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A CLARIFICATION AND A NEW PROOF OF
THE CERTAINTY EQUIVALENCE THEOREM

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Simon and Theil's certainty equivalence theorem [9, 10] states that for a certain class of stochastic control models, the optimal first period decision can be obtained by replacing all stochastic variables by their expected values and then finding the optimal decision for the resulting deterministic model. One of the assumptions needed for the theorem to hold is that the decision maker cannot affect the probability distribution of the stochastic elements in the model. In the literature on the theorem, attempts at making this rather informal statement more precise have been misleading. Because the theorem has been widely used in applied studies and has stimulated many theoretical papers, it is important that the assumption in question be clarified.

The main purpose of this paper is to show that the most recent formal statement of the assumption [12, p. 130] is stronger than needed. Indeed, were the conditions given in [12] necessary ones, many applied studies which use the theorem would be in error (e.g., [2, 3]). In order to demonstrate that the certainty equivalence theorem holds under a weaker condition than the one given in [12], we prove the theorem by a method different from Simon and Theil's. Besides showing exactly what formal assumption is needed, our proof serves another purpose--out of it falls a computationally convenient expression for the optimal decision.

In the next section, we review the Theil-Simon model after which we look at Theil's discussion of the above assumption. A new, formal statement of the assumption is presented in Section III together with our proof of the certainty equivalence theorem. Finally, we compare, from the

point of view of computational ease and generality, various representations of the optimal strategy.

II. The Theil-Simon Model

Consider a decision maker (for brevity, an agent) who, in each of T successive periods, has control over an $m_t \times 1$ vector of variables, x_t, $t = 1,\ldots,T$. Following Theil, these will be called instruments. The agent's choice for the instruments influences an $n \times 1$ state or non-controlled vector, y_t, related to the instruments by

$$(2.1) \qquad y_t = \sum_{i=1}^{t} R_{ti}x_i + s_t, \qquad t = 1,\ldots,T.$$

In (2.1), R_{ti} is an $n \times m_i$ matrix of known constants and s_t is an $n \times 1$ random vector with known mean vector and finite variance-covariance matrix. Furthermore, and this is the assumption that this paper clarifies, the decision made by the agent has no effect on the distribution of s_t.

In matrix notation, (2.1) is

$$(2.2) \qquad \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_T \end{pmatrix} = \begin{pmatrix} R_{11} & 0 & \cdots & 0 \\ R_{21} & R_{22} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ R_{T1} & R_{T2} & & R_{TT} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_T \end{pmatrix} + \begin{pmatrix} s_1 \\ s_2 \\ \vdots \\ s_T \end{pmatrix}$$

or

$$(2.3) \qquad y = Rx + s,$$

where y and s are $nT \times 1$, and, if we let $m = \frac{1}{T}\sum_{t=1}^{T} m_t$, R is $nT \times mT$ and x is $mT \times 1$. Most discussions of this model assume that the number

of instruments in each period is a constant; i.e., $m_1 = m_2 = \cdots = m_T$. Letting the number of instruments vary adds no complexity to the problem.

The agent's utility is measured by the scalar

$$w = a'x + b'y + \frac{1}{2}\ (x'Ax + y'By + x'Cy + y'C'x) \tag{2.4}$$

where A, B, C, a, and b are matrices and vectors of known constants, and A and B are symmetric. The agent's goal is to find a strategy or policy for x that maximizes the expected value of w subject to the constraints $y = Rx + s$.

A strategy is a rule giving each x_t as a function of the information the agent has at the end of period t-1. Theil [12, pp. 130-131] delibrately keeps the notion of information vague. For some problems the information available at the end of period t-1 will consist of the realized values of s_1 through s_{t-1}. However, the idea of information is much more general. Any knowledge that causes the decision maker to revise his assessment of s would be included in the information that he has at the beginning of period t. We assume here that all of the information available at the beginning of period t is contained in the realization of a random vector, z_{t-1}. For consistency in notation, we denote the information available at the beginning of period 1 by z_o.

It is of interest to ask whether the certainty equivalence theorem holds if there is information loss. By information loss, we mean that there is some t and $t' \leq t$ such that $z_{t'}$ is not a subvector of z_t. The proof in the next section shows that the theorem holds even with

information loss provided the agent's decisions do not affect the amount of loss. This result is in contrast with Porter [8, p. 114] who, when discussing the theorem, explicitly assumes no information loss; and to Malinvaud [6] who assumes no information loss in order to prove a result related to the certainty equivalence theorem.

We turn now to a close examination of the assumption that the agent cannot affect the distribution of s. Theil presents three proofs of the certainty equivalence theorem [10, 11, 12]. In each succeeding one, his framing of the assumption is longer and more precise. In his first proof, he writes only that "the distribution of s is independent of the instruments" [10]. His second proof enlarges on the last statement:

(2.5) s is independent of the strategy chosen. . . . Note however, that the optimal strategy $\tilde{x}$ is not independent of s; . . . s is independent of the strategy in the sense that its distribution is the same for whatever x [11, p. 512].

While the intention of (2.5) is clear, it can be argued that it is imprecise because "independent" is used in two different senses. <u>Before</u> a strategy is chosen, s is independent of x in the sense that the strategy selected does not affect s. <u>After</u> a strategy is chosen, x is a random vector and x and s are stochastically dependent.

In his most recent proof, Theil is much more precise:

(2.6) The decision maker can in principle make x_{t+1} dependent on s_t, so that there is then a stochastic dependence between s-subvectors and <u>later</u> x-subvectors. This possibility will and, in fact, must be accepted; but we

> shall exclude the possibility that s_t depends on $x_{t'}$ where $t \geq t'$. More precisely: . . . the distribution of the subvector s_t is independent of $x_{t'}$ for $t, t' = 1,\ldots,T$ and $t \geq t'$ [12, p. 130].[1]

While (2.6) may appear to be nothing more than a careful restatement of (2.5), in actuality, it is much stronger and, as our proof shows, stronger than needed to prove the theorem. For (2.6) to hold, either the subvectors for different time periods must be temporally independent or there must be severe restrictions on the set of admissable strategies.[2] To see this, consider a two-period problem in which the only information available at the end of the first period is the realization of s_1 so that a strategy for x_2 is just some function of s_1. When s_1 and s_2 are dependent, there is no particular reason to believe that the certainty equivalence strategy will be such that x_2 and s_2 are independent. In fact, for many applied studies using the theorem, the certainty equivalence strategy violates (2.6). See, e.g., [2, 3].

III. A New Assumption and a New Proof

Informally, the assumption needed to prove the theorem is that the agent cannot affect the probability density function (pdf) of s. In general, the pdf of s at the beginning of period t can depend upon z_{t-1} (information available at the end of period t-1), $x_1,\ldots,x_{t-1}$ (past decisions), and x_t (the decision for period t). To say that the agent cannot affect the pdf of s is to say that the pdf of s conditional on z_{t-1} is independent of $x_1,\ldots,x_{t-1}$ and of x_t. Formally, we require that

(3.1) $$f(s|z_{t-1}, x_1,\ldots,x_t) = f(s|z_{t-1}), \qquad t = 1,\ldots,T.$$

Assumption (3.1) is all that is needed if the only information the agent has at the end of period t-1 is $s_1,\ldots,s_{t-1}$. If he has other information, we also need to assume that he cannot affect the amount of information available to him. Formally,

$$f(z_t,\ldots,z_{T-1}|z_{t-1}, x_1,\ldots,x_t) = f(z_t,\ldots,z_{T-1}|z_{t-1}), \quad t = 1,\ldots,T-1. \tag{3.2}$$

The last assumption says that conditional on information available at the end of period t-1, the information the agent will have at the beginning of any future period is independent of decisions the agent has made $(x_1,\ldots,x_{t-1})$ and the decision he is about to make (x_t).

Taken together, (3.1) and (3.2) say that the agent's past and present decisions affect neither the present distribution of s nor information about s that will be available in the future. Although (3.1) and (3.2) could be combined, it is didactically useful to keep them separate.

We now make use of (3.1) and (3.2) to prove the certainty equivalence theorem. We find both the optimal strategy and the certainty equivalence strategy by backward induction (i.e., by working backward from period T; see [5]) and show that the first period decisions for the two strategies are the same. The following notation will prove useful. Let

$$x^t = \begin{pmatrix} x_1 \\ \vdots \\ x_t \end{pmatrix}. \tag{3.3}$$

Note that x^t can be partitioned as $x^t = (x^{t-1'} ; x_t')'$. To avoid having to write lengthy expressions for conditional expectations, let

(3.4) $$E_t(\circ) = E(\circ | z_{t-1}, x^{t-1}; x_t),$$

where x_t has been separated by a semicolon to stress that it is x_t that the agent must decide upon at the beginning of period t.

It will be convenient to use the constraints to eliminate y from the welfare function. Substituting (2.3) into (2.4), we have

(3.5) $$w = p_T + q_T'x + \frac{1}{2} x'Q_T x,$$

where

(3.6) $$\begin{aligned} p_T &= b's + \frac{1}{2}s'Bs, \\ q_T &= a + R'b + (C + R'B)s, \\ Q_T &= A + R'BR + CR + R'C'. \end{aligned}$$

To find the optimal first period decision, we work backwards from period T. At the beginning of period T only x_T remains undetermined and the agent faces the static problem of choosing x_T to minimize $E_T(w) = E_T(p_T) + E_T(q_T')x + \frac{1}{2}x'Q_T x$. Since p_T and q_T are functions of the random vector s and of known, constant matrices and since, from (3.1), $f(s|z_{T-1}, x^{T-1}; x_T) = f(s|z_{T-1})$, we have

(3.7) $$E_T(w) = E(p_T|z_{T-1}) + E(q_T'|z_{T-1})x + \frac{1}{2} x'Q_T x.$$

The important point to note about (3.7) is that $E(p_T|z_{T-1})$ and $E(q_T|z_{T-1})$ are independent of x^{T-1} and of x_T.

Next, we need $\partial E_T(w)/\partial x_T$. To find this derivative, it is convenient to write $E_T(w)$ as a function of x_T plus terms that are constant at the beginning of period T. Partition q_T and Q_T as

$$(3.8) \qquad q_T = \begin{pmatrix} q_{T1} \\ q_{T2} \end{pmatrix} ; \qquad Q_T = \begin{pmatrix} Q_{T1} & Q_{T2} \\ Q'_{T2} & Q_{T3} \end{pmatrix} ,$$

where q_{T1} is $(mT-m_T) \times 1$, q_{T2} is $m_T \times 1$, Q_{T1} is $(mT-m_T) \times (mT-m_T)$, Q_{T2} is $(mT-m_T) \times m_T$, and Q_{T3} is $m_T \times m_T$. Then

$$(3.9) \qquad E_T(w) = E(p_T|z_{T-1}) + E(q'_{T1}|z_{T-1})x^{T-1} + \frac{1}{2} x^{T-1'} Q_{T1} x^{T-1}$$
$$+[E(q'_{T2}|z_{T-1}) + x^{T-1'} Q_{T2}]x_T + \frac{1}{2} x'_T Q_{T3} x_T,$$

and

$$(3.10) \qquad \frac{\partial E_T(w)}{\partial x_T} = E(q_{T2}|z_{T-1}) + Q'_{T2}x^{T-1} + Q_{T3}x_T.$$

The optimal strategy for x_T, obtained by setting $\partial E_T(w)/\partial x_T = 0$, is

$$(3.11) \qquad \hat{x}_T = -Q_{T3}^{-1}[E(q_{T2}|z_{T-1}) + Q'_{T2}x^{T-1}].$$ [3]

The last expression gives the optimal decision for period T as a function of x^{T-1} and z_{T-1}.

Consider now the decision problem facing the agent at the beginning of period T-1. By substituting (3.11) into (3.5), x_T can be eliminated from the welfare function:

$$(3.12) \qquad w_{T-1} = p_{T-1} + q'_{T-1}x^{T-1} + \frac{1}{2} x^{T-1'} Q_{T-1} x^{T-1},$$

where a subscript has been used on w to indicate that (3.12) is the welfare function facing the agent at the beginning of period T-1 provided he follows the strategy given by (3.11) for x_T; and where

(3.13)
$$p_{T-1} = p_T - q_{T2}'Q_{T3}^{-1}E(q_{T2}|z_{T-1}) + \frac{1}{2} E(q_{T2}'|z_{T-1})Q_{T3}^{-1}E(q_{T2}|z_{T-1}),$$
$$q_{T-1} = q_{T1} - Q_{T2}Q_{T3}^{-1}q_{T2},$$
$$Q_{T-1} = Q_{T1} - Q_{T2}Q_{T3}^{-1}Q_{T2}'.$$

At this point, we need $E_{T-1}(w_{T-1})$. Consider each random vector on the right hand side of (3.13). First, p_T, q_{T1}, and q_{T2} are functions of s; by (3.1), $f(s|z_{T-2}, x^{T-2}; x_{T-1}) = f(s|z_{T-2})$ so that the pdf's of p_T, q_{T1}, and q_{T2} are independent of x^{T-2} and x_{T-1}. Second, $E(q_{T2}|z_{T-1})$ is a function only of z_{T-1}; by (3.2), $f(z_{T-1}|z_{T-2}, x^{T-2}; x_{T-1}) = f(z_{T-1}|z_{T-2})$ so that $E(q_{T2}|z_{T-1})$ is independent of x^{T-2} and x_{T-1}. It follows immediately that $E_{T-1}(p_{T-1}) = E(p_{T-1}|z_{T-2})$, $E_{T-1}(q_{T-1}) = E(q_{T-1}|z_{T-2})$, and

(3.14)
$$E_{T-1}(w_{T-1}) = E(p_{T-1}|z_{T-2}) + E(q_{T-1}'|z_{T-2})x^{T-1} + \frac{1}{2} x^{T-1'}Q_{T-1}x^{T-1}.$$

Note that both $E_{T-1}(w_{T-1})$ and $E_T(w)$ have the same form.

By induction, it is easy to show that $E_{T-2}(w_{T-2}),\ldots, E_1(w_1)$ also have this form. As a preliminary step, for $t = 2,\ldots,T$, define

$$(3.15)\qquad p_{t-1} = p_t - q_{t2}'Q_{t3}^{-1}E(q_{t2}|z_{t-1}) + \frac{1}{2}E(q_{t2}'|z_{t-1})Q_{t3}^{-1}E(q_{t2}|z_{t-1}),$$

$$q_{t-1} = q_{t1} - Q_{t2}Q_{t3}^{-1}q_{t2},$$

$$Q_{t-1} = Q_{t1} - Q_{t2}Q_{t3}^{-1}Q_{t2}',$$

where q_{t1}, q_{t2}, Q_{t1}, Q_{t2}, and Q_{t3} are submatrices of q_t and Q_t [cf. (3.8)]. The partitioning of q_t and Q_t is

$$(3.16)\qquad q_t = \begin{pmatrix} q_{t1} \\ q_{t2} \end{pmatrix} ; \qquad Q_t = \begin{pmatrix} Q_{t1} & Q_{t2} \\ Q_{t2}' & Q_{t3} \end{pmatrix}$$

where q_{t1} is $(m_1+\cdots+m_{t-1}) \times 1$, q_{t2} is $m_t \times 1$, Q_{t1} is $(m_1+\cdots+m_{t-1}) \times (m_1+\cdots+m_{t-1})$, Q_{t2} is $(m_1+\cdots+m_{t-1}) \times m_t$, and Q_{t3} is $m_t \times m_t$.

We now show that for t = 1,..., T-1,

$$(3.17)\qquad w_t = p_t + q_t'x^t + \frac{1}{2} x^{t'} Q_t x^t$$

and

$$(3.18)\qquad E_t(w_t) = E(p_t|z_{t-1}) + E(q_t'|z_{t-1})x^t + \frac{1}{2} x^{t'} Q_t x^t$$

where the subscript on t indicates that w_t is the welfare function facing the agent at the beginning of period t provided he follows the optimal strategy for periods t+1,...,T. Suppose that (3.17) and (3.18) hold for t=k. Using (3.16) to write (3.18) as a function of x^{k-1} and x_k;

and setting $\partial E_k(w_k)/\partial x_k = 0$, we obtain the optimal decision for period k:

$$\hat{x}_k = -Q_{k3}^{-1}[E(q_{k2}|z_{k-1}) + Q_{k2}' x^{k-1}]. \tag{3.19}$$

Next, substituting (3.19) into (3.17), we can write w_{k-1} as

$$w_{k-1} = p_{k-1} + q_{k-1}' x^{k-1} + \frac{1}{2} x^{k-1'} Q_{k-1} x^{k-1}, \tag{3.20}$$

with p_{k-1}, q_{k-1}, and Q_{k-1} defined by (3.16).

Finally, we find $E_{k-1}(w_{k-1})$. By repeated use of (3.15), it is evident that p_{k-1} and q_{k-1} can be written as functions of p_T, q_T, and $E(q_{k2}|z_{k-1}),\ldots, E(q_{T2}|z_{T-1})$. Now p_T and q_T are functions of s and by (3.1), the pdf of s (conditional on z_{k-2}) is independent of x^{k-2} and x_{k-1}. Further, $E(q_{k2}|z_{k-1}),\ldots, E(q_{T2}|z_{T-1})$ are functions of $z_{k-1},\ldots,$ z_{T-1}. By (3.2), the joint pdf of $z_{k-1},\ldots, z_{T-1}$ (conditional on z_{k-2}) is independent of x^{k-2} and x_{k-1}. It follows that (3.18) holds for $t = k-1$. Since we have already shown that (3.17) and (3.18) are true for $t = T-1$, we have shown that they are true for $t = 1, 2,\ldots, T-1$. In particular,

$$E_1(w_1) = E(p_1|z_o) + E(q_1'|z_o)x^1 + \frac{1}{2} x^{1'} Q_1 x^1. \tag{3.21}$$

The optimal decision for the first period, obtained by recognizing that $x^1 = x_1$ and setting $\partial E_1(w_1)/\partial x_1 = 0$, is

$$\hat{x}_1 = -Q_1^{-1} E(q_1|z_o). \tag{3.22}$$

Consider next the certainty equivalence decision. This is the decision obtained by replacing s by $E(s|z_o)$ in (3.6) and maximizing (3.5) under certainty. Specifically, let

$$p_T^* = b'E(s|z_o) + \frac{1}{2} E(s'|z_o) BE(s|z_o), \qquad (3.23)$$

$$q_T^* = a + R'b + (C + R'B)E(s|z_o).$$

Then the certainty equivalence decision is the one that maximizes

$$w^* = p_T^* + (q_T^*)'x + \frac{1}{2} x'Q_T x. \qquad (3.24)$$

Partitioning q_T^* the same as q_T was partitioned in (3.8), and setting $\partial w^*/\partial x_T = 0$, we find that the certainty equivalence decision for period T, say $\tilde{x}_T$, is

$$\tilde{x}_T = -Q_{T3}^{-1}(q_{T2}^* + Q_{T2}'x^{T-1}). \qquad (3.25)$$

At this point, it is not hard to see that the inductive proof used to find the optimal first period decision can be used to find the certainty equivalence first period decision. The result is

$$\tilde{x}_1 = -Q_1^{-1}q_1^*, \qquad (3.26)$$

where q_1^* is defined recursively by

$$q_{t-1}^* = q_{t1}^* - Q_{t2}Q_{t3}^{-1}q_{t2}^*, \qquad t = 2,\ldots,T. \qquad (3.27)$$

Comparing (3.26) and (3.22), we see that to complete the proof of the

certainty equivalence theorem, we need only show that $q_1^* = E(q_1|z_o)$.

Let $D_t = [I \; ; \; -Q_{t2}Q_{t3}^{-1}]$. Then from (3.15), $E(q_1|z_o) = D_2D_3\cdots D_{T-1}D_TE(q_T|z_o)$.

Similarly, from (3.27), $q_1^* = D_2D_3\cdots D_{T-1}D_Tq_T^*$. Since, from (3.6) and (3.23), $q_T^* = E(q_T|z_o)$, we have the desired result that $q_1^* = E(q_1|z_o)$. Thus the first period certainty equivalence decision, $\tilde{x}_1$, equals the optimal first period decision, $\hat{x}_1$, completing the proof.

IV. Concluding remarks

In this section, we compare several representations of the optimal strategy from the point of view of computational convenience and generality.

In our algorithm, finding the optimal first period decision requires inverting the matrices Q_{T3}, $Q_{T-1,3}, \ldots, Q_{13}$ which are of order m_T, $m_{T-1}, \ldots, m_1$. Once these inverses are found, the optimal decisions for periods 2 through T are easily obtained with no further matrix inversions needed. Algorithms with similar computational requirements appear in Chow [4] and Aoki [1]. Aoki and Chow use a formulation of the constraints favored by engineers in which y_t is expressed as a function of y_{t-1} and x_t. Those economists who are used to Theil's formulation as given by (2.2) may find our algorithm more appealing. In contrast to our algorithm, Theil's seems relatively difficult. Finding the first period decision by his method requires inverting Q_T which is of order mT. To find the second period decision requires inverting a matrix of order mT - m_1; in general, the t^{th} period decision uses the inverse of

a matrix of order $mT - (m_1 + \cdots + m_{t-1})$. Panne [7] develops an algorithm in which the t^{th} period decision can be found by inverting t-1 matrices with orders m_1, $m_1 + m_2, \ldots, m_1 + m_2 + \cdots + m_{t-1}$.[4] Combining his algorithm with Theil's would be more efficient than either one alone, with Panne's used for the earlier periods and Theil's for the later ones.

As regards generality, Chow's algorithm allows for delays in observing y_t or in executing decisions. These same features can be incorporated into both our algorithm and Theil's. Delays in observing y_t can be accounted for by defining z_t appropriately, as discussed in Theil [12]. Delays in carrying out decisions can be accounted for by defining x_t as the set of control variables that must be decided upon at time t. Whether those decisions are carried out at time t or at some later time is immaterial.

One small advantage that our algorithm and Theil's have over Chow's is that in his, the s-subvectors for different time periods must be independent or generated by an autoregressive process [4, p. 19]. No such requirement is needed for our algorithm or Theil's although we do require that $E(q_t | z_{t-1})$ be known at the beginning of period t.

To summarize, we have presented a new proof of the certainty equivalence theorem which clarifies an important assumption needed for the theorem to hold. In addition, the proof led to a computationally convenient algorithm for the optimal strategy. As somewhat minor by-products, the proof shows that the number of instruments can vary from period to period and that the agent can lose information provided he cannot control the loss.

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Footnotes

[1]Panne [7] states the assumption in essentially the same way as (2.6).

[2]This was pointed out to me by A. S. Goldberger.

[3]If Q_{T3} is not of full rank, Q_{T3}^{-1} can be interpreted as a generalized inverse. This remark also holds for (3.19), below.

[4]Panne assumes that the number of instruments per period is a constant, m. In his notation the largest matrix to be inverted for the t^{th} period's decision is of order m(t-1). His claim [7, p. 314] that it is or order t-1 appears to be a typographical error.