the side BC fall along the side EF.

Since BC is equal to EF, the point C will fall upon the point F.

Since the angle B is equal to the angle E, by hypothesis, and the side BC has been made to coincide with the side EF, BA must fall along ED, and the point A will fall somewhere on the side ED, or on that side extended.

Since the angle C is equal to the angle F, by hypothesis, and BC coincides with EF, CA must fall along FD, and the point A will fall somewhere on the line FD, or on that line extended.

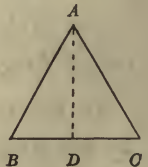
Since A has been proved to lie upon ED, and also upon FD, it must coincide with the only point they have in common, the point D.

Hence the triangles coincide throughout, and are equal.

PROPOSITION VIII.—THEOREM.

23. *In an isosceles triangle the angles opposite the equal sides are equal.*

Let AB and AC be the equal sides of the isosceles triangle ABC; then the angles B and C are equal.



Through the vertex A draw a line AD, bisecting the angle BAC, and meeting the side BC at D.

In the triangles ABD and ACD the side AB is equal to the side AC by hypothesis, the side AD is common, and the included angle BAD is equal to the included angle CAD by construction. The triangles are therefore equal, by Proposition VI., and the angle C of the one is equal to the angle B of the other, by (21).

24. Corollary. *The straight line bisecting the vertical angle of an isosceles triangle bisects the base, and is perpendicular to the base.*

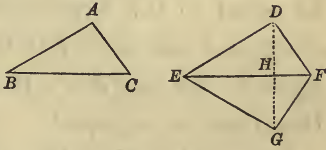
EXERCISE.

Theorem.—An equilateral triangle is also equiangular.

PROPOSITION IX.—THEOREM.

25. *Two triangles are equal when the three sides of the one are respectively equal to the three sides of the other.*

In the triangles ABC, DEF, let AB be equal to DE, AC to DF, and BC to EF; then the triangles are equal.



For, suppose the triangle ABC to be placed so that its base BC coincides with its equal EF, but so that the vertex A falls on the opposite side of EF from D, as at G, and join D and G by a straight line.

The triangle EDG is isosceles, since the side ED is equal to the side EG by hypothesis; therefore the angles EDG and EGD are equal, by Proposition VIII.

The triangle FDG is isosceles, since the side FD is equal to the side FG by hypothesis; therefore the angles FDG and FGD are equal, by Proposition VIII.

If to the equal angles EDG and EGD we add the equal angles FDG and FGD, the sums will be equal, and we have the whole angle EDF equal to the whole angle EGF.

The two triangles EDF and EGF have now the side ED equal to the side EG by hypothesis, the side DF equal to the side FG by hypothesis, and the included angle EDF proved equal to the included angle EGF. Hence the triangles are equal, by Proposition VI.

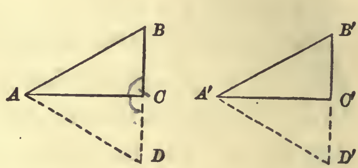
EXERCISE.

Theorem.—A line drawn from the vertex of an isosceles triangle to the middle point of the base is perpendicular to the base, and bisects the vertical angle.

PROPOSITION X.—THEOREM.

26. *Two right triangles are equal when they have the hypotenuse and a side of the one respectively equal to the hypotenuse and a side of the other.*

In the right triangles ABC, $A'B'C'$, let the hypotenuse AB be equal to the hypotenuse $A'B'$, and the side BC be equal to the side $B'C'$; then the triangles are equal.



Extend the side BC to D, making CD equal to BC, and join A and D; and extend $B'C'$ to D', making $C'D'$ equal to $B'C'$, and join A' and D'.

The triangle ADC and the triangle ABC having the side AC in common; the side CD equal to the side CB by construction; and the included angle ACD equal to the included angle ACB, since they are adjacent angles and ACB is a right angle by hypothesis, are equal, by Proposition VI.

In like manner the triangle $A'D'C'$ may be proved equal to the triangle $A'B'C'$.

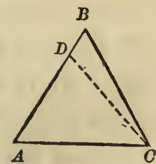
The triangles BAD and $B'A'D'$ having the side AB equal to the side $A'B'$ by hypothesis; the side BD equal to the side $B'D'$, because they were constructed the doubles of BC and $B'C'$, which were equal by hypothesis; and the side AD equal to the side $A'D'$, since they have been proved to be equal respectively to the sides AB and $A'B'$; are equal to each other, by Proposition IX., and B is equal to B'.

The triangles ABC and $A'B'C'$ have now been proved to have two sides and the included angle of the one respectively equal to two sides and the included angle of the other, and are equal, by Proposition VI.

PROPOSITION XI.—THEOREM.

27. *If two angles of a triangle are equal, the sides opposite to them are equal, and the triangle is isosceles.*

Let the angles BAC and BCA of the triangle ABC be equal, then are the sides AB and BC equal.



For, if AB and BC are not equal, one must be greater than the other. Suppose AB greater than BC.

Then cut off from AB a part AD equal to BC, and join D and C. Compare now the triangle ADC with the whole triangle ABC, of which it is a part.

The two triangles have the side AC in common; the side AD equal to the side BC by construction; and the included angle A equal to the included angle BCA by hypothesis. Therefore, by Proposition VI., the triangles ADC and ABC are equal, which is impossible. Consequently, AB could not have been greater than BC.

In like manner we can prove that BC cannot be greater than AB.

Therefore, since neither can be greater than the other, AB and BC are equal.

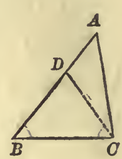
EXERCISE.

Theorem.—An equiangular triangle is also equilateral.

PROPOSITION XII.—THEOREM.

28. *If two angles of a triangle are unequal, the side opposite the greater angle is greater than the side opposite the less angle.*

In the triangle ABC let the angle C be greater than the angle B; then AB is greater than AC.

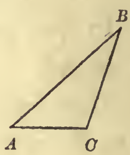


For, suppose the line CD to be drawn, cutting off from the greater angle a part $BCD = B$. Then BDC is an isosceles triangle, by Proposition XI., and $DC = DB$. But in the triangle ADC we have $AD + DC > AC$, by Axiom I.; or, putting DB for its equal DC, $AD + DB > AC$; or $AB > AC$.

PROPOSITION XIII.—THEOREM.

29. *If two sides of a triangle are unequal, the angle opposite the greater side is greater than the angle opposite the less side.*

In the triangle ABC let the side AB be greater than the side BC; then will the angle C be greater than the angle A.



For, if C is not greater than A, it must be equal to A or less than A.

C cannot be equal to A, for in that case AB and BC would be equal, by Proposition XI.

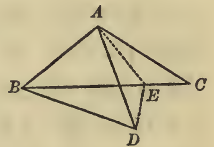
C cannot be less than A, for in that case AB would be less than BC, by Proposition XII.

Therefore C is greater than A.

PROPOSITION XIV.—THEOREM.

30. *If two triangles have two sides of the one respectively equal to two sides of the other, and the included angles unequal, the triangle which has the greater included angle has the greater third side.*

Let ABC and ABD be the two triangles in which the sides AB, AC are respectively equal to the sides AB, AD, but the included angle BAC is greater than the included angle BAD; then BC is greater than BD.



For, suppose the line AE to be drawn, bisecting the angle CAD and meeting BC in E; join DE. The triangles AED and AEC are equal, by Proposition VI., and therefore $ED = EC$. But in the triangle BDE we have

$$BE + ED > BD, \text{ by Axiom I.,}$$

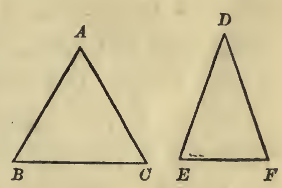
and substituting EC for its equal ED,

$$BE + EC > BD, \text{ or } BC > BD.$$

PROPOSITION XV.—THEOREM.

31. *If two triangles have two sides of the one respectively equal to two sides of the other, and the third sides unequal, the triangle which has the greater third side has the greater included angle.*

In the triangles ABC and DEF let $AB = DE$, $AC = DF$, and let BC be greater than EF; then will the angle A be greater than the angle D.



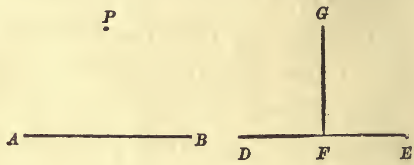
For, if A were equal to D, BC would be equal to EF, by Proposition VI.; and if A were less than D, BC would be less than EF, by Proposition XIV.

PROPOSITION XVI.—THEOREM.

32. *From a given point, without a straight line, one perpendicular can be drawn to the line, and but one.*

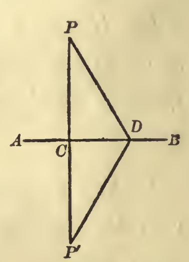
Let AB be the given line and P the given point.

Take a second line DE, and at some point F of DE let a perpendicular be erected (Proposition I.). Superpose this second figure upon the first, placing the line DE upon the line AB, and then move the figure along, keeping DE always in coincidence with AB, until the perpendicular FG passes through P; we shall then have a perpendicular to AB drawn through P. Let PC in the figure below be this perpendicular.



No other perpendicular from P can be drawn to the line AB. For, suppose that a second perpendicular PD could be drawn.

Extend PC to P', making CP' equal to PC, and join D and P'.



The two triangles PCD and $P'CD$ have the side PC equal to the side $P'C$ by construction; the side CD common; and the included angle PCD equal to the included angle $P'CD$, by Proposition III. Therefore, by Proposition VI., the triangles are equal, and the angle PDC is equal to the angle $P'DC$. But PDC is a right angle by hypothesis; therefore $P'DC$ must be a right angle, and PD and DP' must lie in the same straight line, by Proposition IV.; and we have two straight lines drawn between P and P', which, by Postulate I., is impossible.

Since this impossible result follows necessarily from the assumption that a second perpendicular can be drawn from P to AB, that assumption must be false.

EXERCISES.

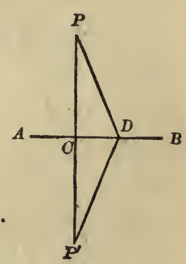
1. *Theorem.—A perpendicular let fall from the vertex of an isosceles triangle upon the base bisects the base and bisects the vertical angle.*

2. *Theorem.—Two right triangles are equal when they have the hypotenuse and an adjacent angle of the one respectively equal to the hypotenuse and an adjacent angle of the other.*

Suggestion. Superpose the second triangle upon the first, making the given equal angles coincide.

PROPOSITION XVII.—THEOREM.

33. *The perpendicular is the shortest line that can be drawn from a point to a straight line.*



Let PC be the perpendicular and PD any oblique line from the point P to the line AB. Then PC is shorter than PD.

Extend PC to P', making CP' equal to PC, and join D and P'.

The triangles PCD and $P'CD$ are equal, by Proposition VI. Therefore $P'D = PD$.

$$PP' < PD + DP', \text{ by Axiom I.}$$

Therefore PC, the half of PP', is less than PD, the half of PDP'.

EXERCISES.

1. *Theorem.—Two oblique lines drawn from a point to a line, and meeting the line at equal distances from the foot of the perpendicular from the given point, are equal.*

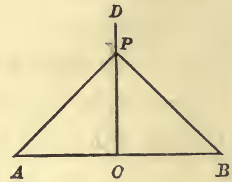
2. *Theorem.—Two equal oblique lines drawn from a point to a line meet it at equal distances from the foot of the perpendicular.*

PROPOSITION XVIII.—THEOREM.

34. *If a perpendicular is erected at the middle of a straight line, then,*

1st. *Every point in the perpendicular is equally distant from the extremities of the line;*

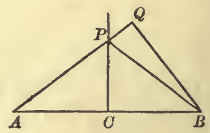
2d. *Every point without the perpendicular is unequally distant from the extremities of the line.*



Let AB be a finite straight line and CD a perpendicular at its middle point.

1st. Then is any point P on CD equidistant from A and B. For, join P and A and P and B.

The triangles PCA and PCB are equal, by Proposition VI.; therefore PA and PB are equal.



2d. Any point Q without the perpendicular is unequally distant from A and B. For, Q being on one side or the other of the perpendicular, one of the lines QA, QB must cut the perpendicular; let it be QA and let it cut in P; join PB. The straight line QB is less than the broken line QPB, by Axiom I.; that is, $QB < QP + PB$. But $PB = PA$; therefore $QB < QP + PA$, or $QB < QA$.

35. *Definition.* A geometric *locus* is the geometric figure containing all the points which possess a common property, and no others.

In this definition, points are understood to have a common property when they satisfy the same geometrical *conditions.*

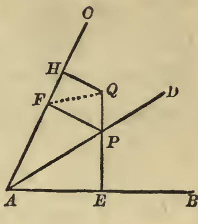
Thus, since all the points in the perpendicular erected at the middle of a line possess the common property of being equally distant from the extremities of the line (that is, satisfy the *condition* that they shall be equally distant from those extremities), and *no other points possess this property*, the perpendicular is the locus of these points; so that the preceding proposition is fully covered by the following brief statement:

The perpendicular erected at the middle of a straight line is the locus of the points which are equally distant from the extremities of that line.

PROPOSITION XIX.—THEOREM.

36. *Every point in the bisector of an angle is equally distant from the sides of the angle; and every point not in the bisector, but within the angle, is unequally distant from the sides of the angle; that is, the bisector of an angle is the locus of the points within the angle and equally distant from its sides.*

1st. Let AD be the bisector of the angle BAC, P any point in it, and PE, PF, the perpendicular distances of P from AB and AC; then $PE = PF$.

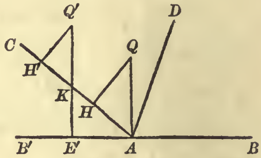


For, the right triangles APE, APF, having the angles PAE and PAF equal, and AP common, are equal (32, Exercise 2); therefore $PE = PF$.

2d. Let Q be any point not in the bisector, but within the angle; then the perpendicular distances QE and QH are unequal.

For, suppose that one of these distances, as QE, cuts the bisector in some point P; from P let PF be drawn perpendicular to AC, and join QF. We have $QH < QF$; also $QF < QP + PF$, or $QF < QP + PE$, or $QF < QE$; therefore $QH < QE$.

When the angle BAC is obtuse, the point Q, not in the bisector, may be so situated that the perpendicular on one of the sides, as AB, will fall at the vertex A; the perpendicular QH is then less than the oblique line QA. Or, a point Q' may be so situated that the perpendicular $Q'E'$, let fall on one of the sides, as AB, will meet that side produced through the vertex A; this perpendicular must cut the side AC in some point, K, and we then have $Q'H' < Q'K < Q'E'$.

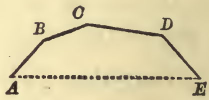


EXERCISE.

Theorem.—The locus of the points equally distant from two intersecting straight lines is the pair of lines which bisect all the angles formed by the given lines.

(*v.* 19, Exercise 2.)

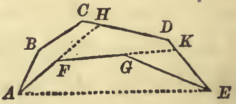
37. *Definition.* A broken line, as $ABCDE$, is called *convex* when no one of its component straight lines, if produced, can enter the space enclosed by the broken line, and the straight line joining its extremities.



PROPOSITION XX.—THEOREM.

38. *A convex broken line is less than any other line which envelops it and has the same extremities.*

Let the convex broken line $AFGE$ have the same extremities A, E, as the line $ABCDE$, and be *enveloped* by it; that is, wholly included within the space bounded by $ABCDE$ and the straight line AE. Then $AFGE < ABCDE$.



For, produce AF and FG to meet the enveloping line in

H and *K*. Imagine *ABCDE* to be the path of a point moving from *A* to *E*. If the straight line *AH* be substituted for *ABCH*, the path *AHDE* will be shorter than the path *ABCDE*, the portion *HDE* being common to both. If, further, the straight line *FK* be substituted for *FHDK*, the path *AFKE* will be a still shorter path from *A* to *E*. And if, finally, *GE* be substituted for *GKE*, *AFGE* will be a still shorter path. Therefore *AFGE* is less than any enveloping line.

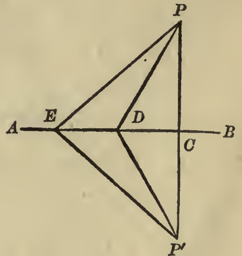
39. *Scholium.* The preceding demonstration applies when the enveloping line is a *curve*, or any species of line whatever.

PROPOSITION XXI.—THEOREM.

40. *If two oblique lines drawn from a point to a line meet the line at unequal distances from the foot of the perpendicular, the more remote is the greater.*

1st. If the lines lie on the same side of the perpendicular.

Let *PC* be the perpendicular and *PD* and *PE* the two oblique lines, *EC* being greater than *DC*; then is *PE* greater than *PD*.



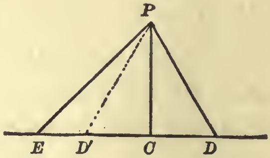
For, produce *PC* to *P′*, making *CP′* equal to *PC*, and join *P′* with *D* and with *E*.

The triangles *PDC* and *P′DC* and the triangles *PEC* and *P′EC* are equal, by Proposition VI. Hence $PD = P'D$, and $PE = P'E$.

PDP′ is less than *PEP′*, by Proposition XX.; therefore *PE*, the half of *PEP′*, is greater than *PD*, the half of *PDP′*.

2d. If the lines lie on opposite sides of the perpendicular.

Let PC be the perpendicular and PD and PE the oblique lines, EC being greater than CD.

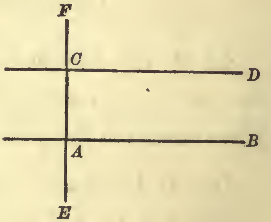


Lay off CD' equal to CD, and join P and D'. Then the triangles PDC and $PD'C$ are equal, by Proposition VI., and $PD' = PD$.

But PD' is less than PE, by the proof given above. Hence its equal PD is less than PE.

PROPOSITION XXII.—THEOREM.

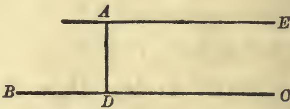
41. *Two straight lines perpendicular to the same straight line are parallel.*



Let AB and CD be two lines perpendicular to the same line EF; then are they parallel. For, if AB and CD are not parallel, they must meet if produced; but this is impossible, for in that case we should have two perpendiculars from their point of meeting to the same straight line EF, which is contrary to Proposition XVI.

PROPOSITION XXIII.—THEOREM.

42. *Through a given point one line, and only one, can be drawn parallel to a given line.*



Let A be the given point and BC the given line.

From A draw AD perpendicular to BC, and through A draw AE perpendicular to AD. AE and DC, being perpendicular to the same line AD, are parallel, by Proposition XXII. No other line can be drawn through A parallel to BC, for, by Axiom II., it would have the same direction as AE, and therefore, by Postulate II., it would coincide with AE.

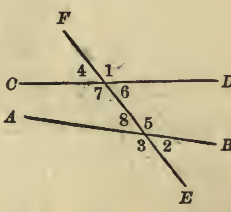
EXERCISES.

1. *Theorem.—Lines having the same direction are parallel.*

Suggestion. Suppose them to meet; *v.* Postulate II.

2. *Theorem.—Lines parallel to the same line are parallel to each other.*

43. *Definitions.* When two straight lines AB, CD, are cut by a third EF, the eight angles formed at their points of intersection are named as follows:



The four angles, 1, 2, 3, 4, without the two lines, are called *exterior* angles.

The four angles, 5, 6, 7, 8, within the two lines, are called *interior* angles.

Two exterior angles on opposite sides of the secant line and not adjacent—as 1, 3—or 2, 4—are called *alternate-exterior* angles.

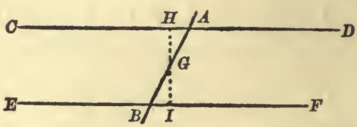
Two interior angles on opposite sides of the secant line and not adjacent—as 5, 7—or 6, 8—are called *alternate-interior* angles.

Two angles similarly situated with respect both to the secant and to the line intersected by it are called *corresponding* angles; as 1, 5—2, 6—3, 7—4, 8.

PROPOSITION XXIV.—THEOREM.

44. *When two straight lines are cut by a third, if the alternate-interior angles are equal, the two straight lines are parallel.*

Let the line AB cut the lines CD and EF, making the alternate-interior angles CAB and ABF equal; then are CD and EF parallel.



Through G, the middle point of AB, draw GH perpendicular to CD, and cutting EF in I.

Then the triangles AGH and BGI, having the side AG equal to the side GB by construction, the angle GAH equal to the angle GBI by hypothesis, and the angle AGH equal to the angle IGB by Proposition V., are equal, by Proposition VII. Therefore the angle GIB is equal to the angle GHA. But GHA is a right angle by construction; hence GIB is a right angle, and CD and EF are perpendicular to the same line HI, and are therefore parallel, by Proposition XXII.

If EBA and BAD are the given equal alternate angles, their supplements CAB and ABF are equal, and the proof just given is valid.

If the given alternate-interior angles are right angles, the lines are parallel, by Proposition XXII.

45. Corollary I. *When two straight lines are cut by a third, if a pair of corresponding angles are equal, the lines are parallel.*

Suggestion. Show that in that case a pair of alternate-interior angles are equal.

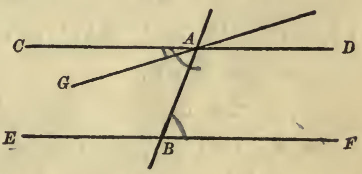
46. Corollary II. *When two straight lines are cut by a third, if the sum of two interior angles on the same side of the secant line is equal to two right angles, the two lines are parallel.*

Suggestion. Show that a pair of alternate-interior angles are equal.

PROPOSITION XXV.—THEOREM.

47. *If two parallel lines are cut by a third straight line, the alternate-interior angles are equal.*

Let the parallel lines CD and EF be cut by the line AB; then will the angles CAB and ABF be equal.



For, if they are not equal, draw through A a line AG, making the angles GAB and ABF equal; then, by Proposition XXIV., GA and EF are parallel, and we have two parallels to the same line EF drawn through the same point A, which is contrary to Proposition XXIII., and therefore impossible. Hence the angles CAB and ABF are equal.

48. Corollary I. *If two parallel lines are cut by a third straight line, any two corresponding angles are equal.*

49. Corollary II. *If two parallel lines are cut by a third straight line, the sum of the two interior angles on the same side of the secant line is equal to two right angles.*

EXERCISE.

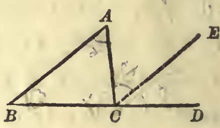
Theorem.—A line perpendicular to one of two parallel lines is perpendicular to the other.

PROPOSITION XXVI.—THEOREM.

50. *The sum of the three angles of any triangle is equal to two right angles.*

Let ABC be any triangle; then the sum of its three angles is equal to two right angles.

Produce BC, and through C draw CE parallel to BA.



Since the line AC meets the parallel lines AB and EC, the alternate-interior angles ACE and BAC are equal, by Proposition XXV.

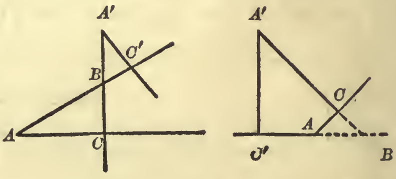
Since the line BD cuts the parallel lines AB and EC, the corresponding angles ECD and ABC are equal, by Proposition XXV., Corollary I.

Therefore the sum of the angles of the triangle is equal to $BCA + ACE + ECD$, which is two right angles, by Proposition III., Corollary I.

51. Corollary. *If a side of a triangle is extended, the exterior angle is equal to the sum of the two interior opposite angles.*

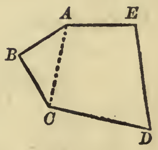
EXERCISE.

Theorem.—If the sides of an angle are respectively perpendicular to the sides of a second angle, the angles are equal, or supplementary.



POLYGONS.

52. *Definitions.* A *polygon* is a portion of a plane bounded by straight lines; as $ABCDE$. The bounding lines are the *sides;* their sum is the *perimeter* of the polygon. The angles which the adjacent sides make with each other are the angles of the polygon; and the vertices of these angles are called the vertices of the polygon.



Any line joining two vertices not consecutive is called a *diagonal;* as AC.

53. *Definitions.* Polygons are classed according to the number of their sides:

A *triangle* is a polygon of three sides.

A *quadrilateral* is a polygon of four sides.

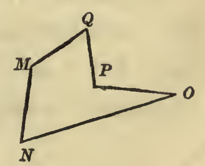
A *pentagon* has five sides; a *hexagon*, six; a *heptagon*, seven; an *octagon*, eight; an *enneagon*, nine; a *decagon*, ten; a *dodecagon*, twelve; etc.

An *equilateral* polygon is one all of whose sides are equal; an *equiangular* polygon, one all of whose angles are equal.

54. *Definition.* A *convex* polygon is one no side of which, when produced, can enter within the space enclosed by the perimeter; as $ABCDE$ in (52). Each of the angles of such a polygon is less than two right angles.

It is also evident from the definition that the perimeter of a convex polygon cannot be intersected by a straight line in more than two points.

A *concave* polygon is one of which two or more sides, when produced, will enter the space enclosed by the perimeter; as $MNOPQ$, of which OP and QP, when produced, will enter within the polygon.



The angle OPQ, formed by two adjacent re-entrant sides, is called a *re-entrant angle;* and hence a concave polygon is sometimes called a *re-entrant polygon.*

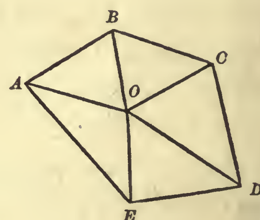
All the polygons hereafter considered will be understood to be convex.

PROPOSITION XXVII.—THEOREM.

55. *The sum of all the angles of any polygon is equal to twice as many right angles, less four, as the figure has sides.*

Join any point O within the polygon to each of the vertices, thus dividing the polygon into as many triangles as it has sides.

The sum of the angles of these triangles will, by Proposition XXVI., be twice as many right angles as the figure has sides. But the angles of the triangles form the angles of the polygon plus the angles at O, which are equal to four right angles, by Proposition III., Corollary II.



EXERCISE.

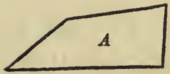
1. *Theorem.—If each side of a polygon is extended, the sum of the exterior angles is four right angles.*

Suggestion. The sum of all the angles, exterior and interior, is obviously twice as many right angles as the figure has sides.

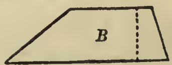
QUADRILATERALS.

56. *Definitions.* Quadrilaterals are divided into classes, as follows:

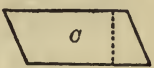
1st. The *trapezium* (A), which has no two of its sides parallel.



2d. The *trapezoid* (B), which has two sides parallel. The parallel sides are called the *bases*, and the perpendicular distance between them the *altitude* of the trapezoid.



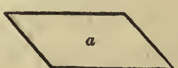
3d. The *parallelogram* (C), which is bounded by two pairs of parallel sides.



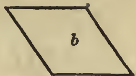
The side upon which a parallelogram is supposed to stand and the opposite side are called its lower and upper *bases.* The perpendicular distance between the bases is the *altitude.*

57. *Definitions.* Parallelograms are divided into species, as follows:

1st. The *rhomboid* (a), whose adjacent sides are not equal and whose angles are not right angles.



2d. The *rhombus*, or *lozenge* (b), whose sides are all equal.



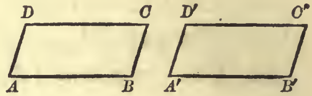
3d. The *rectangle* (c), whose angles are all equal, and therefore right angles.

4th. The *square* (d), whose sides are all equal and whose angles are all equal.

The square is at once a rhombus and a rectangle.

PROPOSITION XXVIII.—THEOREM.

58. *Two parallelograms are equal when two adjacent sides and the included angle of the one are equal to two adjacent sides and the included angle of the other.*



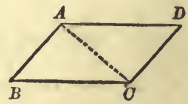
Let AC, $A'C'$, have $AB = A'B'$, $AD = A'D'$, and the angle $BAD = B'A'D'$; then these parallelograms are equal.

For they may evidently be applied the one to the other, so as to coincide throughout. (*v.* Proposition XXIII.)

59. Corollary. *Two rectangles are equal when they have equal bases and equal altitudes.*

PROPOSITION XXIX.—THEOREM.

60. *The opposite sides of a parallelogram are equal and the opposite angles are equal.*



Suggestion. Draw a diagonal AC. ACB and CAD are equal, by Proposition XXV.

CAB and ACD are equal, by Proposition XXV.

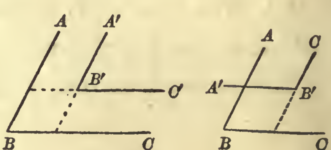
AD = BC

Hence the triangles ABC and ADC are equal, by Proposition VII.

EXERCISES.

1. *Theorem.—If one angle of a parallelogram is a right angle, all the angles are right angles, and the figure is a rectangle.*

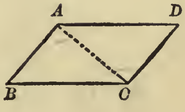
2. *Theorem.—If two angles have the sides of one respectively parallel to the sides of the other, they are equal, or supplementary.*



3. *Theorem. — Two parallel lines are everywhere equidistant.*

PROPOSITION XXX.—THEOREM.

61. *If two opposite sides of a quadrilateral are equal and parallel, the figure is a parallelogram.*



Suggestion. Let AD be equal and parallel to BC. Draw a diagonal AC.

The triangles ABC and ADC are equal, by Proposition VI. Therefore the angles BAC and ACD are equal, and AB and CD are parallel, by Proposition XXIV.

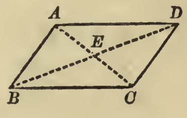
PROPOSITION XXXI.—THEOREM.

62. *If the opposite sides of a quadrilateral are equal, the figure is a parallelogram.*

Suggestion. Draw a diagonal, and prove the two triangles equal.

PROPOSITION XXXII.—THEOREM.

63. *The diagonals of a parallelogram bisect each other.*



Suggestion. The triangles AED and BEC are equal, by Proposition VII.

EXERCISES.

1. *Theorem.—The diagonals of a rectangle are equal.*

2. *Theorem.—The diagonals of a rhombus are perpendicular to each other.*

3. *Theorem.—If the diagonals of a quadrilateral bisect each other, the figure is a parallelogram.*

4. *Theorem.—If the diagonals of a parallelogram are equal, the figure is a rectangle.*

5. *Theorem.—If the diagonals of a parallelogram are perpendicular to each other, the figure is a rhombus.*

ARRANGEMENT OF WRITTEN EXERCISES.

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64. In writing out a demonstration, brevity of statement and clearness of arrangement should be carefully studied, and symbols and abbreviations may be used with profit. The following list is recommended:

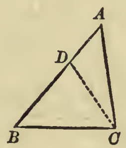
SYMBOLS AND ABBREVIATIONS.

$\therefore$	therefore.	*Def.*	definition.
$=$	equal to.	*Post.*	postulate.
$\Bumpeq$	equivalent to.	*Ax.*	axiom.
$>$	greater than.	*Prop.*	proposition.
$<$	less than.	*Cor.*	corollary.
$\parallel$	parallel to.	*Hyp.*	hypothesis.
$\perp$	perpendicular to.	*Cons.*	construction.
$\angle$	angle.	*Adj.*	adjacent.
$\angle s$	angles.	*Inc.*	included.
rt. $\angle$	right angle.	*Alt.-int.*	alternate-interior.
$\triangle$	triangle.	*Sup.*	supplementary.
$\triangle s$	triangles.	*Comp.*	complementary.
rt. $\triangle$	right triangle.	*Q. E. D.*	quod erat demonstrandum (= which was to be proved).
$\square$	parallelogram.		
$\square s$	parallelograms.		
$\odot$	circle.		
$\odot s$	circles.		

65. In arranging a written demonstration, it is well to begin each statement on a separate line, giving the reason for the statement at the end of the line, if it can be written briefly, or in parenthesis immediately below the line, if it cannot be written briefly. The following examples of demonstrations prepared as written exercises, or for a written examination, will serve as illustrations.

(1) PROPOSITION XII.—THEOREM.

If two angles of a triangle are unequal, the side opposite the greater angle is greater than the side opposite the less angle.



In $\triangle ABC$, let it be given that $\angle ACB > \angle B$,
we are to prove $AB > AC$.

Draw CD, cutting off from $\angle ACB$ a part

$$\angle BCD = \angle B.$$

Then in $\triangle BCD$ we have

$\angle BCD = \angle B$. Cons.

$\therefore \quad BD = CD$, Prop. XI.

and $BD + DA = CD + DA$.

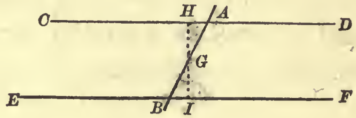
But $AC < CD + DA$. Ax. I.

$\therefore \quad AC < BD + DA$,

and $AC < AB$. Q. E. D.

(2) PROPOSITION XXIV.—THEOREM.

When two straight lines are cut by a third, if the alternate interior angles are equal, the two straight lines are parallel.



Let AB cut CD and EF in the points A and B, making

$$\angle BAC = \angle ABF,$$

we are to prove $CD \parallel$ to EF.

Through G, the middle point of AB, draw $HI \perp$ to CD.

Then in the $\triangle$s AGH and BGI

we have	$AG = BG$,	Cons.
	$\angle GAH = \angle GBI$,	Hyp.
	$\angle AGH = \angle BGI$.	Prop. V.
$\therefore$	$\triangle AGH = \triangle BGI$,	Prop. VII.
and	$\angle GIB = \angle GHA$	
	(homologous angles of $=$ $\triangle$s).	
But	$\angle GHA$ is a *rt.* $\angle$.	Cons.
$\therefore$	$\angle GIB$ is a *rt.* $\angle$,	
and	HI is $\perp$ to EF.	
But	HI is $\perp$ to CD.	Cons.
$\therefore$	CD and EF are $\parallel$.	Prop. XXII.

Q. E. D.

If the given equal $\angle$s are ABE and BAD,

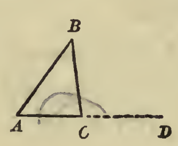
we have	$\angle ABE = \angle BAD$,	Hyp.
	$\angle ABE + \angle ABF = 2$ *rt.* $\angle$s,	Prop. III.
	$\angle BAD + \angle BAC = 2$ *rt.* $\angle$s.	Prop. III.
$\therefore$	$\angle ABF = \angle BAC$,	

and the proof given above applies.

If the given equal alt.-int. $\angle s$ are *rt.* $\angle s$ the two given lines are $\perp$ to the third line and are $\parallel$, by Proposition XXII.

(3) PROPOSITION XXVI. COROLLARY.

If one side of a triangle is extended, the exterior angle is equal to the sum of the two interior opposite angles.



In the $\triangle$ ABC let the side AC be extended,

we are to prove $\angle BCD = \angle A + \angle B$.

We have the sum of the adj. $\angle s$

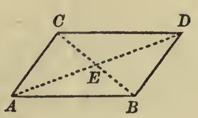
$BCD + BCA = 2$ *rt.* $\angle s$. Prop. III.

But $\angle A + \angle B + \angle BCA = 2$ *rt.* $\angle s$. Prop. XXVI.

$\therefore$ $\angle BCD = \angle A + \angle B$. Q. E. D.

(4) EXERCISE 3, PAGE 43.—THEOREM.

If the diagonals of a quadrilateral bisect each other, the figure is a parallelogram.



In the quadrilateral $ABCD$, let the diagonals AD and BC bisect each other.

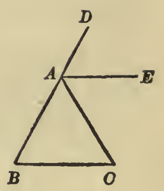
We are to prove $ABCD$ a $\square$.

In the $\triangle$s AEB and CED

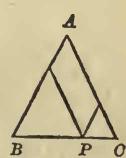
we have	$CE = EB$,	Hyp.
	$ED = AE$,	Hyp.
	$\angle CED = \angle AEB$.	Prop. V.
$\therefore$	$\triangle AEB = \triangle CED$,	Prop. VI.
and	$CD = AB$	
	(homologous sides of equal $\triangle$s),	
and	$\angle EDC = \angle EAB$	
	(homologous $\angle$s of equal $\triangle$s).	
But	EDC and EAB are alt.-int. $\angle$s.	
$\therefore$	CD is $\parallel$ to AB,	Prop. XXIV.
and since	$CD = AB$,	Proved above.
	$ABCD$ is a $\square$.	Prop. XXX.

Q. E. D.

EXERCISES ON BOOK I.



1. THE straight line AE which bisects the angle exterior to the vertical angle of an isosceles triangle ABC is parallel to the base BC.



2. If from a *variable point* in the base of an isosceles triangle parallels to the sides are drawn, a parallelogram is formed whose perimeter is *constant*.

3. The sum of the four lines drawn to the vertices of a quadrilateral from any point except the intersection of the diagonals, is greater than the sum of the diagonals.

4. The lines drawn from the extremities of the base of an isosceles triangle to the middle points of the opposite sides are equal.

5. The perpendiculars from the extremities of the base of an isosceles triangle upon the opposite sides are equal.

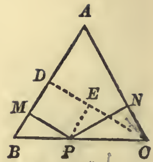
6. The bisectors of the base angles of an isosceles triangle are equal.

7. A perpendicular let fall from one end of the base of an isosceles triangle upon the opposite side makes with the base an angle equal to one-half the vertical angle.

8. If the vertical angle of an isosceles triangle is one-half as great as an angle at the base, a line bisecting a base angle will divide the given triangle into two isosceles triangles.

9. If two isosceles triangles have the sides of one equal to the sides of the other, and the base of one double the altitude of the other, the two triangles are of the same size.

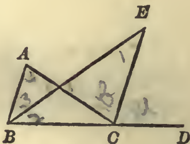
10. If from a variable point P in the base of an isosceles triangle ABC, perpendiculars, PM, PN, to the sides are drawn, the sum of PM and PN is constant, and equal to the perpendicular from C upon AB.



Suggestion. The triangles PNC and PEC are equal, by Proposition VII.

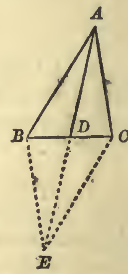
11. The line joining the feet of perpendiculars let fall from the extremities of the base of an isosceles triangle upon the opposite sides is parallel to the base.

12. If BE bisects the angle B of a triangle ABC, and CE bisects the exterior angle ACD, the angle E is equal to one-half the angle A.

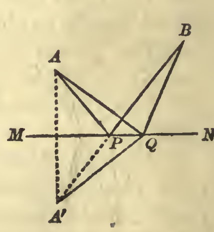


13. The medial line to any side of a triangle is less than the half sum of the other two sides.

Definition. A line joining a vertex of a triangle with the middle point of the opposite side is called a *medial line.*



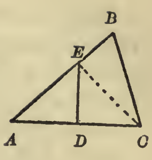
14. If from two points, A and B, on the same side of a straight line MN, straight lines, AP, BP, are drawn to a point P in that line, making with it equal angles APM and BPN, the sum of the lines AP and BP is less than the sum of any other two lines, AQ and BQ, drawn from A and B to any other point Q in MN.



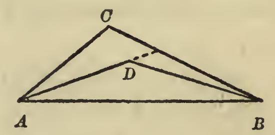
15. If the medial line from the vertex of a triangle to the base is equal to one-half the base, the vertical angle is a right angle.

16. The altitude of a triangle divides the vertical angle into two parts, whose difference is equal to the difference of the base angles of the triangle.

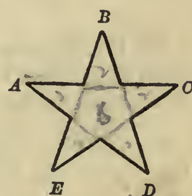
17. The perpendicular erected at the middle point of one side of a triangle meets the longer of the other two sides.



18. Lines drawn from a point within a triangle to the extremities of the base include an angle greater than the vertical angle of the triangle. (*v.* Proposition XXVI., Corollary.)

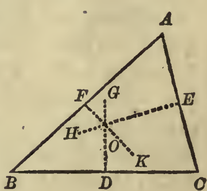


19. The sum of the angles at the vertices of a five-pointed star (pentagram) is equal to two right angles.



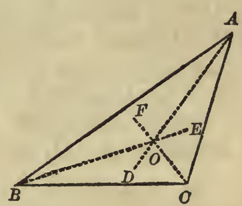
20. The three perpendiculars erected at the middle points of the sides of a triangle meet in the same point.

Suggestion. The point of intersection of EH and FK is equidistant from the three vertices, and therefore must lie on DG (Proposition XVIII.).



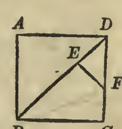
21. The three bisectors of the three angles of a triangle meet in the same point.

Suggestion. The point of intersection of BE and CF is equidistant from the three sides, and therefore must lie on AD (Proposition XIX.).

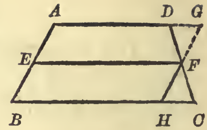


22. The bisectors of two external angles of a triangle and the bisector of the remaining internal angle meet in a point.

23. If from the diagonal BD of a square $ABCD$, BE is cut off equal to BC, and EF is drawn perpendicular to BD, then $DE = EF = FC$.

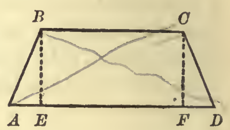


24. In a trapezoid, the straight line joining the middle points of the non-parallel sides is parallel to the bases, and is equal to one-half their sum.



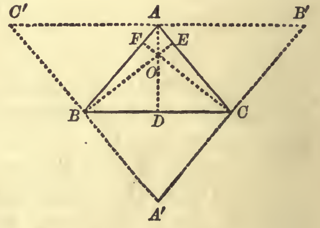
Suggestion. Draw HG parallel to AB, and extend AD. $DGF = CHF$ (Proposition VII.), and $EFHB$ is a parallelogram, by Proposition XXX.

25. If the sides of a trapezoid which are not parallel are equal, the base angles are equal and the diagonals are equal.



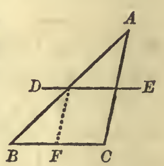
26. If through the four vertices of a quadrilateral lines are drawn parallel to the diagonals, they will form a parallelogram twice as large as the quadrilateral.

27. The three perpendiculars from the vertices of a triangle to the opposite sides meet in the same point.



Suggestion. Draw through the three vertices lines parallel to the opposite sides of the triangle. By the aid of the three parallelograms $ABCB'$, $ABA'C$, and $ACBC'$, prove that the sides of $A'B'C'$ are bisected by A, B, and C. See now Exercise 20.

28. If a straight line drawn parallel to the base of a triangle bisects one of the sides, it also bisects the other side; and the portion of it intercepted between the two sides is equal to one-half the base.

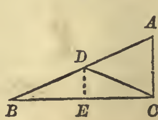


Suggestion. Draw DF parallel to AC. See now Proposition VII. and Proposition XXIX.

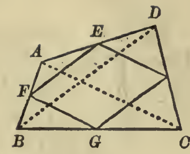
29. The straight line joining the middle points of two sides of a triangle is parallel to the third side. (v. Exercise 28.)

30. The three straight lines joining the middle points of the sides of a triangle divide the triangle into four equal triangles.

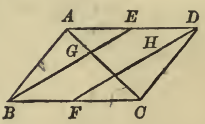
31. In any right triangle, the straight line drawn from the vertex of the right angle to the middle of the hypotenuse is equal to one-half the hypotenuse. (v. Exercise 28.)



32. The straight lines joining the middle points of the adjacent sides of any quadrilateral form a parallelogram whose perimeter is equal to the sum of the diagonals of the quadrilateral (Exercise 29).

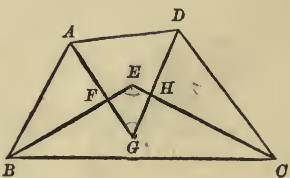


33. If E and F are the middle points of the opposite sides, AD, BC, of a parallelogram $ABCD$, the straight lines BE, DF, trisect the diagonal AC (Exercise 28).



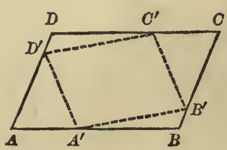
34. The four bisectors of the angles of a quadrilateral form a second quadrilateral, the opposite angles of which are supplementary.

If the first quadrilateral is a parallelogram, the second is a rectangle. If the first is a rectangle, the second is a square.

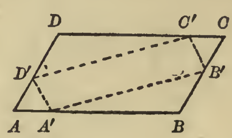


35. The point of intersection of the diagonals of a parallelogram bisects every straight line drawn through it and terminated by the sides of the parallelogram.

36. If from each vertex of a parallelogram the same given distance is laid off on a side of the parallelogram, care being taken that no two distances are laid off on the same side, the points thus obtained will be the vertices of a new parallelogram.

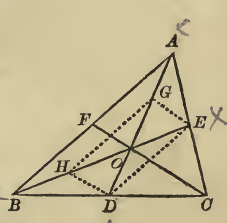


37. If from two opposite vertices of a parallelogram equal distances are laid off on the sides adjacent to those vertices, the points thus obtained will be the vertices of a parallelogram.

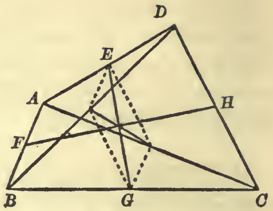


38. The three medial lines of a triangle meet in the same point.

Suggestion. Let O be the point of intersection of AD and BE, and H and G the middle points of OB and OA. Hence prove $OD = \frac{1}{3}AD$ and $OE = \frac{1}{3}BE$. In like manner the point of intersection of AD and CF can be shown to cut off one-third of AD.



39. The intersection of the straight lines which join the middle points of opposite sides of any quadrilateral is the middle point of the straight line which joins the middle points of the diagonals (Exercise 29).



SYLLABUS TO BOOK I.

POSTULATES, AXIOMS, AND THEOREMS.

POSTULATE I.

Through any two given points one straight line, and only one, can be drawn.

POSTULATE II.

Through a given point one straight line, and only one, can be drawn having any given direction.

AXIOM I.

A straight line is the shortest line that can be drawn between two points.

AXIOM II.

Parallel lines have the same direction.

PROPOSITION I.

At a given point in a straight line one perpendicular to the line can be drawn, and but one.

Corollary. Through the vertex of any given angle one straight line can be drawn bisecting the angle, and but one.

PROPOSITION II.

All right angles are equal.

PROPOSITION III.

The two adjacent angles which one straight line makes with another are together equal to two right angles.

Corollary I. The sum of all the angles having a common vertex, and formed on one side of a straight line, is two right angles.

Corollary II. The sum of all the angles that can be formed about a point in a plane is four right angles.

PROPOSITION IV.

If the sum of two adjacent angles is two right angles, their exterior sides are in the same straight line.

PROPOSITION V.

If two straight lines intersect each other, the opposite (or vertical) angles are equal.

PROPOSITION VI.

Two triangles are equal when two sides and the included angle of the one are respectively equal to two sides and the included angle of the other.

PROPOSITION VII.

Two triangles are equal when a side and the two adjacent angles of the one are respectively equal to a side and the two adjacent angles of the other.

PROPOSITION VIII.

In an isosceles triangle the angles opposite the equal sides are equal.

Corollary. The straight line bisecting the vertical angle of an isosceles triangle bisects the base, and is perpendicular to the base.

PROPOSITION IX.

Two triangles are equal when the three sides of the one are respectively equal to the three sides of the other.

PROPOSITION X.

Two right triangles are equal when they have the hypotenuse and a side of the one respectively equal to the hypotenuse and a side of the other.

PROPOSITION XI.

If two angles of a triangle are equal, the sides opposite to them are equal, and the triangle is isosceles.

PROPOSITION XII.

If two angles of a triangle are unequal, the side opposite the greater angle is greater than the side opposite the less angle.

PROPOSITION XIII.

If two sides of a triangle are unequal, the angle opposite the greater side is greater than the angle opposite the less side.

PROPOSITION XIV.

If two triangles have two sides of the one respectively equal to two sides of the other, and the included angles unequal, the triangle which has the greater included angle has the greater third side.

PROPOSITION XV.

If two triangles have two sides of the one respectively equal to two sides of the other, and the third sides unequal, the triangle which has the greater third side has the greater included angle.

PROPOSITION XVI.

From a given point, without a straight line, one perpendicular can be drawn to the line, and but one.

PROPOSITION XVII.

The perpendicular is the shortest line that can be drawn from a point to a straight line.

PROPOSITION XVIII.

If a perpendicular is erected at the middle of a straight line, then every point on the perpendicular is equally distant from the extremities of the line; and every point not on the perpendicular is unequally distant from the extremities of the line.

PROPOSITION XIX.

Every point in the bisector of an angle is equally distant from the sides of the angle; and every point not in the bisector, but within the angle, is unequally distant from the sides of the angle; that is, the bisector of an angle is the locus of the points within the angle and equally distant from its sides.

PROPOSITION XX.

A convex broken line is less than any other line which envelops it and has the same extremities.

PROPOSITION XXI.

If two oblique lines drawn from a point to a line meet the line at unequal distances from the foot of the perpendicular, the more remote is the greater.

PROPOSITION XXII.

Two straight lines perpendicular to the same straight line are parallel.

PROPOSITION XXIII.

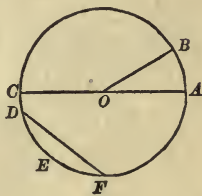
Through a given point one line, and only one, can be drawn parallel to a given line.

BOOK II.

THE CIRCLE.

1. *Definitions.* A *circle* is a portion of a plane bounded by a curve, all the points of which are equally distant from a point within it called the *centre.*

The curve which bounds the circle is called its *circumference.*



Any straight line drawn from the centre to the circumference is called a *radius.*

Any straight line drawn through the centre and terminated each way by the circumference is called a *diameter.*

In the figure, O is the centre, and the curve $ABCEA$ is the circumference of the circle; the circle is the space included within the circumference; OA, OB, OC, are radii; AOC is a diameter.

By the definition of a circle, all its radii are equal; also all its diameters are equal, each being double the radius.

If one extremity, O, of a line OA is fixed, while the line revolves in a plane, the other extremity, A, will *describe* a circumference, whose radii are all equal to OA.

2. *Definitions.* An *arc* of a circle is any portion of its circumference; as DEF.

A *chord* is any straight line joining two points of the circumference; as DF. The arc DEF is said to be *subtended* by its chord DF.

PROPOSITION XV.

If two triangles have two sides of the one respectively equal to two sides of the other, and the third sides unequal, the triangle which has the greater third side has the greater included angle.

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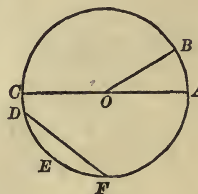


The curve which bounds the circle is called its *circumference.*

Any straight line drawn from the centre to the circumference is called a *radius.*

Any straight line drawn through the centre and terminated each way by the circumference is called a *diameter.*

In the figure, O is the centre, and the curve $ABCEA$ is the circumference of the circle; the circle is the space included within the circumference; OA, OB, OC, are radii; AOC is a diameter.

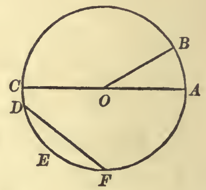
By the definition of a circle, all its radii are equal; also all its diameters are equal, each being double the radius.

If one extremity, O, of a line OA is fixed, while the line revolves in a plane, the other extremity, A, will *describe* a circumference, whose radii are all equal to OA.

2. *Definitions.* An *arc* of a circle is any portion of its circumference; as DEF.

A *chord* is any straight line joining two points of the circumference; as DF. The arc DEF is said to be *subtended* by its chord DF.

Every chord subtends two arcs, which together make up the whole circumference. Thus, DF subtends both the arc DEF and the arc $DCBAF$. When an arc and its chord are spoken of, the arc less than a semi-circumference, as DEF, is always understood, unless otherwise stated.



A *segment* is a portion of the circle included between an arc and its chord; thus, by the segment DEF is meant the space included between the arc DF and its chord.

A *sector* is the space included between an arc and the two radii drawn to its extremities; as AOB.

3. From the definition of a circle it follows that every point *within* the circle is at a distance from the centre which is less than the radius; and every point *without* the circle is at a distance from the centre which is greater than the radius. Hence *the* locus *of all the points in a plane which are at a given distance from a given point is the circumference of a circle described with the given point as a centre and with the given distance as a radius.*

4. Postulate. *A circumference may be described with any point as centre and any distance as radius.*

PROPOSITION I.—THEOREM.

5. *Two circles are equal when the radius of the one is equal to the radius of the other.*

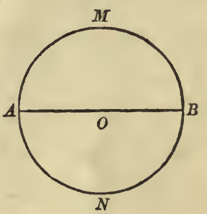
Let the second circle be superposed upon the first, so that its centre falls upon the centre of the first; then will the two circumferences coincide throughout.

For, if any point of either circumference falls outside of the other circle, the line joining it with the common centre must cross the circumference of that circle.

The whole line will be a radius of one circle, the portion of it within the other circle will be a radius of that other circle, and we shall have two unequal radii, which is contrary to our hypothesis.

PROPOSITION II.—THEOREM.

6. *Every diameter bisects the circle and its circumference.*



Let $AMBN$ be a circle whose centre is O; then any diameter AOB bisects the circle and its circumference.

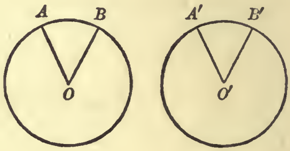
For, if the figure ANB be turned about AB as an axis and superposed upon the figure AMB, the curve ANB will coincide with the curve AMB, since all the points of both are equally distant from the centre. (*v.* Proof of Proposition I.) The two figures then coincide throughout, and are therefore equal in all respects. Therefore AB divides both the circle and its circumference into equal parts.

7. *Definitions.* A *segment* equal to one-half the circle, as the segment AMB, is called a *semicircle.* An *arc* equal to half a circumference, as the arc AMB, is called a *semi-circumference.*

PROPOSITION III.—THEOREM.

8. *In equal circles, or in the same circle, equal angles at the centre intercept equal arcs on the circumference.*

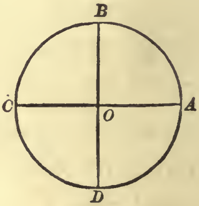
Let O, O', be the centres of equal circles, and AOB, $A'O'B'$, equal angles at these centres; then the intercepted arcs AB, $A'B'$, are equal.



For the angle O' may be superposed upon, and made to coincide with, its equal O. The extremities of the arc $A'B'$ will then fall on the extremities of the arc AB, and the arcs must coincide throughout and be equal, since the radii are equal. (*v.* Proof of Proposition I.)

9. Corollary. *Conversely, in the same circle, or in equal circles, equal arcs subtend equal angles at the centre.*

10. *Definition.* A fourth part of a circumference is called a *quadrant.* It is evident from the preceding theorem that a right angle at the centre intercepts a quadrant on the circumference.



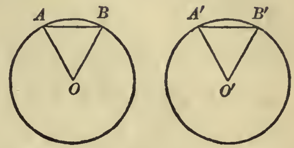
Thus, two perpendicular diameters, AOC, BCD, divide the circumference into four quadrants, AB, BC, CD, DA.

PROPOSITION IV.—THEOREM.

11. *In equal circles, or in the same circle, equal arcs are subtended by equal chords.*

Let O, O', be the centres of equal circles, and AB, $A'B'$, equal arcs; then the chords AB, $A'B'$, are equal.

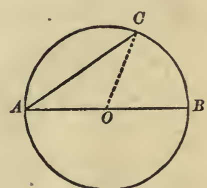
For, drawing the radii to the extremities of the arcs, the angles O and O' are equal (Proposition III., Corollary), and consequently the triangles AOB, $A'O'B'$, are equal (Proposition VI., Book I.). Therefore $AB = A'B'$.



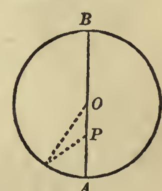
If the arcs are in the same circle the demonstration is similar.

12. Corollary. *Conversely, in equal circles, or in the same circle, equal chords subtend equal arcs.*

EXERCISES.



1. *Theorem.—A diameter is greater than any other chord.*

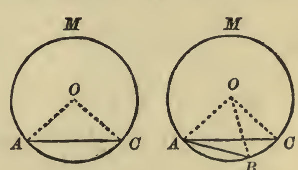


2. *Theorem.—The shortest line that can be drawn from a point within a circle to the circumference is a portion of the diameter drawn through the point.*

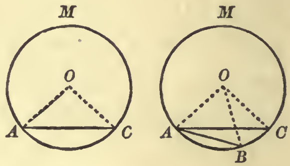
PROPOSITION V.—THEOREM.

13. *In equal circles, or in the same circle, the greater of two unequal arcs is subtended by the greater chord, the arcs being each less than a semi-circumference.*

Let the arc AC be greater than the arc AB; then the chord AC is greater than the chord AB.



Superpose the arc AB upon the arc AC, placing centre upon centre and A upon A; B must fall between A and C, since AB is less than AC. Draw now the radii OA, OB, OC. In the triangles AOC and AOB the angle AOC is obviously greater than the angle AOB; therefore, by Proposition XIV., Book I., the chord AC is greater than the chord AB.



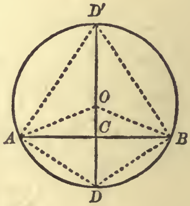
14. Corollary. *Conversely, in equal circles, or in the same circle, the greater of two unequal chords subtends the greater arc.*

Suggestion. v. Proposition XV., Book I.

PROPOSITION VI.—THEOREM.

15. *The diameter perpendicular to a chord bisects the chord and the arcs subtended by it.*

The triangles ACO, BCO, are equal, by Proposition X., Book I. Therefore $AC = CB$.

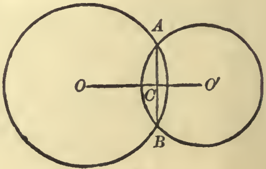


The triangles AOD, BOD, are equal, by Proposition VI., Book I. Therefore $AD = BD$, and hence, by Proposition IV., Corollary, the arc AD is equal to the arc BD.

In the same way we can prove the arc AD' equal to the arc BD'.

16. Corollary I. *The perpendicular erected at the middle point of a chord passes through the centre of the circle.* (*v.* Proposition XVIII., Book I.)

17. Corollary II. *When two circumferences intersect, the straight line joining their centres bisects their common chord at right angles.*



Suggestion. Erect a perpendicular at the middle point of the common chord. (*v.* Corollary I.)

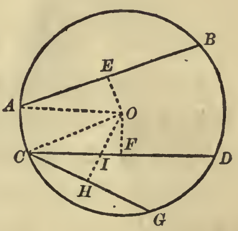
EXERCISE.

Theorem.—The locus of the middle points of a set of parallel chords is the diameter perpendicular to the chords.

PROPOSITION VII.—THEOREM.

18. *In the same circle, or in equal circles, equal chords are equally distant from the centre; and of two unequal chords, the less is at the greater distance from the centre.*

1st. Let AB, CD, be equal chords; OE, OF, the perpendiculars which measure their distances from the centre O; then $OE = OF$.



For, since the perpendiculars bisect the chords, $AE = CF$; hence (Proposition X., Book I.) the right triangles AOE and COF are equal, and $OE = OF$.

2d. Let CG, AB, be unequal chords; OE, OH, their distances from the centre; and let CG be less than AB; then $OH > OE$.

For, since chord $AB >$ chord CG, we have arc $AB >$ arc CG; so that if from C we draw the chord $CD = AB$, its subtended arc CD, being equal to the arc AB, will be greater than the arc CG. Therefore the perpendicular OH will intersect the chord CD in some point I. Drawing the perpendicular OF to CD, we have, by the first part of the demonstration, $OF = OE$. But $OH > OI$, and $OI > OF$ (Proposition XVII., Book I.); still more, then, is $OH > OF$, or $OH > OE$.

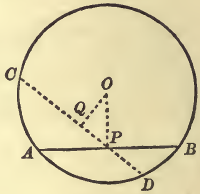
If the chords be taken in two equal circles, the demonstration is the same.

19. Corollary. *Conversely, in the same circle, or in equal circles, chords equally distant from the centre are equal; and of two chords unequally distant from the centre, that is the greater whose distance from the centre is the less.*

EXERCISE.

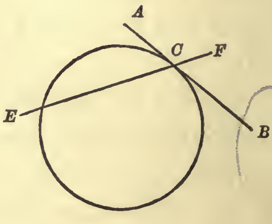
Theorem.—The least chord that can be drawn in a circle through a given point is the chord perpendicular to the diameter through the point.

Suggestion. v. Proposition XVII., Book I.



TANGENTS AND SECANTS.

20. *Definitions.* A *tangent* is an indefinite straight line which has but one point in common with the circumference; as ACB. The common point, C, is called the *point of contact*, or the *point of tangency.* The circumference is also said to be tangent to the line AB at the point C.

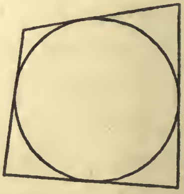


A *secant* is a straight line which meets the circumference in two points; as EF.

Two circumferences are tangent to each other when they are both tangent to the same straight line at the same point.

21. *Definition.* A rectilinear figure is said to be *circumscribed* about a circle when all its sides are tangents to the circumference.

In the same case, the circle is said to be *inscribed* in the figure.



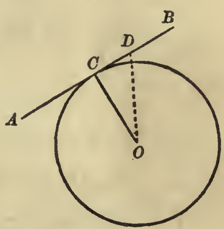
PROPOSITION VIII.—THEOREM.

22. *A straight line cannot intersect a circumference in more than two points.*

For, if the line could intersect the circumference in three points, the radii drawn to these points would meet the line at unequal distances from the perpendicular let fall from the centre of the circle upon the line, and would be unequal, by Proposition XXI., Book I.

PROPOSITION IX.—THEOREM.

23. *A straight line tangent to a circle is perpendicular to the radius drawn to the point of contact.*



For any other point of the tangent, as D, must lie outside of the circle, and therefore the line OD, joining it with the centre, must be greater than the radius OC, drawn to the point of contact.

OC is, then, the shortest line that can be drawn from O to the tangent AB, and is therefore perpendicular to AB, by Proposition XVII., Book I.

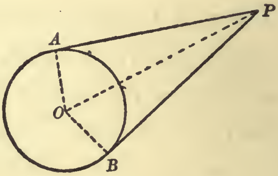
24. Corollary I. *A perpendicular to a tangent line drawn through the point of contact must pass through the centre of the circle.*

25. Corollary II. *If two circumferences are tangent to each other, their centres and their point of contact lie in the same straight line.*

Suggestion. Through their point of contact draw a line perpendicular to the tangent at that point. (*v.* Corollary I.)

PROPOSITION X.—THEOREM.

26. *When two tangents to the same circle intersect, the distances from their point of intersection to their points of contact are equal.*



For the right triangles OAP and OBP (Proposition IX.) are equal, by Proposition X., Book I.

EXERCISES.

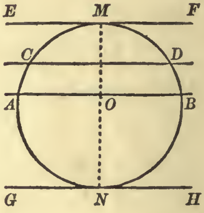
1. *Theorem.—In any circumscribed quadrilateral, the sum of two opposite sides is equal to the sum of the other two opposite sides.*

2. *Theorem.—If two circumferences are tangent, and from any point, P, of the tangent at their point of contact, tangents are drawn to the two circles, the points of contact of these tangents are equally distant from P.*

PROPOSITION XI.—THEOREM.

27. *Two parallels intercept equal arcs on a circumference.*

We may have three cases:



1st. When the parallels AB, CD, are both secants, then the intercepted arcs AC and BD are equal. For, let OM be the radius drawn perpendicular to the parallels. By Proposition VI., the point M is at once the middle of the arc AMB and of the arc CMD, and hence we have

$$AM = BM \text{ and } CM = DM,$$

whence, by subtraction,

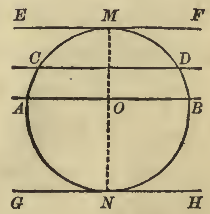
$$AM - CM = BM - DM,$$

that is,

$$AC = BD.$$

2d. When one of the parallels is a secant, as AB, and the other is a tangent, as EF at M, then the intercepted arcs AM and BM are equal. For the radius OM drawn to the point of contact is perpendicular to the tangent (Proposition IX.), and consequently perpendicular also to its parallel AB; therefore, by Proposition VI., $AM = BM$.

3d. When both the parallels are tangents, as EF at M, and GH at N, then the intercepted arcs MAN and MBN are equal. For, drawing any secant AB parallel to the tangents, we have, by the second case,



$$AM = BM \text{ and } AN = BM,$$

whence, by addition,

$$AM + AN = BM + BN,$$

that is,

$$MAN = MBN;$$

and each of the intercepted arcs in this case is a semi-circumference.

MEASURE OF ANGLES.

As the measurement of magnitude is one of the principal objects of geometry, it will be proper to premise here some principles in regard to the measurement of quantity in general.

28. *Definition.* To *measure* a quantity of any kind is to find how many times it contains another quantity of the same kind, called the *unit.*

Thus, to measure a line is to find the number expressing how many times it contains another line, called the *unit of length*, or the *linear unit.*

The number which expresses how many times a quantity contains the unit is called the *numerical measure* of that quantity.

29. *Definition.* The *ratio* of two quantities is the quotient arising from dividing one by the other: thus, the ratio of A to B is $\frac{A}{B}$.

To find the ratio of one quantity to another is, then, to find how many times the first contains the second; therefore it is the same thing as to *measure* the first by the second taken as the unit (28). It is implied in the definition of ratio that the quantities compared are of the *same kind.*

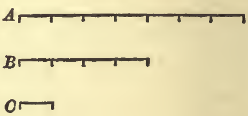
Hence, also, instead of the definition (28), we may say that to measure a quantity is to find its ratio to the unit.

The ratio of two quantities is the same as the ratio of their numerical measures. Thus, if P denotes the unit, and if P is contained m times in A and n times in B, then

$$\frac{A}{B}=\frac{mP}{nP}=\frac{m}{n}.$$

30. *Definition.* Two quantities are *commensurable* when there is some third quantity of the same kind which is contained a whole number of times in each. This third quantity is called the *common measure* of the proposed quantities.

Thus, the two lines A and B are commensurable if there is some line, C, which is contained a whole number of times in each, as, for example, 7 times in A, and 4 times in B.

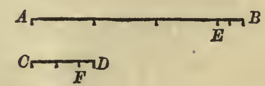


The ratio of two commensurable quantities can, therefore, be *exactly* expressed by a number, whole or fractional (as in the preceding example by $\frac{7}{4}$), and is called a *commensurable ratio.*

31. *Definition.* Two quantities are *incommensurable* when they have no common measure. The ratio of two such quantities is called an *incommensurable ratio.*

If A and B are two incommensurable quantities, their ratio is still expressed by $\frac{A}{B}$.

32. *Problem. To find the greatest common measure of two quantities.* The well-known arithmetical process may be extended to quantities of all kinds. Thus, suppose AB and CD are two straight lines whose common measure is required. Their greatest common measure cannot be greater than the less line CD. Therefore let CD be applied to AB as many times as possible, suppose three times, with a remainder EB less than CD.



Any common measure of AB and CD must also be a common measure of CD and EB; for it will be contained a whole number of times in CD, and in AE, which is a multiple of CD, and therefore to measure AB it must also measure the part EB. Hence the *greatest* common measure of AB and CD must also be the *greatest* common measure of CD and EB. This greatest common measure of CD and EB cannot be greater than the less line EB; therefore let EB be applied as many times as possible to CD, suppose twice, with a remainder FD. Then, by the same reasoning, the greatest common measure of CD and EB, and consequently also that of AB and CD, is the greatest common measure of EB and FD. Therefore let FD be applied to EB as many times as possible: suppose it is contained exactly twice in EB without remainder; the process is then completed, and we have found FD as the required greatest common measure.

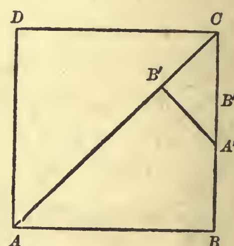
The measure of each line, referred to FD as the unit, will then be as follows: we have

$$EB = 2FD,$$
$$CD = 2EB + FD = 4FD + FD = 5FD,$$
$$AB = 3CD + EB = 15FD + 2FD = 17FD.$$

The proposed lines are therefore numerically expressed, in terms of the unit FD, by the numbers 17 and 5; and their ratio is $\frac{17}{5}$.

33. When the preceding process is applied to two quantities and no remainder can be found which is exactly contained in a preceding remainder, however far the process be continued, the two quantities have no common measure; that is, they are *incommensurable*, and their ratio cannot be exactly expressed by any number, whole or fractional.

34. As the student often has difficulty in realizing the possibility of an *incommensurable ratio*, and imagines that if two lines are given it must be possible to take a divisor so small that it will go exactly into each of them, it seems worth while to consider at some length an important example,—namely, the ratio of the diagonal of a square to one of the sides.



Let the method of (32) be applied to finding the common measure of the diagonal and a side of the square $ABCD$.

AC is clearly less than twice AB; *i.e.*, than $AB + BC$. Lay off on AC AB', equal to AB. Our problem is now reduced to finding a common measure of $B'C$ and AB, or its equal, CB.

Erect at B' a perpendicular $B'A'$ to AC. $A'B'$, $B'C$, and $A'B$ are all equal (*v.* Exercise 23, Book I.). If, then, we lay off CB'' equal to CB', $B'C$ goes into BC twice, with a remainder $B''A'$, by which we must proceed to divide $B'C$. But $A'B'C$ is half a square, precisely similar to ABC, and in performing the division of $B'C$, or its equal, $A'B'$, by $A'B''$, we are merely repeating, on a smaller scale, the process just performed in dividing BC by $B'C$. This will lead us to another repetition, on a still smaller scale, and so on indefinitely, and we shall never reach an exact division. The diagonal and the side of a square have then no common divisor, and are absolutely incommensurable.

35. Although an incommensurable ratio cannot be exactly expressed by a number, a number can be found by the following method that will approximately express it, and the approximation may be made as close as we choose.

Suppose that $\frac{A}{B}$ denotes the ratio of two incommensurable quantities, A and B. Let B be divided into n equal parts, n being some number taken at pleasure; and then let A be divided by one of these parts. Suppose A is found to contain this divisor m times, with a remainder, which, of course, is less than the divisor; then $\frac{m}{n}$ is an approximation to the value of $\frac{A}{B}$, and an approximation that may be made as close as we please by taking a sufficiently great value of n.

For, if x is the magnitude of one of the parts into which B is divided, we have

$$B = nx, \text{ while } A > mx \text{ and } < (m+1)x.$$

Hence

$$\frac{A}{B} > \frac{mx}{nx} \text{ and } < \frac{(m+1)x}{nx};$$

that is,

$$\frac{A}{B} \text{ lies between } \frac{m}{n} \text{ and } \frac{m}{n} + \frac{1}{n};$$

and by increasing n we may make $\frac{1}{n}$, which is the difference between two numbers, one less and one greater than $\frac{A}{B}$, as small as we choose, and may thus make the less number $\frac{m}{n}$ as close an approximation to the value of $\frac{A}{B}$ as we please.

As a numerical example, take the ratio of the diagonal of a square to one of the sides (34). If the side is divided into three equal parts, the diagonal will contain one of these parts four times, with a remainder less than the divisor. $\frac{4}{3}$ is then an approximation, though a very rough one, to the value of the ratio in question, which must lie between $\frac{4}{3}$ and $\frac{5}{3}$.

If the side is divided into five equal parts, the diagonal will contain seven of them, and $\frac{7}{5}$ is a closer approximation. $\frac{141}{100}$ and $\frac{1414}{1000}$ are still closer approximations.

36. *Definition.* A *proportion* is an equation between two ratios. Thus, if the ratio $\frac{A}{B}$ is equal to the ratio $\frac{A'}{B'}$, the equation

$$\frac{A}{B} = \frac{A'}{B'}$$

is a proportion. It may be read, "Ratio of A to B equals ratio of A' to B'," or, "A is to B as A' is to B'."

A proportion is often written as follows:

$$A : B = A' : B',$$

where the notation $A : B$ is equivalent to $A \div B$. When thus written, A and B' are called the *extremes*, B and A' the *means*, and B' is called a fourth proportional to A, B and A'; the first terms, A and A', of the ratios are called the *antecedents*; the second terms, B and B', the *consequents*.

When the means are equal, as in the proportion

$$A : B = B : C,$$

the middle term B is called a *mean proportional* between A and C, and C is called a *third proportional* to A and B.

37. In cases where it is necessary to prove the equality of incommensurable ratios, it is usually best to employ what is called the *method of limits*.

38. *Definitions.* A *variable quantity*, or simply a *variable*, is a quantity whose value is supposed to change.

A *constant quantity*, or simply a *constant*, is a quantity whose value is fixed.

The value of a variable may be changed at pleasure, in which case it is called an *independent* variable; or it may be changed by changing at pleasure the value of some other variable or variables on which it depends, and in this case it is called a *dependent* variable.

39. *Definition.* If, by changing in some specified way the variable on which it depends, we can make a dependent variable approach as near as we please to some given constant, but can never make the values of the variable and the constant exactly coincide; or, in other words, if we can make the difference between the variable and the constant as small as we please, but cannot make it absolutely zero, the constant is called the *limit* of the variable under the circumstances specified.

40. For example, consider the fraction $\frac{1}{n}$, where n is supposed to be an independent variable,—*i.e.*, one whose value may be changed arbitrarily and to any extent,—the fraction $\frac{1}{n}$ is then a *dependent variable.* By increasing n at pleasure, $\frac{1}{n}$ may be made to approach as near as we please to the value zero, but can never be made exactly equal to zero.

We say, therefore, that zero is the *limit* of $\frac{1}{n}$, as n is indefinitely increased.

Again, the numerical approximation to the value of an incommensurable ratio (*v.* 35) is a *dependent variable*, depending upon the arbitrarily chosen number, n, of equal parts into which the denominator of the ratio is divided, and it has been shown to differ from the actual value of the ratio by an amount less than $\frac{1}{n}$. By increasing n at pleasure we can make this difference as small as we please, but can never make it absolutely zero, for in that case we should have found a common measure of the incommensurable numerator and denominator of the given ratio.

The actual value of an incommensurable ratio is, then, the *limit* approached by the approximation described in (35), as n is indefinitely increased.

41. The usefulness of the *method of limits* flows entirely from the following fundamental theorem, the truth of which is almost axiomatic.

Theorem.—If two variables dependent upon the same variable are so related that they are always equal, no matter what value is given to the variable on which they depend, and if, as the independent variable is changed in some specified way, each of them approaches a limit, the two limits must be absolutely equal.

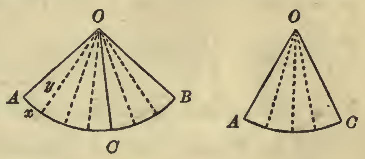
For, in considering two variables that are and that always remain equal to each other, we are dealing with a single varying value,—*i.e.*, their common value,—and it is clear that a single variable cannot be made to approach as near as we please to two different constant values at the same time, as if it is once brought between the two values in question, afterward, in approaching nearer to one, it must inevitably recede from the other.

The student should study this demonstration in connection with that of Proposition XII., which follows.

PROPOSITION XII.—THEOREM.

42. *In the same circle, or in equal circles, two angles at the centre are in the same ratio as their intercepted arcs.*

Let AOB and AOC be two angles at the centre of the same circle, or at the centres of equal circles; AB and AC, their intercepted arcs; then



$$\frac{AOB}{AOC}=\frac{AB}{AC}.$$

1st. Suppose the arcs to have a common measure, x, which is contained m times in AB and n times in AC. Then $AB = mx$ and $AC = nx$, and

$$\frac{AB}{AC}=\frac{mx}{nx}=\frac{m}{n}.$$

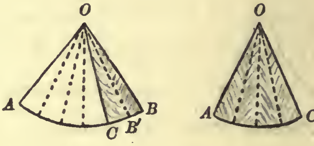
Apply the measure x to the arcs AB and AC, and draw radii to the points of division. The angle AOB is thus divided into m parts, and the angle AOC into n parts, all of which are equal, by Proposition III., Corollary. Call any one of these smaller angles y; then $AOB = my$ and $AOC = ny$, and

$$\frac{AOB}{AOC}=\frac{my}{ny}=\frac{m}{n}.$$

Therefore $$\frac{AOB}{AOC}=\frac{AB}{AC},$$

or (v. 36) $$AOB : AOC = AB : AC.$$

2d. If the arcs are incommensurable, suppose the arc AC to be divided into any arbitrarily chosen number, n, of equal parts, and let one of the parts be applied as many times as possible to the arc AB; let B' be the last point of division, and draw the radius OB'.



By construction, the arcs AB' and AC are commensurable. Therefore, by the proof above,

$$\frac{AOB'}{AOC} = \frac{AB'}{AC}.$$

If, now, we change n the number of parts into which AC is divided, AB' and AOB' will change, and consequently $\frac{AOB'}{AOC}$ and $\frac{AB'}{AC}$ will change. $\frac{AOB'}{AOC}$ and $\frac{AB'}{AC}$ are then variables depending upon the same variable, n.

By increasing n at pleasure we can make each of the equal parts into which AC is divided as small as we please, and consequently the remainder $B'B$, which is necessarily less than one of these parts, can be made as small as we please. It cannot, however, be made zero, for the arcs AB and AC are incommensurable, by hypothesis.

It is clear, then, that if n is indefinitely increased, AB' will have AB for its *limit*, and AOB' will have AOB for its *limit*. Hence

$$\frac{AOB}{AOC} \text{ is the limit of } \frac{AOB'}{AOC},$$

and

$$\frac{AB}{AC} \text{ is the limit of } \frac{AB'}{AC},$$

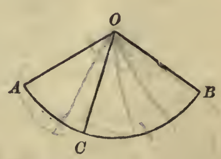
as n is indefinitely increased.

As the two variables $\frac{AOB'}{AOC}$ and $\frac{AB'}{AC'}$, both depending upon n, are always equal, no matter what the value of n, and each approaches a limit as n is indefinitely increased, the two limits in question are absolutely equal (41). Hence

$$\frac{AOB}{AOC}=\frac{AB}{AC}.$$

PROPOSITION XIII.—THEOREM.

43. *The numerical measure of an angle at the centre of a circle is the same as the numerical measure of its intercepted arc, if the adopted unit of angle is the angle at the centre which intercepts the adopted unit of arc.*



Let AOB be an angle at the centre O, and AB its intercepted arc. Let AOC be the angle which is adopted as the *unit of angle*, and let its intercepted arc AC be the arc which is adopted as the *unit of arc*. By Proposition XII., we have

$$\frac{AOB}{AOC}=\frac{AB}{AC}.$$

But the first of these ratios is the measure (28) of the angle AOB referred to the unit AOC; and the second ratio is the measure of the arc AB referred to the unit AC. Therefore, with the adopted units, the numerical measure of the angle AOB is the same as that of the arc AB.

44. *Scholium* I. This theorem, being of frequent application, is usually more briefly, though less accurately, expressed by saying that *an angle at the centre is measured by its inter-*

cepted arc. In this conventional statement of the theorem, the condition that the adopted units of angle and arc correspond to each other is understood; and the expression "is measured by" is used for "has the same numerical measure as."

45. *Scholium* II. The right angle is, by its nature, the most simple unit of angle; nevertheless custom has sanctioned a different unit.

The unit of angle generally adopted is an angle equal to $\frac{1}{90}$ part of a right angle, called a *degree*, and denoted by the symbol °. The corresponding unit of arc is $\frac{1}{90}$ part of a quadrant (10), and is also called a degree.

A right angle and a quadrant are therefore both expressed by 90°. Two right angles and a semi-circumference are both expressed by 180°. Four right angles and a whole circumference are both expressed by 360°.

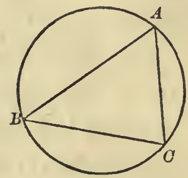
The degree (either of angle or arc) is subdivided into *minutes* and *seconds*, denoted by the symbols ′ and ″: a minute being $\frac{1}{60}$ part of a degree, and a second being $\frac{1}{60}$ part of a minute. Fractional parts of a degree less than one second are expressed by decimal parts of a second.

An angle, or an arc, of any magnitude is, then, numerically expressed by the unit degree and its subdivisions. Thus, for example, an angle equal to $\frac{1}{7}$ of a right angle, as well as its intercepted arc, will be expressed by $12° \ 51' \ 25''.714 \ldots$

46. *Definition.* When the sum of two arcs is a quadrant (that is, 90°), each is called the *complement* of the other.

When the sum of two arcs is a semi-circumference (that is, 180°), each is called the *supplement* of the other. See (I., 16).

47. *Definitions.* An *inscribed angle* is one whose vertex is on the circumference and whose sides are chords; as BAC.



In general, any rectilinear figure, as ABC, is said to be inscribed in a circle when its angular points are on the circumference; and the circle is then said to be *circumscribed* about the figure.

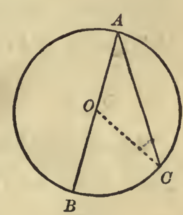
An angle is said to be *inscribed in a segment* when its vertex is in the arc of the segment, and its sides pass through the extremities of the subtending chord. Thus, the angle BAC is inscribed in the segment BAC.

PROPOSITION XIV.—THEOREM.

48. *An inscribed angle is measured by one-half its intercepted arc.*

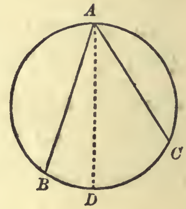
There may be three cases:

1st. Let one of the sides AB of the inscribed angle BAC be a diameter; then the measure of the angle BAC is one-half the arc BC.



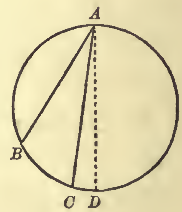
For, draw the radius OC. Then, AOC being an isosceles triangle, the angles OAC and OCA are equal (I., Proposition VIII.). The angle BOC, an exterior angle of the triangle AOC, is equal to the sum of the interior angles OAC and OCA (I., Proposition XXVI., Corollary), and therefore double either of them. But the angle BOC, at the centre, is measured by the arc BC (44); therefore the angle OAC is measured by one-half the arc BC.

2d. Let the centre of the circle fall within the inscribed angle BAC; then the measure of the angle BAC is one-half of the arc BC.

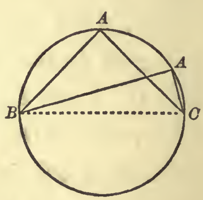


For, draw the diameter AD. The measure of the angle BAD is, by the first case, one-half the arc BD; and the measure of the angle CAD is one-half the arc CD; therefore the measure of the sum of the angles BAD and CAD is one-half the sum of the arcs BD and CD; that is, the measure of the angle BAC is one-half the arc BC.

3d. Let the centre of the circle fall without the inscribed angle BAC; then the measure of the angle BAC is one-half the arc BC.



For, draw the diameter AD. The measure of the angle BAD is, by the first case, one-half the arc BD; and the measure of the angle CAD is one-half the arc CD; therefore the measure of the difference of the angles BAD and CAD is one-half the difference of the arcs BD and CD; that is, the measure of the angle BAC is one-half the arc BC.



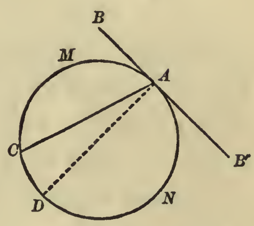
49. Corollary. *An angle inscribed in a semicircle is a right angle.*

EXERCISE.

Theorem.—The opposite angles of an inscribed quadrilateral are supplements of each other.

PROPOSITION XV.—THEOREM.

50. *An angle formed by a tangent and a chord is measured by one-half the intercepted arc.*

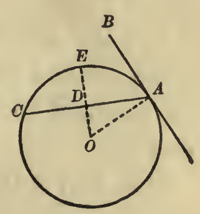


Let the angle BAC be formed by the tangent AB and the chord AC; then it is measured by one-half the intercepted arc AMC.

For, draw the diameter AD. The angle BAD, being a right angle (Proposition IX.), is measured by one-half the semi-circumference AMD; and the angle CAD is measured by one-half the arc CD; therefore the angle BAC, which is the difference of the angles BAD and CAD, is measured by one-half the difference of AMD and CD; that is, by one-half the arc AMC.

Also, the angle $B'AC$ is measured by one-half the intercepted arc ANC. For, it is the sum of the right angle $B'AD$ and the angle CAD, and is measured by one-half the sum of the semi-circumference AND and the arc CD; that is, by one-half the arc ANC.

EXERCISE.



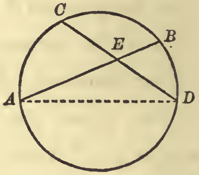
Prove Proposition XV. by the aid of this figure, OE being a radius perpendicular to AC.

Suggestion. Complements of the same angle are equal.

PROPOSITION XVI.—THEOREM.

51. *An angle formed by two chords, intersecting within the circumference, is measured by one-half the sum of the arcs intercepted between its sides and between the sides of its vertical angle.*

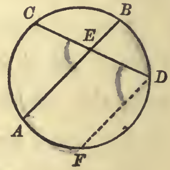
Let the angle *AEC* be formed by the chords *AB*, *CD*, intersecting within the circumference; then will it be measured by one-half the sum of the arcs *AC* and *BD*, intercepted between the sides of *AEC* and the sides of its vertical angle *BED*.



For, join *AD*. The angle *AEC* is equal to the sum of the angles *EDA* and *EAD*, and these angles are measured by one-half of *AC* and one-half of *BD*, respectively; therefore the angle *AEC* is measured by one-half the sum of the arcs *AC* and *BD*.

EXERCISE.

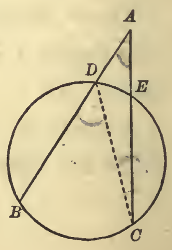
Prove Proposition XVI. by the aid of this figure, *DF* being drawn parallel to *AB*. (*v.* Proposition XI.)



PROPOSITION XVII.—THEOREM.

52. *An angle formed by two secants, intersecting without the circumference, is measured by one-half the difference of the intercepted arcs.*

Let the angle *BAC* be formed by the secants *AB* and *AC*; then will it be measured by one-half the difference of the arcs *BC* and *DE*.



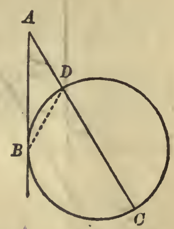
For, join *CD*. The angle *BDC* is equal to the sum of the angles *DAC* and *ACD*; therefore the angle *A* is equal to the difference of the angles BDC and ACD. But these angles are measured by one-half of BC and one-half of DE respectively; hence the angle A is measured by one-half the difference of BC and DE.

EXERCISE.

Prove Proposition XVII. by the aid of Proposition XI., drawing a suitable figure.

PROPOSITION XVIII.—THEOREM.

53. *An angle formed by a tangent and a secant is measured by half the difference of the intercepted arcs.*



For the angle A is equal to BDC minus ABD, by I., Proposition XXVI., Corollary.

54. COROLLARY. *An angle formed by two tangents is measured by half the difference of the intercepted arcs.*

EXERCISE.

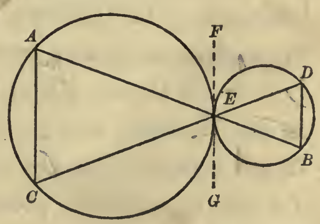
1. Prove Proposition XVIII. and its Corollary by the aid of Proposition XI.

2. *Theorem.—If, through the point of contact of two tangent circles, two secants are drawn, the chords joining the points where the secants cut the circles are parallel.*

Suggestion. $FED = CEG$,

$\therefore \ DBE = CAE.$

Consider, also, the case where the given circles are internally tangent.



PROBLEMS OF CONSTRUCTION.

Heretofore our figures have been assumed to be constructed under certain conditions, although methods of constructing them have not been given. Indeed, the precise construction of the figures was not necessary, inasmuch as they were only required as *aids* in following the demonstration of *principles.* We now proceed, first, to apply these principles in the solution of the simple problems necessary for the construction of the plane figures already treated of, and then to apply these simple problems in the solution of more complex ones.

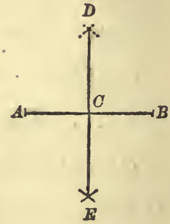
All the constructions of elementary geometry are effected solely by the straight line and the circumference, these being the only lines treated of in the elements; and these lines are practically *drawn*, or *described*, by the aid of the ruler and compasses, with the use of which the student is supposed to be familiar.

PROPOSITION XIX.—PROBLEM.

55. *To bisect a given straight line.*

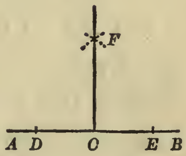
Let AB be the given straight line.

With the points A and B as centres, and with a radius greater than the half of AB, describe arcs intersecting in the two points D and E. Through these points draw the straight line DE, which bisects AB at the point C. For, D and E being equally distant from A and B, the straight line DE is perpendicular to AB at its middle point (I., Proposition XVIII.).



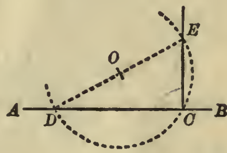
PROPOSITION XX.—PROBLEM.

56. *At a given point in a given straight line, to erect a perpendicular to that line.*



Let AB be the given line and C the given point.

Take two points, D and E, in the line and at equal distances from C. With D and E as centres, and a radius greater than DC or CE, describe two arcs intersecting in F. Then CF is the required perpendicular (I., Proposition XVIII.).

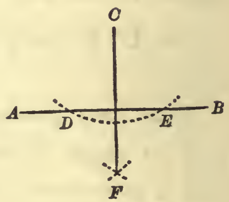


57. *Another solution.* Take any point O, without the given line, as a centre, and with a radius equal to the distance from O to C, describe a circumference intersecting AB in C and in a second point D. Draw the diameter DOE, and join EC. Then EC will be the required perpendicular; for the angle ECD, inscribed in a semicircle, is a right angle (Proposition XIV., Corollary).

This construction is often preferable to the preceding, especially when the given point C is at, or near, one extremity of the given line, and it is not convenient to produce the line through that extremity. The point O must evidently be so chosen as not to lie in the required perpendicular.

PROPOSITION XXI.—PROBLEM.

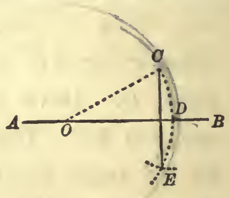
58. *From a given point without a given straight line, to let fall a perpendicular to that line.*



Let AB be the given line and C the given point.

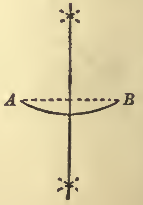
With C as a centre, and with a radius sufficiently great, describe an arc intersecting AB in D and E. With D and E as centres, and a radius greater than the half of DE, describe two arcs intersecting in F. The line CF is the required perpendicular (I., Proposition XVIII.).

59. *Another solution.* With any point O in the line AB as a centre, and with the radius OC, describe an arc CDE intersecting AB in D. With D as a centre, and a radius equal to the distance DC, describe an arc intersecting the arc CDE in E. The line CE is the required perpendicular. For, the point D is the middle of the arc CDE, and the radius OD drawn to this point is perpendicular to the chord CE (Proposition VI.).



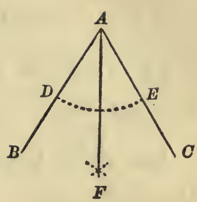
PROPOSITION XXII.—PROBLEM.

60. *To bisect a given arc or a given angle.*



1st. Let AB be a given arc.

Bisect its chord AB by a perpendicular, as in (55). This perpendicular also bisects the arc (Proposition VI.).

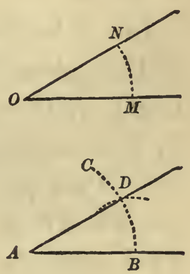


2d. Let BAC be a given angle. With A as a centre, and with any radius, describe an arc intersecting the sides of the angle in D and E. With D and E as centres, and with equal radii, describe arcs intersecting in F. The straight line AF bisects the arc DE, and consequently also the angle BAC.

61. *Scholium.* By the same construction, each of the halves of an arc, or an angle, may be bisected; and thus, by successive bisections, an arc, or an angle, may be divided into 4, 8, 16, 32, etc., equal parts.

PROPOSITION XXIII.—PROBLEM.

62. *At a given point in a given straight line, to construct an angle equal to a given angle.*

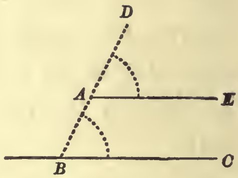


Let A be the given point in the straight line AB, and O the given angle.

With O as a centre, and with any radius, describe an arc MN terminated by the sides of the angle. With A as a centre, and with the same radius, OM, describe an indefinite arc BC. With B as a centre, and with a radius equal to the chord of MN, describe an arc intersecting the indefinite arc BC in D. Join AD. Then the angle BAD is equal to the angle O. For the chords of the arcs MN and BD are equal; therefore these arcs are equal, and consequently also the angles O and A.

PROPOSITION XXIV.—PROBLEM.

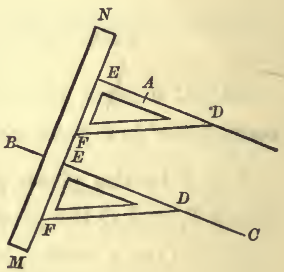
63. *Through a given point, to draw a parallel to a given straight line.*



Let A be the given point, and BC the given line.

From any point B in BC draw the straight line BAD through A. At the point A, by the preceding problem, construct the angle DAE equal to the angle ABC. Then AE is parallel to BC (I., Proposition XXIV., Corollary I.).

64. *Scholium.* This problem is, in practice, more accurately solved by the aid of a *triangle*, constructed of wood or metal. This triangle has one right angle, and its acute angles are usually made equal to 30° and 60°.



Let A be the given point, and BC the given line. Place the triangle, EFD, with one of its sides in coincidence with the given line BC. Then place the straight edge of a ruler, MN, against the side EF of the triangle. Now, keeping the ruler firmly fixed, slide the triangle along its edge until the side ED passes through the given point A. Trace the line EAD along the edge ED of the triangle; then it is evident that this line will be parallel to BC.

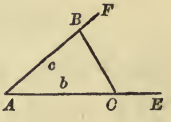
EXERCISE.

Problem. Two angles of a triangle being given, to find the third. (*v.* I., Proposition XXVI., and I., Proposition III., Corollary I.)

PROPOSITION XXV.—PROBLEM.

65. *Two sides of a triangle and their included angle being given, to construct the triangle.*

Let b and c be the given sides, and A their included angle.



Draw an indefinite line AE, and construct the angle $EAF = A$. On AE take $AC = b$, and on AF take $AB = c$; join BC. Then ABC is the triangle required; for it is formed with the data.

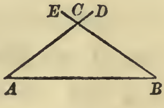
With the data, two sides and the included angle, only one triangle can be constructed; that is, *all* triangles constructed with these data are equal, and thus only repetitions of the same triangle (I., Proposition VI.).

66. *Scholium.* It is evident that *one* triangle is always possible, whatever may be the magnitude of the proposed sides and their included angle.

PROPOSITION XXVI.—PROBLEM.

67. *One side and two angles of a triangle being given, to construct the triangle.*

Two angles of the triangle being given, the third angle can be found; and we shall therefore always have given the two angles adjacent to the given side. Let, then, c be the given side, A and B the angles adjacent to it.



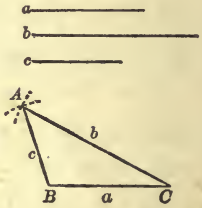
Draw a line $AB = c$; at A make an angle $BAD = A$, and at B an angle $ABE = B$. The lines AD and BE intersecting in C, we have ABC as the required triangle.

With these data but one triangle can be constructed (I., Proposition VII.).

68. *Scholium.* If the two given angles are together equal to or greater than two right angles, the problem is *impossible;* that is, no triangle can be constructed with the data; for the lines AD and BC will not intersect on that side of AB on which the angles have been constructed.

PROPOSITION XXVII.—PROBLEM.

69. *The three sides of a triangle being given, to construct the triangle.*



Let a, b, and c be the three given sides.

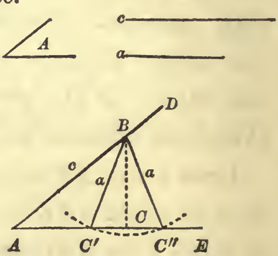
Draw $BC = a$; with C as a centre and a radius equal to b describe an arc; with B as a centre and a radius equal to c describe a second arc intersecting the first in A. Then ABC is the required triangle.

With these data but one triangle can be constructed (I., Proposition IX.).

70. *Scholium.* The problem is impossible when one of the given sides is equal to or greater than the sum of the other two (I., Axiom I.).

PROPOSITION XXVIII.—PROBLEM.

71. *Two sides of a triangle and the angle opposite to one of them being given, to construct the triangle.*



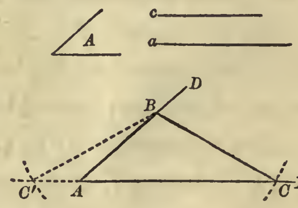
We shall consider two cases.

1st. When the given angle A is acute, and the given side a, opposite to it in the triangle, is less than the other given side c.

Construct an angle $DAE = A$. In one of its sides, as AD, take $AB = c$; with B as a centre and a radius equal to a, describe an arc which (since $a < c$) will intersect AE in two points, C' and C'', on the same side of A. Join BC' and BC''. Then either ABC' or ABC'' is the required triangle, since each is formed with the data; and the problem has two solutions.

There will, however, be but one solution, even with these data, when the side a is so much less than the side c as to be just equal to the perpendicular from B upon AE. For then the arc described from B as a centre, and with the radius a, will touch AE in a single point, C, and the required triangle will be ABC, right angled at C.

2d. When the given angle A is either acute, right, or obtuse, and the side a opposite to it is greater than the other given side c.



The same construction being made as in the first case, the arc described with B as a centre, and with a radius equal to a, will intersect AE in only one point, C, on the same side of A. Then ABC will be the triangle required, and will be the only possible triangle with the data.

The second point of intersection, C', will fall in EA produced, and the triangle ABC' thus formed will not contain the given angle.

72. *Scholium.* The problem is impossible when the given angle A is acute and the proposed side opposite to it is less than the perpendicular from B upon AE; for then the arc described from B will not intersect AE.

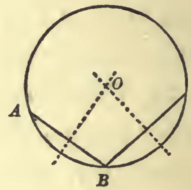
The problem is also impossible when the given angle is right, or obtuse, if the given side opposite to the angle is less than the other given side; for either the arc described from B would not intersect AE, or it would intersect it only when produced through A.

EXERCISE.

Problem.—The adjacent sides of a parallelogram and their included angle being given, to construct the parallelogram.

PROPOSITION XXIX.—PROBLEM.

73. *To find the centre of a given circumference, or of a given arc.*



Take any three points, A, B, and C, in the given circumference or arc, and join them by chords AB, BC. The perpendiculars erected at the middle points of these chords will intersect in the required centre (Proposition VI., Corollary I.).

74. *Scholium* I. Only one solution is possible; for, since the centre is equidistant from B and C, it must lie in the perpendicular erected at the middle point of BC (I., Proposition XVIII.), and since it is equidistant from A and B, it must lie in the perpendicular erected at the middle point of AB; and these perpendiculars can have but one point in common.

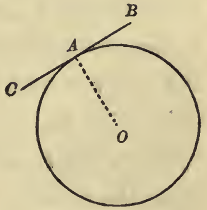
75. *Scholium* II. The same construction serves to describe a circumference which shall pass through three given points, A, B, C; or to *circumscribe* a circle about a given triangle, ABC; that is, to describe a circumference in which the given triangle shall be *inscribed* (47).

76. *Scholium* III. It follows from *Scholium* I. that three points not in the same straight line will *determine* a circumference,—*i.e*, through three points not in the same straight line one circumference, and only one, can be drawn.

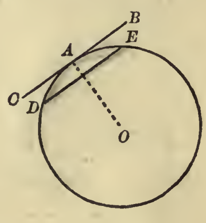
Hence two circumferences cannot intersect in more than two points; for if they had three points in common they would coincide throughout.

PROPOSITION XXX.—PROBLEM.

77. *At a given point in a given circumference, to draw a tangent to the circumference.*



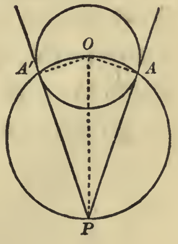
Let A be the given point in the given circumference. Draw the radius OA, and at A draw BAC perpendicular to OA; BC will be the required tangent (Proposition IX.).



If the centre of the circumference is not given, it may first be found by the preceding problem, or we may proceed more directly as follows: take two points, D and E, equidistant from A; draw the chord DE, and through A draw BAC parallel to DE. Since A is the middle point of the arc DE, the radius drawn to A will be perpendicular to DE (Proposition VI.), and consequently also to BC; therefore BC is a tangent at A.

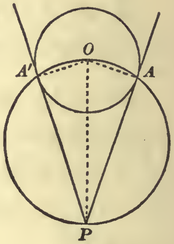
PROPOSITION XXXI.—PROBLEM.

78. *Through a given point without a given circle, to draw a tangent to the circle.*



Let O be the centre of the given circle and P the given point.

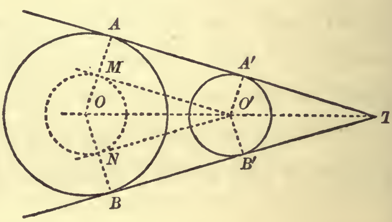
Upon OP, as a diameter, describe a circumference intersecting the circumference of the given circle in two points, A and A'. Draw PA and PA', both of which will be tangent to the given circle. For, drawing the radii OA and OA', the angles OAP and $OA'P$ are right angles (Proposition XIV., Corollary); therefore PA and PA' are tangents (Proposition IX.).



In practice, this problem is accurately solved by placing the straight edge of a ruler through the given point and tangent to the given circumference, and then tracing the tangent by the straight edge. The precise point of tangency is then determined by drawing a perpendicular to the tangent from the centre.

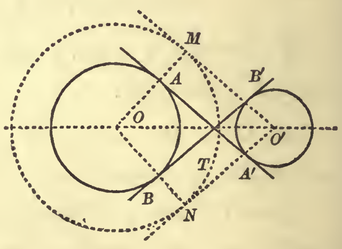
79. *Scholium.* This problem always admits of two solutions.

EXERCISE.



Problem.—To draw a common tangent to two given circles.

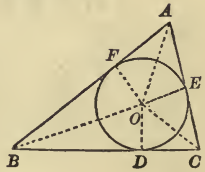
Suggestion. For an *exterior* common tangent, in the larger circle draw a concentric circle whose radius is the difference of the radii of the given circles. For an *interior* common tangent, about one of the circles draw a concentric circle whose radius is the sum of the radii of the given circles.



PROPOSITION XXXII.—PROBLEM.

80. *To inscribe a circle in a given triangle.*

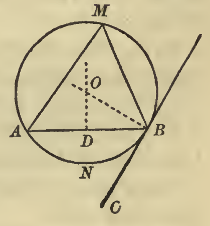
Let ABC be the given triangle. Bisect any two of its angles, as B and C, by straight lines meeting in O. From the point O let fall perpendiculars OD, OE, OF, upon the three sides of the triangle; these perpendiculars will be equal to each other (I., Proposition XIX.). Hence the circumference of a circle, described with the centre O, and a radius $= OD$, will pass through the three points D, E, F, will be tangent to the three sides of the triangle at these points (Proposition IX.), and will therefore be inscribed in the triangle.



EXERCISE.

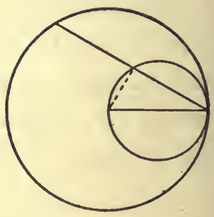
Problem.—Upon a given straight line, to describe a segment which shall contain a given angle.

Suggestion. Through one end of the given line AB draw a line BC, making with it the given angle. The two lines will be one a chord and the other a tangent. Hence the centre of the circle can be found.



EXERCISES ON BOOK II.

THEOREMS.



1. If two circumferences are tangent internally, and the radius of the larger is the diameter of the smaller, then any chord of the larger drawn from the point of contact is bisected by the circumference of the smaller (*v.* Proposition XIV., Corollary, and Proposition VI.).

2. If two equal chords intersect within a circle, the segments of one are respectively equal to the segments of the other. What is the corresponding theorem for the case where the chords meet when produced?

3. A circumference described on the hypotenuse of a right triangle as a diameter passes through the vertex of the right angle. (*v.* Proposition XIV., Corollary.)

4. The circles described on two sides of a triangle as diameters intersect on the third side.

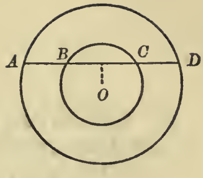
Suggestion. Drop a perpendicular from the opposite vertex upon the third side.

5. The perpendiculars from the angles upon the opposite sides of a triangle are the bisectors of the angles of the triangle formed by joining the feet of the perpendiculars.

Suggestion. On the three sides of the given triangle as diameters describe circumferences. (*v.* Exercise 3, Proposition XIV., and I., Proposition XXVI.).

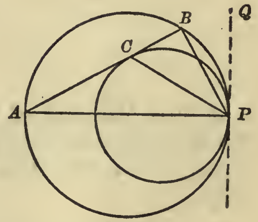
6. If a circle is circumscribed about an equilateral triangle, the perpendicular from its centre upon a side of the triangle is equal to one-half of the radius.

7. The portions of any straight line which are intercepted between the circumferences of two concentric circles are equal.



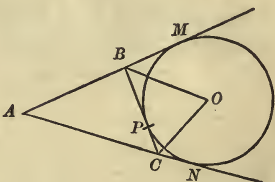
8. Two circles are tangent internally at P, and a chord AB of the larger circle touches the smaller at C; prove that PC bisects the angle APB.

Suggestion. $CPQ = BCP$, $BPQ = BAP$, $BCP - BAP = APC$ (I., Proposition XXVI., Corollary).



9. If a triangle ABC is formed by the intersection of three tangents to a circumference, two of which, AM and AN, are fixed, while the third, BC, touches the circumference at a variable point P, prove that the perimeter of the triangle ABC is constant, and equal to $AM + AN$, or $2AN$ (Proposition X.).

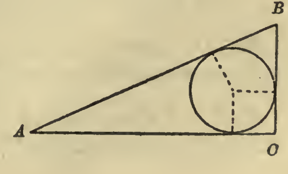
Also, prove that the angle BOC is constant.



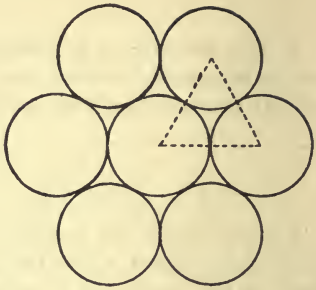
10. If through one of the points of intersection of two circumferences a diameter of each circle is drawn, the straight line which joins the extremities of these diameters passes through the other point of intersection, and is parallel to the line joining the centres.

Suggestion. Draw the common chord and the line joining the centres. (*v.* Proposition VI., Corollary II., and Exercise 29, Book I.)

11. The difference between the hypotenuse of a right triangle and the sum of the other two sides is equal to the diameter of the inscribed circle.



12. A circle can be entirely surrounded by six circles having the same radius with it.



13. The bisectors of the vertical angles of all triangles having the same base and equal vertical angles have a point in common.

Suggestion. The triangles may all be inscribed in the same circle.

14. If the hypotenuse of a right triangle is double one of the sides, the acute angles of the triangle are 30° and 60° respectively.

15. If, from a point whose distance from the centre of a given circle is equal to a diameter, tangents are drawn to the circle, they will make with each other an angle of 60°.

LOCI.

16. Find the locus of the centre of a circumference which passes through two given points. (*v.* I., Proposition XVIII.)

17. Find the locus of the centre of a circumference which is tangent to two given straight lines. (*v.* I., Proposition XIX.)

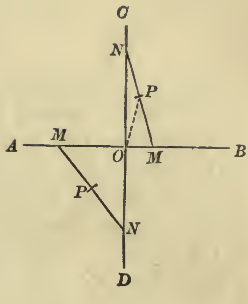
18. Find the locus of the centre of a circumference which is tangent to a given straight line at a given point of that line, or to a given circumference at a given point of that circumference.

19. Find the locus of the centre of a circumference passing through a given point and having a given radius.

20. Find the locus of the centre of a circumference tangent to a given straight line and having a given radius.

21. Find the locus of the centre of a circumference of given radius, tangent externally or internally to a given circumference.

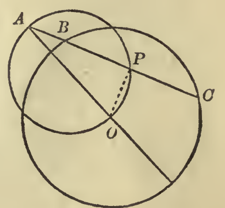
22. A straight line MN, of given length, is placed with its extremities on two given perpendicular lines AB, CD; find the locus of its middle point P (Exercise 31, Book I.).



23. A straight line of given length is inscribed in a given circle; find the locus of its middle point. (v. Proposition VII.)

24. A straight line is drawn through a given point A, intersecting a given circumference in B and C; find the locus of the middle point, P, of the intercepted chord BC.

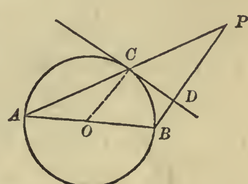
Note the special case in which the point A is on the given circumference.



25. From any point A in a given circumference, a straight line AP of fixed length is drawn parallel to a given line MN; find the locus of the extremity P. (v. I., Proposition XXX.)

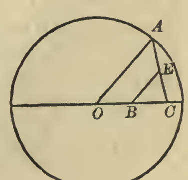
26. From one extremity A of a fixed diameter AB, any chord AC is drawn, and at C a tangent CD. From B, a perpendicular BD to the tangent is drawn, meeting AC in P. Find the locus of P.

Suggestion. (Draw radius OC. v. I., Exercise 28.)



27. The base BC of a triangle is given, and the medial line BE, from B, is of a given length. Find the locus of the vertex A.

Suggestion. Draw AO parallel to EB. Since $BO = BC$, O is a fixed point; and since $AO = 2BE$, OA is a constant distance.



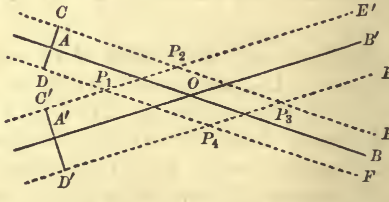
PROBLEMS.

The most useful general precept that can be given, to aid the student in his search for the solution of a problem, is the following: Suppose the problem solved, and construct a figure accordingly; study the properties of this figure, drawing auxiliary lines when necessary, and endeavor to discover the dependence of the problem upon previously solved problems. This is an *analysis* of the problem. The reverse process, or *synthesis*, then furnishes a construction of the problem. In the analysis, the student's ingenuity will be exercised especially in drawing useful auxiliary lines; in the synthesis, he will often find room for invention in combining in the most simple form the several steps suggested by the analysis.

The analysis frequently leads to the solution of a problem by the *intersection of loci.* The solution may turn upon the determination of the position of a particular point. By one condition of the problem it may appear that this required point is necessarily one of the points of a certain line; this line is a locus of the point satisfying that condition. A second condition of the problem may furnish a second locus of the point; and the point is then fully determined, being the intersection of the two loci.

Some of the following problems are accompanied by an analysis to illustrate the process.

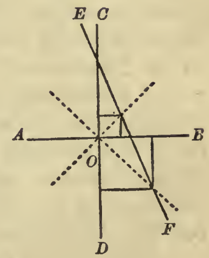
28. To determine a point whose distances from two given intersecting straight lines, AB, $A'B'$, are given.



Analysis. The locus of all the points which are at a given distance from AB consists of two parallels to AB, CE, and DF, each at the given distance from AB. The locus of all the points at a given distance from $A'B'$ consists of two parallels, $C'E'$ and $D'F'$, each at the given distance from $A'B'$. The required point must be in both loci, and therefore in their intersection. There are in this case four intersections of the loci, and the problem has four solutions.

Construction. At any point of AB, as A, erect a perpendicular CD, and make $AC = AD =$ the given distance from AB; through C and D draw parallels to AB. In the same manner, draw parallels to $A'B'$ at the given distance $A'C' = A'D'$. The intersection of the four parallels determines the four points P_1, P_2, P_3, P_4, each of which satisfies the conditions.

29. Given two perpendiculars, AB and CD, intersecting in O, to construct a square, one of whose angles shall coincide with one of the right angles at O, and the vertex of the opposite angle of the square shall lie on a given straight line EF. (Two solutions.)



30. In a given straight line, to find a point equally distant from two given points without the line.

31. To construct a square, given its diagonal.

32. Through a given point P within a given angle, to draw a straight line, terminated by the sides of the angle, which shall be bisected at P. (*v.* Exercise 28, Book I.)

33. Given two straight lines which cannot be produced to their intersection, to draw a third which would pass through their intersection and bisect their contained angle.

Suggestion. Find two points equidistant from the two lines. (*v.* I., Proposition XIX.)

34. Given the middle point of a chord in a given circle, to draw the chord.

35. To draw a tangent to a given circle which shall be parallel to a given straight line.

36. To draw a tangent to a given circle, such that its segment intercepted between the point of contact and a given straight line shall have a given length.

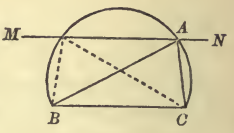
Suggestion. The tangent, the radius drawn to the point of contact, and a line drawn from the centre to the end of the tangent form a right triangle, two of whose sides are known. A simple construction gives the hypotenuse.

In general there are four solutions. Show when there will be but two; also, when no solution is possible.

37. Through a given point within or without a given circle, to draw a straight line, intersecting the circumference, so that the intercepted chord shall have a given length. (Two solutions.) (*v.* Exercise 23 and Section 78.)

38. Construct an angle of 60°, one of 120°, one of 30°, one of 150°, one of 45°, and one of 135°.

39. Construct a triangle, given the base, the angle opposite to the base, and the altitude.



Analysis. Suppose BAC to be the required triangle. The side BC being fixed in position and magnitude, the vertex A is to be determined. One locus of A is an arc of a segment, described upon AB, containing the given angle. Another locus of A is a straight line MN drawn parallel to BC, at a distance from it equal to the given altitude. Hence the position of A will be found by the intersection of these two loci, both of which are readily constructed.

Limitation. If the given altitude were greater than the perpendicular distance from the middle of BC to the arc BAC, the arc would not intersect the line MN, and there would be no solution possible.

The limits of the data within which the solution of any problem is possible should always be determined.

40. Construct a triangle, given the base, the medial line to the base, and the angle opposite to the base.

41. With a given radius, describe a circumference, 1st, tangent to two given straight lines; 2d, tangent to a given straight line and to a given circumference; 3d, tangent to two given circumferences; 4th, passing through a given point and tangent to a given straight line; 5th, passing through a given point and tangent to a given circumference; 6th, having its centre on a given straight line, or a given circumference, and tangent to a given straight line, or to a given circumference. (Exercises 19, 20, 21.)

42. Describe a circumference, 1st, tangent to two given straight lines, and touching one of them at a given point (Exercises 17, 18); 2d, tangent to a given circumference at a given point and tangent to a given straight line; 3d, tangent to a given straight line at a given point and tangent to a given circumference (Exercise 18); 4th, passing through a given point and tangent to a given straight line at a given point; 5th, passing through a given point and tangent to a given circumference at a given point.

43. Draw a straight line equally distant from three given points. When will there be but three solutions, and when an indefinite number of solutions?

44. Inscribe a straight line of given length between two given circumferences, and parallel to a given straight line. (*v.* Exercise 25.)

BOOK III.

PROPORTIONAL LINES. SIMILAR FIGURES.

THEORY OF PROPORTION.

1. *Definition.* One quantity is said to be *proportional* to another when the ratio of any two values, A and B, of the first, is equal to the ratio of the two corresponding values A' and B', of the second; so that the four values form the proportion (II., 36)

$$A : B = A' : B',$$

or

$$\frac{A}{B} = \frac{A'}{B'}.$$

This definition presupposes two quantities, each of which can have various values, so related to each other that each value of one corresponds to a value of the other. An example occurs in the case of an angle at the centre of a circle and its intercepted arc. The angle may *vary*, and with it also the arc; but to each value of the angle there corresponds a certain value of the arc. It has been proved (II., Proposition XII.) that the ratio of any two values of the angle is equal to the ratio of the two corresponding values of the arc; and, in accordance with the definition just given, this proposition would be briefly expressed as follows: "The angle at the centre of a circle is proportional to its intercepted arc."

2. *Definition.* One quantity is said to be *reciprocally proportional* to another when the ratio of two values, A and B, of the first, is equal to the reciprocal of the ratio of the two corresponding values, A' and B', of the second, so that the four values form the proportion

$$A : B = B' : A',$$

or

$$\frac{A}{B} = \frac{B'}{A'} = 1 \div \frac{A'}{B'}.$$

For example, if the product p of two numbers, x and y, is given, so that we have

$$xy = p,$$

then x and y may each have an indefinite number of values, but as x increases y diminishes. If, now, A and B are two values of x, while A' and B' are the two corresponding values of y, we must have

$$A \times A' = p,$$
$$B \times B' = p,$$

whence, by dividing one of these equations by the other,

$$\frac{A}{B} \times \frac{A'}{B'} = 1,$$

and therefore

$$\frac{A}{B} = \frac{1}{\frac{A'}{B'}} = \frac{B'}{A'};$$

that is, *two numbers whose product is constant are reciprocally proportional.*

3. Let the quantities in each of the couplets of the proportion

$$\frac{A}{B} = \frac{A'}{B'}, \text{ or } A : B = A' : B', \qquad [1]$$

be measured by a unit of their own kind, and thus expressed by numbers (II., 28); let a and b denote the numerical measures of A and B, a' and b' those of A' and B'; then (II., 29)

$$\frac{A}{B}=\frac{a}{b}, \qquad \frac{A'}{B'}=\frac{a'}{b'},$$

and the proportion [1] may be replaced by the *numerical* proportion

$$\frac{a}{b}=\frac{a'}{b'}, \text{ or } a:b=a':b'.$$

4. Conversely, if the numerical measures a, b, a', b', of four quantities, A, B, A', B', are in proportion, these quantities themselves are in proportion, *provided* that A and B are quantities of the same kind, and A' and B' are quantities of the same kind (though not necessarily of the same kind as A and B); that is, if we have

$$a:b=a':b',$$

we may, under these conditions, infer the proportion

$$A:B=A':B'.$$

5. Let us now consider the numerical proportion

$$a:b=a':b'.$$

Writing it in the form

$$\frac{a}{b}=\frac{a'}{b'},$$

and multiplying both members of this equality by bb', we obtain

$$ab'=a'b,$$

whence the theorem: *the product of the extremes of a (numerical) proportion is equal to the product of the means.*

Corollary. If the means are equal, as in the proportion $a : b = b : c$, we have $b^2 = ac$, whence $b = \sqrt{ac}$; that is, *a mean proportional* (II., 36) *between two numbers is equal to the square root of their product.*

6. Conversely, *if the product of two numbers is equal to the product of two others, either two may be made the extremes, and the other two the means, of a proportion.* For, if we have given

$$ab' = a'b,$$

then, dividing by bb', we obtain

$$\frac{a}{b} = \frac{a'}{b'}, \text{ or } a : b = a' : b'.$$

Corollary. The terms of a proportion may be written in any order which will make the product of the extremes equal to the product of the means. Thus, any one of the following proportions may be inferred from the given equality $ab' = a'b$:

$$\begin{aligned} a : b &= a' : b', \\ a : a' &= b : b', \\ b : a &= b' : a', \\ b : b' &= a : a', \\ b' : a' &= b : a, \text{ etc.} \end{aligned}$$

Also, any one of these proportions may be inferred from any other.

7. *Definitions.* When we have given the proportion

$$a : b = a' : b',$$

and infer the proportion

$$a : a' = b : b',$$

the second proportion is said to be deduced *by alternation.*

When we infer the proportion

$$b : a = b' : a',$$

this proportion is said to be deduced *by inversion.*

8. It is important to observe that when we speak of the products of the extremes and means of a proportion, it is implied that at least two of the terms are numbers. If, for example, the terms of the proportion

$$A : B = A' : B'$$

are all *lines*, no meaning can be directly attached to the products $A \times B'$, $B \times A'$, since in a product the multiplier at least must be a number.

But if we have a proportion such as

$$A : B = m : n,$$

in which m and n are numbers, while A and B are any two quantities of the same kind, then we may infer the equality $nA = mB$.

Nevertheless, we shall, for the sake of brevity, often speak of *the product of two lines*, meaning thereby *the product of the numbers which represent those lines when they are measured by a common unit.*

9. If A and B are any two quantities of the same kind, and m any number whole or fractional, we have, identically,

$$\frac{mA}{mB} = \frac{A}{B};$$

that is, *equimultiples of two quantities are in the same ratio as the quantities themselves.*

Similarly, if we have the proportion

$$A : B = A' : B',$$

and if m and n are any two numbers, we can infer the proportions

$$mA : mB = nA' : nB',$$
$$mA : nB = mA' : nB'.$$

10. *Composition and Division.* If we have given the proportion $\frac{A}{B} = \frac{A'}{B'}$, we have, by alternation,

$$\frac{A}{A'} = \frac{B}{B'}.$$

Let r be the common value of these two ratios; then

$$\frac{A}{A'} = r, \text{ and } \frac{B}{B'} = r,$$

and

$$A = rA', \text{ and } B = rB'.$$

Adding the second equation to the first, we have

$$A + B = r(A' + B'),$$

or

$$\frac{A + B}{A' + B'} = r = \frac{A}{A'} = \frac{B}{B'}.$$

The proportions $\frac{A + B}{A' + B'} = \frac{A}{A'}$, and $\frac{A + B}{A' + B'} = \frac{B}{B'}$ are said to be formed from the given proportion

$$A : B = A' : B', \text{ by } \textit{composition}.$$

If we subtract the equation $B = rB'$ from $A = rA'$, we have

$$A - B = r(A' - B'),$$

whence, as above,

$$\frac{A - B}{A' - B'} = \frac{A}{A'},$$

and

$$\frac{A - B}{A' - B'} = \frac{B}{B'},$$

two proportions which are said to be formed from the given proportion

$$A : B = A' : B', \text{ by } \textit{division}.$$

11. *Definition.* A *continued proportion* is a series of equal ratios, as

$$A : B = A' : B' = A'' : B'' = A''' : B''' = \text{etc.}$$

12. Let r denote the common value of the ratio in the continued proportion of the preceding article; that is, let

$$r = \frac{A}{B} = \frac{A'}{B'} = \frac{A''}{B''} = \frac{A'''}{B'''} = \text{etc.};$$

then we have

$$A = Br, \quad A' = B'r, \quad A'' = B''r, \quad A''' = B'''r, \text{ etc.},$$

and, adding these equations,

$$A + A' + A'' + A''' + \text{etc.} = (B + B' + B'' + B''' + \text{etc.})\, r,$$

whence

$$\frac{A + A' + A'' + A''' + \text{etc.}}{B + B' + B'' + B''' + \text{etc.}} = r = \frac{A}{B} = \frac{A'}{B'} = \text{etc.};$$

that is, *the sum of any number of the antecedents of a continued proportion is to the sum of the corresponding consequents as any antecedent is to its consequent.*

In this theorem the quantities A, B, C, etc., must all be quantities of the same kind.

If, instead of a continued proportion, we have an ordinary proportion, the theorem just proved obviously holds good.

13. If we have any number of proportions, as

$$a : b = c : d,$$
$$a' : b' = c' : d',$$
$$a'' : b'' = c'' : d'', \text{ etc.};$$

then, writing them in the form

$$\frac{a}{b} = \frac{c}{d}, \quad \frac{a'}{b'} = \frac{c'}{d'}, \quad \frac{a''}{b''} = \frac{c''}{d''}, \text{ etc.},$$

and multiplying these equations together, we have

$$\frac{a\, a'\, a'' \ldots}{b\, b'\, b'' \ldots} = \frac{c\, c'\, c'' \ldots}{d\, d'\, d'' \ldots},$$

or

$$a\, a'\, a'' \ldots : b\, b'\, b'' \ldots = c\, c'\, c'' \ldots : d\, d'\, d'' \ldots;$$

that is, *if the corresponding terms of two or more proportions are multiplied together, the products are in proportion.*

If the corresponding terms of the several proportions are equal, that is, if $a = a' = a''$, $b = b' = b''$, etc., then the multiplication of two or more proportions gives

$$a^2 : b^2 = c^2 : d^2,$$
$$a^3 : b^3 = c^3 : d^3;$$

that is, *if four numbers are in proportion, like powers of these numbers are in proportion.*

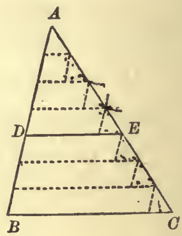
PROPORTIONAL LINES.

PROPOSITION I.—THEOREM.

14. *A parallel to the base of a triangle divides the other two sides proportionally.*

Let DE be a parallel to the base, BC, of the triangle ABC; then

$$AB : AD = AC : AE.$$



1st. Suppose the lines AB, AD, to have a common measure which is contained m times in AB and n times in AD. Then $\frac{AB}{AD} = \frac{m}{n}$.

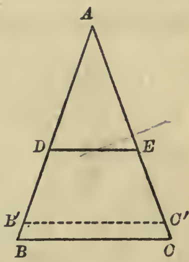
Apply this measure to AB, and through the points of division draw lines parallel to the base BC of the triangle; then through the points of intersection of these lines with AC draw lines parallel to AB. The small triangles thus formed are all equal, by Propositions XXIX. and VII., Book I. Hence the m parts into which AC is divided are all equal, and, as AE contains n of these parts,

$$\frac{AC}{AE} = \frac{m}{n}.$$

Therefore

$$\frac{AB}{AD} = \frac{AC}{AE}.$$

2d. If AB and AD are incommensurable, let AD be divided into any arbitrarily chosen number n of equal parts, and let AB be divided by one of these parts. Let B' be the last point of division, $B'B$ being of course less than the divisor. Through B' draw $B'C'$ parallel to DE.



Since AB' and AD are commensurable, $\frac{AB'}{AD}=\frac{AC'}{AE}$, and this holds true no matter what value may be given to n. By taking a sufficiently great value for n, we can make B' come as near as we please to B; but we cannot make B' and B coincide, since no divisor of AD can divide AB without remainder.

AB' and AC', and consequently $\frac{AB'}{AD}$ and $\frac{AC'}{AE}$, are then variables dependent upon the same variable, n; and, as we have seen above, they are equal, no matter what value is given to n. If n is indefinitely increased,

$$\frac{AB'}{AD} \text{ has the limit } \frac{AB}{AD},$$

and

$$\frac{AC'}{AE} \text{ has the limit } \frac{AC}{AE}.$$

Therefore, by the fundamental theorem in the Doctrine of Limits (41, Book II.), these limits are equal, and therefore

$$\frac{AB}{AD}=\frac{AC}{AE}.$$

Compare this reasoning with that in II., 42.

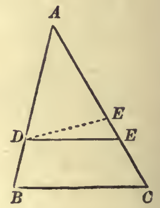
EXERCISE.

Show that in Proposition I. $AD : DB = AE : EC$ (*v.* 10), and also that

$$\frac{AB}{AC}=\frac{AD}{AE}=\frac{DB}{EC}.$$

PROPOSITION II.—THEOREM.

15. Conversely, *if a straight line divides two sides of a triangle proportionally, it is parallel to the third side.*



Let DE divide the sides AB, AC, of the triangle ABC, proportionally; then DE is parallel to BC.

For, if DE is not parallel to BC, let some other line DE', drawn through D, be parallel to BC. Then, by Proposition I.,

$$AB : AD = AC : AE'.$$

But, by hypothesis, we have

$$AB : AD = AC : AE.$$

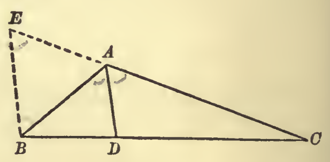
Hence

$$\frac{AC}{AE'} = \frac{AC}{AE},$$

whence it follows that $AE' = AE$, which is impossible unless DE' coincides with DE. Therefore DE is parallel to BC.

EXERCISE.

1. *Theorem.—The line bisecting the vertical angle of a triangle divides the base into segments proportional to the adjacent sides of the triangle.*



Suggestion. Through B draw a line parallel to the bisector and extend the side CA to meet it. The triangle EAB is isosceles. $\therefore AE = AB$. CE and CB are divided proportionally (Proposition I.). Hence $CD : DB = CA : AB$.

2. Prove the converse of Exercise I. (*v.* Proposition II.)

SIMILAR POLYGONS.

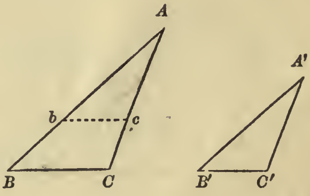
16. *Definitions.* Two polygons are *similar* when they are mutually equiangular and have their homologous sides proportional.

In similar polygons, any points, angles, or lines, similarly situated in each, are called *homologous.*

The ratio of a side of one polygon to its homologous side in the other is called the *ratio of similitude* of the polygons.

PROPOSITION III.—THEOREM.

17. *Two triangles are similar when they are mutually equiangular.*



Let ABC, $A'B'C'$, be mutually equiangular triangles, in which $A = A'$, $B = B'$, $C = C'$; then these triangles are similar.

For, superpose the triangle $A'B'C'$ upon the triangle ABC, making the angle A' coincide with its equal, the angle A, and let B' fall at b and C' at c. Since the angle Abc is equal to B, bc is parallel to BC (I., Proposition XXIV., Corollary I.), and we have (Proposition I.)

$$AB : Ab = AC : Ac,$$

or

$$AB : A'B' = AC : A'C'.$$

If, now, we superpose $A'B'C'$ upon ABC, making B' coincide with B, we may prove, in the same manner, that

$$AB : A'B' = BC : B'C';$$

and, combining these proportions,

$$\frac{AB}{A'B'} = \frac{AC}{A'C'} = \frac{BC}{B'C'}. \qquad [1]$$

Therefore the homologous sides are proportional, and the triangles are similar (16).

18. *Scholium* I. The homologous sides lie opposite to equal angles.

19. *Scholium* II. The ratio of similitude (16) of the two similar triangles is any one of the equal ratios in the continued proportion [1].

EXERCISE.

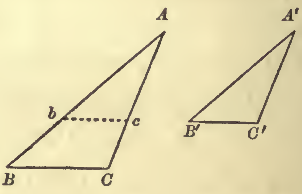
Theorem.—The altitudes of two similar triangles are to each other in the ratio of similitude of the triangles.

PROPOSITION IV.—THEOREM.

20. *Two triangles are similar when an angle of the one is equal to an angle of the other, and the sides including these angles are proportional.*

In the triangles ABC, $A'B'C'$, let $A = A'$, and

$$\frac{AB}{A'B'} = \frac{AC}{A'C'};$$



then these triangles are similar.

For, place the angle A' upon its equal angle A; let B' fall at b, and C' at c. Then, by the hypothesis,

$$\frac{AB}{Ab} = \frac{AC}{Ac}.$$

Therefore bc is parallel to BC (Proposition II.), and the triangle Abc is similar to ABC (Proposition III.). But Abc is equal to $A'B'C'$; therefore $A'B'C'$ is also similar to ABC.

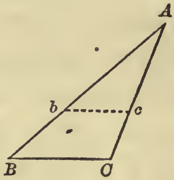
PROPOSITION V.—THEOREM.

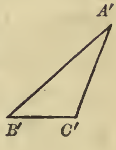
21. *Two triangles are similar when their homologous sides are proportional.*

In the triangles ABC, $A'B'C'$, let

$$\frac{AB}{A'B'} = \frac{AC}{A'C'} = \frac{BC}{B'C'}; \qquad [1]$$

then these triangles are similar.





For, take $Ab = A'B'$, and $Ac = A'C'$, and join b and c.

Abc is similar to ABC, by Proposition IV. Therefore

$$\frac{AB}{Ab} = \frac{BC}{bc}, \text{ or } \frac{AB}{A'B'} = \frac{BC}{bc}.$$

But, by hypothesis,

$$\frac{AB}{A'B'} = \frac{BC}{B'C'}.$$

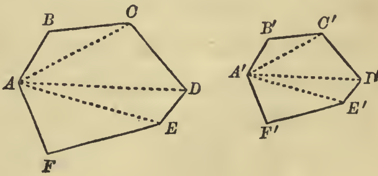
Hence

$$\frac{BC}{B'C'} = \frac{BC}{bc}, \text{ and } B'C' = bc.$$

The triangle $A'B'C'$ is then equal to Abc, by I., Proposition IX., and is consequently similar to ABC.

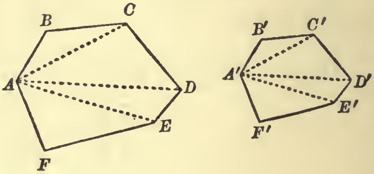
PROPOSITION VI.—THEOREM.

22. *If two polygons are composed of the same number of triangles similar each to each and similarly placed, the polygons are similar.*



Let the polygon $ABCD$, etc., be composed of the triangles ABC, ACD, etc.; and let the polygon $A'B'C'D'$, etc., be composed of the triangles $A'B'C'$, $A'C'D'$, etc., similar to ABC, ACD, etc., respectively, and similarly placed; then the polygons are similar.

1st. The polygons are mutually equiangular. For, the homologous angles of the similar triangles are equal; and any two corresponding angles of the polygons are either homologous angles of two similar triangles, or sums of homologous angles of two or more similar triangles. Thus, $B = B'$; $BCD = BCA + ACD = B'C'A' + A'C'D' = B'C'D'$; etc.



2d. Their homologous sides are proportional. For, from the similar triangles, we have

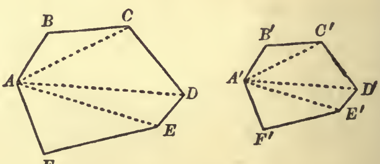
$$\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{AC}{A'C'} = \frac{CD}{C'D'} = \frac{AD}{A'D'} = \frac{DE}{D'E'} = \text{etc.}$$

Therefore the polygons fulfil the two conditions of similarity (16).

PROPOSITION VII.—THEOREM.

23. Conversely, *two similar polygons may be decomposed into the same number of triangles similar each to each and similarly placed.*

Let $ABCD$, etc., $A'B'C'D'$, etc., be two similar polygons. From two homologous vertices, A and A', let diagonals be drawn in each polygon; then the polygons will be decomposed as required.



For, 1st. We have, by the definition of similar polygons,

$$\text{Angle } B = B', \text{ and } \frac{AB}{A'B'} = \frac{BC}{B'C'};$$

therefore the triangles ABC and $A'B'C'$ are similar (Proposition IV.).

2d. Since ABC and $A'B'C'$ are similar, the angles BCA and $B'C'A'$ are equal; subtracting these equals from the equals BCD and $B'C'D'$, respectively, there remain the equals ACD and $A'C'D'$. Also, from the similarity of the triangles ABC and $A'B'C'$, and from that of the polygons, we have

$$\frac{AC}{A'C'} = \frac{BC}{B'C'} = \frac{CD}{C'D'};$$

therefore the triangles ACD and $A'C'D'$ are similar (Proposition IV.).

Thus, successively, each triangle of one polygon may be shown to be similar to the triangle similarly situated in the other.

PROPOSITION VIII.—THEOREM.

24. *The perimeters of two similar polygons are in the same ratio as any two homologous sides.*

For we have (see preceding figures)

$$\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'} = \text{etc.};$$

whence (12)

$$\frac{AB + BC + CD + \text{etc.}}{A'B' + B'C' + C'D' + \text{etc.}} = \frac{AB}{A'B'} = \frac{BC}{B'C'} = \text{etc.}$$

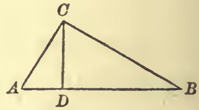
PROPOSITION IX.—THEOREM.

25. *If a perpendicular is drawn from the vertex of the right angle to the hypotenuse of a right triangle:*

1st. *The two triangles thus formed are similar to each other and to the whole triangle;*

2d. *The perpendicular is a mean proportional between the segments of the hypotenuse;*

3d. *Each side about the right angle is a mean proportional between the hypotenuse and the adjacent segment.*



Let C be the right angle of the triangle ABC, and CD the perpendicular to the hypotenuse; then,

1st. The triangles ACD and CBD are similar to each other and to ABC. For the triangles ACD and ABC have the angle A common, and the right angles ADC, ACB, equal; therefore they are mutually equiangular, and consequently similar (Proposition III.). For a like reason CBD is similar to ABC, and consequently also to ACD.

2d. The perpendicular CD is a mean proportional between the segments AD and DB. For the similar triangles ACD, CBD, give

$$AD : CD = CD : BD.$$

3d. The side AC is a mean proportional between the hypotenuse AB and the adjacent segment AD. For the similar triangles, ACD, ABC, give

$$AB : AC = AC : AD.$$

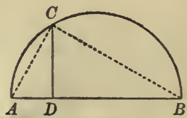
In the same way the triangles CBD and ABC give

$$AB : BC = BC : BD.$$

26. *Scholium.* If the lengths of the lines in the figure above are expressed in terms of the same unit, the results just obtained may be written (5, Corollary)

$$\overline{CD}^2 = AD \times DB,$$
$$\overline{AC}^2 = AB \times AD,$$
$$\overline{BC}^2 = AB \times BD.$$

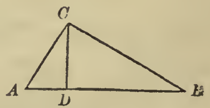
27. Corollary. *If from any point in the circumference of a circle a perpendicular is let fall upon a diameter, the perpendicular is a mean proportional between the segments of the diameter.*



Suggestion. Draw the chords AC and CB. (*v.* II., Proposition XIV., Corollary.)

PROPOSITION X.—THEOREM.

28. *The square of the length of the hypotenuse of a right triangle is the sum of the squares of the lengths of the other two sides, the three lengths being expressed in terms of the same unit.*



Let ABC be right angled at C; then

$$\overline{AB}^2 = \overline{AC}^2 + \overline{BC}^2.$$

For, by Proposition IX., we have

$$\overline{AC}^2 = AB \times AD, \text{ and } \overline{BC}^2 = AB \times BD,$$

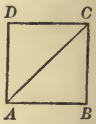
the sum of which is

$$\overline{AC}^2 + \overline{BC}^2 = AB \times (AD + BD) = AB \times AB = \overline{AB}^2.$$

29. *Scholium* I. By this theorem, if the numerical measures of two sides of a right triangle are given, that of the third is found. For example, if $AC = 3$, $BC = 4$; then $AB = \sqrt{[3^2 + 4^2]} = 5$.

If the hypotenuse, AB, and one side, AC, are given, we have $\overline{BC}^2 = \overline{AB}^2 - \overline{AC}^2$; thus, if there are given $AB = 5$, $AC = 3$, then we find $BC = \sqrt{[5^2 - 3^2]} = 4$.

30. *Scholium* II. If AC is the diagonal of a square $ABCD$, we have, by the preceding theorem,



$$\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2 = 2\overline{AB}^2,$$

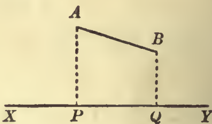
whence

$$\frac{\overline{AC}^2}{\overline{AB}^2} = 2,$$

and, extracting the square root,

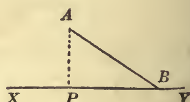
$$\frac{AC}{AB} = \sqrt{2} = 1.41421 + ad\ inf.$$

Since the square root of 2 is an incommensurable number, it follows that *the diagonal of a square is incommensurable with its side.* (v. II., 34.)



31. *Definition.* The *projection of a point* A upon an indefinite straight line XY is the foot P of the perpendicular let fall from the point upon the line.

The *projection of a finite straight line* AB upon the line XY is the distance PQ between the projections of the extremities of AB.



If one extremity B of the line AB is in the line XY, the distance from B to P (the projection of A) is the projection of AB on XY; for the point B is in this case its own projection.

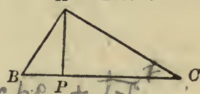
EXERCISES.

1. *Theorem.—In any triangle, the square of the side opposite to an acute angle is equal to the sum of the squares of the other two sides diminished by twice the product of one of these sides and the projection of the other upon that side.*

$\overline{bc}^2 = \overline{ac}^2 + \overline{ab}^2 - a^2bh.$

Suggestion. Let C be the acute angle in question in Fig. 1 or Fig 2.

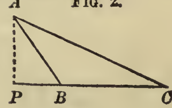
Fig. 1.



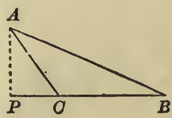
$$
\begin{aligned}
\overline{AB}^2 &= \overline{AP}^2 + \overline{BP}^2, \\
&= \overline{AP}^2 + (BC - PC)^2, \text{ Fig. 1, or} \\
&= \overline{AP}^2 + (PC - BC)^2, \text{ Fig. 2,} \\
&= \overline{AP}^2 + \overline{PC}^2 + \overline{BC}^2 - 2BC \times PC, \\
&= \overline{AC}^2 + \overline{BC}^2 - 2BC \times PC.
\end{aligned}
$$

$c^2 - 2bcpc + bc$

Fig. 2.



2. *Theorem.—In an obtuse angled triangle, the square of the side opposite to the obtuse angle is equal to the sum of the squares of the other two sides, increased by twice the product of one of these sides and the projection of the other upon that side.*



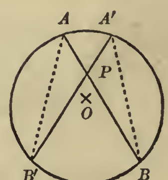
$ac^2 = bc^2 + ab^2 - 2\,cb.p$

PROPOSITION XI.—THEOREM.

32. *If two chords intersect within a circle, their segments are reciprocally proportional.*

For the triangles APB' and $A'PB$ are mutually equiangular (II., Proposition XIV.), and therefore similar (Proposition III.). Hence

$$AP : A'P = PB' : PB.$$



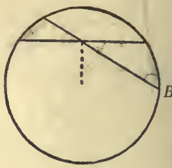
33. *Scholium.* If the lengths of the lines in question are expressed in terms of the same unit, the result above can be written

$$AP \times PB = A'P \times PB',$$

and the proposition may be stated: *if through a fixed point within a circle any chord is drawn, the product of the lengths of its segments is the same whatever its direction.*

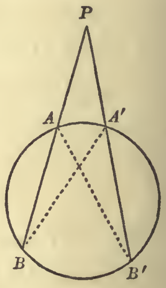
EXERCISE.

Theorem.—Either segment of the least chord that can be drawn through a fixed point is a mean proportional between the segments of any other chord drawn through that point. (v. II., 19, Exercise.)



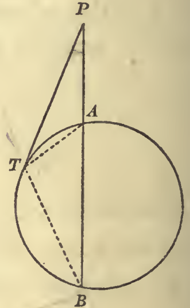
PROPOSITION XII.—THEOREM.

34. *If two secants intersect without a circle, the whole secants and their external segments are reciprocally proportional.*



For the triangles PAB' and $PA'B$ are mutually equiangular (II., Proposition XIV.), and therefore similar. Consequently

$$PB : PB' = PA' : PA.$$



35. Corollary. *If a tangent and a secant intersect, the tangent is a mean proportional between the whole secant and its external segment.*

Suggestion. Show that the triangles PAT and PTB are similar.

36. *Scholium.* If the lengths of the lines are expressed in terms of the same unit, the result of (34) can be written $PB \times PA = PB' \times PA'$, and Proposition XII. can be stated: *if through a fixed point without a circle a secant is drawn, the product of the lengths of the whole secant and its external segment has the same value in whatever direction the secant is drawn.*

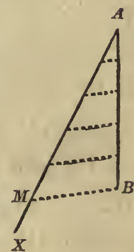
EXERCISE.

Theorem.—If from any point on the common chord of two intersecting circles, produced, tangents are drawn to the two circles, the lengths of these tangents are equal.

PROBLEMS OF CONSTRUCTION.

PROPOSITION XIII.—PROBLEM.

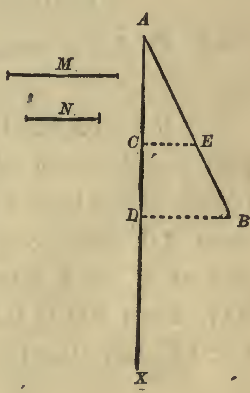
37. *To divide a given straight line into any given number of equal parts.*



Let AB be the given line. Through A draw an indefinite line AX, upon which lay off the given number of equal distances, each distance being of any convenient length; through M the last point of division on AX draw MB, and through the other points of division of AX draw parallels to MB, which will divide AB into the required number of equal parts. This follows from the first part of the proof of Proposition I.

PROPOSITION XIV.—PROBLEM.

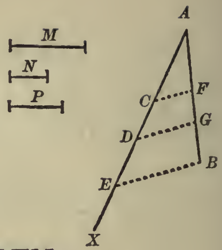
38. *To divide a given straight line into parts proportional to two given straight lines.*



Let it be required to divide AB into parts proportional to M and N. From A draw the indefinite line AX, upon which lay off $AC = M$ and $CD = N$. Join DB, and through C draw CE parallel to DB. Then we shall have $AE : EB = AC : CD$, by Proposition I.

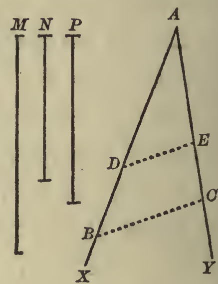
EXERCISE.

Problem.—To divide a given straight line into parts proportional to given straight lines.



PROPOSITION XV.—PROBLEM.

39. *To find a fourth proportional to three given straight lines.*



Let it be required to find a fourth proportional to M, N, and P. Draw the indefinite lines AX, AY, making an angle with each other. Upon AX lay off $AB = M$, $AD = N$; and upon AY lay off $AC = P$; join BC, and draw DE parallel to BC; then AE is the required fourth proportional.

For we have (Proposition I.)

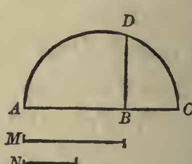
$$AB : AD = AC : AE, \text{ or } M : N = P : AE.$$

EXERCISE.

Problem.—To find a third proportional to two given straight lines.

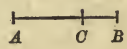
PROPOSITION XVI.—PROBLEM.

40. *To find a mean proportional between two given straight lines.*



Let it be required to find a mean proportional between M and N. Upon an indefinite line lay off $AB = M$, $BC = N$; upon AC describe a semi-circumference, and at B erect a perpendicular, BD, to AC. Then BD is the required mean proportional (Proposition IX., Corollary).

41. *Definition.* When a given straight line is divided into two segments such that one of the segments is a mean proportional between the given line and the other segment, it is said to be divided *in extreme and mean ratio.*

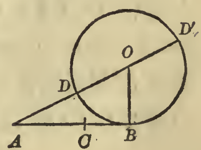


Thus, AB is divided in extreme and mean ratio at C, if $AB : AC = AC : CB$.

PROPOSITION XVII.—PROBLEM.

42. *To divide a given straight line in extreme and mean ratio.*

Let AB be the given straight line. At B erect the perpendicular BO equal to one-half of AB. With the centre O and radius OB, describe a circumference, and through A and O draw AO cutting the circumference first in D and a second time in D'. Upon AB lay off $AC = AD$. Then AB is divided at C in extreme and mean ratio.



For we have (Proposition XII., Corollary)

ad : ab

$$AD' : AB = AB : AD \text{ or } AC, \qquad [1]$$

whence, by division (10),

$$AD' - AB : AB = AB - AC : AC,$$

or, since $DD' = 2OB = AB$, and therefore $AD' - AB = AD' - DD' = AD = AC$,

$$AC : AB = CB : AC,$$

and, by inversion (7),

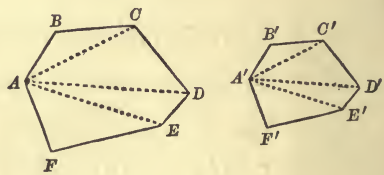
$$AB : AC = AC : CB;$$

that is, AB is divided at C in extreme and mean ratio.

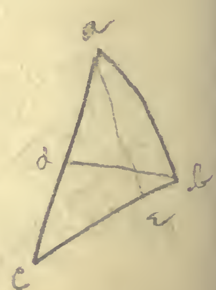
PROPOSITION XVIII.—PROBLEM.

43. *On a given straight line, to construct a polygon similar to a given polygon.*

Let it be required to construct upon $A'B'$ a polygon similar to $ABCDEF$.



Divide $ABCDEF$ into triangles by diagonals drawn from A. Make the angles $B'A'C'$ and $A'B'C'$ equal to BAC and ABC respectively; then the triangle $A'B'C'$ will be similar to ABC (Proposition III.). In the same manner construct the triangle $A'D'C'$ similar to ADC, $A'E'D'$ similar to AED, and $A'E'F'$ similar to AEF. Then $A'B'C'D'E'F'$ is the required polygon (Proposition VI.).



EXERCISES ON BOOK III.

THEOREMS.

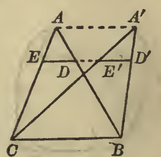
1. If two straight lines are intersected by any number of parallel lines, the corresponding segments of the two lines are proportional. (*v.* Proposition I.)

2. The diagonals of a trapezoid divide each other into segments which are proportional.

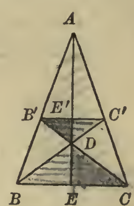
3. In a triangle any two sides are reciprocally proportional to the perpendiculars let fall upon them from the opposite vertices.

4. The perpendiculars from two vertices of a triangle upon the opposite sides divide each other into segments which are reciprocally proportional.

5. If the three sides of a triangle are respectively perpendicular to the three sides of a second triangle, the triangles are similar.



6. If ABC and $A'BC$ are two triangles having a common base BC and their vertices in a line AA' parallel to the base, and if any parallel to the base cuts the sides AB and AC in D and E, and the sides $A'B$ and $A'C$ in D' and E', then $DE = D'E'$ (Proposition III.).



7. If two sides of a triangle are divided proportionally, the straight lines drawn from corresponding points of section to the opposite angles intersect on the line joining the vertex of the third angle and the middle of the third side.

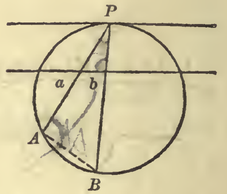
Suggestion. Draw the line ADE through the intersection of $B'C$ and BC'. $B'E'D$ and CED are similar; $\therefore \frac{EC}{B'E'} = \frac{DC}{B'D}$. $B'DC'$ and BDC are similar; $\therefore \frac{DC}{B'D} = \frac{BC}{B'C'}$. $AB'C'$ and ABC are similar; $\therefore \frac{BC}{B'C'} = \frac{AB}{AB'}$. $AB'E'$ and ABE are similar; $\therefore \frac{AB}{AB'} = \frac{BE}{B'E'}$. Hence $\frac{EC}{B'E'} = \frac{BE}{B'E'}$ and $BE = EC$.

8. The difference of the squares of two sides of any triangle is equal to the difference of the squares of the projections of these sides on the third side (Proposition X.).

9. If from any point in the plane of a polygon perpendiculars are drawn to all the sides, the two sums of the squares of the alternate segments of the sides are equal.

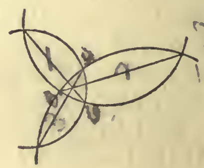
10. If through a point P in the circumference of a circle two chords are drawn, the chords and the segments cut from them by a line parallel to the tangent at P are reciprocally proportional.

Suggestion. Prove PAB and Pba similar.



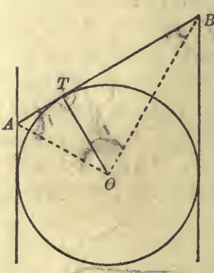
— Wednesday —

11. If three circles intersect, their three common chords pass through the same point. (*v.* Proposition XI.)



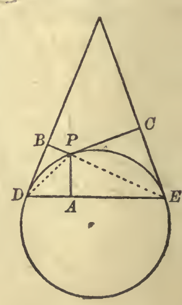
12. If two tangents are drawn to a circle at the extremities of a diameter, the portion of any third tangent intercepted between them is divided at its point of contact into segments whose product is equal to the square of the radius.

Suggestion. Prove AOB a right triangle.

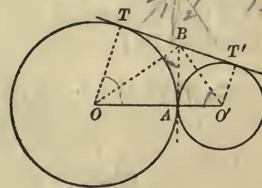


13. The perpendicular from any point of a circumference upon a chord is a mean proportional between the perpendiculars from the same point upon the tangents drawn at the extremities of the chord.

Suggestion. PBD and PAE are similar; $\therefore$ $\dfrac{PB}{PA} = \dfrac{PD}{PE}$. PCE and PAD are similar; $\therefore$ $\dfrac{PA}{PC} = \dfrac{PD}{PE}$. Hence $\dfrac{PB}{PA} = \dfrac{PA}{PC}$.



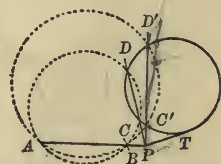
14. If two circles touch each other, secants drawn through their point of contact and terminating in the two circumferences are divided proportionally at the point of contact. (*v.* II., 54, Exercise 2.)



15. If two circles are tangent externally, the portion of their common tangent included between the points of contact is a mean proportional between the diameters of the circles.

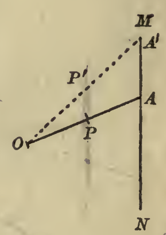
Suggestion. Show that OBO' is a right triangle.

16. If a fixed circumference is cut by any circumference which passes through two fixed points, the common chord passes through a fixed point.

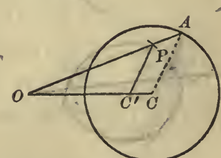


Suggestion. $PA \,.\, PB = PC \,.\, PD = \overline{PT}^2$, by Proposition XII. and Corollary. Join P with C', and show that PC' will cut both circles at the same distance from P, and will be their common chord.

LOCI.



17. From a fixed point O, a straight line OA is drawn to any point in a given straight line MN, and divided at P in a given ratio $m:n$ (*i.e*, so that $OP:PA = m:n$); find the locus of P. (*v.* Proposition II.)



18. From a fixed point O, a straight line OA is drawn to any point in a given circumference, and divided at P in a given ratio; find the locus of P.

Suggestion. PC' is a fixed length.

19. Find the locus of a point whose distances from two given straight lines are in a given ratio. (The locus consists of two straight lines.)

20. Find the locus of the points which divide the various chords of a given circle into segments whose product is equal to a given constant, k^2 (33, Exercise).

21. Find the locus of a point the sum of whose distances from two given straight lines is equal to a given constant, k. (*v.* I., Exercise 10.)

22. Find the locus of a point the difference of whose distances from two given straight lines is equal to a given constant, k.

Suggestion. Reduce it to I., Proposition XIX., by drawing a third line parallel to the more distant of the given lines at a distance from it equal to k.

PROBLEMS.

23. To divide a given straight line into three segments, A, B, and C, such that A and B shall be in the ratio of two given straight lines M and N, and B and C shall be in the ratio of two other given straight lines P and Q.

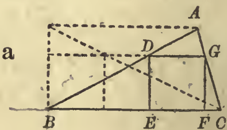
24. Through a given point, to draw a straight line so that the portion of it intercepted between two given straight lines shall be divided at the point in a given ratio.

Suggestion. Through the point draw a line parallel to one of the given lines. (*v.* II., Exercise 32.)

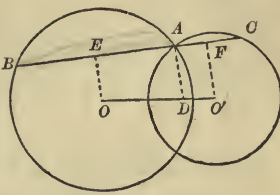
25. Through a given point, to draw a straight line so that the distances from two other given points to this line shall be in a given ratio.

Suggestion. Divide the line joining the two other given points in the given ratio.

26. To determine a point whose distances from three given indefinite straight lines shall be proportional to three given straight lines. (Exercise 19.)



27. In a given triangle ABC, to inscribe a square $DEFG$. (Exercises 6 and II., 29.)

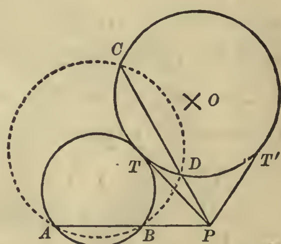


28. Given two circumferences intersecting in A, to draw through A a secant, BAC, such that AB shall be to AC in a given ratio.

Suggestion. Divide OO' in the given ratio. (*v.* Exercise 1.)

Wednesday

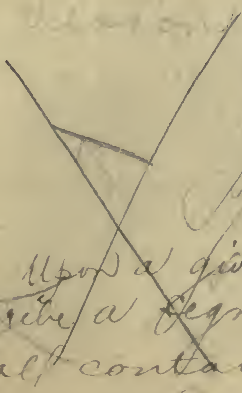
29. To describe a circumference passing through two given points A and B and tangent to a given circumference O.



Analysis. Suppose ATB is the required circumference tangent to the given circumference at T, and $ACDB$ any circumference passing through A and B and cutting the given circumference in C and D. The common chords AB and CD, and the common tangent at T, all pass through a common point P (Exercise 16); from which a simple construction may be inferred. There are two solutions given by the two tangents that can be drawn from P.

30. To describe a circumference passing through two given points and tangent to a given straight line. (Two solutions.) (*v.* Proposition XII., Corollary.)

31. To describe a circumference passing through a given point and tangent to two given straight lines. (*v.* Exercise 13.)



I

Friday Analysis

Upon a given straight line, to describe a segment of a circle which shall contain a given angle

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Study these problems from first for Monday

BOOK IV.

COMPARISON AND MEASUREMENT OF THE SURFACES OF RECTILINEAR FIGURES.

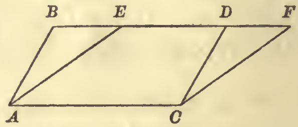
1. *Definition.* The *area* of a surface is its numerical measure, referred to some other surface as the unit; in other words, it is the *ratio* of the surface to the *unit of surface* (II., 29).

The unit of surface is called the *superficial unit.* The most convenient superficial unit is the square whose side is the linear unit.

2. *Definition. Equivalent figures* are those whose areas are equal.

PROPOSITION I.—THEOREM.

3. *Parallelograms having equal bases and equal altitudes are equivalent.*



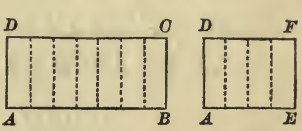
Let $ABCD$ and $AECF$ be two parallelograms having equal bases and equal altitudes.

Superpose the second upon the first, making the equal bases coincide. Since the altitudes are equal, the upper bases will lie in the same straight line. The triangles ABE and CDF are equal (I., Proposition VI.). If the triangle CDF is taken from the whole figure, $ABFC$, the first parallelogram $ABCD$ is left; if the equal triangle ABE is taken from the same figure, the second parallelogram $AECF$ is left. The magnitudes of the two parallelograms are therefore equal, and the parallelograms are equivalent.

4. Corollary. *Any parallelogram is equivalent to a rectangle having the same base and the same altitude.*

PROPOSITION II.—THEOREM.

5. *Two rectangles having equal altitudes are to each other as their bases.*



Let $ABCD$ and $AEFD$ be two rectangles having equal altitudes; then are they to each other as $AB : AE$.

1. Suppose the bases have a common measure which is contained m times in AB and n times in AE. Then we have

$$\frac{AB}{AE} = \frac{m}{n}.$$

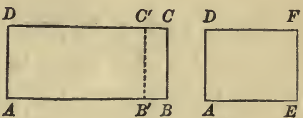
Apply this measure to the two bases, and through the points of division draw perpendiculars to the bases. The two rectangles are thus divided into smaller rectangles, all of which are equal, by I., Proposition XXVIII., Corollary, and of which $ABCD$ contains m and $AEFD$ contains n. Then

$$\frac{ABCD}{AEFD} = \frac{m}{n},$$

and consequently

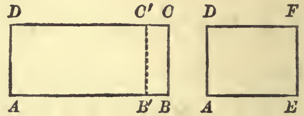
$$\frac{ABCD}{AEFD} = \frac{AB}{AE}.$$

2. If the bases are incommensurable, divide AE in any arbitrarily chosen number n of equal parts, and apply one of these parts to AB. Let B' be the last point of division, $B'B$ being of course less than the divisor.



Since AB' and AE are commensurable, we have $\frac{AB'C'D}{AEFD} = \frac{AB'}{AE}$, and this holds, no matter what the value of n. If, now,

n is increased at pleasure, we can make $B'B$, and consequently $B'BCC'$, as small as we please, but cannot make them absolutely zero. Hence, as n is indefinitely increased, AB' has AB for its limit, $AB'C'D$ has $ABCD$ for its limit,



$$\frac{AB'C'D}{AEFD} \text{ has } \frac{ABCD}{AEFD} \text{ for its limit,}$$

and

$$\frac{AB'}{AE} \text{ has } \frac{AB}{AE} \text{ for its limit.}$$

Therefore, by II., *Theorem, Doctrine of Limits,*

$$\frac{ABCD}{AEFD} = \frac{AB}{AE}. \quad (v.\ \text{II., 42, and III., 14.})$$

6. Corollary. *Two rectangles having equal bases are to each other as their altitudes.*

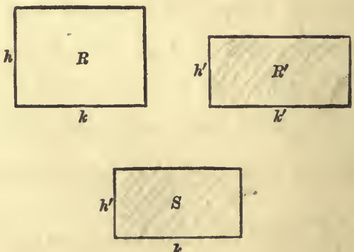
Note. In these propositions, by "rectangle" is to be understood "surface of the rectangle."

PROPOSITION III.—THEOREM.

7. *Any two rectangles are to each other as the products of their bases by their altitudes.*

Let R and R' be two rectangles, k and k' their bases, h and h' their altitudes; then

$$\frac{R}{R'} = \frac{k \times h}{k' \times h'}.$$



For, let S be a third rectangle, having the same base k

as the rectangle R, and the same altitude h' as the rectangle R'; then we have, by Proposition II., Corollary, and Proposition II.,

$$\frac{R}{S} = \frac{h}{h'}, \qquad \frac{S}{R'} = \frac{k}{k'},$$

and multiplying these ratios, we find

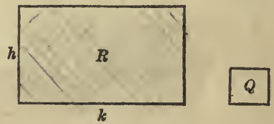
$$\frac{R}{R'} = \frac{k \times h}{k' \times h'}.$$

8. *Scholium.* It must be remembered that by the product of two lines is to be understood the product of the numbers which represent them when they are measured by the linear unit (III., 8).

PROPOSITION IV.—THEOREM.

9. *The area of a rectangle is equal to the product of its base and altitude.*

Let R be any rectangle, k its base, and h its altitude numerically expressed in terms of the linear unit; and let Q be the square whose side is the linear unit; then, by Proposition III.,

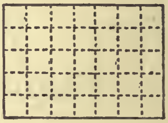


$$\frac{R}{Q} = \frac{k \times h}{1 \times 1} = k \times h.$$

But since Q is the unit of surface, $\frac{R}{Q}$ = the numerical measure, or area, of the rectangle, R, (1); therefore

$$\text{Area of } R = k \times h.$$

10. *Scholium* I. When the base and altitude are exactly divisible by the linear unit, this proposition is rendered evident by dividing the rectangle into squares each equal to the superficial unit. Thus, if the base contains 7 linear units and the altitude 5, the rectangle can obviously be divided into 35 squares each equal to the superficial unit; that is, its area $= 5 \times 7$. The proposition, as above demonstrated, is, however, more general, and includes also the cases in which either the base or the altitude, or both, are incommensurable with the unit of length.



11. *Scholium* II. The area of a square, being the product of two equal sides, is the *second power* of a side. Hence it is that in arithmetic and algebra the expression "square of a number" has been adopted to signify "second power of a number."

We may also here observe that many writers employ the expression "rectangle of two lines" in the sense of "product of two lines," because the rectangle constructed upon two lines is measured by the product of the numerical measures of the lines.

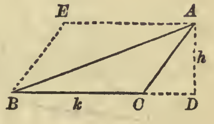
PROPOSITION V.—THEOREM.

12. *The area of a parallelogram is equal to the product of its base and altitude.*

For, by Proposition I., the parallelogram is equivalent to a rectangle having the same base and the same altitude.

PROPOSITION VI.—THEOREM.

13. *The area of a triangle is equal to half the product of its base and altitude.*



Let ABC be a triangle, k the numerical measure of its base BC, h that of its altitude AD, and S its area; then

$$S = \tfrac{1}{2} k \times h.$$

For, through A draw AE parallel to CB, and through B draw BE parallel to CA. The triangle ABC is one-half the parallelogram $AEBC$ (I., Proposition IX.); but the area of the parallelogram $= k \times h$; therefore, for the triangle, we have $S = \tfrac{1}{2} k \times h$.

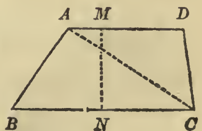
14. Corollary I. *A triangle is equivalent to one-half of any parallelogram having the same base and the same altitude.*

15. Corollary II. *Triangles having equal bases and equal altitudes are equivalent.*

16. Corollary III. *Triangles having equal altitudes are to each other as their bases; and triangles having equal bases are to each other as their altitudes.*

PROPOSITION VII.—THEOREM.

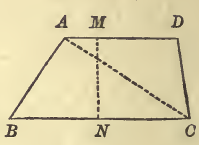
17. *The area of a trapezoid is equal to the product of its altitude by half the sum of its parallel bases.*



Let $ABCD$ be a trapezoid; $MN = h$, its altitude; $AD = a$, $BC = b$, its parallel bases; and let S denote its area; then

$$S = \tfrac{1}{2}(a + b) \times h.$$

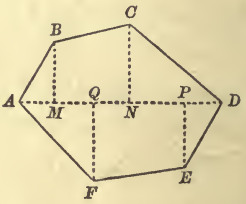
For, draw the diagonal AC. The altitude of each of the triangles ADC and ABC is equal to h, and their bases are respectively a and b; the area of the first is $\frac{1}{2}a \times h$, that of the second is $\frac{1}{2}b \times h$; and the trapezoid being the sum of the two triangles, we have



$$S = \tfrac{1}{2}a \times h + \tfrac{1}{2}b \times h = \tfrac{1}{2}(a + b) \times h.$$

18. *Scholium.* The area of any polygon may be found by finding the areas of the several triangles into which it may be decomposed by drawing diagonals from any vertex.

The following method, however, is usually preferred, especially in surveying. Draw the longest diagonal AD of the proposed polygon $ABCDEF$; and upon AD let fall the perpendiculars BM, CN, EP, FQ. The polygon is thus decomposed into right triangles and right trapezoids, and by measuring the lengths of the perpendiculars and also of the distances AM, MN, ND, AQ, QP, PD, the bases and altitudes of these triangles and trapezoids are known. Hence their areas can be computed by the preceding theorems, and the sum of these areas will be the area of the polygon.

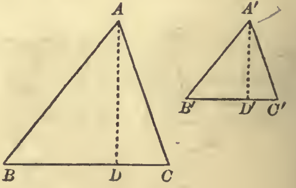


PROPOSITION VIII.—THEOREM.

19. *Similar triangles are to each other as the squares of their homologous sides.*

Let ABC, $A'B'C'$, be similar triangles; then

$$\frac{ABC}{A'B'C'} = \frac{\overline{AB}^2}{\overline{A'B'}^2}.$$



Let AD and $A'D'$ be the altitudes; then

$$\frac{ABC}{A'B'C'} = \frac{\frac{1}{2} AD \times BC}{\frac{1}{2} A'D' \times B'C'} = \frac{AD}{A'D'} \times \frac{BC}{B'C'}.$$

But the triangles ADB and $A'D'B'$ are similar (III., Proposition III.); therefore

$$\frac{AD}{A'D'} = \frac{AB}{A'B'},$$

and from the similarity of ABC and $A'B'C'$,

$$\frac{BC}{B'C'} = \frac{AB}{A'B'};$$

hence

$$\frac{AD}{A'D'} \times \frac{BC}{B'C'} = \frac{\overline{AB}^2}{\overline{A'B'}^2},$$

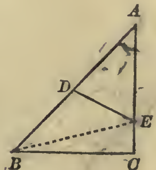
and we have

$$\frac{ABC}{A'B'C'} = \frac{\overline{AB}^2}{\overline{A'B'}^2}.$$

EXERCISE.

Theorem.—Two triangles having an angle of the one equal to an angle of the other are to each other as the products of the sides including the equal angles.

Suggestion. Let ADE and ABC be the two triangles. Draw BE, and compare the two triangles with AEB. (*v.* Proposition VI., Corollary III.)

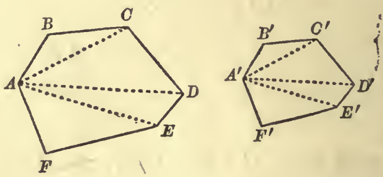


PROPOSITION IX.—THEOREM.

20. *Similar polygons are to each other as the squares of their homologous sides.*

Let $ABCDEF$, $A'B'C'D'E'F'$, be two similar polygons, and denote their surfaces by S and S'; then

$$\frac{S}{S'} = \frac{\overline{AB}^2}{\overline{A'B'}^2}.$$



For, let the polygons be decomposed into homologous similar triangles (III., Proposition VII.). The ratio of the surfaces of any pair of homologous triangles, as ABC and $A'B'C'$, ACD and $A'C'D'$, etc., will be the square of the ratio of two homologous sides of the polygons (Proposition VIII.); therefore we shall have

$$\frac{ABC}{A'B'C'} = \frac{ACD}{A'C'D'} = \frac{ADE}{A'D'E'} = \frac{AEF}{A'E'F'} = \frac{\overline{AB}^2}{\overline{A'B'}^2}.$$

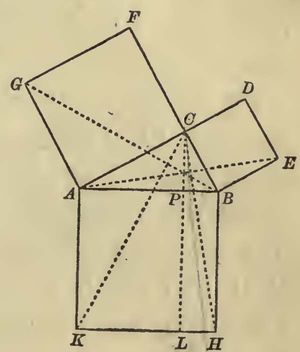
Therefore, by addition of antecedents and consequents (III., 12).

$$\frac{ABC + ACD + ADE + AEF}{A'B'C' + A'C'D' + A'D'E' + A'E'F'} = \frac{S}{S'} = \frac{\overline{AB}^2}{\overline{A'B'}^2}.$$

PROPOSITION X.—THEOREM.

21. *The square described upon the hypotenuse of a right triangle is equivalent to the sum of the squares described on the other two sides.*

Let the triangle ABC be right angled at C; then the square AH, described upon the hypotenuse, is equal in area to the sum of the squares AF and BD, described on the other two sides.



For, from C draw CP perpendicular to AB and produce it to meet KH in L. Join CK, BG. Since ACF and ACB are right angles, CF and CB are in the same straight line (I., Proposition IV.); and for a similar reason AC and CD are in the same straight line.

In the triangles CAK, GAB, we have AK equal to AB, being sides of the same square; AC equal to AG, for the same reason; and the angles CAK, GAB, equal, being each equal to the sum of the angle CAB and a right angle; therefore these triangles are equal (I., Proposition VI.).

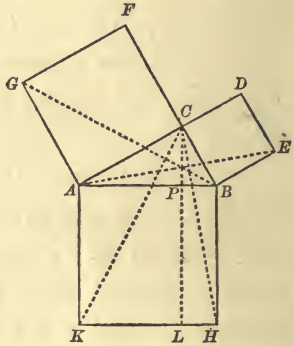
The triangle CAK and the rectangle AL have the same base AK; and since the vertex C is upon LP produced, they also have the same altitude; therefore the triangle CAK is equivalent to one-half the rectangle AL (Proposition VI., Corollary I.).

The triangle GAB and the square AF have the same base AG; and since the vertex B is upon FC produced, they also have the same altitude; therefore the triangle GAB is equivalent to one-half the square AF (Proposition VI., Corollary I.).

But the triangles CAK, GAB, have been shown to be equal; therefore the rectangle AL is equivalent to the square AF.

In the same way it is proved that the rectangle BL is equivalent to the square BD.

Therefore the square AH, which is the sum of the rectangles AL and BL, is equivalent to the sum of the squares AF and BD.



22. *Scholium.* This theorem is ascribed to Pythagoras (born about 600 B.C.), and is commonly called the *Pythagorean Theorem.* The preceding demonstration of it is that which was given by Euclid, in his Elements (about 300 B.C.).

It is important to observe that we may deduce the same result from the *numerical* relation $\overline{AB}^2 = \overline{AC}^2 + \overline{BC}^2$, already established in III., Proposition X. For, since the measure of the area of a square is the second power of the number which represents its side, it follows directly from this numerical relation that the area of which $\overline{AB}^2$ is the measure is equal to the sum of the areas of which $\overline{AC}^2$ and $\overline{BC}^2$ are the measures.

EXERCISES.

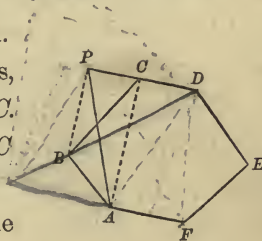
1. *Theorem.—If the three sides of a right triangle be taken as the homologous sides of three similar polygons constructed upon them, then the polygon constructed upon the hypotenuse is equivalent to the sum of the polygons constructed upon the other two sides.* (*v.* Proposition IX.)

2. *Theorem.—The squares on the sides of a right triangle are proportional to the segments into which the hypotenuse is divided by a perpendicular let fall from the vertex of the right angle.* (*v.* Figure, Proposition X.)

PROBLEMS OF CONSTRUCTION.

PROPOSITION XI.—PROBLEM.

23. *To construct a triangle equivalent to a given polygon.*



Let $ABCDEF$ be the given polygon.

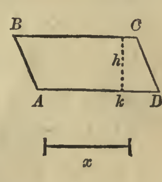
Take any three consecutive vertices, as A, B, C, and draw the diagonal AC. Through B draw BP parallel to AC meeting DC produced in P; join AP.

The triangles APC, ABC, have the same base AC; and since their vertices, P and B, lie on the same straight line BP parallel to AC, they also have the same altitude; therefore they are equivalent. Therefore the pentagon $APDEF$ is equivalent to the hexagon $ABCDEF$. Now, taking any three consecutive vertices of this pentagon, we shall, by a precisely similar construction, find a quadrilateral of the same area; and, finally, by a similar operation upon the quadrilateral we shall find a triangle of the same area.

Thus, whatever the number of the sides of the given polygon, a series of successive steps, each step reducing the number of sides by one, will give a series of polygons of equal areas, terminating in a triangle.

PROPOSITION XII.—PROBLEM.

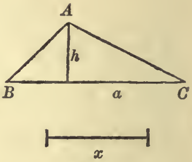
24. *To construct a square equivalent to a given parallelogram or to a given triangle.*



1st. Let AC be a given parallelogram, k its base, and h its altitude.

Find a mean proportional x between h and k, by III., 40. The square constructed upon x will be equivalent to the parallelogram, since $x^2 = h \times k$.

2d. Let ABC be a given triangle, a its base, and h its altitude.

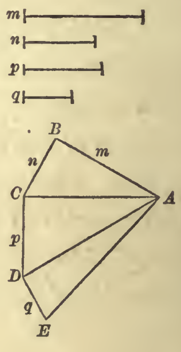


Find a mean proportional x between a and $\frac{1}{2}h$; the square constructed upon x will be equivalent to the triangle, since $x^2 = a \times \frac{1}{2}h = \frac{1}{2}ah$.

25. *Scholium.* By means of this problem and the preceding, a square can be found equivalent to any given polygon.

PROPOSITION XIII.—PROBLEM.

26. *To construct a square equivalent to the sum of two or more given squares, or to the difference of two given squares.*



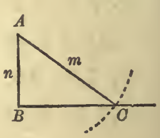
1st. Let m, n, p, q, be the sides of given squares.

Draw $AB = m$, and $BC = n$, perpendicular to each other at B; join AC. Then (Proposition X.) $\overline{AC}^2 = m^2 + n^2$.

Draw $CD = p$ perpendicular to AC, and join AD. Then $\overline{AD}^2 = \overline{AC}^2 + p^2 = m^2 + n^2 + p^2$.

Draw $DE = q$ perpendicular to AD, and join AE. Then $\overline{AE}^2 = \overline{AD}^2 + q^2 = m^2 + n^2 + p^2 + q^2$. Therefore the square constructed upon AE will be equivalent to the sum of the squares constructed upon m, n, p, q.

In this manner may the areas of any number of given squares be added.



2d. Construct a right angle ABC, and lay off $BA = n$. With the centre A and a radius $= m$, describe an arc cutting BC in C. Then

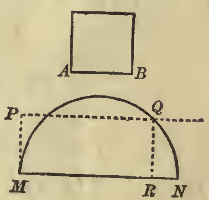
$\overline{BC}^2 = \overline{AC}^2 - \overline{AB}^2 = m^2 - n^2$; therefore the square constructed upon BC will be equivalent to the difference of the squares constructed upon m and n.

EXERCISE.

Problem.—Upon a given straight line, to construct a rectangle equivalent to a given rectangle.

PROPOSITION XIV.—PROBLEM.

27. *To construct a rectangle, having given its area and the sum of two adjacent sides.*

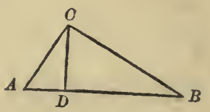


Let MN be equal to the given sum of the adjacent sides of the required rectangle; and let the given area be that of the square whose side is AB.

Upon MN as a diameter describe a semicircle. At M erect $MP = AB$ perpendicular to MN, and draw PQ parallel to MN, intersecting the circumference in Q. From Q let fall QR perpendicular to MN; then MR and RN are the base and altitude of the required rectangle. For, by III., Proposition IX., Corollary, $MR \times RN = \overline{QR}^2 = \overline{PM}^2 = \overline{AB}^2$.

PROPOSITION XV.—PROBLEM.

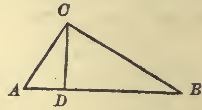
28. *To find two straight lines in the ratio of the areas of two given polygons.*



Let squares be found equal in area to the given polygons respectively (23 and 24). Upon the sides of the right angle ACB, take CA and CB equal to the sides of these squares, join AB, and let fall CD perpendicular to

AB. Then, by (III., 26), we have $\overline{AC}^2 = AD \times AB$, $\overline{CB}^2 = DB \times AB$. Hence

$$\frac{\overline{AC}^2}{\overline{CB}^2} = \frac{AD}{DB};$$



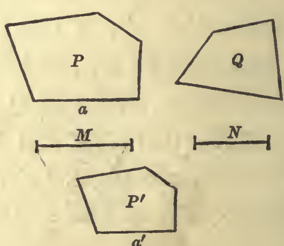
therefore AD and DB are in the ratio of the areas of the given polygons.

EXERCISE.

Problem.—To find a square which shall be to a given square in the ratio of two given straight lines. (v. 28.)

PROPOSITION XVI.—PROBLEM.

29. *To construct a polygon similar to a given polygon P and equivalent to a given polygon Q.*



Find M and N, the sides of squares respectively equal in area to P and Q (23 and 24).

Let a be any side of P, and find a fourth proportional a' to M, N, and a; upon a', as a homologous side to a, construct the polygon P' similar to P; this will be the required polygon. For, by construction,

$$\frac{M}{N} = \frac{a}{a'};$$

therefore, taking the letters P, Q, and P', to denote the areas of the polygons,

$$\frac{P}{Q} = \frac{M^2}{N^2} = \frac{a^2}{a'^2}.$$

But, the polygons P and P' being similar, we have, by (Proposition IX.),

$$\frac{P}{P'} = \frac{a^2}{a'^2};$$

and comparing these equations, we have $P' = Q$.

Therefore the polygon P' is similar to the polygon P and equivalent to the polygon Q, as required.

EXERCISE.

Problem.—To construct a polygon similar to a given polygon, and whose area shall be in a given ratio to that of the given polygon. (*v.* 28, Exercise, and III., 43.)

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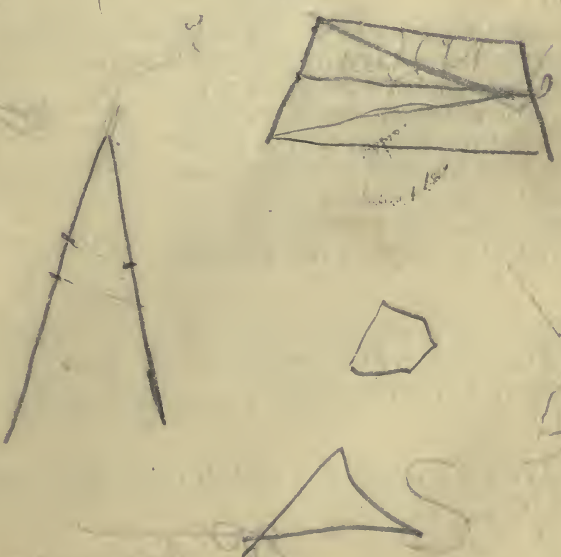


EXERCISES ON BOOK IV.

THEOREMS.

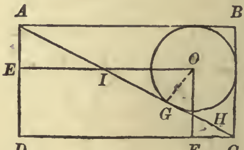
1. Two triangles are equivalent if they have two sides of the one respectively equal to two sides of the other, and the included angle of the one equal to the supplement of the included angle of the other.

2. The two opposite triangles formed by joining any point in the interior of a parallelogram to its four vertices are together equivalent to one-half the parallelogram.

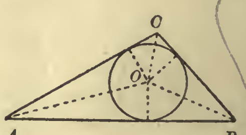
3. The triangle formed by joining the middle point of one of the non-parallel sides of a trapezoid to the extremities of the opposite side is equivalent to one-half the trapezoid. (*v.* I., Exercise 24.)

4. The figure formed by joining consecutively the four middle points of the sides of any quadrilateral is equivalent to one-half the quadrilateral. (*v.* I., Exercise 32.)

5. If in a rectangle $ABCD$ we draw the diagonal AC, inscribe a circle in the triangle ABC, and from its centre draw OE and OF perpendicular to AD and DC respectively, the rectangle OD will be equivalent to one-half the rectangle $ABCD$.



6. The area of a triangle is equal to one-half the product of its perimeter by the radius of the inscribed circle.



7. The area of a rhombus is one-half the product of the diagonals.

8. The straight line joining the middle points of the parallel sides of a trapezoid divides it into two equivalent figures.

9. Any line drawn through the point of intersection of the diagonals of a parallelogram divides it into two equal quadrilaterals.

10. In an isosceles right triangle either leg is a mean proportional between the hypotenuse and the perpendicular upon it from the vertex of the right angle.

11. If two triangles have an angle in common, and have equal areas, the sides about the equal angles are reciprocally proportional.

12. The perimeter of a triangle is to a side as the perpendicular from the opposite vertex is to the radius of the inscribed circle. (*v.* Exercise 6.)

13. Two quadrilaterals are equivalent when the diagonals of one are respectively equal and parallel to the diagonals of the other.

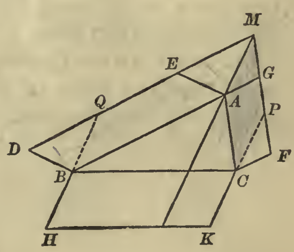
14. The sum of the perpendiculars from any point within an equilateral convex polygon upon the sides is constant.

Suggestion. Join the point with the vertices of the polygon.

15. The lines joining two opposite vertices of a parallelogram with the middle points of the sides form a parallelogram whose area is one-third the area of the given parallelogram.

16. The sum of the squares on the segments of two perpendicular chords in a circle is equivalent to the square on the diameter.

17. Let ABC be any triangle, and upon the sides AB, AC, construct parallelograms AD, AF, of any magnitude or form. Let their exterior sides DE, FG, meet in M; join MA, and upon BC construct a parallelogram BK, whose side BH is equal and parallel to MA. Then the parallelogram BK is equivalent to the sum of the parallelograms AD and AF. (*v.* Proposition I.)



From this deduce the Pythagorean Theorem.

18. Prove, geometrically, that the square described upon the sum of two straight lines is equivalent to the sum of the squares described on the two lines *plus* twice their rectangle.

Note. By the "rectangle of two lines" is here meant the rectangle of which the two lines are the adjacent sides.

19. Prove, geometrically, that the square described upon the difference of two straight lines is equivalent to the sum of the squares described on the two lines *minus* twice their rectangle.

20. Prove, geometrically, that the rectangle of the sum and the difference of two straight lines is equivalent to the difference of the squares of those lines.

PROBLEMS.

21. To construct a triangle, given its angles and its area (equal to that of a given square).

Suggestion. Construct any triangle having the given angles. The problem then reduces to (29).

22. Given any triangle, to construct an isosceles triangle of the same area, whose vertical angle is an angle of the given triangle. (*v.* 19, Exercise.)

23. Given any triangle, to construct an equilateral triangle of the same area. (*v.* Exercise 21.)

24. Bisect a given triangle by a parallel to one of its sides. (*v.* Proposition VIII. and 28.)

25. Bisect a triangle by a straight line drawn through a given point in one of its sides. (*v.* 19, Exercise.)

26. Inscribe a rectangle of a given area in a given circle.

Suggestion. Draw a diagonal of the rectangle. The problem can then be reduced to inscribing in the given circle a right triangle of given area.

27. Given three points, A, B, and C, to find a fourth point P, such that the areas of the triangles APB, APC, BPC, shall be equal. (Four solutions.) (*v.* III., Exercise 19.)

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BOOK V.

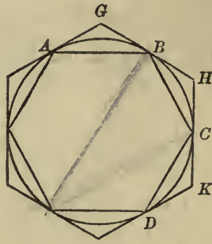
REGULAR POLYGONS. MEASUREMENT OF THE CIRCLE.

1. *Definition.* A *regular polygon* is a polygon which is at once equilateral and equiangular.

The equilateral triangle and the square are simple examples of regular polygons. The following theorem establishes the possibility of regular polygons of any number of sides.

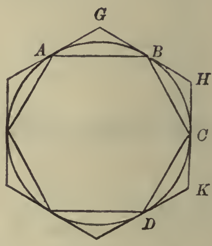
PROPOSITION I.—THEOREM.

2. *If the circumference of a circle be divided into any number of equal parts, the chords joining the successive points of division form a regular polygon inscribed in the circle; and the tangents drawn at the points of division form a regular polygon circumscribed about the circle.*



Let the circumference be divided into the equal arcs AB, BC, CD, etc.; then, 1st, drawing the chords AB, BC, CD, etc., $ABCD$, etc., is a regular inscribed polygon. For its sides are equal, being chords of equal arcs; and its angles are equal, being inscribed in equal segments.

2d. Drawing tangents at A, B, C, etc., the polygon GHK, etc., is a regular circumscribed polygon. For, in the triangles AGB, BHC, CKD, etc., we have $AB = BC = CD$, etc., and the angles GAB, GBA, HBC, HCB, etc., are equal, since each is formed by a tangent and chord and is measured by half of one of the equal parts of the circumference (II., Proposition XV.); therefore these triangles are all isosceles and equal to each other. Hence we have the angles $G = H = K$, etc., and $AG = GB = BH = HC = CK$, etc., from which, by the addition of equals, it follows that $GH = HK$, etc.



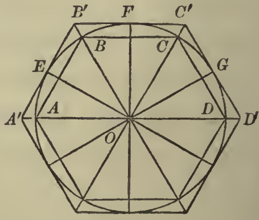
3. Corollary I. *If the vertices of a regular inscribed polygon are joined with the middle points of the arcs subtended by the sides of the polygon, the joining lines will form a regular inscribed polygon of double the number of sides.*

4. Corollary II. *If at the middle points of the arcs joining adjacent points of contact of the sides of a regular circumscribed polygon tangents are drawn, a regular circumscribed polygon of double the number of sides will be formed.*

5. *Scholium.* It is evident that the area of an inscribed polygon is less than that of the inscribed polygon of double the number of sides; and the area of a circumscribed polygon is greater than that of the circumscribed polygon of double the number of sides.

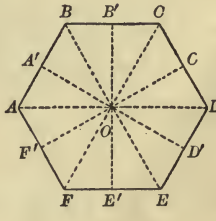
EXERCISE.

Theorem.—If a regular polygon is inscribed in a circle, the tangents drawn at the middle points of the arcs subtended by the sides of the inscribed polygon form a circumscribed regular polygon, whose sides are parallel to the sides of the inscribed polygon, and whose vertices lie on the radii drawn to the vertices of the inscribed polygon.



PROPOSITION II.—THEOREM.

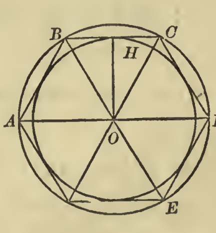
6. *A circle may be circumscribed about any regular polygon; and a circle may also be inscribed in it.*



Let $ABCD\ldots$ be a regular polygon. Through A' and B', the middle points of AB and BC, draw perpendiculars, and connect O, their point of intersection, with all the vertices of the polygon and with the middle points of all the sides.

The triangles $OA'B$ and $OB'B$ are equal, by I., Proposition X. $OB'B$ and $OB'C$ are equal, by I., Proposition VI. The angle OBB' is one-half of ABC; $\therefore$ OCB' is one-half of the equal angle BCD. Hence the triangles $OB'C$ and OCC' are equal, by I., Proposition VI. By continuing this process we may prove all the small triangles equal. O, then, is equidistant from all the vertices, and therefore with O as a centre a circle may be circumscribed about the polygon. O is also equidistant from all the sides, and therefore with O as a centre a circle may be inscribed in the polygon.

7. *Definitions.* The *centre of a regular polygon* is the common centre, O, of the circumscribed and inscribed circles.

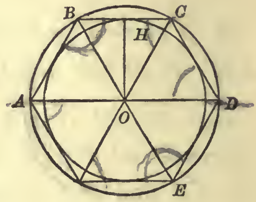


The *radius* of a regular polygon is the radius, OA, of the circumscribed circle.

The *apothem* is the radius, OH, of the inscribed circle.

The *angle at the centre* is the angle, AOB, formed by radii drawn to the extremities of any side.

8. The angle at the centre is equal to four right angles divided by the number of sides of the polygon.

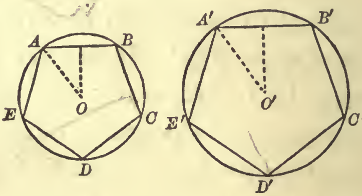


9. Since the angle ABC is equal to twice ABO, or to $ABO + BAO$, it follows that the angle ABC of the polygon is the supplement of the angle at the centre.

PROPOSITION III.—THEOREM.

10. *Regular polygons of the same number of sides are similar.*

Let $ABCDE$, $A'B'C'D'E'$, be regular polygons of the same number of sides; then they are similar.



For, 1st, they are mutually equiangular, since the magnitude of an angle of either polygon depends only on the number of the sides (8 and 9), which is the same in both.

2d. The homologous sides are proportional, since the ratio $AB : A'B'$ is the same as the ratio $BC : B'C'$, or $CD : C'D'$, etc.

Therefore the polygons fulfil the two conditions of similarity.

11. Corollary. *The perimeters of regular polygons of the same number of sides are to each other as the radii of the circumscribed circles, or as the radii of the inscribed circles; and their areas are to each other as the squares of these radii.* (*v.* III., Proposition VIII., and IV., Proposition IX.)

PROPOSITION IV.—THEOREM.

12. *The area of a regular polygon is equal to half the product of its perimeter and apothem.*

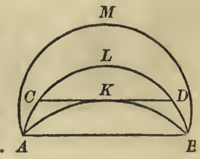
For straight lines drawn from the centre to the vertices of the polygon divide it into equal triangles whose bases are the sides of the polygon and whose common altitude is the apothem. The area of one of these triangles is equal to half the product of its base and altitude; therefore the sum of their areas, or the area of the polygon, is half the product of the sum of the bases by the common altitude; that is, half the product of the perimeter and apothem.

EXERCISE.

Theorem.—The area of any polygon circumscribed about a circle is half the product of its perimeter by the radius of the circle.

PROPOSITION V.—THEOREM.

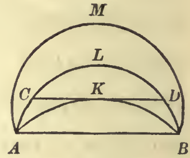
13. *An arc of a circle is less than any line which envelops it and has the same extremities.*



Let AKB be an arc of a circle, AB its chord; and let ALB, AMB, etc., be any lines enveloping it and terminating at A and B.

Of all the lines AKB, ALB, AMB, etc., which can be drawn (each including between itself and the chord AB the segment, or *area*, AKB), there must be at least one *minimum* or shortest line, since all the lines are obviously not equal. Now, no one of the lines ALB, AMB, etc., enveloping AKB, can be such a minimum; for, drawing a tangent CKD to the arc AKB, the line $ACKDB$ is less than $ACLDB$;

therefore ALB is not the minimum; and in the same way it is shown that no other enveloping line can be the minimum. Therefore the arc AKB is the minimum.

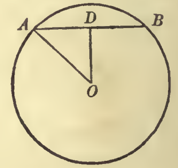


14. Corollary. *The circumference of a circle is less than the perimeter of any polygon circumscribed about it.*

15. *Scholium.* The demonstration is applicable when AKB is any convex curve whatever.

PROPOSITION VI.—THEOREM.

16. *If the number of sides of a regular polygon inscribed in a circle be increased indefinitely, the apothem of the polygon will approach the radius of the circle as its limit.*



Let AB be a side of a regular polygon inscribed in the circle whose radius is OA; and let OD be its apothem.

Whatever the number of sides of the polygon $OD < OA$, by I., Proposition XVII. $OA < AD + OD$ (I., Axiom I.); $\therefore OA - OD < AD$, and consequently $OA - OD < AB$.

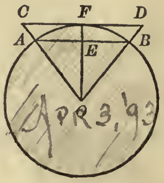
The perimeter of the polygon is manifestly less than the circumference of the circle. If n is the number of sides of the polygon, AB is less than one-nth of the circumference. Therefore, by taking a sufficiently great value of n, we can make AB, and consequently $OA - OD$, as small as we please.

Since $OA - OD$ can be made as small as we please by increasing the number of sides of the polygon, but cannot be made absolutely zero, OA is the limit of OD, as the number of sides of the polygon is indefinitely increased (II., 39).

PROPOSITION VII.—THEOREM.

17. *The circumference of a circle is the limit which the perimeters of regular inscribed and circumscribed polygons approach when the number of their sides is increased indefinitely; and the area of the circle is the limit of the areas of these polygons.*

Let AB and CD be sides of a regular inscribed and a similar circumscribed polygon (*v.* Proposition I., Exercise); let r denote the apothem OE, R the radius OF, p the perimeter of the inscribed polygon, P the perimeter of the circumscribed polygon. Then we have (Proposition III., Corollary)



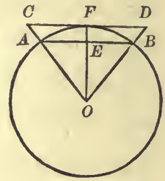
$$\frac{P}{p} = \frac{R}{r},$$

whence, by division (III., 10),

$$\frac{P-p}{P} = \frac{R-r}{R}, \text{ or } P - p = \frac{P}{R} \times (R-r).$$

Now, we have seen in Proposition VI. that by increasing the number of sides of the polygons the difference $R - r$ may be decreased at pleasure; consequently, since $\frac{P}{R}$ does not increase, $\frac{P}{R} \times (R - r)$, or $P - p$, may be decreased at pleasure. But P being always greater, and p always less, than the circumference of the circle, the difference between this circumference and either P or p is less than the difference $P - p$, and consequently may be made as small as we please by increasing the number of sides of the polygons, and since it obviously cannot be made absolutely zero, the circumference is the common limit of P and p, as the number of sides of the polygons is indefinitely increased.

Again, let s and S denote the areas of two similar inscribed and circumscribed polygons. The difference between the triangles COD and AOB is the trapezoid $CABD$, the measure of which is $\frac{1}{2}(CD + AB) \times EF$; therefore the difference between the areas of the polygons is



$$S - s = \tfrac{1}{2}(P + p) \times (R - r);$$

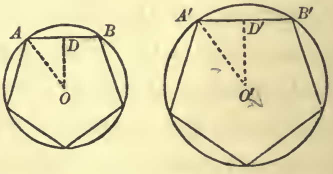
consequently,

$$S - s < P \times (R - r).$$

Now, by increasing the number of sides of the polygons the quantity $P \times (R - r)$, and consequently also $S - s$, may be decreased at pleasure. But S being always greater, and s always less, than the area of the circle, the difference between the area of the circle and either S or s is less than the difference $S - s$, and consequently may also be made as small as we please by increasing the number of sides of the polygons, and since it obviously cannot be made absolutely zero, the area of the circle is the common limit of S and s, as the number of sides of the polygons is indefinitely increased.

PROPOSITION VIII.—THEOREM.

18. *The circumferences of two circles are to each other as their radii, and their areas are to each other as the squares of their radii.*



Let R and R' be the radii of the circles, C and C' their circumferences, S and S' their areas.

Inscribe in the two circles similar regular polygons of any arbitrarily chosen number, n, of sides; let P and P' denote the perimeters, A and A' the areas, of these polygons; then, the polygons being similar, we have (Proposition III., Corollary)

$$\frac{P}{P'} = \frac{R}{R'}, \qquad \frac{A}{A'} = \frac{R^2}{R'^2},$$

or

$$P = \frac{R}{R'} \times P', \qquad A = \frac{R^2}{R'^2} \times A',$$

$\frac{1}{2} C \times R$

But $C = 2\pi R$

$\frac{1}{2}(2\pi R \times R) = \pi R^2$

no matter what the value of n.

As we change n, P and $\frac{R}{R'} \times P'$ change, but remain always equal to each other.

As n is indefinitely increased, P approaches the limit C, and $\frac{R}{R'} \times P'$ approaches the limit $\frac{R}{R'} \times C'$. Therefore, by II., Theorem of Limits, these limits are equal, and we have

$$C = \frac{R}{R'} \times C',$$

or

$$\frac{C}{C'} = \frac{R}{R'}.$$

In the same way we may prove

$$\frac{S}{S'} = \frac{R^2}{R'^2}.$$

19. Corollary I. *The circumferences of circles are to each other as their diameters, and their areas are to each other as the squares of their diameters.*

Suggestion. $D = 2R$, if D is the diameter and R the radius.

20. Corollary II. *The ratio of the circumference of a circle to its diameter is constant;* that is, it is the same for all circles.

For, from $\frac{C}{C'} = \frac{2R}{2R'}$, we have at once

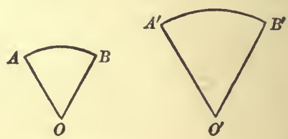
$$\frac{C}{2R} = \frac{C'}{2R'}.$$

This constant ratio is usually denoted by π, so that for any circle whose diameter is $2R$ and circumference C we have

$$\frac{C}{2R} = \pi, \text{ or } C = 2\pi R.$$

21. *Scholium.* The ratio π is incommensurable (as can be proved by the higher mathematics), and can therefore be expressed in numbers only approximately. The letter π, however, is used to *symbolize* its *exact* value.

22. *Definitions. Similar arcs* are those which subtend equal angles at the centres of the circles to which they belong.



Similar sectors are sectors whose bounding radii include equal angles.

Sector is space between the arc and two radii.

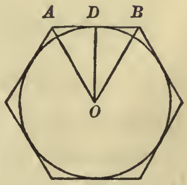
EXERCISE.

Theorem.—Similar arcs are to each other as their radii, and similar sectors are to each other as the squares of their radii.

Suggestion. The arcs are like parts of their respective circumferences, and the sectors like parts of their circles.

PROPOSITION IX.—THEOREM.

23. *The area of a circle is equal to half the product of its circumference by its radius.*



Let the area of any regular polygon circumscribed about the circle be denoted by A, its perimeter by P, and its apothem, which is equal to the radius of the circle, by R; and let S be the area and C the circumference of the circle. Then $A = \frac{1}{2} P \times R$ (Proposition IV.), no matter what the number of sides of the polygon. If we change the number of sides of the polygon, A and $\frac{1}{2} P \times R$ change, but remain always equal to each other.

As the number of sides is indefinitely increased, A approaches the limit S, and $\frac{1}{2} P \times R$ the limit $\frac{1}{2} C \times R$. Therefore, by II., Theorem of Limits, these limits are equal, and we have

$$S = \tfrac{1}{2} C \times R. \qquad [1]$$

24. Corollary. *The area of a circle is equal to the square of its radius multiplied by the constant number π.*

Suggestion. If we substitute for C in [1] its value $2\pi R$ (20), we have

$$S = \pi R^2.$$

EXERCISE.

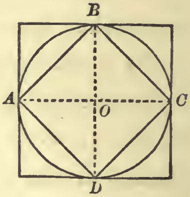
Theorem.—The area of a sector is equal to half the product of its arc by the radius.

Suggestion. Compare the sector with the whole circle.

PROBLEMS OF CONSTRUCTION AND COMPUTATION.

PROPOSITION X.—PROBLEM.

25. *To inscribe a square in a given circle.*



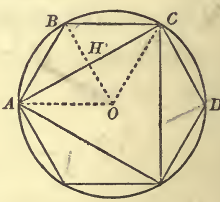
Draw any two diameters AC, BD, perpendicular to each other, and join their extremities by the chords AB, BC, CD, DA; then $ABCD$ is an inscribed square.

26. Corollary. *To circumscribe a square about a circle,* draw tangents at the extremities of two perpendicular diameters AC, BD.

27. *Scholium.* In the right triangle ABO we have $\overline{AB}^2 = \overline{OA}^2 + \overline{OB}^2 = 2\overline{OA}^2$, whence $AB = OA . \sqrt{2}$, by which the side of the inscribed square can be computed, the radius being given.

PROPOSITION XI.—PROBLEM.

28. *To inscribe a regular hexagon in a given circle.*



Suppose the problem solved, and let $ABCDEF$ be a regular inscribed hexagon.

Draw the radii OA and OB. The angle AOB is measured by $\frac{1}{6}$ of the circumference, and therefore contains 60°. OAB and OBA are therefore together equal to 180° — 60°, or 120°; and, since they are equal, each is 60°, and the triangle OAB is equilateral, and therefore the side of the inscribed regular hexagon is equal to the radius of the circle.

Consequently, to inscribe a regular hexagon, apply the radius to the circle six times as a chord.

29. Corollary. *To inscribe an equilateral triangle*, join the alternate vertices of the regular hexagon.

30. *Scholium.* Since OB bisects the arc ABC, it bisects the chord AC at right angles; and since in the isosceles triangle AOB, AH is perpendicular to the base, it bisects the base, and

$$OH = \tfrac{1}{2}OB = \tfrac{1}{2}OA;$$

that is, the *apothem* of an inscribed regular triangle is equal to one-half the radius.

In the right triangle AHO, $\overline{AH}^2 = \overline{OA}^2 - \overline{OH}^2 = \overline{OA}^2 - (\tfrac{1}{2}OA)^2 = \tfrac{3}{4}\overline{OA}^2$, and

$$AH = \frac{OA}{2}\sqrt{3};$$

whence $AC = OA\sqrt{3}$, by which the side of the inscribed triangle can be computed from the radius.

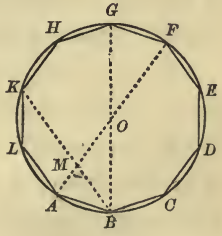
The apothem of the regular inscribed hexagon is equal to

$$AH = \frac{OA}{2}\sqrt{3}.$$

PROPOSITION XII.—PROBLEM.

31. *To inscribe a regular decagon in a given circle.*

Suppose the problem solved, and let $ABC \ldots L$ be a regular inscribed decagon.



Join AF, BG; since each of these lines bisects the circumference, they are diameters and intersect in the centre O. Draw BK intersecting OA in M.

The angle AMB is measured by half the sum of the arcs KF and AB (II., Proposition XVI.),—

by divisions is meant – length of one of the sides

that is, by two divisions of the circumference; the inscribed angle MAB is measured by half the arc BF,—that is, also, by two divisions; therefore AMB is an isosceles triangle, and $MB = AB$.

Again, the inscribed angle MBO is measured by half the arc KG,—that is, by one division; and the angle MOB at the centre has the same measure; therefore OMB is an isosceles triangle, and $OM = MB = AB$.

The inscribed angle MBA, being measured by half the arc AK,—that is, by one division,—is equal to the angle AOB. Therefore the isosceles triangles AMB and AOB are mutually equiangular and similar, and give the proportion

$$OA : AB = AB : AM;$$

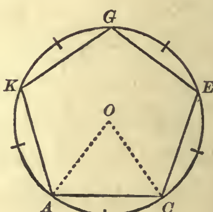
whence

$$OA \times AM = \overline{AB}^2 = \overline{OM}^2;$$

that is, the radius OA is divided in extreme and mean ratio at M (III., 41); and the greater segment OM is equal to the side AB of the inscribed regular decagon.

Consequently, to inscribe a regular decagon, divide the radius in extreme and mean ratio (III., 42), and apply the greater segment ten times as a chord.

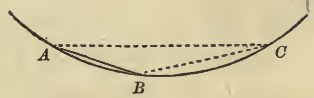
32. Corollary. *To inscribe a regular pentagon*, join the alternate vertices of the regular inscribed decagon.



PROPOSITION XIII.—PROBLEM.

33. *To inscribe a regular pentedecagon in a given circle.*

Suppose AB is the side of a regular inscribed pentedecagon, or that the arc AB is $\frac{1}{15}$ of the circumference.



Now, the fraction $\frac{1}{15} = \frac{1}{6} - \frac{1}{10}$; therefore the arc AB is the difference between $\frac{1}{6}$ and $\frac{1}{10}$ of the circumference. Hence, if we inscribe the chord AC equal to the side of the regular inscribed hexagon, and then CB equal to that of the regular inscribed decagon, the chord AB will be the side of the regular inscribed pentedecagon required.

34. *Scholium.* Any regular inscribed polygon being given, a regular inscribed polygon of double the number of sides can be formed by bisecting the arcs subtended by its sides and drawing the chords of the semi-arcs (Proposition I., Corollary I.). Also, any regular inscribed polygon being given, a regular circumscribed polygon of the same number of sides can be formed (Proposition I.). Therefore, by means of the inscribed square, we can inscribe and circumscribe, successively, regular polygons of 8, 16, 32, etc., sides; by means of the hexagon, those of 12, 24, 48, etc., sides; by means of the decagon, those of 20, 40, 80, etc., sides; and, finally, by means of the pentedecagon, those of 30, 60, 120, etc., sides.

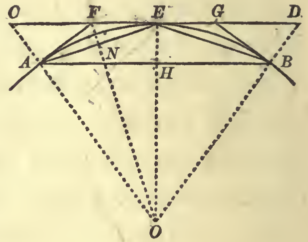
Until the beginning of the present century, it was supposed that these were the only polygons that could be constructed by elementary geometry; that is, by the use of the straight line and circle only. GAUSS, however, in his *Disquisitiones Arithmeticæ*, Lipsiæ, 1801, proved that it is possible, by the use of the straight line and circle only, to construct regular polygons of 17 sides, of 257 sides, and in general of

Study this well

any number of sides which can be expressed by $2^n + 1$, n being an integer, provided that $2^n + 1$ is a prime number.

PROPOSITION XIV.—PROBLEM.

35. *Given the perimeters of a regular inscribed and a similar circumscribed polygon, to compute the perimeters of the regular inscribed and circumscribed polygons of double the number of sides.*



Let AB be a side of the given inscribed polygon, and CD a side of the similar circumscribed polygon, tangent to the arc AB at its middle point E. Join AE, and at A and B draw the tangents AF and BG; then AE is a side of the regular inscribed polygon of double the number of sides, and FG is a side of the circumscribed polygon of double the number of sides.

Denote the perimeters of the given inscribed and circumscribed polygons by p and P, respectively; and the perimeters of the required inscribed and circumscribed polygons of double the number of sides by p' and P', respectively.

Since OC is the radius of the circle circumscribed about the polygon whose perimeter is P, we have (Proposition III., Corollary)

$$\frac{P}{p} = \frac{OC}{OA} \text{ or } \frac{OC}{OE};$$

and since OF bisects the angle COE, we have (III., 15, Exercise)

$$\frac{OC}{OE} = \frac{CF}{FE};$$

therefore

$$\frac{P}{p} = \frac{CF}{FE},$$

whence, by composition,

$$\frac{P+p}{2p}=\frac{CF+FE}{2FE}=\frac{CE}{FG}.$$

Now, FG is a side of the polygon whose perimeter is P', and is contained as many times in P' as CE is contained in P; hence (III., 9)

$$\frac{CE}{FG}=\frac{P}{P'},$$

and therefore

$$\frac{P+p}{2p}=\frac{P}{P'},$$

whence

$$P'=\frac{2pP}{P+p}. \qquad [1]$$

Again, the right triangles AEH and EFN are similar, since their acute angles EAH and FEN are equal, and give

$$\frac{AH}{AE}=\frac{EN}{EF}.$$

Since AH and AE are contained the same number of times in p and p', respectively, we have

$$\frac{AH}{AE}=\frac{p}{p'};$$

and since EN and EF are contained the same number of times in p' and P', respectively, we have

$$\frac{EN}{EF}=\frac{p'}{P'};$$

therefore we have

$$\frac{p}{p'}=\frac{p'}{P'},$$

whence

$$p'=\sqrt{p\times P'}. \qquad [2]$$

Therefore, from the given perimeters p and P we compute P' by the equation [1], and then with p and P' we compute p' by the equation [2].

$\frac{a}{b}=\frac{b}{c}$ $\quad b=\frac{c\times a}{b}$

$b^2=c\times a$

PROPOSITION XV.—PROBLEM.

36. *To compute the ratio of the circumference of a circle to its diameter, approximately.*

METHOD OF PERIMETERS.—In this method, we take the diameter of the circle as given and compute the perimeters of some inscribed and a similar circumscribed regular polygon. We then compute the perimeters of inscribed and circumscribed regular polygons of double the number of sides, by Proposition XIV. Taking the last-found perimeters as given, we compute the perimeters of polygons of double the number of sides by the same method; and so on. Each computation gives us, of course, a pair of values between which the value of the circumference must lie. As we continue the process, these values will come nearer and nearer to the actual value of the circumference (Proposition VII.), and we may thus obtain as close an approximation to that value as we please.

Taking, then, the diameter of the circle as given $= 1$, let us begin by inscribing and circumscribing a square. The perimeter of the inscribed square $= 4 \times \frac{1}{2} \times \sqrt{2} = 2\sqrt{2}$ (27); that of the circumscribed square $= 4$; therefore, putting

$$P = 4,$$
$$p = 2\sqrt{2} = 2.8284271,$$

we find, by Proposition X., for the perimeters of the circumscribed and inscribed regular octagons,

$$P' = \frac{2p \times P}{P + p} = 3.3137085,$$

$$p' = \sqrt{p \times P'} = 3.0614675.$$

Then, taking these as given quantities, we put

$$P = 3.3137085,\ p = 3.0614675,$$

and find by the same formulæ for the polygons of 16 sides

$$P' = 3.1825979,\ p' = 3.1214452.$$

Continuing this process, the results will be found as in the following

TABLE.*

Number of sides.	Perimeter of circumscribed polygon.	Perimeter of inscribed polygon.
4	4.0000000	2.8284271
8	3.3137085	3.0614675
16	3.1825979	3.1214452
32	3.1517249	3.1365485
64	3.1441184	3.1403312
128	3.1422236	3.1412773
256	3.1417504	3.1415138
512	3.1416321	3.1415729
1024	3.1416025	3.1415877
2048	3.1415951	3.1415914
4096	3.1415933	3.1415923
8192	3.1415928	3.1415926

From the last two numbers of this table we learn that the circumference of the circle whose diameter is unity is less than 3.1415928 and greater than 3.1415926; and since, when the diameter $= 1$, we have $C = \pi$ (20), it follows that

$$\pi = 3.1415927$$

within a unit of the seventh decimal place.

* The computations have been carried out with ten decimal places in order to insure the accuracy of the seventh place, as given in the table.

37. *Scholium.* ARCHIMEDES (born 287 B.C.) was the first to assign an approximate value of π. By a method similar to that above, he proved that its value is between $3\frac{1}{7}$ and $3\frac{10}{71}$, or, in decimals, between 3.1428 and 3.1408; he therefore assigned its value correctly within a unit of the third decimal place. The number $3\frac{1}{7}$, or $\frac{22}{7}$, usually cited as Archimedes' value of π (although it is but one of the two limits assigned by him), is often used as a sufficient approximation in rough computations.

METIUS (A.D. 1640) found the much more accurate value $\frac{355}{113}$, which correctly represents even the sixth decimal place. It is easily remembered by observing that the denominator and numerator written consecutively, thus, 113|355, present the first three odd numbers each written twice.

More recently, the value has been found to a very great number of decimals, by the aid of series demonstrated by the Differential Calculus. CLAUSEN and DASE, of Germany (about A.D. 1846), computing independently of each other, carried out the value to two hundred decimal places, and their results agree to the last figure. The mutual verification thus obtained stamps their results as thus far the best established value to the two-hundredth place. (See SCHUMACHER'S *Astronomische Nachrichten*, No. 589.) Other computers have carried the value to over five hundred places, but it does not appear that their results have been verified.

The value to fifteen decimal places is

$$\pi = 3.141592653589793.$$

For the greater number of practical applications, the value $\pi = 3.1416$ is sufficiently accurate.

EXERCISES ON BOOK V.

THEOREMS.

1. An equilateral polygon inscribed in a circle is regular.

2. An equilateral polygon circumscribed about a circle is regular if the number of its sides is *odd*.

3. An equiangular polygon inscribed in a circle is regular if the number of its sides is *odd*.

4. An equiangular polygon circumscribed about a circle is regular.

5. The area of the regular inscribed triangle is one-half the area of the regular inscribed hexagon.

6. The area of the regular inscribed hexagon is three-fourths of that of the regular circumscribed hexagon.

7. The area of the regular inscribed hexagon is a mean proportional between the areas of the inscribed and circumscribed equilateral triangles.

8. A plane surface may be entirely covered (as in the construction of a pavement) by *equal* regular polygons of either three, four, or six sides.

9. A plane surface may be entirely covered by a combination of squares and regular octagons having the same side, or by dodecagons and equilateral triangles having the same side.

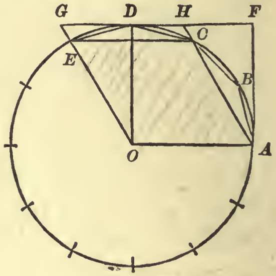
10. If squares be described on the sides of a regular hexagon, and their adjacent external vertices be joined, a regular dodecagon will be formed.

11. The diagonals of a regular pentagon form a regular pentagon.

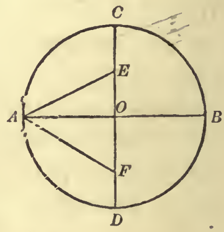
12. The diagonals joining alternate vertices of a regular hexagon enclose a regular hexagon one-third as large as the original hexagon.

13. The area of the regular inscribed octagon is equal to the product of the side of the inscribed square by the diameter.

Suggestion. A quarter of the octagon is the sum of two triangles having as a common base the side of the inscribed square, and having the radius as the sum of their altitudes.



14. The area of a regular inscribed dodecagon is equal to three times the square of the radius.



15. Prove the correctness of the following construction:

If AB and CD are two perpendicular diameters in a circle, and E the middle point of the radius OC, and if EF is taken equal to EA, then OF is equal to the side of the regular inscribed decagon, and AF is equal to the side of the regular inscribed pentagon. (*v.* III., 42.)

16. From any point within a regular polygon of n sides, perpendiculars are drawn to the several sides; prove that the sum of these perpendiculars is equal to n times the apothem.

Suggestion. Join the point with the vertices of the polygon, and obtain an expression for the area in terms of the perpendiculars: then see Proposition IV.

17. The side of the regular inscribed triangle is equal to the hypotenuse of a right triangle of which the sides of the inscribed square and of the regular inscribed hexagon are the sides. (*v.* IV., Proposition X.)

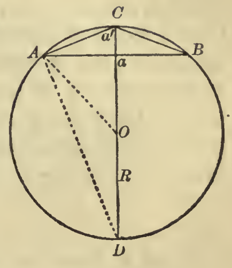
18. If a is the side of a regular decagon inscribed in a circle whose radius is R,

$$a = \frac{R}{2}(\sqrt{5} - 1).$$

L

Suggestion. By (31), $\frac{R}{a} = \frac{a}{R-a}$.

19. If $a =$ the side of a regular polygon inscribed in a circle whose radius is R, and $a' =$ the side of the regular inscribed polygon of double the number of sides, then



$$a'^2 = R\,(2R - \sqrt{4R^2 - a^2}),$$
$$a^2 = \frac{a'^2\,(4R^2 - a'^2)}{R^2}.$$

Suggestion. ABC and ADO are similar. Hence $\frac{a}{a'} = \frac{AD}{R}$ and $\frac{a^2}{a'^2} = \frac{\overline{AD}^2}{R^2}$; but $\overline{AD}^2 = (2R)^2 - a'^2$.

20. If $a =$ the side of a regular pentagon inscribed in a circle whose radius is R, then

$$a = \frac{R}{2}\sqrt{10 - 2\sqrt{5}}.$$

21. If $a =$ the side of a regular octagon inscribed in a circle whose radius is R, then

$$a = R\sqrt{2 - \sqrt{2}}.$$

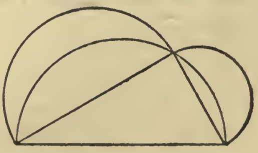
22. If $a =$ the side of a regular dodecagon inscribed in a circle whose radius is R, then

$$a = R\sqrt{2 - \sqrt{3}}.$$

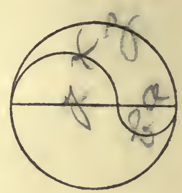
23. The side of the regular inscribed pentagon is equal to the hypotenuse of a right triangle whose sides are the radius and the side of the regular inscribed decagon.

24. The area of a ring bounded by two concentric circumferences is equal to the area of a circle having for its diameter a chord of the outer circumference tangent to the inner circumference.

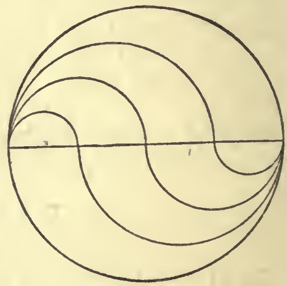
25. If on the legs of a right triangle, as diameters, semicircles are described external to the triangle, and from the whole figure a semicircle on the hypotenuse is subtracted, the remainder is equivalent to the given triangle.



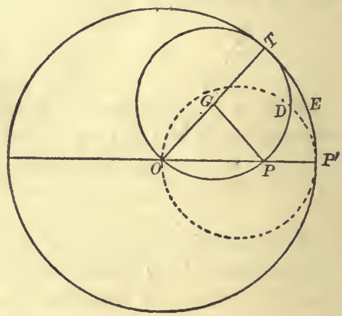
26. If on the two segments into which a diameter of a given circle is divided by any point, as diameters, semi-circumferences are described lying on opposite sides of the given diameter, the sum of their lengths is equal to the length of a semi-circumference of the given circle, and a line which they form divides the circle into two parts whose areas are to each other as the segments of the given diameter.



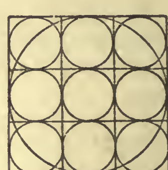
27. If a diameter of a given circle is divided into n equal parts, and through each point of division a curved line of the sort described in the last problem is drawn, these lines will divide the circle into n equivalent parts.



28. If a circle rolls around the circumference of a circle of twice its radius, the two circles being always tangent internally, the locus of a fixed point on the circumference of the rolling circle is a diameter of the fixed circle.



29. If a given square is subdivided into n^2 equal squares, n being any given number, and in each of these smaller squares a circle is inscribed, the sum of their areas is equal to the area of the circle inscribed in the original square.



MISCELLANEOUS EXERCISES

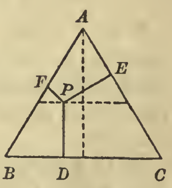
ON

PLANE GEOMETRY.

THEOREMS.

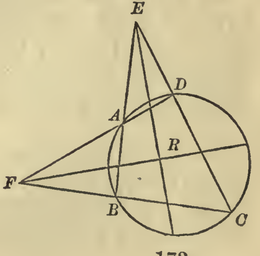
1. THE sum of the three straight lines drawn from any point within a triangle to the three vertices is less than the sum and greater than the half sum of the three sides of the triangle.

2. If one of the acute angles of a right triangle is double the other, the hypotenuse is double the shortest side.

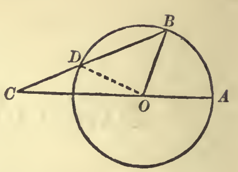


3. If from any point within an equilateral triangle perpendiculars to the three sides are drawn, the sum of these lines is constant, and equal to the perpendicular from any vertex upon the opposite side.

4. Lines drawn from one vertex of a parallelogram to the middle points of the opposite sides trisect a diagonal.

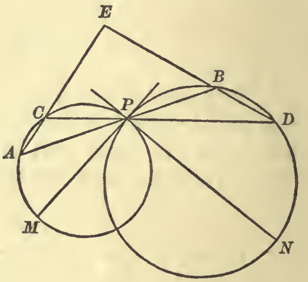


5. The bisectors of the angles contained by the opposite sides (produced) of an inscribed quadrilateral intersect at right angles.



6. If AOB is any given angle at the centre of a circle, and if BC can be drawn meeting AO produced in C, and the circumference in D, so that CD shall be equal to the radius of the circle, then the angle C will be equal to one-third the angle AOB.

Note. There is no method known of drawing BC, under these conditions, and with the use of straight lines and circles only, AOB being *any* given angle; so that the *trisection of an angle*, in general, is a problem that cannot be solved by elementary geometry.



7. If through P, one of the points of intersection of two circumferences, any two secants, APB, CPD, are drawn, the straight lines, AC, DB, joining the extremities of the secants, make a constant angle E, equal to the angle MPN formed by the tangents at P.

8. If a figure is moved in a plane, it may be brought from one position to any other by revolving it about a certain fixed point; that is, by causing each point of the figure to move in the circumference of a circle whose centre is the fixed point.

9. If a square $DEFG$ is inscribed in a right triangle ABC, so that a side DE coincides with the hypotenuse BC (the vertices F and G being in the sides AC and AB), then the side DE is a mean proportional between the segments BD and EC of the hypotenuse.

10. If the middle points of the sides of a triangle are joined by straight lines, the medial lines of the triangle thus formed are the medial lines of the original triangle, and the perpendiculars from the vertices upon the opposite sides are the perpendiculars at the middle points of the sides of the original triangle.

11. If O is the centre of the circle circumscribed about a triangle ABC, and P is the intersection of the perpendiculars from the angles upon the opposite sides, the perpendicular from O upon the side BC is equal to one-half the distance AP.

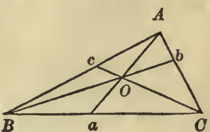
√

12. In any triangle, the centre of the circumscribed circle, the intersection of the medial lines, and the intersection of the perpendiculars from the angles upon the opposite sides, are in the same straight line; and the distance of the first point from the second is one-half the distance of the second from the third.

13. If two circles intersect in the points A and B, and through A any secant CAD is drawn terminated by the circumferences at C and D, the straight lines BC and BD are to each other as the diameters of the circles.

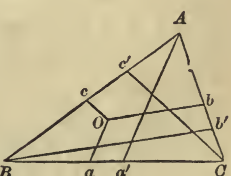
14. If through the middle point of each diagonal of any quadrilateral a parallel is drawn to the other diagonal, and from the intersection of these parallels straight lines are drawn to the middle points of the four sides, these straight lines divide the quadrilateral into four equivalent parts.

15. If three straight lines Aa, Bb, Cc, drawn from the vertices of a triangle ABC to the opposite sides, pass through a common point O within the triangle, then



$$\frac{Oa}{Aa} + \frac{Ob}{Bb} + \frac{Oc}{Cc} = 1.$$

16. If from any point O within a triangle ABC any three straight lines, Oa, Ob, Oc, are drawn to the three sides, and through the vertices of the triangle three straight lines, Aa', Bb', Cc', are drawn parallel respectively to Oa, Ob, Oc, then



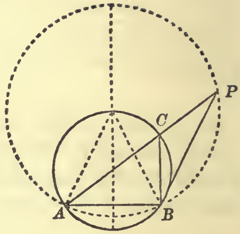
$$\frac{Oa}{Aa'} + \frac{Ob}{Bb'} + \frac{Oc}{Cc'} = 1.$$

17. The area of a circle is a mean proportional between the areas of any two similar polygons, one of which is circumscribed about the circle and the other isoperimetrical with the circle. (*Galileo's Theorem.*)

18. Two diagonals of a regular pentagon, not drawn from a common vertex, divide each other in extreme and mean ratio.

LOCI.

19. The angle ACB is any inscribed angle in a given segment of a circle; AC is produced to P, making CP equal to CB; find the locus of P.

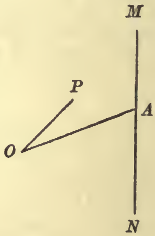


20. The hypotenuse of a right triangle is given in magnitude and position; find the locus of the centre of the inscribed circle.

21. The base BC of a triangle ABC is given in position and magnitude, and the vertical angle A is of a given magnitude; find the locus of the centre of the inscribed circle.

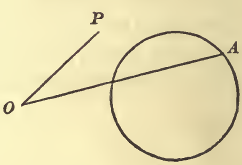
22. From a given point O, any straight line OA is drawn to a given straight line MN, and OP is drawn making a given angle with OA, and such that OP is to OA in a given ratio; find the locus of P.

With the same construction, if OP is so taken that the product $OP.OA$ is equal to a given constant; find the locus of P.



23. From a given point O, any straight line OA is drawn to a given circumference, and OP is drawn making a given angle with OA, and such that OP is to OA in a given ratio; find the locus of P.

With the same construction, if OP is so taken that the product $OP.OA$ is equal to a given constant; find the locus of P.



24. One vertex of a triangle whose angles are given is fixed, while the second vertex moves on the circumference of a given circle; what is the locus of the third vertex?

25. Through A, one of the points of intersection of two given circles, any secant is drawn cutting the two circumferences in the points B and C; find the locus of the middle point of BC.

PROBLEMS.

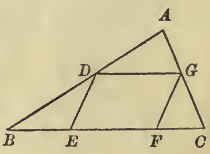
26. Describe a circle through two given points which lie outside a given line, the centre of the circle to be in that line. Show when no solution is possible.

27. In a given circle, inscribe a chord of a given length which produced shall be tangent to another given circle.

28. Through P, one of the points of intersection of two circumferences, draw a straight line, terminated by the circumferences, which shall be bisected in P.

29. Through one of the points of intersection of two circumferences draw a straight line, terminated by the circumferences, which shall have a given length.

30. In a given triangle ABC, to inscribe a parallelogram $DEFG$, such that the adjacent sides DE and DG shall be in a given ratio and contain a given angle.



31. Construct a triangle, given its base, the ratio of the other two sides, and one angle.

32. To determine a point in a given arc of a circle, such that the chords drawn from it to the extremities of the arc shall have a given ratio.

33. To find a point within a given triangle, such that the three straight lines drawn from it to the vertices of the triangle shall make three equal angles with each other.

34. Inscribe a trapezoid in a given circle, knowing its area and the common length of its inclined sides.

35. To construct a triangle, given one angle, the side opposite to that angle, and the area (equal to that of a given square).

36. Divide a given circle into a given number of equivalent parts, by concentric circumferences.

Also, divide it into a given number of parts proportional to given lines, by concentric circumferences.

37. A circle being given, to find a given number of circles whose radii shall be proportional to given lines, and the sum of whose areas shall be equal to the area of the given circle.

38. In a given equilateral triangle, inscribe three equal circles tangent to each other and to the sides of the triangle.

Determine the radius of these circles in terms of the side of the triangle.

39. In a given circle, inscribe three equal circles tangent to each other and to the given circle.

Determine the radius of these circles in terms of the radius of the given circle.

Friday. Mar. 16

NUMERICAL EXAMPLES.

NOTE.—The following approximate values are close enough for ordinary purposes: $\pi = \frac{22}{7}$, $\sqrt{2} = \frac{17}{12}$, $\sqrt{3} = \frac{19}{11}$, $\sqrt{5} = \frac{38}{17}$. Radius of earth $= 3960$ miles.

40. The vertical angle of an isosceles triangle is 36°, and the length of the base is 2 feet; find the base angles, the length of the bisector of a base angle, and the length of a side of the given triangle. *Ans.* 72°, 2 feet, $(1 + \sqrt{5})$ feet.

41. One angle of a triangle is 60°, the including sides are 3 feet and 8 feet; find the area and the third side.

Ans. $6\sqrt{3}$ square feet, 7 feet.

omit

42. The three sides of a triangle are 9 inches, 10 inches, and 17 inches, its area is 36 square inches; find the area of the inscribed circle. *Ans.* 4π.

43. The adjacent sides of a parallelogram are 12 feet and 14 feet, the area is 120 square feet; find the long diagonal.

Ans. 24 feet.

44. The area of a right triangle is 6 square feet, the length of the hypotenuse is 5 feet; find the other sides.

Ans. 3 feet, 4 feet.

45. Obtain a formula connecting the length of a chord l, its distance from the centre d, and the radius r.

Ans. $\frac{l^2}{4} = r^2 - d^2$.

46. Obtain a formula for the length t of a common tangent to two circles, given the radii r, r', and the distance between the centres d. *Ans.* $(r - r')^2 + t^2 = d^2$ for external tangent.
$(r + r')^2 + t^2 = d^2$ for internal tangent.

47. Through what angle does the hour-hand of a clock move in 1 hour? in 1 minute? Through what angle does the minute-hand move in 1 minute?

What angle do the hands of a clock make with each other at ten minutes past three? at quarter of six? *Ans.* 35°, 97° 30′.

48. Two secants cut each other without a circle, the intercepted arcs are 12° and 48°; what is the angle between the secants?

Two chords intersect within the circle, a pair of opposite intercepted arcs are 12° and 48°; what is the angle between the chords?

49. Two tangents make with each other an angle of 60°; required the lengths of the arcs into which their points of contact divide the circle, given radius equals 7 inches.

Ans. $14\frac{2}{3}$ inches, $29\frac{1}{3}$ inches.

50. A swimmer whose eye is at the surface of the water can just see the top of a stake a mile distant; the stake proves to be 8 inches out of water; required the radius of the earth.

Ans. 3960 miles.

51. A passenger standing on the deck of a steamer about to start observes that his eye is on a level with the top of the wharf, which he knows to be 12 feet high; when they have steamed $8\frac{1}{2}$ miles the wharf disappears below the horizon; required the radius of the earth. *Ans.* 3974 miles.

52. How many miles is the light of a light-house 150 feet high visible at sea? *Ans.* 15.

53. On approaching Portland from the sea, Mount Washington is first visible 12 miles from shore; Portland is 85 miles from Mount Washington; required the height of the mountain.

Ans. 6270 feet.

54. The latitude of Leipsic is 51° 21′, that of Venice 45° 26′, and Venice is due south of Leipsic; how many miles are they apart? Use 4000 miles as the earth's radius. *Ans.* 413 miles.

55. The latitude of the Peak of Teneriffe is about 30° N.; the rising sun shines on its summit on the 21st of March 9 minutes before it shines on its base; required the height of the mountain.

Ans. About 12,000 feet.

56. A quarter-mile running-track 10 feet wide, with straight parallel sides and semicircular ends, is to be laid out in a rectangular field 220 feet wide. How long must the field be in order that a runner, keeping in the middle of the track, may have one-quarter of a mile to cover? how much can he gain by keeping close to the inner edge of the track? what is the area of the field? of the portion encircled by the track? of the track itself?

Ans. 550 feet; $31\frac{3}{7}$ feet; 121,000 square feet; 97,$428\frac{4}{7}$ square feet; 13,200 square feet.

57. The fly-wheel of an engine is connected by a belt with a smaller wheel driving the machinery of a mill. The radius of the

fly-wheel is 7 feet; of the small wheel, 21 inches. How many revolutions does the small wheel make to one of the fly-wheel? The distance between the centres of the two wheels is $10\frac{1}{2}$ feet. What is the length of the connecting band?

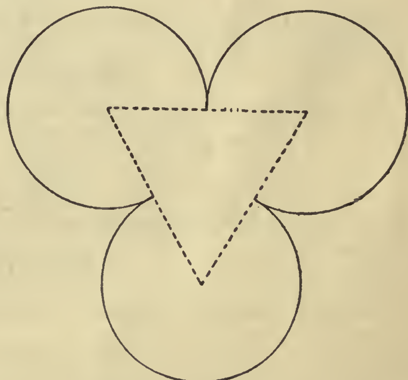
Ans. 51 feet 2 inches.

58. If from each vertex of a regular polygon as a centre, with a radius equal to one-half the side, an arc is described outward from side to side of the polygon, an ornamental figure much used in architecture is formed. Such a figure formed on a polygon of numerous sides is often used as a rose-window.

The figure bounded by three arcs is called a trefoil; by four arcs, a quatre-foil; by five arcs, a cinque-foil.

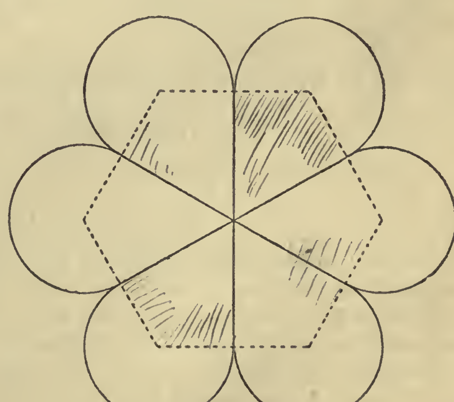
Find the area of a trefoil, given the distance between the centres of adjacent arcs equal to 21 inches.

Ans. 7.338 square feet.



59. A rose-window of six lobes is to be placed in a circular space 42 feet in diameter. How many square feet of glass will it contain?

Ans. 1123.8 square feet.



SYLLABUS OF PLANE GEOMETRY.

POSTULATES, AXIOMS, AND THEOREMS.

BOOK I.

Postulate I.

Through any two points one straight line, and only one, can be drawn.

Postulate II.

Through a given point one straight line, and only one, can be drawn having any given direction.

Axiom I.

A straight line is the shortest line that can be drawn between two points.

Axiom II.

Parallel lines have the same direction.

Proposition I.

At a given point in a straight line one perpendicular to the line can be drawn, and but one.

Corollary. Through the vertex of any given angle one straight line can be drawn bisecting the angle, and but one.

Proposition II.

All right angles are equal.

Proposition III.

The two adjacent angles which one straight line makes with another are together equal to two right angles.

Corollary I. The sum of all the angles having a common vertex, and formed on one side of a straight line, is two right angles.

Corollary II. The sum of all the angles that can be formed about a point in a plane is four right angles.

Proposition IV.

If the sum of two adjacent angles is two right angles, their exterior sides are in the same straight line.

Proposition V.

If two straight lines intersect each other, the opposite (or vertical) angles are equal.

Proposition VI.

Two triangles are equal when two sides and the included angle of the one are respectively equal to two sides and the included angle of the other.

Proposition VII.

Two triangles are equal when a side and the two adjacent angles of the one are respectively equal to a side and the two adjacent angles of the other.

Proposition VIII.

In an isosceles triangle the angles opposite the equal sides are equal.

Corollary. The straight line bisecting the vertical angle of an isosceles triangle bisects the base, and is perpendicular to the base.

Proposition IX.

Two triangles are equal when the three sides of the one are respectively equal to the three sides of the other.

Proposition X.

Two right triangles are equal when they have the hypotenuse and a side of the one respectively equal to the hypotenuse and a side of the other.

Proposition XI.

If two angles of a triangle are equal, the sides opposite to them are equal, and the triangle is isosceles.

Proposition XII.

If two angles of a triangle are unequal, the side opposite the greater angle is greater than the side opposite the less angle.

Proposition XIII.

If two sides of a triangle are unequal, the angle opposite the greater side is greater than the angle opposite the less side.

Proposition XIV.

If two triangles have two sides of the one respectively equal to two sides of the other, and the included angles unequal, the triangle which has the greater included angle has the greater third side.

Proposition XV.

If two triangles have two sides of the one respectively equal to two sides of the other, and the third sides unequal, the triangle which has the greater third side has the greater included angle.

Proposition XVI.

From a given point without a straight line one perpendicular can be drawn to the line, and but one.

Proposition XVII.

The perpendicular is the shortest line that can be drawn from a point to a straight line.

Proposition XVIII.

If a perpendicular is erected at the middle of a straight line, then every point on the perpendicular is equally distant from the extremities of the line, and every point not on the perpendicular is unequally distant from the extremities of the line.

Proposition XIX.

Every point in the bisector of an angle is equally distant from the sides of the angle; and every point not in the bisector is unequally distant from the sides of the angle; that is, the bisector of an angle is the locus of the points within the angle and equally distant from its sides.

Proposition XX.

A convex broken line is less than any other line which envelops it and has the same extremities.

Proposition XXI.

If two oblique lines drawn from a point to a line meet the line at unequal distances from the foot of the perpendicular, the more remote is the greater.

Proposition XXII.

Two straight lines perpendicular to the same straight line are parallel.

Proposition XXIII.

Through a given point one line, and only one, can be drawn parallel to a given line.

Proposition XXIV.

When two straight lines are cut by a third, if the alternate-interior angles are equal, the two straight lines are parallel.

Corollary I. When two straight lines are cut by a third, if a pair of corresponding angles are equal, the lines are parallel.

Corollary II. When two straight lines are cut by a third, if the sum of two interior angles on the same side of the secant line is equal to two right angles, the two lines are parallel.

Proposition XXV.

If two parallel lines are cut by a third straight line, the alternate-interior angles are equal.

Corollary I. If two parallel lines are cut by a third straight line, any two corresponding angles are equal.

Corollary II. If two parallel lines are cut by a third straight line, the sum of the two interior angles on the same side of the secant line is equal to two right angles.

Proposition XXVI.

The sum of the three angles of any triangle is equal to two right angles.

Corollary. If one side of a triangle is extended, the exterior angle is equal to the sum of the two interior opposite angles.

Proposition XXVII.

The sum of all the angles of any convex polygon is equal to twice as many right angles, less four, as the figure has sides.

Proposition XXVIII.

Two parallelograms are equal when two adjacent sides and the included angle of the one are equal to two adjacent sides and the included angle of the other.

Corollary. Two rectangles are equal when they have equal bases and equal altitudes.

Proposition XXIX.

The opposite sides of a parallelogram are equal, and the opposite angles are equal.

Proposition XXX.

If two opposite sides of a quadrilateral are equal and parallel, the figure is a parallelogram.

Proposition XXXI.

If the opposite sides of a quadrilateral are equal, the figure is a parallelogram.

Proposition XXXII.

The diagonals of a parallelogram bisect each other.

BOOK II.

PROPOSITIONS.

Postulate.

A circumference may be described with any point as centre and any distance as radius.

Proposition I.

Two circles are equal when the radius of the one is equal to the radius of the other.

Proposition II.

Every diameter bisects the circle and its circumference.

Proposition III.

In equal circles, or in the same circle, equal angles at the centre intercept equal arcs on the circumference.

Corollary. Conversely, in the same circle, or in equal circles, equal arcs subtend equal angles at the centre.

PROPOSITION IV.

In equal circles, or in the same circle, equal arcs are subtended by equal chords.

Corollary. Conversely, in equal circles, or in the same circle, equal chords subtend equal arcs.

PROPOSITION V.

In equal circles, or in the same circle, the greater of two unequal arcs is subtended by the greater chord, the arcs being each less than a semi-circumference.

Corollary. Conversely, in equal circles, or in the same circle the greater of two unequal chords subtends the greater arc.

PROPOSITION VI.

The diameter perpendicular to a chord bisects the chord and the arcs subtended by it.

Corollary I. The perpendicular erected at the middle point of a chord passes through the centre of the circle.

Corollary II. When two circumferences intersect, the straight line joining their centres bisects their common chord at right angles.

PROPOSITION VII.

In the same circle, or in equal circles, equal chords are equally distant from the centre; and of two unequal chords the less is at the greater distance from the centre.

Corollary. Conversely, in the same circle, or in equal circles, chords equally distant from the centre are equal; and of two chords unequally distant from the centre, that is the greater whose distance from the centre is the less.

PROPOSITION VIII.

A straight line cannot intersect a circle in more than two points.

PROPOSITION IX.

A straight line tangent to a circle is perpendicular to the radius drawn to the point of contact.

Corollary I. A perpendicular to a tangent line drawn through the point of contact must pass through the centre of the circle.

Corollary II. If two circumferences are tangent to each other, their centres and their point of contact lie in the same straight line.

Proposition X.

When two tangents to the same circle intersect, the distances from their point of intersection to their points of contact are equal.

Proposition XI.

Two parallels intercept equal arcs on a circumference.

Doctrine of Limits.—Theorem.

If two variables dependent upon the same variable are so related that they are always equal, no matter what value is given to the variable on which they depend, and if, as the independent variable is changed in some specified way, each of them approaches a limit, the two limits must be absolutely equal.

Proposition XII.

In the same circle, or in equal circles, two angles at the centre are in the same ratio as their intercepted arcs.

Proposition XIII.

The numerical measure of an angle at the centre of a circle is the same as the numerical measure of its intercepted arc, if the unit of angle is the angle at the centre which intercepts the adopted unit of arc.

Proposition XIV.

An inscribed angle is measured by one-half its intercepted arc.

Corollary. An angle inscribed in a semicircle is a right angle.

Proposition XV.

An angle formed by a tangent and a chord is measured by one-half the intercepted arc.

Proposition XVI.

An angle formed by two chords intersecting within the circumference is measured by one-half the sum of the arcs intercepted between its sides and between the sides of its vertical angle.

Proposition XVII.

An angle formed by two secants intersecting without the circumference is measured by one-half the difference of the intercepted arcs.

Proposition XVIII.

An angle formed by a tangent and a secant is measured by one-half the difference of the intercepted arcs.

Corollary. An angle formed by two tangents is measured by one-half the difference of the intercepted arcs.

BOOK III.

THEOREMS.

Proposition I.

A parallel to the base of a triangle divides the other two sides proportionally.

Proposition II.

If a straight line divides two sides of a triangle proportionally, it is parallel to the third side.

Proposition III.

Two triangles are similar when they are mutually equiangular.

Proposition IV.

Two triangles are similar when an angle in the one is equal to an angle in the other, and the sides including these angles are proportional.

Proposition V.

Two triangles are similar when their homologous sides are proportional.

Proposition VI.

If two polygons are composed of the same number of triangles, similar each to each and similarly placed, the polygons are similar.

Proposition VII.

Two similar polygons may be decomposed into the same number of triangles, similar each to each and similarly placed.

Proposition VIII.

The perimeters of two similar polygons are in the same ratio as any two homologous sides.

Proposition IX.

If a perpendicular is drawn from the vertex of the right angle to the hypotenuse of a right triangle:

1st. The two triangles thus formed are similar to each other and to the whole triangle;

2d. The perpendicular is a mean proportional between the segments of the hypotenuse;

3d. Each side about the right angle is a mean proportional between the whole hypotenuse and the adjacent segment.

Corollary. If from any point in the circumference of a circle a perpendicular is let fall upon a diameter, the perpendicular is a mean proportional between the segments of the diameter.

Proposition X.

The square of the length of the hypotenuse of a right triangle is the sum of the squares of the lengths of the other two sides, the three lengths being expressed in terms of the same unit.

Proposition XI.

If two chords intersect within a circle, their segments are reciprocally proportional.

Proposition XII.

If two secants intersect without a circle, the whole secants and their external segments are reciprocally proportional.

Corollary. If a tangent and a secant intersect, the tangent is a mean proportional between the whole secant and its external segment.

BOOK IV.

THEOREMS.

Proposition I.

Parallelograms having equal bases and equal altitudes are equivalent.

Corollary. Any parallelogram is equivalent to a rectangle having the same base and the same altitude.

Proposition II.

Two rectangles having equal altitudes are to each other as their bases.

Corollary. Two rectangles having equal bases are to each other as their altitudes.

Proposition III.

Any two rectangles are to each other as the products of their bases by their altitudes.

Proposition IV.

The area of a rectangle is equal to the product of its base and altitude.

Proposition V.

The area of a parallelogram is equal to the product of its base and altitude.

Proposition VI.

The area of a triangle is equal to half the product of its base and altitude.

Corollary I. A triangle is equivalent to one-half of any parallelogram having the same base and the same altitude.

Corollary II. Triangles having equal bases and equal altitudes are equivalent.

Corollary III. Triangles having equal altitudes are to each other as their bases, and triangles having equal bases are to each other as their altitudes.

Proposition VII.

The area of a trapezoid is equal to the product of its altitude by half the sum of its parallel bases.

Proposition VIII.

Similar triangles are to each other as the squares of their homologous sides.

Proposition IX.

Similar polygons are to each other as the squares of their homologous sides.

Proposition X.

The square described upon the hypotenuse of a right triangle is equivalent to the sum of the squares described on the other two sides.

BOOK V.

THEOREMS.

PROPOSITION I.

If the circumference of a circle be divided into any number of equal parts, the chords joining the successive points of division form a regular polygon inscribed in the circle; and the tangents drawn at the points of division form a regular polygon circumscribed about the circle.

Corollary I. If the vertices of a regular inscribed polygon are joined with the middle points of the arcs subtended by the sides of the polygon, the joining lines will form a regular inscribed polygon of double the number of sides.

Corollary II. If at the middle points of the arcs joining adjacent points of contact of the sides of a regular circumscribed polygon tangents are drawn, a regular circumscribed polygon of double the number of sides will be formed.

PROPOSITION II.

A circle may be circumscribed about any regular polygon, and a circle may also be inscribed in it.

PROPOSITION III.

Regular polygons of the same number of sides are similar.

Corollary. The perimeters of regular polygons of the same number of sides are to each other as the radii of the circumscribed circles, or as the radii of the inscribed circles; and their areas are to each other as the squares of these radii.

PROPOSITION IV.

The area of a regular polygon is equal to half the product of its perimeter and apothem.

PROPOSITION V.

An arc of a circle is less than any line which envelops it and has the same extremities.

Corollary. The circumference of a circle is less than the perimeter of any polygon circumscribed about it.

Proposition VI.

If the number of sides of a regular polygon inscribed in a circle be increased indefinitely, the apothem of the polygon will approach the radius of the circle as its limit.

Proposition VII.

The circumference of a circle is the limit which the perimeters of regular inscribed and circumscribed polygons approach when the number of their sides is increased indefinitely; and the area of the circle is the limit of the areas of these polygons.

Proposition VIII.

The circumferences of two circles are to each other as their radii, and their areas are to each other as the squares of their radii.

Corollary I. The circumferences of circles are to each other as their diameters, and their areas are to each other as the squares of their diameters.

Corollary II. The ratio of the circumference of a circle to its diameter is constant.

Proposition IX.

The area of a circle is equal to half the product of its circumference by its radius.

Corollary. The area of a circle is equal to the square of its radius multiplied by the constant number π.

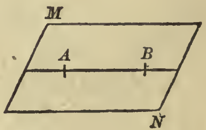
GEOMETRY OF SPACE.

In *Plane Geometry* we have considered merely figures composed of lines and points, all of which are supposed to lie in the same plane (*v.* Introduction, 5 and 6), and in the propositions and definitions of the preceding five books it has been tacitly assumed that the figures in question are plane figures. In many of the propositions and definitions this limitation is essential to the truth of the proposition; for example, Propositions I. and XXII., Book I., and Definition 20, Book II. In others the demonstration given is inconclusive without the limitation in question, although the proposition is true even when the limitation is removed; for example, Exercise 1, Proposition XXIII., Book I. While in propositions concerning equal polygons, which depend for their proof directly or indirectly upon a superposition of one polygon upon the other, the limitation is obviously of no importance; for example, Propositions VI., VII., and IX., Book I. It is, then, important, when we use the theorems of Plane Geometry in proving theorems of the Geometry of Space, to satisfy ourselves that they are still true in the figures with which we are concerned.

BOOK VI.

THE PLANE. POLYEDRAL ANGLES.

1. *Definition.* A *plane* has already been defined as a surface such that the straight line joining any two points in it lies wholly in the surface.



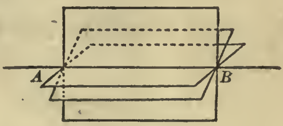
Thus, the surface MN is a plane, if, A and B being *any two* points in it, the straight line AB lies wholly in the surface.

The plane is understood to be indefinite in extent, so that, however far the straight line is produced, all its points lie in the plane. But to represent a plane in a diagram, we are obliged to take a limited portion of it, and we usually represent it by a parallelogram supposed to lie in the plane.

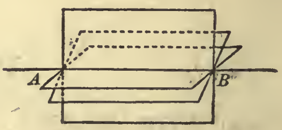
2. *Definition.* A plane is said to be *determined* by given lines or points when one plane, and only one, can be drawn containing the given lines or points.

PROPOSITION I.—THEOREM.

3. *Through any given straight line a plane may be passed; but the line will not determine the plane.*



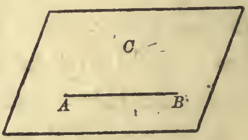
Let AB be a given straight line. A straight line may be drawn in any plane, and the position of that plane may be changed until the line drawn in it is brought into coincidence with AB. We shall then have a plane passed through AB; and this plane may be turned upon AB as an axis, and made to occupy as many different positions as we choose, and as in each of these positions it is a plane through AB, we may have as many planes as we choose through AB; consequently AB does not determine (2) a plane.



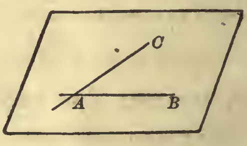
PROPOSITION II.—THEOREM.

4. *A plane is determined*, 1st, *by a straight line and a point without that line;* 2d, *by two intersecting straight lines;* 3d, *by three points not in the same straight line;* 4th, *by two parallel straight lines.*

1st. Through a given line AB a plane may be passed, and may then be turned upon AB as an axis, until it contains a given point C. If it is then turned by the smallest amount in either direction, it ceases to contain C. Therefore one plane, and only one, can be drawn containing a given line and a given point without that line.

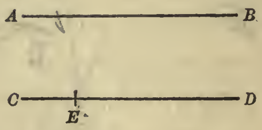


2d. Through one of the lines AB, and a point C of the other, one plane, and only one, can be drawn. This plane will contain AC (1), and no other plane through AB will.



3d. If three points, A, B, C, are given, any plane containing them must contain one of them and the line joining the other two (1); and one, and only one, such plane can be drawn.

4th. Two parallels, AB, CD, lie in the same plane, by Definition (I., 5); there is, then, one plane containing them. There is only one, for through AB and a point E of CD only one plane can be passed.



5. Corollary. *The intersection of two planes is a straight line.* For if two points of the intersection be joined by a straight line, that line must lie in both planes, by (1); and no point outside of this line can be common to the two planes, by Proposition II.; therefore the straight line in question is the line of intersection of the two planes.

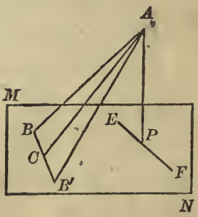
PERPENDICULARS AND OBLIQUE LINES TO PLANES.

6. *Definition.* A straight line is *perpendicular to a plane* when it is perpendicular to every straight line drawn in the plane through its *foot;* that is, through the point in which it meets the plane.

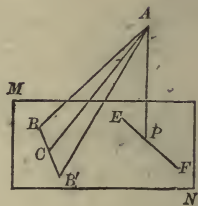
In the same case, the plane is said to be perpendicular to the line.

PROPOSITION III.—THEOREM.

7. *From a given point without a plane, one perpendicular to the plane can be drawn, and but one; and the perpendicular is the shortest line that can be drawn from the point to the plane.*

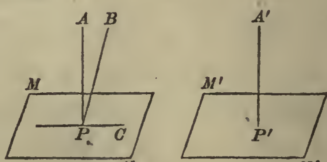


Let A be the given point, and MN the plane. Consider the various lines that can be drawn from A to MN. These lines are obviously not all of the same length; there must, then, be among them either one *minimum* line, or a set of equal shortest lines. There cannot be a set of equal shortest lines. For, suppose that AB and AB' are two such lines. Join BB'. Then, since AB and AB' are equal lines drawn from A to BB', they cannot be perpendicular to BB' (I., Proposition XVI.), and consequently they are longer than the perpendicular AC from A to BB', by I., Proposition XVII., which is contrary to the hypothesis that they were shorter than any other lines that could be drawn from A to MN. There is therefore one, and but one, minimum line from A to the plane. Let AP be that minimum line; then AP is perpendicular to any straight line EF drawn in the plane through its foot P. For, in the plane of the lines AP and EF, AP is the shortest line that can be drawn from A to any point in EF, since it is the shortest line that can be drawn from A to any point in the plane MN; therefore AP is perpendicular to EF (I., Proposition XVII.). Thus AP is perpendicular to *any*, that is, to *every*, straight line drawn in the plane through its foot, and is therefore perpendicular to the plane.



There can be no other perpendicular from A to the plane MN; for, if there were, both lines would be perpendicular to the line joining the points where they met the plane, and we should have two perpendiculars from a point to a line, which is contrary to I., Proposition XVI.

8. Corollary. *At a given point in a plane, one perpendicular can be erected to the plane, and but one.*



Let MN be the plane and P the point.

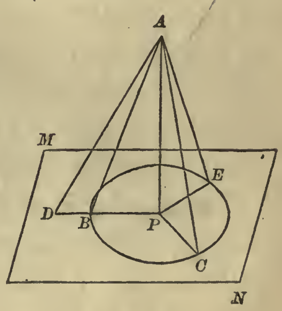
Let $M'N'$ be any other plane, A' any point without it, and $A'P'$ the perpendicular from A' to this plane. Suppose the plane $M'N'$ to be applied to the plane MN with the point P' upon P, and let AP be the position then occupied by the perpendicular $A'P'$. We then have one perpendicular, AP, to the plane MN, erected at P. There can be no other: for let PB be another perpendicular at P. Then AP and PB are both perpendicular to PC, the line of intersection of MN with the plane determined by the two lines AP and BP, at the same point, and lie in the same plane with PC, and this is contrary to I., Proposition I. (At a given pt. in a str. line 1 ⊥ to the line can be drawn, & only one)

9. *Scholium.* By the *distance of a point from a plane* is meant the *shortest distance;* hence it is the perpendicular distance from the point to the plane.

EXERCISES.

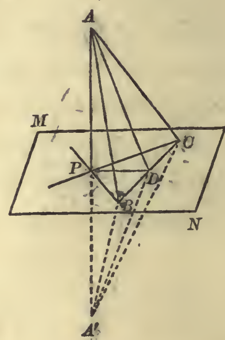
1. *Theorem.—Oblique lines drawn from a point to a plane, and meeting the plane at equal distances from the foot of the perpendicular, are equal; and of two oblique lines meeting the plane at unequal distances from the foot of the perpendicular the more remote is the greater.*

2. *Theorem.—Equal oblique lines from a point to a plane meet the plane at equal distances from the foot of the perpendicular; and of two unequal oblique lines the greater meets the plane at the greater distance from the foot of the perpendicular.*



PROPOSITION IV.—THEOREM.

10\. *If a straight line is perpendicular to each of two straight lines at their point of intersection, it is perpendicular to the plane of those lines.*



Let AP be perpendicular to PB and PC, at their intersection P; then AP is perpendicular to the plane MN which contains those lines.

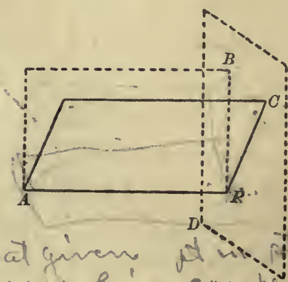
For, let PD be any other straight line drawn through P in the plane MN. Draw any straight line BDC intersecting PB, PC, PD, in B, C, D; produce AP to A', making $PA' = PA$, and join A and A' to each of the points B, C, D.

Since BP is perpendicular to AA', at its middle point, we have $BA = BA'$ (I., Proposition XVIII.), and for a like reason $CA = CA'$; therefore the triangles ABC, $A'BC$, are equal (I., Proposition IX.), and the angle ABD is equal to the angle $A'BD$. The triangles ABD and $A'BD$ are equal (I., Proposition VI.), and $AD = A'D$. Hence the triangles APD and $A'PD$ are equal (I., Proposition IX.). Therefore the adjacent angles APD and $A'PD$ are equal, and PD is perpendicular to AP. AP, then, is perpendicular to *any*, that is, to *every*, line passing through its foot in the plane MN, and is consequently perpendicular to the plane.

11\. Corollary I. *At a given point of a straight line one plane can be drawn perpendicular to the line, and but one.*

Let AP be the line, and P the point. Through AP pass two planes, and in each of these planes draw through P a line perpendicular to AP. The plane determined by these two lines is perpendicular to AP at P, by Proposition IV.

No other perpendicular plane can be drawn through P; for, if it could, a plane containing AP would intersect the two perpendicular planes in lines which would lie in the same plane with AP, and be perpendicular to AP at the same point, which is contrary to I., Proposition I.



(at given pt in the line ⊥ plane can be drawn & only 1)

12. Corollary II. *Through a given point without a straight line one plane can be drawn perpendicular to the line, and but one.*

In the plane determined by the point and the line draw a perpendicular from the point to the line, and through the foot of this perpendicular draw, in any second plane passing through the given line, a second perpendicular to the line. The plane of these two perpendiculars is obviously a plane passing through the given point and perpendicular to the given line.

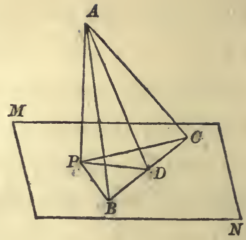
No second perpendicular plane can be drawn through the given point, for the plane determined by the line and the point would cut the two perpendicular planes in lines which would be two perpendicular lines from the given point to the given line, which is contrary to I., Proposition XVI.

(From a given pt, without a st. line, 1 ⊥ can be drawn to line, & only 1).

EXERCISES.

1. *Theorem.—All the perpendiculars that can be drawn to a straight line at the same point lie in a plane perpendicular to the line at the point.*

2. *Theorem.—If from the foot of a perpendicular to a plane a straight line is drawn at right angles to any line of the plane, and its intersection with that line is joined to any point of the perpendicular, this last line will be perpendicular to the line of the plane.*



PARALLEL STRAIGHT LINES AND PLANES.

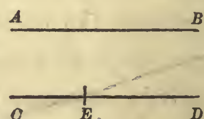
13. *Definitions.* A straight line is *parallel to a plane* when it cannot meet the plane, though both be indefinitely produced.

In the same case, the plane is said to be parallel to the line.

Two planes are parallel when they do not meet, both being indefinite in extent.

PROPOSITION V.—THEOREM.

14. *Two lines in space having the same direction are parallel.*



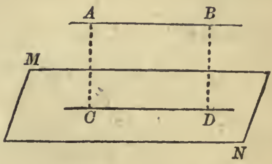
Let AB and CD be two lines having the same direction. Through AB and any point E of CD pass a plane, and in this plane draw through E a line parallel to AB. This line will have the same direction as AB (I., Axiom II.), and consequently the same direction as CD, and must therefore coincide with CD, by I., Postulate II. Hence AB and CD are parallel.

15. Corollary. *Two lines parallel to the same line are parallel to each other.* For they have the same direction.

PROPOSITION VI.—THEOREM.

16. *If two straight lines are parallel, every plane passed through one of them and not coincident with the plane of the parallels is parallel to the other.*

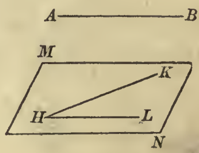
Let AB and CD be parallel lines, and MN any plane passed through CD; then the line AB and the plane MN are parallel.



For the parallels AB, CD, are in the same plane, $ACDB$, which intersects the plane MN in the line CD; and since AB cannot leave this plane, if it meets MN at all it must meet it in some point common to the two planes; that is, in some point of CD, which is contrary to the hypothesis that AB and CD are parallel.

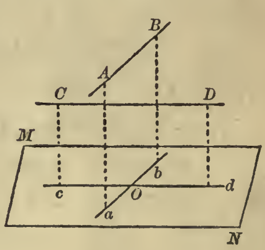
17. Corollary I. *Through any given straight line a plane can be passed parallel to any other given straight line.*

Let HK and AB be the two given lines. In the plane determined by AB and any point H of HK let HL be drawn parallel to AB; then the plane MN, determined by HK and HL, is parallel to AB, by Proposition VI.



18. Corollary II. *Through any given point a plane can be passed parallel to any two given straight lines in space.*

Let O be the given point, and AB and CD the given straight lines. In the plane determined by the given point O and the line AB let aOb be drawn through O parallel to AB; and in the plane determined by the point O and the line CD let cOd be drawn through O parallel to CD; then the plane determined by the lines ab and cd is parallel to each of the lines AB and CD, by Proposition VI.

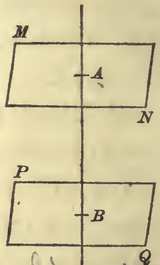


EXERCISE.

Theorem.—If a straight line and a plane are parallel, the intersection of the plane and a plane passed through the given line is parallel to the given line. (*v.* Figure of Proposition VI.)

PROPOSITION VII.—THEOREM.

19. *Planes perpendicular to the same straight line are parallel to each other.*

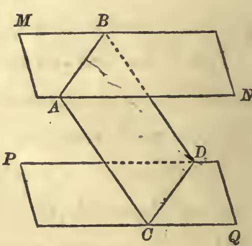


The planes MN, PQ, perpendicular to the same straight line AB, cannot meet; for, if they met, we should have through a point of their intersection two planes perpendicular to the same straight line, which is impossible (Proposition IV., Corollary II.); therefore these planes are parallel.

(Thru a given pt. without or. the line one pl. can be drawn ⊥ to the line, & only 1.)

PROPOSITION VIII.—THEOREM.

20. *The intersections of two parallel planes with any third plane are parallel.*



Let MN and PQ be parallel planes, and AD any plane intersecting them in the lines AB and CD; then AB and CD are parallel.

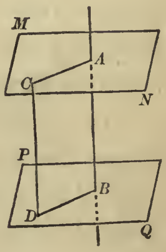
For the lines AB and CD cannot meet, since the planes in which they are situated cannot meet, and they are lines in the same plane AD; therefore they are parallel.

EXERCISE.

Theorem.—Parallel lines intercepted between parallel planes are equal.

PROPOSITION IX.—THEOREM.

21. *A straight line perpendicular to one of two parallel planes is perpendicular to the other.*



Let MN and PQ be parallel planes, and let the straight line AB be perpendicular to PQ; then it will also be perpendicular to MN.

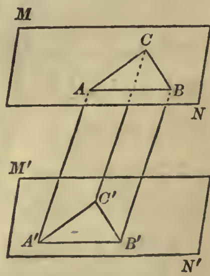
For through A draw any straight line AC in the plane MN, pass a plane through AB and AC, and let BD be the intersection of this plane with PQ. Then AC and BD are parallel (Proposition VIII.); but AB is perpendicular to BD (6), and consequently also to AC; therefore AB, being perpendicular to any line AC which it meets in the plane MN, is perpendicular to the plane MN.

22. Corollary. *Through any given point one plane can be passed parallel to a given plane, and but one.*

Suggestion. Drop a perpendicular line from the point to the plane, and then pass a plane through the point perpendicular to this line.

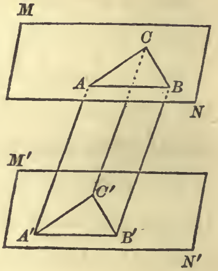
PROPOSITION X.—THEOREM.

23. *If two angles, not in the same plane, have their sides respectively parallel and lying in the same direction, they are equal and their planes are parallel.*



Let BAC, $B'A'C'$, be two angles lying in the planes MN, $M'N'$; and let AB, AC, be parallel respectively to $A'B'$, $A'C'$, and in the same directions.

1st. The angles BAC and $B'A'C'$ are equal. Take the distances $A'B'$ and AB equal, and $A'C'$ and AC equal, and join BC and $B'C'$. Draw now the lines AA', BB', and CC'. The quadrilaterals AB' and AC' are parallelograms, by I., Proposition XXX. Therefore BB' and CC' are equal and parallel to AA'. They are then equal and parallel to each other (Proposition V., Corollary), and BC' is a parallelogram. Hence $BC = B'C'$, and the triangles ABC and $A'B'C'$ are equal, by I., Proposition IX. Consequently the angles BAC and $B'A'C'$ are equal.

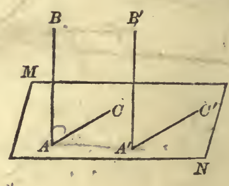


2d. The planes MN and $M'N'$ are parallel. For MN is parallel to $A'B'$ and $A'C'$, by Proposition VI., and therefore, since if it met $M'N'$ the line of intersection would have to cut one or the other of the intersecting lines $A'B'$, $A'C'$, it is parallel to $M'N'$.

PROPOSITION XI.—THEOREM.

24. *If one of two parallel lines is perpendicular to a plane, the other is also perpendicular to that plane.*

Let AB, $A'B'$, be parallel lines, and let AB be perpendicular to the plane MN; then $A'B'$ is also perpendicular to MN.



For, let A and A' be the intersections of these lines with the plane; through A' draw any line $A'C'$ in the plane MN, and through A draw AC parallel to $A'C'$ and in the same direction. The angles BAC, $B'A'C'$, are equal (Proposition X.); but BAC is a right angle, since BA is perpendicular to the plane; hence $B'A'C'$ is a right angle; that is, $B'A'$ is perpendicular to any line $A'C'$ drawn through its foot in the plane MN, and is consequently perpendicular to the plane.

(If 2 ∠s not in same plane, have their sides respectively ∥ & lying in same
X direction, the ∠s are = & their planes

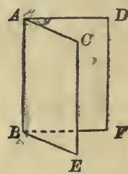
25. Corollary. *Two straight lines perpendicular to the same plane are parallel to each other.*

EXERCISE.

Theorem.—Two parallel planes are everywhere equally distant.

DIEDRAL ANGLES.—ANGLE OF A LINE AND PLANE, ETC.

26. *Definition.* When two planes meet and are terminated by their common intersection, they form a *diedral angle.*



Thus, the planes AE, AF, meeting in AB, and terminated by AB, form a diedral angle.

The planes AE, AF, are called the *faces*, and the line AB the *edge*, of the diedral angle.

A diedral angle may be named by four letters, one in each face and two on its edge, the two on the edge being written between the other two; thus, the angle in the figure may be named $DABC$.

When there is but one diedral angle formed at the same edge, it may be named by two letters on its edge; thus, in the preceding figure, the diedral angle $DABC$ may be named the diedral angle AB.

27. *Definition.* The angle CAD formed by two straight lines AC, AD, drawn, one in each face of the diedral angle, perpendicular to its edge AB at the same point, is called the *plane angle* of the diedral angle.

EXERCISES.

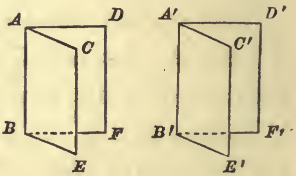
1. *Theorem.—All plane angles of the same diedral angle are equal.* (*v.* Proposition X.)

2. *Theorem.—If a plane is drawn perpendicular to the edge of a diedral angle, its intersections with the faces of the diedral angle form the plane angle of the diedral angle.*

28. A diedral angle $DABC$ may be conceived to be *generated* by a plane, at first coincident with a fixed plane AE, revolving upon the line AB as an axis until it comes into the position AF. In this revolution a straight line CA, perpendicular to AB, generates the plane angle CAD.

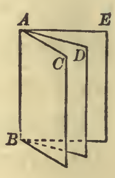
29. *Definition.* Two diedral angles are equal when they can be placed so that their faces shall coincide.

Thus, the diedral angles $CABD$, $C'A'B'D'$, are equal, if, when the edge $A'B'$ is applied to the edge AB and the face $A'F'$ to the face AF, the face $A'E'$ also coincides with the face AE.



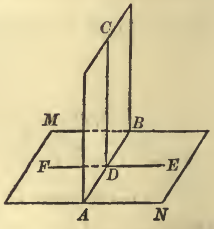
Since the faces continue to coincide when produced indefinitely, it is apparent that the *magnitude* of the diedral angle does not depend upon the extent of its faces, but only upon their relative position.

30. *Definition.* Two diedral angles $CABD$, $DABE$, which have a common edge AB and a common plane BD between them, are called *adjacent*.



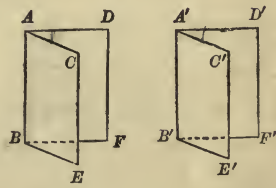
Two diedral angles are added together by placing them adjacent to each other. Thus, the diedral angle $CABE$ is the sum of the two diedral angles $CABD$ and $DABE$.

31. *Definition.* Two planes are perpendicular when the plane angle of the diedral angle which they form is a right angle. The diedral angle is then called a *right diedral angle.*



PROPOSITION XII.—THEOREM.

32. *Two diedral angles are equal if their plane angles are equal.*

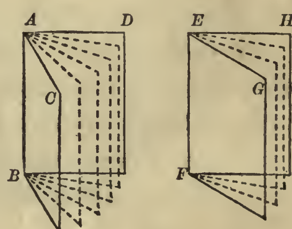


Let the plane angles CAD and $C'A'D'$ of the diedral angles $CABD$, $C'A'B'D'$, be equal; then are the diedral angles equal.

For, superpose $C'A'B'D'$ upon $CABD$, making the plane angle $C'A'D'$ coincide with its equal CAD; then the planes of these angles will coincide (Proposition II.). $A'B'$ and AB, being now perpendicular to the same plane at the same point, must coincide (Proposition III., Corollary); and, finally, the planes $B'C'$ and BC will coincide, and $B'D'$ and BD (Proposition II.). Therefore the diedral angles are equal.

PROPOSITION XIII.—THEOREM.

33. *Two diedral angles are in the same ratio as their plane angles.*

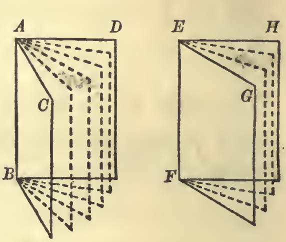


Let $CABD$ and $GEFH$ be two diedral angles; and let CAD and GEH be their plane angles.

Suppose the plane angles have a common measure, contained m times in CAD and n times in GEH; we have, then,

$$\frac{CAD}{GEH} = \frac{m}{n}.$$

Apply this measure to CAD and GEH, and through the lines of division and the edges of the given diedral angles pass planes, thus dividing $CABD$ into m and $GEFH$ into n smaller diedral angles. Each of these small diedral angles has one of the parts into which CAD is divided, or one of the parts into which GEH is divided, as its plane angle, because AB is perpendicular to the plane of CAD, and EF to the plane of GEH, by Proposition IV. These small diedral angles are, then, all equal, by Proposition XII., and we have



$$\frac{CABD}{GEFH} = \frac{m}{n}.$$

Therefore

$$\frac{CABD}{GEFH} = \frac{CAD}{GEH}.$$

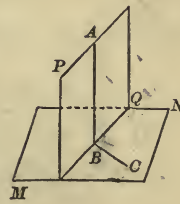
The proof is extended to the case where the given plane angles are incommensurable, by the method exemplified in the proof of II., Proposition XII., of III., Proposition I., and of IV., Proposition II.

34. *Scholium.* Since the diedral angle is proportional to its plane angle (that is, varies proportionally with it), the plane angle is taken as the *measure* of the diedral angle, just as an arc is taken as the measure of a plane angle. Thus, a diedral angle will be expressed by 45° if its plane angle is expressed by 45°, etc.

PROPOSITION XIV.—THEOREM.

35. *If a straight line is perpendicular to a plane, every plane passed through the line is also perpendicular to that plane.*

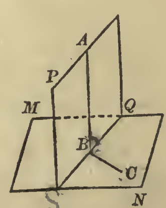
Let AB be perpendicular to the plane MN; then any plane PQ, passed through AB, is also perpendicular to MN.



For, at B draw BC, in the plane MN, perpendicular to the intersection BQ. Since AB is perpendicular to the plane MN, it is perpendicular to BQ and BC; therefore the angle ABC is the plane angle of the diedral angle formed by the planes PQ and MN; and since the angle ABC is a right angle, the planes are perpendicular to each other.

PROPOSITION XV.—THEOREM.

36. *If two planes are perpendicular to each other, a straight line drawn in one of them, perpendicular to their intersection, is perpendicular to the other.*



Let the planes PQ and MN be perpendicular to each other; and at any point B of their intersection BQ let BA be drawn, in the plane PQ, perpendicular to BQ; then BA is perpendicular to the plane MN.

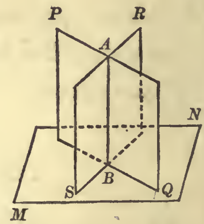
For, drawing BC, in the plane MN, perpendicular to BQ, the angle ABC is a right angle, since it is the plane angle of the right diedral angle formed by the two planes; therefore AB, perpendicular to the two straight lines BQ, BC, is perpendicular to their plane MN (Proposition IV.).

37. Corollary I. *If two planes are perpendicular to each other, a straight line drawn through any point of their intersection perpendicular to one of the planes will lie in the other.* (*v.* Proposition III., Corollary.)

38. Corollary II. *If two planes are perpendicular, a straight line let fall from any point of one plane perpendicular to the other will lie in the first plane.* (*v.* Proposition III.)

PROPOSITION XVI.—THEOREM.

39. *If two intersecting planes are each perpendicular to a third plane, their intersection is also perpendicular to that plane.*

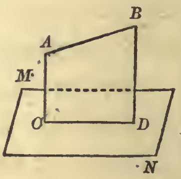


Let the planes PQ, RS, intersecting in the line AB, be perpendicular to the plane MN; then AB is perpendicular to the plane MN.

For, if from any point A of AB a perpendicular be drawn to MN, this perpendicular will lie in each of the planes PQ and RS (Proposition XV., Corollary II.), and must therefore be their intersection AB.

PROPOSITION XVII.—THEOREM.

40. *Through any given straight line a plane can be passed perpendicular to any given plane.*

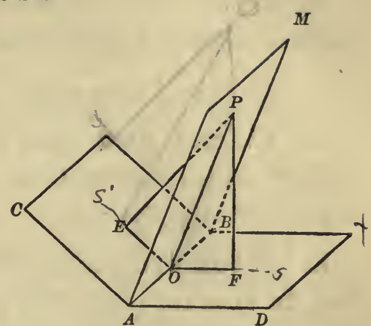


Let AB be the given straight line, and MN the given plane. From any point A of AB let AC be drawn perpendicular to MN, and through AB and AC pass a plane AD. This plane is perpendicular to MN (Proposition XIV.).

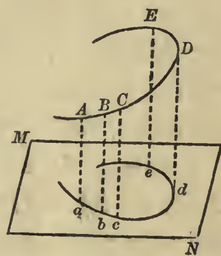
Moreover, since, by Proposition XV., Corollary II., any plane passed through AB perpendicular to MN must contain the perpendicular AC, the plane AD is the only plane perpendicular to MN that can be passed through AB, unless AB is itself perpendicular to MN, in which case every plane through AB is perpendicular to MN.

EXERCISE.

Theorem.—The locus of the points equally distant from two given planes is the plane bisecting the diedral angle between the given planes. (*v.* I., Proposition XIX.)



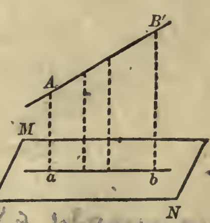
41. *Definitions.* The *projection of a point A upon a plane MN* is the foot *a* of the perpendicular let fall from *A* upon the plane.



The *projection of a line ABCDE . . . upon a plane MN* is the line *abcde . . .* containing the projections of all the points of the line *ABCDE . . .* upon the plane.

PROPOSITION XVIII.—THEOREM.

42. *The projection of a straight line upon a plane is a straight line.*

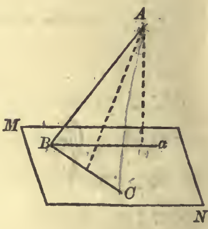


Let *AB* be the given straight line, and *MN* the given plane. The plane *Ab*, passed through *AB* perpendicular to the plane *MN*, contains all the perpendiculars let fall from points of *AB* upon *MN* (Proposition XV., Corollary II.); therefore these perpendiculars all meet the plane *MN* in the intersection *ab* of the perpendicular plane with *MN*. The projection of *AB* upon the plane *MN* is, consequently, the straight line *ab*.

43. *Scholium.* The plane Ab is called the *projecting plane* of the straight line AB upon the plane MN.

PROPOSITION XIX.—THEOREM.

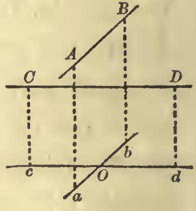
44. *The acute angle which a straight line makes with its own projection upon a plane is the least angle which it makes with any line of that plane.*



Let Ba be the projection of the straight line BA upon the plane MN, the point B being the point of intersection of the line BA with the plane; let BC be any other straight line drawn through B in the plane; then the angle ABa is less than the angle ABC.

For, take $BC = Ba$, and join AC. In the triangles ABa, ABC, we have AB common, and $Ba = BC$; but $Aa < AC$, since the perpendicular is less than any oblique line; therefore the angle ABa is less than the angle ABC (I., Proposition XV.).

45. *Definition.* The acute angle which a straight line makes with its own projection upon a plane is called the *inclination of the line to the plane*, or the *angle of the line and plane.*

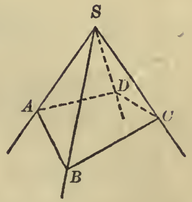


46. *Definition.* Two straight lines AB, CD, not in the same plane, are regarded as making an angle with each other which is equal to the angle between two straight lines Ob, Od, drawn through any point O in space, parallel respectively to the two lines and in the same directions.

47. From the preceding definition, it follows that *when a straight line is perpendicular to a plane, it is perpendicular to all the lines of the plane, whether the lines pass through its foot or not.*

POLYEDRAL ANGLES.

48. *Definition.* When three or more planes meet in a common point, they form a *polyedral angle.*



Thus the figure $S-ABCD$, formed by the planes ASB, BSC, CSD, DSA, meeting in the common point S, is a polyedral angle.

The point S is the vertex of the angle; the intersections of the planes SA, SB, etc., are its *edges;* the portions of the planes included between the edges are its *faces;* the angles ASB, BSC, etc., formed by the edges, are its *face angles.*

A *triedral* angle is a polyedral angle having but three faces, which is the least number of faces that can form a polyedral angle.

49. In a polyedral angle every pair of adjacent edges form a *face angle*, and every pair of adjacent faces form a *diedral angle.* These face angles and diedral angles are the *parts* of the polyedral angle.

50. *Definition.* Two polyedral angles are *equal* when their faces and edges can be made to coincide, if one angle is suitably superposed upon the other.

Of course it follows that corresponding parts of two equal polyedral angles are equal.

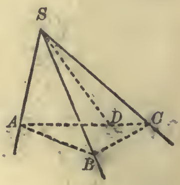
51. *Definition.* Two polyedral angles are *symmetrical* if the parts of one are respectively equal to the parts of the other; but the corresponding parts succeed each other in the two angles in inverse order. When two polyedral angles are symmetrical, it is impossible to superpose one upon the other in such a way as to bring corresponding parts together. One figure is, so to speak, right-handed and the other left-handed.

52. *Definition.* A polyedral angle $S\text{–}ABCD$ is *convex*, when any section, $ABCD$, made by a plane cutting all its faces, is a convex polygon (I., 54).

PROPOSITION XX.—THEOREM.

53. *The sum of any two face angles of a triedral angle is greater than the third.*

The theorem requires proof only when the third angle considered is greater than each of the others.



Let $S\text{–}ABC$ be a triedral angle in which the face angle ASC is greater than either ASB or BSC; then $ASB + BSC > ASC$.

For in the face ASC draw SD making the angle ASD equal to ASB, and through any point D of SD draw any straight line ADC cutting SA and SC; take $SB = SD$, and join AB, BC.

The triangles ASD and ASB are equal, by the construction (I., Proposition VI.), whence $AD = AB$. Now, in the triangle ABC, we have

$$AB + BC > AC,$$

and, subtracting the equals AB and AD,

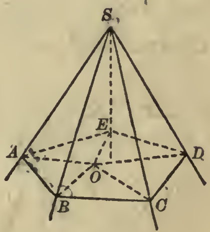
$$BC > DC;$$

therefore, in the triangles BSC and DSC, we have the angle $BSC > DSC$ (I., Proposition XV.), and adding the equal angles ASB and ASD, we have $ASB + BSC > ASC$.

PROPOSITION XXI.—THEOREM.

54. *The sum of the face angles of any convex polyedral angle is less than four right angles.*

Let the polyedral angle S be cut by a plane, making the section $ABCDE$, by hypothesis, a convex polygon. From any point O within this polygon draw OA, OB, OC, OD, OE.



The sum of the angles of the triangles ASB, BSC, etc., which have the common vertex S, is equal to the sum of the angles of the same number of triangles AOB, BOC, etc., which have the common vertex O. But in the triedral angles formed at A, B, C, etc., by the faces of the polyedral angle and the plane of the polygon, we have (Proposition XX.)

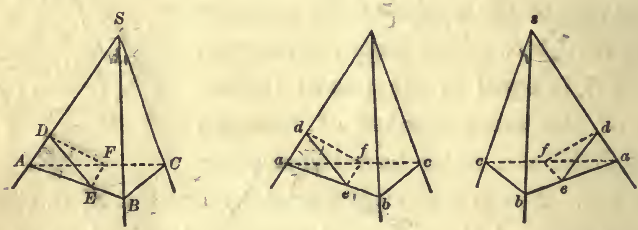
$$SAE + SAB > EAB,$$
$$SBA + SBC > ABC, \text{ etc.};$$

hence, taking the sum of all these inequalities, it follows that the sum of the angles at the bases of the triangles whose vertex is S is greater than the sum of the angles at the bases of the triangles whose vertex is O; therefore the sum of the angles at S is less than the sum of the angles at O; that is, less than four right angles.

PROPOSITION XXII.—THEOREM.

55. *If two triedral angles have the three face angles of the one respectively equal to the three face angles of the other, the corresponding diedral angles are equal.*

In the triedral angles S and s, let $ASB = asb$, $ASC = asc$, and $BSC = bsc$; then the diedral angle SA is equal to the diedral angle sa.



On the edges of these angles take the six equal distances SA, SB, SC, sa, sb, sc, and draw AB, BC, AC, ab, bc, ac. The isosceles triangles SAB and sab are equal, having an equal angle included by equal sides, hence $AB = ab$; and for the same reason, $BC = bc$, $AC = ac$; therefore the triangles ABC and abc are equal.

At any point D in SA, draw DE in the face ASB and DF in the face ASC, perpendicular to SA; these lines meet AB and AC, respectively, for, the triangles ASB and ASC being isosceles, the angles SAB and SAC are acute; let E and F be the points of meeting, and join EF. Now on sa take $sd = SD$, and repeat the same construction in the triedral angle s.

The triangles ADE and ade are equal, since $AD = ad$, and the angles at A and D are equal to the angles at a and d; hence $AE = ae$ and $DE = de$. In the same manner we have

$AF = af$ and $DF = df$. Therefore the triangles AEF and aef are equal (I., Proposition VI.), and we have $EF = ef$. Finally, the triangles EDF and edf, being mutually equilateral, are equal; therefore the angle EDF, which measures the diedral angle SA, is equal to the angle edf, which measures the diedral angle sa, and the diedral angles SA and sa are equal (Proposition XII.). In the same manner it may be proved that the diedral angles SB and SC are equal to the diedral angles sb and sc, respectively.

Scholium. It follows that the polyedral angles S and s are either equal or symmetrical. Both cases are represented in the figure.

EXERCISES ON BOOK VI.

THEOREMS.

1. If a straight line AB is parallel to a plane MN, any plane perpendicular to the line AB is perpendicular to the plane MN. (*v.* Proposition VI., Exercise.)

2. If a plane is passed through one of the diagonals of a parallelogram, the perpendiculars to this plane from the extremities of the other diagonal are equal.

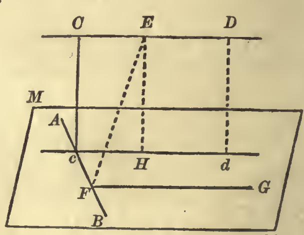
3. If the intersections of a number of planes are parallel, all the perpendiculars to these planes, drawn from a common point in space, lie in one plane.

Suggestion. Through the common point pass a plane perpendicular to one of the intersections. (*v.* Proposition XV., Corollary II.)

4. If the projections of a number of points on a plane are in a straight line, these points are in one plane.

5. If each of the projections of a line AB upon two intersecting planes is a straight line, the line AB is a straight line.

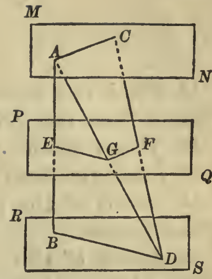
6. Two straight lines not in the same plane being given: 1st, a common perpendicular to the two lines can be drawn; 2d, the common perpendicular is the shortest distance between the two lines.



Suggestion. Let AB and CD be the two given lines. Pass through AB a plane MN parallel to CD, and through AB and CD pass planes perpendicular to MN. Their intersection Cc is the required common perpendicular. CD and cd are parallel, by 18, Exercise.

2d. Any other line EF joining AB and CD is greater than EH, the perpendicular from E to cd (Proposition XV.), and therefore greater than Cc.

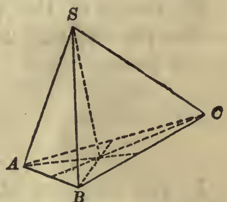
7. If two straight lines are intersected by three parallel planes, their corresponding segments are proportional. (*v.* Proposition VIII.)



8. A plane passed through the middle point of the common perpendicular to two straight lines in space, and parallel to both these lines, bisects every straight line joining a point of one of these lines to a point of the other. (*v.* Exercise 7.)

9. In any triedral angle, the three planes bisecting the three diedral angles intersect in the same straight line. (*v.* 40, Exercise.)

10. In any triedral angle, the three planes passed through the edges and the bisectors of the opposite face angles respectively intersect in the same straight line.

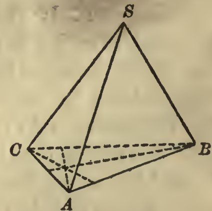


Suggestion. Lay off equal distances SA, SB, SC, on the three edges, and pass a plane through A, B, C. The intersections of the three planes in question with ABC are the medial lines of ABC, and have a common intersection, and the line joining this common intersection with S lies in the three planes.

11. In any triedral angle, the three planes passed through the bisectors of the face angles, and perpendicular to these faces respectively, intersect in the same straight line.

Suggestion. Use the same construction as in Exercise 10. Then the intersections of the three planes with ABC are perpendicular to the sides of ABC at their middle points, and have a common intersection.

12. In any triedral angle, the three planes passed through the edges, perpendicular to the opposite faces respectively, intersect in the same straight line.



Suggestion. At any point A of one of the edges, draw a plane ABC perpendicular to the edge SA. The intersections of the three planes with ABC are the perpendiculars from the vertices of ABC, upon the opposite sides, and have a common intersection. (*v.* Proposition XVI.)

LOCI.

13. Find the locus of the points in space which are equally distant from two given points.

14. Locus of the points which are equally distant from two given straight lines in the same plane.

15. Locus of the points which are equally distant from three given points.

16. Locus of the points which are equally distant from three given planes. (*v.* 40, Exercise.)

17. Locus of the points which are equally distant from three given straight lines in the same plane.

18. Locus of the points which are equally distant from the three edges of a triedral angle (Exercise 11).

19. Locus of the points in a given plane which are equally distant from two given points out of the plane.

20. Locus of the points which are equally distant from two given planes, and at the same time equally distant from two given points.

PROBLEMS.

In the solution of problems in space, we assume,—1st, that a plane can be *drawn* passing through three given points (or two intersecting straight lines) and its intersections with given straight lines or planes determined; and, 2d, that a perpendicular to a given plane can be drawn at a given point in the plane, or from a given point without it. The actual graphic construction of the solutions belongs to *Descriptive Geometry.*

21. Through a given straight line, to pass a plane perpendicular to a given plane. (*v.* Proposition XVII.)

22. Through a given point, to pass a plane perpendicular to a given straight line.

Suggestion. If the given point is in the given line, pass two planes through the given line, and draw in each of them, through the given point, a line perpendicular to the given line. The plane determined by these lines is the perpendicular plane required. (*v.* Proposition IV.)

If the given point is not in the given line, pass a plane through it and the given line, and in this plane, through the given point, draw a line parallel to the given line. A plane through the given point, perpendicular to this second line, is the plane required. (*v.* Proposition XI.)

23. Through a given point, to pass a plane parallel to a given plane. (*v.* Proposition IX., Corollary.)

24. To determine that point in a given straight line which is equidistant from two given points not in the same plane with the given line. (*v.* Exercise 13.)

25. To find a point in a plane which shall be equidistant from three given points in space.

26. Through a given point in space, to draw a straight line which shall cut two given straight lines not in the same plane.

Suggestion. Pass a plane through the given point and through one of the given lines; the line through the given point and the point where the plane cuts the second given line is the solution required.

27. Through a given point, to draw a straight line which shall meet a given straight line and the circumference of a given circle not in the same plane. (Two solutions in general.)

28. In a given plane and through a given point of the plane, to draw a straight line which shall be perpendicular to a given line in space.

Suggestion. Draw a plane through the given point and perpendicular to the given line. Its intersection with the given plane is the solution required.

29. Through a given point A in a plane, to draw a straight line AT in that plane, which shall be at a given distance PT from a given point P without the plane.

Suggestion. Drop a perpendicular from P to the plane, and with the foot of this perpendicular as a centre, and with a radius equal to a side of a right triangle whose hypotenuse is PT, and whose other side is the length of the perpendicular, describe a circumference in the plane. A tangent from A to this circumference is the solution required. (*v.* 12, Exercise 2.)

30. Through a given point A, to draw to a given plane M a straight line which shall be parallel to a given plane N and of a given length.

BOOK VII.

POLYEDRONS.

1. *Definition.* A *polyedron* is a geometrical solid bounded by planes.

The bounding planes, by their mutual intersections, limit each other, and determine the *faces* (which are polygons), the *edges*, and the *vertices* of the polyedron. A *diagonal* of a polyedron is a straight line joining any two of its vertices not in the same face.

The least number of planes that can form a polyedral angle is three; but the space within the angle is indefinite in extent, and it requires a fourth plane to enclose a finite portion of space, or to form a solid; hence the least number of planes that can form a polyedron is four.

2. *Definition.* A polyedron of four faces is called a *tetraedron;* one of six faces, a *hexaedron;* one of eight faces, an *octaedron;* one of twelve faces, a *dodecaedron;* one of twenty faces, an *icosaedron.*

3. *Definition.* A polyedron is *convex* when the section formed by any plane intersecting it is a convex polygon.

All the polyedrons treated of in this work will be understood to be convex.

4. *Definition.* The *volume* of any polyedron is the numerical measure of its magnitude, referred to some other polyedron as the *unit.* The polyedron adopted as the unit is called the *unit of volume.*

To *measure* the volume of a polyedron is, then, to find its ratio to the unit of volume.

The most convenient unit of volume is the cube whose edge is the linear unit.

5. *Definition.* *Equivalent* solids are those which have equal volumes.

PRISMS AND PARALLELOPIPEDS.

6. *Definitions.* A *prism* is a polyedron two of whose opposite faces, called *bases*, are in parallel planes, and whose *lateral edges* (that is, the edges intersecting the bases) are all parallel to the same line.

From this definition we readily deduce the following consequences:

1st. Any two lateral edges of a prism are parallel (VI., Proposition V., Corollary).

2d. All the lateral faces of a prism are parallelograms (VI., Proposition VIII.). Hence all the lateral edges are equal.

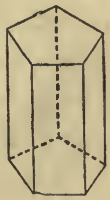
3d. The bases of a prism are equal polygons (VI., Proposition X.).

The lateral faces of a prism constitute its *lateral* or *convex* surface.

The *altitude* of a prism is the perpendicular distance between the planes of its bases (VI., Proposition IX.)

A *triangular* prism is one whose base is a triangle; a *quadrangular* prism, one whose base is a quadrilateral; etc.

7. *Definitions.* A *right prism* is one whose lateral edges are perpendicular to the planes of its faces (VI., Proposition XI.)



In a right prism, any lateral edge is equal to the altitude.

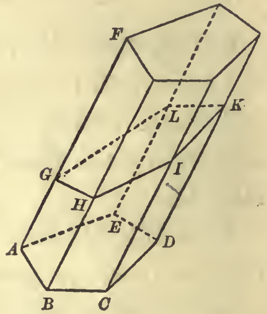
The lateral faces of a right prism are perpendicular to the bases (VI., Proposition XIV.)

An *oblique prism* is one whose lateral edges are oblique to the planes of its bases.

In an oblique prism, a lateral edge is greater than the altitude.

8. *Definition.* A *regular prism* is a right prism whose bases are regular polygons.

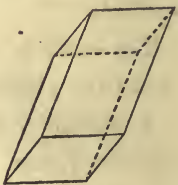
9. *Definition.* If a prism, $ABCDE–F$, is intersected by a plane GK, not parallel to its base, the portion of the prism included between the base and this plane, namely, $ABCDE–GHIKL$, is called a *truncated prism.*



10. *Definition.* A *right section* of a prism is the section made by a plane passed through the prism perpendicular to one of its lateral edges.

A right section is perpendicular to all the lateral edges (VI., Proposition XI.) and to all the lateral faces (VI., Proposition XIV.) of the prism.

11. *Definition.* A *parallelopiped* is a prism whose bases are parallelograms. It is therefore a polyedron all of whose faces are parallelograms.

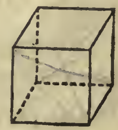


From this definition and VI., Proposition X., it is evident that any two opposite faces of a parallelopiped are equal parallelograms.

12. *Definition.* A *right parallelopiped* is a parallelopiped whose lateral edges are perpendicular to the planes of its bases. Hence, by VI., 6, its lateral faces are rectangles; but its bases may be either rhomboids or rectangles.

A *rectangular parallelopiped* is a right parallelopiped whose bases are rectangles. Hence it is a parallelopiped all of whose faces are rectangles.

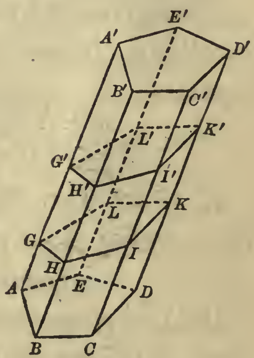
13. *Definition.* A *cube* is a rectangular parallelopiped whose edges are all equal. Hence its faces are all squares.



PROPOSITION I.—THEOREM.

14. *The sections of a prism made by parallel planes are equal polygons.*

For the portion of the prism included between the two sections is a new prism (6). Therefore its bases, which are the sections in question, are equal.



15. Corollary. *Any section of a prism made by a plane parallel to the base is equal to the base.*

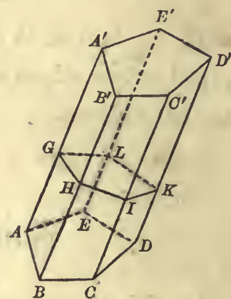
EXERCISE.

Theorem.—In a rectangular parallelopiped, the four diagonals are equal to each other; and the square of a diagonal is equal to the sum of the squares of the three edges which meet at a common vertex.

PROPOSITION II.—THEOREM.

16. *The lateral area of a prism is equal to the product of the perimeter of a right section of the prism by a lateral edge.*

Let AD' be a prism, and $GHIKL$ a right section of it; then the area of the convex surface of the prism is equal to the perimeter $GHIKL$ multiplied by a lateral edge AA'.



For, the sides of the section $GHIKL$, being perpendicular to the lateral edges AA', BB' (VI., 6), etc., are the altitudes of the parallelograms which form the convex surface of the prism, if we take as the bases of these parallelograms the lateral edges AA', BB', etc., which are all equal (6). Hence the area of the sum of these parallelograms is

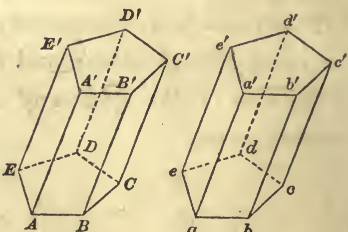
$$GH \times AA' + HI \times BB' + \text{etc.}$$
$$= (GH + HI + \text{etc.}) \times AA'.$$

17. Corollary. *The lateral area of a right prism is equal to the product of the perimeter of its base by its altitude.*

PROPOSITION III.—THEOREM.

18. *Two prisms are equal, if three faces including a triedral angle of the one are respectively equal to three faces similarly placed including a triedral angle of the other.*

Let the triedral angles A and a of the prisms $ABCDE$–A', $abcde$–a', be contained by equal faces similarly placed, namely, $ABCDE$ equal to $abcde$, AB' equal to ab', and AE' equal to ae'; then the prisms are equal.



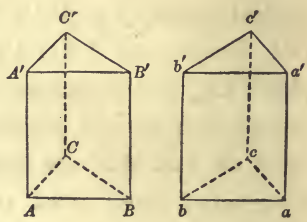
For, superpose the second prism upon the first, making the base $abcde$ coincide with the equal base $ABCDE$. Since the diedral angles ab and AB are equal and ae and AE are equal (VI., Proposition XXII.), the plane ab' will coincide with the plane AB', and the plane ae' with the plane AE'. Hence the intersection aa' will fall along the intersection AA'. As the faces ab' and AB' are equal, and have now been suitably superposed, they must coincide throughout, and $a'b'$ will coincide with $A'B'$. For the same reason, $a'e'$ will coincide with $A'E'$. Consequently, the plane determined by $a'b'$ and $a'e'$, namely, the plane of the upper base of the second prism, will coincide with the plane of the upper base of the first prism. Any lateral edge, as ee', will fall along the corresponding lateral edge EE', for they are now parallel to the same line AA', and have a point e of one coinciding with a point E of the other. They have thus the same direction and a point in common, and must coincide throughout, by I., Postulate II.

Since all the lateral edges of the second prism coincide with the corresponding lateral edges of the first, the planes of all the corresponding lateral faces must coincide. Therefore, as all the corresponding faces of the two prisms coincide (the bases included), the prisms are equal.

19. Corollary I. *Two truncated prisms are equal, if three faces including a triedral angle of the one are respectively equal to three faces similarly placed including a triedral angle of the other.* For the preceding demonstration applies whether the planes $A'B'C'D'E'$ and $a'b'c'd'e'$ are parallel or inclined to the lower bases.

20. Corollary II. *Two right prisms are equal, if they have equal bases and equal altitudes.*

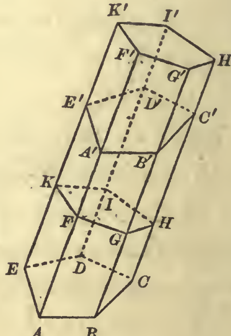
In the case of right prisms, it is not necessary to add the condition that the faces shall be similarly placed; for if the two right prisms $ABC-A'$, $abc-a'$, cannot be made to coincide by placing the base ABC upon the equal base abc; yet, by inverting one of the prisms and applying the base ABC to the base $a'b'c'$, they will coincide.



PROPOSITION IV.—THEOREM.

21. *Any oblique prism is equivalent to a right prism whose base is a right section of the oblique prism, and whose altitude is equal to a lateral edge of the oblique prism.*

Let $ABCDE-A'$ be the oblique prism. At any point F in the edge AA', pass a plane perpendicular to AA' and forming the right section $FGHIK$. Produce AA' to F', making $FF' = AA'$, and through F' pass a second plane perpendicular to the edge AA', intersecting all the faces of the prism produced, and forming another right section $F'G'H'I'K'$ parallel and equal to the first. The prism $FGHIK-F'$ is a right prism whose base is the right section and whose altitude FF' is equal to the lateral edge of the oblique prism.

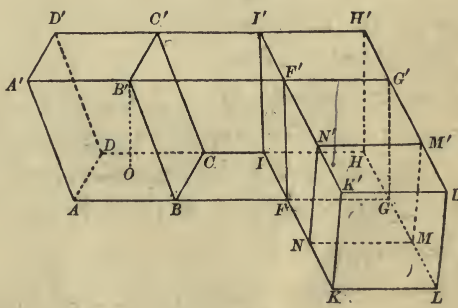


The solid $ABCDE-F$ is a truncated prism which is easily shown to be equal to the truncated prism $A'B'C'D'E'-F'$ (Proposition III., Corollary I.). Taking the first away from the whole solid $ABCDE-F'$, there remains the right prism;

taking the second away from the same solid, there remains the oblique prism; therefore the right prism and the oblique prism have the same volume; that is, they are equivalent.

PROPOSITION V.—THEOREM.

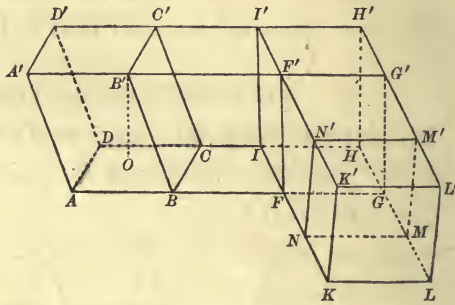
22. *Any parallelopiped is equivalent to a rectangular parallelopiped of the same altitude and an equivalent base.*



Let $ABCD\text{–}A'$ be any oblique parallelopiped whose base is $ABCD$, and altitude $B'O$.

Produce the edges AB, $A'B'$, DC, $D'C'$; in AB produced take $FG = AB$, and through F and G pass planes $FF'I'I$, $GG'H'H$, perpendicular to the produced edges; then the given parallelopiped and the right parallelopiped $FF'I'I\text{–}H$ are equivalent, by Proposition IV.

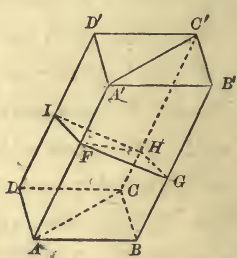
Produce, now, the edges of this second parallelopiped IF, $I'F'$, HG, $H'G'$; in IF produced take $NK = IF$, and through N and K pass planes $KLL'K'$ and $NMM'N'$ perpendicular to the produced edges. Then the second parallelopiped and the parallelopiped $NMM'N'\text{–}K$ are equivalent, by Proposition IV. Consequently, the given parallelopiped and the parallelopiped $NMM'N'\text{–}K$ are equivalent. The last-named parallelopiped is a right parallelopiped, by construction, since the face $KLL'K'$ was drawn perpendicular to the lateral edges. Moreover, as the planes KL' and KN' are perpendicular, the first to KI, the second to AG, they are perpendicular to the plane AHK, by VI., Proposition XIV., and their intersection KK' is perpendicular to AHK (VI., Proposition XVI.), and therefore to KL (VI., 6). Hence the base $KLL'K'$ is a rectangle, and the parallelopiped $KLL'K'$–N is a rectangular parallelopiped. If, now, we take $KLMN$ as its base, its altitude is equal to that of the given parallelopiped, since the planes AHK and $A'H'K'$ are parallel; and its base is equivalent to $ABCD$, since each of them is equivalent to $FGHI$ (IV., Proposition I.).



PROPOSITION VI.—THEOREM.

23. *The plane passed through two diagonally opposite edges of a parallelopiped divides it into two equivalent triangular prisms.*

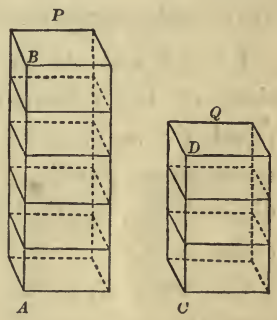
Let $ABCD$–A' be any parallelopiped; the plane $ACC'A'$, passed through its opposite edges AA' and CC', divides it into two equivalent triangular prisms ABC–A' and ADC–A'.



Let $FGHI$ be any right section of the parallelopiped, made by a plane perpendicular to the edge AA'. The intersection, FH, of this plane with the plane AC' is the diagonal of the parallelogram $FGHI$, and divides that parallelogram into two equal triangles, FGH and FIH. The oblique prism ABC–A' is equivalent to a right prism whose base is the triangle FGH and whose altitude is AA' (Proposition IV.); and the oblique prism $ADC–A'$ is equivalent to a right prism whose base is the triangle FIH and whose altitude is AA'. The two right prisms are equal (Proposition III., Corollary II.); therefore the oblique prisms, which are respectively equivalent to them, are equivalent to each other.

PROPOSITION VII.—THEOREM.

24. *Two rectangular parallelopipeds having equal bases are to each other as their altitudes.*



Let P and Q be two rectangular parallelopipeds having equal bases, and let AB and CD be their altitudes.

1st. Suppose the altitudes have a common measure, which is contained m times in AB and n times in CD. Then we have

$$\frac{AB}{CD} = \frac{m}{n}.$$

Apply this measure to AB and CD, and through the points of division draw planes perpendicular to AB and CD. P will thus be divided into m, and Q into n, smaller parallelopipeds, all of which will be equal, by Proposition I., Corollary, and Proposition III., Corollary II. Hence

$$\frac{P}{Q} = \frac{m}{n}.$$

Therefore

$$\frac{P}{Q} = \frac{AB}{CD}.$$

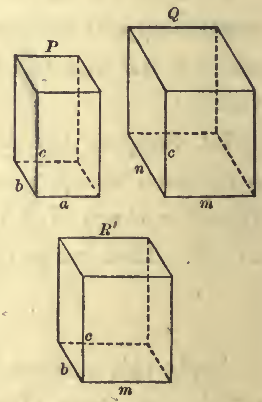
The proof may be extended to the case where the altitudes are incommensurable, by the method exemplified in the proof of II., Proposition XII., of III., Proposition I., and of IV., Proposition II.

25. *Scholium.* The three edges of a rectangular parallelopiped which meet at a common vertex are called its *dimensions*, and the preceding theorem may be expressed as follows:

Two rectangular parallelopipeds which have two dimensions in common are to each other as their third dimensions.

PROPOSITION VIII.—THEOREM.

26. *Two rectangular parallelopipeds having equal altitudes are to each other as their bases.*



Let a, b, and c be the three dimensions of the rectangular parallelopiped P; m, n, and c those of the rectangular parallelopiped Q; the dimension c, or the altitude, being common.

Construct R, a third rectangular parallelopiped, having the dimensions m, b, and c.

If a and m are taken as the altitudes of P and R, their bases are equal, and, by Proposition VII.,

$$\frac{P}{R} = \frac{a}{m}.$$

If b and n are taken as the altitudes of R and Q, their bases are equal, and, by Proposition VII.,

$$\frac{R}{Q} = \frac{b}{n};$$

and, multiplying these ratios together,

$$\frac{P}{Q} = \frac{a \times b}{m \times n}.$$

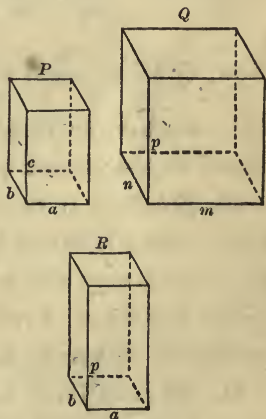
But $a \times b$ is the area of the base of P, and $m \times n$ is the area of the base of Q; therefore P and Q are in the ratio of their bases.

27. *Scholium.* This proposition may also be expressed as follows:

Two rectangular parallelopipeds which have one dimension in common, are to each other as the products of the other two dimensions.

PROPOSITION IX.—THEOREM.

28. *Any two rectangular parallelopipeds are to each other as the products of their three dimensions.*



Let a, b, and c be the three dimensions of the rectangular parallelopiped P; m, n, and p those of the rectangular parallelopiped Q.

Construct R, a third rectangular parallelopiped, having the dimensions a, b, and p. By Proposition VII. we have

$$\frac{P}{R} = \frac{c}{p},$$

and by Proposition VIII.,

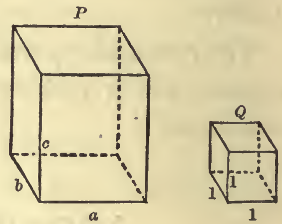
$$\frac{R}{Q} = \frac{a \times b}{m \times n};$$

and, multiplying these ratios together,

$$\frac{P}{Q} = \frac{a \times b \times c}{m \times n \times p}.$$

PROPOSITION X.—THEOREM.

29. *The volume of a rectangular parallelopiped is equal to the product of its three dimensions, the unit of volume being the cube whose edge is the linear unit.*



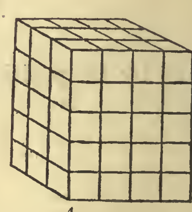
Let a, b, c, be the three dimensions of the rectangular parallelopiped P; and let Q be the cube whose edge is the linear unit. The three dimensions of Q are each equal to unity, and we have, by the preceding proposition,

$$\frac{P}{Q} = \frac{a \times b \times c}{1 \times 1 \times 1} = a \times b \times c.$$

Now, Q being taken as the unit of volume, $\frac{P}{Q}$ is the numerical measure, or volume of P, in terms of this unit (4); therefore the volume of P is equal to the product $a \times b \times c$.

30. *Scholium* I. Since the product $a \times b$ represents the base, when c is called the altitude, of the parallelopiped, this proposition may also be expressed as follows:

The volume of a rectangular parallelopiped is equal to the product of its base by its altitude.



31. *Scholium* II. When the three dimensions of the parallelopiped are each exactly divisible by the linear unit, the truth of the proposition is rendered evident by dividing the solid into cubes, each of which is equal to the unit of volume. Thus, if the three edges which meet at a common vertex A are, respectively, equal to 3, 4, and 5 times the linear unit, these edges may be divided respectively into 3, 4, and 5 equal parts, and then planes passed through the several points of division at right angles to these edges will divide the solid into cubes, each equal to the unit cube, the number of which is evidently $3 \times 4 \times 5$.

But the more general demonstration, above given, includes also the cases in which one of the dimensions, or two of them, or all three, are incommensurable with the linear unit.

32. *Scholium* III. If the three dimensions of a rectangular parallelopiped are each equal to a, the solid is a cube whose edge is a, and its volume is $a \times a \times a = a^3$; or, *the volume of a cube is the third power of its edge.* Hence it is that in arithmetic and algebra the expression "cube of a number" has been adopted to signify the "third power of a number."

PROPOSITION XI.—THEOREM.

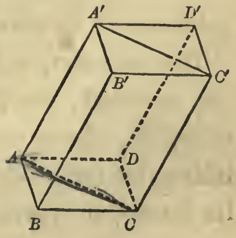
33. *The volume of any parallelopiped is equal to the product of the area of its base by its altitude.*

For, by Proposition V., the volume of any parallelopiped is equal to that of a rectangular parallelopiped having an equivalent base and the same altitude (30).

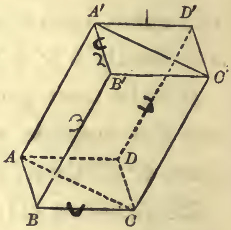
PROPOSITION XII.—THEOREM.

XIII

34. *The volume of a triangular prism is equal to the product of its base by its altitude.*

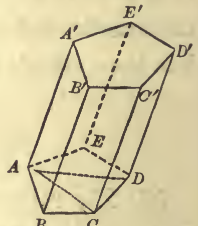


Let ABC–A' be a triangular prism. In the plane of the base complete the parallelogram $ABCD$, and then through D draw a line DD' parallel to AA', and through DD' and CC', and DD' and AA', pass planes, thus constructing the parallelopiped $ABCD\text{–}D'$. The given prism is half of the parallelopiped, by Proposition VI., and it has the same altitude.



The volume of the parallelopiped is equal to its base BD multiplied by its altitude (Proposition XI.); therefore the volume of the triangular prism is equal to its base ABC, the half of BD, multiplied by its altitude.

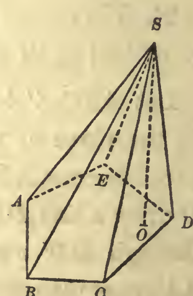
XIV **35.** Corollary. *The volume of any prism is equal to the product of its base by its altitude.*



Let $ABCDE\text{–}A'$ be any prism. It may be divided into triangular prisms by planes passed through a lateral edge AA' and the several diagonals of its base. The volume of the given prism is the sum of the volumes of the triangular prisms, or the sum of their bases multiplied by their common altitude, which is the base $ABCDE$ of the given prism multiplied by its altitude.

PYRAMIDS.

36. *Definitions.* A *pyramid* is a polyedron bounded by a polygon and triangular faces formed by the intersections of planes passed through the sides of the polygon and a common point out of its plane; as $S\text{–}ABCDE$.

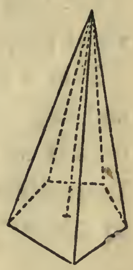


The polygon, $ABCDE$, is the *base* of the pyramid; the point, S, in which the triangular faces meet, is its *vertex;* the triangular faces taken together constitute its *lateral, or convex, surface;* the area of this surface is the *lateral area;* the lines SA, SB, etc., in which the lateral faces intersect, are its *lateral edges.* The *altitude* of the pyramid is the perpendicular distance SO from the vertex to the base.

A *triangular* pyramid is one whose base is a triangle; a *quadrangular* pyramid, one whose base is a quadrilateral; etc.

A triangular pyramid, having but four faces (all of which are triangles), is a tetraedron; and any one of its faces may be taken as its base.

37. *Definitions.* A *regular pyramid* is one whose base is a regular polygon, and whose vertex is in the perpendicular to the base erected at the centre of the polygon. This perpendicular is called the *axis* of the regular pyramid.



From this definition it can be readily shown that the lateral edges of a regular pyramid are all equal, and hence that the lateral faces are equal isosceles triangles.

The *slant height* of a regular pyramid is the perpendicular from the vertex to the base of any one of its lateral faces. It is the common altitude of all the lateral faces.

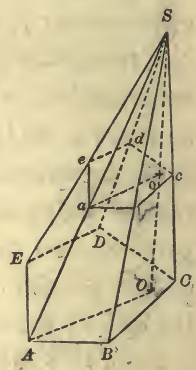
38. *Definitions.* A *truncated pyramid* is the portion of a pyramid included between its base and a plane cutting all its lateral edges.

When the cutting plane is parallel to the base, the truncated pyramid is called a *frustum of a pyramid.* The *altitude* of a frustum is the perpendicular distance between its bases.

In a frustum of a regular pyramid, the lateral faces are equal trapezoids; and the perpendicular distance between the parallel sides of any one of these trapezoids is the slant height of the frustum.

PROPOSITION XIII.—THEOREM.

39. *If a pyramid is cut by a plane parallel to its base:* 1st, *the edges and the altitude are divided proportionally;* 2d, *the section is a polygon similar to the base.*



Let the pyramid $S\text{–}ABCDE$, whose altitude is SO, be cut by the plane $abcde$ parallel to the base, intersecting the lateral edges in the points a, b, c, d, e, and the altitude in o; then

1st. The edges and the altitude are divided proportionally.

Pass a plane through the altitude SO and any lateral edge SA, cutting the base in AO and the section in ao. By VI., Proposition VIII., ab, bc, cd, ... ao are parallel respectively to AB, BC, CD, ... AO. Therefore, by III., Proposition I.,

$$\frac{Sa}{SA}=\frac{Sb}{SB}=\frac{Sc}{SC}=\frac{Sd}{SD}\cdots=\frac{So}{SO}.$$

2d. The section $abcde$ and the base are similar. For they are mutually equiangular, by VI., Proposition X., and by similar triangles we have

$$\frac{ab}{AB}=\frac{Sa}{SA},\quad \frac{bc}{BC}=\frac{Sb}{SB},\quad \frac{cd}{CD}=\frac{Sc}{SC}\cdots;$$

whence

$$\frac{ab}{AB}=\frac{bc}{BC}=\frac{cd}{CD}\cdots,$$

and the homologous sides of the polygons are proportional. Therefore the section and the base are similar.

40. Corollary I. *If a pyramid is cut by a plane parallel to its base, the area of the section is to the area of the base as the square of its distance from the vertex is to the square of the altitude of the pyramid.*

For

$$\frac{abcde}{ABCDE} = \frac{\overline{ab}^2}{\overline{AB}^2}, \text{ by IV., Proposition IX.};$$

but

$$\frac{ab}{AB} = \frac{Sa}{SA} = \frac{So}{SO}.$$

Therefore

$$\frac{abcde}{ABCDE} = \frac{\overline{So}^2}{\overline{SO}^2}.$$

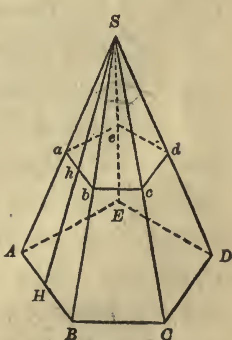
41. Corollary II. *If two pyramids have equal altitudes and equivalent bases, sections made by planes parallel to their bases and at equal distances from their vertices are equivalent.*

PROPOSITION XIV.—THEOREM.

XVI

42. *The lateral area of a regular pyramid is equal to the product of the perimeter of its base by one-half its slant height.*

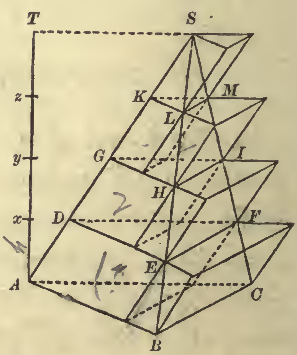
For, let S–$ABCDE$ be a regular pyramid; the lateral faces SAB, SBC, etc., being equal isosceles triangles, whose bases are the sides of the regular polygon $ABCDE$ and whose common altitude is the slant height SH, the sum of their areas, or the lateral area of the pyramid, is equal to the sum of AB, BC, etc., multiplied by $\frac{1}{2}SH$.



43. Corollary. *The lateral area of the frustum of a regular pyramid is equal to the half sum of the perimeters of its bases multiplied by the slant height of the frustum.*

PROPOSITION XV.—THEOREM.

44. *If the altitude of any given triangular pyramid is divided into equal parts, and through the points of division planes are passed parallel to the base of the pyramid, and on the sections made by these planes as upper bases prisms are described having their edges parallel to an edge of the pyramid and their altitudes equal to one of the equal parts into which the altitude of the pyramid is divided, the total volume of these prisms will approach the volume of the pyramid as its limit as the number of parts into which the altitude of the pyramid is divided is indefinitely increased.*



Let S–ABC be the given triangular pyramid, whose altitude is AT. Divide the altitude AT into any number of equal parts Ax, xy, etc., and denote one of these parts by h. Through the points of division x, y, etc., pass planes parallel to the base, cutting from the pyramid the sections DEF, GHI, etc. Upon the triangles DEF, GHI, etc., as *upper* bases, construct prisms whose lateral edges are parallel to SA, and whose altitudes are each equal to h. This is effected by passing planes through EF, HI, etc., parallel to SA. There will thus be formed a series of prisms DEF–A, GHI–D, etc., *inscribed* in the pyramid.

{Prove carefully.}

Again, upon the triangles ABC, DEF, GHI, etc., as *lower* bases, construct prisms whose lateral edges are parallel to SA, and whose altitudes are each equal to h. This also is effected by passing planes through BC, EF, HI, etc., parallel to SA. There will thus be formed a series of prisms ABC–D, DEF–G, etc., which may be said to be *circumscribed* about the pyramid.

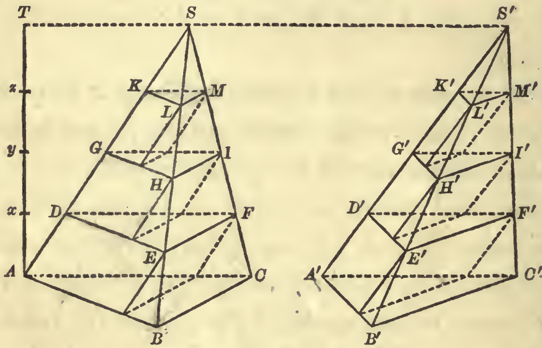
The total volume of the inscribed prisms is obviously less and the total volume of the circumscribed prisms is obviously greater than the volume of the pyramid.

Each inscribed prism is equivalent to the circumscribed prism immediately above it, since they have the same base and equal altitudes. Consequently, the difference between the total volume of the inscribed prisms and the total volume of the circumscribed prisms is the volume of the lowest circumscribed prism ABC–D, and therefore the difference between the total volume of the inscribed prisms and the volume of the pyramid is less than the volume of ABC–D. By increasing at pleasure the number of parts into which the altitude AT is divided, we can make the volume of ABC–D as small as we please, since we diminish its altitude at pleasure without changing its base. Therefore we can make the difference between the total volume of the inscribed prisms and the volume of the pyramid as small as we please; but, as we have seen above, we cannot make it absolutely zero. Hence the volume of the pyramid is the limit of the total volume of the inscribed prisms, as the number of parts into which the altitude AT is divided is indefinitely increased.

PROPOSITION XVI.—THEOREM.

45. *Two triangular pyramids having equivalent bases and equal altitudes are equivalent.*

Let $S\text{–}ABC$ and $S'\text{–}A'B'C'$ be two triangular pyramids



having equivalent bases ABC, $A'B'C'$, in the same plane, and a common altitude AT.

Divide the altitude AT into any arbitrarily chosen number n of equal parts, Ax, xy, yz, etc., and through the points of division pass planes parallel to the plane of the bases, intersecting the two pyramids. In the pyramid $S\text{–}ABC$ inscribe a series of prisms whose upper bases are the sections DEF, GHI, etc., and in the pyramid $S'\text{–}A'B'C'$ inscribe a series of prisms whose upper bases are the sections $D'E'F'$, $G'H'I'$, etc. Since the corresponding sections are equivalent (Proposition XIII., Corollary II.), the corresponding prisms, having equivalent bases and equal altitudes, are equivalent; therefore the sum of the prisms inscribed in the pyramid $S\text{–}ABC$ is equivalent to the sum of the prisms inscribed in the pyramid $S'\text{–}A'B'C'$; that is, if we denote the total volumes of the two series of prisms by V and V', we have

$$V = V',$$

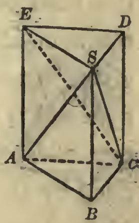
no matter what the value of n. If we vary n, V and V' obviously vary.

If n is indefinitely increased, V has the volume of the pyramid $S\text{–}ABC$, and V' the volume of the pyramid $S'\text{–}A'B'C'$, as its limit (Proposition XV.). Therefore, by III., Theorem of Limits, these volumes are equal.

PROPOSITION XVII.—THEOREM.

46. *A triangular pyramid is one-third of a triangular prism of the same base and altitude.*

Let $S\text{–}ABC$ be a triangular pyramid. Through A and C draw the lines AE and CD parallel to BS. Through AE and CD, which are parallel, by VI., Proposition V., Corollary, pass a plane, and through S pass a second plane parallel to ABC. The prism $ABC\text{–}E$ has the same base and altitude as the given pyramid, and we are to prove that the pyramid is one-third of the prism.



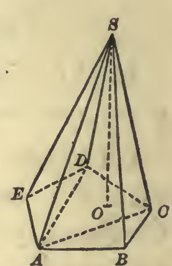
Taking away the pyramid $S\text{–}ABC$ from the prism, there remains a quadrangular pyramid whose base is the parallelogram $ACDE$ and vertex S. The plane SEC, passed through SE and SC, divides this pyramid into two triangular pyramids, $S\text{–}AEC$ and $S\text{–}ECD$, which are equivalent to each other, since their triangular bases AEC and ECD are the halves of the parallelogram $ACDE$, and their common altitude is the perpendicular from S upon the plane $ACDE$ (Proposition XVI.). The pyramid $S\text{–}ECD$ may be regarded as having ESD as its base and its vertex at C; therefore it is equivalent to the pyramid $S\text{–}ABC$, which has an equivalent base and the same altitude. Therefore the three pyramids into which the prism is divided are equivalent to each other, and the given pyramid is one-third of the prism.

47. Corollary. *The volume of a triangular pyramid is equal to one-third of the product of its base by its altitude.*

XVIII

PROPOSITION XVIII.—THEOREM.

48. *The volume of any pyramid is equal to one-third of the product of its base by its altitude.*



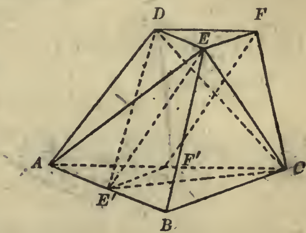
For any pyramid, $S\text{–}ABCDE$, may be divided into triangular pyramids by passing planes through an edge SA and the diagonals AD, AC, etc., of its base. The bases of these pyramids are the triangles which compose the base of the given pyramid, and their common altitude is the altitude SO of the given pyramid. The volume of the given pyramid is equal to the sum of the volumes of the triangular pyramids, which is one-third of the sum of their bases multiplied by their common altitude, or one-third of the product of the base $ABCDE$ by the altitude SO.

49. *Scholium.* The volume of any polyedron may be found by dividing it into pyramids, and computing the volumes of these pyramids separately. The division may be effected by taking a point within the polyedron and joining it with all the vertices. The polyedron will then be decomposed into pyramids whose bases will be the faces of the polyedron, and whose common vertex will be the point taken within it.

PROPOSITION XIX.—THEOREM.

50. *A frustum of a triangular pyramid is equivalent to the sum of three pyramids whose common altitude is the altitude of the frustum, and whose bases are the lower base, the upper base, and a mean proportional between the bases, of the frustum.*

Let $ABC–D$ be a frustum of a triangular pyramid, the plane DEF being parallel to the base ABC.



Through the vertices A, E, and C pass a plane AEC, and through the vertices E, D, and C pass a plane EDC, dividing the frustum into three pyramids.

The first of these, $ABC–E$, has for its base the lower base of the frustum, and for its altitude the altitude of the frustum; the second, $DEF–C$, has for its base the upper base of the frustum, and for its altitude the altitude of the frustum. It remains to show that the third, $ACD–E$, is equivalent to a pyramid having for its altitude the altitude of the frustum, and for its base a mean proportional between the bases of the frustum.

Through E in the plane $ABED$ draw a line EE' parallel to AD, and through E', D, and C pass a plane. EE' is parallel to the plane $ACFD$, by VI., Proposition VI. Therefore the pyramids $ACD–E$ and $ACD–E'$ are equivalent, since they have the same base and equal altitudes. If we take D as the vertex and $AE'C$ as the base of $ACD–E'$, it has for its altitude the altitude of the frustum.

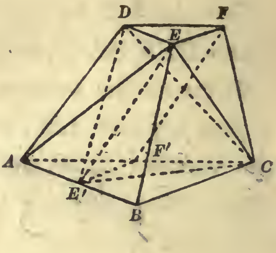
Through F in the plane $ACFD$ draw FF' parallel to AD. $AE'F'–D$ is a prism, and consequently its bases DEF and $AE'F'$ are equal, and $E'F'$ is parallel to EF, and therefore to BC.

$$\frac{AE'F'}{AE'C} = \frac{AF'}{AC},$$

since the triangles $AE'F'$ and $AE'C$ have the same altitude.

$$\frac{AE'C}{ABC} = \frac{AE'}{AB}, \text{ for the same reason.}$$

$$\frac{AF'}{AC} = \frac{AE'}{AB},$$



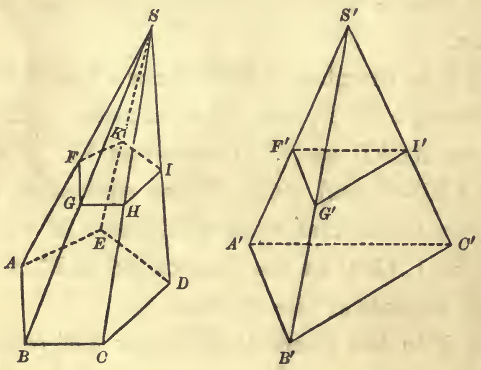
by III., Proposition I.

Therefore

$$\frac{DEF}{AE'C} = \frac{AE'C}{ABC},$$

and the base of the pyramid $AE'C$–D is a mean proportional between the bases of the frustum.

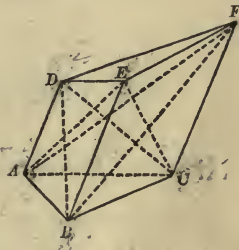
51. Corollary. *A frustum of any pyramid is equivalent to the sum of three pyramids whose common altitude is the altitude of the frustum, and whose bases are the lower base, the upper base, and a mean proportional between the bases, of the frustum.*



Suggestion. Let $ABCDE$–F be a frustum of any pyramid S–$ABCDE$. Construct a triangular pyramid, S'–$A'B'C'$, having the same altitude as S–$ABCDE$, and a base $A'B'C'$ equivalent to $ABCDE$ and in the same plane with it. Let the plane of the upper base of the given frustum be produced to cut the triangular pyramid in $F'G'I'$. The upper bases of the frustums are equivalent, by Proposition XIII., Corollary II., and the frustums themselves are equivalent, since the pyramids are equivalent and the pyramids above the frustums are equivalent.

PROPOSITION XX.—THEOREM.

52. *A truncated triangular prism is equivalent to the sum of three pyramids whose common base is the base of the prism, and whose vertices are the three vertices of the inclined section.*



Let *ABC–DEF* be a truncated triangular prism whose base is *ABC* and inclined section *DEF*.

Pass the planes *AEC* and *DEC*, dividing the truncated prism into the three pyramids *E–ABC*, *E–ACD*, and *E–CDF*.

The first of these pyramids, *E–ABC*, has the base *ABC* and the vertex *E*.

The second pyramid, *E–ACD*, is equivalent to the pyramid *B–ACD*; for they have the same base *ACD*, and the same altitude, since their vertices *E* and *B* are in the line *EB* parallel to this base. But the pyramid *B–ACD* is the same as *D–ABC*; that is, it has the base *ABC* and the vertex *D*.

The third pyramid, *E–CDF*, is equivalent to the pyramid *B–ACF*; for they have equivalent bases *CDF* and *ACF* in the same plane, and also the same altitude, since their vertices *E* and *B* are in the line *EB* parallel to that plane. But the pyramid *B–ACF* is the same as *F–ABC*; that is, it has the base *ABC* and the vertex *F*.

Therefore the truncated prism is equivalent to three pyramids whose common base is *ABC* and whose vertices are *E*, *D*, and *F*.

THE REGULAR POLYEDRONS.

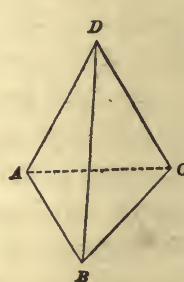
53. *Definition.* A *regular polyedron* is one whose faces are all equal regular polygons and whose polyedral angles are all equal to each other.

PROPOSITION XXI.—THEOREM.

54. *Only five regular (convex) polyedrons are possible.*

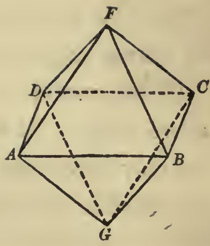
The faces of a regular polyedron must be regular polygons, and at least three faces are necessary to form a polyedral angle; moreover, the sum of the face angles of a polyedral angle must be less than four right angles (VI., Proposition XXI.).

1st. The simplest regular polygon is the equilateral triangle, and, since each angle of an equilateral triangle is an angle of 60°, three equilateral triangles can be combined to form a polyedral angle. It is probable, then, that a regular polyedron can be formed bounded by equilateral triangles and having three at each vertex.



There is such a regular polyedron. It has four faces, and is called the regular *tetraedron.*

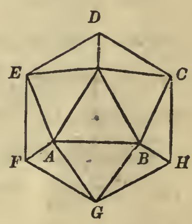
Since four angles of 60° are less than four right angles, four equilateral triangles can be combined to form a polyedral



angle. It is probable, then, that a regular polyedron can be formed bounded by equilateral triangles and having four at each vertex.

There is such a regular polyedron. It has eight faces, and is called the regular *octaedron.*

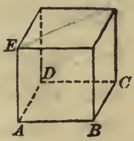
Since five angles of 60° are less than four right angles, five equilateral triangles can be combined to form a polyedral angle. It is probable, then, that a regular polyedron can be formed bounded by equilateral triangles and having five at each vertex.



There is such a regular polyedron. It has twenty faces, and is called the regular *icosaedron.*

No regular polyedrons bounded by equilateral triangles and having more than five at a vertex are possible. For six or more angles of 60° cannot form a polyedral angle.

2d. The next regular polygon to the equilateral triangle, in order of simplicity, is the square, each of whose angles is a right angle.



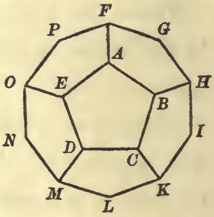
Three right angles can be combined to form a polyedral angle. It is probable, then, that a regular polyedron can be formed bounded by squares and having three at each vertex.

There is such a regular polyedron. It has six faces, and is called the *cube*, or the regular *hexaedron.*

No regular polyedrons bounded by squares and having more than three at a vertex are possible. For four or more right angles cannot form a polyedral angle.

3d. The next regular polygon is the regular pentagon, each of whose angles contains 108° (I., Proposition XXVII.).

Three angles of 108° each can be combined to form a polyedral angle. It is probable, then, that a regular polyedron can be formed bounded by regular pentagons and having three at each vertex.



There is such a regular polyedron. It has twelve faces, and is called the regular *dodecaedron.*

No regular polyedrons bounded by regular pentagons and having more than three at a vertex are possible. For four or more angles of 108° cannot be combined to form a polyedral angle.

4th. Each angle of the regular hexagon contains 120°. No regular polyedron can be formed bounded by hexagons. For three or more angles of 120° cannot be combined to form a polyedral angle.

No regular polyedron can be formed bounded by regular polygons of more than six sides. For it follows, from I., 55, Exercise, that the greater the number of sides in a regular polygon the greater the magnitude of its angles, and since, as we have seen, the angles of a hexagon are too great to allow the existence of a polyedral angle whose plane faces are regular hexagons, those of any regular polygon of more than six sides will be too great.

Therefore the only possible regular polyedrons are the five we have figured.

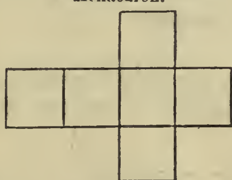
55. *Scholium* I. It must be observed that we have not attempted to prove that the five regular polyedrons are possible. This can be done by showing how to construct them; but the investigation is difficult and tedious.

56. *Scholium* II. The student may derive some aid in comprehending the preceding discussion of the regular polyedrons by constructing models of them, which he can do in a very simple manner, and at the same time with great accuracy, as follows.

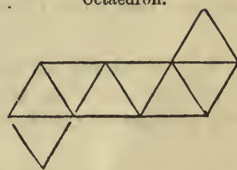
Draw on card-board the following diagrams; cut them out entire, and at the lines separating adjacent polygons cut the card-board half through; the figures will then readily bend into the form of the respective surfaces, and can be retained in that form by gluing the edges.

Tetraedron.

Hexaedron.

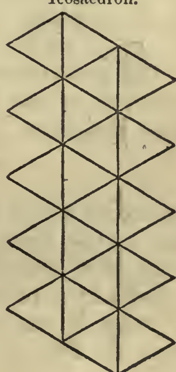


Octaedron.



Dodecaedron.

Icosaedron.



EXERCISES ON BOOK VII.

THEOREMS.

1. THE volume of a triangular prism is equal to the product of the area of a lateral face by one-half the perpendicular distance of that face from the opposite edge.

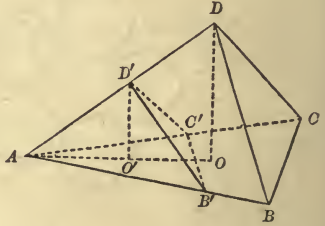
2. The lateral surface of a pyramid is greater than the base.

Suggestion. Join the projection of the vertex on the base with the corners of the base.

3. At any point in the base of a regular pyramid a perpendicular to the base is erected which intersects the several lateral faces of the pyramid, or these faces produced. Prove that the sum of the distances of the points of intersection from the base is constant.

Suggestion. The distances in question are proportional to the distances of the foot of the perpendicular from the sides of the base, and these distances have a constant sum. (*v.* V., Exercise 16.)

4. Two tetraedrons which have a triedral angle of the one equal to a triedral angle of the other, are to each other as the products of the three edges of the equal triedral angles. (*v.* IV., 19, Exercise.)

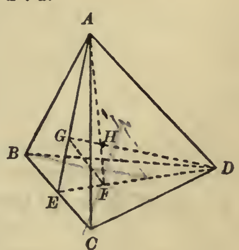


5. In a tetraedron, the planes passed through the three lateral edges and the middle points of the edges of the base intersect in a straight line.

Suggestion. The intersections of the planes with the base are medial lines of the base. Therefore they intersect in the line joining the vertex with the point of intersection of the medial lines of the base.

6. The lines joining each vertex of a tetraedron with the point of intersection of the medial lines of the opposite face all meet in a point, which divides each line in the ratio 1 : 4.

Note. This point is the *centre of gravity* of the tetraedron.



Suggestion. If AF and DG are two of the lines in question, they must intersect, since they both lie in the plane passed through AD and the middle point E of the opposite edge. Moreover, since $EF = \frac{1}{3}ED$ and $EG = \frac{1}{3}EA$ (I., Exercise 38), GF is parallel to AD and is equal to $\frac{1}{3}AD$. Whence $HF = \frac{1}{3}HA$ and $GH = \frac{1}{3}HD$. The lines through C and B will also each cut off $\frac{1}{4}$ of AF. Hence the four lines have a common intersection.

7. The straight lines joining the middle points of the opposite edges of a tetraedron all pass through the centre of gravity of the tetraedron, and are bisected by the centre of gravity. (*v.* III., Exercise 7.)

8. The plane which bisects a diedral angle of a tetraedron divides the opposite edge into segments which are proportional to the areas of the adjacent faces.

Suggestion. Consider the volumes of the two parts into which the tetraedron is divided.

9. If a, b, c, d, are the perpendiculars from the vertices of a tetraedron upon the opposite faces, and a', b', c', d', the perpendiculars from any point within the tetraedron upon the same faces respectively, then

$$\frac{a'}{a} + \frac{b'}{b} + \frac{c'}{c} + \frac{d'}{d} = 1.$$

Suggestion. Join the point in question with the vertices of the tetraedron, and compare the volumes of the four tetraedrons thus obtained with the volume of the given tetraedron.

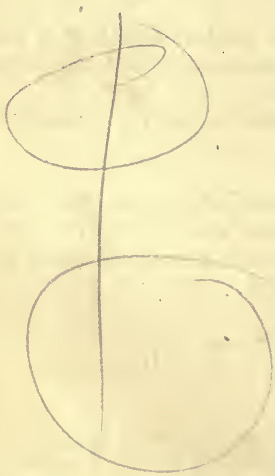
10. The altitude of a regular tetraedron is equal to the sum of the four perpendiculars let fall from any point within it upon the four faces.

11. Any lateral face of a prism is less than the sum of the other lateral faces. (*v.* Proposition II.)

PROBLEMS.

12. Given three indefinite straight lines in space which do not intersect, to construct a parallelopiped which shall have three of its edges on these lines. (*v.* VI., Exercise 8.)

13. Within a given tetraedron, to find a point such that planes passed through this point and the edges of the tetraedron shall divide the tetraedron into four equivalent tetraedrons. (*v.* Exercise 6.)



BOOK VIII.

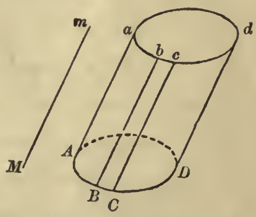
THE THREE ROUND BODIES.

1. Of the various solids bounded by curved surfaces, but three are treated of in Elementary Geometry,—namely, the *cylinder*, the *cone*, and the *sphere*, which are called the THREE ROUND BODIES.

THE CYLINDER.

2. *Definitions.* A *cylindrical surface* is a curved surface generated by a moving straight line which continually touches a given curve, and in all of its positions is parallel to a given fixed straight line not in the plane of the curve.

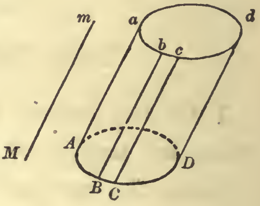
Thus, if the straight line Aa moves so as continually to touch the given curve $ABCD$, and so that in any of its positions, as Bb, Cc, Dd, etc., it is parallel to a given fixed straight line Mm, the surface $ABCDdcba$ is a cylindrical surface. If the moving line is of indefinite length, a surface of indefinite extent is generated.



The moving line is called the *generatrix;* the curve which it touches is called the *directrix.* Any straight line in the surface, as Bb, which represents one of the positions of the generatrix, is called an *element* of the surface.

To draw an element through any given point of a cylindrical surface, it is sufficient to draw a line through the point parallel to the given fixed straight line, or parallel to an element (I., Postulate II.).

In this general definition of a cylindrical surface, the directrix may be any curve whatever. Hereafter we shall assume it to be a *closed* curve, and usually a circle, as this is the only curve whose properties are treated of in elementary geometry.

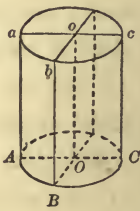


3. *Definition.* The solid *Ad* bounded by a cylindrical surface and two parallel planes, *ABD* and *abd*, is called a *cylinder;* its plane surfaces *ABD*, *abd*, are called its *bases;* the curved surface is sometimes called its *lateral surface;* and the perpendicular distance between its bases is its *altitude.*

The elements of a cylinder are all equal.

A cylinder whose base is a circle is called a *circular cylinder.*

4. *Definition.* A *right cylinder* is one whose elements are perpendicular to its base.



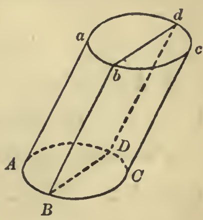
5. *Definition.* A *right cylinder with a circular base*, as *ABCa*, is called a *cylinder of revolution*, because it may be generated by the revolution of a rectangle *AOoa* about one of its sides, *Oo*, as an axis; the side *Aa* generating the curved surface, and the sides *OA* and *oa* generating the bases. The fixed side *Oo* is the *axis* of the cylinder. The radius of the base is called the *radius of the cylinder.*

PROPOSITION I.—THEOREM.

6. *Every section of a cylinder made by a plane passing through an element is a parallelogram.*

Let *Bb* be an element of the cylinder *Ac;* then the section *BbdD*, made by a plane passed through *Bb*, is a parallelogram.

The line Dd in which the cutting plane intersects the curved surface a second time is an element. For, if through any point D of this intersection a straight line is drawn parallel to Bb, this line, by the definition of a cylindrical surface, is an element of the surface, and it must also lie in the plane Bd; therefore this element, being common to both surfaces, is their intersection.



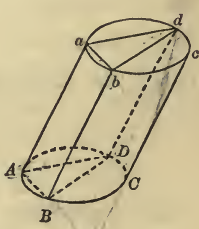
The lines BD and bd are parallel (VI., Proposition VIII.), and the elements Bb and Dd are parallel; therefore Bd is a parallelogram.

7. Corollary. *Every section of a right cylinder made by a plane perpendicular to its base is a rectangle.*

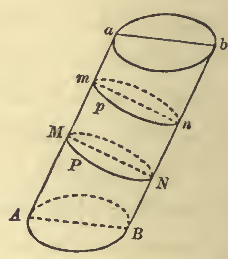
PROPOSITION II.—THEOREM.

8. *The bases of a cylinder are equal.*

Let BD be the straight line joining two points of the perimeter of the lower base, and let a plane passing through BD and the element Bb cut the upper base in the line bd; then $BD = bd$ (Proposition I.).



Let A be any third point in the perimeter of the lower base, and Aa the corresponding element. Through the parallels Aa and Bb pass a plane, and through Aa and Dd pass a plane. Then $AB = ab$ and $AD = ad$ (Proposition I.); and the triangles ABD, abd, are equal. Therefore, if the upper base be applied to the lower base with the line bd in coincidence with its equal BD, the triangles will coincide and the point a will fall upon A; that is, *any* point a of the upper base will fall on the perimeter of the lower base, and consequently the perimeters will coincide throughout. Therefore the bases are equal.



9. Corollary I. *Any two parallel sections of a cylindrical surface are equal.*

For these sections are the bases of a cylinder.

10. Corollary II. *All the sections of a circular cylinder parallel to its bases are equal circles; and the straight line joining the centres of the bases passes through the centres of all the parallel sections.* This line is called the *axis* of the cylinder.

Suggestion. In the base draw two diameters, and through these diameters and elements of the cylinder pass planes. They will cut all the sections in diameters, and their line of intersections will pass through all the centres.

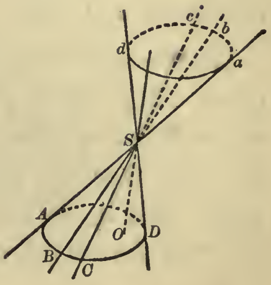
11. *Definition.* A *tangent plane* to a cylinder is a plane which passes through an element of the curved surface without cutting the surface. The element through which it passes is called the *element of contact.*

THE CONE.

12. *Definition.* A *conical surface* is a curved surface gener ated by a moving straight line which continually touches a given curve, and passes through a given fixed point not in the plane of the curve.

Thus, if the straight line SA moves so as continually to touch the given curve $ABCD$, and in all its positions, SB, SC, SD, etc., passes through the given fixed point S, the surface S–$ABCD$ is a conical surface.

The moving line is called the *generatrix;* the curve which it touches is called the *directrix.* Any straight line in the surface, as SB, which represents one of the positions of the generatrix, is called an *element* of the surface. The point S is called the *vertex.*



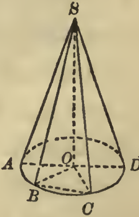
The straight line joining any point of a conical surface with the vertex is obviously an element.

If the generatrix is of indefinite length, as ASa, the whole surface generated consists of two symmetrical portions, each of indefinite extent, lying on opposite sides of the vertex, as $S\text{–}ABCD$ and $S\text{–}abcd$, which are called *nappes;* one the *upper*, the other the *lower*, *nappe.*

13. *Definition.* The solid $S\text{–}ABCD$, bounded by a conical surface and a plane ABD cutting the surface, is called a *cone;* its plane surface ABD is its *base*, the point S is its *vertex*, and the perpendicular distance SO from the vertex to the base is its *altitude.*

A cone whose base is a circle is called a *circular cone.* The straight line drawn from the vertex of a circular cone to the centre of its base is the *axis* of the cone.

14. *Definition.* A *right circular cone* is a circular cone whose axis is perpendicular to its base, as $S\text{–}ABCD$.

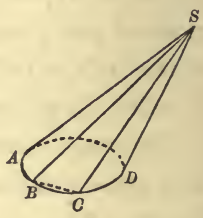


The right circular cone is also called a *cone of revolution*, because it may be generated by the revolution of a right triangle, SAO, about one of its perpendicular sides, SO, as an axis; the hypotenuse SA generating the curved surface, and the remaining perpendicular side OA generating the base.

PROPOSITION III.—THEOREM.

15. *Every section of a cone made by a plane passing through its vertex is a triangle.*

Let the cone S–$ABCD$ be cut by a plane SBC, which passes through the vertex S and cuts the base in the straight line BC; then the section SBC is a triangle; that is, the intersections SB and SC with the curved surface are straight lines.

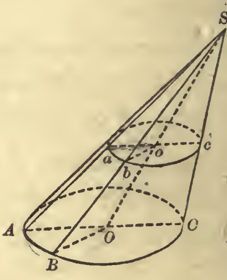


For the straight lines joining S with B and C are elements of the surface, by the definition of a cone, and they also lie in the cutting plane; therefore they coincide with the intersections of that plane with the curved surface; and BC, being the intersection of two planes, is a straight line.

PROPOSITION IV.—THEOREM.

16. *If the base of a cone is a circle, every section made by a plane parallel to the base is a circle.*

Let the section abc, of the circular cone S–ABC, be parallel to the base.



Let O be the centre of the base, and let o be the point in which the axis SO cuts the plane of the parallel section. Through SO and any number of elements SA, SB, etc., pass planes cutting the base in the radii OA, OB, etc., and the parallel section in the straight lines oa, ob, etc. Since oa is parallel to OA, and ob to OB, we have

$$\frac{oa}{OA} = \frac{So}{SO} \text{ and } \frac{ob}{OB} = \frac{So}{SO}, \text{ whence } \frac{oa}{OA} = \frac{ob}{OB}.$$

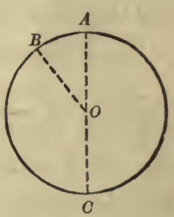
But $OA = OB$, therefore $oa = ob$; hence all the straight lines drawn from o to the perimeter of the section are equal, and the section is a circle.

17. Corollary. *The axis of a circular cone passes through the centres of all the sections parallel to the base.*

18. *Definition.* A *tangent plane* to a cone is a plane which passes through an element of the curved surface without cutting this surface. The element through which it passes is called the *element of contact.*

THE SPHERE.

19. *Definition.* A *sphere* is a solid bounded by a surface all the points of which are equally distant from a point within, called the *centre.*



A sphere may be generated by the revolution of a semicircle ABC about its diameter AC as an axis; for the surface generated by the curve ABC will have all its points equally distant from the centre O.

A *radius* of the sphere is any straight line drawn from the centre to the surface. A *diameter* is any straight line drawn through the centre and terminated both ways by the surface.

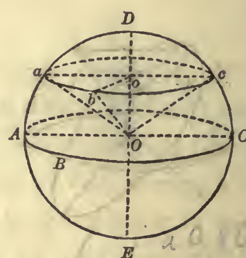
Since all the radii are equal and every diameter is double the radius, all the diameters are equal.

20. *Definition.* It will be shown that every section of a sphere made by a plane is a circle; and, as the greatest possible section is one made by a plane passing through the centre, such a section is called a *great circle.* Any section made by a plane which does not pass through the centre is called a *small circle.*

21. *Definition.* The *poles* of a circle of the sphere are the extremities of the diameter of the sphere which is perpendicular to the plane of the circle; and this diameter is called the *axis* of the circle.

PROPOSITION V.—THEOREM.

22. *Every section of a sphere made by a plane is a circle.*



1st. If the plane passes through the centre of the sphere, the lines joining points on the perimeter of the section with the centre O of the sphere are radii of the sphere, and are therefore all equal. Consequently it is a circle with its centre at O.

2d. If the plane does not pass through the centre of the sphere, as abc, draw a diameter EOD perpendicular to the section and meeting it at o. If points a, b, c, of the perimeter are joined with o, and also with O, the triangles aoO, boO, coO, are all equal (I., Proposition X.). Therefore ao, bo, co, etc., are all equal, and the section is a circle with its centre at o.

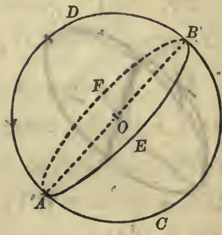
23. Corollary I. *The axis of a circle on a sphere passes through the centre of a circle.*

24. Corollary II. *All great circles of the same sphere are equal.*

25. Corollary III. *Every great circle divides the sphere into two equal parts.*

Suggestion. Superpose one part upon the other. (*v.* II., Proposition II.)

26. Corollary IV. *Any two great circles on the same sphere bisect each other;* for the common intersection AB of their planes passes through the centre of the sphere and is a diameter of each circle.



27. Corollary V. *An arc of a great circle may be drawn through any two given points of the surface of the sphere, and, unless the points are the opposite extremities of a diameter, only one such arc can be drawn;* for the two points, together with the centre O, determine the plane of a great circle whose circumference passes through the points.

If, however, the two given points are the extremities A and B of a diameter of the sphere, the position of the circle is not determined, for the points A, O, and B, being in the same straight line, will not determine a plane (VI., Proposition I.).

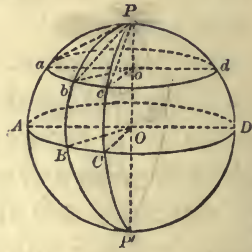
28. Corollary VI. *An arc of a circle may be drawn through any three given points on the surface of the sphere;* for the three points determine a plane which cuts the sphere in a circle.

EXERCISE.

Theorem.—The greater the distance of the plane of a small circle from the centre of the sphere, the less the circle.

PROPOSITION VI.—THEOREM.

29. *All the points in the circumference of a circle of the sphere are equally distant from either of its poles.*



Let *abcd* be any circle of the sphere, and PP' the diameter of the sphere perpendicular to its plane; then, by the definition (21), P and P' are the poles of the circle *abcd*, and, by Proposition V., Corollary I., PP' passes through o, the centre of *abcd*. Join P with any points, a, b, c, on the circumference of the circle. Then Pa, Pb, Pc, are equal, since the triangles Poa, Pob, Poc, are equal, by I., Proposition VI. Hence all the points of the circumference *abcd* are equally distant from the pole P. For the same reason, they are equally distant from the pole P'.

30. Corollary I. *All the arcs of great circles drawn from a pole of a circle to points in its circumference are equal,* since their chords are equal chords in equal circles.

By the distance of two points on the surface of a sphere is usually understood the arc of a great circle joining the two points. The arc of a great circle drawn from any point of a given circle *abc*, to one of its poles, as the arc Pa, is called the *polar distance* of the given circle, and the distance from the *nearest* pole is usually understood.

31. Corollary II. *The polar distance of a great circle is a quadrant;* thus, PA, PB, etc., $P'A$, $P'B$, etc., polar distances of the great circle $ABCD$, are quadrants; for they are the measures of the right angles AOP, BOP, AOP', BOP', etc., whose vertices are at the centre of the great circles PAP', PBP', etc.

In connection with the sphere, by a *quadrant* is usually to be understood a *quadrant of a great circle.*

32. COROLLARY III. *If a point on the surface of a sphere is at a quadrant's distance from each of two given points of the surface which are not opposite extremities of a diameter, it is the pole of the great circle passing through them.*

Suggestion. Let P be at a quadrant's distance from B and C; then POB and POC are right angles, and PO is perpendicular to the plane $ABCD$.

33. *Scholium.* By means of poles, arcs of circles may be drawn upon the surface of a sphere with the same ease as upon a plane surface. Thus, by revolving the arc Pa about the pole P, its extremity a will describe the small circle abd; and by revolving the quadrant PA about the pole P, the extremity A will describe the great circle ABD.

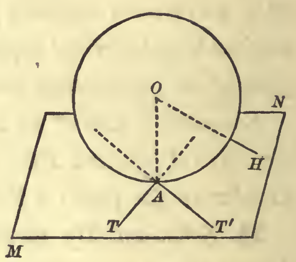
If two points, B and C, are given on the surface, and it is required to draw the arc BC, of a great circle, between them, it will be necessary first to find the pole P of this circle; for which purpose, take B and C as poles, and at a quadrant's distance describe two arcs on the surface intersecting in P. The arc BC can then be described with a pair of compasses, placing one foot of the compasses on P and tracing the arc with the other foot. The opening of the compasses (distance between their feet) must in this case be equal to the chord of a quadrant; and to obtain this it is necessary to know the radius of the sphere.

34. *Definition.* A plane is *tangent* to a sphere when it has but one point in common with the surface of the sphere.

35. *Definition.* Two spheres are tangent to each other when their surfaces have but one point in common.

PROPOSITION VII.—THEOREM.

36. *A plane tangent to a sphere is perpendicular to the radius drawn to the point of contact.*



For any other line drawn from the centre of the sphere to the plane must reach beyond the surface of the sphere, and therefore must be greater than the radius. The radius is, then, the shortest line that can be drawn from the centre of the sphere to the plane, and is consequently perpendicular to the plane (VI., Proposition III.).

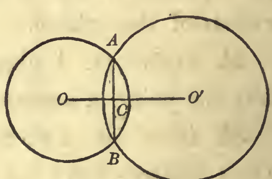
37. Corollary. Conversely, *a plane perpendicular to a radius of a sphere at its extremity is tangent to the sphere.*

38. *Scholium.* Any straight line AT, drawn in the tangent plane through the point of contact, is tangent to the sphere.

Any two straight lines, AT, AT', tangent to the sphere at the same point A, determine the tangent plane at that point.

PROPOSITION VIII.—THEOREM.

39. *The intersection of two spheres is a circle whose plane is perpendicular to the straight line joining the centres of the spheres, and whose centre is in that line.*



Through the centres O and O' of the two spheres let any plane be passed, cutting the spheres in great circles which intersect each other in the points A and B; the chord AB is bisected at C by the line OO' at right angles (II., Proposition VI., Corollary II.). If we now revolve the plane of these two circles about the line OO', the circles will generate the two spheres, and the point A will describe the line of intersection of their surfaces. Moreover, since the line AC will, during this revolution, remain perpendicular to OO', it will generate a circle whose plane is perpendicular to OO', and whose centre is C.

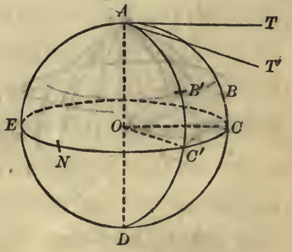
SPHERICAL ANGLES.

40. *Definition.* The *angle of two curves* passing through the same point is the angle formed by the two tangents to the curves at that point.

This definition is applicable to any two intersecting curves in space, whether drawn in the same plane or upon a surface of any kind.

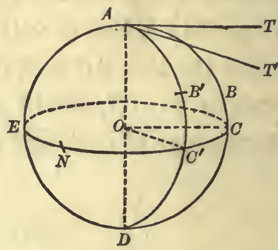
PROPOSITION IX.—THEOREM.

41. *The angle of two arcs of great circles is equal to the angle of their planes, and is measured by the arc of a great circle described from its vertex as a pole and included between its sides (produced if necessary).*



Let AB and AB' be two arcs of great circles, AT and AT' the tangents to these arcs at A, and O the centre of the sphere. TA and $T'A$ lie in the planes of their arcs, and are perpendicular to the radius OA drawn to their point of contact. They form, then, the plane angle measuring the diedral angle formed by the planes of the arcs; but, by (40), the angle which they form is equal to the angle of the two arcs.

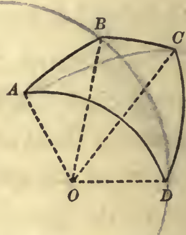
Now let CC' be the arc of a great circle described from A as a pole and intersecting the arcs AB, AB' (produced if necessary), in C and C'. The radii OC and OC' are perpendicular to AO, since the arcs AC, AC', are quadrants (Proposition VI., Corollary II.); therefore the angle COC' is a plane angle of the diedral angle AO, and is equal to TAT', or to BAB', and it is obviously measured by the arc CC'.



42. Corollary. *All arcs of great circles drawn through the pole of a given great circle are perpendicular to its circumference;* for their planes are perpendicular to its plane (VI., Proposition XIV.).

SPHERICAL POLYGONS.

43. *Definition.* A *spherical polygon* is a portion of the surface of a sphere bounded by three or more arcs of great circles, as $ABCD$.



Since the planes of all great circles pass through the centre of the sphere, the planes of the sides of a spherical polygon form, at the centre O, a polyedral angle of which the edges are the radii drawn to the vertices of the polygon, the face angles are angles at the centre measured by the sides of the polygon, and the diedral angles are equal to the angles of the polygon (Proposition IX.).

Since in a polyedral angle each face angle is assumed to be less than two right angles, each side of a spherical polygon will be assumed to be less than a semi-circumference.

A spherical polygon is *convex* when its corresponding polyedral angle at the centre is convex (VI., 52).

A *diagonal* of a spherical polygon is an arc of a great circle joining any two vertices not consecutive.

44. *Definition.* A *spherical triangle* is a spherical polygon of three sides. It is called *right angled*, *isosceles*, or *equilateral*, in the same cases as a plane triangle.

45. In consequence of the relation established between polyedral angles and spherical polygons (43), it follows that from any property of polyedral angles we may infer an analogous property of spherical polygons.

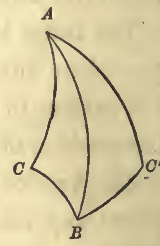
Reciprocally, from any property of spherical polygons we may infer an analogous property of polyedral angles.

The latter is in almost all cases the more simple mode of procedure, inasmuch as the comparison of figures drawn on the surface of a sphere is nearly if not quite as simple as the comparison of plane figures.

46. Arcs of great circles on the same sphere can be superposed and made to coincide just as straight lines are superposed and made to coincide. We have merely to place one point of the first arc on some given point of the second, and, keeping this point fixed, to turn the first arc about it as a pivot, until some second point in the arc not diametrically opposite the fixed point falls on the second arc. The two arcs must then coincide throughout, by Proposition V., Corollary V.

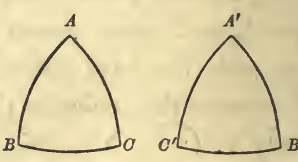
Equal angles formed by arcs of great circles on the surface of the same sphere can be superposed and made to coincide just as equal plane angles are superposed and made to coincide; that is, if the vertex of the first angle is placed upon the vertex of the second, and one side of the first placed upon the corresponding side of the second, the other side of the first will coincide with the other side of the second. For, if the two given angles are equal, their diedral angles are equal (Proposition IX.). If the vertices of the angles coincide, the edges of the diedral angles coincide; if a side of the first angle is placed on a side of the second, one face of the first diedral angle coincides with one face of the second. The remaining faces of the diedral angles must then coincide, and consequently the remaining sides of the given angles coincide.

47. *Definition.* Two spherical triangles are *symmetrical* if all the parts of one are respectively equal to the parts of the other, but the corresponding parts are arranged in opposite orders in the two triangles.

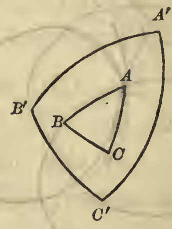


Two symmetrical triangles, as ABC, ABC', in the figure cannot be made to coincide; for, to bring the vertex C' upon the corresponding vertex C, the second triangle would have to be turned over, and the two convex surfaces would thus be brought together.

48. There is, however, one exception to the last remark,—namely, the case of *symmetrical isosceles triangles.* For, if ABC is an isosceles spherical triangle and $AB = AC$, then, in its symmetrical triangle, we have $A'B' = A'C'$, and consequently $AB = A'C'$, $AC = A'B'$, and, since the angles A and A' are equal, if AB be placed on $A'C'$, AC will fall on its equal $A'B'$, and the two triangles will coincide throughout.



49. *Definition.* If from the vertices of a spherical triangle as poles, arcs of great circles are described, these arcs form by their intersection a second triangle, which is called the *polar triangle* of the first.

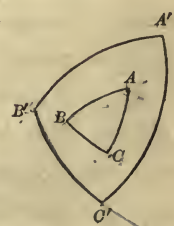


Thus, if A, B, and C are the poles of the arcs of great circles, $B'C'$, $A'C'$, and $A'B'$, respectively, $A'B'C'$ is the polar triangle of ABC.

Since all great circles, when completed, intersect each other in two points, the arcs $B'C'$, $A'C'$, $A'B'$, if produced, will form three other triangles; but the triangle which is taken as the polar triangle is that whose vertex A', homologous to A, lies on the same side of the arc BC as the vertex A; and so of the other vertices.

PROPOSITION X.—THEOREM.

50. *If the first of two spherical triangles is the polar triangle of the second, then, reciprocally, the second is the polar triangle of the first.*



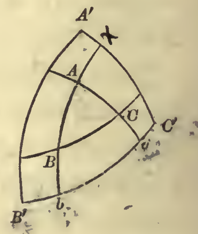
Let $A'B'C'$ be the polar triangle of ABC; then is ABC the polar triangle of $A'B'C'$.

For, since A is the pole of the arc $B'C'$, the point B' is at a quadrant's distance from A; and, since C is the pole of the arc $A'B'$, the point B' is at a quadrant's distance from C; therefore B' is the pole of the arc AC (Proposition VI., Corollary III.). In the same manner it is shown that A' is the pole of the arc BC, and C' the pole of the arc AB. Moreover, A and A' are on the same side of $B'C'$, B and B' on the same side of $A'C'$, C and C' on the same side of $A'B'$; therefore ABC is the polar triangle of $A'B'C'$.

PROPOSITION XI.—THEOREM.

51. *In two polar triangles, each angle of one is measured by the supplement of the side lying opposite to it in the other.*

Let ABC and $A'B'C'$ be two polar triangles.



Let the sides AB and AC, produced if necessary, meet the side $B'C'$ in the points b and c. The vertex A being the pole of the arc bc, the angle A is measured by the arc bc (Proposition IX.).

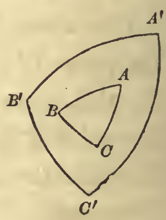
Now, B' being the pole of the arc Ac, and C' the pole of the arc Ab, the arcs $B'c$ and $C'b$ are quadrants; hence we have

$$B'C' + bc = B'c + C'b = \text{a semi-circumference.}$$

Therefore bc, which measures the angle A, is the supplement of the side $B'C'$.

In the same manner it can be shown that each angle of either triangle is measured by the supplement of the side lying opposite to it in the other triangle.

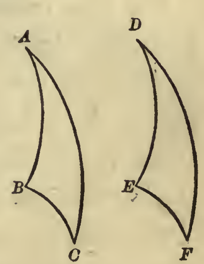
52. *Scholium.* Let the angles of the triangle ABC be denoted by A, B, and C, and let the sides opposite to them, namely, BC, AC, and AB, be denoted by a, b, and c, respectively. Let the corresponding angles and sides of the polar triangle be denoted by A', B', C', a', b', and c'. Also let both angles and sides be expressed in degrees. Then the preceding theorem gives the following relations:



$$A + a' = B + b' = C + c' = 180^\circ.$$
$$A' + a = B' + b = C' + c = 180^\circ.$$

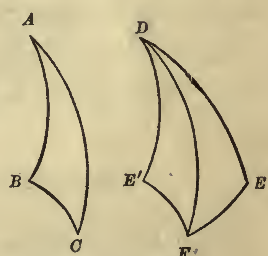
PROPOSITION XII.—THEOREM.

53. *Two triangles on the same sphere are either equal or symmetrical, when two sides and the included angle of one are respectively equal to two sides and the included angle of the other.*



In the triangles ABC and DEF, let the angle A be equal to the angle D, the side AB equal to the side DE, and the side AC equal to the side DF.

1st. When the parts of the two triangles are in the same order, ABC can be applied to DEF, as in the corresponding case of plane triangles (I., Proposition VI.), and the two triangles will coincide; therefore they are equal.

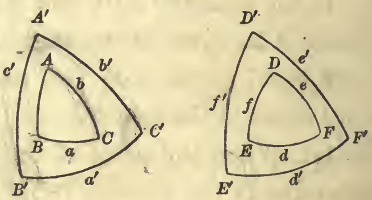


2d. When the parts of the two triangles are in inverse order, let $DE'F$ be the symmetrical triangle of DEF, and therefore having its angles and sides equal, respectively, to those of DEF. Then, in the triangles ABC and $DE'F$, we shall have the angle BAC equal to the angle $E'DF$, the side AB to the side DE', and the side AC to the side DF, and these parts arranged in the same order in the two triangles; therefore the triangle ABC is equal to the triangle $DE'F$, and consequently symmetrical with DEF.

54. *Scholium.* In this proposition, and in the propositions which follow, the two triangles may be supposed on the same sphere, or on two equal spheres.

PROPOSITION XIII.—THEOREM.

55. *Two triangles on the same sphere are either equal or symmetrical, when a side and two adjacent angles of one are equal respectively to a side and two adjacent angles of the other.*



Let the triangles ABC and DEF have the side a equal to the side d, and the angles B, C, equal respectively to the angles E, F; then are the triangles equal.

Construct the polar triangles of ABC and DEF. We have $b' = e'$, $c' = f'$, and $A' = D'$, by Proposition XI. Then $A'B'C'$ and $D'E'F'$ are equal or symmetrical, by Proposition XII. Therefore their polar triangles ABC, DEF, are equal or symmetrical.

56. *Scholium.* The proposition might be proved by direct superposition, as in I., Proposition VII.

PROPOSITION XIV.—THEOREM.

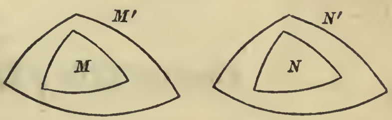
57. *Two triangles on the same sphere are either equal or symmetrical, when the three sides of one are respectively equal to the three sides of the other.*

For if their vertices are joined with the centre of the sphere, the triedral angles thus formed have the three face angles of the one respectively equal to the three face angles of the other, and consequently, by VI., Proposition XXII., their corresponding diedral angles are equal. The given triangles are, then, mutually equilateral and mutually equiangular, and are equal or symmetrical.

58. *Scholium.* The proposition can be proved as in I., Proposition IX.

PROPOSITION XV.—THEOREM.

59. *If two triangles on the same sphere are mutually equiangular, they are also mutually equilateral, and are either equal or symmetrical.*



Let the spherical triangles M and N be mutually equiangular.

Let M' be the polar triangle of M, and N' the polar triangle of N. Since M and N are mutually equiangular, their polar triangles M' and N' are mutually equilateral (Proposition XI.); therefore, by Proposition XIV., the triangles M' and N' are mutually equiangular. But M' and N' being mutually equiangular, their polar triangles M and N are mutually equilateral. Consequently, M and N are either equal or symmetrical.

60. *Scholium.* It may seem to the student that the preceding property destroys the analogy which subsists between plane and spherical triangles, since two mutually equiangular plane triangles are not necessarily mutually equilateral. But in the case of spherical triangles the equality of the sides follows from that of the angles only upon the condition that the triangles are constructed upon the same sphere or on equal spheres; if they are constructed on spheres of different radii, the homologous sides of two mutually equiangular triangles will no longer be equal, but will be proportional to the radii of the sphere; the two triangles will then be *similar*, as in the case of plane triangles.

EXERCISES.

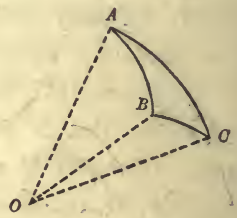
1. *Theorem.—In an isosceles spherical triangle the angles opposite the equal sides are equal.*

2. *Theorem.—The arc drawn from the vertex of an isosceles spherical triangle to the middle point of the base is perpendicular to the base, and bisects the vertical angle.*

3. *Theorem.—If two angles of a spherical triangle are equal, the triangle is isosceles.*

PROPOSITION XVI.—THEOREM.

61. *Any side of a spherical triangle is less than the sum of the other two.*



Let ABC be a spherical triangle; then any side, as AC, is less than the sum of the other two, AB and BC.

For, in the corresponding triedral angle formed at the centre O of the sphere, we have the angle AOC less than the sum of the angles AOB and BOC (VI., Proposition XX.); and since the sides of the triangle measure these angles, respectively, we have $AC < AB + BC$.

EXERCISES.

1. *Theorem.—If two angles of a spherical triangle are unequal, the side opposite the greater angle is greater than the side opposite the less angle.* (*v.* I., Proposition XII.)

2. *Theorem.—If two sides of a spherical triangle are unequal, the angle opposite the greater side is greater than the angle opposite the less side.* (*v.* I., Proposition XIII.)

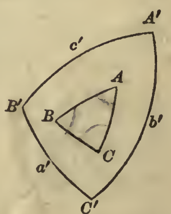
PROPOSITION XVII.—THEOREM.

62. *The sum of the sides of a convex spherical polygon is less than the circumference of a great circle.*

For the sum of the face angles of the corresponding polyedral angle at the centre of the sphere is less than four right angles (VI., Proposition XXI.).

PROPOSITION XVIII.—THEOREM.

63. *The sum of the angles of a spherical triangle is greater than two, and less than six, right angles.*



For, denoting the angles of a spherical triangle by A, B, C, and the sides respectively opposite to them in its polar triangle by a', b', c', we have (Proposition XI.)

$$A = 180° - a',\ B = 180° - b',\ C = 180° - c',$$

the sum of which is

$$A + B + C = 540° - (a' + b' + c').$$

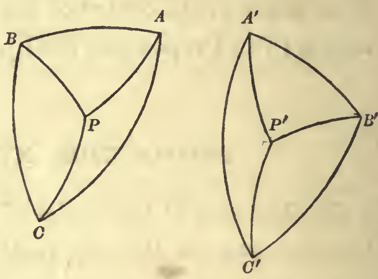
But $a' + b' + c' < 360°$ (Proposition XVII.); therefore $A + B + C > 180°$; that is, the sum of the three angles is greater than two right angles. Also, since each angle is less than two right angles, their sum is less than six right angles.

64. *Scholium.* A spherical triangle may have two or even three right angles; also two or even three obtuse angles.

PROPOSITION XIX.—THEOREM.

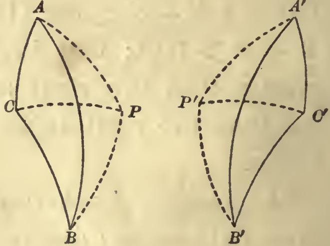
65. *Two symmetrical spherical triangles are equivalent.*

Let ABC and $A'B'C'$ be symmetrical spherical triangles. Let P be the pole of the small circle passing through A, B, and C (Proposition V., Corollary VI.). Then the arcs PA, PB, PC, are equal (Proposition VI., Corollary I.), and divide ABC into three isosceles triangles.

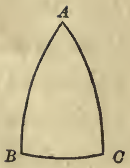


Through A' and B' in the triangle $A'B'C'$ draw arcs making with $A'B'$ angles equal respectively to PAB and PBA, and join P', their point of intersection, with C'. The isosceles triangle PAB is equal to the triangle $P'A'B'$, by Proposition XIII. and (48). The isosceles triangle PBC is equal to the triangle $P'B'C'$, by Proposition XII. and (48). The isosceles triangle PCA is equal to the triangle $P'C'A'$, by Proposition XIV. and (48). Hence ABC and $A'B'C'$ are equivalent.

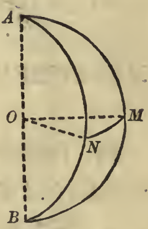
If the pole P should fall without the triangle ABC, the triangle would be equivalent to the sum of two of the isosceles triangles diminished by the third; but, as the same thing would occur for the symmetrical triangle, the conclusion would be the same.



66. *Definition.* If a spherical triangle ABC has two right angles, B and C, it is called a *bi-rectangular* triangle; and, since the sides AB and AC must each pass through the pole of BC (Proposition IX., Corollary), the vertex A is that pole, and therefore AB and AC are quadrants.



67. *Definition.* A *lune* is a portion of the surface of a sphere included between two semicircumferences of great circles; as $AMBNA$.



The two angles of a lune are equal, since each is equal to the diedral angle formed by the planes of the arcs of the lune; and the lune is equal to the sum of two equal bi-rectangular triangles, each of which has the angle of the lune for its third angle.

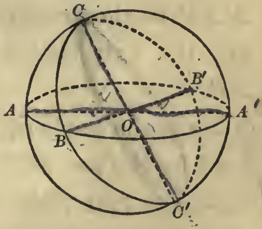
EXERCISE.

Theorem.—Two lunes on the same sphere or on equal spheres are equal if their angles are equal.

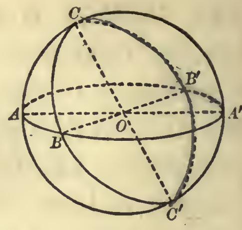
PROPOSITION XX.—THEOREM.

68. *If two arcs of great circles intersect on the surface of a hemisphere, the sum of the opposite spherical triangles which they form is equivalent to a lune whose angle is the angle between the arcs in question.*

Let the arcs ACA', BCB', intersect on the surface of the hemisphere $ABA'B'C$. Then will the triangles ABC, $A'B'C$, be together equivalent to a lune whose angle is ACB.



For, continue the arcs ACA', BCB', until they intersect in C'. $A'C' = AC$, $B'C' = BC$, and $A'B' = AB$, since they subtend equal angles.



The triangles $A'B'C'$ and ABC are then equal or symmetrical, by Proposition XIV., and are in either case equivalent (Proposition XIX.). Therefore ABC and $A'B'C$ are together equivalent to $A'B'C + A'B'C'$; that is, to the lune $CA'C'B'$.

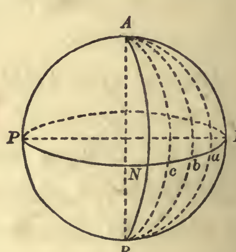
MEASUREMENT OF THE SURFACES OF SPHERICAL FIGURES.

69. *Definition.* A *degree of spherical surface*, or, more briefly, a *spherical degree*, is $\frac{1}{360}$ of the surface of a hemisphere. It is a convenient unit in measuring the surfaces of spherical figures. Like the degree of arcs, it is not a unit of absolute magnitude, but depends upon the size of the sphere on which the figures are drawn.

It may be conveniently conceived as a bi-rectangular spherical triangle whose third angle is an angle of one degree.

PROPOSITION XXI.—THEOREM.

70. *A lune is to the surface of the sphere as the angle of the lune is to four right angles.*



Let $ANBMA$ be a lune, and let MNP be the great circle whose poles are the extremities of the diameter AB.

Since the angle of the lune is measured by the arc MN, the angle of the lune is to four right angles as the arc MN is to the whole circumference $MNPM$.

1st. Suppose that MN and the circumference have a common measure which is contained m times in MN and n times in $MNPM$. Then

$$\frac{MN}{MNPM}=\frac{m}{n}.$$

Apply the measure to the circumference, and through the points of division and the axis AB pass planes; they will divide the whole surface of the sphere into n equal lunes (67, Exercise), of which the given lune $ANBMA$ will contain m.

Therefore

$$\frac{ANBMA}{\text{surface of sphere}}=\frac{m}{n},$$

and we have

$$\frac{ANBMA}{\text{surface of sphere}}=\frac{MN}{MNPM}.$$

2d. We can extend the proof to the case where MN and $MNPM$ are incommensurable by our usual method. (*v.* VII., Proposition VII.)

71. Corollary. *The area of a lune is expressed by twice its angle, the angular unit being the degree, and the unit of surface the spherical degree.*

For, by (69), the area of the surface of the sphere is 720 spherical degrees. We have, then, if S is the area and A the angle of the lune,

$$\frac{S}{720}=\frac{A}{360};$$

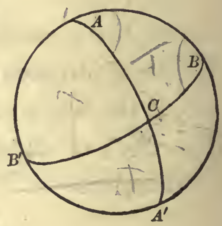
whence

$$S=2A.$$

72. *Scholium.* If the angle A contains a whole number of degrees, and each of the parts of the arc MN in the figure above is one degree, each of the small lunes is made up of two spherical degrees, and the lune $AMBN$ obviously contains twice as many spherical degrees as the arc MN contains degrees of arc.

PROPOSITION XXII.—THEOREM.

73. *The area of a spherical triangle is equal to the excess of the sum of its angles over two right angles.*



For, let ABC be a spherical triangle. Complete the great circle $ABA'B'$, and produce the arcs AC and BC to meet this circle in A' and B'.

We have, by the figure,

$$ABC + A'BC = \text{lune } A,$$
$$ABC + AB'C = \text{lune } B,$$

and, by Proposition XX.,

$$ABC + A'B'C = \text{lune } C.$$

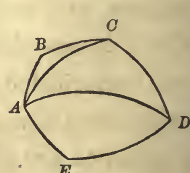
The sum of the first members of these equations is equal to twice the triangle ABC, *plus* the four triangles ABC, $A'BC$, $AB'C$, $A'B'C$, which compose the surface of the hemisphere, whose area is 360 spherical degrees.

Therefore, denoting the area of the triangle ABC by T, we have (Proposition XXI., Corollary)

$$2T + 360° = 2A + 2B + 2C,$$
$$T + 180° = A + B + C,$$
$$T = A + B + C - 180°.$$

74. *Scholium.* The excess of the sum of the angles of a spherical triangle over two right angles is sometimes called its *spherical excess*.

EXERCISE.



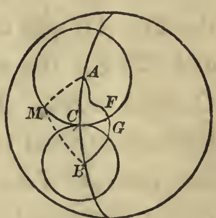
Theorem.—The area of a spherical polygon is measured by the sum of its angles minus *the product of two right angles multiplied by the number of sides of the polygon less two.*

75. *Scholium.* It must not be forgotten that Propositions XXI. and XXII. merely enable us to express our areas in spherical degrees; that is, in terms of $\frac{1}{720}$ of the surface of the whole sphere. If the area is required in terms of the ordinary unit of surface (IV., 1), the area of the surface of the sphere must first be given in terms of the unit in question.

SHORTEST LINE ON THE SURFACE OF A SPHERE BETWEEN TWO POINTS.

PROPOSITION XXIII.—THEOREM.

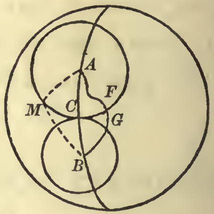
76. *The shortest line that can be drawn on the surface of a sphere between two points is the arc of a great circle, not greater than a semi-circumference, joining the two points.*



Let AB be an arc of a great circle, less than a semi-circumference, joining any two points A and B of the surface of a sphere; and let C be any arbitrary point taken in that arc. Then we say that the shortest line from A to B, on the surface of the sphere, must pass through C.

From A and B as poles, with the polar distances AC and BC, describe circumferences on the surface; these circumferences touch at C and lie wholly without each other. For, let M be any point other than C in the circumference whose pole is A, and draw the arcs of great circles AM, BM, forming the spherical triangle AMB. We have, by Proposition XVI., $AM + BM > AB$, and subtracting from the two members of this inequality the equal arcs AM and AC, we have $BM > BC$; therefore M lies without the circumference whose pole is B.

Now let $AFGB$ be any line from A to B, on the surface of the sphere, which does not pass through the point C, and which therefore cuts the two circumferences in different points, one in F, the other in G. Then a shorter line can be drawn from A to B, passing through C. For, whatever may be the nature of the line AF, an equal line can be drawn from A to C; since, if AC and AF be conceived to be drawn on two equal spheres having a common diameter passing through A, and therefore having their surfaces in coincidence, and if one of these spheres be turned upon the common diameter as an axis, the point A will be fixed and the point F will come into coincidence with C; the surfaces of the two spheres continuing to coincide, the line AF will then lie on the common surface between A and C. For the same reason, a line can be drawn from B to C, equal to BG. Therefore a line can be drawn from A to B, through C, equal to the sum of AF and BG, and consequently less than $AFGB$. The shortest line from A to B therefore passes through C; that is, through any, or every, point in AB. Consequently it must be the arc AB itself.

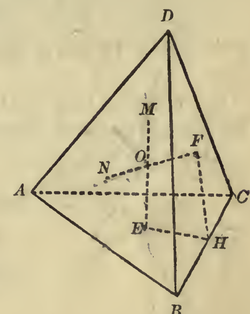


EXERCISES ON BOOK VIII.

THEOREMS.

1. A SPHERE can be circumscribed about any tetraedron.

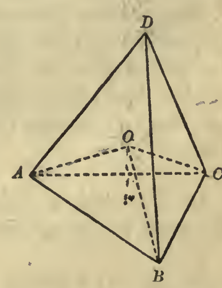
Suggestion. The locus of the points equally distant from A, B, and C is the perpendicular EM erected at the centre of the circle circumscribed about ABC (VI., Exercise 15.) The locus of the points equally distant from B, C, and D is the perpendicular FN, and both EM and FN lie in the plane perpendicular to BC at its middle point, since that plane contains all the points equally distant from B and C. EM and FN therefore intersect, and O, their point of intersection, is equally distant from the four vertices of the tetraedron. There is only one such point. Therefore only one sphere can be circumscribed about a tetraedron.



2. The perpendiculars erected at the centres of the four faces of a tetraedron meet in a point.

3. A sphere can be inscribed in any tetraedron.

Suggestion. The locus of the points equally distant from two faces of the tetraedron is the plane bisecting the diedral angle between them.



4. The planes bisecting the six diedral angles of a tetraedron intersect in a point.

LOCI.

5. Locus of the points in space which are at a given distance from a given straight line.

6. Locus of the points which are at the distance a from a point A, and at the distance b from a point B.

7. Locus of the centres of the spheres which are tangent to three given planes.

8. Locus of the centres of the sections of a given sphere made by planes passing through a given straight line.

Suggestion. Pass a plane through the centre of the sphere perpendicular to the given straight line. Then see II., Exercise 24.

9. Locus of the centres of the sections of a given sphere made by planes passing through a given point.

PROBLEMS.

10. Through a given point on the surface of a sphere, to pass a plane tangent to the sphere. (*v.* Proposition VII., Corollary.)

11. Through a given straight line without a sphere, to pass a plane tangent to the sphere.

Suggestion. Through the centre of the sphere pass a plane perpendicular to the given line. In this plane, from its point of intersection with the line, draw a line tangent to the circle in which the plane cuts the sphere. A plane through the tangent line and the given line is the tangent plane required. (Two solutions.)

12. Through a given point without a sphere, to pass a plane tangent to the sphere.

13. To cut a given sphere by a plane passing through a given straight line so that the section shall have a given radius.

Suggestion. Pass a plane through the centre of the sphere perpendicular to the given line. Then *v.* II., Exercise 37.

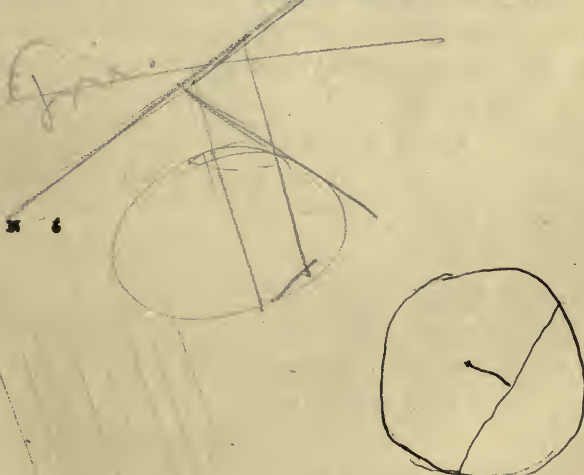
14. To construct a spherical surface with a given radius—1st, passing through three given points; 2d, passing through two given points and tangent to a given plane, or to a given sphere; 3d, passing through a given point and tangent to two given planes, or to two given spheres, or to a given plane and a given sphere; 4th, tangent to three given planes, or to three given spheres, or to two given planes and a given sphere, or to a given plane and two given spheres.

15. Through a given point on the surface of a sphere, to draw a great circle tangent to a given small circle.

Suggestion. With the pole of the small circle as a pole, and with a polar distance equal to the polar distance of the small circle plus a quadrant, describe a second small circle. With the given point as a pole describe a great circle. A point of intersection of this great circle with the second small circle will be the pole of the great circle required.

16. To draw a great circle tangent to two given small circles.

17. At a given point in a great circle, to draw an arc of a great circle which shall make a given angle with the first.

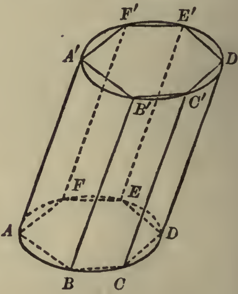


BOOK IX.

MEASUREMENT OF THE THREE ROUND BODIES.

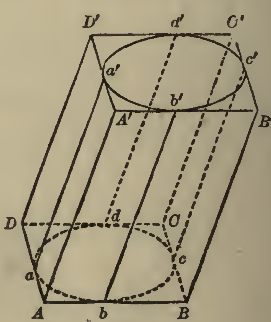
THE CYLINDER.

1. *Definition.* The area of the convex, or lateral, surface of a cylinder is called its *lateral area.*



2. *Definition.* A prism is *inscribed in a cylinder* when its base is inscribed in the base of the cylinder and its lateral edges are elements of the cylinder. It follows that the upper base of the prism is inscribed in the upper base of the cylinder.

To inscribe, then, a prism of any given number of lateral faces in a cylinder, we have merely to inscribe in the base a polygon of the given number of sides, and through the vertices of the polygon to draw elements of the cylinder. Planes passed through adjacent elements will form the lateral faces of the prism which is obviously wholly contained in the cylinder.



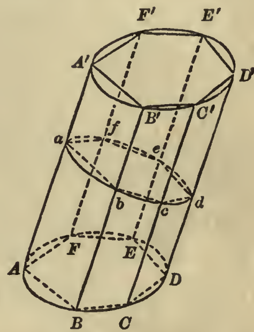
3. *Definition.* A prism is *circumscribed about a cylinder* when its base is circumscribed about the base of the cylinder and its lateral edges are parallel to elements of the cylinder.

It follows that its lateral faces are tangent to the lateral faces of the cylinder (VIII., 11); for any face, as AB', contains the element bb', since it contains the parallel line AA' and the point b (VI., Proposition II.), and, by VIII., Proposition I., it cannot cut the surface of the cylinder again unless AB cuts the base again; and that its upper base is circumscribed about the upper base of the cylinder.

The cylinder is obviously wholly contained in the prism.

4. *Definition.* A *right section* of a cylinder is a section made by a plane perpendicular to its elements; as *abcdef*.

The intersection of the same plane with an inscribed or circumscribed prism is a right section of the prism.



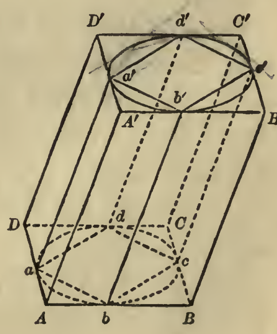
5. *Definition. Similar cylinders of revolution* are those which are generated by similar rectangles revolving about homologous sides.

PROPOSITION I.—THEOREM.

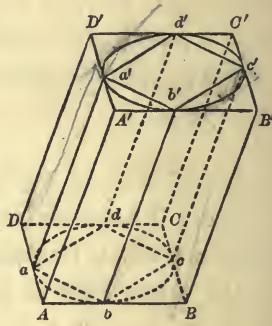
6. *If a prism whose base is a regular polygon be inscribed in or circumscribed about a given cylinder, its volume will approach the volume of the cylinder as its limit, and its lateral surface will approach the lateral surface of the cylinder as its limit as the number of sides of its base is indefinitely increased.*

For, if we could make the base of the prism exactly coincide with the base of the cylinder, the prism and the cylinder would coincide throughout, and their volumes would be equal and their lateral surfaces equal.

But, by increasing the number of sides of the base of the prism,



we can make it come as near as we please to coinciding with the base of the cylinder (V., Proposition VII.); we can then make the prism and cylinder fail of coincidence by as small an amount as we choose. Consequently, by increasing at pleasure the number of sides of the base of the circumscribed or inscribed prism, we can make the difference between the volumes of prism and cylinder, and between the lateral surfaces of prism and cylinder, as small as we choose, but cannot make it absolutely zero.

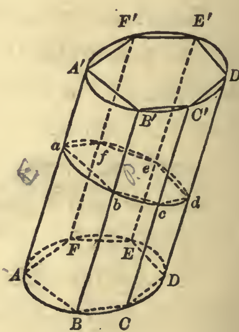


7. *Scholium.* The proposition just proved is true when the base of the prism is not a regular polygon; but it is only for the case of the regular polygon that a rigorous proof has been given in Book V.

PROPOSITION II.—THEOREM.

8. *The lateral area of a cylinder is equal to the product of the perimeter of a right section of the cylinder by an element of the surface.*

Let *ABCDEF* be the base and *AA'* any element of a cylinder, and let the curve *abcdef* be any right section of the surface. Denote the perimeter of the right section by *P*, the element *AA'* by *E*, and the lateral area of the cylinder by *S*.



Inscribe in the cylinder a prism *ABCDEFA'* of any arbitrarily chosen number n of faces. The right section, *abcdef*, of this prism will be a polygon inscribed in the right section of the cylinder formed by the same plane (4). Denote the lateral area of the prism by s, and the perimeter of its right section by p; then, the lateral edge of the prism being equal to E, we have (VII., Proposition II.)

$$s = p \times E,$$

no matter what the value of n. If n is indefinitely increased, s approaches the limit S (Proposition I.), and $p \times E$, the limit $P \times E$. Therefore, by III., Theorem of Limits,

$$S = P \times E.$$

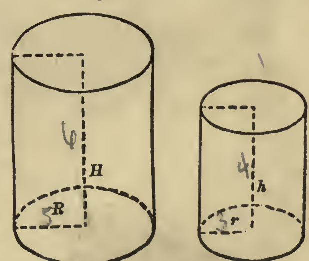
9. Corollary I. *The lateral area of a cylinder of revolution is equal to the product of the circumference of its base by its altitude.*

This may be formulated,

$$S = 2\pi R.H,$$

if R is the radius of the base and H the altitude.

10. Corollary II. *The lateral areas of similar cylinders of revolution are to each other as the squares of their altitudes, or as the squares of the radii of their bases.*

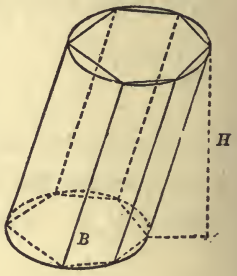


Suggestion. $\dfrac{S}{s} = \dfrac{2\pi R.H}{2\pi r.h} = \dfrac{R}{r} \cdot \dfrac{H}{h} = \dfrac{R^2}{r^2} = \dfrac{H^2}{h^2}$, since $\dfrac{R}{r} = \dfrac{H}{h}$, by (5).

PROPOSITION III.—THEOREM.

11. *The volume of a cylinder is equal to the product of its base by its altitude.*

Let the volume of the cylinder be denoted by V, its base by B, and its altitude by H. Let the volume of an inscribed prism be denoted by V', and its base by B'; its altitude will also be H, and we shall have (VII., Proposition XII., Corollary)



$$V' = B' \times H,$$

no matter what the number of faces of the prism.

If the number of faces of the prism is indefinitely increased, V' has the limit V, and $B' \times H$ the limit $B \times H$. Therefore

$$V = B \times H.$$

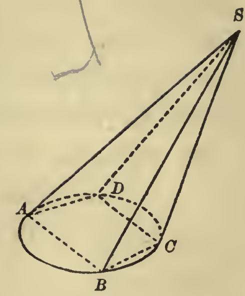
12. Corollary I. *For a cylinder of revolution this proposition may be formulated,* $V = \pi R^2 . H$. (V., Proposition IX., Corollary.)

13. Corollary II. *The volumes of similar cylinders of revolution are to each other as the cubes of their altitudes, or as the cubes of their radii.*

THE CONE.

14. *Definition.* The area of the convex, or lateral, surface of a cone is called its *lateral area.*

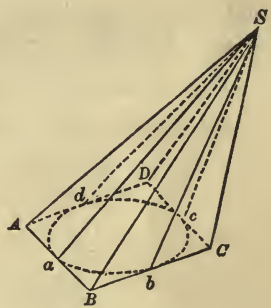
15. *Definition.* A pyramid is *inscribed in a cone* when its base is inscribed in the base of the cone and its vertex coincides with the vertex of the cone.



It follows that the lateral edges of the pyramid are elements of the cone.

An inscribed pyramid is wholly contained within the cone.

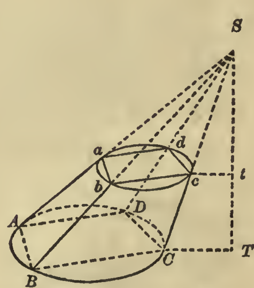
16. *Definition.* A pyramid is *circumscribed about a cone* when its base is circumscribed about the base of the cone and its vertex coincides with the vertex of the cone.



Any lateral face, as SAB, of the pyramid is tangent to the cone; for, since it passes through a and S, it contains the element Sa, and it cannot cut the convex surface again without cutting the perimeter of the base again (VIII., Proposition III.).

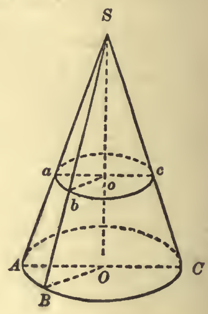
The cone is then wholly contained within the pyramid.

17. *Definition.* A *truncated cone* is the portion of a cone included between its base and a plane cutting its convex surface.



When the cutting plane is parallel to the base, the truncated cone is called a *frustum of a cone;* as *ABCD–abcd.* The *altitude* of a frustum is the perpendicular distance *Tt* between its bases.

If a pyramid is inscribed in the cone, the cutting plane determines a truncated pyramid inscribed in the truncated cone; and if a pyramid is circumscribed about the cone, the cutting plane determines a truncated pyramid circumscribed about the truncated cone.



18. *Definition.* In a cone of revolution all the elements are equal, and any element is called the *slant height of the cone.*

In a cone of revolution the portion of an element included between the parallel bases of a frustum, as Aa, or Bb, is called the *slant height of the frustum.*

19. *Definition. Similar cones of revolution* are those which are generated by similar right triangles revolving about homologous sides.

PROPOSITION IV.—THEOREM.

20. *If a pyramid be inscribed in or circumscribed about a given cone, its volume will approach the volume of the cone as its limit, and its lateral surface will approach the convex surface of the cone as a limit, as the number of faces of the pyramid is indefinitely increased.*

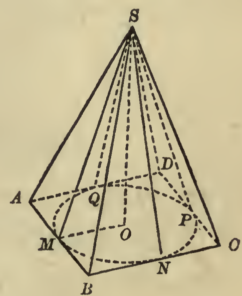
The demonstration is precisely the same as that of Proposition I., substituting cone for cylinder and pyramids for prisms.

21. Corollary. *A frustum of a cone is the limit of the inscribed and circumscribed frustums of pyramids, the number of whose faces is indefinitely increased.*

PROPOSITION V.—THEOREM.

22. *The lateral area of a cone of revolution is equal to the product of the circumference of its base by half its slant height.*

Suggestion. Circumscribe a regular pyramid about the cone, and then suppose the number of its faces to be indefinitely increased. (*v.* VII., Proposition XIV.)



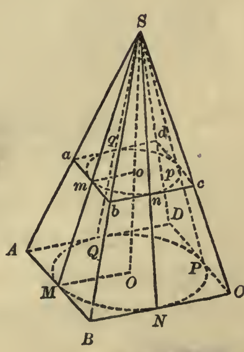
23. Corollary I. *The proposition may be formulated,* $S = \pi RL$, where R is the radius of the base and L the slant height.

24. Corollary II. *The lateral areas of similar cones of revolution are to each other as the squares of their slant heights, or as the squares of their altitudes, or as the squares of the radii of their bases.*

PROPOSITION VI.—THEOREM.

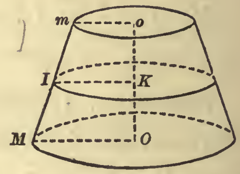
25. *The lateral area of a frustum of a cone of revolution is equal to the half sum of the circumferences of its bases multiplied by its slant height.*

Suggestion. Circumscribe the frustum of a regular pyramid about the frustum of the cone (17), and suppose the number of its faces indefinitely increased. (*v.* VII., Proposition XIV., Corollary.)



26. Corollary I. *The proposition may be formulated,* $S = \pi(R + r)L$, if R and r are the radii of the bases and L is the slant height.

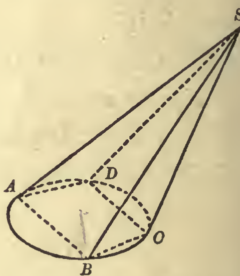
27. Corollary II. *The lateral area of a frustum of a cone of revolution is equal to the circumference of a section equidistant from its bases multiplied by its slant height.*



Suggestion. $IK = \frac{1}{2}(om + OM)$. (*v.* Exercise 24, Book I.)

PROPOSITION VII.—THEOREM.

28. *The volume of any cone is equal to one-third of the product of its base by its altitude.*

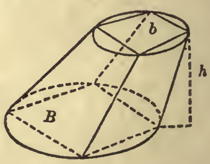


Suggestion. Inscribe a pyramid in the cone, and suppose the number of its faces to be indefinitely increased. (*v.* VII., Proposition XVIII.)

29. Corollary I. *For a cone of revolution, the proposition may be formulated,* $V = \frac{1}{3}\pi R^2 . H$.

30. Corollary II. *Similar cones of revolution are to each other as the cubes of their altitudes, or as the cubes of the radii of their bases.*

EXERCISE.



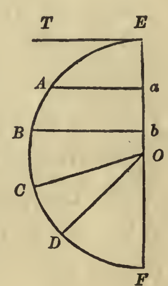
Theorem.—A frustum of any cone is equivalent to the sum of three cones whose common altitude is the altitude of the frustum, and whose bases are the lower base, the upper base, and a mean proportional between the bases of the frustum. (*v.* VII., Proposition XIX., Corollary.)

2π R · 14.

THE SPHERE.

31. *Definition.* A *spherical segment* is a portion of a sphere included between two parallel planes.

The sections of the sphere made by the parallel planes are the *bases* of the segment; the distance between the planes is the *altitude* of the segment.



Let the sphere be generated by the revolution of the semicircle *EBF* about the axis *EF;* and let *Aa* and *Bb* be two parallels, perpendicular to the axis. The solid generated by the figure *ABba* is a spherical segment; the circles generated by *Aa* and *Bb* are its bases; and *ab* is its altitude.

If two parallels *Aa* and *TE* are taken, one of which is a tangent at *E*, the solid generated by the figure *EAa* is a spherical segment having but one base, which is the section generated by *Aa*. The segment is still included between two parallel planes, one of which is the tangent plane at *E*, generated by the line *ET*.

32. *Definition.* A *zone* is a portion of the surface of a sphere included between two parallel planes.

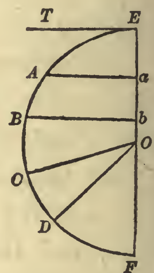
The circumferences of the sections of the sphere made by the parallel planes are the *bases* of the zone; the distance between the planes is its *altitude*.

A zone is the curved surface of a spherical segment.

In the revolution of the semicircle *EBF* about *EF*, an arc *AB* generates a zone; the points *A* and *B* generate the bases of the zone; and the altitude of the zone is *ab*.

An arc, *EA*, one extremity of which is in the axis, generates a zone of one base, which is the circumference described by the extremity *A*.

33. *Definition.* When a semicircle revolves about its diameter, the solid generated by any sector of the semicircle is called a *spherical sector.*



Thus, when the semicircle EBF revolves about EF, the circular sector COD generates a spherical sector.

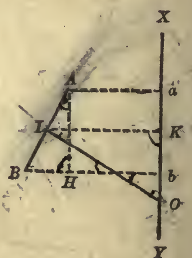
The spherical sector is bounded by three curved surfaces; namely, the two conical surfaces generated by the radii OC and OD, and the zone generated by the arc CD. This zone is called the *base* of the spherical sector.

OD may, however, coincide with OF, in which case the spherical sector is bounded by a conical surface and a zone of one base.

Again, OC may be perpendicular to OF, in which case the spherical sector is bounded by a plane, a conical surface, and a zone.

PROPOSITION VIII.—LEMMA.

34. *The area of the surface generated by a straight line revolving about an axis in its plane, is equal to the projection of the line on the axis multiplied by the circumference of the circle whose radius is the perpendicular erected at the middle of the line and terminated by the axis.*



Let AB be the straight line revolving about the axis XY; ab its projection on the axis; OI the perpendicular to it, at its middle point I, terminating in the axis; then *area* $AB = ab \times$ *circ.* OI.

For, draw IK perpendicular and AH parallel to the axis. The area generated by AB is that of the frustum of a cone; hence (Proposition VI., Corollary II.)

$$area\ AB = AB \times circ.\ IK.$$

The triangles ABH and IOK are similar, being mutually equiangular, and we have Why?

$$\frac{AH}{AB} = \frac{IK}{OI}, \text{ or } \frac{ab}{AB} = \frac{IK}{OI};$$

but

$$\frac{circ.\ IK}{circ.\ OI} = \frac{IK}{OI} \text{ (V., Proposition VIII.)},$$

$$\frac{ab}{AB} = \frac{circ.\ IK}{circ.\ OI},$$

and

$$ab \times circ.\ OI = AB \times circ.\ IK.$$

Therefore

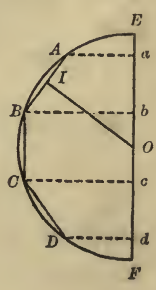
$$area\ AB = ab \times circ.\ OI.$$

If AB meets XY, the surface generated is a conical surface; but the proposition still holds, as may be easily proved. (*v.* Proposition V.)

If AB is parallel to the axis, the result is the same. (*v.* Proposition II., Corollary I.)

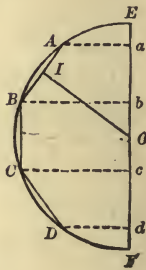
PROPOSITION IX.—THEOREM.

35. *The area of a zone is equal to the product of its altitude by the circumference of a great circle.*



Let the sphere be generated by the revolution of the semicircle EBF about the axis EF; and let the arc AD generate the zone whose area is required.

Let the arc AD be divided into any number of equal parts, AB, BC, CD, and draw the chords AB, BC, etc. These chords are all equal, since they subtend equal arcs; and the perpendiculars at their middle points all pass through the centre O of the semicircle, and are equal (II., Proposition VII.).



Let ab, bc, etc., be the projections of these chords on the axis. Then, by Proposition VIII.,

$$\text{area } AB = ab \times \text{circ. } OI,$$
$$\text{area } BC = bc \times \text{circ. } OI,$$
$$\text{area } CD = cd \times \text{circ. } OI.$$

Hence the sum of these areas, which is the area generated by the broken line $ABCD$, is equal to

$$(ab + bc + cd) \times \text{circ. } OI;$$

that is, to $ad \times$ *circ.* OI.

Calling the area generated by the broken inscribed line, S, we have

$$S = ad \times \text{circ. } OI,$$

no matter what the number of the equal parts into which the arc AD is divided. If, now, we increase the number of parts indefinitely, OI will approach the radius of the sphere, and *circ.* OI the circumference of a great circle as its limit, and S will approach the surface of the zone as its limit. Therefore

surface of zone $= ad \times$ *circumference of great circle.*

36. Corollary. *The proposition may be formulated,*

$$S = 2\pi R . H,$$

where R is the radius of the sphere and H the altitude of the zone.

S = 2π R·H   4πR²

PROPOSITION X.—THEOREM.

37. *The area of the surface of a sphere is equal to the product of its diameter by the circumference of a great circle.*

This follows directly from Proposition IX., since the surface of the whole sphere may be regarded as a zone whose altitude is the diameter of the sphere.

38. Corollary I. *This may be formulated,*

$$S = 2\pi R \times 2R = 4\pi R^2.$$

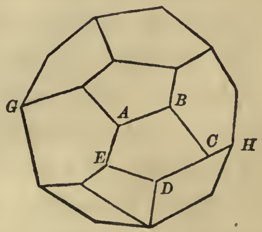
Hence the surface of a sphere is equivalent to four great circles.

39. Corollary II. *The surfaces of two spheres are to each other as the squares of their diameters, or as the squares of their radii.*

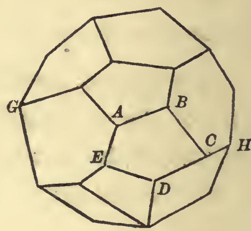
40. *Scholium.* The area of a spherical degree on a sphere whose radius is R is $\frac{4\pi R^2}{720}$ (VIII., 69), and, by the aid of this value, we may readily reduce the area of a spherical polygon to ordinary square measure.

PROPOSITION XI.—THEOREM.

41. *The volume of a sphere is equal to the area of its surface multiplied by one-third of its radius.*



Circumscribe a polyedron about the sphere. This may be done by taking at pleasure points on the surface of the sphere, and drawing tangent planes at these points. The circumscribed polyedron wholly contains the sphere, and is greater than the sphere. Join all the vertices of the polyedron with the centre of the sphere, and pass planes through the edges of the polyedron and these lines, thus dividing the polyedron into pyramids, each of which has its vertex at the centre of the sphere, and has a face of the polyedron as its base, and has, therefore, the radius of the sphere for its altitude (VIII., Proposition VII.). The volume of any one of these pyramids is then one-third of the product of a face of the polyedron by the radius of the sphere, and the sum of the volumes of the pyramids, or the whole volume of the polyedron, is one-third of the product of the sum of the faces of the polyedron by the radius of its sphere; that is, one-third of the product of the whole surface of the polyedron by the radius of the sphere. Representing the surface of the polyedron by s, and its volume by v, we have



$$v = \tfrac{1}{3}Rs,$$

and this equation holds no matter what the number of the faces of the polyedron.

If, now, we increase the number of faces of the polyedron by drawing additional tangent planes to the sphere, we decrease the volume v, for each new tangent plane cuts off a corner of the polyedron. We may carry on indefinitely this process of shaving down the polyedron, and may thus make the difference between its volume and the volume of the sphere as small as we please; but we cannot make the two volumes absolutely coincide. As the two volumes approach coincidence, the two surfaces also approach coincidence. If, now, S is the surface and V the volume of the sphere, S is the limit of s and V the limit of v, as the number of faces of the circumscribed polyedron is indefinitely increased. Therefore, by III., Theorem of Limits,

$$V = \tfrac{1}{3}RS.$$

42. Corollary I. *The result of this proposition may be formulated,*

$$V = \frac{4}{3}\pi R^3.$$

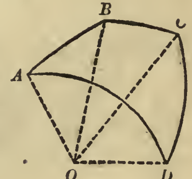
43. Corollary II. *The volumes of two spheres are to each other as the cubes of their radii, or as the cubes of their diameters.*

PROPOSITION XII.—THEOREM.

44. *The volume of a spherical sector is equal to the area of the zone which forms its base multiplied by one-third the radius of the sphere.*

The proof is analogous to the proof of Proposition XI. The form of the circumscribed polyedron is, however, somewhat more complicated, as it will be bounded by a surface made up of plane faces tangent to the zone of the spherical sector, and by two pyramidal faces tangent to, or inscribed in, the two conical surfaces of the spherical sector.

45. *Definition.* A *spherical pyramid* is a solid bounded by a spherical polygon and the planes of the sides of the polygon; as O–$ABCD$. The centre of the sphere is the *vertex* of the pyramid; the spherical polygon is its base.



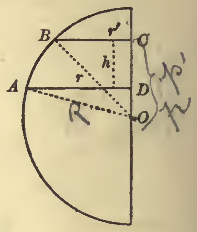
EXERCISE.

Theorem.—The volume of a spherical pyramid is equal to the area of its base multiplied by one-third of the radius of the sphere.

PROPOSITION XIII.—PROBLEM.

46. *To find the volume of a spherical segment.*

Any spherical segment may be obtained from a spherical sector by adding to it, or subtracting from it, cones having as bases the bases of the segment.



For example, let us consider a segment of two bases which does not contain the centre of the sphere. The segment generated by the revolution of $ABCD$ about OC may be obtained by taking the cone generated by OAD from the sum of the cone generated by OBC and the spherical sector generated by OAB.

Call OC p', OD p, DC h, AD r, BC r', and OA R, and the volume of the segment V. Then we have the simple relations

$$h = p' - p,$$
$$r^2 + p^2 = R^2,\ r'^2 + p'^2 = R^2.$$

The area of the zone of the segment is $2\pi R.h$ (Proposition IX., Corollary). Hence

$$V = \tfrac{2}{3}\pi h R^2 + \tfrac{1}{3}\pi p' r'^2 - \tfrac{1}{3}\pi p r^2 \text{ (Proposition XII., and Proposition VII., Corollary I.),}$$
$$V = \tfrac{2}{3}\pi(p' - p)R^2 + \tfrac{1}{3}\pi p'(R^2 - p'^2) - \tfrac{1}{3}\pi p(R^2 - p^2),$$
$$V = (p' - p)\pi R^2 - \tfrac{1}{3}\pi(p'^3 - p^3), \qquad [1]$$

a convenient formula when the distances of the bases of the segment from the centre of the sphere are given.

Another convenient formula can be obtained by introducing in [1] h, r, and r' in place of p and p'. We have

$$V = (p' - p)\frac{\pi}{3}[3R^2 - (p'^2 + p'p + p^2)].$$

Now

$$h^2 = p'^2 - 2p'p + p^2.$$

Hence

$$p'p = \frac{p'^2 + p^2 - h^2}{2},$$

and

$$p'^2 + p'p + p^2 = \tfrac{3}{2}(p'^2 + p^2) - \frac{h^2}{2},$$

$$= \tfrac{3}{2}(R^2 - r'^2 + R^2 - r^2) - \frac{h^2}{2} = 3R^2 - \tfrac{3}{2}(r'^2 + r^2) - \frac{h^2}{2},$$

and we have

$$V = \frac{h\pi}{3}[\tfrac{3}{2}(r'^2 + r^2) + \frac{h^2}{2}],$$

$$V = \frac{h}{2}(\pi r'^2 + \pi r^2) + \tfrac{1}{6}\pi h^3. \qquad [2]$$

This formula is convenient when the areas of the bases of the segment are given, and it may be put into words as follows:

The volume of a spherical segment is equal to the half sum of its bases multiplied by its altitude plus *the volume of a sphere of which that altitude is the diameter.*

This is a Corker!
You bet your
sweet life!
You Don't say so!!

EXERCISES ON BOOK IX.

THEOREMS.

1. GIVE a strict proof of Proposition I. and Proposition IV. for the volumes of cylinder and cone, by showing that the difference between the volumes of the inscribed and circumscribed figures can be decreased at pleasure.

2. Assuming that if a solid has a plane face the area of that face is less than the rest of the surface of the solid, prove, first, that if two convex solids have a plane face in common, and one solid is wholly included by the other, its surface is less than that of the other (*v.* V., 13), and then give a strict proof of Proposition I. and Proposition IV. for the surfaces of cylinder and cone.

3. The volumes of a cone of revolution, a sphere, and a cylinder of revolution are proportional to the numbers 1, 2, 3 if the bases of the cone and cylinder are each equal to a great circle of the sphere, and their altitudes are each equal to a diameter of the sphere.

4. An *equilateral cylinder* (of revolution) is one a section of which through the axis is a square. An *equilateral cone* (of revolution) is one a section of which through the axis is an equilateral triangle. These definitions premised, prove the following theorems:

I. The total area of the equilateral cylinder inscribed in a sphere is a mean proportional between the area of the sphere and the total area of the inscribed equilateral cone. The same is true of the volumes of these three bodies.

II. The total area of the equilateral cylinder circumscribed about a sphere is a mean proportional between the area of the sphere and the total area of the circumscribed equilateral cone. The same is true of the volumes of these three bodies.

5. If h is the altitude of a segment of one base in a sphere whose radius is r, the volume of the segment is equal to $\pi h^2(R - \frac{1}{3}h)$.

6. The volumes of polyedrons circumscribed about the same sphere are proportional to their surfaces.

MISCELLANEOUS EXERCISES

ON THE

GEOMETRY OF SPACE.

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1. A PERPENDICULAR let fall from the middle point of a line upon any plane not cutting the line is equal to one-half the sum of the perpendiculars let fall from the ends of the line upon the same plane.

2. The perpendicular let fall from the point of intersection of the medial lines of a given triangle upon any plane not cutting the triangle is equal to one-third the sum of the perpendiculars from the vertices of the triangle upon the same plane.

3. The perpendicular from the centre of gravity of a tetraedron upon any plane not cutting the tetraedron is equal to one-fourth the sum of the perpendiculars from the vertices of the tetraedron upon the same plane.

4. The volume of a truncated triangular prism is equal to the product of the area of its lower base by the perpendicular upon the lower base let fall from the intersection of the medial lines of the upper base.

5. The volume of a truncated parallelopiped is equal to the product of the area of its lower base by the perpendicular from the centre of the upper base upon the lower base.

6. If $ABCD$ is any tetraedron, and O any point within it, and if the straight lines AO, BO, CO, DO, are produced to meet the faces in the points a, b, c, d, respectively, then

$$\frac{Oa}{Aa}+\frac{Ob}{Bb}+\frac{Oc}{Cc}+\frac{Od}{Dd}=1.$$

7. If the three face angles of the vertical triedral angle of a tetraedron are right angles, and the lengths of the lateral edges are represented by a, b, and c, and of the altitude by p, then

$$\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}.$$

8. If the three face angles of the vertical triedral angle of a tetraedron are right angles, the square of the area of the base is equal to the sum of the squares of the areas of the lateral faces.

9. The perpendicular from the middle point of the diagonal of a rectangular parallelopiped upon a lateral edge bisects the edge, and is equal to one-half of the projection of the diagonal upon the base.

10. A straight line of a given length moves so that its extremities are constantly upon two given perpendicular but non-intersecting straight lines: what is the locus of the middle point of the moving line?

PROBLEMS.

11. To cut a given polyedral angle of four faces by a plane so that the section shall be a parallelogram.

12. To cut a cube by a plane so that the section shall be a regular hexagon.

13. To find the ratio of the volumes generated by a rectangle revolving successively about its two adjacent sides.

SYLLABUS OF PROPOSITIONS

IN

SOLID GEOMETRY.

BOOK VI.

THEOREMS.

Proposition I.

Through any given straight line a plane may be passed, but the line will not determine the plane.

Proposition II.

A plane is determined, 1st, by a straight line and a point without that line; 2d, by two intersecting straight lines; 3d, by three points not in the same straight line; 4th, by two parallel straight lines.

Corollary. The intersection of two planes is a straight line.

Proposition III.

From a given point without a plane one perpendicular to the plane can be drawn, and but one; and the perpendicular is the shortest line that can be drawn from the point to the plane.

Corollary. At a given point in a plane one perpendicular can be erected to the plane, and but one.

Proposition IV.

If a straight line is perpendicular to each of two straight lines at their point of intersection, it is perpendicular to the plane of those lines.

Corollary I. At a given point of a straight line, one plane can be drawn perpendicular to the line, and but one.

Corollary II. Through a given point without a straight line, one plane can be drawn perpendicular to the line, and but one.

Proposition V.

Two lines in space having the same direction are parallel.

Corollary. Two lines parallel to the same line are parallel to each other.

Proposition VI.

If two straight lines are parallel, every plane passed through one of them and not coincident with the plane of the parallels is parallel to the other.

Corollary I. Through any given straight line a plane can be passed parallel to any other given straight line.

Corollary II. Through any given point a plane can be passed parallel to any two given straight lines in space.

Proposition VII.

Planes perpendicular to the same straight line are parallel to each other.

Proposition VIII.

The intersections of two parallel planes with any third plane are parallel.

Proposition IX.

A straight line perpendicular to one of two parallel planes is perpendicular to the other.

Corollary. Through any given point one plane can be passed parallel to a given plane, and but one.

Proposition X.

If two angles, not in the same plane, have their sides respectively parallel and lying in the same direction, they are equal and their planes are parallel.

Proposition XI.

If one of two parallel lines is perpendicular to a plane, the other is also perpendicular to that plane.

Corollary. Two straight lines perpendicular to the same plane are parallel to each other.

Proposition XII.

Two diedral angles are equal if their plane angles are equal.

Proposition XIII.

Two diedral angles are in the same ratio as their plane angles.

Proposition XIV.

If a straight line is perpendicular to a plane, every plane passed through the line is perpendicular to the plane.

Proposition XV.

If two planes are perpendicular to each other, a straight line drawn in one of them, perpendicular to their intersection, is perpendicular to the other.

Corollary I. If two planes are perpendicular to each other, a straight line drawn through any point of their intersection perpendicular to one of the planes will lie in the other.

Corollary II. If two planes are perpendicular, a straight line let fall from any point of one plane perpendicular to the other will lie in the first plane.

Proposition XVI.

If two intersecting planes are each perpendicular to a third plane, their intersection is also perpendicular to that plane.

Proposition XVII.

Through any given straight line a plane can be passed perpendicular to any given plane.

Proposition XVIII.

The projection of a straight line upon a plane is a straight line.

Proposition XIX.

The acute angle which a straight line makes with its own projection upon a plane is the least angle it makes with any line of that plane.

Proposition XX.

The sum of any two face angles of a triedral angle is greater than the third.

Proposition XXI.

The sum of the face angles of any convex polyedral angle is less than four right angles.

Proposition XXII.

If two triedral angles have the three face angles of the one respectively equal to the three face angles of the other, the corresponding diedral angles are equal.

BOOK VII.

THEOREMS.

Proposition I.

The sections of a prism made by parallel planes are equal polygons.

Corollary. Any section of a prism made by a plane parallel to the base is equal to the base.

Proposition II.

The lateral area of a prism is equal to the product of the perimeter of a right section of the prism by a lateral edge.

Corollary. The lateral area of a right prism is equal to the product of the perimeter of its base by its altitude.

Proposition III.

Two prisms are equal, if three faces including a triedral angle of the one are respectively equal to three faces similarly placed including a triedral angle of the other.

Corollary I. Two truncated prisms are equal, if three faces including a triedral angle of the one are respectively equal to three faces similarly placed including a triedral angle of the other.

Corollary II. Two right prisms are equal if they have equal bases and equal altitudes.

Proposition IV.

Any oblique prism is equivalent to a right prism whose base is a right section of the oblique prism, and whose altitude is equal to a lateral edge of the oblique prism.

Proposition V.

Any parallelopiped is equivalent to a rectangular parallelopiped of the same altitude and an equivalent base.

Proposition VI.

The plane passed through two diagonally opposite edges of a parallelopiped divides it into two equivalent triangular prisms.

Proposition VII.

Two rectangular parallelopipeds having equal bases are to each other as their altitudes.

Proposition VIII.

Two rectangular parallelopipeds having equal altitudes are to each other as their bases.

Proposition IX.

Any two rectangular parallelopipeds are to each other as the products of their three dimensions.

Proposition X.

The volume of a rectangular parallelopiped is equal to the product of its three dimensions, the unit of volume being the cube whose edge is the linear unit.

Proposition XI.

The volume of any parallelopiped is equal to the product of the area of its base by its altitude.

Proposition XII.

The volume of a triangular prism is equal to the product of its base by its altitude.

Corollary. The volume of any prism is equal to the product of its base by its altitude.

Proposition XIII.

If a pyramid is cut by a plane parallel to its base, 1st, the edges and the altitude are divided proportionally ; 2d, the section is a polygon similar to the base.

Corollary I. If a pyramid is cut by a plane parallel to its base, the area of the section is to the area of the base as the square of its distance from the vertex is to the square of the altitude of the pyramid.

Corollary II. If two pyramids have equal altitudes and equivalent bases, sections made by planes parallel to their bases and at equal distances from their vertices are equivalent.

Proposition XIV.

The lateral area of a regular pyramid is equal to the product of the perimeter of its base by half its slant height.

Corollary. The lateral area of the frustum of a regular pyramid is equal to the half sum of the perimeters of its bases multiplied by the slant height of the frustum.

Proposition XV.

If the altitude of any given triangular pyramid is divided into equal parts, and through the points of division planes are passed parallel to the base of the pyramid, and on the sections made by these planes as upper bases prisms are described having their edges parallel to an edge of the pyramid and their altitudes equal to one of the equal parts into which the altitude of the pyramid is divided, the total volume of these prisms will approach the volume of the pyramid as its limit as the number of parts into which the altitude of the pyramid is divided is indefinitely increased.

Proposition XVI.

Two triangular pyramids having equivalent bases and equal altitudes are equivalent.

Proposition XVII.

A triangular pyramid is one-third of a triangular prism of the same base and altitude.

Corollary. The volume of a triangular pyramid is equal to one-third of the product of its base by its altitude.

Proposition XVIII.

The volume of any pyramid is equal to one-third of the product of its base by its altitude.

Proposition XIX.

A frustum of a triangular pyramid is equivalent to the sum of three pyramids whose common altitude is the altitude of the frustum, and whose bases are the lower base, the upper base, and a mean proportional between the bases of the frustum.

Corollary. A frustum of any pyramid is equivalent to the sum of three pyramids whose common altitude is the altitude of the frustum, and whose bases are the lower base, the upper base, and a mean proportional between the bases of the frustum.

Proposition XX.

A truncated triangular prism is equivalent to the sum of three pyramids whose common base is the base of the prism and whose vertices are the three vertices of the inclined section.

Proposition XXI.

Only five regular (convex) polyedrons are possible.

BOOK VIII.

THEOREMS.

Proposition I.

Every section of a cylinder made by a plane passing through an element is a parallelogram.

Corollary. Every section of a right cylinder made by a plane perpendicular to its base is a rectangle.

Proposition II.

The bases of a cylinder are equal.

Corollary I. Any two parallel sections of a cylindrical surface are equal.

Corollary II. All the sections of a circular cylinder parallel to its bases are equal circles, and the straight line joining the centres of the bases passes through the centres of all the parallel sections.

Proposition III.

Every section of a cone made by a plane passing through its vertex is a triangle.

Proposition IV.

If the base of a cone is a circle, every section made by a plane parallel to the base is a circle.

Corollary. The axis of a circular cone passes through the centres of all the sections parallel to the base.

Proposition V.

Every section of a sphere made by a plane is a circle.

Corollary I. The axis of a circle on a sphere passes through the centre of the circle.

Corollary II. All great circles of the same sphere are equal.

Corollary III. Every great circle divides the sphere into two equal parts.

Corollary IV. Any two great circles on the same sphere bisect each other.

Corollary V. An arc of a great circle may be drawn through any two given points on the surface of a sphere, and, unless the points are the opposite extremities of a diameter, only one such arc can be drawn.

Corollary VI. An arc of a circle may be drawn through any three given points on the surface of a sphere.

Proposition VI.

All the points in the circumference of a circle on a sphere are equally distant from either of its poles.

Corollary I. All the arcs of great circles drawn from a pole of a circle to points in its circumference are equal.

Corollary II. The polar distance of a great circle is a quadrant.

Corollary III. If a point on the surface of a sphere is at a quadrant's distance from each of two given points of the surface, which are not opposite extremities of a diameter, it is the pole of the great circle passing through them.

Proposition VII.

A plane tangent to a sphere is perpendicular to the radius drawn to the point of contact.

Corollary. A plane perpendicular to a radius of a sphere at its extremity is tangent to the sphere.

Proposition VIII.

The intersection of two spheres is a circle whose plane is perpendicular to the straight line joining their centres, and whose centre is in that line.

Proposition IX.

The angle of two arcs of great circles is equal to the angle of their planes, and is measured by the arc of a great circle described from its vertex as a pole and included between its sides (produced if necessary).

Corollary. All arcs of great circles drawn through the pole of a given great circle are perpendicular to its circumference.

Proposition X.

If the first of two spherical triangles is the polar triangle of the second, then, reciprocally, the second is the polar triangle of the first.

Proposition XI.

In two polar triangles, each angle of one is measured by the supplement of the side lying opposite to it in the other.

Proposition XII.

Two triangles on the same sphere are either equal or symmetrical when two sides and the included angle of one are respectively equal to two sides and the included angle of the other.

Proposition XIII.

Two triangles on the same sphere are either equal or symmetrical when a side and the two adjacent angles of one are respectively equal to a side and the two adjacent angles of the other.

Proposition XIV.

Two triangles on the same sphere are either equal or symmetrical when the three sides of one are respectively equal to the three sides of the other.

Proposition XV.

If two triangles on the same sphere are mutually equiangular, they are mutually equilateral, and are either equal or symmetrical.

Proposition XVI.

Any side of a spherical triangle is less than the sum of the other two.

Proposition XVII.

The sum of the sides of a convex spherical polygon is less than the circumference of a great circle.

Proposition XVIII.

The sum of the angles of a spherical triangle is greater than two, and less than six, right angles.

Proposition XIX.

Two symmetrical spherical triangles are equivalent.

Proposition XX.

If two arcs of great circles intersect on the surface of a hemisphere, the sum of the opposite spherical triangles which they form is equivalent to a lune whose angle is the angle between the arcs in question.

Proposition XXI.

A lune is to the surface of the sphere as the angle of the lune is to four right angles.

Corollary. The area of a lune is expressed by twice its angle, the angular unit being the degree, and the unit of surface the spherical degree.

Proposition XXII.

The area of a spherical triangle is equal to the excess of the sum of its angles over two right angles.

Proposition XXIII.

The shortest line that can be drawn on the surface of a sphere between two points is the arc of a great circle, not greater than a semi-circumference, joining the two points.

BOOK IX.

THEOREMS.

Proposition I.

If a prism whose base is a regular polygon be inscribed in or circumscribed about a given cylinder, its volume will approach the volume of the cylinder as its limit, and its lateral surface will approach the lateral surface of the cylinder as its limit as the number of sides of its base is indefinitely increased.

Proposition II.

The lateral area of a cylinder is equal to the product of the perimeter of a right section of the cylinder by an element of the surface.

Corollary I. The lateral area of a cylinder of revolution is equal to the product of the circumference of its base by its altitude. This may be formulated,

$$S = 2\pi RH.$$

Corollary II. The lateral areas of similar cylinders of revolution are to each other as the squares of their altitudes, or as the squares of the radii of their bases.

Proposition III.

The volume of a cylinder is equal to the product of its base by its altitude.

Corollary I. For a cylinder of revolution this may be formulated,

$$V = \pi R^2 H.$$

Corollary II. The volumes of similar cylinders of revolution are to each other as the cubes of their altitudes, or as the cubes of their radii.

Proposition IV.

If a pyramid be inscribed in or circumscribed about a given cone, its volume will approach the volume of the cone as its limit, and its lateral surface will approach the convex surface of the cone as its limit as the number of faces of the pyramid is indefinitely increased.

Corollary. A frustum of a cone is the limit of the inscribed and circumscribed frustums of pyramids, the number of whose faces is indefinitely increased.

Proposition V.

The lateral area of a cone of revolution is equal to the product of the circumference of its base by half its slant height.

Corollary I. This proposition may be formulated,

$$S = \pi RL.$$

Corollary II. The lateral areas of similar cones of revolution are to each other as the squares of their slant heights, or as the squares of their altitudes, or as the squares of the radii of their bases.

Proposition VI.

The lateral area of a frustum of a cone of revolution is equal to the half sum of the circumferences of its bases multiplied by its slant height.

Corollary I. This proposition may be formulated,

$$S = \pi(R + r)L.$$

Corollary II. The lateral area of a frustum of a cone of revolution is equal to the circumference of a section equidistant from its bases multiplied by its slant height.

Proposition VII.

The volume of any cone is equal to one-third the product of its base by its altitude.

Corollary I. For a cone of revolution this proposition may be formulated,

$$V = \tfrac{1}{3}\pi R^2 H.$$

Corollary II. Similar cones of revolution are to each other as the cubes of their altitudes, or as the cubes of the radii of their bases.

Proposition VIII.

The area of the surface generated by a straight line revolving about an axis in its plane is equal to the projection of the line on the axis multiplied by the circumference of the circle whose radius is the perpendicular erected at the middle of the line and terminated by the axis.

Proposition IX.

The area of a zone is equal to the product of its altitude by the circumference of a great circle.

Corollary. This proposition may be formulated,

$$S = 2\pi RH.$$

Proposition X.

The area of the surface of a sphere is equal to the product of its diameter by the circumference of a great circle.

Corollary I. This may be formulated,

$$S = 2\pi R \times 2R = 4\pi R^2.$$

Hence the surface of a sphere is equivalent to four great circles.

Corollary II. The surfaces of two spheres are to each other as the squares of their diameters, or as the squares of their radii.

Proposition XI.

The volume of a sphere is equal to the area of its surface multiplied by one-third of its radius.

Corollary I. This proposition may be formulated,

$$V = \tfrac{4}{3}\pi R^3.$$

Corollary II. The volumes of two spheres are to each other as the cubes of their radii, or as the cubes of their diameters.

Proposition XII.

The volume of a spherical sector is equal to the area of the zone which forms its base multiplied by one-third the radius of the sphere.

THE END.

Triangle = $\frac{1}{2}$ base x alt.

area of trapezoid = alt. x $\frac{1}{2}$ sum of ll sides

circumf. of c = diameter x 3.1416

area of c = " ^{2} x .7854

or radius^2 x 3.1416

Lateral surf. of cylinder = circum. of base x alt.

Vol. of cylinder = area of base x alt.

surf. of sphere = diam. x circum.

Vol. of sphere = $\frac{4}{3} \pi\ 2\ R^3$

Surf. of cone = $\frac{1}{2}$ slant ht. x circum. of base

Vol. of cone = $\frac{1}{3}$ base x alt.