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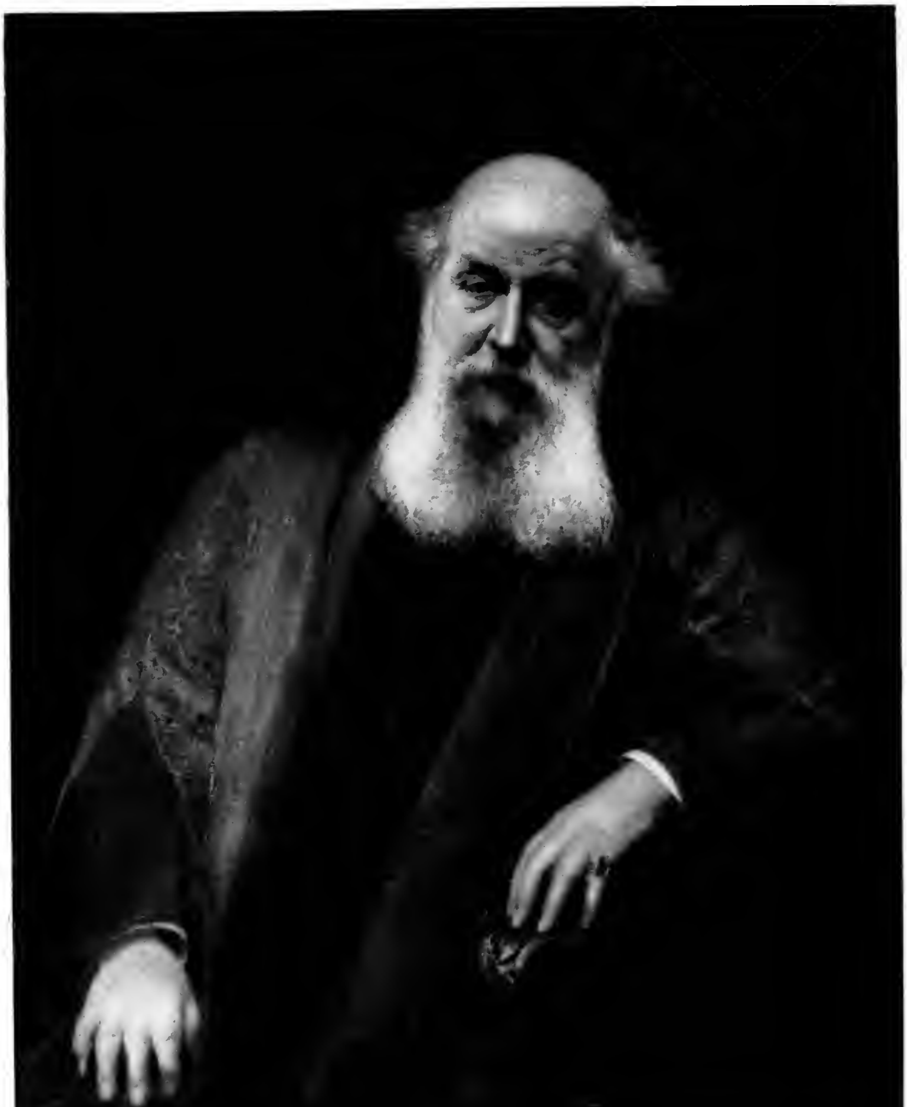

Yours faithfully
J. J. Sylvester

THE COLLECTED MATHEMATICAL PAPERS

OF

JAMES JOSEPH SYLVESTER

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PREFATORY NOTE

THE present volume contains Sylvester's Constructive Theory of Partitions, papers on Binary Matrices, and the Lectures on the Theory of Reciprocants. There is added an Index to the four volumes, and a Biographical Notice of Sylvester. The Mathematical Questions in the *Educational Times* are as yet unedited, but an Index to them is appended here. I have to acknowledge the kindness of Dr J. E. McTaggart, F.B.A., who secured for me the loan of the Essay on Canonical Forms, from the Library of Trinity College, Cambridge, for Vol. I, and that of Mr R. F. Scott, M.A., Master of St John's College, Cambridge, for the use of the volume called *The Laws of Verse,* from which the matter contained in the Appendix to Vol. II was reprinted, who supplied also the Autograph on the Frontispiece of this Volume. To the latter gentleman, as well as to Major P. A. MacMahon, Professor E. B. Elliott and Sir Joseph Larmor, I owe my best thanks for reading through the Biographical Notice. In carrying through the task of editing the Papers, I have, in general, thought it most fitting not to offer any remarks of my own in regard to Sylvester's text, though many times at a loss to know how best to act. In the Appendix to Vol. I I have departed from this rule, giving there an account of Sylvester's chief theorems in regard to determinants. For two other cases the reader may find notes, *Proceedings of the London Mathematical Society,* Vol. IV, Ser. II (1907), pp. 131—135, and Vol. VI (1908), pp. 122—140; these refer respectively to the paper No. 36, p. 229, and to the paper No. 74, p. 452, both in Vol. II of the Reprint. Many corrections of errors in the printing of algebraical formulae have been introduced, though many, it is to be feared, still remain; but no alterations of Sylvester's statements have been made without definite indication, by square brackets or otherwise. To the Readers and Staff of the University Press the very greatest credit and gratitude for their watchful carefulness are assuredly due, many of the corrections in the volumes being due to them.

H. F. BAKER.

June 1912.

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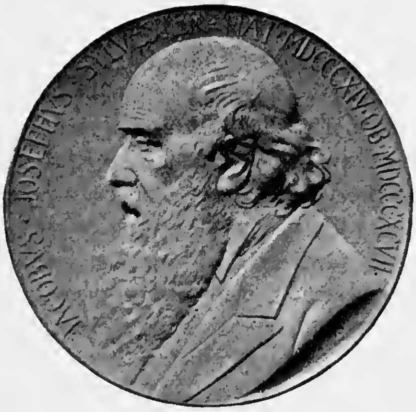

BIOGRAPHICAL NOTICE*.

Lord of himself and blest shall prove
He who can boast "I've lived to-day,
To-morrow let dispensing Jove
Cast o'er the skies what tint he may.

"Sunshine or cloud! the work begun
And ended may his power defy,
He cannot change nor make undone
What once swift Time has hurried by."

Laws of Verse, p. 73 (from Horace).

JAMES JOSEPH SYLVESTER was born in London on 3 September 1814, **1814**
of a family said to have been originally resident in Liverpool. He was among the youngest of several brothers and sisters, and the last to survive. His father, whose name was Abraham Joseph, died while he was young. His eldest brother early in life established himself in America and assumed the name of Sylvester, an example followed by all the brothers.

If we attempt to realise the scientific circumstances of the time of Sylvester's birth by recalling the dates of some of those whose work might

* The chief authority for the outward facts of Sylvester's life used in this record is the Obituary Notice by Major P. A. MacMahon, R.A., F.R.S., *Royal Society Proceedings*, LXIII, 1898, p. ix. There is also an article in the *Dictionary of National Biography*, by Professor E. B. Elliott, F.R.S. and Mr P. E. Matheson, M.A., which gives a list of authorities, and an earlier article by Major MacMahon, *Nature*, 25 March 1897. Other sources of information are referred to in the course of the following.

naturally come before him, either in connexion with his subsequent career at Cambridge, or with his own later investigations, we find it difficult to make a choice. Of Englishmen Henry Cavendish (1731—1810) was dead, Thomas Young (1773—1829) was forty-one, Faraday (1791—1867) was twenty-three, and had just exchanged (in 1813) a bookbinder's workshop for the laboratory of the Royal Institution, Sir John Herschel (1792—1871) was twenty-two, and George Green (1793—1841), who was afterwards to be examined with Sylvester at Cambridge, was twenty-one. Cayley, with whom he was to be so much associated, was born in 1821, and was Senior Wrangler in 1842. The year 1814 was "the year of peace," and was the year in which Poncelet (1788—1867) returned to Paris from the Russian prison in which he had reconstructed the theory of conic sections; Lagrange (1736—1813) had just died, but there were living Laplace (1749—1827), Legendre (1752—1833), Fourier (1768—1830), Ampère (1775—1836), Poisson (1781—1840), Fresnel (1788—1827), Cauchy (1789—1857). J. C. F. Sturm (1803—1855), whose theorem was to have such an importance for Sylvester, was eleven years his senior; Hermite's life extended from 1822 to 1901. In Germany there were Gauss (1777—1855), whose *Disquisitiones Arithmeticae* is dated 1801, Steiner (1796—1863), von Staudt (1798—1867), Jacobi (1804—1851), W. Weber (1804—1891), Dirichlet (1805—1859), Kummer (1810—1893), while Weierstrass was born in 1815; and then there were Helmholtz (1821—1894), Kirchhoff (1824—1886), Riemann (1826—1866), and Clebsch (1833—1872). In Italy Brioschi, who took part in the development of the theory of invariants, was born in 1824 and died in 1897; and the name of Abel (1802—1829) cannot be omitted. All these, and many others, went to form the atmosphere in which Sylvester's life was spent.

Until Sylvester was fifteen years of age he was educated in London—from the age of six to the age of twelve with Mr Neumegen, at Highgate, subsequently, for a year and a half, with Mr Daniell at Islington, then, for five months, at the University of London (afterwards University College), where apparently he met Professor De Morgan, who (except from 1831 to 1835) taught at this institution from 1828 to 1867; for Sylvester speaks in 1840 (I 53) of having been a pupil of De Morgan's. His gift for Mathematics seems undoubtedly to have been apparent at this time; for Mr Neumegen sent him at the age of eleven to be examined in Algebra by Dr Olinthus Gregory, at the Royal Military Academy, Woolwich, and it is recorded that this gentleman was writing to Sylvester's father two years later to enquire for him, with a view to testing his progress in the interval.

1829 In 1829, at the age of fifteen, Sylvester went to Liverpool; here he attended the school of the Royal Institution, residing with aunts. The Institution, it appears, was founded in 1814, largely by the exertions of William Roscoe (1753—1831), and its school in 1819; it must not be confounded with the Liverpool Institute, which grew out of the Mechanics Institute, founded in

1825, by Mr Huskisson. The Head-master at this time was the Rev. T. W. Peile, afterwards Head-master of Repton, and the mathematical master was Mr Marratt. A contemporary at the school was Sir William Leece Drinkwater, afterwards First Deemster, Isle of Man. At this school Sylvester remained less than two years. In February 1830 he was awarded the first prize in the Mathematical School, and was so far beyond the other scholars that he could not be included in any class. While here, also, he was awarded a prize of 500 dollars for solving a question in arrangements, to the great satisfaction of the Contractors of Lotteries in the United States, the question being referred to him by the intervention of his elder brother in New York. At this early period of his life, too, he seems to have suffered for his Jewish faith at the hands of his young contemporaries; possibly this may account for the episode recorded, of his running away from school and sailing to Dublin. Here, with only a few shillings in his pocket, he was accidentally accosted by the Right Hon. R. Keatinge, Judge of the Prerogative Court of Ireland, who, having discovered him to be a first cousin of his wife, entertained him, and sent him back to Liverpool.

The indications were by now sufficient to encourage him to a mathe- **1831**
matical career. After reading for a short time with the Rev. Dr Richard Wilson, sometime Fellow of St John's College, Cambridge, afterwards Head-master of St Peter's Collegiate School, Eaton Square, London, Sylvester was entered* at St John's College on 7 July, as a Sizar, commencing residence on 6 October 1831, when just over seventeen, his tutor being Mr Gwatkin. He resided continuously till the end of the Michaelmas Term, 1833, though he seems to have been seriously ill in June of this year. For two years from the beginning of 1834 his name does not appear as a member of the College, and apparently he was at home on account of illness. In January 1836 he was readmitted, this time as a Pensioner, and resided during the Lent and Michaelmas Terms, being also incapacitated in the intervening term. In January 1837 he underwent his final University examination, the Mathematical Tripos, and was placed second on the list. The first six names of that year were Griffin, St John's; Sylvester, St John's; Brumell, St John's; Green, Gonville and Caius; Gregory, Trinity, and Ellis, Trinity. Of these, George Green, born at Sneinton, near Nottingham, in 1793, was already the author of the famous paper, "An essay on the application of Mathematical Analysis to the theories of Electricity and Magnetism," which was published at Nottingham, by subscription, in 1828. He died in 1841, more than fifty years before Sylvester.

Of the general impression which Sylvester produced upon his contemporaries at Cambridge, it is difficult to judge. It is recorded that he attended the lectures of J. Cumming, Professor of Chemistry in the

* *The Eagle*, the College Magazine, XIX (1897), p. 603. A list of Sylvester's scientific distinctions is given in this place (p. 600).

University from 1815 to 1861, and, as required by College regulations, the Classical lectures of Bushby. We know how keen was his interest in Chemistry many years later in Baltimore (cf. his paper on *The New Atomic Theory*, III 148): and his writings furnish evidence of the pleasure he took in introducing a Classical allusion. When he became Editor of the *Quarterly Journal of Mathematics* in 1855 he secured the printing of a Greek motto on its title-page:

> *ὅ τι οὐσία πρὸς γένεσιν, ἐπιστημὴ πρὸς πίστιν*
> *καὶ διάνοια πρὸς εἰκασίαν ἔστι;*

later on, the *American Journal* under his care also had (IV 298) a Greek motto:

> *πραγμάτων ἔλεγχος οὐ βλεπομένων;*

in his older age the reading and translation of Classical authors was one of his resources.

He was, in later life at least, well acquainted with French, German and Italian, and rejoices (II 563) because these with Latin and English "may happily at the present day be regarded as the common property and inheritance of mathematical Europe." He was also much interested in Music. We are told that at one time he took lessons in singing from Gounod, and was known to sing at entertainments given to working men. "May not Music," he asks (II 419), "be described as the Mathematic of sense, Mathematic as Music of the reason?..." Or again (III 123), "It seems to me that the whole of aesthetic (...) may be regarded as a scheme having four centres, ..., namely Epic, Music, Plastic and Mathematic"; and he advocated "a new method of learning to read on the pianoforte" (III 8).

Of his interest in general literature, and his keen relish for a striking phrase, no reader of his papers needs to be reminded. To his first long paper on Syzygetic Relations, published in the *Philosophical Transactions of the Royal Society* (I 429), he prefixes the words

> How charming is divine philosophy!
> Not harsh and crabbed as dull fools suppose,
> But musical as is Apollo's lute
> And a perpetual feast of nectar'd sweets,
> Where no crude surfeit reigns!

In his paper on Newton's rule, also in the publications of the Royal Society (II 380), he quotes

> Turns them to shapes and gives to airy nothing
> A local habitation and a name.

In his *Constructive Theory of Partitions* (IV 1) he leads off with

> seeming parted,
> But yet a union in partition;

the Second Act, in which the Partitions are transformed by cunning operations performed on the diagrams which represent them, is introduced by

> Naturelly, by composiciouns
> Of anglis, and slie reflexiouns;

as the plot thickens he begins to feel more need of apology, and Act III begins with

> mazes intricate,
> Eccentric, intervolved, yet regular
> Then most, when most irregular they seem;

while, when he comes to the Exodion, and feels that, after fifty-eight pages, direct appeal may have lost its power, he takes refuge in Spenser's fairyland with the lines

> At which he wondred much and gan enquere
> What stately building durst so high extend
> Her lofty towres, unto the starry sphere.

Of his clever sayings we all remember many: "Symmetry, like the grace of an Eastern robe, has not unfrequently to be purchased at the expense of some sacrifice of freedom and rapidity of action" (I 309); or again, in support of the contention, that to say that a proposition is *little to the point* is not to be taken as *demurring to its truth* (II 725), "I should not hesitate to say, if some amiable youth wished to entertain his partner in a quadrille with agreeable conversation, that it would be *little to the point*, according to the German proverb, to regale her with such information as how

> Long are the days of summer-tide
> And tall the towers of Strasburg's fane,

but should be surprised to have it imputed to me on that account that I *demurred to the proposition* of the length of the days in summer, or the height of Strasburg's towers." More direct still (III 9), disclaiming the idea that the simplicity of Peaucellier's linkwork should discredit the difficulty of its discovery, "The idea of the facility of the result, by a natural mental illusion, gets transferred to the process of conception, as if a healthy babe were to be accepted as proof of an easy act of parturition." Some others will be found referred to in the index.

It is also recorded that among the friends of his earlier life was H. T. Buckle, author of the *History of Civilisation*, with whom, in addition to more serious reasons for sympathy, chess playing was a link of friendship.

Whether the many sides of Sylvester's character, indicated by these gleanings from his later life, were much in evidence at Cambridge, we do not know. The intellectual atmosphere of the place at the time was extremely vigorous in some ways. The Philosophical Society was founded in 1819, largely on the initiative of Adam Sedgwick and J. S. Henslow, and obtained a Charter in 1832; its early volumes are evidence of the great width and alertness of scientific interest in Cambridge at this time; papers of George Green were read at the Society in 1832, 1833, 1837 and 1839; James Cumming, whose chemical lectures Sylvester attended, Sir John Herschel, De Morgan, and Whewell are among the early contributors. Sir John Herschel's *Preliminary Discourse on the Study of Natural Philosophy* is dated 1831. The third meeting of the British Association was in Cambridge, on 24 June 1833. Whewell's *History of the Inductive Sciences* was published at Cambridge in 1837, the *Philosophy of the Inductive Sciences* in 1840. But we find* that in 1818 Sedgwick gave up his assistant tutorship, whose duties were mainly those of teaching the mathematical students of Trinity College, on the ground that "as far as the improvement of the mind is considered, I am at this moment doing nothing....I am...very sensibly approximating to that state of fatuity to which we must all come if we remain here long enough." This was before Sylvester's student time, and while mathematics at Cambridge was still suffering, partly from the long consequences of the controversy in regard to Leibniz and Newton, and more immediately from the loss of communication with the mathematicians of the Continent due to the war. Yet Sir John Herschel†, writing in 1833, feels compelled to speak very decidedly of the long-subsisting superiority of foreign mathematics to our own, as he phrases it, and there seems to be no doubt that mathematics, as distinct from physics, was then at a very low ebb in Cambridge, notwithstanding the success of the struggle, about a quarter of a century before, to introduce the analytical methods then in use on the Continent. C. Babbage, in his amusing *Passages from the Life of a Philosopher*, describes how he went (about 1812) to his public tutor to ask the solution of one of his mathematical difficulties and received the answer that it would not be asked in the Senate House, and was of no sort of consequence, with the advice to get up the earlier subjects of the university studies; and how, after two further attempts and similar replies from other teachers, he acquired a distaste for the routine of the place. His connexion with the translation of Lacroix's *Elementary Differential Calculus* (1816), and his association with George Peacock, Sir John Herschel and others in the Analytical Society, is well known; the title proposed by him for a volume of their *Transactions*, "The principles of pure D-ism in opposition to the Dot-age of the University," has often been quoted.

In addition to the better known accounts, there is an echo of what is usually said about Cambridge in this connexion in an Éloge on Sir John Herschel, read at the Royal Astronomical Society, 9 February 1872, by a writer who compares the work of Lagrange on the theory of equations with that of Waring, who was born in the same year, and was Senior Wrangler at Cambridge in 1757. We may add to this the bare titles of two continental

* *Life of Adam Sedgwick*, by J. W. Clark, I, p. 154.

† *Collected Essays*, Longmans, 1857, pp. 30—39.

publications of 1837, the year of Sylvester's Tripos Examination:—C. Lejeune Dirichlet, *Beweis des Satzes, dass jede unbegrenzte arithmetische Progression, deren erstes Glied und Differenz ganze Zahlen ohne gemeinschaftlichen Factor sind, unendlich viele Primzahlen enthält*; E. Kummer, *De aequatione* $x^{2\lambda} + y^{2\lambda} = z^{2\lambda}$ *per numeros integros resolvenda.* Augustus De Morgan, who was fourth Wrangler in 1827, speaking in 1865, at the inaugural meeting of the London Mathematical Society, pronounces that "The Cambridge Examination is nothing but a hard trial of what we must call problems—since they call them so—between the Senior Wrangler that is to be of this present January, and the Senior Wrangler of some three or four years ago. The whole object seems to be to produce problems—or, as I should prefer to call them, hard ten-minute conundrums....It is impossible in such an examination to propose a matter that would take a competent mathematician two or three hours to solve, and for the consideration of which it would be necessary for him to draw his materials from different sources, and see how he can put together his previous knowledge, so as to bring it to bear most effectually on this particular subject." This is the mathematician's criticism of the system then, and, to a large extent, still in vogue. A criticism from another point of view is found in a letter* of Sir Frederick Pollock, written in 1869, to De Morgan: "I believe the most valuable qualities for practical life cannot be got at by any examination—such as steadiness and perseverance....I think a Cambridge education has for its object to make good members of society—not to extend science and make profound mathematicians...." These criticisms appear to agree in one implication, the dominance of the examination in the training offered by the University; and they are necessary to a right appreciation of Sylvester's university life and subsequent work. Accordingly, we do not hear, as frequently we do in the case of young students at continental universities, of Sylvester being led to study for himself the great masters in Mathematics. We find him, in 1839 (I 39), disclaiming a first-hand knowledge of Gauss's works; there is no anecdote, known to me, to put with that he himself tells of Riemann. In a sheet of verses issued by himself, in February 1896—one of many such sheets, I believe—there is a footnote containing the following: "...the hotel on the river at Nuremberg, where I conversed outside with a Berlin bookseller, bound, like myself, for Prague....He told me he was formerly a fellow pupil of Riemann, at the University, and that, one day, after receipt of some numbers of the *Comptes rendus* from Paris, the latter shut himself up for some weeks, and when he returned to the society of his friends, said (referring to newly-published papers of Cauchy), 'This is a new mathematic.'" We find Sylvester, however, writing in 1839 of "the reflexions which Sturm's memorable theorem had originally excited" (I 44), and we know how much of his subsequent thought was given to this matter. Whether he read Sturm's paper of

* W. W. R. Ball, *History of Mathematics at Cambridge*, 1889, p. 113.

23 May 1829 (*Bulletin de Férussac,* XI, 1829, p. 419; *Mémoires par divers Savans,* VI, 1835, pp. 273—318), or in what way he learnt of the theorem, there seems to be no record. It is not referred to in the Report on Analysis by George Peacock, *Cambridge British Association Report,* 1833, pp. 185—352, which deals at length with Fourier's method. Sylvester records (II 655—6) that Sturm told him that the theorem originated in the theory of compound pendulums, but he makes no reference to Sturm's recognition of the application of his principles to certain differential equations of the second order.

Another aspect of Sylvester's time at Cambridge must be referred to. At this time, and indeed until 1871, it was necessary, in order to obtain the Cambridge degree, to subscribe to the Articles of the Church of England; one of the attempts, in 1834, to remove the restriction, is recorded in the *Life* of Adam Sedgwick, already referred to (I 418; Sedgwick writes a letter to the *Times,* 8 April 1834). Sylvester was, in his own subsequent bitter phrase (III 81), one of the first holding "the faith in which the Founder of Christianity was educated" to compete for high honours in the Mathematical Tripos; not only could he not obtain a degree, but he was excluded from the examination for Dr Smith's mathematical prizes, which, founded in 1769, was usually taken by those who had been most successful in the Mathematical Tripos. Most probably, too, had the facts been otherwise, he would have been shortly elected to a Fellowship at St John's College. To obtain a degree he removed to Trinity College, Dublin, from which, it appears, he received in turn the B.A. and the M.A. (1841). He finally received the B.A. degree at Cambridge, 29 February 1872, the M.A. (*honoris causa*) following 25 May of the same year.

1838 In the year succeeding his Tripos examination at Cambridge, he was elected to the Professorship of Natural Philosophy at (what is now) University College, London, and so became a colleague of Professor De Morgan. The list of the supporters of his candidature includes the names of Dr Olinthus Gregory, who had examined him in Algebra when a schoolboy of eleven, of Dr Richard Wilson, who had taught him before his entrance at St John's College, of the Senior Moderator and Senior Examiner in his Tripos examination, of Philip Kelland, of Queens' College, Senior Wrangler in 1834, afterwards Professor at Edinburgh, and of J. W. Colenso, afterwards Bishop of Natal; the two last had been private tutors of Sylvester at some portions of his career at Cambridge. He held the post of Professor of Natural Philosophy for a few years only; Professor G. B. Halsted (*Science,* 11 April 1897) makes a statement suggesting that the examination papers set by him during his tenure of the office are of a nature to indicate that he did not find his subject congenial. During these years he was elected a Fellow of the Royal Society (25 April 1839), at the early age of twenty-five. About this time also an oil-painting of him was made by Patten, of the Royal Scottish Academy, from the recorded description of which it appears that he had dark curly hair and wore spectacles. It has been said that he took his Tripos examination in January 1837; he at once began to publish, in the *Philosophical Magazine* of 1837—38. The first four of his papers are on the analytical development of Fresnel's optical theory of crystals, and on the motion and rest of fluids and rigid bodies; but the papers immediately following contain the dialytic method of elimination, and the expression of Sturm's functions in terms of the roots of the equation, as well as many results afterwards included in the considerable memoir on the theory of the syzygetic relations of two polynomials, published in the *Philosophical Transactions* of 1853.

Leaving University College in the session of 1840—41, he proceeded **1841**
as Professor of Mathematics across the Atlantic, to the University of Virginia, founded in 1824 at Charlottesville, Albemarle Co., where* his colleague, Key, of University College, had previously occupied the chair of Mathematics. Such a considerable change deserved a better fate than befell; in Virginia at this time the question of slavery was a subject of bitter contention, and Sylvester had a horror of slavery. The outcome was his almost immediate return; apparently he had intervened vigorously in a quarrel between two of his students.

On his return from America Sylvester seems to have abandoned mathe- **1844**
matics for a time. In 1844 he accepted the post of Actuary to the Legal and Equitable Life Assurance Company, and threw himself into the work with great energy. He did not accept another teaching post for ten years, until 1854, but seems to have given some private instruction, as it is related† that he had, what was unusual at that time, a lady among his pupils—whose name was afterwards famous—Miss Florence Nightingale. He entered at the Inner Temple 29 July 1846, and was called to the Bar 22 November 1850. He also founded the Law Reversionary Interest Society. It was in 1846 **1846**
that Cayley, who had been Senior Wrangler in 1842, left Cambridge and became a pupil of the famous conveyancer, Mr Christie, entering at Lincoln's Inn. He was already an author, and had in fact entered upon one of the main activities of his life; for in 1845 he had published his fundamental paper "On the Theory of Linear Transformations," in which he discusses Boole's discovery of the invariance of a discriminant. To us, knowing how pregnant with consequences the meeting was, it would be interesting to have some details of the introduction of Cayley and Sylvester; the latter lived, then or soon after, in Lincoln's Inn Fields, and we are told‡ that during the following years they might often be found walking together round the Courts of Lincoln's Inn, discussing no doubt many things but among them assuredly the Theory of Invariants. Perhaps it was particularly of this time that Sylvester was thinking when he described Cayley (I 376) as "habitually

* J. J. Walker, *Proc. Lond. Math. Soc.* XXVIII (1896—97), p. 582.

† *The Eagle*, XIX (1897), p. 597.

‡ Biographical notice of Arthur Cayley, Cayley's *Collected Papers*, Volume VIII.

1846 discoursing pearls and rubies," or when, much later (IV 300), he spoke of "Cayley, who, though younger than myself, is my spiritual progenitor—who first opened my eyes and purged them of dross so that they could see and accept the higher mysteries of our common mathematical faith." It is in a paper published in 1851 (I 246) that we find him saying, "The theorem above enunciated was in part suggested in the course of a conversation with Mr Cayley (to whom I am indebted for my restoration to the enjoyment of mathematical life)"; and Sylvester's productiveness during the latter part of this period is remarkable. In particular there are seven papers whose date of publication is 1850, including the paper on the intersections, contacts and other correlations of two conics, wherein he was on the way to establish the properties of the invariant factors of a determinant, afterwards recognised by Weierstrass; and there are thirteen papers whose date is 1851, including the sketch of a memoir on elimination, transformation and canonical forms, in which the remarkable expression of a cubic surface by five cubes is given, the essay on Canonical Forms, and the paper on the relation between the minor determinants of linearly equivalent quadratic functions, in which the notion of invariant factors is implicit; while in 1852 is dated the first of the papers "On the principles of the Calculus of Forms." Dr Noether remarks* how important for the history of mathematics these years were in other respects; Kummer's memoir, "Ueber die Zerlegung der aus Wurzeln der Einheit gebildeten complexen Zahlen in ihre Primfactoren," appeared in 1847 (*Crelle*, XXXV); Weierstrass's "Beitrag zur Theorie der Abel'schen Integrale" (*Beilage zum Jahresbericht über das Gymnasium zu Braunsberg*) is dated 1849; Riemann's Inaugural-dissertation, "Grundlagen für eine allgemeine Theorie der Functionen einer veränderlichen complexen Grösse," is dated 1851. Referring to the discovery of the Canonical Forms in order to enforce the statement that observation, induction, invention and experimental verification all play a part in mathematical discovery (II 714), Sylvester tells an anecdote which has a personal interest: "I discovered and developed the whole theory of canonical binary forms for odd degrees, and, as far as yet made out, for even degrees too, at one evening sitting, with a decanter of port wine to sustain nature's flagging energies, in a back office in Lincoln's Inn Fields. The work was done, and well done, but at the usual cost of racking thought—a brain on fire, and feet feeling, or feelingless, as if plunged in an ice-pail. *That night we slept no more.*"

To Englishmen, in whose minds the modern developments of physical mathematics are associated with many familiar names, who recall Thomas Young, Faraday, Herschel, George Green, Stokes, Adams, Kelvin, Maxwell, the activity of Cayley and Sylvester may at first sight seem very natural. But in fact the aim of such men as those first named was primarily the coordination of the phenomena of Nature, not the development of any

* Charles Hermite, *Math. Annalen*, LV, p. 343.

mathematical theory. And if we think of such names as those of De Morgan, **1846** Warren, Peacock, their interest perhaps was either systematic or didactic; their endeavours were necessarily largely directed to criticising, and expounding to their countrymen, the proposals of continental mathematicians. But Cayley and Sylvester were in a different position at the time of which we are speaking; neither of them had any official duties as teacher of mathematics; to Cayley, as he afterwards said (in 1883) to the British Association, mathematics was "a tract of beautiful country seen at first in the distance, but which will bear to be rambled through and studied in every detail of hillside and valley, stream, rock, wood, and flower." To him and to Sylvester, Pure Mathematics was an opportunity for unceasing exploration; or, in another figure, a challenge to carve from the rough block a face whose beauty should for all time tell of the joy there was in the making of it; or again, it was the discernment and identification of high peaks of which the climbing might be in the years to come the task of those to whom strenuous labour is a delight and fine air an intoxication. And this spirit was a new one in England at this time, of which we may easily miss the significance. It may therefore help if we quote, without expressing any opinion as to its proportionate justice, the impression of an American observer, Dr Fabian Franklin, who succeeded Sylvester as Professor at Baltimore. Speaking* at the memorial meeting held immediately after Sylvester's death, 2 May 1897, he says of Sylvester, "His influence upon the development of mathematical science rests chiefly, of course, upon his work in the Theory of Invariants. Apart from Sir William Rowan Hamilton's invention and development of Quaternions, this theory is the one great contribution made by British thought to the progress of Pure Mathematics in the present century, or indeed since the days of the contemporaries of Newton. From about the middle of the eighteenth century, until near the middle of the nineteenth, English mathematics was in a condition of something like torpor....And, accordingly, it proved to be the case that in the magnificent extension of the bounds of mathematics which was effected by the continental mathematicians during the first four decades of the present century, England had no share. It is almost literally correct to say that the history of mathematics for about a hundred years might be written without serious defect with English mathematics left entirely out of account.

"That a like statement cannot be made in regard to the past fifty years is due pre-eminently to the genius and labours of three men: Hamilton, Cayley and Sylvester....Not only did other English mathematicians join in the work, but Hermite in France, Aronhold and Clebsch in Germany, Brioschi in Italy, and other continental mathematicians, seized upon the new ideas, and the theory of invariants was for three decades one of the leading objects of mathematical research throughout Europe. It is impossible to apportion

* *Johns Hopkins University Circulars*, June 1897.

between Cayley and Sylvester the honour of the series of brilliant discoveries which marked the early years of the theory of invariants...."

It would not be right to omit reference to another factor in the mathematical life of the time we are dealing with—the influence of George Salmon. At what time Sylvester first became acquainted with him, I have not ascertained; but we know that the theory of the straight lines lying upon a cubic surface was worked out in a correspondence between Cayley and Salmon in 1849. Readers of Salmon are aware of the intimate way in which he followed Sylvester's work, while Sylvester, in his papers, makes frequent reference to Salmon's books. There is a personal letter* from Salmon to Sylvester, of date 1 May 1861, which exhibits the relations of the two men in an interesting light, "...I should be very glad if there was any chance of your preparing an edition of your opuscula. There have been, of course, occasional little statements in your papers requiring verification. Written, as they were, in the very heat of discovery, they are rather to be compared to the hurried bulletins written by a general on the field of battle than to the cool details of the historian. Honestly, however, I don't think there is the least chance of your going back to these former studies. I shall be content to let you off some of these if you will do justice to what you have done on the subject of partitions. I wish you would seriously consider whether it is not a duty everyone owes to Society, when one brings a child into the world, to look to the decent rearing of it. I must say that you have to a reprehensible degree, a cuckoo-like fashion of dropping eggs and not seeming to care what becomes of them. Your procreative instincts ought to be more evenly balanced by such instincts as would inspire greater care of your offspring and more attention to providing for them in life, and producing them to the world in a presentable form.

"Hoping you will meditate on this homily and be the better for it, I remain, yours sincerely, GEO. SALMON."

Salmon himself did a great deal for the rearing of many of Sylvester's offspring, and I suppose it would be hard to estimate how much of Sylvester's and Cayley's reputation in their lifetime was due to his large-minded and genial exposition.

Sylvester himself, in a paper of 1863 (II 337), supplies some answer to such criticisms as this of Salmon's: "in consequence of the large arrears of algebraical and arithmetical speculations waiting in his mind their turn to be called into outward existence, he [the author] is driven to the alternative of leaving the fruits of his meditations to perish...or venturing to produce from time to time such imperfect sketches as the present, calculated to evoke the mental cooperation of his readers...."

1854 It was not until 10 June 1863 that Cayley returned to Cambridge, as Sadlerian Professor of Pure Mathematics. In 1854, Sylvester was a

* Printed in the *Eagle*, the Magazine of St John's College, XXIX (1908), p. 380.

candidate for the Professorship of Mathematics at the Royal Military Academy, Woolwich. At this time he had published the papers now reprinted in Volume I, the Theory of Invariants had an existence firmly established, and Sylvester had an European reputation. But his candidature was unsuccessful. This was in August of 1854. In December of the same year he gave his Probationary lecture on Geometry before the Electors to the Professorship of Geometry in Gresham College, London (II 2). In this he was also unsuccessful. Professor G. B. Halsted has recorded that Sylvester often deplored the time he had lost "fighting the world," and he would feel these disappointments keenly. However, the successful candidate at Woolwich died a few months after being appointed, and Sylvester was again a candidate. A letter on his behalf by Lord Brougham, of date 28 August 1855, speaks of him as my "learned and excellent friend and brother mathematician Mr Sylvester." This time he was elected. He took up the appointment on 15 September 1855, being, for a year, lecturer in Natural Philosophy as well as Professor of Mathematics. There is record of the exact emoluments of the post, a salary of £550, a Government Residence (K Quarters, Woolwich Common), medical attendance and right of pasturage on the Common. The house was a pleasant one, with a good garden, in which he could enjoy the shade of his own walnut tree, we are told, and he was able to entertain his scientific friends. The conversations with Cayley still went on; we hear of them walking to meet one another, Cayley from 2 Stone Buildings and he from his home, their meeting point falling near Lewisham. Sylvester retained this post until July 1870, sometimes justifying, we are led to believe, the original hesitation of the electors in regard to his efficiency as an elementary teacher; there are stories such as that of his housekeeper pursuing him from home carrying his collar and necktie. His publications during this time are, approximately, those reprinted in Volume II.

Sylvester gave seven lectures on the Theory of Partitions at King's College, London, in 1859 (II 119), not published until 1897, and then only from outlines privately circulated at the time of delivery; Capt. (now Sir Andrew) Noble collaborated with him in an important degree in his work on the Theory of Partitions. He wrote the paper on the involution of lines in space considered as axes of rotation (II 236). The long paper on Newton's rule and the invariantive discrimination of the roots of a quintic was published in the *Philosophical Transactions*, 1864 (II 376). His work on the proof of Newton's rule made its appeal in various directions—Todhunter remarks in his *Theory of Equations*, "If we consider the intrinsic beauty of the theorem, the interest which belongs to the rule associated with the great name of Newton, and the long lapse of years during which the reason and extent of that rule remained undiscovered by mathematicians—among whom Maclaurin, Waring and Euler are explicitly included—we must regard

Professor Sylvester's investigations as among the most important contributions made to the Theory of Equations in modern times, justly to be ranked with those of Fourier, Sturm and Cauchy."

1855 Sylvester's outward life also contained points to be remarked. In April 1855 appeared the first number of the *Quarterly Journal of Pure and Applied Mathematics,* edited by J. J. Sylvester, M.A., F.R.S. and N. M. Ferrers, M.A.; this replaced the *Cambridge and Dublin Mathematical Journal* which had first been edited by W. Thomson, M.A. (the late Lord Kelvin) and then by W. Thomson, M.A. and N. M. Ferrers, M.A. In the Preface, the plea is put forward that a more ambitious journal was necessary in view of the growing state of the subject, and might render British mathematicians less dependent on the courtesy of the editors of Foreign journals. Assisted by Stokes, Cayley and Hermite, this joint editorship continued unchanged until June 1877.

1856 In 1856 Sylvester was elected* to the Athenaeum Club, under the special Rule II. The fact is worth recording. Sylvester was never married, and in subsequent years this was the address he frequently appended to his writings.

1859 In 1859 he delivered seven lectures on the Partition of Numbers, at King's College, London, as noted above.

1861 In 1861 he was awarded a Royal Medal by the Royal Society, Cayley having received that honour in 1859.

1863 On 7 Dec.† 1863 he was chosen correspondent in mathematics by the French Academy of Sciences, in place of the great geometer Steiner, who had died in the preceding April. We notice that he had just commenced (in 1861) what was to be a long series of communications to the Academy, and his paper on Involutions of lines in space had been presented to the Academy by M. Chasles (II 236). His closely following paper on the Double Sixes of lines on a Cubic surface (II 242) he himself afterwards (II 451) notes as being an unconscious plagiarism from a paper of Schläfli, which he had read as editor before its publication in the *Quarterly Journal* (Vol. II (1858), p. 116).

1864 His memoir in the *Phil. Trans.* on Newton's rule is of date 1864 (II 376). In 1865 he delivered a lecture on the subject at King's College, London (II 498). A syllabus of this lecture forms the first mathematical paper published by the London Mathematical Society. This Society was inaugurated by a speech of Professor De Morgan 16 Jan. 1865, with "the great aim of the cultivation of pure Mathematics and their most immediate applications." The Society consisted at its formation of twenty-seven members, nearly all of whom were members of University College. Sylvester was elected the second President at the Annual General Meeting held at Burlington

* As I have been able to verify by the courtesy of the Secretary.

† J. J. Walker, *Proc. Lond. Math. Soc.* XXVIII (1896—97), p. 585.

House on 8 November 1866 (in the rooms of the Chemical Society), and held office until November 1868. He served on the Council for many years.

In 1869 Sylvester was President of the Mathematical and Physical **1869**
Section of the British Association at Exeter. He took as the subject of his Presidential address the charge that Huxley had brought against Mathematics, of being the study that knew nothing of observation or induction (II 650), nothing of experiment or causation. An incidental reference in this address to Kant's doctrine of space and time led to a lively controversy in the pages of *Nature*, in which Sylvester's trenchant style and wide range of intellectual alertness may be well seen (II Appendix). Characteristically enough Sylvester reprinted the address, with annotations, and the correspondence in regard to Kant, as an Appendix to his volume on the *Laws of Verse* (Longmans, 1870)—a volume which should be consulted for an appreciation of a side of Sylvester's activity which occupied him to the end of his life.

In 1870 Sylvester retired from his post at Woolwich, in consequence **1870**
of what he regarded as an unfair change in the regulations. As may be seen in the article of G. B. Halsted, above quoted, *Science*, 11 April 1897, and in the Leading Article which appeared in the *Times*, 17 August 1871 (see also Sylvester's own letter to the *Times*, 24 August 1871, and *Nature*, Vol. IV (1871), pp. 324, 326), there was much bitterness as to the question of pension, which was however finally secured to him, if not on the scale desired. For the next few years Sylvester resided near the Athenaeum Club, apparently somewhat undecided as to his course in life. We hear of him as reciting and singing at Penny Readings (cf. his remarks on the utility of these in the *Laws of Verse*, p. 70), and as being a candidate for the London School Board*, and, in *The Gentleman's Magazine* for February 1871, there appears "The Ballad of Sir John de Courcy," translated from the German by "Syzygeticus."

In 1874 Sylvester gave a Friday evening discourse at the Royal Insti- **1874**
tution on Peaucellier's link bar motion. He was then sixty years old, yet, even in the abstract of the lecture which remains (III 7), the vivacity with which he dealt with the matter is very striking. His enthusiasm evoked a wide interest in the subject.

In 1875 the Johns Hopkins University was founded at Baltimore. A **1875**
letter to Sylvester from the celebrated Joseph Henry, of date 25 August 1875, seems to indicate that Sylvester had expressed at least a willingness to share in forming the tone of the young university; the authorities seem to have felt that a Professor of Mathematics and a Professor of Classics could inaugurate the work of an University without expensive buildings or

* Sylvester's election address as candidate for the London School Board for Marylebone in the place of Professor Huxley, with a list of his scientific supporters, is found in *Nature*, 21 March 1872, p. 410.

elaborate apparatus. It was finally agreed that Sylvester should go, securing, besides his travelling expenses, an annual stipend of 5000 dollars "paid in gold." And so, at the age of sixty-one, still full of fire and enthusiasm, as appears abundantly from the work he devoted to the papers here reprinted in Volume III, he again crossed the Atlantic, and did not relinquish the post for eight years, until 1883. It was an experiment in educational method; Sylvester was free to teach whatever he wished in the way he thought best; so far as one can judge from the records, if the object of an University be to light a fire of intellectual interests, it was a triumphant success. His foibles no doubt caused amusement, his faults as a systematic lecturer must have been a sore grief to the students who hoped to carry away note-books of balanced records for future use; but the moral effect of such earnestness as we see him shewing for instance in the papers 19—21 of Volume III (on the true number of irreducible concomitants for the cubic and biquadratic), and in paper 34 (on the system for two cubics), must have been enormous. "His first pupil, his first class," was Professor George Bruce Halsted; he it was who, as recorded in the Commemoration-day Address (III 76) "would have the New Algebra." How the consequence was that Sylvester's brain "took fire," is recorded in the pages of the *American Journal of Mathematics*. Others have left records of his influence and methods. Major MacMahon quotes the impressions of Dr E. W. Davis, Mr A. S. Hathaway and Dr W. P. Durfee. Professor Halsted's Article in *Science* has already been quoted. From Dr Fabian Franklin's long commemorative address*, already referred to, another paragraph may be given: "One of the most striking of Sylvester's achievements was his demonstration and extension of Newton's improved rule concerning the number of the imaginary roots of an algebraic equation....We who knew him well in later years can find no difficulty in understanding the hold this problem had upon him. It was the good fortune of his early hearers in this University to be present when he came into the lecture-room, flushed with the achievement of a somewhat similar task. A certain fundamental theorem in the Theory of Invariants (III 117, 232), which had formed the basis of an important section of Cayley's work, had never been completely demonstrated. The lack of this demonstration had always been, to Sylvester's mind, a most serious blemish in the structure. He had, however, he told us, years ago given up the attempt to find the proof, as hopeless. But, upon coming fresh to the subject in connection with his Baltimore Lectures, he again grappled with the problem, and by a fortunate inspiration, succeeded in solving it. It was with a thrill of sympathetic pleasure that his young hearers thus found themselves in some measure associated with an intellectual feat, by which had been overcome a difficulty that had successfully resisted assault for a quarter of a century."

* *Johns Hopkins University Circulars*, June 1897.

The same writer gives an anecdote illustrating another side of the picture, which may be repeated here. "The reading of the Rosalind poem at the Peabody Institute was the occasion of an amusing exhibition of absence of mind. The poem consisted of no less than 400 lines, all rhyming with the name Rosalind (the long and short sound of i both being allowed). The audience quite filled the hall, and expected to find much interest or amusement in listening to this unique experiment in verse. But Professor Sylvester had found it necessary to write a large number of explanatory footnotes, and he announced that in order not to interrupt the poem he would read the footnotes in a body, first. Nearly every footnote suggested some additional extempore remark, and the reader was so interested in each one that he was not in the least aware of the flight of time, or of the amusement of the audience. When he had dispatched the last of the notes, he looked up at the clock, and was horrified to find that he had kept the audience an hour and a half before beginning to read the poem they had come to hear. The astonishment on his face was answered by a burst of good-humoured laughter from the audience; and then, after begging all his hearers to feel at perfect liberty to leave if they had engagements, he read the Rosalind poem." It may be noted here that it was at Baltimore he wrote "Spring's Début, a Town Idyll," two centuries of lines all rhyming with "Winn." (January 1880.)

Sylvester's own account of his life at Baltimore, and many other matters, are sufficiently given in the Commemoration-day Address, 22 February 1877 (III 72); it is not necessary to dwell on this further here.

In 1878 appeared the first volume of the *American Journal of Mathematics* established by the University under Sylvester's care. His first paper is a long account of the application of the new atomic theory to the graphical representation of the concomitants of binary quantics (III 148). **1878**

In 1880 he was awarded by the Royal Society the highest honour possible, the Copley Medal; on 11 June 1880, he was elected Honorary Fellow of his old College of St John at Cambridge, Benjamin Hall Kennedy, the famous schoolmaster and Greek scholar, being elected on the same day. Their portraits are now both preserved in the College. **1880**

It is to this period of his life we must refer also the beginning of his investigations in regard to matrices, especially binary matrices. He says (IV 209) "my memoir on Tchebycheff's method concerning the totality of prime numbers within certain limits, was the indirect cause of turning my attention to the subject, as (through the systems of difference equations therein employed to contract Tchebycheff's limits) I was led to the discovery of the properties of the latent roots of matrices, and had made considerable progress in developing the theory of matrices considered as quantities, when on writing to Professor Cayley upon the subject he referred me to [his own] memoir." Here also, in the interesting communications to the Mathematical

Club reprinted in the *Johns Hopkins University Circulars*, arose a new interest in developing the Theory of Partitions, which issued in the *Constructive Theory of Partitions* (IV 1—83) printed in the *American Journal* (1883). In the course of the year 1883 the University of Oxford conferred upon Sylvester the honorary degree of D.C.L.; and in December of that year, soon after his sixty-ninth birthday, his great distinction was recognised further in the same University by his election to succeed the illustrious H. J. S. Smith as occupant of the chair of Savilian Professor of Geometry. The Professorship had been founded in 1619 by Sir Henry Savile, Warden of Merton College, the first professor being obtained by promoting Henry Briggs from the post which Sylvester had vainly sought in 1854, that of Gresham Professor of Geometry in London, so that, as Mr Rouse Ball remarks, Briggs held in succession the two earliest chairs of mathematics that were founded in England—the college founded by Sir Thomas Gresham having been opened in 1596. Other holders of the Savilian chair were John Wallis, 1649, and Edmund Halley, 1704. The companion chair at Oxford, of Savilian Professorship of Astronomy, was held from 1870 to 1893 by the Rev. Charles Pritchard, who was also an alumnus at St John's College, Cambridge. These two were now to be again members of the same house, as Fellows of New College.

The election of Sylvester to Oxford was a matter of importance at Baltimore. On 20 December 1883, a goodbye meeting was held in Hopkins' Hall, Baltimore, by invitation of the President, the guests including Mr Matthew Arnold, Professor Newcomb and others. The following address was agreed to, in Professor Sylvester's presence*.

"The teachers of the Johns Hopkins University, in bidding farewell to their illustrious colleague, Professor Sylvester, desire to give united expression to their appreciation of the eminent services he has rendered the University from the beginning of its actual work. To the new foundation he brought the assured renown of one of the great mathematical names of our day, and by his presence alone made Baltimore a great center of mathematical research.

"To the work of his own department he brought an energy and a devotion that have quickened and informed mathematical study not only in America, but all over the world; to the workers of the University, whether within his own field or without, the example of reverent love of truth and of knowledge for its own sake, the example of a life consecrated to the highest intellectual aims. To the presence, the work, the example of such a master as Professor Sylvester, the teachers of the Johns Hopkins University all owe, each in his own measure, guidance, help, inspiration; and in grateful recognition of all that he has done for them and through them for the University, they wish for him a long and happy continuance of his work in his native land, for

* *Johns Hopkins University Circulars*, January 1884, p. 31.

themselves the power of transmitting to others that reverence for the ideal which he has done so much to make the dominant characteristic of this University."

And thus at length, crowned with the gratitude of his American colleagues, **1884**
Sylvester was acknowledged in one of the two ancient English Universities, though not his own. And to this, at the age of seventy years, he did not come without something new to say! On 12 December 1885, he delivered an Inaugural lecture, On the Method of Reciprocants (IV 278), that is of functions of differential coefficients whose form is unaltered by certain linear transformations of the variables. This he followed up by a course of lectures which, as finally edited, extend over more than two hundred pages of the present Reprint. The matter evidently appealed to him as a generalisation of the theory of concomitants, and he worked hard and enthusiastically at the relations of the two theories, gathering round him a school of advanced students. This was the last great continent of thought to be won by him, though he wrote, in 1886, for the centenary volume of "the leading Mathematical Journal in the world," *Crelle's Journal*, a paper on the so-called Tschirnhausen Transformation, which he ascribed to the inspiration of Bring (1786) (IV 531), and a paper on a funicular solution of Buffon's "problem of the needle" in 1890 (IV 663), besides other papers. In the Theory of Reciprocants he had been anticipated in detail by Halphen (*Thèse*, 1878), as he acknowledges. The general idea of differential invariants had been already formulated by Sophus Lie (see his paper on Differential Invariants, *Math. Ann.* XXIV (1884) in which he states that his investigations go back to 1869—72), as an application of his theory of Continuous Groups; to this Sylvester paid but scant attention. On the other hand it may be recalled that Sylvester had himself in cooperation with Cayley long before stated and frequently employed the principle of infinitesimal transformations, and in his first paper on Schwarzian Derivatives (IV 252) he employs the idea of "extended" infinitesimal variations without remark.

One striking note in his Inaugural address at Oxford is the fulness of his references to his colleagues in mathematical work—and of these, what he said about Hammond, fully borne out as it was by the help he gave in the Theory of Reciprocants, seems worthy of being recalled: "I should not do justice to my feelings if I did not acknowledge my deep obligations to Mr Hammond for the assistance which he has rendered me, not only in preparing this lecture which you have listened to with such exemplary patience, but in developing the theory;...saving only our Cayley (...) there is no one I can think of with whom I ever have conversed, from my intercourse with whom I have derived more benefit..." (IV 300)*.

* Another worker to whom he referred in warm terms was Arthur Buchheim. It was his melancholy duty a few years later to write an Obituary Notice of this distinguished young mathematician, who died at the age of twenty-nine. *Nature*, 27 September 1888, p. 515.

1887 In 1887 the Council of the London Mathematical Society made the second award of the De Morgan medal to Sylvester, the first award (in 1884) having been made to Cayley.

1889 In 1889, at the request of a few College friends at Cambridge and elsewhere, he sat to A. E. Emslie for an oil-painting, now hanging in the Hall of St John's College, which was exhibited in the Academy of that year*. It is stated to be a good portrait, though, as he himself writes (*Eagle*, Vol. XIX, 1897, p. 604), "I was in much trouble at that time...and could scarcely keep awake from the effect of the light on my wearied eyes." This portrait is reproduced at the commencement of the present volume. A copy of it is at New College, Oxford. An oil-painting by Patten, made when he was twenty-six, has already been referred to. An engraving by G. J. Stodart, from a photograph by Messrs I. Stilliard & Co., Oxford, appeared in *Nature*, accompanying an appreciation by Cayley (*Nature*, Vol. XXXIX, 1889; Cayley's *Collected Papers*, XIII, p. 43 gives the appreciation); he himself is said to have much prized a particular photograph taken at Venice. On the occasion of his leaving Baltimore a medal was struck in his honour, of which an exemplar is in the library of St John's College, Cambridge, giving in profile an idea of powerful features. Another medal, struck shortly after his death, is now awarded triennially by the Royal Society of London, for the encouragement of Mathematical Research. This also is a profile with the same impression of strength. It is one side of this medal which is reproduced at the beginning of this Notice (p. xv).

1890 On 10 June 1890 he was awarded the Honorary Degree of Sc.D. by the University of Cambridge. Honorary degrees were conferred on this occasion upon Benjamin Jowett, Henry Parry Liddon, Andrew Clark, Jonathan Hutchinson, George Richmond, John Evans, James Joseph Sylvester and Alexander John Ellis. The speech delivered upon Sylvester by the Public Orator, with his own footnotes, is as follows (*Orationes et Epistolae Cantabrigienses* (1876—1909), Macmillan, 1910, p. 83):

"Plus quam tres et quinquaginta anni interfuerunt, ex quo Academiae nostrae inter silvas adulescens quidem errabat, populi sacri antiquissima stirpe oriundus, cuius maiores ultimi, primum Chaldaeorum in campis, deinde Palestinae in collibus, caeli nocturni stellas innumerabiles, prolis futurae velut imaginem referentes†, non sine reverentia quadam suspiciebant. Ipse numerorum peritia praeclarus, primum inter Londinienses Academiae nostrae studia praecipua ingenii sui lumine illustrabat. Postea trans aequor Atlanticum plus quam semel honorifice vocatus, fratribus nostris transmarinis doctrinae mathematicae facem praeferebat‡. Nuper professoris insignis in locum electus, et Britanniae non sine laude redditus, in Academia Oxoniensi

* Graves' *Catalogue of the Royal Academy*, 1769—1904.

† *Genesis*, xv. 5.

‡ University of Virginia, 1841—45; Johns Hopkins University, 1877—83.

scientiae flammam indies clariorem excitat*. Ubicumque incedit, exemplo suo nova studia semper accendit. Sive numerorum *θεωρίαν* explicat, sive Geometriae recentioris terminos extendit, sive regni sui velut in puro caelo regiones prius inexploratas pererrat, scientiae suae inter principes ubique conspicitur. Nonnulla quae Newtonus noster, quae Fresnelius, Iacobius, Sturmius, alii, imperfecta reliquerunt, Sylvester noster aut elegantius explicavit, aut argumentis veris comprobavit. Quam parvis ab initiis argumenta quam magna evolvit; quotiens res prius abditas exprimere conatus, sermonem nostrum ditavit, et nova rerum nomina audacter protulit†! Arte quali numerorum leges non modo poëtis antiquis interpretandis sed etiam carminibus novis pangendis accommodat‡! Neque surdis canit, sed 'respondent omnia silvae§,' si quando, inter rerum graviorum curas, aevi prioris pastores aemulatus,

'Silvestrem tenui musam meditatur avena‖.'

Duco ad vos Collegii Divi Ioannis Socium, trium simul Academiarum Senatorem, quattuor deinceps Academiarum Professorem, *Iacobum Iosephum Sylvester.*"

During his residence at Oxford he founded the Oxford Mathematical Society. "Members of that Society, even more perhaps than the attendants at his formal lectures, have been impressed and excited to emulation as they have seen his always commanding face grow handsome with enthusiasm, and his eyes flash out irresistible fascination, while he expounded his latest discovery or brilliant anticipation," writes the *Oxford Magazine* (5 May 1897). From the same source we gather that, "despondent over his lecturing work he undoubtedly was, and the feeling of discouragement grew upon him." In 1893 his eyesight began to be a serious trouble to him, and in 1894 he applied **1893**
for leave to resign the active duties of his chair. After that he lived mainly in London or at Tunbridge Wells, sad and dejected because his mathematical power was failing. About August 1896 a revival of energy took place and **1896**
he worked at the theory of Compound Partitions, and the Goldbach-Euler conjecture of the expression of every even number as a sum of two primes. He was present at a meeting of the London Mathematical Society on 14 January 1897, and spoke at some length of his work, answering questions put to him in regard to it. On 12 February he sent a paper, on the number of fractions in their lowest terms that can be formed with limited integers, to the editor of the *Messenger of Mathematics,* and corrected the proofs about the end of the month (iv 742). At the beginning of March, he had a paralytic seizure **1897**
while working in his rooms at Hertford Street, Mayfair. He never spoke again, and died 15 March 1897. He was buried with simple ceremonial at

* Succeeded H. J. S. Smith as Savilian Professor, 1883—97.

† Prof. Cayley in *Nature*, 3 Jan. 1889.

‡ *The Laws of Verse*, 1870; *Eagle*, xiv 251, xv 251, xix 601 f., 604.

§ Virgil, *Ecl.* x 8. ‖ *ib. Ecl.* i 2.

the Jewish Cemetery at Dalston on March 19, the Royal Society, the London Mathematical Society, and New College, Oxford, being represented (*Nature*, 25 March 1897).

One rises from the task of editing Sylvester's mathematical writings for the Press, with a feeling that here was a great personality as well as a remarkable mathematician, wide and accurate in thought, deep and sensitive in feeling, and inspired with a great faith in things spiritual. "...is a very great genius," he is reported to have said when pressed on one occasion, "I only wish he would stick to mathematics, instead of talking atheism."

Of the detailed relations of his work with that of contemporary writers, especially for the Theory of Equations, Dr M. Noether has written a masterly and easily accessible account (*Math. Annalen*, Bd L, 1898). In his Presidential address to the London Mathematical Society (*Proceedings*, XXVIII, 1896—97) Major MacMahon has given an appreciation of his work on the Theory of Partitions, which should be consulted. Sylvester's long devotion to the Theory of Invariants, in conjunction with Cayley, transforming the whole analysis of Projective Geometry, has left an ineffaceable mark on Mathematics; but in all questions of algebraical form, working more often by divination than by computation, he is wonderful—his theorems in regard to Sturm's Functions, Canonical Forms, and Determinants suggest themselves at once. So general are some of his results that even the recognition of other theorems as particular cases of them may sometimes be difficult, as very distinguished writers have found.

But another aspect of his mathematical work must, I think, be referred to, if only to place in due proportion what has been said already. It would seem that the multiplicity of the ideas which pressed upon Sylvester's mind left him little leisure to read, more than cursorily, the writings of other mathematicians. He gives a proof of the theorem for six points lying upon a conic section, known as Pascal's theorem, by the method of indeterminate coordinates, and no theorem of analytical geometry seems strange to him, but he makes no reference to the philosophical interest of Poncelet's imaginary elements at infinity. He deals with von Staudt's formulae for the mensuration of pyramids, but von Staudt's scheme for dispensing with the notion of length in geometrical theory does not attract him. The ferment and broad conclusions as to the foundations of geometry, surely one of the most important of nineteenth century speculations, stir no echo in his pages. Again, he gives remarkable formulae in the Theory of Numbers, but Kummer's investigations in regard to ideal numbers, and the vast new regions opened by them, even Gauss's consideration of complex integers, he does not speak of. His silence as to Lie's theory of continuous groups has already been remarked; he is also silent as to the theory of systems of linear partial differential equations; and though he gives important results as to the permutations of an assigned number of elements, he does not refer to the question of the algebraic solution of the quintic equation, and writes nothing as to the abstract theory of groups. Most remarkable of all, though he gives, and evidently values, an evaluation of an elliptic integral, and proves, in a wonderful way, by partitions, formulae of theta-functions, the majesty of the new world which we associate with such names as those of Cauchy, Abel and Jacobi, of Riemann and Weierstrass and others, does not greatly stir his longing, so far as his writings declare. Indeed the abstract notion of a function whether for a real, or a complex variable, never occurs in his papers; such a definite instance as Fourier's use of trigonometric series in the Theory of Heat, of 1822, fails to draw him from his combinatorial standpoint; to him the solution of a differential equation is its solution in explicit form; and his formula for the quotity of a partition is an isolated result. For an ordinary man, trained in a country where the importance attached to time examinations tends to discourage the study of all mathematical doctrine, this might be easy to understand; but in Sylvester's case it is very noticeable, and should not be passed over without mention.

Sylvester's position however is secure. As the physicist glories in the interest of his contact with concrete things, so Sylvester loved to mark his progress with definite formulae. He was however before all an abstract thinker, his admiration was ever for intellectual triumphs, his constant worship was of the things of the mind. This it was which seems to have most impressed those who knew him personally. And because of this, his work will endure, according to its value,—mingling with the stream fed by the toil of innumerable men,—of which the issue is as the source. He is of those to whom it is given to renew in us the sanity which is called faith.

H. F. BAKER.

1.

A CONSTRUCTIVE THEORY OF PARTITIONS, ARRANGED IN THREE ACTS, AN INTERACT AND AN EXODION.

[*American Journal of Mathematics*, v. (1882), pp. 251—330; vi. (1884), pp. 334—336.]

Act I. On Partitions Regarded as Entities.

. . . seeming parted,
But yet a union in partition.
Twelfth-night.

(1) In the new method of partitions it is essential to consider a partition as a *definite thing,* which end is attained by regularization of the succession of its parts according to some prescribed law. The simplest law for the purpose is that the arrangement of the parts shall be according to their order of magnitude. A leading idea of the method is that of correspondence between different complete systems of partitions regularized in the manner aforesaid. The perception of the correspondence is in many cases greatly facilitated by means of a graphical method of representation, which also serves *per se* as an instrument of transformation.

(2) The most obvious mode of graphically representing a partition is by means of a network or web formed by two systems of parallel lines or filaments. Any continuous portion of such web will serve to represent a partition, as for example the graph

```
     *    *    *
*    *    *    *    *
*    *    *    *    *
*    *    *    *
     *    *    *
```

will represent the partition 3 5 5 4 3 of 20 by reading off the successive numbers of nodes parallel to the horizontal lines of the web. This, however, is not a regularized partition; the partition will be represented in its regularized form by such a graph as the following:

```
*    *    *    *    *
     *    *    *    *    *
     *    *    *    *
               *    *    *
                    *    *    *
```

which corresponds to the order 5 5 4 3 3, but it may be represented much more advantageously by the figure

```
*   *   *   *   *
*   *   *   *   *
*   *   *   *
*   *   *
*   *   *
```

which is a portion of the web bounded by lines of nodes belonging to the two systems of parallel filaments. Any such portion becomes then subject to the important condition that the two transverse parallel readings will each give a regularized partition, one being in the present example 5 5 4 3 3, and the other 5 5 5 3 2. Any such graph as this will be termed a *regular* partition-graph, and the two partitions which it represents will be said to be conjugate to one another. The mere conception of a regular graph serves at once by effecting an interchange (so to say) between the warp and the woof, through the principle of correspondence, to establish a well-known fundamental theorem of reciprocity. In the last figure, the extent* of (meaning the number of nodes contained by) the uppermost horizontal line or filament is the maximum magnitude of any element (or part) of the partition, and the extent of the first vertical line is the number of the parts. Hence, every regularized partition beginning with i and containing j parts is conjugate to another beginning with j and containing i parts. The content of the graph (that is, the sum of the parts) of the partition is the same in both cases (it will sometimes be convenient to speak of the *partible number* as the content of the elements of the partition). From the above correspondence it is clear that if two complete partition-systems be formed with the same content in one of which the largest part is i and the number of parts j, and in the other the largest part is j and the number of the parts i, the order (that is, the number of partitions) of the first system will be identical with the order of the second: so that calling the content n, it follows that $n-i$ may be decomposed in as many ways into $j-1$ parts as $n-j$ into $i-1$ parts.

(3) This, however, is not the usual nor the more convenient mode of expressing the reciprocity in question. We may, for the two partition systems spoken of, substitute two others of larger inclusion, taking for the first, all partitions of n in which no one part is greater than i, and the number of parts is not greater than j (that is, is j or fewer), and for the second system, one subject to the same conditions as just stated, but with i and j (as before) interchanged: it is obvious that each regularized partition

* *Extent* may be used to denote the number of nodes on a line or column or angle of a graph; *content* the number of nodes in the graph itself; but I have by inadvertence in what follows frequently applied *content* alike to designate areal and linear numerosity.

of one system will be conjugate to one regularized partition of the other system, and accordingly the order of the two systems will be the same*.

(4) When $i=\infty$ it follows from the general theorem of reciprocity last established, that the number of partitions of n into j parts or fewer will be the same as the number of ways of composing n with the integers 1, 2, ... j, and is therefore the coefficient of x^n in the expansion of

$$\frac{1}{1-x\,.\,1-x^2\,...\,1-x^j}.$$

Thus, then, we can at once find the general term in

$$\frac{1}{(1-a)(1-ax)(1-ax^2)\,...},$$

expanded according to ascending powers of a; for, if the above fraction be regarded as the product of an infinite number of infinite series arising from the expansion of the several factors

$$\frac{1}{1-a},\ \frac{1}{1-ax},\ \frac{1}{1-ax^2},\ ...$$

it will readily be seen that the coefficient of $x^n a^j$ will be the number of ways in which n can be resolved into j parts or fewer, that is, by what has been just shown is the coefficient of x^n in

$$\frac{1}{1-x\,.\,1-x^2\,...\,1-x^j};$$

and this being true for all values of n, it follows that the entire coefficient of a^j is the fraction last written developed in ascending powers of x; so that

$$\frac{1}{(1-a)(1-ax)(1-ax^2)\,...}$$
$$=1+\frac{1}{1-x}a+\frac{1}{1-x\,.\,1-x^2}a^2+\frac{1}{1-x\,.\,1-x^2\,.\,1-x^3}a^3\,...$$

as is well known.

The general term in

$$\frac{1}{(1-a)(1-ax)\,...\,(1-ax^i)}$$

is also well known to be

$$\frac{1-x^{i+1}\,.\,1-x^{i+2}\,...\,1-x^{i+j}}{1-x\,.\,1-x^2\,...\,1-x^j}a^j,$$

* The above proof of the theorem of reciprocity is due to Dr Ferrers, the present head of Gonville and Caius College, Cambridge. It possesses the double merit of having set the first example of graphical construction and of putting into salient relief the principle of correspondence, applied to the theory of partitions. It was never made public by its author, but first promulgated by myself in the *Lond. and Edin. Phil. Mag.* for 1853. [Vol. I. of this Reprint, p. 597.]

or in other words, the number of ways of resolving n into j parts none greater than i is the coefficient of x^n in the fraction

$$\frac{1-x^{i+1} \,.\, 1-x^{i+2} \ldots 1-x^{i+j}}{1-x \,.\, 1-x^2 \ldots 1-x^j},$$

which [denoting $1-x^q$ by (q)] is the same as

$$\frac{(1)\,(2)\ldots(i+j)}{(1)\,(2)\ldots(i)\,.\,(1)\,(2)\ldots(j)},$$

and furnishes, if I am not mistaken, Euler's proof of the theorem of reciprocity already established by means of the correspondence of conjugate partitions.

(5) [It may be as well to advert here to the practical method of obtaining the conjugate to a given partition. For this purpose it is only necessary to call a_i the number of parts in the given partition not less than i; a_1, a_2, a_3, ... a_i ... continued to infinity (or which comes to the same thing until i is equal to the maximum part), will be the required conjugate.]

(6) The following very beautiful method of obtaining the general term in question by the constructive method is due to Mr F. Franklin of the Johns Hopkins University* :

He, as it were, interpolates between the theorem to be established in general and the theorem for $i=\infty$, and attaches a definite meaning to the above fraction regarded as a generating function when the factors in the numerator are limited to the first q of them, q being any number not exceeding i, so that in fact the theorem to be proved, according to this view, is only the extreme case of (the last link in the chain to) a new and more general one with which he has enriched the theory of partitions. The method will be most easily understood by means of an example or two: the proof and use to be made of the construction will be given towards the end of the Act.

Let $n=10$, $i=5$, $j=4$.

Write down the indefinite partitions of 10 into 4 or fewer parts, or say rather into 4 parts, among which zeros are admissible: they will be

(1)	10.0.0.0	5.5.0.0
(1)	9.1.0.0	5.4.1.0
(1)	8.2.0.0	5.3.2.0
(1)	8.1.1.0	5.3.1.1
(2)	7.3.0.0	5.2.2.1
(2)	7.2.1.0	4.4.2.0
(1)	7.1.1.1	4.4.1.1
(2)	6.4.0.0	4.3.3.0
(3)	6.3.1.0	4.3.2.1
(3)	6.2.2.0	4.2.2.2
(4)	6.2.1.1	3.3.3.1
		3.3.2.2

* For a vindication of the constructive method applied to this and an allied theorem, see p. [18] *et seq.*

The partitions to which (1) is prefixed are those in which the *first excess*, that is, the excess of the first (the dominant) part over the next is *too great* (meaning greater than i, here 5); those to which (2) is prefixed are those in which after the batch marked with (1) are removed, the second excess, that is, the excess of the first over the third element is "too great"; those to which (3) is prefixed are those in which after the batches marked (1) and (2) are removed, the third excess is "too great," and lastly those (only one as it happens) marked with j (here 4) are those in which, so to say, the *absolute excess* of the dominant, that is its actual value, is "too great," that is, exceeding i (here 5); the partitions that are left over will be the partitions of n (here 10) into 4 parts, none exceeding i (here 5) in magnitude.

It is easy to see from this how to *construct* the partitions which are to be *eliminated* from the indefinite partitions of the n (10) into 4 (j) parts so as to obtain the quaternary partitions in which no part superior to 5 (i) appears. To obtain the first batch we must subtract $i+1$ (6) from n (10) and form the system of indefinite partitions of 4 into four parts, namely:

4 . 0 . 0 . 0
3 . 1 . 0 . 0
2 . 2 . 0 . 0
2 . 1 . 1 . 0
1 . 1 . 1 . 1

and adding to each of these 6 . 0 . 0 . 0 (term-to-term addition) batch (1) will be obtained.

To obtain the second batch, form the quaternary partitions of $n-(i+2)$, that is, 3, namely:

3 . 0 . 0 . 0
2 . 1 . 0 . 0
1 . 1 . 1 . 0

[but omit those in which the first excess is "too great" (greater than i); here there are none such to be omitted] and bring the second element into the first place; thus we shall obtain the system

0 3 0 0
1 2 0 0
1 1 1 0

The *augments* of those obtained by adding 6 . 1 . 0 . 0 to each of them will reproduce batch (2).

Again, form the quaternary partition-system of $n-(i+3)$, rejecting all those (here there are none such) in which the second excess is "too great." We thus obtain

2 0 0 0
1 1 0 0

and now bringing the third element in each of these into the first place so as to obtain

0 2 0 0
0 1 1 0

The augments of these last partitions obtained by adding 6 . 1 . 1 . 0 to each of them will give the third batch, and finally taking the quaternary partition-system to $n-(i+j)$, that is, 1, rejecting (if there should be any such) those in which the third excess is "too great," we obtain 1 . 0 . 0 . 0, and bringing the fourth element to the first place so as to get 0 . 1 . 0 . 0, and adding 6 . 1 . 1 . 1, the fourth batch 6 . 2 . 1 . 1 is reconstructed.

As another example take $n=15$, $i=3$, $j=3$.

The indefinite ternary partitions of 15 are

15.0.0	(1)	9.4.2	(1)
14.1.0	(1)	9.3.3	(1)
13.2.0	(1)	8.7.0	(2)
13.1.1	(1)	8.6.1	(2)
12.3.0	(1)	8.5.2	(2)
12.2.1	(1)	8.4.3	(1)
11.4.0	(1)	7.7.1	(2)
11.3.1	(1)	7.6.2	(2)
11.2.2	(1)	7.5.3	(2)
10.5.0	(1)	7.4.4	(3)
10.4.1	(1)	6.6.3	(3)
10.3.2	(1)	6.5.4	(3)
9.6.0	(2)	5.5.5	(3)
9.5.1	(1)		

There are, of course, no partitions left in which no part exceeds 3, as the maximum content subject to that condition would be only 9.

The partitions marked (1) (2) (3) are those in which the first, second and absolute excess respectively exceed 3.

Firstly, the indefinite ternary partitions of $15-4$ or 11 augmented by 4 . 0 . 0 will obviously reproduce the system of partitions marked (1).

Secondly, taking the indefinite ternary partitions of 10 in which the *first* excess, and those of 9 in which the *second* excess, does not exceed 3, we shall obtain

6.4.0	and 5.2.2
6.3.1	4.4.1
5.5.0	4.3.2
5.4.1	3.3.3
5.3.2	
4.4.2	
4.3.3	

which by *metastasis* become

4.6.0	2.5.2
3.6.1	1.4.4
5.5.0	2.4.3
4.5.1	3.3.3
3.5.2	
4.4.2	
3.4.3	

and adding to each term of these two groups 4.1.0 and 4.1.1 respectively, the systems of partitions marked (2) and (3) respectively result.

(7) It may, I think, be desirable to give here my own construction for the case of repeated partitions, which, having regard to its features of resemblance to the one preceding, it is proper to state preceded it in the date of its discovery and promulgation. The problem which I propose to myself is to construct a system of partitions of a given number into parts limited in number and magnitude, by means of partitions of itself and other numbers into parts limited in number but not in magnitude.

As before, let i be the limit of magnitude, j the number of parts (zeros admissible), and n the partible number; form a square matrix of the jth order in which the diagonal elements are all $i+1$, the elements below the diagonal all of them unity, and those above the diagonal all of them zero, say M_1.

From this matrix construct $M_1, M_2, M_3, \ldots M_j$, such that the lines in M_q (q being any integer from 1 to j inclusive) are the sums of those in M_1, added (term-to-term) q and q together.

Let (r, q) be the rth line in M_q and $[r, q]$ the sum of the numbers which it contains.

Form the complete system of the partitions of $n-[r, q]$ into j parts, and to each such add (term-to-term) (r, q).

In this way, by giving r all possible values we shall obtain a system of partitions of n into j parts corresponding to M_q, which may be called P_q. I say that $P_1 - P_2 + P_3 \ldots + (-)^{j-1} P_j$ will be the complete system of partitions of n into j parts in which one element at least exceeds i; where it is to be observed that having regard to the effect of the − and + signs (which are used here to indicate the addition and subtraction, or say rather the adduction and sub-duction not of numbers but of things), each such partition will occur once and once only; so that calling P the complete system of indefinite partitions of n into j parts, the complete system of partitions of n into j parts in which no part exceeds i in magnitude will be

$$P - P_1 + P_2 \ldots + (-)^j P_j\,*.$$

* It must, however, be understood that the same partition is liable to appear in more than one, and not exclusively in its regularized phase, or as it may be expressed, the regularized partition undergoes *metastasis*.

(8) This construction, which I will illustrate by two examples, proceeds upon the fact which, although confirmed by a multitude of instances, *remains to be proved,* that if $k_1, k_2, \dots k_j$ be any partition of n into j parts and the number of *unequal* parts greater than i be μ, then the number of times in which this partition, in its regular or any other phase, appears in P_q is $\frac{\mu(\mu-1)\dots(\mu-q+1)}{1.2\dots q}$ (interpreted to mean 1 when $q=0$), and consequently its total number of appearances in $P-P_1+P_2\dots$ is $(1-1)^\mu$, that is, is 0.

From this it follows that the total number of partitions of n into j parts none exceeding i in magnitude will be $C-C_1+C_2-\dots$, where C_q is the sum of the number of ways in which the various numbers $n_1, n_2, n_3 \dots$ can be decomposed into j parts, the numbers $n_1, n_2, n_3, \dots$ being n diminished by the sums of the quantities $i+1, i+2, \dots, i+j$ added q and q together; C_q is therefore the coefficient of x^n in $\frac{x^{n-n_1}+x^{n-n_2}+x^{n-n_3}+\dots}{(1-x)(1-x^2)\dots(1-x^j)}$; and consequently the number of partitions of n into j parts none exceeding i in magnitude will be the coefficient of x^n in $\frac{1-x^{i+1}.1-x^{i+2}\dots1-x^{i+j}}{1-x.1-x^2\dots1-x^j}$ as was to be shown.

(9) As a first example let $i=2$, $j=3$, $n=12$, the matrices and the partitions corresponding to their several lines will be as underwritten; the indefinite partitions of the reduced contents, $n-[r, q]$, are written opposite to the respective matrix lines to which they correspond, and their augments, found by adding the line to this partition system, are written immediately under them. The zeros are omitted for the sake of brevity.

3.0.0	9	8.1	7.2	7.1.1	6.3	6.2.1	5.4	5.3.1	5.2.2	4.4.1	4.3.2	3.3.3
	12	11.1	10.2	10.1.1	9.3	9.2.1	8.4	8.3.1	8.2.2	7.4.1	7.3.2	6.3.3
1.3.0	8	7.1	6.2	6.1.1	5.3	5.2.1	4.4	4.3.1	4.2.2	3.3.2		
	9.3	8.4	7.5	7.4.1	6.6	6.5.1	5.7	5.6.1	5.5.2	4.6.2		
1.1.3	7	6.1	5.2	5.1.1	4.3	4.2.1	3.3.1	3.2.2				
	8.1.3	7.2.3	6.3.3	6.2.4	5.4.3	5.3.4	4.4.4	4.3.5				
—												
4.3.0	5	4.1	3.2	3.1.1	2.2.1							
	9.3	8.4	7.5	7.4.1	6.5.1							
4.1.3	4	3.1	2.2	2.1.1								
	8.1.3	7.2.3	6.3.3	6.2.4								
2.4.3	3	2.1	1.1.1									
	5.4.3	4.5.3	3.5.4									
—												
5.4.3	0											
	5.4.3											

In 6.3.3 there are two unlike elements greater than 2; accordingly 6.3.3 occurs 2 times in P_1 and 1 time in P_2.

In 7.3.2 there are again two unlike elements greater than 2, and 7.3.2, 7.2.3 (the metastatic equivalent to the former) are found in P_1 and 7.2.3 in P_2.

Again, in 5.4.3 there are 3 unlike elements greater than 2, and we find

5.4.3	5.3.4	4.3.5	in P_1
5.4.3	4.5.3	3.5.4	„ P_2
5.4.3			„ P_3.

But such terms as 11.1 10.1.1 9.2.1 8.2.2 in which there is only one distinct element greater than 2 are found 1 time only in P_1 and not at all in P_2 or P_3.

As another example let $n = 12$, $i = 4$, $j = 3$, then a similarly constructed table to the foregoing will be as follows, in which, however, all matrices or lines of matrices which have a sum too large to give rise to partition systems are omitted.

5.0.0	7	6.1	5.2	5.1.1	4.3	4.2.1	3.3.1	3.2.2
	12	11.1	10.2	10.1.1	9.3	9.2.1	8.3.1	8.2.2
1.5.0	6	5.1	4.2	4.1.1	3.3	3.2.1	2.2.2	
	7.5	6.6	5.7	5.6.1	4.8	4.7.1	3.7.2	
1.1.5	5	4.1	3.2	3.1.1	2.2.1			
	6.1.5	5.2.5	4.3.5	4.2.6	3.3.6			
—								
6.5.0	1							
	7.5							
6.1.5	0							
	6.1.5							

7.5 and 6.5.1 are the only two partitions of 12 into 3 parts in which there are two unlike parts greater than 4; each of these accordingly is found twice (in one or another phase) in P_1 and once in P_2. Every other partition of 12 into 3 parts in which one of them at least is greater than 4 will be found exclusively and only once in P_1.

(10) The two expansions for $(1 - ax)(1 - ax^2)\dots(1 - ax^i)$ and its reciprocal may readily be obtained from one another by the method of correspondence.

The coefficient of $x^n a^j$ in the former is the number of partitions of n into j *unequal*, and in the latter into j *equal or unequal* parts none greater than i or less than unity. The correspondence to be established has been given by Euler for the case of $i = \infty$ (*Comm. Arith.*, 1849, Tom. I. p. 88), and is probably known for the general case, but as coming strictly within the purview of the essay, seems to deserve mention here.

If $k_1, k_2, k_3, \ldots, k_j$ be a partition of n into j equal or unequal parts written in ascending order, none exceeding i, on adding to it 0, 1, 2 ... $(j-1)$, it becomes a partition of $n+\frac{j^2-j}{2}$ into j parts none exceeding $i+j-1$, and conversely, if $\lambda_1, \lambda_2, \ldots, \lambda_j$ be a partition of $n+\frac{j^2-j}{2}$ into j unequal parts none exceeding $i+j-1$, written in ascending order, on subtracting from it 0, 1, 2 ... $(j-1)$, it becomes a partition of n into equal or unequal (say relatively independent) parts none exceeding i.

Hence the complete system of partitions of n into j unlike parts none exceeding i has a one-to-one correspondence with the complete system of the partitions of $n-\frac{j^2-j}{2}$ into j parts none exceeding $i-j+1$. Consequently the coefficient of a^j in the expansion of $(1-ax)\ldots(1-ax^i)$ may be found from that of a^j in the expansion of its reciprocal by changing i into $i-j+1$ and introducing the factor $x^{\frac{j^2-j}{2}}$.

(11) The expansion of the reciprocal may of course be found algebraically from the multiplication of the expansion which has been given of $\frac{1}{(1-a)(1-ax)\ldots(1-ax^i)}$ by $(1-a)$, or immediately by the correspondence between partitions into an exact number j of parts limited not to exceed i, and partitions into j or fewer parts so limited.

By subtracting a unit from each term of $k_1, k_2, \ldots, k_j$, a partition of n where no k exceeds i, results a partition $q_1, q_2, \ldots q_j$, a partition of $n-j$ where no q exceeds $i-1$. Hence the coefficient of a^j in

$$\frac{1}{1-ax\,.\,1-ax^2\ldots 1-ax^i}$$

may be found from that in

$$\frac{1}{1-a\,.\,1-ax\ldots 1-ax^i}$$

by introducing the factor x^j and changing i into $i-1$, so that choosing for the latter the alternative form

$$\frac{1-x^{j+1}\,.\,1-x^{j+2}\ldots 1-x^{j+i}}{1-x\,.\,1-x^2\ldots 1-x^i},$$

the former becomes

$$\frac{1-x^{j+1}\,.\,1-x^{j+2}\ldots 1-x^{j+i-1}}{1-x\,.\,1-x^2\ldots 1-x^{i-1}}\,x^j,$$

and consequently the coefficient of a^j in $1-ax\,.\,1-ax^2\ldots 1-ax^i$ will be

$$\frac{1-x^{j+1}\,.\,1-x^{j+2}\ldots 1-x^i}{1-x\,.\,1-x^2\ldots 1-x^{i-j}}\,x^{\frac{j^2+j}{2}}.$$

(12) Before quitting this part of the subject it is desirable to make mention of Dr F. Franklin's remarkable method of proving Euler's celebrated expansion of $(1-x)(1-x^2)(1-x^3)\ldots$ *ad inf.* by the method of correspondence. This has been given by Dr Franklin himself in the *Comptes Rendus* of the Institut (1880), and by myself in some detail in the last February Number of the *J. H. U. Circular**. The method is in its essence absolutely independent of graphical considerations, but as it becomes somewhat easier to apprehend by means of graphical description and nomenclature, I shall avail myself here of graphical terminology to express it.

If a regular graph represent a partition with unequal elements, the lines of magnitude must continually increase or decrease. Let the annexed figures be such graphs written in ascending order from above downwards.

```
                 *  *
                 *  *  *
                 *  *  *  *  *                 (A)
                 *  *  *  *  *  *
                 *  *  *  *  *  *  *

      *  *                      *  *  *
(B)   *  *  *                   *  *  *  *              (C)
      *  *  *  *  *  *          *  *  *  *  *  *
      *  *  *  *  *  *  *       *  *  *  *  *  *  *
```

In A and B the graphs may be transformed without altering their content or regularity by removing the nodes at the summit and substituting for them a new slope line at the base. In C the slope line at the base may be removed and made to form a new summit; the graphs so transformed will be as follows:

```
                *  *  *
                *  *  *  *  *                    (A')
                *  *  *  *  *  *  *
                *  *  *  *  *  *  *  *

                                 *  *
                                 *  *  *
(B')   *  *  *                   *  *  *  *            (C')
       *  *  *  *  *  *  *       *  *  *  *  *
       *  *  *  *  *  *  *  *    *  *  *  *  *  *
```

A' and B' may be said to be derived from A, B by a process of contraction, and C' from C by one of protraction.

Contraction could not now be applied to A' and B', nor protraction to C' without destroying the regularity of the graph; but the inverse processes may of course be applied, namely, of protraction to A' and B' and contraction to C', so as to bring back the original graph A, B, C.

In general (but as will be seen not universally), it is obvious that when the number of nodes in the summit is inferior or equal to the number in the base-slope, contraction may be applied, and when superior to that number, protraction: each process alike will alter the number of parts from even to

[* Vol. III. of this Reprint, p. 664.]

odd or from odd to even, so that barring the exceptional cases which remain to be considered where neither protraction nor contraction is feasible, there will be a one-to-one correspondence between the partitions of n into an odd number and the partitions of n into an even number of unrepeated parts; the exceptional cases are those shown below where the summit meets the base-slope line, and contains either the same number or one more than the number of nodes in that line; in which case neither protraction nor contraction will be possible, as seen in the annexed figures which are written in regular order of succession, but may be indefinitely continued:

```
*      * *     * *         * * *         * * *
               * * *       * * * *       * * * *
                                         * * * * *

      * * * *              * * * *
      * * * * *            * * * * *
      * * * * * *          * * * * * *
                           * * * * * * *

            * * * * *
            * * * * * *
            * * * * * * *
            * * * * * * * *
```

for the protraction process which *ought*, for example, according to the general rule, to be applicable to the last of the above graphs, cannot be applied to it, because on removing the nodes in the slope line and laying them on the summit, in the very act of so doing the summit undergoes the loss of a node and is thereby incapacitated to be surmounted by the nodes in the slope, which will have not now a less, but the same number of nodes as itself; and in like manner, in the last graph but one, the nodes in the summit cannot be removed and a slope line be added on containing the same number of nodes without the transformed graph ceasing to be regular, in fact it would take the form

```
      * *
* * * * * *
* * * * * * *
* * * * * * *
```

and so the last graph transformed according to rule [by protraction] would become:

```
* * * *
* * * *
* * * * *
* * * * * *
* * * * * * *
```

which, although regular, would cease to represent a partition into unlike numbers.

The excepted cases then or unconjugate partitions are those where the number of parts being j, the successive parts form one or the other of the two arithmetical series

$$j, j+1, j+2, \ldots 2j-1 \text{ or } j+1, j+2, \ldots 2j,$$

in which cases the contents are $\frac{3j^2-j}{2}$ and $\frac{3j^2+j}{2}$ respectively, and consequently since in the product of $1-x\,.\,1-x^2\,.\,1-x^3\ldots$ the coefficient of x^n is the number of ways of composing n with an even less the number of ways of composing it with an odd number of parts, the product will be completely represented by $\sum_{j=+\infty}^{j=-\infty} (-)^j x^{\frac{3j^2+j}{2}}$ *.

(13) It has been well remarked by Prof. Cayley that barring the unconjugate partitions, the rest really constitute 4 classes, which using c and x to signify contractile and extensile and e and o to signify of-an-even or of-an-odd order, may be denoted by

$$\begin{matrix} c\,.\,e & c\,.\,o \\ x\,.\,e & x\,.\,o. \end{matrix}$$

Hence as each $c\,.\,e$ is conjugate to an x o and *vice versâ*, and each $c\,.\,o$ to an $x\,.\,e$ and *vice versâ*, the theorem established really splits up into two, one affirming that the number of contractile partitions of an odd order is the same as the number of extensile ones of an even order, the other that the number of contractiles of an even is equal to the number of extensiles of an odd order. It might possibly be worth while to investigate the difference between the number of partitions which each set of one couple and the number of partitions which each set of the sub-contrary couple contain : the sets which belong to the same couple and contain the same number of partitions being those *both* of whose characters are dissimilar.

(14) There are one or two other simple cases of correspondence which are interesting, inasmuch as the construction employed to effect the correspondence involves the operations of division and multiplication, which have not occurred previously.

If $$fx = (1-x)(1-x^3)(1-x^5)(1-x^7)(1-x^9)\ldots$$
and $$\phi x = (1+x)(1+x^2)(1+x^3)(1+x^4)(1+x^5)\ldots$$
$$fx\,.\,\phi x = 1,$$
from which we obtain $\phi x = 1/fx$ and $1/\phi x = fx$.

The first of these equations has been noticed by Euler as involving the elegant theorem that a number may be partitioned in as many ways into only-once-occurring odd-or-even integers as into any-number-of-times-occurring only-odd integers.

* Another proof of this theorem, deduced as an immediate algebraical consequence of a more general one, obtained by graphical dissection, will be given in Act 2; and in the Exodion I furnish a purely arithmetical proof by the method of correspondence of Jacobi's series for
$$(1 \pm x^{n-m})(1 \pm x^{n+m})(1-x^{2n})(1 \pm x^{3n-m})(1 \pm x^{3n+m})(1-x^{4n})\ldots$$
(which includes Euler's theorem as a particular case). I prove this theorem in a more extended sense than was probably intended by its immortal author, inasmuch as I regard m and n as absolutely general symbols.

The second, which I think he does not dwell upon, expresses that the difference between the number of partitions with an even number of parts and that of partitions with an odd number of parts of the same number n is the same as the number of partitions of n into exclusively odd [unrepeated] numbers (such difference being in favour of the partitions of even or of odd order, according as the partible number is even or odd).

This latter theorem brings out a point of analogy between repetitional and non-repetitional partition systems which appears to me worthy of notice.

Any one of the former contains a class of what may be termed singular partitions, in the sense that they are their own associates, or more briefly, *self-conjugate* in respect to the Ferrers transformation. Any one system of the latter may also be *said* to contain a set of singular partitions (0 or 1 in number) in the sense of being *unconjugate* in respect to the Franklin process of transformation. Since then in this case the difference between the number of partitions of an odd and those of an even order of the same number is equal to the number (1 or 0) of singular partitions of that number, so we might anticipate as not improbable that the like difference for the repetitional partitions of a number should be equal to the number of singular partitions of that number—and such is actually the case; for it will be shown in a future section that the number of self-conjugate partitions of a number is the same as the number of ways in which it can be composed with odd integers.

(15) The correspondence indicated by the equation $\phi x = 1/fx$ can be established as follows:

Let $2^\lambda . l$, $2^\mu . m$, $2^\nu . n$, ... be any partition of unrepeated general numbers, where l, m, n ... are any odd integers not exceeding unity; and let $k^{[q]}$ in general denote q parts k, then without changing its content the above partition can be converted into $l^{[2^\lambda]}$, $m^{[2^\mu]}$, $n^{[2^\nu]}$, ... which consists exclusively of odd numbers.

It will of course be understood that the original partition may contain any the same odd number as l multiplied by different powers 2^λ, $2^{\lambda'}$, $2^{\lambda''}$ ... of 2, with the sole restriction that the λ, λ', λ'', ... must be all unequal.

Conversely, any such partitions as $l^{[\sigma]}$, $m^{[\tau]}$, $n^{[\upsilon]}$ may be converted back into one and only one partition of the former kind. For there will be one and but one way of resolving σ into the sum of powers of 2 (the zero power not excluded), and supposing σ to be equal to $2^\lambda + 2^{\lambda'} + 2^{\lambda''} + \ldots$, $l^{[\sigma]}$ may be replaced by $2^\lambda l$, $2^{\lambda'} l$, $2^{\lambda''} l$, and the same process of conversion may be simultaneously applied to each of the other products $m^{[\tau]}$, $n^{[\upsilon]}$, ...

Hence each partition of either one kind is conjugate to one of the other, and the number of partitions in the two systems will be the same, as was to be shown.

(16) But we have here another example of the fact that the theory of correspondence reaches far deeper than that of mere numerical congruity with which it is associated as the substance with the shadow. For a correspondence exists of a much more refined nature than that above demonstrated between the two systems, and which, moreover (it is important to notice) does not bring the same individuals into correlation as does the former method.

The partition system made up of unrepeated general numbers may be divided into groups of the first, second, ... ith ... class respectively, those of the ith class containing i distinct sequences of consecutive numbers having no term in common, with the understanding that no two sequences must form part of a single sequence (so that the largest term of one sequence and the smallest one of the next sequence must differ by more than a single unit), and that a single number unpreceded and unfollowed by a consecutive number is to count as a sequence.

The partition system, made up of repeatable odd numbers may, in like manner, be resolved into groups of the 1st, 2nd, ... ith, ... class respectively, those of the ith class containing i distinct numbers; and the new theorem of correspondence is that there is a correlation between the numbers of the ith class of one system and the ith class of the other; so that the number of partitions in a class of the same name must be the same to whichever system it belongs; and thus Euler's theorem becomes a corollary to this deeper-reaching one, obtained from it by *adding together* the number of partitions in all the several classes in the one system and in the other.

(17) As regards the first class, the theorem amounts to the statement that the number of single sequences of consecutive numbers into which n may be resolved is equal to the number of odd factors which n contains; so that if $N = 2^c . l^\lambda . m^\mu . n^\nu \ldots$ where $l, m, n, \ldots$ are odd numbers, N can be represented by $(\lambda+1)(\mu+1)(\nu+1)\ldots$ such sequences; thus, for example, if $N = 15 = 3 . 5$ we have

$$1+2+3+4+5 = 4+5+6 = 7+8 = 15.$$

So
$$30 = 4+5+6+7+8 = 6+7+8+9 = 9+10+11,$$
$$27 = 2+3+4+5+6+7 = 8+9+10 = 13+14,$$
$$45 = 1+2+3+\ldots+9 = 5+6+7+8+9+10$$
$$= 7+8+9+10+11 = 14+15+16 = 22+23.$$

So too if N is a prime number it can only be resolved into the two sequences $\frac{N-1}{2} + \frac{N+1}{2}$ and N. More generally N can be resolved into as many different sets of i distinct sequences as there are solutions in positive integers of the equation $2(x_1y_1 + x_2y_2 + \ldots + x_iy_i) + x_1 + x_2 + \ldots + x_i = N$, of the truth of which remarkable theorem, in its general form, I have for the present only obtained empirical evidence, but may possibly be able to discover the proof in time to annex it in the form of a note at the end, so as not to keep the press waiting*.

(18) The proof for the case of the first class and the mode of establishing the correspondence between the partitions of this class of the two kinds is not far to seek. I use as previously $a^{(b)}$ to signify a repeated b times.

Consider then any sequence of consecutive numbers for the cases where the number of terms is odd and where it is even separately, calling s the sum of the first and last terms, and i the number of terms; where i is odd, so that s is even, the sequence may be replaced by $i^{\left(\frac{s}{2}\right)}$, and where i is even (so that s is odd) by $s^{\left(\frac{i}{2}\right)}$. Hence each partition of the first class of the first kind may be transformed into one of the first class of the second kind.

It is necessary to show the converse of this, which may be done as follows: Let λ^{μ} be any partition of the second kind so that λ is necessarily odd. I say that this must be transformable into one or the other (but not into both) of two sequences, namely, one of λ terms of which the sum of the first and last is 2μ, the other of which the sum of the first and last terms is λ and the number of terms 2μ. The former supposition is admissible if 2μ is equal to or greater than $\lambda + 1$, inadmissible if 2μ is less than $\lambda + 1$. The second supposition is admissible if λ is equal to or greater than $2\mu + 1$, inadmissible if λ is less than $2\mu + 1$.

The two conditions of admissibility coexisting would imply that 2μ is equal to or greater than $2\mu + 2$; the two conditions of inadmissibility the one that 2μ is equal to or less than $\lambda - 1$, the other that λ is equal to or less than $2\mu - 1$, that is, $\lambda - 1$ equal to or less than $2\mu - 2$, which are inconsistent. Hence one of the two transformations is always possible and the other impossible to be effected; which proves the correlation that was to be established. A single example will serve to show that this correspondence is entirely different from that offered by the first and (so to say) grosser method; suppose $N = 15$, then $1.2.3.4.5$ will be a partition of the first kind and will be converted by the new rule into $5.5.5$, whereas, by the former rule, it would be inverted into $1.1.1.3.1.1.1.1.5$, that is, into $1^7.3.5$ belonging to the third class instead of to the first.

(19) I will now pass on to the conjugate theorem corresponding to $fx = 1/\phi x$.

* A complete proof of the general theorem will be given in the 3rd Act.

It may be well here to recall that this identity essentially depends upon the identity $1-x=1/(1+x)(1+x^2)(1+x^4)\ldots$ which, interpreted*, signifies that any number greater than unity may be made up in as many ways with an odd as with an even number of numbers restricted to the geometrical progression 1, 2, 4, 8 .... This may be called, for brevity, a geometric partition. The correspondence to which this points is itself worthy of notice; one mode of establishing it would be to proceed to decompose N into such parts in regular dictionary order—it would easily be seen that each pair of partitions thus deduced would be of contrary *parities*, but it would not be easy, or at all events evident, how to determine at once the conjugate to a given partition by reference to this principle; but if we observe that it is possible to pass from the geometric partitions of n immediately to those of $n+1$ by the addition of a unit to each of the former, and consequently to those of $n+2$ from the partitions of $E\frac{n}{2}, E\frac{n-2}{2}, E\frac{n-4}{2}, \ldots 2, 1$, by an obvious process of doubling and adding complementary units, another rule or law of correspondence, which proves itself as soon as stated (and is not identical in effect with that supplied by the dictionary-order method), looms into the field of vision, than which nothing can be simpler. Hence we may derive a transcendental equation in differences for u_n, the number of geometric partitions (with radix 2) to n, namely, to find the conjugate of any geometric partition, look at its greatest part—if it is repeated add two of them together: if it is unrepeated split it into two equal parts; these processes are obviously reversible, just as in Dr Franklin's method of correspondence for the pentagonal-series-theorem; and the method is equally open to the remark made thereon by Prof. Cayley; that is to say, there will be four classes, extensile even, extensile odd, contractile even and contractile odd, and the number of partitions in any class will be the same as in the class in which both the characters are reversed.

The application of this transformation to the construction indicated by the equation $fx=1/\phi x$ will be obvious. Let any partition containing only unrepeated numbers consist of odd numbers $p, q, r, \ldots t$, each multiplied by one or more powers of 2; form batches of these terms which have the same greatest odd divisor $(p, q, r, \ldots t)$, and arrange those batches in a line according to the order of magnitude of $p, q, r, \ldots t$. Then we may agree to proceed either from left to right or from right to left in reading off the batches, and that convention being established once for all, as soon as a batch is reached which does not consist of a single odd term, if it contain one term larger than all the rest that term is to be split into two equal parts, but if it contain two terms not less than any

* Just so the equation $1/(1-x)=(1+x)(1+x^2)(1+x^4)\ldots$ teaches that there is one and only one way of effecting the unrepetitional geometric partition of any number—a theorem which has been applied in the preceding theory.

others in the batch, those two are to be amalgamated into one. In this way the order of a partition consisting of terms not all of them distinct odd numbers, will have its *parity* (quality of being odd or even) reversed, and it is obvious that if A has been under the operation of the rule converted into B, B by the operation of the same rule will be converted back into A. Hence it follows that (making abstraction of the partitions consisting exclusively of unrepeated odd numbers) all the rest will be separable into as many contractile of an odd as into extensile of an even order, and into as many extensile of an odd as into contractile of an even order, so that the difference between the entire number of the partitions of N into an odd and those of an even order of repeatable numbers (odd or even) will be the number of partitions of N into unrepeated odd numbers, and those of an odd or of an even order will be in the majority according as N itself is odd or even*.

It will be convenient to interpolate here Dr F. Franklin's constructive proof of the theorems referred to in p. [4] of what precedes, as there will be frequent occasion to refer to them in what follows. The theory is thus made completely self-contained. I give the proofs in the author's own words, which I think cannot be bettered.

(20) *Constructive Proof of the Formula for Partitions into Repeatable Parts, limited in Number and Magnitude.* The partitions herein spoken of are always partitions into a fixed number, j, of parts, *written in descending order.*

Take any partition of w in which the first excess† is greater than i; subtracting $i+1$ from the first part we get a partition of $w-(i+1)$; and conversely if to the first part in a partition of $w-(i+1)$ we add $i+1$ we get a partition of w in which the first excess is greater than i. Hence the number of partitions of w in which the first excess is greater than i is equal to the whole number of partitions of $w-(i+1)$; so that if the generating

* Dr F. Franklin has remarked that "the theorem admits of the following extensions," which the method employed in the text naturally suggests, and "which are very easily obtained either by the constructive proof or by generating functions":

1. The number of ways in which w can be made up of any number of odd and k distinct even parts is equal to the number of ways in which it can be made up of any number of unrepeated and k distinct repeated parts.

2. The number of ways in which w can be made up of parts not divisible by m is equal to the number of ways in which it can be made up of parts not occurring as many as m times.

3. The number of ways in which w can be made up of an infinite number of parts not divisible by m, together with k parts divisible by m, is equal to the number of ways in which it can be made up of an indefinite number of parts occurring less than m times, together with k parts occurring m or more times. (3) of course comprehends (1) and (2) as special cases.

Dr Franklin adds, "another extension is naturally contained in the mode of proof, which it is perhaps not worth while to state." See *Johns Hopkins Circular* for March, 1883.

† The first excess signifies the excess of the largest part over the next largest; the second excess the excess of the largest over the next part but one, and so on.

function for the partitions of w is $f(x)$, that for those partitions in which the first excess is *not greater* than i is $(1-x^{i+1})f(x)$. Confining ourselves now to this class of partitions, consider any one of them in which the second excess is greater than i; subtracting $i+1$ from the first part and 1 from the next, and putting the reduced first part into the second place we have a partition of $w-(i+2)$ in which the first excess is not greater than i; and conversely if in any partition of $w-(i+2)$ in which the first excess is not greater than i, we add $i+1$ to the second part and 1 to the first part and transfer the augmented second part to the first place, we get a partition of w in which the first excess is not greater than i and the second excess *is* greater than i. Hence the generating function for those partitions in which the second excess is *not* greater than i is $(1-x^{i+1})(1-x^{i+2})f(x)$. Considering now exclusively the partitions last mentioned, any one of them in which the third excess is greater than i may be converted into a partition of $w-(i+3)$ in which the second excess is not greater than i, by subtracting $i+1$ from the first part, 1 from the second part, and 1 from the third part, and removing the reduced first part to the third place, and, as before, by the reverse operation, the latter class of partitions are converted into the former. Hence the generating function for the partitions in which the third excess is not greater than i is

$$(1-x^{i+1})(1-x^{i+2})(1-x^{i+3})f(x).$$

So in like manner, the generating function for the partitions in which the k-th excess is not greater than i is

$$(1-x^{i+1})(1-x^{i+2})(1-x^{i+3})\dots(1-x^{i+k})f(x);$$

and for the partitions in which the j-th or absolute excess is not greater than i, that is in which the greatest part does not exceed i, the generating function is

$$(1-x^{i+1})(1-x^{i+2})(1-x^{i+3})\dots(1-x^{i+j})f(x).$$

(21) *Constructive Proof of the Formula for Partitions into Unrepeated Parts, limited in Number and Magnitude.* All the partitions to be considered consist of a fixed number, j, of *unrepeated* parts, written in *descending order*.

Take any partition of w in which the first excess is greater than $i+1$; subtracting $i+1$ from the first part we get a partition of $w-(i+1)$; conversely, if to the first part in any partition of $w-(i+1)$ we add $i+1$, we get a partition of w in which the first excess is greater than $i+1$; hence the number of partitions of w in which the first excess is greater than $i+1$ is equal to the whole number of partitions of $w-(i+1)$; so that, if the generating function for all the partitions is $\phi(x)$, the generating function for partitions whose first excess is *not* greater than $i+1$ is $(1-x^{i+1})\phi(x)$.

Considering now only partitions subject to this condition, if in any such partition of w the second excess is greater than $i+2$, we obtain by subtracting $i+2$ from the first part and removing the part so diminished to the second place a partition of $w-(i+2)$ subject to the condition; and conversely from any partition of $w-(i+2)$ in which the first excess is not greater than $i+1$, we obtain, by adding $i+2$ to the second part and removing the augmented part to the first place, a partition of w, in which the first excess is not greater than $i+1$ and the second excess *is* greater than $i+2$; hence the generating function for the partitions in which the second excess is *not* greater than $i+2$ (which restriction includes the condition that the first excess is not greater than $i+1$) is

$$(1-x^{i+1})(1-x^{i+2})\,\phi(x).$$

Confining ourselves now to this class of partitions, and taking any partition of w in which the third excess is greater than $i+3$, we obtain, by subtracting $i+3$ from the first part and removing the diminished part to the third place, a partition of $w-(i+3)$ belonging to the class now under consideration; and reversely. Hence the number of partitions in which the third excess is not greater than $i+3$ is given by the generating function

$$(1-x^{i+1})(1-x^{i+2})(1-x^{i+3})\,\phi(x).$$

Proceeding in this manner, we have finally that the generating function giving the number of partitions into j unrepeated parts, in which the absolute excess, that is the magnitude of the greatest part, is not greater than $i+j$, is

$$(1-x^{i+1})(1-x^{i+2})(1-x^{i+3})\dots(1-x^{i+j})\,\phi(x).$$

For example, if $w=18$, $j=3$, $i=4$, the partitions

15, 2, 1   14, 3, 1   13, 4, 1   13, 3, 2   12, 5, 1   12, 4, 2   11, 5, 2   11, 4, 3

in which the first excess is greater than 5, become by subtraction of 5 from their first part,

10, 2, 1   9, 3, 1   8, 4, 1   8 3, 2   7, 5, 1   7, 4, 2   6, 5, 2   6, 4, 3

which are *all* the partitions of 13; the partitions

11, 6, 1   10, 7, 1   10, 6, 2   10, 5, 3   9, 8, 1   9, 7, 2

in which the first excess is not greater than 5, but the second excess is greater than 6 become, by the subtraction of 6 from the first part and its removal to the second place,

6, 5, 1   7, 4, 1   6, 4, 2   5, 4, 3   8, 3, 1   7, 3, 2

which are all the partitions of 12 whose first excess is not greater than 5; the partitions

9, 6, 3   9, 5, 4   8, 7, 3   8, 6, 4

in which the second excess is not greater than 6, but the third excess (the greatest part) is greater than 7, become, by the subtraction of 7 from the first part and its removal to the last place,

6, 3, 2   5, 4, 2   7, 3, 1   6, 4, 1

which are all partitions of 11 whose second excess is not greater than 6. The only remaining partition of 18 is 7, 6, 5.

Interact.

Notes on certain Generating Functions and their Properties.

(22) (A) It may be as well to reproduce here (so as to keep the whole subject together) the entire proof of the well-known expansions of

$$1 + ax \,.\, 1 + ax^2 \,.\, 1 + ax^3 \dots 1 + ax^i,$$

and of the reciprocal of

$$1 - a \,.\, 1 - ax \,.\, 1 - ax^2 \,.\, 1 - ax^3 \dots 1 - ax^i,$$

which appeared in *part* in the *Johns Hopkins Circular* for February* last. This is, I think, distinguishable from the ordinary proofs as being, so to say, *classical* in form (using the word in an algebraical sense), inasmuch as it establishes the identity of two rational integral functions, one explicitly, the other implicitly given, by comparison of their zeros.

Let the coefficient of a^j in the expansion of

$$(1 + ax)(1 + ax^2) \dots (1 + ax^i),$$

say in the expansion of $F(x, a)$, be called J_x, and

$$\frac{1 - x^i \,.\, 1 - x^{i-1} \dots 1 - x^{i-j+1}}{1 - x \,.\, 1 - x^2 \dots 1 - x^j}$$

be called X_j.

J_x being the sum of the j-ary combinations of $x, x^2, \dots x^i$ will necessarily contain $x^{1+2+\dots+j}$, that is $x^{\frac{j^2+j}{2}}$, and will be of the degree

$$i + (i - 1) + \dots + (i - j + 1)$$

in x, and therefore of the same degree as $X_j x^{\frac{j^2+j}{2}}$.

All the linear factors of X_j are obviously of the form $x - \rho$, where $x - \rho$ is a primitive factor of some binomial expression $x^r - 1$: the number of times that any $x - \rho$ occurs in X_j will obviously be equal to $E\frac{i}{r} - E\frac{j}{r} - E\frac{i-j}{r}$ which is either 1 or 0. Now consider $F(\rho, a)$, the value of $F(x, a)$ when x becomes ρ. Let $i = kr + \delta$, where $\delta < r$; then $F(\rho, a) = (1 \pm a^r)^k$ multiplied

[* Vol. III. of this Reprint, p. 677.]

by δ linear functions of a, and consequently if $j = k'r + \delta'$, where $\delta' < r$, J_x vanishes when $\delta' > \delta$, in which case

$$E\frac{i}{r} - E\frac{j}{r} - E\frac{i-j}{r} = 1.$$

Hence any linear factor $x - \rho$ of X_j possesses the two-fold property of being unrepeated and of being contained in J_x. Hence J_x must contain $X_j x^{\frac{j^2+j}{2}}$, and being of the same degree as it is in x must bear to it a constant ratio, which, by making $x = 1$, is seen to be that of the coefficient of a^j in $(1+a)^i$, that is of $\frac{i(i-1)(i-2)\ldots(i-j+1)}{1.2.3\ldots j}$ to the product of the fractions in their vanishing state

$$\frac{1-x^i}{1-x}, \frac{1-x^{i-1}}{1-x^2}, \ldots, \frac{1-x^{i-j+1}}{1-x^j},$$

that is, is a ratio of equality, so that $J_x = X_j x^{\frac{j^2+j}{2}}$. Q.E.D.

(23) Again let X_j and J_x now stand respectively for

$$\frac{1-x^{i+1}.1-x^{i+2}\ldots 1-x^{i+j}}{1-x.1-x^2\ldots 1-x^j}$$

and the coefficient of a^j in the reciprocal of $1-a.1-ax\ldots 1-ax^i$ (say $F(x, a)$); this latter is the sum of homogeneous products of the jth order of $1, x, x^2, \ldots x^i$, and is therefore of the degree ij which is also the degree (as is obvious) of X_j in x. For like reason as in what precedes $x - \rho$, any linear factor of $x^r - 1$, is contained 1 or 0 times in X_j according as

$$E\frac{i+j}{r} - E\frac{i}{r} - E\frac{j}{r} = 1 \text{ or } 0.$$

Let the minimum negative residue of $i+1$ to modulus r be $-\delta$; $F(\rho, a)$ may be expressed as the product of δ linear functions of a, divided by a power of $1 - a^r$, and the only power of a (say a^θ) which appears in its development will accordingly be those for which the residue of θ in respect to r is $0, 1, 2, \ldots \delta$, and consequently if a^θ appears in the development

$$E\frac{i+\theta}{r} - E\frac{i}{r} - E\frac{\theta}{r} = 0,$$

or conversely if $x - \rho$ is a factor of X_j so that

$$E\frac{i+\theta}{r} - E\frac{i}{r} - E\frac{\theta}{r} = 1,$$

J_x vanishes. Hence J_x contains each linear factor of X_j, and these being simple, contains X_j itself, and on account of their degrees in x being the same must bear to it a ratio independent of x, which, by making $x = 1$, so that the things to be compared are the coefficient of a^j in $\dfrac{1}{(1-a)^{i+1}}$ and the product of the vanishing fractions $\dfrac{1-x^{i+1}}{1-x}, \dfrac{1-x^{i+2}}{1-x^2}, \ldots, \dfrac{1-x^{i+j}}{1-x^j}$, is readily seen to be a ratio of equality, so that $J_x = X_j$. Q.E.D.

(24) (B) *On the General Term in the Generating Function to Partitions into parts limited in number and magnitude*, by Dr F. FRANKLIN.

To prove that the coefficient of a^j in the development of

$$\frac{1}{(1-a)(1-ax)(1-ax^2)\ldots(1-ax^i)} \text{ is } \frac{(1-x^{j+1})(1-x^{j+2})\ldots(1-x^{j+i})}{(1-x)(1-x^2)\ldots(1-x^i)},$$

I showed that the number of partitions of w into i or fewer parts, subject to the condition that the first excess (the excess of the first part over the second) is not greater than j, is the coefficient of x^w in

$$\frac{1-x^{j+1}}{(1-x)(1-x^2)\ldots(1-x^i)},$$

and in general that the number of partitions in which the rth excess (the excess of the first part over the $(r-1)$th) is not greater than j, is the coefficient in

$$\frac{(1-x^{j+1})(1-x^{j+2})\ldots(1-x^{j+r})}{(1-x)(1-x^2)\ldots(1-x^i)}.$$

If we look at the question reversely, namely, the coefficient of a^j in

$$\frac{1}{(1-a)(1-ax)(1-ax^2)\ldots(1-ax^i)}$$

being known to be

$$\frac{(1-x^{j+1})(1-x^{j+2})\ldots(1-x^{j+i})}{(1-x)(1-x^2)\ldots(1-x^i)},$$

if we ask what is the significance of the fractions

$$\frac{1-x^{j+1}}{(1-x)(1-x^2)\ldots(1-x^i)}, \ldots, \frac{(1-x^{j+1})(1-x^{j+2})\ldots(1-x^{j+r})}{(1-x)(1-x^2)\ldots(1-x^i)},$$

the answer is immediately given by the generating function itself. For

$$\frac{1-x^{j+1}}{(1-x)(1-x^2)\ldots(1-x^i)}$$

$$= \frac{1}{(1-x^2)(1-x^3)\ldots(1-x^i)} \cdot \frac{1-x^{j+1}}{1-x}$$

$$= \frac{1}{(1-x^2)(1-x^3)\ldots(1-x^i)} \left(\text{co. of } a^j \text{ in } \frac{1}{(1-a)(1-ax)}\right)$$

$$= \text{co. of } a^j \text{ in } \frac{1}{(1-a)(1-ax)(1-x^2)(1-x^3)\ldots(1-x^i)}.$$

But the coefficient of $a^j x^w$ in the last written fraction is obviously the number of ways in which w can be composed of the numbers 1, 2, 3, ... i, using not more than j 1's. And the number of 1's in a given partition is equal to the excess of the first part over the second part in its conjugate. In like manner

$$\frac{(1-x^{j+1})(1-x^{j+2})\dots(1-x^{j+r})}{(1-x)(1-x^2)\dots(1-x^i)}$$

$$= \text{co. of } a^j \text{ in } \frac{1}{(1-a)(1-ax)\dots(1-ax^r)(1-x^{r+1})\dots(1-x^i)};$$

and the coefficient of $a^j x^w$ in the fraction on the right is the number of ways in which w can be composed of the parts 1, 2, 3, ... i, not more than j of the parts being as small as r. But the number of 1's in a given partition is equal to the excess of the first part over the second in its conjugate; the number of 2's to the excess of the second part over the third, and so on. Hence the number of 1's plus the number of 2's ... plus the number of r's in a given partition is equal to the excess of the first part over the rth part in its conjugate; and we have thus proved that the coefficient of x^w in the development of

$$\frac{(1-x^{j+1})(1-x^{j+2})\dots(1-x^{j+r})}{(1-x)(1-x^2)\dots(1-x^i)}$$

may be indifferently regarded as the number of partitions of w into parts none greater than i and not more than j of them as small as r or as the number of partitions of w into j or fewer parts, the excess of the first part over the rth part being as small as j. These results may obviously be extended by introducing the a in non-consecutive factors of the product

$$(1-x)(1-x^2)\dots(1-x^i).$$

(25) (C) *On the theorem of one-to-one and class-to-class correspondence between partitions of n into uneven and its partitions into unequal parts,* by Dr F. Franklin.

The number of partitions of w into k distinct odd numbers, each repeated an indefinite number of times, is evidently the coefficient of $a^k x^w$ in the development of

$$\left(1+a\frac{x}{1-x}\right)\left(1+a\frac{x^3}{1-x^3}\right)\left(1+a\frac{x^5}{1-x^5}\right)\dots.$$

It is not easy to form the generating function for the number of partitions containing k sequences, but it is plain that the number of partitions of w containing *one* sequence is the coefficient of x^w in

$$S_1 + S_2 + S_3 + \dots,$$

where

$$S_1 = x + x^2 + x^3 + x^4 + x^5 + \dots = \frac{x}{1-x}$$

$$S_2 = x^3 + x^5 + x^7 + x^9 + x^{11} + \dots = \frac{x^3}{1-x^2}$$

$$S_3 = x^6 + x^9 + x^{12} + x^{15} + x^{18} + \dots = \frac{x^6}{1-x^3}$$

$$S_4 = x^{10} + x^{14} + x^{18} + x^{22} + x^{26} + \dots = \frac{x^{10}}{1-x^4}$$

$$S_5 = x^{15} + x^{20} + x^{25} + x^{30} + x^{35} + \dots = \frac{x^{15}}{1-x^5},$$

and in general

$$S_r = x^{1+2+3+\dots+r} + x^{2+3+4+\dots+(r+1)} + \dots = \frac{x^{\frac{1}{2}r(r+1)}}{1-x^r}.$$

So much of Prof. Sylvester's theorem as relates to a single sequence follows from inspection of the above scheme. For $S_1 = \frac{x}{1-x}$; adding to S_3 the first term of S_2, we get $\frac{x^3}{1-x^3}$; adding to S_5 the first term of S_4 and the second term of S_2, we get $\frac{x^5}{1-x^5}$; adding to S_{2m+1} the first term of S_{2m}, the second term of $S_{2(m-1)}$, the third term of $S_{2(m-2)}, \dots$, and the mth term of S_1, we get $\frac{x^{2m+1}}{1-x^{2m+1}}$; thus the proposition is proved. The fact is made more evident to the eye if we write the scheme as follows:

$$\begin{array}{ll}
S_1 = x + x^2 + x^3 + x^4 + x^5 + \dots & S_2 = x^3 + x^5 + x^7 + x^9 + x^{11} + \dots \\
S_3 = x^6 + x^9 + x^{12} + x^{15} + x^{18} + \dots & S_4 = \quad x^{10} + x^{14} + x^{18} + x^{22} + \dots \\
S_5 = x^{15} + x^{20} + x^{25} + x^{30} + x^{35} + \dots & S_6 = \qquad x^{21} + x^{27} + x^{33} + \dots \\
S_7 = x^{28} + x^{35} + x^{42} + x^{49} + x^{56} + \dots & S_8 = \qquad\quad x^{36} + x^{44} + \dots \\
S_9 = x^{45} + x^{54} + x^{63} + x^{72} + x^{81} + \dots & S_{10} = \qquad\qquad x^{55} + \dots
\end{array}$$

Here $\frac{x^9}{1-x^9}$, for instance, is obtained by adding the fourth column on the right to the fifth row on the left.

It may be noted that we have thus found that

$$\frac{x}{1-x} + \frac{x^3}{1-x^3} + \frac{x^5}{1-x^5} + \dots + \frac{x^{2m+1}}{1-x^{2m+1}} + \dots$$

$$= \frac{x}{1-x} + \frac{x^3}{1-x^2} + \frac{x^6}{1-x^3} + \dots + \frac{x^{\frac{1}{2}n(n+1)}}{1-x^n} + \dots.$$

(26) [Compare Jacobi's theorem contained in the last-but-one two lines of the last but one page of the *Fundamenta Nova*, which may be easily reduced to the form

$$\frac{x}{1+x}-\frac{x^3}{1+x^3}+\frac{x^5}{1+x^5}\ldots=\frac{x}{1+x}-\frac{x^3}{1+x^2}+\frac{x^5}{1+x^3}-\ldots.\quad \text{J. J. S.]}$$

Act II. On the Graphical Conversion of Continued Products into Series.

> Naturelly, by composiciouns
> Of anglis, and slie reflexiouns.
> *The Squieres Tale.*

(27) The method about to be explained of representing the elements of partitions by means of a succession of angles fitting into one another arose out of an investigation (instituted for the purpose of facilitating the arrangement of tables of symmetric functions)* as to the number of partitions of n which are their own conjugates. The ordinary graphs to such partitions must obviously be symmetrical in respect to the two nodal boundaries, as seen below.

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Let the above figure be any such graph; it may be dissected into a square (which may contain one or any greater square number) of say i^2 nodes, and two perfectly similar appended graphs, each having the content $\frac{n-i^2}{2}$, and subject to the sole condition that the number of its lines (or columns), that is that the number (or magnitude) of the parts in the partition which it represents, shall be i or less; such number is the coefficient of $x^{\frac{n-i^2}{2}}$ in $\frac{1}{1-x\,.\,1-x^2\ldots1-x^i}$, which is the same as that of x^{n-i^2} in

$$\frac{1}{1-x^2\,.\,1-x^4\ldots1-x^{2i}}$$

or of x^n in

$$\frac{x^{i^2}}{1-x^2\,.\,1-x^4\ldots1-x^{2i}}.$$

* By Mr Durfee, of California (Fellow of the Johns Hopkins University), to whom I suggested the desirability of investigating more completely than had been done the method of arrangement of such tables founded upon the notion of self-conjugate partitions, which M. Faà de Bruno had the honour of initiating. The very valuable results of Mr Durfee's inquiries, embodying, systematising and completing the theory of arrangement originated by Professor Cayley, and further illustrated by the labours of Professors Betti and De Bruno, will probably appear in the next number of the *Journal*.

Hence giving i all possible values we see that the coefficient of x^n in the infinite series

$$1+\frac{x}{1-x^2}+\frac{x^4}{1-x^2.1-x^4}+\frac{x^9}{1-x^2.1-x^4.1-x^6}+\ldots$$

is the number of self-conjugate partitions of n, or which is the same thing of symmetrical groups whose content is n.

(28) But any such graph, in which there is a square of i^2 nodes with its two appendices, may be dissected in another manner into i angles or bends, each containing any continually decreasing odd number of nodes, and *vice versâ*, any set of equilateral angles of nodes continually decreasing in number (which condition is necessary in order that the lower lines and posterior columns may not protrude beyond the upper lines and anterior columns) when fitted into one another in the order of their magnitudes will form a regular graph. Thus the actual figure (where there is a square of 9 nodes) formed by the intersections of the lines and columns may be dissected into 3 angles containing respectively 13, 11, 3 nodes; and so in general the number of ways in which n can be made up of odd and unrepeated parts will be the same as the number of ways in which $\frac{n-j^2}{2}$ can be partitioned into not more than j parts; hence we see that the coefficients of $x^n a^j$ in

$$(1+ax)(1+ax^3)\ldots(1+ax^{2j-1})\ldots$$

and in

$$\frac{x^{j^2}}{1-x^2.1-x^4\ldots 1-x^{2j}}$$

are the same, so that the continued product above written is equal to

$$1+\frac{x}{1-x^2}a+\ldots+\frac{x^{j^2}}{1-x^2.1-x^6\ldots 1-x^{2j}}a^j+\ldots$$

as is well known.

(29) In like manner if the expansion in a series of ascending powers of a of the finite continued product

$$(1+ax)(1+ax^3)\ldots(1+ax^{2i-1})$$

be required, the coefficient of x^n in the coefficient of a^j will be the number of ways in which n can be made up with j of the unrepeated numbers $1, 3, \ldots 2i-1$, and as $2i-1$ is the number of nodes in an equilateral angle whose sides contain i nodes, it follows that this coefficient will be the number of ways in which $\frac{n-j^2}{2}$ can be composed with parts none *exceeding* $i-j$ in magnitude, and will therefore be the same as the coefficient of $x^{\frac{n-j^2}{2}}$ in

$$\frac{1-x^{i-j+1}.1-x^{i-j+2}\ldots 1-x^i}{1-x.1-x^2\ldots 1-x^j},$$

and consequently the finite continued product above written is equal to

$$1+\ldots+\frac{1-x^{2i-2j+2}.1-x^{2i-2j+4}\ldots 1-x^{2i}}{1-x^2.1-x^4\ldots 1-x^{2j}}x^{j^2}a^j+\ldots$$

(30) If it be required to ascertain how many self-conjugate partitions of n there are containing exactly i parts, this may be found by giving j all possible values and making p_j equal to the number of ways in which $\frac{n-j^2}{2}$ can be composed with j or fewer parts the greatest of which is $i-j$, that is $(n-j^2+2j-2i)/2$ with $j-1$ or fewer parts none greater than $i-j$, so that p_j will be the coefficient of $x^{(n-j^2+2j-2i)/2}$ in

$$\frac{1-x^{i-j+1}\,.\,1-x^{i-j+2}\ldots 1-x^{i-1}}{1-x\,.\,1-x^2\ldots 1-x^{j-1}}$$

or of x^n in

$$\frac{1-x^{2i-2j+2}\,.\,1-x^{2i-2j+4}\ldots 1-x^{2i-2}}{1-x^2\,.\,1-x^4\ldots 1-x^{2j-2}}\,x^{j^2-2j+2i};$$

the sum of the values of p_j for all values of j will be the number required: this number, therefore, writing ω for $2i-1$, will be the coefficient of x^n in

$$x^\omega+\frac{1-x^{\omega-1}}{1-x^2}\,x^{\omega+1}+\frac{1-x^{\omega-1}\,.\,1-x^{\omega-3}}{1-x^2\,.\,1-x^4}\,x^{\omega+4}+\text{etc.};$$

the outstanding factor in the qth term in this series being $x^{\omega+(q-1)^2}$ we may suppose q the least integer number not less than $1+\sqrt{(n-\omega)}$ and then the subsequent term to the $(q+1)$th being inoperative may be neglected.

(31) In order to see how any self-conjugate graph may be recovered, so to say, from the corresponding partition consisting of unrepeated odd numbers, consider the diagrammatic case of the partition 17, 9, 5, 1 represented by the angles of the graph below written

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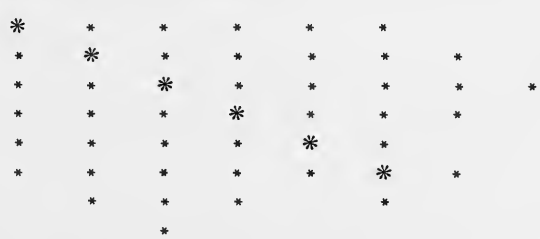
The number of angles is the number of the given parts, that is 4, and the first four lines of the graph will be obtained by adding 0, 1, 2, 3 to the major half (meaning the integer next above the half) of 17, 9, 5, 1, that is will be 9, 6, 5, 4, the total number of lines will be the major half of the highest term (17) and the remaining lines will have the same contents, namely 3, 2, 1, 1, 1, as the columns of the graph found by subtracting 4 (the number of the parts) from the numbers last found, that is will be the lines of the graph which is conjugate to 5, 2, 1. And so in general the self-conjugate graph corresponding to any partition of unrepeated odd numbers $q_1, q_2, \ldots q_j$ will be found by the following rule:

Let P be the system of partitions $k_1, k_2, \dots k_j$, in which any term k_θ is the major half of q_θ augmented by $\theta - 1$, and P' another system of $k_1', k_2', \dots k_j'$, obtained by subtracting j from each term in P, then P and the conjugate to P' will be the self-conjugate partition corresponding to the given q partition. Thus as an example, 19, 11, 7, 5 being given, P, P' will be 10, 7, 6, 6; 6, 3, 2, 2 respectively, and the self-conjugate system required will be 10, 7, 6, 6, 4, 4, 2, 1, 1, 1. Of course P' might also be obtained by taking the minor halves of the given parts in inverse (ascending) order and subtracting from them the numbers 0, 1, 2, ... respectively.

To pass from a given self-conjugate to the corresponding unrepeated odd numbers-partition is a much simpler process, the rule being to take the numbers in descending order and from their doubles subtract the successive odd numbers in the natural scale until the point is reached at which the difference is about to become negative; thus the partition 6 6 5 4 3 2 is self-conjugate, and the correspondent to it is 11 9 5 1.

(32) The expansion of the reciprocal to $(1-ax)(1-ax^3)\dots(1-ax^{2i-1})$ may be read off with the same facility as the direct product. In this case we are concerned with partitions of odd numbers capable of being repeated in the same partition; now, therefore, if we use the same method of equilateral angles as before, and fit them into one another in regular order of magnitude, it will no longer be the case that their sum will form a regular graph, for if there be θ parts alike, each line and column which ranges with either side of any (but the first one) of these will jut out one step beyond the anterior line and column (respectively), so that the line joining the extremities of the lines or columns will be parallel to the axis of symmetry. The figure then corresponding to i odd parts can no longer be dissected into a *square* of nodes and two equal regular graphs, but it may be dissected into a *line* of nodes lying in the axis of symmetry, and two regular graphs one of which has for its boundaries one of the original boundaries and a line of nodes parallel to the axis of symmetry, and the other one the other original boundary and a line of nodes parallel to the same axis, as seen in the annexed figure, where the axial points are distinguished by being made larger than the rest.



The graph read off in angles represents the partition 11 11 11 7 3 3. On removing the six diagonal nodes it breaks up into two regular graphs, of which one is 5 5 5 3 1 1, and the other the conjugate thereto; hence the coefficient of x^n in the coefficient of a^j in the expansion of the reciprocal of $1-ax\,.\,1-ax^3\ldots.\,1-ax^{2i-1}$ in ascending powers of a is the number of ways in which $\frac{n-j}{2}$ can be resolved into j parts limited not to exceed $i-1$, which is the coefficient of $x^{\frac{n-j}{2}}$ in

$$\frac{1-x^i\,.\,1-x^{i+1}\ldots 1-x^{i+j-1}}{1-x\,.\,1-x^2\ldots 1-x^j}$$

or of x^n in

$$\frac{1-x^{2i}\,.\,1-x^{2i+2}\ldots 1-x^{2i+2j-2}}{1-x^2\,.\,1-x^4\ldots 1-x^{2j}}\,x^j.$$

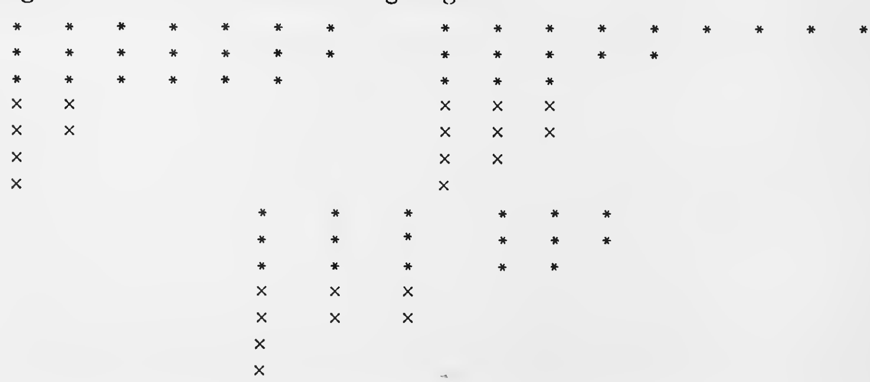
(33) Although I shall not require any intermediate expansion whatever in order to obtain the transcendant $\Theta_1 x$ product in the form of a series, I will give another of those which are sometimes employed together in combination (see Cayley, *Elliptic Functions*, pp. 296—7) to obtain this result: thus to prove that the continued product of the reciprocal of

$$(1-ax)(1-ax^2)(1-ax^3)\ldots$$

is identical with

$$1+\frac{x}{1-x}\cdot\frac{a}{1-xa}+\frac{x^4}{1-x\,.\,1-x^2}\cdot\frac{a^2}{1-xa\,.\,1-x^2a}$$
$$+\frac{x^9}{1-x\,.\,1-x^2\,.\,1-x^3}\cdot\frac{a^3}{1-xa\,.\,1-x^2a\,.\,1-x^3a}+\text{etc.}$$

if n is partitioned into j parts, the regular graph which represents the result of any such partition must consist either of $1, 2, 3, \ldots j-1$ or of not less than j columns, and its graph may accordingly in these several cases be dissected into a square of $1, 4, 9, \ldots j^2$ nodes; suppose that such square consists of θ parts, then there will be $n-\theta^2$ nodes remaining over subject to distribution into two groups limited by the condition as to one of the groups that it may contain an unlimited number of parts none exceeding θ in magnitude, and as to the other that it must contain $j-\theta$ parts none exceeding θ in magnitude, as seen in the following diagrams:



in all of which the partible number is 26, and j and θ are 7 and 3 respectively. Now the number of such distributions is the coefficient of $x^{n-\theta^2}a^{j-\theta}$ in

$$\frac{1}{1-x\,.\,1-x^2\ldots 1-x^\theta}\cdot\frac{1}{1-ax\,.\,1-ax^2\ldots 1-ax^\theta},$$

that is of $x^n a^j$ in

$$\frac{x^{\theta^2}}{1-x\,.\,1-x^2\ldots 1-x^\theta}\cdot\frac{a^\theta}{1-ax\,.\,1-ax^2\ldots 1-ax^\theta},$$

and consequently giving θ all values from 1 to ∞, the proposed equation is verified.

(34) It may be desired to apply the same method to obtain a similar development for the reciprocal of the limited product

$$(1-ax)(1-ax^2)\ldots(1-ax^i);$$

the construction will be the same as in the last case; the distribution into two groups can be made as before; the second group will remain subject to the same condition as in the preceding case (seeing that the number of parts being less than $j-\theta$, will necessarily be less than $i-\theta$, for j cannot exceed i), but the first group will be subject to the condition of being partitioned not now into an unlimited but into $i-\theta$ (or fewer) parts none exceeding θ in magnitude, and the number of such distributions into the two groups will accordingly become the coefficient of $x^{n-\theta^2}a^{j-\theta}$ in

$$\frac{1-x^{i-\theta+1}\,.\,1-x^{i-\theta+2}\ldots 1-x^i}{1-x\,.\,1-x^2\ldots 1-x^\theta}\cdot\frac{1}{1-ax\,.\,1-ax^2\ldots 1-ax^\theta}$$

or of $x^n a^j$ in the last written fraction multiplied by $x^{\theta^2}\,.\,a^\theta$, so that the required expansion will be

$$1+\frac{1-x^i}{1-x}\cdot\frac{xa}{1-ax}+\frac{1-x^i\,.\,1-x^{i-1}}{1-x\,.\,1-x^2}\cdot\frac{x^4a^2}{1-ax\,.\,1-ax^2}$$
$$+\frac{1-x^i\,.\,1-x^{i-1}\,.\,1-x^{i-2}}{1-x\,.\,1-x^2\,.\,1-x^3}\cdot\frac{x^9a^3}{1-ax\,.\,1-ax^2\,.\,1-ax^3}+\ldots.$$

(35) It is interesting to investigate what will be the form of the mixed development resulting from an application of the same method to the *direct* product

$$1+ax\,.\,1+ax^2\ldots 1+ax^i.$$

For greater clearness I shall first suppose i indefinitely great. Consider the diagram:

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In the above graph j and θ used in the same sense as *ante* are 5 and 3 respectively, so that there is a square of 9 points; an appendage to the right of and another appendage below the square, which I shall call the lateral and subjacent appendages respectively. The content of the graph being 25, there are 16 points to be distributed between these two appendages. What now are the conditions of the distribution of the $n-\theta^2$ points between them?

I say that there will be two sorts of such distribution—one in which the lateral appendage will consist of θ unrepeated parts, none of them zero, as in the graph above, and the subjacent appendage of $j-\theta$ unrepeated parts, limited not to exceed θ in magnitude, and another sort as in the graph below written,

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*  *  *  *  *  *  *  *  *  *  *  *
*  *  *  *  *  *  *
*  *  *
*  *
*
```

in which the jth line of the lateral appendage is missing, and consequently the subjacent graph will consist of $j-\theta$ unrepeated parts limited not to exceed $\theta-1$ in magnitude, for there could not be a part so great as θ without the last line of the square having the same content as the first line of the subjacent appendage.

It should be observed that only the *last* admissible line of the lateral appendage can be wanting, for if more than this were wanting, two lines of the square would belong to the graph, and consequently there would be two equal parts θ.

Hence there are two kinds of association of the appendages, one leading to a distribution of $n-\theta^2$ between one group of θ unrepeated but unlimited parts, and another of $j-\theta$ unrepeated parts limited not to exceed θ; the other to a distribution of $n-\theta^2$ between one group of $\theta-1$ unrepeated but unlimited parts, and another of $j-\theta$ unrepeated parts limited not to exceed $\theta-1$.

The number of distributions of the first kind is the coefficient of $x^{n-\theta^2}.\, a^{j-\theta}$ in

$$\frac{x^{\frac{\theta^2+\theta}{2}}}{1-x\,.\,1-x^2\ldots 1-x^\theta}\,.\,(1+ax)(1+ax^2)\ldots(1+ax^\theta),$$

the other of $x^{n-\theta^2}.\, a^{j-\theta}$ in

$$\frac{x^{\frac{\theta^2-\theta}{2}}}{1-x\,.\,1-x^2\ldots 1-x^{\theta-1}}\,.\,(1+ax)(1+ax^2)\ldots(1+ax^{\theta-1});$$

hence the sum of the distributions of the two kinds is the coefficient of the same argument in

$$\frac{x^{\frac{\theta^2-\theta}{2}}}{1-x\,.\,1-x^2\,...\,1-x^\theta}\{x^\theta(1+ax^\theta)+(1-x^\theta)\}\{1+ax\,.\,1+ax^2\,...\,1+ax^{\theta-1}\},$$

that is of $x^n a^j$ in

$$x^{\frac{3\theta^2-\theta}{2}}a^\theta\left(\frac{1+ax\,.\,1+ax^2\,...\,1+ax^{\theta-1}}{1-x\,.\,1-x^2\,...\,1-x^{\theta-1}}\cdot\frac{1+ax^{2\theta}}{1-x^\theta}\right),$$

and consequently we obtain the equation

$$1+ax\,.\,1+ax^2\,.\,1+ax^3\,...=1+\frac{1+ax^2}{1-x}xa+\frac{1+ax\,.\,1+ax^4}{1-x\,.\,1-x^2}x^5a^2+...$$

$$+\frac{1+ax\,.\,1+ax^2\,...\,1+ax^{j-1}\,.\,1+ax^{2j}}{1-x\,.\,1-x^2\,...\,1-x^{j-1}\,.\,1-x^j}x^{\frac{3j^2-j}{2}}a^j+...,$$

and thus by a very unexpected route we arrive at a proof of Euler's celebrated pentagonal-number theorem; for on making $a=-1$ the above equation becomes

$$1-x\,.\,1-x^2\,.\,1-x^3\,...=1-(1+x)\,x+(1+x^2)\,x^5\,...+(-)^j(1+x^j)\,x^{\frac{3j^2-j}{2}}+....$$

Such is one of the fruits among a multitude arising out of Mr Durfee's ever-memorable example of the dissection of a graph (in the case of a symmetrical one) into a square, and two regular graph appendages.

Even the trifling algebraical operation above employed to arrive at the result might have been spared by expressing the continued product as the sum of the two series (which flow immediately from the graphical dissection process), left uncombined, namely,

$$1+\frac{1+ax}{1-x}x^2a+\frac{1+ax\,.\,1+ax^2}{1-x\,.\,1-x^2}x^7a^2+\frac{1+ax\,.\,1+ax^2\,.\,1+ax^3}{1-x\,.\,1-x^2\,.\,1-x^3}x^{15}a^3+...,$$

together with

$$+xa+\frac{1+ax}{1-x}x^5a^2+\frac{1+ax\,.\,1+ax^2}{1-x\,.\,1-x^2}x^{12}a^3+...,$$

which for $a=-1$ unite into the single series

$$1-x-x^2+x^5+x^7-x^{12}-x^{15}\text{ etc.}$$

(36) I will now proceed to find the expression in a mixed series of the limited product

$$1+ax\,.\,1+ax^2\,...\,1+ax^i.$$

In each of the two systems of distribution (as shown already in the theory of the reciprocal of such product) the second group will remain unaffected by the new limitation, but the first group will now consist of partitions (limited in number as before), but in magnitude instead of being unlimited, limited not to exceed $(i-\theta)$, so that we will have to take the coefficient of $x^{n-\theta^2}.a^{j-\theta}$ in the sum of

$$x^{\frac{\theta^2+\theta}{2}}\frac{1-x^{i-\theta}.1-x^{i-\theta-1}\ldots 1-x^{i-2\theta+1}}{1-x.1-x^2\ldots 1-x^\theta}.(1+ax)(1+ax^2)\ldots(1+ax^\theta)$$

and

$$x^{\frac{\theta^2-\theta}{2}}\frac{1-x^{i-\theta}.1-x^{i-\theta-1}\ldots 1-x^{i-2\theta+2}}{1-x.1-x^2\ldots 1-x^{\theta-1}}.(1+ax)(1+ax^2)\ldots(1+ax^{\theta-1}).$$

This will be the same as the coefficient of $x^n a^j$ in

$$x^{\frac{3\theta^2-\theta}{2}}a^\theta(1+ax)(1+ax^2)\ldots(1+ax^{\theta-1})\frac{1-x^{i-\theta}.1-x^{i-\theta-1}\ldots 1-x^{i-2\theta+2}}{1-x.1-x^2\ldots 1-x^{\theta-1}.1-x^\theta}$$
$$\times\{1-x^\theta+(1-x^{i-2\theta+1})(x^\theta+ax^{2\theta})\},$$

where the quantity within the final bracket is equal to

$$1-x^{i+1}a-x^{i-\theta+1}+x^{2\theta}a.$$

Hence the required series is

$$\left\{1+\frac{1-x^i}{1-x}xa+\frac{1-x^{i-1}.1-x^{i-2}}{1-x.1-x^2}(1+ax)x^5a^2\right.$$
$$\left.+\frac{1-x^{i-2}.1-x^{i-3}.1-x^{i-4}}{1-x.1-x^2.1-x^3}.1+ax.1+ax^2.x^{12}a^3+\ldots\right\}$$
$$+\left\{\frac{1-x^{i-1}}{1-x}x^3a^2+\frac{1-x^{i-2}.1-x^{i-3}}{1-x.1-x^2}(1+ax)x^9a^3\right.$$
$$\left.+\frac{1-x^{i-3}.1-x^{i-4}.1-x^{i-5}}{1-x.1-x^2.1-x^3}.1+ax.1+ax^2.x^{18}a^4+\ldots\right\},$$

the indices in the outstanding powers of x being the pentagonal numbers in the first, and the triangular numbers trebled, in the second of the above series.

In obtaining in the preceding articles mixed series for continued products, it will be noticed that the graphical method has been employed, not to exhibit correspondence, but as an instrument of transformation. The graphs are virtually segregated into classes, and the number of them contained in each class separately determined. (The magnitude of the square in the Durfee-dissection serves as the basis of the classification.)

(37) Now let us consider the famous double product of

$$(1+ax)(1+ax^3)(1+ax^5)\ldots$$

by

$$(1+a^{-1}x)(1+a^{-1}x^3)(1+a^{-1}x^5)\ldots.$$

Here it will be expedient to introduce a new term and to explain the meaning of a bi-partition and a system of parallel bi-partitions of a number. The former indicates that the elements are to be distributed into two groups, say into a left and right-hand group: the latter that the number of the elements (on one, say) on the left-hand side of each bi-partition of the system is to be equal to or exceed by a constant difference the number (on the other, say) on the right-hand side of the same bi-partition. If we use dots, regularly spaced, to represent the elements (themselves numbers and not units), we get a figure or pair of figures such as the following:

*	*	*	*	*	*		*	*	*	*
*	*	*	*	*	*		*	*	*	*
*	*	*	*	*			*	*	*	
*	*	*	*				*	*		

for which the corresponding lines of the contour are respectively parallel—hence the name. When the numbers of elements on the two sides are identical, I call the system an equi-bi-partition-system—in the general case, a parallel bi-partition-system to a constant difference j, where j is the excess of the number of elements in the left-hand over that in the right-hand part of any of the bi-partitions.

(38) Consider now the given double product—it is obvious that it may be expanded in terms of paired powers $a^j + a^{-j}$ of a, and the coefficient of x^n in the term not involving a will evidently be the number of equi-bi-partitions of n that can be formed with unrepeated odd numbers; and so the coefficient of x^n associated with a^j or a^{-j} will be the number of parallel bi-partitions of n to the constant difference j that can be so formed.

For the equi-bi-partitions; suppose $l_1, l_2 \dots l_i, \lambda_1, \lambda_2 \dots \lambda_i$ is an equi-bi-partition, all the elements being odd and unrepeated; take successive angles whose (say horizontal and vertical) sides are the major halves of l_1, λ_1; $l_2, \lambda_2 \dots$; l_i, λ_i; these angles will fit on to one another so as to form a regular graph by reason of the relations

$$l_1 > l_2 + 1, \quad l_2 > l_3 + 1 \dots l_{i-1} > l_i + 1,$$
$$\lambda_1 > \lambda_2 + 1, \quad \lambda_2 > \lambda_3 + 1 \dots \lambda_{i-1} > \lambda_i + 1.$$

Conversely any regular graph may be resolved into angles whose horizontal sides shall be the major halves of one set of odd numbers, and their vertical sides the major halves of another set of as many odd numbers, and these two sets of odd numbers will each form a decreasing series; hence there is a one-to-one conjugate correspondence between any bi-partition of n written in regular order, and the totality of regular graphs whose content is $\frac{n}{2}$, so that the number of the equi-bi-partitions of n will be the coefficient of $x^{\frac{n}{2}}$ in

$$\frac{1}{1 - x \,.\, 1 - x^2 \,.\, 1 - x^3 \dots},$$

that is of x^n in
$$\frac{1}{1 - x^2 \,.\, 1 - x^4 \,.\, 1 - x^6 \dots},$$

which fraction is therefore equal to the totality of the terms not involving a.

(39) Next for the coefficient of a^j.

Let $l_1, l_2, \ldots l_j, l_{j+1}, l_{j+2}, \ldots l_{j+\theta}; \lambda_1, \lambda_2, \ldots \lambda_\theta$ be an equi-parallel bi-partition to the difference j (with the elements on each side written in descending order); with the equi-bi-partition $l_{j+1}, l_{j+2}, \ldots l_{j+\theta}; \lambda_1, \lambda_2, \ldots \lambda_\theta$, form a graph, as in the preceding case; say, for distinctness, with major halves of the l series horizontal and of the λ series vertical; over the highest horizontal line the successive quantities*

$$\frac{l_j - 1}{2}, \ \frac{l_{j-1} - 3}{2}, \ \frac{l_{j-2} - 5}{2}, \ldots \frac{l_1 - (2j-1)}{2}$$

may be laid so as to form a regular graph of which the content will be $\frac{n - j^2}{2}$.

Conversely every regular graph whose content is $\frac{n-j^2}{2}$ will correspond to a parallel bi-partition of unrepeated odd numbers to a difference j; to obtain the bi-partition the first j lines of the graph must be abstracted†, and the graph thus diminished resolved into angles; the doubles of the contents of each vertical side of these angles diminished by unity will constitute the right-hand side of the bi-partition, and the doubles of the contents of each horizontal side preceded by the doubles of the lines of the abstracted portion of the graph increased by 1, 3, 5, ... $2j-1$ respectively, will form the left-hand portion. Hence the number of such bi-partitions will be the number of ways of resolving $\frac{n-j^2}{2}$ into unrestricted parts, that is, will be the coefficient of x^n in

$$\frac{1}{1-x^2 \,.\, 1-x^4 \,.\, 1-x^6 \ldots} x^{j^2},$$

and this being true for all values of n and j, we see that the double product in question will be identical with the infinite series

$$\frac{1}{1-x^2 \,.\, 1-x^4 \,.\, 1-x^6 \ldots} \{1 + x(a + a^{-1}) + x^4(a^2 + a^{-2}) + x^9(a^3 + a^{-3}) + \ldots\}.$$

(40) To expand the limited double product

$$(1 + ax)(1 + ax^3) \ldots (1 + ax^{2i-1})$$

into

$$(1 + a^{-1}x)(1 + a^{-1}x^3) \ldots (1 + a^{-1}x^{2i-1})$$

the procedure and reasoning will be precisely the same as in the extreme case of i infinite, the only difference being that the elements of the bi-partition instead of being unlimited odd numbers will be limited not to exceed $2i-1$. In the case of $j = 0$ the equi-bi-partition will furnish a series of nodal angles in which neither side can exceed the major half of $2i-1$,

* Any number of these quantities may happen to become zero.

† If the actual number of horizontal lines in the graph is less than j, it must be made to count as j, by understanding lines of zero content to be supplied underneath the graph.

that is i, and the coefficient of x^n in the term not containing any power of a will consequently be the number of ways in which n can be divided into parts limited as well in number as in magnitude not to exceed i, and will therefore be the same as the coefficient of $x^{\frac{1}{2}n}$ in the development of

$$\frac{1-x^{i+1}\,.\,1-x^{i+2}\ldots 1-x^{2i}}{1-x\,.\,1-x^2\ldots 1-x^i},$$

or, which is the same thing, of x^n in the development of

$$\frac{1-x^{2i+2}\,.\,1-x^{2i+4}\ldots 1-x^{4i}}{1-x^2\,.\,1-x^4\ldots 1-x^{2i}},$$

and when the bi-partition system has a constant difference j, the corresponding graph will be of the same form, except that it will be overlaid with j lines, obtained as in the preceding case by subtracting $1, 3, \ldots 2j-1$ from the first j left-hand elements, and taking the halves of the remainders; the graphs thus formed will be subject to the condition of having a content $\frac{n-j^2}{2}$, and parts limited not to exceed $i-j$ in magnitude nor $i+j$ in number [$i-j$ in magnitude because the topmost line cannot exceed $\frac{(2i-1)-(2j-1)}{2}$ in content; $i+j$ in number because without reckoning the j superimposed lines the subjacent portion of the graph cannot contain more than i lines]. The converse that out of every regular graph fulfilling these conditions may be spelled out a parallel bi-partition with a difference j, and containing only unrepeated odd numbers limited not to exceed $2i-1$ in magnitude may be shown as in the preceding case. Hence the coefficient of x^n in the coefficient of a^j+a^{-j} in the expansion, is the number of ways of resolving $\frac{n-j^2}{2}$ into parts none exceeding $i-j$ in magnitude nor $i+j$ in number, that is, is the coefficient of x^n in

$$\frac{1-x^{2i+2j+2}\,.\,1-x^{2i+2j+4}\ldots 1-x^{4i}}{1-x^2\,.\,1-x^4\ldots 1-x^{2i-2j}}\,x^{j^2}.$$

Hence by the process of reasoning, which has been so often applied, we see that the finite double product

$$1+ax\,.\,1+ax^3\ldots 1+ax^{2i-1}$$

into

$$1+a^{-1}x\,.\,1+a^{-1}x^3\ldots 1+a^{-1}x^{2i-1}$$

$$=\frac{1-x^{2i+2}\,.\,1-x^{2i+4}\ldots 1-x^{4i}}{1-x^2\,.\,1-x^4\ldots 1-x^{2i}}\left\{1+\frac{1-x^{2i}}{1-x^{2i+2}}x+\frac{1-x^{2i}\,.\,1-x^{2i-2}}{1-x^{2i+2}\,.\,1-x^{2i+4}}x^4\right.$$
$$\left.+\frac{1-x^{2i}\,.\,1-x^{2i-2}\,.\,1-x^{2i-4}}{1-x^{2i+2}\,.\,1-x^{2i+4}\,.\,1-x^{2i+6}}x^9+\ldots\right\}.$$

Compare Hermite, *Note sur les fonctions elliptiques*, p. 35, where Cauchy's method is given of arriving at this and the preceding identity.

Act III. On the One-to-one and Class-to-class Correspondence between Partitions into Uneven and Partitions into Unequal Parts.

. . . mazes intricate,
Eccentric, intervolved, yet regular
Then most, when most irregular they seem.
Paradise Lost, v. 622.

(41) It has been already shown that any partition of n into unequal parts may be converted into a partition consisting of odd numbers equal or unequal by, first, expressing any even part by its longest odd divisor, say its nucleus and a power of 2, and, second, adding together the powers of 2 belonging to the same nucleus, so that there will result a sum of odd nuclei, each occurring one or more times; a like process is obviously applicable to convert a partition in which any number occurs 1, 2,... or $(r-1)$ times into one in which only numbers not divisible by r occur with unrestricted liberty of recurrence. The nuclei will here be numbers not divisible by r multiplied by powers of r, and by adding together the powers of r belonging to the same nucleus there results a series of nuclei, each occurring one or more times. Conversely when the nuclei and the number of occurrences of each are given, there being only one way in which any such number can be expressed in the scale whose radix is r, it follows that there is but one partition of the previous kind in which one of the latter kind can originate, and there is thus a one-to-one correspondence, and consequently equality of content between the two systems of partitions.

(42) To return to the case of $r = 2$, with which alone we shall be here occupied, we see that the number of parts in the unequal partition which corresponds after this fashion with a partition made up of given odd numbers depends on the sum of the places occupied when the number of occurrences of each of the odd numbers is expressed in the notation of dual arithmetic. Such correspondence then is eminently arithmetical and transcendental in its nature, depending as it does on the forms of the numbers of repetitions of each different integer with reference to the number 2.

Very different is the kind of correspondence which we are now about to consider between the self-same two systems, as well in its nature, which is essentially graphical, as in its operation, which is to bring into correspondence the two systems, not as wholes but as separated each of them into distinct classes; and it is a striking fact that the pairs arithmetically and graphically associated will be entirely different, thus evidencing that correspondence is rather a creation of the mind than a property inherent in the things associated*.

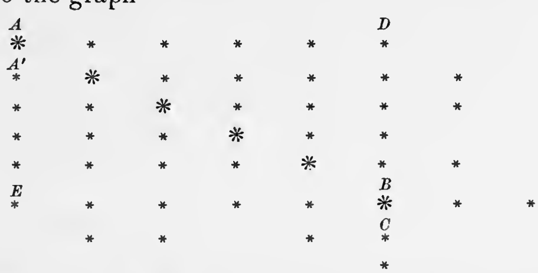
* Just so it is possible for two triangles to stand in a treble perspective relation to each other, as I have had previous occasion to notice in this *Journal*.

(43) I shall call the totality of the partitions of n consisting of odd numbers the U, and that consisting of unequal numbers the V system.

I say that any U may be converted into a V by the following rule: Let each part of the given U be converted into an equilateral bend, and these bends fitted into one another as was done in the problem of converting the reciprocal of

$$(1-ax)(1-ax^3)(1-ax^5)\ldots$$

into an infinite series, considered in the preceding section. We thus form what may be called a bent graph. Then, as there shown, such graph may be dissected into a diagonal line of points and two precisely similar regular graphs. The graph compounded of the diagonal and one of these, it is obvious, will also be regular, and I shall call it the major component of the bent graph; the remaining portion may be called the minor component. Each of these graphs will be bounded by lines inclined to each other at an angle one-half of that contained between the original bounding lines, and each may be regarded as made up of bends fitting into one another. The contents of these bends taken in alternate succession, commencing with the major graph, will form a series of continually decreasing numbers, that is to say, a V partition. As an example let 11 11 9 5 5 5 be the given U partition; this gives rise to the graph



Reading off the bends on the major and minor graphs alternately, commencing with BAD, $CA'E$ respectively, there results the regularized partition into unequal numbers

11 10 9 8 6 2.

(44) The application of the rule is facilitated to the eye by at once constructing a graph, the number of points in whose horizontal lines are the major halves of the given parts, and construing this to signify two graphs, one the graph actually written down, the other the same graph with its first column omitted; for instance in the case before us the graph will be*

* This may be regarded as a parallel-ruler form of dislocation of the figure produced by making the portion to the right of the diagonal of larger asterisks revolve about that diagonal

If we call the lines and columns in the directions of the lines and columns of the Durfee-square appurtenant to the graph $a_1a_2 \dots a_i$, $\alpha_1\alpha_2 \dots \alpha_i$ [i (here 3) being the extent of the side of the square], the partition given by the rule will be

$$a_1 + \alpha_1 - 1, \quad a_1 + \alpha_2 - 2, \quad a_2 + \alpha_2 - 3, \quad a_2 + \alpha_3 - 4, \quad a_3 + \alpha_3 - 5, \dots$$

$$\dots [a_{i-1} + \alpha_{i-1} - (2i-3)], \quad [a_{i-1} + \alpha_i - (2i-2)], \quad [a_i + \alpha_i - (2i-1)], \quad [a_i - i],$$

and inasmuch as

$$a_1 = \text{ or } > a_2 = \text{ or } > a_3 \dots \quad \text{and} \quad \alpha_1 = \text{ or } > \alpha_2 = \text{ or } > \alpha_3 \dots$$

the above series is necessarily made up of continually decreasing numbers, at all events until the last term is reached. But this term will form no exception, for the fact of i being the content of the side of the square belonging to the transverse graph $\alpha_1, \alpha_2 \dots, \alpha_i, \alpha_{i+1} \dots$ implies that $\alpha_i =$ or $> i$, hence

$$[a_i + \alpha_i - (2i-1)] - (a_i - i) = \alpha_i - i + 1 > 0.$$

In the above example the side of the square *nucleus* in the original total graph was supposed to be the same for the major and minor graphs of which it is composed. If we suppose that graph to contain only i nodes in the ith line, then the side of the square to the minor graph which it contains will be $i - 1$, and the number of parts given by the angular readings of the two graphs combined will be $2i - 1$ instead of $2i$, as for example if the 3rd line in the graph above written be 3 instead of 5, the resulting partition will be 11 10 9 8 2, but we may, if we please, regard this as 11 10 9 8 2 0 and the last term will then still be $a_i - i$, and the general expression will remain unchanged from what it was before.

Next I proceed to the converse of what has been established, namely, that every U may be transformed by the rule into a V, and shall show that any V may be derived from some one (and only one) U.

Whether the number of effective parts in the given V be odd or even, we may always suppose it to be even by supplying a zero part if necessary, and may call the parts $l_1, \lambda_1, l_2, \lambda_2 \dots, l_i, \lambda_i$. Suppose that it is capable of being derived from a certain U: form with the parts of U a graph expressed in the usual way by equilateral bends or elbows, then the side of the square appurtenant to the regular graph formed by the major half of this, say G, must have for content the given number i.

---

until it coincides with the portion to the left of the diagonal; the graph thus formed (merely as a matter of convenience to the eye) may be then made to revolve about an axis perpendicular to the plane, so as to bring the diagonal out of its oblique into the more usual horizontal position. All this trouble of description might have been saved by beginning not with a bent graph but with a graph formed with *straight* lines of points written *symmetrically* under each other, which is made possible by the fact of there being an *odd* number of points in each line. The graph so formed then resolves itself naturally into a major and minor regular graph.

Let $a_1, a_2 \ldots a_i, \alpha_1, \alpha_2 \ldots \alpha_i$ be the contents of the first i rows and first i columns respectively of G, then the equations to be satisfied are

$$a_1 + \alpha_1 - 1 = l_1, \quad a_2 + \alpha_2 - 3 = l_2, \quad a_3 + \alpha_3 - 5 = l_3 \ldots, \quad a_i + \alpha_i - (2i-1) = l_i,$$
$$a_1 + \alpha_2 - 2 = \lambda_1, \quad a_2 + \alpha_3 - 4 = \lambda_2, \quad a_3 + \alpha_4 - 6 = \lambda_3 \ldots, \quad a_i \quad - i \quad = \lambda_i.$$

Hence

$$a_1 - a_2 = \lambda_1 - l_2 - 1 \quad a_2 - a_3 = \lambda_2 - l_3 - 1 \ldots$$
$$a_{i-1} - a_i = \lambda_{i-1} - l_i - 1 \quad a_i = \lambda_i + i,$$
$$\alpha_1 - \alpha_2 = l_1 - \lambda_1 - 1 \quad \alpha_2 - \alpha_3 = l_2 - \lambda_2 - 1 \ldots$$
$$\alpha_{i-1} - \alpha_i = l_{i-1} - \lambda_{i-1} - 1 \quad \alpha_i = l_i - \lambda_i + i - 1,$$

and for all values of θ,

$$l_\theta > \lambda_\theta > l_{\theta+1}.$$

Hence $a_1, a_2 \ldots a_i$ are all positive, and $\alpha_1, \alpha_2 \ldots \alpha_i$ are all at least equal to i. There will therefore be one and only one graph G satisfying the required conditions, namely a graph the contents of whose lines are

$$a_1, a_2, \ldots a_i, \quad A_1, A_2, \ldots A_{\alpha_i} - i$$

[where $A_1, A_2, \ldots A_{\alpha_i} - i$ is the conjugate partition to $\alpha_1 - i, \alpha_2 - i, \ldots \alpha_i - i$]; the partition U will be found by subtracting unity from the doubles of each of those parts. Thus then it has been shown that every U will give rise to some one V, and every V be derived from a determinate U; hence there must exist a one-to-one correspondence between the U and V systems. In a certain sense it is a work of supererogation to show that there is a U corresponding to each V; it would have been sufficient to infer from the linear form of the equations that there could not be more than one U transformable into a V; for each U being associated with a distinct V it would follow that there could be no V's not associated with a U, since otherwise there would be more V's than U's, which we know *aliunde* is impossible.

As an example of what precedes let the partible number be 12. The U system computed exhaustively will be

$11.1 \quad 9.3 \quad 9.1^3 \quad 7.5 \quad 7.3.1^2 \quad 7.1^5 \quad 5^2.1^2 \quad 5.3.1^4$
$5.3^2.1 \quad 5.1^7 \quad 3^4 \quad 3^3.1^3 \quad 3^2.1^6 \quad 3.1^9 \quad 1^{12}$

Underneath of these partitions I will write the major component graph, and underneath this again the corresponding V; we shall thus have the table

11.1	9.3	9.1^3	7.5	$7.3.1^2$	7.1^5
* * * * * *	* * * * *	* * * * *	* * * *	* * * *	* * * *
*	* *	*	* * *	* *	*
		*		*	*
		*		*	*
					*
					*
7.5	6.5.1	8.4	5.4.2.1	7.4.1	9.3

$5^2.1^2$	$5.3.1^4$	$5.3^2.1$	5.1^7	3^4	$3^3.1^3$	$3^2.1^6$	3.1^9	1^{12}
* * *	* * *	* * *	* * *	* *	* *	* *	* *	$(*)^{12}$
* * *	* *	* *	$(*)^7$	* *	* *	* *	$(*)^9$	
*	*	* *		* *	* *	$(*)^6$		
*	*	*		* *	$(*)^3$			
	*							
	*							
6.3.2.1	8.3.1	6.4.2	10.2	5.4.3	7.3.2	9.2.1	11.1	12

Thus we obtain for the V system:

7.5 6.5.1 8.4 5.4.2.1 7.4.1 9.3 6.3.2.1 8.3.1
6.4.2 10.2 5.4.3 7.3.2 9.2.1 11.1 12

which are all the ways in which 12 can be broken up into unequal parts*.

The U's corresponding to those given by the arithmetical method of effecting correspondence would be:

$7.5 \quad 1.3^2.5 \quad 1^{12} \quad 1^7.5 \quad 1^5.7 \quad 3.9 \quad 1^3.3^3 \quad 1^9.3 \quad 1^6.3^2$
$1^2.5^2 \quad 3.1^4.5 \quad 1^2.3.7 \quad 1^3.3^3 \quad 11.1 \quad 3^4$

instead of

$11.1 \quad 9.3 \quad 9.1^3 \quad 7.5 \quad 7.3.1^2 \quad 7.1^5 \quad 5^2.1^2 \quad 5.3.1^4$
$5.3^2.1 \quad 5.1^7 \quad 3^4 \quad 3^3.1^3 \quad 3^2.1^6 \quad 3.1^9 \quad 1^{12}$

so that there is absolutely not a single pair the same in the two methods of conjugation.

(45) The object, however, of instituting the graphical correspondence is not to exhibit this variation, however interesting to contemplate, but to find a correspondence between the two systems which shall resolve itself into correspondences between the classes into which each may be subdivided.

Thus we may call U_i that class of U's in which there are i distinct odd numbers, and V_i that class of V's in which there are i sequences with a gap between each two successive ones: the theorem now to be established is that the V corresponding to any U_i is a V_i, so that class corresponds with class, and as a corollary, that the number of ways in which n can be made up by a series of ascending numbers constituting i distinct sequences is the same as the number of ways in which it can be composed with any i distinct odd numbers each occurring any number of times. This part of the investigation which I will presently enter upon is purely graphical. A few remarks and illustrations may usefully precede.

In the example above worked out it will be observed that there are three classes of U's, namely,

$1^{12} \quad 3^4: \quad 11.1 \quad 9.3 \quad 9.1^3 \quad 7.5 \quad 7.1^5 \quad 5^2.1^2$
$3^3.1^3 \quad 3^2.1^6 \quad 3.1^9: \quad 7.3.1^2 \quad 5.3.1^4 \quad 5.3^2.1$

* In Note D, *Interact*, Part 2, I show how this transformation can be accomplished by the continual doubling of a string on itself.

and three classes of V's agreeing with those above in the number of partitions in each, namely,

12 3.4.5: 11.1 9.3 10.2 8.4 7.5 9.2.1
7.3.2 6.5.1 5.4.2.1: 8.3.1 7.4.1 6.4.2.

So again for $n=16$ there will be found to be eleven partitions into odd parts of the third class, which, with their quasi-graphs and corresponding partitions into unequal parts are exhibited below:

$11.3.1^2$	$9.5.1^2$	$9.3^2.1$	$9.3.1^4$	$7.5.1^4$
* * * * * * * * (*)^2	* * * * * * * * (*)^2	* * * * * * * * * *	* * * * * * * (*)^4	* * * * * * * (*)^4
9.6.1	8.5.2.1	8.6.2	10.5.1	9.4.2.1

$7.3.1^6$	$7.3^2.1^3$	$5^2.3.1^3$	$5.3^3.1^2$	$5.3^2.1^5$	$5.3.1^8$
* * * * * * (*)^6	* * * * * * * * (*)^3	* * * * * * * * (*)^3	* * * * * * * * * (*)^2	* * * * * * * (*)^5	* * * * * (*)^8
11.4.1	9.5.2	8.4.3.1	8.5.3	10.4.2	12.3.1

The transformed partitions above written are all of them of the third class (that is consist of three distinct sequences) and comprise all that exist of that class. 16 will correspond to 1^{16} and 1.3.5.7 to itself. All the other partitions of each of the two systems will be of the second class, and will necessarily have a one-to-one graphical correspondence inasmuch as the entire systems have been proved to have such correspondence.

It is worthy of preliminary remark that the association of the first classes of U's and V's given in the previous section will be identical with the association furnished by the graphical method—but whereas in converting V into U by the antecedent process, the two cases of the sequence being of an odd or even order had to be separately considered, the graphical method is uniform in its operation.

Thus 9 8 7 6 a sequence of an even order will be given graphically by

* * * * * * * *
* * * * * * * *

corresponding to 15^2, and 9 8 7 6 5 a sequence of an odd order will be given graphically by

* * *
* * *
* * *
* * *
* * *
* * *
* * *

corresponding to 5^7, whereas it will be observed that $15^2 = (9+6)^{\frac{4}{2}}$ and $5^7 = 5^{\frac{9+5}{2}}$.

It may be noticed that when the major component is an oblate rectangle it gives rise to a sequence of an even order, and when a quadrate or prolate rectangle to one of an odd order.

I subjoin an example of the algorithm by means of which a given V can be transformed into its corresponding U, taking as a first example $V = 10\ 9\ 8\ 5\ 4\ 1$.

The process of finding U is exhibited below:

3	3	5	5	(9)
2	2	3	3	(8)
	4	4	2	(7)
	1	3	3	(6)
	10	8	4	(1)
	9	5	1	(2)
	1	1	1	(3)
	4	4	4	(4)
	7	7	7	(5)

$3^2 . 5^2 . 7^3$ will be the U required.

As a second example let $V = 12\ 10\ 9\ 8\ 5\ 4\ 1$; the algorithm will be as shown below:

			1	(9)
			1	(8)
1	0	0	0	(7)
2	1	1	1	(6)
12	9	5	1	(1)
10	8	4	0	(2)
1	3	3	0	(3)
8	8	6	4	(4)
15	15	11	7	(5)

1 7 11 15 15 will be the U required. Lines (1) and (2) are the parts of the given V written alternately in the upper and lower line; lines (3) and (6) are obtained by oblique and direct subtraction performed between (1) and (2); line (4) is obtained from (3) by adding the number of terms in (1) to the last term in (3) which gives the last term in (4) and then adding in successively the other terms in (3) each diminished by one unit; (7) is derived from (6) by diminishing each term in the latter by a unit and taking the continued sum of the terms thus diminished; (8) is found by the usual

rule of "calling"* from its conjugate (7); and finally (5) and (9) are obtained by subtracting a unit from the doubles of the several terms in (4) and (8).

It thus becomes apparent that the passage back from a V to a U is a much more complicated operation than that of making the passage from a U to a V, so much more so that it would seemingly have been labour in vain to have attacked the problem of transformation by beginning from the V end.

(46) I now proceed to the main business, which is to show that any U containing i distinct odd numbers will, by the method described, be graphically converted into a V containing i distinct sequences.

Let G be any regular graph; H what G becomes when the first column of G is removed; a, b, c, d ... the contents of the angles of G, H taken in succession.

Also let i be the number of lines of unequal content in G, j the number of distinct sequences in $a, b, c, d, e, \ldots$.

The two first lines of G, say L, L', and also the two first columns, say K, K', may be equal or unequal†.

$$\text{If } L = L' \text{ and } K = K',\ a - 1 = b,\ b - 1 = c.$$
$$\text{If } L = L' \text{ and } K > K',\ a - 1 = b,\ b - 1 > c.$$
$$\text{If } L > L' \text{ and } K = K',\ a - 1 > b,\ b - 1 = c.$$
$$\text{If } L > L' \text{ and } K > K',\ a - 1 > b,\ b - 1 > c.$$

Let G', H' represent what G, H become on removing the first bend, that is the first line and the first column, and let i', j' be the values of i, j for G', H', so that j' is the number of sequences in $c, d, e \ldots$.

It is obvious from what precedes that in the four cases considered $j' = j$, $j' = j - 1$, $j' = j - 1$, $j' = j - 2$ respectively. But in these four cases $i' = i$, $i' = i - 1$, $i' = i - 1$, $i' = i - 2$ respectively.

Hence on each supposition $i - j = i' - j'$, and continuing the process by removing each bend in succession, $i - j$ must for any number of bends have the same values as it has for *one* bend; but in that case if h and k are the contents of the line and column of the bend, the reading of the corresponding G, G' will be $h + k - 1, h - 1$, so that for that case j will be 1 or 2 according as h and k are not or are both greater than 1, that is according as i is 1 or 2‡.

* I borrow this term from the vernacular of the American Stock Exchange.

† For brevity I use line and column to signify the extent of (that is, the number of nodes in) either.

‡ The final graph after denudation pushed as far as it will go must be either a single bend, a column, a line or a single node. In the first case $i=2, j=2$, in each of the remaining three cases $i=1, j=1$.

Hence $i-j$ is always equal to zero, consequently a U of the ith class will be transformed by the graphical process into a V of the ith class, as was to be proved.

(47) I have previously noticed [p. 25 above] that the simplest case of $i=j=1$ leads to the formula

$$\frac{q}{1-q}+\frac{q^3}{1-q^3}+\frac{q^5}{1-q^5}+\frac{q^7}{1-q^7}+\ldots=\frac{q}{1-q}+\frac{q^3}{1-q^2}+\frac{q^6}{1-q^3}+\frac{q^{10}}{1-q^4}+\ldots,$$

which is a sort of pendant to Jacobi's formula

$$\frac{q}{1+q}-\frac{q^3}{1+q^3}+\frac{q^5}{1+q^5}-\frac{q^7}{1+q^7}+\ldots=\frac{q}{1+q}-\frac{q^3}{1+q^2}+\frac{q^6}{1+q^3}-\frac{q^{10}}{1+q^4}+\ldots\text{*}.$$

These formulae may be derived from one another or both obtained simultaneously as follows: From addition of the left-hand sides of the two equations there results the double of

$$\frac{q}{1-q^2}+\frac{q^6}{1-q^6}+\frac{q^5}{1-q^{10}}+\frac{q^{14}}{1-q^{14}}+\ldots \text{ or of } \sum_{i=1}^{i=\infty}\left(\frac{q^{4i-3}}{1-q^{8i-6}}+\frac{q^{8i-2}}{1-q^{8i-2}}\right),$$

and from addition of the right-hand sides of the same there results the double of

$$\frac{q}{1-q^2}+\frac{q^5}{1-q^4}+\frac{q^6}{1-q^6}+\frac{q^{14}}{1-q^8}+\ldots \text{ or of } \sum_{i=1}^{i=\infty}\left(\frac{q^{i(2i-1)}}{1-q^{4i-2}}+\frac{q^{i(2i+3)}}{1-q^{4i}}\right).$$

Consequently in order by the operation of addition of the two equations to deduce one from the other we must be able to show that these expressions are identical: observing then that $4i-3$ and $8i-2$ are odd and even respectively for all values of i, but $i(2i-1)$ and $i(2i+3)$ odd or even, according as for i, $2i-1$ or $2i$ be written, it has to be shown that

$$\sum_1^\infty \frac{q^{4i-3}}{1-q^{8i-6}}=\sum_1^\infty\left(\frac{q^{2i-1.4i-3}}{1-q^{8i-6}}+\frac{q^{2i-1.4i+1}}{1-q^{8i-4}}\right) \qquad (A)$$

and

$$\sum_1^\infty \frac{q^{8i-2}}{1-q^{8i-2}}=\sum_1^\infty\left(\frac{q^{i(8i-2)}}{1-q^{8i-2}}+\frac{q^{i(8i+6)}}{1-q^{8i}}\right). \qquad (B)$$

(A) is equivalent to

$$\sum_1^\infty q^{4i-3}\frac{1-q^{i-1.8i-6}}{1-q^{8i-6}}=\sum_1^\infty \frac{q^{2i-1.4i+1}}{1-q^{8i-4}}$$

or

$$\sum_1^\infty q^{4i+1}\frac{1-q^{i(8i+2)}}{1-q^{8i+2}}=\sum_1^\infty \frac{q^{2i-1.4i+1}}{1-q^{8i-4}}.$$

Hence if i signify any number from 1 to ∞ and k signify any number from 0 to $i-1$, it has to be shown that $(4i+1)(2k+1)$ contains the same integers and each taken the same number of times as $(2m-1)(4m+1+4n)$, where m is any number from 1 to ∞ and n is any number from 0 to ∞. But the $(4i+1)(2k+1)$ is the same as $(2k+1)\{4(k+l+1)+1\}$ where k and l

* My formula is what Jacobi's becomes when every middle *minus* sign in it is changed into *plus* and every inferior *plus* sign into *minus*.

each extend from 0 to ∞, and the $(2m-1)(4m+4n+1)$ is the same as $(2m+1)\{4(m+n+1)+1\}$ where m and n each extend from 0 to ∞, and the two latter expressions on writing $k=m$, $l=n$ become identical.

Again (B) is equivalent to

$$\sum_1^\infty q^{8i-2}\frac{1-q^{i-1.8i-2}}{1-q^{8i-2}} = \sum_1^\infty \frac{q^{i(8i+6)}}{1-q^{8i}}.$$

Hence we have to show that $(8i-2)(1+j)$ when $i=2, 3, \ldots \infty$ and $j=0, 1, 2, \ldots, (i-2)$, or say $(8i+6)(1+j)$, where $i=1, 2, \ldots \infty$ and $j=0, 1, 2, \ldots (i-1)$ is identical with $l(8l+6+8m)$, where $l=1, 2, \ldots \infty$ and $m=0, 1, 2, \ldots \infty$; the former of these is identical with

$$(1+j)\{8(j+k+1)+6\},$$

where $j=0, 1, \ldots \infty$; $k=0, 1, \ldots \infty$, and the latter is identical with

$$(1+l)\{8(l+m+1)+6\},$$

where $l=0, 1, \ldots \infty$; $m=0, 1, \ldots \infty$, consequently the two expressions are coextensive, which proves (B), and (A) has been already proved. Hence we see that either of the two original equations can be deduced from the other from the fact that their sum leads to an identity.

In like manner subtraction performed between the two allied equations leads to the fissiparous equation

$$\sum_0^\infty\left\{\frac{x^{8i+2}}{1-x^{8i+2}}+\frac{x^{4i+3}}{1-x^{8i+6}}\right\} = \sum_0^\infty\left\{\frac{x^{(i+2)(2i+1)}}{1-x^{4i+2}}+\frac{x^{i+1.2i+3}}{1-x^{4i+4}}\right\},$$

which gives birth to the pair

$$\sum_0^\infty \frac{x^{4i+3}}{1-x^{8i+6}} = \sum_0^\infty\left\{\frac{x^{2i+3.4i+3}}{1-x^{8i+6}}+\frac{x^{2i+1.4i+3}}{1-x^{8i+4}}\right\} \qquad (C)$$

and

$$\sum_0^\infty \frac{x^{8i+2}}{1-x^{8i+2}} = \sum_0^\infty\left\{\frac{x^{2i+2.4i+1}}{1-x^{8i+2}}+\frac{x^{2i+2.4i+5}}{1-x^{8i+8}}\right\}. \qquad (D)$$

(C) is equivalent to

$$\sum_0^\infty \frac{x^{4i+3}(1-x^{i+1.8i+6})}{1-x^{8i+6}} = \sum_0^\infty \frac{x^{2i+1.4i+3}}{1-x^{8i+4}},$$

which is an identity by virtue of the equivalence of $(4i+3)[1+2\{j<(i+1)\}]$ that is $(4j+4k+3)(1+2j)$ to $(2\lambda+1)(4\lambda+3+4\mu)$ where j, k, λ, μ each extend from zero to infinity, and

(D) is equivalent to

$$\sum_0^\infty \frac{x^{8i+2}(1-x^{i(8i+2)})}{1-x^{8i+2}} = \sum_0^\infty \frac{x^{2i+2.4i+5}}{1-x^{8i+8}},$$

which is an identity by virtue of the equivalence of $(8i+2)\{1+(j<i)\}$ that is $\{8(j+k+1)+2\}(1+j)$ to $(2\lambda+2)(4\lambda+5+4\mu)$, each symbol j, k, μ having as before the same range, namely from zero to infinity. Thus then the difference of the two allied equations (as previously their sum) is reduced to an identity which establishes the validity of each of them.

INTERACT, PART 2.

With notes of many a wandering bout,
Of linkèd sweetness long drawn out.
L'Allegro.

(48) D. *Transformation of Partitions by the Cord Rule.*—The figures below are designed to show how it is possible by means of the continuous doubling of a string upon itself to pass from an arrangement of groups of repetitions of r distinct odd integers to the corresponding one with like sum, made up of r distinct sequences. Each of the two figures duplicated by rotation about its upper horizontal boundary of nodes through two right angles will represent an arrangement of repeated odd numbers, the parts being represented by the contents of the *vertical* lines in the figures so duplicated.

Fig. 1.

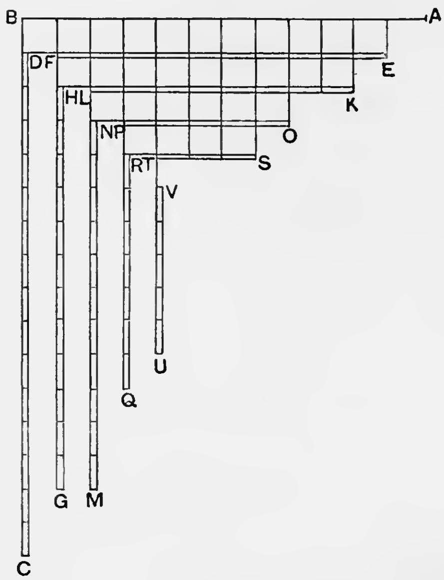
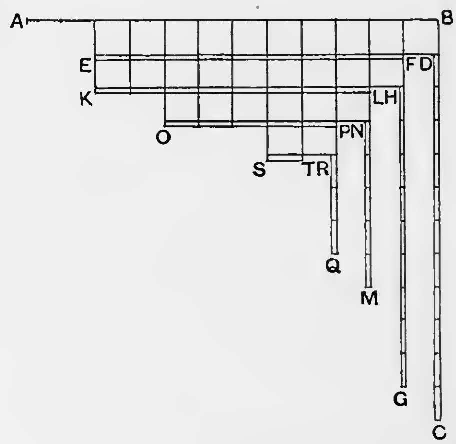


Fig. 2.



The first duplicated figure represents the arrangement 33, 29^2, 23, 21, 9^3, 7, 5^2, 3, 1 whose sum is 183; its correspondent will be the contents of the lengths of * *ABC, CDE, EFG, GHK, KLM, MNO, OPQ, QRS, STU, UV*, namely the arrangement 29, 27, 24 (22, 21,), 18, 14, 12, 10, 6 which is the same number 183 partitioned into (ten parts but) nine sequences: the second duplicated figure represents the arrangement 25, 23, 17, 15, 9^2, 7^3, 5^2, 1^2, whose sum is 130; its correspondent is represented by the lengths of *ABC, CDE, DEF, FGH, HKL, LMN, NOP, PQR, RST, TU*, which is the same number 130 partitioned into the (nine parts but) eight sequences 25, 22 (20, 19,), 15, 12, 10, 6, 1.

* A line containing i units of length represents $(i+1)$ nodes.

(49) E. *On Graphical Dissection.*—It may be not unworthy of notice that there is a sort of potential anticipation of Mr Durfee's dissection of a symmetrical graph, in a method which, whether it is generally known or not I cannot say, but is substantially identical with Dirichlet's for finding approximately $\sum_1^n \left[\frac{n}{i}\right]$ and other such like series (a bracketed quantity being used to signify that quantity's integer part). Constructing the hyperbola $xy = n$, drawing its ordinates to the abscissas 1, 2, 3, ... n, and in each of them planting nodes to mark the distances 1, 2, 3, ... from its foot, there results a *symmetrical graph* included between one branch of the curve, its two asymptotes, and lines parallel to and cutting each of them at the distance n from the original. Its content will be the sum in question. The Durfee-square to it will be limited by the square whose side is $[\sqrt{n}]$, and this added to the original area gives twice over the area in which the number of nodes is $\sum_1^{\sqrt{n}} \left[\frac{n}{i}\right]$, and consequently neglecting magnitudes of the order $\sqrt{n}$,

$$\sum_1^n \left[\frac{n}{i}\right] = 2n \sum_1^{\sqrt{n}} \frac{1}{i} - i^2 = n(\log n + 2C - 1)$$

and as a corollary

$$\sum_1^{\sqrt{n}} \left\{\frac{n}{i} - \left[\frac{n}{i}\right]\right\} = n(C - 2C + 1) = (1 - C)n,$$

where C is Euler's number ·57721, so that $1 - C$ for large values of n will be the average value of the fractional part of n divided by an inferior number. Furthermore a similar graph, but with $xy = 2n$ diminished by the portion contained between a branch of the new curve, one of its asymptotes and two parallel ordinates cutting that asymptote at distances n and $2n$ from the origin (which portion obviously contains $(2n - n)$ that is n nodes) will represent $\sum_1^n \left[\frac{2n}{i}\right]$, and consequently the sum $\sum_1^n \left\{\left[\frac{2n}{i}\right] - 2\Sigma \left[\frac{n}{i}\right]\right\}$, that is (see *Berl. Abhand.* 1849, p. 75) the number of times that $\frac{n}{i} - \left[\frac{n}{i}\right]$ equals or exceeds $\frac{1}{2}$, as i progresses from 1 to n (within the same limits of precision as previously) $= 2n(\log 2n + 2C - 1) - n$ less $2n(\log n + 2C - 1)$, that is $= (\log 4 - 1)n$, so that the probability of the fractional part of n divided by an inferior number not falling under $\frac{1}{2}$ is $\log 4 - 1$*.

* What precedes I recall as having been orally communicated to me many years ago by the late ever to be regretted Prof. Henry Smith, so untimely snatched away when in the very zenith of his powers, and so to say, in the hour of victory, at the moment when his intellectual eminence was just beginning to be appreciated at its true value, by the outside world. I was under the impression until lately that he was quoting literally from Dirichlet when so communicating with me, but as the geometrical presentation given in the text is not to be found in the

(50) F. *Mr Ely's method of finding the asymptotic value of the number of improper fractions with a very large given numerator which are nearer to the integer below than to the integer above**.

"Let a number n be divided by all the numbers from 1 to n; then a value is required for the number of residues which are equal to or greater than $\frac{1}{2}$. An example will make evident a method by which we may obtain limits to the value sought. If n be 100 the residues $= > \frac{1}{2}$ are

$$(1)\quad \frac{49}{51}\ \frac{48}{52}\ \frac{47}{53}\ \frac{46}{54}\ \frac{45}{55}\ \frac{44}{56}\ \frac{43}{57}\ \frac{42}{58}\ \frac{41}{59}\ \frac{40}{60}\ \frac{39}{61}\ \frac{38}{62}\ \frac{37}{63}\ \frac{36}{64}\ \frac{35}{65}\ \frac{34}{66}$$

$$(2)\quad \frac{32}{34}\ \frac{30}{35}\ \frac{28}{36}\ \frac{26}{37}\ \frac{24}{38}\ \frac{22}{39}\ \frac{20}{40}$$

$$(3)\quad \frac{22}{26}\ \frac{19}{27}\ \frac{16}{28}$$

$$(4)\quad \frac{16}{21}\ \frac{12}{22}$$

$$(5)\quad \frac{15}{17}\ \frac{10}{18}$$

$$(6)\quad \frac{10}{15}$$

$$(a)\quad \frac{4}{6}\ \frac{4}{8}\ \frac{9}{13}$$

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memoir cited from the *Berlin Transactions*, I infer that it originated with himself. In comparing Mertens' memoir, *Crelle*, 1874, with Dirichlet's (1849), upon which it is a decided step in advance, one cannot fail to be struck with surprise that the point to which the closer drawing of the limits to the values of certain transcendental arithmetical functions achieved by the former is owing, should have escaped the notice of so profound and keen an intellect as Dirichlet's, and those who came after him in the following quarter of a century. The point I refer to is the almost self-evident fact that if in the cases under consideration

$$\Sigma\phi\,(Fi\,.\,x) = \psi x \text{ then } \phi x = \Sigma\mu\,(i)\,\psi\,(Fi\,.\,x)$$

where $\mu\,(i)$ means 0, if i contains any repeated prime factors, but otherwise 1 or $\bar{1}$ according as the number of prime factors in i is even or odd. Dirichlet works with a function given implicitly by an equation, Mertens with the same function expressed in a series, wherein exclusively lies the secret of his success.

* It is proper to state that what follows in the text was handed in to me by Mr Ely on the morning after I had proposed to my class to think of some "common sense method" to explain the somewhat startling fact brought to light by Dirichlet, of more than three-fifths of the residues of n in regard to $i = 1, 2, 3, \ldots n$ being less than $\frac{i}{2}$. Mr Ely's method shows at once, in a very common sense manner, why the proportion must be considerably greater than the half, inasmuch as whilst the terms in the first few harmonic ranges are approximately $\frac{n}{1\,.\,2}, \frac{n}{2\,.\,3}, \frac{n}{3\,.\,4}$, etc., in number, the number of them which employed as denominators to n give fractional parts greater than $\frac{1}{2}$, instead of being the halves of these are only $\frac{n}{2\,.\,3}, \frac{n}{3\,.\,5}, \frac{n}{4\,.\,7}$, etc. The mean value in both methods to quantities of the order of $\sqrt{n}$ *inclusive*, turns out to be the same, whichever method is employed, but the margin of unascertained error by the use of Mr Ely's method (as compared with Dirichlet's) is reduced in the proportion of $1 : 1 + \sqrt{2}$, that is, nearly $2 : 5$.

In which it will be observed that the residues $= > \frac{1}{2}$ occur in batches. Let X be the whole number, and x_i the number in batch i. In batch i the numerators decrease by i and the denominators increase by 1. (Those marked (a) of which the denominators are less than $\sqrt{200}$ are left out of account for the present.) It is evident for the general case we have approximately

$$\frac{\left[\frac{n}{i+1}\right] - ix_i}{\left[\frac{n}{i+1}\right] + x_i} = \frac{1}{2}$$

or accurately

$$x_i = \left[\frac{n}{(i+1)(2i+1)}\right] \text{ or } \left[\frac{n}{(i+1)(2i+1)}\right] + 1\text{*."}$$

Mr Ely is then able to show that by limiting the calculation of x_i to the values of i which do not exceed $[\sqrt{n}/2]$, so that roughly speaking the character of $\sqrt{2n}$ of the remainders is left undetermined (and no account taken of them in finding the value of X), and giving to x_i its approximate value $\frac{n}{(i+1)(2i+1)}$, and then extending the series $\frac{n}{2.3} + \frac{n}{3.5} + \frac{n}{4.7}$ beyond the $[\sqrt{n}/2]$th term, where it ought to stop, to infinity, the errors arising from each of these three sources† and therefore their combined effect will be of the order $\sqrt{n}$, so that the asymptotic value of X will be

$$\left(\frac{1}{2.3} + \frac{1}{3.5} + \frac{1}{4.7} + \dots\right) n,$$

which is $(2 \log 2 - 1) n$, with an uncertainty of the order $\sqrt{n}$, as was to be shown.

(51) It may be seen that Mr Ely's method consists in distributing the n numbers from n to 1 into what I have elsewhere termed *harmonic ranges* and determining what portions of the several ranges employed as denominators to n give fractional parts, greater or less than $\frac{1}{2}$. It may assist in forming a more vivid idea of this kind of distribution, if the reader takes a definite case, say of $n = 121$, the first (10) harmonic ranges will then comprise

* I find by an exact calculation that if R is the remainder of n in regard to $(i+1)(2i+1)$ and $R = \lambda(i+1) + \mu$, where $\lambda < 2i+1$ and $\mu < i+1$, then for $\lambda = 2\theta - 1$ or 2θ, $x_i = \left[\frac{n}{(i+1)(2i+1)}\right] + 1$ if $\mu = i-1$ or $i-2$ ... or $i-\theta$, and $x_i = \left[\frac{n}{(i+1)(2i+1)}\right]$ for all other values of μ. Hence it follows that out of $(2i^2+3i+1)$ successive values of n, (i^2+i) and (i^2+2i+1) will be the respective numbers of the cases for which the one or the other of these two values of x_i is employed, so that for larger values of i the chances for the two values are nearly the same, but with a slight preponderance in favour of the smaller value. See p. [54].

† The error from the first cause makes the determination of X too small by an unknown amount, that from the third cause too large by a known amount, and that from the second too large or too small (as it may happen) by an unknown amount.

all the numbers from 121 to 12 inclusive, and the remaining 111 harmonic ranges will comprise the remaining 11 numbers from 11 to 1; that is to say 11 of them will contain a single number, and the remaining 100 ranges be vacant of content.

So again if $n = 20$ the first four ranges will contain all the numbers from 20 to 5 inclusive; the 5th, 6th, 9th and 20th range will consist of the sole numbers 4, 3, 2, 1, and the remaining 12 ranges will be vacant. I shall proceed to compare the precision of Mr Ely's result with that of Dirichlet's—for this purpose it will be enough to determine the asymptotic value of the uncertainty and to take no account of quantities of a lower order than $\sqrt{n}$.

Let us then suppose that $\sqrt{(kn)}$ ranges are preserved, and consequently $\sqrt{\left(\frac{n}{k}\right)}$ fractions left out (k being an arbitrary constant which will eventually be determined so as to make the uncertainty a minimum).

The first cause of error necessitates a correction of which the limits are 0 and $\sqrt{\left(\frac{n}{k}\right)}$; the second cause a correction of which the limits are $\sqrt{(kn)}$ and $-\sqrt{(kn)}$; and the third, namely the overreckoning of

$$\frac{n}{(j+1)(2j+1)} + \frac{n}{(j+2)(2j+3)} + \cdots$$

where $j = \sqrt{(kn)}$, a correction of which the value is $-\frac{n}{2j}$ or $-\frac{1}{2}\sqrt{\left(\frac{n}{k}\right)}$.

Hence making $(\log 4 - 1)\, n = U$, the superior limit of X is

$$U + \frac{1}{2}\sqrt{\left(\frac{n}{k}\right)} + \sqrt{(kn)},$$

and the inferior limit $U - \frac{1}{2}\sqrt{\left(\frac{n}{k}\right)}\sqrt{(kn)}$. Consequently $X = U + \rho n^{\frac{1}{2}}$ where $\rho < \sqrt{k} + \frac{1}{2}\sqrt{\left(\frac{1}{k}\right)}$, of which the minimum value is found by making $k = \frac{1}{2}$, so that $\rho < \sqrt{2}$ and the *uncertainty* is $\sqrt{2} \,.\, n^{\frac{1}{2}}$. Adopting Mertens' asymptotic value of the *uncertainty* of $\sum_1^n \left[\frac{n}{i}\right]$, namely $\sqrt{n}$, and using Dirichlet's formula, $\sum_n^1 \left[\frac{2n}{i}\right] - 2\sum_n^1 \left[\frac{n}{i}\right]$, X has the same mean value as above, but the uncertainty becomes $(\sqrt{2} + 2)\, n^{\frac{1}{2}}$ which is nearly two and a half times as great as that given by the direct method employed by Mr Ely.

I use the word *uncertainty*, it will be noticed, in a different sense from *error*; the latter is objective, referring to fact, the former subjective, referring to knowledge. Both methods in the case here presented give the same mean value, and therefore the *error* is the same, but the uncertainty is widely different according to the method made use of. Of two formulae referring to the same fact one might very well give the smaller error and the other the smaller uncertainty.

I have shown above that for considerable values of i, the average value of x_i is $\frac{n}{(i+1)(2i+1)}+\frac{1}{2}$; if then it may be assumed (and there seems no reason for suspecting the contrary) that for $i=1, 2, \ldots, \sqrt{2n}$, the mean value of $\frac{n}{i}-\left[\frac{n}{i}\right]$ is $\frac{1}{2}$, U will not only be the mean value of the known limits of X but also the mean value of X itself. The value found for k shows that the most advantageous mode of employing Mr Ely's method is to make the series $\frac{n}{2.3}+\frac{n}{3.5}+\ldots\frac{n}{(i+1)(2i+1)}+\ldots$ stop at one of the terms which is approximately equal to unity.

(52) It is not without interest to consider the exact law for the extent of a harmonic range of a given denomination, say i: this it is easily seen will be always equal to $\left[\frac{n}{i^2+i}\right]$ or $\left[\frac{n}{i^2+i}\right]+1$.

I shall regard i as given and determine the values of n which correspond to the one or the other of the two formulae: this will depend not on the absolute value of n but on its remainder in respect to the modulus i^2+i. To fix the ideas, let $i=4$ so that $i^2+i=20$, and let n take in successively all values from 40 to 59 inclusive.

Then corresponding to n equal to

40	44	48	52	56
41	45	49	53	57
42	46	50	54	58
43	47	51	55	59

the fourth range will be

10, 9	11, 10, 9	12, 11, 10	13, 12, 11	14, 13, 12
10, 9	11, 10	12, 11, 10	13, 12, 11	14, 13, 12
10, 9	11, 10	12, 11	13, 12, 11	14, 13, 12
10, 9	11, 10	12, 11	13, 12	14, 13, 12

that is in half the terms of the period $\left[\frac{n}{i^2+i}\right]$ and in the other half $\left[\frac{n}{i^2+i}\right]+1$ gives the extent of the range.

So in general, if $n=k(i^2+i)+\lambda i+\mu$, where $\lambda=0, 1, 2, \ldots i$, and $\mu\equiv 0, 1, 2, \ldots (i-1)$, when the remainder of n to modulus (i^2+i) is of the form

$\lambda(i^2+i)+\{0, 1, 2, \ldots (\lambda-1)\}$ that is in $\frac{i^2+i}{2}$ cases the extent of the ith harmonic range to n is $\left[\frac{n}{i^2+i}\right]+1$, and when of the form

$$\lambda(i^2+i)+\{\lambda, \lambda+1, \ldots (i-1)\},$$

that is in the remaining $\frac{i^2+i}{2}$ cases it is $\left[\frac{n}{i^2+i}\right]$.

As the sum of the harmonic ranges to n is n itself, and

$$\frac{n}{1.2}+\frac{n}{2.3}+\ldots+\frac{n}{n(n+1)}=n-\frac{n}{n+1},$$

it follows that if we separate all the numbers from 1 to n into two classes, say i's and j's, i being any number for which n is of the form

$$k(i^2+i)+\lambda i+0, 1, 2, \ldots (\lambda-1),$$

and j any other number within the prescribed limits, then

$$\sum_1^n \frac{n}{t}-\sum_1^n\left[\frac{n}{t}\right]=\text{number of } i\text{'s}-\frac{n}{n+1},$$

and consequently the number of the i terms has $(1-C)\,n$ for its asymptotic value.

(53) In like manner the law previously stated in a footnote, p. [51], for giving the extent of that portion of the ith range for which $\frac{n}{t}$ contains a fractional part not less than $\frac{1}{2}$ may be verified. Thus let $i=3$ then $(i+1)(2i+1)=28$, let $n=56, 57, \ldots 83$. Then for the values of n

28	32	36	40	44	48	52
29	33	37	41	45	49	53
30	34	38	42	46	50	54
31	35	39	43	47	51	55

the portion of the third range having the required character will contain the numbers

8	9	10	11	12	13	14
8	9	10	11	12	14, 13	15, 14
8	9	10	12, 11	12, 11	14, 13	15, 14
8	10, 9	11, 10	12, 11	13, 12	14, 13	15, 14

so that there are $2(1+2+3)$, that is 3.4 forms of n out of 7.4 for which the formula $\left[\frac{n}{4.7}\right]+1$ has to be employed, and so in general if R is the residue of n in respect to $(i+1)(2i+1)$, there are i^2+i cases where the formula $\left[\frac{n}{(i+1)(2i+1)}\right]+1$ and $(i+1)^2$ where the formula $\left[\frac{n}{(i+1)(2i+1)}\right]$ has to be employed.

G. *On Farey Series.*

(54) This note is a natural sequel to and has grown out of the two which precede; it has also a collateral affinity with the subject-matter of the Acts, inasmuch as a graph affords the most simple mode of viewing and stating the fundamental property of an ordinary Farey series, and any series *ejusdem generis*. For instance, let A, B, C be a reticulation in the form of an equilateral triangle, where B is a right angle, and n the number of nodes in the base or height of the triangle; if the hypothenuse be made to revolve in the plane of the triangle about (either end say about) A, the triangle formed by joining A with any two consecutive nodes of greatest proximity to the centre of rotation traversed by the rotating line will be equal in area to the minimum triangle which has any three nodes for its apices, that is its double will be equal to unity. This law of uniform description of areas (say of *equal areas in equal jerks*) is identical with the characteristic law of an ordinary Farey series which deals with terms whose number is the sum-totient τn: but it will also hold good if the triangle be scalene instead of equilateral, which corresponds to Glaisher's extension of a Farey series, to the case where the numerator and denominator of each term has its own separate limit (*Phil. Mag.* 1879), or again, when the rotation takes place about the right angle B as centre, which gives rise to a Farey series of a totally different species, defined by the inequality $ax + by < n$, or again when the hypothenuse is replaced by the quadrant of a circle or ellipse, and in an infinite variety of other cases, as for example when the graph is contained between a branch of an equilateral hyperbola and the asymptotes, which case corresponds to the subject-matter of the theory of Dirichlet (*Berl. Abhand.* 1844) concerning the sum of the number of ways in which all integers up to n can be resolved into the product of two relative primes, which is the same thing as the half of the number of divisors (containing no repeated prime factors) which enter into the several integers up to n, or as the entire number of solutions in relative primes of the inequality $xy =$ or $< n$. The law of equal description of areas ($pq' - p'q = \pm 1$), Mr Glaisher has shown very acutely, is an immediate inference (by an obvious induction) from the well-known fact that between a fraction and its two nearest convergents (namely the one ordinarily so called and that which is obtained by substituting $\delta - 1$ and 1 for the last partial quotient), no other fraction can be interposed whose denominator is not greater than that of the one first named.

From the areal-law obviously follows the equation $\frac{p''}{q''} = \frac{xp' - p}{xq' - q}$ (where $\frac{p}{q}, \frac{p'}{q'}, \frac{p''}{q''}$ are any three consecutive terms of the series), so that in order to construct explicitly such a series from the two first terms, all we have to do is to give to x at each step the highest value it can assume, consistent with the imposed limit or limits. Thus for example I have found by this method when the limiting inequality is $x + y =$ or < 15, the series

$$\frac{0}{1}\ \frac{1}{15}\ \frac{1}{14}\ \frac{1}{13}\ \frac{1}{12}\ \frac{1}{11}\ \frac{1}{10}\ \frac{1}{9}\ \frac{1}{8}\ \frac{1}{7}\ \frac{1}{6}\ \frac{2}{11}\ \frac{1}{5}$$
$$\frac{2}{9}\ \frac{1}{4}\ \frac{3}{11}\ \frac{2}{7}\ \frac{3}{10}\ \frac{1}{3}\ \frac{4}{11}\ \frac{3}{8}\ \frac{2}{5}\ \frac{3}{7}\ \frac{4}{9}\ \frac{1}{2}\ \frac{1}{1}{}^{*}$$

and the complements in respect to unity of the several terms which precede $\frac{1}{2}$ taken in reverse order, and again for $xy =$ or < 15 the series (which might be called the Dirichlet-Farey series)

$$\frac{0}{1}\ \frac{1}{15}\ \frac{1}{14}\ \frac{1}{13}\ \frac{1}{12}\ \frac{1}{11}\ \frac{1}{10}\ \frac{1}{9}\ \frac{1}{8}\ \frac{1}{7}\ \frac{1}{6}\ \frac{1}{5}$$
$$\frac{1}{4}\ \frac{2}{7}\ \frac{1}{3}\ \frac{2}{5}\ \frac{1}{2}\ \frac{3}{5}\ \frac{2}{3}\ \frac{3}{4}\ \frac{1}{1}.$$

In general if we agree to understand respectively by the *decernent* and the *secernent* to x, the number of divisors without restriction, and the number of divisors restricted to contain no square number, that go into x, and denote the sum-secernent and sum-decernent of n by Sn and Dn respectively, Dirichlet's mode of looking at the question leads immediately to the equation $\sum_1^n S\frac{n}{i^2} = Dn$. Mertens' equation $\left[Sn = \sum_1^n \mu i D\frac{n}{i^2}\right]$ obtained by a longer and somewhat more difficult process is in point of fact merely that equation *reverted*. On pointing out to Mr F. Franklin this elegant passage in Dirichlet's memoir, he remarked to me to the effect that it was an example, which might admit of wide generalization, of a concept resembling that inherent in the subject-matter of the ordinary Farey series; which excellent and keen-witted observation led me to look into the subject from the point of view herein explained. The present theory diverges from the ordinary one in quite another and more natural direction (I imagine) than that pursued by M. Darboux, whose article on the subject of quasi-Farey series (*Bulletin de la Société Mathématique de France*, tome VI.) I have not been able to obtain sight of, and can only conjecture its purport through the reference made to it in a subsequent article which I *have* been able to procure in the same journal by M. Edouard Lucas.

* It is advisable for the purpose of securing generality in reasoning upon Farey series not to omit the initial and final terms $\frac{0}{1}$, $\frac{1}{1}$ which seem generally to have been lost sight of by previous writers on the subject Even then the series is only half complete, for after $\frac{1}{1}$ should follow the reciprocals of the preceding terms until $\frac{1}{0}$ is reached. Thus a complete *ordinary* Farey series beginning with $\frac{0}{1}$ and ending with $\frac{1}{0}$ consists of two symmetrical branches with $\frac{1}{1}$ as their point of junction, each made up of two symmetrical sub-branches meeting respectively in the terms $\frac{1}{2}$ and $\frac{2}{1}$, and such that the sum of a corresponding pair of fractions on the one side of $\frac{1}{1}$ and of their reciprocals on the other side is equal to unity: whereas in the two complete branches the *product* of each corresponding pair is unity.

(55) I prove the persistency of the fundamental property of ordinary Farey series for such series generalized in the manner supposed above, as follows.

Let us use $O.\ F.\ S_i$ to denote an ordinary Farey series for which the limit is i, and $G.\ F.\ S.$ a Farey series in which, calling the numerator and denominator of any term x, y, $\phi(x, y) <= i$, $\phi(x, y)$ meaning a rational function which increases when either x or y increases. If in an $O.\ F.\ S_i$ any two consecutive terms be $\frac{a}{b}$, $\frac{c}{d}$, and in an $O.\ F.\ S_{i+1}$ $\frac{p}{q}$ intervenes between $\frac{a}{b}$, $\frac{c}{d}$ we know, p being greater than b and d, the two nearest convergents to $\frac{p}{q}$ must be contained in $O.\ F.\ S_i$, and consequently must be $\frac{a}{b}$, $\frac{c}{d}$ themselves, so that $p = a + c$, $q = b + d$, and as a corollary if $\frac{a}{b}$, $\frac{c}{d}$ be consecutive terms in any $O.\ F.\ S.$, and $\frac{p}{q}$ be any one of the terms which subsequently intervene between $\frac{a}{b}$, $\frac{c}{d}$, we must have $p =$ or $> a + c$, $q =$ or $> b + d$. In order to fix the ideas let us suppose $\phi(x, y)$ to represent $x + y$, so that $x + y <= n$.

For the values 2, 3, 4, 5, 6, 7, 8, 9 ... of n, the $G.\ F.\ S.$ will be

$$\frac{0}{1}\ \frac{1}{1};\quad \frac{0}{1}\left(\frac{1}{2}\right)\frac{1}{1};\quad \frac{0}{1}\left(\frac{1}{3}\right)\frac{1}{2}\ \frac{1}{1};\quad \frac{0}{1}\left(\frac{1}{4}\right)\frac{1}{3}\ \frac{1}{2}\left(\frac{2}{3}\right)\frac{1}{1};\quad \frac{0}{1}\left(\frac{1}{5}\right)\frac{1}{4}\ \frac{1}{3}\ \frac{1}{2}\ \frac{2}{3}\ \frac{1}{1};$$

$$\frac{0}{1}\left(\frac{1}{6}\right)\frac{1}{5}\ \frac{1}{4}\ \frac{1}{3}\left(\frac{2}{5}\right)\frac{1}{2}\ \frac{2}{3}\left(\frac{3}{4}\right)\frac{1}{1};\quad \frac{0}{1}\left(\frac{1}{7}\right)\frac{1}{6}\ \frac{1}{5}\ \frac{1}{4}\ \frac{1}{3}\ \frac{2}{5}\ \frac{1}{2}\left(\frac{3}{5}\right)\frac{2}{3}\ \frac{3}{4}\ \frac{1}{1};$$

$$\frac{0}{1}\left(\frac{1}{8}\right)\frac{1}{7}\ \frac{1}{6}\ \frac{1}{5}\ \frac{1}{4}\left(\frac{2}{7}\right)\frac{1}{3}\ \frac{2}{5}\ \frac{1}{2}\ \frac{3}{5}\ \frac{2}{3}\ \frac{3}{4}\left(\frac{4}{5}\right)\frac{1}{1};\ldots$$

where the terms in parenthesis are the new terms which intervene as n increases from any value to the next following integer, and where it will be noticed that if $\frac{p}{q}$ be any such parenthesised fraction lying between $\frac{a}{b}$ and $\frac{c}{d}$, $p = a + c$ and $q = b + d$, just as in the successive form of an $O.\ F.\ S.$ The theorem to be proved may be made to depend on the following lemma.

If for any given value of n every two consecutive terms in a $G.\ F.\ S.$ appear as consecutive terms in an $O.\ F.\ S.$ for the same or any smaller value of n; this will continue to be true for all superior values of n.

The proof is immediate, for let $\frac{a}{b}$, $\frac{c}{d}$ be any two consecutive terms in the $G.\ F.\ S_j$ which are also consecutive terms in $O.\ F.\ S_i$ where $i =$ or $< j$.

If a term $\frac{p}{q}$ intervene between $\frac{a}{b}, \frac{c}{d}$ in *G. F.* S_{j+1}, $p =$ or $> a + c$, $q =$ or $> b + d$, by virtue of the remark made. But if $p > a + c$ and $q > b + d$,

$$\phi(a + c, b + d) < \phi(p, q) < j + 1,$$

but $\frac{a+c}{b+d}$ is intermediate in value between $\frac{a}{b}, \frac{c}{d}$, hence $\frac{a+c}{b+d}$ must have appeared in a *G. F.* $S_{j'}$, where $j' < j$, which is contrary to hypothesis.

Hence $\frac{a}{b}, \frac{p}{q}, \frac{c}{d}$ will have been consecutive terms in some *O. F. S.*, and in like manner any two consecutive terms in *G. F. S.* either remain consecutive in *G. F.* S_{j+1} or admit a new term between them which is consecutive to each of them in some *O. F. S.*, so that the supposed relation if it holds good for j is true for all superior values of j; but $\frac{0}{1}, \frac{1}{1}$ will in any of the supposed cases be a *G. F. S.*; consequently in all these cases no two terms are consecutive in any *G. F. S.* which are not so in some *O. F. S.*, and therefore the law of equal description of areas will apply to them equally as to the *O. F. S.*, as was to be proved.

The theory may be extended to *G. F. S.*, defined by several concurrent limiting equations. Thus for example Mr Glaisher has proved this for the case of $x <= m$, $y <= n$: I have not had time as yet to consider what are the restrictions to which the limiting functions may be subject, but the theorem is obviously an extremely elastic one, and the above proof suffices for all the special cases which I have enumerated*.

(56) I am indebted to Mr Ely for the following additional examples of Farey series, in the enlarged sense, which may interest some of my readers.

Ex. (1). $x + y =$ or < 20

$$\frac{0}{1}\ \frac{1}{19} \cdots \frac{1}{9}\ \frac{2}{17}\ \frac{1}{8}\ \frac{2}{15}\ \frac{1}{7}\ \frac{2}{13}\ \frac{1}{6}\ \frac{3}{17}\ \frac{2}{11}\ \frac{3}{16}$$

$$\frac{1}{5}\ \frac{3}{14}\ \frac{2}{9}\ \frac{3}{13}\ \frac{1}{4}\ \frac{4}{15}\ \frac{3}{11}\ \frac{2}{7}\ \frac{3}{10}\ \frac{4}{13}\ \frac{1}{3}\ \frac{5}{14}\ \frac{4}{11}\ \frac{3}{8}\ \frac{5}{13}\ \frac{2}{5}$$

$$\frac{5}{12}\ \frac{3}{7}\ \frac{4}{9}\ \frac{5}{11}\ \frac{6}{13}\ \frac{1}{2}\ \frac{7}{13}\ \frac{6}{11}\ \frac{5}{9}\ \frac{4}{7}\ \frac{7}{12}\ \frac{3}{5}\ \frac{5}{8}\ \frac{7}{11}\ \frac{2}{3}\ \frac{7}{10}\ \frac{5}{7}$$

$$\frac{8}{11}\ \frac{3}{4}\ \frac{7}{9}\ \frac{4}{5}\ \frac{9}{11}\ \frac{5}{6}\ \frac{6}{7}\ \frac{7}{8}\ \frac{8}{9}\ \frac{9}{10}.$$

Ex. (2). $x^2 + y =$ or < 20

$$\frac{1}{19} \cdots \frac{1}{9}\ \frac{1}{8}\ \frac{2}{15}\ \frac{1}{7}\ \frac{2}{13}\ \frac{1}{6}\ \frac{2}{11}\ \frac{1}{5}\ \frac{2}{9}\ \frac{1}{4}\ \frac{3}{11}\ \frac{2}{7}\ \frac{3}{10}\ \frac{1}{3}\ \frac{3}{8}\ \frac{2}{5}\ \frac{3}{7}\ \frac{1}{2}\ \frac{2}{3}\ \frac{3}{4}.$$

* Since the above was in type I have discovered the true principle of Farey series, for which see Note H following the Exodion.

Ex. (3). $y - \sqrt{x} =$ or < 15

$$\frac{1}{16} \cdots \frac{1}{8}\ \frac{2}{15}\ \frac{1}{7}\ \frac{2}{13}\ \frac{1}{6}\ \frac{2}{11}\ \frac{3}{16}\ \frac{1}{5}\ \frac{3}{14}\ \frac{2}{9}\ \frac{3}{13}\ \frac{4}{17}\ \frac{1}{4}\ \frac{4}{15}\ \frac{3}{11}\ \frac{2}{7}$$

$$\frac{5}{17}\ \frac{3}{10}\ \frac{4}{13}\ \frac{5}{17}\ \frac{1}{3}\ \frac{6}{17}\ \frac{5}{14}\ \frac{4}{11}\ \frac{3}{8}\ \frac{5}{13}\ \frac{2}{5}\ \frac{7}{17}\ \frac{5}{12}\ \frac{3}{7}\ \frac{7}{17}\ \frac{4}{9}\ \frac{5}{11}\ \frac{6}{13}$$

$$\frac{7}{15}\ \frac{8}{17}\ \frac{1}{2}\ \frac{9}{17}\ \frac{8}{15}\ \frac{7}{13}\ \frac{6}{11}\ \frac{5}{9}\ \frac{9}{16}\ \frac{4}{7}\ \frac{7}{12}\ \frac{10}{17}\ \frac{3}{5}\ \frac{11}{18}\ \frac{8}{13}\ \frac{5}{8}\ \frac{7}{11}$$

$$\frac{9}{14}\ \frac{11}{17}\ \frac{2}{3}\ \frac{11}{16}\ \frac{9}{13}\ \frac{7}{10}\ \frac{12}{17}\ \frac{5}{7}\ \frac{13}{18}\ \frac{8}{11}\ \frac{11}{15}\ \frac{3}{4}\ \frac{13}{17}\ \frac{10}{13}\ \frac{7}{9}\ \frac{11}{14}$$

$$\frac{4}{5}\ \frac{13}{16}\ \frac{9}{11}\ \frac{14}{17}\ \frac{5}{6}\ \frac{16}{19}\ \frac{11}{13}\ \frac{6}{7}\ \frac{13}{15}\ \frac{7}{8}\ \frac{15}{17}\ \frac{8}{9}\ \frac{17}{19}\ \frac{9}{10}\ \frac{10}{11}\ \frac{11}{12}\ \frac{12}{13}\ \frac{13}{14}$$

$$\frac{14}{15}\ \frac{15}{16}\ \frac{16}{17}\ \frac{17}{18}\ \frac{18}{19}\ \frac{1}{1}.$$

EXODION. *On the Correspondence between certain Arrangements of Complex Numbers.*

At which he wondred much and gan enquere
What stately building durst so high extend
Her lofty towres, unto the starry sphere.

Faerie Queene I. x. 56.

(57) Starting from the expansion in a series of $\Theta_1 x$, multiplying in the usual notation both sides of the equation by

$$(1-q^2)(1-q^4)(1-q^6)\ldots,$$

and intercalating the factors of this product between those of

$$(1-qz)(1-q^3z)\ldots(1-qz^{-1})(1-q^3z^{-1})\ldots$$

taken in alternate order, there results the equation

$$(1-qz^{-1})(1-qz)(1-q^2)(1-q^3z^{-1})(1-q^3z)(1-q^4)\ldots = \sum_{i=-\infty}^{i=+\infty}(-)^i q^{i^2} z^i,$$

and writing q^n in place of q and making $z = \mp q^m$, Jacobi (*Crelle*, Vol. XXXII. p. 166) derives the identity

$$(1 \pm q^{n-m})(1 \pm q^{n+m})(1-q^{2n})(1 \pm q^{3n-m})(1-q^{3n+m})(1-q^{4n})\ldots = \sum_{-\infty}^{+\infty}(\pm)^i q^{ni^2+mi}.$$

From this equation, using the lower sign and making $n = \frac{3}{2}$, $m = \frac{1}{2}$, he observes, may be deduced Euler's expression in a series for

$$(1-q)(1-q^2)(1-q^3)\ldots,$$

and using the *upper* sign and making $n = \frac{1}{2}$, $m = \frac{1}{2}$, another known series "given by Gauss in the first volume of the *Göttingen Commentaries* for the years 1808–11."

It is not without interest, I think, to observe that by making $n=\frac{1}{2}$, $m=\frac{1}{2}+\epsilon$ (where ϵ is an infinitesimal), and using the *lower* sign we may immediately deduce Jacobi's own celebrated postscript (so to say) to Euler's equation, namely,

$$(1-q)^3(1-q^2)^3(1-q^3)^3\ldots=\sum_{-\infty}^{+\infty}(-)^i q^{\frac{i^2+i}{2}+i\epsilon}\div(1-q^{-\epsilon})$$
$$=1-3q+5q^3-7q^6\ldots,$$

the general term being

$$\sum_0^\infty(-)^i\left\{(q^{\frac{i^2+i}{2}+i\epsilon}-q^{\frac{i^2+i}{2}-(i+1)\epsilon})\div\frac{1}{1-q^{-\epsilon}}\right\},$$

which is $$(-)^i(2i+1)\,q^{\frac{i^2+i}{2}}.$$

(58) It is obvious, that by the same right and within the same limits of legitimacy as the equation involving q, n, m (or if we please to say so in q, m) has been derived from the equation in (q, z), the equation in q, z may be recovered from the equation in q and m, if this latter can be shown to be true, morphologically interpreted for general values of m. I shall show that regarding m and n as absolutely general symbols, such as $\sqrt{(-1)}$ or $\sqrt{2}$ or ρ or the quaternion units, or any other heterogeneous or homogeneous units we please, the equation in question which I shall write under the equivalent form

$$(1\mp q^a)(1\mp q^b)(1-q^c)(1\mp q^{a+c})(1\mp q^{b+c})(1-q^{2c})\ldots=\sum_{i=-\infty}^{i=+\infty}(\mp)^i q^{\frac{i^2}{2}c+\frac{i}{2}(a-b)}$$

[where $c=a+b$, and a, b are absolutely general symbols or species of units entirely independent of one another] does hold good as a morphological identity*. Thus interpreted, it amounts to a theorem in complex quantities, dealing with arrangements of three sorts of elements which I shall call C's, B's, A's respectively, meaning by a C any non-negative integer (that is zero or any positive integer) multiple of c, by a B such multiple augmented by a single b, and by an A such multiple augmented by a single a.

The C's, the B's and the A's in any such arrangement will be regarded as three separate series, the terms in each of which flow from left to right in descending order, that is the multiples of c which represent totally or with the exception of a single b or a single a, the terms in each such series taken in severalty are to form a continually decreasing series.

* This theorem is less transcendental than Newton's binomial theorem when the same latitude is given to the meaning of the symbols in either case: for $(1+x)^m=1+mx+\frac{m^2-m}{2}x^2+\ldots$ does not admit of *direct* interpretation when m is a general symbol. The passage from numerical proximate equality to absolute identity, prepared but not perfected nor capable of being explained by infinitesimal gradation, brings to mind the analogous transfiguration of sensibility into sensation, or of sensation into consciousness, or of consciousness into thought.

The total number of elements and the number of C's will be called the major and minor parameters respectively—the relation to the modulus 2 (that is the parity or imparity) of either one of them its character: and for brevity, the terms major and minor character will be used to signify the character of the major or minor parameter. The totality of all arrangements whatever of A's, B's, C's in which *no element is repeated,* will constitute the sphere of the investigation, limited only by the absence of what I term the exceptional or isolated arrangements, consisting exclusively of a series of *consecutive* B's ending in b, or of consecutive A's ending in a. Within the prescribed sphere I shall prove that a process may be instituted for transforming any arrangement which shall satisfy the five following conditions:

(1) That it shall be capable of acting on every licit and unexceptional arrangement.

(2) That it shall transform it into another such arrangement.

(3) That operating once upon an arrangement, and then again upon the operate, it brings back the original arrangement.

(4) That it leaves the sum of the elements in the arrangement unaltered.

(5) That it reverses each of its two characters*.

From (3) it will follow that all the arrangements within the prescribed sphere are associated in pairs, and from (1) that the sum of the elements in each such pair is the same. This being so, it is obvious from the fact of the parity of the total number of elements being opposite for any pair of associated arrangements, that in the development in a series of

$$(1-q^a)(1-q^b)(1-q^c)(1-q^{a+c})\ldots,$$

no term will appear in which the index of q is other than the sum of the terms in one of the exceptional (we may now call them unconjugated or unconjugable) arrangements, and from the fact of the parity of the number of the C's being opposite in any pair, the same will be true of the development in a series of

$$(1+q^a)(1+q^b)(1-q^c)(1+q^{a+c})\ldots.$$

As regards the coefficient in this latter series of any term whose index is

* It will presently be seen that all the licit and unexceptional arrangements will be divided into 3 classes and a specific operator be found for each class capable of acting on each arrangement of that class and converting it into another of the same class, and which will satisfy also the 3rd, 4th and 5th of the enumerated conditions. The total operator contemplated in the text may then be regarded as the sum of these specific ones, each of which, within its own sphere, will have to fulfil the five conditions of Catholicity, Homoeogenesis, Mutuality, Inertia and Enantiotropy (the last a word used in the school of Heraclitus to signify "the conversion of the primeval being into its opposite"). See Kant's *Critique of Pure Reason* by Max Müller, Vol. I., p. 18.

the sum of the elements in an unconjugate arrangement it will manifestly be the number of ways in which the same complex number can be thrown under the form of a sum of the arithmetical series

$$a,\ a+c, \ldots,\ a+(i-1)c,$$

which is
$$\frac{i^2-i}{2}c+ia,$$

that is
$$\frac{i^2}{2}c+\frac{i}{2}(a-b),$$

or of
$$b,\ b+c, \ldots,\ b+(i-1)c,$$

which is
$$\frac{i^2}{2}c-\frac{i}{2}(a-b).$$

If
$$\frac{i^2}{2}c+\frac{i}{2}(a-b)=\frac{j^2}{2}c+\frac{j}{2}(a-b),$$

then
$$\frac{i^2+i}{2}a+\frac{i^2-i}{2}b=\frac{j^2+j}{2}a+\frac{j^2-j}{2},$$

which necessitates $i=j$, and if

$$\frac{i^2}{2}c+\frac{i}{2}(a-b)=\frac{j^2}{2}c-\frac{j}{2}(a-b),$$

then
$$\frac{i^2+i}{2}a+\frac{i^2-i}{2}b=\frac{j^2-j}{2}a+\frac{j^2+j}{2}b,$$

so that
$$i^2+i-(i^2-i)=(j^2-j)-(j^2+j) \text{ or } i=-j.$$

Hence the general term is $q^{\frac{i^2}{2}c\pm\frac{i}{2}(a-b)}$, where i is an integer stretching from zero to infinity, and in like manner, and for the same reason, the general term in the former series will be $(-)^i q^{\frac{i^2}{2}c\pm\frac{i}{2}(a-b)}$ with the like interpretation: or which is the same thing, comprising both cases in one and interpreting i to be integer stretching from $-\infty$ to $+\infty$, the general term will be $(\mp)^i q^{\frac{i^2}{2}c+\frac{i}{2}(a-b)}$.

(59) The task before us then is to show the *possibility* of instituting, by *actually* instituting, a law of operation which shall satisfy the five preliminary conditions of catholicity, homoeogenesis, reciprocity, reversal of characters and conservation of sum.

The following notation will be found greatly to conduce to clearness in effecting the needful separation into classes or species. A capital letter with a point above, as $\dot{X}$, will be used to signify the greatest value, and with a point below, as $\underset{\cdot}{X}$, the least value of any term in a series which that letter is used to denote. $X=0$, $X>0$, $X+Y=0$, $X+Y>0$ will signify respectively that there are no X's, that there are X's, that there are no X's and no Y's, that there are either X's or Y's or both in any arrangement under consideration. B's will be separated into $'B$ and B''s, or as we may write it $B = 'BB'$, where $'B$ is the general name for all the B's, which beginning with the highest term $\dot{B}$ form an arithmetical series of which c is the common difference. If there is a gap of more than one c between $\dot{B}$ and the next lowest B, $'B$ is of course the single term $\dot{B}$: B' is any B which is not a $'B$.

So again, A_1 is any A which belongs to a series of A's forming an arithmetical series whose constant difference is c and lowest term a, so that unless $\underset{.}{A} = a$, $A_1 = 0$: any other A will be designated by $_1A$. The signs of accent and point may of course be separate or combined: thus for example $\underset{.}{C}$ will mean the smallest C in any given arrangement, $\dot{B}$ will mean the greatest B, $\underset{.}{A}$ will mean the lowest A, $_1\underset{.}{A}$ will mean the lowest of the $_1A$'s and $\dot{A}_1$ the highest of the A_1's. Every $'B$ is necessarily greater than any B', and every $_1A$ than any A_1. If $'\underset{.}{B} - b = 0$, this will indicate that all the B''s will form a consecutive series of terms (that is having a constant difference c) and ending in b, so that here $B' = 0$, that is there are no B's except those that belong to the regular arithmetical progression ending in b. If $_1A = 0$, all the A's will form an arithmetical progression ending in a. Thus we see that the arrangements belonging to the 1st terms (those that I have called exceptional) will consist of two species denoted respectively by

$$_1A + B + C = 0 \quad \text{and} \quad ('\underset{.}{B} - b) + A + C = 0.$$

It may sometimes be found convenient to use a point to the left centre of a quantitative letter to signify that the quantity denoted is to be increased, and a point to the right centre to signify that the quantity denoted is to be diminished, by c. Thus $\dot{B}\cdot$ will mean $\dot{B} - c$, and $\cdot\underset{.}{A}_1$ will mean $\underset{.}{A}_1 + c$, the first signifying the greatest B diminished by and the second the smallest A_1 increased by c. When any general letter, say X, is wanting as indicated by the equation $X = 0$, $\underset{.}{X}$ must be understood to mean zero. So for instance if $A = 0$, and consequently $_1A = 0$ and $A_1 = 0$, $_1\underset{.}{A} = 0$. Again, when there is a gap between the highest B and the one that follows it in any arrangement, the arithmetical progression of $'B$'s reduces as above remarked to a single term and there results $'\dot{B} = '\underset{.}{B}$. It may be noticed also that always $'\dot{B} = \dot{B}$, and $\underset{.}{A}_1 = \underset{.}{A}$.

The arrangements which are comprised under the forms

$$(\alpha)\quad A,\ A - c,\ A - 2c, \ldots,\ a,$$
$$(\beta)\quad B,\ B - c,\ B - 2c, \ldots,\ b,$$

may be regarded as belonging to what I shall term the first genus.

The second genus, namely that consisting of unexceptional combinations of unrepeated A's, B's, C's, may then be divided into the following three species, the conditions by which they are severally distinguished being attached to each in its proper place.

1st Species. Conditions (γ) $'\underset{\cdot}{B} - b > 0$,

or (γ') $'\underset{\cdot}{B} - b = 0,\ \ C > 0,\ \ \underset{\cdot}{C} - c <= '\dot{B} - b$.

2nd Species. (δ) $'\underset{\cdot}{B} - b = 0,\ A + C > 0,\ C = 0$ or $\underset{\cdot}{C} - c > '\dot{B} - b$,

or (δ') $B = 0,\ \ C > 0,\ \ A = 0$, or ${}_1\underset{\cdot}{A} - a => C$.

3rd Species. (ϵ) $B = 0,\ \ A > 0,\ \ {}_1A + C > 0,\ \ C = 0$, or $\underset{\cdot}{C} > {}_1\underset{\cdot}{A} - a$.

Where it is to be understood that the conditions set out in the same line are simultaneous conditions. Thus for example the conditions of an arrangement being of the second species are when all the conditions of the upper or else all the conditions of the lower of the two lines written under that species are fulfilled: the conditions of the upper line (be it noticed) are that $'\underset{\cdot}{B}$ is b, and that there are either some A's or some C's, and that if there are some C's, $\underset{\cdot}{C} - c > '\dot{B} - b$, and of the lower line, that there are no B's and some C's, and that if there are A's, $\underset{\cdot}{A} - a => \underset{\cdot}{C}$, and so for the interpretation of the conditions of the existence of each of the other two species.

To these (7) systems of conditions α, β, γ, γ', δ, δ', ϵ may be joined the trivial system (ω) $A = 0$, $B = 0$, $C = 0$*; the (8) systems thus constituted will easily be seen to be mutually exclusive and between them to comprehend the entire sphere of possibility, leaving no space vacant to be occupied by any other hypothesis. I will now proceed to assign the operators ϕ, ψ, ϑ appropriate to the three species of the second genus.

Office of the Operator ϕ. $\phi = '\phi + \phi'$.

When in Genus 2, Species 1, $C = 0$ or $\underset{\cdot}{C} - c > '\dot{B} - '\underset{\cdot}{B}$, $'\phi$ is to be performed, meaning that for each $'B$, $'B\cdot$ is to be substituted, and the inertia kept constant by forming a new $\underset{\cdot}{C}$ with the sum of the c's thus abstracted. In the contrary case ϕ' is to be performed, meaning that $\underset{\cdot}{C}$ is to be resolved into simple c's and as many of the $'B$'s, commencing with $'\dot{B}$ and taken in regular order to be converted into $'\cdot B$ as are required to maintain the inertia constant, that is c is to be added to each B in succession, until all the c's which together make up $\underset{\cdot}{C}$ are absorbed.

Office of the Operator ψ. $\psi = '\psi + \psi'$.

When in Genus 2, Species 2, $C = 0$ or $\underset{\cdot}{C} > '\dot{B} + \underset{\cdot}{A}$, $'\psi$ is to be performed, meaning that for $'\dot{B}$ and A their sum is to be substituted, producing a C [which, on the second hypothesis, will be a new $\underset{\cdot}{C}$]. In the contrary case ψ' is to be performed, meaning that for $\underset{\cdot}{C}$ is to be substituted $'\cdot\dot{B}$ (which will form a new $'\dot{B}$) and $\underset{\cdot}{C} - '\cdot\dot{B}$ which will form a new $\underset{\cdot}{A}_1$.

* It would be perfectly logical, and indeed is necessary to regard the trivial case as belonging to the cases of exception, and then we might say that there are two genera, each containing three species, those of the first genus solitary, and those of the second, each of them comprising two sub-species, namely the sub-species subject to the action of the left-accented and that subject to the operation of the right-accented operators. The *trivial* species of the first genus consists of a single individual.

Office of the Operator ϑ. $\vartheta = {}'\vartheta + \vartheta'$.

When $C > 0$ and $\underset{\cdot}{C} + \dot{A}_1 < {}_1\underset{\cdot}{A}$, $'\vartheta$ is to be performed, meaning that for $\underset{\cdot}{C}$ and $\dot{A}_1$ their sum is to be substituted, producing a new ${}_1\underset{\cdot}{A}$. In the contrary case ϑ' is to be performed, meaning that for ${}_1\underset{\cdot}{A}$, $\cdot\dot{A}_1$ forming a new $\dot{A}_1$ and ${}_1\underset{\cdot}{A} - \dot{A}_1$ forming a new C are to be substituted.

(60) It will be seen that every species of the second genus consists of two contrary sub-species having opposite characters, and it will presently appear that any arrangement belonging to one of these sub-species under the effect of its appropriate operator passes over into the other, which operated upon in its turn by its appropriate operator becomes identical with the original one, so that any two contrary sub-species may be said to be of equal extent: in fact if the sum of the parts is supposed to be given there will be as many arrangements in any sub-species as in its opposite, for each one will be conjugated with some one of the others.

It may not be amiss to call attention here to the fact that the scheme of classification adopted is, in a certain sense, artificial. Thus, for instance, it proceeds upon an arbitrary choice between which shall be regarded as the A and which as the B series, so that by an interchange of these letters a totally different correspondence would be brought about between the arrangements of the second genus, those of the first genus remaining unaltered. Nor is there any reason for supposing that these are the only two correspondences capable of being instituted between the arrangements of the second genus—in particular there is great reason to suspect that a symmetrical mode of procedure might be adopted, remaining unaffected by the interchange between A and B. As a simple example of the effect of interchange, applying the method here given, suppose $A = 0$, $B = 0$, a case belonging to the second species and that sub-species thereof to which ψ' is applicable, and imagine further that the C series is monomial. Then C will be associated according to the scheme here given with b, $C - b$, but in the correlative scheme it would be associated with a, $C - a$.

(61) I need hardly say that so highly organized a scheme, although for the sake of brevity presented in a synthetical form, has not issued from the mind of its composer in a single gush, but is the result of an analytical process of continued residuation or successive heaping of exception upon exception in a manner dictated at each point in its development by the nature of the process and the resistance, so to say, of its subject-matter. The initial step (that applicable to species γ) is akin to the procedure applied by Mr F. Franklin to the pentagonal-number theorem of Euler, of which I shall have more to say presently. It will facilitate the comprehension of the scheme to take as an example the particular case where a and b represent actual and real quantities, say, to fix the ideas, $b = 1$, $a = 2$. Nothing, it will

be noticed, turns upon the fact of this specialization, which is adopted solely for the purpose of greater concision and to afford more ready insight into the *modus operandi.*

To illustrate the classes and laws of transformation consider (with $b=1$, $a=2$*, $c=a+b=3$) all the arrangements, the sum of whose parts is 12, namely 12, 11.1, 10.2, 9.2.1, 8.4, 8.3.1, 7.5, 7.4.1, 7.3.2, 6.5.1, 6.4.2, 5.4.3, 5.4.2.1.

One of these, 7.4.1, belongs to the exceptional genus. The rest will be conjugated and fall into species in the manner shown below, where the first species means where the conditions (γ) or (γ'), the second that where (δ) or (δ'), and the third where the conditions (ϵ) are satisfied. The C's, B's, A's are now numbers whose residues are 0, 1 or 2 in respect to the modulus 3. For greater clearness in each arrangement, numbers belonging to the same series are kept together, the law of descent only applying in this theory to elements belonging to the same series.

Species 1. 10.2 3.7.2; 4.8 3.1.8; 7.5; 3.4.5; 6.4.2 6.3.1.2; 5.7 3.2.7.

Species 2. 9.1.2 9.3; 6.1.5 4.1.5.2;

Species 3. Caret.

Or again let the collection of arrangements be one in which the sum is 18. The partitions of 18 are 18 17.1 16.2 15.3 15.2.1 14.4 14.3.1 13.5 13.4.1 13.3.2 12.6 12.5.1 12.4.2 12.3.2.1 11.7 11.6.1 11.5.2 11.4.3 11.4.2.1 10.8 10.7.1 10.6.2 10.5.3 10.5.2.1 10.4.3.1 9.8.1 9.7.2 9.6.3 9.6.2.1 9.5.4 9.5.3.1 9.4.3.2 8.7.3 8.7.2.1 8.6.4 8.6.3.1 8.5.4.1 8.5.3.2 8.4.3.2.1 7.6.5 7.6.4.1 7.6.3.2 7.5.4.2 7.5.3.2.1 6.5.4.3 6.5.4.2.1. In this case there are no exceptional arrangements.

1st Species. 16.2 3.13.2; 4.14 3.1.14; 13.5 3.10.5; 13.4.1 3.10.4.1; 7.11 3.4.11; 10.8 3.7.8; 12.4.2 12.3.1.2; 10.7.1 6.7.4.1; 6.10.2 6.3.7.2; 10.1.5.2 3.7.1.5.2; 9.4.5 9.3.1.5; 6.7.5 6.3.4.5; 7.1.8.2 3.4.1.8.2; 6.4.8 6.3.1.8; 7.4.5.2 6.4.1.5.2;

2nd Species. 18 17.1; 15.3 15.1.2; 12.6 12.5.1; 6.1.11 4.1.11.2; 9.1.8 4.1.8.5; 9.7.2 9.3.4.2; 9.6.3 9.6.1.2; 11.5.2 3.8.5.2.

3rd Species. Caret.

If the partible number is 11, of which the partitions are 11 10.1 9.2 8.3 8.2.1 7.4 7.3.1 6.5 6.4.1 6.3.2 5.4.2 5.3.2.1, there will be no exceptional arrangements and the pairs of unexceptional ones will be as below.

* No use it will be seen is made of the *accidental* relation $a=b+b$.

1st Species. 10.1 3.7.1; 7.4 6.4.1; 4.5.2 3.1.5.2.

2nd Species. 3.8 1.8.2.

3rd Species. 11 9.2; 6.5 6.3.2.

By interchanging a and b, that is making $a=1$, $b=2$, the correspondence changes into the following:

1st Species. 11, 3.8; 6.3.2, 6.5; 8.2.1, 3.5.2.1; 7.4, 6.4.1.

2nd Species. Caret.

3rd Species. 10.1, 6.4.1; 7.4, 3.7.1.

According to Mr Franklin's process the correspondence takes a form quite distinct from either of the above, namely 11, 10.1; 9.2, 8.2.1; 8.3, 7.3.1; 7.4, 6.4.1; 6.5, 5.4.2; 6.3.2, 5.3.2.1, all these arrangements constituting one single species.

A careful study of the preceding examples will sufficiently explain to the reader the ground of the divisions into species with their appropriate rules of transformation, and might almost supersede the necessity of a formal proof of the operator supplying the conditions of catholicity, homoeogenesis and mutuality; from their very definition they are seen to comply with the other two essential conditions of inertia and enantiotropy.

Signifying by Ω the total operator $\phi+\psi+\vartheta$, it has been already remarked that Ω will in the general case have two values which only come together when $a=b$, or which is the same thing, each of them is 1; a special case of the special case when the complex reduces to simple numbers, namely, it is the case indicated in the well-known equation

$$(1-q)^2(1-q^3)^2(1-q^5)^2\ldots=\frac{1}{(1-q^2)(1-q^4)\ldots}\sum_{i=-\infty}^{i=\infty} q^{i^2}.$$

But besides the two correspondences given by the two values of Ω, if we take the actual (no longer a diagrammatic case) $b=2$, $a=1$, we revert to Euler's theorem concerning the partitions of all pentagonal and non-pentagonal numbers, and can obtain by Dr Franklin's process, given in Art. (12), a totally different distribution into genera and species, namely the first genus instead of containing arrangements of the species

$$1, 4, 7, \ldots 3i-2;\quad 2, 5, 8, \ldots 3i-1$$

will, as previously shown, consist of the very different arrangements (giving the same infinite series of numbers as those for other sums)

$$i, i+1, i+2, \ldots 2i-1;\quad i+1, i+2, i+3\ldots;\ 2i.$$

The character of each arrangement in the new solution depends in part on the relation to the modulus 2 of the whole number of parts and of the number of parts which are divisible by 3, so that we may divide the conjugate arrangements into four groups* designated respectively by *Oo, Oe*; *Eo, Ee,* using the capital letters to signify the oddness or evenness of the whole set of parts, and the small letters the same for the parts divisible by 3. There will thus be a cross classification of the arrangements of the second genus into groups over and above that into species, each species in fact consisting of four groups, which may be denoted as above, and of which *Oo* and *Ee* are one associative couple, and *Oe, Eo* the other†.

(62) The following elegant investigation has been handed in to me by Arthur S. Hathaway, fellow and one of my hearers at the Johns Hopkins University, to which, although it does not exactly strike at the object of the constructive theory here expounded, I gladly give hospitality in these pages.

"The theorem to be proved is as follows:

$$\begin{aligned} &1+\epsilon x^a \,.\, 1+\epsilon x^{a+h} \,.\, 1+\epsilon x^{a+2h} \ldots \\ \times\, &1+\epsilon x^b \,.\, 1+\epsilon x^{b+h} \,.\, 1+\epsilon x^{b+2h} \ldots \\ \times\, &1-x^h \,.\, 1-x^{2h} \,.\, 1-x^{3h} \ldots = \sum_{\delta=-\infty}^{\delta=+\infty} \epsilon^\delta \,.\, x^{\frac{a+b}{2}\delta^2+\frac{a-b}{2}\delta}, \end{aligned}$$

where $\epsilon^2=1$ and $h=a+b$, a and b being any quantities whatever.

"The general term contains, say i exponents of x selected from the first line, j from the second line, and k from the third line, namely

$$\begin{gathered} a+\alpha_0 h, \ldots a+\alpha_{i-1} h, \\ b+\beta_0 h, \ldots b+\beta_{j-1} h, \\ \gamma_1 h, \ldots \gamma_k h, \end{gathered}$$

where $\alpha_0 \ldots \alpha_{i-1}$, $\beta_0 \ldots \beta_{j-1}$, $\gamma_1 \ldots \gamma_k$ are respectively sets of i, j, k unequal integers arranged in ascending order, none representing a less integer than its subscript. This term is (remembering that $h=a+b$)

$$\epsilon^{i+j}(-)^k x^{ma+nb},$$

where

$$m=[(\alpha_0+1)+\ldots(\alpha_{i-1}+1)]+[\beta_0+\ldots\beta_{j-1}]+[\gamma_1+\ldots\gamma_k] \qquad (1)$$

$$n=[\alpha_0+\ldots\alpha_{i-1}]+[(\beta_0+1)+\ldots(\beta_{j-1}+1)]+[\gamma_1+\ldots\gamma_k] \qquad (2)$$

* It will be seen later on that there is a division into sixteen groups analogous to the division into four groups first noticed by Prof. Cayley arising under the Franklin process.

† The *Oe* and *Eo* conjugation has a very striking analogue in nature (as I am informed) in the existence of dissimilar hermaphrodite characters in two sorts of the wild English *primrose* and the American flower *Spring-beauty* or *Quaker-lady*—it being the law of nature that only those of different sorts can fertilize one another. Possibly the double symbolic character of *Oo* and *Ee* will justify or suggest the inquiry whether there may not be a latent duality in the unisexual specimens of such flowers as those just mentioned, where male and female are found codomiciled with the bisexual florets. There is also, it seems, a trace of analogy to the sparsely distributed unconjugate individuals of my first genus in Darwin's "complemental males."

In addition to these we obtain by subtraction

$$m-n=i-j\equiv i+j \bmod 2. \tag{3}$$

Whence (since $\epsilon^2=1$) $\epsilon^{i+j}=\epsilon^{m-n}$.

"Thus all the above general terms having the same m and the same n divide themselves into positive and negative groups (corresponding to even and odd values of k), a term from one group cancelling a term from the other group. I propose to prove that the number of terms in each of these groups are equal, except when a certain relation exists between m and n, namely

$$m-\frac{(m-n)(m-n+1)}{2}=0, \text{ (or } m=0 \text{ if } m=n),$$

corresponding to which there is but one general term having the same m and the same n which falls into the positive group ($k=0$). This establishes the theorem in question, as we see by putting $m-n=\delta$.

"It is sufficient to consider (1) in connection with (3). In the first place the first two partitions in (1) may be converted by a (1 : 1) correspondence into an indefinite partition (bearing in mind (3)) with a decrease ($m-n>0$) in the sum or content of the integers by $\frac{1}{2}(m-n)(m-n+1)$, as follows: extend α_0+1 in a horizontal line of dots, and *under* the first dot extend β_0 in a vertical line of dots, thus forming an *elbow*; in a similar manner form elbows out of α_1+1, β_1 &c. until one of the partitions is exhausted; this will be according to (3), the first or the second, according as $m<$ or $>n$, leaving in the inexhausted partition $m-n$ integers; place these elbows successively one without the other, and place on top ($m-n>0$) horizontal lines of dots corresponding to the successive unmatched integers decreased respectively by $0, 1, \ldots (n-m-1)$ or $1, 2, \ldots (m-n)$, according as $m<$ or $>n$; in either case the total decrease is $\frac{1}{2}(m-n)(m-n+1)$. In other words, the above tri-partition of m has a (1 : 1) correspondence with a bi-partition of

$$m-\frac{(m-n)(m-n+1)}{2}, \text{ (or } m \text{ if } m=n),$$

consisting of an indefinite partition on one side and a partition of unrepeated integers on the other ($\gamma_1, \ldots \gamma_k$). Such a bi-partition (on removing the line of demarcation) is an indefinite partition; and, conversely, every indefinite partition involving θ different integers gives rise as follows to $(1+1)^\theta$ such bi-partitions, the number of those involving even and odd values of k being respectively the positive and negative parts of the expansion of $(1-1)^\theta$, which are equal: namely, first, the indefinite partition itself ($k=0$); second, the θ bi-partitions obtained by placing each of the θ integers successively on the k side ($k=1$); third, the $\frac{1}{2}\theta(\theta-1)$ bi-partitions obtained by placing the $\frac{1}{2}\theta(\theta-1)$ pairs of the θ integers successively on the k side ($k=2$), and so on.

The only exception to this equality of the number of partitions for even and odd values of k is when the partible number,

$$m - \frac{(m-n)(m-n+1)}{2} \text{ or } m,$$

is zero, for which case there is but one bi-partition $[0]+[0]$ $(k=0)$. Q.E.D. The tri-partition of m corresponding to the celibate case reduces to the natural sequence above subtracted whose content is

$$\frac{(m-n)(m-n+1)}{2} \text{ (or 0)},$$

which is the second or the first partition (according as $m <$ or $> n$), the others being wanting."

(63) The same infinitesimal method which applied to the expansion of $\Theta_1 x$ gives rise as was shown to the expression for the cubes of the successive rational binomial functions may be applied to the development of

$$(1+ax)(1+ax^2)(1+ax^3)\ldots$$

given in Art. (35), but will not lead to any new result. Making $a = -x^{-1-\epsilon}$, where ϵ is infinitesimal, we obtain from the general theorem

$$(1-x^\epsilon)(1-x)(1-x^2)(1-x^3)\ldots$$

$$= 1 - \frac{1-x^\epsilon}{1-x}x + \frac{1-x^\epsilon\,.\,1-x}{1-x\,.\,1-x^2}x^5 - \frac{1-x^\epsilon\,.\,1-x\,.\,1-x^2}{1-x\,.\,1-x^2\,.\,1-x^3}x^{12}\ldots$$

$$-x^\epsilon + \frac{1-x^\epsilon}{1-x}x^3 - \frac{1-x^\epsilon\,.\,1-x}{1-x\,.\,1-x^2}x^9\ldots,$$

or $$(1-x)(1-x^2)(1-x^3)\ldots = 1 - \frac{x-x^3}{1-x} + \frac{x^5-x^9}{1-x^2}\ldots$$

$$= 1 - x(1+x) + x^5(1+x^2)\ldots,$$

the same equation as results from writing $a=-1$.

To arrive at any new result it would be necessary to have recourse to processes of differentiation; the above calculation serves, however, as a verification if any were needed of the accuracy of the theorem to which it refers.

(64) Since sending what precedes to press I have thought it would be desirable in the interest of sound logic to set out the marks or conditions of the several species of the arrangements of unrepeated A, B, C's, somewhat more fully and explicitly than before. And first, I may observe that since it has been convenient to understand that when there are no X terms $\underset{\cdot}{X}$ shall signify zero, the quantitative equation $\underset{\cdot}{X}=0$ dispenses with the necessity of

using the symbolical one $X = 0$, and in like manner $\underset{\cdot}{X} > 0$ supersedes the symbolical inequality $X > 0$, and, of course, the same remark extends to the equality or inequality $X + Y =$ or > 0.

We have then for what I shall term the first, second and third species of genus 1, the conditions

$$\underset{\cdot}{C} + \underset{\cdot}{B} + \underset{\cdot}{A} = 0, \quad \underset{\cdot}{C} + {}'\underset{\cdot}{B} + \underset{\cdot}{A} = b, \quad \underset{\cdot}{C} + \underset{\cdot}{B} + \underset{\cdot}{A}_1 = 0$$

respectively—the first, the trivial case of vacuous content; the second, of only a complete natural B progression, that is, one ending with b (the minimum value of B), and the third, the same for A similarly ending with the minimum a. In what follows the conditions in each separate *line* are to be understood to be not disjunctive but simultaneous or accumulative; they of course refer to the species of the second genus.

Marks of species (1) (α) $\underset{\cdot}{B} - b > 0$,
or (β) $\underset{\cdot}{B} - b = 0$, ${}'\dot{B} - {}'\underset{\cdot}{B} = > \underset{\cdot}{C} - c$, $\underset{\cdot}{C} > 0$.
" " (2) (α) $\underset{\cdot}{B} - b = 0$, $\underset{\cdot}{C} - c > {}'\dot{B} - {}'\underset{\cdot}{B}$,
or (β) $\underset{\cdot}{B} - b = 0$, $\underset{\cdot}{C} = 0$ [$\underset{\cdot}{A} > 0$],
or (γ) $\underset{\cdot}{B} = 0$, $\underset{\cdot}{A} - a = > \underset{\cdot}{C}$, $\underset{\cdot}{C} > 0$,
or (δ) $\underset{\cdot}{B} = 0$, $\underset{\cdot}{A} = 0$ [$\underset{\cdot}{C} > 0$].
" " (3) (α) $\underset{\cdot}{B} = 0$, $\underset{\cdot}{C} > \underset{\cdot}{A} - a$, $\underset{\cdot}{A} > 0$,
or (β) $\underset{\cdot}{B} = 0$, $\underset{\cdot}{C} = 0$ [${}_1\underset{\cdot}{A} - a > 0$].

The three inequalities included in brackets are only required in order to exclude arrangements belonging to the first genus. Leaving these out of account for the moment, merely for the sake of greater concision of statement, it is easy to see by mere inspection of the above table that the three species are mutually exclusive and share between them the total sphere of possibility, for (1) α exhausts the hypothesis of there being other B's besides those forming a complete natural progression, (1) β and (2) α of the B's forming such progression when there are existent C's, and (2) β when there are not. Also ((2) γ, (2) δ), (3) α exhaust between them the hypothesis of there being no B's when there are some existent C's, and (3) β of neither B's nor C's appearing in an arrangement.

Thus all unexceptional arrangements must bear the marks occurring in one or the other of the first four lines of the table, and all those where no B's occur, either of the last line when there are neither B's nor C's, and of the three preceding ones when there are no B's but some C's, and the total sum of these hypotheses plus the hypothesis of the first genus together make up necessity, as was to be shown.

The convention $\underset{\cdot}{X} = 0$ when an arrangement contains no X with the consequent reduction of the conditions to a purely quantitative form has lent itself very advantageously to the above bird's-eye view of the completeness of the scheme (as covering the whole ground of possibility); it also will be found to simplify the expression of the proof. I did not employ it until the necessity for so doing forced itself upon my notice, for a very obvious reason, namely that X is a B (or an A), which is defined to be congruous to b (or a) [mod c], which zero is not: there is thus an apparent paralogism in admitting that any X of these two *where there is a* B (or when there is an A) is congruent to b (or to a), but that *when there is no* B (or no A) then the conventional least B (or A) is zero. It will be seen, however, *ex post facto*, that no inconvenience in working the scheme results from this extended definition which constitutes an important gain to the perfect evolution of the method. It is usually in the form of some apparent contradiction or paradox that a scientific advance makes its first appearance.

(65) Aided by this clearer and fuller expression of the definitions of the genera and species, I will now set out a logical proof that the respective operators fulfil the three additional necessary conditions. I may observe preliminarily that the Greek letterings α, β; $\alpha, \beta, \gamma, \delta$; α, β, do not express sub-species, for one distinguishing mark of species (or sub-species) may be taken to be that conjugation cannot take place except between individuals of the same species or sub-species, but it will be presently seen that individuals belonging to the differently lettered divisions of the above species are susceptible of mutual conjugation—and are therefore in conformity with biological precedent to be regarded as mere varieties. Besides these varieties of each of the species there is another entirely different principle of cross classification applicable to each of them, namely in general an arrangement must belong to one of sixteen groups designated by combining together one out of each of the four pairs of opposite symbols X, C; x, c; O, E; o, e, where the large O, E refer to the oddness or evenness of the major, and the small o, e to the same for the minor parameter; and in like manner the large X and large C to the result of the operation appropriate to any arrangement, being to extend or contract the major, and x, c to extend or contract the minor parameter. There are thus eight pairs of groups, and conjugation can only take place between individuals belonging to the same pair.

The pairs are as follows:

$$\begin{pmatrix} XxOo \\ CcEe \end{pmatrix}, \begin{pmatrix} XxOe \\ CcEo \end{pmatrix}, \begin{pmatrix} XxEo \\ CcOe \end{pmatrix}, \begin{pmatrix} XxEe \\ CcOo \end{pmatrix};$$

and

$$\begin{pmatrix} XcOo \\ CxEe \end{pmatrix}, \begin{pmatrix} XcOe \\ CxEe \end{pmatrix}, \begin{pmatrix} XcEo \\ CxOe \end{pmatrix}, \begin{pmatrix} XcEe \\ CxOo \end{pmatrix}.$$

Species (1) and species (3) it will be seen may each be separately divided into four sub-species denoted by the upper four, and species (2) into the four sub-species denoted by the lower four pairs of combined characters, so that there will be in all twelve (and not as might at first be supposed twenty-four) sub-species of conjugable arrangements. The different sub-species of the same species do not admit of cross-conjugation ; it is the property which they have in common of being subject to the same law of transformation when passage is made from an individual to its conjugate, which binds them together into a single species. In the arrangements peculiar to Euler's problem, we see that there was no division of the second genus at the outset, but that a separation would be made of it into two pairs of groups with conjugation possible only between individuals belonging to the same pair, and consequently there may be said in this case to be two species of the second genus, analogous, however, not to the species but the sub-species in the more general theory. The final separation of a pair of groups into its component elements has nothing to do with the concept of species, sub-species or variety, but may be regarded as similar to the separation of the sexes.

In what follows, a bracket enclosing a letter will be used to denote that it belongs to an arrangement after it has been operated upon by its appropriate operator, or what may be called its operate.

Species (1). When $\underset{.}{B} - b > 0$, if $\underset{.}{C} - c > {}'\dot{B} - {}'\underset{.}{B}$ or $\underset{.}{C} = 0$, $'\phi$ may be performed, giving $[\underset{.}{C}] = {}'\underset{.}{B} - {}'B + C < \underset{.}{C}$ so that the law of descending magnitude is maintained ; we have then $[\dot{B}] - ['\underset{.}{B}] =$ or $> \dot{B} - {}'\underset{.}{B} = > [\underset{.}{C}] - c$; hence ϕ' has to be performed and will obviously restore the original arrangement. Again if in the original arrangement $\dot{B} - {}'\underset{.}{B} = > \underset{.}{C} - c$ and $\underset{.}{C} > 0$, ϕ' has to be applied ; a resolution of $\underset{.}{C}$ can take place into c's and the $\underset{.}{C}/c$ first $'B$'s, and will each be increased by c and $[\dot{B}] - {}'[\underset{.}{B}] = \underset{.}{C} - c$, so that either $[\underset{.}{C}] = 0$ or $[\underset{.}{C}] - c < \underset{.}{C} - c < [\dot{B}] - {}'[\underset{.}{B}]$, and $'\phi$ being applicable to the new arrangement will convert it back to the original one.

First Species (β). When $\underset{.}{B} - b = 0$ and $\dot{B} - {}'\underset{.}{B} = > \underset{.}{C} - c$ and $\underset{.}{C} > 0$, ϕ' can be performed, and the new arrangement as before may be operated upon by ϕ' and so brought back to its original value. If $\underset{.}{C} = 0$ or $\underset{.}{C} - c > \dot{B} - {}'\underset{.}{B}$, $'\phi$ could not be performed, for then $\underset{.}{B} = b$ and has no c to part with to help make up $[\underset{.}{C}]$.

These two hypotheses belong to Species (2), which we will now proceed to consider throughout its full extent. When $\underset{.}{B} - b = 0$, then $'\underset{.}{B} = b$, and I shall first suppose [(α) and (β)] that $\underset{.}{C} = 0$ or $\underset{.}{C} - c > B - b$. When $\underset{.}{C} = 0$ or $\dot{B} + \underset{.}{A} > \underset{.}{C}$, then $'\psi$ will be applicable, making $[\underset{.}{C}] = \dot{B} + \underset{.}{A}$; if now $[\underset{.}{B}] > 0$ and $[\underset{.}{A}] > 0$, $[\underset{.}{B}] + [\underset{.}{A}] = > (\dot{B} - c) + (\underset{.}{A} + c) = > \dot{B} + \underset{.}{A} = > [\underset{.}{C}]$, and

$$[\underset{.}{C}] - c = \dot{B} + \underset{.}{A} - c = [\underset{.}{B}] + \underset{.}{A} > [\underset{.}{B}] - b.$$

Hence we are still within Species 2 and have fallen upon the case to which the reversing operator ψ' has to be applied. If $[\underset{.}{B}] = 0$, $[\underset{.}{A}] = 0$ we must have $B[\underset{.}{C}] > 0$, inasmuch as the original content (or inertia) is originally greater than zero and is kept constant, and this is a case which still belongs to Species 2 and falls under the operation of ψ'.

If $[\underset{\cdot}{B}]=0$ so that $\dot{B}=\underset{\cdot}{B}=b$ and $[\underset{\cdot}{A}]>0$, then

$$[\underset{\cdot}{A}]-a=>\underset{\cdot}{A}+c-a=>\underset{\cdot}{A}+\dot{B}=>\underset{\cdot}{C},$$

which also falls within the second species and is amenable to the reversing operator ψ'.

Finally, if $[\underset{\cdot}{B}]>0$, that is $\underset{\cdot}{B}-b=0$ and $[\underset{\cdot}{A}]=0$,

$$[\underset{\cdot}{C}]-c=\dot{B}+\underset{\cdot}{A}-c=>[\dot{B}]-b,$$

that is $=>[\dot{B}]-{}'\underset{\cdot}{B}$, and we are still within Species (2) and in the case amenable to the reversing operator ψ'.

If now on the other hand we begin with an arrangement of the second species in the case amenable to ψ' we must suppose either $\underset{\cdot}{B}=0$ or $\underset{\cdot}{A}=0$, or else $\underset{\cdot}{C}>0$ and $\underset{\cdot}{C}<=\dot{B}+\underset{\cdot}{A}$.

Take first this last supposition. The operation of ψ' gives $[\underset{\cdot}{C}]=>\underset{\cdot}{C}+c$,

$$[\dot{B}]=\dot{B}+c \text{ and } [\underset{\cdot}{A}]=\dot{C}-c-\dot{B}>\dot{B}-b-\dot{B}>-b=>c-b=>a.$$

And $\quad [\dot{B}]+[\underset{\cdot}{A}]=\dot{B}+\underset{\cdot}{C}-\dot{B}=\underset{\cdot}{C}<[\underset{\cdot}{C}]$,

$$[\underset{\cdot}{C}]-c=>(\underset{\cdot}{C}-c)+c=>B-b+c=>[B]-[\underset{\cdot}{B}].$$

Hence the operate is licit, belongs to the second species and is amenable to the reversing operator ${}'\psi$.

If $\underset{\cdot}{B}=0$ and $\underset{\cdot}{A}=0$, $[\dot{B}]=[\underset{\cdot}{B}]=b$ and $[\underset{\cdot}{A}]=\underset{\cdot}{C}-b$ and $[\underset{\cdot}{C}]=0$ or $>\underset{\cdot}{C}$.

If $[\underset{\cdot}{C}]=0$ since $[\underset{\cdot}{A}]>0$, the operate is included in variety (β) of the second species and amenable to the reversing operator ${}'\psi$, and if

$$[\underset{\cdot}{C}]>\underset{\cdot}{C}\,[\underset{\cdot}{C}-c]>\underset{\cdot}{C}-c>0,$$

that is $>[\dot{B}]-\underset{\cdot}{B}$ which belongs to variety (α) of the second species; and since $[\underset{\cdot}{C}]>\underset{\cdot}{C}>[\dot{B}]+[\underset{\cdot}{A}]$ is amenable to the reversing operator ${}'\psi$.

If $\underset{\cdot}{B}>0$ and $\underset{\cdot}{A}=0$, then $\underset{\cdot}{C}>0$ [otherwise it would be an arrangement in Genus 1, Species 2] $[\underset{\cdot}{C}]=0$ or $>\underset{\cdot}{C}$, $[\dot{B}]=\dot{B}+c$,

$$[\underset{\cdot}{A}]=\underset{\cdot}{C}-[\dot{B}]>(c+\dot{B}-b)-(c+\dot{B})=>a,$$

and either $[\underset{\cdot}{C}]=0$ and $[\underset{\cdot}{A}]>0$ or

$$[\underset{\cdot}{C}]-c>(\underset{\cdot}{C}-c)+c>\dot{B}+c-b>[\dot{B}]-{}'\underset{\cdot}{B}$$

and $[\underset{\cdot}{A}]+[\dot{B}]=\underset{\cdot}{C}>[\underset{\cdot}{C}]$. Hence in either hypothesis the operate is still in Species (2) and amenable to the reversing operator ${}'\psi$.

Lastly, if $\underset{\cdot}{B}=0$, $\underset{\cdot}{A}-a=>\underset{\cdot}{C}$ and $\underset{\cdot}{C}>0$, the arrangement is amenable to the operator ψ', which will make $[\underset{\cdot}{B}]=b$, $[\underset{\cdot}{A}]=\underset{\cdot}{C}-b<\underset{\cdot}{C}+a<\underset{\cdot}{A}$. We have then $[\underset{\cdot}{B}]-b=0$ and $[\underset{\cdot}{C}]=0$, and consequently also $\underset{\cdot}{A}>0$ or

$$[\underset{\cdot}{C}]-c>\underset{\cdot}{C}-c>0,$$

that is $>[\dot{B}]-{}'[\underset{\cdot}{B}]$, and the result is still contained within Species (2) and is amenable to the reversing operator ${}'\psi$.

(66) The following are examples of paired arrangements belonging to the first species, adapted to the case of $a=2$, $b=1$. The C and B terms are

expressed; the A line is the same for each of any pair of this species, and may be filled in at will.

$$\phi'\left\{\begin{matrix} X.9. & \\ 16.13.10.Y \end{matrix}\right\} = \left\{\begin{matrix} X. & \\ 19.16.13.Y \end{matrix}\right\}$$

where X, Y represent any licit series of C's and B's respectively.

$$'\phi\left\{\begin{matrix} X.9 & \\ 16.13.7.Y \end{matrix}\right\} = \left\{\begin{matrix} X.9.6. & \\ 13.10.7.Y \end{matrix}\right\} \qquad \phi'\left\{\begin{matrix} X.9 & \\ 16.13.10.4 \end{matrix}\right\} = \left\{\begin{matrix} X. & \\ 19.16.13.4 \end{matrix}\right\}$$

$$\phi'\left\{\begin{matrix} X.9 \\ 7.4.1 \end{matrix}\right\} = \left\{\begin{matrix} X. \\ 10.7.4 \end{matrix}\right\} \qquad '\phi\left\{\begin{matrix} 10.7.4 \\ \end{matrix}\right\} = \left\{\begin{matrix} 9. \\ 7.4.1 \end{matrix}\right\}$$

$$\phi'\left\{\begin{matrix} 3. \\ 13.7.4.1 \end{matrix}\right\} = \left\{\begin{matrix} 16.7.4.1 \\ \end{matrix}\right\}.$$

The following are examples of paired arrangements of the second species with $a = 2$ and $b = 1$ as usual.

$$'\psi\left\{\begin{matrix} X.12. \\ 7.4.1. \\ Y.2 \end{matrix}\right\} = \left\{\begin{matrix} X.12.9. \\ 4.1 \\ Y \end{matrix}\right\} \qquad \psi'\left\{\begin{matrix} X.12. \\ 7.4.1. \\ Y.5 \end{matrix}\right\} = \left\{\begin{matrix} X \\ 10.7.4.1. \\ Y.5.2 \end{matrix}\right\}$$

$$'\psi\left\{\begin{matrix} 7.4.1. \\ Y.5. \end{matrix}\right\} = \left\{\begin{matrix} 12. \\ 4.1. \\ Y \end{matrix}\right\} \qquad \psi'\left\{\begin{matrix} X.15 \\ 7.4.1 \\ Y.8 \end{matrix}\right\} = \left\{\begin{matrix} X. \\ 10.7.4.1 \\ Y.8.5 \end{matrix}\right\}$$

$$\psi'\left\{\begin{matrix} X.9. \\ .... \\ .... \end{matrix}\right\} = \left\{\begin{matrix} X. \\ 1. \\ 8 \end{matrix}\right\} \qquad \psi'\left\{\begin{matrix} 6. \\ 1. \\ 8 \end{matrix}\right\} = \left\{\begin{matrix} ... \\ 4.1. \\ 8.2 \end{matrix}\right\}$$

$$\psi'\left\{\begin{matrix} X.9. \\ ... \\ Y.11 \end{matrix}\right\} = \left\{\begin{matrix} X. \\ 1. \\ Y.11.8 \end{matrix}\right\}.$$

We come now to the third species. Here, I think, the reader will find it a great relief to the strain upon his attention if I invite him before attacking the demonstration to consider the annexed diagrammatic cases accommodated to the supposition $a = 2$, $b = 1$. The B's it will be remembered in this species do not exist, and the action neither of $'\vartheta$ nor ϑ' introduces any B into the transformed arrangement. In the examples given below the C and A terms occupy the higher and lower lines respectively—the comma is used in the latter to mark off the ${}_1A$'s from the A_1's.

$$'\vartheta\left\{\begin{matrix} 9.6. \\ 14.11.8.5, \end{matrix}\right\} = \begin{matrix} 9.6.3. \\ 14.11.8,\ 2 \end{matrix} \qquad \vartheta'\left\{\begin{matrix} 6.3. \\ 14.11.8,\ 2 \end{matrix}\right\} = \begin{matrix} 6. \\ 14.11.8.5, \end{matrix}$$

$$'\vartheta\left(17.8.5\right) = \begin{matrix} 3. \\ 17.8,\ 2 \end{matrix} \qquad '\vartheta\left(17.8.5,\right) = \begin{matrix} 3. \\ 17.8,\ 2 \end{matrix}$$

$$'\vartheta\left(17,\ 8.5.2\right) = \begin{matrix} 6. \\ ,\ 11.8.5.2 \end{matrix} \qquad '\vartheta\left(17.14,\ 8.5.2\right) = \begin{matrix} 3. \\ 17,\ 11.8.5.2 \end{matrix}$$

$$'\vartheta\, 11, = \begin{matrix} 9. \\ ,\ 2 \end{matrix} \qquad \vartheta'\left\{\begin{matrix} 12.9.3. \\ ,\ 11.8.5.2 \end{matrix}\right\} = \begin{matrix} 12.9. \\ 14,\ 8.5.2 \end{matrix}$$

$$\vartheta'\left\{\begin{matrix} 9.6.3. \\ ,\ 11.8.5.2 \end{matrix}\right\} = \begin{matrix} 9.6 \\ 14,\ 8.5.2 \end{matrix}.$$

The left-hand accent is used here as elsewhere to signify that phase of the operator which brings about an increase and the right-hand one a decrease in the number of C's. It will readily be seen that the action of the operator in each of the above examples prepares the arrangement for the action of the contrary one which will restore it to its original value. It is worthy of notice that in any two associated arrangements above, an a (here 2) *may* appear in each and *must* appear in one of them. I will now proceed to the general demonstration.

(67) Let us first suppose $\underset{\cdot}{A}_1 = 0$, then ${}_1\underset{\cdot}{A} > 0$, otherwise we shall be dealing with the antecedent species and $'\vartheta$ will be applicable, making $[\underset{\cdot}{A}] = [\dot{A}_1] = a\,[\underset{\cdot}{C}] = \underset{\cdot}{A} - a < \underset{\cdot}{C}$ and $> (\underset{\cdot}{A} - a)$. Thus the generated arrangement is licit and belongs still to the third species; but now $[\underset{\cdot}{C}] + [\dot{A}_1] = \underset{\cdot}{A}$ and $[{}_1\underset{\cdot}{A}] = 0 > \underset{\cdot}{A}$. Hence the reversing operator ϑ' is applicable to the new arrangement; the remaining cases to consider (in which $\underset{\cdot}{A} = a$ for the arrangement as well before as after being operated upon) may be separated into those where $\underset{\cdot}{C} > 0$, and at the same time either $\underset{\cdot}{C} + \dot{A}_1 < {}_1\dot{A}$ or ${}_1\dot{A} = 0$, which are amenable to the operator ϑ' and the complementary cases which are amenable to $'\vartheta$.

In the cases first considered $[\dot{A}_1] = \dot{A}_1 - c$, $[{}_1\dot{A}] = \underset{\cdot}{C} = \underset{\cdot}{A}_1\,\vartheta\,[\underset{\cdot}{C}] + 0$ or $> \underset{\cdot}{C}$ (and *à fortiori* > 0), consequently the new arrangement is licit and still belongs to the third species, and since either $[\underset{\cdot}{C}] = 0$ or else

$$[\underset{\cdot}{C}] + [\dot{A}_1] > \underset{\cdot}{C} + \dot{A}_1 - \underset{\cdot}{C} = > [{}_1\dot{A}]$$

and $[{}_1\dot{A}] > 0$, it is one of the complementary cases and is subject to the reversing operator $'\vartheta$.

Again, any arrangement for which $\underset{\cdot}{A} = a$ belonging to the complementary cases is defined by the conditions ${}_1\dot{A} > 0$ and $\underset{\cdot}{C} + \dot{A}_1 = > {}_1\underset{\cdot}{A}$ and is by hypothesis to be subjected to the operator $'\vartheta$ which will make $[\dot{A}_1] = \dot{A}_1 + c$, $[{}_1\underset{\cdot}{A}] = 0$ or $> {}_1\underset{\cdot}{A}\ [\underset{\cdot}{C}] = {}_1\underset{\cdot}{A} - \dot{A}_1 - c$, and since $\underset{\cdot}{C} = > {}_1\underset{\cdot}{A} - \dot{A}_1$, $[\underset{\cdot}{C}] < \underset{\cdot}{C}$, so that the operation leads to a licit new arrangement.

Also $[\underset{\cdot}{C}] + [\dot{A}_1] = {}_1\underset{\cdot}{A}$, and consequently either $[{}_1\underset{\cdot}{A}] = 0$ or $[\underset{\cdot}{C} + \dot{A}_1] < [{}_1\underset{\cdot}{A}]$, which is a condition belonging to the first considered class of cases, subject to the reversing operator ϑ', and thus for the third as for both the antecedent species of the second genus, it has been proved that each designated operator prior to any arrangement being performed does not take away its licit character nor carry it out of the species to which it belongs, and on being repeated brings it back to its original form, and that the effect of any single operation is to maintain the content (or inertia) of the arrangement constant but to reverse each of its characters. This is the thing that was to be proved and brings my wearisome but indispensable task to an end.

(68) Another and perhaps somewhat clearer image of the classification of the numbers of the second Genus may be presented as follows: The combinations of the characters $XCOExcoe$ give rise to eight pairs of groups, say eight classes. Of these classes four belong to Species 2, and may be represented by four indefinite vertical parallelograms, set side to side, and subdivided each of them into four, (say) black, white, grey and tawny stripes, corresponding to the four varieties of the second species. The other four classes may be similarly represented by four such parallelograms as before, but separated by a transverse horizontal line into eight sub-classes, four corresponding to the first species and four to the second. The upper parallelograms may then be each divided into blue and green, the lower into yellow and red stripes to represent the respective couples of varieties of the first and third species. There will thus be in all thirty-two stripes, namely four blue, green, yellow and red, and four black, white, grey and tawny, each of which is bifid, representing two groups of opposite sexual characters, which may be fittingly represented by the upper and under sides of the sixteen unlimited single-coloured stripes of the first and the eight unlimited double-coloured stripes of the second set of parallelograms.

The above logical scheme is not intended to convey any notion of the relative frequency of the three species. The general case is that of the first species. The second is conditioned by $'\underset{.}{B} = b$ or $\underset{.}{B} = 0$, and the third by $\underset{.}{B} = 0$. When $'\underset{.}{B} = b$ it is about an even chance whether the arrangement is of the second or first species, and when $\underset{.}{B} = 0$ of the second or third. Either equality is a particularization of the B series, the latter signifying that there are no B's in the arrangement, the former that there are B's descending in rational progression down to b: this supposition is apparently infinitely more general than the former, because there is no limit to the number of terms in the progression, and the case of a natural progression of B's of the kind mentioned with any given number of terms as regards the probability of its occurring in an arrangement seems to be on a par with the case of the B's being all wanting. Hence the first species is infinitely more frequent than the second, and the second than the third. According to Prof. Max Müller's theory of the relation of thought to language (if I interpret it rightly) I ought to have thought out my divisions and schemes of operation in language, but I certainly had formed in my mind a dim abstract of them before I had found the language that was competent to give them expression.

In conclusion, I may remark that whilst the experience of the past indicated the probability that there did exist (if one could find it) a method of distributing the arrangements of the second genus into pairs, in such a way that in each pair the total or partial character should be reversed in passing from the one to the other, there was nothing to induce a reasonable degree of assurance that both those characters should be found simultaneously reversed in one and the same distribution; for aught that could have been foreseen to the contrary, it might very well have happened that one mode of distribution might have been needed to prove Jacobi's theorem for the case of only negative signs appearing in the factors on the left-hand side of the equation, and a different one for the other case where only every third factor contains such sign—indeed upon the principle of *divide et impera* or doing one thing at a time (as invaluable a maxim to the algebraist as to the politician) I had completed the proof for the former case without thinking of the latter, and only when on the point of attacking it was agreeably surprised to find that there was nothing left to be done, for that the proof found for the one extended to the other—in familiar phrase, I had hit two birds with one stone. We may now ask whether this was a happily found chance solution or was predestined by the nature of things, and that *simple* necessarily implies *double* enantiotropy of conjugation. Probably I think not, and if so, a question arises as to the number of solutions for each of the two sorts of enantiotropy and whether the number of each kind of simply-enantiotropic conjugations is the same.

Viewed merely as a question of direct multiplication, I think it must be allowed that what I have here called Jacobi's theorem (including Euler's marvellous one, as the ocean a drop of water) is the most surprising revelation that has been made in elementary algebra since the discovery of the general binomial theorem, and that the space devoted to its independent, and so to say, materialistic proof in these pages, although considerable, is not out of proportion to its intrinsic importance.

H. *Intuitional Exegesis of Generalized Farey Series*.*

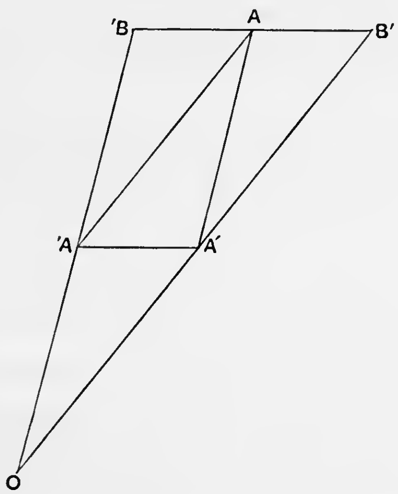
(69) The demands of the press will only admit of a rapid sketch of what appears to me to be the true underlying principles of the theory initiated by Farey, honoured by the notice of Cauchy, and to a certain extent generalized by Mr Glaisher, whose inductive method in the cases treated by him finds its full development in the method of continuous change of boundary, explained in the course of what follows. Let us start from the conception of an infinite cross-grating formed by two orthogonal systems of parallel lines in a plane, the distance between any two parallels being made equal to unity. The intersections of any two lines of the grating may, as heretofore, be termed nodes. A triangle which has nodes at its apices and at no other point on or within its periphery, may be termed an elementary triangle, and the double of the area of any such triangle will be unity. If any finite aggregate of nodes be given it must be possible to pick out a certain number of them which may be formed together by right lines so as to form a sort of ring-fence, within which all the rest are included: the area thus formed, if it

* Continued from note G, Interact, Part 2.

admits of being mapped out into elementary triangles, may be termed a *complete* nodal aggregate. Any other contour consisting of lines of any form (curved or straight) drawn outside of this ring-fence in such a manner that no nodes occur between the two, may be termed a regular contour.

If any node O be taken as origin and any nodal lines through O as axes of coordinates, and if $'A$, A' are the nearest nodes to O in the radial lines on which they lie, and if no nodes of the given aggregate are passed over as an indefinite line rotating round O, passes from one of these radial lines to the other, $'AOA$ is an elementary triangle, and if $'p, 'q$; p, q be the coordinates of $'A$, A respectively, $'pq - p'q = \epsilon$ where ϵ is $+1$ or -1 but is fixed in sign when the direction of the rotation is given.

When the aggregate is *complete*, if the values of the coordinates of the successive points passed over by the rotating line be called ... $''p, ''q$; $'p, 'q$; p, q; p', q', p'', q''; ..., we shall have a Farey series formed by the successive couples p, q, that is $p''\,'q - p'q'' = \epsilon$; $p'q - pq' = \epsilon$; $pq' - p'q = \epsilon$.... Thus we see that the Farey property is invariantive in the sense of being independent of the position of the origin.



Next I say, that if any contour to a given aggregate is regular, every contour similar thereto in respect to any node of the aggregate regarded as the centre of similitude is also regular, provided the boundary is simple; meaning that there are no interior limiting lines giving rise to holes or perforations in aggregate, and no loops formed by the boundary cutting itself.

In the above figure $'BOB'$ is any triangle whose sides are bisected in $'A$, A, A'. Suppose O to be the origin, $'A$, A' two nodes of greatest proximity to O successively passed over by the rotating line for a given contour. As this contour expands uniformly in all directions through O, the line $'AA'$ remains parallel to itself. Since $'AOA'$ is an elementary triangle so also must the similar triangles $'AAA'$, $A'AB'$, $'AA'B$ be all elementary, consequently A will be the first new node intervening between $'A, A'$ brought into the enlarged aggregate as $'AA'$ moves continuously parallel to itself, and $'AOA$, AOA' will be elementary triangles; it may be noticed in order to bring this method into relation with that indicated by Mr Glaisher, that the coordinates of this new node A are the sums of the coordinates of its neighbours $'A, A'$. If the contour were not supposed to be simple, this condition could not be drawn; for if there were a hole round the middle point of $'AA'$ the node A would be missing in the enlarged aggregate, and if the first node to intervene as the contour went on enlarging be called (A), $'AO(A)$ or $(A)OA'$ or each of them would be a multiple of the elementary triangle, so that the constancy of the value of the successive determinants would no longer hold. In like manner it will be seen that on the same supposition as above made, if in consequence of the contour contracting about O as the centre of similitude, two points $'A, A'$ which originally are non-contiguous, at any moment become contiguous, at the moment previous to this taking place A (and no other point) must have intervened, and after A has disappeared from the reduced aggregate, no other point can make its appearance between $'A, A'$.

(70) Hence we may contract at pleasure the given contour about any node as origin, and if the contour so contracted contains at least one node besides the origin, it will suffice to determine whether the given contour is or is not regular.

Thus for example in the case of a triangle limited by the axes and by the right line $x+y=n$, we may make $n=1$ and the trial series will then become $\frac{0}{1}\ \frac{1}{1}\ \frac{1}{0}$ which possesses the Farey property. Hence this will hold good for a triangular boundary of any size and wherever the origin is situated: this includes the case of the ordinary Farey series when the origin is taken at either extremity of the hypothenuse. So again for the area contained within the axes and the hyperbola $xy=n$, we may take $xy=1$ and the trial series is the same as before.

(71) It is easy to form *unperforated* areas of any magnitude which shall not satisfy the Farey law: for example we may as in the annexed figure draw a curve passing through the origin, the point (0, 1), and the point (2, 3), $\frac{0}{1}, \frac{2}{3}$ does not satisfy the Farey law, and consequently no similar contour obtained by treating any one of the three nodes which it contains as a centre of similitude will be a "complete contour," and the successive values of (p, q)

obtained by the rotation of a line round the origin in such contour will not constitute a Farey series.

The theory will, I believe, admit of being extended to solid reticulations, formed by the intersections of three systems of equidistant parallel planes, determinants of the third order between the three coordinates of successive points, replacing the $pq' - p'q$ of the plane theory. The chief difference will consist in the introduction of a new element in the multiplicity of the "normal orders" in which a given set (of points in a plane or) of radii *in solido* may be taken. (Points in a plane arranged in any order of sequence, such that the successive determinants formed by their trilinear coordinates are of uniform sign, are said to be in a normal order. Rays of a conical pencil arranged in any order of sequence, such that their intersections by a plane satisfy the above condition, are also said to be in a normal order: see privately printed syllabus* of my lectures on Partitions, 1859, or M. Halphen's theory of *Aspects.*) But as far as I can see this will in no way militate against the existence of the laws of invariance and similitude established for the case of a plane reticulation, but will only introduce a further principle of invariance, namely that the law of unit-determinants if satisfied by one normal arrangement of the points of the solid reticulation will be satisfied by every other.

APPENDIX†.

LIST OF CORRECTIONS SUGGESTED BY M. JENKINS TO PROFESSOR SYLVESTER'S CONSTRUCTIVE THEORY OF PARTITIONS.

Page 5, 5 lines from end, $2n - (i+3)$ should be $n - (i+3)$.
" 6, between 2nd and 3rd rows of sinister table insert 13 . 2 . 0.
" " " 7th and 8th " " " " 11 . 2 . 2.
" " in 6th row of dexter table, for 8 . 4 . 3 (2) write 8 . 4 . 3 (1).
" 11, line 8 from the end, interchange protraction and contraction so as to read "*contraction* could not now be applied to A' and B' nor *protraction* to C'."
" 13, line 25. If $f(x) = (1-x)(1-x^3)(1-x^3)(1-x^7)(1-x^9)$, for the second x^3 read x^5.

[* Vol. II. of this Reprint, p. 119.]

[† These corrections have been included in those made in the text preceding.]

Page 13, line 29, for "latter" read "former."

„ 15, line 11 from end, for l^{r} read l^{λ}.

„ 20, line 4, for $1+2$ read $i+2$.

„ „ line 5, for $1+2$ read $i+2$.

„ 22, line 11, for $X_j x^{\frac{i^2+i}{2}}$ read $X_j x^{\frac{j^2+j}{2}}$.

„ „ line 20, for "the minimum negative residue of $i-1$" read $i+1$.

„ 25, line 7, for $\dfrac{x^{\frac{1}{2}n(n+1)}}{1-x^n}$ read $\dfrac{x^{\frac{1}{2}r(r+1)}}{1-x^r}$.

„ „ line 4 from the end, for "to the 5th now" read "to the 5th row now."

„ 27, line 15, for 15, 7, 3 read 13, 11, 3.

„ „ line 19, for $(1+ax)(1-ax^3)(1-ax^j)\ldots$ read
$(1+ax)(1+ax^3)\ldots(1+ax^{2j-1})$.

„ „ line 22, for $\dfrac{x}{1-x}a$ read $\dfrac{x}{1-x^2}a$.

„ „ line 30, for "angle whose *nodes* contain i nodes" read whose *sides*.

„ 28, line 5, for "with $j-i$ or fewer parts" read $j-1$.

„ „ line 12, for $1+\dfrac{1-x^{\omega+1}}{1-x^2}x^{\omega}+\dfrac{1-x^{\omega+1}.1-x^{\omega+3}}{1-x.1-x^4}x^{\omega+1}$ etc.

read $x^{\omega}+\dfrac{1-x^{\omega-1}}{1-x^2}x^{\omega+1}+\dfrac{1-x^{\omega-1}.1-x^{\omega-3}}{1-x^2.1-x^4}x^{\omega+4}+$ etc.

If in the expression in line 9, namely in

$$\frac{1-x^{2i-2j+2}.1-x^{2i-2j+4}\ldots1-x^{2i-2}}{1-x^2.1-x^4\ldots1-x^{2j-2}}x^{j^2-2j+2i},$$

we put $j=3$ we obtain

$$\frac{1-x^{2i-4}.1-x^{2i-2}}{1-x^2.1-x^4}.x^{9-6+2i}=\frac{1-x^{2i-2}.1-x^{2i-4}}{1-x^2.1-x^4}.x^{2i+3}$$

$$=\frac{1-x^{\omega-1}.1-x^{\omega-3}}{1-x^2.1-x^4}.x^{\omega+4},$$

since $\omega=2i-1$, and similarly for other terms when we put $j=2$ and $j=1$.

The correction which I offer seems to me to be right, and the expression in the paper to give a wrong result in the case when n happens to be equal to $\omega+2$; for then the number of parts being supposed to be exactly i, the first bend contains $2i-1$ or ω nodes, and there is then no way of placing the remaining 2 nodes so as to make the partition a conjugate partition—supposing I have not misunderstood the article.

Page 29, line 8, for 19, 7, 6, 6 read 10, 7, 6, 6.

„ „ figure, either insert a node at junction of 5th column and 7th row or remove a node from junction of 7th column and 5th row.

„ „ lines 7 and 8 from the bottom, if we remove a node from the figure no change is required in these two lines; but if we insert a node in the figure, then 11 11 11 7 3 3 should be 11 11 11 7 5 3 and 5 5 5 3 1 1 should be 5 5 5 3 2 1.

Page 31, line 15 from end, after $\dfrac{1}{1-ax\,.\,1-ax^2\ldots 1-ax^\theta}$ insert "or of $x^n a^j$."

„ 34, line 7, for a^j read a^θ.

„ „ line 8, for $(x^\theta + ax^{1\theta})\}$ read $(x^\theta + x^{2\theta})$.

„ 36, line 8, for $\dfrac{l_1(2-j-1)}{2}$ read $\dfrac{l_1-(2j-1)}{2}$.

„ 37, line 4, for x^n read $x^{\frac{n}{2}}$.

„ „ line 7, for x^{2i+1} read x^{2i+2}.

„ 40, line 6, $a_i - i$ is, I believe, the right final term; but it appears as if it were the first of a pair instead of the last of a pair, $a_i - i$ being a quantity which may vanish.

If the pair of expressions which in the text precede $a_i - i$, if definitely expressed and not left to be understood, should be

$$[a_{i-1} + \alpha_{i-1} - (2i-3)], \quad [a_{i-1} + \alpha_{i-2} - (2i-2)],$$

and not as in the text

$$[a_{i-1} + \alpha_{i-1} - (2i-1)], \quad [a_{i-1} + \alpha_i - 2i],$$

the factor which should precede $a_i - i$ is $[a_i + \alpha_i - (2i-1)]$.

I do not quite follow lines 9—13 of p. 40, possibly from the oversight in the subscripts I do not see what is intended. But it seems to me the following proof would be right:

The expressions of the same form succeeding $a_1 + \alpha_1 - 1$ and $a_1 + \alpha_2 - 2$ must be continued so long as they are positive, and must be rejected when they become negative.

Now from the fact of i being the content of the side of the square belonging to the transverse graph $a_i =$ or $> i$, $\alpha_i =$ or $> i$, therefore $a_i + \alpha_i - (2i-1)$ is positive and is therefore one of the terms of the series. Also $a_{i+1} =$ or $< i$ and $\alpha_{i+1} =$ or $< i$, therefore $a_{i+1} + \alpha_{i+1} - (2i+1)$ is negative and must consequently be rejected.

The intermediate expression is $a_i + \alpha_{i+1} - 2i$; and for this we may in all cases put $a_i - i$ as the last term of the series for the following reason:

If the extreme inside bend have more than one node in the row, then $\alpha_{i+1} = i$ and $a_i + \alpha_{i+1} - 2i$ is $= a_i - i$, which is not negative since $a_i =$ or $> i$. If the extreme inside bend degenerate, so that it consists only of a vertical line or of a single point, then $a_i = i$; and since $\alpha_{i+1} < i$ in this case, therefore $a_i + \alpha_{i+1} - 2i$ is negative and inadmissible as a term in the series; but since $a_i - i = 0$ there is no harm in putting it as the final term in the series.

Page 601, *Vol.* III. *of this Reprint*, line 6 from the end, for 3100 read 3110.

2.

SUR LES NOMBRES DE FRACTIONS ORDINAIRES INÉGALES QU'ON PEUT EXPRIMER EN SE SERVANT DE CHIFFRES QUI N'EXCÈDENT PAS UN NOMBRE DONNÉ.

[*Comptes Rendus*, XCVI. (1883), pp. 409—413.]

DANS le *Philosophical Magazine*, 1881, p. 175, M. Airy, associé étranger de l'Institut, annonce qu'il a calculé, pour l'usage de l'Institution of civil Engineers, à Londres, les valeurs logarithmiques de toutes les fractions ordinaires $\frac{m}{n}$, dans lesquelles m et n ne contiennent nul facteur commun et n'excèdent pas 100, arrangées dans l'ordre de leurs grandeurs, et que le nombre de ces fractions est 3043.

Je vais montrer qu'on peut appliquer la méthode dont M. Tchebycheff s'est servi dans sa théorie célèbre sur les nombres premiers, avec l'addition que j'y ai faite*, pour trouver des limites supérieures et inférieures au nombre d'un système pareil de fonctions quand la limite des valeurs de m et de n est un nombre quelconque donné.

1. Je dis que si Ti signifie le nombre de nombres inférieurs et premiers à i, nombre entier (ce que nous nommons, à Baltimore, le *totient* de i), on aura l'identité

$$\sum_{r=\infty}^{r=1}\left(E\frac{i}{r}\,Tr\right)=\frac{i^2+i}{2}.$$

C'est une conséquence du théorème plus général que "si $a_1, a_2, \ldots, a_i$ sont des nombres entiers quelconques, et si l'on nomme le nombre des a qui contiennent r la fréquence de r par rapport au système des a, et qu'on prenne le produit de la fréquence de r par son totient, la somme de ces produits (quand r prend toutes les valeurs de 1 jusqu'à l'infini) sera la somme des a."

* Voir *American Journal of Mathematics.* [Vol. III. of this Reprint, pp. 530, 605, 672.]

2. Nommons Jx la somme-totient de x, c'est-à-dire la somme des totients de tous les nombres qui n'excèdent pas la valeur de Ex (la partie entière de x).

Je me servirai désormais de $\left(\frac{p}{q}\right)$ pour signifier la partie entière de $\frac{p}{q}$.

Or écrivons les suites successives

$$\begin{array}{llllllll}
x, & x-1, & \ldots, & \left(\frac{x}{2}\right)+1; & \left(\frac{x}{2}\right), & \left(\frac{x}{2}\right)-1, & \ldots, & \left(\frac{x}{3}\right)+1; \\
\left(\frac{x}{3}\right), & \left(\frac{x}{3}\right)-1, & \ldots, & \left(\frac{x}{4}\right)+1; & \left(\frac{x}{4}\right), & \left(\frac{x}{4}\right)-1, & \ldots, & \left(\frac{x}{5}\right)+1; \\
\ldots, & \ldots\ldots, & \ldots, & \ldots\ldots; & \ldots, & \ldots\ldots, & \ldots, & \ldots\ldots; \\
\left(\frac{x}{2q-1}\right), & \left(\frac{x}{2q-1}\right)-1, & \ldots, & \left(\frac{x}{2q}\right)+1; & \left(\frac{x}{2q}\right), & \left(\frac{x}{2q}\right)-1, & \ldots, & \left(\frac{x}{2q+1}\right)+1; \\
\ldots, & \ldots\ldots, & \ldots, & \ldots\ldots; & \ldots, & \ldots\ldots, & \ldots, & \ldots\ldots;
\end{array}$$

q augmentant *ad libitum*.

Je dis que, "si r est un nombre entier quelconque qui se trouve dans les suites d'ordre impair, c'est-à-dire commençant avec x, $\left(\frac{x}{3}\right)$, $\left(\frac{x}{5}\right)$, …, et si $j = 2i$ ou $2i+1$, on aura

$$E\left(\frac{j}{r}\right) - 2E\left(\frac{i}{r}\right) = 1,$$

et que, si r appartient à une suite quelconque d'ordre pair, on aura

$$E\left(\frac{j}{r}\right) - 2E\left(\frac{i}{r}\right) = 0."$$

Conséquemment, en appliquant le théorème précédent, on aura

$$\frac{j(j+1)}{2} - 2\frac{i(i+1)}{2} = S_1 + S_3 + \ldots + S_{2q-1} + \ldots,$$

où S_{2q-1} est la somme des totients des nombres qui sont en même temps égaux ou inférieurs à $E\frac{j}{2q-1}$ et plus grands que $E\frac{j}{2q}$, c'est-à-dire

$$S_{2q-1} = J\left(\frac{j}{2q-1}\right) - J\left(\frac{j}{2q}\right).$$

Si donc on écrit

$$\theta x = Jx - J\frac{x}{2} + J\frac{x}{3} - J\frac{x}{4} + J\frac{x}{5} - J\frac{x}{6} + \ldots,$$

on aura, quand $x =$ un nombre entier pair (soit $2i$),

$$\theta x = (2i^2 + i) - (i^2 + i) = i^2 = \frac{x^2}{4},$$

et, quand $x =$ un nombre entier impair (soit $2i+1$),

$$\theta x = (i+1)(2i+1) - (i^2+i) = \frac{(x+1)^2}{4}.$$

Avec l'aide de ces égalités, si x est un nombre positif quelconque entier ou fractionnel, on obtient facilement les inégalités

$$\theta x = \text{ou} > \frac{x^2 - 2x}{4}$$

$$\theta x = \text{ou} < \frac{x^2 + 2x + 1}{4}.$$

En appliquant à ces deux inégalités la méthode d'approximation successive que j'ai appliquée, dans* le Mémoire cité, aux inégalités auxquelles est assujettie la fonction $\psi(x)$ (*voir* Serret, *Algèbre supérieure*, édition de 1879, t. II. p. 233), je parviens facilement et rigoureusement à démontrer que, étant donnée une quantité ϵ aussi petite qu'on veut, on peut trouver une limite supérieure L et une limite inférieure Λ à Jx, où

$$L = \left(\frac{3}{\pi^2} + \eta\right) x^2 - Ax + R(\log x)$$

$$\Lambda = \left(\frac{3}{\pi^2} - \eta'\right) x^2 - A'x + R'(\log x),$$

où $R(\log x)$, $R'(\log x)$ sont tous les deux fonctions rationnelles et entières de $\log x$ d'un degré fini, dont les coefficients aussi bien que A et A' restent toujours finis et où η, η' sont tous les deux plus petits que ϵ.

Il s'ensuit que la fraction $\frac{J(x)}{x^2}$ possède une valeur asymptotique $\frac{3}{\pi^2}$ (ce qui n'est pas démontré pour la fraction analogue $\frac{\psi x}{x}$, dans la théorie parallèle de M. Tchebycheff) et que la valeur de $\frac{Jx}{x^2}$ approche indéfiniment près quand x est pris suffisamment grand de $\frac{3}{\pi^2}$, c'est-à-dire de $\cdot 30396\ldots$.

Il est facile de voir que la quantité Jx diminuée de l'unité n'est autre chose que le nombre des fractions dans les Tables pareilles à celles de M. Airy. Ainsi, pour le cas de $x = 100$ selon M. Airy, $Jx = 3044$. Pour ce cas $\frac{3}{\pi^2} x^2 = 3039 \cdot 6$.

Avec l'aide de ces limites on peut calculer la probabilité que deux nombres dont la limite supérieure est très grande soient premiers entre eux. Car si cette limite est x, le nombre total des cas qui peuvent arriver est x^2, et le nombre des cas pour lesquels les nombres choisis sont premiers entre eux sera $2Jx - 1$. Conséquemment, la probabilité en question sera $\frac{6}{\pi^2}$.

M. Franklin, l'auteur de la belle démonstration, insérée dans les *Comptes rendus*, du théorème d'Euler sur le produit $(1 - x)(1 - x^2)(1 - x^3)\ldots$, a bien

[* Vol. III. of this Reprint, p. 532.]

voulu m'adresser la remarque que cette conclusion peut être au moins confirmée, peut-être même absolument démontrée, de la manière suivante :

x étant pris très grand, la probabilité que deux nombres inférieurs à x, pris au hasard, ne contiennent pas tous les deux le nombre premier p, sera $1-\frac{1}{p^2}$. Donc, la probabilité cherchée sera

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{5^2}\right)\left(1-\frac{1}{7^2}\right)\ldots,$$

qui est la réciproque de

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{6^2}+\frac{1}{7^2}+\ldots,$$

c'est-à-dire est égal à $\frac{6}{\pi^2}$.

Il y a une suite doublement infinie d'équations fonctionnelles exactes qu'on peut former avec les $J(x)$. En particulier, il y a une série simplement infinie de telles fonctions où les signes sont alternativement positifs et négatifs, et conséquemment peuvent servir chacun à donner une suite infinie de limites à Jx.

Ainsi, si l'on écrit

$$\theta x = Jx - J\frac{x}{2} + J\frac{x}{3} - J\frac{x}{4} + J\frac{x}{5} - J\frac{x}{6} + \ldots\ldots + \ldots\ldots$$

$$\theta_2 x = 2J\frac{x}{2} - 3J\frac{x}{3} + 2J\frac{x}{4} - J\frac{x}{6} + 2J\frac{x}{8} - 3J\frac{x}{9} + 2J\frac{x}{10} - J\frac{x}{12} + 2J\frac{x}{14} - 3J\frac{x}{15} + 2J\frac{x}{16} - J\frac{x}{18} + \ldots\ldots + \ldots\ldots$$

$$\theta_3 x = 3J\frac{x}{3} - 4J\frac{x}{4} + 3J\frac{x}{6} - 4J\frac{x}{8} + 3J\frac{x}{9} - J\frac{x}{12} + 3J\frac{x}{15} - 4J\frac{x}{16} + 3J\frac{x}{18} - 4J\frac{x}{20} + 3J\frac{x}{21} - J\frac{x}{24} + 3J\frac{x}{27} - 4J\frac{x}{28} + 3J\frac{x}{30} - 4J\frac{x}{32} + 3J\frac{x}{33} - J\frac{x}{34} + \ldots\ldots + \ldots\ldots$$

on aura toujours, quand

$$x=(k^2+k)\,i, \qquad \theta_k x = \frac{x^2}{2(k^2+k)},$$

et quand

$$x=(k^2+k)\,i-1, \qquad \theta_k x = \frac{(x+1)^2}{2(k^2+k)},$$

et, quel que soit le résidu de x par rapport au module k^2+k, on peut calculer la valeur de $\theta_k x$. Enfin, si x est une quantité positive quelconque, on trouvera

$$\theta_k x = \text{ou} > \frac{x^2-x}{2(k^2+k)}, \qquad \theta_k x = \text{ou} < \frac{x^2+2x+1}{2(k^2+k)}.$$

3.

NOTE SUR LE THÉORÈME DE LEGENDRE CITÉ DANS UNE NOTE INSÉRÉE DANS LES *COMPTES RENDUS*.

[*Comptes Rendus*, XCVI. (1883), pp. 463—465.]

Le théorème de Legendre, cité par MM. de Jonquières et Lipschitz, est une conséquence immédiate d'un théorème logique bien connu, lequel, *mis sous forme sensible*, équivaut à dire que, si A, B, C, ... sont des corps avec la faculté de s'entrecouper, contenus dans un vase d'eau, et si a, ab, abc, ... représentent symboliquement les volumes de A, de la partie commune à A et à B, de la partie commune à A, B, C, ..., alors le volume du liquide déplacé par la totalité des corps sera

$$\Sigma a - \Sigma ab + \Sigma abc - \ldots.$$

Conséquemment, ce théorème admet une généralisation infinie dont je donnerai un seul exemple.

Nommons les nombres premiers qui n'excèdent pas n, nombres premiers subordonnés à n, et distinguons entre eux ceux qui sont plus grands que $\sqrt{n}$ comme supérieurs.

Le théorème de Legendre équivaut à dire que, si p_1, p_2, ..., p_i sont les nombres premiers subordonnés à $\sqrt{n}$, le nombre des nombres premiers subordonnés à n du genre supérieur augmenté de l'unité est égal à

$$n - \Sigma\left(\frac{n}{p_1}\right) + \Sigma\left(\frac{n}{p_1 p_2}\right) - \Sigma\left(\frac{n}{p_1 p_2 p_3}\right) + \ldots.$$

Or, représentons la fonction $\frac{1}{2}x(x+1)$ par Δx; alors on aura le théorème que la *somme* des nombres premiers subordonnés à n du genre supérieur augmenté de l'unité sera égale à

$$\Delta n - \Sigma p_1 \Delta\left(\frac{n}{p_1}\right) + \Sigma p_1 p_2 \Delta\left(\frac{n}{p_1 p_2}\right) - \ldots.$$

Par exemple, si $n = 11$, les nombres premiers subordonnés à 11 du genre supérieur seront 5, 7, 11, et les nombres premiers subordonnés à $\sqrt{n}$ sont 2, 3.

On doit donc trouver, et en effet on trouve

$$(11 \,.\, 12) - 2\,(5 \,.\, 6) - 3\,(3 \,.\, 4) + 6\,(1 \,.\, 2) = 2\,(1 + 5 + 7 + 11).$$

Je saisis cette occasion pour dire que j'ai fait calculer la valeur de $J(n)$, "somme-totient de n," pour toutes les valeurs entières de n jusqu'à 500, et je trouve que sans aucune exception $J(n)$ est toujours plus grand que $\frac{3}{\pi^2}(n^2)$ et plus petit que $\frac{3}{\pi^2}(n+1)^2$.

Il reste à démontrer que ces limites sont d'application universelle pour un nombre entier quelconque n.

On peut faire une extension illimitée du théorème donné dans le numéro précédent des *Comptes rendus* sur les *sommes-totients,* tout à fait analogue à l'extension ci-dessus donnée au théorème de Legendre sur les nombres premiers. Nommons, par exemple, $u(j)$ la somme de tous les nombres premiers et inférieurs à j, et Uj la somme

$$u(1) + u(2) + \ldots + u(j).$$

On établit facilement* l'identité

$$\sum_{r=\infty}^{r=1} \Delta\left(E\frac{j}{r}\right) u\left(\frac{j}{r}\right) = \tfrac{1}{6} j\,(j+1)(j+2),$$

où Δx signifie le nombre triangulaire $\frac{1}{2}x(x+1)$, et avec ce théorème, en se servant, comme dans la théorie des sommes-totients, du principe† de la division harmonique et en écrivant

$$Vj = Uj - 2U\frac{j}{2} + 3U\frac{j}{3} - 4U\frac{j}{4} + 5U\frac{j}{5} - \ldots,$$

on en déduit facilement $Vj = \frac{j^3}{12} - \frac{j}{3}$ quand j est pair,

$$Vj = \frac{(j+1)^3}{12} + \frac{j+1}{6} \text{ quand } j \text{ est impair, etc.}$$

Dans ma Note‡ *Sur le nombre des fractions ordinaires inégales,* etc., j'ai omis de dire que l'équation

$$\sum_r E\frac{j}{r}\, Tr = \frac{j^2+j}{2}$$

peut être écrite sous la forme

$$Jj + J\frac{j}{2} + J\frac{j}{3} + J\frac{j}{4} + \ldots = \frac{j^2+j}{2}. \qquad (1)$$

[* With $u(r) = \frac{1}{2} r T(r)$, $u(1) = \frac{1}{2}$, $T(r)$ being the totient of r, we have

$$2 \sum_{r=1} \Delta\left(E\frac{i}{r}\right) u(r) = \tfrac{1}{6} i\,(i+1)\,(2i+1).]$$

[† Vol. III. of this Reprint, p. 673.] [‡ p. 84 above.]

De même, l'équation

$$\Sigma\Delta E\frac{j}{r}\,u\,\frac{j}{r}=\frac{j(j+1)(j+2)}{6}$$

équivaut à l'équation *

$$Uj+2U\frac{j}{2}+3U\frac{j}{3}+4U\frac{j}{4}+\ldots=\frac{j(j+1)(j+2)}{6}. \qquad (2)$$

Il est facile de démontrer, avec l'aide des équations (1) et (2), que les valeurs asymptotiques de $\frac{Jj}{j^2}$ et $\frac{Uj}{j^3}$ pour j indéfiniment grand sont $\frac{3}{\pi^2}$ et $\frac{1}{\pi^2}$ respectivement.

Cauchy, MM. Halphen et Lucas ont écrit sur *les suites de Farey.* Il est donc bon de faire remarquer que Jj est le nombre des fractions et Uj la somme des numérateurs des fractions dans une telle suite pour laquelle la limite donnée est j.

[* For $\frac{1}{6}j(j+1)(j+2)$ *read* $\frac{1}{12}j(j+1)(2j+1)$.]

4.

SUR LE PRODUIT INDÉFINI $1 - x . 1 - x^2 . 1 - x^3 \ldots$

[*Comptes Rendus*, XCVI. (1883), p. 674.]

DANS le *Johns Hopkins Circular*, numéro de février*, on trouvera l'explication d'une méthode *graphique* pour convertir les produits continus en séries. J'ai appliqué cette méthode pour obtenir la formule connue (Cayley, *Elliptic Functions*, p. 296)

$$\frac{1}{1-ax.1-ax^2.1-ax^3\ldots}$$

$$= 1 + \frac{xa}{1-x.1-ax} + \frac{x^4a^2}{1-x.1-x^2.1-ax.1-ax^2}$$

$$+ \frac{x^9a^3}{1-x.1-x^2.1-x^3.1-ax.1-ax^2.1-ax^3} + \ldots.$$

Je me suis demandé quelle serait l'expression obtenue en appliquant la même construction (ou dissection) graphique (qui fournit la formule citée en haut), au produit $1+ax.1+ax^2.1+ax^3\ldots$, et j'ai trouvé sans aucune difficulté l'expression suivante:

$$1 + xa\frac{1+ax^2}{1-x} + x^5a^2\frac{1+ax.1+ax^4}{1-x.1-x^2} + \ldots$$

$$+ x^{\frac{3j^2-j}{2}}a^j\frac{1+ax.1+ax^2\ldots 1+ax^{j-1}}{1-x.1-x^2\ldots 1-x^{j-1}}\cdot\frac{1+ax^{2j}}{1-x^j} + \ldots.$$

En faisant $a = -1$, on obtient

$$1-x.1-x^2.1-x^3\ldots$$

$$= 1 - x(1+x) + x^5(1+x^2) + \ldots + (-)^j x^{\frac{3j^2-j}{2}}(1+x^j) + \ldots.$$

C'est le théorème bien connu d'Euler, lequel, sous ce point de vue, n'est qu'un corollaire d'un théorème plus général.

Par la même méthode, j'obtiens la série pour les *thêta* fonctions et d'autres séries beaucoup plus générales, sans calcul algébrique aucun.

[* Vol. III. of this Reprint, pp. 669, 686; and above pp. 30, 33.]

5.

SUR UN THÉORÈME DE PARTITIONS.

[*Comptes Rendus*, XCVI. (1883), pp. 674, 675.]

SOIENT $s_1, s_2, \ldots, s_i$ des suites de nombres consécutifs, telles que le plus petit terme dans aucune d'elles n'excède de plus de l'unité le plus grand terme dans la suite qui précède ; bien entendu que i peut devenir l'unité et qu'une suite quelconque peut se réduire à un seul terme. On peut envisager ce système de suites comme une partition de la somme des nombres contenus dans leur totalité : alors on aura le théorème suivant :

Le nombre de systèmes de i suites de nombres consécutifs dont la somme est N est le même que le nombre de partitions de N qu'on peut former avec les repétitions de i nombres impairs. Comme exemple, en faisant $N = 10$ et $i = 1, 2, 3$ successivement, on aura d'un côté les divers groupes de partitions

10	9, 1	1, 2, 7	1, 3, 6
1, 2, 3, 4	8, 2	2, 3, 5	
	7, 3	1, 4, 5	
	6, 4		

et de l'autre (en se servant d'un indice supérieur pour signifier le nombre des réflexions de sa base),

5^2	9, 1	$3^3, 1$	$1^2, 3, 5$
1^{10}	7, 3	$3^2, 1^4$	
	$7, 1^3$	$3, 1^7$	
	$5, 1^5$		

En ajoutant ensemble les équations qui, pour la même valeur de N, répondent à toutes les valeurs possibles de i, on retombe sur le théorème bien connu d'Euler que *le nombre des partitions de N, en excluant seulement les répétitions, est le même que le nombre de ses partitions en excluant seulement les nombres pairs.* Ainsi, on peut envisager ce dernier théorème comme un corollaire d'un théorème bien autrement profond et qui n'est pas du tout facile à démontrer, sinon pour le cas le plus simple, c'est-à-dire quand il n'y a qu'une seule suite. Pour ce cas, le théorème peut s'exprimer en disant que *le nombre de suites de nombres consécutifs dont la somme est N est égal au nombre de diviseurs impairs de N.*

6.

PREUVE GRAPHIQUE* DU THÉORÈME D'EULER SUR LA PARTITION DES NOMBRES PENTAGONAUX.

[*Comptes Rendus*, XCVI. (1883), pp. 743—745.]

UNE partition quelconque de n peut être représentée par un assemblage de points uniformément distribués sur un plan et limités par deux lignes droites. Ainsi, par exemple, l'arrangement suivant:

. . . .
. . . . .
. . . . .
. . .
. .

sera la représentation graphique de la partition du nombre 22 dans les parties

$$7, \quad 5, \quad 5, \quad 3, \quad 2.$$

Mais, de plus, un tel arrangement de points peut être distribué dans un carré et deux groupes que je nommerai *latéral* et *inférieur*. Ainsi, l'arrangement écrit ci-dessus peut être décomposé dans un carré de neuf points, dans un groupe latéral de huit et dans un groupe inférieur de cinq points.

Considérons les partitions de n dans j parties *inégales*. Tous les arrangements de points qui correspondent à ces partitions peuvent être classifiés selon la valeur du côté du carré qui y correspond et que je nommerai θ. Alors, pour une valeur donnée de θ, le groupe latéral contiendra nécessairement ou θ ou $\theta-1$ lignes de points, car autrement il y aurait des parties égales dans l'arrangement. Dans le premier cas, le nombre de colonnes dans ce groupe inférieur peut être un nombre quelconque, mais pas plus grand que θ; dans le second cas, pas plus grand que $\theta-1$. Donc, en se rappelant que le nombre de partitions de ν en θ parties inégales est le coefficient de x dans le développement de

$$\frac{x^{\frac{\theta^2+\theta}{2}}}{1-x \,.\, 1-x^2 \ldots 1-x^\theta}$$

et que le nombre de partitions de ν dans $j-\theta$ parties inégales et pas plus grandes que θ est le coefficient de $x^\nu a^{j-\theta}$ dans le développement de

$$(1+ax)(1+ax^2)\ldots(1+ax^\theta),$$

on voit que, quand le nombre de lignes dans le groupe latéral est θ, le nombre

[* See p. 32 above.]

total d'arrangements de n dans j parties inégales qui correspondent à cette espèce de distribution sera le coefficient de $x^{n-\theta^2}a^{j-\theta}$ dans le développement de

$$\frac{1+ax.1+ax^2\ldots1+ax^\theta}{1-x.1-x^2\ldots1-x^\theta}x^{\frac{\theta^2+\theta}{2}}.$$

De même, le nombre des partitions qui correspondent à la seconde hypothèse sera le coefficient de $x^{n-\theta^2}a^{j-\theta}$ dans le développement de

$$\frac{1+ax.1+ax^2\ldots1+ax^{\theta-1}}{1-x.1-x^2\ldots1-x^{\theta-1}}x^{\frac{\theta^2-\theta}{2}}.$$

En donnant à θ toutes les valeurs depuis 1 jusqu'à l'infini, on obtiendra toutes les partitions de n dans j parties inégales. Les cas où θ excède j n'offrent rien d'exceptionnel, car, pour ces cas, le coefficient de $a^{j-\theta}$ dans les deux fonctions génératrices sera nul.

Or le coefficient de $x^{n-\theta^2}a^{j-\theta}$ dans chacune de ces deux fonctions est le même que le coefficient de $x^n a^j$ dans les produits qui résultent de leur multiplication par $x^{\theta^2}a^\theta$.

En comparant les coefficients de $x^n a^j$ pour toute valeur de n et i, on trouve donc

$$\begin{aligned}&(1+xa)(1+x^2a)(1+x^3a)+\ldots\\&=1+\frac{1+ax}{1-x}x^2a+\frac{1+ax.1+ax^2}{1-x.1-x^2}x^7a^2+\ldots\\&\quad+\frac{1+ax.1+ax^2\ldots1+ax^\theta}{1-x.1-x^2\ldots1-x^\theta}x^{\frac{3\theta^2+\theta}{2}}a^\theta+\ldots\\&\quad+xa+\frac{1+ax}{1-x}x^5a^2+\ldots\\&\quad+\frac{1+ax.1+ax^2\ldots1+ax^{\theta-1}}{1-x.1-x^2\ldots1-x^{\theta-1}}x^{\frac{3\theta^2-\theta}{2}}a^\theta+\ldots.\end{aligned}$$

En mettant $a=-1$, on obtient ainsi

$$1-x.1-x^2.1-x^3-\ldots=1-x-x^2-\ldots+(-)^\theta\left(x^{\frac{3\theta^2-\theta}{2}}+x^{\frac{3\theta^2+\theta}{2}}\right)+\ldots,$$

ce qui est le théorème d'Euler.

En réunissant les deux séries dans une seule, on obtient, pour le cas général,

$$\begin{aligned}&(1+xa)(1+x^2a)(1+x^3a)+\ldots\\&=1+\frac{1+ax^2}{1-x}xa+\frac{1+ax.1+ax^4}{1-x.1-x^2}x^5a^2+\frac{1+ax.1+ax^2.1+ax^6}{1-x.1-x^2.1-x^3}x^{12}a^3+\ldots,\end{aligned}$$

c'est-à-dire l'équation que j'ai donnée dans la Note précédente [p. 91].

Je dois dire que c'est M. Durfee, étudiant à Baltimore, qui, le premier (dans un tout autre problème), a fait usage du genre de décomposition d'une *assemblée régulière* de points dans un carré et deux groupes supplémentaires dont j'ai profité dans l'analyse précédente (voir *Johns Hopkins Circular*, [Vol. III. of this Reprint, pp. 661 ff.]).

7.

DÉMONSTRATION GRAPHIQUE* D'UN THÉORÈME D'EULER CONCERNANT LES PARTITIONS DES NOMBRES.

[*Comptes Rendus*, XCVI. (1883), pp. 1110—1112.]

COMME confirmation de la puissance de la méthode graphique appliquée à la théorie des partitions, la preuve suivante d'un théorème que je crois être nouveau ne sera pas, je l'espère, tout à fait dépourvue d'intérêt pour les géomètres; car il serait, il me semble, assez difficile d'en trouver une preuve directe analytique au moyen de la comparaison de fonctions génératrices, comme on le fait ordinairement pour des théorèmes de ce genre.

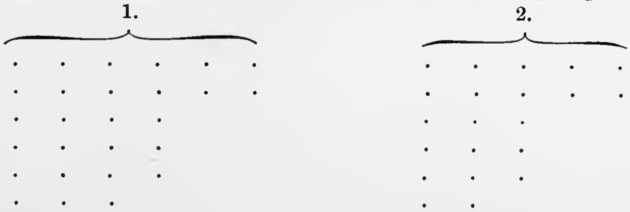
Euler a trouvé facilement, par une comparaison de telles fonctions, que le nombre de partitions de n en nombres impairs est le même que le nombre de partitions de n en nombres inégaux; je précise ce théorème en ajoutant que le nombre de partitions de n en nombres impairs, qui se divisent en i groupes de nombres distincts, est égal au nombre de partitions de n en i suites tout à fait distinctes de nombres consécutifs.

Nommons U une partition en nombres impairs et V une partition en nombres inégaux.

Je dis qu'on peut passer de U à V par la méthode suivante. Supposons, par exemple, que U soit la partition 11.11.7.7.7.5.

Je forme deux assemblages réguliers de points en prenant dans l'un d'eux, sur chaque ligne, un nombre de points égal à $\frac{11+1}{2}$, $\frac{11+1}{2}$, $\frac{7+1}{2}$, $\frac{7+1}{2}$, $\frac{7+1}{2}$, $\frac{5+1}{2}$, et l'autre assemblage en diminuant de l'unité chacun de ces nombres de points. On forme ainsi ces deux assemblages:

1. 2.



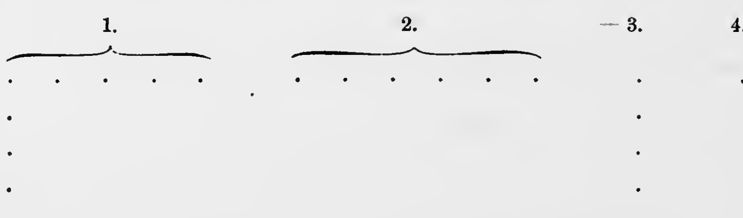
et, en comptant le nombre de points dans les *angles* successifs de chaque figure, on obtient, dans l'un, 11, 9, 5, 2, et, dans l'autre, 10, 8, 3; en les réunissant, on obtient la partition

$$11.10.9.8.5.3.2,$$

qui est un V.

[* See p. 39 above.]

Or il est facile de voir que dans cette méthode de transformation U devient V, et l'on démontre (en construisant un certain système d'équations linéaires) que, pour un V quelconque donné, on peut trouver un et un seul U qui se transformera dans ce V, de sorte qu'il y a correspondance un à un entre la totalité des U et la totalité des V, ce qui sert à démontrer le théorème original d'Euler. Mais si tel était le but de cette recherche, cette méthode de transformation serait peine perdue, car il existe une tout autre méthode, infiniment plus simple, d'établir une telle correspondance: on la trouvera expliquée dans le cahier de l'*American Journal of Mathematics* qui va paraître. L'utilité de cette méthode spéciale de créer la correspondance consiste en ceci: que le V ainsi conjugué avec un U contiendra le même nombre de suites distinctes de nombres consécutifs que le U contient de nombres impairs distincts: cela veut dire que le nombre des lignes inégales (disons i) dans l'un ou l'autre assemblage de points est toujours égal à j, nombre de suites distinctes obtenu en opérant de la manière expliquée ci-dessus. La preuve en est facile; car si l'on enlève l'angle extérieur à l'un et à l'autre des assemblages, on verra facilement que quatre cas se présenteront: pour un de ces cas, j ne change pas de valeur, à cause du changement opéré dans les deux assemblages; dans un autre cas, j subira une diminution de deux unités, et dans les deux cas intermédiaires d'une seule unité. Ces cas correspondent aux quatre suppositions qui résultent de la combinaison des hypothèses que les deux premières lignes ou les deux premières colonnes dans l'un ou l'autre des assemblages sont ou ne sont pas égales entre elles: de sorte qu'on verra facilement que le j et le i seront toujours diminués de la même quantité, ou 0, ou 1 ou 2, et conséquemment on aura $i-j$ *constant*; si l'on enlève l'un après l'autre les angles des deux assemblages jusqu'à ce qu'on arrive à un assemblage qui sera de l'une ou l'autre des quatre formes suivantes:



pour lesquels cas $i=2$, $j=2$; $i=1$, $j=1$; $i=1$, $j=1$; $i=1$, $j=1$; respectivement on aura toujours ainsi $i=j$, de sorte qu'il y a correspondance une à une entre les partitions du même nombre n qui contiennent justement i nombres impairs répétés (ou non) à volonté, et celles qui contiennent justement i suites distinctes de nombres consécutifs, et conséquemment il y aura le même nombre des unes et des autres: ce qui est le théorème que j'ai voulu démontrer.

8.

SUR UN THÉORÈME DE PARTITIONS* DE NOMBRES COMPLEXES CONTENU DANS UN THÉORÈME DE JACOBI.

[*Comptes Rendus*, XCVI. (1883), pp. 1276—1280.]

DANS le *Journal de Crelle*, t. XXXII. p. 166, Jacobi fait la remarque que le développement en série de $\Theta_1 x$ donne lieu à un théorème que j'exprime de la manière suivante.

Soient a et b deux quantités $c = a + b$; alors le produit infini

$$(1 \mp q^a)(1 \mp q^b)(1 - q^c)(1 \mp q^{a+c})(1 \mp q^{b+c})(1 - q^{2c}) \ldots = \sum_{-\infty}^{+\infty} (\mp)^i q^{\frac{i^2c + i(a-b)}{2}}.$$

Ce théorème étant vrai pour un nombre infini de valeurs de $\frac{a}{b}$ sera, par sa forme même, nécessairement vrai quand a et b sont de symboles absolument arbitraires, et l'on voit facilement que, pour le montrer dans ce sens universel, il suffira d'énoncer un certain théorème sur les nombres complexes dont voici l'énoncé:

Désignons par C, B, A des nombres complexes de la forme fc, $fc + b$, $fc + a$, où f est ou zéro ou un nombre entier et positif quelconque.

Considérons un arrangement composé avec des C, des B et des A non répétés ou avec des C, B, A pris seuls ou combinés deux à deux, en excluant les arrangements (que je nomme exceptionnels) qui ne contiennent que des B formant une série arithmétique dont b est le dernier terme et c la différence constante, ou des A formant une série semblable dont a est le dernier terme.

Par le caractère majeur et le caractère mineur d'un tel arrangement, je désigne la parité ou l'imparité du nombre total *des termes et du nombre des C qu'il contient. Je dis qu'à chaque arrangement (non exceptionnel) on peut en associer un autre pareil dont la somme totale des éléments (les A, B, C) sera la même, mais dont les caractères seront tous les deux opposés.*

La démonstration deviendra plus claire en se servant de la notation suivante. En désignant par X un symbole d'une série de termes, je me servirai de X et de X pour signifier le terme le plus haut et le terme le plus

[* See above, p. 59 ff.]

bas de la série, et en me servant de Y ou Z pour signifier un symbole ou simple ou affecté de marques quelconques, j'emploie les notations

$$Y = 0, \quad Y + Z = 0, \quad Y > 0, \quad Y + Z > 0,$$

pour signifier que les Y manquent, que les Y et les Z manquent tous les deux, que les Y ne manquent pas, que les Y et les Z ne manquent pas tous les deux.

Je divise les B (d'un arrangement quelconque) en deux espèces, $'B$ et B', dont $'B$ représente un B appartenant à la série arithmétique (la plus grande qu'on puisse former) commençant avec le plus grand B, et B' les autres B qui se trouvent dans l'arrangement.

Ainsi je divise les A en ${}_{,}A$ et en $A_{,}$; $A_{,}$ signifie un A appartenant à la série arithmétique la plus grande qu'on puisse former, dont a est le terme minimum (de sorte que, si l'arrangement ne contient pas un a, $A_{,}$ manque) et ${}_{,}A$ signifie les autres A de l'arrangement.

Finalement un point au centre d'un symbole à droite ou à gauche signifiera ce symbole diminué ou augmenté respectivement de c.

On voit que dans cette notation les arrangements exceptionnels seront exprimés ainsi : ceux qui appartiennent à l'une des deux classes par les conditions $'B - b = 0$ avec $A + C = 0$, et les autres par les conditions $B = 0$ avec ${}_{,}A + C = 0$.

Je divise les arrangements non exceptionnels en trois classes, dont les conditions seront respectivement les suivantes :

Première classe :

$$'\underset{\cdot}{B} - b > 0 \quad \text{ou} \quad ('\underset{\cdot}{B} - b = 0 \quad \text{avec} \quad \underset{\cdot}{C} - c \leqq '\dot{B} - b).$$

Deuxième classe :

$$'B - b = 0 \quad \text{avec} \quad (\underset{\cdot}{C} - c > '\dot{B} - b \quad \text{ou} \quad C = 0, \quad \text{mais} \quad A + C > 0),$$

ou

$$B = 0 \quad \text{avec} \quad (A = 0 \quad \text{ou} \quad \underset{\cdot}{A} - a \gneqq C).$$

Troisième classe :

$$B = 0 \quad \text{avec} \quad A > 0 \quad \text{et} \quad \underset{\cdot}{A} - a < \underset{\cdot}{C} \quad \text{et} \quad {}_{,}A + C > 0.$$

Toutes les hypothèses possibles se trouvent comprises dans ces tableaux des arrangements exceptionnels et non exceptionnels.

A chacune des trois classes des derniers je vais assigner un opérateur qui peut être appliqué à chaque arrangement de cette classe et qui le transformera dans un autre arrangement appartenant à la même classe ; cette disposition, appliquée deux fois successivement, reproduira l'arrangement sur lequel on opère, lequel ne changera pas la somme des éléments, mais changera chacun des deux caractères en sens opposé : c'est-à-dire que chacun des trois opérateurs que je vais définir, et que je nommerai ϕ, ψ, ϑ, doit satisfaire à cinq conditions qu'on peut nommer *catholicité, homœogénèse, mutualité, inertie* et *énantiotropie.*

1. ϕ signifie que, si $C=0$ ou $\underset{.}{C}-c>{}'\dot{B}-{}'\underset{.}{B}$, on doit former un nouveau C, en substituant, pour chaque ${}'B$, ${}'B\cdot$ (c'est-à-dire sa valeur diminuée de c), et reconstituer l'inertie originale en ajoutant ensemble les c ainsi soustraits pour former un nouveau C, et que, dans le cas contraire, $\underset{.}{C}$ doit être décomposé en simples c, dont on ajoutera un au premier ${}'B$ (le B le plus grand), un au second ${}'B$, etc., jusqu'à ce que tous les c dont on a à disposer soient épuisés.

2. ψ signifie que, si $B>0$ ou $C=0$, ou $\underset{.}{C}>{}'\dot{B}+A$, on doit former un nouveau C en substituant à ${}'\dot{B}$ et A leur somme et que, dans le cas contraire, $\underset{.}{C}$ doit être décomposé en ${}'\cdot\dot{B}$ et A si $B>0$ et en b et A si $B=0$.

3. ϑ signifie que, si $C=0$ ou $\underset{.}{C}+\dot{A}_{,}=>\underset{.}{A}$, il faut décomposer ${}_{,}A$ en $\cdot\dot{A}_{,}$ et C ou en a et C, selon que $A_{,}=$ ou >0, et que, dans le cas contraire, pour $\underset{.}{C}$ et $\dot{A}_{,}$, il faut substituer leur somme. On sera satisfait en étudiant les conditions des trois classes que les ϕ, ψ, ϑ possèdent tous les trois cinq attributs voulus: la preuve en est facilitée en supposant que, dans chaque série des C, des B et des A, prise séparément, on suit un ordre régulier de grandeur dans l'arrangement de ces termes respectivement au multiple de c qui entre dans chacun d'eux.

Si l'on donne à a et à b des valeurs quantitatives (ce qui est toujours permis), et en particulier les valeurs 1 et 2 respectivement, on retombe sur le théorème d'Euler, mais (chose à noter) la correspondance donnée par le procédé général appliqué à ce cas ne sera nullement identique à la correspondance donnée par le procédé de Franklin. En effet, les arrangements exceptionnels ne seront pas les mêmes dans les deux méthodes: selon le procédé de Franklin, les arrangements non conjugables sont de la forme

$$i,\ i+1,\ \ldots,\ 2i-1 \quad \text{ou} \quad i+1,\ i+2,\ \ldots,\ 2i,$$

tandis que la méthode actuelle donnera, comme non conjugués, les arrangements de la forme

$$1,\ 4,\ \ldots,\ 3i-2 \quad \text{ou} \quad 2,\ 5,\ \ldots,\ 3i-1.$$

La méthode employée ici fournira elle-même toujours deux systèmes de correspondance absolument distincts, dont on obtient l'un, qui n'est pas exprimé, en échangeant entre eux les a, A et les b, B, car la méthode n'est pas symétrique dans son opération sur ces deux systèmes de lettres.

Ce cas est analogue à celui de la correspondance perspective entre deux triangles, laquelle peut être simple ou triple, comme je l'ai montré ailleurs. Jacobi, dans l'endroit cité, a fait la remarque que, pour $a=1$, $b=2$, en se servant du signe supérieur ($\mp$) dans son théorème, on retombe sur le théorème d'Euler et que, pour le cas de $a=1$, $b=1$, en se servant du signe inférieur, sur un théorème donné (il y a longtemps par Gauss). On peut ajouter que, si avec cette supposition on se sert du signe supérieur, on obtient $0=0$, mais si l'on écrit $a=1-\epsilon$, $b=1$, en faisant ϵ infinitésimal, on tombe (chose singulière) sur l'équation de Jacobi elle-même,

$$(1-q)^3(1-q^2)^3(1-q^3)^3+\ldots=1-3q+5q^3-7q^6+\ldots.$$

Puisque j'ai introduit le nom de l'auteur des *Fundamenta nova*, qu'on me permette la remarque que, dans les deux avant-dernières lignes de l'avant-dernière page de cet immortel Ouvrage, on trouve un théorème qui équivaut à l'équation

$$\frac{q}{1+q}-\frac{q^3}{1+q^3}+\frac{q^5}{1+q^5}-\ldots=\frac{q}{1+q}-\frac{q^{1+2}}{1+q^2}+\frac{q^{1+2+3}}{1+q^3}-\ldots;$$

or, le premier cas du théorème intitulé: *Sur un théorème d'Euler*, contenu dans une Note précédente des *Comptes rendus**, affirme que le nombre des séries arithmétiques avec lesquelles on peut exprimer n est égal au nombre des diviseurs impairs de n, laquelle considération mène immédiatement à une conséquence qu'on ne pourrait manquer d'observer (mais que M. Franklin, effectivement, a remarquée le premier) et qui s'exprime par l'équation

$$\frac{q}{1-q}+\frac{q^3}{1-q^3}+\frac{q^5}{1-q^5}+\ldots=\frac{q}{1-q}+\frac{q^{1+2}}{1-q^2}+\frac{q^{1+2+3}}{1-q^3}+\ldots,$$

équation très ressemblante à l'autre et qui peut être combinée avec elle de manière à donner naissance à quatre autres équations de la même espèce.

On n'a pas besoin de dire que le théorème qui constitue la matière principale de cette Note, en faisant $a=1$ et en considérant b comme une quantité arbitraire, contient ou au moins conduit immédiatement au développement de $\Theta_1 x$ dont Jacobi l'a traité comme conséquence.

[* p. 95 above. Cf. p. 25 above.]

9.

ON THE NUMBER OF FRACTIONS CONTAINED IN ANY "FAREY SERIES" OF WHICH THE LIMITING NUMBER IS GIVEN.

[*Philosophical Magazine*, XV. (1883), pp. 251—257; XVI. (1883), pp. 230—233.]

A *Farey series* ("suite de Farey") is a system of all the unequal vulgar fractions arranged in order of magnitude, the numerator and denominator of which do not exceed a given number.

The first scientific notice of these series appeared in the *Philosophical Magazine*, Vol. XLVII. (1816), pp. 385, 386. In 1879 Mr Glaisher published in the *Philosophical Magazine* (pp. 321—336) a paper on the same subject containing a proof of their known properties, an important extension of the subject to series in which the numerators and denominators are subject to distinct limits, and a bibliography of Mr Goodwyn's tables of such series. Finally, in 1881 Sir George Airy contributed a paper also to the *Philosophical Magazine* of that year, in which he refers to a table calculated by him "some years ago," and printed in the Selected Papers of the *Transactions* of the Institution of Civil Engineers, which is in fact a Farey table with the logarithms of the fractions appended to each of them. Previous tables had only given the decimal values of such fractions. The drift of this paper is to point out a caution which it is necessary to observe in the use of such tables, and which limits their practical utility: this arises from the fact of the differences receiving a very large augmentation in the immediate neighbourhood of the fractions which are a small aliquot part of unity—a fact which may be inferred *à priori* from the well-known law discovered by Farey applicable to those differences, but to which the author of the paper makes no allusion.

In addition to the tables of Farey series by Goodwyn, Wucherer, an anonymous author mentioned in the Babbage Catalogue, and Gauss, referred to by Mr Glaisher in his Report to the Bradford Meeting of the British Association (1873), may be mentioned one contained in Herzer's *Tabellen*

(Basle, 1864) with the limit 57, and another in Hrabak's *Tabellen-Werk* (Leipsic, 1876), in which the limit is taken at 50.

The writers on the theory are:—Cauchy (as mentioned by Mr Glaisher), who inserted a communication relating to it in the *Bulletin des Sciences par la Société Philomathique de Paris*, republished in his *Exercices de Mathématiques*; Mr Glaisher himself (*loc. cit.*); M. Halphen, in a recent volume of the Proceedings of the Mathematical Society of France; and M. Lucas, in the next following volume of the same collection. I am indebted to my friend and associate Dr Story for these later references.

For theoretical purposes it is desirable to count $\frac{1}{1}$ as one of the fractions in a Farey series. The number of such fractions for the limit j then becomes identical with the sum of the *totients* of all the natural numbers up to j inclusive—a totient to x (which I denote by τx) meaning the number of numbers less than x and prime to it. Such sum, that is, $\overset{x=1}{\underset{x=j}{\Sigma}} \tau x$, I denote by Tj. My attention was called to the subject by this number Tj expressing the number of terms in a function whose residue (in Cauchy's sense) is the generating function to any given simple denumerant (see *American Journal of Mathematics*, [Vol. III. of this Reprint, p. 605]); and I became curious to know something about the value of Tj. I had no difficulty in finding a functional equation which serves to determine its limits (see *Johns Hopkins University Circular*, Jan. and Feb. 1883*). The most simple form of that equation (omitted to be given in the *Circular*) is

$$Tj + T\frac{j}{2} + T\frac{j}{3} + T\frac{j}{4} + T\frac{j}{5} + \ldots = \frac{j^2 + j}{2},$$

(where, when x is a fraction, Tx is to be understood to mean Tj, j being the integer next below x); and from this it is not difficult to deduce by strict demonstration that Tj/j^2, when j increases indefinitely, approximates indefinitely near to $3/\pi^2$.

I have subsequently found that if ux be used to denote the sum of all the numbers inferior and prime to x, and $Uj = \overset{x=1}{\underset{x=j}{\Sigma}} ux$, then †

$$Uj + 2U\frac{j}{2} + 3U\frac{j}{3} + 4U\frac{j}{4} + \ldots = \frac{j(j+1)(j+2)}{3}$$

(where Ux, when x is a fraction, means the U of the integer next inferior to x). From this equation it is also possible to prove that Uj/j^3, when j becomes indefinitely great, approximates to $1/\pi^2$. Uj, it may be well to notice, is the sum of all the numerators of the fractions in a Farey series whose limit is j, just as Tj is the number of these fractions.

In the annexed Table the value of τx (the totient), of Tx (the sum-totient), and of $3/\pi^2 . x^2$ is calculated for all the values of x from 1 to 1000; and the

[* See pp. 84, 89 above.]

[† The right side should be $\frac{1}{12}j(j+1)(2j+1)$.]

remarkable fact is brought to light that Tx is always greater than $3/\pi^2 . x^2$ (the number opposite to it), and less than $3/\pi^2 . (x+1)^2$, the number which comes after the following one in the same table.

I have calculated in my head the first few values of Ux, and find (if I have made no mistake) that it obeys an analogous law, namely is always intermediate between $1/\pi^2 . x^3$ and $1/\pi^2 . (x+1)^3$.

It may also be noticed that when n is a prime number, Tn is always nearer, and usually very much nearer, to the superior than to the inferior limit—as might have been anticipated from the circumstance that, when this is the case, in passing from $n-1$ to n the T receives an augmentation of $n-1$, whereas its average augmentation is only $\frac{3}{\pi^2}(2n-1)$.

In like manner and for a similar reason, when n contains several small factors Tn is nearer to the inferior than to the superior limit. For instance, when $n=210$, $Tn=13414$ and $3/\pi^2 . n^2 = 13404{\cdot}79$.

TABLE *of Totients, of Sum-totients, and of* $3/\pi^2$ *into the Squares of all the Numbers from* 1 *to* 1000 *inclusive.*

$$\left[\frac{3}{\pi^2} = {\cdot}30396355\right].$$

n	$\tau(n)$	$T(n)$	$\frac{3}{\pi^2}n^2$	n	$\tau(n)$	$T(n)$	$\frac{3}{\pi^2}n^2$	n	$\tau(n)$	$T(n)$	$\frac{3}{\pi^2}n^2$
1	1	1	·30	27	18	230	221·59	53	52	882	853·83
2	1	2	1·22	28	12	242	238·31	54	18	900	886·36
3	2	4	2·74	29	28	270	255·63	55	40	940	919·49
4	2	6	4·86	30	8	278	273·56	56	24	964	953·23
5	4	10	7·60	31	30	308	292·11	57	36	1000	987·58
6	2	12	10·94	32	16	324	311·26	58	28	1028	1022·54
7	6	18	14·90	33	20	344	331·01	59	58	1086	1058·10
8	4	22	19·46	34	16	360	351·38	60	16	1102	1094·27
9	6	28	24·62	35	24	384	372·35	61	60	1162	1131·05
10	4	32	30·40	36	12	396	393·93	62	30	1192	1168·44
11	10	42	36·78	37	36	432	416·12	63	36	1228	1206·43
12	4	46	43·77	38	18	450	438·92	64	32	1260	1245·03
13	12	58	51·37	39	24	474	462·32	65	48	1308	1284·25
14	6	64	59·58	40	16	490	486·34	66	20	1328	1324·07
15	8	72	68·39	41	40	530	510·96	67	66	1394	1364·49
16	8	80	77·81	42	12	542	536·19	68	32	1426	1405·53
17	16	96	87·84	43	42	584	562·02	69	44	1470	1447·17
18	6	102	98·48	44	20	604	588·47	70	24	1494	1489·42
19	18	120	109·73	45	24	628	615·52	71	70	1564	1532·28
20	8	128	121·58	46	22	650	643·19	72	24	1588	1575·75
21	12	140	134·05	47	46	696	671·45	73	72	1660	1619·82
22	10	150	147·12	48	16	712	700·33	74	36	1696	1664·51
23	22	172	160·79	49	42	754	729·82	75	40	1736	1709·80
24	8	180	175·08	50	20	774	759·91	76	36	1772	1755·69
25	20	200	189·98	51	32	806	790·61	77	60	1832	1802·20
26	12	212	205·48	52	24	830	821·92	78	24	1856	1849·31

TABLE (*continued*).

n	$\tau(n)$	$T(n)$	$\frac{3}{\pi^2}n^2$	n	$\tau(n)$	$T(n)$	$\frac{3}{\pi^2}n^2$	n	$\tau(n)$	$T(n)$	$\frac{3}{\pi^2}n^2$
79	78	1934	1897·04	134	66	5498	5457·97	189	108	10904	10857·88
80	32	1966	1945·37	135	72	5570	5539·74	190	72	10976	10973·09
81	54	2020	1994·31	136	64	5634	5622·11	191	190	11166	11088·90
82	40	2060	2043·85	137	136	5770	5705·09	192	64	11230	11205·31
83	82	2142	2094·01	138	44	5814	5788·68	193	192	11422	11322·34
84	24	2166	2144·77	139	138	5952	5872·88	194	96	11518	11439·97
85	64	2230	2196·14	140	48	6000	5957·69	195	96	11614	11558·21
86	42	2272	2248·12	141	92	6092	6043·10	196	84	11698	11677·06
87	56	2328	2300·70	142	70	6162	6129·12	197	196	11894	11796·52
88	40	2368	2353·90	143	120	6282	6215·75	198	60	11954	11916·59
89	88	2456	2407·70	144	48	6330	6302·99	199	198	12152	12037·26
90	24	2480	2462·10	145	112	6442	6390·83	200	80	12232	12158·54
91	72	2552	2517·12	146	72	6514	6479·29	201	132	12364	12280·43
92	44	2596	2572·75	147	84	6598	6568·35	202	100	12464	12402·93
93	60	2656	2628·98	148	72	6670	6658·02	203	168	12632	12526·03
94	46	2702	2685·82	149	148	6818	6748·29	204	64	12696	12649·75
95	72	2774	2743·27	150	40	6858	6839·18	205	160	12856	12774·07
96	32	2806	2801·33	151	150	7008	6930·67	206	102	12958	12899·00
97	96	2902	2860·00	152	72	7080	7022·77	207	132	13090	13024·54
98	42	2944	2919·27	153	96	7176	7115·48	208	96	13186	13150·68
99	60	3004	2979·15	154	60	7236	7208·80	209	180	13366	13277·43
100	40	3044	3039·64	155	120	7356	7302·72	210	48	13414	13404·79
101	100	3144	3100·73	156	48	7404	7397·26	211	210	13624	13532·76
102	32	3176	3162·44	157	156	7560	7492·40	212	104	13728	13661·34
103	102	3278	3224·75	158	78	7638	7588·15	213	140	13868	13790·52
104	48	3326	3287·67	159	104	7742	7684·51	214	106	13974	13920·32
105	48	3374	3351·20	160	64	7806	7781·47	215	168	14142	14050·72
106	52	3426	3415·34	161	132	7938	7879·04	216	72	14214	14181·73
107	106	3532	3480·08	162	54	7992	7977·22	217	180	14394	14313·34
108	36	3568	3545·44	163	162	8154	8076·01	218	108	14502	14445·57
109	108	3676	3611·40	164	80	8234	8175·41	219	144	14646	14578·40
110	40	3716	3677·96	165	80	8314	8275·41	220	80	14726	14711·84
111	72	3788	3745·14	166	82	8396	8376·02	221	192	14918	14845·89
112	48	3836	3812·92	167	166	8562	8477·24	222	72	14990	14980·54
113	112	3948	3881·31	168	48	8610	8579·07	223	222	15212	15115·81
114	36	3984	3950·31	169	156	8766	8681·50	224	96	15308	15251·68
115	88	4072	4019·92	170	64	8830	8784·55	225	120	15428	15388·16
116	56	4128	4090·14	171	108	8938	8888·20	226	112	15540	15525·25
117	72	4200	4160·96	172	84	9022	8992·46	227	226	15766	15662·94
118	58	4258	4232·39	173	172	9194	9097·33	228	72	15838	15801·24
119	96	4354	4304·43	174	56	9250	9202·80	229	228	16066	15940·15
120	32	4386	4377·08	175	120	9370	9308·88	230	88	16154	16079·67
121	110	4496	4450·33	176	80	9450	9415·57	231	120	16274	16219·80
122	60	4556	4524·19	177	116	9566	9522·87	232	112	16386	16360·53
123	80	4636	4598·66	178	88	9654	9630·78	233	232	16618	16501·87
124	60	4696	4673·74	179	178	9832	9739·29	234	72	16690	16643·82
125	100	4796	4794·43	180	48	9880	9848·42	235	184	16874	16786·38
126	36	4832	4825·72	181	180	10060	9958·15	236	116	16990	16929·55
127	126	4958	4902·63	182	72	10132	10068·49	237	156	17146	17073·32
128	64	5022	4980·14	183	120	10252	10179·44	238	96	17242	17217·70
129	84	5106	5058·26	184	88	10340	10290·99	239	238	17480	17362·70
130	48	5154	5136·98	185	144	10484	10403·15	240	64	17544	17508·30
131	130	5284	5216·32	186	60	10544	10515·92	241	240	17784	17654·51
132	40	5324	5296·26	187	160	10704	10629·30	242	110	17894	17801·32
133	108	5432	5376·81	188	92	10796	10743·29	243	162	18056	17948·74

TABLE (*continued*).

n	$\tau(n)$	$T(n)$	$\frac{3}{\pi^2}n^2$	n	$\tau(n)$	$T(n)$	$\frac{3}{\pi^2}n^2$	n	$\tau(n)$	$T(n)$	$\frac{3}{\pi^2}n^2$
244	120	18176	18096·77	299	264	27318	27174·65	354	116	38174	38091·50
245	168	18344	18245·41	300	80	27398	27356·72	355	280	38454	38307·01
246	80	18424	18394·66	301	252	27650	27539·40	356	176	38630	38523·12
247	216	18640	18544·51	302	150	27800	27722·69	357	192	38822	38739·85
248	120	18760	18694·97	303	200	28000	27906·59	358	178	39000	38957·18
249	164	18924	18846·04	304	144	28144	28091·10	359	358	39358	39175·13
250	100	19024	18997·72	305	240	28384	28276·21	360	96	39454	39393·68
251	250	19274	19150·01	306	96	28480	28461·93	361	342	39796	39612·83
252	72	19346	19302·90	307	306	28786	28648·26	362	180	39976	39832·60
253	220	19566	19456·40	308	120	28906	28835·20	363	220	40196	40052·97
254	126	19692	19610·51	309	204	29110	29022·75	364	144	40340	40273·95
255	128	19820	19765·23	310	120	29230	29210·90	365	288	40628	40495·54
256	128	19948	19920·56	311	310	29540	29399·66	366	120	40748	40717·74
257	256	20204	20076·49	312	96	29636	29589·03	367	366	41114	40940·55
258	84	20288	20233·03	313	312	29948	29779·01	368	176	41290	41163·96
259	216	20504	20390·18	314	156	30104	29969·59	369	240	41530	41387·98
260	96	20600	20547·94	315	144	30248	30160·79	370	144	41674	41612·61
261	168	20768	20706·30	316	156	30404	30352·59	371	312	41986	41837·85
262	130	20898	20865·28	317	316	30720	30545·00	372	120	42106	42063·69
263	262	21160	21024·86	318	104	30824	30738·01	373	372	42478	42290·15
264	80	21240	21185·05	319	280	31104	30931·64	374	160	42638	42517·21
265	208	21448	21345·84	320	128	31232	31125·87	375	200	42838	42744·87
266	108	21556	21507·25	321	212	31444	31320·71	376	184	43022	42973·15
267	176	21732	21669·26	322	132	31576	31516·16	377	336	43358	43202·04
268	132	21864	21831·88	323	288	31864	31712·22	378	108	43466	43431·53
269	268	22132	21995·11	324	108	31972	31908·88	379	378	43844	43661·63
270	72	22204	22158·95	325	240	32212	32106·15	380	144	43988	43892·34
271	270	22474	22323·39	326	162	32374	32304·03	381	252	44240	44123·65
272	128	22602	22488·44	327	216	32590	32502·52	382	190	44430	44355·58
273	144	22746	22654·10	328	160	32750	32701·62	383	382	44812	44588·11
274	136	22882	22820·37	329	276	33026	32901·32	384	128	44940	44821·25
275	200	23082	22987·25	330	80	33106	33101·63	385	240	45180	45055·00
276	88	23170	23154·73	331	330	33436	33302·55	386	192	45372	45289·35
277	276	23446	23322·82	332	164	33600	33504·08	387	252	45624	45524·32
278	138	23584	23491·52	333	216	33816	33706·22	388	192	45816	45759·89
279	180	23764	23660·83	334	166	33982	33908·96	389	388	46204	45996·07
280	96	23860	23830·75	335	264	34246	34112·31	390	96	46300	46232·86
281	280	24140	24001·27	336	96	34342	34316·27	391	352	46652	46470·25
282	92	24232	24172·40	337	336	34678	34520·84	392	168	46820	46708·25
283	282	24514	24344·14	338	156	34834	34726·01	393	260	47080	46946·87
284	140	24654	24516·49	339	224	35058	34931·80	394	196	47276	47186·09
285	144	24798	24689·44	340	128	35186	35138·19	395	312	47588	47425·91
286	120	24918	24863·00	341	300	35486	35345·19	396	120	47708	47666·35
287	240	25158	25037·18	342	108	35594	35552·80	397	396	48104	47907·39
288	96	25254	25211·96	343	294	35888	35761·01	398	198	48302	48149·04
289	272	25526	25387·34	344	168	36056	35969·83	399	216	48518	48391·30
290	112	25638	25563·34	345	176	36232	36179·26	400	160	48678	48634·17
291	192	25830	25739·94	346	172	36404	36389·30	401	400	49078	48877·64
292	144	25974	25917·15	347	346	36750	36599·95	402	132	49210	49121·73
293	292	26266	26094·97	348	112	36862	36811·21	403	360	49570	49366·42
294	84	26350	26273·40	349	348	37210	37023·07	404	200	49770	49611·72
295	232	26582	26452·43	350	120	37330	37235·54	405	216	49986	49857·62
296	144	26726	26632·07	351	216	37546	37448·61	406	168	50154	50104·14
297	180	26906	26812·32	352	160	37706	37662·30	407	360	50514	50351·26
298	148	27054	26993·18	353	352	38058	37876·59	408	128	50642	50598·99

TABLE (*continued*).

n	$\tau(n)$	$T(n)$	$\frac{3}{\pi^2}n^2$	n	$\tau(n)$	$T(n)$	$\frac{3}{\pi^2}n^2$	n	$\tau(n)$	$T(n)$	$\frac{3}{\pi^2}n^2$
409	408	51050	50847·33	464	224	65630	65442·14	519	344	82028	81875·93
410	160	51210	51096·27	465	240	65870	65724·52	520	192	82220	82191·75
411	272	51482	51345·83	466	232	66102	66007·51	521	520	82740	82508·18
412	204	51686	51595·99	467	466	66568	66291·11	522	168	82908	82825·21
413	348	52034	51846·76	468	144	66712	66575·31	523	522	83430	83142·85
414	132	52166	52098·14	469	396	67108	66860·13	524	260	83690	83461·10
415	328	52494	52350·12	470	184	67292	67145·55	525	240	83930	83779·95
416	192	52686	52602·72	471	312	67604	67431·58	526	262	84192	84099·42
417	276	52962	52855·92	472	232	67836	67718·22	527	480	84672	84419·49
418	180	53142	53109·73	473	420	68256	68005·46	528	160	84832	84740·17
419	418	53560	53364·15	474	156	68412	68293·32	529	506	85338	85061·46
420	96	53656	53619·17	475	360	68772	68581·78	530	208	85546	85383·36
421	420	54076	53874·80	476	192	68964	68870·85	531	348	85894	85705·87
422	210	54286	54131·04	477	312	69276	69160·52	532	216	86110	86028·98
423	276	54562	54387·89	478	238	69514	69450·81	533	480	86590	86352·70
424	208	54770	54645·35	479	478	69992	69741·70	534	176	86766	86677·03
425	320	55090	54903·42	480	128	70120	70033·20	535	424	87190	87001·97
426	140	55230	55162·09	481	432	70552	70325·31	536	264	87454	87327·51
427	360	55590	55421·39	482	240	70792	70618·03	537	356	87810	87653·66
428	212	55802	55681·26	483	264	71056	70911·35	538	268	88078	87980·42
429	240	56042	55941·76	484	220	71276	71205·29	539	420	88498	88307·79
430	168	56210	56202·86	485	384	71660	71499·83	540	144	88642	88635·77
431	430	56640	56464·57	486	162	71822	71794·98	541	540	89182	88964·35
432	144	56784	56726·89	487	486	72308	72090·73	542	270	89452	89293·54
433	432	57216	56989·82	488	240	72548	72387·10	543	360	89812	89623·34
434	180	57396	57253·36	489	324	72872	72684·07	544	256	90068	89953·75
435	224	57620	57517·50	490	168	73040	72981·65	545	432	90500	90284·77
436	216	57836	57782·26	491	490	73530	73279·84	546	144	90644	90616·39
437	396	58232	58047·62	492	160	73690	73578·63	547	546	91190	90948·62
438	144	58376	58313·58	493	448	74138	73878·04	548	272	91462	91281·46
439	438	58814	58580·16	494	216	74354	74178·05	549	360	91822	91614·91
440	160	58974	58847·34	495	240	74594	74478·67	550	200	92022	91948·97
441	252	59226	59115·14	496	240	74834	74779·90	551	504	92526	92283·64
442	192	59418	59383·54	497	420	75254	75081·73	552	176	92702	92618·91
443	442	59860	59652·54	498	164	75418	75384·18	553	468	93170	92954·79
444	144	60004	59922·16	499	498	75916	75687·23	554	276	93446	93291·28
445	352	60356	60192·38	500	200	76116	75990·89	555	288	93734	93628·38
446	222	60578	60463·22	501	332	76448	76295·15	556	276	94010	93966·08
447	296	60874	60734·66	502	250	76698	76600·03	557	556	94566	94304·39
448	192	61066	61006·70	503	502	77200	76905·52	558	180	94746	94643·31
449	448	61514	61279·36	504	144	77344	77211·61	559	504	95250	94982·84
450	120	61634	61552·62	505	400	77744	77518·31	560	192	95442	95322·98
451	400	62034	61826·49	506	220	77964	77825·62	561	320	95762	95663·72
452	224	62258	62100·97	507	312	78276	78133·54	562	280	96042	96005·07
453	300	62558	62376·06	508	252	78528	78442·06	563	562	96604	96347·03
454	226	62784	62651·75	509	508	79036	78751·19	564	184	96788	96689·60
455	288	63072	62928·05	510	128	79164	79060·93	565	448	97236	97032·77
456	144	63216	63204·97	511	432	79596	79371·28	566	282	97518	97376·55
457	456	63672	63482·48	512	256	79852	79682·23	567	324	97842	97720·94
458	228	63900	63760·61	513	324	80176	79993·79	568	280	98122	98065·94
459	288	64188	64039·35	514	256	80432	80305·96	569	568	98690	98411·55
460	176	64364	64318·69	515	408	80840	80618·74	570	144	98834	98757·76
461	460	64824	64598·64	516	168	81008	80932·13	571	570	99404	99104·58
462	120	64944	64879·20	517	460	81468	81246·12	572	240	99644	99452·01
463	462	65406	65160·36	518	216	81684	81560·72	573	380	100024	99800·05

TABLE (*continued*).

n	$\tau(n)$	$T(n)$	$\frac{3}{\pi^2}n^2$	n	$\tau(n)$	$T(n)$	$\frac{3}{\pi^2}n^2$	n	$\tau(n)$	$T(n)$	$\frac{3}{\pi^2}n^2$
574	240	100264	100148·70	629	576	120544	120260·45	684	216	142380	142211·17
575	440	100704	100497·95	630	144	120688	120643·14	685	544	142924	142627·30
576	192	100896	100847·81	631	630	121318	121026·44	686	294	143218	143044·03
577	576	101472	101198·28	632	312	121630	121410·35	687	456	143674	143461·37
578	272	101744	101549·36	633	420	122050	121794·86	688	536	144010	143879·32
579	384	102128	101901·05	634	316	122366	122179·98	689	624	144634	144297·88
580	224	102352	102253·34	635	504	122870	122565·71	690	176	144810	144717·05
581	492	102844	102606·24	636	208	123078	122952·05	691	690	145500	145136·82
582	192	103036	102959·75	637	504	123582	123338·00	692	344	145844	145557·20
583	520	103556	103313·87	638	280	123862	123726·55	693	360	146204	145978·19
584	288	103844	103668·60	639	420	124282	124114·71	694	346	146550	146399·79
585	288	104132	104023·93	640	256	124538	124503·48	695	552	147102	146821·99
586	292	104424	104379·87	641	640	125178	124892·86	696	224	147326	147244·80
587	586	105010	104736·42	642	212	125390	125282·85	697	640	147966	147668·22
588	168	105178	105093·58	643	642	126032	125673·44	698	348	148314	148092·25
589	540	105718	105451·35	644	264	126296	126064·64	699	464	148778	148516·89
590	232	105950	105809·72	645	336	126632	126456·45	700	240	149018	148942·14
591	392	106342	106168·70	646	288	126920	126848·87	701	700	149718	149367·99
592	288	106630	106528·29	647	646	127566	127241·89	702	216	149934	149794·45
593	592	107222	106888·49	648	216	127782	127635·52	703	648	150582	150221·52
594	180	107402	107249·29	649	580	128362	128029·76	704	320	150902	150649·20
595	384	107786	107610·70	650	240	128602	128424·60	705	368	151270	151077·48
596	296	108082	107972·72	651	360	128962	128820·06	706	352	151622	151506·37
597	396	108478	108335·35	652	324	129286	129216·12	707	600	152222	151935·87
598	264	108742	108698·59	653	652	129938	129612·79	708	232	152454	152365·98
599	598	109340	109062·43	654	216	130154	130010·07	709	708	153162	152796·70
600	160	109500	109426·88	655	520	130674	130407·96	710	280	153442	153228·02
601	600	110100	109791·94	656	320	130994	130806·46	711	468	153910	153659·95
602	252	110352	110157·61	657	432	131426	131205·56	712	352	154262	154092·49
603	396	110748	110523·89	658	276	131702	131605·27	713	660	154922	154525·64
604	300	111048	110890·77	659	658	132360	132005·59	714	192	155114	154959·40
605	440	111488	111258·26	660	160	132520	132406·52	715	480	155594	155393·76
606	200	111688	111626·36	661	660	133180	132808·06	716	356	155950	155828·73
607	606	112294	111995·07	662	330	133510	133210·20	717	476	156426	156264·31
608	288	112582	112364·39	663	384	133894	133612·95	718	358	156784	156700·50
609	336	112918	112734·31	664	328	134222	134016·31	719	718	157502	157137·30
610	240	113158	113104·84	665	432	134654	134420·28	720	192	157694	157574·70
611	552	113710	113475·98	666	216	134870	134824·86	721	612	158306	158012·71
612	192	113902	113847·73	667	616	135486	135230·04	722	342	158648	158451·33
613	612	114514	114220·09	668	332	135818	135635·83	723	480	159128	158890·56
614	306	114820	114593·05	669	444	136262	136042·23	724	360	159488	159330·40
615	320	115140	114966·62	670	264	136526	136449·24	725	560	160048	159770·84
616	240	115380	115340·80	671	600	137126	136856·86	726	220	160268	160211·89
617	616	115996	115715·59	672	192	137318	137265·08	727	726	160994	160653·55
618	204	116200	116090·99	673	672	137990	137673·91	728	288	161282	161095·82
619	618	116818	116466·99	674	336	138326	138083·35	729	486	161768	161538·69
620	240	117058	116843·60	675	360	138686	138493·40	730	288	162056	161982·17
621	396	117454	117220·82	676	312	138998	138904·05	731	672	162728	162426·26
622	310	117764	117598·65	677	676	139674	139315·31	732	240	162968	162870·96
623	528	118292	117977·08	678	224	139898	139727·18	733	732	163700	163316·27
624	192	118484	118356·12	679	576	140474	140139·66	734	366	164066	163762·18
625	500	118984	118735·77	680	256	140730	140552·75	735	336	164402	164208·70
626	312	119296	119116·03	681	452	141182	140966·44	736	352	164754	164655·83
627	360	119656	119496·90	682	300	141482	141380·74	737	660	165414	165103·57
628	312	119968	119878·37	683	682	142164	141795·65	738	240	165654	165551·92

TABLE (*continued*).

n	$\tau(n)$	$T(n)$	$\frac{3}{\pi^2}n^2$	n	$\tau(n)$	$T(n)$	$\frac{3}{\pi^2}n^2$	n	$\tau(n)$	$T(n)$	$\frac{3}{\pi^2}n^2$
739	738	166392	166000·87	794	396	191870	191629·56	849	564	219340	219097·23
740	288	166680	166450·43	795	416	192286	192112·56	850	320	219660	219613·66
741	432	167112	166900·60	796	396	192682	192596·17	851	792	220452	220130·71
742	312	167424	167351·38	797	796	193478	193080·39	852	280	220732	220648·36
743	742	168166	167802·77	798	216	193694	193565·21	853	852	221584	221166·62
744	240	168406	168254·76	799	736	194430	194050·64	854	360	221944	221685·48
745	592	168998	168707·36	800	320	194750	194536·67	855	432	222376	222204·96
746	372	169370	169160·57	801	528	195278	195023·32	856	424	222800	222725·04
747	492	169862	169614·39	802	400	195678	195510·57	857	856	223656	223245·73
748	320	170182	170068·82	803	720	196398	195998·43	858	240	223896	223767·03
749	636	170818	170523·85	804	264	196662	196486·90	859	858	224754	224288·93
750	200	171018	170979·50	805	528	197190	196975·98	860	336	225090	224811·44
751	750	171768	171435·75	806	360	197550	197465·66	861	480	225570	225334·56
752	368	172136	171892·61	807	536	198086	197955·96	862	430	226000	225858·29
753	500	172636	172350·07	808	400	198486	198446·86	863	862	226862	226382·62
754	336	172972	172808·14	809	808	199294	198938·37	864	288	227150	226907·57
755	600	173572	173266·82	810	216	199510	199430·48	865	688	227838	227433·12
756	216	173788	173726·11	811	810	200320	199923·21	866	432	228270	227959·28
757	756	174544	174186·01	812	336	200656	200416·54	867	544	228814	228486·05
758	378	174922	174646·52	813	540	201196	200910·48	868	360	229174	229012·43
759	440	175362	175107·63	814	360	201556	201405·03	869	780	229954	229541·41
760	288	175650	175569·35	815	648	202204	201900·19	870	224	230178	230070·01
761	760	176410	176031·68	816	256	202460	202395·95	871	792	230970	230599·21
762	252	176662	176494·62	817	756	203216	202892·32	872	432	231402	231129·02
763	648	177310	176958·16	818	408	203624	203389·30	873	576	231978	231659·43
764	380	177690	177422·31	819	432	204056	203886·89	874	396	232374	232190·46
765	384	178074	177887·07	820	320	204376	204385·09	875	600	232974	232722·09
766	382	178456	178352·44	821	820	205196	204883·89	876	288	233262	233254·33
767	696	179152	178818·42	822	272	205468	205383·30	877	876	234138	232787·18
768	256	179408	179285·00	823	822	206290	205883·32	878	438	234576	234320·64
769	768	180176	179752·19	824	408	206698	206383·95	879	584	235160	234854·70
770	240	180416	180219·99	825	400	207098	206885·19	880	320	235480	235389·37
771	512	180928	180688·40	826	348	207446	207387·03	881	880	236360	235924·65
772	384	181312	181157·42	827	826	208272	207889·48	882	252	236612	236460·54
773	772	182084	181627·04	828	264	208536	208392·54	883	882	237494	236997·04
774	252	182336	182097·27	829	828	209364	208896·21	884	384	237878	237534·14
775	600	182936	182568·11	830	328	209692	206400·49	885	464	238342	238071·85
776	384	183320	183039·56	831	552	210244	209905·37	886	442	238784	238610·17
777	432	183752	183511·61	832	384	210628	210410·86	887	886	239670	239149·10
778	388	184140	183984·28	833	672	211300	210916·96	888	288	239958	239688·64
779	720	184860	184457·55	834	276	211576	211423·67	889	756	240714	240228·78
780	192	185052	184931·43	835	664	212240	211930·98	890	352	241066	240769·53
781	700	185752	185405·92	836	360	212600	212438·91	891	540	241606	241310·89
782	352	186104	185881·01	837	540	213140	212947·44	892	444	242050	241852·86
783	504	186608	186356·71	838	418	213558	213456·58	893	828	242878	242395·43
784	336	186944	186833·02	839	838	214396	213966·32	894	296	243174	242938·62
785	624	187568	187309·94	840	192	214588	214476·68	895	712	243886	243482·41
786	260	187828	187787·47	841	812	215400	214987·64	896	384	244270	244026·81
787	786	188614	188265·60	842	420	215820	215499·21	897	528	244798	244571·81
788	392	189006	188744·34	843	560	216380	216011·39	898	448	245246	245117·43
789	524	189530	189223·69	844	420	216800	216524·18	899	840	246086	245663·65
790	312	189842	189703·65	845	624	217424	217037·57	900	240	246326	246210·48
791	672	190514	190184·22	846	276	217700	217551·58	901	832	247158	246757·91
792	240	190754	190665·39	847	660	218360	218066·19	902	400	247558	247305·96
793	720	191474	191147·17	848	416	218776	218581·40	903	504	248062	247854·61

TABLE* (*continued*).

n	$\tau(n)$	$T(n)$	$\frac{3}{\pi^2}n^2$	n	$\tau(n)$	$T(n)$	$\frac{3}{\pi^2}n^2$	n	$\tau(n)$	$T(n)$	$\frac{3}{\pi^2}n^2$
904	448	248510	248403·88	937	936	267256	266870·57	970	384	286076	285999·30
905	720	249230	248953·75	938	396	267652	267440·51	971	970	287046	286589·30
906	300	249530	249504·22	939	624	268276	268011·05	972	324	287370	287179·90
907	906	250436	250055·31	940	368	268644	268582·19	973	828	288198	287771·11
908	452	250888	250607·00	941	940	269584	269153·95	974	486	288684	288362·92
909	600	251488	251159·31	942	312	269896	269726·31	975	480	289164	288955·35
910	288	251776	251712·22	943	880	270776	270299·28	976	480	289644	289548·39
911	910	252686	252265·73	944	464	271240	270872·86	977	976	290620	290142·03
912	288	252974	252819·86	945	432	271672	271447·05	978	324	290944	290736·28
913	820	253794	253374·59	946	420	272092	272021·84	979	880	291824	291331·13
914	456	254250	253929·93	947	946	273038	272597·25	980	336	292160	291926·60
915	480	254730	254485·88	948	312	273350	273173·26	981	648	292808	292522·67
916	456	255186	255042·44	949	864	274214	273749·88	982	490	293298	293119·35
917	780	255966	255599·61	950	360	274574	274327·10	983	982	294280	293716·64
918	288	256254	256157·38	951	632	275206	274905·94	984	320	294600	294314·54
919	918	257172	256715·76	952	384	275590	275483·38	985	784	295384	294913·04
920	352	257524	257274·75	953	952	276542	276062·43	986	448	295832	295512·15
921	612	258136	257834·34	954	312	276854	276642·09	987	552	296384	296111·87
922	460	258596	258394·55	955	760	277614	277222·36	988	432	296816	296712·20
923	840	259436	258955·36	956	476	278090	277803·23	989	924	297740	297313·14
924	240	259676	259516·78	957	560	278650	278384·71	990	240	297980	297914·68
925	720	260396	260078·81	958	478	279128	278966·80	991	990	298970	298516·83
926	462	260858	260641·45	959	816	279944	279549·50	992	480	299450	299119·59
927	612	261470	261204·69	960	256	280200	280132·81	993	660	300110	299722·96
928	448	261918	261768·55	961	930	281130	280716·72	994	420	300530	300326·94
929	928	262846	262333·01	962	432	281562	281301·24	995	792	301322	300931·52
930	240	263086	262898·07	963	636	282198	281886·37	996	328	301650	301536·71
931	756	263842	263463·75	964	480	282678	282472·11	997	996	302646	302142·51
932	464	264306	264030·03	965	768	283446	283058·46	998	498	303144	302748·92
933	620	264926	264596·93	966	264	283710	283645·41	999	648	303792	303355·93
934	466	265392	265164·43	967	966	284676	284232·97	1000	400	304192	303963·55
935	640	266032	265732·53	968	440	285116	284821·14				
936	288	266320	266301·25	969	576	285692	285409·92				

* In the extended as well as in the original Table it will be seen that the sum-totient is always intermediate between $3/\pi^2 . n^2$ and $3/\pi^2 . (n+1)^2$.

The formula of verification applied at every tenth step to the T column precludes the possibility of the existence of other than typographical errors or errors of transcription. Accumulative errors are rendered impossible.

10.

ON THE EQUATION TO THE SECULAR INEQUALITIES IN THE PLANETARY THEORY.

[*Philosophical Magazine*, XVI. (1883), pp. 267—269.]

A VERY long time ago I gave, in this *Magazine**, a proof of the reality of the roots in the above equation, in which I employed a certain property of the square of a symmetrical matrix which was left without demonstration. I will now state a more general theorem concerning the *product* of *any* two matrices of which that theorem is a particular case. In what follows it is of course to be understood that the product of two matrices means the matrix corresponding to the *combination* of two *substitutions* which those matrices represent.

It will be convenient to introduce here a notion (which plays a conspicuous part in my new theory of multiple algebra), namely that of the *latent roots* of a matrix—latent in a somewhat similar sense as vapour may be said to be latent in water or smoke in a tobacco-leaf. If from each term in the diagonal of a given matrix, λ be subtracted, the determinant to the matrix so modified will be a rational integer function of λ; the roots of that function are the latent roots of the matrix; and there results the important theorem that the latent roots of any function of a matrix are respectively the same functions of the latent roots of the matrix itself: for example, the latent roots of the square of a matrix are the squares of its latent roots.

The latent roots of the product of two matrices, it may be added, are the same in whichever order the factors be taken. If, now, m and n be any two matrices, and $M = mn$ or nm, I am able to show that the sum of the products of the latent roots of M taken i together in every possible way is equal to the sum of the products obtained by multiplying every minor determinant of the ith order in one of the two matrices m, n by its *altruistic opposite* in the other: the reflected image of any such determinant, in respect to the principal diagonal of the matrix to which it belongs, is its *proper* opposite, and the corresponding determinant to this in the other matrix is its *altruistic opposite.*

[* Vol. I. of this Reprint, p. 378.]

The proof of this theorem will be given in my large forthcoming memoir on Multiple Algebra designed for the *American Journal of Mathematics*.

Suppose, now, that m and n are transverse to one another, that is, that the lines in the one are identical with the columns in the other, and *vice versâ*, then any determinant in m becomes identical with its altruistic opposite in n; and furthermore, if m be a symmetrical matrix, it is its own transverse. Consequently we have the theorem (the one referred to at the outset of this paper) that the sum of the i-ary products of the latent roots of the square of a symmetrical matrix (that is, of the squares of the roots of the matrix itself) is equal to the sum of the squares of all the minor determinants of the order i in the matrix; whence it follows, from Descartes's theorem, that when all the terms of a symmetrical matrix are real, none of its latent roots can be *pure* imaginaries, and, as an easy inference, cannot be *any kind* of imaginaries; or, in other words, all the latent roots of a symmetrical matrix are real, which is Laplace's theorem.

I may take this opportunity of stating the important theorem that if $\lambda_1, \lambda_2, \ldots \lambda_i$ are the latent roots of any matrix m, then

$$\phi m = \Sigma \frac{(m-\lambda_2)(m-\lambda_3)\ldots(m-\lambda_i)}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)\ldots(\lambda_1-\lambda_i)}\phi\lambda.$$

This theorem of course presupposes the rule first stated by Prof. Cayley (*Phil. Trans.* 1857) for the addition of matrices.

When any of the latent roots are equal, the formula must be replaced by another obtained from it by the usual method of infinitesimal variation. If $\phi m = m^{\frac{1}{\omega}}$, it gives the expression for the ωth root of the matrix; and we see that the number of such roots is ω^i, where i is the order of the matrix. When, however, the matrix is *unitary*, that is, all its terms except the diagonal ones are *zeros*, or *zeroidal*, that is, when all its terms are *zeros*, this conclusion is no longer applicable, and a certain definite number of arbitrary quantities enter into the general expressions for the roots.

The case of the extraction of any root of a unitary matrix of the second order was first considered and successfully treated by the late Mr Babbage; it reappears in M. Serret's *Cours d'Algèbre supérieure*. This problem is of course the same as that of finding a function $\frac{ax+b}{cx+d}$ of any given order of periodicity. My memoir will give the solution of the corresponding problem for a matrix of any order. Of the many unexpected results which I have obtained by my new method, not the least striking is the *rapprochement* which it establishes between the theory of Matrices and that of Invariants. The theory of invariance relative to associated Matrices includes and transcends that relative to algebraical functions.

11.

ON THE INVOLUTION AND EVOLUTION OF QUATERNIONS.

[*Philosophical Magazine,* XVI. (1883), pp. 394—396.]

THE subject-matter of quaternions is really nothing more nor less than that of substitutions of the second order, such as occur in the familiar theory of quadratic forms. A linear substitution of the second order is in essence identical with a square matrix of the second order, the law of multiplication between one such matrix and another being understood to be the same as that of the composition of one substitution with another, and therefore depending on the order of the factors; but as regards the multiplication of three or more matrices, subject to the same associative law as in ordinary algebraical multiplication.

Every matrix of the second order may be regarded as representing a quaternion, and *vice versâ*; in fact if, using i to denote $\sqrt{(-1)}$, we write a matrix m of the second order under the form

$$\begin{matrix} a+bi, & c+di, \\ -c+di, & a-bi, \end{matrix}$$

we have by definition,

$$m = a\alpha + b\beta + c\gamma + d\delta,$$

where $$\alpha = \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}, \quad \beta = \begin{matrix} i & 0 \\ 0 & -i \end{matrix}, \quad \gamma = \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix}, \quad \delta = \begin{matrix} 0 & i \\ i & 0 \end{matrix}.$$

Now $$\alpha^2 = \alpha, \quad \beta^2 = \gamma^2 = \delta^2 = -\alpha,$$
$$\alpha\beta = \beta\alpha = \beta, \quad \alpha\gamma = \gamma\alpha = \gamma, \quad \alpha\delta = \delta\alpha = \delta,$$
$$\beta\gamma = -\gamma\beta = \alpha, \quad \gamma\delta = -\delta\gamma = \beta, \quad \delta\beta = -\beta\delta = \gamma;$$

so that we may for $\alpha, \beta, \gamma, \delta$, substitute $1, h, k, l$, four symbols subject to the same laws of self-operation and mutual interaction as unity and the three Hamiltonian symbols. Now I have given the universal formula for expressing any given function of a matrix of *any* order as a rational function of that matrix and its latent roots; and consequently the qth power or root of any quadratic matrix, and therefore of any quaternion, is known. As far as I am informed, only the square root of a quaternion has been given in the text-books on quaternions, notably by Hamilton in his Lectures on Quaternions.

The latent roots of m are the roots of the quadratic equation

$$\lambda^2 - 2a\lambda + a^2 + b^2 + c^2 + d^2 = 0.$$

The general formula

$$\phi m = \Sigma\phi\lambda_1 \frac{(m-\lambda_2)(m-\lambda_3)\dots(m-\lambda_i)}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)\dots(\lambda_1-\lambda_i)};$$

where i is the order of the matrix m, when $i=2$ and $\phi m = m^{\frac{1}{q}}$, becomes

$$m^{\frac{1}{q}} = \frac{\lambda_1^{\frac{1}{q}} - \lambda_2^{\frac{1}{q}}}{\lambda_1 - \lambda_2} m - \frac{\lambda_2\lambda_1^{\frac{1}{q}} - \lambda_1\lambda_2^{\frac{1}{q}}}{\lambda_1 - \lambda_2},$$

where λ_1, λ_2 are the roots of the above equation. If μ is the modulus of the quaternion, namely is $\sqrt{(a^2+b^2+c^2+d^2)}$, and $\mu\cos\theta = a$, the latent roots λ_1, λ_2 assume the form

$$\mu(\cos\theta \pm i\sin\theta).$$

When the modulus is zero the two latent roots are equal to one another, and to a, the scalar of the quaternion; so that in this case the ordinary theory of vanishing fractions shows that

$$m^{\frac{1}{q}} = a^{\frac{1}{q}}\left(\frac{m}{a} + \frac{q-1}{q}\right).$$

In the general case there are q^2 roots of the qth order to a quaternion. Calling $\frac{\pi}{q} = \omega$, and writing $m^{\frac{1}{q}} = Am + B$,

$$A = \frac{\mu^{\frac{1}{q}}}{\mu} \frac{\cos\left(\frac{\theta}{q} + 2k\omega\right) + i\sin\left(\frac{\theta}{q} + 2k\omega\right) - \cos\left(\frac{\theta}{q} + 2k'\omega\right) + i\sin\left(\frac{\theta}{q} + 2k'\omega\right)}{2i\sin\theta},$$

$$B = -\mu^{\frac{1}{q}} \frac{\cos\left(\frac{q-1}{q}\theta + 2k\omega\right) + i\sin\left(\frac{q-1}{q}\theta + 2k\omega\right) - \cos\left(\frac{q-1}{q}\theta + 2k'\omega\right) + i\sin\left(\frac{q-1}{q}\theta + 2k'\omega\right)}{2i\sin\theta}.$$

For the q system of values $k = k' = 1, 2, 3 \dots q$, the coefficients A and B will be real, for the other $q^2 - q$ systems of values imaginary; so that there are q quaternion-proper qth roots of a quaternion-proper in Hamilton's sense, and $q^2 - q$ of the sort which, by a most regrettable piece of nomenclature, he terms bi-quaternions. The real or proper-quaternion values of $m^{\frac{1}{q}}$ are

$$\frac{\mu^{\frac{1}{q}}}{\sin\theta}\left\{\sin\left(\frac{\theta}{q} + 2k\omega\right)\frac{m}{\mu} + \sin\left(\frac{q-1}{q}\theta + 2k\omega\right)\right\},$$

$\mu^{\frac{1}{q}}$ meaning *the* or (when there is an alternative) *either* real value of the qth root of the modulus.

In the qth root (or power) of a quaternion m, the form $Am + B$ shows that the vector-part remains constant to an ordinary algebraical factor *près*; and we know *à priori* from the geometrical point of view that this ought to be the case. When the vector disappears a porism starts into being; and besides the values of the roots given by the general formula, there are others involving arbitrary parameters. Babbage's famous investigation of the form of the homographic function of $\frac{px + q}{rx + s}$ of x, which has a periodicity of any given degree q, is in fact (surprising as such a statement would have appeared to Babbage and Hamilton) one and the same thing as to find the qth root of unity under the form of a quaternion!

It is but justice to the eminent President of the British Association to draw attention to the fact that the substance of the results here set forth (although arrived at from an independent and more elevated order of ideas) may be regarded as a statement (reduced to the explicit and most simple form) of results capable of being extracted from his memoir on the Theory of Matrices, *Phil. Trans.* Vol. CXLVIII. (1858) (*vide* pp. 32—34, arts. 44—49).

12.

ON THE INVOLUTION OF TWO MATRICES OF THE SECOND ORDER.

[*British Association Report, Southport* (1883), pp. 430—432.]

IF m, n be two matrices of any order i, then, taking the determinant of the matrix $z + yn + xm$, there results a ternary quantic in the variables x, y, z, which may be termed the quantic of the corpus m, n.

In what follows I confine myself almost exclusively to the case of a corpus of the second order; the quantic may be written

$$z^2 + 2bzx + 2cyz + dx^2 + 2exy + fy^2:$$

it is then easy to establish the identical relations

$$m^2 - 2bm + d = 0,$$
$$mn + nm - 2bn - 2cm + 2e = 0,$$
$$n^2 - 2cn + f = 0.$$

It hence easily appears that any given function of m, n can, by aid of the five parameters b, c, d, e, f, be expressed in the form $A + Bm + Cn + Dmn$.

This form containing four arbitrary constants, it follows that in general any given matrix of the second order can be expressed as a function of m and n; for there will be four linear equations between A, B, C, D and the four elements of the given matrix. But this statement is subject to two cases of exception.

The first of these is when n and m are functions of one another: for in this case $A + Bm + Cn + Dmn$ is reducible to the form $P + Qm$, and there will be only two disposable constants wherewith to satisfy the four linear equations.

The second case is when the determinant of the fourth order formed by the elements of the four matrices $\begin{matrix} 1, & m \\ n, & mn \end{matrix}\Bigg|$ vanishes; writing

$$m, n = \begin{vmatrix} t_1, & t_2 \\ t_3, & t_4 \end{vmatrix}, \quad \begin{vmatrix} \tau_1, & \tau_2 \\ \tau_3, & \tau_4 \end{vmatrix}$$

respectively, it is not difficult to show that the value of this determinant is

$$(t_2\tau_3 - \tau_2 t_3)^2 + \{(t_1 - t_4)\,\tau_2 - (\tau_1 - \tau_4)\,t_2\}\,\{(t_1 - t_4)\,\tau_3 - (\tau_1 - \tau_4)\,t_3\}.$$

This expression is a function of the five parameters b, c, d, e, f, as may be shown in a variety of ways.

Thus it is susceptible of easy proof that if μ_1, μ_2 are the roots of the equation $\mu^2 - 2b\mu + d = 0$, and ν_1, ν_2 the roots of the equation $\nu^2 - 2c\nu + f = 0$, then, the two matrices being related as above, we must have

$$(m - \mu_1)(n - \nu_1) = 0,$$
$$(n - \nu_2)(m - \mu_2) = 0,$$

and consequently, by virtue of the middle one of the three identities,

$$\mu_1\nu_1 + \mu_2\nu_2 - 2e = 0.$$

Writing this in the form

$$(\mu_1\nu_1 + \mu_2\nu_2 - 2e)(\mu_1\nu_2 + \mu_2\nu_1 - 2e) = 0,$$

this is $$4e^2 - 2e\,.\,4bc + (\mu_1^2 + \mu_2^2)\,\nu_1\nu_2 + (\nu_1^2 + \nu_2^2)\,\mu_1\mu_2 = 0,$$

which gives $$e^2 - 2bce + b^2f + c^2d - df = 0\,;$$

the function on the left hand is the invariant (discriminant) of the ternary quantic appurtenant to the corpus, and we have this invariant $= 0$ as the necessary and sufficient condition of the involution of the elements of the corpus; the invariant in question is for this reason called the involutant.

Expressing the values of the coefficients in terms of the elements of the two matrices, namely

$$2b = t_1 + t_4, \quad 2c = \tau_1 + \tau_4,$$
$$d = t_1t_4 - t_2t_3, \quad 2e = t_1\tau_4 + \tau_1t_4 - t_2\tau_3 - t_3\tau_2, \quad f = \tau_1\tau_4 - \tau_2\tau_3,$$

it at once appears that the two expressions for the involutant are, to a numerical factor *près*, identical.

It can be shown *à priori* that the involutant of a corpus of the second order must be expressible in terms of the coefficients of the function; and therefore, being obviously invariantive in regard to linear substitutions impressed on m, n, it must be also invariantive for linear substitutions impressed on z, x, y, and must therefore be the invariant of the function. The corresponding theorem is not true, it should be observed, for the involutant of a corpus beyond the second order; for such involutant cannot in general be expressed in terms of the coefficients of the function.

The expression for the involutant in terms of the t's and τ's may also be obtained directly from the equation $(m - \mu_1)(n - \nu_1) = 0$. To this end it is only necessary to single out any term of the matrix represented by the left-hand side of the equation and equate it to zero: the resulting equation rationalised will be found to reproduce the expression in question.

I have thus indicated four methods of obtaining the involutant to a matrix-corpus of the second order; but there is yet a fifth, the simplest of all, and the most suggestive of the course to be pursued in investigating the higher order of involutants.

I observe that for a corpus of any order the function $mn - nm$ is invariantive for any linear substitution impressed on m and n. Its determinant will therefore be an invariant for any substitution impressed on m and n. When m and n are of the second order, reducing each term of $(mn - nm)^2$, that is $mnmn - mn^2m - nm^2n + nmnm$, and of $mn - nm$, by means of the three identical equations, to the form of a linear function of mn, m, n, 1, it will be found without difficulty that there results the identical equation

$$(mn - nm)^2 + I = 0,$$

the coefficient of $mn - nm$ vanishing. Consequently the determinant of the matrix $mn - nm$ is equal to I, which on calculation will be found to be identical with the invariant of the ternary quadric function.

It is obvious from the three identical equations that if m, n are in involution—that is, if their involutant is zero—every rational and integral function of m, n will be in involution with every other rational and integral function of m, n. Hence follows this new and striking theorem concerning matrices of the second order: If $f(m, n)$ and $\phi(m, n)$ are any rational functions whatever of m, n, the determinant to the matrix $mn - nm$ is contained as a factor in the determinant to the matrix $f\phi - \phi f$.

It may be noticed that f, ϕ need not be integer functions *by stipulation*, because any linear function of mn, m, n, 1, divided anteriorly or posteriorly by a second like function, can itself be expressed as a linear function of the same four terms.

As a very simple example of the theorem, observe that the determinant of $m^2n - mnm$ will contain as a factor the determinant of $mn - nm$.

13.

SUR LES QUANTITÉS FORMANT UN GROUPE DE NONIONS ANALOGUES AUX QUATERNIONS DE HAMILTON.

[*Comptes Rendus*, XCVII. (1883), pp. 1336—1340.]

On sait qu'on peut tout à fait (et très avantageusement) changer la base de la théorie des quaternions en considérant les trois symboles i, j, k de Hamilton comme des matrices binaires.

Si h, j sont des matrices binaires qui satisfont à l'équation $hj = -jh$, on démontre facilement que, en écartant le cas où $hj = jh = 0$, h^2 et k^2 seront de la forme

$$\begin{matrix} c & 0 \\ 0 & c \end{matrix}, \quad \begin{matrix} \gamma & 0 \\ 0 & \gamma \end{matrix}$$

c'est-à-dire cu, γu, où u est l'unité binaire

$$\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}$$

On peut ajouter, si l'on veut, les deux conditions $c^2 = \bar{1}$, $\gamma^2 = \bar{1}$; alors, en supprimant, pour plus de brièveté, le u, qui jouit de propriétés tout à fait analogues à celles de l'unité ordinaire, on obtient facilement les équations connues

$$h^2 = \bar{1}, \quad j^2 = \bar{1}, \quad k^2 = \bar{1},$$

$$hj = -jh = k, \quad jk = -kj = i, \quad ki = -ik = j.$$

De plus, en supposant que (i, j) soit un système particulier qui satisfait à l'équation $ij = -ji$, on peut déduire les valeurs universelles de I, J qui satisfont à l'équation $IJ = -JI$ en termes de i, j. En effet, on démontre rigoureusement que, en écartant toujours la solution $mn = nm = 0$, on aura

$$I = ai + bj + cij,$$

$$J = \alpha i + \beta j + \gamma ij,$$

avec la seule condition $a\alpha + b\beta + c\gamma = 0$. De plus, si l'on suppose $i^2 = j^2 = \bar{u}$ et aussi $I^2 = J^2 = \bar{u}$, on aura

$$a^2 + b^2 + c^2 = 1, \quad \alpha^2 + \beta^2 + \gamma^2 = 1,$$

de sorte que, en écrivant $ij = k$, $IJ = K$ et $K = Ai + Bj + Ck$, la matrice

$$\begin{matrix} a & b & c \\ \alpha & \beta & \gamma \\ A & B & C \end{matrix}$$

formera une matrice orthogonale. Une solution, parmi les plus simples, des équations $ij = -ji$, $i^2 = \bar{u}$, $j^2 = \bar{u}$, est la suivante:

$$i = \begin{vmatrix} \theta & 0 \\ 0 & -\theta \end{vmatrix}, \quad j = \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix}$$

et conséquemment

$$k = ij = \begin{vmatrix} 0 & -\theta \\ -\theta & 0 \end{vmatrix},$$

où $\theta = \sqrt{(-1)}$.

En écrivant une quantité binormale quelconque (c'est-à-dire une matrice binaire) sous la forme

$$\begin{matrix} a + b\theta, & -c - d\theta, \\ c - d\theta, & a - b\theta, \end{matrix}$$

on voit qu'elle peut être mise sous la forme $au + bi + cj + dk$, où il est souvent commode de supprimer (c'est-à-dire de sous-entendre) sans écrire l'unité binaire u.

On peut construire d'une manière tout à fait analogue un système de nonions en considérant l'équation $m = \rho n$, où m, n sont des matrices ternaires et ρ une racine cubique primitive de l'unité (voir* la *Circular* du *Johns Hopkins University* qui va prochainement paraître), en prenant pour les nonions fondamentaux u (l'unité ternaire)

$$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$$

et les huit matrices m, m^2; n, n^2; m^2n, mn^2; mn, m^2n^2 construites avec les valeurs les plus simples de m, n qui satisfont aux équations

$$nm = \rho mn, \quad m^3 = u, \quad n^3 = u.$$

Les valeurs

$$m = \begin{vmatrix} 1 & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & \rho^2 \end{vmatrix} \quad \text{et} \quad n = \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & \rho \\ \rho^2 & 0 & 0 \end{vmatrix}$$

peuvent être prises pour les valeurs basiques du système de nonions.

Une quantité ternaire (c'est-à-dire une matrice) quelconque s'exprime alors sous la forme

$$a + bm + \beta m^2 + cn + \gamma n^2 + dm^2n + \delta mn^2 + emn + \epsilon m^2n^2;$$

[* Vol. III. of this Reprint, p. 647. Also below, p. 122.]

mais, quand cette matrice M est capable de s'associer avec une autre N dans l'équation $NM = \rho MN$, alors il devient nécessaire que

$$a = 0, \quad b\beta + c\gamma + d\delta + e\epsilon = 0.$$

Je n'entrerai pas ici dans les détails de la méthode d'associer la solution générale de l'équation $NM = \rho MN$ avec une solution quelconque particulière de cette équation, mais je me bornerai à expliquer quelles sont les conditions auxquelles les éléments de M et de N doivent satisfaire afin que cette équation ait lieu.

M. Cayley a résolu la question analogue pour les matrices binaires dans le beau Mémoire, qu'il a publié dans les *Transactions of the Royal Society* de 1858. En supposant que m et n sont les matrices

$$\begin{matrix} a & b \\ c & d \end{matrix} \qquad \begin{matrix} a' & b' \\ c' & d' \end{matrix}$$

il trouve que, afin que $nm = -mn$, il faut avoir

$$a + d = 0, \quad a' + d' = 0, \quad aa' + bc' + cb' + dd' = 0.$$

Au lieu de cette troisième équation (en la combinant avec les deux précédentes), on peut écrire

$$ad' + a'd - bc' - b'c = 0.$$

Alors ces trois conditions équivalent à dire que le déterminant de la matrice $xu + my + nz$ (u étant l'unité binaire), qui, en général, est de la forme

$$x^2 + 2Bxy + 2Cxz + Dy^2 + 2Eyz + Fz^2,$$

se réduira à la forme

$$x^2 + Dy^2 + Fz^2,$$

car, dans le déterminant de $xu + my + nz$, c'est-à-dire de

$$\begin{vmatrix} x + ay + a'z & by + b'z \\ cy + c'z & x + dy + d'z \end{vmatrix},$$

les coefficients de xy, xz, yz seront évidemment

$$a + d, \quad a' + d', \quad ad' + a'd - bc' - b'c$$

respectivement.

Passons au cas de m et n, matrices ternaires qui satisfont à l'équation

$$nm = \rho mn.$$

Formons le déterminant de $xu + ym + zn$, où u représente l'unité ternaire

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}.$$

Ce déterminant sera de la forme

$$x^3+3Bx^2y+3Cx^2z+3Dxy^2+6Exyz+3Fxz^2+Gy^3+3Hy^2z+3Kyz^2+Lz^3,$$

et je trouve que, dans le cas supposé, il faut que les sept conditions souscrites soient satisfaites; $B=0$, $C=0$, $D=0$, $E=0$, $F=0$, $H=0$, $K=0$, de sorte que la fonction en x, y, z devient une somme de trois cubes, mais ces sept conditions, qu'on pourrait nommer *conditions paramétriques*, quoique nécessaires, ne sont pas suffisantes; il faut y ajouter une huitième condition que je nommerai $Q=0$.

Pour former Q, voici la manière de procéder:

En supposant que

$$m=\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix} \quad \text{et} \quad n=\begin{vmatrix} a' & b' & c' \\ d' & e' & f' \\ g' & h' & k' \end{vmatrix},$$

on écrit, au lieu de m, son transversal

$$\begin{vmatrix} a' & d' & g' \\ b' & e' & h' \\ c' & f' & k' \end{vmatrix},$$

et l'on forme neuf produits en multipliant chaque déterminant mineur du second ordre contenu dans m avec le déterminant mineur semblablement posé dans le transversal de n: la somme de ces neuf produits est Q.

Ces huit conditions que je démontre sont suffisantes et nécessaires (en écartant comme auparavant le cas où $nm=mn=0$) pour que $nm=\rho mn$.

On pourrait très bien se demander ce qui arrive dans le cas où les sept conditions paramétriques sont satisfaites, mais non pas la huitième condition supplémentaire.

Dans ce cas, je trouve* que mn et nm restent fonctions l'une et l'autre et qu'on aura

$$nm = A + B_1 mn + C(mn)^2,$$
$$mn = -A + B_2 nm + C(nm)^2,$$

où B_1, B_2 sont les racines de l'équation algébrique

$$B^2+B+1=0,$$

A, C étant deux quantités arbitraires et indépendantes, sauf que l'une d'elles ne peut pas s'évanouir sans l'autre, les deux s'évanouissant ensemble pour le cas (et seulement pour le cas) où Q (qui fournit la condition supplémentaire) s'évanouit.

[* See footnote [†], p. 154 below.]

14.

ON QUATERNIONS, NONIONS, SEDENIONS, ETC.

[*Johns Hopkins University Circulars*, III. (1884), pp. 7—9.]

(1) SUPPOSE that m and n are two matrices of the second order. Then if we call the determinant of the matrix $x + my + nz$,

$$x^2 + 2bxy + 2cxz + dy^2 + 2eyz + fz^2,$$

the necessary and sufficient conditions for the subsistence of the equation $nm = -mn$ is that $b = 0$, $c = 0$, $e = 0$, and if we superadd the equations $m^2 + 1 = 0$, $n^2 + 1 = 0$, then $d = 1$ and $f = 1$, or in other words in order to satisfy the equations $mn = -nm$, $m^2 = -1$, $n^2 = -1$, where it will of course be understood that in these (as in the equations $m^2 + 1 = 0$, $n^2 + 1 = 0$) 1 is the abbreviated form of the matrix $\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}$ and $\bar{1}$ of* the form $\begin{matrix} \bar{1} & 0 \\ 0 & \bar{1} \end{matrix}$, the necessary and sufficient condition is that the determinant of $x + my + nz$ shall be equal to $x^2 + y^2 + z^2$.

The simplest mode of satisfying this condition is to write $m = \begin{matrix} i & 0 \\ 0 & -i \end{matrix}$, $n = \begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix}$, i meaning $\sqrt{(-1)}$, which gives $mn = \begin{matrix} 0 & -i \\ -i & 0 \end{matrix}$ and $nm = \begin{matrix} 0 & i \\ i & 0 \end{matrix}$.

It is easy to express any matrix of the second order as a linear function of 1 $\left(\text{meaning } \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}\right)$ m, n, p, where p stands for mn.

For if $\begin{matrix} a, & b \\ c, & d \end{matrix}$ be any such matrix it is only necessary to write

$$a = f + ig, \quad b = -h - ki,$$
$$d = f - ig, \quad c = -h + ki;$$

and then $\begin{matrix} a, & b \\ c, & d \end{matrix} = f + gm + hn + kp$.

The most general solution of the equations $MN = -NM$, $M^2 = N^2 = -1$, must contain three arbitrary constants, namely, the difference between the number of terms in m and n, and the number of conditions $b = 0$, $c = 0$, $e = 0$, $d = 1$, $f = 1$, which are to be satisfied.

[* $\bar{1}$ denotes -1.]

Suppose M, N to be the most general solution fulfilling these conditions; we may write

$$M = f + gm + hn + kp,$$

$$N = f' + g'm + h'n + k'p,$$

where m, n is any particular solution and $p = mn$, and we shall have inasmuch as $M^2 = \bar{1}$,

$$f^2 - g^2 - h^2 - k^2 + 2fgm + 2fhn + 2fkp = \text{the matrix } \bar{1},$$

and consequently $$g^2 + h^2 + k^2 = 1 + f^2,$$

$$fg = 0, \quad fh = 0, \quad fk = 0.$$

Hence $f = 0$ and $$g^2 + h^2 + k^2 = 1.$$

Similarly $f' = 0$ and $$g'^2 + h'^2 + k'^2 = 1,$$

and also inasmuch as $MN = -NM$,

$$gg' + hh' + kk' = 0,$$

and since the equations $M^2 = \bar{1}$, $N^2 = \bar{1}$, $MN = -NM$ imply if we make $MN = P$ that $P^2 = -1$, and $MP = -PM$, and $NP = -PN$, it follows that M, N, P, are connected with m, n, p, in the same way as the coordinates of a point referred to one set of rectangular coordinates in space are connected with the coordinates of the same point referred to any other set of the same*.

Herein lies the ground of the geometrical interpretation to which quaternions lend themselves and it is hardly necessary to do more than advert to the fact that the theory of Quaternions is one and the same thing as that of Matrices of the second order viewed under a particular aspect †.

(2) Let m, n now denote matrices of the third order.

We might propose to solve the equation $mn = -nm$.

The result of the investigation is that we must have $m^2 = n^2$, $m^3 = 0$, $n^3 = 0$, and writing $mn = p$, $m^2 = n^2 = q$, there results a set of *quinions*, 1, m, n, p, q, for which the multiplication is that marked (a_5) p. 144 of the late Prof. Peirce's invaluable memoir in Vol. IV. of the *American Journal of Mathematics*.

* There is another solution possible, obtained by writing

$$-\frac{f}{f'} = \frac{g}{g'} = \frac{h}{h'} = \frac{k}{k'}, \quad f^2 + g^2 + h^2 + k^2 = 0$$

but this leads to a linear relation between m and n, so that $mn = nm$ and consequently $mn = nm = 0$ which is not the kind of solution proposed in the question.

† See my article in the *Lond. and Edin. Phil. Mag.* on "Involution and Evolution of Quaternions," November, 1883. [Above, p. 112.]

But instead of this let us propose the equation $mn = \rho nm$, where ρ is one of the imaginary roots of unity; if now we write the determinant of $x + my + nz$ under the form

$$x^3 + 3bx^2y + 3cx^2z + 3dxz^2 + 6exyz + 3fyz^2 + gy^3 + 3hy^2z + 3kyz^2 + lz^3,$$

it may be shown [cf. p. 126, below] that we must have

$$b = 0, \quad c = 0, \quad d = 0, \quad e = 0, \quad f = 0, \quad h = 0, \quad k = 0,$$

and if we superadd the conditions $m^3 = 1$, $n^3 = 1$, we must also have $g = 1$, $l = 1$, or in other words the determinant to $x + my + nz$ must take the form $x^3 + y^3 + z^3$; but this condition (or system of conditions) although necessary *is not sufficient* (a point which I omitted to notice in my article entitled "A Word on Nonions" inserted* in a previous *Circular*).

It is obviously necessary that we must have $(mn)^3 = 1$.

Now if the *identical equation* to mn be written under the form

$$(mn)^3 - 3B(mn)^2 + 3Dmn - E = 0,$$

B may be shown to be a linear homogeneous function of b, c, and e; also $E = gl = 1$; but D is not a function of $b, c, d, e, f, g, h, k, l$, and will not in general vanish (as it is here required to do) when b, c, d, e, f, h, k vanish. Its value is the sum of the products obtained on multiplying each quadratic minor of m by its *altruistic* opposite in n: (the *proper* opposite to a minor of m means the minor which is the reflected image of such minor viewed in the Principal Diagonal of m regarded as a mirror; and the *altruistic* opposite is the minor which occupies in n a position precisely similar to that of the proper opposite in m). There are, therefore, 10 equations in all to be satisfied between the coefficients of m and n when $m^3 = n^3 = 1$ and $nm = \rho mn$.

These *ten* conditions I have demonstrated are *sufficient* as well as necessary. There remains then $18 - 10$ or 8 arbitrary constants in the general solution. If m, n is a particular solution we may take for M, N (the matrices of the general solution),

$$M = \alpha + \beta m + \gamma m^2 + \alpha' n + \beta' mn + \gamma' m^2 n + \alpha'' n^2 + \beta'' mn^2 + \gamma'' m^2 n^2,$$

$$N = \alpha_1 + \beta_1 m + \gamma_1 m^2 + \alpha_1' n + \beta_1' mn + \gamma_1' m^2 n + \alpha_1'' n^2 + \beta_1'' mn^2 + \gamma_1'' m^2 n^2,$$

and 10 relations between the 18 coefficients *must* be sufficient to enable to be satisfied the equations $M^3 = N^3 = 1$, $NM = \rho MN$: but what these relations are and how they may most simply be expressed I am not at present in a condition to state†.

[* Vol. III. of this Reprint, p. 647.]

† The solution of this problem would seem to involve some unknown expansion of the idea of orthogonalism. Unless $MN = NM = 0$, a solution to be neglected, it may be proved that $\alpha = 0$, $\alpha_1 = 0$.

I showed in "A Word on Nonions" that the 9 first conditions are satisfied by taking

$$m = \begin{matrix} 1 & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & \rho^2, \end{matrix} \qquad n = \begin{matrix} 0 & 0 & 1 \\ \rho & 0 & 0 \\ 0 & \rho^2 & 0. \end{matrix}$$

The 10th condition is also satisfied; for the only quadratic minors (not having a zero determinant) in m are $\begin{matrix} 1 & 0 \\ 0 & \rho \end{matrix}$, $\begin{matrix} \rho & 0 \\ 0 & \rho^2 \end{matrix}$, $\begin{matrix} 1 & 0 \\ 0 & \rho^2 \end{matrix}$; the *altruistic opposites* to which in n are $\begin{matrix} 0 & \rho \\ 0 & 0 \end{matrix}$, $\begin{matrix} 0 & \rho^2 \\ 0 & 0 \end{matrix}$, $\begin{matrix} 0 & 0 \\ 1 & 0 \end{matrix}$, the determinants to each of which are zeros, and accordingly we find

$$m^3 = n^3 = \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1, \end{matrix}$$

$$nm = \begin{matrix} 0 & 0 & \rho^2 \\ \rho & 0 & 0 \\ 0 & 1 & 0, \end{matrix} \qquad mn = \begin{matrix} 0 & 0 & 1 \\ \rho^2 & 0 & 0 \\ 0 & \rho & 0, \end{matrix}$$

so that $mn = \rho nm$ and $m^3 = n^3 = 1$ as required.

I subjoin an outline proof of the fundamental portion of the theory of Quaternions and Nonions above stated as it will serve to throw much light upon the nature of the processes employed in that new world of thought to which I gave the name of Universal Algebra or the Algebra of multiple quantity: a fuller explanation will be found in the long memoir which I am preparing on the entire subject for the *American Journal of Mathematics.*

(1) As regards the equation $nm = -mn$, where m, n are matrices of the second order.

As before let the determinant of $(x + ym + zn)$ be

$$x^2 + 2bxy + 2cxz + dy^2 + 2eyz + fz^2.$$

I may observe here parenthetically that the Invariant of the above Quantic is equal to the determinant of $mn - nm$, and that when it vanishes $1, m, n, mn$, as also $1, n, m, nm$ are linearly related—or, as I express it, this Invariant is the Involutant of the system m, n or n, m. When m, n are of higher than the second order, the Involutant of m, n, say I, is that function whose vanishing implies that the 9 matrices $(1, m, m^2 \between 1, n, n^2)$ are linearly related, and the Involutant of n, m, say J, that function whose vanishing implies that the 9 quantities $(1, n, n^2 \between 1, m, m^2)$ are so related (I, J being two distinct functions), and so for matrices of any order higher than the second.

By virtue of a general theorem for any two matrices m, n of the second order, the following identities are satisfied:

$$m^2 - 2bm + d = 0,$$
$$mn + nm - 2bn - 2cm + 2e = 0,$$
$$n^2 - 2cn + f = 0.$$

If then $mn + nm = 0$, since m and n cannot be functions of one another (for then $mn = nm$), the second equation shows that $b = 0$, $c = 0$, $e = 0$, and conversely if $b = 0$, $c = 0$, $e = 0$, $mn + nm = 0$, and $m^2 + d = 0$, $n^2 + f = 0$, where, if we please, we may make $d = 1$, $f = 1$.

(2) Let m, n be matrices of the third order, and write as before,

$$\text{Det. } (x + ym + zn) = x^3 + 3bx^2y + 3cx^2z + 3dxy^2 + 6exyz + 3fxz^2 + gy^3 + 3hy^2z + 3kyz^2 + lz^3.$$

Then by virtue of the general theorem last referred to* there exist the identical equations

$$m^3 - 3bm^2 + 3dm - g = 0,$$
$$m^2n + mnm + nm^2 - 3b(mn + nm) - 3cm^2 + 3dn + 6em - 3h = 0,$$
$$mn^2 + nmn + n^2m - 3c(mn + nm) - 3bn^2 + 3fm + 6en - 3k = 0,$$
$$n^3 - 3cn^2 + 3fn - l = 0.$$

Let now $nm = \rho mn$, where ρ is either imaginary cube root of unity, then

$$(1)\ m^2n + mnm + nm^2 = 0 \text{ and } (2)\ mn^2 + nmn + n^2m = 0;$$

for greater simplicity suppose also that $m^3 = n^3 = 1$, where 1 means the matrix

$$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0. \\ 0 & 0 & 1 \end{matrix}$$

From the 1st and 2nd of the four identical equations combined it may be proved that $b = 0$, $d = 0$; I do not produce the proof here because to make it *rigorous*, the theory of Nullity would have to be gone into which would occupy too much space; and in like manner from the 3rd and 4th it may be shown that $c = 0$, $f = 0$†. Hence returning to the two middle equations it follows that $e = 0$, $h = 0$, $k = 0$, and from the two extremes that $g = 1$, $l = 1$.

If then $nm = \rho mn$, $m^3 = 1$, and $n^3 = 1$, it is necessary that

$$b = 0, \quad c = 0, \quad d = 0, \quad e = 0, \quad f = 0, \quad g = 1, \quad h = 0, \quad k = 0, \quad l = 1.$$

But these equations although necessary are manifestly insufficient; for they lead to the equations $m^3 - 1 = 0$, $n^3 - 1 = 0$, and

$$(1)\ m^2n + mnm + nm^2 = 0; \ (2)\ mn^2 + nmn + n^2m = 0,$$

[* By Cayley's theorem, if in Det. $(x + ym + zn)$ we replace x by $-ym - zn$, the result vanishes identically in regard to y and z.]

† Except when m, n are functions of one another, so that mn and nm are identical and consequently are each of them zero.

but not necessarily to $nm = \rho mn$. In fact the supposed equations between m and n involve as a consequence the equation $(mn)^3 = 1$. Now the general identical equation to (mn) is

$$(mn)^3 - 3B(mn)^2 + 3D(mn) - F = 0,$$

where B is the sum of each term in m by its altruistic opposite in n $= 3bc - 2e = 0$, $F = gl = 1$, and D is the sum of each first minor in m by its altruistic opposite in n which sum does not necessarily vanish when b, c, d, e, f, h, k, all vanish. Hence there is a 10th condition necessary not involved in the other 9, namely, $D = 0$. These 10 conditions I shall show are sufficient as well as necessary. For when they are satisfied since $(mn)^3 = 1$, $mn \,.\, mn = n^2m^2$.

Hence from (1) $\quad m^2n^2 + n^2m^2 + nm^2n = 0,$

and from (2) $\quad m^2n^2 + n^2m^2 + mn^2m = 0.$

Hence $nm \,.\, mn = mn \,.\, nm$ *, and consequently nm is a function of mn [cf. p. 149, below]. Hence we may write

$$nm = A + Bmn + C(mn)^2.$$

But the *latent roots* of mn and nm (which are always identical) are $1, \rho, \rho^2$, hence

$$A + B + C, \quad A + B\rho + C\rho^2, \quad A + B\rho^2 + C\rho,$$

must be equal to $1, \rho, \rho^2$, each to each taken in some one of the 6 orders in which these quantities can be written †.

Solving these 6 systems of linear equations there results:

$$A = 0, \qquad B = 0, \qquad C = 1, \ \rho \text{ or } \rho^2$$

or $$A = 0, \qquad B = 1, \ \rho \text{ or } \rho^2, \qquad C = 0.$$

Hence $nm = \theta mn$, or $\theta(mn)^2$ where $\theta = 1, \rho, \rho^2$.

If $\quad nm = \theta(mn)^2, \quad nmmn = \theta(mn)^3 = \theta.$

Hence $\quad m^2 = \theta n^2 \,.\, \theta n^2 = \theta^2 n;$

and $\quad m^2n + mnm + nm^2 = 3\theta m^4 = 3\theta m = 0,$

so that $m = 0$, and $m^3 = 0 = 1$; and again if $nm = mn$,

$$m^2n + mnm + nm^2 = 2m^2n + mnm = 3m^2n = 0,$$

* This equation is independent of the equation $(mn)^3 = 1$; for

$$nm^2n - mn^2m = (m^2n + mnm + nm^2)\,n - m\,(mn^2 + nmn + n^2m) = 0$$

by virtue of equations (1) and (2) above: accordingly these equations taken alone imply the equations

$$nm = A + B_1 mn + C(mn)^2, \quad mn = -A + B_2 nm - C(nm)^2$$

where B_1, B_2 are the roots of $B^2 + B + 1 - \frac{AC}{2} = 0$; A, C being arbitrary and independent except that each vanishes when and only when the cube of mn and (as a consequence) of nm, is a scalar matrix. [See below, p. 154. Footnote [†].]

† By virtue of the general theorem that the latent roots of any function of a matrix are the like functions of the latent roots of the original matrix.

so that $m^3n = 0$, $n = 0$, and $n^3 = 0 = 1$ as before, where it should be noticed that $0 = 1$ means that $\begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix}$ is identical with $\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$.

Hence the only available hypothesis remaining is the equation $nm = v \,.\, mn$, where v is one of the imaginary cube-roots of unity as was to be proved.

(3) It remains to say a few words on the general equation $nm = kmn$, where m, n are matrices of any order ω. To avoid prolixity I shall confine my remarks to the general case, which is, that where the determinants (or as I am used to say the contents) of m and n are each of them finite; with this restriction, the proposed equation is impossible for general values of k as will be at once obvious from the fact that the totalities of the latent roots of mn and of nm are always identical, but the individual latent roots are by virtue of the proposed equation in the ratio to one another of $1 : k$, which, since by hypothesis no root is zero, is only possible when $k^\omega = 1$.

When the above equation is satisfied the ω^2 equations arising from the identification of nm with kmn cease to be incompatible and (as is necessary or at all events usual in such a contingency) become mutually involved. Thus, for example, when $\omega = 1$ and $k = 1$, the number of independent equations is 0, that is, $1 - 1$, when $\omega = 2$ and $k = -1$ the number is 3, that is, $4 - 1$, when $\omega = 3$ and $k = \rho$ or ρ^2 the number is 8, that is, $9 - 1$; it is fair therefore to presume (although the assertion requires proof) that for any value of ω when k is a primitive ωth root of unity the number of conditions to be satisfied when $nm = kmn$ is $\omega^2 - 1$. Of these the condition that the content of $x + my + nz$ shall be of the form $x^\omega + cy^\omega + c'z^\omega$ will supply

$$\frac{(\omega + 1)(\omega + 2)}{2} - 3, \text{ that is, } \frac{\omega^2 + 3\omega}{2} - 2,$$

and there will therefore be

$$\frac{\omega^2 - 3\omega}{2} + 1 \text{ or } \frac{(\omega - 1)(\omega - 2)}{2}$$

to be supplied from some other source.

When k is a non-primitive ωth root of unity, the number of equations of condition is no longer the same. Thus when $k = 1$ we know that n may be of the form

$$A + Bm + Cm^2 + \ldots + Lm^{\omega - 1},$$

where $A, B, \ldots L$, and all the ω^2 terms in m are arbitrary, and consequently the number of conditions for that case is $2\omega^2 - (\omega^2 + \omega)$ or $\omega^2 - \omega$. It seems then very probable that if k is a qth power of a primitive ωth root of unity the number of conditions required to satisfy $nm = kmn$ is $\omega^2 - \delta$ where δ is the greatest common measure of q and ω: but, of course, this assertion awaits confirmation.

When $\omega = 4$ besides the case of $nm = mn$, that is, of n being a function of m of which the solution is known, there will be two other cases to be considered, namely, $nm = -mn$ and $nm = imn$: the former probably requiring 14 and the latter 15 conditions to be satisfied between the coefficients of m, the coefficients of n and the two sets of coefficients combined.

It is worthy of notice that the conditions resulting from the content of $x + my + nz$ becoming a sum of 3 powers are incompatible with the equation $nm = vmn$ when v is other than a primitive ωth root of unity (ω being of course the order of m or n).

Thus suppose $\omega = 4$; the conditions in question applied to the middle one of the 5 identical equations give

$$m^2n^2 + n^2m^2 + mn^2m + nm^2n + mnmn + nmnm = 0;$$

when $nm = imn$ the left-hand side of this equation becomes

$$(1 + i^4 + i^2 + i^2 + i + i^3)\, m^2n^2,$$

that is, is zero, but when $nm = -mn$, the value is

$$(1 + 1 - 1 - 1 - 1 - 1)\, m^2n^2$$

which is not zero, and so in general. Thus the pure power form of the content of $x + my + nz$ is a condition applicable to the case of $\frac{nm}{mn}$ being a primitive root of unity and to no other.

The case of nm being a primitive root of ordinary unity is therefore the one which it is most interesting to thrash out.

There are in this case, we have seen, $\frac{1}{2}(\omega^2 + 3\omega - 4)$ simple conditions expressible by the vanishing of that number of coefficients in the content of $x + my + nz$ and $\frac{1}{2}(\omega - 1)(\omega - 2)$ supplemental ones. What are these last? I think their constitution may be guessed at with a high degree of probability. For revert to the case of $\omega = 3$ in which there is one such found by equating to zero the second coefficient in the identical equation

$$(mn)^3 - 3B(mn)^2 + 3Dmn - G = 0.$$

Suppose now $\quad (m^2n^2)^3 - 3B'(m^2n^2)^2 + 3D'm^2n^2 - G' = 0$

is the identical equation to m^2n^2. By virtue of the 8 conditions supposed to be satisfied we *know* that $nm = \rho mn$ as well as $m^3 = 1$, $n^3 = 1$, and consequently that $(m^2n^2)^3 = 1$. Hence $B' = 0$, $D' = 0$, by virtue of the 7 parameters in the oft-quoted content and of D being all zero, and thus the evanescence of B' or D' imports no new condition.

Now suppose $\omega = 4$, and that

$$(mn)^4 - 4B(mn)^3 + 6D(mn)^2 - 4Gmn + M = 0,$$
$$(m^2n^2)^4 - 4B'(m^2n^2)^3 + 6D'(m^2n^2)^2 - 4G'm^2n^2 + M' = 0.$$

Here we know that B vanishes by virtue of b, c and e vanishing, but $D = 0$, $G = 0$, which must be satisfied if $nm = imn$, will be two new conditions not implied in those which precede. It seems then, although not certain, highly probable that $B' = 0$, $D' = 0$, will be implied in the satisfaction of the antecedent conditions but that $G' = 0$ will be an independent condition, so that $D = 0$, $G = 0$, $G' = 0$, will be the three supplemental conditions: and again when $\omega = 5$ forming the identical equations to mn, m^2n^2, m^3n^3, and using an analogous litteration to what precedes, the supplemental conditions will be

$$\begin{array}{lll} D=0, & G=0, & M=0, \\ & G'=0, & M'=0, \\ & & M''=0, \end{array}$$

and so in general for any value of ω.

The functions D, G, M, etc., above equated to zero are known from the following theorem of which the proof will be given in the forthcoming memoir*.

If $$(\overline{mn})^\omega + k_1(\overline{mn})^{\omega-1} + \ldots + k_i(\overline{mn})^{\omega-i} + \ldots = 0$$

is the *identical equation* to mn, then k_i is equal to the sum of the product of each minor of order i in m multiplied by its *altruistic* opposite in n.

The annexed example will serve to illustrate in the case of $\omega = 3$ that unless the supplemental condition is satisfied we cannot have $nm = \rho mn$.

Write $$m = \begin{matrix} 1 & 0 & 0, \\ 0 & \rho & 0, \\ 0 & 0 & \rho^2, \end{matrix} \qquad n = \begin{matrix} 0 & c & k, \\ k & 0 & c\rho, \\ c\rho^2 & k & 0, \end{matrix}$$

then the determinant to $x + my + nz$ will be easily found to be

$$x^3 + y^3 + (c^3 + k^3) z^3;$$

but D becomes $-3\rho ck$, and does not vanish unless $c = 0$ or $k = 0$, and accordingly we find

$$nm = \begin{matrix} 0 & \rho c & \rho^2 k, \\ k & 0 & c, \\ \rho^2 c & \rho k & 0, \end{matrix} \qquad mn = \begin{matrix} 0 & c & k, \\ \rho k & 0 & \rho^2 c, \\ \rho c & \rho^2 k & 0. \end{matrix}$$

When $k = 0$ $mn = \rho^2 nm$, when $c = 0$ $nm = \rho^2 mn$, but on no other supposition will $\frac{nm}{mn}$ be a primitive cube root of unity.

* This theorem furnishes as a Corollary the principle employed to prove the stability of the Solar System. (See *Lond. and Edin. Phil. Mag.*, October, 1883.) [Above, p. 110.]

ADDENDUM.

Referring to the equation $MN = -NM$, and to the eight equations expressing M and N in terms of the combinations of the powers of m with those of n, in which it is to be understood that M and N are *non-vacuous*, we know that the sums of the latent roots of M and of N must each vanish and consequently, as may be proved, that $a = 0$, $a' = 0$, leaving $8 - 2$ or 6 conditions to be satisfied. If we further stipulate that $M^3 = 1$, $N^3 = 1$, there will be 8 relations connecting the coefficients $b, c, \ldots k$ and $b', c', \ldots k'$, so that the 64 coefficients in the 8 equations connecting M, M^2; N, N^2; MN, M^2N^2; M^2N, MN^2, or say rather M, M^2; N, N^2; $\rho^2 MN, \rho^2 M^2N^2$; $\rho M^2 N, \rho MN^2$, with like combinations or multiples of combinations of powers of m, n * will be connected together by 56 equations; the coefficients in the expression for any one of the above 8 terms may then be arranged in pairs f_i, f_i'; g_i, g_i'; h_i, h_i'; k_i, k_i'; and in the expression for its fellow by F_i, F_i'; G_i, G_i'; H_i, H_i'; K_i, K_i'; so that the Matrix is resolved as it were into 4 sets of paired columns and 4 sets of paired lines; the 4 different sets of paired lines being found by writing successively $i = 1, 2, 3, 4$.

It is then easy to see that there will be 4 equations of the form

$$\Sigma (f_\alpha G_\alpha' + f_\alpha G_\alpha') = 1,$$

and 6 quaternary groups (that is, 24 equations) of the form

$$\Sigma (f_\alpha G_\beta' + f_\alpha G_\beta') = 0,$$

with liberty to change f into F or G into g or each into each: together then the above are 28 of the 56 conditions required. But inasmuch as the 8 $[m, n]$ arguments may be interchanged with the 8 $[M, N]$ ones, we may transform the above equations by substituting for each letter f its conjugate $\frac{d \log \Delta}{df}$ (where Δ is the content of the Matrix) and thus obtain 28 others, giving in all (if the two sets as presumably is the case are independent) the required 56 conditions: the latter 28, however, may be replaced by others of much simpler form †.

* It is easy to see that the sum of the latent roots of M^iN^j must be zero for all values of i, j so that it is a homogeneous linear function of the 8 quantities $m, m^2, \ldots, mn, m^2n^2$.

† I am still engaged in studying this matrix, which possesses remarkable properties. Is it orthogonal? I rather think not, but that it is allied to a system of 4 pairs of somethings drawn in four mutually perpendicular hyperplanes in space of 4 dimensions. In the general case of $MN = \rho NM$ where ρ is a primitive ωth root of unity, there will be an analogous matrix of the order $\omega^2 - 1$ where each line and each column will consist of $\omega + 1$ groups of $\omega - 1$ associated terms.

The value of the cube of any one of the 8 matrices M, M^2; ...; MN, M^2N^2 may be expressed as follows: It is P into ternary unity. Such a quantity may be termed by analogy a Scalar. To find $P_{i,j}$ I imagine the 8 letters corresponding to M^iN^j (but without powers of ρ attached) to be set over 8 of the 9 points of inflexion to any cubic curve, the paired letters being made suitably

To me it seems that this vast new science of multiple quantity soars as high above ordinary or quaternion Algebra as the Mécanique Céleste above the "Dynamics of a Particle" or a pair of particles, (if a new Tait and Steele should arise to write on the Dynamics of such pair,) and is as well entitled to the name of Universal Algebra as the Algebra of the past to the name of Universal Arithmetic.

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collinear with the missing 9th point. Then among themselves the 8 letters may be taken in 8 different ways to form collinear triads and the product of the letters in each triad may be called a collinear product; $P_{i,j}$ (which is identical with the Determinant to M^iN^j) will be the sum of the cubes of the 8 letters less 3 times the sum of their 8 collinear products, and its 8 values will be analogous to the 3 values of the sum of 3 squares in the Quaternion Theory. Each of these 8 values is assumed equal to unity.

It may be not amiss to add that the product of four squares by four is representable rationally as a sum of four squares, so if we place (not now 8 specially related but) nine perfectly arbitrary letters over the nine points of inflexion of a cubic curve the sum of their 9 cubes less three times their 12 collinear products multiplied by a similar function of 9 other letters may be expressed by a similar function of 9 quantities lineo-linear functions of the two preceding sets of 9 terms.

By the 8 letters of any set as, for example, $b, \ldots, h'$ being "specialized," I mean that they are subject to the condition $bb' + dd' + ff' + hh' = 0$. When this equation is satisfied, and not otherwise, M^3 will be a Scalar, and it *must* be satisfied when $MN = \rho NM$.

15.

ON INVOLUTANTS AND OTHER ALLIED SPECIES OF INVARIANTS TO MATRIX SYSTEMS.

[*Johns Hopkins University Circulars*, III. (1884), pp. 9—12, 34, 35.]

To make what follows intelligible I must premise the meaning and laws of vacuity and nullity.

A matrix is said to be vacuous when its content (the determinant of the matrix) is zero, but it may have various degrees of vacuity from 0 up to ω the order of the matrix.

If from each term in the principal diagonal of a matrix λ be subtracted, the content of the resulting matrix is a function of degree ω in λ; the ω values of λ which make this content vanish are called its latent roots, and if i of these roots are zero, the vacuity (treated as a number) is said to be i. This comes to the same thing as saying that the vacuity is i when the determinant, and the sums of the determinants of the principal minors of the orders $\omega-1$, $\omega-2$, ... $(\omega-i+1)$ are each zero. A principal minor of course means one which is divided into 2 [equal] triangles by the principal diagonal of the parent matrix.

Again the nullity is said to be i when *every* minor of the order $(\omega-i+1)$, and consequently of each superior order, is zero. It follows therefore that it means the same thing to predicate a vacuity 1 and a nullity 1 of any matrix, but for any value of i greater than 1, a nullity i implies a vacuity i but not *vice versâ*; the vacuity may be i, whilst the nullity may have any value from 1 up to i inclusive.

The law of nullity which I am about to enunciate is one of paramount importance in the theory of matrices*.

* The three cardinal laws or landmarks in the science of multiple quantity are (1) the law of *nullity*, (2) the law of *latency*, namely, that if $\lambda_1, \lambda_2, \ldots \lambda_\omega$ are the latent roots of m, then $f\lambda_1, f\lambda_2, \ldots f\lambda_\omega$ are those of fm, including as a consequence that

$$fm = \Sigma f\lambda_1 \frac{(m-\lambda_2)(m-\lambda_3)\ldots(m-\lambda_\omega)}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)\ldots(\lambda_1-\lambda_\omega)},$$

and (3) the law of *identity*, namely, that the powers and combinations of powers of two matrices m, n of the order ω are connected together by $(\omega+1)$ equations whose coefficients are all included among the coefficients of the determinant to the Matrix

$$x+ym+zn.$$

The law is that the nullity of the product of *two* (and therefore of any number of) matrices cannot be less than the nullity of any factor nor greater than the sum of the nullities of the several factors which make up the product.

Suppose now that $\lambda_1, \lambda_2, \ldots \lambda_\omega$ are the latent roots of any matrix with unequal latent roots of the order ω. It is obvious that any such term as $m - \lambda_1$ will have the nullity 1, for its latent roots will be $0, \lambda_2 - \lambda_1, \lambda_3 - \lambda_1, \ldots \lambda_\omega - \lambda_1$, and consequently its *vacuity* is 1.

Moreover we know from Cayley's famous identical equation that the nullity of the product of all the ω factors is ω.

Hence it follows that if M_i contains i, and M_j the remaining $\omega - i$ of these factors (so that $i + j = \omega$), the nullity of M_i must be exactly i and of M_j exactly j.

For the theorem above stated shows that M_i cannot have a nullity greater than i, nor M_j a nullity greater than j.

Hence if the nullity of the one were less than i or of the other less than j, the nullity of M_iM_j would be less than $i + j$, that is, less than ω, whereas its nullity is ω; hence the two nullities are respectively i and j as was to be shown.

Furthermore we know that the latent roots of $(m - \lambda_1)^a$ are $(\lambda_1 - \lambda_1)^a$; $(\lambda_2 - \lambda_1)^a$; ... $(\lambda_\omega - \lambda_1)^a$.

Hence if the latent roots of m are all distinct, the nullity of $(m - \lambda_1)^a$ is unity and consequently by the same reasoning as that above employed it follows that the nullity of

$$(m - \lambda_1)^{a_1} . (m - \lambda_2)^{a_2} \ldots (m - \lambda_i)^{a_i}$$

is exactly i.

I will now explain what is meant by the Involutant or Involutants of a system of two matrices of like order.

It will be convenient here to introduce the term "topical resultant" of a system of ω^2 matrices each of order ω.

We may denote any matrix say

$$\begin{matrix} a_{1,1} & a_{1,2} & \ldots & a_{1,\omega} \\ a_{2,1} & a_{2,2} & \ldots & a_{2,\omega} \\ \cdots & \cdots & \cdots & \cdots \\ a_{\omega,1} & a_{\omega,2} & \ldots & a_{\omega,\omega} \end{matrix}$$

by the linear form

$$\begin{aligned} & a_{1,1}\, t_{1,1} + a_{1,2}\, t_{1,2} + \ldots + a_{1,\omega}\, t_{1,\omega} \\ + & a_{2,1}\, t_{2,1} + a_{2,2}\, t_{2,2} + \ldots + a_{2,\omega}\, t_{2,\omega} \\ & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ + & a_{\omega,1}\, t_{\omega,1} + a_{\omega,2}\, t_{\omega,2} + \ldots + a_{\omega,\omega}\, t_{\omega,\omega} \end{aligned}$$

where the t system is the same for all matrices of the order ω. If, then, we have ω^2 such matrices, their topical resultant is the Resultant in the ordinary sense of the ω^2 linear forms above written, proper to each of them respectively.

Suppose now that m, n are two independent matrices of the order ω, we may form ω^2 matrices by taking each power of m from 0 to $\omega - 1$ as an antecedent factor, and can combine it with similar powers of n as a consequent factor, and in this way obtain ω^2 matrices, of which the first will be the ω-ary unity, that is, a matrix of the order ω in which the principal diagonal terms are all units and the other terms all zero. The topical resultant of these ω^2 matrices I shall for brevity denote as the Involutant to m, n.

In like manner, inverting the position of the powers of m and of n so as to make the latter precede instead of following the former in the ω^2 products above referred to, we shall obtain another topical resultant which may be termed the Involutant to n, m.

The reason why I speak of these topical resultants as involutants to m, n or n, m is the following:

In general if m, n are two independent matrices, any other matrix p, by means of solving ω^2 linear equations, may obviously be expressed as a linear function of the ω^2 products

$$(1, m, m^2, \dots, m^{\omega-1})(1, n, n^2, \dots, n^{\omega-1}).$$

There are, however, exceptions to this fact.

The most obvious exception is that which takes place when n is a function of m; for then any ω of the ω^2 products will be linearly related, and there will be substantially only ω disposable quantities to solve ω^2 equations.

Another exception is when the m, n Involutant, that is, the topical resultant of the ω^2 matrices, is zero; in which case the general values of the ω^2 disposable quantities each becomes infinite. So that m, n may be said to be in a kind of mutual involution with one another. So, again, p may in general be expressed as a linear function of the ω^2 matrices

$$(1, n, n^2, \dots, n^{\omega-1})(1, m, m^2, \dots, m^{\omega-1}),$$

but when the n, m Involutant vanishes this is no longer possible.

When $\omega = 2$ the two involutants, considered as definite determinants, are absolutely equal in magnitude and in Algebraical sign, but when ω exceeds 2 this is no longer the case; the two Involutants are then entirely distinct functions of the elements of m and n.

Thus to take a simple example: if $m = \begin{matrix} 1 & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & \rho^2 \end{matrix}$ and $n = \begin{matrix} 0 & \rho & k \\ k & 0 & \rho^2 \\ 1 & k & 0 \end{matrix}$ it will be found by direct calculation of two topical resultants of the 9th order, that the two involutants will be

$$81\,(\rho - \rho^2)\,(k^3 - \rho)^3 \text{ and } 81\,(\rho^2 - \rho)\,(k^3 - \rho^2)^3$$

respectively. The reason why the two involutants coincide in the case of $\omega = 2$ is not far to seek. It depends upon the fact of the existence of the mixed identical equation

$$mn + nm - 2bn - 2cm + 2e = 0\,;$$

from which it is obvious that the topical resultant of 1, m, n, mn is the negative of that of 1, m, n, nm or identical with that of 1, n, m, nm.

By direct calculation it will be found that the Involutant m, n, or n, m, where $m = \begin{matrix} f & g \\ h & k \end{matrix}$ $n = \begin{matrix} f' & g' \\ h' & k' \end{matrix}$ is

$$-(gh' - g'k)^2 + \{(f-k)\,g' - (f' - k')\,g\}\,\{(f-k)\,h' - (f' - k')\,h\},$$

which is the same thing as the content of the matrix $(mn - nm)$. It may also be shown *à priori* or by direct comparison to be identical (to a numerical factor *près*) with the Discriminant of the Determinant to the matrix $(x + ym + zn)$ which is a ternary quantic of the second order. Its actual value is 4 times that discriminant.

Let us consider the analogous case of Mechanical Involution of lines in a plane or in space. There are two questions to be solved. The one is to find the condition that the Involution may exist, that is, that a set of equilibrating forces admit of being found to act along the lines; the second, to determine the relative magnitudes of the forces when the involution exists, and this is the simpler question of the two.

In like manner we may consider two questions in the case of m, n being in either of the two kinds of involution; the one being to find what the condition is of such involution existing, the other what are the coefficients of the ω^2 coefficients in the equation which connects the ω^2 products, when the involution exists.

This latter part of the question (surprising as the assertion may appear and is) admits of a very simple and absolutely general direct and almost instantaneous solution by means of the Law of Nullity, above referred to, as I will proceed to show.

The determination of the Involutants, or at all events of their product, will then be seen to follow as an immediate consequence from this prior determination of the form of the equations which express the involutions of the two kinds respectively.

But first it may be well to explain why and in what sense I refer in the title to Involutants as belonging to a class of invariants. I say, then, that universally involutants are invariants in this sense, that if for m and for n, any function of m, or any function of n be substituted, the ratio of the two Involutants, say I and J, remains unaltered. By virtue of the Identical Equation $(m)^i$ will be of the form of

$$A_i + B_i + C_i m^2 + \ldots + L_i m^{\omega-1}$$

and as a consequence it is easy to see that when m^i is substituted for m, I and J will become respectively PI, PJ where P is the ωth power of the determinant to the matrix formed by writing under one another the $(\omega - 1)$ lines of terms, of which the line $B_i, C_i, \ldots; L_i$ is the general expression.

Moreover, in the particular case where $\omega = 2$ and $I = J$*, besides being an Invariant in this modified sense, I will be an invariant in a sense including but transcending the more ordinary conception of an Invariant; for if when, for m and n, $f(m, n)$ and $\phi(m, n)$ are substituted, I becomes I', then I' will contain I as a factor; this is a consequence of the fact that when m and n are in involution $f(m, n)$ and $\phi(m, n)$ will also be in involution, for in consequence of the identical equation

$$mn + nm - 2bn - 2cm + 2e = 0$$

f and ϕ and $f\phi$ will each be reducible to the form

$$A + Bm + Cn + Dmn$$

and it is obvious from the ordinary theory of the determinants that the topical resultant of 1, $\left(\text{meaning } \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}\right)$, and three linear functions of 1, m, n, nm, will contain as a factor the topical resultant of 1, m, n, mn.

Nor must it be supposed that Involutants are the only species of invariants in the modified sense first described which appertain to the

* I for some time had imagined, and indeed thought I had proved, that the two involutants were always identical. When crossing the Atlantic last month on board the "Arizona," having hit upon a pair of matrices of the third order, for which the two topical resultants admitted of easy calculation, I found, to my surprise, that they were perfectly distinct. The cause of the failure of the supposed proof constitutes a paradox which will form the subject of a communication to a future meeting of the Johns Hopkins Mathematical Society.

I will here only premise that the seeming contradiction between the logical conclusion and the facts of the case takes its rise in a sort of mirage with which invariantists are familiar, namely: the apparent *à priori* establishment of algebraical forms as the result of perfectly valid processes, which forms have no more real existence in nature than the Corona of the Sun under our Dr Hastings' scrutinizing gaze: the contradiction between the logical inference and the truth being accounted for by the circumstance that any such supposed form on actual performance of the operations indicated, turns out to be a congeries of terms, each affected with a null coefficient; we are thus taught the lesson that all *à priori* reasoning until submitted to the test of experience, is liable to be fallacious, and it is impossible to prove that a proof may not be erroneous by any other method than that of actual trial of the results which it is supposed to yield.

system m and n. Thus, for example, when $\omega=2$ it is not only true that the determinant of the matrix $mn-nm$ is such a kind of Invariant (which for greater clearness it may be desirable to denote by the term Perpetuitant*), but each element of that matrix will also be a perpetuitant, and these 4 perpetuitants, when for m, n $\rho m, \phi n$ are substituted, will be in an invariable ratio to one another and to either square root of the Involutant.

In like manner it will eventually be seen that for two matrices m, n of any order ω, it is possible to form a matrix of the order ω analogous to $mn-nm$ $\left(\text{which be it observed may be regarded as the Determinant of the matrix } \begin{matrix} m & n \\ m & n \end{matrix}\right)$ each of whose ω^2 terms will be in a constant ratio to each other and to any ωth root of I and of J.

I will now return to the problem of finding what is the form of the equation which connects the ω^2 matrices denoted by

$$(1, m, m^2, \ldots m^{\omega-1})(1, n, n^2, \ldots n^{\omega-1})$$

when such an equation admits of being formed, that is, $I=0$.

To fix the ideas let us suppose that m, n are matrices of the 3rd order of perfectly general form so that the m, n involution necessitates the satisfaction of one single condition, $I=0$.

Let $A+Bn+Cn^2=0$ be the equation whose form is to be determined where A, B, C, are each of them quadratic functions of m. I say that neither A, B, nor C, can contain a non-vacuous linear factor. For suppose that any one of them as A should contain the non-vacuous factor $m+q$, and that

$$A=(m+q)(am+p).$$

Then we may multiply the equation by $(m+q)^{-1}$ and thus obtain the equation

$$(am+p)+B'n+C'n^2=0,$$

that is, we have an equation in which not all 9 but only 8 of the terms signified by $(1, m, m^2)(1, n, n^2)=0$ are linearly related. But this obviously implies, contrary to the hypothesis, the existence of two equations of condition instead of one.

Hence then A must be of the form $c(m-\lambda)(m-\lambda')$ where λ, λ' are each of them a latent root of m; whether the same or different remains to be determined.

In like manner it may be shown that B is of the form $c_1(m-\lambda_1)(m-\lambda_1')$ and C of the form $c_2(m-\lambda_2)(m-\lambda_2')$. But now I say further that

$$(m-\lambda)(m-\lambda'), \quad (m-\lambda_1)(m-\lambda_1'), \quad (m-\lambda_2)(m-\lambda_2')$$

must be identical.

* *Perpetuitant* formed from *perpetuity* by analogy to Annuitant from Annuity. Perpetuant would have been better, but that it has already been applied by myself in the theory of Invariants in a sense recognized and adopted by Cayley, Hammond, and MacMahon.

For, firstly, suppose that any one pair of the λ's, say λ, λ', are distinct. If any other pair, say λ_2, λ_2', is not identical with this pair, on multiplying the equation by $m-\lambda''$, where λ'' is the 3rd latent root of M, the term containing the term $A(\lambda \ldots \lambda'')$ will vanish, but $B(\lambda \ldots \lambda'')$ will not vanish and consequently there will be an equation, if $C(\lambda \ldots \lambda'')$ does not vanish, between 6 only, and if $C(\lambda \ldots \lambda'')$ does vanish, between 3 only of the 9 terms denoted by $(1, m, m^2)(1, n, n^2)$, contrary to hypothesis.

The only remaining supposition is that A, B, C are each perfect squares. Suppose, then, that any one of them as A is a multiple of $(m-\lambda)^2$; unless B, C are each of them also multiples of the same, on multiplying the equation by $(m-\lambda')(m-\lambda'')$, one of the three coefficients of $1, n, n^2$ will vanish but one at least of the other two will not vanish, which is impossible for the same reason as before. Hence the left-hand side of the equation of involution must contain $(m-\lambda)(m-\lambda')$ as a sinister factor where λ, λ' (whether the same or different) are latent roots of λ. And in like manner precisely, by arranging the equation of involution under the form $A'+mB'+m^2C'$ where A', B', C' are quadratic functions of n, it may be found that the same function must contain $(n-\mu)(n-\mu')$ where μ, μ' are latent roots of n as a dexter factor.

Hence the form of the equation must be

$$(m-\lambda)(m-\lambda')(n-\mu)(n-\mu')=0.$$

It is easy to see that we cannot have λ and λ' the same latent root of m and at the same time μ, μ' the same latent root of n, for then the above product would have at most the nullity 2 whereas it is an absolute null, that is, has the nullity 3.

But I will now show that λ, λ' and μ, μ' must each consist of unlike roots. Let t be any term of the matrix

$$(m-\lambda)(m-\lambda')(n-\mu)(n-\mu'),$$

where t will be a known function of the elements of m, n, of λ, λ' entering symmetrically, and of μ, μ' also entering symmetrically: this is the same thing as saying that t will be a function of the elements of m and n, of λ'', μ'', and of the coefficients of the equations which contain the 3 latent roots of λ and μ respectively.

Consequently the product of the 9 values of t found by writing $\lambda'', \lambda', \lambda$ for λ'', and μ'', μ', μ for μ'', will be a rational integer function of the elements of m, n which vanishes when the Involutant I vanishes and must consequently contain I as a factor. If then, *in any single instance*, the matrix

$$(m-\lambda)^2(n-\mu')(n-\mu'')$$

does not vanish for some one value of λ and μ when I vanishes, it cannot be the form, or *one of two* conceivably possible coexisting forms, of the left-hand side of the general equation of involution. A similar remark of course applies to

$$(m-\lambda_1)(m-\lambda_2)(n-\mu_1)^2.$$

Let now
$$m = \begin{matrix} 1 & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & \rho^2 \end{matrix}, \quad n = \begin{matrix} 0 & \rho & k \\ k & 0 & \rho^2 \\ 1 & k & 0 \end{matrix}.$$

The latent roots of m are $1, \rho, \rho^2$, and of n are $\theta, \rho\theta, \rho^2\theta$, where $\theta = \sqrt[3]{(1+k^3)}$; we have also

$$m^2 = \begin{matrix} 1 & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & \rho \end{matrix}, \quad n^2 = \begin{matrix} -\rho^2 k & k^2 & 1 \\ \rho^2 & -k & k^2 \\ k^2 & \rho & -\rho k \end{matrix}.$$

The three values of $(m-\lambda')(m-\lambda'')$ are

$$\begin{matrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix}, \quad \begin{matrix} 0 & 0 & 0 \\ 0 & 3\rho^2 & 0 \\ 0 & 0 & 3\rho \end{matrix}, \quad \begin{matrix} 0 & 0 & 0 \\ 0 & 3\rho & 0 \\ 0 & 0 & 3\rho^2 \end{matrix},$$

and the three values of $(n-\mu_1)(n-\mu_2)$ are

$$\begin{vmatrix} -\rho^2 k + \theta^2 & k^2 + \rho\theta & 1 + \theta k \\ \rho^2 + \theta k & -k + \theta^2 & k^2 + \rho^2\theta \\ k^2 + \theta & \rho + \theta k & -\rho k + \theta^2 \end{vmatrix} \quad \begin{vmatrix} -\rho^2 k + \rho^2\theta^2 & k^2 + \rho^2\theta & 1 + \rho\theta k \\ \rho^2 + \rho\theta k & -k + \rho^2\theta^2 & k^2 + \theta \\ k^2 + \rho\theta & \rho + \rho\theta k & -\rho k + \rho^2\theta^2 \end{vmatrix}$$

$$\begin{vmatrix} -\rho^2 k + \rho^2\theta & k^2 + \theta & 1 + \rho^2\theta k \\ \rho^2 + \rho^2\theta k & -k + \rho\theta^2 & k^2 + \rho\theta \\ k^2 + \rho^2\theta & \rho + \rho^2\theta k & -\rho k + \rho\theta^2 \end{vmatrix}.$$

The general value of

$$(m-\lambda_1)(m-\lambda_2)(n-\mu_1)(n-\mu_2)$$

will (to a numerical factor *près*) be a matrix consisting of a single column accompanied by two columns of zeros, the non-zero column being some one of the 9 columns found in the above 3 matrices.

Now by direct calculation we know that the n, m Involutant in this case is a numerical multiple of $(k^3-\rho^2)^3$ and vanishes when $k^3 = \rho^2$, which gives $\theta = \sqrt[3]{(1+\rho^2)}$, that is, $-\rho = \theta^3$, and if we please $k = \theta^2$.

Hence not merely one but three of the products of

$$(m-\lambda')(m-\lambda'')(n-\mu')(n-\mu'')$$

will in this case vanish, for the above equations will cause the 2nd, 4th and 9th columns all to become columns of nulls.

If now instead of the factor $(m-\lambda')(m-\lambda'')$ we substitute the factor $(m-\lambda)^2$, the three values of $(m-\lambda)^2$ will become

$$\begin{matrix} 0 & 0 & 0 \\ 0 & -3\rho & 0 \\ 0 & 0 & -3\rho^2 \end{matrix} \quad \begin{matrix} -3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -3\rho^2 \end{matrix} \quad \begin{matrix} -3 & 0 & 0 \\ 0 & -3\rho & 0 \\ 0 & 0 & 0 \end{matrix}$$

so that if $$(m-\lambda)^2(n-\mu')(n-\mu'')$$
is to vanish, it will readily be seen that each of two columns of one or the other of the two matrices representing $(n-\mu')(n-\mu'')$ will have to vanish simultaneously, and that this cannot be brought to pass when $\theta^3=-\rho$ and $k^3=\rho^2=\theta^6$ whether we make $k=\theta^2$ or $-\theta^5$ or θ^8.

Hence $$(m-\lambda)^2(n-\mu')(n-\mu'')=0$$
is not an admissible general involution form of equation. Similarly by interchanging the above special values assigned to m and n, it may be shown that
$$(m-\lambda')(m-\lambda'')(n-\mu)^2=0$$
is not an admissible form, and consequently that the one universal form of the involution equation is expressed by saying that
$$(m-\lambda')(m-\lambda'')(n-\mu')(n-\mu'')$$
is an absolute null. If no connexion exists between the elements of m and n, we know from the law of nullity that the above matrix has a nullity 2, that is, that all its minors except the elements themselves have zero contents. The effect of the vanishing of I is to make the elements themselves one and all vanish when the two sets of latent roots are duly selected.

So in general if
$$F=\lambda^\omega-A_1\lambda^{\omega-1}+A_2\lambda^{\omega-2}-A_3\lambda^{\omega-3}\ldots=0,$$
and $$G=\mu^\omega-B_1\mu^{\omega-1}+B_2\mu^{\omega-2}-B_3\mu^{\omega-3}\ldots=0,$$
are the two equations to the latent roots of m, n matrices of order ω, and if
$$M=m^{\omega-1}-(A_1-\lambda)m^{\omega-2}+(A_2-A_1\lambda+\lambda^2)m^{\omega-3}\ldots$$
and $$N=n^{\omega-1}-(B_1-\mu)n^{\omega-2}+(B_2-B_1\lambda+\lambda^2)n^{\omega-3}\ldots,$$
$MN=0$ for some value of λ and of μ is the one equation of involution, and $NM=0$ for some value of λ and some value of μ is the other such equation.

I will now show how to deduce from the above statement the following marvellous theorem.

Let H represent the sum of the product of each term in the matrix M by its *altruistic opposite* in N (so that H is a function of λ and μ and of degree $\omega-1$ in each of them) then will the ordinary Algebraical Resultant of F, G, H* be exactly equal (in magnitude as well as form) to the product of the two involutants to the *corpus* m, n†.

* The system of equations whose resultant expresses the undifferentiated condition of involution, may be written under the form $(x,y)^\omega=0$; $(z,t)^\omega=0$; $(x,y)^{\omega-1}=0$. *Quære* whether such a resultant may not be written under the form of a determinant by an application of the Dialytic Method?

† If I and J be the two involutants, $I=0$ will be the condition of left-handed involution of m, n or right-handed of n, m, and $J=0$ of right-handed involution of m, n or left-handed of n, m, for Involution, like light, "has sides." But $IJ=0$ will be the condition of *one* or *the other* kind, or so to say of undifferentiated Involution.

By the theorem proved at the beginning of this note, the nullity of M and that of N are each $\omega - 1$, hence the nullity of MN and consequently *à fortiori* its vacuity cannot be less than $\omega - 1$, and accordingly the identical equation to MN may be written under the form

$$(MN)^{\omega} - H(MN)^{\omega-1} = 0,$$

where H is the sum of the product of each element in the Matrix M or the Matrix N multiplied by its altruistic opposite in the other. Suppose now that $I = 0$ then for some one system of λ, μ out of the ω^2 systems given by the equations $F = 0$, $G = 0$, H must vanish (for the nullity and *à fortiori* the vacuity of MN in that case becomes ω); hence the *double norm* of H, that is, the product of the ω^2 values of H, or, which comes to the same thing, the resultant of F, G, H, must vanish when I vanishes and must therefore contain I; in like manner because the nullity of NM and *à fortiori* its vacuity is ω when $J = 0$, it follows that the same resultant, say R, must contain also J; R will therefore contain IJ, from which it may readily be concluded that it can differ from IJ, if it differ at all, only by a numerical factor.

I need hardly pause to defend the assumption that I, J have no common factor, and that it is the first and not necessarily any higher power of R which contains IJ; the single instance, when

$$m = \begin{matrix} 1 & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & \rho^2 \end{matrix}, \qquad n = \begin{matrix} 0 & \rho & k \\ k & 0 & \rho^2 \\ 1 & k & 0 \end{matrix},$$

of I, J being respectively (to a numerical factor *près*) the cubes of $k^3 - \rho$ and $k^3 - \rho^2$ which have no common factor, settles the first part of this assumption at all events for the case of $\omega = 3$, and as regards the second, it is only necessary to show that neither I nor J is equal to, or contains a square or higher power of a function of the letters in m and n as may be done easily enough when $\omega = 3$ by another simple instance*. We may then at once proceed to compare the dimensions of R with those of I and J.

* Limiting ourselves to the case of matrices of the third order, if we take for m, n the matrices

$$\begin{matrix} 0 & b & 0 \\ d & 0 & f \\ 0 & h & 0 \end{matrix}, \qquad \begin{matrix} 0 & B & 0 \\ D & 0 & F \\ 0 & H & 0 \end{matrix},$$

it may be shown by direct computation that one of the Involutants becomes

$$(bH - hB)^2 (fD - dF)^2 (bd + fh)(BD - FH)(dB - fH) . \{(hF + bD)^2 - (bd + fh)(BD + FH)\},$$

and consequently if there were any square factor in either involutant such factor would contain the elements belonging to the two sets indecomposably blended, but on the other hand, if we take for m, n the matrices

$$\begin{matrix} 1 & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & \rho^2 \end{matrix}, \qquad \begin{matrix} 0 & f & F \\ G & 0 & g \\ h & H & 0 \end{matrix},$$

either involutant to m, n may easily be shown (also by direct computation) to be made up of three factors, each of which is an indecomposable cubic function of f, g, h, F, G, H. Hence it follows that neither involutant can in its general

R being the product of ω^2 values of $\lambda^{\omega-1}\mu^{\omega-1}+$ etc., where λ, μ are codimensional with the elements in m and n respectively, is obviously of the degree $\omega^2.(\omega-1)$ in regard to each set of elements, that is, of the degree $2\omega^2(\omega-1)$ in regard to the two sets taken together.

Consider now the degree of I; this is the topical resultant of ω^2 matrices of the form $m^i.n^j$, where

$$i=0,\ 1,\ 2,\ \ldots\ \omega-1,\quad j=0,\ 1,\ 2,\ \ldots\ \omega-1,$$

so that each term in I will consist of a combination of ω^2 elements selected respectively from these ω^2 matrices. If ω is even, there will be $\frac{\omega^2}{2}$ pairs of matrices, one of any such pair of the form m^in^j, the other of form $m^{\omega-1-i}.n^{\omega-1-j}$, and the combination of elements taken from any such pair will be of the collective degree $2(\omega-1)$ in the two sets of elements, so that the total degree of the Involutant will be $\frac{\omega^2}{2}.2(\omega-1)$ or $\omega^2(\omega-1)$. If again ω is odd, there will be $\frac{1}{2}(\omega^2+1)$ such pairs, and one factor (unpaired) belonging to the matrix $m^{\frac{\omega-1}{2}}.n^{\frac{\omega-1}{2}}$ of the collective degree $(\omega-1)$. Hence the degree of the involutant will be

$$(\omega^2-1)(\omega-1)+(\omega-1) \text{ or } \omega^2(\omega-1)$$

as before.

Hence the product of IJ is of the degree $2\omega^2(\omega-1)$, or the same as R, and consequently (at all events to a numerical factor *près*) R and IJ coincide, which is the essential thing to be proved.

N.B. As regards $\omega=3$, the above proof is exact; for higher values of ω to make it valid, it must be demonstrated as a Lemma that the two general twin involutants (even were they decomposable forms, which they undoubtedly are not) could not have any common factor, nor either of them contain any square factor. The Resultant of F, G, H may be compared to a cradle just large enough to contain the twin forms in question, so as to give assurance that no other form is mixed up with them; and the proof given above shows that this must be the case if neither twin is doubled

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form contain any square factor. As a matter of fact, not only for ternary matrices but for matrices of any order, there can be no reasonable doubt whatever in any sane mind that every Involutant is *absolutely* indecomposable. One must try, however, to obtain a strict proof of this upon the general principle of crushing every logical difficulty regarded as a challenge to the human reason, which falls in our way; it is in overcoming the difficulties attendant upon the proof of negative propositions that the mind acquires new strength and accumulates the materials for future and more significant conquests. To prove that involutants in their general form are indecomposable may possibly, I imagine, prove to be a hard nut to crack, or it may be exceedingly easy.

up upon itself, and if the two do not grow into one another, but like such creatures each possesses a perfectly distinct organization.

A single instance will serve to establish the fact that the Resultant of F, G, H is the very product IJ itself, without any numerical multiplier. I have made this verification for binary and ternary matrices, and as the point is not one of an essential importance need not dwell here further upon it.

To pass to a much more important subject, I am inclined to anticipate as the result of a long and interesting investigation into the relations of the involutants of a certain particular *corpus* of the third order that the *sum* of the two involutants of any *corpus* admits of being represented by means of invariants similar in kind to that which expresses the single involutant to a binary *corpus* (m, n), namely, the content of (that is, the determinant to) the matrix $mn - nm$, which itself (as previously observed) may be written as the determinant to the matrix $\begin{Bmatrix} m & n \\ m & n \end{Bmatrix}$, or say $(m, n)_2$; and in some similar way it is, I think, not unlikely that the *product* also of the two involutants (the resultant of F, G, H) is capable of being expressed; but I must for the present content myself with exhibiting the bare fact of the existence of invariants of the kind referred to for matrices of any order.

Suppose then that m, n is a *corpus* of the third order. Form the determinant

$$\left.\begin{matrix} m & n & m^2 & n^2 \\ m & n & m^2 & n^2 \\ m & n & m^2 & n^2 \\ m & n & m^2 & n^2 \end{matrix}\right\}, \text{ say } (m,\ n,\ m^2,\ n^2)_4.$$

The number of terms, half of them positive and half of them negative, in such determinant is 24; but of these, all but 8 will obviously appear as pairs of equal terms affected with opposite signs and so cancel one another: the 8 excepted ones are those in which no m and n come together, *to wit*:

$$mnm^2n^2 + nmn^2m^2 + m^2n^2mn + n^2m^2nm$$
$$- m^2nmn^2 - nm^2n^2m - mn^2m^2n - n^2mnm^2.$$

The determinant to this matrix will be of the total degree 18 in the two sets of elements belonging to m and n respectively, that is, of the degree 9 in respect to each set of elements *per se*. And so in general if m, n be of the order ω the determinant

$$(m,\ m^2,\ \ldots\ m^{\omega-1},\ n,\ n^2,\ \ldots\ n^{\omega-1})_{2\omega}$$

will contain only $2(\pi\omega)^2$ effective terms, of which half will bear the positive and the others the negative sign.

The determinant to this matrix will be of the order

$$\omega[2\{1+2+\ldots+(\omega-1)\}], \text{ that is, } (\omega-1)\omega^2,$$

in regard to the combined elements in m and n, that is, equi-dimensional with either involutant to the *corpus* m, n.

Whatever else may be its properties (on which I do not dare yet to pronounce), it is certain that such determinant (and over and above that, every term in the matrix of which it is the content) will be an Invariant to the *corpus* in the same sense in which either Involutant has been previously shown to be entitled to bear that name. And here for the present it becomes necessary for me to break off, bidding *au revoir* to any reader who may peruse this sketch, and trusting to meet him again in the broader field of the *American Journal of Mathematics*, where I hope to be spared to set out this portion of the theory with more certainty, and the whole doctrine of multiple quantity with much greater completeness and in more ample detail than is possible within the limits of the *Circulars* and in the short interval remaining between the present time and the date of my intended departure for Europe.

16.

ON THE THREE LAWS OF MOTION IN THE WORLD OF UNIVERSAL ALGEBRA.

[*Johns Hopkins University Circulars*, III. (1884), pp. 33, 34, 57.]

IN the preceding *Circular* allusion was made to the three cardinal principles or conspicuous landmarks in Universal Algebra; these may be called, it seems to me (without impropriety), its Laws of Motion, on the ground that as motion is operation in the world of pure space, so operation is motion in the world of pure order, and without claiming any exact analogy between these and Newton's laws, it will be seen that there is an element in each of the former which matches with a similar element in the latter, so that there is no difficulty in pairing off the two sets of laws and determining which in one set is to be regarded as related by affinity with which in the other. They may be termed the law of *concomitance* or *congruity*, the law of *consentaneity* and the law of *mutuality* or *community*.

The law of congruity is that which affirms that the latent roots of a matrix follow the march of any functional operation performed upon the matrix, not involving the action of any foreign matrix; it is the law which asserts that any function of a latent root to a matrix is a latent root to that same function of the matrix; in so far as it regards a matrix *per se*, or with reference solely to its environment, it obviously pairs off with Newton's first law.

The law of *consentaneity*, which is an immediate inference from the rule for combining or multiplying substitutions or matrices, is that which affirms that a given line (or parallel of latitude) can be followed out in the matrices resulting from the continued action of a matrix upon a fixed matrix of the same order, that is, in the series $M, mM, m^2M, m^3M, \ldots$ (which may be regarded as so many modified states of the original matrix) without reference to any other of the lines or parallels of latitude in the series, or again any column or parallel of longitude in the correlated series $M, Mm, Mm^2, \ldots$ without reference to any other such column or parallel of longitude.

An immediate consequence of this obvious fact (a direct consequence for the rule of multiplication) obtained by dealing at will with either of the systems of parallels referred to, is that a system of simultaneous linear equations in differences may be formed for finding each term in any given line or in any given column at any point in the series, and the integration of these equations leads at once to the conclusion that any term of given latitude and longitude in the ith term of either series is a syzygetic function of the ith powers of the latent roots of m.

If, then, M be made equal to multinomial unity, this at once shows that supposing ω to be the order of m, on substituting m for the *carrier* (or latent variable) in the latent function to m, and multiplying the last term by the proper multinomial unit, the matrix so formed is an absolute null, which proves the proposition concerning the "identical equation" first enunciated by Professor Cayley in his great paper on Matrices in the *Philosophical Transactions* for 1858.

This proposition admits of augmentation, (1), from within, as shown in a former note, by applying to it the limiting law of the nullity of a product (a branch of the 3rd law), which leads to the very important conclusion that the nullity of any factor of the function of a matrix which is an absolute null, or more generally of any product of powers of its linear factors, is exactly equal to the number of distinct linear factors which such factor or product contains, at all events, in the general case where the latent roots are all unequal; and (2), from without, by substituting for m, $m + \epsilon n$ where n is any second matrix whatever and ϵ is an infinitesimal. This leads to the *catena* of identities, to which allusion has been made in the preceding *Circular*. Then, again, the *endogenous* growth of the theorem (that which determines the exact nullity of any factor of the left-hand side of the identical equation) in its turn seems to lead to a remarkable theorem concerning the form of the general term of any power of m into M.

Observe that every such term is expressed as a syzygetic function of powers of the ω latent roots, and contains, therefore, ω constants, so that the total number of syzygetic multipliers is ω^3; but the number of variables in m and M together is $2\omega^2$; and, consequently, apart from the ω arbitrary latent roots the number of independent constants in m^iM should be $2\omega^2 - \omega$. The ω^3 syzygetic multipliers ought then to contain only $\omega(2\omega - 1)$ arbitrary constants, and such will be found to be the case by virtue of the following hypothetical theorem: Calling λ any one of the latent roots, the multipliers of λ^i in m^iM will form a square of ω^2 quantities; the theorem in question* is that every minor of the second order in such square is zero, so that the ω^2 terms in the square is given when the bounding angle containing

* I have not had leisure of mind, being much occupied in preparing for my departure, to reduce this theorem to apodictic certainty. I state it therefore with all due reserve.

$2\omega - 1$ terms is given; and the same being true for the multipliers of each latent root (which resolve themselves into ω squares) the number of arbitrary quantities in all is $\omega(2\omega - 1)$ as has to be shown.

The law of *consentaneity* in so far as it relates to the decomposition of the motion of a matrix into a set of parallel motions, has an evident affinity with Newton's second law*.

Remains the law of *mutuality*, which is concerned with the effect of the mutual action upon one another of two matrices, and so claims kindred with Newton's third law.

This law branches off into two, one of which may be termed the law of reversibility, the other that of co-occupancy or permeability.

The law of reversibility affirms that the latent function of the product of two matrices is independent of the sense in which either of them operates upon the other, that is, is the same for mn as for nm, just as the kinetic energy developed by the mutual action of two bodies is not affected by their being supposed to change places.

As regards the second branch of the third law, the word co-occupancy refers to the fact that although the space occupied by two similarly shaped figures (say two spheres) is not absolutely determined (in the absence of other data) by the spaces occupied by them each separately (for they may intersect or one of them coincide with or contain the other), a superior as well as an inferior limit to such joint occupation is so determined; the inferior limit being the space occupied by either such figure, that is, the *dominant* of these two given spaces, and the superior limit their arithmetical sum. So the nullity resulting from the action in either sense of two matrices upon one another is not given when their separate nullities are assigned, but has for an inferior limit the dominant of these two nullities and for a superior limit their sum; the nullities of the two component matrices may also be conceived under the figure of two gases or other fluids which are mutually *permeable* and capable of occupying each other's pores.

Although the limits spoken of are independent of the sense in which the two matrices act on one another, it must not however be supposed that the actual resultant nullity is unaffected by that circumstance; thus, for example, if the latent roots of a ternary matrix m are $\lambda, \lambda', \lambda''$, the nullity resulting from $(m-\lambda)(m-\lambda')$ acting sinistrally upon $(m-\lambda'')n$, that is, of $(m-\lambda)(m-\lambda')(m-\lambda'')n$ is 3, but from the same acting dextrally upon the same, that is, of $(m-\lambda'')n(m-\lambda)(m-\lambda')$, need not necessarily exceed 2.

* For another and closer bond of affinity between the two laws see concluding paragraph of this note.

Such then are the three primary Laws of Algebraical Motion; but as Conservation of areas, *Vis viva*, D'Alembert's Principle, the principle of Synchronous Vibrations, of Least action, and various other general laws may be deduced from Newton's three ground laws, so, of course, various subordinate but very general laws may be deduced from the interaction of the above stated three ground laws, namely, the law of Congruity, the law of Consentaneity, and the law of Mutuality.

The deduction of the catena of identical equations connecting two matrices m and n from the second and third laws combined, affords an instance of such derivative general laws. Another instance of the same is the theorem that when the product resulting from the action upon one another of two matrices, is the same in whichever of the two senses the action takes place, the matrices must be functionally related, unless one of them is a scalar, that is, a multiple of multinomial unity, at all events when neither m nor n possesses a pair of equal latent roots.

This very important and almost fundamental law (seemingly so simple and yet so hard to prove) may be obtained as an immediate inference from that identical equation in the catena of such equations connecting the matrices m and n, in which one of the two enters only singly at most in any term. As for example if m and n are of the 3rd order, beside the identical equation $m^3 - 3bm^2 + 3dm - g = 0$ we have* the identity

$$m^2n + mnm + nm^2 - 3b(mn + nm) - 3cm^2 + 3dn + 6em - 3h = 0.$$

But if $nm = mn$ then $mnm = m^2n, nm^2 = mnm = m^2n$, so that this equation becomes

$$m^2n - 2bmn + dn = m^2c - 2em + h, \quad \text{or} \quad n = \frac{cm^2 - 2em + h}{m^2 - 2bm + d}\dagger,$$

unless $m^2 - 2bm + d$ is vacuous.

The first branch of the third law, namely, the law of *reversibility*, is an almost immediate inference from the rule for the multiplication of matrices, and becomes intuitively evident when the process of multiplication in each of the two senses between m and n is actually set out. The second branch, namely, the law of co-occupancy or permeability, as it is the most far-reaching so it is the most deep seated (the most *caché*) of all the primary laws of

[* See p. 126 above.]

† Whence it follows that n must be a function of m convertible into an integral polynomial form, unless the numerator and denominator of the fraction to which n is equated vanish simultaneously, which is what happens when m is scalar. If the numerator exactly contains the denominator n becomes a scalar. Seeing that a constant c is a specialized case of a function of a variable x although the converse is not true, we may say that whenever $nm = mn$, *one at least* of the two matrices m and n is a function of the other, and that *each* is a function of the other unless that other is a scalar. Compare Clifford's "Fragment on Matrices" in the posthumous edition of his collected works.

motion. I found my proof of it upon the fact that the value of any minor determinant, say of the ith order, in either product of m and n (two matrices of the order ω) may be expressed as the quantitative product of a certain couple of rectangular matrices (in Cauchy's sense of the term), of which one is formed by i columns and the other by i lines in the two given matrices respectively. Such rectangle as shown by Cauchy (and as may be intuitively demonstrated by the simplest of my umbral theorems on compound determinants) is the sum of the

$$\frac{\pi(\omega)}{\pi(\omega - i)\,\pi i}$$

complete determinants of the one rectangle multiplied respectively by the corresponding complete determinants of the other rectangle.

This shows at once the truth of the proposition in so far as relates to the lower limit, that is, that if $mn = p$, and m, n have the nullities ϵ, ζ, and p the nullity θ, then θ must be at least as great as ϵ and at least as great as ζ. As regards the superior limit the proof is also founded on the theorem in determinants already cited, and the form of it is as follows. If ϵ be any number r, it may be shown that ζ must be at least as great as $\theta - r$; hence giving r all values successively from 0 to $\zeta - 1$, it follows that $\epsilon + \zeta$ cannot be less than θ, that is, that θ cannot be greater than $\epsilon + \zeta$.

The proof of the first law, that of concomitance or congruity, I ought to have stated antecedently, is a deduction from the theory of resultants and the well-known fact that the determinant of a product of matrices is the product of their determinants. Thus each of the three laws of motion is deduced independently of the two others.

As another example of a derivative law of motion, I may quote the very notable one which results from the interaction of the first and second fundamental laws upon one another, and which gives the general expression for any function whatever of a matrix in the form of a rational polynomial function of the same and of its latent roots, to wit, the magnificent theorem that whatever the form of the functional symbol ϕ, and whether it be a single or many valued function, if $\lambda_1, \lambda_2, \ldots \lambda_\omega$ be the latent roots of m,

$$\phi m = \Sigma \phi\lambda_1 \frac{(m-\lambda_2)(m-\lambda_3)\ldots(m-\lambda_\omega)}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)\ldots(\lambda_1-\lambda_\omega)}.$$

As for example if $\phi m = m^{\frac{p}{q}}$, $m^{\frac{p}{q}}$ will have q^ω roots which are completely determined by the above formula.

The first law, as already stated, regards a single body or matrix, uninfluenced by the action of any external force. The second law regards the effect upon a single matrix, subject to external impulses, taking their rise in an external source; whilst the third law has regard to the mutual action or joint effect of two bodies or matrices simultaneously operating upon one another.

Note. Making [in p. 149] $m^3 - 3bm^2 + 3dm - g = F(m)$, we found

$$(F'm)\,n = cm^2 - 2em + g.$$

When two of the latent roots of m are equal, it is easy to prove that $F'm$ is vacuous, and conversely, that when $F'm$ is vacuous, two of the latent roots of m are equal; but when $F'm$ is vacuous it is no longer permissible to drive it out of the equation, and accordingly the true statement of the theorem in question is that when m, n are two matrices of (any) the same order, such that $mn = nm$, n must *in general* be a function of m, but that this ceases to be true, when and only when m has two equal roots. The theorem requires further investigation in order to make out what happens when, or how it can happen that, two of the latent roots of one and only one of the two convertible matrices are equal; for supposing this to happen it would seem to lead to the conclusion that n may be a function of m, but m not a function of n; which, however, is not quite so paradoxical as it looks, inasmuch as in ordinary algebra a constant may be regarded as a specialized function of a variable, whilst a variable in no sense can be regarded as a function of a constant. The following example of two matrices not functions of one another, but forming commutable products, has recently occurred to me in practice, and led to the discovery of the oversight I had committed in stating the theorem in question in too absolute terms.

If $x = \begin{matrix} 0 & \rho & \rho^2 \\ 1 & 0 & 1 \\ \rho^2 & \rho & 0 \end{matrix}$, $y = \begin{matrix} 0 & 1 & 1 \\ \rho & 0 & \rho^2 \\ \rho & \rho^2 & 0 \end{matrix}$ where $\rho^2 + \rho + 1 = 0$, it will be found that $xy = yx$, but that neither x nor y is a function of the other; this may easily be deduced from the fact that $x^2 - \rho^2 x - 2\rho = 0$, so that if y were any function of x, it would be reducible to the form of a linear function thereof, and consequently (on account of the zeros in the two matrices) y must be a multiple of x, which is absurd.

In like manner it will be found that $y^2 - \rho^2 y - 2\rho = 0$, and that consequently x cannot be a function of y.

17.

EQUATIONS IN MATRICES.

[*Johns Hopkins University Circulars*, III. (1884), p. 122.]

I HAVE been lately considering the subject of equations in matrices. Sir William Hamilton in his *Lectures on Quaternions* has treated the case of what I call unilateral equations of the form $x^2 + px + q = 0$, or $x^2 + xp + q = 0$, where we may, if we please, regard x, p, q as general matrices of the second order. He has found there are six solutions, which may be obtained by the solution of an ordinary cubic equation. In a paper now in print and which will probably appear in the May number of the *Philosophical Magazine*, I have discussed by my own methods the general *unilateral* equation, say

$$x^\omega + px^{\omega-1} + qx^{\omega-2} + \ldots + l = 0,$$

where $x, p, q \ldots l$, are quaternions or matrices of the second order, and have shown, by a method satisfactory if not absolutely rigorous, that the number of solutions is $\omega^3 - \omega^2 + \omega$, that is to say, the nearest superior integer to the general maximum number of roots (ω^4) divided by the augmented degree $(\omega + 1)$.

But after I had done this it occurred to me that there were multitudinous failing cases of which neither Hamilton nor myself had taken account, as for example $x^2 + px = 0$, besides the solutions $x = 0$, $x = -p$, will admit of a solution containing an arbitrary constant, I think; but that is a matter which I shall have to look further into before committing myself to a positive assertion about it. I have only had time to pass in review the more elementary case of a unilateral simple equation, say $px = q$, where p, q are matrices of any order ω.

If p is non-vacuous there is one solution, namely, $x = p^{-1}q$; but suppose p is vacuous: what is the condition that the equation may be soluble?

(1) Suppose $q = 0$, p being vacuous has for its identical equation $pP = 0$, and consequently we may make $x = \lambda P$ where λ is an arbitrary constant.

(2) Suppose q is finite and that $x = r$ is one solution, then obviously the general solution is $x = r + \lambda P$.

We have now to inquire what is the condition that r may exist. I find from the mere fact of x being indeterminate (and confirm the result by another order of considerations) that the determinant of $q + \lambda p$ must vanish identically; so that for instance when p, q are of the second order and $\begin{matrix} b' c \\ d e f \end{matrix}$ are the *parameters to the corpus* (p, q), we must have when $d = 0$, which is implied in the vacuity of p, $f = 0$ and $e = 0$. The first of these conditions is known *à priori* immediately from my third law of motion; but not so, without introducing a slight intervening step, the intermediate one (I mean the connective to d and f, namely) $e = 0$.

So in general in order that $px + q = 0$ may be soluble, that is, in order that $p^{-1}q$ where p is simply vacuous may be *Actual* and not Ideal, q must satisfy as many conditions as there are units in the order of p or q, all implied in the fact that the determinant to $p + \lambda q$, where λ is an arbitrary constant, vanishes identically. When these conditions are satisfied $p^{-1}q$ becomes actual but indeterminate. (This, by the way, shows the disadvantage of calling a *vacuous* matrix *indeterminate*, as was done in the infancy of the theory by Cayley and Clifford—for we want this word as you see to signify a combination of the inverse of a vacuous matrix with another which takes the combination out of the ideal sphere and makes it actual.)

So in general in order that $p^{-1}q$ where p is a null of the ith order (that is where all the $(i+1)$th but not all the ith minors of p are zero) shall be an actual (although indeterminate) matrix, it is necessary and sufficient that $p + \lambda q$, where λ is arbitrary, shall be a null of the same (ith) order. What will be the degree of indeterminateness in $p^{-1}q$, that is, how many arbitrary constants are contained in the value of x which satisfies the equation $px = 0$ remains to be considered.

The law as to the conditions is an immediate *corollary* to my third law of motion, for if $px = q$ then $p + \lambda q = p(1 + \lambda x)$; consequently $p + \lambda q$, whatever λ may be, must have at least as high a degree of nullity as p. Q.E.D.

18.

SUR LES QUANTITÉS FORMANT UN GROUPE DE NONIONS ANALOGUES AUX QUATERNIONS DE HAMILTON.

[*Comptes Rendus*, XCVIII. (1884), pp. 273—276, 471—475.]

DANS une Note précédente*, j'ai fait allusion au cas où le déterminant de $x + ym + zn$ devient une fonction linéaire de x^3, y^3, z^3 sans que la quantité nommée Q s'évanouisse. Dans ce cas, on aura

$$(mn)^3 + Q(mn) - R = 0, \tag{1}$$

R étant le déterminant de mn. C'est bien la peine, comme on va le voir, de donner plus de précision aux équations qui lient ensemble mn et nm pour ce cas.

En suivant la même marche que pour le cas particulier où $Q = 0$, on trouvera sans difficulté les résultats suivants:

$$nm = -\frac{3Q}{\zeta}(mn)^2 - \frac{\zeta + 9R}{2\zeta}\,mn - \frac{2Q^2}{\zeta}, \tag{2}$$

$$mn = \frac{3Q}{\zeta}(nm)^2 - \frac{\zeta - 9R}{2\zeta}\,nm + \frac{2Q^2}{\zeta}, \tag{3}$$

ζ étant le produit des différences des racines de la fonction $\lambda^3 + Q\lambda - R$, de sorte que $\zeta^2 = -(4Q^3 + 27R^2)$.

Conséquemment on peut écrire

$$nm = A(mn)^2 + Bmn + C, \tag{4}$$

$$mn = -A(nm)^2 + B'nm - C, \tag{5}$$

où A et C peuvent être tous les deux zéro, ou tous les deux des quantités finies quelconques, mais non pas l'un d'entre eux une quantité finie et l'autre zéro, et B, B' les deux racines par rapport à B de l'équation

$$B^2 + B + 1 + \frac{AC}{2} = 0\dagger. \tag{6}$$

* *Comptes rendus*, t. XCVII. p. 1336.

[† It follows from $n(mn \div \theta) = (nm + \theta)n$ that M, $= mn$ and N, $= nm$ both satisfy equation (1); further $MN = NM$ (footnote * p. 127 above), so that (p. 149 above) there exists an equation $N = pM^2 + qM + r$; from (1), if $|M - N| \neq 0$, follows $M^2 + MN + N^2 + Q = 0$. Hence (2), (3) can be deduced.]

On peut vérifier, comme je l'ai fait, par un calcul algébrique direct, que les équations (4) et (5), en vertu des équations (1) et (6), sont compatibles.

Or une chose digne de remarque, c'est ce qui arrive quand $\zeta = 0$, car cela servira à révéler un phénomène d'Algèbre universelle d'un genre que personne n'avait encore même soupçonné.

Dans ce cas, les deux équations (4) et (5) changent leur caractère et deviennent

$$Q(mn)^2 + 3Rmn + \tfrac{2}{3}Q^2 = 0,$$
$$Q(nm)^2 + 3Rnm + \tfrac{2}{3}Q^2 = 0,$$

de sorte que mn et nm cessent d'être fonctions l'un de l'autre.

Nommons, pour le moment, $mn = u$, $nm = v$; on aura, comme auparavant, $uv = vu$, sans que v et u soient fonctionnellement liés ensemble. Dans le *Johns Hopkins Circular* de janvier 1884 (dans l'article intitulé *On the three laws of motion in the world of universal Algebra*, [above p. 146]), on trouvera le moyen d'établir qu'en général cette équation amène à la conclusion que ou u doit être un *scalar*, c'est-à-dire de la forme $\begin{matrix} C & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & C \end{matrix}$, ou bien v un scalar, ou sinon que nm, mn doivent être fonctions l'un de l'autre ; mais on remarquera (ce qui m'avait alors échappé) que, si $Fu = 0$ est l'équation identique en u et que la dérivée fonctionnelle $F'u$ est une matrice *vide* (*vacuous*), c'est-à-dire dont le déterminant est zéro, le raisonnement est en défaut ; cette vacuité a lieu dans le cas, et seulement dans le cas, où deux des racines latentes (lambdaïques) de m sont égales. On peut généraliser cette conclusion et l'étendre à deux matrices u et v d'un ordre quelconque au-dessus du deuxième ; c'est-à-dire quand les racines latentes de u (ou bien de v) ne sont pas toutes inégales, *il est des cas* où $uv = vu$, sans que u ou v soient des scalars et sans que v et u soient fonctions l'un de l'autre. Par exemple, si l'on fait

$$u = \begin{vmatrix} 0 & \rho & \rho^2 \\ 1 & 0 & 1 \\ \rho^2 & \rho & 0 \end{vmatrix}, \quad v = \begin{vmatrix} 0 & 1 & 1 \\ \rho & 0 & \rho^2 \\ \rho & \rho^2 & 0 \end{vmatrix},$$

on trouvera

$$uv = \begin{vmatrix} -\rho & \rho & 1 \\ \rho & -\rho & 1 \\ \rho^2 & \rho^2 & -\rho \end{vmatrix} = vu.$$

Mais on démontrera sans difficulté que v ne peut pas s'exprimer comme somme de puissances de u, ni *vice versa* v comme somme de puissances de u.

On n'a pas besoin de remarquer que la seule condition de l'existence de racines latentes égales en u ou en v ne peut pas suffire en elle-même pour assurer que $uv = vu$, mais il faut réserver pour une autre occasion la pleine discussion de la totalité des solutions de cette équation importante.

J'ajouterai seulement cette remarque, qui est essentielle. En supposant l'existence des équations

$$m^2n + mnm + nm^2 = 0,$$
$$n^2m + nmn + mn^2 = 0,$$
$$(mn)^3 + Qmn - R = 0,$$
$$(nm)^3 + Qnm - R = 0,$$

qui ont lieu nécessairement quand le déterminant de $x + ym + zn$ devient une fonction linéaire de x^3, y^3, z^3, et en regardant nm comme fonction de mn (en vertu de l'équation $mn \,.\, nm = nm \,.\, mn$), alors, en additionnant aux deux valeurs de nm (exprimé comme fonction de mn) données ci-dessus, qui correspondent aux deux valeurs de ζ, c'est-à-dire $\sqrt{-(4Q^3 + 27R^2)}$, on a à considérer quatre autres valeurs, le nombre total en étant six. Car si l'on suppose $nm = A(mn)^2 + Bmn + C$ et si $\lambda_1, \lambda_2, \lambda_3$ sont les trois racines de $\lambda^3 + Q\lambda - R = 0$, les valeurs de A, B, C sont déterminées en mettant

$$A\lambda_1^2 + B\lambda_1 + C = \lambda_i,$$
$$A\lambda_2^2 + B\lambda_2 + C = \lambda_j,$$
$$A\lambda_3^2 + B\lambda_3 + C = \lambda_k,$$

où i, j, k sont respectivement

$$1\ 2\ 3 \quad \text{ou} \quad \begin{matrix} 2 & 3 & 1 \\ 3 & 1 & 2 \end{matrix} \quad \text{ou bien} \quad \begin{matrix} 1 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{matrix}$$

Les valeurs de A, B, C données ci-dessus correspondent au deuxième de ces groupes de valeurs de i, j, k.

Si l'on écrit $i = 1, j = 2, k = 3$, on trouvera $nm = mn$.

Si l'on écrit $i = 1, j = 3, k = 2$, en faisant $\lambda_1 = \Lambda$, on trouvera

$$nm = \frac{3\Lambda(mn)^2 - Qmn + 2\Lambda Q}{3\Lambda^2 + Q}.$$

Dans le cas critique où $\zeta = 0$, de sorte que $3\Lambda^2 + Q = 0$, l'équation devient $(mn)^2 + \Lambda mn - 2\Lambda^2 = 0$, comme dans le cas déjà traité. Quand on suppose Q égal à zéro et R (c'est-à-dire le déterminant de mn) fini, les seules solutions possibles avec ces conditions sont celles fournies en écrivant $i, j, k = 2, 3, 1$, ou $3, 1, 2$; mais, pour le cas général, il n'y a pas de raison (au moins *très évidente*) pour exclure aucune des trois classes de solution. Si l'on admet la légitimité des solutions de la troisième classe, en écrivant

$$nm = A(mn)^2 + Bmn + C,$$

on trouvera
$$B^2 + B + \frac{AC}{2} = 0$$

au lieu de l'équation

$$B^2 + B + 1 + \frac{AC}{2} = 0,$$

qui est applicable aux solutions de la deuxième classe.

Avant de considérer l'équation $xy = yx$, il importe d'avoir une idée nette d'une certaine classe de matrices que je nomme *privilégiées* ou *dérogatoires*, en tant qu'elles dérogent à la loi générale que toute matrice est assujettie à satisfaire à une équation identique dont le degré ne peut pas être moindre que l'ordre de la matrice.

Les matrices dérogatoires sont justement celles qui satisfont à une équation d'un ordre inférieur à leur ordre propre; on peut les nommer *simplement, doublement, triplement, ... dérogatoires*, selon que le degré de l'équation identique à laquelle elles satisfont diffère par une, deux, trois, ... unités du degré minimum ordinaire.

Pour le cas des matrices du deuxième ordre, il n'y a que les *scalars* $\begin{matrix} a & 0 \\ 0 & a \end{matrix}$ qui soient dérogatoires.

Pour le cas des matrices du troisième ordre, en écartant les *scalars* de la forme $\begin{matrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{matrix}$, toute matrice x dérogatoire peut être ramenée ou à la forme

$$a + b(\epsilon + \epsilon^2),$$

où ϵ est une matrice qui satisfait à l'équation $\epsilon^3 = 1$, c'est-à-dire une matrice dont les racines latentes sont 1, ρ, ρ^2, ou à la forme

$$a + b(1 + \epsilon + \epsilon^2)\zeta,$$

où $$\epsilon^3 = 1, \quad \zeta^3 = 1 \quad \text{et} \quad \zeta\epsilon = \rho\epsilon\zeta,$$

ρ signifiant une racine cubique primitive de l'unité. Dans le premier cas,

$$x^2 - (2a + b)x + (a^2 + ab - 2b^2) = 0,$$

et dans le second

$$x^2 - 2ax + a^2 = 0,$$

car on trouvera facilement que

$$(1 + \epsilon + \epsilon^2)\zeta(1 + \epsilon + \epsilon^2)\zeta = 0.$$

Pour le cas du quatrième ordre, en écartant les scalars et en se bornant au cas où l'équation identique dérogée (vue pour le moment comme une équation ordinaire en x) ne contient pas des racines égales, toute matrice x peut être ramenée à l'une ou à l'autre des deux formes suivantes :

$$a + b(U + U^3) \quad \text{ou bien} \quad a + b\left(U + \frac{1 + ki}{1 + i}U^2 + kU^3\right),$$

où U est une matrice du quatrième ordre telle que $U^4+1=0$; a, b, k sont des scalars arbitraires et i est une racine primitive biquadratique de l'unité; quand, pour la seconde forme $k=1$, on trouvera qu'il y aura une dérogation double de l'ordre de l'équation satisfaite par x, l'équation identique pour x ne sera que du deuxième degré.

En réservant les détails du calcul, voici le résultat général que j'ai démontré rigoureusement (en m'aidant de la notation des nonions) pour les matrices du troisième degré qui satisfont à l'équation $xy=yx$.

A moins que x ne soit une matrice privilégiée ou dérogatoire, y sera toujours une fonction rationnelle et entière quadratique de x, et de même, à moins que y ne soit privilégiée, x sera une fonction pareille de y.

Il est bien entendu que le caractère dérogatoire d'une seule des deux matrices n'empêche pas qu'elle ne soit une fonction entière et rationnelle quadratique de l'autre. Dans le cas où x et y sont tous les deux dérogatoires, ni l'un ni l'autre ne peut être exprimé comme fonction explicite l'un de l'autre, mais ils seront liés ensemble par une équation linéo-linéaire.

Il paraît peu douteux qu'une règle semblable doive être applicable à l'équation $xy=yx$, quel que soit l'ordre des matrices x et y, sauf quand l'équation qui lie ensemble x et y pourra être d'un degré moindre que l'ordre de chacune d'elles.

Il est bon de remarquer que nulle matrice ne peut être dérogatoire, sauf pour le cas où il existe des égalités entre ses racines latentes; mais ces égalités peuvent parfaitement subsister sans que la matrice à laquelle elles appartiennent soit dérogatoire. En général, si $x=a+by+cy^2$, on peut, par une formule générale que j'ai déjà donnée, exprimer y sous la forme

$$\alpha+\beta x+\gamma x^2;$$

avec l'aide des racines latentes de x, cette formule ne cesse pas en général d'être valable, même pour le cas où x contient des racines égales, en regardant leur différence comme une quantité infinitésimale; seulement le nombre des racines finies subira dans ce cas une diminution; mais, dans le cas où l'équation $xy=yx$ (x étant dérogatoire) mènerait à l'équation

$$x=a+by+cy^2,$$

on trouverait que nulle fonction explicite de x avec des coefficients finis ne peut exprimer le y cherché.

Il est à peine nécessaire d'ajouter que rien n'empêche, dans le cas où l'un ou l'autre de x et y ou tous les deux sont dérogatoires, qu'on puisse satisfaire à $xy=yx$, en supposant que x et y soient des fonctions explicites chacune l'une de l'autre: tout ce qu'on affirme, c'est que, dans le cas admis, cette supposition cesse d'être obligatoire; c'est un cas très semblable à ce qui arrive dans le cas de défaut (*failing case*) du théorème de Maclaurin: c'est celui où une variable est une fonction sans pouvoir être développée dans une série de puissances d'une autre variable.

Dans ce qui précède, on a vu un exemple du fait général que, m étant une matrice donnée, l'équation $\phi(x, m) = 0$, pour certaines valeurs de m, cesse d'admettre la solution ordinaire $x = Fm$.

Mais il existe encore une classe assez étendue d'équations entre x et m pour lesquelles, quand m prend certaines valeurs, x n'a aucune existence actuelle; par exemple, m étant une matrice *vide* d'un ordre quelconque, si $mx = 1$, la matrice x devient inexprimable et n'a, pour ainsi dire, qu'une existence idéale.

Je citerai encore l'exemple $x^2 = m$, m étant une matrice du deuxième ordre; si les racines latentes de m sont inégales, on trouvera, par la formule générale, quatre valeurs de x. Si les deux racines latentes sont égales et finies, ces quatre valeurs se réduisent à deux; mais, si les deux racines sont toutes les deux égales à zéro, il n'y aura aucune valeur de x qui satisfasse à l'équation donnée, c'est-à-dire si $m = \begin{matrix} a & -\frac{a}{k} \\ ka & -a \end{matrix}$; l'équation devient absolument insoluble, ou, si l'on peut s'exprimer ainsi, les quatre racines carrées de m sont toutes idéales.

Dans le cas supposé, on vérifiera aisément que $m^2 = 0$ et, *vice versa*, toute racine carrée du zéro binomial est de la forme $\begin{matrix} a & -\frac{a}{k} \\ ka & -a \end{matrix}$, de sorte que l'on peut dire qu'une racine carrée quelconque du zéro binomial ne possède pas elle-même des racines algébriques quelconques, ou, en d'autres termes, une racine algébrique quelconque du quaternion $i + \sqrt{(-1)}\,j$ est purement idéale et n'admet pas d'être représentée sous la forme d'un quaternion. Finalement je remarque que toute matrice est d'un certain ordre et d'une certaine classe; l'ordre, c'est le nombre total de ses racines latentes; la classe, c'est le degré minimum de l'équation latente (c'est-à-dire de l'équation identique à laquelle la matrice satisfait), lequel ne peut être plus petit que le nombre des racines latentes inégales.

Je dois ajouter (ce que j'aurais dû dire auparavant) que, quand x est une matrice ternaire dérogatoire dont *toutes les racines latentes sont égales*, l'équation $xy = yx$ peut subsister sans que ni x ni y ne soit une fonction explicite l'un de l'autre, même quand y n'est pas une matrice privilégiée; c'est le cas où, ϵ et ζ faisant partie d'un groupe de nonions élémentaires, on a $x = a + b(1 + \epsilon + \epsilon^2)\,\zeta$. Les calculs sont un peu compliqués pour ce cas spécial, mais je crois ne pas me tromper en faisant cette correction. Le champ de la théorie de la quantité multiple est tellement nouveau et inexploité que, sans les plus grandes précautions, on est toujours en danger de se heurter contre quelque cause imprévue d'incertitude ou même d'erreur.

19.

SUR UNE NOTE RÉCENTE DE M. D. ANDRÉ*.

[*Comptes Rendus*, XCVIII. (1884), pp. 550, 551.]

Le théorème de M. André est une conséquence immédiate de la généralisation que j'ai donnée du théorème de Newton (*Arithmétique universelle*, 2e Partie, Ch. II.) sur les racines imaginaires des équations.

On verra, en consultant mon travail† sur ce sujet (*Proceedings of the London Mathematical Society*, No. 2), que si $u_0, u_1, u_2, \ldots u_m$ sont les coefficients d'une équation du degré m et si

$$G_r' = ru_r^2 - (r+1)\gamma_r u_{r-1} u_{r+1}$$

ou

$$\gamma_r = \frac{v+r-1}{v+r},$$

γ_r étant une quantité réelle quelconque qui n'est pas intermédiaire entre 0 et $-m$, l'équation aura nécessairement au moins autant de racines imaginaires qu'il y a de variations de signes dans la série $G_0, G_1, G_2, \ldots, G_m$.

En faisant $v = -m$, on a le théorème de Newton; en faisant $v = 1$, on voit qu'on peut prendre $G_r = u_r^2 - u_{r-1} u_{r+1}$. Conséquemment le théorème de M. André subsiste, quel que soit le signe de la quantité qu'il nomme α et quels que soient les signes des quantités qu'il nomme $u_0, u_1, \ldots, u_m$.

De plus, le théorème subsistera encore quand, outre ces modifications, au lieu de l'équation

$$u_n = \alpha u_{n-1} + \beta u_{n-2},$$

on écrit

$$v_n = \alpha v_{n-1} + \beta v_{n-2}$$

ou

$$v_0, v_1, v_2, \ldots, v_m,$$

identiques avec

$$u_0, \frac{u_1}{m}, \frac{u_2}{\frac{1}{2}(m.m-1)}, \frac{u_3}{\frac{1}{1.2.3}m(m-1)(m-2)}, \ldots.$$

* *Comptes rendus*, séance du 18 février 1884.

[† Vol. II. of this Reprint, pp. 501, 507.]

Il y a encore une autre extension importante à ajouter, en considérant l'équation

$$u_{n-1}u_{n+1} - u_n^2 = A\alpha^x + B\beta^x + C\gamma^x,$$

dont j'ai donné une solution particulière dans l'*American Mathematical Journal*, Vol. IV. [Vol. III. of this Reprint, pp. 546, 633.]

Il est peut-être digne de remarque que si, dans la formule établie pour γ_r, on fait v infini, la règle calquée sur celle de Newton (mais plus générale) enseigne que, quels que soient a, b, c ou m, l'équation

$$a\left(1 + x + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \ldots + \frac{x^m}{1.2\ldots m}\right) + b\left(1 - x + \frac{x^2}{2} - \frac{x^3}{1.2.3} + \ldots \pm \frac{x^m}{1.2\ldots m}\right) + c = 0$$

ne peut jamais avoir plus de deux racines réelles.

20.

SUR LA SOLUTION D'UNE CLASSE TRÈS ÉTENDUE D'ÉQUATIONS EN QUATERNIONS.

[*Comptes Rendus*, XCVIII. (1884), pp. 651, 652.]

L'ÉQUATION parfaitement générale du deuxième degré en quaternions sera de la forme

$$\Sigma\,(axbxc + dxe) + f = 0$$

et admettra seize solutions, qu'on pourrait obtenir d'une manière directe au moyen de quatre équations, chacune du deuxième degré, contenant les quatre éléments de x comme inconnus. De même, l'équation en quaternions ou en matrices du deuxième ordre du degré ω admettra ω^4 solutions. Parmi ces formes générales, on peut distinguer celles dans lesquelles tous les quaternions donnés se trouvent du même côté du quaternion cherché, par exemple $ax^2 + bx + c = 0$. On peut nommer de telles équations *équations unilatérales*. Hamilton a considéré le seul cas de l'équation quadratique (voir *Lectures on Quaternions*, art. 636, pp. 631—2), et a déterminé le nombre (6) des racines.

Or, je trouve que ma méthode générale de traiter les matrices amène directement à la solution d'une équation unilatérale d'un ordre quelconque ω (c'est-à-dire la fait dépendre de la solution d'une équation algébrique ordinaire) et donne sans la moindre difficulté et sans aucun effort d'invention le nombre des racines. Ce nombre est exprimé par la fonction $\omega^3 - \omega^2 + \omega$, de sorte que le nombre des racines, pour ainsi dire évanouies par suite de l'unilatéralisme de la forme, est $\omega^4 - \omega^3 + \omega^2 - \omega$, c'est-à-dire $(\omega^2 - \omega)(\omega^2 + 1)$. On comprend bien qu'en certains cas le nombre des racines subit une réduction; par exemple, le nombre des racines de $x^\omega + l = 0$ est ω^2 et celui de $x^\omega + kx + l = 0$ est $2\omega^2 - \omega$. Il semble que le nombre, pour l'équation

$$x^\omega + p_\theta x^\theta + p_{\theta-1} x^{\theta-1} + \ldots + p_0 = 0,$$

doit être $(\theta + 1)\,\omega^2 - \theta\omega$, lequel, quand $\theta = \omega - 1$, devient le nombre général $\omega^3 - \omega^2 + \omega$. Les détails de ce petit travail seront donnés dans un prochain numéro du *London and Edinburgh Philosophical Magazine*.

21.

SUR LA CORRESPONDANCE ENTRE DEUX ESPÈCES DIFFÉRENTES DE FONCTIONS DE DEUX SYSTÈMES DE QUANTITÉS, CORRÉLATIFS ET ÉGALEMENT NOMBREUX.

[*Comptes Rendus*, XCVIII. (1884), pp. 779—781.]

VOICI le théorème à démontrer, dans lequel, par *somme-puissance*, on sous-entend une somme de puissances de quantités données:

A i quantités on peut en associer i autres telles, que chaque fonction symétrique (qui est une fonction des différences) des premières sera une fonction des sommes-puissances du 2^e, *du* 3^e, ..., *du* $i^{\text{ième}}$ *ordre des dernières.*

Faisons, pour plus de clarté, $i = 3$.

Soient r_1, r_2, r_3 les racines de l'équation

$$fr = ar^3 + br^2 + cr + d = 0.$$

En prenant b, c, d; r_1, r_2, r_3 comme deux systèmes corrélatifs de variables indépendants, on trouve

$$\delta_b = -\Sigma \frac{r^2}{f'r}\delta_r, \quad \delta_c = -\Sigma \frac{r}{f'r}\delta_r, \quad \delta_d = -\Sigma \frac{1}{f'r}\delta_r.$$

Donc

$$3a\delta_b + 2b\delta_c + c\delta_d = -\Sigma\,\delta_r,$$

$$a\delta_b + b\delta_c + c\delta_d = d\,\Sigma \frac{1}{rf'r}\delta_r.$$

Soients $a = \alpha$, $b = 3\beta$, $c = 3.2.\gamma$, $d = 3.2.1.\delta$, et soient ρ_1, ρ_2, ρ_3 les racines de l'équation

$$\alpha\rho^3 + \beta\rho^2 + \gamma\rho + \delta = 0.$$

Alors, si $\Sigma\,\delta_r\,\phi = 0$, on aura $(\alpha\delta_\beta + \beta\delta_\gamma + \gamma\delta_\delta)\,\phi = 0$. C.Q.F.D.

L'intégrale générale de la première équation est

$$\phi = \mathfrak{F}(r_1 - r_2, r_1 - r_3),$$

et celle de la dernière est

$$\phi = \mathfrak{F}_1(\rho_1^2 + \rho_2^2 + \rho_3^2, \rho_1^3 + \rho_2^3 + \rho_3^3).$$

Ces deux intégrales sont donc identiques, et, le raisonnement étant général pour une valeur quelconque de i, on voit que chaque fonction des différences des r doit pouvoir s'exprimer comme une fonction de $i-1$ sommes-puissances consécutives des ρ (commençant avec la seconde), les r et les ρ étant liés ensemble par les équations

$$ar^i + br^{i-1} + cr^{i-2} + dr^{i-3} + \ldots = 0,$$

$$a\rho^i + \frac{b}{i}\rho^{i-1} + \frac{c}{i(i-1)}\rho^{i-2} + \frac{d}{i(i-1)(i-2)}\rho^{i-3} + \ldots = 0,$$

et conséquemment une fonction *symétrique* des différences des r sera une fonction rationnelle et entière des $i-1$ puissances consécutives (dont on a déjà fait mention) des ρ.

En prenant $i = \infty$, on voit que le théorème équivaut à dire que tous les *sous-invariants,* sources des covariants de $(a, b, c\between x, y)^2$, $(a, b, c, d\between x, y)^3, \ldots$ (à l'infini), seront des fonctions des sommes-puissances prises à l'infini, avec la seule exception de la somme linéaire, des racines de l'équation

$$a + bx + \frac{c}{1.2}x^2 + \frac{d}{1.2.3}x^3 + \ldots \text{ (à l'infini).}$$

Tel est le théorème capital découvert par M. le capitaine Mac-Mahon, de l'Artillerie royale anglaise, dont il a fait le plus heureux usage en développant la théorie des perpétuants (voir *American Journal of Mathematics*). Il est évident que le même principe peut être appliqué aux invariants de toute espèce, de sorte que, grâce à la belle découverte de M. Mac-Mahon, avec la généralisation (qui en sort presque intuitivement) que j'ai donnée, on est aujourd'hui en état de traiter les parties les plus difficiles et les plus essentielles de la théorie des formes algébriques, comme M. Schubert l'a fait avec sa *Zahl-Geometrie* pour les figures dans l'espace, en faisant abstraction, pour ainsi dire, de toute question de substance (de matière contenue dans les formes), et en se bornant à un calcul purement arithmétique.

Je dois avertir que le théorème de correspondance, tel que M. Mac-Mahon l'a donné, a paru dans l'*American Journal of Mathematics* (Vol. VI. p. 131). M. Mac-Mahon affirme (mais sans aucune preuve) que, si ($\alpha, \beta, \gamma, \ldots$ étant des nombres entiers plus grands chacun que l'unité) ϕ est de la forme $\Sigma r^\alpha s^\beta t^\gamma, \ldots$, où $r, s, t, \ldots$ sont les racines de l'équation

$$\left(a_0, a_1, \frac{a_2}{1.2}, \frac{a_3}{1.2.3}, \ldots\right)(x, 1)^n = 0,$$

alors $$(a_0\delta_{a_1} + a_1\delta_{a_2} + a_2\delta_{a_3} + \ldots)\,\phi = 0,$$

et il donne à ϕ le nom de *fonction symétrique non unitaire* des racines. Ce théorème est vrai seulement pour le cas où n est infini (ce que M. Mac-

Mahon a oublié de dire), et dans ce cas il conduit à la conséquence que les *différentiants* (c'est-à-dire les sous-invariants) de

$$(a_0, a_1, a_2, \ldots)(x, 1)^{\infty}$$

sont des *fonctions symétriques non unitaires* des racines de l'équation

$$a_0 + a_1 x^{-1} + \frac{a_2}{1.2} x^{-2} + \frac{a_3}{1.2.3} x^{-3} + \ldots = 0$$

et *vice versâ*. Or il est évident que chaque fonction *symétrique non unitaire* d'un nombre *infini* de quantités n'est autre chose qu'une fonction des sommes de toutes les puissances de ces quantités au delà de la première. Voilà pourquoi j'ai attribué à M. Mac-Mahon, dans ce qui précède (pour le cas d'une équation dont le degré est infini), la connaissance du théorème que j'ai démontré dans toute sa généralité.

22.

SUR LE THÉORÈME DE M. BRIOSCHI, RELATIF AUX FONCTIONS SYMÉTRIQUES.

[*Comptes Rendus*, XCVIII. (1884), pp. 858—862.]

DANS la démonstration du théorème sur une correspondance algébrique, inséré dans les *Comptes rendus* de la semaine dernière [p. 163 above], j'ai eu occasion de considérer l'intégrale de l'équation

$$\left(a_0 \frac{d}{da_1} + a_1 \frac{d}{da_2} + \ldots + a_{n-1} \frac{d}{da_n}\right) \phi = 0.$$

Je me suis aperçu depuis que cette intégrale peut se déduire immédiatement du beau théorème de M. Brioschi, *sur les fonctions symétriques*, à savoir que :

$$r \frac{d\phi}{ds_r} + a_0 \frac{d\phi}{da_r} + a_1 \frac{d\phi}{da_{r+1}} + \ldots + a_{n-r} \frac{d\phi}{da_n} = 0.$$

On en tire cette conséquence immédiate que, si ϕ est une fonction des n premières sommes-puissances des racines de l'équation

$$a_0 x^n + a_1 x^{n-1} + \ldots = 0,$$

avec exclusion de la puissance $r^{\text{ième}}$, on aura

$$a_0 \frac{d\phi}{da_r} + \ldots + a_{n-r} \frac{d\phi}{da_n} = 0,$$

et conséquemment $F(s_1, s_2, \ldots, s_{r-1}, s_{r+1}, \ldots, s_n)$ sera l'équivalent complet de l'expression

$$\left(a_0 \frac{d}{da_r} + a_1 \frac{d}{da_{r+1}} + \ldots + a_{n-r} \frac{d}{da_n}\right)^{-1} . \, 0.$$

Dans le cas que j'ai considéré, $r = 1$, et nous avons trouvé

$$\left(a_0 \frac{d}{da_1} + a_1 \frac{d}{da_2} + \ldots + a_{n-1} \frac{d}{da_n}\right)^{-1} . \, 0 = F(s_2, s_3, \ldots, s_n).$$

On peut trouver aussi facilement l'intégrale complète de l'équation

$$\left(a_0 \frac{d}{da_1} + a_1 \frac{d}{da_2} + \ldots + a_{n-1} \frac{d}{da_n}\right)^{*i} \phi = 0,$$

où l'astérisque signifie qu'on doit prendre le *produit complet* de l'action de la forme linéaire agissant $i-1$ fois sur elle-même. Ainsi, par exemple,

$$\left(a\frac{d}{db}+b\frac{d}{dc}\right)^{*2} \text{signifie } a^2\left(\frac{d}{db}\right)^2+2ab\frac{d}{db}\frac{d}{dc}+b^2\left(\frac{d}{dc}\right)^2+a\frac{d}{dc}.$$

On trouvera sans difficulté que la valeur de cette intégrale est

$$F+s_1F_1+s_1{}^2F_2+\ldots+s_1{}^{i-1}F_{i-1},$$

où chaque F est une fonction exclusivement de $s_2, s_3, \ldots, s_n$.

Conséquemment le $i^{\text{ième}}$ coefficient d'un covariant quelconque de

$$(\alpha_0, \alpha_1, \ldots, \alpha_n)(x, y)^n$$

peut être mis sous cette forme, si l'on se sert de s_ω pour exprimer la somme des $\omega^{\text{ièmes}}$ puissances des racines de

$$x^n+\alpha_1x^{n-1}+\frac{\alpha_2}{1.2}x^{n-2}+\frac{\alpha_3}{1.2.3}x^{n-3}+\ldots=0.$$

En effet, en écrivant $\dfrac{s_1}{n}=s$, tout covariant de degré arbitraire ν appartenant à ce quantic sera de la forme

$$[u_0,\ (u_0, u_1 \between s, 1),\ (u_0, u_1, u_2 \between s, 1)^2,\ (u_0, u_1, u_2, u_3 \between s, 1)^3, \ldots](x, y)^\nu,$$

où, en général,

$$u_{\theta+1}=\frac{du_\theta}{ds_2}v_2+\frac{du_\theta}{ds_3}v_3+\ldots+\frac{du_\theta}{ds_n}v_n,$$

v_ω étant une fonction exclusivement de ω, n; $s_2, s_3, \ldots, s_n$ du poids $\omega+1$. J'ajoute encore cette observation que tout différentiant (c'est-à-dire *sous-invariant* ou *seminvariant*) d'un système de i quantics des degrés $m, \mu, \ldots, M$ sera fonction exclusivement de $s_2, s_3, \ldots, s_m$; $\sigma_2, \sigma_3, \ldots, \sigma_\mu, \ldots, S_2, S_3, \ldots, S_M$ et de $i-1$ fonctions linéaires indépendantes de la forme

$$ls_1+\lambda\sigma_1+\ldots+LS_1,$$

soumises à la condition que $l+\lambda+\ldots+L=0$.

Je ne sais s'il vaut la peine de dire, comme conclusion, qu'en combinant le théorème de M. Brioschi avec le mien sur les puissances (*avec astérisque*) on trouve, pour l'équation

$$\left(a_0\frac{d}{da_1}+a_1\frac{d}{da_2}+a_2\frac{d}{da_3}+\ldots\right)^i\phi=0$$

(où le i est *sans astérisque*), l'intégrale partielle

$$\phi=F+F_1s_1+F_2s_1{}^2+\ldots+F_{i-1}s_1{}^{i-1},$$

où chaque F est une fonction arbitraire de $s_{i+1}, s_{i+2}, \ldots, s_n$.

En effet, cette expression est l'intégrale complète du système formé par l'équation supposée conjointe avec les équations

$$\left(a_0\frac{d}{da_2}+\ldots\right)\phi=0,\quad \left(a_0\frac{d}{da_3}+\ldots\right)\phi=0,\quad \ldots,\quad \left(a_0\frac{d}{da_r}+\ldots\right)\phi=0.$$

On voit aussi facilement que l'intégrale de

$$\left(a_0 \frac{d}{da_r} + a_1 \frac{d}{da_{r+1}} + \ldots\right)^{*i} \phi = 0$$

est
$$\phi = U_0 + U_1 s_r + U_2 s_r^2 + \ldots + U_{r-1} s_r^{i-1},$$
où chaque U est une fonction arbitraire de $s_1, s_2, \ldots, s_{r-1}, s_{r+1}, \ldots s_n$.

On peut former un nombre infini de systèmes construits au moyen des opérateurs $\left(a_0 \frac{d}{da_r} + \ldots\right)$ dont on connaîtra d'avance les intégrales; ainsi, par exemple, le système de r équations

$$\left(a_0 \frac{d}{da_2} + \ldots\right)^i \phi = 0, \quad \left(a_0 \frac{d}{da_r} + \ldots\right) \phi = 0, \quad \ldots, \quad \left(a_0 \frac{d}{da_{2r}} + \ldots\right) \phi = 0$$

aura pour intégrale complète

$$\phi = U_0 + s_2 U_1 + s_2^2 U_2 + \ldots + s_2^{i-1} U_{i-1},$$

où chaque U représente une fonction arbitraire de $(s_1 s_3 s_5 \ldots s_{2i-1} s_{2i} \ldots s_n)$, en omettant celles des quantités $s_1, s_3, \ldots, s_{2i-1}$ dont les sous-indices excèdent n.

Pour indiquer le moyen de justifier ces énoncés, prenons comme exemple le cas des équations simultanées

$$(a_0 \delta a_1 + \ldots + a_{n-1} \delta a_n)^3 \phi = 0, \quad \text{ou} \quad E_1^3 \phi = 0,$$
$$(a_0 \delta a_2 + \ldots + a_{n-2} \delta a_n) \phi = 0, \quad \text{ou} \quad E_2 \phi = 0,$$
$$(a_0 \delta a_3 + \ldots + a_{n-3} \delta a_n) \phi = 0, \quad \text{ou} \quad E_3 \phi = 0.$$

On trouvera facilement qu'en général $E_1^3 = E^{*}{}_1^3 - 2E^{*}{}_1^1 E_2 + E_3$, de sorte que le système donné équivaut au système

$$E^{*}{}_1^3 \phi = 0, \quad E_2 \phi = 0, \quad E_3 \phi = 0.$$

Pour que ces équations soient satisfaites séparément, il faut et il suffit que ϕ soit respectivement de la forme

$$F(s_2 s_3 s_4 \ldots s_n) + s_1 F_1 (s_2 s_3 s_4 \ldots s_n) + s_1^2 F_2 (s_2 s_3 s_4 \ldots s_n),$$
$$G(s_1 s_3 s_4 \ldots s_n), \quad H(s_1 s_2 s_4 \ldots s_n).$$

Conséquemment, afin que les trois équations soient toutes satisfaites simultanément, la condition suffisante et nécessaire sera que ϕ soit de la forme

$$F(s_4 \ldots s_n) + s_1 F_1 (s_4 \ldots s_n) + s_1^2 F_2 (s_4 \ldots s_n),$$

laquelle est conséquemment l'intégrale complète du système donné. De même, on démontre facilement que l'intégrale complète des équations

$$(a_0 \delta a_1 + \ldots + a_{n-1} \delta a_n)^2 \phi = 0,$$
$$(a_0 \delta a_2 + \ldots + a_{n-2} \delta a_n) \phi = 0,$$
$$(a_0 \delta a_2 + \ldots + a_{n-2} \delta a_n)^2 \phi = 0$$

sera

$$\phi = F(s_3 s_5 s_6 \ldots s_n) + s_1 F_1 (s_3 s_5 s_6 \ldots s_n).$$

23.

SUR UNE EXTENSION DE LA LOI DE HARRIOT RELATIVE AUX ÉQUATIONS ALGÉBRIQUES.

[*Comptes Rendus*, XCVIII. (1884), pp. 1026—1030.]

On peut envisager la loi de Harriot comme une loi qui affirme la possibilité de décomposer d'une seule manière un polynôme en x dans un produit de facteurs linéaires composés avec les différences entre x et les racines du polynôme. En réfléchissant sur la cause de cette possibilité et la manière de la démontrer, on voit facilement que le même principe doit, avec une certaine modification, s'appliquer à toute équation en matrices d'un ordre quelconque dont les coefficients sont transitifs entre eux-mêmes, c'est-à-dire qui agissent les uns sur les autres exactement comme les quantités de l'Algèbre ordinaire, si chaque coefficient, par exemple, est une fonction rationnelle de la même matrice. On peut nommer les équations dont les coefficients satisfont à cette condition équations *monothétiques:* on remarquera que de telles équations forment une classe spéciale des équations que j'ai nommées *unilatérales* dans une Note précédente.

Pour fixer les idées, prenons comme exemple une équation monothétique du second degré en matrices binaires, laquelle peut toujours être ramenée à la forme

$$x^2 - 2px + Ap + B = 0.$$

En supposant que $p^2 - (\alpha + \beta)p + \alpha\beta = 0$ soit l'*équation identique* de p, on aura

$$x = \frac{p-\beta}{\alpha-\beta}\{\alpha \pm \surd(\alpha^2 - A\alpha - B)\} + \frac{p-\alpha}{\beta-\alpha}\{\beta \pm \surd(\beta^2 - A\beta - B)\}.$$

Faisons $\dfrac{p-\beta}{\alpha-\beta}\surd(\alpha^2 - A\alpha - B) = u$, $\dfrac{p-\alpha}{\beta-\alpha}\surd(\beta^2 - A\beta - B) = v$.

Alors les quatre racines de p seront

$$p+u+v,\quad p-u-v;\quad p+u-v,\quad p-u+v.$$

Disons r_1, r_2, r_3, r_4.

On trouve

$$(p-\beta)^2=(p-\beta)(p-\alpha)+(\alpha-\beta)(p-\beta)=(\alpha-\beta)(p-\beta),$$

et de même
$$(p-\alpha)^2=(\beta-\alpha)(p-\alpha),$$

de sorte que

$$u^2+v^2=\frac{p-\beta}{\alpha-\beta}(\alpha^2-A\alpha-B)+\frac{p-\alpha}{\beta-\alpha}(\beta^2-A\beta-B)$$
$$=(\alpha+\beta)p-\alpha\beta-Ap-B=p^2-Ap-B.$$

On a aussi $uv=0$ et conséquemment $(u+v)^2=u^2+v^2=(u-v)^2$. Donc

$$(x-r_1)(x-r_2)=(x-p)^2-(u+v)^2=x^2-2px+Ap+B,$$
$$(x-r_3)(x-r_4)=(x-p)^2-(u-v)^2=x^2-2px+Ap+B.$$

Or considérons le cas général d'une équation monothétique du degré n en matrices de l'ordre ω.

Cette équation (que j'écrirai $fx=0$), en vertu de ce que j'ai nommé la seconde loi de mouvement algébrique (c'est-à-dire la formule

$$\phi m=\Sigma\frac{(m-b)(m-c)\ldots(m-l)}{(a-b)(a-c)\ldots(a-l)}\phi a,$$

où $a, b, c, \ldots, l$ sont les *racines latentes* de la matrice m), aura n^ω racines qu'on peut représenter par les symboles composés

$$r_1,\quad r_2,\quad \ldots,\quad r_\omega,$$

où chaque r parcourt les valeurs 1, 2, 3, ..., n.

En réfléchissant sur la manière de démontrer le principe de Harriot, on arrivera facilement à la conclusion suivante : en prenant une combinaison quelconque de n symboles $r_1, r_2, \ldots, r_\omega$, de telle manière que chaque r parcoure toutes ses n valeurs, $R_1, R_2, \ldots, R_n$, on aura

$$fx=(x-R_1)(x-R_2)\ldots(x-R_n).$$

Ainsi on arrive au théorème suivant :

Toute fonction monothétique rationnelle et entière de x du degré n en matrices de l'ordre ω peut être représentée de $(1\,.\,2\,.\,3\,\ldots\,n)^{\omega-1}$ *manières différentes comme un produit de n facteurs linéaires dont chacun sera la différence entre x et une des racines de la fonction donnée.*

Telle est la loi de Harriot, étendue au cas des quantités multiirrationnelles.

Dans le cas de l'Algèbre ordinaire, $\omega=1$, et le nombre des décompositions de fx en facteurs, selon la formule, devient unique, comme il doit être.

De même, pour les quaternions, le nombre des décompositions d'une fonction monothétique du degré n en facteurs linéaires sera πn. Par exemple, si $n = 3$, les racines de fx peuvent être exprimées par les neuf symboles

$$\begin{array}{ccc} 0.0 & 0.1 & 0.2 \\ 1.0 & 1.1 & 1.2 \\ 2.0 & 2.1 & 2.2 \end{array}$$

La fonction (comme on le démontrera facilement) peut être mise sous la forme $x - 0.0$ multipliée par une fonction quadratique dont les racines seront des racines de fx, et conséquemment, par raison de symétrie, seront les quatre racines

$$\begin{array}{cc} 1.1 & 1.2, \\ 2.1 & 2.2; \end{array}$$

donc la fonction quadratique dont j'ai parlé sera égale à

$$(x-1.1)(x-2.2)$$

et à

$$(x-1.2)(x-2.1).$$

Ainsi il y aura deux décompositions de fx qui correspondent aux deux diagonales 0.0, 1.1, 2.2; 0.0, 1.2, 2.1, et de même il y aura des décompositions qui répondent aux diagonales 0.1, 1.2, 2.0; 0.1, 1.0, 2.2; 0.2, 1.0, 2.1; 0.2, 1.1, 2.0, de sorte que le nombre total est égal à 1.2.3.

De même, quand fx est monothétique et matrice du troisième ordre, on peut prendre les diagonales d'un cube. Par exemple, les racines de l'équation monothétique du second degré en matrices du troisième ordre peuvent être représentées par

$$\begin{array}{cccc} 0.0.0 & 0.0.1 & 0.1.0 & 0.1.1 \\ 1.1.1 & 1.1.0 & 1.0.1 & 1.0.0 \end{array}$$

et l'on aura les quatre décompositions

$$(x-0.0.0)(x-1.1.1); \quad (x-0.0.1)(x-1.1.0);$$
$$(x-0.1.0)(x-1.0.1); \quad (x-0.1.1)(x-1.0.0);$$

et de même, en général, pour le degré n, le nombre des diagonales (en se servant de ce mot dans le sens analytique, bien entendu) sera

$$(1.2.3 \ldots n)^2.$$

C'est ainsi qu'on trouve l'expression générale que j'ai donnée $(\pi n)^{\omega-1}$ pour le nombre des décompositions quand le degré est n et que l'ordre des matrices est ω.

En multipliant ensemble toutes les équations de décomposition, et en nommant v chacune des n^ω racines, on parvient à l'équation

$$\pi(x-v)^{\pi(n-1)^{\omega-1}} = (fx)^{\pi n^{\omega-1}};$$

donc, quoiqu'on ne puisse pas en général conclure que, si $X^j = Y^j$ (X et Y

étant des matrices), X est nécessairement égal à Y, il y a toute raison de croire qu'on pourra démontrer que, dans le cas actuel, on aura

$$\pi(x - v) = (fx)^{n^{\omega-1}}.$$

Ainsi la règle de Harriot se reproduira de nouveau sous la forme très peu modifiée qu'un polynôme (monothétique) en x (élevé à une puissance convenable) est égal au produit des différences entre x et toutes les racines en succession de ce polynôme.

On aura remarqué, dans ce qui précède, qu'en appliquant la seconde des trois lois du mouvement algébrique aux équations monothétiques, on a trouvé que le nombre des racines est n^{ω}, et conséquemment est n^2 dans le cas des quaternions, tandis que le nombre des racines pour la classe des équations en quaternions unilatérales (à laquelle les formes monothétiques appartiennent) est en général $n^3 - n^2 + n$ (*voir* le numéro d'avril 1884 du *London and Edinburgh Phil. Mag.*), de sorte qu'il y a une élimination $n(n-1)^2$ de racines en passant du cas général au cas particulier.

Il reste à examiner s'il n'est pas possible d'étendre la loi de Harriot aux équations unilatérales polythétiques. C'est ce que je vais étudier, mais sans cela, et en me bornant au cas monothétique, il me semble qu'en attribuant aux éléments des matrices des valeurs entières (simples ou complexes), comme le fait M. le professeur Lipschitz pour les quaternions, on voit s'ouvrir un nouveau champ immense de recherches arithmétiques fondées sur la loi fondamentale de Harriot généralisée de la manière indiquée dans ce qui précède.

24.

SUR LES ÉQUATIONS MONOTHÉTIQUES.

[*Comptes Rendus*, XCIX. (1884), pp. 13—15.]

DANS une Note précédente sur une extension de la loi de Harriot, j'ai eu occasion de considérer les équations dites *monothétiques* dont tous les coefficients sont des fonctions d'une seule matrice. Or il y a une circonstance très intéressante et importante relative aux équations de cette forme qu'il est essentiel de faire connaître ; car, à défaut d'une telle explication, le lecteur de la Note citée pourrait facilement être induit dans une erreur très grave. Voici en quoi consiste l'addition à faire.

Supposons que tous les coefficients d'une équation donnée soient des fonctions d'une seule matrice m. En appelant x l'inconnue, on peut résoudre l'équation en regardant x comme fonction de m, et l'on trouvera ainsi n^ω racines, en supposant que n soit le degré de l'équation et ω l'ordre de m. Ces racines seront parfaitement déterminées : mais on n'a nullement le droit de supposer qu'il n'y a pas d'autres racines qui ne sont pas des fonctions de m, qu'on peut nommer racines *aberrantes*, et un exemple, des plus simples qu'on puisse imaginer, suffira à démontrer que de telles racines, en effet, existent ; je me servirai, pour cet objet, de l'équation en quaternions (ou matrices binaires) $x^2 - px = 0$.

En effet, on connaît déjà, *a priori*, la possibilité de l'existence des racines aberrantes, car l'équation en matrices $x^n + q = 0$, quand q est une matrice *scalar* $\left(\text{comme si, par exemple, } q = \begin{Bmatrix} q & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & q \end{Bmatrix}\right)$, possède, on le sait, bien des racines qui ne sont pas scalars et conséquemment ne sont pas des fonctions de q, et, de plus, ces racines contiennent des constantes *arbitraires*. Comme on va le voir, c'est aussi le cas pour l'équation $x^2 - px = 0$, qui possède une seule constante.

Si l'on veut trouver ses racines normales (ou non aberrantes), on n'a qu'à résoudre cette équation comme une équation ordinaire, et l'on trouve ainsi

$$x = \tfrac{1}{2}\{p + \sqrt{(p^2)}\}.$$

En nommant r et s les racines latentes de p, on obtient par ma formule d'interpolation (pour ainsi dire), récemment citée par M. Weyr,

$$x = \frac{1}{2}\left(p \pm \frac{p-s}{r-s} r \pm \frac{p-r}{s-r} s\right),$$

c'est-à-dire $x = 0$, p, $\frac{r(p-s)}{r-s}$, $\frac{s(p-r)}{s-r}$, et il n'y a pas d'autres racines de ce caractère. Mais sortons de cette restriction arbitraire (produit de la paresse de l'esprit humain, qui se fatigue enfin en voyant sans cesse se reproduire des horizons nouveaux et inattendus), et posons hardiment

$$x = \begin{matrix} \alpha & \beta \\ \gamma & \delta \end{matrix}, \quad p = \begin{matrix} a & b \\ c & d \end{matrix},$$

où α, β, γ, δ sont les quantités à déterminer.

Puisqu'on fait abstraction des solutions $x = 0$, $x = p$, on sent, en vertu de la *troisième loi du mouvement algébrique*, que x et $x - p$ auront chacun un degré de nullité (car leur produit possède deux degrés); ainsi, si $\alpha + \delta = 0$, on aura

$$x^2 = 0,$$

donc aussi

$$px = 0,$$

et p sera aussi une matrice vide, c'est-à-dire qu'on aura

$$ad - bc = 0.$$

La solution pour ce cas (dont, dans ce qui suit, je veux faire abstraction) sera

$$x = \lambda \begin{Bmatrix} ac & -a^2 \\ a^2 & -ac \end{Bmatrix},$$

λ étant arbitraire.

Dans tout autre cas, en égalant la raison du second au troisième membre de x^2 avec la même pour px, on trouve sans difficulté que x sera de la forme

$$\begin{matrix} -\lambda(d-r) & \lambda b \\ \mu c & -\mu(a-r) \end{matrix}$$

où r et s sont les racines latentes de p, c'est-à-dire les racines de l'équation

$$r^2 - (a+d)r + ad - bc = 0.$$

Alors, en calculant x^2 et px, et en les égalant terme à terme, on obtient les quatre équations suivantes:

$$\begin{aligned} &\lambda(d-r)^2 + \mu bc = bc - a(d-r), \\ &b[\lambda(d-r) + \mu(a-r)] = -br, \\ &c[\lambda(d-r) + \mu(a-r)] = -cr, \\ &\lambda bc + \mu(a-r)^2 = bc - d(a-r). \end{aligned}$$

En écartant le cas spécial pour lequel $b=0$ et $c=0$, on voit (et c'est M. Franklin, de Baltimore, qui le premier s'est aperçu de cette conclusion capitale) que toutes ces équations seront satisfaites avec la seule supposition

$$\lambda(d-r)+\mu(a-r)+r=0,$$

de sorte qu'une constante reste parfaitement libre dans la solution aberrante de l'équation $x^2-px=0$.

Dans le cas où $p=\begin{matrix} a & 0 \\ 0 & d \end{matrix}$ on trouvera facilement les deux solutions déterminées

$$x=\begin{matrix} a & 0 \\ 0 & 0 \end{matrix} \quad \text{et} \quad x=\begin{matrix} 0 & 0 \\ 0 & d \end{matrix}.$$

Dans ses *Lectures sur les quaternions,* Hamilton n'a pas mis le doigt sur les cas véritablement singuliers des équations quadratiques unilatérales. La condition de singularité, c'est-à-dire de la présence de l'un ou de l'autre des cas où une ou plusieurs des trois paires de racines de l'équation $px^2+qx+r=0$ disparaissent ou deviennent indéterminées (c'est-à-dire affectées de constantes arbitraires), peut se résumer dans la seule équation $I=0$, où I est l'invariant quartique ternaire quadratique (en u, v, w) qui exprime le déterminant d'une matrice $up+vq+wr$.

25.

SUR L'ÉQUATION EN MATRICES $px = xq$.

[*Comptes Rendus*, XCIX. (1884), pp. 67—71; 115, 116.]

SOIENT p et q deux matrices de l'ordre ω.

Pour résoudre l'équation $px = xq$, on obtiendra ω^2 équations homogènes linéaires entre les ω^2 éléments de l'inconnue x et les éléments de p et de q, de sorte que, afin que l'équation donnée soit résoluble, les éléments de p et de q doivent être liés ensemble par une et une seule équation.

Mais, si l'*équation identique* en p est écrite sous la forme

$$p^\omega + Bp^{\omega-1} + Cp^{\omega-2} + \ldots + L = 0,$$

on aura apparemment, en vertu de l'équation $p = xqx^{-1}$,

$$xq^\omega x^{-1} + Bxq^{\omega-1}x^{-1} + Cxq^{\omega-2}x^{-1} + \ldots + L = 0$$

ou bien
$$q^\omega + Bq^{\omega-1} + Cq^{\omega-2} + \ldots + L = 0\,;$$

donc les ω racines de q seront identiques avec celles de p et, au lieu d'une seule équation, on aura en apparence (*au moins*) ω équations entre les éléments de p et de q.

Pour faire disparaître ce paradoxe, il n'y a qu'une seule supposition à faire : c'est que x, sous les suppositions faites, devient une matrice vide, car alors x^{-1} n'a plus une existence actuelle, et l'équation $p = xqx^{-1}$ n'aura pas lieu ; c'est ce qu'on va voir arriver *dans le cas général*, où $px = xq$.

Pour fixer les idées, supposons $\omega = 1$ et faisons

$$p = \begin{vmatrix} a & b \\ c & d \end{vmatrix}, \quad q = \begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix}, \quad x = \begin{vmatrix} \lambda & \mu \\ \nu & \pi \end{vmatrix}.$$

En égalant px à xq, on obtient les quatre équations simultanées et homogènes entre λ, μ, ν, π suivantes :

$$\begin{aligned} (a-\alpha)\,\lambda + c\mu - \beta\nu + 0\pi &= 0, \\ b\lambda + (d-\alpha)\,\mu + 0\nu - \beta\pi &= 0, \\ -\gamma\lambda + 0\mu + (a-\delta)\,\nu + c\pi &= 0, \\ 0\lambda + \gamma\mu + b\nu + (d-\delta)\,\pi &= 0, \end{aligned}$$

et conséquemment on aura*

$$b^2c^2 + \beta^2\gamma^2 - 2bc\beta\gamma - 2abcd - 2\alpha\beta\gamma\delta + (bc + \beta\gamma)(a + d)(\alpha + \delta)$$
$$- bc(\alpha^2 + \delta^2) - \beta\gamma(a^2 + d^2) + \alpha\delta(a^2 + d^2) + ad(\alpha^2 + \delta^2)$$
$$+ 2ad\alpha\delta + a^2d^2 + \alpha^2\delta^2 - (a + d)(\alpha + \delta)(ad + \alpha\delta) = 0,$$

ou, en écrivant $a + d = B,\ ad - bc = D,\ \alpha + \delta = C,\ \alpha\delta - \beta\gamma = F,$

$$(D - F)^2 + (B - C)(BF - CD) = 0\,;$$

c'est-à-dire, si R est le résultant de $X^2 - Bx + D$, $X^2 - Cx + F$, $R = 0$ sera la condition générale de la possibilité de satisfaire à l'équation $px = xq$.

Il est facile de faire voir que ce résultat peut être étendu au cas général où p et q sont des matrices de l'ordre ω : on n'a qu'à démontrer que si une des racines latentes de p est égale à une de q, l'équation $px = xq$ est résoluble; et de plus, sans que cette condition soit satisfaite, l'équation est irrésoluble. Soient donc $\lambda_1, \lambda_2, \ldots, \lambda_\omega$ les racines latentes de p et $\mu_1, \mu_2, \ldots, \mu_\omega$ de q et supposons que $\lambda_1 = \mu_1$, alors

$$(p - \lambda_1)\,x = x\,(q - \mu_1),$$

et l'on peut satisfaire à cette équation en écrivant

$$x = (p - \lambda_2)(p - \lambda_3)\ldots(p - \lambda_\omega)(q - \mu_2)(q - \mu_3)\ldots(q - \mu_\omega).$$

Conséquemment, si les racines latentes de p et de q sont les racines des deux formes algébriques $X^\omega + BX^{\omega-1} + \ldots + L$, $X^\omega + CX^{\omega-1} + \ldots + M$, quand R (le résultant de ces deux formes) s'évanouit, le résultant des ω^2 équations homogènes linéaires obtenues en égalant $px = xq$ s'évanouira ; mais R est indécomposable et du même degré (ω^2) que ce dernier résultant dans les éléments de p et q. Conséquemment les deux résultants (à un facteur numérique près) sont identiques : ce qui démontre que la condition $R = 0$ est non pas seulement nécessaire, mais de plus suffisante afin que $px = xq$ soit résoluble.

Pour ce qui regarde la valeur de x, posons $x = UV$, où

$$U = (p - \lambda_2)(p - \lambda_3)\ldots(p - \lambda_\omega)\,;\quad V = (q - \mu_2)(q - \mu_3)\ldots(q - \mu_\omega),$$

le seul fait que x contient U comme facteur ou que x contient V comme facteur suffit à constater que x n'est pas seulement vide, mais de plus possède au moins $\omega - 1$ degrés de nullité, c'est-à-dire que tous ses déterminants mineurs du second ordre sont des zéros.

Cela est la conséquence d'un théorème que j'ai démontré dans le *Johns Hopkins Circular*† relatif au degré de nullité des combinaisons des *facteurs latents* d'une matrice, dont le théorème relatif à l'*équation* dite *identique* de Cayley ou de Hamilton n'est qu'un cas particulier, ou pour mieux dire le cas extrême; seulement il faut y ajouter un théorème qui fait partie de ma troisième loi de mouvement algébrique, c'est-à-dire que le degré de nullité d'un facteur ne peut jamais excéder le degré de nullité du produit auquel il appartient.

[* The expressions for p, q in line 7 from the bottom of p. 176 should be interchanged; in the last line of p. 176, for $+\gamma\mu$ *read* $-\gamma\mu$.]

[† p. 134 above.]

Nous avons donc complètement résolu le paradoxe qui était à expliquer. Mais, sur-le-champ, une nouvelle contradiction surgit, car il semble que nous avons démontré que, dans *tout* cas sans exception, si $px=xq$, x est nécessairement une matrice vide, ce qui est évidemment faux, car on sait bien que, si, ω étant de l'ordre de p et de q, $q=\sqrt[\omega]{(1)}p$, alors, afin que l'équation $px=xq$ soit résoluble, il n'est jamais nécessaire que x soit vide. Ainsi, par exemple, pour les matrices binaires, l'équation $qx=xq$ est satisfaite quand x est une fonction quelconque de q, et l'équation $qx=-xq$ est résoluble, pourvu que q^2 soit *scalar*, en imposant deux conditions (dont une que son carré soit *scalar*) sur x. Pour lever cette contradiction, revenons au cas où $\omega=2$ et aux équations fondamentales

$$(a-\alpha)\lambda+c\mu-\beta\nu=0,$$
$$b\lambda+(d-\alpha)\mu-\beta\pi=0,$$
$$-\gamma\lambda+(a-\delta)\nu+c\pi=0,$$
$$-\gamma\mu+b\nu+(d-\delta)\pi=0.$$

Certes, si ces équations donnent des valeurs *déterminées* aux rapports λ, μ, ν, π, le raisonnement précédent rend certain que x doit être vide, c'est-à-dire que $\lambda\pi-\mu\nu=0$, mais cette conclusion devient fausse aussitôt que p et q sont pris tels que ces rapports deviennent indéterminés, ce qui arrive quand tous les premiers déterminants mineurs de la matrice

$$\begin{vmatrix} (a-\alpha) & c & -\beta & 0 \\ b & (d-\alpha) & 0 & -\beta \\ -\gamma & 0 & (a-\delta) & c \\ 0 & -\gamma & b & (d-\delta) \end{vmatrix}$$

s'évanouissent simultanément.

Dans ce cas, quoique la solution générale qui donne *x vide* tienne bon, rien n'empêche qu'il n'existe d'autres valeurs de x, c'est-à-dire de $\begin{matrix}\lambda & \mu \\ \nu & \pi\end{matrix}$, pour lequels cela n'est pas vrai.

La matrice écrite en haut doit posséder et possède, en effet, la propriété remarquable que, en supprimant une ligne horizontale quelconque et en nommant A, B, C, D les quatre déterminants mineurs de la matrice rectangulaire qui survient, affectés de signes convenables, la quantité $AD-BC$ contiendra le déterminant complet comme facteur. Il sera peut-être utile, avant de conclure, de donner un exemple d'un genre nouveau de subsistance de l'équation $px=xq$ avec une valeur finie du déterminant de x. Faisons donc

$$a-\delta=0, \quad d-\alpha=0, \quad bc-\beta\gamma=0,$$

on aura

$$(a-d)\lambda+c\mu-\beta\nu=0,$$
$$b\lambda-\beta\pi=0, \quad -\gamma\lambda+c\pi=0,$$
$$-\gamma\mu+b\nu+(d-a)\pi=0,$$

équations qui n'équivalent qu'à deux,

$$b\lambda - \beta\pi = 0, \quad (a-d)\lambda + (c\mu - \beta\nu) = 0,$$

et le déterminant de x, c'est-à-dire $\lambda\pi - \mu\nu$, aura en général une valeur finie.

Dans la dernière Note (insérée dans les *Comptes rendus**) qui roule sur l'équation en matrices binaires $x^2 - px = 0$, j'ai remarqué qu'en addition aux solutions normales

$$x = 0, \quad x = p, \quad x = r\frac{p-s}{r-s}, \quad x = s\frac{p-r}{s-r}$$

(où r, s sont les racines latentes de p), on a la solution indéterminée (due en grande partie à la sagacité de M. Franklin)

$$x = \begin{Bmatrix} -\lambda(d-r) & \lambda b \\ \mu c & -\mu(a-r) \end{Bmatrix}$$

avec la condition $\lambda(d-r) + \mu(a-r) + r = 0$. Évidemment on a aussi la solution tout à fait distincte

$$x = \begin{Bmatrix} -\lambda(d-s) & \lambda b \\ \mu c & -\mu(a-s) \end{Bmatrix}$$

avec la condition $\lambda(d-s) + \mu(a-s) + s = 0$; mais on doit noter que, quand on prend $\lambda = \mu$, on reprend les deux valeurs normales $x = r\frac{p-s}{r-s}$, $x = s\frac{p-r}{s-r}$; le fait curieux que, quand $b = 0$ et $c = 0$, les deux solutions aberrantes forment un troisième couple tout à fait déterminé a été déjà noté, et l'on peut y ajouter la remarque que si, en addition à $b = 0$, $c = 0$, on a aussi

$$a - d = 0,$$

alors l'indétermination reparaît à pas redoublé, la solution entière étant dans ce cas extra-spécialement constituée par une paire de solutions dont l'une et l'autre contiennent *deux* constantes arbitraires au lieu d'une seule.

Je dois ajouter que, dans le cas où i racines de p $(\lambda_1, \lambda_2, \ldots, \lambda_i)$ sont identiques avec i de q $(\mu_1, \mu_2, \ldots, \mu_i)$, l'équation

$$px = xq,$$

qui amène à

$$p^2x = xq^2, \ldots, p^ix = xq^i$$

et, par conséquent, à

$$(p-\lambda_1)\ldots(p-\lambda_i)\,x = x\,(q-\mu_1)\ldots(q-\mu_i),$$

sera satisfaite si l'on fait $x = UV$, où

$$U = (p-\lambda_{i+1})\ldots(p-\lambda_\omega), \quad V = (q-\mu_{i+1})\ldots(q-\mu_\omega),$$

[* p. 174 above.]

de sorte que x (en vertu du théorème déjà cité) aura au moins $\omega - \theta$ degrés de nullité, c'est-à-dire tous ses déterminants mineurs de l'ordre $\theta + 1$ s'évanouiront. Mais on sait, pour le cas où $\theta = \omega$ (et l'on a toute raison de croire pour le cas où θ a une valeur quelconque au-dessus de l'unité), qu'il existe pour des valeurs spéciales de p et de q des solutions singulières de l'équation $px = xq$, lesquelles (comme dans le cas de l'equation de Riccati) sont bien autrement intéressantes et beaucoup plus importantes que la solution générale.

On remarquera que, quand $\theta = \omega$, la solution générale disparaît, tandis que les solutions singulières pour des valeurs particulières de p et de q, ayant toutes les racines latentes de l'un identiques avec celles de l'autre, forment la base de la présentation des matrices sous la forme de quaternions, nonions, etc.

26.

SUR LA SOLUTION DU CAS LE PLUS GÉNÉRAL DES ÉQUATIONS LINÉAIRES EN QUANTITÉS BINAIRES, C'EST-À-DIRE EN QUATERNIONS OU EN MATRICES DU SECOND ORDRE.

[*Comptes Rendus*, XCIX. (1884), pp. 117, 118.]

SOIENT p, q deux matrices d'un ordre donné et servons-nous du symbole $p(\ )q$ pour signifier l'opérateur, lequel, appliqué à une autre matrice x du même ordre, donne pxq.

Alors, si l'on pose

$$p_1(\ )q_1 + p_2(\ )q_2 + \ldots + p_n(\ )q_n = \phi,$$

ϕx sera une matrice dont chaque élément sera une fonction linéaire des éléments de x; conséquemment, en supposant que les matrices p, q sont de l'ordre ω, on parvient ainsi à une matrice de l'ordre ω^2, et conséquemment ϕ sera assujetti à une équation identique de l'ordre ω^2; disons $F=0$.

Je vais donner la valeur de F pour le cas où $\omega = 2$, c'est-à-dire où F sera une fonction du quatrième degré. Supposons que P et P' sont deux quantics du second ordre dans les deux systèmes de variables $x_1, x_2, \ldots, x_n$; $\xi_1, \xi_2, \ldots, \xi_n$ contragredients. Alors, si l'on représente par $\dot{P}'$ ce que devient P' quand on écrit $\delta_{x_1}, \delta_{x_2}, \ldots, \delta_{x_n}$ au lieu de $\xi_1, \xi_2, \ldots, \xi_n$, $(\dot{P}')^i$. P^i sera un invariant du système donné pour toute valeur de i.

Considérons le cas où $P = ax^2 + bxy + cy^2$ et $P' = \alpha\xi^2 + \beta\xi\eta + \gamma\eta^2$. Dans ce cas, on trouvera que $\frac{1}{8}[(\dot{P}')^2 P^2 - 4(\dot{P}' . P)^2]$ sera identique avec le résultat de $ax^2 + bxy + cy^2$, $\gamma x^2 - \beta xy + \alpha y^2$, de sorte qu'on peut le nommer le *contra-résultant* des formes (a, b, c), (α, β, γ). Je nommerai donc, en général, l'invariant $\frac{1}{8}[(\dot{P}')^2 P^2 - 4(\dot{P}' . P)^2]$ le *quasi contra-résultant* des deux formes P, P' quand elles contiennent un nombre quelconque de variables.

Or, en revenant à l'expression ϕ, nommons P le déterminant de

$$u_1 p_1 + u_2 p_2 + \ldots + u_n p_n + \phi . v$$

et Q le déterminant de

$$u_1 q_1 + u_2 q_2 + \ldots + u_n q_n - v,$$

où ϕ, pour le moment, est traité comme une quantité ordinaire. J'ai trouvé que le quasi contra-résultant de P, Q, quand ϕ appartient à des matrices du second ordre (lequel sera une fonction biquadratique de ϕ), égalé à zéro, est l'équation identique cherchée en ϕ.

Il est probable, mais je n'en suis pas encore absolument convaincu, qu'une méthode analogue donnera l'équation identique de ϕ pour des matrices d'un ordre quelconque.

Si l'on suppose que les p et les q sont des quaternions, rien ne change avec l'exception que P et Q seront définis comme étant les modules (les *tensors* carrés) au lieu d'être les déterminants de $\phi v + \Sigma pu$, $-v + \Sigma qu$ respectivement.

Connaissant ainsi l'équation identique de ϕ, on peut résoudre immédiatement l'équation

$$\Sigma\,(pxq) = T,$$

car, en écrivant $p\,(\quad)\,q = \phi$, on a l'équation connue

$$\phi^4 + B\phi^3 + C\phi^2 + D\phi + E = 0,$$

et, conséquemment, en exceptant toujours le cas où $E = 0$ (dans lequel cas l'équation devient ou impossible ou indéterminée), on trouve

$$x = \phi^{-1}T = -\frac{D + C\phi + B\phi^2 + \phi^3}{E}T.$$

Par exemple, si l'équation donnée est $pxq + rxs = T$,

$$\begin{aligned}
\phi T &= pTq + rTs,\\
\phi^2 T &= p^2Tq^2 + prTsq + rpTqs + r^2Ts^2,\\
\phi^3 T &= p^3Tq^3 + p^2rTsq^2 + prpTqsq\\
&\quad + rp^2Tq^2s + pr^2Ts^2q + rprTsqs + r^2pTqs^2 + r^3Ts^3,
\end{aligned}$$

et, éventuellement, en ne se servant que des coefficients qui entrent dans les fonctions P et Q par le moyen de formules connues, on réduit x à une somme de multiples de termes de la forme

$$pT,\ rT,\ prT;\ pTq,\ rTq,\ prTq;\ pTqs,\ rTqs,\ prTqs,$$

et ainsi en général. Donc le problème de la résolution des équations linéaires est complètement résolu ; seulement il reste à traiter en détail le cas singulier où la matrice appartenant à ϕ est *vide.*

27.

SUR LES DEUX MÉTHODES, CELLE DE HAMILTON ET CELLE DE L'AUTEUR, POUR RÉSOUDRE L'ÉQUATION LINÉAIRE EN QUATERNIONS.

[*Comptes Rendus*, XCIX. (1884), pp. 473—476, 502—505.]

UN célèbre quaternioniste m'ayant demandé de lui expliquer la portée de ma solution de l'équation linéaire en matrices sur la solution du même problème en quaternions, il me semble désirable de donner explicitement le moyen de passer d'une solution à l'autre. Préalablement, il sera bon cependant de remarquer que, faute d'un examen suffisamment attentif de la forme du résultat obtenu ou plutôt indiqué par Hamilton (*Lectures on Quaternions*, pp. 559—561), on pourrait attribuer à sa solution une propriété qu'elle ne possède pas, celle de fournir le moyen de trouver la solution de l'équation linéaire en quaternions *sous une forme réduite* semblable à celle que fournit ma méthode: mais, en effet, l'examen d'un seul terme de m (voir au bas de la page 561), par exemple $SrJr^2$, suffit à montrer que le dénominateur m de Hamilton est du douzième degré dans les éléments des quaternions (b et a) de son équation $\Sigma bqa = c$ (p. 559), tandis que le degré pour la forme réduite n'est que huit. Il s'ensuit que le numérateur (si l'on avait la patience de le déduire des formules de Hamilton), aussi bien que le dénominateur obtenu par ce moyen, serait affecté d'un facteur étranger à la question, du quatrième degré, dans les éléments nommés.

J'ajoute qu'il est parfaitement possible de donner la valeur de x dans l'équation $\Sigma pxp' = T$ comme fonction seulement des p et p' et des coefficients des *deux formes associées* sans aucune irrationnalité. Car le déterminant du nivellateur $\Sigma p(\ )p'$, disons N, étant obtenu sous la forme $\Omega_2 + \sqrt{(\Omega_4)}$, le déterminant du nivellateur

$$\Sigma p(\ )p' + \begin{matrix} -1 & 0 \\ 0 & -1 \end{matrix}\ (\ )\ \begin{matrix} N & 0 \\ 0 & N \end{matrix}$$

(disons FN) sera aussi exprimé sous une forme semblable à celle-là, disons $\Phi_2 + \sqrt{(\Phi_4)}$.

Or, au lieu de l'équation identique $FN = 0$, on peut se servir d'un multiple quelconque de cette équation pour obtenir l'inverse de N comme fonction de puissances positives de N. Ainsi l'on peut, dans ce but, se servir de l'équation $\Phi_2^2 - \Phi_4^4 = 0$, au lieu de $FN = 0$, et, avec l'aide de cette équation, on obtiendra x exprimé en fonction des p et p' et de fonctions rationnelles des coefficients des deux formes associées ; mais alors, au lieu d'être obtenu sous sa forme la plus simple, son numérateur et son dénominateur contiendront un facteur commun qui sera une fonction du huitième degré des éléments des p et des p'.

Je passe à la règle pour traduire ma solution de l'équation en matrices $\Sigma pxp' = T$ en solution de cette même équation quand les p, les p' et le T, au lieu d'être matrices, sont donnés comme quaternions. Évidemment tout ce qui est nécessaire, c'est de connaître l'équation qui serait identique pour $\Sigma p(\ )p'$; je vais donner la règle pour l'obtenir.

Sous le signe Σ, je suppose compris $p, q, r, \ldots, p', q', r', \ldots$.

Écrivons la forme symbolique $[Nx + (p)y + (q)z + \ldots]^2$, disons X ; les coefficients de $xy, xz, \ldots$, symboliquement écrits, sont

$$2(p)N,\ 2(q)N,\ \ldots;$$

à $(p), (q), \ldots$ il *faut substituer* $Sp, Sq, \ldots$; le coefficient de y^2 est $(p)^2$ auquel il faut substituer Tp^2 ; finalement le coefficient de yz est $(p)(q)$, auquel il faut substituer $S(Vp.Vq)$*.

De même, on construit et l'on interprète la forme

$$[-x' + (p')y' + (q')z' + \ldots]^2$$

(disons X').

On calcule† la valeur de $\dot{X}'^2X^2 - 4(\dot{X}'X)^2$. Ce résultat (une fonction du quatrième degré en N) (disons ΩN) sera une partie de la fonction qui doit être identiquement zéro. Le reste de cette fonction (disons $64\Omega_1 N$) sera

$$[\Sigma S(VpVqVr)\,S(Vp'Vq'Vr')]\,N - \Sigma SpSp'\,S(VpVqVr)\,S(Vp'Vq'Vr'),$$

et je dis que

$$\Omega N + 64\Omega_1 N = 0$$

sera l'équation identique en N, et servira pour trouver la valeur de x, c'est-à-dire $N^{-1}T$ comme fonction du quaternion T, des quaternions $p, q, \ldots, p', q', \ldots$ et des symboles S, V, T ; de plus la valeur ainsi obtenue sera x *sous sa forme réduite.*

Il y a encore une petite observation à ajouter à mes remarques sur la solution de Hamilton de l'équation $\Sigma bqa = c$ (*Lectures*, p. 559). Il divise q en deux parties, le scalar w et le vecteur ρ.

C'est cette dernière quantité (ρ) qu'il exprime sous la forme $\frac{R}{m}$; alors $w = \frac{S(c) - S\eta'\rho}{\Sigma S(ab)}$, de sorte que, à défaut d'avoir recours à des réductions

[* See first note on p. 191 below.] [† See p. 181 above and p. 202 below.]

ultérieures, le dénominateur de q contiendra, non seulement le facteur étranger du quatrième degré dans les éléments des a et des b dont j'ai déjà parlé, mais encore le facteur étranger $\Sigma S(ab)$.

On remarquera que, dans cette solution, on aura des combinaisons des b avec des a et des fonctions quaternionistiques de ces combinaisons, tandis que, dans la solution infiniment plus simple que je donne du problème, il ne se trouve nulle part des mélanges de cette nature, mais seulement des fonctions quaternionistiques de combinaisons des a entre eux-mêmes et des b entre eux-mêmes. Le vice fondamental de la méthode de Hamilton, c'est la réduction du problème donné à un autre, où, au lieu de q, il n'entre que sa partie vectorielle. Néanmoins le travail de Hamilton (quoique sa raison d'être ne subsiste plus) méritera toujours d'être regardé comme un monument du génie de son grand et admirable auteur.

C'est là, pour la première fois dans l'histoire des Mathématiques, qu'on rencontre la conception de l'équation identique (voir *Lectures*, pp. 566, 567) qui est la base de tout ce qu'on a fait depuis et de tout ce qui reste à faire dans l'évolution de la Science vivante et remuante de la quantité multiple, c'est-à-dire l'*Algèbre universelle,* née à peu près 250 ans après l'organisation définitive de sa sœur aînée l'*Arithmétique universelle,* dans le Mémoire de M. Cayley sur les matrices, dans les *Philosophical Transactions,* vol. 148.

Dans une Note précédente, on a vu que dans la nouvelle et seule *bonne* méthode pour résoudre, par rapport à x, l'équation en quaternions

$$pxp' + qxq' + rxr' + sxs' + \ldots = \Gamma,$$

on fait trois opérations. La première, à laquelle on peut donner le nom de *nivellation,* consiste à trouver le nivellant, c'est-à-dire le déterminant de la matrice du quatrième ordre appartenant à un nivellateur donné du second ordre. La seconde, qu'on peut appeler *déduction,* consiste à obtenir l'équation identique, à laquelle un nivellateur correspond au moyen d'un autre nivellateur qu'on obtient du nivellateur donné en y adjoignant un couple de plus de la forme $-N(\ )\,\delta s$, ou, ce qui revient au même, le couple $\sqrt{(-N)}(\ )\sqrt{(-N)}$, où N est considéré comme un *scalar.* Finalement, on arrive à la dernière opération, que je nommerai *substitution et réduction,* et qui consiste à substituer à l'inverse du nivellateur sa valeur en fonction rationnelle du troisième ordre de lui-même, puis à faire des réductions dont je parlerai tout à l'heure.

Au moyen de ces opérations, on arrive à la valeur de l'inconnue de l'équation sous sa forme réduite la plus simple qu'elle puisse prendre.

Pour obtenir la forme de l'équation identique, voici ce que j'ai trouvé en appliquant la méthode indiquée dans la Note précédente.

Pour plus de simplicité, je me sers de la notation suivante, qui s'applique à des lettres quelconques, accentuées ou non, représentant des quaternions.

Je pose

$$Sp = (p), \quad Tp^2 = p_2, \quad S(VpVq) = (pq), \quad S(VpVqVr) = (pqr).$$

Alors, en écrivant

$$p(\;)p' + q(\;)q' + r(\;)r' + s(\;)s' + \ldots = N,$$

on aura

$$\begin{aligned}
&N^4 - 4\Sigma(p)(p')N^3 + \Sigma[4(p)^2p'_2 + 4(p')^2p_2 - 2p_2p'_2]N^2 \\
&\quad - \Sigma\{4(p)(p')p_2p'_2 \\
&\qquad + 8[(p)(q')(pq).p'_2 + (p')(q)p'q'.p_2] - 4(p)(p')q_2q'_2 \\
&\qquad + 4[(q)(p')p_2q'_2 + (p)(q')p'_2q_2] - 8pp'(qr)(q'r') \\
&\qquad + 8[(p)(q')(qr)p'r' + (p')(q)(q'r')(pr)] + 8\Sigma(pqr)(p'q'r')\}N \\
&\quad + \Sigma\{p_2{}^2p'_2{}^2 - 2p_2p'_2.q_2q'_2 \\
&\qquad + 4[p_2q_2(p'q')^2 + p'_2q'_2(pq)^2] - 4p_2p'_2pq.p'q' \\
&\qquad + 4p_2p'_2qr.q'r' + 8[p_2(qr)(p'q')(p'r') + p'_2(q'r')(pq)(pr)] \\
&\qquad + 8[pq.rs.p'r'.q's' + p'q'.r's'.pr.qs] - 8(p)(p')(qrs)(q'r's')\} = 0,
\end{aligned}$$

où le dernier terme de la partie fonctionnelle de l'équation est le nivellant de N.

Quant à la substitution, si, dans l'équation précédente

$$N^4 - AN^3 + BN^2 + CN - D = 0\ *,$$

on remplace $N^{-1}\Gamma$ par la fraction

$$\frac{N^3\Gamma - AN^2\Gamma + BN\Gamma - C\Gamma}{D},$$

tous les termes du numérateur de cette fraction seront des multiples connus de la forme $P\Gamma P'$, où P est de l'une des formes suivantes : p^3; p^2q, pqp, qp^2; p^2, pq; p; ..., et où de même P' a des types semblables avec des lettres accentuées. Il ne reste plus qu'à réduire chaque P à sa forme la plus simple, c'est-à-dire à l'exprimer comme fonction linéaire de $1, p, q, pq - qp$, et de même pour P'. Alors le numérateur de x ne contiendra plus que des termes dont les arguments seront tous d'un des types suivants (je remplace la moitié de $pq - qp$ par $[pq]$) :

$$\Gamma,\ p\Gamma,\ \Gamma p',\ p\Gamma p',\ p\Gamma q',$$
$$[pq]\Gamma,\ \Gamma[p'q'],\ p\Gamma[p'q'],\ [pq]\Gamma p',\ [pq]\Gamma[p'q'];$$

il faut y ajouter le type $pqr\Gamma r'q'p'$, qui est déjà sous sa forme la plus simple et n'exige aucune formule de réduction.

* D est le déterminant de la matrice qui appartient au nivellement N. Quand $D = 0$, la solution de l'équation $Nx = \Gamma$ devient ou *idéale* (ce qui a lieu en général), ou (ce qui a lieu pour des cas particuliers) *actuelle*, mais indéterminée.

Je n'entreprendrai pas pour le moment de calculer les coefficients de ces arguments, mais j'indiquerai du moins les formules de réduction qui seules sont nécessaires pour effectuer ce calcul. Ce travail, bien digne d'attirer l'attention de quelque jeune géomètre, peut très probablement amener à des résultats qui, à l'aide d'une notation symbolique, pourront être présentés sous une forme d'une simplicité tout à fait inattendue et pour ainsi dire providentielle. J'en ai eu l'expérience pareille dans d'autres recherches du même genre, dans la solution de certains cas d'équations quaternionistiques du second degré.

Voici toutes les formules de réduction dont on aura besoin :

$$p^2 = 2(p)p - p_2, \quad p^3 = [4(p)^2 - p_2]p - 2(p)p_2,$$
$$pq = [pq] + (p)q + (q)p - (pq),$$
$$qp = -[pq] + (p)q + (q)p - (pq),$$
$$p^2q = 2(p)[pq] + 2(p)(q)p + (2p^2 - p_2)q - 2(p)(pq),$$
$$pqp = 4(p)[pq] + [8(p)(q) - 2(pq)]p$$
$$- [4(p)^2 + p_2]q - [2(q)p_2 + 4(p)(pq)];$$

dans les formules on peut, au lieu de $[pq]$, écrire $V(VpVq)$.

Remarque.— Quand un nivellateur devient *symétrique*, c'est-à-dire quand $p = p'$, $q = q'$, ..., alors les *deux formes associées* coïncident en une seule dont le nivellant devient un *invariant orthogonal*.

Qu'il me soit permis, avant de conclure, d'ajouter encore une petite réflexion sur l'importance de la question traitée ici. Elle constitue, pour ainsi dire, un canal qui, comme celui de Panama, sert à unir deux grands océans, celui de la théorie des invariants et celui des quantités complexes ou multiples : dans l'une de ces théories, en effet, on considère l'action des substitutions sur elles-mêmes, et dans l'autre, leur action sur les formes ; de plus, on voit que la théorie *analytique* des quaternions, étant un cas particulier de celle des matrices, cesse d'exister comme une science indépendante ; ainsi, de trois branches d'analyse autrefois regardées comme étant indépendantes, en voilà une abolie ou absorbée, et les deux autres réunies en une seule de substitution algébrique.

28.

SUR LA SOLUTION EXPLICITE DE L'ÉQUATION QUADRATIQUE DE HAMILTON EN QUATERNIONS OU EN MATRICES DU SECOND ORDRE.

[*Comptes Rendus*, XCIX. (1884), pp. 555—558, 621—631.]

HAMILTON, dans ses *Lectures on quaternions* (p. 632), a fourni un moyen de résoudre l'équation (en quaternions ou en matrices binaires) de la forme

$$x^2 - 2px + q = 0\,;$$

mais les circonstances les plus intéressantes de la solution ne se font pas voir dans sa méthode de traiter la question. Voici la manière analytique directe que nous employons pour obtenir x sous sa forme explicite.

On suppose $$x^2 - 2Bx + D = 0$$

l'équation identique pour x, où B et D sont des *scalars* à trouver.

En combinant ces deux équations en x, on obtient

$$2x = (p - B)^{-1}(q - D),$$

et, en supposant que la *forme associée* à [1], p, q, c'est-à-dire le déterminant de $\lambda + \mu p + \nu q$, soit

$$\lambda^2 + 2b\lambda\mu + 2c\lambda\nu + d\mu^2 + 2e\mu\nu + f\nu^2,$$

on aura*

$$4(d - 2bB + B^2)\,x_0{}^2 - 4(e - bD - cB + BD)\,x_0 + f - 2cD + D^2 = 0.$$

Conséquemment, en écrivant $u = B - b$, $v = D - c$,

$$d - b^2 = \alpha, \quad e - bc = \beta, \quad f - c^2 = \gamma,$$

et, en comparant cette équation avec l'équation donnée, on voit qu'on peut écrire

$$u^2 + \alpha = \lambda, \quad uv + \beta = 2\lambda(u + b), \quad v^2 + \gamma = 4\lambda(v + c).$$

De plus, puisque $p^2 - 2bp + d = 0$, on aura

$$x = \frac{(p - b + u)(q - c - v)}{2(b^2 - d - u^2)} = -\frac{(p - b + u)(q - c - v)}{2\lambda}.$$

[* The determinant of $2Bx_0 - D - 2x_0 p + q$ being zero, if x_0 is a latent root of x.]

En éliminant u, v entre les trois équations qui les lient avec b, c, α, β, γ, on trouvera l'équation bien remarquable

$$e^{\lambda(2\delta_c - \delta_d)} . I = 0,$$

où I est le discriminant de la forme associée donnée plus haut, c'est-à-dire

$$I = \begin{vmatrix} 1 & b & c \\ b & d & e \\ c & e & f \end{vmatrix} = df + 2bce - dc^2 - e^2 - fb^2,$$

de sorte que la quantité exponentielle symbolique représente une fonction cubique et donne lieu à une équation cubique en λ.

A chaque valeur de λ correspondent les deux valeurs $\pm\sqrt{(\lambda - \alpha)}$ de u et à chaque valeur de u (autre que $u = 0$) correspondra la seule valeur $2\lambda + \dfrac{(2\lambda + c)\,b - e}{u}$ de v.

Quand $u = 0$, $\lambda = \alpha = d - b^2$, et l'équation

$$v^2 - 4\lambda v + \gamma - 4\lambda c = 0$$

a ses deux racines finies. Donc, quand $u = 0$, il faut que $\dfrac{(2\lambda + c)\,b - e}{u}$ prenne la forme $\dfrac{0}{0}$, et à cette valeur de u (qu'on peut envisager comme deux valeurs de u réunies en une) correspondront pour v les deux valeurs données par l'équation quadratique ci-dessus.

Ainsi l'on voit qu'en général x a trois paires de valeurs déterminées et qu'aucune de ces valeurs ne cesse d'être *actuelle* et *déterminée* que pour le seul cas où l'une des trois valeurs de λ est égale à zéro, c'est-à-dire où I, l'invariant de la *pleine** forme associée à (p, q), s'évanouit.

Cela revient à dire que I est le critérium de la normalité de l'équation donnée.

Si l'on regarde p et q comme des quaternions, on aura

$$b = Vp, \quad c = Vq, \quad d = Tp^2, \quad e = SpSq - S(VpVq), \quad f = Tq^2.$$

Il est bien digne de remarque que $4I$ est identique avec $(pq - qp)^2$.

On peut démontrer que, si p et q sont des matrices d'un ordre quelconque, les racines de l'équation $x^2 - 2px + q = 0$ seront toujours (comme ici) associées en paires; car, si l'on écrit $x + x_1 = 2p$, on aura

$$x_1^2 - 2x_1 p + q = 0,$$

et conséquemment, si $p^\omega - \omega b p^{\omega-1} + \ldots = 0$ est l'équation identique connue en p et $x^\omega - \omega B x^{\omega-1} + \ldots = 0$ l'équation identique à trouver en x, à chaque valeur

* Nous avons déjà défini la *forme associée* au *corps* p, q, r, ... Par la *pleine* forme, on peut sous-entendre ce que devient la forme associée quand on adjoint au corps une matrice unitaire.

de $B-b$ correspondra une valeur égale de $b-B$, c'est-à-dire que l'équation pour trouver B sera de la forme $F(B-b)^2=0$.

En se servant de l'équation conjuguée (c'est-à-dire en x_1) dont la somme des racines sera évidemment la même que pour l'équation en x, on obtient immédiatement, dans le cas où p et q sont du second ordre, par le moyen de la formule

$$x=-\frac{(p+b-u)(q-c-v)}{2\lambda}$$

et de l'équation en λ, la valeur de Σx*.

Cette valeur sera $6[p+(2\delta_c-\delta_d)I^{\frac{1}{2}}]$, de sorte que la valeur moyenne d'une racine de l'équation $x^2-2px+q=0$ est p (la valeur moyenne pour le cas où p et q sont *scalars*), augmentée de $(2\delta_c-\delta_d)I^{\frac{1}{2}}$, où $I^{\frac{1}{2}}$ doit avoir le signe qui le rend égal à $\frac{1}{2}(pq-qp)$. De même on trouve

$$\Sigma x^2=2p\Sigma x-6q,$$

et ainsi la valeur moyenne de x^2 sera

$$2p^2-q+(4\delta_c-2\delta_d)I^{\frac{1}{2}}p,$$

et l'on peut trouver successivement, par la même méthode, la valeur moyenne d'une puissance quelconque de x. Les détails du calcul précédent, et encore d'autres propriétés de l'équation en x, seront donnés prochainement dans le *Quarterly mathematical Journal* ou quelque autre recueil mathématique. Ici on n'a voulu que produire les résultats principaux obtenus par notre méthode.

L'équation de Hamilton en quaternions ou en matrices binaires est celle que nous avons traitée dans une Note précédente. C'est l'équation

$$x^2+2qx+r=0.$$

Nous avons trouvé que la solution de cette équation dépend d'une équation cubique ordinaire en λ, à chaque valeur de laquelle correspondent deux valeurs de x, et qu'elle est normale ou régulière quand le dernier terme de cette équation diffère de zéro. L'équation est dite *régulière* ou *normale* quand sa solution dépend du nombre maximum de racines déterminées, c'est-à-dire de trois paires de racines déterminées; chaque paire est alors connue comme fonction de λ, q, r et des paramètres b, c, d, e, f qui dépendent de q

* On aura $$\Sigma x=-\Sigma\frac{(p-b+u)(q-c-v)}{2\lambda},$$

$$\Sigma(2p-x)=-\Sigma\frac{(q-c-v)(p-b+u)}{2\lambda}.$$

On retranche une équation de l'autre, on substitue pour $\Sigma\frac{1}{\lambda}$ sa valeur tirée de l'équation cubique en λ, et on écrit $pq-qp=2I^{\frac{1}{2}}$.

et r et sont définis au moyen du déterminant de $u+vq+wr$* qu'on a supposé être mis sous la forme

$$u^2+2buv+2cuw+dv^2+2evw+fw^2,$$

d'où

$$b=Sq, \quad c=Sr, \quad d=Tq^2, \quad f=Tr^2 e=SqSr-S(Vq\,.\,Vr)\text{*}.$$

Dans ce cas, on peut dire que la solution elle-même est régulière.

En nommant I l'invariant de la forme ternaire, écrite plus haut, c'est-à-dire en posant

$$I=df+2bce-b^2f-c^2d-e^2,$$

nous avons trouvé que l'équation en λ peut être mise sous la forme

$$e^{\lambda\Omega}I=0,$$

où

$$\Omega=2\delta_c-\delta_d,$$

c'est-à-dire qu'on aura

$$4\lambda^3+(4c-4d)\lambda^2+(4be-4cd+c^2-f)\lambda+I=0.$$

Ainsi, afin que la solution soit régulière, il faut et il suffit que I diffère de zéro†.

De là il suit que, dans le cas d'une équation régulière, deux x ne peuvent être égaux, à moins qu'ils n'appartiennent à la même paire ou bien que deux λ ne deviennent égaux; car x peut être exprimé comme une fonction linéaire de qr, q, r, 1, dans laquelle le coefficient de qr est $-\dfrac{1}{2\lambda}$.

Donc, si deux des x sont égaux sans que deux λ le soient, une équation linéaire subsistera entre pq, p, q, 1, mais dans ce cas nous avons trouvé ailleurs que $I=0$, et la solution cesse d'être régulière.

Nous allons pour le moment nous borner au cas où l'équation est régulière, et conséquemment nous n'aurons qu'à considérer les cas où il y a égalité ou entre deux racines de λ ou bien entre deux valeurs de x qui correspondent à la même valeur de λ.

Si l'on suppose que deux valeurs de λ soient égales, il en résultera que deux des paires de valeurs de x deviendront identiques, de sorte qu'une seule condition suffira à réduire le nombre des racines distinctes de 6 à 4, c'est-à-

* Par un oubli très regrettable nous avons pris, dans une Note précédente, pour le coefficient de $2xy$ dans la forme associée à

$$(up+vq+wr+\ldots),$$

$S(VpVq)$ au lieu de sa vraie valeur, $\quad SpSq-S(VpVq)$,

et de même pour les autres coefficients des termes mixtes, de sorte que le calcul du déterminant du *nivellateur* $\Sigma p(\quad)p'$ dans la Note sur l'achèvement de la solution de l'équation linéaire en quaternions est erroné et a besoin d'être fait de nouveau.

† Conséquemment, quand l'équation est régulière, ni q ni v ne peut devenir zéro; car, dans l'un et l'autre de ces deux cas, $I=0$; aussi, pour la même raison, r ne peut pas être une fonction de q.

dire que les valeurs de x, qui, en général, sont de la forme $m, m'; n, n'; p, p'$, deviendront de la forme $m, m'; n, n'; n, n'$.

Au lieu de calculer directement le discriminant de l'équation en λ, qui donnera un résultat très compliqué, nous allons montrer qu'on peut substituer le discriminant de la forme très simple biquadratique

$$\left(1,\ b,\ \frac{c+2d}{3},\ e,\ f\right)(r,\ s)^4.$$

Mais préalablement il sera utile d'opérer une transformation linéaire sur l'équation en λ.

Écrivons $\lambda = \mu + d$; l'équation en μ sera

$$4\mu^3 + 4(c+2d)\mu^2 + [(c+2d)^2 + 4be - f]\mu + 2b(c+2d)e - b^2 f - e^2 = 0.$$

On voit donc que le discriminant qu'on veut calculer est une fonction complète de $b,\ c+2d,\ e,\ f$.

Nous avons trouvé $u^2 = \lambda - d + b^2$, c'est-à-dire $\mu + b^2$. On aura donc

$$\begin{aligned}&4u^6 + 4(c+2d-3b^2)u^4\\ &\quad + [12b^4 - 8(c+2d)b^2 + (c+2d)^2 + (4be - f)]u^2\\ &\quad - [2b^3 - b(c+2d) + e]^2 *.\end{aligned}$$

Dans l'équation donnée, substituons $x + \epsilon$, où ϵ est un infinitésimal (*scalar* si l'on parle de quaternions ou représentant la matrice $\begin{matrix}\epsilon & 0\\ 0 & \epsilon\end{matrix}$ si l'on parle de matrices); alors p sera augmenté par ϵ et q par $2\epsilon p$, et ainsi $(\lambda + \mu p + \nu q)$ deviendra $(\lambda + \epsilon\mu) + (\mu + 2\epsilon\nu)p + \nu q$, de sorte qu'en désignant le discriminant cherché par D, l'accroissement de D est nul quand λ et μ deviennent $\lambda + \epsilon\mu,\ \mu + \epsilon\nu$ simultanément, c'est-à-dire quand la forme ternaire en u, v, w devient

$$\begin{aligned}u^2 + 2(b+\epsilon)uv + 2(c+2\epsilon b)uw + (d+2\epsilon b)v^2&\\ + (2e + 2\epsilon c + 4\epsilon d)vw + (f + 4\epsilon e)w^2&.\end{aligned}$$

Donc $$[a\delta_b + 2b\delta_c + 2c\delta_d + (c+2d)\delta_e + 4e\delta_f]D = 0.$$

Écrivons $c + 2d = 3m$. On sait que D est une fonction complète de b, m, e, f, de sorte que, par rapport à D (comme opérande), $\delta_c + \delta_d = \delta_m$; ainsi, en écrivant $1 = a$, on aura

$$(a\delta_b + 2b\delta_m + 3m\delta_e + 4e\delta_f)D = 0.$$

D sera donc ou un invariant ou un sous-invariant de la forme biquadratique (a, b, m, e, f).

* u sera la partie scalar de x si l'équation est donnée sous la forme quaternionique, ou bien la moitié de la somme du premier et du quatrième élément de x si l'équation est donnée entre des matrices. Hamilton a trouvé l'équation équivalente à celle donnée pour u dans le texte; mais, dans sa formule, les coefficients sont exprimés sous une forme compliquée et assez difficile à débrouiller.

Mais, en faisant attention à l'équation en μ, on voit que D sera de l'ordre 6 dans les coefficients et du poids 12; il est donc un invariant et une fonction linéaire de s^3 et t^2 (où s et t sont les deux invariants irréductibles) de la forme biquadratique.

En nommant Δ le discriminant de cette forme, on a

$$\Delta = s^3 - 27t^2,$$

dont une partie sera $$f^3 - 27b^4 f^2;$$

mais on voit, par l'examen de l'équation en μ, qu'une partie de D sera

$$16b^4 f^2 - \frac{16 f^3}{27}$$

et, conséquemment, $$D = -\tfrac{16}{27}\Delta.$$

Il s'ensuit que la condition nécessaire et suffisante pour l'égalité de deux des racines de l'équation donnée avec deux autres est tout simplement $\Delta = 0$, comme nous l'avons déjà énoncé.

Cherchons la condition pour laquelle les trois paires coïncideront toutes dans une seule paire; alors les trois racines de μ deviennent toutes égales, et l'on a non seulement

$$\Delta = 0,$$

mais encore $$(12m^2) - (9m^2 + 4be - f) = 0,$$

c'est-à-dire $$f - 4be + 3m^2 = 0 \quad \text{ou} \quad s = 0.$$

Donc les conditions nécessaires et suffisantes, pour qu'il n'y ait que deux racines distinctes chacune, prises trois fois dans la solution de l'équation donnée, seront

$$s = 0, \quad t = 0.$$

On peut aussi demander quelle est la condition ou plutôt quelles sont les équations de condition pour que deux racines de la même paire soient égales.

Dans ce cas, nous avons trouvé que $u = 0$; cela exige que le dernier terme dans l'équation à u^2 devienne zéro. On aura donc, en vertu de l'équation en u^2,

$$ae - 3bm + 2b^3 = 0,$$

c'est-à-dire que le sous-invariant gauche ou bien le premier coefficient du Hessien à la forme biquadratique s'évanouit. Mais cela ne suffit pas pour que les deux x d'une paire deviennent parfaitement identiques. Il faut aussi que les deux valeurs de v, qui correspondent à la valeur zéro de u, ou que les deux racines de l'équation

$$v^2 - 4\lambda(v + c) + \gamma = 0,$$

où $$\lambda = \alpha = d - b^2,$$

deviennent égales, c'est-à-dire que

$$\gamma + c^2 - (2\alpha + c^2) = 0,$$

ou bien, puisque $\gamma = f - c^2$, que

$$f - (3m - 2b^2)^2 = 0\,;$$

à cette équation il faut joindre l'équation déjà trouvée

$$ae - 3bm + 2b^3 = 0\,;$$

le système de ces deux équations exprime la condition de la coïncidence des deux x d'une paire. Quoique $f - (3m - 2b^2)^2 = 0$ ne soit pas en elle-même un sous-invariant, les deux équations ci-dessus constituent (comme elles doivent le faire) un *plexus* sous-invariantif; car on trouvera

$$(a\delta_b + 2b\delta_m + 3m\delta_e + 4e\delta_f)\,[af - (3am - 2b^2)^2] = 4\,(ae - 3bm + 2b^3) = 0.$$

En effet, puisque $f - (3m - 2b^2)^2$ ne diffère de $f - 9m^2 + 2abe + 6b^2m$ (le second coefficient du Hessien) que par $-\,2b\,(ae - 3bm + 2b^3)$, on peut substituer, pour le plexus écrit plus haut, le plexus $H_1 = 0$, $H_2 = 0$, où H_1, H_2 sont le premier et le second coefficient du Hessien de la forme quadratique.

Or il est facile de démontrer que, quand dans la forme $(a, b, m, e, f)(x, y)$ a n'est pas zéro, mais que les deux premiers coefficients du covariant irréductible gauche le sont, le covariant s'évanouit complètement*, et la forme biquadratique a deux paires de racines égales.

On sait aussi que, quand les deux invariants irréductibles s'évanouissent, il y a trois racines égales, et, quand en même temps les deux invariants et le covariant gauche s'évanouissent, toutes les racines de la biquadratique sont égales.

Ainsi on voit que les seuls cas d'égalité possibles entre les racines de l'équation quadratique donnée, quand sa solution est régulière, correspondent aux quatre cas d'égalité entre les racines de la biquadratique ordinaire qui s'y est associée.

En prenant les quatre cas: 1° ou la quadratique a deux racines égales; 2° ou elle a deux paires de racines égales; 3° trois racines égales; 4° toutes ses racines égales; alors la quadratique donnée aura, dans le premier cas, deux paires de racines égales; dans le deuxième, quatre racines égales; dans le troisième, trois paires de racines égales, et dans le dernier cas toutes ses racines seront égales.

Quant au rapport de la biquadratique binaire à la forme ternaire quadratique, on passe de la seconde à la première, en se servant de la substitution dont s'est servi notre très honoré collègue, M. Darboux, dans sa belle Note sur la résolution de l'équation biquadratique (*Journal de Liouville*, t. XVIII. p. 220). On n'a qu'à faire $x = u^2$, $y = 2uv$, $z = v^2$, et la forme ternaire passe dans la forme binaire biquadratique. On voit ainsi que les genres de solutions régulières de l'équation en quaternions donnée dépendent ex-

* Quand les deux premiers coefficients du covariant irréductible gauche d'une biquadratique binaire s'évanouissent, le discriminant s'évanouit nécessairement: nous avons trouvé que ce discriminant pris négativement égale 16 fois le produit des coefficients extrêmes, moins le produit du second et l'avant-dernier coefficient du covariant gauche.

clusivement de la relation entre la conique qui s'y est associée avec la conique absolue $y^2 - 4xz$. Dans le cas le plus général, les deux courbes se coupent en quatre points; dans les quatre autres cas, il y aura l'une ou l'autre des quatre espèces de contact entre les deux coniques.

Mais, de plus, on voit évidemment que cette idée des deux coniques peut être étendue à l'équation de Hamilton, même pour le cas où la solution devient irrégulière.

Dans ce cas, la forme ternaire, associée à l'équation $x^2 + qx + r$, perdra sa forme de conique et deviendra un système de deux lignes droites qui se croisent ou de deux lignes coïncidentes. Dans la première supposition, il y aura le cas où les deux droites toutes les deux coupent et les cas où l'une ou toutes les deux touchent la conique fixe; il y aura aussi les cas où la conique fixe passe par le point d'intersection des deux droites en les coupant toutes les deux ou en touchant une. Dans la seconde supposition, il y aura les deux cas où les droites coïncidentes coupent ou touchent la conique fixe.

Ainsi donc il nous paraît qu'on peut affirmer avec pleine confiance que, dans l'équation de Hamilton*, il y a exactement douze cas, ou au moins douze cas principaux, à considérer†. Nous devons cette méthode si simple

* Quant à l'équation plus générale $px^2+qx+r=0$, dans le cas où le discriminant ou le tenseur de p devient zéro et que, par conséquent, la forme ne rentre pas dans celle de Hamilton (puisqu'on ne peut plus diviser l'équation par p), il peut se présenter encore un grand nombre de cas singuliers que nous n'avons pas encore étudiés à fond.

† Cela donne lieu à une réflexion curieuse. Si l'on considère tous les genres de rapports qui peuvent avoir lieu entre une vraie conique et une conique variable et capable de dégénérer en n'excluant pas les deux cas où la conique variable coïncide avec l'autre ou s'évanouit tout à fait, le nombre de ces genres sera 14, qui est le nombre de doubles décompositions du nombre 4, savoir :

4: 3, 1: 2, 2: 2, 1, 1: 1, 1, 1, 1: 3 : 1 2, 1 : 1 1, 1, 1 : 1 2 : 2 1, 1 : 2 1, 1 : 1, 1
2 : 1 : 1 1, 1 : 1 : 1 1 : 1 : 1 : 1.

De même on trouvera facilement que, pour le cas de formes binaires, le nombre de genres semblables sera 6, car, ayant sur une ligne droite deux points fixes et deux points variables, ces derniers peuvent être distincts entre eux-mêmes en coïncidant avec un ou tous les deux ou avec ni l'un ni l'autre des deux premiers, ou bien ils peuvent être réunis dans un seul point qui peut coïncider ou ne pas coïncider avec un des points fixes, et finalement ils peuvent disparaître; or le nombre de décompositions doubles du nombre 3, c'est-à-dire

3: 2, 1: 1, 1, 1: 2 : 1 1, 1 : 1 1 : 1 : 1,

est aussi 6.

Mais nous avons démontré autrefois, dans le *Philosophical Magazine*, que pour le cas de deux formes quadratiques de n variables dont chacune reste générale, c'est-à-dire n'a pas le discriminant zéro, le nombre des genres de rapport est exactement le nombre de doubles décompositions du nombre n. C'est une question qui mérite d'être examinée, si cette identité entre le nombre de genres pour n variables dans le second cas avec celui pour le nombre $n-1$ dans le premier, reste vraie pour toute valeur de n. Une considération qui s'y oppose, c'est que, dans le premier cas, quand $(n-1=1)$ le nombre de genres, au lieu d'être 3 (le nombre de décompositions doubles de 2), n'est que 2, mais il peut arriver que pour ce cas (le cas d'une seule variable), la forme générale étant la même que la forme de coïncidence parfaite, ce genre doit compter pour deux, et ainsi la loi se maintiendra.

de dénombrement à la connaissance que nous avons acquise du Mémoire ci-dessus cité de M. Darboux*.

Mais ce qui plus est, on peut beaucoup simplifier, comme on va voir, la solution de l'équation quadratique $fx = px^2 + qx + r = 0$.

En regardant pour le moment x comme une quantité ordinaire, soient Fx le déterminant de la matrice $x^2p + xq + r$ et ϕx un quelconque des six facteurs quadratiques de Fx; alors $\phi x = 0$ sera l'équation identique d'une des racines de $fx = 0$, et ces deux équations, en éliminant x^2, donneront la valeur précise de cette racine†. De même nous ferons voir qu'en général, quel que soit le degré (n) de fx (fonction rationnelle entière et unilatérale de x), lequel, comme aussi chaque coefficient, est une matrice d'un ordre donné (ω) quelconque, en prenant le déterminant Fx de fx (où pour le moment on regarde x comme une quantité ordinaire), chaque facteur du degré ω de Fx sera la fonction identiquement zéro d'une des racines (prise négativement) de l'équation $fx = 0$, et réciproquement.

Ce beau théorème‡, *pulcherrima regula*, repose sur les considérations suivantes:

Soit $\phi\lambda$ le déterminant de $\lambda + x$; alors on peut démontrer facilement que $\phi x = 0$ sera l'équation identique de x.

Or soit $fx = 0$, alors $f(-\lambda) = f(-\lambda) - f(x)$ et conséquemment contiendra le facteur $x + \lambda$. Donc le déterminant de $f(-\lambda)$ contiendra le déterminant de $(\lambda + x)$, c'est-à-dire contiendra $\phi\lambda$, où $\phi x = 0$ est l'équation identique.

Ainsi ϕx (la fonction de x qui est identiquement zéro) ne peut qu'être un facteur du déterminant de $f(-x)$ pris comme si x était une quantité ordinaire. De plus, puisqu'en général ce déterminant sera une fonction irréductible de x, de sorte qu'on ne peut plus distinguer une racine d'avec une autre, tout facteur qu'il contient dont le degré est égal à l'ordre de x sera la fonction identiquement nulle d'une des racines de l'équation $fx = 0$.

* On doit remarquer que le discriminant de l'équation en λ ou μ ou u^2 est le même que celui de la biquadratique associée à l'équation donnée; en effet, l'équation en μ a pour racines $\frac{(\alpha+\beta)(\gamma+\delta)}{4}$, $\frac{(\alpha+\gamma)(\beta+\delta)}{4}$, $\frac{(\alpha+\delta)(\beta+\gamma)}{4}$, où α, β, γ, δ sont les racines de cette biquadratique; ainsi on peut dire que les six racines cherchées sont associées respectivement aux six côtés du quadrangle complet formé par les quatre points d'intersection de la conique appartenant aux coefficients de l'équation donnée avec la conique absolue $y^2 - 4xz$.

On comprend que la forme appartenant à p, q, r veut dire le déterminant de la matrice $xp + yq + zr$ qui est une courbe dont l'ordre sera toujours celui des matrices p, q, r.

† Ainsi on possède une méthode immédiate, et qui s'applique à tous les cas qui peuvent se présenter pour résoudre l'équation de Hamilton. L'analyse précédente suffit pour en donner une démonstration qui a été passée dans le texte.

‡ On peut donner à cet énoncé une autre forme, à savoir: *Toute racine latente de chaque racine de fx* (fonction rationnelle entière et unilatérale par rapport à x) *est une racine* (prise négativement) *du déterminant de fx* (où x est traité comme une quantité ordinaire) *et réciproquement chaque racine ainsi prise de ce déterminant est une racine latente d'une des racines de fx.*

Il paraît donc (s'il n'y a aucune erreur dans ce dernier raisonnement) que le nombre des racines de fx sera le nombre exact de combinaisons de $n\omega$ choses prises ω à ω ensemble, où n est le degré de fx en x et ω l'ordre des matrices qui paraissent là-dedans; conséquemment le nombre des racines sera

$$\frac{\pi n\omega}{\pi(n-1)\,\omega\,.\,\pi\omega}\,*;$$

ainsi, par exemple, le nombre des racines dans le cas d'une équation du degré n en quaternions sera $2n^2 - n$†.

Pour trouver ces racines, on n'a qu'à combiner les deux équations $fx = 0$ qui ne change pas, avec $\phi x = 0$, qui varie avec chaque combinaison des racines de Fx [c'est-à-dire le déterminant de $f(-x)$], et, en éliminant les puissances supérieures de x, on trouvera une équation linéaire qui sert à donner x sous la forme d'une fraction : par des procédés qui ne présentent nulle difficulté, cette fraction peut être ramenée (au moins pour le cas des matrices binaires) à la forme d'une autre fraction dont le dénominateur sera une fonction exclusivement des coefficients de la *forme* associée à l'ensemble des coefficients de l'équation donnée dont nous nous proposons d'essayer de trouver la valeur générale. Ce dénominateur sera toujours (comme dans le cas que nous avons traité en détail dans ce qui précède) le *criterium* de la *régularité* de l'équation donnée. Quand ce *criterium* s'évanouit (et pas autrement), quelques-unes des racines vont à l'infini, c'est-à-dire cessent d'être actuelles et deviennent purement conceptuelles.

En général, pour résoudre l'équation unilatérale du degré n et l'ordre ω, on n'aura besoin que de résoudre une équation ordinaire du degré $n\omega$. Si une racine de l'équation donnée est connue, on n'aura qu'à résoudre deux équations ordinaires des degrés ω et $(n-1)\omega$ respectivement. Dans le cas d'une équation quadratique, quand une racine est donnée, on peut trouver immédiatement l'équation identique d'une seule autre qui y est associée, et conséquemment en déterminer la valeur sans résoudre une équation d'un degré supérieur au premier. Quand deux racines de l'équation résolvante (celle du degré $n\omega$) sont égales, on a $\dfrac{\pi(n\omega-2)}{\pi(\omega-1)\,.\,\pi[(n-1)\,\omega-1]}$ paires de racines égales dans l'équation du degré n qui est à résoudre.

* Dans le cas le plus général d'une équation en x du degré n et de l'ordre ω par rapport aux matrices, on peut supposer un nombre indéfini de termes dans l'équation. Chacun de ces termes sera composé d'un nombre pas plus grand que n des x dont chacun sera suivi et précédé par une matrice multiplicatrice. En appliquant la méthode algébrique directe pour résoudre cette équation, on sera amené à un système de ω^2 équations du degré n chacune. Ainsi le nombre des racines sera en général n^{ω^2}.

† Cela démontre que le nombre 21 que nous avions trouvé pour le cas de $n=3$ dans le *Philosophical Magazine* (mai 1884) [p. 229 below] et la formule générale que nous avons basée là-dessus sont erronés ; la raison en est évidemment que l'ordre *apparent* du système d'équations qui nous a fourni ce résultat surpasse l'ordre *actuel* de 6 unités.

Nous n'avions pas discuté en détail ces équations, et ainsi cet abaissement du degré nous a échappé. C'est un point curieux qui reste à discuter.

Prenons comme exemple de l'application de la méthode l'équation en quaternions

$$q_3x^3 + q_2x^2 + q_1x + q_0 = 0.$$

La fonction résolvante sera

$$(3.3)x^6 + (3.2)x^5 + (3.1 + 2.2)\,x^4\,(3.0 + 2.1)x^3$$
$$+ (2.0 + 1.1)\,x^2\,(1.0)x + (0.0) = 0,$$

où en général $i.i$ et $i.j$ signifient

$$Tq_i^2, \quad 2\,[Sq_iq_j - S\,(Vq_iVq_j)]$$

respectivement.

Les quinze facteurs quadratiques de cette fonction égalés à zéro donneront chacun une équation quadratique à laquelle doit satisfaire une des quinze racines de l'équation donnée, et, en combinant séparément chacune de ces équations avec la cubique donnée, on peut éliminer x^3 et x^2 et obtenir ainsi quinze équations linéaires pour déterminer les quinze racines voulues.

29.

SUR LA RÉSOLUTION GÉNÉRALE DE L'ÉQUATION LINÉAIRE EN MATRICES D'UN ORDRE QUELCONQUE.

[*Comptes Rendus*, XCIX. (1884), pp. 409—412, 432—436.]

Ce qui intéresse le plus dans les résultats nouvellement acquis que j'ai l'honneur de présenter à l'Académie, c'est l'union ou bien l'anastomose dont ils offrent un exemple frappant et tout à fait inattendu entre les deux grandes théories de l'*Algèbre moderne* et de l'*Algèbre nouvelle*, dont l'une s'occupe des transformations linéaires, et l'autre de la quantité généralisée, de sorte qu'au même titre que Newton définit l'Algèbre ordinaire comme étant l'Arithmétique universelle, on pourrait très bien caractériser cette Algèbre-ci comme étant l'Algèbre universelle, ou au moins une de ses branches les plus importantes.

En général, un invariant de deux formes signifie une fonction de deux systèmes de coefficients qui reste invariable, à un facteur près, quand les deux systèmes des variables sont ou identiques ou assujettis à des substitutions semblables; mais rien n'empêche qu'on n'applique ce même mot au cas où les substitutions sont réciproques: ainsi, sans parler du cas de deux formes mixtes, on aura des invariants de deux formes données à mouvement semblable et des invariants à mouvement contraire; on peut très bien nommer ces derniers (comme titre distinctif) *contrariants*. C'est à une classe spéciale de contrariants que nous aurons affaire dans la solution de l'équation générale linéaire en matrices d'un ordre quelconque.

En supposant que chaque p et p' soit une matrice de l'ordre ω, l'opérateur qui contient i couples

$$p_1(\ )p'_1+p_2(\ )p'_2+\ldots+p_i(\ )p'_i$$

peut être nommé provisoirement un *nivellateur* de l'ordre ω et de l'étendue i, et on peut le caractériser par le symbole $\Omega_{\omega,i}$. Servons-nous toujours du symbole 0 pour signifier une matrice dont tous les éléments sont des zéros, et désignons par 1 (ou bien par υ indifféremment) une matrice dont tous les éléments sont zéro, à l'exception des éléments de la diagonale qui seront des unités: ce sont les matrices nommées *matrice nulle* et *matrice unitaire* respectivement.

J'ai déjà expliqué comment un nivellateur général, de l'ordre ω, donne naissance à une matrice de l'ordre ω^2: je nomme le déterminant de cette matrice le *déterminant du nivellateur**. Ces déterminants possèdent des propriétés tout à fait analogues à celles des déterminants des matrices simples; ainsi, par exemple, je démontre la propriété dont je me suis servi avec grand avantage dans les recherches actuelles, que le déterminant du produit de deux *nivellateurs* est égal au produit de leurs déterminants séparés, et que le déterminant d'une fonction rationnelle d'un nivellateur, disons $F\Omega$, est égal au résultant (par rapport à Ω regardé comme une quantité ordinaire) de $F\Omega$ et $I\Omega$, où $I\Omega = 0$ représente l'équation identique du degré ω^2 à laquelle Ω est assujetti.

En général, à un système ou *corps* de matrices $p_1, p_2, \ldots, p_i$ de l'ordre ω correspond un quantic de l'ordre ω, c'est-a-dire le déterminant de

$$x_1 p_1 + x_2 p_2 + \ldots + x_i p_i.$$

Je nomme les coefficients de ce quantic les *paramètres du corps.* Ces paramètres doivent être regardés comme des quantités connues. Ainsi, par exemple, si au *corps* p, q (deux matrices binaires) on adjoint la matrice unitaire v, et qu'on forme le déterminant de la matrice $x + yp + zq$, on obtiendra un quantic

$$x^2 + Bxy + Cxz + Dy^2 + Eyz + Fz^2,$$

où, si l'on regarde p, q comme des *quaternions,* on aura, dans le langage du grand Hamilton,

$$B = Sp, \quad C = Sq, \quad D = T^2 p, \quad F = T^2 q, \quad E = S(Vp \,.\, Vq).$$

Il résulte de cette définition qu'à chaque nivellateur $\Omega_{\omega,i}$ appartiennent deux quantics de l'ordre ω et avec i variables, dont l'un appartient au corps $p_1, p_2, \ldots, p_i$ et l'autre au corps $p'_1, p'_2, \ldots, p'_i$.

Si l'on connaît l'équation identique $I\Omega = 0$ à laquelle le nivellateur Ω obéit, on peut immédiatement, comme je l'ai déjà montré, résoudre l'équation $\Omega x = T$.

Mais il est très facile de voir que $I\Omega$ n'est autre chose que le déterminant du nivellateur $\Omega - \lambda v(\ )v$, quand dans ce résultat on substitue Ω à λ. Donc la question de la solution linéaire la plus générale est ramenée à ce seul problème:

Exprimer le déterminant d'un nivellateur en termes de quantités connues.

Or la première conclusion et la plus difficile à établir dans cette recherche, mais que j'ai enfin réussi à démontrer, c'est que ce déterminant est toujours

* Quelquefois ce déterminant sera nommé un *nivellant.*

une *fonction* entière, mais pas nécessairement rationnelle, des *coefficients des deux quantics* qui sont associés au nivellateur.

Cela étant convenu, on démontre avec une extrême facilité que ce déterminant est un *contrariant* du degré ω dans chaque système de coefficients des deux quantics associés.

Cela ne suffit pas ou peut ne pas suffire en soi-même à définir complètement le contrariant cherché; nommons, en général, ce contrariant le *nivellant* des deux quantics.

Supposons que $N_{x, y, \dots z, t}$ soit le nivellant pour deux quantics d'un ordre donné ω, et représentons par $N_{x, y, \dots z, 0}$ ce que ce nivellant devient quand on réduit à zéro tous les coefficients qui appartiennent aux termes dans les deux quantics qui contiennent t; alors il est facile de voir que

$$N_{x, y, \dots, z, 0} = N_{x, y, \dots z}.$$

Cette propriété seule est suffisante (avec l'aide d'un quelconque des opérateurs différentiels qui servent pour annuler un contrariant) pour préciser le contrariant (nivellant) dans le cas de deux quantics du second ordre, et c'est ainsi que j'ai obtenu la solution de l'équation linéaire pour le cas des matrices binaires donné dans la Note précédente. Or il est bien concevable que cette loi ne peut pas suffire à déterminer les paramètres arbitraires qui entrent dans le contrariant d'ordre (ω, ω) appartenant à deux quantics de l'ordre ω.

Mais il y a encore une autre loi (constituant par elle-même un très beau théorème) qui doit suffire surabondamment à cette fin.

C'est une loi qui établit une liaison entre les nivellants de deux systèmes de quantics contenant chacun le même nombre de variables, mais dont l'un est d'un ordre plus grand par unité que l'ordre de l'autre.

Supposons que N soit le nivellant de deux quantics de l'ordre ω,

$$F(x, y, \dots, z) \text{ et } G(x, y, \dots, z);$$

soit N' ce que devient N quand

$$F(x, y, \dots, z) = (lx + my + \dots + nz)\, F_1(x, y, \dots, z)$$

et

$$G(x, y, \dots, z) = (\lambda x + \mu y + \dots + \nu z)\, G_1(x, y, \dots, z);$$

alors je dis que, quand

$$l\lambda + m\mu + \dots + n\nu = 0,$$

le nivellant de (F_1, G_1) sera contenu comme facteur dans le nivellant modifié N'.

A l'aide de ces principes, je me propose de calculer les nivellants pour les degrés supérieurs au second. On voit par ce qui précède que la solution de l'équation linéaire $\Sigma pxp' = T$ sera alors connue en termes des p, des p', de T et des paramètres des deux corps $p_1, p_2, \dots, p_i, p'_1, p'_2, \dots, p'_i$, augmentés l'un et l'autre d'une matrice unitaire.

C'est dans les *Lectures*, publiées en 1844, que pour la première fois a paru la belle conception de l'équation identique appliquée aux matrices du troisième ordre, enveloppée dans un langage propre à Hamilton, après lui mise à nu par M. Cayley dans un très important Mémoire sur les matrices dans les *Philosophical Transactions* pour 1857 ou 1858, et étendue par lui aux matrices d'un ordre quelconque, mais sans démonstration; cette démonstration a été donnée plus tard per feu M. Clifford (*voir* ses œuvres posthumes), par M. Buchheim dans le *Mathematical Messenger* (marchant, comme il l'avoue, sur les traces de M. Tait, d'Édimbourg), par M. Ed. Weyr, par nous-même, et probablement par d'autres; mais les quatre méthodes citées plus haut paraissent être tout à fait distinctes l'une de l'autre.

Par le moyen d'une chaîne de matrices couplées (disons N), opérant non pas sur une matrice générale, mais sur une matrice x (disons du degré ω) d'une forme spéciale suivie par un autre opérateur V qui aura l'effet de réduire la matrice du degré ω de Nx (dont les éléments sont des fonctions linéaires des éléments de x) à une forme identique à celle de x, il est facile de voir qu'à l'opérateur composé VN on peut faire correspondre une matrice d'un *ordre quelconque* non supérieur à ω^2, et c'est ainsi virtuellement que Hamilton, à cause d'une transformation qu'il effectue sur l'équation linéaire générale, est tombé dans ses *Lectures* sur la matrice du troisième ordre, et ce n'est que dans les *Éléments* publiés en 1866 (après sa mort) qu'on trouve quelque allusion à l'équation identique pour les matrices du quatrième ordre.

On pourrait nommer l'opérateur composé VN, pour lequel l'équation identique est d'un degré moindre que ω^2, *nivellateur qualifié*, mais il est essentiel de remarquer que ces opérateurs ne posséderont pas les propriétés analogues à celles des matrices que possèdent ces nivellateurs purs dont il est question dans ma méthode. Comme exemple d'un nivellateur qualifié, on pourrait admettre que le x (matrice du deuxième ordre), sur lequel opère le N, aura son quatrième élément zéro, et que l'effet du V sera d'abolir le quatrième élément dans Nx, où l'on peut supposer (et cette supposition est, dans son essence, à peu près identique à la méthode des vecteurs de Hamilton) que le premier et le quatrième élément de x sont égaux, mais de signes contraires, et que l'effet de V est de substituer dans la matrice du second ordre $N(x)$ la moitié de la différence entre le premier et le quatrième élément au lieu du premier et, au lieu du quatrième, cette même quantité avec le signe algébrique contraire.

Évidemment un tel opérateur donnera naissance à une matrice et sera assujetti à une équation identique du troisième ordre. Avant de conclure, pour convaincre de la justesse de la formule importante

$$\tfrac{1}{8}[(\dot{P}')^2 P^2 - 4(\dot{P}'.P)^2] - \tfrac{1}{2}\sqrt{(I.I')}*,$$

* Pour rendre intelligible cette formule, il est nécessaire de dire que l'expression

$$\tfrac{1}{8}[(\dot{P}')^2 P^2 - 4(\dot{P}'.P)^2],$$

applicable au cas d'un nivellateur du second ordre à quatre couples de matrices, il sera bon d'en donner une démonstration parfaite *a posteriori*, ce qu'une transformation légitime rend très facile à faire. Remarquons que le déterminant du nivellateur du second ordre $\Sigma\,{}^{a\ b}_{c\ d}(\ )^{\alpha\ \beta}_{\gamma\ \delta}$ est le déterminant de la matrice suivante:

$$\begin{matrix} \Sigma a\alpha & \Sigma c\alpha & \Sigma a\beta & \Sigma c\beta \\ \Sigma b\alpha & \Sigma d\alpha & \Sigma b\beta & \Sigma d\beta \\ \Sigma a\gamma & \Sigma c\gamma & \Sigma a\delta & \Sigma c\delta \\ \Sigma b\gamma & \Sigma d\gamma & \Sigma b\delta & \Sigma d\delta \end{matrix}$$

laquelle contiendra dans le cas supposé 144 termes, puisque chaque Σ comprend 4 produits: mais, sans perdre en généralité, on peut prendre une forme de nivellateur dont le déterminant ne comprendra pas plus de 24 termes; car il est facile de démontrer que, si aux 4 matrices de gauche on substitue 4 fonctions linéaires quelconques, pourvu que sur les 4 de droite on opère une substitution contragrédiente à la substitution précédente, la valeur du déterminant ne subira nul changement. On peut donc supposer que les 4 matrices de gauche sont

$$\begin{matrix} 10 & 01 & 00 & 00 \\ 00 & 00 & 10 & 01 \end{matrix}$$

respectivement, et, si la formule est vérifiée dans cette supposition (vu que les *contravariants* des deux quantics associés ne sont pas affectés par les substitutions contragrédientes opérées sur les deux systèmes de matrices), elle

donnée dans la Note du 21 juillet [pp. 181, 184 above], a besoin d'une correction (dont je pensais avoir fait mention dans le texte): il fait lui ajouter la *racine carrée* d'un contrariant connue du quatrième degré (appartenant aux deux *formes associées*), laquelle sera une fonction rationnelle des éléments des matrices du nivellateur. Pour le cas d'un nivellateur à quatre couples de matrices, c'est la racine carrée du produit de I et I', les discriminants des deux formes associées prises séparément; en nommant les quatre matrices à gauche

$$\begin{matrix} a & b \\ c & d \end{matrix} \quad \begin{matrix} a' & b' \\ c' & d' \end{matrix} \quad \begin{matrix} a'' & b'' \\ c'' & d'' \end{matrix} \quad \begin{matrix} a''' & b''' \\ c''' & d''' \end{matrix},$$

la racine carrée de I sera égale au déterminant

$$\begin{vmatrix} a & b & c & d \\ a' & b' & c' & d' \\ a'' & b'' & c'' & d'' \\ a''' & b''' & c''' & d''' \end{vmatrix},$$

qu'on peut nommer le développant de ces quatre matrices; de même la racine carrée de I' sera égale au développant des quatre matrices correspondantes à droite, de sorte que le terme irrationnel dans la formule pour le nivellant à quatre couples de matrices est égal au produit de ces deux développants; dans le cas général, la partie *relativement* irrationnelle de la formule pour un nivellant sera égale à la somme de tous les produits de développants accouplés qu'on peut former en combinant quatre à quatre, ensemble, les couples de matrices qui en dépendent. Dans le cas où le nivellateur contient moins de quatre couples, la racine carrée disparaît entièrement de la formule pour le nivellant. Je nommerai $\dot{P}'.\,P$ et $(\dot{P}')^2 P^2$, ϑ_1 et ϑ_2 respectivement.

sera non pas seulement *vérifiée*, mais absolument *démontrée* pour les valeurs parfaitement générales des deux systèmes.

Avec ces valeurs des matrices gauches, la matrice écrite plus haut, en prenant

$$\begin{matrix}\alpha & \beta \\ \gamma & \delta\end{matrix} \quad \begin{matrix}\alpha' & \beta' \\ \gamma' & \delta'\end{matrix} \quad \begin{matrix}\alpha_1 & \beta_1 \\ \gamma_1 & \delta_1\end{matrix} \quad \begin{matrix}\bar\alpha & \bar\beta \\ \bar\gamma & \bar\delta\end{matrix}$$

pour les matrices à droite, devient

$$\begin{matrix}\alpha & \alpha_1 & \beta & \beta_1 \\ \alpha' & \bar\alpha & \beta' & \bar\beta \\ \gamma & \gamma_1 & \delta & \delta_1 \\ \gamma' & \bar\gamma & \delta' & \bar\delta\end{matrix}$$

dont je nommerai le déterminant Q.

De plus, le quantic à gauche deviendra $xt - yz$, et le quantic à droite

$$(\alpha\delta - \beta\gamma)\,x^2 + (\bar\alpha\bar\delta - \bar\beta\bar\gamma)\,t^2 + (\alpha'\delta' - \beta'\gamma')\,y^2 + (\alpha_1\delta_1 - \beta_1\gamma_1)\,z^2$$
$$+ (1\,.\,2)\,xy + (3\,.\,4)\,zt + (1\,.\,3)\,xz + (2\,.\,4)\,yt + (1\,.\,4)\,xt + (2\,.\,3)\,yz,$$

où $\qquad (1\,.\,2) = \alpha\delta' + \delta\alpha' - \beta\gamma' - \beta'\gamma, \quad (3\,.\,4) = \alpha_1\bar\delta + \delta_1\bar\alpha - \beta_1\bar\gamma - \gamma_1\bar\beta,$

....

Donc
$$\vartheta_1 = (\alpha\bar\delta + \bar\alpha\delta - \beta\bar\gamma - \bar\beta\gamma) - (\alpha'\delta_1 + \alpha_1\delta' - \beta'\gamma_1 - \beta_1\gamma'),$$
$$\tfrac{1}{4}\vartheta_2 = (\alpha\bar\delta + \bar\alpha\delta - \beta\bar\gamma - \bar\beta\gamma)^2 + (\alpha'\delta_1 + \alpha_1\delta' - \beta'\gamma_1 - \beta_1\gamma')^2$$
$$+ 2\,(\alpha\delta - \beta\gamma)\,(\bar\alpha\bar\delta - \bar\beta\bar\gamma) + 2\,(\alpha'\delta' - \beta'\gamma')\,(\alpha_1\delta_1 - \beta_1\gamma_1)$$
$$- (\alpha\delta' + \delta\alpha' - \beta\gamma' - \beta'\gamma)\,(\alpha_1\bar\delta + \delta_1\bar\alpha - \beta_1\bar\gamma - \bar\beta\gamma_1)$$
$$- (\alpha\delta_1 + \delta\alpha_1 - \beta\gamma_1 - \beta_1\gamma)\,(\alpha'\bar\delta + \delta'\bar\alpha - \beta'\bar\gamma - \bar\beta\gamma')$$
$$- (\alpha\bar\delta + \bar\alpha\delta - \beta\bar\gamma - \bar\beta\gamma)\,(\alpha'\delta_1 + \alpha_1\delta' - \beta'\gamma_1 - \beta_1\gamma'),$$

et $\sqrt{(I\,.\,I')}$ (pris avec le signe convenable) sera le déterminant de la matrice

$$\begin{matrix}\alpha & \beta & \gamma & \delta \\ \alpha' & \beta' & \gamma' & \delta' \\ \alpha_1 & \beta_1 & \gamma_1 & \delta_1 \\ \bar\alpha & \bar\beta & \bar\gamma & \bar\delta.\end{matrix}$$

En faisant les multiplications nécessaires, on trouvera que

$$\tfrac{1}{4}\vartheta_2 - \vartheta_1{}^2 - \sqrt{(I\,.\,I')} = 2Q,$$

ce qui démontre l'exactitude de la formule donnée pour un nivellateur du deuxième ordre à quatre couples de matrices.

D'ici à peu de temps, j'espère avoir l'honneur de soumettre à l'Académie la valeur du déterminant du nivellateur du troisième ordre à trois couples de matrices. Pour présenter l'expression générale de ce déterminant pour une matrice d'un ordre et d'une étendue quelconques*, il faudrait avoir une connaissance des propriétés des formes qui va beaucoup au delà

* C'est-à-dire pour résoudre l'équation linéaire en matrices dans toute sa généralité.

des limites des facultés humaines, telles qu'elles ne sont manifestées jusqu'au temps actuel et qui, dans mon jugement, ne peut appartenir qu'à l'intelligence suprême.

Post-scriptum.—Qu'on me permette d'ajouter une petite observation qui fournit, il me semble, une raison suffisante *a priori* pour le signe ambigu du terme $\sqrt{(I \,.\, I')}$ qui entre dans la formule donnée pour un nivellant (c'est-à-dire déterminant d'un nivellateur) du deuxième ordre.

Les déterminants d'un nivellateur et de son *conjugué* étant *identiques* en signe algébrique tout autant qu'en grandeur, ce n'est pas dans cette direction qu'on peut chercher l'origine de l'ambiguïté.

Mais, si, en se bornant aux matrices correspondantes d'un nivellateur *de la même espèce*, c'est-à-dire à main droite ou à main gauche du symbole ( ), on échange entre eux, dans chacune de ces matrices, le premier terme avec le quatrième et le deuxième avec le troisième, on verra facilement que le nivellant et en même temps les deux *quantics associés* restent absolument sans altération; mais, si l'on exécute l'une ou l'autre de ces substitutions séparément, alors, tandis que les deux quantics associés restent constants, le nivellant (quand son nivellateur possède plus de trois couples) subira un changement de valeur (et, pour l'une et l'autre substitution, le *même* changement), de sorte que pour les quatre positions qu'on peut assigner simultanément aux éléments des matrices de la même espèce sans changer en rien les quantics associés, le nivellant aura deux valeurs distinctes. Voilà, il me semble, l'explication suffisante et la véritable origine de l'ambiguïté dont il est question.

A peine est-il nécessaire de remarquer qu'on peut faire 4 autres dispositions semblables et simultanées des matrices à l'un ou l'autre côté du symbole ( ), dispositions qui donneront naissance à des nivellants identiques en valeur avec les deux dont j'ai parlé (c'est-à-dire deux à une valeur et deux à l'autre), et pour lesquelles les deux quantics associés seront sans autre changement que celui du signe algébrique.

En combinant les 24 dispositions semblables des matrices d'un côté d'un nivellateur donné avec les 24 de l'autré côté, on obtiendra un système de 576 nivellateurs corrélatifs dont les déterminants ne prendront que 3 paires de valeurs; de plus, les deux valeurs d'une quelconque de ces paires seront les racines d'une équation quadratique dont les coefficients seront des contrariants rationnels et entiers d'une des trois paires de formes quadratiques; mais le discriminant de ces trois équations sera le même *certainement* quand les nivellateurs du système seront formés avec quatre couples de matrices et *probablement* quel que soit le nombre de ces couples. Quand ce nombre est moindre que 4, le discriminant de ces trois quadratiques devient nul pour toutes les trois.

30.

SUR L'ÉQUATION LINÉAIRE TRINÔME EN MATRICES D'UN ORDRE QUELCONQUE.

[*Comptes Rendus*, XCIX. (1884), pp. 527—529.]

POUR résoudre l'équation trinôme $pxp' + qxq' + r = 0$ (où toutes les lettres désignent des matrices du même ordre ω) sous sa forme symétrique, on a besoin de connaître l'équation identique à un nivellateur de cet ordre à deux couples de matrices, ce qui équivaut virtuellement à connaître le déterminant d'un nivellateur à trois de ces couples. Mais, sans avoir recours à cette méthode générale, il existe, comme on va le voir, un moyen plus court et plus direct pour résoudre l'équation et exprimer x sous la forme essentiellement bonne d'une fraction réduite, si l'on est d'accord à se dispenser de la condition que le numérateur soit symétrique.

A cet effet, on peut multiplier l'équation, à volonté, ou par $q^{-1}(\ )p'^{-1}$ ou par $p^{-1}(\ )q'^{-1}$. Choisissons le premier de ces deux multiplicateurs et écrivons $q^{-1}p = \phi$, $q'p'^{-1} = -\psi$, $-q^{-1}rp'^{-1} = \mu$; alors on obtient l'équation $\phi x - x\psi = \mu$ (mais déjà avec une brèche de symétrie, par la raison du choix d'une entre deux choses pareilles). En multipliant cette équation par le nivellateur $\phi^i(\ ) + \phi^{i-1}(\ )\psi + \phi^{i-2}(\ )\psi^2 + \ldots + (\ )\psi^i$ (disons U_i) et en écrivant $U_i\mu = \mu_{i+1}$, on obtient la suite d'équations

$$\phi x - x\psi = \mu, \quad \phi^2 x - x\psi^2 = \mu_2, \quad \phi^3 x - x\psi^3 = \mu_3, \ldots, \quad \phi^\omega x - x\psi^\omega = \mu_\omega.$$

Soient $B_0, B_1, \ldots, B_\omega$ et $C_0, C_1, \ldots, C_\omega$ les coefficients des deux formes associées aux deux systèmes p, q et p', q' respectivement; alors, en vertu d'un théorème général en matrices*, on aura

$$C_\omega\psi^\omega + C_{\omega-1}\psi^{\omega-1} + \ldots + C_0 = 0, \quad B_0 - B_1\phi + \ldots + (-)^\omega B_\omega\phi^\omega = 0.$$

Avec l'aide de ces deux équations et de la suite précédente, on peut déduire une équation de l'une ou de l'autre des deux formes $Mx = N$ ou $xM = N$. Faisons le choix (qui amène encore une fois une brèche de symétrie) de la première.

On aura $(C_\omega\phi^\omega + C_{\omega-1}\phi^{\omega-1} + \ldots + C_1\phi + C_0)\,x = C_\omega\mu_\omega + C_{\omega-1}\mu_{\omega-1} + \ldots + C_1\mu$. Or, selon la théorie ordinaire d'élimination, on peut déterminer ϑ et H deux fonctions chacune du degré $(\omega - 1)$ en ϕ (traité comme une quantité ordinaire), telles que

$$\vartheta\,[B_0 - B_1\phi + \ldots + (-)^\omega B_\omega\phi^\omega] + H\,(C_\omega\phi^\omega + C_{\omega-1}\,\phi^{\omega-1} + \ldots + C_0)$$

* Ainsi, par exemple, si p, q sont des quaternions, on a

$$Tp^2\,(p^{-1}q)^2 - 2S\,(VpVq)\,(p^{-1}q) + Tq^2 = 0.$$

sera égal à R, le contre-résultant des deux formes associées à (p, q) et (p', q')* respectivement, et l'on aura

$$x = \frac{C_1 H\mu + C_2 H\mu_2 + \ldots + C_\omega H\mu_\omega}{R},$$

et ainsi x sera déterminé.

Si μ est zéro, alors, afin que x ne soit pas zéro, le R doit devenir zéro, comme nous avons déjà trouvé dans une Note précédente. En général, si R (le contre-résultant des deux formes adjointes à p, q et p', q' dans l'équation $pxp' + qxq' + r = 0$) s'évanouit, l'équation ne peut pas admettre une solution en même temps actuelle et déterminée; sans autres conditions, la solution deviendra *idéale*; avec conditions convenables, elle peut redevenir *actuelle*, mais contiendra (selon les circonstances) une ou plusieurs constantes arbitraires.

Hamilton, dans ses *Lectures*, a considéré l'équation trinôme pour les quaternions, mais il n'en a pas poussé la solution, c'est-à-dire la valeur de l'inconnue, à sa forme finale dans laquelle le dénominateur doit être un scalar (je dis *doit* être), parce que, ici comme dans toutes les équations en matrices, c'est le dénominateur de l'inconnue convenablement exprimé dont l'évanouissement est le *critérium* pour distinguer le cas où la solution est actuelle et déterminée d'avec les cas où elle devient ou idéale ou indéterminée.

En combinant le résultat ici obtenu avec celui de notre Note précédente, on voit qu'on est entré en pleine possession de la solution de l'équation $Nx = \Gamma$ dans les deux cas où le nivellateur N est de l'ordre 2 et d'une étendue quelconque ou bien de l'étendue 2 et d'un ordre quelconque.

Remarque. — On peut objecter que le numérateur de l'expression trouvée pour x dans l'équation trinôme contient des combinaisons de $q^{-1}p$, $q'p'^{-1}$, $q^{-1}rp'^{-1}$ et que, conséquemment, x pourrait devenir idéal à cause de l'évanouissement du déterminant de p' ou de q sans que le contre-résultant R s'évanouisse. Pour répondre à cette objection, soient D', Δ les déterminants de p' et de q^{-1}; alors, en se servant des équations identiques à p' et à q, on peut substituer pour leurs inverses des fonctions rationnelles de l'un et de l'autre divisées respectivement par D' et Δ, et alors le numérateur de x sera une quantité incapable de devenir infinie, tandis que son dénominateur sera R multiplié par des puissances de D' et de Δ; mais, vu qu'on peut représenter x tout aussi bien par une autre fraction dont le numérateur sera aussi incapable de devenir infini et dont le dénominateur sera R multiplié par des puissances de D' et de Δ (les déterminants de p et de q'), il est évident que ces deux fractions doivent toutes les deux admettre d'être simplifiées et que dans leurs formes réduites le dénominateur sera tout simplement R et qu'ainsi ce contre-résultant est le seul critérium pour distinguer le cas de l'actuel et déterminé d'avec le cas de l'idéal ou indéterminé.

* C'est-à-dire le *résultant* des fonctions multipliées par ϑ et H ci-dessus.

31.

LECTURES ON THE PRINCIPLES OF UNIVERSAL ALGEBRA.

[*American Journal of Mathematics*, VI. (1884), pp. 270—286.]

LECTURE I.

Preliminary Conceptions and Definitions.

Apotheosis of Algebraical Quantity.

A matrix of a quadrate form historically takes its rise in the notion of a linear substitution performed upon a system of variables or carriers; regarded apart from the determinant which it may be and at one time was almost exclusively used to represent, it becomes an empty *schema* of operation, but in conformity with Hegel's principle that the Negative is the course through which thought arrives at another and a fuller positive, only for a moment loses the attribute of quantity to emerge again as quantity, if it be allowed that that term is properly applied to whatever is the subject of functional operation, of a higher and unthought of kind, and so to say, in a glorified shape,—as an organism composed of discrete parts, but having an essential and undivisible unity as a whole of its own. *Naturam expellas furcâ, tamen usque recurret**. The conception of multiple quantity thus rises upon the field of vision.

At first undifferentiated from their content, matrices came to be regarded as susceptible of being multiplied together; the word multiplication, strictly applicable at that stage of evolution to the content alone, getting transferred by a fortunate confusion of language to the schema, and superseding, to some extent, the use of the more appropriate word composition applied to the reiteration of substitution in the Theory of Numbers. Thus there came into view a process of multiplication which the mind, almost at a glance, is able to recognize must be subject to the associative law of ordinary

* *Chassez le naturel, il revient au galop*, a familiar quotation which I thought was from Boileau, but my friend Prof. Rabillon informs me is from a comedy of Destouches (born in 1680, died 1754).

multiplication, although not so to the commutative law; but the full significance of this fact lay hidden until the subject-matter of such operations had dropped its provisional mantle, its aspect as a mere schema, and stood revealed as *bona-fide* multiple quantity subject to all the affections and lending itself to all the operations of ordinary numerical quantity. This revolution was effected by a forcible injection into the subject of the concept of addition, that is, by choosing to regard matrices as susceptible of being added to one another; a notion, as it seems to me, quite foreign to the idea of substitution, the *nidus* in which that of multiple quantity was laid, hatched and reared. This step was, as far as I know, first made by Cayley in his Memoir on Matrices, in the *Phil. Trans.* 1858, wherein he may be said to have laid the foundation-stone of the science of multiple quantity. That memoir indeed (it seems to me) may with truth be affirmed to have ushered in the reign of Algebra the 2nd; just as Algebra the 1st, in its character, not as mere art or mystery, but as a science and philosophy, took its rise in Harriot's *Artis Analyticae Praxis*, published in 1631, ten years after his death, and exactly 250 years before I gave the first course of lectures ever delivered on Multinomial Quantity, in 1881, at the Johns Hopkins University. Much as I owe in the way of fruitful suggestion to Cayley's immortal memoir, the idea of subjecting matrices to the additive process and of their consequent amenability to the laws of functional operation was not taken from it, but occurred to me independently before I had seen the memoir or was acquainted with its contents; and indeed forced itself upon my attention as a means of giving simplicity and generality to my formula for the powers or roots of matrices, published in the *Comptes Rendus* of the Institute for 1882 (Vol. XCIV. pp. 55, 396). My memoir on Tchebycheff's method concerning the totality of prime numbers within certain limits, was the indirect cause of turning my attention to the subject, as (through the systems of difference-equations therein employed to contract Tchebycheff's limits) I was led to the discovery of the properties of the latent roots of matrices, and had made considerable progress in developing the theory of matrices considered as quantities, when on writing to Prof. Cayley upon the subject he referred me to the memoir in question: all this only proves how far the discovery of the quantitative nature of matrices is removed from being artificial or factitious, but, on the contrary, was bound to be evolved, in the fulness of time, as a necessary sequel to previously acquired cognitions.

Already in Quaternions (which, as will presently be seen, are but the simplest order of matrices viewed under a particular aspect) the example had been given of Algebra released from the yoke of the commutative principle of multiplication—an emancipation somewhat akin to Lobatchewsky's of Geometry from Euclid's noted empirical axiom; and later on, the Peirces, father and son (but subsequently to 1858) had prefigured the universalization of Hamilton's theory, and had emitted an opinion to the effect that probably all systems of algebraical symbols subject to the associative law of multiplication would be eventually found to be identical with linear transformations of schemata susceptible of matricular representation.

That such must be the case it would be rash to assert; but it is very difficult to conceive how the contrary can be true, or where to seek, outside of the concept of substitution, for matter affording pabulum to the principle of free consociation of successive actions or operations.

Multiplication of Matrices.

A matrix written in the usual form may be regarded as made up of parallels of latitude and of longitude, so that to every term in one matrix corresponds a term of the same latitude and longitude in any other of the same order.

Every matrix possesses a principal axis, namely, the diagonal drawn from the intersection of the first two parallels to the intersection of the last two of latitude and longitude; and by a symmetrical matrix is always to be understood one in which the principal diagonal is the axis of symmetry. If there were ever occasion to consider a symmetrical matrix in which this coincidence does not exist, it might be called improperly symmetrical. This designation might and probably ought to be extended to matrices symmetrical, not merely in regard to the second visible diagonal, but to all the $(\omega - 1)$ rational diagonals of a matrix of the order ω, a rational diagonal being understood to mean any line straight or broken, drawn through ω elements, of which no two have the same latitude or longitude.

The composition of substitutions directly leads to the following rule for the multiplication of matrices. If m, n, be matrices corresponding to substitutions in which m is the antecedent or passive, and n the consequent or active, their product may be denoted by mn (that is, m multiplied by n), and then any term in the product of the two matrices will be equal to its parallel of latitude taken in the antecedent or passive and multiplied by its parallel of longitude taken in the consequent or active matrix. Cauchy has taught us what is to be understood by the product of one rectangular array or matrix by another of the same length and breadth, and we have only to consider the case of rectangles degenerating each to a single line and column respectively, to understand what is meant by the product of the multiplication of the two parallels spoken of above. It may, however, be sometimes convenient to speak of the *disjunctive product* of two sets of the same number of elements, meaning by this the sum of the products of each element in the one by the corresponding element in the other. Thus $(\lambda l)\,mn$ denoting the term in mn of latitude λ and longitude l, we have the equation

$$(\lambda l)\,mn = \lambda m \times ln,$$

where, of course, λm means the λth parallel of latitude, and ln the lth parallel of longitude, in m and n respectively. This notation may be extended so as to express the value of any minor determinant of mn; such minor may obviously be denoted by

$$\begin{array}{l} \lambda_1 l_1, \quad \lambda_1 l_2, \ldots \lambda_1 l_i, \\ \lambda_2 l_1, \quad \lambda_2 l_2, \ldots \lambda_2 l_i, \\ \ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots \\ \lambda_i l_1, \quad \lambda_i l_2, \ldots \lambda_i l_i, \end{array}$$

and its value will be the product of the two rectangles (in Cauchy's sense) formed respectively by the $\lambda_1, \lambda_2, \ldots \lambda_i$ parallels of latitude in m, and the $l_1, l_2, \ldots l_i$ parallels of longitude in n.

Any other definition of multiplication of matrices, such as the rule for multiplying lines by lines, or columns by columns, sins against good method, as being incompatible with the law of consociation, and ought to be inexorably banished from the text-books of the future. It is almost unnecessary to add that by a pth power of a matrix m is to be understood the result of multiplying p m's together; and by the qth root of m, a matrix which multiplied by itself q times produces m: hence we can attach a clear idea to any positive integral or fractional power. The complete extension of the ordinary theory of surds to multinomial quantity will appear a little further on. But it is well at this point to draw attention to the fact that at all events, if M, M' are positive integer powers of the same matrix m, the factors M, M' are convertible, that is, $MM' = M'M$, this commutative law being an immediate consequence (too obvious to insist upon) of the associative law of multiplication.

On Zero and Nullity.

The absolute zero for matrices of any order is the matrix all of whose elements are zero. It possesses so far as regards multiplication (and as will presently be evident as regards addition also) the distinguishing property of the ordinary zero, namely, that when entering into composition with any other matrix, either actively or passively, the product of such composition is itself over again; so that it may be said to absorb into itself any foreign matrix (of its own order) with which it is combined. This is the highest degree of nullity which any matrix can possess, and (regarded as an integer) will be called ω, the order of the matrix. On the other hand, if the matrix has finite content, its nullity will be regarded as zero. Between these two limits the nullity may have any integer value; thus, if its content, that is, its determinant, vanishes without any other special relation existing between its elements, the nullity will be called 1; if all the first minors vanish, 2; and, in general and more precisely, if all the minors of order $\omega - i + 1$ vanish, but the minors of order $\omega - i$ do not all vanish, the nullity will be said to be i: as an example, if the elements are not all zero, but every minor of the second order vanishes, the nullity is $\omega - 1$.

In general, a substitution impressed on a set of variables may be reversed, and the problem of reversal is perfectly determinate; but when the matrix—the schema of the substitution—is affected with any degree of nullity, such reversal becomes indeterminate. Hence the use of the word indeterminate employed by Cayley to characterize matrices affected with any degree of nullity, in which he has been followed by Clifford, who goes a step further in distinguishing the several degrees of indeterminateness from one another.

On Addition and Monomial Multiplication of Matrices.

The sum of two matrices of like order is the matrix of which each element is the sum of the elements of the same latitude and longitude as its own in the component matrices; thus, as stated by anticipation in what precedes, the addition of a zero matrix to any matrix of like order leaves the latter entirely unchanged.

Addition of matrices obviously will be subject to the same two associative and commutative laws as the addition of monomial quantities. This seems to me a sufficient ground for declining to accept *associative* as the distinguishing name of the algebra of multinomial quantity; for the emphasis thereby laid on association would seem to imply the entire absence of the commutative principle from the theory, whereas, although not having a place in multinomial multiplication, it flourishes in full vigour in the not less important, and, so to say, collateral process of multinomial addition. If k is any positive integer, the addition of the same matrix taken k times obviously leads to a matrix of which each element is k times the corresponding element of the given one; and if p times one matrix is q times another, the elements of the first are obviously $\frac{q}{p}$ into the corresponding ones of the other: hence, if k is any positive monomial quantity, k times a given matrix, by a legitimate use of language, should and will be taken to mean the matrix obtained by multiplying each element in the given one by k. And as the negative of a given matrix ought to mean the matrix which added to the given one should produce the zero-matrix previously defined, the meaning of multiplying a matrix by k may be extended, with the certainty of leading to no contradiction, to the case of any commensurable value of k positive or negative, and consequently, by the usual and valid course of inference, to the case of k being any monomial symbol whatever, whether possessing arithmetical content or not.

On the Multinomial Unit and Scalar Matrix.

On subjecting a matrix of any order ω to a resolution similar to that by which one of the second order may be resolved into a scalar and a vector, it will be shown hereafter that the ω^2 components separate into a group of $\omega^2 - 1$ terms analogous to the vector and to a single term analogous to the scalar of a quaternion. This outstanding single term is of an invariable form, namely, its principal diagonal consists of elements having the same value, which may be called its parameter, and all the other elements are zeros.

A matrix of such form I shall call a scalar. When the parameter is unity it may be termed a multinomial unity and denoted by $\overset{\omega}{\Upsilon}$*, or in place of ω we may write ω dots over Υ, or for greater simplicity when desirable write simply Υ. Any scalar, by virtue of what precedes, is a mere monomial multiplier of some such Υ.

Let $k\Upsilon$ be any scalar of order ω. It will readily be seen, by applying the laws of multiplication and addition previously laid down, that

$$\phi(k\Upsilon) = \phi(k) \,.\, \Upsilon, \text{ and that } k\Upsilon \,.\, m = m \,.\, k\Upsilon = km.$$

Thus a scalar possesses all the essential properties of a monomial quantity, and a multinomial unity of ordinary unity; in particular, the faculty of being absorbed in any other coordinate matrix with which it comes in contact. A scalar whose parameter vanishes of course becomes a zero-matrix.

The properties stated of a scalar $k\Upsilon$ serve to show that in all operations into which it enters the Υ may be dropped, and supplied or understood to be supplied at the end of the operations when needed to give homogeneity to an expression. Thus, for example,

$$(m + h\Upsilon)(m + k\Upsilon) = m^2 + (h + k)\Upsilon m + hk\Upsilon^2 = m^2 + (h + k)m + hk\Upsilon;$$

but this result may be obtained by the multiplication of $(m + h)(m + k)$, and supplying Υ (or imagining it to be supplied) to the final term in order to preserve the homogeneity of the form. In like manner, 0_ω or 0 with ω points over it may be used to denote the absolute zero of the order ω; but it will be more convenient to use the ordinary 0, having only recourse to the additional notation when thought necessary or desirable in order to make obvious the homogeneity of the terms in any equation or expression. Thus, for example, such an expression as $m^2 + 2bm + d = 0$, where m is a matrix, say of the 2nd

* Perhaps more advantageously by 1_ω. I shall hold myself at liberty in what follows to use whichever of these two notations may appear most convenient in any case as it arises.

order, and b and d monomials, set out in full would read $m^2 + 2bm + d\ddot{\Upsilon} = \ddot{0}$, meaning $m.m + 2bm + \begin{matrix} d & 0 \\ 0 & d \end{matrix} = \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix}$.

On the Inverse and Negative Powers of a Matrix.

The inverse of a matrix, denoted by m^{-1}, means the matrix which multiplied by m on either side produces multinomial unity. It is a matter of demonstration that when a matrix is non-vacuous (that is, has a finite content or determinant appertaining to it), an inverse to it fulfilling this *double* condition can always be found, and that if the product of mn is unity, so also must be that of nm.

It is a well-known fact, proved in the ordinary theory of determinants, that if every element in the first of two matrices is the logarithmic differential derivative, in respect to its correspondent in the second, of the content of that second, so conversely, every element of the second is the logarithmic derivative, in respect to its correspondent in the first, of the content of the first.

But two such matrices multiplied together in either sense would not give for their product multinomial unity; to obtain this product either matrix must be multiplied indifferently into or by the *transverse* of the other (meaning by the transverse of a matrix, the new matrix obtained by rotating the original one through 180° about its principal diagonal). In other words, if m be a given matrix and n be obtained from it by substituting for each element the logarithmic derivatives of its content in respect to its opposite, then $mn = \overset{\omega}{\Upsilon}$ and $nm = \overset{\omega}{\Upsilon}$, where ω means (as will always be the case throughout these lectures) the order of the matrices concerned. The n which satisfies these two equations (and it cannot satisfy the one without satisfying the other) will be called the inverse of m and be denoted by m^{-1}.

For brevity and suggestiveness it will be advantageous to write in general 1 for $\overset{\omega}{\Upsilon}$ as we write 0 for 0_ω, so that $mn = 1$ will imply $nm = 1 = mn$ and $n = m^{-1}$.

We may define in general (as in monomial algebra) m^{-i} to mean the inverse of m^i, that is, $(m^i)^{-1}$. We shall then have $(m^{-1})^i = m^{-i}$, for $mn . mn = 1$ implies $m . mn . n = mn = 1$ or $m^2n^2 = 1$. Hence $n^2 = m^{-2}$, that is, $(m^{-1})^2 = m^{-2}$. Also since $m^2n^2 = 1$, $m^3n^3 = mn = 1$ or $n^3 = m^{-3}$, that is, $(m^{-1})^3 = m^{-3}$, and so in general for all positive integer values of i, $(m^{-1})^i = m^{-i}$. And, as in monomial algebra, it may now be proved and taken as proved that, for all real values of i and j, whether positive or negative, $m^i . m^j = m^{i+j}$, and the same relation may be assumed to continue when i, j become general quantities. The elements in the inverse to any matrix m all involving the reciprocal of the determinant to m, if D be the content of m we may write $m^{-1} = \frac{1}{D}\mu$, where μ is a matrix all of whose elements are always finite. Hence we come to the important conclusion that for vacuous matrices inverses only exist in idea and are incapable of being realized so as to have an actual existence. In the sequel it will be shown that the inverse is only a single instance of an infinite class of matrices which exist ideally as functions of actual matrices, but are incapable of realization.

Suppose now that M, N are any two matrices such that $MN = 0$ or that $NM = 0$; multiplying each side of the equation by M^{-1} if such expression has an actual existence (that is, if M is non-vacuous), we obtain, from the known properties of zero, $N = 0$, but if M is vacuous no such conclusion can be drawn. So further if $m^i = 0$ (i being any positive integer), it will be seen under the third law of motion that m is necessarily vacuous. Hence from this equation it cannot be inferred that any lower power than the ith of m is necessarily zero.

On the Latent Roots and Different Degrees of Vacuity of Matrices.

If m be any matrix, the augmented matrix $m - \lambda\Upsilon$ or $m - \lambda . 1_\omega$ or $m - \lambda$ will be found simply by subtracting λ from each element in the principal diagonal of m. The content of this matrix or the same multiplied by -1 or any other constant, I term the latent function to m, which will be an algebraical function of the degree ω in λ (which may be termed the latent variable or carrier); and the ω roots of this function (that is, the ω values of the carrier which annihilate the latent function) I call the latent roots of the unaugmented matrix m. It is obvious from this definition that if λ_1 be any latent root of m, the content of $m - \lambda_1$ will vanish, that is, $m - \lambda_1$ will be vacuous, and conversely that if $m - \lambda_1$ is vacuous, λ_1 must be one of the latent roots to m. Thus if m is vacuous, one of the latent roots must be zero; if only one of them is zero I call m simply vacuous and say that its vacuity is 1: thus zero vacuity and simple vacuity mean the same thing as zero nullity and simple nullity respectively. More generally if any number i, but not $i+1$, of the latent roots of m are all of them zero, m will be said to have the vacuity i.

By a principal minor determinant to any matrix I mean any minor determinant whose matrix is divided by the principal diagonal into two triangles. It will then easily be seen that if s_i means in general the sum of the principal ith minors to m, and s_0 means the complete determinant, the assertion of m having the vacuity i is exactly coextensive with the assertion that

$$s_0 = 0, \quad s_1 = 0, \quad s_2 = 0, \ \dots \ s_{i-1} = 0.$$

If the nullity of m is i, *every* qth minor of m is zero when $q < i$. Hence the vacuity cannot fall short of the nullity, but the converse is not true.

A matrix may not have any vacuity up to ω inclusive without the nullity being greater than 1. It will hereafter be shown, under the 2nd law of motion, that if $\lambda_1, \lambda_2, \ldots \lambda_\omega$ are the ω latent roots of m, then

$$(m-\lambda_1)(m-\lambda_2)\ldots(m-\lambda_\omega)=0 \text{ or say } M=0.$$

But it will be interesting even at this early stage to show that a theorem closely approaching this may be deduced from the distinction drawn between vacuous and non-vacuous matrices as regards their possession of real inverses.

I propose to prove instantaneously by this means that at all events $M^{\omega-1}=0$. It is obvious from any single instance of multiplication that mn and nm are not in general coincident. But if n could be expressed as a linear function of powers of m (including m^0 or 1_ω among such powers), mn and nm must be coincident. If now we take the ω^2 matrices

$$1, m, m^2, \ldots m^{\omega^2-1},$$

n at first blush one would say ought to be expressible as a linear function of these ω^2 quantities determinable by means of the solution of ω^2 linear equations, and can only escape being so expressible in consequence of the fact that these ω^2 powers of m are linearly related. Hence we must have an identical equation of the form

$$Am^{\omega^2-1}+Bm^{\omega^2-2}+Cm^{\omega^2-3}\ldots+Gm+H=0_\omega \text{ or say } Fm=0.$$

If now Fm were supposed to contain any factor other than

$$m-\lambda_1,\ m-\lambda_2, \ldots m-\lambda_\omega,$$

such factors being non-vacuous may be expelled from Fm; consequently the equation in question must be of the form

$$(m-\lambda_1)^{\alpha_1}(m-\lambda_2)^{\alpha_2}(m-\lambda_\omega)^{\alpha_\omega}=0,$$

and as the coefficients of the equation in m are necessarily rational we must have $\alpha_1=\alpha_2, \ldots, \alpha_\omega=\alpha$. Hence $\omega\alpha=\alpha_1+\alpha_2+\ldots\alpha_\omega<\omega^2$, and consequently $\alpha<\omega$.

Hence, at all events (since $M^{\omega-1-\theta}=0$ on multiplication by M^θ gives $M^{\omega-1}=0$),

$$\{(m-\lambda_1)(m-\lambda_2)\ldots(m-\lambda_\omega)\}^{\omega-1}=M^{\omega-1}=0. \quad \text{Q.E.D.}$$

LECTURE II.

On Reduction.

It follows from what has been already shown in Lecture I, when m is a matrix of the second order ($\omega-1$ being here unity) that $(m-\lambda_1)(m-\lambda_2)=0$.

Understanding by m the matrix $\begin{matrix} t_1, & \tau_1, \\ t_2, & \tau_2, \end{matrix}$ the latent equation to m is

$$\begin{vmatrix} t_1-\lambda, & \tau_1 \\ t_2 \quad , & \tau_2-\lambda \end{vmatrix}=0,$$

that is, $$\lambda^2 - (t_1 + \tau_2)\lambda + (t_1\tau_2 - t_2\tau_1) = 0,$$
so that $$m^2 - (t_1 + \tau_2)m + (t_1\tau_2 - t_2\tau_1) = 0,$$
or, using the literation applied to the parametric triangle,
$$m^2 - 2bm + d = 0; \qquad (1)$$
for since the content of $x + ym + zn$ is supposed to be
$$x^2 + 2bxy + 2cxz + dy^2 + 2eyz + fz^2,$$
that of $-\lambda + m$ will be found by making $z = 0$, $x = -\lambda$, $y = 1$. The variation of equation (1) obtained by taking ϵn for the increment of m (remembering that the variation of m^2 is $(m + \epsilon n)(m + \epsilon n) - m^2$, that is, $\epsilon(mn + nm)$) gives rise to the identical equation
$$mn + nm - 2bn - 2cm + 2e = 0, \qquad (2)$$
and the variation of this again gives
$$n^2 + n^2 - 2cn - 2cn + 2f = 0,$$
or $n^2 - 2cn + f = 0$, as of course will be obtained immediately from (1) by substituting n, c, f in place of m, b, d.

The parameters c, f, if n represents $\begin{matrix} u_1 & v_1 \\ u_2 & v_2 \end{matrix}$ are the sum of the principal diagonal elements and the content of u, just as b, d are such sum and content in respect to m.

The parameter e (the connective to d and f) or rather its double $2e$ is obviously the emanant of d in respect to the operator
$$u_1\delta_{t_1} + u_2\delta_{t_2} + v_1\delta_{\tau_1} + v_2\delta_{\tau_2},$$
or, if we please, of f in respect to the inverse operator
$$t_1\delta_{u_1} + t_2\delta_{u_2} + \tau_1\delta_{v_1} + \tau_2\delta_{v_2},$$
that is, $$t_1v_2 + u_1\tau_2 - t_2v_1 - u_2\tau_1.$$

With the aid of the *catena* of equations in m, in m and n, and in n, any combination of functions of m and n may be reduced to the standard form
$$Amn + Bm + Cn + D.$$

For, in the first place,
$$\phi m = P(m^2 - 2bm + d) + rm + s = rm + s,$$
and similarly $$\psi n = \rho n + \sigma.$$
Hence the most general combination referred to is expressible as the product of alternating linear functions of m and n, and may therefore be reduced to a sum of terms of which each is a product of alternate powers of m and of n, each of which powers may again be reduced to the form of linear functions, and this process admits of being continually repeated.

Suppose then, at any stage of it, that the greatest number of occurrences of linear functions of m and n in the aggregate of terms is i; then at the next stage of the process the new aggregate will consist of monomial multipliers of one or more *simple* successions of m and n, and of terms in which the number of alternating linear functions never exceeds $i-1$; hence, eventually we must arrive at a stage when the aggregate will be reduced to a sum of monomial multipliers of simple successions of m and n, every such succession being of the form

$$(mn)^q \text{ or } m^{-1}(mn)^q \text{ or } (mn)^q n^{-1} \text{ or } m^{-1}(mn)^q n^{-1}.$$

But
$$\begin{aligned}(mn)^2 &= m\,.\,nm\,.\,n = -\,m\,(mn - 2bn - 2cm + 2e)\,n\\ &= -\,m^2n^2 + 2bmn^2 + 2cm^2n - 2emn\\ &= -\,(2bm-d)(2cn-f) + 2bm\,(2cn-f) + 2c\,(2bm-d)\,n - 2emn\\ &= -\,(2e-4bc)\,mn - df.\end{aligned}$$

Hence
$$(mn)^2 + 2\,(e-2bc)\,mn + df = 0.$$

Hence $\quad (mn)^q = P\,\{(mn)^2 + 2\,(e-2bc)\,mn + df\} + Amn + B = Amn + B,$

where A and B are known functions of $(e-2bc)$ and f; and therefore

$$m^{-1}(mn)^q = An + Bm^{-1} = An - \frac{B}{d}\,m + \frac{2Bb}{d}.$$

Similarly
$$(mn)^q n^{-1} = Am - \frac{B}{f}\,n + \frac{2Bc}{f},$$

and
$$m^{-1}(mn)^q n^{-1} = A + B\,(mn)^{-1} = -\,\frac{B}{df}\,mn + \left(A - B\,\frac{2e-4bc}{df}\right).$$

And this being true (*mutatis mutandis*) for all values q, it follows that the function expressed by any succession of products of functions of m and n is reducible to the form of a linear expression in m, n, mn, in which the 4 monomial coefficients are known or determinable functions of the parameters to the corpus m, n.

The latent function to any such linear expression, say

$$Amn + Bm + Cn + D,$$

may be found in the same way as the latent function to mn has been found, namely, as follows:

$$\begin{aligned}(Amn + Bm + Cn + D)^2 &= A^2(mn)^2 + AB\,(mnm + mmn) + AC\,(mnn + nmn)\\ &\quad + 2ADmn + B^2m^2 + BC\,(mn + nm) + C^2n^2 + 2BDm + 2CDn + D^2\\ &= A^2(-\,2e + 4bc)\,mn - A^2df + ABm\,(2bn + 2cm - 2e)\\ &\quad + AC\,(2bn + 2cm - 2e)\,n + 2ADmn + B^2m^2 + BC\,(2bn + 2cm - 2e) + C^2n^2\\ &\quad + 2BDm + 2CDn + D^2.\end{aligned}$$

Let $\quad (Amn + Bm + Cn + D)^2 - 2P\,(Amn + Bm + Cn + D) + Q = 0$

be the identical equation to $Amn + Bm + Cn + D$.

The coefficient of mn in the development of the first term being

$$(4bc - 2e)\,A^2 + 2bAB + 2cAC + 2AD,$$

and m^2, n^2 being reducible to linear functions of m, n respectively, it follows that

$$P = A\,(2bc - e) + Bb + Cc + D.$$

To find Q it is only needful to fasten the attention upon the constant terms in the before named development reduced to the standard form. These will be

$$-A^2df - 2ABcd - 2ACbf - B^2d - 2BCe - C^2f + D^2, \text{ say } K,$$

and the constant part in $-2P(Amn+Bm+Cn+D)$ being $-2DP$, it follows that

$$\begin{aligned} Q &= 2AD\,(2bc - e) + 2BDb + 2CDc + D^2 - K \\ &= A^2df + 2ABcd + 2ACbf + 2AD\,(2bc - e) \\ &\quad + B^2d + 2BCe + C^2f + 2BDb + 2CDc, \end{aligned}$$

and consequently the latent function $\Lambda^2 - 2P\Lambda + Q$, of which the algebraical roots are the latent roots of $Amn + Bm + Cn + D$, is completely determined. Thus, for example, if the latent function of $m + n$ is required, making $A = D = 0$, $B = C = 1$, its value will be seen to be $\Lambda^2 - 2\,(b + c)\,\Lambda + d + 2e + f = 0$, so that the roots will be $b + c \pm \sqrt{\{(b + c)^2 - (d + 2e + f)\}}$.

On Involution.

In general, if m and n be two given binary matrices, and p any third matrix, say

$$m = \begin{matrix} t_1 & t_2 \\ t_3 & t_4 \end{matrix}, \quad n = \begin{matrix} \tau_1 & \tau_2 \\ \tau_3 & \tau_4 \end{matrix}, \quad p = \begin{matrix} T_1 & T_2 \\ T_3 & T_4 \end{matrix},$$

p may be expressed as a linear function of $\ddot{\Upsilon}$, m, n, mn or of $\ddot{\Upsilon}$, m, n, nm. For in order that p may be expressible under the form $A + Bm + Cn + Dnm$, observing that

$$nm = \begin{matrix} t_1\tau_1 + t_3\tau_2 & t_2\tau_1 + t_4\tau_2 \\ t_1\tau_3 + t_3\tau_4 & t_2\tau_3 + t_4\tau_4 \end{matrix},$$

and that $\ddot{\Upsilon} = \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}$, it is only necessary to write

$$\begin{aligned} A + Bt_1 + C\tau_1 + D\,(t_1\tau_1 + t_3\tau_2) &= T_1, \\ Bt_2 + C\tau_2 + D\,(t_2\tau_1 + t_4\tau_2) &= T_2, \\ Bt_3 + C\tau_3 + D\,(t_1\tau_3 + t_3\tau_4) &= T_3, \\ D + Bt_4 + C\tau_4 + D\,(t_2\tau_3 + t_4\tau_4) &= T_4, \end{aligned}$$

and then A, B, C, D may be found by the solution of these four linear equations: and this solution must always be capable of being effected unless the determinant

$$\begin{vmatrix} 1, & t_1, & \tau_1, & t_1\tau_1 + t_3\tau_2 \\ 0, & t_2, & \tau_2, & t_2\tau_1 + t_4\tau_2 \\ 0, & t_3, & \tau_3, & t_1\tau_3 + t_3\tau_4 \\ 1, & t_4, & \tau_4, & t_2\tau_3 + t_4\tau_4 \end{vmatrix}$$

vanishes.

When this is the case the matrices m, n, in the order in which they are written, will be said to be in sinistral involution. In like manner, if 1, n, m, mn are linearly related, m, n may be said to be in dextral involution. But it is very easy to see from the identical equation (2) that in this case these two involutions are really identical, for, since $A + Bm + Cn + Dmn = 0$, by subtraction

$$A + Bm + Cn - Dnm + 2Dcm + 2Dbn - 2De = 0,$$

that is, $$(A - 2eD) + (B + 2cD)\,m + (C + 2bD)\,n - Dnm = 0.$$

The above determinant then will be called the involutant to m, n or n, m, indifferently, for it will be seen, and indeed may be shown, *à priori*, that its value remains absolutely unaltered (not merely to a numerical factor *près*, but in sign and in arithmetical magnitude as well) when the Latin and Greek letters, or which is the same thing, when the matrices m and n are interchanged.

On the Linearform or Summatory Representation of Matrices, and the Multiplication Table to which it gives rise.

This method by which a matrix is robbed as it were of its areal dimensions and represented as a linear sum, first came under my notice incidentally in a communication made some time in the course of the last two years to the Mathematical Society of the Johns Hopkins University, by Mr C. S. Peirce, who, I presume, had been long familiar with its use. Each element of a matrix in this method is regarded as composed of an ordinary quantity and a symbol denoting its place, just as 1883 may be read

$$1\theta + 8h + 8t + 3u,$$

where θ, h, t, u, mean thousands, hundreds, tens, units, or rather, the places occupied by thousands, hundreds, tens, units, respectively.

Take as an example matrices of the second order, as

$$\begin{matrix} \alpha & \beta \\ \gamma & \delta \end{matrix} \qquad \begin{matrix} a & b \\ c & d. \end{matrix}$$

These may be denoted respectively by

$$\alpha\lambda + \beta\mu + \gamma\nu + \delta\pi, \quad a\lambda + b\mu + c\nu + d\pi;$$

their product by

$$(a\alpha + c\beta)\,\lambda + (b\alpha + d\beta)\,\mu + (a\gamma + c\delta)\,\nu + (b\gamma + d\delta)\,\pi,$$

which therefore must be capable of being made identical with

$$\begin{aligned} &a\alpha\lambda^2 + a\beta\lambda\mu + a\gamma\lambda\nu + a\delta\lambda\pi \\ &+ b\alpha\mu\lambda + b\beta\mu^2 + b\gamma\mu\nu + b\delta\mu\pi \\ &+ c\alpha\nu\lambda + c\beta\nu\mu + c\gamma\nu^2 + c\delta\nu\pi \\ &+ d\alpha\pi\lambda + d\beta\pi\mu + d\gamma\pi\nu + d\delta\pi^2, \end{aligned}$$

when a proper system of relations is established between the quadric combinations and the simple powers of λ.

The arguments of like coefficients in the two sums being equated together, there result the equations

$$\lambda^2 = \lambda, \quad \lambda\nu = \nu, \quad \mu\lambda = \mu, \quad \mu\nu = \pi,$$
$$\nu\mu = \lambda, \quad \nu\pi = \nu, \quad \pi\mu = \mu, \quad \pi^2 = \pi,$$

and again, the arguments to the 8 coefficients in the second sum which are not included among the coefficients of the first, being equated to zero, there result the equations

$$\lambda\mu = 0, \quad \lambda\pi = 0, \quad \mu^2 = 0, \quad \mu\pi = 0,$$
$$\nu\lambda = 0, \quad \nu^2 = 0, \quad \pi\lambda = 0, \quad \pi\nu = 0.$$

These 16 equalities may be brought under a single *coup d'œil* by the following multiplication table:

	λ	ν	μ	π
λ	λ	ν	0	0
ν	0	0	λ	ν
μ	μ	π	0	0
π	0	0	μ	π

In like manner it will be found that any matrix of the 3rd order as $\begin{matrix} a & b & c \\ d & e & f, \\ g & h & k \end{matrix}$ regarded as a quantity, may be expressed linearformly by the sum

$$a\lambda + b\mu + c\nu + d\pi + e\rho + f\sigma + g\tau + hv + k\phi,$$

where the topical symbols are subject to the multiplication table below written:

	λ	π	τ	μ	ρ	υ	ν	σ	ϕ
λ	λ	π	τ	0	0	0	0	0	0
π	0	0	0	λ	π	τ	0	0	0
τ	0	0	0	0	0	0	λ	π	τ
μ	μ	ρ	υ	0	0	0	0	0	0
ρ	0	0	0	μ	ρ	υ	0	0	0
υ	0	0	0	0	0	0	μ	ρ	υ
ν	ν	σ	ϕ	0	0	0	0	0	0
σ	0	0	0	ν	σ	ϕ	0	0	0
ϕ	0	0	0	0	0	0	ν	σ	ϕ

And, in like manner, matrices of any order ω may be expressed linearformly as the sum of ω^2 terms, each consisting of a monomial multiplier of a topical symbol, the entire ω^2 symbols being subject to a multiplication table containing ω^4 places, of which ω^3 will be occupied by the ω^2 simple symbols, each appearing ω times, and the remaining $\omega^4 - \omega^3$ places by the ordinary zero.

This conception applied to quadratic matrices might have served to establish the connection between them and Hamilton's quaternions, regarded as homogeneous functions of 1, i, j, k, themselves linear functions of the topical symbols λ, μ, ν, π; but the same result may be arrived at somewhat more simply by a method given in a *subsequent* lecture.

On the Corpus *formed by two Independent Matrices of the same order, and the Simple* Parameters *of such Corpus.*

By the latent function of a *corpus* (m, n) we may understand the content or any numerical multiplier of the content of (that is, the determinant to) the matrix $x + ym + zn$, where x, y, z are monomial carriers. This function will be a quantic of the order ω in x, y, z, and in the standard form the coefficient of x^ω may be supposed to be unity, so that it will contain $\frac{1}{2}(\omega^2 + 3\omega)$ coefficients, which may be termed the parameters of the corpus.

To fix the ideas, suppose $\omega = 3$ and let the latent function to

$$\begin{matrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'', \end{matrix} \qquad \begin{matrix} \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \\ \alpha'' & \beta'' & \gamma'' \end{matrix}$$

be called F, where

$$F = x^3 + 3bx^2y + 3cx^2z + 3dxy^2 + 6exyz + 3fxz^2 + gy^3 + 3hy^2z + 3kyz^2 + lz^3.$$

Let m become $m + \epsilon n$, where ϵ is a monomial infinitesimal. Then the function to the corpus becomes the content of

$$x + y(m + \epsilon n) + zn, \text{ that is, } x + ym + (z + \epsilon y)n,$$

and consequently the variation of the function to (m, n) is $\epsilon y \delta_z F$. If then the *rate* of variation of any of the parameters, when n is the rate of variation of m, be denoted by prefixing to such parameter the symbol E, we shall find

$$Eb = c; \quad Ed = 2e; \quad Ee = f; \quad Eg = 3h; \quad Eh = 2k; \quad Ek = l;$$

and similarly, if $\mathfrak{Z}$, preceding a parameter, be used to indicate its rate of variation corresponding to n's rate of variation being m, then

$$\mathfrak{Z}c = b; \quad \mathfrak{Z}f = 2e; \quad \mathfrak{Z}e = d; \quad \mathfrak{Z}l = 3k; \quad \mathfrak{Z}k = 2h; \quad \mathfrak{Z}h = g;$$

and the variations of c, f, l, as regards E, and of b, d, g, as regards $\mathfrak{Z}$, are of course zero.

By forming the triangle of parameters

$$\begin{array}{c} 1 \\ b\ \ c \\ d\ \ e\ \ f \\ g\ \ h\ \ k\ \ l \\ p\ \ q\ \ r\ \ s\ \ t \end{array}$$

the law of variations of the parameters of the function to (m, n) (expressed in the ordinary manner by a ternary quantic affected with the proper numerical multipliers) becomes evident, whatever may be the order of the corpus (that is, of the matrices m and n, of which it is constituted): thus, for example, when $\omega = 4$, in addition to the previous expressions we shall find

$$Ep = 4q, \quad Eq = 3r, \quad Er = 2s, \quad Es = t, \quad Et = 0,$$

$$\exists t = 4s, \quad \exists s = 3r, \quad \exists r = 2q, \quad \exists q = p, \quad \exists p = 0.$$

By means of the above relations, any identical equation, into which enters one or more matrices, admits of being varied, so as to give rise to an identical equation connecting one additional number of the same.

Scholium.—In what precedes it will have been observed that the matter under consideration has always regard to matrices, or, as we may say, quantities of a fixed order ω, combined exclusively with one another and with ordinary monomial quantities. Every such combination forms as it were a *clausum* or world of its own, lying completely outside and having no relations with any other. It is, however, possible, and even probable, that as the theory is further evolved, this barrier may be found to give way and the worlds of all the various orders of quantity be brought into relation and intercommunion with one another.

LECTURE III.

On Quantity of the Second Order.

The theory of matrices of the second order seems to me to deserve a special preliminary investigation on various grounds. First, as affording a facile and natural introduction to the general theory (as the study of Conic Sections is usually made to precede that of universal Geometry); secondly, because it presents certain very special features distinguishing it from all other kinds of quantity, such as the coincidence of the two involutants (reminding one of the single image in the case of ordinary refraction as contrasted with the double image seen through iceland spar), or, again, the rational relation between the products of matrices of the second order, in whatever order the factors are introduced in the performance of the multiplication; and thirdly, because the theory of this kind of quantity has already been extensively studied and developed under the name or aspect of Quaternions. Hence it may not be out of place to make the remark that, as it surely would not be logical to seek for the origin of the conception included in the symbol $\sqrt{(-1)}$ in geometrical considerations, however important its application to geometrical exegesis, so now that an independent algebraical foundation has been discovered for the introduction and use of the symbols employed in Hamilton's theory, it would (it seems to me) be exceedingly illogical and contrary to good method to build the pure theory of the same upon space conceptions; the more so, as it will hereafter be shown that quantities of every order admit of being represented in a mode strictly analogous to that in which quantity of the second order is represented by quaternions, namely, if the order is ω, by ω^2-ions, or as I shall in future say, by *Ions*, of which the geometrical interpretation, although there is little doubt that it exists, is not yet discovered, and it must, it is certain, draw upon the resources of inconceivable space before it can be effected.

32.

ON THE SOLUTION OF A CLASS OF EQUATIONS IN QUATERNIONS.

[*Philosophical Magazine*, XVII. (1884), pp. 392—397.]

THE general equation of the degree ω in Quaternions or Binary Matrices is obviously ω^4, but in certain cases some of these roots evaporate and go off to infinity. The only equation considered by Sir William Hamilton in his Lectures is the Quadratic Equation of a form which I call unilateral, because the quaternion coefficients in it are supposed all to lie on the same side of the unknown quantity. I propose here to show how Hamilton's equation, and indeed a unilateral one of any order, may be solved by a general algebraical method and the number of its roots determined.

It will be convenient to begin by setting out certain general equations relating to any two binary matrices m, n.

Writing the determinant of $x+ym+zn$ under the form

$$x^2+2bxy+2cxz+dy^2+2eyz+fz^2$$

(b, c, d, e, f, thus constituting what I call the parameters of the *corpus* m, n), we have universally

$$m^2-2bm+d=0,\quad n^2-2cn+f=0,\quad d(m^{-1}n)^2-2e(m^{-1}n)+f=0.$$

Moreover if m, n receive the scalar increments μ, ν; d, e, f become respectively

$$d-2\mu b+\mu^2,\quad e-\mu c-\nu b+\mu\nu,\quad f-2\nu c+\nu^2.$$

Let us begin with Hamilton's form, say

$$x^2-2px+q=0,$$

and suppose

$$x^2-2Bx+D=0,$$

where B, D are scalars to be determined.

Let b, c, d, e, f be the five known parameters of the *corpus* p, q. Then, since

$$(p-B)^{-1}(q-D)=2x,$$

we shall have [cf. p. 188 above]

$$4(d-2bB+B^2)x^2-4(e-bD-cB+BD)x+f-2cD+D^2=0.$$

Hence, writing $$B-b=u,\quad D-c=v,$$

$$d-b^2=\alpha,\quad e-bc=\beta,\quad f-c^2=\gamma,$$

we have $$u^2+\alpha=\lambda,\quad uv+\beta=2\lambda(u+b),\quad v^2+\gamma=4\lambda(v+c).$$

From the last two equations, eliminating v, there results

$$(2\lambda u-2b\lambda-\beta)^2-4\lambda(2\lambda u-2b\lambda-\beta)u+(\gamma-4c\lambda)u^2=0.$$

Hence substituting $\lambda-\alpha$ for u^2,

$$(4\lambda^2+4c\lambda-\gamma)(\lambda-\alpha)-(2b\lambda-\beta)^2=0.$$

We have thus six values of u, namely

$$\pm\sqrt{(\lambda-\alpha)}$$

(where λ has three values), to which correspond six values of v, namely

$$2\lambda\pm\frac{2\lambda b-\beta}{\sqrt{(\lambda-\alpha)}};$$

and, finally, $$2x=(p-u-b)^{-1}(q-v+c)$$

$$=\{(p-b)^2-u^2\}^{-1}(p-b+u)(q-c-v),$$

or $$x=\frac{pq-(c+v)p-(b-u)q+(b-u)(c+v)}{2(b^2-d-u^2)};$$

which equation gives six values for x, and shows that ten have evaporated.

It is easy to account *à priori* for the solution depending only upon a cubic in u^2.

For $x^2-2px+q=0$ is the same as $y^2-2yp+q=0$, where $y=-x+2p$. But obviously, from the nature of the process for determining them, B and C are independent of the *side* of the unknown on which the first coefficient lies. Hence the actual B will be associated with B', B' being what B becomes when x becomes $-x+2p$, which is obviously $-B+2b$.

Hence with any value of $B-b$, which is u, is associated a corresponding $B-b$, which is $-u$.

I will now proceed to apply a similar or the same method to the trinomial cubic equation in quaternions (or binary quantity) $x^3+px-q=0$, with a view to ascertain the number of its roots.

Retaining the same notation as before, and still supposing

$$x^2-2Bx+D=0,$$

we obtain $$x^3 + (D - 4B^2)x + 2BD = 0,$$

and $$x = \frac{:q + 2BD}{p + 4B^2 - D}\text{*}.$$

Hence $$\{(4B^2 - D)^2 - 2b(4B^2 - D) + d\}x^2$$
$$- 2\{2(4B^2 - D)BD - c(4B^2 - D) - 2bBD + e\}x$$
$$+ 4B^2D^2 - 2cBD + f = 0.$$

Hence we may write

$$(4B^2 - D)^2 - 2b(4B^2 - D) + d = \lambda,$$
$$2(4B^2 - D)BD - c(4B^2 - D) - 2bBD + e = \lambda B,$$
$$4B^2D^2 - 2cBD + f = \lambda D;$$

from which equations B and D are to be determined. Eliminating λ between the first and second and between the first and third of these equations, we obtain two equations, of which the arguments are

$$D^3; \quad B^2D^2, D^2; \quad B^4D, B^2D, BD, D; \quad 1$$

for the one,

$$BD^2; \quad B^3D, BD, D; \quad B^5, B^3, B^2, B; \quad 1$$

for the other.

Eliminating D by the Dialytic method between these two equations, we shall have (using points to signify unexpressed coefficients) the following three linear equations in D^2, D, 1, namely:

$$\cdot BD^2 + (\cdot B^3 + \&c.)D + (\cdot B^5 + \&c.) = 0,$$
$$\cdot B^3D^2 + (\cdot B^5 + \&c.)D + (\cdot B^7 + \&c.) = 0,$$
$$\cdot B^5D^2 + (\cdot B^7 + \&c.)D + (\cdot B^9 + \&c.) = 0.$$

Hence in the final equation B rises to the 15th power; and by combining any two of the above equations, D is given linearly in terms of B; and, finally, x is known from the equation

$$x = \frac{:(p + D - 4B^2 - 2b)(q + 2BD)}{-(4B^2 - D)^2 - 2(4B^2 - D) + d},$$

and has 15 values.

A like process may be extended to a unilateral equation (of the Jerrardian form) of any degree, say $x^\omega + qx + r = 0$.

Introducing the auxiliary equation with scalar coefficients as before, namely

$$x^2 - 2Bx + D = 0,$$

x may be expressed as a function of q, r, B, D; and the term containing the

* I use $\frac{:L}{M}$ and $\frac{L:}{M}$ to signify $M^{-1}L$ and LM^{-1} respectively.

highest power of B in the equation for determining B (of which D is a one-valued function), when $\omega = 4$, will be found to be the determinant

$$\begin{matrix} \cdot B & \cdot B^3 & \cdot B^5 & \cdot B^7 \\ \cdot B^3 & \cdot B^5 & \cdot B^7 & \cdot B^9 \\ \cdot B^5 & \cdot B^7 & \cdot B^9 & \cdot B^{11} \\ \cdot B^7 & \cdot B^9 & \cdot B^{11} & \cdot B^{13} \end{matrix} *$$

and a similar determinant will fix the degree of B in the resolving equation for any value of ω. Hence the number of solutions of the unilateral equation in quaternions of the Jerrardian form of the degree ω is $\omega(2\omega - 1)$ or $2\omega^2 - \omega$, and the evaporation will accordingly be $\omega^4 - 2\omega^2 + \omega$, or

$$(\omega^2 - \omega)(\omega^2 + \omega - 1).$$

Moreover the same method with a slight addition will serve to determine the roots of the general unilateral equation in quaternions, the number of which will be a cubic function of ω, as I propose to show and to give its precise value in some future communication, either in this Journal, or at all events in the memoir on Universal Algebra now in the course of publication, under the form of lectures, in the *American Journal of Mathematics*†.

I very much question whether the old method of Hamilton, as taught by its most consummate masters, Tait in this country, or the late Prof. Benjamin Peirce in America, would be found sufficiently plastic to deal effectually with an analytical investigation in quaternions of this degree of complexity, so as to lead to the formula for the number of solutions of the unilateral equation of the Jerrardian form above given.

I invite my much esteemed and most capable former colleague and former pupil, Dr Story, of the Johns Hopkins, and Prof. Stringham, of the University of California, who carry on the traditions of the Harvard School, to put the power of the old method as compared with the new to this practical test.

Postscript.—If $$x^3 - 3px^2 + 3qx - r = 0,$$

(where p, q, r are perfectly general matrices of the second order which satisfy the general equations

$$q^2 - 2bq + d = 0, \quad qr + rq - 2bq - 2b_1q + 2\epsilon = 0, \quad r^2 - 2b_1q + d_1 = 0,$$
$$pq + qp - 2bp - 2\beta q + 2e = 0, \quad p^2 - 2\beta p + \delta = 0,$$
$$pr + rp - 2b_1p - 2\beta r + 2e_1 = 0),$$

* It may readily be seen that the highest term in the equation for finding B is identical with the resultant of

$$D^4 - 24B^2D^3 + 80B^4D^2 \text{ and } 4BD^3 - 40B^3D^2 + 64B^5D - 64B^7,$$

that is, will be $2^{18}.3.7.19B^{28}$; and that the last term (at all events to the sign *près*) will be $b^4\delta^2$, which is of $4.3+2.2.4$ (that is of 28) dimensions in x, and is therefore codimensional (as it ought to be) with B^{28}.

† It is given in the Postscript below.

and if we write $$x^2 - 2Bx + D = 0,$$

then $$px = \frac{:r + 3Dp - BD}{3q - 3Bp + B^2 - D};$$

and I find by perfectly easy and straightforward work that B, D may be determined by means of the following equations:

$$\frac{(B^2-D)^2}{9} + 2(b - \beta B)\frac{B^2 - D}{3} + (d - 2eB + 4\delta B^2) = 9\lambda,$$

$$\frac{B^3D - BD^2}{3} + (b_1 + 3\beta D)\frac{B^2 - D}{3} + (\epsilon - e_1 B + 3eD - 6\delta BD) = 3B\lambda,$$

$$B^2D^2 - 2(b_1 + 3\beta D)BD + d_1 + 6De_1 + 9\delta D^2 = D\lambda.$$

The order (by which I mean the number of solutions of this system of equations) is readily seen to be the same as that of

$$\cdot D^3 \;+ \cdot B^3D + \cdot B^4D = 0$$
$$\cdot BD^2 + \cdot B^3D + \cdot B^5 \;= 0;$$

that is, is the same as the degree in B of $B^3(B^5)^2 . R$, where R is the resultant of

$$\cdot D^2 + \cdot B^3 + \cdot B^4 \text{ and } \cdot D^2 + \cdot B^2D + \cdot B^4.$$

Hence* the number of solutions is $3 + 10 + 8$, that is, is 21.

Practically, therefore, we have now sufficient data to determine the number of solutions of a unilateral equation in quaternions of any order ω; for it is morally certain that such number is a rational function of ω; and as it cannot but be of a lower order than ω^4, we have only to determine a cubic function of ω whose values for $\omega = 0, 1, 2, 3$ are 0, 1, 6, 21, which is easily found to be $\omega^3 - \omega^2 + \omega$; so that the evaporation is $\omega^4 - \omega^3 + \omega^2 - \omega$, that is

$$(\omega^2 + 1)(\omega^2 - \omega).$$

Practically also we can solve (subject to hardly needful verification) the number of roots of a unilateral equation of the special form

$$x^\omega + q_\theta x^\theta + q_{\theta-1}x^{\theta-1} + \ldots + q_0 = 0.$$

For when $\theta = \omega$, we know the number is ω^2; and when $\theta = 1$, the number is $\omega^3 + \omega^2 - \omega$; consequently if the second differences of the function of (ω, θ) which expresses the number of roots are constant, the value of this function when $\theta = \omega - 1$ is $\omega^3 - \omega^2 + \omega$, which we have found to be the actual number; and consequently, if the second differences are not constant, they must be sometimes positive and sometimes negative, which is in the highest degree improbable. Hence in all probability it will be found that the required number of solutions in the form supposed is $(1 + \theta)\omega^2 - \theta\omega$.

I need hardly add that the nine quantities $2b, 2b_1, 2\beta$; $2e, 2e_1, 2\epsilon$; d, δ, d_1, which occur in the discussion above given of the general unilateral cubic, or, say, rather the ten quantities obtained by adding on to these *unity*, are the

[* See footnote † p. 197 above.]

ten coefficients of the determinant to the binary matrix $(x+py+qz+rt)$, which of course there is not the slightest difficulty in expressing in terms of scalar and vector affections of p, q, r and their combinations, if any one chooses to regard them as given in quaternion form.

Scholium. In what precedes it is very requisite to notice that only *general* cases are considered; and that there are multitudinous others which escape the direct application of this method, and do not conform to the rule which assigns the number of solutions. Thus, for example, the equation $x^2+px=0$, besides the solutions $x=0$, $x=-p$, will have two others which will require the method of the text to be modified in order to determine. Or take the most elementary case of all, the simple equation $px=q$. If p is not vacuous (that is, if its determinant when regarded as a matrix, or its modulus when regarded as a quaternion, is finite), there is the one solution $x=p^{-1}q$. But if p is vacuous, then, unless q is also vacuous, the equation is insoluble. If $q=0$, there will be two solutions; one of them $x=0$, the other $x=$ conjugate of p in quaternion terminology; or

$$x = \begin{matrix} -d; & b \\ c; & -a \end{matrix}, \text{ when } p = \begin{matrix} a; & b \\ c; & d \end{matrix}$$

in the language of matrices. If, p still remaining vacuous, q is vacuous but not zero, a further condition must be satisfied, namely, if

$$p = \begin{matrix} a; & b \\ c; & d \end{matrix} \text{ and } q = \begin{matrix} \alpha; & \beta \\ \gamma; & \delta \end{matrix},$$

the condition is $$a\delta + \alpha d - b\gamma - c\beta = 0;$$

or if $$p = a + bi + cj + dk \text{ and } q = \alpha + \beta i + \gamma j + \delta k,$$

the condition is $$a\alpha + b\beta + c\gamma + d\delta = 0.$$

When this condition (besides that of q being vacuous) is satisfied, the equation $px=q$ is soluble, and $p^{-1}q$ becomes finite but indeterminate, containing two arbitrary constants*.

* So in general if p, q be two simply vacuous matrices of any order, the condition that the equation $px=q$ may be soluble, or, in other words, that $p^{-1}q$ (a combination of an ideal with a vacuous matrix) may be non-ideal, may be shown to be that the determinant to the matrix $\lambda p+\mu q$ (where λ, μ are scalar quantities) shall vanish identically—which (p being supposed already to be vacuous) involves just as many additional conditions as there are units in the order of the matrix.

33.

ON HAMILTON'S QUADRATIC EQUATION AND THE GENERAL UNILATERAL EQUATION IN MATRICES.

[*Philosophical Magazine*, XVIII. (1884), pp. 454—458.]

In the *Philosophical Magazine* of May last I gave a purely algebraical method of solving Hamilton's equation in Quaternions, but did not carry out the calculations to the full extent that I have since found is desirable. The completed solution presents some such very beautiful features, that I think no apology will be required for occupying a short space of the *Magazine* with a succinct account of it.

Hamilton was led to this equation as a means of calculating a continued fraction in quaternions, and there is every reason for believing that the Gaussian theory of Quadratic Forms in the theory of numbers may be extended to quaternions or binary matrices, in which case the properties of the equation with which I am about to deal will form an essential part of such extended theory*. Let us take a form slightly more general than that before considered, namely, the form

$$px^2 + qx + r = 0,$$

with the understanding that the determinant of p (if we are dealing with matrices), or its tensor if with quaternions, differs from zero. Let us construct the ternary quadratic

$$au^2 + 2buv + 2cuw + dv^2 + 2evw + fw^2,$$

defined as the determinant of $up + vq + wr$, on the one supposition, or by means of the equations

$$a = Tp^2, \quad d = Tq^2, \quad f = Tr^2, \quad b = SpSq - SVpVq,$$
$$c = SpSr - SVpVr, \quad e = SqSr - SVqVr,$$

on the other supposition.

* I have found, and stated, I believe, in the form of a question in the *Educational Times* some years ago, that any fraction whose terms are real integer quaternions may be expressed as a finite continued fraction, the greatest-common-measure process being applicable to its two terms, provided *both* their Moduli are not odd multiples of an odd power of 2, which can always be guarded against by a previous preparation of the fraction.

On referring to the article of May [p. 226 above], it will be seen that the solution of the equation may be made to depend on the roots of a cubic equation in the quantity therein called λ. When fully worked out, this equation will be found to take the remarkable form $e^{\lambda\Omega}.I=0$, where I is the invariant of the ternary quadratic above written, and $\Omega = 2a\delta_c - a\delta_d$. It may also be shown that

$$x = -\frac{(p+b-u)(q-c-v)}{2\lambda},$$

where u is a two-valued function of λ, and v a linear function of u.

I shall suppose that I, the final term in the equation in λ, differs from zero: the solution of the given equation in x will then be what may be termed *regular*, and will consist of three pairs of actual and determinate roots. When $I=0$, the solution ceases to be regular; some of the roots may disappear from the sphere of actuality, or may remain actual but become indeterminate, or these two states of things may coexist. The first coefficient of the equation in λ is a, the determinant of p (or its squared tensor), which also must not be zero, as in that case one root at least of λ would be infinite. Let us suppose, then, that neither a nor I vanishes. The very interesting question presents itself as to what kind of equalities can arise among the *three* pairs of roots, and what are the conditions of such arising.

This equation admits of an extremely interesting and succinct answer as follows:—Let m represent $\frac{c+2d}{3}$; the equalities between the roots of the given equation in x will be completely governed, and are definable by the equalities existing between those of the biquadratic binary form

$$(a, b, m, e, f)(X, Y)^4*.$$

* If the equation is regarded as one in quaternions, the determining biquadratic is the modulus of x^2+xp+q; from which it follows immediately that, if p, q are *real* quaternions, all the four roots, say α, β, γ, δ, are imaginary. It may be shown that the roots of Hamilton's determining cubic are

$$d-\frac{(\alpha+\beta)(\gamma+\delta)}{4},\quad d-\frac{(\alpha+\gamma)(\beta+\delta)}{4},\quad d-\frac{(\alpha+\delta)(\beta+\gamma)}{4},$$

and these therefore are (as shown also by Hamilton) all of them real. The biquadratic serves to determine the points in which the variable conic associated to the equation px^2+qx+r (that is, the determinant to $xp+yq+zr$) is intersected by the absolute conic $xz-y^2$. Each root of the given equation corresponds to a side of the complete quadrilateral formed by the four points of intersection of these two conics; and thus we see that there are five cases to consider when the variable conic is a conic proper, according as it intersects or touches the fixed conic (which can happen in four different ways); and seven other cases where the conic degenerates into two intersecting or two coincident lines (in which cases the solution becomes irregular); namely, the intersecting lines may cut or touch in one or two points the fixed one, and may cut or touch the conic at their point of intersection, which gives five cases; and the coincident lines may cut or touch the fixed conic, which gives two more. Hence there are in all twelve principal cases to consider in Hamilton's form of the Quadratic Equation in Quaternions: or rather thirteen, for the case of the variable and fixed conics coinciding must not be lost sight of.

If the biquadratic has two equal roots, the given quadratic will have two pairs of equal roots.

If the biquadratic has two pairs of equal roots, the given quadratic will have four equal roots.

If the biquadratic has three equal roots, the quadratic will have three pairs of equal roots.

If the biquadratic has all its roots equal, the quadratic will have all its roots equal.

In the first case two of the three pairs of roots of the given quadratic coincide, or merge into a single pair.

In the second case, not only two pairs merge into one pair, but the two roots of that pair coincide with one another.

In the third case the three pairs merge into a single pair.

In the fourth case the two members of that single pair coincide with one another.

So long as the equation in x remains regular, no kind of equalities can exist between the roots other than those above specified.

For instance, let us consider the possibility of two values of x, and no more, becoming equal. First, let us inquire what is the condition to be satisfied in order that the scalar parts of two roots which belong to the same pair shall become equal. It may be shown that the sufficient and necessary condition that this may take place is that the irreducible sub-invariant of degree 3 and weight 6 (that is, the first coefficient of the irreducible skew-covariant of the associated biquadratic form $[a, b, m, e, f]$) shall vanish.

If, now, the *vectors* as well as the *scalars* of the two roots are to be equal, it may be shown that the *second* as well as the first coefficient of the skew-covariant must vanish. But this cannot happen without the discriminant vanishing*; for it may easily be seen that the discriminant of a binary biquadratic with its sign changed is equal to sixteen times the product of the first and last coefficients, less the product of the second and penultimate coefficients of its irreducible skew-covariant. Hence when two roots belonging to the same pair of the given quadratic coincide, two values of λ become equal, and therefore all four roots belonging to two pairs merge into one.

Again, it is not possible for two roots belonging to two pairs corresponding to two different values of λ to coincide; for in such case the expression

* The first two coefficients of the skew-covariant vanishing implies the existence of two pairs of equal roots and *vice versâ*. This is on the supposition made that a, the first coefficient of the given quartic, is not zero.

given for x shows that $pq, p, q, 1$ would be connected by a linear equation. But when this happens (as has been shown by me elsewhere), the invariant of the associated ternary quartic vanishes and the equation ceases to be regular. Thus, then, it appears that it is impossible for a single relation of equality (*and no more*) to exist between the roots of the given equation when its form is regular. So, again, it may be shown that it is impossible for four, and no more, relations of equality to exist between the roots.

It need hardly be added, that the equation $px^2+qx+r=0$ ceases to be regular when q or r vanishes.

The reader may satisfy himself as to the truth of what has been alleged as to the relation of the discriminant of a binary biquadratic to the coefficients of its skew-covariant by simple verification of the identity

$$\begin{aligned}16\,(a^2d-3abc+2b^3)\,(e^2b-3edc+2d^3)\\ -(a^2e+2abd-9c^2a+6b^2c)\,(e^2a+2edb-9ec^2+6d^2c)\\ =27\,(ace+2bcd-c^3-b^2e-ad^2)^2-(ae-4bd+3c^2)^3.\end{aligned}$$

The biquadratic equation in X, Y is what the determinant of $\lambda p+\mu q+\nu r$ becomes when X^2, XY, Y^2 are substituted therein for λ, μ, ν; so that we may say that $(a, b, m, e, f)(x, 1)^4$ is the determinant of px^2+qx+r, when x is regarded as an ordinary quantity. Let ϕx be any quadratic factor of this biquadratic function in x: I have found that $\phi x=0$ will be the *identical* equation to one of the roots of the given equation $fx=0$, where

$$fx=px^2+qx+r.$$

Between the two equations $fx=0$, $\phi x=0$, x^2 may be eliminated and x found in terms of known quantities: ϕx will have six different values, which will give the six roots of $fx=0$. It is far from improbable that a similar solution applies to a unilateral equation $fx=0$ of any degree n in matrices of any order ω.

Call Fx the determinant of fx when x is regarded as an ordinary quantity; then, if ϕx is an algebraical factor of the degree ω in x contained in Fx, it would seem to be in all probability true that $\phi x=0$ is the identical equation to one of the roots of $fx=0$; and, *vice versâ*, that the function identically zero of any such root is a factor of Fx. By combining the equations $fx=0$, $\phi x=0$, all the powers of x except the first may be eliminated, and thus every root of x determined. The solution of the given equation will depend upon the solution of an ordinary equation of the degree $n\omega$, and the number of roots will be the number of ways of combining $n\omega$ things ω and ω together. Thus, for a cubic equation in quaternions the number of roots would be $\frac{1}{2}6\,.\,5$, or 15. In the May number of this *Magazine* [p. 229 above] it was supposed to be shown to be 21; but it is quite conceivable that this determination may be erroneous, especially as it was deduced from general considerations of the degrees of a certain system of equations without attention being paid to their particular form, which might very well be such as to occasion a fall in the *order* of the system. I am strongly inclined, with the new light I have gained on the subject, to believe that such must be the case, and that the true number of roots for a unilateral equation in quaternions of the degree n is $2n^2 - n$*; in which case the theorem above stated, and which may be viewed as a marvellous generalization of the already marvellous Hamilton-Cayley Theorem of the identical equation, will be undoubtedly true for all values of n and ω. But I can only assert positively at present that it is true for the case of $n = 1$ whatever ω may be, and for the case of $n = 2$, $\omega = 2$†.

* From the number 21 above referred to, now known to be erroneous, the general value was inferred to be $n^3 - n^2 + n$, whereas it is demonstrably $2n^2 - n$ only for the *general* unilateral equation of degree n in quaternions, as I proved it to be for the *Jerrardian* form of that equation.

† I have since obtained an easy proof of the truth of the conjectural theorem for all values of n and ω; see the *Comptes Rendus* of the Institute of France for October 20th last [p. 197 above].

34.

NOTE ON CAPTAIN MACMAHON'S TRANSFORMATION OF THE THEORY OF INVARIANTS.

[*Messenger of Mathematics,* XIII. (1884), pp. 163—165.]

THE whole question as is well known consists in finding the free forms of $\Omega^{-1}0$, where

$$\Omega = a_0\delta a_1 + 2a_1\delta a_2 + \ldots + ia_{i-1}\delta a_i;$$

but, as long ago noticed by me* in the *Am. Math. Journal,* $\Omega^{-1}0$ is only a deformation of $V^{-1}0$, where

$$V = a_0\delta a_1 - a_1\delta a_2 + \ldots \pm a_{i-1}\delta a_i,$$

$\Omega^{-1}0$ being deducible from $V^{-1}0$ by altering the dimensions of the a elements which it contains in known numerical proportions, so that $\Omega^{-1}0$ may be said to be $V^{-1}0$ subjected to a known *strain*†.

To fix the ideas let $i = 3$ and call the a's by the names a, b, c, d or, for greater simplicity, $1, b, c, d$.

Let
$$b = r + s + t,$$
$$c = rs + rt + st,$$
$$d = rst.$$

Then the matrix

$$\frac{D(b, c, d)}{D(r, s, t)} = \begin{matrix} 1 & 1 & 1, \\ s+t & t+r & r+s, \\ st & tr & rs, \end{matrix}$$

so that

$$\frac{D(r, s, t)}{D(b, c, d)} = \begin{matrix} \dfrac{r^2}{(r-s)(r-t)} & \dfrac{s^2}{(s-r)(s-t)} & \dfrac{t^2}{(t-r)(t-s)}, \\ \dfrac{r}{(r-s)(r-t)} & \dfrac{s}{(s-r)(s-t)} & \dfrac{t}{(t-r)(t-s)}, \\ \dfrac{1}{(r-s)(r-t)} & \dfrac{1}{(s-r)(s-t)} & \dfrac{1}{(t-r)(t-s)}. \end{matrix}$$

[* Vol. III. of this Reprint, p. 570.]

† In fact the numerical multipliers of the terms in Ω may be taken perfectly arbitrary without producing any effect upon the form $\Omega^{-1}0$ than what may be represented by a *strain.*

Consequently

$$V = \Sigma \frac{r^2 - (r+s+t)r + (rs+rt+st)}{(r-s)(r-t)} \delta_r = \Sigma \frac{st}{(r-s)(r-t)} \delta_r.$$

In like manner in general for 1, $a_1, a_2, \ldots a_i$ we shall find, on writing

$$a_1 = r_1 + r_2 + \ldots + r_i,$$

$$a_2 = r_1 r_2 + r_2 r_3 + \ldots + r_{i-1} r_i,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

$$a_i = r_1 r_2 \ldots r_i,$$

$$V = \delta a_1 - a_1 \delta a_2 + \ldots \pm a_{i-1} \delta a_i = \Sigma \frac{r_2 r_3 \ldots r_i}{(r_1 - r_2)(r_1 - r_3) \ldots (r_1 - r_i)} \delta r_1.$$

Hence $$V^{-1} 0 = F(s_1, s_2, \ldots s_i),$$

where, in general, $$s_\omega = r_1^\omega + r_2^\omega \ldots + r_i^\omega;$$

and consequently the theory of invariants, which endoscopically treated in the ordinary way hinges upon symmetrical functions of the differences of a set of letters, is made to depend upon functions of the simple sums of powers commencing with the second power and ending with a power whose index is the order of any given finite quantic, but in the case of *perpetuants* taking in all the powers except the first.

It goes without saying that the same method applied to the *constrained* V will show that it is equal to $\Sigma \delta r_1$, so that V_0^{-1} is an arbitrary function of the differences of the r's corresponding to that hypothesis, as we know ought to be the case.

What has been established in the foregoing investigation is a principle of correspondence whose importance as a simplifying agent recalls Ivory's use of such principle in Attractions, namely, the remarkable algebraical law that any symmetrical function of the differences of a set of i quantities is a symmetrical function of the sums of the 2nd, 3rd, ..., ith powers of another equi-numerous set.

By virtue of this principle the numerical part of the Calculus of Invariants is capable of being entirely divorced from all question of algebraical content and a Zahl-Invariant theory comes into being, in its fundamental conception analogous to the Zahl-Geometrie of Schubert.

Further remarks on this subject will be found in the *Comptes Rendus de l'Institut* presumably for March 31 and April 7 of this year [p. 163 above].

35.

ON THE D'ALEMBERT-CARNOT GEOMETRICAL PARADOX AND ITS RESOLUTION.

[*Messenger of Mathematics,* XIV. (1885), pp. 92—96.]

I WILL presently state the simple geometrical problem which led D'Alembert to call into question the validity of the received Cartesian doctrine of positive and negative geometrical magnitudes, and which, according to Carnot, furnishes an unanswerable argument against it. See Mouchot, *La réforme Cartésienne,* pp. 74, 75.

Against this doctrine, presented in its crude form, the objections of these illustrious impugners of it are unquestionably well founded and unanswerable; but the inference to be drawn from this is not that no such or such-like doctrine reposing on an unassailable logical basis exists or is capable of being established (woe worth the day! when such a conclusion should be admitted), but that the doctrine as usually stated is incomplete and requires a supplement.

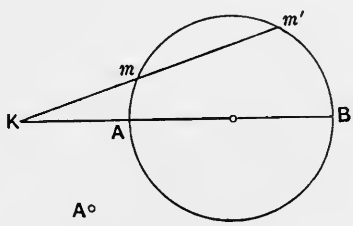
This has been anticipatively furnished by me many years ago in this very *Journal,* and in conjunction with the substitution of positive and negative indefinite rotation in lieu of Euclid's positive and limited angular magnitude, made the basis of a strictly logical deduction (which was before wanting) of the trigonometrical canon.

It consists in the notion of a line having, so to say, sides (returning upon itself at its two semi-points at infinity), or to put the matter in a more practical form, in regarding an Euclidean indefinite straight line as representing two distinct lines locally coincident, but running in contrary directions, and in referring the algebraical sign of any rectilinear segment to the concurrence or discordance of its flow (which is represented by the order in which its two extremities are named or written down) with that of the indefinite line, upon which it is supposed to be carried.

Thus, for example, AB taken on the upper side of a line or line-pair will be the negative of AB taken on the same side, but the same as BA taken on the under side.

I will now state the D'Alembert-Carnot problem. "Voici" says Carnot, "un exemple aussi simple que frappant, qui seul suffit pour renverser toute cette doctrine" of positive and negative magnitudes.

"D'un point K, pris hors d'un cercle donné, soit proposé de mener une droite Kmm', telle que la portion mm', interceptée dans le cercle, soit égale à une droite donnée.



"Du point K, et par le centre du cercle menons une droite KAB qui rencontre la circonférence en A et B. Supposons $KA = a$, $KB = b$, $mm' = c$, $Km = x$. On aura donc par les propriétés du cercle

$$ab = x(c + x) = cx + x^2$$

donc
$$x^2 + cx - ab = 0$$

ou
$$x = -\tfrac{1}{2}c \pm \sqrt{(\tfrac{1}{4}c^2 + ab)}.$$

x a deux valeurs: la première, qui est positive, satisfait sans difficulté à la question; mais que signifie la seconde, qui est négative? Il paraît qu'elle ne peut répondre qu'au point m', qui est le second de ceux où Km coupe la circonférence; et, en effet, si l'on cherche directement Km', en prenant cette droite pour l'inconnue x, on aura

$$x(x - c) = ab$$

ou
$$x = \tfrac{1}{2}c \pm \sqrt{(\tfrac{1}{4}c^2 + ab)}$$

dont la valeur positive est précisément la même que celle qui s'était présentée dans le premier cas avec le signe négatif. Donc, quoique les deux racines de l'équation

$$x = -\tfrac{1}{2}c \pm \sqrt{(\tfrac{1}{4}c^2 + ab)}$$

soient l'une positive et l'autre négative, elles doivent être prises toutes les deux dans le même sens par rapport au point fixe K. Ainsi, la règle qui veut que ces racines soient prises en sens opposés porte à faux. Si au contraire le point fixe K était pris sur le diamètre même AB et non sur le prolongement, on trouverait pour x deux valeurs positives et cependant elles devraient être prises en sens contraires l'une de l'autre. La règle est donc encore fausse pour ce cas.

"Si l'on dit que ce n'est pas ainsi qu'il faut entendre ce principe, que les racines positives et négatives doivent être prises en sens opposés, je demanderai comment il faut l'entendre? et j'en conclurai par là même qu'il faut une explication pour empêcher qu'il ne soit pris dans l'acceptation la plus naturelle. Il suit que ce principe est obscur et vague."

The answer has been already given to the question, "comment il faut entendre ce principe," and it will be seen in such a way as to remove all grounds for the charge of its being *any longer* "obscur et vague."

This is how the problem set out in full ought to be enunciated:

A complete line (that is, a line-pair or two-sided line) drawn from K cuts the circle in the points m, m'; mm' measured on either side of the line (and of course denoted quantitatively by the number of units of given length which it contains) is to be equal to c a given positive or negative number. Required the value of Am.

(1) Suppose K to be exterior to the circle as in the diagram above.

I distinguish the two sides of the complete line, as the under and upper line, and suppose the flow of the under one to make an acute Euclidean angle with the flow from K to the centre of the circle. In all cases

$$Km' = Km + mm',$$

and consequently the equation for finding x remains always $x^2 + cx = ab$, of which the two roots are $-\frac{1}{2}c + \sqrt{(\frac{1}{4}c^2 + ab)}$ and $-\frac{1}{2}c - \sqrt{(\frac{1}{4}c^2 + ab)}$.

Adhering to the letters of the diagram, if c is positive the two values of x will correspond to Am on the under line and Am' on the upper line of the line-pair. If, again, c is negative, the two values of x will correspond to Am on the upper and Am' on the under one.

(2) Suppose K to be within the circle.

It will still be true (paying attention to the signs) that $Km' = Km + mm'$ (that being a universal identity in algebraical geometry), but the algebraical values of KA, KB being contrary, we may regard KA as positive and equal to a, KB as negative and equal to $-b$, and shall have the equation

$$x^2 + cx = -ab,$$

of which the two roots are

$$-\tfrac{1}{2}c + \sqrt{(\tfrac{1}{4}c^2 - ab)}, \quad -\tfrac{1}{2}c - \sqrt{(\tfrac{1}{4}c^2 - ab)}.$$

Understand by the *two segments* Km and Km'.

We may suppose the indefinite line-pair mKm' to swing round K, its under-side in the position of coincidence with the diameter having the same flow as KA; then, if c is positive, until the swinging line revolving with the sun has described a right angle, the first root will be the *infrà*-diametral segment taken on the lower line (or side), and the second root the *suprà*-diametral segment taken on the upper line (or side) of the line-pair (or complete line); in the next quadrant of rotation the first root will be the *suprà*-diametral segment on the under and the second root the *infrà*-diametral segment on the upper side of the complete line. When c is negative a similar statement may be made if only the words *under* and *upper* are interchanged. In the critical position, when the swinger is at right angles to the diameter, the two roots become equal and undistinguishable; but throughout and subject to no exception, the complex of the two roots contains the complete solution of the problem, and the complete solution of the problem necessitates the retention of the complex of the two roots.

Thus, then, as in the preceding case, it has been shown that the Cartesian view of the equipollence of positive and negative roots (the latter Descartes influenced by hereditary prepossessions calls *radices falsae*) is made exact through the intermediation of the conception of sides to a line. D'Alembert and Carnot are entitled to the gratitude of Geometers and all lovers of truth for raising objections so perfectly well founded to the then, and even now, too prevalent interpretation of the meaning of the geometrical positive and negative, but the difficulty which they so justly appreciated and so clearly expressed is overcome and exists no longer.

P.S. I am informed that M. Laguerre has emitted the same view as that I have set forth relative to the sign to be given to geometrical distances, and made use of the same conception of the double or complete line-carrier.

My note on the subject appeared before my exodus across the Atlantic, probably nine or ten years ago. M. Laguerre's publication must have been many years posterior to this. The references to the reappearance of the theory on the other side of the Channel, obligingly furnished to me by M. Mannheim in Paris, have unfortunately got mislaid. I believe the communication containing it was made by M. Laguerre within the last three or four years, but it has already had time to find its way into some of the most esteemed French text-books. Being not only true but *the* truth, it must eventually find universal acceptance. It is not without interest (it seems to me) that we may regard a double or complete right line as a sort of embryonic embodiment of the idea of a Riemann Surface.

36.

SUR UNE NOUVELLE THÉORIE DE FORMES ALGÉBRIQUES*.

[*Comptes Rendus*, CI. (1885), pp. 1042—1046, 1110—1111, 1225—1229, 1461—1464.]

Si l'on imagine une fonction de dérivées différentielles (toutes d'un ordre supérieur à l'unité) de y par rapport à x, qui, sauf l'introduction d'un facteur multiple numérique, d'une puissance de $\frac{dy}{dx}$, ne change pas sa valeur quand on remplace x par y et y par x, il est évident qu'une telle fonction restera invariable (sauf l'introduction d'une constante comme facteur) quand pour x et y on substitue des fonctions linéaires quelconques, homogènes ou non homogènes de y et x. Ainsi une telle fonction conduira immédiatement à la connaissance d'un point singulier d'une courbe d'un degré quelconque. Le seul exemple d'une telle fonction, traité jusqu'à ce jour, est la simple fonction $\frac{d^2y}{dx^2}$ qui, par cette seule propriété, sans aucune autre considération, sert à démontrer l'existence d'une propriété projective de courbes dont la condition est $\frac{d^2y}{dx^2}=0$. Il nous paraît donc très utile de chercher un moyen de produire toutes les fonctions de cette espèce auxquelles nous donnerons le nom de *réciprocants purs* ou simplement *réciprocants*. On verra qu'il existe des réciprocants mixtes, c'est-à-dire contenant des puissances de $\frac{dy}{dx}$ $\left(\text{comme la forme bien connue de M. Schwarz, } \frac{dy}{dx}\frac{d^3y}{dx^3}-\frac{3}{2}\frac{d^2y}{dx^2}\frac{d^2y}{dx^2}\right)$ qui possèdent la même faculté d'invariance par rapport à l'échange de y avec x, comme les réciprocants purs, mais qui évidemment ne peuvent pas indiquer l'existence de points singuliers dans les courbes.

Nous écrirons, au lieu de $\delta_x y$, $\delta_x^2 y$, $\delta_x^3 y$, $\delta_x^4 y, \ldots$, les lettres $t, a, b, c, \ldots$, et pour leurs réciproques $\delta_y x$, $\delta_y^2 x$, $\delta_y^3 x, \ldots$, $\tau, \alpha, \beta, \gamma, \ldots$. On verra facilement que, pour que $F(t, a, b, c, \ldots)$ soit un réciprocant pur, F doit être d'un degré et d'un poids constant dans les lettres de chaque terme; de plus (pour un

[* See the lectures, below p. 303.]

réciprocant F d'une nature quelconque), on aura $F(a\ldots)/F(\alpha\ldots) = (-1)^\theta t^{2\lambda}$, où θ sera le plus petit nombre des lettres $a, b, c, \ldots$ dans un terme quelconque de F, et λ sera la moyenne arithmétique entre le poids et trois fois le degré de F, en comptant le poids de $t, a, b, c, \ldots$ comme étant $-1, 0, 1, 2, \ldots$. Cela donne lieu à une remarque importante par rapport aux *réciprocants mixtes*: pour qu'on puisse additionner deux formes mixtes afin de former un nouveau réciprocant, il faut non seulement que le degré et le poids soient les mêmes pour tous les deux, mais aussi le *caractère* qui dépend de la valeur de θ et que l'on peut qualifier comme caractère pair ou impair selon la parité de θ. Ainsi, par exemple, $2tb - 3a^2$ et a^2 sont tous deux réciprocants, mais $2tb$ ne le sera pas, parce que les *caractères* des deux données sont contraires. Il est facile de démontrer que, si R est un réciprocant quelconque,

$$(2tb - 3a^2)\,\delta_a R + (2tc - 4ab)\,\delta_b R + (2td - 5ac)\,\delta_c R + \ldots$$

sera aussi un réciprocant de même caractère que R. Ainsi, en commençant avec le réciprocant a, on peut obtenir une suite infinie de réciprocants mixtes : ces réciprocants ainsi obtenus ne seront pas en général irréductibles; mais, sans les réduire, leur forme fait voir immédiatement que tout réciprocant, qu'il soit pur ou mixte, peut être exprimé comme une fonction rationnelle et aussi (si l'on regarde t comme unité) entière de combinaisons *légitimes** de ces quantités.

Pour obtenir tous les réciprocants purs de poids, degré et ordre (c'est-à-dire nombre de lettres) donnés, linéairement indépendants les uns des autres, on peut former une équation partielle différentielle, linéaire, où R est la variable dépendante, et $a, b, c, \ldots$ les variables indépendantes; elle exprimera la condition nécessaire et suffisante pour que R soit un tel réciprocant et fournira un moyen sûr de résoudre le problème proposé. Voici la manière de démontrer ce théorème fondamental.

Si, dans l'équation

$$F(a, b, c, \ldots) = (-1)^\theta t^{2\lambda} F(\alpha, \beta, \gamma, \ldots),$$

on donne à y la variation ϵx, on voit que $a, b, c, \ldots$, et conséquemment F, restent invariables. Les variations de $\alpha, \beta, \gamma, \ldots$ sont faciles à déterminer, et la variation de t est donnée.

Ainsi, après quelques calculs faciles, en égalant à zéro, séparément, dans la variation de $t^{2\lambda}F(\alpha, \beta, \ldots)$, les termes qui contiennent t et ceux qui ne le contiennent pas, on arrive à deux équations dont l'une sera

$$\left(3a\frac{d}{da} + 4b\frac{d}{db} + 5c\frac{d}{dc} + \ldots\right) F(a, b, \ldots) = 2\lambda F,$$

* Je nomme *légitime* une combinaison quelconque de réciprocants où l'on évite *d'additionner* ceux dont le poids, le degré, l'ordre et le *caractère* ne sont pas les mêmes pour tous.

qui exprime la valeur numérique de λ, comme fonction du poids et du degré de F; l'autre équation, en écrivant

$$V = 3a^2\delta_b + 10ab\delta_c + (15ac + 10b^2)\,\delta_d + (21ad + 35bc)\,\delta_e + (28ae + 56bd + 35c^2)\,\delta_f + \ldots,$$

sera

$$VR = 0.$$

Pour voir la loi des chiffres arithmétiques dans V, formons les suites des coefficients de $(1+x)^i$ en commençant avec $i = 4$; divisons chaque coefficient *central* en deux parties égales, et supprimons la dernière moitié des séries numériques ainsi formées; on obtiendra ainsi la Table:

1	4	3		
1	5	10		
1	6	15	10	
1	7	21	35	
1	8	28	56	35
......				

En négligeant les deux premières colonnes, on trouve les nombres qui paraissent dans la formule.

On démontre ainsi que $VR = 0$ est une condition nécessaire pour que R soit un réciprocant. Mais il faut aussi démontrer que cette condition est suffisante. Soit donc D la valeur de $F(a, b, \ldots) - t^{2\lambda} F(\alpha, \beta, \ldots)$, exprimée comme une fonction de $a, b, c, \ldots$ seulement. D sera donc une fonction de la même forme que $F(a, b, \ldots)$.

On suppose que

$$\Delta D = 0\,;$$

c'est-à-dire que la variation de D produite par la substitution de $x + \epsilon y$ à x est égale à zéro, en vertu de l'équation $VR = 0$.

Donnons à y une variation arbitraire $y + \eta u$; alors, si D devient D', la variation de D' sera nulle, quand on substitue, pour x, $x + \epsilon y + \epsilon\eta u$, et, conséquemment, quand on substitue $x + 2y$ pour x; on aura donc

$$\Delta D' = 0,$$

et, en prenant la différence des variations de D et D', on obtient

$$\Delta\left(u'\frac{d}{da}R + u''\frac{d}{db}R + u'''\frac{d}{dc}R + \ldots\right) = 0.$$

Donc, à cause de la forme arbitraire de u, il faut que

$$\Delta\frac{d}{da}D = 0, \quad \Delta\frac{d}{db}D = 0, \quad \ldots;$$

et, en raisonnant sur $\frac{d}{da}D, \frac{d}{db}D, \ldots$ comme on a raisonné sur D, on voit que le Δ de chacune des dérivées secondes différentielles de D sera zéro; en poursuivant le même calcul, on trouve évidemment que le Δ d'une dérivée de D d'un ordre quelconque par rapport à $a, b, c, \ldots$ sera nul.

Donc D est nul ; car, dans le cas contraire, s'il contient un terme quelconque, dont les lettres peuvent être distinctes ou identiques, en isolant une seule de ces lettres et prenant la dérivée de D par rapport à toutes les autres lettres, on aura le Δ de la lettre isolée, c'est-à-dire de $\delta_x y, \delta_x{}^2 y, \ldots$, zéro quand on substitue $x + \epsilon y$ pour x, ce qui est absurde. Ainsi l'on voit que, quand $\Delta D = 0$, c'est-à-dire quand $VR = 0$, $D = 0$, ce qui était à démontrer.

Soient ω, i, j le poids, le degré et l'ordre d'un réciprocant quelconque : de même que pour les sous-invariants, le nombre de formes linéairement indépendantes s'exprime par $(\omega;\ i, j) - (\omega - 1;\ i, j)$, où, en général, $(\omega;\ i, j)$ signifie le nombre de partitions de ω en i parties dont nulle n'excède j ; ainsi l'on voit que, en vertu de l'équation $VR = 0$, on aura, pour le nombre des réciprocants linéairement indépendants, la formule

$$(\omega;\ i, j) - (\omega - 1;\ i + 1, j).$$

Mon long exil en Amérique expliquera, je l'espère, comment j'ai pu ignorer l'identité des invariants différentiels de M. Halphen avec les formes que j'ai nommées *réciprocants purs.* Les travaux vraiment remarquables de M. Halphen n'ont pas besoin de mes éloges et auront été couronnés par l'admiration de tous les géomètres dignes de ce nom.

Je crois cependant qu'il y a assez de différence entre le but et la marche de mes recherches sur ce terrain et ceux de M. Halphen pour justifier l'insertion dans les *Comptes rendus* de ma discussion de la théorie regardée comme une théorie de formes algébriques. Si je ne me trompe pas, M. Halphen, s'il l'a découverte, n'a fait nul usage de l'équation partielle différentielle que j'ai donnée et qui sert à établir le parallélisme merveilleux entre les invariants différentiels et les semi-invariants ordinaires.

De plus, il n'a pas eu occasion de faire allusion aux formes que j'appelle *réciprocants mixtes orthogonaux,* qui ne sont point compris dans la définition des *invariants différentiels,* et qui sont essentiels pour expliquer les singularités quasi-métriques des courbes.

Nous rappelons que par le mot *réciprocant* (sans qualification) il a été convenu de sous-entendre une forme de cette espèce qui ne contient pas t $\left(\text{c'est-à-dire } \frac{dy}{dx}\right)$ et nous avons trouvé que le nombre de ces réciprocants linéairement indépendants, du degré i, de l'étendue j (c'est-à-dire contenant $j + 1$ lettres distinctes) et du poids ω, s'exprime par la formule

$$(\omega;\ i, j) - (\omega - 1;\ i + 1, j),$$

où en général $(l;\ m, n)$ signifie le nombre de partitions de l en m ou un plus petit nombre que m de parties dont aucune n'excède n en grandeur; de sorte que $(l;\ m,\ n)$, quand m est plus grand que l, signifie la même chose que $(l;\ l,\ n)$, car tous les deux sont équivalents à $(l;\ \infty,\ n)$. Conséquemment

$$(i;\ i, j)-(i-1;\ i+1, j)=(i;\ i, j)-(i-1;\ i, j),$$

lequel sera toujours positif quand i et j sont tous les deux plus grands que l'unité; et, puisque a, qui est du degré 1, est un réciprocant, il s'ensuit que, pour un degré quelconque donné, il existe toujours des réciprocants (car on peut faire $\omega = i$), mais en nombre fini, car, en faisant croître ω, $(\omega - 1;\ i+1, \infty)$, au delà d'une certaine valeur de ω, deviendra nécessairement plus grand que $(\omega;\ i, \infty)$. On peut exprimer par $(l : m)$ ce que devient $(l;\ m, n)$ quand $n=\infty$, et alors $(\omega : i)-(\omega-1 : i+1)$ exprimera le nombre de réciprocants linéairement indépendants du poids ω et du degré i sans autre limitation. Ainsi on trouvera que du degré 1 il n'existe qu'un seul réciprocant du poids 0; pour le degré 2, un seul du poids 2; pour le degré 3, deux qui seront respectivement du poids 3 et du poids 4; etc.

On trouvera qu'étant donné j il existe toujours, sauf pour le cas où $j=1$, un réciprocant qui contient toutes les $j+1$ lettres et qui de plus contiendra un terme qui est un produit de la dernière lettre par une puissance de a. Ces formes, qu'on peut nommer les *protomorphes*, sont les analogues des formes a, $ac-b^2$, $a^2d+\ldots$, $ae+\ldots$, qu'on connaît dans la théorie des sous-invariants. Dans le cas des réciprocants, ces protomorphes seront a, ac, ..., $a^2d, \ldots$, $a^2e, \ldots$, $a^3f, \ldots$, $a^3g, \ldots$, etc.

Évidemment une fonction rationnelle *quelconque* des lettres peut, au moyen de substitutions successives, être exprimée comme une fonction rationnelle des protomorphes et de b divisée par une puissance de a. Soit donc R un réciprocant quelconque; on aura

$$a^2R+P+Qb+\ldots+Jb^i=0,$$

où P, Q, ..., J sont eux-mêmes des réciprocants. En opérant i fois sur cette équation avec notre opérateur V, on voit qu'on obtient $a^{2i}J=0$; donc J est nul, et l'on voit ainsi que tous les termes Q, ..., J disparaissent et que R (en faisant $a=1$) devient une fonction rationnelle et entière des protomorphes. Nous allons appliquer ce principe fondamental, commun aux deux théories des sous-invariants et des réciprocants, pour obtenir les formes irréductibles (les *Grundformen*) des réciprocants pour les ordres 2, 3, 4.

Faisons $j=2$, $i=2$, $\omega=2$ et supposons que le réciprocant R soit $\lambda ac+\mu b^2$; on obtient

$$VR=(3a^2\delta_b+10ab\delta_c)\,R=(6\mu+10\lambda)\,a^2b=0.$$

Donc $-\lambda : \mu :: 3 : 5$ et nous obtenons le réciprocant $3ac-5b^2$*.

* Il est bon de remarquer que $3ac-5b^2=0$, c'est-à-dire

$$3\frac{d^2y}{dx^2}\frac{d^4y}{dx^4}-5\left(\frac{d^2y}{dx^2}\right)^2=0,$$

indique que le point (x, y), quand cette équation est satisfaite par telles coordonnées d'une courbe quelconque, est un point supra-parabolique, c'est-à-dire où une parabole passe par 5 au lieu de 4 points consécutifs seulement.

Passons au cas $j=3$, $i=3$, $\omega=3$, et posons

$$R = \lambda a^2 d + \mu abc + \nu b^3.$$

On aura

$$VR = (3a^2\delta_b + 10ab\delta_c + 15ac + 10b^2\delta_d)\,R$$
$$= (3\mu + 15\lambda)\,a^3c + (9\nu + 10\mu + 10\lambda)\,a^2b^2 = 0.$$

On aura donc

$$\mu = -5\lambda, \quad 9\nu = 40\lambda,$$

de sorte qu'on peut écrire

$$R = 9a^2d - 45abc + 40b^3.$$

On reconnaîtra immédiatement que $R=0$ est l'équation différentielle donnée par Monge et retrouvée par M. Halphen à une conique et que

$$9\,(\delta_x{}^2y)^2\,(\delta_x{}^5y) - 45\delta_x{}^2y\,\delta_x{}^3y\,\delta_x{}^4y + 40\,(\delta_x{}^3y)^3 = 0$$

exprime la condition que le point $(x,\ y)$ d'une courbe quelconque sera un point d'inflexion du second ordre, c'est-à-dire un point où une conique passe par six points consécutifs. Le nombre de ces points peut être trouvé en fonction linéaire de n, ordre d'une courbe donnée, en opérant sur cette équation une transformation analogue à celle au moyen de laquelle on passe du système $y=0$, $\dfrac{d^2y}{dx^2}=0$ au système équivalent, mais épuré, $\phi=0$; $H\phi=0$*.

Passons au cas où $j=4$, $i=3$, $\omega=4$, et écrivons

$$R = \lambda a^2e + \mu abd + \nu ac^2 + \pi b^2c.$$

On aura

$$V = 3a^2\delta_b + 10ab\,\delta_c + (15ac + 10b^2)\,\delta_d + (21ad + 35bc)\,\delta_e,$$

et, en posant $RV=0$, on obtient, en égalant séparément à zéro les coefficients de a^3d, a^2bc, ab^3, les équations

$$21\lambda + 3\mu = 0, \quad 35\lambda + 15\mu + 20\nu + 6\pi = 0, \quad 10\mu + 10\pi = 0,$$

* Pour le cas d'une cubique, le nombre de ces points d'inflexion du second ordre est vingt-sept; on démontre facilement que ce sont les intersections de la courbe avec son covariant du *degré-ordre* 12 . 9.

On voit immédiatement, au moyen de notre théorie connue de *résidus géométriques*, que ces vingt-sept points sont les points de la cubique où elle est rencontrée par les neuf faisceaux des tangentes qu'on peut mener des neuf points d'inflexion ordinaire. Car un quelconque de ces points doit être tel que sa dérivée à l'indice 5 sera coïncidente avec le point lui-même. On aura donc 1, 1=1, 5, c'est-à-dire 2=4, ce qui veut dire que le tangentiel du point est un point d'inflexion ; ce qui était à démontrer.

Soit dit, par parenthèse, que la même théorie de résiduation enseigne que le point fixe Q, où une cubique donnée sera coupée par une autre cubique quelconque qui a en commun avec la première 8 points consécutifs à un point donné P, sera le troisième tangentiel de P et peut être nommé son *satellite* ; quand le satellite coïncide avec son primaire, en se servant pour le moment de la forme canonique pour exprimer la cubique donnée, et en nommant x, y, z les coordonnées du primaire, celles du satellite seront (d'après notre théorie exposée dans l'*American Journal of Mathematics*) x, y, z multipliés respectivement par des fonctions rationnelles de x^3, y^3, z^3, chacune du degré 21. [Vol. III. of this Reprint, p. 339.]

C'est un fait depuis longtemps connu que les points primaires qui coïncident avec leurs satellites (en ne tenant pas compte des neuf inflexions) sont en nombre 72.

et ainsi on peut écrire

$$R = 5a^2e - 35abd + 7ac^2 + 35b^2c.$$

Voici donc le système de protomorphes pour tous les ordres jusqu'au quatrième inclusivement :

$$a, \qquad (1)$$

$$3ac - 5b^2, \qquad (2)$$

$$9a^2d - 45abc + 40b^3, \qquad (3)$$

$$5a^2e - 35abd + 7ac^2 + 35b^2c. \qquad (4)$$

En combinant le cube du deuxième avec le carré du troisième, et en divisant par a, on obtient la forme (analogue au discriminant) de la cubique, mais d'un degré plus élevé,

$$\left.\begin{array}{l} 405a^3d^2 - 4050a^2bcd + 1728a^2c^3 \\ \qquad + 1585ab^2c^2 + 3600ab^3d - 18000b^4c^*. \end{array}\right\} \qquad (5)$$

En combinant le produit de (2) et de (4), linéairement, avec (5), on obtient

$$\left.\begin{array}{l} 4800a^2ce - 8000ab^2e - 2835a^2d^2 - 5376ac^3 \\ \qquad - 5250abcd + 30800b^3d + 11305b^2c^2. \end{array}\right\} \qquad (6)$$

Si l'on se borne aux lettres a, b, c, d, les formes (1), (2), (3), (5) formeront un système complet de *Grundformen* : si on laisse entrer la nouvelle lettre e, (5) n'est plus irréductible, et le système complet de *Grundformen* est constitué par les formes (1), (2), (3), (4), (6).

Tout cela se passe précisément comme avec les sous-invariants avec les mêmes lettres : les poids des formes sont les mêmes pour les deux systèmes, et la seule différence essentielle entre les deux consiste en ce fait, que les trois dernières formes subissent chacune une élévation d'une unité de degré en passant du système des sous-invariants à celui des réciprocants.

Il est nécessaire d'ajouter quelques mots sur les réciprocants mixtes, qui se distinguent en deux espèces, homogènes et hétérogènes. Comme exemple des premiers, on a la dérivée Schwarzienne $2tb - 3a^2$, laquelle, égalée à zéro, ne donne aucune espèce de singularité, mais signifie seulement qu'au point (x, y) on peut mener une conique qui passera par cinq points consécutifs, en ayant ses deux asymptotes parallèles aux axes, ou bien la forme $tc - 5ab$. Comme exemple de l'autre classe, on a la forme connue $(1 + t^2)b - 3ta^2$, dont l'évanouissement (pourvu que x, y soient des coordonnées *rectangulaires*) signifie que le point (x, y) est un point de courbure maximum ou minimum.

* Cette fonction, égalée à zéro, exprime que x, y sont les coordonnées d'un point par où l'on peut faire passer une parabole *cubique* ayant 5 points consécutifs communs à la courbe dont x, y sont les coordonnées.

Nous avons remarqué, par parenthèse, que l'équation

$$(1+t^2)\,b-3ta^2=0$$

indique l'existence d'une singularité au point dont les coordonnées sont les x, y sous-entendus dans t, a, b de l'équation.

Mais, pour que cela soit vrai, il faut introduire la restriction que x, y sont des coordonnées *rectangulaires.*

On peut donner le nom de *réciprocant orthogonal* à tout réciprocant mixte qui jouit de la propriété de rester invariable (sauf l'introduction d'une puissance de t) quand on opère sur x et y une transformation linéaire orthogonale. Cela étant convenu, on peut démontrer facilement que le coefficient différentiel par rapport à t d'un réciprocant est lui-même un réciprocant ou pur ou mixte. La proposition réciproque est aussi vraie, de sorte qu'on a le beau théorème suivant:

Si R et $\frac{dR}{dt}$ sont tous les deux réciprocants, alors R est un réciprocant orthogonal.

Par exemple, le réciprocant que nous avons cité plus haut a pour coefficient différentiel par rapport à t la Schwarzienne $2tb-3a^2$; donc c'est un réciprocant orthogonal; et, en effet, il exprime qu'au point (x, y), où l'équation $2tb-3a^2=0$ est satisfaite, on peut appliquer un cercle qui aura un contact du troisième ordre avec la courbe dont x et y sont les coordonnées; au contraire, la Schwarzienne elle-même ne correspond pas à une singularité quelconque, car sa dérivée par rapport à t, c'est-à-dire $2b$, n'est pas un réciprocant.

De même nous avons trouvé qu'en intégrant le réciprocant $2tc-10ab$ par rapport à t, entre les limites t et $-c-15a^3$, la forme résultante

$$(t^2+1)\,c-10abt+15a^3$$

sera un réciprocant et conséquemment un réciprocant orthogonal, de sorte que l'équation

$$(1+t^2)\,c-10abt+15a^3=0$$

sera la condition d'une singularité de la courbe $f(y, x)=0$ qui se rapporte aux points circulaires à l'infini*. Peut-être trouvera-t-on que l'intégrale, par rapport à t, d'un réciprocant mixte quelconque, prise entre des limites convenables, conduira nécessairement à un réciprocant orthogonal. Les singularités d'une courbe peuvent être partagées en trois classes: celles de la première classe seront projectives et peuvent être définies indifféremment au moyen de covariants de formes ternaires ou par des réciprocants purs;

* M. James Hammond, dont on connaît les belles et importantes découvertes dans la théorie invariantive des formes binaires, a trouvé l'intégrale de cette équation, que nous avons donnée dans un discours inaugural, prononcé devant l'Université d'Oxford, lequel va être publié dans le journal anglais *Nature*. [p. 278 below.]

celles de la deuxième classe seront non projectives, mais n'auront affaire qu'avec la ligne à l'infini ; les singularités de cette classe seront exprimables au moyen de réciprocants purs, mais non pas au moyen de covariants de formes ternaires. Restent celles de la troisième classe qui non seulement ne sont pas projectives, mais sont quasi métriques en caractère, c'est-à-dire ont des rapports avec les points circulaires à l'infini ; les singularités de cette classe sont signalées par l'évanouissement de réciprocants orthogonaux. Les réciprocants mixtes, qui ne sont ni purs ni orthogonaux, comme celui, par exemple, de M. Schwarz, ne répondront à aucune de ces trois espèces de singularités ; mais, quoique ne servant pas à représenter une propriété invariable d'une courbe, ils serviront souvent, peut-être toujours, comme bases des réciprocants orthogonaux, c'est-à-dire qu'ils seront les coefficients différentiels par rapport à t de ces derniers.

L'échelle des *protomorphes*, aussi bien dans la théorie des réciprocants purs que dans celle des sous-invariants, joue un rôle si capital, en ce qui concerne la détermination des formes irréductibles, qu'il nous semble indispensable de donner une démonstration rigoureuse de son existence dans l'une et l'autre théorie.

1° Quant aux sous-invariants, soit j l'ordre (c'est-à-dire $j+1$ le nombre des lettres que l'on considère). Si j est pair, on connaît les formes invariantives $ac + \ldots$, $ae + \ldots$, $ag + \ldots$, et l'on peut passer au cas où j est impair. Dans ce cas, le nombre de sous-invariants du poids j et du degré 3 sera

$$(j;\ 3, j) - (j-1;\ 3, j).$$

Mais il faut démontrer qu'il existe une forme de ce type, dans laquelle le coefficient du produit de a^2 et de la dernière lettre n'est pas nul.

Or je dis que le nombre des formes du type supposé, qui ne contiennent pas cette lettre, sera

$$(j;\ 3, j-1) - (j-1;\ 3, j-1).$$

Mais $$(j-1;\ 3, j) = (j-1;\ 3, j-1)$$

et, évidemment, $$(j;\ 3, j) - (j;\ 3, j-1) = 1\ ;$$

car les partitions dont le nombre est $(j;\ 3, j)$ contiendront toutes les partitions dont le nombre est $(j;\ 3, j-1)$ et en plus la partition constituée par j combiné avec des zéros.

Conséquemment il existe un sous-invariant dont un terme sera le produit de a^2 par la dernière des lettres que l'on considère.

2° Quant aux réciprocants purs de l'ordre j, nous avons déjà démontré qu'on peut satisfaire à l'inégalité

$$(j;\ x, j) - (j-1;\ x+1, j) > 0$$

en donnant à x une certaine valeur pas plus grande que $j-1$; et, pour démontrer qu'il y aura un réciprocant pur qui contient actuellement un terme

a^{x-1} multiplié par la dernière lettre, on pourrait faire précisément le même raisonnement que nous avons fait ci-dessus pour le cas précédent, et, puisque

$$(j; x, j) - (j-1; x+1, j)$$

excède de l'unité la valeur de $(j; x, j-1) - (j; x+1, j-1)$, on conclura avec certitude l'existence d'un protomorphe pour l'ordre j.

On peut, en général, trouver plusieurs valeurs de x qui rendent $(j; x, j) - (j-1; x+1, j)$ positif; parmi ces valeurs, il est commode d'adopter, comme *protomorphe* par excellence, une quelconque de celles pour lesquelles la valeur de x qui satisfait à cette inégalité est un minimum. Quand la lettre la plus avancée est inférieure à h, il n'y en a qu'un seul qui réponde à cette définition. Ainsi, par exemple, si $j=5$, l'inégalité

$$(5:x) - (4:x+1) > 1$$

donne pour x la valeur minimum $x=4$ et, avec l'aide de l'anéantisseur

$$3a^2\delta_b + 10ab\,\delta_c + (15ac + 10b^2)\,\delta_d + (21ad + 35bc)\,\delta_e + (28ae + 56bd + 35c^2)\,\delta_f,$$

on obtient le protomorphe

$$45a^3f - 420a^2be - 42a^2cd + 1120ab^2d - 315abc^2 - 1120b^3c.$$

Cela servira pour conduire à la connaissance de tous les réciprocants purs de l'ordre 5, dont le nombre sera au moins égal à celui des *Grundformen* du quantic binaire.

Dans une Communication qui suivra celle-ci, nous nous proposons de donner la théorie des réciprocants doubles ou multiples dont ceux de l'espèce pure sont précisément analogues aux invariants ou sous-invariants de systèmes de formes binaires.

La théorie des doubles réciprocants purs comprend nécessairement, comme cas particulier, l'étude des formes qui déterminent la position des tangentes communes à deux courbes et les points bitangentiels d'une seule.

Dans la remarque que nous avons faite, dans la première Note, sur le même sujet que la Note actuelle, à propos des réciprocants mixtes de la forme

$$[(2tb - 3a^2)\,\delta_a + (2tc - 4ab)\,\delta_b + (2td - 5ac)\,\delta_c + \ldots]^i\, a,$$

nous avons affirmé que tout réciprocant pur ou mixte peut être exprimé en fonction rationnelle et, de plus (quand on fait t égal à l'unité), entière de réciprocants de cette famille ; nous n'avons pas limité, comme nous aurions dû le faire, cette affirmation au cas de réciprocants homogènes : la proposition a besoin d'une certaine modification si on veut la rendre applicable au cas de réciprocants non homogènes; mais nous ne croyons pas nécessaire d'y insister en ce moment. Seulement, il est bon de remarquer que l'existence d'une équation partielle différentielle linéaire, que nous avons trouvée pour les réciprocants *purs*, suffit à établir immédiatement que ces réciprocants seront nécessairement, et sans exception aucune, ou homogènes ou séparables en parties homogènes, dont chacune sera elle-même un réciprocant.

37.

NOTE ON SCHWARZIAN DERIVATIVES.

[*Messenger of Mathematics*, XV. (1886), pp. 74—76.]

READING with great pleasure and profit Mr Forsyth's masterly treatise on Differential Equations (in my opinion the best written mathematical book extant in the English language), it occurred to me to find an easy proof of the fundamental and striking identity concerning Schwarzian derivatives, from which all others are immediate consequences, namely $(y, x) - (z, x) = \left(\frac{dz}{dx}\right)^2 (y, z)$, where one of which is, it may be observed, that (y, x) like y'' has the property of remaining a factor of what it becomes when x and y are interchanged; a persistent factor, so to say, of its altered self. I will return to this point subsequently, my present concern is to give a natural proof of the above striking identity; to do this, it will be sufficient to show that (considering y, z, x, the two former as fixed, and the last as a variable function of a common variable) $\dfrac{(y, x) - (z, x)}{\left(\frac{dz}{dx}\right)^2}$ does not vary when x becomes $x + \epsilon\phi(x)$ where ϵ may be regarded as infinitesimal*. For then this must remain true by successive accumulation when x becomes any function whatever of itself, and accordingly making $x = z$ we obtain (y, z) as the value of the invariable quotient as was to be shown. Call† $\epsilon\delta\phi x = \theta$, then using dashes to denote differentiation *quâ* x, and a parenthesis to signify the augmented value of the derivatives, we obtain

$$(y') = y' - \theta y',$$
$$(y'') = y'' - 2\theta y'' - \theta' y',$$
$$(y''') = y''' - 3\theta y''' - 3\theta' y'' - \theta'' y'.$$

* It is easy to see *à priori* that if the theorem is true, it can only be so in virtue of (y, x) when x receives an infinitesimal, becoming of the form

$$(1 - 2\theta)(y, x) + \lambda\theta'',$$

as is subsequently shown to be the case in the text.

[† Cf. p. 306 below.]

Hence
$$(y'y''') = y'y''' - 4\theta y'y''' - 3\theta' y'y'' - \theta'' y'^2,$$
$$\tfrac{3}{2}(y''^2) = \tfrac{3}{2}y''^2 - 6\theta y''^2 - 3\theta' y'y'',$$
$$(y')^2 = y'^2 - 2\theta y'^2,$$
$$((y, x)) = (1 + 2\theta)\{(y, x) - 4\theta(y, x)\} - \theta''$$
$$= (1 - 2\theta)(y, x) - \theta'',$$

and
$$((y, x) - (z, x)) = (1 - 2\theta)[(y, x) - (z, x)].$$

Hence
$$\left(\frac{(y, x) - (z, x)}{\left(\frac{dz}{dx}\right)^2}\right) = \frac{(y, x) - (z, x)}{\left(\frac{dz}{dx}\right)^2},$$

that is, the right-hand expression does not change, when y, z remaining fixed forms of function, x passes from one form of function of the independent variable to another; as was to be shown.

From what precedes, it appears that if y, z, x be regarded as functions of t, then $\{(y, x) - (z, x)\}\left(\frac{dx}{dt}\right)^2$ is a constant function in the sense that it remains unaltered, whatever function x may be of t, or which is the same thing if y and z functions of x when expressed as functions of x' (any function of x) are written y', z', then $(y', x') - (z', x')$ is identical with $(y, x) - (z, x)$, save as to a *factor* which depends only on the form of the *substitution* of x' for x. Hence to all intents and purposes, any function of the differences of the Schwarzian derivatives of any system of functions of the same variable, in respect thereto, is (in a sense comprising, but infinitely transcending the sense in which that wòrd is used in Algebra) a *covariant* of the system.

Addendum.—Let us for the moment call functions of x, y which either remain unaltered or only change their sign when x and y are interchanged self-reciprocating functions.

The first case of the kind is $\frac{y''}{y'^{\frac{3}{2}}}$, the next is $\frac{y'y''' - y''^2}{y'^2}$, and obviously a very general one of this sort will be the function
$$\left(\frac{1}{y'^{\frac{1}{2}}}\frac{d}{dx}\right)\log y'.$$

For greater simplicity, let us call the *numerator* of any such function when expanded and brought to the lowest possible common denominator, a *reciprocant*, the highest index of differentiation which such reciprocant contains its *order*, and the number of factors in each term its *degree*. Then in any reciprocant so formed the degree is always just one unit less than the order: but as a matter of fact the function so obtained is in general not irreducible, so that its degree may be depressed, and it becomes a question of much interest to form the scale of degrees of reciprocants of this sort. For the orders 2, 3, 4, 5, 6 the degrees in question are respectively 1, 2, 2, 3, 3. Calling the successive derivatives of y, a, b, c, d, ..., they will be found to be

$$a,$$
$$b,$$
$$2ac - 3b^2,$$
$$ad - 5bc,$$
$$2a^2e - 15acd - 10ad^2 + 35b^2c,$$
$$2a^2f - 21abc - 35acd + 60ab^2d + 110bc^2,$$

where each form is obtained by operating upon the preceding one with the operator $a(b\delta c + c\delta d + d\delta e + \ldots) - \lambda b$ (λ meaning half the weight + the degree of the operand), combining the result of this operation in each *alternate* case with a *legitimate* combination of those that precede, and in that case dividing out by a. I have proved that in this way can be obtained an infinite progression of reciprocants, of which the leading terms (substituting numbers for letters), will be alternately of the forms $1^i.(2i+1)$ and $1^i.(2i+2)$. Every other reciprocant can be formed algebraically from these primordial forms, as every seminvariant can be obtained from the primordial forms a, $ac - b^2$, $a^2d - 3abc + 2b^3$, ... The two theories run in parallel courses, but their relationship is that which naturalists call homoplazy as distinguished from homogeny; I propose to give further developments of this new algebraical theory in a subsequent Note.

38.

ON RECIPROCANTS.

[*Messenger of Mathematics*, XV. (1886), pp. 88—92.]

IN a note on Invariant Derivatives in the September number of the *Messenger* I have given a definition and examples of reciprocants.

If in any of the forms at the end of the postscript to the note we restore to $a, b, c, \ldots$ their values $\delta_x y, \delta_x^2 y, \delta_x^3 y, \ldots$ any such function divided by a certain power of $\delta_x y$ will change its sign, but otherwise remain unaltered when x and y are interchanged. The index of that power is the degree added to half the weight and will be called the index of the reciprocant. Any product of i of such reciprocants will be a reciprocant of the same kind or contrary kind to those in the table (subsequent to a) according as i is odd or even. In the latter case the interchange of x and y will leave the function absolutely unaltered. Reciprocants which cause a change of sign will be said to be of an odd, those which cause no change of sign of an even character. Any linear function of reciprocants of the same weight, degree, and *character* will be itself a reciprocant of that character, but reciprocants of opposite characters cannot be combined to form a new reciprocant: those of an odd character may be regarded as analogous to skew, those of an even character to non-skew seminvariants; the rule against combining forms of opposite characters becomes superfluous in the case of seminvariants, because those that offer themselves for combination as having the same weight and degree must of necessity be of like character. Any reciprocant being given there is a simple *ex post facto* rule for assigning its character without any knowledge of the mode of its genesis, namely its character is odd or even according as the smallest number of letters other than a in any of its terms is odd or even. Thus the *character* of a reciprocant whose leading term is a^2e, or ab^2e, or $abce$ is odd; that of one whose *leading* term is abe or abf is even, as is also that of the remarkable reciprocant $bd - 5c^2$ in which no power of a appears.

A further important distinction between the two theories* is that there are two linear reciprocants a and b but only one linear seminvariant. As an illustration of the combinatorial *law* of *like character* it will be seen that if we operate upon $2ac - 3b^2$ with the operator

$$a(b\delta a + c\delta b) - 3b,$$

* That is of reciprocants and invariants.

we obtain a new reciprocant

$$2ad - 10bc + 9b^3,$$

of which the character is the same as that of b^3, namely both are odd; we may therefore add $-9b^3$ to the latter expression, and then dividing out by $2a$ there results the reciprocant $ad - 5bc$, but we cannot combine $2ac - 3b^2$ with b^2 because these two reciprocants are of opposite characters.

Again, remembering that a is of an even and b of an odd character, the three reciprocants

$$-\tfrac{45}{4}b^4, \quad 5(ac - \tfrac{3}{2}b^2)^2, \quad 3ab(ad - 5bc)$$

are all of an even order, hence we may add them together and divide the sum by a^2, which gives the new reciprocant $3bd - 5c^2$ a form not containing the first letter a.

No seminvariant exists, nor, except the one just given $bd - 5c^2$, have I been able to discover any other reciprocant in which the first letter does not make its appearance†.

The infinite progression of odd reciprocants with the leading terms

$$ac, \quad ad, \quad a.a.e, \quad a.a.f, \quad a.a.a.g, \quad a.a.a.h, \quad \ldots$$

will easily be seen to exist by virtue of the general theorem that any reciprocant of degree, extent, and weight (say briefly of *dew* i, j, w) gives birth to two others of the same character as its own, one of *dew* $i+1, j+1, w+2$, the other of *dew* $i+1, j+2, w+3$.

For let

$$\tfrac{1}{2}w + i = \lambda,$$

then denoting the operator

$$b\delta_a + c\delta_b + \ldots \text{ by } \Omega,$$

and the result of the action of Ω upon itself $(\Omega*)^2$, which is in fact $\Omega^2 + \Omega_2$ (Ω_2 meaning $c\delta_a + d\delta_b + \ldots$); $(a\Omega - \lambda b)R$ will obviously be a reciprocant of *dew* $i+1, j+1, w+2$, and will give rise to a second reciprocant

$$\{a\Omega - (\lambda + \tfrac{3}{2})b\}(a\Omega - \lambda b)R,$$

which is

$$a^2(\Omega*)^2 - (2\lambda + \tfrac{1}{2})ab\Omega R - \lambda acR + (\lambda^2 + \tfrac{3}{2}\lambda)b^2R;$$

the last term of this being a reciprocant of the same character as the entire expression may be omitted, and dividing out the residue by a we obtain the second new reciprocant

$$\{a(\Omega*)^2 - (2\lambda + \tfrac{1}{2})b - \lambda c\}R,$$

which will be of *dew* $i+1, j+2, w+3$, as was to be shown.

It is easy to see that every reciprocant must be a rational integral function of the forms above stated commencing with a, b, $2ac - 3b^2$ (whose *dew*'s are alternately of the form $i, 2i-1, 3i-2$; $i, i-2, 3i-1$) divided by some power of a. For if any reciprocant contains only the letters $a, b, \ldots$

† Since the above went to press I have made the capital discovery that there are an infinite number of such reciprocants, and that all those of a given weight, extent and degree may be obtained by aid of a certain quadratico-linear partial differential equation.

h, k, l, it may be expressed as a rational integral function of the protomorph in which l first appears and of the letters $a, b, \ldots k$ divided by a power of a, and consequently the reciprocant may be so expressed, and continually repeating this process of substitution it follows that the reciprocant will be a rational integer of the protomorphs exclusively divided by a power of a*: this of course will necessarily be found only to contain combinations of like character; we already know the converse that the sum of all combinations of like character of the protomorphs is a reciprocant†. If any homogeneous reciprocant consists of portions of unlike degree (although of the same index) it is obvious that each portion must be itself a reciprocant, for if $P, P', P'' \ldots$ be such portions, $P + P' + P'' \ldots$ must be identical with $\Pi + \Pi' + \Pi'' + \ldots$ when $\Pi, \Pi', \Pi'' \ldots$ are the same functions of $\alpha, \beta, \gamma \ldots$ (that is, $\delta_y x, \delta_y{}^2 x, \delta_y{}^3 x \ldots$) that $P, P', P'' \ldots$ are of $a, b, c \ldots$. If then we make

$$P - a^{2\lambda}\Pi = \Delta, \quad P' - a^{2\lambda}\Pi' = \Delta' \ldots,$$

we have $\Delta + \Delta' + \Delta'' + \ldots$ identically zero.

But $P, P' \ldots$ being of the same index but different degrees must be of different weights, and consequently $\Delta, \Delta', \ldots$ are of different weights. Hence we must have $\Delta = 0$, $\Delta' = 0$, &c., as was to be shown.

It follows from this that every reciprocating function whatever may be obtained by an algebraical combination of the protomorphs, and consequently by an algebraical combination of the forms

$$\left(\frac{1}{y'^{\frac{1}{2}}}\delta_x\right)^i \log y',$$

* The proof that every seminvariant is a rational integral function of the protomorphs is very similar: any proposed seminvariant is by the method employed in the text shown to be at worst a function of the protomorphs and of b; but the terms involving any power of b must disappear because no identical equation can connect seminvariants with a non-seminvariant b. In the text we see in like manner that any given reciprocant may be reduced to the form $H + K$, where H and K are protomorphic combinations of opposite character, so that one of them will disappear.

† Another general mode of generating a class of reciprocants would be to express any function of $a, b, c, \ldots$ say $\phi(a, b, c, \ldots)$ under the form $\psi(\alpha, \beta, \gamma, \ldots)$. The product $\phi(a, b, c, \ldots)\,\psi(a, b, c, \ldots)$, or its numerator, will then obviously be a reciprocant. To take a simple example,

$$c = \frac{d^3y}{dx^3} = -\frac{\dfrac{dx}{dy}\cdot\dfrac{d^3x}{dy^3} - 3\left(\dfrac{d^2x}{dy^2}\right)^2}{\left(\dfrac{dx}{dy}\right)^5} = -\alpha\gamma + 3\beta^2 \div \alpha^5.$$

Hence, by the rule laid down, $c(ac - 3b^2)$, that is, $ac^2 - 3b^2c$ ought to be a reciprocant, which is right, for it is equal to $(2ac - 3b^2)^2 - 9b^4$ divided by a multiple of a. The law that the factors of seminvariants must be seminvariants cannot be extended to the theory of reciprocants. In this case the factors may some or none of them be reciprocants, and the others on reciprocation exchange forms monocyclically or polycyclically with one another. *I add the remark that this is not true of pure reciprocants, that is, those in which* $\frac{dy}{dx}$ *does not appear. Every factor of a pure reciprocant must be itself a reciprocant.*

and that we should gain nothing in generality by operating with successive operators of the form

$$\left(\frac{1}{y'^{\frac{1}{2}}}\delta_x\phi_1\cdot\right),\quad \left(\frac{1}{y'^{\frac{1}{2}}}\delta_x\phi_2\cdot\right),\ \ldots$$

where $\phi_1, \phi_2, \ldots$ are arbitrary functions of $y' \pm \frac{1}{y'}$, instead of with the simple operator $\frac{1}{y'^{\frac{1}{2}}}\delta_x$ continually repeated.

The results of using the more general operators would only amount to algebraical combinations of the results obtained from the simple forms

$$\left(\frac{1}{y'^{\frac{1}{2}}}\delta_x\right)^i \log y',$$

where i may take all values from zero to infinity*.

As in the case of seminvariants so also reciprocants would in extent contain only a finite number of ground-forms; but furthermore for reciprocants limited in degree the number of ground-forms will also be finite. Whether reciprocants which are irreducible for a given extent ever cease to be so and become reducible when the order is increased, as is the case with seminvariants, remains to be seen†.

In order to facilitate the verification of the results obtained and to be obtained it may be well to express the successive derivatives of x in regard to y in terms of those of y in regard to x, that is, of $\alpha, \beta, \gamma, \ldots$ in terms of $a, b, c, \ldots$ as shown in the following short table.

$$\begin{array}{ll}
 & \div \\
a = \alpha & \alpha^2, \\
b = -\beta & \alpha^3, \\
c = -\alpha\gamma + 3\beta^2 & \alpha^5, \\
d = -\alpha^2\delta + 10\alpha\beta\gamma - 15\beta^3 & \alpha^7, \\
e = -\alpha^3\epsilon + 15\alpha^2\beta\delta + 10\alpha^2\gamma^2 - 105\alpha\beta^2\gamma + 105\beta^4 & \alpha^9, \\
f = -\alpha^4\zeta + 21\alpha^3\beta\epsilon + 35\alpha^3\gamma\delta - 210\alpha^2\beta^2\delta - 280\alpha^2\beta\gamma^2 + 1260\alpha\beta^3\gamma - 945\beta^5 & \alpha^{11}, \\
\left.\begin{array}{l} g = -\alpha^5\eta + 28\alpha^4\beta\zeta + 56\alpha^4\gamma\epsilon + 35\alpha^4\delta^2 - 378\alpha^3\beta^2\epsilon - 1260\alpha^3\beta\gamma\delta + 3150\alpha^2\beta^3\delta \\ \qquad - 280\alpha^3\gamma^3 + 6300\alpha^2\beta^2\gamma^2 - 17325\alpha\beta^4\gamma + 10395\beta^6 \end{array}\right\} & \alpha^{13},
\end{array}$$

where $a, b, c, d, e, f, \ldots$ represent the successive derivatives of y with respect to x; and $\alpha, \beta, \gamma, \delta, \epsilon, \zeta, \ldots$ of x with respect to y.

In any subsequent paper on reciprocants in this Journal, I shall make the absolutely necessary transliteration referred to in a preceding footnote, replacing the present letters $a, b, c, d, \ldots$ by the letters $t, a, b, c, \ldots$ or possibly, for reasons which carry great weight, by the expressions

$$t,\quad 2a,\quad 2.3b,\quad 2.3.4c, \ldots$$

* This is not true of homogeneous reciprocants.

† I have since found that this *is* true for reciprocants, as for seminvariants.

39.

NOTE ON CERTAIN ELEMENTARY GEOMETRICAL NOTIONS AND DETERMINATIONS.

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A CURVE, as every one knows, may be regarded as a *locus* of points or as an *assembly* of directions, every point being common to two consecutive directions of the assembly, and every direction to two consecutive points of the locus; the locus is called the *envelop* of the assembly (that is part of the accepted language of geometry), and, conversely, the assembly may be called the *environment* of the locus. So we may regard a surface as an assembly of tangent planes or as a locus of points standing to each other in the relation of envelop and environment, and extend these definitions to space of any number of dimensions.

By a *plasm*, waiting a better word, we may understand a figure analogous to a point-pair in a line, a triangle in a plane, a pyramid in space, etc.; and an n-gonal plasm or n-gon will signify a plasm having n vertices and n faces themselves $(n-1)$-gons.

It is easy and desirable to find the general value of the content of a regular n-gon, say $abcde$, all whose edges we may call unity.

If $$b\beta = \tfrac{1}{2}ab, \quad c\gamma = \tfrac{2}{3}c\beta, \quad d\delta = \tfrac{3}{4}d\gamma \ldots,$$

it is easily seen by an elementary process of integration that $\beta, \gamma, \delta \ldots$ are the centres of figure to the successive plasms ab, abc, $abcd$, ..., and, making

$$ba = p_1, \quad c\beta = p_2, \quad d\gamma = p_3 \ldots,$$

each term in $p_1, p_2, p_3 \ldots$ will be perpendicular to the one which precedes it, so that, if V_n is the content of the plasm,

$$(1, 2, 3 \ldots n)^2\, V_n = p_1 p_2 \ldots p_n.$$

Moreover, we shall have

$$p_n^2 = 1 - \left(\frac{n-1}{n}\right)^2 p^2_{n-1},$$

of which the general integral is

$$p_n^2 = \frac{n+1}{2 \,.\, n} + C(-)^n \frac{1}{n^2};$$

in the present case, since $p_1 = 1$, $C = 0$, so that

$$V_n^2 = \frac{n+1}{(1 \,.\, 2 \ldots n)^2\, 2^n}.$$

If a, b, c be the angles of a fixed triangle, and A, B, C are proportional to the distances of a variable line from a, b, c, respectively, we may denote the line by $A : B : C$; as regards a variable point, it will presently be seen to be advantageous to denote its proportional coordinates, not, as is rather more usually done, by equimultiples of its distances from the three sides, but as equimultiples of these distances multiplied by the sides of the triangle from which they are measured*; so that, calling these coordinates a, b, c, the image† of the line at infinity becomes $a + b + c$.

Consider now the universal mixed concomitant (which it will be convenient to call a *mutuant*) $Aa + Bb + Cc$ (where a, b, c, A, B, C are used in lieu of the more usual letters $x, y, z, \xi, \eta, \zeta$); it will readily be seen that, when a, b, c vary, and A, B, C are fixed, the mutuant images the line $A : B : C$, and that, when A, B, C vary and a, b, c are fixed, the mutuant images the *radiant* point $a : b : c$; that is to say, $Aa + Bb + Cc = 0$ is true for every point in the point-containing line $A : B : C$ in the one case, and to every line through the *radiant point* $a : b : c$ in the other.

Supposing, then, that the two kinds of coordinates are chosen in this manner, we see (what would not be the case if the simple distances were taken) that a form F and its "polar-reciprocal" ϕ image the self-same curve referred to the self-same fundamental triangle.

These consequences would moreover continue to subsist if, calling the distances of a line from the vertices P, Q, R, and of a point from the sides p, q, r, we took $\Lambda P : \mathrm{M}Q : \mathrm{N}R$, $\lambda p : \mu q : \nu r$ for the two sets of coordinates, provided only that $\lambda\Lambda F = \mu \mathrm{M} G = \nu \mathrm{N} H$; F, G, H being the distances of the sides from the vertices of the fundamental triangle, in which case the line at infinity would no longer be imaged by $a + b + c$. I shall, however, adhere in what follows to the convention above laid down. I need hardly add that in like manner, in space taking $A : B : C : D$ (the distances of a plane from the

* Or rather divided by the distances of these sides from the opposite angles of the fundamental triangle, whose coordinates thus become 1, 0, 0, 0, 1, 0, 0, 0, 1.

† If $F=0$ is the equation to any *locus* or *assembly*, I call F the *image*, and such locus or assembly the *object*; to a given image responds in general an absolutely definite object, but, when the object is given, the image is only determined to a *constant* factor *près*.

vertices of a fundamental pyramid) as the coordinate-representation of such plane, and $a:b:c:d$ (the contents of the volumes which any variable point makes with the respective faces) as the coordinate-representation of such point, the mutuant $aA+bB+cC+dD$ will be the image of the radiant point $a:b:c:d$ when the capital letters are the variables, and of the plane $A:B:C:D$ when the small letters are the variables, meaning of course that $Aa+Bb+Cc+Dd=0$ will be true of every point in the plane $A:B:C:D$ and of every plane through the point $a:b:c:d$, and, as before, F and ϕ polar-reciprocals to each other will image the self-same surface (referred to the self-same fundamental pyramid) viewed as a locus or envelop on the one hand, as an assembly or environment on the other.

If a, b, c, d be used to signify the actual as distinguished from the proportional coordinates of a point, a linear function of these is constant, whereas it is a quadratic function of $A, B, C, D \ldots$, when used to signify the actual distances of a variable line, plane, &c., from the vertices of the fundamental plasm which is constant; and it is the principal object of this note to determine the form of this quadratic function, which, as Prof. Cayley was the first to show, may be expressed by the determinant to a matrix standing in close relation to the well-known "invertebrate symmetrical matrix," the determinant to which represents a numerical multiple of any plasm in terms of its edges, as, for example:

$$\begin{vmatrix} . & ab & ac & ad & 1 \\ ba & . & bc & bd & 1 \\ ca & cb & . & cd & 1 \\ da & db & dc & . & 1 \\ 1 & 1 & 1 & 1 & . \end{vmatrix}$$

where $ab, ac, bc \ldots$ are used for brevity to signify the measure of absolute distance between a, b, a, c, $b, c \ldots$, that is, stand for what in ordinary notation would be denoted by $(ab)^2$, $(ac)^2$, $(bc)^2$, ... This may be quoted as the mutual-distance matrix; its determinant, besides representing a numerical multiplier of the squared content of the pyramid when equated to zero, expresses the conditions of the four points a, b, c, d lying in a plane, the former property being a consequence immediately deducible by *strict* algebraical reasoning from the latter.

That this determinant does image the condition of the plasm to which the points $a, b, c, d \ldots$ are the vertices, losing one dimension of space, may be shown in a somewhat striking manner as follows. If for a moment we use x, y, z, the distances of any point in the plane of abc from bc, ca, ab as coordinates, the equation to a circle circumscribed about abc will be of the form $fyz+gzx+hxy$, and, calling the sides of the triangle a, b, c respectively,

$ax+by+cz$ is constant. Hence, substituting for z its value in terms of x and y, the image of the circle may be put under a form in which fb and ga will be the coefficients of y^2 and x^2 respectively; but, since x and y are *proportional* to the Cartesian coordinates y and x respectively, the coefficients of x^2 and y^2 must be equal. Hence $f:g:h::a:b:c$, and if now ax, by, cz, instead of x, y, z, be used as the coordinates of the variable point, the image to the circumscribing circle becomes $\Sigma \frac{ayz}{bc}$, or if we please Σa^2yz, that is, $\Sigma bcyz$, where bc stands as convened for $(bc)^2$.

Hence, if a, b, c, d be the vertices of a pyramid, $\Sigma abyz$ will be the image of the circumscribing sphere, for when any coordinate t is made zero the image becomes that of a circle; and so universally for a plasm of any number of dimensions.

Consider the case of a circle, and suppose that

$$\begin{vmatrix} . & ab & ac & 1 \\ ba & . & bc & 1 \\ ca & cb & . & 1 \\ 1 & 1 & 1 & . \end{vmatrix}$$

vanishes; this means that the line $x+y+z$ touches the circle

$$abxy+bcyz+cazx.$$

But, if $x+y+z$ images the line at infinity, it must *cut* this (as it cuts any other circle) in two distinct points, namely, the so-called circular points at infinity. Hence $x+y+z$ must, when the above determinant vanishes, cease to be the line at infinity, which can only come to pass by the triangle abc losing a dimension of space, and a, b, c coming into a straight line, in which case $x+y+z=0$, instead of being true of a particular line, is true of every point in the plane.

Just in like manner, if

$$\begin{vmatrix} . & ab & ac & ad & 1 \\ ba & . & bc & bd & 1 \\ ca & cb & . & cd & 1 \\ da & db & dc & . & 1 \\ 1 & 1 & 1 & 1 & . \end{vmatrix}$$

vanishes, unless $x+y+z+t$ ceases to image the plane at infinity, this plane would touch the sphere $\Sigma abxy$, that is, would cut it in a pair of straight lines, whereas it intersects it in a circle. Consequently the plasm $abcd$ must, as before, lose one dimension, and so in general. The content of a plasm vanishes when the mutual-distance determinant does so, and the latter as well as the former may be expressed rationally in terms of ordinary Cartesian coordinates; but the expression for the content (being linear in each set of coordinates) is obviously indecomposable, and must therefore be a numerical multiple of some power of the mutual-distance determinant; a comparison of dimensions shows at once that this power is the square root.

As regards the numerical multiplier, when the plasm has all its edges equal to unity (say a triangle, for example), the mutual-distance determinant becomes

$$\begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix},$$

which is easily transformable into

$$\begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & \bar{1} & 0 & 0 \\ 1 & 0 & \bar{1} & 0 \\ 1 & 0 & 0 & \bar{1} \end{vmatrix},$$

of which the value is -3; and so in general for a regular plasm with $(n+1)$ vertices; that is, in space of n dimensions the mutual-distance determinant, say D_n, becomes $(-)^{n+1}(n+1)$, whereas the (volume)^2, say V_n^2, has been shown to be $\dfrac{n+1}{2^n(1\,.\,2\,\ldots\,n)^2}$.

Hence, universally,

$$D_n = (-)^{n+1}\, 2^n\, (1\,.\,2\,\ldots\,n)^2\, V_n^2.$$

It may be here noticed that, if p be the perpendicular from any vertex on an opposite face of the plasm whose content is V_{n-1}, we shall have

$$V_{n-1}p = nV_n.$$

Consequently,
$$D_{n-1}p^2 = (-)^n\, 2^{n-1}\, \{1\,.\,2\,\ldots\,(n-1)\}^2\, V^2_{n-1}\, p^2$$
$$= (-)^n\, 2^{n-1}\, (1\,.\,2\,\ldots\,n)^2\, V_n^2 = -\tfrac{1}{2}D_n.$$

I now pass on to the leading motive of this note, namely, the determination of the connection between the coordinates $A, B, C \ldots$ drawn from $a, b, c \ldots$.

It is clear *à priori* that the form of the condition will be in all cases that a homogeneous quadratic function of the distances must be constant. Thus, for example, when there are four points, if A, B, C be assumed, we may describe three spheres with these quantities as radii, and the fourth point will be determined by means of one of the pairs of tangent planes drawn to them, the particular pair depending on the relative signs attributed to

A, B, C. Hence, if $F(A, B, C, D) = \infty$ be the general equation, each of the quantities must enter in the second and no higher degree; moreover, since by transporting the plane from which the distances are measured parallel to itself, A, B, C, D will be all increased by the same quantity, F must express a function of their differences, and consequently, since any two distances may be interchanged, F can contain no terms of the first order in the variables, so that $F = 0$ must amount to the predication of a homogeneous quadratic function of the distances being constant.

Thus, for example, in the case of three points, we have the well-known equation

$$\Sigma (ab)(A - C)(B - C) = \tfrac{1}{4}(abc)^2.$$

Suppose now that A, B, C are taken in proportions consistent with making

$$\Sigma (ab)^2 (A - C)(B - C) = 0.$$

Let $\Sigma (ab)^2 (A - C)(B - C) = P \,.\, Q$, where P, Q are two linear functions of A, B, C; then P, Q image two radiant points, each of which will have the property that any of its rays is at an infinite distance from a, b, c, or at all events, if it should pass through one of them, from the other two, and it is easy to anticipate that these two points must be the circular points at infinity. That such is the fact is obvious, because (using Cartesian co-ordinates) the perpendicular distance from any point upon $x \pm \sqrt{-1} \,.\, y$ contains zero in its denominator; so that the two points of the absolute may be regarded as the centres of two points of rays, all of them infinitely distant from the finite region.

But these two points are the intersections of the circumscribing circle with the line at infinity, and consequently their collective equation will be found by taking the resultant of $\Sigma ab\, xy$, Σx, ΣAx, which is well known to be the determinant of the quadratic function bordered by the coefficients of the two linear ones. Hence the constant quadratic function in A, B, C, namely, $\Sigma ab(A - B)(A - C)$, ought to be a numerical multiple of the determinant

$$\begin{vmatrix} . & A & B & C & . \\ A & . & ab & ac & 1 \\ B & ba & . & bc & 1 \\ C & ca & cb & . & 1 \\ . & 1 & 1 & 1 & . \end{vmatrix},$$

as is the case, the value of this determinant being

$$-2\Sigma ab(A - C)(B - C).$$

The same thing may be shown in a more elementary manner as follows. Combining

$$x + y + z = 0, \quad abxy + bcyz + cazx = 0,$$

we have $$acx^2 + (bc + ca - ab)\,xy + bcy^2 = 0,$$

at each point of the absolute. And, taking $x_1y_1z_1$, $x_2y_2z_2$ as the coordinates at these two points, it follows that

$$x_1x_2 : y_1y_2 : z_1z_2 : x_1y_2 + x_2y_1 : y_1z_2 + y_2z_1 : z_1x_2 + z_2x_1$$
$$:: bc : ca : ab : -bc - ca + ab : -ca - ab + bc : -ab - bc + ca.$$

And, as the two points will be imaged by

$$x_1A + y_1B + z_1C, \quad x_2A + y_2B + z_2C,$$

respectively, it follows that their collective image will be

$$\Sigma\,\{bcA^2 + (bc - ab - ac)\,BC\},$$

which is easily seen to be identical with

$$\Sigma bc\,(A - B)\,(A - C).$$

The universal algebraical theorem upon which the first method of proof depends is the well-known one that, if Q is a quadratic function and $L_1, L_2, \ldots L_i$ i linear functions of j variables, and if Q' (where j is not less than $i + 1$) is what Q becomes when i of its variables are expressed in terms of the rest, then the necessary and sufficient condition of the discriminant of *every* such Q' vanishing is that the determinant to Q bordered by the coefficients of the i linear functions shall vanish. When j is equal to $i + 1$, the theorem shows that the resultant of the quadratic and its i attendant linear functions will be the bordered determinant in question. In the above example we had $j = 3$, $i = 2$.

Let us now proceed to apply a similar principle to the case of four points a, b, c, d in space.

If we take the case $x^2 + y^2 + z^2 + t^2 = 0$, any tangent plane to it at x', y', z', t' will be

$$x'x + y'y + z'z + t't,$$

and, as $$x'^2 + y'^2 + z'^2 + t'^2 = 0,$$

it follows that every tangent plane will be at infinite distance from any point external to it; and, as this is true wherever the centre of the cone be placed, and all the cones so obtained have the "circle at infinity" in common, —it follows that every tangent plane to the circle at infinity is infinitely distant from any external point in the finite region,—the infinitely-infinite system of planes thus obtained one may regard, if one pleases, as consisting of sheaves of planes whose axes form the environment to the circle at infinity, and will be the correlative to the infinitely-infinite system of points in the plane at infinity, which are infinitely distant from all external planes in the finite region. We see, then, that the coordinates to each such plane must satisfy the condition that, on making $\Sigma x = 0$ and $\Sigma Ax = 0$, and expressing any two of the variables x, y, z, t in terms of the two others, the discriminant of the form then assumed by $\Sigma abxy$ must vanish, and consequently, as before, the mutual-distance determinant to the points a, b, c, d, bordered with a row and column of units and a row and column consisting of the letters A, B, C, D, will represent to a numerical factor *près* the constant quadratic function of distances, that is, this function will be

$$\begin{vmatrix} . & A & B & C & D & . \\ A & . & ab & ac & ad & 1 \\ B & ba & . & bc & bd & 1 \\ C & ca & cb & . & cd & 1 \\ D & da & db & dc & . & 1 \\ . & 1 & 1 & 1 & 1 & . \end{vmatrix},$$

and obviously a similar algebraical conclusion will continue to apply, whatever may be the number of points n in a space of $n-1$ dimensions.

As regards the value of the constant, in any case, that may be obtained by taking a face of the plasm as the *term* (line, plane, etc.) from which the distances A, B, C ..., are measured; that is, we may make $B=0$, $C=0$, $D=0$ ..., provided we make A equal to the perpendicular from a on the opposite face. The value of the bordered determinant then becomes the *negative* of the squared perpendicular from a on bcd ... multiplied by the mutual-distance determinant to bcd ...; that is, by virtue of what has previously been shown, will be half of the mutual-distance determinant of $abcd$ ....

Hence the complete relation between A, B, C, D may be exhibited by making

$$\begin{vmatrix} -\frac{1}{2} & A & B & C & D & . \\ A & . & ab & ac & ad & 1 \\ B & ba & . & bc & bd & 1 \\ C & ca & cb & . & cd & 1 \\ D & da & db & dc & . & 1 \\ . & 1 & 1 & 1 & 1 & . \end{vmatrix} = 0,$$

and similarly for any number of points.

Professor Cayley has obtained the same result by a more direct but not more instructive process, as follows. Taking, by way of example, three points, $A+k$, $B+k$, $C+k$, (where k is infinite,) may be regarded as the distances of a, b, c from a fourth point at an infinite distance, and accordingly we may write

$$\begin{vmatrix} . & ab & ac & (A+k)^2 & 1 \\ ba & . & bc & (B+k)^2 & 1 \\ ca & cb & . & (C+k)^2 & 1 \\ (A+k)^2 & (B+k)^2 & (C+k)^2 & . & 1 \\ 1 & 1 & 1 & 1 & . \end{vmatrix} = 0.$$

For the *gnomon* bordering the square formed by the small letters and dots, we may substitute

$$\begin{vmatrix} . & . & . & 2kA+A^2 & 1 \\ . & . & . & 2kB+B^2 & 1 \\ . & . & . & 2kC+C^2 & 1 \\ 2kA+A^2 & 2kB+B^2 & 2kC+C^2 & -2k^2 & 1 \\ 1 & 1 & 1 & 1 & . \end{vmatrix},$$

without altering the value of the determinant, which therefore, remembering that k is infinite, is in a ratio of equality to $(2k)^2$ multiplied into the determinant

$$\begin{vmatrix} . & ab & ac & A & 1 \\ ba & . & bc & B & 1 \\ ca & cb & . & C & 1 \\ A & B & C & -\frac{1}{2} & . \\ 1 & 1 & 1 & . & . \end{vmatrix}.$$

This last determinant therefore must vanish, agreeing with what has been shown above by a more purely geometrical method*. I will now proceed to develop this determinant deprived of its constant term, expressing it as a function of the differences of the capital letters.

It is obvious that it may be expressed as a sum of terms of which each variable part will be of one or the other of these three forms

$$(A-B)^2, \quad (A-B)(A-C), \quad (A-B)(C-D);$$

and accordingly we may distribute the totality of the terms of the constant function of difference into three families depending on the form of the variable argument.

In general, if we consider any *invertebrate* symmetrical determinant expressed by the *umbral* notation

$$\begin{vmatrix} aa & ab & ac \ldots al \\ ba & bb & bc \ldots bl \\ \ldots & \ldots & \ldots \quad \ldots \\ la & lb & lc \ldots ll \end{vmatrix},$$

* As a corollary, we may infer, from the vanishing of this determinant, that, using the notation previously employed,

$$\frac{D_n}{V_n^2} = -\tfrac{1}{2}n^2 \frac{D_{n-1}}{V^2_{n-1}},$$

and consequently that $D_n = -(2)^n (1.2 \ldots n)^2 V_n^2$,

and that thus the content of a regular plasm with unit edges and $(n+1)$ vertices is

$$\frac{n+1}{2^n(1.2\ldots n)^2}, \text{ namely, } \frac{3}{16}, \frac{1}{72}, \frac{5}{9.2^{10}} \cdots$$

for triangle, pyramid, plu-pyramid, etc.

where $aa = bb = cc = ll \ldots = 0$ and $pq = qp$, we have this simple rule of proceeding:

Divide the letters $a \ldots l$ in every possible manner into cyclical sets, each set containing at least two letters.

Any cycle $a_1 a_2 \ldots a_i$ is to be interpreted as meaning

$$a_1 a_2 \,.\, a_2 a_3 \ldots a_{i-1} a_i \,.\, a_i a_1,$$

which, by virtue of the supposed condition $ab = ba$, will be the same in whichever direction the cycle is read, the effect of the inversion of the cycle being merely to give the same product over again, written under the form $a_1 a_i \,.\, a_2 a_1 \ldots a_1 a_{i-1}$.

The cycle of two letters $a_1 a_2$ must be interpreted to mean $(a_1 a_2)^2$. If now $C_1 C_2 \ldots C_i$ are cycles of two letters each, and $\chi_1 \chi_2 \ldots \chi_j$ cycles of three or more letters, the total value of the determinant will be

$$\Sigma\,(-)^{n+i+j}\, 2^j\, C_1 C_2 \ldots C_i \chi_1 \chi_2 \ldots \chi_j.$$

If, the principal diagonal terms remaining zero, the other terms were general, then the expression of the value of the determinant, calling the cycles $C_1 C_2 \ldots C_\nu$, and making no distinction between the case of their being binary or super-binary, would be $\Sigma\,(-)^{n+\nu} C_1 C_2 \ldots C_\nu$; only it would have to be understood that each cycle of two letters, as (ab), would mean $(ab)^2$, but a cycle of three or more letters, as (abc), would mean $ab \,.\, bc \,.\, ca + ac \,.\, cb \,.\, ba$.

This being premised, it is easy to deduce the following rule for the determination of the *three* different families of terms belonging to the constant determinant of distances, which, to avoid prolixity, must be left to the reader to verify.

Family I.—Omitting any two letters, and forming all possible cyclical products with the remaining $(n-2)$ letters, if $C_1 C_2 \ldots C_\nu$ be any set thereof, and ν' the number of them containing more than two letters, the general term will be $\Sigma\Sigma\,(-)^{n+\nu} 2^{\nu'} C_1 \,.\, C_2 \ldots C_\nu\,(A-B)^2$, a, b being the two letters which do not occur in the cycles $C_1 C_2 \ldots C_\nu$.

Family II.—Omitting any one letter, and forming with the remaining $n-1$ letters, in every possible way, *a chain* χ containing two or more letters, and cycles $C_1 C_2 \ldots C_\nu$, then, supposing the chain to be $bcd \ldots kl$, and understanding by (χ) the product $bc \,.\, cd \ldots kl$, the general term will be

$$\Sigma\Sigma\,(-)^{n+\nu}\, 2^{\nu'+1} C_1 C_2 \ldots C_\nu\,(\chi)\,(A-B)\,(A-L),$$

a being the letter which does not appear in the chain or any of the cycles, and ν' meaning as before the number of the cycles which contain at least three elements.

Family III.—Form all the letters in every possible way into two *chains* (each containing two or more letters) χ, χ', and into cycles C_1, C_2, $\ldots$ C_ν;

then, supposing the initial and final letters of χ to be a, h, and of χ' to be k, l, the general term of this family will be

$$2\Sigma(-)^{n+\nu}2^{\nu'+1}C_1C_2\ldots C_\nu(\chi)(\chi')\{(A-K)(H-L)+(A-L)(H-K)\}.$$

I subjoin in the following table the *types* of the coefficients of the several families for all the values of n from 2 up to 7; the vacant cycle ( ) of course means unity, and a cycle (ab) means $(ab)^2$; that is, the *fourth* power of the length ab.

Every cycle enclosed in a parenthesis of three or more letters, will be understood to be affected with a coefficient 2, and for greater brevity the variable part of each term is *left to be supplied.* A round parenthesis indicates a cycle, a square parenthesis a chain.

Number of Letters	Types	Name of Family
2	$(\ )$	1st
3	(bc)	2nd
4	$-(cd)$	1st
,,	$2[bcd]$	2nd
,,	$2[ab].[cd]$	3rd
5	(cde)	1st
,,	$-2[bcde]:2(bc)[de]$	2nd
,,	$-2[ab][cde]$	3rd
6	$-(cdef):(cd)(ef)$	1st
,,	$-2(bcd)[ef]:-2(bc)[def]:-2[bcdef]$	2nd
,,	$-2(ab)[cd][ef]:[abc][def]:[ab][cdef]$	3rd
7	$(cdefg)-(cd)(efg)$	1st
,,	$2(bcde)[fg]:2(bcd)[efg]:-2(bc)(de)[fg]\ 2(bc)[defg]:-2[bcdefg]$	2nd
,,	$2(abc)[de][fg]:2(ab)[cd][efg]:-2[abc][defg]:-2[abcdefg]$	3rd

Thus, for example, the constant function of distances for three points in a plane is $2\Sigma bc(A-B)(A-C)$; for four points in space is

$$-\Sigma cd(A-B)^2+2\Sigma bc.cd(A-B)(A-D)$$
$$+2\Sigma ab.cd\{(A-C)(B-D)+(A-D)(B-C)\};$$

for five points in hyper-space is

$$2\Sigma(cd.de.ec)(A-B)^2-2\Sigma(bc.cd.de)(A-B)(A-E)$$
$$+2(bc)^2(de)(A-D)(B-E)$$
$$-2\Sigma ab.cd.de.ec\{(A-C)(B-E)+(A-E)(B-C)\}.$$

The part of the constant function of distances for seven points belonging to the 2nd family of terms will be

$$4\Sigma bc.cd.de.eb.fg(A-B)(A-E)+4\Sigma bc.cd.db.ef.fg(A-E)(A-G)$$
$$-2(bc)^2(de)^2fg(A-F)(A-G)+2(bc)^2(de.ef.fg)(A-D)(A-G)$$
$$-2\,bc.cd.de.ef.fg(A-B)(A-G).$$

The number of types in each family for n points is easily expressible by a generating function.

Obviously in the 1st family this number is the number of ways of resolving n into parts none less than 2; that is, it is the coefficient of x^{n-2} in

$$\frac{1}{1-x^2 \,.\, 1-x^3 \,.\, 1-x^4 \ldots}.$$

In the 2nd family, it is the sum of the number of ways of decomposing $n-3$, $n-4$, ... into parts none less than 2; that is, it is the coefficient of x^{n-3} in

$$\frac{1+x+x^2+\ldots}{(1-x^2)(1-x^3)\ldots}, \text{ that is, in } \frac{1}{(1-x)(1-x^2)(1-x^3)\ldots}.$$

In the 3rd family, if the number of ways of dividing r into two parts, neither of them less than 2, is called (r), and of dividing $(n-r)$ into any number of parts, none less than 2, is called $[n-r]$, the number of types is $\Sigma\,(r)\,[n-r]$; that is, it is the coefficient of x^{n-4} in

$$\frac{1+x+2x^2+2x^3+3x^4+3x^5+\ldots}{(1-x^2)(1-x^3)(1-x^4)\ldots}, \text{ that is, in } \frac{1}{(1-x)(1-x^2)^2(1-x^3)(1-x^4)\ldots}.$$

Hence the total number of types in all three families combined will be the coefficient of x^{n-2} in

$$\frac{(1-x)(1-x^2)+x(1-x^2)+x^2}{1-x \,.\, 1-x^2 \,.\, 1-x^3 \ldots}, \text{ that is, in } \frac{1}{1-x \,.\, (1-x^2)^2 \,.\, 1-x^3 \,.\, 1-x^4 \ldots}.$$

Consequently, the indefinite partitions of 0, 1, 2, 3, 4, 5, 6, 7, ... being 1, 1, 2, 3, 5, 7, 11, 15, ..., the series for the type-number will be found by summing all the terms in the odd and even places successively. We thus obtain the series 1, 1, 3, 4, 8, 11, 19, 26, ... for the number of types in the constant-distance function for 2, 3, 4, 5, 6, 7, 8, 9, ... points respectively.

It may be worth while to exhibit the rule for the formation of the constant function of distances under a slightly different aspect.

As before, by the reading of any cycle, understand the product of its successive duads affected with the multiplier -1 or -2, according as the number of letters in the cycle is two or more than two.

By a *modified* reading of a cycle, understand what the reading becomes on substituting for any two duads pq, rs the product $(P-Q)(R-S)$, as for instance $(A-B)(C-D)$ in lieu of $ab\,.\,cd$, $(A-B)(B-C)$ in lieu of $ab\,.\,bc$, and (which can only happen in the case of a cycle of two letters), $(A-B)(B-A)$, that is, $-(A-B)^2$ in lieu of $ab\,.\,ba$.

Then, to find the constant function of distances to any given set of letters, we must begin with distributing the letters in every possible way into cycles containing between them two or more letters. Each such combination of cycles we may call a distribution.

In each distribution the cycle is to be taken (each in its turn), and the *sum* of its modified readings is to be multiplied by the product of the readings of the remaining cycles, if there are any. The sum of these sums (or the single sum, if there is but one cycle) is the portion of the quadratic function sought, due to the particular distribution dealt with; and the sum of these double sums, taken for each distribution in succession, is the total value of the function, and will be equal exactly to its representative determinant when the number of letters is odd, and to the same with its sign changed when that number is even.

As an example for five letters a, b, c, d, e, there will be ten distributions of the form $(ab)\,(cde)$, and twelve distributions of the form $(abcde)$.

From any one of the first ten distributions, as $(ab)(cde)$, by modifying first (ab) and then (cde), we obtain

(1) $2\,(cd\,.\,de\,.\,ec)\,(A-B)\,(B-A)$,

(2) $2\,(ab)^2\,\{ce\,(C-D)\,(D-E)+dc\,(D-E)\,(E-C)+ed\,(E-C)\,(C-D)\}$.

And from a distribution of the form $(abcde)$ we obtain, by operating on consecutive duads,

$$5 \text{ terms of the form } -2\,\{cd\,.\,de\,.\,ea\,(A-B)\,(B-C)\},$$

and, by operating on non-consecutive duads,

$$5 \text{ terms of the form } -2\,\{bc\,.\,de\,.\,ea\,(A-B)\,(C-D)\}\,*.$$

The sum of all the sums of terms due to the twenty-two distributions is the constant function of distances for the five given letters.

In the case of six letters the distributions into cycles will be of four kinds, corresponding to the partitions 6; 4, 2; 3, 3; 2, 2, 2.

The first kind will contain two types of the 3rd family and one of the 2nd family; the second kind will contain one type of each of the three families, and the third and fourth kinds single types of the 2nd and 1st families respectively, thus giving eight distinct types of terms in all, as should be the case according to the rule.

* It will be observed that the distribution $(acbde)$ will give a term

$$-2\,\{cb\,.\,de\,.\,ea\,(A-C)\,(B-D)\},$$

in which the literal part $cb\,.\,de\,.\,ea$ is equal to the literal part $bc\,.\,de\,.\,ea$ in the term above expressed. This is how it comes to pass that the terms of the 3rd family may be grouped in pairs, as stated in the prior mode of arranging the result according to families instead of according to cycles.

40.

ON THE TRINOMIAL UNILATERAL QUADRATIC EQUATION IN MATRICES OF THE SECOND ORDER.

[*Quarterly Journal of Mathematics*, XX. (1885), pp. 305—312.]

In the May number [p. 225 above] of the present year of the *London and Edinburgh Philosophical Magazine* (disfigured by numerous errors or inaccuracies) I investigated the number of the solutions of an equation in quaternions or matrices of the second order, belonging to what I term the unilateral class, meaning one in which the coefficients of any actual power of the unknown quantity lie on the same side of it; this number for the Jerrardian Trinomial form I proved *strictly* is $2i^2 - i$ (i being the degree of the equation) and with evidence little short of moral certainty $i^3 - i^2 + i$* in the general case where none of the terms are wanting†.

But it must be well borne in mind that these numbers only apply when the coefficients are left general, and that for special relations between them some or all of the roots may become either ideal or indeterminate, or some the one and some the other. In all cases of equations in matrices one principal feature of the investigation is, or should be, to determine the equation of condition between the coefficients, in order that the solution may lose or retain its normal form; if we wish to avoid being compelled to enter upon a complicated consideration of exceptions piled upon exceptions, it is necessary to presuppose a certain criterion function to be other than zero; otherwise it is like the opening of Pandora's box, letting loose an almost incalculable train of vexatious inquiries scarcely worth the trouble they give to answer correctly.

* This article was written and sent to the press many months ago. I have since shown that the number of roots of a general unilateral equation of degree i in matrices of the order ω is the number of combinations of $i\omega$ things taken ω and ω together, and consequently for the case of quaternions is $2i^2 - i$ for the general and not merely for the Jerrardian form. See [above, pp. 197, 233. Also] *Nature*, Nov. 13, 1884.

† I made the assumption that the required number is an analytical function of ω.

Take as an instance the subject of monothetic equations. I have defined a monothetic equation to be one in which all the coefficients are functions of a single matrix, which may be called the base. In such an equation of the degree i and of the order ω in the matrices, we may suppose the unknown quantity to be a function of the base, and then the general formula for expressing a function of a matrix as a rational and integral function of the matrix with the aid of its latent roots, shows that i^ω and no more of such roots exist. But this in no manner precludes the possibility of the existence of other roots which are not functions of the base. Thus, for example, in the very simple case of the equation $x^2+px=0$, where x and p are quaternions or matrices of the second order, I have shown in the *Comptes Rendus* [pp. 174, 179 above] that besides the four determinate ones, all of which (0 included) may be regarded as functions of p, there are two other *indeterminate* ones, each one containing an arbitrary constant, and neither of them (to use quaternion language) coplanar with the base. Here there is a sort of reversion to the normal case of 3 pairs of roots to an unilateral quadratic, with the modification of two of them having become indeterminate. It becomes then of importance to fix accurately the condition of this normal state of things ceasing to exist. Being intent on the Denumeration theory of the roots in the general quadratic, I did not in the paper cited do this explicitly for the unilateral quadratic, although I gave there my own form of solution. Moreover, there are other features of much interest belonging to the question, which, for the same reason, I omitted to notice. These omissions and shortcomings it is the object of this present article to supply.

Starting with the form $x^2 - 2px + q = 0$, and for convenience of comparison with Hamilton's formulæ treating p, q indifferently as matrices or as quaternions, and forming the equation $x^2 - 2Bx + D = 0$, where B, D are scalars to be determined, so that $B = Sx$ and $D = Tx^2$, we shall have

$$2x = (p-B)^{-1}(q-D).$$

If now we understand by b, c, d, e, f

$$Sp,\ Sq,\ Tp^2,\ S(Vp\,Vq),\ Tq^2 \text{ respectively,}$$

by means of the general formula

$$T\pi^2.(\pi^{-1}\chi)^2 - 2S(V\pi\,V\chi)(\pi^{-1}\chi) + T\chi^2 = 0\,*,$$

[remembering that

$$T(p-B)^2 = d^2 - 2bB + B^2,$$

$$T(q-D)^2 = f^2 - 2cD + D^2,$$

* This formula, which I have not met with in Treatises on Quaternions, is a particular case only of the general Theorem in Matrices, that if

$$A\lambda^\omega + B\lambda^{\omega-1}\mu + \ldots + L\mu^\omega$$

is the determinant to $(\lambda L + \mu M)$, where L and M are two matrices of the order ω and λ and μ two ordinary quantities, then

$$A(L^{-1}M)^\omega - B(L^{-1}M)^{\omega-1}\ldots + (-)^\omega L = 0.$$

and $$S\{V(p-B)\,V(q-D)\} = e - bD - cB + BD],$$

we shall obtain [see p. 188 above]

$$4(d-2bB+B^2)\,x^2 - 4(e-bD-cB+BD)\,x + (f-2cD+D^2) = 0.$$

Hence, writing $$B - b = u, \quad D - c = v,$$

$$d - b^2 = \alpha, \quad e - bc = \beta, \quad f - c^2 = \gamma,$$

and comparing with each other the two quadratic equations in x, we may write

$$u^2 + \alpha = \lambda, \quad uv + \beta = 2\lambda(u+b), \quad v^2 + \gamma = 4\lambda(v+c).$$

Eliminating v from the latter two equations there results

$$-(2\lambda u + 2b\lambda - \beta)^2 + 4\lambda(2\lambda u + 2b\lambda - \beta)\,u - (\gamma - 4c\lambda)\,u^2 = 0,$$

and finally writing $\lambda - \alpha$ for u^2, we obtain

$$(4\lambda^2 + 4c\lambda - \gamma)(\lambda - \alpha) - (2b\lambda - \beta)^2 = 0.$$

There are thus 3 pairs of roots, for to each of the three values of λ correspond two values of u, namely

$$\pm(\lambda - d + b^2)^{\frac{1}{2}},$$

and to each value of λ and u one value of v, namely

$$2\lambda + \frac{2b\lambda + bc - e}{u}.$$

We have also $$x = \tfrac{1}{2}\{(p-b-u)^{-1}(q-D)\},$$

consequently, since $p^2 - 2bp + d = 0$,

$$x = \frac{(p-b+u)(q-c-v)}{2(b^2-d-u^2)} = -\frac{(p-b+u)(q-c-v)}{2\lambda}.$$

Thus then we see that x can only cease to have 6 determinate values when $\lambda = 0$, and consequently the *Criterion of Normality* is the last term in the equation to λ.

This equation, written out at length, is

$$4\lambda^3 + 4(c - b^2 - \alpha)\lambda^2 + (-4c\alpha + 4b\beta - \gamma)\lambda + \alpha\gamma - \beta^2 = 0,$$

that is, $4\lambda^3 + 4(c-d)\lambda^2 + (-4cd + 4be - f + c^2)\lambda + (d-b^2)(f-c^2) - (e-bc)^2$.

Hence the Criterion in question is $(d-b^2)(f-c^2) - (e-bc)^2$ or $df - c^2d - b^2f - e^2 + 2bce$, which is the discriminant to the quadratic form

$$\lambda^2 + 2b\lambda\nu + 2c\mu\nu + d\mu^2 + 2e\mu\nu + f\nu^2;$$

this, as I have elsewhere shown, is the Criterion of the matrices p, q* being *in involution*†, that is, of a linear equation existing between the matrices 1, p, q, pq; or if p, q are regarded as quaternions, it is the condition of the square of

* When p, q are regarded as matrices, then

$$p^2 - 2bp + d = 0, \quad q^2 - 2cq + f = 0, \quad \tfrac{1}{2}(pq + qp) - bq - cp + e = 0,$$

where $$\lambda^2 + 2b\lambda\nu + 2c\mu\nu + d\mu^2 + 2e\mu\nu + f\nu^2$$

is the determinant to $\lambda + \mu p + \nu q$.

[† Above p. 116.]

the sine of the angle between the vectors of p and q vanishing; a condition which of course does not imply the coincidence of the vectors unless accompanied by the futile limitation of such vectors being real.

It admits of easy demonstration by virtue of the foregoing that in the case of the more general equation

$$px^2 + qx + r = 0,$$

the Criterion of Normality will be the discriminant of the ternary quadratic, which is the determinant of

$$pu + qv + rw;$$

this seems to me a very remarkable and noteworthy theorem. When this Criterion does not vanish, the quadratic equation above written must have 3 pairs of determinate roots.

Why they go in pairs and can be found by solving only a cubic instead of a sextic is best seen *à priori* by reverting to the original form $x^2 - 2px + q = 0$.

It follows from the nature of the process for finding B and D that they will be the same for that equation as for the equation $y^2 - 2yp + q = 0$.

But on writing $x + y = 2p$ these two equations pass into one another.

Hence each value of B, say B_1, will be associated with another value, say B', where $B_1 + B' = 2b$*, that is to say, if u_1, namely $B - b$ is one value of u, then $b - B$, that is, $-u_1$ will be another value of u, so that the equation in u^2 ought to be (as it has been shown to be) a cubic.

It might for a moment be supposed that $\lambda = \alpha = d - b^2$ would lead to a breach of normality on account of the equation $v - 2\lambda = \dfrac{2b\lambda + bc - e}{u}$, where $u^2 = 0$.

This, however, is not the case. For the equation

$$v^2 + \gamma = 4\lambda(v + c)$$

becomes, when $\lambda = \alpha$,

$$v^2 - 4(d - b^2)v + f - c^2 - 4cd + 4b^2c = 0,$$

so that v remains *finite*; consequently $2b\lambda + bc - e$, that is, $2bd - 2b^3 + bc - e$, must vanish when $\lambda = d - b^2$, and $v - 2\lambda$ assumes the form $\dfrac{0}{0}$. Obviously then in this case, to the one value $u = 0$ will be associated the two values of v, say v_1 and v_2, given by the above quadratic, and to $\lambda = \alpha$ will still correspond two values of (u, v), namely $(0, v_1)$, $(0, v_2)$; where, ideally speaking, the two zeros may be regarded as the same infinitesimal affected with opposite signs.

* In quaternion phrase, if $x + y = 2p$, $Sx + Sy = 2Sp$.

It should be observed, in order to understand what follows in the text, that $b - B_1 = B' - b$, and that the values of B must obviously be the same in the equation $x^2 - 2px + q = 0$ as in the equation $x^2 - 2xp + q = 0$.

The equation in λ may be made to undergo a useful linear transformation.

Let $\lambda = \mu + \alpha$, so that $\mu = u^2$.

Then

$$\mu\{4\mu^2 + (8\alpha + 4c)\mu + 4\alpha^2 + 4c\alpha - \gamma\} - (2b\mu + 2b\alpha - \beta)^2 = 0,$$

that is
$$4\mu^3 + \{4(c+2d) - 12b^2\}\mu^2 + \{(c+2d)^2 - 8(c+2d)b^2 + 12b^4 + 4be - f\}\mu - \{b(c+2d) - 2b^3 - e\}^2 = 0,$$

where it is noticeable that the number of parameters is reduced from 5 to 4, c and d only appearing together in the linear combination $c + 2d$. This is tantamount to the form obtained by Hamilton.

Let us make another linear transformation suggested by the preceding remark. Write $c + 2d = g$, and $\mu - b^2 = \gamma = \lambda - d$, the equation becomes

$$4\gamma^3 + 4g\gamma^2 + (g^2 + 4be - f)\gamma + 2beg - b^2f - e^2 = 0.$$

But obviously, notwithstanding this reduction of the parameters, λ itself is the most natural quantity to employ as the base of the solution, or, so to say, as the independent variable, and this admits of being determined by an equation of extraordinary simplicity.

For, let I be the discriminant of

$$\text{det.}\ (\lambda + \mu p + \nu q) = I = df + 2bce - c^2d - b^2f - e^2.$$

Then it will be seen by actual inspection that the equation found for λ takes the following form

$$e^{\lambda(2\delta_c - \delta_d)}\, I = 0,$$

that is

$$I + \left(2\frac{d}{dc} - \frac{d}{d\,.\,d}\right) I\,.\,\lambda + \frac{1}{2}\left(2\frac{d}{dc} - \frac{d}{d\,.\,d}\right)^2 I\,.\,\lambda^2 + \frac{1}{1\,.\,2\,.\,3}\left(2\frac{d}{dc} - \frac{d}{d\,.\,d}\right)^3 I\,.\,\lambda^3 = 0,$$

(the terms in the exponential function subsequent to the fourth term adding nothing to the value of the series).

If in the equation $x^2 - 2px + q = 0$, p and q be regarded as quaternions, then $\lambda = Sx^2 + Ip^2 - (Sp)^2$, $c = Sq$, $d = Ip^2$, and $I = \frac{1}{4}(pq - qp)^2$, which is a scalar quantity, and is to be regarded as an explicit function of Sp, Sq; Tp^2, $S(Vp\,Vq)$, Tq^2; it is in fact the discriminant of the form

$$X^2 + 2SpXY + 2SqXZ + Tp^2Y^2 + 2S(Vp\,Vq)\,YZ + Tq^2Z^2,$$

an identity unknown I believe to the geometrical quaternionists.

[As an example of it, let $p = i$, $q = j$, then

$$Sp = 0,\quad Sq = 0,\quad S(Vp\,Vq) = 0,\quad Tp^2 = -1,\quad Tq^2 = -1,$$

$\frac{1}{4}(pq - qp)^2 = 1 =$ the discriminant of $X^2 - Y^2 - Z^2$.]

With these definitions $e^{\lambda(2\delta_c - \delta_d)}\, I$ becomes identically zero.

The equation $x^2 - 2px + q = 0$ having six roots it is natural to inquire as to the value of their sum. This may be readily found as follows. We have found

$$x = -\frac{(p-b+u)(q-c-v)}{2\lambda}.$$

Also, if $$x + x' = 2p,$$

$$x'^2 - 2x'p + q = 0,$$

and obviously $$\Sigma x = \Sigma x'.$$

Hence $$\Sigma x = -\Sigma \frac{(p-b+u)(q-c-v)}{2\lambda},$$

and $$12p - \Sigma x = -\Sigma \frac{(q-c-v)(p-b+u)}{2\lambda}.$$

Therefore
$$\begin{aligned}\Sigma x &= 6p - \Sigma \frac{3}{2\lambda}(pq - qp)\\ &= 6p - 3I^{\frac{1}{2}}\Sigma \frac{1}{\lambda}\\ &= 6p + 3\frac{I^{\frac{1}{2}}(2\delta_c - \delta_d) I}{I}\\ &= 6\{p + (2\delta_c - \delta_d) I^{\frac{1}{2}}\},\end{aligned}$$

where the sign of $I^{\frac{1}{2}}$ must be so taken that it shall be equal to $\frac{1}{2}(pq - qp)$.

So again
$$\begin{aligned}\Sigma x^2 &= 2p\Sigma x - 6q\\ &= 12p^2 - 6q + 12(2\delta_c - \delta_d) I^{\frac{1}{2}}p.\end{aligned}$$

Thus the mean value of each root is ϵ in excess, and that of each square root ϵp in excess, of what these means would be if p and q were nominal quantities, ϵ denoting $(2\delta_c - \delta_d) I^{\frac{1}{2}}p$. Of course Σx^i may be found by the formula of derivation

$$\Sigma x^{i+1} = 2p\Sigma x^i - 9\Sigma x^{i-1}.$$

In conclusion it may be observed in regard to the equation $x^2 - 2px + q = 0$, (since in writing $x + x_1 = 2p$, we have $x_1^2 - 2x_1 p + q = 0$) it follows that (whatever be the order of the quantities p and q) the roots of either equation must be associated in pairs; because, if the identical equation to p is $p^\omega - \omega b p^{\omega-1} + \ldots$ and to x is $x^\omega - \omega B x^{\omega-1} + \ldots$, the equation for finding B must be of the form $T(B-b)^2 = 0$.

P.S.—Since the above was sent to press I have discovered the general solution of the unilateral equation of any degree in matrices of any order; see the *Comptes Rendus* of the Institute for Oct. 20, 1884 [pp. 197, 233 above], and *Nature* for Nov. 13, 1884*.

[* This paper contains the Theorem "Every latent root of every root of a given unilateral function in matrices of any order, is an algebraical root of the determinant of that function taken as if the unknown were an ordinary quantity, and conversely every algebraical root of the determinant so taken is a latent root of one of the roots of the given function."]

41.

INAUGURAL LECTURE AT OXFORD
12 December 1885.

ON THE METHOD OF RECIPROCANTS AS CONTAINING AN EXHAUSTIVE THEORY OF THE SINGULARITIES OF CURVES*.

[*Nature*, XXXIII. (1886), pp. 222—231.]

It is now two years and seven days since a message by the Atlantic cable containing the single word "Elected" reached me in Baltimore informing me that I had been appointed Savilian Professor of Geometry in Oxford, so that for three weeks I was in the unique position of filling the post and drawing the pay of Professor of Mathematics in each of two Universities: one, the oldest and most renowned, the other—an infant Hercules—the most active and prolific in the world, and which realises what only existed as a dream in the mind of Bacon—the House of Solomon in the New Atlantis.

To Johns Hopkins, who endowed the latter, and in conjunction with it a great Hospital and Medical School, between which he divided a vast fortune accumulated during a lifetime of integrity and public usefulness, I might address the words familiarly applied to one dear to all Wykehamists:—

> "Qui condis lævâ, condis collegia dextrâ,
> Nemo tuarum unam vicit utraque manú."

The chair which I have the honour to occupy in this University is made illustrious by the names and labours of its munificent and enlightened founder, Sir Henry Saville; of Thomas Briggs, the second inventor of logarithms; of Dr Wallis, who, like Leibnitz, drove three abreast to the temple of fame—being eminent as a theologian, and as a philologer, in addition to being illustrious as the discoverer of the theorem connected with the quadrature of the circle named after him, with which every schoolboy is supposed to be familiar, and as the author of the *Arithmetica Infinitorum*, the precursor of Newton's *Fluxions*; of Edmund Halley, the trusted friend and counsellor of Newton, whose work marks an epoch in the history of astronomy, the reviver of the study of Greek geometry and discoverer of the proper motions of the so-

[* The tables referred to in the text are given pp. 301, 302 below.]

called fixed stars; and by one in later times not unworthy to be mentioned in connection with these great names, my immediate predecessor, the mere allusion to whom will, I know, send a sympathetic thrill through the hearts of all here present, to whom he was no less endeared by his lovable nature than an object of admiration for his vast and varied intellectual acquirements, whose untimely removal, at the very moment when his fame was beginning to culminate, cannot but be regarded as a loss, not only to his friends and to the University for which he laboured so strenuously, but to science and the whole world of letters.

As I have mentioned, the first to occupy this chair was that remarkable man Thomas Briggs, concerning whose relation to the great Napier of Merchiston, the fertile nursery of heroes of the pen and the sword, an anecdote, taken from the *Life* of Lilly, the astrologer, has lately fallen under my eyes, which, with your permission, I will venture to repeat:—

"I will acquaint you (says Lilly) with one memorable story related unto me by John Marr, an excellent mathematician and geometrician, whom I conceive you remember. He was servant to King James and Charles the First. At first, when the lord Napier, or Marchiston, made public his logarithms, Mr Briggs, then reader of the astronomy lectures at Gresham College, in London, was so surprised with admiration of them, that he could have no quietness in himself until he had seen that noble person the lord Marchiston, whose only invention they were: he acquaints John Marr herewith, who went into Scotland before Mr Briggs, purposely to be there when those two so learned persons should meet. Mr Briggs appoints a certain day when to meet at Edinburgh; but failing thereof, the lord Napier was doubtful he would not come. It happened one day as John Marr and the lord Napier were speaking of Mr Briggs: 'Ah John (said Marchiston), Mr Briggs will not now come.' At the very moment one knocks at the gate; John Marr hastens down, and it proved Mr Briggs to his great contentment. He brings Mr Briggs up into my lord's chamber, where almost *one quarter of an hour was spent,* each beholding other almost with admiration *before one word was spoke.* At last Mr Briggs began: 'My lord, I have undertaken this long journey purposely to see your person, and to know by what engine of wit or ingenuity you came first to think of this most excellent help unto astronomy, namely, the logarithms; but, my lord, being by you found out, *I wonder nobody else found it out before,* when now known it is so easy.' He was nobly entertained by the lord Napier; and every summer after that, during the lord's being alive, this venerable man Mr Briggs went purposely into Scotland to visit him*."

* A very similar story is told of the meeting of Leopardi and Niebuhr in Rome. What Briggs said of logarithms may be said almost in the same words of the subject of this lecture:—"This most excellent help to geometry which, being found out, one wonders nobody else found it out

Some apology may be needed, and many valid reasons might be assigned, for the departure, in my case, from the usual course, which is that every professor on his appointment should deliver an inaugural lecture before commencing his regular work of teaching in the University. I hope that my remissness, in this respect, may be condoned if it shall eventually be recognised that I have waited, before addressing a public audience, until I felt prompted to do so by the spirit within me craving to find utterance, and by the consciousness of having something of real and more than ordinary weight to impart, so that those who are qualified by a moderate amount of mathematical culture to comprehend the drift of my discourse, may go away with the satisfactory feeling that their mental vision has been extended and their eyes opened, like my own, to the perception of a world of intellectual beauty, of whose existence they were previously unaware.

This is not the first occasion on which I have appeared before a general mathematical audience, as the messenger of good tidings, to announce some important discovery. In the year 1859 I gave a course of seven or eight lectures at King's College, London, at each of which I was honoured by the attendance of my lamented predecessor, on the subject of "The Partitions of Numbers and the Solution of Simultaneous Equations in Integers," in which it fell to my lot to show how the difficulties might be overcome which had previously baffled the efforts of mathematicians, and especially of one bearing no less venerable a name than that of Leonard Euler, and also laid the basis of a method which has since been carried out to a much greater extent in my "Constructive Theory of Partitions," published in the *American Journal of Mathematics,* in writing which I received much valuable co-operation and material contributions from many of my own pupils in the Johns Hopkins University*. Several years later, in the same place, I delivered a lecture on the well-known theorem of Newton, which fills a chapter in the *Arithmetica Universalis,* where it was stated without proof, and of which many celebrated mathematicians, including again the name of Euler, had sought for a proof in vain. In that lecture I supplied the missing demonstration, and owed my success, I believe, chiefly to merging the theorem to be proved, in one of

before; when, now known, it is so easy." I quite entered into Briggs's feelings at his interview with Napier when I recently paid a visit to Poincaré in his airy perch in the Rue Gay-Lussac in Paris (will our grandchildren live to see an Alexander Williamson Street in the north-west quarter of London, or an Arthur Cayley Court in Lincoln's Inn, where he once abode?). In the presence of that mighty reservoir of pent-up intellectual force my tongue at first refused its office, my eyes wandered, and it was not until I had taken some time (it may be two or three minutes) to peruse and absorb as it were the idea of his external youthful lineaments that I found myself in a condition to speak.

* In one of those lectures, two hundred copies of the notes for which were printed off and distributed among my auditors, I founded and developed to a considerable extent the subject since rediscovered by M. Halphen under the name of the Theory of Aspects.

greater scope and generality. In mathematical research, reversing the axiom of Euclid, and converting the proposition of Hesiod, it is a continual matter of experience, as I have found myself over and over again, that the whole is less than its part. On a later occasion, taking my stand on the wonderful discovery of Peaucellier, in which he had realised that exact parallel motion which James Watt had believed to be impossible, and exhausted himself in contrivances to find an imperfect substitute for, in the steam-engine, I think I may venture to say that I brought into being a new branch of mechanico-geometrical science, which has been, since then, carried to a much higher point by the brilliant inventions of Messrs Kempe and Hart. I remember that my late lamented friend, the Lord Almoner's Reader of Arabic in this University, subsequently editor of the *Times*, Mr Chenery, who was present on that occasion in an unofficial capacity, remarked to me after the lecture, which was delivered before a crowded auditory at the Royal Institution, that when they saw two suspended opposite Peaucellier cells, coupled toe-and-toe together, swing into motion, which would have been impossible had not the two connected moving points each described an accurate straight line, "the house rose at you." (The lecture merely illustrated experimentally two or three simple propositions of Euclid, Book III.)

The matter that I have to bring before your notice this afternoon is one far bigger and greater, and of infinitely more importance to the progress of mathematical science, than any of those to which I have just referred. No subject during the last thirty years has more occupied the minds of mathematicians, or lent itself to a greater variety of applications, than the great theory of Invariants. The theory I am about to expound, or whose birth I am about to announce, stands to this in the relation not of a younger sister, but of a brother, who, though of later birth, on the principle that the masculine is more worthy than the feminine, or at all events, according to the regulations of the Salic law, is entitled to take precedence over his elder sister, and exercise supreme sway over their united realms. Metaphor apart, I do not hesitate to say that this theory, *minor natu potestate major*, infinitely transcends in the extent of its subject-matter, and in the range of its applications, the allied theory to which it stands in so close a relation. The very same letters of the alphabet which may be employed in the two theories, in the one may be compared to the dried seeds in a botanical cabinet, in the other to buds on the living branch ready to burst out into blossom, flower and fruit, and in their turn supply fresh seed for the maintenance of a continually self-perpetuating cycle of living forms. In order that I may not be considered to have lost myself in the clouds in making such a statement, let me so far anticipate what I shall have to say on the meaning of Reciprocants and their relation to the ordinary Invariantive or Covariantive forms by taking an instance which happens to be common (or at least, by a slight geometrical adjustment, may be made so) to the two theories. I ask you to compare the form

$$a^2d - 3abc + 2b^3$$

as it is read in the light of the one and in that of the other. In the one case the a, b, c, d stand for the coefficients of a so-called Binary Quantic, and its evanescence serves to express some particular relation between three points lying in a right line. In the other case the letters are interpreted* to mean the successive differential derivatives of the 2nd, 3rd, 4th, 5th orders of one Cartesian co-ordinate of a curve in respect to the other. The equation expressing this evanescence is capable of being integrated, and this integral will serve to denote a relation between the two co-ordinates which furnishes the necessary and sufficient condition in order that the point of the curve of any or no specified order (for it may be transcendental) to which the co-ordinates may refer, may admit of having, at the point where the condition is satisfied, a contact with a conic of a higher order than the common. In the one case the letters employed are dead and inert atoms; in the other they are germs instinct with motion, life, and energy.

A curious history is attached to the form which I have just cited, one of the simplest in the theory, of which the narrative may not be without interest to many of my hearers, even to those whose mathematical ambition is limited to taking a high place in the schools.

At pp. 19 and 20 of Boole's *Differential Equations* (edition of 1859) the author cites this form as the left-hand side of an equation which he calls the "Differential Equation of lines of the second order," and attributes it to Monge, adding the words, "But here our powers of geometrical interpretation fail, and results such as this can scarcely be otherwise useful than as a registry of integrable forms." In this vaticination, which was quite uncalled for, the eminent author, now unfortunately deceased, proved himself a false prophet, for the form referred to is among the first that attracts notice in crossing the threshold of the subject of Reciprocants, and is but one of a crowd of similar and much more complicated expressions, no less than it susceptible of geometrical interpretation and of taking their place on the register of integrable forms. A friend, with whom I was in communication on the subject, and whom I see by my side, remarked to me, in reference to this passage:—"I cannot help comparing a certain passage in Boole to Ezekiel's valley of the dry bones: 'The valley was full of bones, and lo, they were very dry.' The answer to the question, 'Can these bones live?' is supplied by the advent of the glorious idea of the Reciprocants; and the grand invocation, 'Come from the four winds, O breath, and breathe upon these slain, that they may live,' may well be used here. That they will

[* $a=\frac{1}{2!}\frac{d^2y}{dx^2}$, $b=\frac{1}{3!}\frac{d^3y}{dx^3}$, $c=\frac{1}{4!}\frac{d^4y}{dx^4}$, $d=\frac{1}{5!}\frac{d^5y}{dx^5}$.]

'live and stand up upon their feet an exceeding great army' is what we may expect to happen." This, as you will presently see, is just what actually has happened.

Not knowing where to look in Monge for the implied reference, I wrote to an eminent geometer in Paris to give me the desired information; he replied that the thing could not be in Monge, for that M. Halphen, who had written more than one memoir on the subject of the differential equation of a conic, had made nowhere any allusion to Monge in connection with the subject. Hereupon, as I felt sure that a reference contained in repeated editions of a book in such general use as Boole's *Differential Equations* was not likely to be erroneous, I addressed myself to M. Halphen himself, and received from him a reply, from which I will read an extract:—

"En premier lieu, c'est une chose nouvelle pour moi que l'équation différentielle des coniques se trouve dans Boole, dont je ne connais pas l'ouvrage. Je vais, bien entendu, le consulter avec curiosité. Ce fait a échappé à tout le monde ici, et l'on a cru généralement que j'avais le premier donné cette équation. *Nil sub sole novi!* Il m'est naturellement impossible de vous dire où la même équation est enfouie parmi les œuvres de Monge. Pour moi, c'est dans *Le Journal de Math.* (1876), p. 375, que j'ai eu, je crois, la première occasion de développer cette équation sous la forme même que vous citez; et c'est quand je l'ai employée, l'année suivante, pour le problème *sur les lois de Kepler* (*Comptes rendus*, 1877, t. LXXXIV. p. 939), que M. Bertrand l'a remarquée comme neuve. Ce qui vous intéresse plus, c'est de connaître la forme simplifiée sous laquelle j'ai donné plus tard cette équation dans le *Bulletin* de la Société Mathématique. C'est sous cette dernière forme que M. Jordan la donne dans son cours de l'École Polytechnique (t. I. p. 53)."

All my researches to obtain the passage in Monge referred to by Boole have been in vain*.

I will now proceed to endeavour to make clear to you what a Reciprocant means: the above form, which may be called the *Mongian*, would afford an example by which to illustrate the term; but I think it desirable to begin with a much easier one. Consider then the simple case of a single term, the second derivative of one variable, y, in respect to another, x. Every tyro in algebraical geometry knows that this, or rather the fact of its evanescence, serves to characterise one or more points in a curve which possess, so to say,

* Search has been made in the collected works of Monge and in manuscripts of his own or Prony in the library of the Institute, but without effect. I have also made application to the Universal Information Society, who undertake to answer "every conceivable question," but nothing has so far come of it. Perhaps until the citation from Monge is verified it will be safer in future to refer to the so-called Mongian as the Boole-Mongian. It may be regarded as the starting-point of the Differential Invariant Theory, as the Schwarzian is of the deeper-lying and more comprehensive Reciprocant Theory.

a certain indelible and intrinsic character, or what is technically called a singularity; in this case an inflexion such as exists in a capital letter *S*, or Hogarth's line of beauty.

If we invert the two variables, exchanging, that is to say, one with the other, the fact of this indelibility draws with it the consequence that in general these two reciprocal functions must vanish together, and as a fact each is the same as the other multiplied or divided by the third power of the first derivative of the one variable with respect to the other taken negatively. In this case we are dealing with a single derivative and its reciprocal. The question immediately presents itself whether there may not be a combination of derivatives possessing a similar property. We know that no *single* derivative except the second does.

Such a combination actually presents itself in a form which occurs in the solution of Differential Equations of the second order, the form

$$\frac{dy}{dx} \cdot \frac{d^3y}{dx^3} - \frac{3}{2}\left(\frac{d^2y}{dx^2}\right)^2,$$

which, after the name of its discoverer, Schwarz, we may agree to call a Schwarzian (Cayley's "Schwarzian Derivative*"). If in this expression the x and y be interchanged, its value, barring a factor consisting of a power of the first derivative, remains unaltered, or, to speak more strictly, merely undergoes a change of algebraical sign. We may now arrive at the generalised conception of an algebraical function of the derivatives of one variable in respect to another, which, if we agree to pay no regard to the algebraical sign, or to any power of the first derivative that may appear as a factor, will remain unaltered when the dependent and independent variables are interchanged one with another; and we may agree to call any such function a Reciprocant.

But here an important distinction arises—there are Reciprocants such as the one I first mentioned, $\frac{d^2y}{dx^2}$, or such as the Mongian to which allusion has

* More strictly speaking this is Cayley's Schwarzian derivative cleared of fractions—it may well be called the Schwarzian (see my note on it in the *Mathematical Messenger* for September or October past). Prof. Greenhill in regard to the Schwarzian derivative proper writes me as follows:—

"I found the reference in a footnote to p. 74 of Klein's *Vorlesungen über das Ikosaëder*, &c., in which Klein thanks Schwarz for sending him the reference to a paper by Lagrange, 'Sur la construction des cartes géographiques' in the *Nouveaux Mémoires de l'Académie de Berlin*, 1779. Compare also Schwarz's paper in Bd. 75 of *Borchardt's Journal*, where further literary notices are collected together. Klein says further that in the 'Sächsischen Gesellschaft von Januar 1883,' he has considered the inner meaning (*innere Bedeutung*) of the differential equation

$$\frac{\eta'''}{\eta} - \frac{3}{2}\left(\frac{\eta''}{\eta'}\right)^2 = f(\eta), \text{ where } \eta' = \frac{d\eta}{dz} \ldots."$$

There are two papers by Lagrange, one immediately following the other, "Sur la construction des cartes géographiques," but I have not been able to discover the Schwarzian derivative in either of them.

been made in the letter from M. Halphen, in which the second and higher differential derivatives alone appear, the first differential derivative not figuring in the expression. These may be termed Pure Reciprocants. Thus I repeat $\frac{d^2y}{dx^2}$, and

$$9\left(\frac{d^2y}{dx^2}\right)^2 \cdot \frac{d^5y}{dx^5} - 45\frac{d^2y}{dx^2} \cdot \frac{d^3y}{dx^3} \cdot \frac{d^4y}{dx^4} + 40\left(\frac{d^3y}{dx^3}\right)^3$$

are pure reciprocants. Those from which the first derivative $\frac{dy}{dx}$ is not excluded may be called Mixed Reciprocants. An example of such kind of Reciprocants is afforded by the Schwarzian above referred to. This distinction is one of great moment, for a little attention will serve to make it clear that every pure reciprocant expressed in terms of x and y marks an intrinsic feature or singularity in the curve, whatever its nature may be, of which x and y are the co-ordinates; for if in place of the variables (x, y) any two linear functions of these variables be substituted, a pure reciprocant, by virtue of its reciprocantive character, must remain unaltered save as to the immaterial fact of its acquiring a factor containing merely the constants of substitution*.

The consequence is that every pure reciprocant corresponds to, and indicates, some singularity or characteristic feature of a curve, and *vice versâ* every such singularity of a general nature and of a descriptive (although not necessarily of a projective) kind, points to a pure reciprocant.

Such is not the case with mixed reciprocants. They will not in general remain unaltered when linear substitutions are impressed upon the variables. Is it then necessary, it may be asked, to pay any attention to mixed reciprocants; or may they not be formally excluded at the very threshold of the inquiry? Were I disposed to put the answer to this question on mere personal grounds, I feel that I should be guilty of the blackest ingratitude, that I should be kicking down the ladder by which I have risen to my present commanding point of view, if I were to turn my back on these humble mixed reciprocants, to which I have reason to feel so deeply indebted; for it was the putting together of the two facts of the substantial permanence under linear substitutions impressed upon the variables of the Schwarzian form and the simpler one which marks the inflexions of a curve—it was, if I may so say, the collision in my mind of these two facts—that kindled the spark and fired the train which set my imagination in a blaze by the light of which the whole horizon of Reciprocants is now illumined.

* The form as it stands shows that for y a linear function of x and y may be substituted; and the form *reciprocated* (by the interchange of x and y) shows that a similar substitution may be made for x. Hence arbitrary linear substitutions may be simultaneously impressed on x and y without inducing any change of form.

But it is not necessary for me to defend the retention of mixed reciprocants on any such narrow ground of personal predilection. The whole body of Reciprocants, pure and mixed, form one complete system, a single garment without rent or seam, a complex whole in which all the parts are inextricably interwoven with each other. It is a living organism, the action of no part of which can be thoroughly understood if dissevered from connection with the rest.

It was in fact by combining and interweaving mixed reciprocants that I was led to the discovery of the pure binomial reciprocant, which comes immediately after the trivial monomial one,—the earliest with which I became acquainted, and of the existence of compeers to which I was for some time in doubt, and only became convinced of the fact after the discovery of the Partial Differential Equation, the master-key to this portion of the subject, which gives the means of producing them *ad libitum* and ascertaining all that exist of any prescribed type. Of this partial differential equation I shall have occasion hereafter to speak; but this is not all, for, as we shall presently see, mixed reciprocants are well worthy of study on their own account, and lead to conclusions of the highest moment, whether as regards their applications to geometry or to the theory of transcendental functions and of ordinary differential equations.

The singularities of curves, taking the word in its widest acceptation, may be divided into three classes: those which are independent of homographic deformation and which remain unaltered in any perspective picture of the curve; those which, having an express or tacit reference to the line at infinity, are not indelible under perspective projection, but using the word descriptive with some little latitude may, in so far as they only involve a reference to the line at infinity as a line, be said to be of a purely descriptive character; and, lastly, those which are neither projective nor purely descriptive, having relation to the points termed, in ordinary parlance, "circular points at infinity"—for which the proper name is "centres of infinitely distant pencils of rays," that is, pencils, every ray of which is infinitely distant from every point external to it. Such, for instance, would be the character of points of maximum or minimum curvative, which, as we shall see, indicate, or are indicated by, that particular class of Mixed to which I give the name of "Orthogonal Reciprocants." All purely descriptive singularities alike, whether projective or non-projective, are indicated by pure reciprocants, and are subject to the same Partial Differential Equation; just as, in the Theory of Binary Quantics, Invariants, although under one aspect they may be regarded as a self-contained special class, admit of being and are most advantageously studied in connection with, and as forming a part of, the whole family of forms commonly known by the name of "semi-, or subinvariants," but which I find it conduces to much greater clearness of expression and avoidance of ambiguity or periphrasis to designate as Binariants.

The question may here be asked, How, then, are projective and non-projective pure reciprocants to be discriminated by their external characters?

I believe that I know the answer to this question, which is, that the former are subject to satisfy a second partial differential equation of a certain simple and familiar type, but this is a matter upon which it is not necessary for me to enter on the present occasion*. It is enough for our present purpose to remark that every projective pure reciprocant must, so to say, be in essence a masked ternary covariant. For instance, if we take the simplest of all such, namely, a, that is $\frac{d^2y}{dx^2}$, we have, if $\phi(x, y) = 0$,

$$\frac{d^2y}{dx^2} \cdot \left(\frac{d\phi}{dy}\right)^3 = \begin{vmatrix} \frac{d^2\phi}{dx^2} & \frac{d^2\phi}{dxdy} & \frac{d\phi}{dx} \\ \frac{d^2\phi}{dxdy} & \frac{d^2\phi}{dy^2} & \frac{d\phi}{dy} \\ \frac{d\phi}{dx} & \frac{d\phi}{dy} & \cdot \end{vmatrix}$$

which, for facility of reference, let me call M. Obviously we might instead of $a = 0$ substitute $M = 0$ to mark an inflexion. And now if we write Φ as the completed form of ϕ, when made homogeneous by the substitution of z for unity; and if we suppose it to be of n dimensions in x, y, z, and call its Hessian H, we shall obtain the syzygy

$$(n-1)^2 \left(\frac{d\phi}{dy}\right)^3 a + H + \left\{\frac{d^2\Phi}{dx^2} \cdot \frac{d^2\Phi}{dy^2} - \left(\frac{d^2\Phi}{dxdy}\right)^2\right\} \Phi = 0.$$

Hence the system $\Phi = 0$, $a = 0$, will be in effect the same as the system $\Phi = 0$, $H = 0$, and in this sense a may be said to carry H as it were in its bosom. And so in general every pure projective reciprocant may, in the language of insect transformation, be regarded as passing, so to say, first from the grub to the pupa or chrysalis, and from this again, divested of all superfluous integuments, to the butterfly or imago state.

Non-projective pure reciprocants undergo only one such change. There is no possibility of their ever emerging into the imago—their development being finally arrested at the chrysalis stage.

It would, I think, be an interesting and instructive task to obtain the imago or Hessianised transformation of the Mongian, but I am not aware

* In Paris, from which I correct the proofs, I have succeeded in reducing this conjecture to a certainty and in establishing the marvellous fact that every Projective Reciprocant, or, which is the same thing, every Differential Invariant, is, at the same time, an Ordinary Subinvariant. Thus a differential invariant (or projective reciprocant) may be regarded as *a single personality clothed with two distinct natures*—that of a reciprocant and that of a subinvariant.

that anyone has yet done, or thought of doing, this*. It seems to me that by substituting Reciprocants in lieu of Ternary Covariants we are as it were stealing a dimension from space, inasmuch as Reciprocants, that is, Ternary Covariants in their undeveloped state, are closely allied to, and march *pari passu* with, the familiar forms which appertain to merely binary quantics.

I will now proceed to bring before your notice the general partial differential equation which supplies the necessary and sufficient condition to which all pure reciprocants are subject.

It is highly convenient to denote the successive derivatives

$$\frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \frac{d^4y}{dx^4}, \ldots$$

by the simple letters $a, b, c, \ldots$.

The first derivative $\frac{dy}{dx}$ plays so peculiar a part in this theory that it is necessary to denote it by a letter standing aloof from the rest, and I call it t. This last letter, I need not say, does not make its appearance in any pure reciprocant. This being premised, I invite your attention to the equation in question, in which you will perceive the symbols of operation are separated from the object to be operated upon.

Writing $V = 3a^2\delta_b + 10ab\delta_c + (15ac + 10b^2)\delta_d + \ldots$ and calling any pure reciprocant R,

$$VR = 0$$

is the equation referred to.

I cannot undertake, within the brief limits of time allotted to this lecture, to explain how this operation, or, as it may be termed, this annihilator V is arrived at. The table of binomial coefficients, or rather half series of binomial coefficients, shown† in Chart 4, will enable you to see what is the law of the numerical coefficients of its several terms. Let the words *weight, degree, extent* (extent, you will remember, means the number of places by which the most remote letter in the form is separated from the first letter in the alphabet) of a pure reciprocant signify the same things as they would do if the letters $a, b, c, \ldots$ referred, according to the ordinary notation, to Binariants instead of to Reciprocants. The number of binariants linearly independent of each other whose weight, degree and extent or order are w, i, j is given by the partition formula $(w; i, j) - (w-1; i, j)$ where in general $(w; i, j)$ means the number of ways of partitioning w into i or fewer parts none greater than j.

* M. Halphen informs me that this has been done by Cayley in the *Phil. Trans.* for 1865, and subsequently in a somewhat simplified form by Painvin, *Comptes Rendus*, 1874. But neither of these authors seems to have had the Boole-Mongian objectively before him, so that a slight supplemental computation is wanting to establish the equation between it and the function which either of them finds to vanish at a *sextactic* point.

[† p. 302 below.]

It follows immediately from the mere form of V that the corresponding formula in the case of Reciprocants of a given type $w.i.j$ will be

$$(w; i, j) - (w-1; i+1, j)$$

the augmentation of i in the second term of the formula being due to the fact that, whereas in the partial differential equation for Binariants it is the letters themselves which appear as coefficients, it is quadratic functions of these in the case of Reciprocants. From the form of V we may also deduce a rigorous demonstration of the existence of Reciprocants strictly analogous to those with which you are familiar in the Binariant Theory, which are pictured in Chart 2, and are now usually designated as Protomorphs, as being the forms by the interweaving of which with one another (or rather by a sort of combined process of mixture and precipitation), all others, even the irreducible ones, are capable of being produced. The corresponding forms for Reciprocants you will see exhibited in the same table. Each series of Protomorphs may of course be indefinitely extended as more and more letters are introduced. In the table I have not thought it necessary to go beyond the letter g. You also know that besides Protomorphs there are other irreducible forms, the organic radicals, so to say, into which every compound form may be resolved, always limited in number, whatever the number of letters or primal elements we may be dealing with. The same thing happens to Reciprocants as you will notice in the comparative table in Chart 2. Without going into particulars, I will ask you to take from me upon faith the assurance that there is no single feature in the old familiar theory, whether it relates to Protomorphs, to Ground-forms, to Perpetuants, to Factorial constitution, to Generating Functions, or whatever else sets its stamp upon the one, which is not counterfeited by and reproduced in the parallel theory.

So much—for time will not admit of more—concerning pure reciprocants.

Let me now say a few words *en passant* on Mixed Reciprocants.

Pure Reciprocants, we have seen, are the analogues of Invariants, or else of the leading terms, for that is what are Semi- or Subinvariants, of Covariantive expansions; each is subject to its own proper linear partial differential equation. Mixed Reciprocants are the exact analogues of the coefficients in such expansions other than those of the leading terms. Starting from the leading terms as the unit point, the coefficients of rank ω are subject to a partial differential equation of order ω; and just so, mixed reciprocants, if involving t up to the power ω, are subject to a partial differential equation of that same order.

I have alluded to a peculiar class of mixed under the name of "Orthogonal Reciprocants." They are distinguished, as I have proved, by the beautiful property that, if differentiated with respect to t, the result must be itself a Reciprocant. In Chart 1 you will see this illustrated in the case of a mixed reciprocant $(1+t^2)b-3ta^2$, which serves to indicate the existence of points of maximum and minimum curvature. Its differential coefficient with respect to t is the oft-alluded-to Schwarzian, transliterated into the simpler notation. Proceeding in the inverse order—of Integration instead of Differentiation—I call your attention to a mixed reciprocant, of a very simple character, one which presents itself at the very outset of the theory, namely

$$tc-5ab,$$

which, integrated in respect to t between proper limits, yields the elegant orthogonal reciprocant

$$(t^2+1)c-10abt+15a^3.$$

Expressed in the ordinary notation, this, equated to zero, takes the form

$$\left\{\left(\frac{dy}{dx}\right)^2+1\right\}\frac{d^4y}{dx^4}-10\frac{dy}{dx}\cdot\frac{d^2y}{dx^2}\cdot\frac{d^3y}{dx^3}+15\left(\frac{d^2y}{dx^2}\right)^3=0.$$

Mr Hammond has integrated this, treated as an ordinary differential equation, and has obtained the complete primitive expressed through the medium of two related Hyper-Elliptic Functions connecting the variables x and y (see* Chart 3). It may possibly turn out to be the case that every mixed reciprocant is either itself an Orthogonal Reciprocant, or by integration, in respect to t, leads to one.

It will of course be understood that, in interpreting equations obtained by equating to zero an Orthogonal Reciprocant, the variables must be regarded as representing not general but rectangular Cartesian co-ordinates.

Here seems to me to be the proper place for pointing out to what extent I have been anticipated by M. Halphen in the discovery of this new world of Algebraical Forms. When the subject first dawned upon my mind, about the end of October or the beginning of November last, I was not aware that it had been approached on any side by any one before me, and believed that I was digging into absolutely virgin soil. It was only when I received M. Halphen's letter, dated November 25, in relation to the Mongian business already referred to, accompanied by a presentation of his memoirs on Differential Invariants, that I became aware of there existing any link of connection between his work and my own. A Differential Invariant, in the sense in which the term is used by M. Halphen, is not what at first blush I supposed it to be, and as in my haste to repair what seemed to me an omission to be without loss of time supplied, I wrote to M. Hermite it was, in a letter which has been or is about to be inserted in the *Comptes Rendus* of the Institute of France; it is not, I say, identical with what I have termed a general pure reciprocant, but only with that peculiar species of Pure Reciprocants to which I have in a preceding part of this lecture referred as corresponding and pointing to Projective Singularities. In his

[* p. 302 below.]

splendid labours in this field Halphen has had no occasion to construct or concern himself with that new universe of forms viewed as a whole, whether of Pure or Mixed Reciprocants, which it has been the avowed and principal object of this lecture to bring under your notice.

I anticipate deriving much valuable assistance in the vast explorations remaining to be made in my own subject from the new and luminous views of M. Halphen, and possibly he may derive some advantage in his turn from the larger outlook brought within the field of vision by my allied investigations.

Let me return for a moment to that simplest class of pure reciprocants which I have called protomorphs. Each of these will be found (as may be shown either by a direct process of elimination, or by integrating the equations obtained by equating them severally to zero, regarded as ordinary differential equations between x and y) each of these, I say, will be found to represent some simple kind of singularity at the point (x, y) of the curve to which these co-ordinates are supposed to refer. Thus, for instance, No. 1 marks a single point of inflexion; No. 2, points of closest contact with a common parabola; No. 3, what our Cayley has called sextactic points, referring to a general conic; No. 4, points of closest contact with a common cubical parabola; and so on. The first and third, it will be noticed, represent projective singularities, and as such, in M. Halphen's language, would take the name of Differential Invariants. The second and fourth, having reference to the line at infinity in the plane of the curve, are of a non-projective character, and as such would not appear in M. Halphen's system of Differential Invariants. It is an interesting fact that every simple parabola, meaning one whose equation can be brought under the form $y = x^{\frac{m}{n}}$, corresponds to a linear function of a square of the third, and the cube of the second protomorph, and consequently will in general be of the sixth degree. In the particular case of the cubical parabola, the numerical parameter of this equation is such that the highest powers of b cancel each other so that the form sinks one degree, and becomes represented by the *Quasi-Discriminant*, No. 4.

This simple instance will serve to illustrate the intimate connection which exists between the projective and non-projective reciprocants, and the advantage, not to say necessity, of regarding them as parts of one organic whole.

It would take me too far to do more than make the most cursory allusion to an extension of this theory similar to that which happens when in the ordinary theory of invariants we pass from the consideration of a single Quantic to that of two or more. There is no difficulty in finding the partial differential equation to double reciprocants which, as far as I have as yet pursued the investigation, appear to be functions of $a, b, c, \ldots$; $a', b', c', \ldots$; and of $(t - t')$.

The theory of double reciprocants will then include as a particular case the question of determining the singularities of paired points of two curves at which their tangents are parallel, and consequently the theory of common tangents to two curves and of bi-tangents to a single one.

I think I may venture to say that a general pure multiple reciprocant which marks off relative singularities, whether projective or non-projective, of a group of curves, is a function of the second and higher differential derivatives appertaining to the several curves of the group, and of the differences of the first derivatives, whereas in a mixed multiple reciprocant these last-named differences are replaced by the first derivatives themselves. As a particular case, when the group dwindles to an individual and there is only one t, this letter disappears altogether from the form, for there are no differences of a single quantity.

In the chart (marked No. 2) you will see the table of Protomorphs carried on as far as the letter g inclusive, and will not fail to notice what may be termed the higher organisation of Reciprocantive as compared with ordinary Invariantive Protomorphs; the degrees of the latter oscillate or librate between the numbers 2 and 3, whereas in the former the degree is variable according to a certain transcendental law dependent on the solution of a problem in the Partition of Numbers. Another interesting difference between general Invariants and general Pure Reciprocants consists in the fact that, whilst the number of the former ultimately (that is, when the extent is indefinitely increased) becomes indefinitely great, that of the latter is determinate for any given degree even for an infinite number of letters.

In carrying on the table of protomorphs up to the letter h (see Chart 6) a new phenomenon presents itself, to which, however, there is a perfect parallel in the allied theory. An arbitrary constant enters into the form, its general value being a linear function of U and W (for which see Chart 6). But this is not all. If you examine the terms in both U and W (there are in all twelve such) you will find that these twelve do not comprise all of the same type to which they belong. There is a Thirteenth (a banished Judas), equally *à priori* entitled to admission to the group, but which does not make its appearance among them, namely, b^4d. I rather believe that a similar phenomenon of one or more terms, whose presence might be expected, but which do not appear, presents itself in the allied invariantive theory, but cannot speak with certainty as to this point, as the circumstance has not received, and possibly does not merit, any very particular attention.

Still, in the case before us, this unexpected absence of a member of the family, whose appearance might have been looked for, made an impression on my mind, and even went to the extent of acting on my emotions. I began to think of it as a sort of lost Pleiad in an Algebraical Constellation, and in the end, brooding over the subject, my feelings found vent, or sought relief, in a rhymed effusion, a *jeu de sottise*, which, not without some apprehension of appearing singular or extravagant, I will venture to rehearse. It will at least serve as an interlude, and give some relief to the strain upon your attention before I proceed to make my final remarks on the general theory.

TO A MISSING MEMBER

Of a Family Group of Terms in an Algebraical Formula.

Lone and discarded one! divorced by fate,
Far from thy wished-for fellows—whither art flown?
Where lingerest thou in thy bereaved estate,
Like some lost star, or buried meteor stone?
Thou mindst me much of that presumptuous one
Who loth, aught less than greatest, to be great,
From Heaven's immensity fell headlong down
To live forlorn, self-centred, desolate:
Or who, new Heraklid, hard exile bore,
Now buoyed by hope, now stretched on rack of fear,
Till throned Astræa, wafting to his ear
Words of dim portent through the Atlantic roar,
Bade him "the sanctuary of the Muse revere
And strew with flame the dust of Isis' shore."

Having now refreshed ourselves and bathed the tips of our fingers in the Pierian spring, let us turn back for a few brief moments to a light banquet of the reason, and entertain ourselves as a sort of after-course with some general reflections arising naturally out of the previous matter of my discourse. It seems to me that the discovery of Reciprocants must awaken a feeling of surprise akin to that which was felt when the galvanic current astonished the world previously accustomed only to the phenomena of machine or frictional electricity. The new theory is a ganglionic one: it stands in immediate and central relation to almost every branch of pure mathematics—to Invariants, to Differential Equations, ordinary and partial, to Elliptic and Transcendental Functions, to Partitions of Numbers, to the Calculus of Variations, and above all to Geometry (alike of figures and of complexes), upon whose inmost recesses it throws a new and wholly unexpected light. The geometrical singularities which the present portion of the theory professes to discuss are in fact the distinguishing *features* of curves; their *technical* name, if applied to the human countenance, would lead us to call a man's eyes, ears, nose, lips, and chin his singularities; but these singularities make up the character and expression, and serve to distinguish one individual from another. And so it is with the so-called singularities of curves.

Comparing the system of ground-forms which it supplies with those of the allied theory, it seems to me clear that some common method, some yet undiscovered, deep-lying, Algebraical principle remains to be discovered, which shall in each case alike serve to demonstrate the finite number of these forms (these organic radicals) for any specified number of letters. The road to it, I believe, lies in the Algebraical Deduction of ground-forms from the Protomorphs*. Gordan's method of demonstration, so difficult and so complicated, requiring the devotion of a whole University semester to master, is inapplicable to reciprocants, which, as far as we can at present see, do not lend themselves to symbolic treatment.

How greatly must we feel indebted to our Cayley, who while he was, to say the least, the joint founder of the symbolic method, set the first, and out of England little if at all followed, example of using as an engine that mightiest instrument of research ever yet invented by the mind of man—a Partial Differential Equation, to define and generate invariantive forms.

With the growth of our knowledge, and higher views now taken of invariantive forms, the old nomenclature has not altogether kept pace, and is in one or two points in need of a reform not difficult to indicate. I think that we ought to give a general name—I propose that of Binariants—to every rational integral form which is nullified by the general operator

$$\lambda a\delta_b + \mu b\delta_c + \nu c\delta_d + \dots,$$

where $\lambda, \mu, \nu, \dots$ are arbitrary numbers.

This operator, I think, having regard to the way in which its segments link on to one another, may be called the Vermicular.

Binariants corresponding to unit values of $\lambda, \mu, \nu, \dots$ may be termed standard binariants. Those for which these numbers are the terms of the natural arithmetical series 1, 2, 3, ... Invariantive binariants, which may be either complete or incomplete invariants; these latter are what are usually termed semi- or sub-invariants. I may presently have to speak of a third class of binariants for which the arbitrary multipliers are the numbers 3, 8, 15, 24 ... (the squares of the natural numbers each diminished by unity) which, if the theorem I have in view is supported by the event, will have to be termed Reciprocantive Binariants. But first let me call attention to what seems a breach of the asserted parallelism between the Invariantive and the

* See the section on the Algebraical Deduction of the Ground-forms of the Quintic in my memoir on Subinvariants in the *American Journal of Mathematics*. [Vol. III. of this Reprint, p. 580.]

Reciprocantive theories. In the former we have complete and incomplete invariants, but we have drawn no such distinction between one set of pure reciprocants and another. A parallel distinction does however exist.

If we use w, i, j to signify the weight, degree, and extent of an invariantive form, w is never less than the half product of ij; when equal to it the form is complete. In the case of reciprocants certain observed facts seem to indicate that there exists an analogous but less simple inequality. If this conjecture is verified it is not merely $\frac{ij}{2} - w$, but $\frac{ij}{2} - (j-2) - w$, which is never negative: and when this is zero, the form may be said to be complete*. There would then be thus complete forms in each of the two theories; in the earlier one they take a special name: this is the only difference.

We have spoken of Pure Reciprocants as being either projective or non-projective, but so far have abstained from particularising the external characters by which the former may be distinguished from the latter. I have good reason to suspect that the former are distinguished from the latter by being Binariants; that, in addition to being subject to annihilation by the operator V, they are also subject to annihilation by the Vermicular operator when made special by the use of the numerical multipliers 3, 8, 15 ... above alluded to, or in other words (as previously mentioned incidentally) are subject to satisfy two simultaneous partial differential equations instead of only one†.

* If this should turn out to be true, the "crude generating function" for reciprocants would be almost identical with that of in- and co-variants of the same extent j. The denominators would be absolutely identical; as regards the numerators, while that for invariantive forms is $1-a^{-1}x^{-2}$ the numerator for reciprocants would be $1-a^{-2}x^{-2j}$. As I write abroad and from memory there is just a chance that the index of a here given may be erroneous.

† As already stated in a previous footnote this conjecture is fully confirmed, my own proof having been corroborated (if it needed corroboration) by another entirely different one invented by M. Halphen, who fully shares my own astonishment at the fact of there being forms (half-horse, half-alligator) at once reciprocants and sub-invariants, and as such satisfying two simultaneous partial differential equations.

If instead of denoting the successive differential derivatives (starting from the second) $a, b, c, \ldots$ we call them $1.2.a$, $1.2.3.b$, $1.2.3.4.c$, ... the two Annihilators will be

$$a\delta_b + 2b\delta_c + 3c\delta_d + 4d\delta_e + \ldots$$

and
$$4\frac{a^2}{2}\delta_b + 5ab\delta_c + 6\left(ac + \tfrac{1}{2}b^2\right)\delta_d + 7\left(ad + bc\right)\delta_e + \ldots$$

the latter being my new operator, the Reciprocator V, accommodated to the above-stated change of notation for the successive differential derivatives.

Hardly necessary is it for me to point out in explanation of the semi-sums $\frac{1}{2}b^2$, ... that we may write the MacMahonised V under the form

$$4a^2\delta_b + 5(ab+ba)\delta_c + 6(ac+b^2+ca)\delta_d + 7(ad+bc+cb+da)\delta_e + \ldots.$$

It is to be presumed that in addition to mixed reciprocants (the ocean into which flows the sea of pure reciprocants, as into that again empties itself the river of projective reciprocants) there may exist a theory of forms in which y as well as $\frac{dy}{dx}$ will appear, or, so to say, doubly mixed reciprocants, the most general of all, in which case we must speak of the content of these as the

Projective Reciprocants we have seen are disguised or masked Ternary Covariants—Covariants in the grub, the first undeveloped state. Now ternary covariants are capable, it may or may not be generally known, of satisfying 6 reducible to 2 simultaneous Partial Differential Equations, and at first sight it might be surmised that nothing would be gained by the substitution of the two new for the two old simultaneous partial differential equations. But the fact is not so, for the old partial differential equations are perfectly unmanageable, or at least have never, as far as I know, been handled by any one, for they have to do with a *triangular heap*, whereas the new ones are solely concerned with a *linear series* of coefficients.

I have alluded to there being a particular form common to the two theories. In the one theory it is the Mongian alluded to in the correspondence, which has been read, with M. Halphen. In the other it is the source of the skew covariant to the cubic. If the latter be subjected to a sort of MacMahonic numerical adjustment, it becomes absolutely identical with the former. Let us imagine that before the invention of Reciprocants an Algebraist happened to have had both forms present to his mind, and had thought of some contrivance for lowering the coefficients of the Mongian written out with the larger coefficients, and had thus stumbled upon this striking fact. It could not have failed to vehemently arouse his curiosity, and he would have set to work to discover, if possible, the cause of this coincidence. He would in all probability have addressed himself to the form which precedes the source alluded to in the natural order of genesis, and have applied a similar adjustment to the much simpler form, $ac-b^2$: having done so he would have tried to discover to what singularity it pointed—but his efforts to do so we know must have been fruitless, and he would have felt disposed to throw down his work in despair, for the intermediate ideas necessary to make out the parallelism would not have been present to his mind. So long as we confine ourselves to Differential Invariants, that is, to projective pure reciprocants, we are like men walking on those elevated ridges, those more than Alpine summits, such as I am told* exist in Thibet, where it may be the labour of days for two men who can see and speak to each other to come together. Reciprocants supply the bridge to span the yawning ravine and to bring allied forms into direct proximity.

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ocean and of the others as sea, river, and brook. Curious is it to reflect that in the theory which as it exists comprises Invariantives, Reciprocants, and Invariantive Reciprocants or Reciprocant Invariantives, the order of discovery was (1) Invariantives (Eisenstein, Boole, &c.); (2) Invariantive Reciprocants (Monge and Halphen); (3) Reciprocants (Schwarz, the author of this lecture).

* I think my informant was my friend Dr Inglis, of the Athenæum Club, who some time ago undertook a journey in the Himalayas in the hopes of coming upon the traces of a lost religion which he thought he had reason to believe existed among mankind in the pre-Glacial period of the earth's history.

I have spoken of mixed reciprocants as being subject to satisfy not a linear partial differential equation, but one of a higher order dependent on the intensity, so to say, of its mixedness—the highest power, that is to say, of the first differential derivative which it contains, and it might therefore be supposed that these forms are much more difficult to be obtained than pure reciprocants. But the fact is just the reverse, for as I discovered in the very infancy of the inquiry, and have put on record in the September or October number* of the *Mathematical Messenger*, mixed reciprocants may be evolved in unlimited profusion by the application of simple and explicit processes of multiplication and differentiation. From any reciprocant whatever, be it mixed or pure, new mixed ones may be educed infinitely infinite in number, inasmuch as at each stage of the process, arbitrary functions of the first differential derivative may be introduced.

The wonderful fertility of this method of generation excited warm interest on the part of one of the greatest of living mathematicians, the expression of which acted as a powerful incentive to me to continue the inquiry. They may be compared with the shower of December meteors shooting out in all directions and covering the heavens with their brilliant trains, all diverging from one or more fixed radiant-points, the radiant-point in the theory before us being the particular form selected to be operated upon.

The new doctrine which I have endeavoured thus imperfectly to adumbrate has taken its local rise in this University, where it has already attracted some votaries to its side, and will, I hope, eventually obtain the cooperation of many more. I have ventured with this view to announce it as the subject of a course of lectures during the ensuing term.

When I lately had the pleasure of attending the new Slade Professor's inaugural discourse, I heard him promise to make his pupils participators in his work, by painting pictures in the presence of his class. I aspire to do more than this—not only to paint before the members of my class, but to induce them to take the palette and brush and contribute with their own hands to the work to be done upon the canvas. Such was the plan I followed at the Johns Hopkins University, during my connection with which I may have published scores of Mathematical articles and memoirs in the journals of America, England, France, and Germany, of which probably there was scarcely one which did not originate in the business of the class-room; in the composition of many or most of them I derived inestimable advantage from the suggestions or contributions of my auditors. It was frequently a chase, in which I started the fox, in which we all took a common interest, and in which it was a matter of eager emulation between my hearers and myself to try which could be first in at the death.

[* p. 255 above.]

During the past period of my professorship here, imperfectly acquainted with the usages and needs of the University, I do not think that my labours have been directed so profitably as they might have been either as regards the prosecution of my own work or the good of my hearers: my attention has been distracted between theories waiting to be ushered into existence and providing for the daily bread of class-teaching. I hope that in future I may be able to bring these two objects into closer harmony and correlation, and I think I shall best discharge my duty to the University by selecting for the material of my work in the class-room any subject on which my thoughts may, for the time being, happen to be concentrated, not too alien to, or remote from, that which I am appointed to teach; and thus, by example, give lessons in the difficult art of mathematical thinking and reasoning—how to follow out familiar suggestions of analogy till they broaden and deepen into a fertilising stream of thought—how to discover errors and to repair them, guided by faith in the existence and unity of that intellectual world which exists within us, and is at least as real as that with which we are environed.

The *American Mathematical Journal*, conducted under the auspices of the Johns Hopkins University, which has gained and retains a leading position among the most important of its class, whether measured by the value of its contents or the estimation in which it is held by the Mathematical world, bears as its motto—

πραγμάτων ἔλεγχος οὐ βλεπομένων.

I have the pleasure of seeing among my audience this day the most distinguished geometer of Holland, Prof. Schoute, who has done me the signal honour of coming over to England to be present at this lecture, who hospitably entertained me at Groningen (in a vacation visit which I recently paid to his country, the classic soil which has given birth to an Erasmus, a Grotius, a Boerhaave, a Spinoza, a Huyghens, and a Rembrandt), and who was kind enough, in proposing my health at a party where many of his colleagues were present, to say that he felt sure "that I should return to England cheered and invigorated, and would, ere long, light on some discovery which would excite the wonder of the Mathematical world."

I do not venture to affirm, nor to think, that this vaticination has been fulfilled in the terms in which it was uttered, but can most truly say that the discovery, which it has been my good fortune to be made the medium of revealing, has excited my own deepest feelings of ever-increasing wonder rising almost to awe, such as must have come over the revellers who saw the handwriting start out more and more plainly on the wall, or the *scienziati* crowding round the blurred palimpsest as they began to be able to decipher characters and piece together the sentences of the long lost and supposed irrecoverable *De Republicâ.*

When I was at Utrecht, on my way to Groningen, Mr Grinwis, the Professor of Mathematics at that University, showed me an English book on "Differential Equations," which had just appeared, of which he spoke in high terms of praise, and said it contained over 800 examples. I wrote at once for the book to England, and on seeing it on my arrival, forgetting that it had been ordered, mistook it for a present from the author or publisher, and, what is unusual with me, read regularly into it, until I came to the section on Hyper-geometrical series, where the Schwarzian Derivative (so named by Cayley after Prof. Schwarz) is spoken of.

Perhaps I ought to blush to own that it was new to me, and my attention was riveted by the property it possesses, in common with the more simple form which points to inflexions on curves, of remaining substantially unaltered, of persisting as a factor at least of its altered self, when the variables which enter it are interchanged. Following out this indication, I at once asked myself the question, "ought there not to exist combinations of derivatives of *all* orders possessing this property of reciprocation?" That question was soon answered, and the universe of mixed reciprocants stood revealed before me. These mixed reciprocants, by simple processes of combination, led me to the discovery of the first pure reciprocant, $3b^2 - 5ac$: whereupon I again put the question to myself, "are there, or are there not, others of this form, and if so, what are they?"

In an unexpected manner the question was answered, and my curiosity gratified to the utmost by the discovery of the partial differential equation which is the central point of the theory, and at once discloses the parallelism between it and the familiar doctrine of Invariants. Two principal exponents of that doctrine, who have infused new blood into it, and given it a fresh point of departure—Capt. MacMahon and Mr Hammond—I have the pleasure of seeing before me. Mr Kempe, who is also present, has lately entered into and signally distinguished himself in the same field, availing himself in so doing of his profound insight into the subject of linkages, his interest in which I believe I may say received its first impulse from the lecture which he heard me deliver upon it at the Royal Institution in January 1874, on the very night when the Prime Minister for the time being sent round letters to his supporters announcing his intention to dissolve Parliament. Of the two events I have ever regarded the lecture as by far the more important to the permanent interests of society. He has lately applied ideas founded upon linkages to produce a most original and remarkable scheme for explaining the nature of the whole pure body of Mathematical truth, under whatever different forms it may be clothed, in a memoir which has been recommended to be printed in the *Transactions* of the Royal Society, and which, I think, cannot fail when published to excite the deepest interest alike in the Mathematical and the Philosophical worlds*.

I also feel greatly honoured by the presence of Prof. Greenhill, who will be known to many in this room from his remarkable contributions to the theory of Hydrodynamics and Vortex Motion, and who has sufficient candour and largeness of mind to be able to appreciate researches of a different character from those in which he has himself gained distinction.

I should not do justice to my feelings if I did not acknowledge my deep obligations to Mr Hammond for the assistance which he has rendered me, not only in preparing this lecture which you have listened to with such exemplary patience, but in developing the theory; I am indebted to him for many valuable suggestions tending to enlarge its bounds, and believe have been saved, by my conversations with him, from falling into some serious errors of omission or oversight. Saving only our Cayley (who, though younger than myself, is my spiritual progenitor—who first opened my eyes and purged them of dross so that they could see and accept the higher mysteries of our common Mathematical faith), there is no one I can think of with whom I ever have conversed, from my intercourse with whom I have derived more benefit. It would be an immense gain to Science, and to the best interests of the University, if something could be done to bring such men as Mr Hammond (and, let me add, Mr Buchheim, who ought never to have been allowed to leave it) to come and live among us. I am sure that with their endeavours added to my own and those of that most able body of teachers and researchers with whom I have the good fortune to be associated—my brother Professors and the Tutorial Staff of the University—we could create such a School of Mathematics as might go some way at least to revive the old scientific renown of Oxford, and to light such a candle in England as, with God's grace, should never be put out†.

* In his memoir for the *Phil. Trans.* Mr Kempe contends that any whatever mathematical proposition or research is capable of being represented by some sort of simple or compound linkage. One would like to know by what sort of linkage he would represent the substance of the memoir itself.

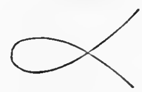
† I have purposely confined myself in my lecture to reciprocants, indicative of properties of plane curves, but had in view to extend the theory to the case of higher dimensions in space leading to reciprocants involving the differential derivatives of any number of variables y, z, ... M. Halphen, with whom I have had the great advantage of being in communication during my stay in Paris, has anticipated me in this part of my plan, and has found that the same method which I have used to obtain the Annihilator V applied to a system of variables leads to an Annihilator of a very similar form to V, and at my request will publish his results in a forthcoming number of the *Comptes Rendus*. Thus the dominion of reciprocants is already extended over the whole range of forms unlimited in their own number as well as in that of the variables which they contain.

Tables of Singularities and Formulæ referred to in the Preceding Lecture.

Chart 1.

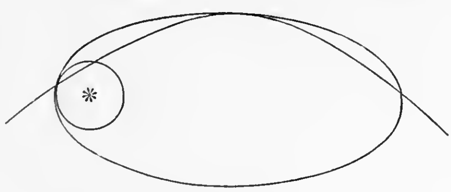
Inflexion

Node



Cusp

Points of maximum and minimum curvature



Bitangent

Chart 2.—Protomorphs.

Binariants.	*Reciprocants.*
a	a
$ac-b^2$	$3ac-5b^2$
$a^2d-3abc+2b^3$	$9a^2d-45abc+40b^3$
$ae-4bd+3c^2$	$5a^2e-35abd+7ac^2+35b^2c$
$a^2f+5abe+2acd+8b^2d-6bc^2$	$45a^3f-420a^2be-42a^2cd+1120ab^2d-315abc^2$ $-1120b^3c$
$ag-6bf+15ce-10d^2$	$a^2g-12abf-450ace+792b^2e+588ade^2$ $-2772bcd+1925c^3$

CHART 3.

No. 1. a

No. 2. $3ac-5b^2$

No. 3. $9a^2d-45abc+40b^3$

No. 4. $45a^3d^2-450a^2bc+192a^2c^3+400ab^3d+165ab^2c^2-400b^4c$

$$x=\int\frac{dt}{\sqrt{\{\kappa(1-15t^2+15t^4-t^6)+\lambda(3t-10t^3+3t^5)\}}}+\mu$$

$$y=\int\frac{tdt}{\sqrt{\{\kappa(1-15t^2+15t^4-t^6)+\lambda(3t-10t^3+3t^5)\}}}+\nu$$

$$V=3a^2\delta_b+10ab\delta_c+(15ac+10b^2)\delta_d+(21ad+35bc)\delta_e$$
$$+(28ae+56bd+35c^2)\delta_f+\ldots$$

CHART 4.—COEFFICIENTS OF ANNIHILATOR V.

1	4	3			
1	5	10			
1	6	15	10		
1	7	21	35		
1	8	28	56	35	
1	9	36	84	126	
1	10	45	120	210	126

CHART 5.—RECIPROCANT TRANSFORMATIONS.

Grub	*Chrysalis*			*Imago*		
$\frac{d^2y}{dx^2}$	$\frac{d^2\phi}{dx^2}$	$\frac{d^2\phi}{dxdy}$	$\frac{d\phi}{dx}$	$\frac{d^2\Phi}{dx^2}$	$\frac{d^2\Phi}{dxdy}$	$\frac{d^2\Phi}{dxdz}$
	$\frac{d^2\phi}{dxdy}$	$\frac{d^2\phi}{dy^2}$	$\frac{d\phi}{dy}$	$\frac{d^2\Phi}{dxdy}$	$\frac{d^2\Phi}{dy^2}$	$\frac{d^2\Phi}{dydz}$
	$\frac{d\phi}{dx}$	$\frac{d\phi}{dy}$	$\bullet$	$\frac{d^2\Phi}{dxdz}$	$\frac{d^2\Phi}{dydz}$	$\frac{d^2\Phi}{dz^2}$
(a)		(M)			(H)	

$$(n-1)^2\left(\frac{d\phi}{dy}\right)^3 a+H+\left\{\frac{d^2\Phi}{dx^2}\cdot\frac{d^2\Phi}{dy^2}-\left(\frac{d^2\Phi}{dxdy}\right)^2\right\}\Phi=0.$$

$\frac{dy}{dx}\frac{d^3y}{dx^3}-\frac{3}{2}\left(\frac{d^2y}{dx^2}\right)^2$ is the Schwarzian, otherwise written $tb-\frac{3a^2}{2}$.

CHART 6.—THE H RECIPROCANTIVE PROTOMORPH.

U	W
$65a^4h$	$120a^3cf$
$-975a^3bg$	$-200a^2b^2f$
$-990a^3cf$	$-195a^3de$
$+6200a^2b^2f$	$-145a^2bce$
$+4690a^2bce$	$+1000ab^3e$
$-1540ab^3e$	$+1365a^2bd^2$
$-2730a^2bd^2$	$-777a^2c^2d$
$+7161a^2c^2d$	$-22260ab^2cd$
$+3080ab^2cd$	$+2485abc^3$
$-24255abc^3$	$+105b^3c^2$
$+25410b^3c^2$	

$H+\Lambda U+MW$

The Vermicular Operator.

$\lambda a\delta_b+\mu b\delta_c+\nu c\delta_d+\pi d\delta_e+\ldots$

Examples.

$a\delta_b+b\delta_c+c\delta_d+d\delta_e+\ldots$
$a\delta_b+2b\delta_c+3c\delta_d+4d\delta_e+\ldots$
$3a\delta_b+8b\delta_c+15c\delta_d+24d\delta_e+\ldots$

☞ b^4d does not appear in either U or W.

Λ and M are arbitrary numbers.

42.

LECTURES ON THE THEORY OF RECIPROCANTS.

[*American Journal of Mathematics*, VIII. (1886), pp. 196—260; IX. pp. 1—37, 113—161, 297—352; X. pp. 1—16. Delivered in Oxford, 1886.]

THE lectures here reproduced were delivered, or are still in the course of delivery, before a class of graduates and scholars in the University of Oxford during the present year. They are to be regarded as easy lessons in the new Theory of Reciprocants of which an outline will be found in *Nature* for January 7, which contains a report of a Public Lecture on the subject delivered before the University of Oxford in December of the preceding year.

They are designed as a practical introduction to an enlarged theory of Algebraical Forms, and as such are not constructed with the rigorous adhesion to logical order which might be properly expected in a systematic treatise. The object of the lecturer was to rouse an interest in the subject, and in pursuit of this end he has not hesitated to state many results, by way of anticipation, which might, with stricter regard to method, have followed at a later period in the course.

There will be found also occasional repetitions and intercalations of allied topics which are to be justified by the same plea, and also by the fact that the lectures were not composed in their entirety previous to delivery, but gradually evolved and written between one lecture and another in the way that seemed most likely to the lecturer to secure the attention of his auditors.

Since the delivery of his public lecture in December last, papers have been contributed on the subject to the *Proceedings of the Mathematical Society of London* by Messrs Hammond, MacMahon, Elliott, Leudesdorf and Rogers, and one to the *Comptes Rendus de l'Institut* by M. George Perrin. It may therefore be inferred that the lectures have not altogether failed in attaining the desired end of drawing attention to a subject which, in the opinion of the lecturer, constitutes a very considerable extension of the previous limits of algebraical science.

LECTURE I.

A new world of Algebraical forms, susceptible of important geometrical applications, has recently come into existence, of which I gave a general account in a public lecture at the end of last term. I propose in the following brief course to go more fully into the subject and lay down the leading principles of the theory so far as they are at present known to me. The parallelism between the theory of what may be called pure reciprocants and that of invariants is so remarkable that it will be frequently expedient to pass from one theory to the other or to treat the two simultaneously. It may be as well therefore at once to give notice that the term invariant will hereafter be applied alike to invariants ordinarily so called and to those more general algebraical forms which have been termed sources of covariants, differentiants, seminvariants, or subinvariants. A form which is an invariant in the old sense will be termed, when necessary to specify it, a satisfied invariant, an expression which the chemico-graphical representation of invariants or covariants will serve to explain and justify.

In an elucidatory course of lectures such as the present, it will be advisable to follow a freer order of treatment than would be suitable to the presentation of it in a systematic memoir. My object is to make the theory known, to excite curiosity regarding it, and to invite co-operation in the task of its development.

By way of introduction to the subject, let us begin with an investigation of the properties of a differential expression involving only the first, second and third differential coefficients of either of two variables in respect to the other. For this purpose let us consider not what I have called the Schwarzian itself, which is an integral rational function of these three quantities, but the fractional expression

$$\frac{\dfrac{d^3y}{dx^3}}{\dfrac{dy}{dx}} - \frac{3}{2}\left(\frac{\dfrac{d^2y}{dx^2}}{\dfrac{dy}{dx}}\right)^2$$

which becomes the Schwarzian when cleared of fractions, and which after Cayley we may call the Schwarzian Derivative and denote by

$$(y, x);$$

(x, y) will then of course mean

$$\frac{\dfrac{d^3x}{dy^3}}{\dfrac{dx}{dy}} - \frac{3}{2}\left(\frac{\dfrac{d^2x}{dy^2}}{\dfrac{dx}{dy}}\right)^2.$$

It is easy to establish the identical equation

$$(y, x) = -\left(\frac{dy}{dx}\right)^2 (x, y). \tag{1}$$

Using for brevity y', y'', y''' to denote, as usual,

$$\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3},$$

and $x_{,}, x_{,,}, x_{,,,}$ to denote

$$\frac{dx}{dy}, \frac{d^2x}{dy^2}, \frac{d^3x}{dy^3},$$

respectively, the relation to be verified is

$$\frac{2y'y''' - 3y''^2}{y'^2} = -y'^2 . \frac{2x_{,}x_{,,,} - 3x_{,,}^2}{x_{,}^2}.$$

Now,
$$x_{,} = \frac{1}{y'},$$

$$x_{,,} = \frac{d}{dy}(x_{,}) = \frac{1}{y'} \cdot \frac{d}{dx}\left(\frac{1}{y'}\right) = -\frac{y''}{y'^3},$$

and
$$x_{,,,} = \frac{d}{dy}(x_{,,}) = \frac{1}{y'} \cdot \frac{d}{dx}\left(-\frac{y''}{y'^3}\right) = -\frac{y'''}{y'^4} + \frac{3y''^2}{y'^5}.$$

Whence we obtain

$$2x_{,}x_{,,,} - 3x_{,,}^2 = \left(-\frac{2y'''}{y'^5} + \frac{6y''^2}{y'^6}\right) - \frac{3y''^2}{y'^6}$$

$$= -\frac{1}{y'^6}(2y'y''' - 3y''^2),$$

and the truth of (1) is manifest.

This may be put under the form

$$\frac{2y'y''' - 3y''^2}{y'^3} = -\frac{2x_{,}x_{,,,} - 3x_{,,}^2}{x_{,}^3},$$

showing that a certain function of the first, second and third derivatives of one variable in respect to another remains unaltered, save as to algebraical sign, when the variables are interchanged. An example of a similar kind with which we are all familiar is presented by the well-known function $\frac{d^2y}{dx^2} \div \left(\frac{dy}{dx}\right)^{\frac{3}{2}}$, which is equal to $-\frac{d^2x}{dy^2} \div \left(\frac{dx}{dy}\right)^{\frac{3}{2}}$.

We are thus led to inquire whether there may not be an infinite number of algebraical functions of differential derivatives which possess a similar property, and by prosecuting this inquiry to lay the foundations of the theory of Reciprocation or Reciprocants.

Having regard to the fact that the present theory originated in that of the Schwarzian Derivative, I shall proceed to demonstrate, although this is not strictly necessary for the theory of Reciprocants, the remarkable identity

$$(y, x) - (z, x) = \left(\frac{dz}{dx}\right)^2 . (y, z).$$

This identical relation is the fundamental property of Schwarzians, and from it every other proposition concerning their form is an immediate deduction.

In the following proof*, y and z are regarded as two given functions of any variable t, and x as a variable function of the same: so that y and z are functions of x for any given function that x is of t.

It will be seen that

$$\{(y, x) - (z, x)\} \left(\frac{dx}{dz}\right)^2$$

remains unaltered by any infinitesimal variation θ of x, that is, when x becomes $x + \epsilon\phi(x)$, ϵ being an infinitesimal constant and $\phi(x)$ an arbitrary finite function of x.

For brevity, let accents denote differential derivation in regard to x, and let any function of x enclosed in a square parenthesis signify the augmented value of that function when x becomes $x+\theta$. In calculating such augmented values, since we suppose that $\theta = \epsilon\phi(x)$, it is clear that $\theta, \theta', \theta'' \ldots$ are each of them infinitesimals of the first order, and consequently that all products, and all powers higher than the first of these quantities, may be neglected.

We have therefore

$$[y'] = \frac{dy}{dx + d\theta} = \frac{y'}{1+\theta'} = y' - \theta' y'$$

$$[y''] = \frac{d\,[y']}{dx + d\theta} = \frac{\frac{d}{dx}(y' - \theta' y')}{1+\theta'} = \frac{y''(1-\theta') - \theta'' y'}{1+\theta'}$$

$$= y'' - 2\theta' y'' - \theta'' y'$$

$$[y'''] = \frac{d\,[y'']}{dx + d\theta} = \frac{\frac{d}{dx}(y'' - 2\theta' y'') - \theta'' y'}{1+\theta'} = \frac{y'''(1-2\theta') - 3\theta'' y'' - \theta''' y'}{1+\theta'}$$

$$= y''' - 3\theta' y''' - 3\theta'' y'' - \theta''' y'.$$

Hence

$$[y'y'''] = y'y''' - 4\theta' y' y''' - 3\theta'' y' y'' - \theta''' y'^2$$

$$\tfrac{3}{2}[y''^2] = \tfrac{3}{2} y''^2 - 6\theta' y''^2 - 3\theta'' y' y''$$

$$[y'^2] = y'^2 - 2\theta' y'^2.$$

And since by definition

$$(y, x) = \frac{y'y''' - \frac{3}{2}y''^2}{y'^2},$$

* As originally given in the *Messenger of Mathematics*, Vol. xv., this was defaced by so many errata as to render expedient its reproduction in a corrected form.

we readily obtain

$$[(y, x)] = \frac{(y, x)}{1-2\theta'} - 4\theta'(y, x) - \theta''' = (y, x)(1-2\theta') - \theta'''.$$

So also $$[(z, x)] = (z, x)(1-2\theta') - \theta'''.$$

Whence by subtraction

$$[(y, x) - (z, x)] = (1-2\theta')\{(y, x) - (z, x)\}.$$

Dividing the left-hand side of this by $[z'^2]$, and the right-hand side by $z'^2(1-2\theta')$ which is the equivalent of $[z'^2]$, our final result is

$$\left[\frac{(y, x) - (z, x)}{z'^2}\right] = \frac{(y, x) - (z, x)}{z'^2}.$$

Thus, then, we have seen that the expression

$$\frac{(y, x) - (z, x)}{\left(\dfrac{dz}{dx}\right)^2}$$

does not vary when x receives an infinitesimal variation $\epsilon\phi(x)$, from which it follows, by the general principle of successive continuous accumulation, that the same will be true when x undergoes any finite arbitrary variation, and consequently this expression has a value which is independent of the form of x regarded as a function of t; it will, of course, be remembered that y and z are supposed to be invariable functions of t. Let x become z, then (y, x) becomes (y, z), while at the same time (z, x) vanishes and $\frac{dz}{dx}$ becomes unity: so that we obtain

$$\frac{(y, x) - (z, x)}{\left(\dfrac{dz}{dx}\right)^2} = (y, z).$$

Hence, *whatever* function x may be of t,

$$(y, x) - (z, x) = \left(\frac{dz}{dx}\right)^2 . (y, z). \qquad (2)$$

To this fundamental proposition the equation marked (1), itself the important point in regard to the Theory of Reciprocants, is an immediate corollary. For if in (2) we interchange y and z, it becomes

$$(z, x) - (y, x) = \left(\frac{dy}{dx}\right)^2 . (z, y);$$

and now, making $x = z$, we have

$$-(y, z) = \left(\frac{dy}{dz}\right)^2 . (z, y),$$

which is the same as (1), except that z occupies the place of x.

But (1) may be obtained more immediately from (2) by substituting in it x for y and y for z, leaving x unaltered; when it becomes

$$-(y, x) = \left(\frac{dy}{dx}\right)^2 . (x, y).$$

This is equivalent to saying that

$$2y'y''' - 3y''^2 = -y'^6 (2x_{,}x_{,,,} - 3x_{,,}^2),$$

a verification of which has been given already.

Observe that $\dfrac{y'y''' - \frac{3}{2}y''^2}{y'^2}$ or (y, x) contains $\left(\dfrac{dy}{dx}\right)^2$ in its denominator and (x, y) contains $\left(\dfrac{dx}{dy}\right)^2$ in its denominator, which is the same as $\left(\dfrac{dy}{dx}\right)^2$ in the numerator. Thus it is that the *square* of $\dfrac{dy}{dx}$ enters three times.

Let me insist for a moment on the import of the fact brought to light in the course of this investigation, that $\dfrac{(y, x) - (z, x)}{\left(\dfrac{dz}{dx}\right)^2}$ is invariable when x, y and z being regarded as functions of t, x alters its form, but y and z retain theirs. Of course we might write $\left(\dfrac{dy}{dx}\right)^2$ in the denominator instead of $\left(\dfrac{dz}{dx}\right)^2$, and then make the same affirmation as before; as will be evident if we only remember that by hypothesis y and z are both of them constant functions of t, and that therefore $\left(\dfrac{dz}{dy}\right)^2$ must also be so. This is tantamount to saying that when the same conditions are fulfilled $\{(y, x) - (z, x)\} (dx)^2$ is invariable, that is, that when x becomes X in virtue of any substitution (including a homographic one) impressed upon it,

$$\{(y, x) - (z, x)\} (dx)^2 = \{(y, X) - (z, X)\} (dX)^2,$$

and thus we see that when x becomes X,

$$(y, x) - (z, x)$$

remains unaltered except that it takes to itself the factor $\left(\dfrac{dX}{dx}\right)^2$ which depends solely on the particular substitution impressed on x.

If $$y = f(x),\ z = \phi(x), \text{ and } X = \omega(x),$$

our formula becomes

$$\{(fx, x) - (\phi x, x)\} (dx)^2 = \{(f\omega^{-1}X, X) - (\phi\omega^{-1}X, X)\} (dX)^2,$$

so that, speaking of Quantics and Covariants with respect to a single variable x, $(fx, x) - (\phi x, x)$ is to all intents and purposes a Covariant to the simultaneous forms $f(x)$ and $\phi(x)$, in a sense comprehending but far transcending that in which the term is ordinarily employed; for it remains a persistent factor of its altered self when for x any arbitrary function of x is substituted, the new factor taken on depending wholly and solely on the particular substitution impressed upon x. In the ordinary theory of invariants, the substitution impressed is limited to be homographic; in this case it is absolutely general. We might, moreover, add as a corollary that if (y, x), (z, x), (u, x) ... are regarded as roots of any Binary Quantic, every invariant of that Binary Quantic is a covariant in the extended sense in which the word has just been used, in respect to the system of simultaneous forms $f(x)$, $\phi(x)$, $\psi(x)$ .... For every such invariant will be a function of

$$(y, x) - (z, x), \quad (y, x) - (u, x), \quad (z, x) - (u, x), \ldots$$

and will therefore remain a persistent factor of its altered self, taking on a power of $\frac{dX}{dx}$ as its extraneous factor.

Calling (fx, x) the Schwarzian Derivative of $f(x)$, our theorem may be stated in general terms as follows:

All invariants of a Binary Quantic whose roots are the Schwarzian Derivatives of a given set of functions of the same variable are Covariants (in an extended sense) of that set of functions.

The theory of the Schwarzian derivative originates in that of the linear differential equation of the second order,

$$u'' + 2Pu' + Qu = 0,$$

which becomes, when we write $u = ve^{-\int P dx}$,

$$v'' + Iv = 0,$$

where

$$I = Q - P^2 - P'.$$

Now, suppose that u_1 and u_2 are any two particular solutions of the first of these equations, and let z denote their mutual ratio; so that, when v_1 and v_2 are the corresponding particular solutions of the second equation, we readily obtain

$$z = \frac{u_2}{u_1} = \frac{v_2}{v_1},$$

and therefore,

$$z' = \frac{v_1 v_2' - v_2 v_1'}{v_1^2}.$$

A second differentiation gives

$$z'' = \frac{v_1 v_2'' - v_2 v_1''}{v_1^2} - \frac{2v_1'(v_1 v_2' - v_2 v_1')}{v_1^3}.$$

But since

$$\frac{v_1''}{v_1} = \frac{v_2''}{v_2} = -I,$$

the first term of the expression just found vanishes identically, and we have

$$z'' = -\frac{2v_1' z'}{v_1},$$

or, $$v_1' = -\frac{z''v_1}{2z'}.$$

Differentiating this again, we find

$$-2v_1'' = \left(\frac{z'''}{z'} - \frac{z''^2}{z'^2}\right)v_1 + \frac{z''}{z'}v_1'$$

$$= \left(\frac{z'''}{z'} - \frac{3}{2}\frac{z''^2}{z'^2}\right)v_1.$$

Hence $$\frac{z'''}{z'} - \frac{3}{2}\frac{z''^2}{z'^2} = 2I,$$

where the left-hand side of the equation is "the Schwarzian Derivative" with z written in the place of y.

LECTURE II.

The expression $2y'y''' - 3y''^2$, which we have called the Schwarzian, may be termed a reciprocant, meaning thereby that on interchanging y', y'', y''' with $x_{,}$, $x_{,,}$, $x_{,,,}$ its form remains unaltered, save as to the acquisition of what may be called an extraneous factor, which, in the case before us, is a power of y' (with a multiplier -1). Before we proceed to consider other examples of reciprocants it will be useful to give formulae by means of which the variables may be readily interchanged in any differential expression.

We shall write t for y' and τ for its reciprocal $x_{,}$, using the letters $a, b, c, \ldots$ to denote the second, third, fourth, etc., differential derivatives of y with respect to x, and α, β, γ, ... to denote those of x with respect to y. The advantage of this notation will be seen in the sequel.

The values of α, β, γ, ... in terms of $t, a, b, c, \ldots$ are given by the formulae

$\alpha = -a \div t^3,$

$\beta = -bt + 3a^2 \div t^5,$

$\gamma = -ct^2 + 10abt - 15a^3 \div t^7,$

$\delta = -dt^3 + (15ac + 10b^2)t^2 - 105a^2bt + 105a^4 \div t^9,$

$\epsilon = -et^4 + (21ad + 35bc)t^3 - (210a^2c + 280ab^2)t^2 + 1260a^3bt - 945a^5 \div t^{11},$

....

If, in these equations, we write

$$a = 1.2.a_0, \quad b = 1.2.3.a_1, \quad c = 1.2.3.4.a_2, \ldots$$

and $$\alpha = 1.2.\alpha_0, \quad \beta = 1.2.3.\alpha_1, \quad \gamma = 1.2.3.4.\alpha_2, \ldots$$

they become

$\alpha_0 = -a_0 \div t^3,$

$\alpha_1 = -a_1 t + 2a_0^2 \div t^5,$

$\alpha_2 = -a_2 t^2 + 5a_0 a_1 t - 5a_0^3 \div t^7,$

$\alpha_3 = -a_3 t^3 + (6a_0 a_2 + 3a_1^2)\, t^2 - 21 a_0^2 a_1 t + 14 a_0^4 \div t^9,$

$\alpha_4 = -a_4 t^4 + (7a_0 a_3 + 7a_1 a_2)\, t^3 - (28 a_0^2 a_2 + 28 a_1^2)\, t^2 + 84 a_0^3 a_1 t - 42 a_0^5 \div t^{11},$

....

Any one of the formulae in either set may be deduced from the formula immediately preceding it by a simple process of differentiation.

Thus, since $\beta = \dfrac{-bt + 3a^2}{t^5}$ and $\dfrac{d}{dy} = \dfrac{1}{t} \cdot \dfrac{d}{dx}$,

we have $$\frac{d\beta}{dy} = \frac{1}{t} \cdot \frac{d}{dx}\left(\frac{-bt + 3a^2}{t^5}\right).$$

But $\dfrac{d\beta}{dy} = \gamma$ and $\dfrac{d}{dx} = a\partial_t + b\partial_a + c\partial_b + \ldots,$

so that $$\gamma = \frac{1}{t}(a\partial_t + b\partial_a + c\partial_b + \ldots)\left(\frac{-bt + 3a^2}{t^5}\right)$$

$$= \frac{1}{t^7}(-ct^2 + 10abt - 15a^3).$$

By continually operating with $\dfrac{1}{t}(a\partial_t + b\partial_a + c\partial_b + \ldots)$ the table may be extended as far as we please, the expressions on the right-hand side being the successive values of

$$\left\{\frac{1}{t}(a\partial_t + b\partial_a + c\partial_b + \ldots)\right\}^n \left(-\frac{a}{t^3}\right)$$

found by giving to n the values 0, 1, 2, 3, ...

Precisely similar reasoning shows that, when the modified letters $a_0, a_1, a_2, \ldots$ are used,

$$(n+2)\,\alpha_n = \frac{1}{t}(2a_0\partial_t + 3a_1\partial_{a_0} + 4a_2\partial_{a_1} + \ldots)\,\alpha_{n-1},$$

and that $$\alpha_n = \frac{\left\{\frac{1}{t}(2a_0\partial_t + 3a_1\partial_{a_0} + 4a_2\partial_{a_1} + \ldots)\right\}^n \left(-\frac{a_0}{t^3}\right)}{3 \,.\, 4 \,.\, 5 \ldots (n+2)}.$$

A proof of the formula

$$\alpha_n = -t^{-n-3}\left(e^{-\frac{V}{t}}\right) a_n,$$

obtained by Mr Hammond, in which

$$V = 4 \,.\, \frac{a_0^2}{2}\partial_{a_1} + 5a_0 a_1 \partial_{a_2} + 6\left(a_0 a_2 + \frac{a_1^2}{2}\right)\partial_{a_3} + 7(a_0 a_3 + a_1 a_2)\,\partial_{a_4} + \ldots,$$

will be given later on, when we treat of this operator, which, in the theory of Reciprocants, is the analogue of the operator $a\partial_b + 2b\partial_c + 3c\partial_d + \ldots$, with which we are familiarly acquainted in the theory of Invariants.

Consider the expression

$$ct - 5ab.$$

If, in $\gamma\tau - 5\alpha\beta$, which may be called its transform, we write

$$\tau = \frac{1}{t}, \quad \alpha = -\frac{a}{t^3}, \quad \beta = \frac{-bt + 3a^2}{t^5}, \quad \gamma = \frac{-ct^2 + 10abt - 15a^3}{t^7},$$

this becomes a fraction whose denominator is t^8, while its numerator is

$$-ct^2 + 10abt - 15a^3 + 5a(-bt + 3a^2) = -ct^2 + 5abt.$$

Removing the common factor t from the numerator and denominator of this fraction, we have

$$\gamma\tau - 5\alpha\beta = -\frac{ct - 5ab}{t^7}.$$

Here, then, as in the case of the well-known monomial for which

$$a = -t^3\alpha,$$

and the Schwarzian for which

$$2bt - 3a^2 = -t^6(2\beta\tau - 3\alpha^2),$$

the expression $\quad ct - 5ab = -t^7(\gamma\tau - 5\alpha\beta)$

changes its sign on reciprocation.

That reciprocation is not always accompanied with a change of sign will be clear if we consider the product of any pair of the three expressions given above. Or we may take, as an example of a reciprocant in which this change of sign does not occur, the form

$$3ac - 5b^2.$$

Here $\quad 3\alpha\gamma - 5\beta^2 = \dfrac{3a(ct^2 - 10abt + 15a^3) - 5(bt - 3a^2)^2}{t^{10}}.$

In the fraction on the right-hand side the only surviving terms of the numerator are those containing the highest power of t, the rest destroying one another. Thus

$$3\alpha\gamma - 5\beta^2 = \frac{1}{t^8}(3ac - 5b^2).$$

Reciprocants which change their sign when the variables x and y are interchanged, will be said to be of odd character; those, on the contrary, which keep their sign unchanged will be said to be of even character. The distinction is an important one, and will be observed in what follows.

Forms such as the one just considered, where t does not appear in the form itself, but only in the extraneous factor, will be called Pure Reciprocants, in order to distinguish them from those forms (of which the Schwarzian $2tb - 3a^2$ is an example) into which t enters, which will be called Mixed Reciprocants. It will be seen hereafter that Pure Reciprocants are the analogues of the invariants of Binary Quantics.

With modified letters (that is, writing $a = 2a_0$, $b = 6a_1$, and $c = 24a_2$)

$$3ac - 5b^2 \text{ becomes } 144a_0a_2 - 180a_1^2 = 36(4a_0a_2 - 5a_1^2).$$

Operating on this with

$$V = 2a^2\partial_{a_1} + 5a_0a_1\partial_{a_2} + \ldots,$$

we have
$$V(4a_0a_2 - 5a_1^2) = 0.$$

We shall prove subsequently that all Pure Reciprocants are, in like manner, subject to annihilation by the operator V.

Hitherto we have only considered homogeneous forms; let us now take as an example of a non-homogeneous reciprocant the expression

$$(1 + t^2)b - 3a^2t.$$

Here
$$(1 + \tau^2)\beta - 3\alpha^2\tau = \left(1 + \frac{1}{t^2}\right)\left(\frac{-bt + 3a^2}{t^5}\right) - \frac{3a^2}{t^7}$$

$$= \frac{(1 + t^2)(-bt + 3a^2) - 3a^2}{t^7}.$$

In the numerator of this fraction the terms $+3a^2$ and $-3a^2$ cancel, a factor t divides out, and we have finally

$$(1 + \tau^2)\beta - 3\alpha^2\tau = -\frac{(1 + t^2)b - 3a^2t}{t^6}.$$

In general, a Reciprocant may be defined to be a function F of such a kind that $F(\tau, \alpha, \beta, \gamma, \ldots)$ contains $F(t, a, b, c, \ldots)$ as a factor. An important special case is that in which the other factor is merely numerical; the function F is then said to be an Absolute Reciprocant.

When we limit ourselves to the case where F is a rational integral function of the letters, it may be proved that

$$F(t, a, b, c, \ldots) = \pm t^\mu F(\tau, \alpha, \beta, \gamma, \ldots).$$

For, in the first place, since any one of the letters $\alpha, \beta, \gamma, \ldots$ is a rational function of $t, a, b, c, \ldots$ and integral with respect to all of them except t, containing only a power of this letter in the denominator, it is clear that any rational integral function of $\tau, \alpha, \beta, \gamma, \ldots$ such as $F(\tau, \alpha, \beta, \gamma, \ldots)$ is supposed to be, must be a rational integral function of $t, a, b, c, \ldots$ divided by some power of t. But since F is a reciprocant, $F(\tau, \alpha, \beta, \gamma, \ldots)$ must contain $F(t, a, b, c, \ldots)$ as a factor; and if we suppose the other factor to be

$$\frac{\phi(t, a, b, c, \ldots)}{t^\lambda},$$

we must have

$$F(\tau, \alpha, \beta, \gamma, \ldots) = \frac{\phi(t, a, b, c, \ldots)}{t^\lambda} F(t, a, b, c, \ldots),$$

where ϕ is rational and integral with respect to all the letters.

Moreover, $F(t, a, b, c, \ldots) = \dfrac{\phi(\tau, \alpha, \beta, \gamma, \ldots)}{\tau^\lambda} F(\tau, \alpha, \beta, \gamma, \ldots)$.

Hence we must have identically

$$\phi(t, a, b, c, \ldots)\, \phi(\tau, \alpha, \beta, \gamma, \ldots) = 1,$$

where, on the supposition that the functions ϕ contain other letters besides t and τ, $\phi(t, a, b, c, \ldots)$ is, and $\phi(\tau, \alpha, \beta, \gamma, \ldots)$ can be expressed as, a rational function integral as regards the letters $a, b, c, \ldots$. But this supposition is manifestly inadmissible, for the product of two integral rational functions of $a, b, c, \ldots$ cannot be identically equal to unity. Hence t is the only letter that can appear in the extraneous factor and we may write

$$F(\tau, \alpha, \beta, \gamma, \ldots) = \frac{\psi(t)}{t^\lambda} F(t, a, b, c, \ldots)$$

where $\psi(t)$ is a rational integral function.

The same reasoning as before shows that we must have identically

$$\psi(t)\, \psi(\tau) = 1.$$

But this cannot be true if $\psi(t)$ has any root different from zero; for if we give t such a value as will make $\psi(t)$ vanish, this value must also make $\psi(\tau)$ infinite; and since

$$\begin{aligned}\psi(\tau) &= A + B\tau + C\tau^2 + \ldots + M\tau^m \\ &= A + \frac{B}{t} + \frac{C}{t^2} + \ldots + \frac{M}{t^m},\end{aligned}$$

the only value of t for which $\psi(\tau)$ becomes infinite is a zero value. Hence $\psi(t)$ is of the form Mt^m, and consequently $\psi(\tau) = M\tau^m$. Thus

$$\psi(t)\, \psi(\tau) = M^2 t^m \tau^m = 1,$$

and therefore $$M^2 = 1.$$

We have now proved that if F is a rational integral reciprocant,

$$F(t, a, b, c, \ldots) = \pm t^\mu F(\tau, \alpha, \beta, \gamma, \ldots),$$

or we may say, $$= (-)^\kappa t^\mu F(\tau, \alpha, \beta, \gamma, \ldots),$$

where $\kappa = 1$ or 0 according as the reciprocant is of odd or even character.

It obviously follows that the product or quotient of any two rational integral reciprocants is itself a reciprocant; but it must be carefully observed that this is not true of their sum or difference unless certain conditions are fulfilled. For if we write

$$F_1(t, a, \ldots) = (-)^{\kappa_1} t^{\mu_1} F_1(\tau, \alpha, \ldots)$$

and $$F_2(t, a, \ldots) = (-)^{\kappa_2} t^{\mu_2} F_2(\tau, \alpha, \ldots),$$

we see that

$$pF_1(t, a, \ldots) + qF_2(t, a, \ldots) = (-)^{\kappa_1} t^{\mu_1} pF_1(\tau, \alpha, \ldots) + (-)^{\kappa_2} t^{\mu_2} qF_2(\tau, \alpha, \ldots),$$

and consequently this expression will be a reciprocant if $\kappa_1 = \kappa_2$ and $\mu_1 = \mu_2$, but not otherwise. If we call the index of t in the extraneous factor the *characteristic*, what we have proved is that no linear function of two reciprocants can be a reciprocant, unless they have the same characteristic and are of the same character. In dealing with Absolute Reciprocants, since the characteristic of these is always zero, we need only attend to their character.

I propose for the present to confine myself to homogeneous and isobaric reciprocants*, that is, to such as are homogeneous and isobaric when the letters $t, a, b, c, \ldots$ are considered to be each of degree 1, their respective weights being $-1, 0, 1, 2, \ldots$. The letter w will be used to denote the weight of such a reciprocant, i its degree, and j its extent, that is, the weight of the most advanced letter which it contains.

Let any such reciprocant $F(t, a, b, c, \ldots)$ contain a term $At^v a^l b^m c^n \ldots$, then

$$v + l + m + n + \ldots = i,$$

and

$$-v + m + 2n + \ldots = w.$$

The corresponding term in $F(\tau, \alpha, \beta, \gamma, \ldots)$ will be $A\tau^v \alpha^l \beta^m \gamma^n \ldots$ where

$$\tau = \frac{1}{t}, \quad \alpha = -\frac{a}{t^3}, \quad \beta = -\frac{b}{t^4} + \ldots, \quad \gamma = -\frac{c}{t^5} + \ldots, \text{ etc.}$$

Now, if no term of F contains a smaller number of the letters $a, b, c, \ldots$ than are found in the term we are considering, the first terms of β, γ, etc., may be taken instead of these quantities themselves and $A\tau^v \alpha^l \beta^m \gamma^n \ldots$ may be replaced by

$$(-)^{l+m+n+\ldots} At^{-v-3l-4m-5n-\ldots} a^l b^m c^n \ldots = (-)^{i-v} At^{v-3i-w} a^l b^m c^n \ldots.$$

But since $\quad F(t, a, b, c, \ldots) = (-)^\kappa t^\mu F(\tau, \alpha, \beta, \gamma, \ldots)$

we must have identically

$$At^v a^l b^m c^n \ldots = (-)^{i-v+\kappa} At^{\mu+v-3i-w} a^l b^m c^n \ldots.$$

Hence the character is even or odd according to the parity of $i - v$ (that is, of the smallest number of letters different from t in any term), and the characteristic $\mu = 3i + w$.

The type of a reciprocant depends on the *character*, weight, degree and extent. As the extraneous factor is always of the form $(-)^\kappa t^\mu$, where κ is 1 or 0, we may define the type of a reciprocant by

$$1 : w : i, j \quad \text{or} \quad 0 : w : i, j,$$

according as its character is odd or even.

For Pure Reciprocants the smallest number of letters different from t in any term is (since all the letters are different from t) the same as its degree.

* Here and elsewhere the word *reciprocant* is used in the sense of *rational integral reciprocant*: this will always be done when there is no danger of confusion arising from it.

Hence the character of a Pure Reciprocant is odd or even according to the parity of i, and for this reason the type of a Pure Reciprocant may be defined by $w : i, j$.

A linear combination of reciprocants of the same type will be a reciprocant, for when the type is known both the character and characteristic are given.

LECTURE III.

Let F be any function (not necessarily homogeneous or even algebraical) of the differential derivatives which acquires a numerical multiplier M, but is otherwise unchanged when the reciprocal substitution of x for y and y for x is effected. A second reciprocation multiplies the function again by M, and thus the total effect of both substitutions is to multiply F by M^2. But since the second reciprocation reproduces the original function, we must have $M^2 = 1$. Functions of this kind are therefore unaltered by reciprocation (except it may be in sign), and for this reason are called *Absolute Reciprocants.* These, as we shall presently see, play an important part in the general theory. Like all other reciprocants, they range naturally in two distinct classes, those of odd and those of even character.

It is perhaps worthy of notice that the extraneous factor of a general reciprocant is the exponential of an absolute reciprocant of odd character. For if

$$F(t, a, b, c, \ldots) = \phi(t, a, b, c, \ldots)\, F(\tau, \alpha, \beta, \gamma, \ldots),$$

we must still have, as before,

$$\phi(t, a, b, c, \ldots)\, \phi(\tau, \alpha, \beta, \gamma, \ldots) = 1\,;$$

that is $$\log \phi(t, a, b, c, \ldots) = -\log \phi(\tau, \alpha, \beta, \gamma, \ldots)\,;$$

or $\log \phi(t, a, b, c, \ldots)$ is an absolute reciprocant of odd character.

An absolute reciprocant may be obtained from any pair of rational integral reciprocants in the same way that an absolute invariant is found from two ordinary invariants. For let

$$F_1(t, a, b, c, \ldots) = (-)^{\kappa_1} t^{\mu_1} F_1(\tau, \alpha, \beta, \gamma, \ldots),$$

and $$F_2(t, a, b, c, \ldots) = (-)^{\kappa_2} t^{\mu_2} F_2(\tau, \alpha, \beta, \gamma, \ldots),$$

then $$\frac{\{F_1(t, a, b, c, \ldots)\}^{\mu_2}}{\{F_2(t, a, b, c, \ldots)\}^{\mu_1}} = (-)^{\kappa_1\mu_2 - \kappa_2\mu_1} \frac{\{F_1(\tau, \alpha, \beta, \gamma, \ldots)\}^{\mu_2}}{\{F_2(\tau, \alpha, \beta, \gamma, \ldots)\}^{\mu_1}}\,;$$

or we may say that $F_1^{\mu_2} \div F_2^{\mu_1}$ is an absolute reciprocant of even or odd character according to the parity of $\kappa_1\mu_2 - \kappa_2\mu_1$.

Thus, for example, from

$$a = -t^3\alpha$$

and $$3ac - 5b^2 = t^8(3\alpha\gamma - 5\beta^2)$$

we form $\dfrac{(3ac-5b^2)^3}{a^8}$, an absolute reciprocant of even character.

From a reciprocant F whose characteristic is μ we obtain an absolute reciprocant of the same character as F by dividing it by $t^{\frac{\mu}{2}}$.

For if we only remember that $\tau = \dfrac{1}{t}$, it obviously follows that

$$F(t, a, b, c, \ldots) = \pm t^{\mu} F(\tau, \alpha, \beta, \gamma, \ldots)$$

can be written in the form

$$\frac{F(t, a, b, c, \ldots)}{t^{\frac{\mu}{2}}} = \pm \frac{F(\tau, \alpha, \beta, \gamma, \ldots)}{\tau^{\frac{\mu}{2}}},$$

where the original character of the reciprocant F is preserved.

It may be noticed that a reciprocant of odd character cannot be divided by $\sqrt{(-1)}\, t^{\frac{\mu}{2}}$ so as to give an absolute reciprocant of even character; for, the reciprocal of F being $-t^{\mu}F'$, that of $F \div \sqrt{(-1)}\, t^{\frac{\mu}{2}}$ will still be $-F' \div \sqrt{(-1)}\tau^{\frac{\mu}{2}}$. The character of a reciprocant is thus seen to be one of its indelible attributes.

As simple examples of absolute reciprocants we may take $\dfrac{3ac-5b^2}{t^4}$, which becomes on reciprocation $\dfrac{3\alpha\gamma - 5\beta^2}{\tau^4}$, and $\dfrac{a}{t^{\frac{3}{2}}}$, which reciprocates into $-\dfrac{\alpha}{\tau^{\frac{3}{2}}}$. The character of the former is even, that of the latter odd.

Observing that

$$\log t = -\log \tau \text{ and } \frac{1}{\sqrt{t}} \cdot \frac{d}{dx} = \frac{1}{\sqrt{\tau}} \cdot \frac{d}{dy},$$

we have $$\left(\frac{1}{\sqrt{t}} \cdot \frac{d}{dx}\right) \log t = -\left(\frac{1}{\sqrt{\tau}} \cdot \frac{d}{dy}\right) \log \tau.$$

From this, in like manner, we obtain

$$\left(\frac{1}{\sqrt{t}} \cdot \frac{d}{dx}\right)^2 \log t = -\left(\frac{1}{\sqrt{\tau}} \cdot \frac{d}{dy}\right)^2 \log \tau;$$

and so, in general,

$$\left(\frac{1}{\sqrt{t}} \cdot \frac{d}{dx}\right)^i \log t = -\left(\frac{1}{\sqrt{\tau}} \cdot \frac{d}{dy}\right)^i \log \tau.$$

Hence $\left(\frac{1}{\sqrt{t}}\cdot\frac{d}{dx}\right)^i \log t$ is an absolute reciprocant, and of an odd character, for all positive integral values of i. We thus obtain a series of fractions with rational integral homogeneous reciprocants in their numerators and powers of $t^{\frac{3}{2}}$ in their denominators. It will be sufficient, before proceeding to the more general theory of *Eduction*, as it may be called, to examine, by way of illustration, the cases in which $i = 1$, 2 and 3.

Let $i = 1$; then

$$\left(\frac{1}{\sqrt{t}}\cdot\frac{d}{dx}\right)\log t = \frac{a}{t^{\frac{3}{2}}}.$$

So that, in the case where $i = 2$, we have

$$\left(\frac{1}{\sqrt{t}}\cdot\frac{d}{dx}\right)^2 \log t = \left(\frac{1}{\sqrt{t}}\cdot\frac{d}{dx}\right)\frac{a}{t^{\frac{3}{2}}} = \frac{b}{t^2} - \frac{3}{2}\cdot\frac{a^2}{t^3}$$

$$= \frac{2bt - 3a^2}{2t^3}.$$

The numerator of this fraction is the Schwarzian.

In like manner, when $i = 3$,

$$\left(\frac{1}{\sqrt{t}}\cdot\frac{d}{dx}\right)^3 \log t = \left(\frac{1}{\sqrt{t}}\cdot\frac{d}{dx}\right)\left(\frac{2bt-3a^2}{2t^3}\right) = \frac{2ct-4ab}{2t^{\frac{7}{2}}} - \frac{6abt-9a^3}{t^{\frac{9}{2}}} = \frac{2ct^2-10abt+9a^3}{2t^{\frac{9}{2}}}.$$

But here a reduction may be effected, for $\left(\frac{a}{t^{\frac{3}{2}}}\right)^3$, as well as $\frac{a}{t^{\frac{3}{2}}}$ itself, is an absolute reciprocant of the same character as the whole of the expression just found. Hence we may reject the term $\frac{9}{2}\cdot\frac{a^3}{t^{\frac{9}{2}}}$ without thereby affecting the reciprocantive property of the form, and thus obtain

$$\frac{ct - 5ab}{t^{\frac{7}{2}}},$$

an absolute reciprocant of odd character. The corresponding rational integral reciprocant is

$$ct - 5ab.$$

We have found that $\frac{a}{t^{\frac{3}{2}}}$ and $\frac{2bt - 3a^2}{t^3}$ are each of them reciprocants. Why, then, by parity of reasoning, is not $\frac{2bt}{t^3}$, and therefore b, a reciprocant? It is because $\frac{a^2}{t^3}$, the square of $\frac{a}{t^{\frac{3}{2}}}$, is of even character, while $\frac{2bt - 3a^2}{t^3}$ is of an odd character, so that no linear combination of the two would be *legitimate*.

If we differentiate any absolute reciprocant with respect to x, we shall obtain another reciprocant of the same character. For let R be any absolute reciprocant and R' its transform, then

$$R = \pm R';$$

and since $\frac{d}{dx} = t\frac{d}{dy}$ may be written in the equivalent but more symmetrical form

$$\frac{1}{\sqrt{t}} \cdot \frac{d}{dx} = \frac{1}{\sqrt{\tau}} \cdot \frac{d}{dy},$$

we have
$$\left(\frac{1}{\sqrt{t}} \cdot \frac{d}{dx}\right) R = \pm \left(\frac{1}{\sqrt{\tau}} \cdot \frac{d}{dy}\right) R'.$$

On one side of this identical equation is a function of the differential derivatives of y with respect to x; on the other, a precisely similar function of those of x with respect to y. Hence $\frac{1}{\sqrt{t}} \cdot \frac{dR}{dx}$ is an absolute reciprocant, and therefore $\frac{dR}{dx}$ is a reciprocant, the character of each being the same as that of R.

I will avail myself of the conclusion just obtained, which is the cardinal property of absolute reciprocants, to give a general method of generating from any given Rational Integral Reciprocant an infinity of others—rational integral educts of it, we may say. Let F be such a reciprocant, and μ its characteristic; then $\frac{F}{t^{\frac{\mu}{2}}}$ is an absolute reciprocant, and consequently $\frac{d}{dx}\left(\frac{F}{t^{\frac{\mu}{2}}}\right)$ is a reciprocant, both of them of the same character as F; that is

$$\frac{t\frac{dF}{dx} - \frac{\mu}{2} \cdot aF}{t^{\frac{\mu}{2}+1}};$$

or we may say
$$2t\frac{dF}{dx} - \mu aF$$

is a reciprocant of the same character as F.

This is even true for non-homogeneous reciprocants, for the only assumption made at present as to the nature of F is that it is a rational integral reciprocant. But if we further assume that it is homogeneous and isobaric*, we know that

$$\mu = 3i + w.$$

Now, Euler's equation gives

$$3i = 3\,(t\partial_t + a\partial_a + b\partial_b + c\partial_c + \ldots),$$

* It will subsequently be proved that every rational integral reciprocant which is homogeneous is also isobaric.

and from the similar equation for isobaric functions (remembering that the weights of $t, a, b, c, \ldots$ are $-1, 0, 1, 2, \ldots$) we obtain

$$w = -t\partial_t + b\partial_b + 2c\partial_c \ldots,$$

so that
$$\mu = 2t\partial_t + 3a\partial_a + 4b\partial_b + 5c\partial_c + \ldots.$$

And since
$$\frac{d}{dx} = a\partial_t + b\partial_a + c\partial_b + d\partial_c + \ldots,$$

we may in $\left(2t\frac{d}{dx} - \mu a\right)F$ replace $2t\frac{d}{dx} - \mu a$ by

$$2t(a\partial_t + b\partial_a + c\partial_b + d\partial_c + \ldots)$$
$$- a(2t\partial_t + 3a\partial_a + 4b\partial_b + 5c\partial_c + \ldots),$$

or by its equivalent

$$(2bt - 3a^2)\partial_a + (2ct - 4ab)\partial_b + (2dt - 5ac)\partial_c + \ldots.$$

The conclusion arrived at is that when F is a rational integral homogeneous reciprocant,

$$\{(2bt - 3a^2)\partial_a + (2ct - 4ab)\partial_b + (2dt - 5ac)\partial_c + \ldots\}F$$

is another, and that both are of the same character.

It will be convenient to use the letter G to denote the operator just found and to speak of it as the generator for mixed reciprocants. By the repeated operation of this generator on a we may obtain the series $Ga, G^2a, G^3a, \ldots$, whose terms will be mixed reciprocants, since each operation increases the highest power of t by unity. The forms thus obtained will, in general, not be irreducible. It is, in fact, easy to see that a reduction must always take place at every second step. Observing that GF only expresses the numerator of the absolute reciprocant $\frac{1}{\sqrt{t}}\cdot\frac{d}{dx}\left(\frac{F}{t^{\frac{\mu}{2}}}\right)$ in a convenient form, and that G^2F is equivalent to the numerator of $\left(\frac{1}{\sqrt{t}}\cdot\frac{d}{dx}\right)^2\left(\frac{F}{t^{\frac{\mu}{2}}}\right)$, we have

$$\frac{1}{\sqrt{t}}\cdot\frac{d}{dx}\left(\frac{F}{t^{\frac{\mu}{2}}}\right) = \frac{t\dfrac{dF}{dx} - \dfrac{\mu}{2}\cdot aF}{t^{\frac{\mu+3}{2}}};$$

so that
$$\left(\frac{1}{\sqrt{t}}\cdot\frac{d}{dx}\right)^2\left(\frac{F}{t^{\frac{\mu}{2}}}\right) = \frac{1}{\sqrt{t}}\cdot\frac{d}{dx}\left(\frac{t\dfrac{dF}{dx} - \dfrac{\mu}{2}\cdot aF}{t^{\frac{\mu+3}{2}}}\right)$$

$$= \frac{t\dfrac{d}{dx}\left(t\dfrac{dF}{dx} - \dfrac{\mu}{2}\cdot aF\right) - \dfrac{\mu+3}{2}\cdot a\left(t\dfrac{dF}{dx} - \dfrac{\mu}{2}\cdot aF\right)}{t^{\frac{\mu}{2}+3}}.$$

The whole of this fraction is an absolute reciprocant of the same character as F; so also is $\dfrac{a^2F}{t^{\frac{\mu}{2}+3}}$ $\left(\text{the product of the } \textit{even} \text{ absolute reciprocant } \dfrac{a^2}{t^3} \text{ by } \dfrac{F}{t^{\frac{\mu}{2}}}\right)$.

We may therefore reject the term $\dfrac{\mu}{2}\cdot\dfrac{\mu+3}{2}\cdot a^2F$ from the numerator, and the remaining fraction

$$\frac{\dfrac{d}{dx}\left(t\dfrac{dF}{dx}-\dfrac{\mu}{2}\cdot aF\right)-\dfrac{\mu+3}{2}\cdot a\dfrac{dF}{dx}}{t^{\frac{\mu}{2}+2}}$$

will still be an absolute reciprocant of the same character as F. Its numerator, which is one degree lower than G^2F, may be written in the form

$$t\frac{d^2F}{dx^2}-(\mu+\tfrac{1}{2})\,a\frac{dF}{dx}-\frac{\mu}{2}\,bF.$$

This, it may be noticed, is a reciprocant of the same character as F, even when F is non-homogeneous.

Starting with a, we have

$Ga = 2bt - 3a^2$ (the Schwarzian),

$G^2a = G(2bt-3a^2) = -6a(2bt-3a^2) + 2t(2ct-4ab) = 4ct^2 - 20abt + 18a^3$.

But, for the reason previously given, $18a^3$ may be removed, so that rejecting this term and dividing out by $4t$ we obtain the form

$$ct - 5ab,$$

which may be called the Post-Schwarzian.

The next form is obtained by operating on the Post-Schwarzian with G; thus, we have to calculate the value of $G(ct-5ab)$, where

$$G = (2bt-3a^2)\,\partial_a + (2ct-4ab)\,\partial_b + (2dt-5ac)\,\partial_c.$$

The working may be arranged as follows:

	dt^2	act	b^2t	a^2b	
$t(2dt-5ac)=$	2	-5	.	.	from $(2dt-5ac)\,\partial_c$
$-5a(2ct-4ab)=$	.	-10	.	20	„ $(2ct-4ab)\,\partial_b$
$-5b(2bt-3a^2)=$	.	.	-10	15	„ $(2bt-3a^2)\,\partial_a$
	2	-15	-10	35	

The result should be read thus:

$$2dt^2 - 15act - 10b^2t + 35a^2b.$$

To obtain the next of this series of reciprocants, we have to operate on this with G and at the same time to take account of the reduction that has to be made at each alternate step. The arrangement of the work is similar to that of the former case.

	et^3	adt^2	bct^2	a^2ct	ab^2t	a^3b	
$2t^2(2et-6ad)=$	4	-12	.	.	.	.	from $(2et-6ad)\,\partial_d$
$-15at(2dt-5ac)=$	.	-30	.	75	.	.	,, $(2dt-5ac)\,\partial_c$
$(35a^2-20bt)(2ct-4ab)=$	.	.	-40	70	80	-140	,, $(2ct-4ab)\,\partial_b$
$(70ab-15ct)(2bt-3a^2)=$	.	.	-30	45	140	-210	,, $(2bt-3a^2)\,\partial_a$
	4	-42	-70	190	220	-350	
$-70a^2(ct-5ab)=$	.	.	.	-70	.	$+350$	
	4	-42	-70	120	220	.	

This divides by $2t$, giving the reduced value

$$2et^2-21adt-35bct+60a^2c+110ab^2.$$

The next obtained by this process will be seen by the following work to be

$$4ft^3-56aet^2-112bdt^2-70c^2t^2+309a^2dt+995abct+220b^3t-660a^3c-1210a^2b^2.$$

	ft^3	aet^2	bdt^2	c^2t^2	a^2dt	$abct$	b^3t	a^3c	a^2b^2	
$2t^2(2ft-7ae)=$	4	-14	.	.	.	.	.	.	.	from $(2ft-7ae)\,\partial_e$
$-21at(2et-6ad)=$	.	-42	.	.	126	.	.	.	.	,, $(2et-6ad)\,\partial_d$
$(-35bt+60a^2)(2dt-5ac)=$	.	.	-70	.	120	175	.	-300	.	,, $(2dt-5ac)\,\partial_c$
$(-35ct+220ab)(2ct-4ab)=$	.	.	.	-70	.	580	.	.	-880	,, $(2ct-4ab)\,\partial_b$
$(-21dt+120ac+110b^2)(2bt-3a^2)=$	.	.	-42	.	63	240	220	-360	-330	,, $(2bt-3a^2)\,\partial_a$
	4	-56	-112	-70	309	995	220	-660	-1210	

This cannot be reduced in the same manner as the preceding form, but it must not be supposed that the forms thus obtained are in general irreducible.

Having regard to the circumstance that the forms of the series

$$a,\ Ga,\ G^2a,\ \ldots$$

occur in the numerators of the successive values of $\left(\frac{1}{\sqrt{t}}\cdot\frac{d}{dx}\right)^n \log t$, they may be called the successive *educts*, and the reduced forms given above may be called the *reduced educts* and denoted by E_1, E_2, E_3 .... Thus

$$E_1=a,$$
$$E_2=2bt-3a^2,$$
$$E_3=ct-5ab,$$
$$E_4=2dt^2-15act-10b^2t+35a^2b,$$
$$E_5=2et^2-21adt-35bct+60a^2c+110ab^2,$$
$$E_6=4ft^3-56aet^2-112bdt^2-70c^2t^2+309a^2dt+995abct$$
$$+220b^3t-660a^3c-1210a^2b^2.$$

LECTURE IV.

We have seen that when F is a rational integral homogeneous and isobaric reciprocant, GF is another of the same character. It will now appear that the condition of isobarism is implied in that of homogeneity; for let F be a rational integral homogeneous reciprocant, μ its characteristic and i its degree in the letters $t, a, b, c, \ldots$, then, in the identical equation

$$F(t, a, b, c, \ldots) = \pm t^{\mu} F(\tau, \alpha, \beta, \gamma, \ldots)$$

both members are homogeneous and of the same degree in the letters $t, a, b, c, \ldots$; that is, if $At^k a^l b^m c^n \ldots$ be any term of $F(t, a, b, c, \ldots)$, its degree must be the same as that of $t^{\mu} A \tau^k \alpha^l \beta^m \gamma^n \ldots$ when $\tau, \alpha, \beta, \gamma, \ldots$ are expressed in terms of $t, a, b, c, \ldots$. But

$$\tau = \frac{1}{t}, \quad \alpha = -\frac{a}{t^3}, \quad \beta = -\frac{b}{t^4} + \ldots, \quad \gamma = -\frac{c}{t^5} + \ldots,$$

and so on. The degrees of $\tau, \alpha, \beta, \gamma, \ldots$ are therefore $-1, -2, -3, -4, \ldots$ respectively. Hence

$$k + l + m + n + \ldots = \mu - k - 2l - 3m - 4n - \ldots,$$

or

$$\mu = 2k + 3l + 4m + 5n + \ldots.$$

And by hypothesis

$$i = k + l + m + n + \ldots,$$

so that

$$\mu - 3i = -k + m + 2n + \ldots.$$

Neither μ nor i is dependent for its value on the selection of a particular term of F, for all terms of $F(\tau, \alpha, \beta, \gamma, \ldots)$ are multiplied by the same extraneous factor $\pm t^{\mu}$, and all terms of $F(t, a, b, c, \ldots)$ are of the same degree i. Hence $-k + m + 2n + \ldots$ must also be the same for each term of F; or, attributing the weights $-1, 0, 1, 2, \ldots$ to the letters $t, a, b, c, \ldots$, the function F is isobaric.

Next, suppose F to be fractional, and let it be the ratio of the two rational integral homogeneous reciprocants F_1 and F_2. The operation of G on F will, in this case also, generate another reciprocant of the same character as F. For, since G is linear in the differential operative symbols $\partial_a, \partial_b, \partial_c, \ldots$, its operation will be precisely analogous to that of differentiation, so that, operating with G on

$$F = \frac{F_1}{F_2},$$

we have

$$GF = \frac{F_2 GF_1 - F_1 GF_2}{F_2^2}.$$

In order to prove that this is a reciprocant, we have to show that the character and characteristic are the same for both terms of the numerator. But GF_1 is a reciprocant of the same character as F_1, and GF_2 is one of the same character as F_2; thus the two terms of the numerator are of the same character as F_1F_2. As regards the characteristic, it should be noticed that G, that is, the operator $(2bt-3a^2)\,\partial_a+(2ct-4ab)\,\partial_b+\dots$, increases the degree by unity, but does not alter the weight, so that it increases the characteristic of any rational integral homogeneous reciprocant by 3. Thus the characteristic of each term in the numerator exceeds by 3 that of F_1F_2. Hence GF is a reciprocant, and, taking account of its denominator as well as its numerator, we see that the operation of G on a rational homogeneous reciprocant, whether fractional or integral, produces another in which the original character is preserved while the characteristic is increased by three units.

More generally, let F_1, F_2, F_3, ... be any rational homogeneous reciprocants whose extraneous factors are $(-)^{\kappa_1}t^{\mu_1}$, $(-)^{\kappa_2}t^{\mu_2}$, $(-)^{\kappa_3}t^{\mu_3}$, ... respectively; and suppose Φ to consist of a series of terms of the form $AF_1^{\lambda_1}F_2^{\lambda_2}F_3^{\lambda_3}\dots$, such that the extraneous factor for each term is $(-)^{\kappa}t^{\mu}$. Then Φ is a reciprocant, but not necessarily a rational one; for the indices λ_1, λ_2, λ_3, ... may be supposed fractional, provided only that they satisfy the conditions

$$\kappa_1\lambda_1+\kappa_2\lambda_2+\kappa_3\lambda_3+\dots-\kappa = \text{a positive or negative } \textit{even} \text{ integer},$$

and

$$\mu_1\lambda_1+\mu_2\lambda_2+\mu_3\lambda_3+\dots-\mu=0.$$

We proceed to show that $G\Phi$ is also a reciprocant, and that its extraneous factor is $(-)^{\kappa}t^{\mu+3}$. Since

$$G\Phi=\frac{d\Phi}{dF_1}\,.\,GF_1+\frac{d\Phi}{dF_2}\,.\,GF_2+\frac{d\Phi}{dF_3}\,.\,GF_3+\dots,$$

we have to prove not only that each term of this expression is a reciprocant, but also that all of them have the same extraneous factor; otherwise their sum would not be a reciprocant.

Now, in
$$\Phi=\Sigma AF_1^{\lambda_1}F_2^{\lambda_2}F_3^{\lambda_3}\dots,$$
the extraneous factor for each term is by hypothesis $(-)^{\kappa}t^{\mu}$, so that the extraneous factor for each term of

$$\frac{d\Phi}{dF_1}=\Sigma A\lambda_1F_1^{\lambda_1-1}F_2^{\lambda_2}F_3^{\lambda_3}\dots,$$

is $(-)^{\kappa-\kappa_1}t^{\mu-\mu_1}$, and therefore $\dfrac{d\Phi}{dF_1}$ is a reciprocant. Also, GF_1 is a reciprocant whose extraneous factor is $(-)^{\kappa_1}t^{\mu_1+3}$. Hence $\dfrac{d\Phi}{dF_1}.GF_1$ is a reciprocant having $(-)^{\kappa}t^{\mu+3}$ for extraneous factor, and in exactly the same way we see that every other term of $G\Phi$ is also a reciprocant with the same extraneous factor.

Thus G, operating on *any* homogeneous reciprocant whose extraneous factor is $(-)^\kappa t^\mu$, generates another whose extraneous factor is $(-)^\kappa t^{\mu+3}$.

If, in the generator for mixed reciprocants,

$$G = (2bt - 3a^2)\,\partial_a + (2ct - 4ab)\,\partial_b + (2dt - 5ac)\,\partial_c + \ldots,$$

we write $\quad a = 1\,.\,2\,.\,a_0,\quad b = 1\,.\,2\,.\,3\,.\,a_1,\quad c = 1\,.\,2\,.\,3\,.\,4\,.\,a_2\ldots,$

(that is, if we use the system of modified letters previously mentioned), its expression assumes a more elegant form. Substituting for $a, b, c, \ldots$ their values in terms of the modified letters, we have

$$2bt - 3a^2 = 2\,.\,1\,.\,2\,.\,3a_1t - 3\,(1\,.\,2)^2a_0^2 = 1\,.\,2^2\,.\,3\,(a_1t - a_0^2),$$

and $$\partial_a = \frac{1}{1\,.\,2}\,.\,\partial_{a_0};$$

so that $$(2bt - 3a^2)\,\partial_a = 1\,.\,2\,.\,3\,(a_1t - a_0^2)\,\partial_{a_0}.$$

Again, $$(2ct - 4ab) = 1\,.\,2^2\,.\,3\,.\,4\,(a_2t - a_0a_1)$$

and $$\partial_b = \frac{1}{1\,.\,2\,.\,3}\,\partial_{a_1};$$

so that $$(2ct - 4ab)\,\partial_b = 1\,.\,2\,.\,4\,(a_2t - a_0a_1)\,\partial_{a_1}.$$

Similarly, $$(2dt - 5ac)\,\partial_c = 1\,.\,2\,.\,5\,(a_3t - a_0a_2)\,\partial_{a_2}.$$

Thus the modified generator for mixed reciprocants is

$$1\,.\,2\,.\,3\,(a_1t - a_0^2)\,\partial_{a_0} + 1\,.\,2\,.\,4\,(a_2t - a_0a_1)\,\partial_{a_1} + 1\,.\,2\,.\,5\,(a_3t - a_0a_2)\,\partial_{a_2} + \ldots,$$

in which the general term is

$$1\,.\,2\,(n+3)\,(a_{n+1}t - a_0a_n)\,\partial_{a_n}.$$

The factor $1\,.\,2$ may, of course, be rejected, and our modified generator may be written in the simple form

$$3\,(a_1t - a_0^2)\,\partial_{a_0} + 4\,(a_2t - a_0a_1)\,\partial_{a_1} + 5\,(a_3t - a_0a_2)\,\partial_{a_2} + \ldots.$$

Operating with this on the homogeneous reciprocant $F(t, a_0, a_1, a_2, \ldots)$, the result will be another homogeneous reciprocant of the same character as F. When we start with a_0 and make the reductions which, as we have seen, occur at every second step, we find a system of reduced educts corresponding in every particular with those formerly given, but expressed in terms of the modified letters $a_0, a_1, a_2, \ldots$ instead of $a, b, c, \ldots$. These are as follows:

$a_0,$

*$a_1t - a_0^2,$

$2a_2t - 5a_0a_1,$

*$2a_3t^2 - 6a_0a_2t - 3a_1^2t + 7a_0^2a_1,$

$2a_4t^2 - 7a_0a_3t - 7a_1a_2t + 8a_0^2a_2 + 11a_0a_1^2,$

*$14a_5t^3 - 56a_0a_4t^2 - 56a_1a_3t^2 - 28a_2^2t^2 + 103a_0^2a_3t + 199a_0a_1a_2t$
$\qquad + 33a_1^3t - 88a_0^3a_2 - 121a_0^2a_1^2.$

....

* It will be observed that in the unreduced forms, marked with an asterisk, the sum of the numerical coefficients is zero. This is a direct consequence, as may be easily seen, of the form of the modified generator, in which the sum of the numerical coefficients in each term is also zero.

It will be found on trial that these modified educts are obtained with greater ease and with less liability to error by a direct application of the generator

$$3(a_1t-a_0^2)\partial_{a_0}+4(a_2t-a_0a_1)\partial_{a_1}+5(a_3t-a_0a_2)\partial_{a_2}+\ldots,$$

than by making the substitution of $1.2.a_0$, $1.2.3.a_1$, $1.2.3.4.a_2$, ... for $a, b, c,$ ... in the system of educts already given. For this reason the working by the former method is here performed, instead of being merely indicated.

From a_0 we obtain immediately

$$a_1t-a_0^2.$$

Operating on this with the generator, there results

$$4t(a_2t-a_0a_1)-6a_0(a_1t-a_0^2)=4a_2t^2-10a_0a_1t+6a_0^3.$$

This, when reduced by removing its last term and dividing the others by $2t$, gives

$$2a_2t-5a_0a_1.$$

The next form is found from this by a simple operation, without subsequent reduction, and is therefore

$$10t(a_3t-a_0a_2)-20a_0(a_2t-a_0a_1)-15a_1(a_1t-a_0^2).$$

Or, collecting the terms and rejecting the numerical factor 5,

$$2a_3t^2-6a_0a_2t-3a_1^2t+7a_0^2a_1.$$

The operation of the generator on this gives

$$12t^2(a_4t-a_0a_3)-30a_0t(a_3t-a_0a_2)+4(7a_0^2-6a_1t)(a_2t-a_0a_1)$$
$$+3(14a_0a_1-6a_2t)(a_1t-a_0^2).$$

The collection of terms and subsequent reduction is shown below:

	a_4t^3	$a_0a_3t^2$	$a_1a_2t^2$	$a_0^2a_2t$	$a_0a_1^2t$	$a_0^3a_1$
	12	− 12	.	.	.	.
	.	− 30	.	30	.	.
	.	.	− 24	28	24	− 28
	.	.	− 18	18	42	− 42
	12	− 42	− 42	76	66	− 70
$-14a_0^2(2a_2t-5a_0a_1)=$	.	.	.	− 28	.	+ 70
	12	− 42	− 42	48	66	.

Removing the factor $6t$, the reduced form is

$$2a_4t^2-7a_0a_3t-7a_1a_2t+8a_0^2a_2+11a_0a_1^2.$$

Operating on this with the generator, we have

$$14t^2(a_5t-a_0a_4)-42a_0t(a_4t-a_0a_3)+5(8a_0^2-7a_1t)(a_3t-a_0a_2)$$
$$+4(22a_0a_1-7a_2t)(a_2t-a_0a_1)+3(11a_1^2+16a_0a_2-7a_3t)(a_1t-a_0^2)$$
$$=14a_5t^3-56a_0a_4t^2-56a_1a_3t^2-28a_2^2t^2+103a_0^2a_3t+199a_0a_1a_2t$$
$$+33a_1^3t-88a_0^3a_2-121a_0^2a_1^2,$$

which cannot be reduced in the same manner as the preceding form.

To obtain a generator for passing from pure to pure reciprocants a process is employed similar to that which gave the generator for mixed reciprocants which we have just been using. I state the results before giving the proof, and then proceed to speak of generators in the theory of Invariants. The generator for pure reciprocants is

$$(3ac - 4b^2)\,\partial_b + (3ad - 5bc)\,\partial_c + (3ae - 6bd)\,\partial_d + \ldots;$$

or, expressed in terms of the modified letters,

$$4\,(a_0a_2 - a_1^2)\,\partial_{a_1} + 5\,(a_0a_3 - a_1a_2)\,\partial_{a_2} + 6\,(a_0a_4 - a_1a_3)\,\partial_{a_3} + \ldots.$$

By operating with this on any pure reciprocant R, we generate another pure reciprocant of opposite character to that of R.

The connection between the two theories of Reciprocants and Invariants is so close, and these brother-and-sister theories throw so much light upon each other, that I began to inquire whether, in the latter, there did not exist a theory of Generators parallel to that of the former.

Fortunately, Mr Hammond was able to recall a correspondence in which Prof. Cayley had given such a theory, which he regarded, and justly, as an important invention. Its substance has been subsequently incorporated in the *Quarterly Journal* (Vol. XX. p. 212). It offers itself spontaneously in the Reciprocantive Theory; in the Invariantive one it calls for a distinct act of invention. Prof. Cayley has discovered two generators similar in form with those for reciprocants, and one of them strikingly so; in a letter to me he calls these P and Q. As given by him,

$$P = ab\partial_a + ac\partial_b + ad\partial_c + \ldots - ib,$$
$$Q = \qquad ac\partial_b + 2ad\partial_c + \ldots - 2wb,$$

where i is the degree and w the weight, the weights of $a, b, c, d, \ldots$ being taken to be $0, 1, 2, 3, \ldots$ (I supply the a which Cayley turns into unity.) As an example he takes the "Invariant" $a^2d - 3abc + 2b^3 = I$, suppose. We have then

$$\begin{aligned} PI &= (ab\partial_a + ac\partial_b + ad\partial_c + ae\partial_d - 3b)\,I \\ &= ab\,(2ad - 3bc) + ac\,(-3ac + 6b^2) - 3a^2bd + a^3e - 3b\,(a^2d - 3abc + 2b^3) \\ &= a^3e - 4a^2bd - 3a^2c^2 + 12ab^2c - 6b^4 \\ &= a^2\,(ae - 4bd + 3c^2) - 6\,(ac - b^2)^2, \end{aligned}$$

and
$$\begin{aligned} QI &= (ac\partial_b + 2ad\partial_c + 3ae\partial_d - 6b)\,I \\ &= ac\,(-3ac + 6b^2) - 6a^2bd + 3a^3e - 6b\,(a^2d - 3abc + 2b^3) \\ &= 3a^3e - 12a^2bd - 3a^2c^2 + 24ab^2c - 12b^4 \\ &= 3a^2\,(ae - 4bd + 3c^2) - 12\,(ac - b^2)^2. \end{aligned}$$

P and Q may be transformed by means of Euler's equation and the similar one for isobaric functions, which enable us to write

$$i = a\partial_a + b\partial_b + c\partial_c + d\partial_d + \ldots,$$

and
$$w = \qquad b\partial_b + 2c\partial_c + 3d\partial_d + \ldots;$$

P thus becomes

$$\begin{aligned} &ab\partial_a + ac\partial_b + ad\partial_c + ae\partial_d + \dots \\ &- ab\partial_a - b^2\partial_b - bc\partial_c - bd\partial_d - \dots \\ &= (ac - b^2)\,\partial_b + (ad - bc)\,\partial_c + (ae - bd)\,\partial_d + \dots, \end{aligned}$$

the same in *form* as either of our generators, except that the arithmetical coefficients are all made units; $a, b, c, \dots$ taking the place of the $t, a, b, \dots$ of the generator for mixed reciprocants.

In like manner, Q becomes

$$(ac - 2b^2)\,\partial_b + 2\,(ad - 2bc)\,\partial_c + 3\,(ae - 2bd)\,\partial_d + \dots,$$

where the arithmetical series 1, 2, 3, ... takes the place of 3, 4, 5, ... or of 4, 5, 6, ... in the two Reciprocant Generators.

The effect of P and of Q is obviously to raise the degree and the weight of the operand I each by one unit. But if we take $R = \frac{1}{a}(2wP - iQ)$, the terms in Cayley's original formulae containing b cancel, so that $2wP - iQ$ divides out by a and the weight is raised one unit without the degree being affected. This is mentioned in the *Quarterly Journal* (*loc. cit.*); but it may also be remarked that when I is a *satisfied invariant*, it is annihilated by the operation of R; when the *invariant* is *unsatisfied*, each of the three operators P, Q and R increases its extent by an unit, that is, introduces an additional letter. For let j denote the extent, then, writing $a_0, a_1, a_2, \dots a_j$ for $a, b, c, \dots$, we have

$$\begin{aligned} P &= a_0a_1\partial_{a_0} + a_0a_2\partial_{a_1} + \dots + a_0a_{j+1}\partial_{a_j} - ia_1, \\ Q &= \qquad\qquad a_0a_2\partial_{a_1} + 2a_0a_3\partial_{a_2} + \dots + ja_0a_{j+1}\partial_{a_j} - 2wa_1; \end{aligned}$$

whence we find

$$\begin{aligned} R &= \frac{1}{a_0}(2wP - iQ) \\ &= 2wa_1\partial_{a_0} + (2w - i)\,a_2\partial_{a_1} + \dots + (2w - ij + i)\,a_j\partial_{a_{j-1}} + (2w - ij)\,a_{j+1}\partial_{a_j}. \end{aligned}$$

But for a *satisfied invariant*

$$2w = ij;$$

and substituting this value for $2w$ in the above expression for R, it becomes

$$i\,\{ja_1\partial_{a_0} + (j-1)\,a_2\partial_{a_1} + \dots + a_j\partial_{a_{j-1}}\},$$

which, as is well known, annihilates any satisfied invariant.

LECTURE V.

It will be desirable to fill up some of the previous investigations by discussing some points in them that have not yet received our consideration.

There may be some to whom it may appear tedious to watch the complete exposition of the algebraical part of the Theory, who are impatient to rush on to its applications. But it is my duty to consider what may be expected to be most useful to the great majority of the class, and for that purpose to make the ground sure under our feet as I proceed. To the greater number it will, I think, be of advantage to have their memories refreshed on the kindred subject of invariants, and probably made acquainted with some important points of that theory which are new to them.

I confess that, to myself, the contemplation of this relationship—the spectacle of a new continent rising from the waters, resembling yet different from the old, familiar one—is a principal source of interest arising out of the new theory. I do not regard Mathematics as a science purely of calculation, but one of ideas, and as the embodiment of a Philosophy. An eminent colleague of mine, in a public lecture in this University, magnifying the importance of classical over mathematical studies, referred to a great mathematician as one who might possibly know every foot of distance between the earth and the moon; and when I was a member, at Woolwich, of the Government Committee of Inventions, one of my colleagues, appealing to me to answer some question as to the number of cubic inches in a pipe, expressed his surprise that I was not prepared with an immediate answer, and said he had supposed that I had all the tables of weights and measures at my fingers' ends.

I hope that in any class which I may have the pleasure of conducting in this University, other ideas will prevail as to the true scope of mathematical science as a branch of liberal learning; and it will be my endeavour to regulate the pace in a manner which seems to me most conducive to real progress in the order of ideas and philosophical contemplation, thus bringing our noble science into harmony and in a line with the prevailing tone and studies of this University. Faraday, at the end of his experimental lectures, was accustomed to say—I have myself heard him do so—"We will now leave that to the calculators." So long as we are content to be regarded as mere calculators we shall be the Pariahs of the University, living here on sufferance, instead of being regarded, as is our right and privilege, as the real leaders and pioneers of thought in it.

That Cayley's two operators, which have been called P and Q, are in fact generators, may be proved as follows †:

Let $$\Omega = a\partial_b + 2b\partial_c + 3c\partial_d + 4d\partial_e + \ldots,$$

and $$\Theta = a(\lambda b\partial_a + \mu c\partial_b + \nu d\partial_c + \ldots) - \kappa b,$$

where $\kappa, \lambda, \mu, \nu, \ldots$ are numbers.

When κ is the degree of the operand, and $\lambda = \mu = \nu = \ldots = 1$, the operator Θ is identical with P; but Θ is identical with Q when κ is twice the weight of the operand and $\lambda = 0$, $\mu = 1$, $\nu = 2, \ldots$.

If now we use $*$ to signify the act of pure differential operation, it is obvious that

$$\Omega\Theta = (\Omega \times \Theta) + (\Omega * \Theta),$$

$$\Theta\Omega = (\Omega \times \Theta) + (\Theta * \Omega),$$

so that $$\Omega\Theta - \Theta\Omega = (\Omega * \Theta) - (\Theta * \Omega).$$

But since $$\Omega a = 0, \quad \Omega b = a, \quad \Omega c = 2b, \ldots$$

we have $$\Omega * \Theta = a(\lambda a\partial_a + 2\mu b\partial_b + 3\nu c\partial_c + \ldots - \kappa)$$

and $$\Theta * \Omega = a(\lambda b\partial_b + 2\mu c\partial_c + 3\nu d\partial_d + \ldots).$$

Hence $$\Omega\Theta - \Theta\Omega = a\{\lambda a\partial_a + (2\mu - \lambda) b\partial_b + (3\nu - 2\mu) c\partial_c + \ldots - \kappa\};$$

now if the operand I be any *invariant* (satisfied or unsatisfied), we have $\Omega I = 0$, and therefore $\Theta\Omega I = 0$; so that we find

$$\Omega\Theta I = a\{\lambda a\partial_a + (2\mu - \lambda) b\partial_b + (3\nu - 2\mu) c\partial_c + \ldots - \kappa\} I.$$

If in this we write $\lambda = \mu = \nu = \ldots = 1$, and $\kappa = i$, where i is the degree of the operand, Θ becomes P and we have

$$\Omega P I = a(a\partial_a + b\partial_b + c\partial_c + \ldots - i) I.$$

But, by Euler's theorem, the right-hand side of this vanishes, and therefore

$$\Omega P I = 0.$$

Similarly, by means of the corresponding theorem for isobaric functions, we may prove that

$$\Omega Q I = 0.$$

For if, in the general formula, we write $\lambda = 0$, $\mu = 1$, $\nu = 2, \ldots$ and $\kappa = 2w$, where w is the weight of the operand, we find

$$\Omega Q I = a(2b\partial_b + 4c\partial_c + 6d\partial_d + \ldots - 2w) I = 0.$$

Thus, when Θ stands either for P or for Q, it is either an annihilator or a generator (that is, ΘI is either identically zero or else an invariant). But if l be the most advanced, or say the *radical letter* of I, no term of $m\partial_l I$ can cancel with any other term of ΘI; and since, for this reason, ΘI cannot vanish identically, it must be an invariant, and the operators P and Q must be generators.

† In the *Quarterly Journal* (Vol. xx. p. 212) Prof. Cayley only considers a special example, and has not given the proof of the general theorem.

The generators previously given for reciprocants also possess this property of introducing a fresh radical letter at each step. The radical letter, on its first introduction, enters in the first degree only, and in the case of the educts of $\log t$, whose values have been calculated, its multiplier is seen to be a power of t. The form of the generator for mixed reciprocants

$$3(a_1t - a_0^2)\partial_{a_0} + 4(a_2t - a_0a_1)\partial_{a_1} + \dots + (n+3)(a_{n+1}t - a_0a_n)\partial_{a_n}$$

shows this, or it may be seen by considering the successive values of

$$\left(\frac{1}{\sqrt{t}} \cdot \frac{d}{dx}\right)^i \log t.$$

For let $\dfrac{F(t, a_0, a_1, a_2, \dots)}{t^\lambda}$ denote this expression, and let its radical letter be a_n; then, on differentiating again with respect to x, the new letter introduced arises solely from a term in the numerator

$$\frac{d}{da_n} F(t, a_0, a_1, a_2, \dots a_n) \cdot \frac{da_n}{dx}.$$

But $\quad a_n = \dfrac{d^ny}{dx^n} \div 2 \cdot 3 \dots n+2$; so that $\dfrac{da_n}{dx} = (n+3)\,a_{n+1}$.

Hence, if when a_n is the radical letter, it occurs in the first degree only and multiplied by a power of t, it follows that, since $\dfrac{dF}{da_n}$ will be a power of t, the derived expression which contains the radical letter a_{n+1} will contain it in the first degree only and multiplied by a power of t. And since this is true for the case $i=1$, when $\dfrac{1}{\sqrt{t}} \cdot \dfrac{d}{dx} \log t = \dfrac{a_0}{t^{\frac{3}{2}}}$, it is true universally.

Observe that for $i = 1, 2, 3, \dots$ the radical letter is $a_0, a_1, a_2, \dots$ respectively.

It will be remembered that $\left(\dfrac{1}{\sqrt{t}} \cdot \dfrac{d}{dx}\right)^i \log t$ is an absolute reciprocant. It may be called the ith absolute educt, to distinguish it from the rational integral educts $E_1, E_2, E_3, \dots$ whose values have already been calculated.

Let $R(t, a_0, a_1, a_2, \dots a_n)$ be any homogeneous rational integral reciprocant, and let the educts be $A_0, A_1, A_2, \dots A_n$; then obviously

a_n may be expressed rationally in terms of A_n and $a_{n-1}, a_{n-2}, \dots a_0, t$,

a_{n-1} ,, ,, ,, ,, A_{n-1} and $a_{n-2}, \dots a_0, t$,

....

a_1 ,, ,, ,, ,, A_1, a_0 and t,

a_0 ,, ,, ,, ,, A_0 and t,

where observe that the denominators in these expressions are all powers of t. Hence, by successive substitutions, $R(t, a_0, a_1, \dots a_n)$ may be expressed rationally in terms of $A_n, \ldots A_1, A_0$, and t. Thus any rational integral homogeneous reciprocant is a rational function of educts, and is of the form $\frac{E}{t^0}$, where E is a rational *integral* function of the educts.

Does not this prove too much, it may be asked, namely, that any function F of the letters is a rational function of the educts, which are themselves reciprocants, and will therefore be a reciprocant? But this is not so; for observe that although F will be expressed as a sum of products of educts, such products will not in general be all of the same character, and their linear combination will be an illicit one, such as is seen in the illicit combination of a_0^2 with the Schwarzian $(a_1 t - a_0^2)$.

We have seen that by differentiating an absolute reciprocant, or by the use of a generator, we obtain a fresh reciprocant. But there are other methods of finding reciprocants; as, for example, if the transform of

$$\phi(t, a, b, c, \ldots)$$

is

$$\psi(\tau, \alpha, \beta, \gamma, \ldots),$$

that is, if

$$\phi(t, a, b, c, \ldots) = \psi(\tau, \alpha, \beta, \gamma, \ldots),$$

then

$$\psi(t, a, b, c, \ldots) = \phi(\tau, \alpha, \beta, \gamma, \ldots).$$

Whence, by multiplication,

$$\phi(t, a, b, c, \ldots)\,\psi(t, a, b, c, \ldots) = \phi(\tau, \alpha, \beta, \gamma, \ldots)\,\psi(\tau, \alpha, \beta, \gamma, \ldots).$$

Thus $\phi\,.\,\psi$ is a reciprocant, and, moreover, an absolute one of even character, although neither ϕ, which is a perfectly arbitrary function, nor ψ, its transform, is a reciprocant.

Herein a mixed reciprocant differs from an invariant, which cannot be resolved into non-invariantive factors. It is worth while to give a proof of this proposition; but first I prove its converse, that if $p, q, r, \ldots$ are all invariants, their product must be so too. This is an immediate consequence of the well-known theorem that

$$\Omega I = 0$$

is the necessary and sufficient condition that I may be an invariant where, as usual, Ω is the operator

$$a\partial_b + 2b\partial_c + 3c\partial_d + \ldots,$$

and the word invariant has been used in the same extended sense as formerly.

For
$$\Omega(pqrs\ldots) = \left(\frac{\Omega p}{p} + \frac{\Omega q}{q} + \frac{\Omega r}{r} + \ldots\right) pqrs \ldots.$$

But since $p, q, r, \ldots$ are all invariants, we have

$$\Omega p = 0, \quad \Omega q = 0, \quad \Omega r = 0, \ldots,$$

and therefore
$$\Omega(pqrs\ldots) = 0.$$

Next, suppose that
$$I = P_1 Q_1,$$
where I is but Q_1 is not an invariant.

To meet the case in which P_1 and Q_1 are not prime to one another, Q_1, if resolved into its factors, must contain one Q^i where Q is not an invariant.

Suppose that P_1 contains Q^j, and let $i+j=k$; then we may write

$$I = PQ^k,$$

where P is prime to Q. But since I is an invariant by hypothesis,

$$\Omega I = 0,$$

and therefore, $$Q^k \Omega P + kPQ^{k-1}\Omega Q = 0;$$

or, $$\frac{Q}{P} = -k\frac{\Omega Q}{\Omega P}.$$

Now P is prime to Q, so that the fraction $\frac{Q}{P}$ is in its lowest terms; therefore ΩQ contains Q; but this is impossible, for the weight of ΩQ is less than that of Q. Hence I cannot contain any non-invariantive factor Q_1.

All this will be equally true for a general function J annihilated by any operator Ω which is *linear* in the differential operators $\partial_a, \partial_b, \partial_c, \ldots$ no matter what its degree in the letters $a, b, c, \ldots$ themselves; that is, we shall still have

$$J = PQ^k$$

and $$\frac{Q}{P} = -k\frac{\Omega Q}{\Omega P},$$

where P and Q are prime to each other, and, as before, ΩQ will contain Q as a factor. But if Ω is an operator which diminishes either the degree or the weight, ΩQ is either of lower degree or of lower weight than Q, and so cannot contain it as a factor. Hence J cannot contain a factor Q not subject to annihilation by Ω.

If, however, Ω does not diminish either the degree or the weight, it may be objected that ΩQ might conceivably contain the factor Q; and were it so, there would be nothing to show the impossibility, in this case, of a function J subject to annihilation by Ω containing a factor Q, which is not so. But *quaere*: Is it possible, when J is a general homogeneous and isobaric function of $a, b, c, \ldots$, for ΩJ to contain J and at the same time the quotient to be other than a number*? *Valde dubitor.* But I reserve the point. Setting aside this doubtful case, and considering only such *linear* partial differential operators as *diminish* either the degree or the weight of the operand, we see that there cannot exist any universal operator of this kind whose effect in annihilating a form is the necessary and sufficient condition of that form being a reciprocant. But this does not preclude the possibility of the existence of such annihilators for special classes of reciprocants, and in fact

* If $\Omega = pa\partial_a + qb\partial_b + rc\partial_c + \ldots$, where $p, q, r, \ldots$ are in Arithmetical Progression, $\frac{\Omega J}{J}$ is a number; but then Ω could not be an annihilator.

(as we have already stated and shall hereafter prove) Pure Reciprocants are definable by means of the Partial Differential Annihilator

$$V = 4 \cdot \frac{a_0^2}{2} \partial_{a_1} + 5a_0a_1\partial_{a_2} + 6\left(a_0a_2 + \frac{a_1^2}{2}\right)\partial_{a_3} + \ldots,$$

which is *linear* in the differential operators, and *diminishes* the weight.

The generator for mixed reciprocants, which we have called G, will not assist us in obtaining pure reciprocants, but generates a mixed reciprocant in every case, even when the one we start with is pure. Thus, starting with the pure reciprocant R, our formula

$$GR = \{3(a_1t - a_0^2)\partial_{a_0} + 4(a_2t - a_0a_1)\partial_{a_1} + 5(a_3t - a_0a_2)\partial_{a_2} + \ldots\} R$$

may be written thus

$$GR = t(3a_1\partial_{a_0} + 4a_2\partial_{a_1} + 5a_3\partial_{a_2} + \ldots) R$$
$$- a_0(3a_0\partial_{a_0} + 4a_1\partial_{a_1} + 5a_2\partial_{a_2} + \ldots) R.$$

Here R being *pure*, that is, a function of $a_0, a_1, a_2, \ldots$ (without t), we see that

$$(3a_0\partial_{a_0} + 4a_1\partial_{a_1} + 5a_2\partial_{a_2} + 6a_3\partial_{a_3} + \ldots) R$$
$$= 3(a_0\partial_{a_0} + a_1\partial_{a_1} + a_2\partial_{a_2} + \ldots) R$$
$$+ (a_1\partial_{a_1} + 2a_2\partial_{a_2} + 3a_3\partial_{a_3} + \ldots) R$$
$$= (3i + w) R,$$

where i is the degree and w the weight of R. Hence

$$GR = t(3a_1\partial_{a_0} + 4a_2\partial_{a_1} + 5a_3\partial_{a_2} + \ldots) R - (3i + w) a_0 R,$$

where it should be noticed that a_0R is of opposite character to R (for a_0 is of odd character), while GR has been proved to be of the same character as R. Thus we cannot infer that $t(3a_1\partial_{a_0} + 4a_2\partial_{a_1} + 5a_3\partial_{a_2} + \ldots) R$ is a reciprocant. The mixed reciprocant GR cannot therefore be resolved into the sum of two terms, one of which is a pure reciprocant and the other a pure reciprocant multiplied by t.

LECTURE VI.

Before proceeding to prove that, as was stated in anticipation in Lecture IV, the operator

$$(3ac - 4b^2)\partial_b + (3ad - 5bc)\partial_c + (3ae - 6bd)\partial_d + \ldots,$$

or, when the modified letters are used,

$$4(a_0a_2 - a_1^2)\partial_{a_1} + 5(a_0a_3 - a_1a_2)\partial_{a_2} + 6(a_0a_4 - a_1a_3)\partial_{a_3} + \ldots,$$

will serve to generate a pure reciprocant from a pure one, it may be useful to briefly recapitulate what has been said concerning the character and characteristic of reciprocants. It will be remembered that the extraneous factor of any rational integral reciprocant is of the form $(-)^\kappa t^\mu$, that the character is determined by the parity (oddness or evenness) of κ, and that μ is what has been called the characteristic.

For homogeneous reciprocants it has been proved that $\mu = 3i + w$, where i is the degree of the reciprocant and w its weight, the weights of the letters $t, a, b, c, \ldots$ being taken to be $-1, 0, 1, 2, \ldots$ respectively. The character is odd or even according as the number of letters other than t in the principal term or terms is odd or even. By a principal term is to be understood one in which t is contained the greatest number of times. So that, in other words, the character is governed by the parity of the smallest number of non-t letters that can be found in any term. For pure reciprocants, there being no t in any term, the character is determined by the parity of the number of letters in any one term.

Let R be any pure reciprocant, and suppose its characteristic to be μ; then $\frac{R}{t^{\frac{\mu}{2}}}$ is an absolute reciprocant. If, however, we differentiate this with respect to x, and thus obtain another reciprocant, the resulting form will not be pure, for its numerator will be identical with the form obtained by the direct operation on R of the generator for mixed reciprocants, and its denominator will be a power of t. But, remembering that $\frac{a}{t^{\frac{3}{2}}}$, and therefore $\frac{a^{\frac{\mu}{3}}}{t^{\frac{\mu}{2}}}$, is an absolute reciprocant, we see that $\frac{R}{a^{\frac{\mu}{3}}}$, which is the quotient of the two absolute reciprocants $\frac{R}{t^{\frac{\mu}{2}}}$ and $\frac{a^{\frac{\mu}{3}}}{t^{\frac{\mu}{2}}}$, is so also. Hence $\frac{d}{dx}\left(\frac{R}{a^{\frac{\mu}{3}}}\right)$ is a reciprocant, and, since it no longer contains t, a pure one. Now,

$$\frac{d}{dx}\left(\frac{R}{a^{\frac{\mu}{3}}}\right) = \frac{a\frac{dR}{dx} - \frac{\mu}{3} \cdot bR}{a^{\frac{\mu}{3}+1}}$$

remains a reciprocant when multiplied by any power of the reciprocant a. Hence the numerator of this expression, or

$$\left(3a\frac{d}{dx} - \mu b\right) R,$$

is a reciprocant. The general value of $\frac{d}{dx}$ has been seen to be

$$a\partial_t + b\partial_a + c\partial_b + d\partial_c + \ldots,$$

but, since R is supposed to be *pure*, $\partial_t R = 0$.

We may therefore, in $3a\frac{d}{dx} - \mu b$, replace $\frac{d}{dx}$ by

$$b\partial_a + c\partial_b + d\partial_c + e\partial_d + \dots .$$

Now, remembering that $\mu = 3i + w$, and that by Euler's theorem and the similar one for isobaric functions

$$i = a\partial_a + b\partial_b + c\partial_c + d\partial_d + \dots$$

and

$$w = \quad b\partial_b + 2c\partial_c + 3d\partial_d + \dots,$$

we see that μ is equivalent to

$$3a\partial_a + 4b\partial_b + 5c\partial_c + 6d\partial_d + \dots .$$

Hence,

$$\begin{aligned} 3a\frac{d}{dx} - \mu b &= 3a(b\partial_a + c\partial_b + d\partial_c + e\partial_d + \dots) \\ &\quad - b(3a\partial_a + 4b\partial_b + 5c\partial_c + 6d\partial_d + \dots) \\ &= (3ac - 4b^2)\partial_b + (3ad - 5bc)\partial_c + (3ae - 6bd)\partial_d + \dots . \end{aligned}$$

Thus, if R be any *pure* reciprocant,

$$\{(3ac - 4b^2)\partial_b + (3ad - 5bc)\partial_c + (3ae - 6bd)\partial_d + \dots\} R$$

is also a pure reciprocant. If the type of R be w; i, j, that of the form derived from it will clearly be $w + 1$; $i + 1$, $j + 1$. Its character (which, for pure reciprocants, depends solely on the degree) will therefore be opposite to that of R, and its characteristic will be $\mu + 4$, that of R being μ.

Beginning with the form $3ac - 5b^2$, which was given as an example in Lecture II, a series of pure "educts" may be obtained by the repeated use of the above generator; and it will be noticed that the successive educts thus formed are alternately of even and odd character, whereas those previously given, namely, a, $2bt - 3a^2 \dots$, were all negative. A reduction similar to that which formerly took place when the generator for mixed reciprocants was used, may be effected at each second step in the present case. For, since the characteristic of $\left(3a\frac{d}{dx} - \mu b\right)R$ is $\mu + 4$, the next operation will give

$$\left(3a\frac{d}{dx} - (\mu + 4)b\right)\left(3a\frac{d}{dx} - \mu b\right)R.$$

Performing the indicated differentiations, this becomes

$$3a\frac{d}{dx}\left(3a\frac{dR}{dx} - \mu bR\right) - 3(\mu + 4)ab\frac{dR}{dx} + \mu(\mu + 4)b^2R$$

$$= 9a^2\frac{d^2R}{dx^2} + 9ab\frac{dR}{dx} - 3\mu ab\frac{dR}{dx} - 3\mu acR - 3(\mu + 4)ab\frac{dR}{dx} + \mu(\mu + 4)b^2R$$

$$= 9a^2\frac{d^2R}{dx^2} - 3(2\mu + 1)ab\frac{dR}{dx} - 3\mu acR + \mu(\mu + 4)b^2R.$$

Adding $\mu(\mu+4)(3ac-5b^2)R$ to 5 times the above expression, we obtain

$$45a^2\frac{d^2R}{dx^2}-15(2\mu+1)ab\frac{dR}{dx}+3\mu(\mu-1)acR,$$

which, when divided by $3a$, gives the pure reciprocant

$$15a\frac{d^2R}{dx^2}-5(2\mu+1)b\frac{dR}{dx}+\mu(\mu-1)cR.$$

This form is one degree lower than the second educt from R, the depression of degree being due to the removal of a factor a by division.

When the modified letters $a_0, a_1, a_2, a_3, \ldots$ are used, the generator

$$(3ac-4b^2)\partial_b+(3ad-5bc)\partial_c+(3ae-6bd)\partial_d+\ldots \qquad (1)$$

is easily transformed by writing in it

$$a=2a_0,\quad b=2.3.a_1,\quad c=2.3.4.a_2,\quad d=2.3.4.5\ a_3\ldots,$$

and consequently

$$\partial_b=\frac{\partial_{a_1}}{2.3},\quad \partial_c=\frac{\partial_{a_2}}{2.3.4},\quad \partial_d=\frac{\partial_{a_3}}{2.3.4.5}\ldots,$$

when it becomes

$$\frac{2^2.3^2.4}{2.3}(a_0a_2-a_1^2)\partial_{a_1}+\frac{2^2.3^2.4.5}{2.3.4}(a_0a_3-a_1a_2)\partial_{a_2}$$
$$+\frac{2^2.3^2.4.5.6}{2.3.4.5}(a_0a_4-a_1a_3)\partial_{a_3}+\ldots.$$

Dividing each term of this by 2.3, and writing the numerical coefficients in their simplest form, we have

$$4(a_0a_2-a_1^2)\partial_{a_1}+5(a_0a_3-a_1a_2)\partial_{a_2}+6(a_0a_4-a_1a_3)\partial_{a_3}+\ldots, \qquad (2)$$

which is the modified generator previously mentioned.

The generators formerly used in the theory of mixed reciprocants were

$$(2tb-3a^2)\partial_a+(2tc-4ab)\partial_b+(2td-5ac)\partial_c+\ldots \qquad (3)$$

and

$$3(ta_1-a_0^2)\partial_{a_0}+4(ta_2-a_0a_1)\partial_{a_1}+5(ta_3-a_0a_2)\partial_{a_2}+\ldots. \qquad (4)$$

The memory will be assisted in retaining these formulae if we observe that (1) is obtainable from (3), or (2) from (4), by increasing at the same time each numerical coefficient and the weight of each letter by unity.

It will, I think, be instructive to see how the form $3ac-5b^2$ was found originally by combining mixed reciprocants. The degree alone of a pure reciprocant suffices, as we have seen, to determine its character; but when we are dealing with mixed reciprocants their character does not depend either on the degree or the weight, so that we require a notation to discriminate between forms of the same degree-weight, but of opposite character. In what follows, (+) placed before any form signifies that it is a reciprocant of *even* character, while (−) signifies that its character is *odd*.

I have previously given the three *odd* reciprocants

$$(-) \quad a, \qquad (A)$$

$$(-) \quad 2bt - 3a^2, \qquad (B)$$

$$(-) \quad ct - 5ab. \qquad (C)$$

From these we obtain *even* reciprocants; thus the product of (A) and (C) is

$$(+) \quad act - 5a^2b, \qquad (D)$$

and the square of (B) is

$$(+) \quad 4b^2t^2 - 12a^2bt + 9a^4.$$

After subtracting the *even* reciprocant $9a^4$ from this, we may remove the factor $4t$ from the remainder without thereby affecting its character. These reductions give

$$(+) \quad b^2t - 3a^2b,$$

which may be combined with the *even* reciprocant (D) in such a manner that the combination contains a factor t. In fact,

$$3\,(act - 5a^2b) - 5\,(b^2t - 3a^2b) = (3ac - 5b^2)\,t,$$

so that a *legitimate* combination of mixed reciprocants can be made to give the pure one

$$3ac - 5b^2.$$

Similarly we might find the known form

$$9a^2d - 45abc + 40b^3,$$

which equated to zero expresses Sextactic Contact at a point x, y. But it is more readily obtained by operating with the generator on $3ac - 5b^2$; thus,

$$\{(3ac - 4b^2)\,\partial_b + (3ad - 5bc)\,\partial_c\}\,(3ac - 5b^2) = -10b\,(3ac - 4b^2) + 3a\,(3ad - 5bc)$$
$$= 9a^2d - 45abc + 40b^3.$$

An *orthogonal reciprocant* may be defined as a mixed reciprocant whose form remains invariable (save as to the acquisition of an extraneous factor when the reciprocant is not absolute) when any orthogonal substitution is impressed on the variables x and y. Concerning such reciprocants, we have the very beautiful theorem: *If R and $\dfrac{dR}{dt}$ are both of them reciprocants, then R is an orthogonal reciprocant.*

For suppose R to be an absolute reciprocant; that is, let

$$R = qR' \ (q = \pm 1),$$

where R is a function of $t, a, b, c, \ldots$ and R' the same function of $\tau, \alpha, \beta, \gamma, \ldots$; then, denoting by ΔR the variation of R due to the variation of y by ϵx, and by DR the variation of R due to the variation of x by $-\epsilon y$, we have

$$\Delta R = \epsilon \frac{dR}{dt}.$$

For the variation of t is ϵ and the variations of $a, b, c, \ldots$ vanish. Similarly

$$DR' = -\epsilon \frac{dR'}{d\tau}.$$

Now, since

$$R = qR',$$

$$DR = qDR' = -\epsilon q \frac{dR'}{d\tau},$$

therefore

$$DR + \Delta R = \epsilon \left(\frac{dR}{dt} - q\frac{dR'}{d\tau}\right);$$

that is, the total variation of R (due to the change of x into $x - \epsilon y$ and of y into $y + \epsilon x$) vanishes if

$$\frac{dR}{dt} = q\frac{dR'}{d\tau}.$$

Hence, if R be an absolute orthogonal reciprocant, $\frac{dR}{dt}$ is also an absolute reciprocant (though it is not orthogonal) of the same character as R.

If R be not absolute, suppose its characteristic to be μ; then it can be made absolute by dividing it by $a^{\frac{\mu}{3}}$. The application of the foregoing method of variations will now prove that $\frac{d}{dt}\left(\frac{R}{a^{\frac{\mu}{3}}}\right)$ is an absolute reciprocant of the same character as $\frac{R}{a^{\frac{\mu}{3}}}$. But $\frac{d}{dt}\left(\frac{R}{a^{\frac{\mu}{3}}}\right) = \frac{1}{a^{\frac{\mu}{3}}}\frac{dR}{dt}$. Hence $\frac{dR}{dt}$ is a reciprocant whose characteristic is μ, and character the same as that of R.

The simplest Orthogonal Reciprocant is the form

$$(1 + t^2)b - 3a^2t,$$

which occurs on p. 19 of Boole's *Differential Equations*. When equated to zero it is the general differential equation of a circle. It is noticeable that although Boole obtains this form by equating to zero the differential of the radius of curvature

$$\frac{(1 + t^2)^{\frac{3}{2}}}{a},$$

he does not recognise the fact that it vanishes at points of maximum or minimum curvature of any plane curve, but says that the "geometrical property which this equation expresses is the invariability of the radius of curvature."

Taking this form as an example of our general theorem, let

$$R = (1 + t^2)b - 3a^2t;$$

then

$$\frac{dR}{dt} = 2bt - 3a^2,$$

which is the familiar Schwarzian. Observe that

$$(1+t^2)\,b - 3a^2t = -t^6\,\{(1+\tau^2)\,\beta - 3\alpha^2\tau\}$$

and

$$2bt - 3a^2 = -t^6\,(2\beta\tau - 3\alpha^2),$$

so that the characteristic and character are the same for both these forms.

The form $ct - 5ab$, which we have called the Post-Schwarzian, when multiplied by 2 and integrated with respect to t, gives

$$ct^2 - 10abt + \phi\,(a, b, \ldots).$$

In order that this may be a reciprocant, we must have

$$\phi\,(a, b, \ldots) = c + 15a^3.$$

In this way the Orthogonal Reciprocant

$$(1+t^2)\,c - 10abt + 15a^3$$

was obtained originally.

It will be easy to verify that this is a reciprocant by means of the identical relations

$$t = \frac{1}{\tau},$$

$$a = -\frac{\alpha}{\tau^3},$$

$$b = -\frac{\beta\tau - 3\alpha^2}{\tau^5},$$

$$c = -\frac{\gamma\tau^2 - 10\alpha\beta\tau + 15\alpha^3}{\tau^7}.$$

We shall find that

$$(1+t^2)\,c - 10abt + 15a^3 = -t^7\,\{(1+\tau^2)\,\gamma - 10\alpha\beta\tau + 15\alpha^3\},$$

and comparing this with

$$ct - 5ab = -t^7\,(\gamma\tau - 5\alpha\beta),$$

it will be noticed that both forms have the same character and the same characteristic.

The complete primitive of the differential equation

$$c\,(1+t^2) - 10abt + 15a^3 = 0$$

has been found by Mr Hammond and Prof. Greenhill. The solution may be written in the following forms:

$$\left.\begin{aligned} x &= \int \frac{dt}{\sqrt{\{\kappa\,(1 - 15t^2 + 15t^4 - t^6) + \lambda\,(6t - 20t^3 + 6t^5)\}}} + \mu \\ y &= \int \frac{t\,dt}{\sqrt{\{\kappa\,(1 - 15t^2 + 15t^4 - t^6) + \lambda\,(6t - 20t^3 + 6t^5)\}}} + \nu \end{aligned}\right\},$$

$$\left.\begin{aligned} x &= \int \frac{\cos(\theta - A)\,d\theta}{\sqrt{\{B\cos 6\,(\theta - A)\}}} + \text{const.} \\ y &= \int \frac{\sin(\theta - A)\,d\theta}{\sqrt{\{B\cos 6\,(\theta - A)\}}} + \text{const.} \end{aligned}\right\}.$$

$$k'^2 tn^2(X, k) = k^2 tn^2(Y, k'),$$

where $$k = \sin 15°, \quad k' = \sin 75°,$$

and $$X = lx + my + n_1,$$

$$Y = mx - ly + n_2,$$

l, m, n_1, n_2 being arbitrary constants.

The last two forms of solution are due to Prof. Greenhill.

LECTURE VII.

I have frequently referred to, and occasionally dilated on, the analogy between pure reciprocants and invariants. A new bond of connection between the two theories has been established by Capt. MacMahon, which I will now explain. Let me, by way of preface, so far anticipate what I shall have to say on the Theorem of Aggregation in Invariants (that is, the theorem concerning the number of linearly independent invariants of a given type) as to remark that the proof of this theorem, first given by me in *Crelle's Journal* and subsequently in the *Phil. Mag.* for March, 1878, depends on the fact that if we take two operators, namely, the Annihilator, say

$$\Omega = a_0\partial_{a_1} + 2a_1\partial_{a_2} + 3a_2\partial_{a_3} + \ldots + ja_{j-1}\partial_{a_j}$$

and its opposite, say

$$O = a_j\partial_{a_{j-1}} + 2a_{j-1}\partial_{a_{j-2}} + 3a_{j-2}\partial_{a_{j-3}} + \ldots + ja_1\partial_{a_0},$$

then $(\Omega O - O\Omega)\, I$ is a multiple of I.

Thus, if I stands for any invariant (that is, if $\Omega I = 0$), it follows immediately that $\Omega O I$ is a multiple of I, and consequently $\Omega^m O^m I$ is also a multiple of I. We may call Ω and O, which are exact opposites to each other, reversing operators.

Now, MacMahon has found out the reversor to V, the Annihilator of pure reciprocants. His reversing operator is no longer of a similar, though opposite, form to V, as O is to Ω, but is simply $\frac{d}{dx}$; nor is the effect of operating with $V\frac{d}{dx}$ on any pure reciprocant R equivalent to multiplication by a merely numerical factor, as was the case with $\Omega O I$, but $\left(V\frac{d}{dx}\right)R$ is a numerical multiple of aR, and as a consequence of this $\left(V^m\frac{d^m}{dx^m}\right)R$ is a numerical multiple of $a^m R$. Thus the parallelism is like that between the two sexes, the same with a difference, as is usually the case in comparing the two theories.

This remarkable relation between the operators V and $\frac{d}{dx}$ may be seen *a priori* if we assume that (as we shall hereafter prove) to each pure reciprocant R there is an annihilator V of the form

$$3a^2\partial_b + (\ldots)\partial_c + (\ldots)\partial_d + (\ldots)\partial_e + \ldots,$$

not containing ∂_a and linear in the remaining differential operators $\partial_b, \partial_c, \partial_d, \ldots$. For if we call the characteristic μ, by differentiating the absolute pure reciprocant $\frac{R}{a^{\frac{\mu}{3}}}$ with respect to x we obtain, as was shown in the last lecture, the pure reciprocant

$$3a\frac{dR}{dx} - \mu bR.$$

Since this is annihilated by V, we have

$$3a\left(V\frac{d}{dx}\right)R - \mu RVb - \mu bVR = 0.$$

But, since R is a pure reciprocant, $VR = 0$; and from the assumed form of V it follows that

$$Vb = 3a^2.$$

Hence $$3a\left(V\frac{d}{dx}\right)R - 3\mu a^2R = 0,$$

or $$\left(V\frac{d}{dx}\right)R = \mu aR.$$

Thus the operation of $V\frac{d}{dx}$ is equivalent to multiplication by μa, so that (barring the introduction of a) V restores to $\frac{dR}{dx}$ the form it had antecedent to the operation of $\frac{d}{dx}$, and may be called a qualified reversor to $\frac{d}{dx}$.

For example, suppose that

$$R = 3ac - 5b^2.$$

Since we are using *natural* letters for the derivatives of y with respect to x, we have

$$\frac{d}{dx} = b\partial_a + c\partial_b + d\partial_c + \ldots,$$

and, as we shall presently see,

$$V = 3a^2\partial_b + 10ab\partial_c + (15ac + 10b^2)\partial_d + \ldots.$$

Now, $\frac{dR}{dx} = (b\partial_a + c\partial_b + d\partial_c)(3ac - 5b^2) = 3bc - 10bc + 3ad = 3ad - 7bc.$

Operating on this with V, we find

$$V\frac{dR}{dx} = V(3ad - 7bc) = -21a^2c - 70ab^2 + 3a(15ac + 10b^2) = 24a^2c - 40ab^2;$$

that is $$V\frac{d}{dx}(3ac - 5b^2) = 8a(3ac - 5b^2).$$

Let us now inquire whether it is possible so to determine an operator V that the relation

$$\left(V\frac{d}{dx}-\frac{d}{dx}V\right)F=(3i+w)\,aF$$

may be satisfied identically when F is any homogeneous isobaric function of the letters $a, b, c, \ldots$ of degree i and weight w. If so, we must be able to satisfy each of the equations

$$\left(V\frac{d}{dx}-\frac{d}{dx}V\right)a=3a^2,$$
$$\left(V\frac{d}{dx}-\frac{d}{dx}V\right)b=4ab,$$
$$\left(V\frac{d}{dx}-\frac{d}{dx}V\right)c=5ac,$$
$$\left(V\frac{d}{dx}-\frac{d}{dx}V\right)d=6ad,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

which are found by writing $a, b, c, d, \ldots$ successively in the place of F.

Now $\frac{da}{dx}=b$, $\frac{db}{dx}=c$, $\frac{dc}{dx}=d$, ... so that the above equations may be written

$$Vb=3a^2+\frac{d}{dx}(Va),$$
$$Vc=4ab+\frac{d}{dx}(Vb),$$
$$Vd=5ac+\frac{d}{dx}(Vc),$$
$$Ve=6ad+\frac{d}{dx}(Vd),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

These equations are sufficient to completely determine V on the supposition previously made that it is linear in the differential operators and does not contain ∂_a; for, since V is linear, it must be of the form

$$(Va)\,\partial_a+(Vb)\,\partial_b+(Vc)\,\partial_c+\ldots,$$

and, since it does not contain ∂_a, we must have $Va=0$, and therefore

$$Vb=3a^2,$$
$$Vc=4ab+\frac{d}{dx}(3a^2)=4ab+6ab=10ab,$$
$$Vd=5ac+\frac{d}{dx}(10ab)=5ac+10b^2+10ac=15ac+10b^2,$$
$$Ve=6ad+\frac{d}{dx}(15ac+10b^2)=6ad+15bc+20bc+15ad=21ad+35bc,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

Hence $\quad V=3a^2\partial_b+10ab\partial_c+(15ac+10b^2)\,\partial_d+(21ad+35bc)\,\partial_e+\ldots.$

When the modified letters $a_0, a_1, a_2, \ldots$ are used, we shall have, in consequence of the change of notation, $\left(V\frac{d}{dx}\right)R = 2\mu a_0 R$ (instead of $\mu a R$). If, as before, we seek to satisfy the equation

$$\left(V\frac{d}{dx} - \frac{d}{dx}V\right)F = 2(3i + w)\,a_0 F, \tag{1}$$

we shall find, on writing a_n in the place of F,

$$\left(V\frac{d}{dx} - \frac{d}{dx}V\right)a_n = 2(3+n)\,a_0 a_n. \tag{2}$$

This condition will be sufficient, as well as necessary, for the satisfaction of (1) when V is linear; for then

$$V\frac{d}{dx} - \frac{d}{dx}V$$

will also be linear, its general term being

$$\left(V\frac{da_n}{dx} - \frac{d}{dx}Va_n\right)\partial_{a_n},$$

which is equal to $2(3+n)\,a_0 a_n \partial_{a_n}$ by equation (2). Hence

$$\begin{aligned}\left(V\frac{d}{dx} - \frac{d}{dx}V\right)F &= \text{a sum of terms of the form } 2(3+n)\,a_0 a_n \partial_{a_n} F\\ &= 2a_0(3a_0\partial_{a_0} + 3a_1\partial_{a_1} + 3a_2\partial_{a_2} + \ldots)F\\ &\quad + 2a_0(a_1\partial_{a_1} + 2a_2\partial_{a_2} + \ldots)F;\end{aligned}$$

that is, equation (1) is satisfied whenever (2) is. Writing in (2)

$$\frac{da_n}{dx} = (n+3)\,a_{n+1},$$

we obtain
$$(n+3)\,Va_{n+1} = 2(n+3)\,a_0 a_n + \frac{d}{dx}(Va_n), \tag{3}$$

from which the values of Va_n may be successively determined.

When $Va_0 = 0$, the value of Va_n, which satisfies (3), is

$$Va_n = \frac{n+3}{2}(a_0 a_{n-1} + a_1 a_{n-2} + \ldots + a_{n-2}a_1 + a_{n-1}a_0);$$

thus
$$Va_1 = \frac{4}{2}\cdot a_0^2,\quad Va_2 = 5a_0 a_1,\quad Va_3 = 6a_0 a_2 + 3a_1^2, \ldots$$

and the value of V is therefore

$$\frac{4}{2}\cdot a_0^2\partial_{a_1} + 5a_0 a_1 \partial_{a_2} + 6\left(a_0 a_2 + \frac{1}{2}a_1^2\right)\partial_{a_3} + 7(a_0 a_3 + a_1 a_2)\,\partial_{a_4} + \ldots.$$

Now that we are on the subject of parallelism between the old and new worlds of Algebraical Form, I feel tempted to point out yet another very interesting bond of connection between them. There is a theorem concerning Invariants which I am not aware that any one but myself has noticed, or at all events I do not remember ever seeing it in print*, which is this: If we take any "invariant" and regard its most advanced letter as a variable, or say rather as the ratio of two variables $u:v$, by multiplying by a proper power of v we obtain a new Quantic in u, v; or, if we take any number of such invariants with the same most advanced letter (or, as we may call it in a double sense, the same radical letter) in common, we shall have a system of binary Quantics in u, v. My theorem is, or was, that an Invariant of any one or more of such Quantics is an Invariant of the original Quantic. I recently found a similar proposition to be true for Reciprocants, namely, forming as before a system of *pure* Reciprocants into Quantics in u, v, any "Invariant" of such system is itself a Reciprocant.

The two theorems may be stated symbolically thus:

$$\left.\begin{array}{l} II' = I'' \\ IR = R' \end{array}\right\}.$$

On mentioning this to Mr L. J. Rogers, he sent me next day a proof which, although only stated as applicable to Reciprocants, is equally so, *mutatis mutandis*, to Invariants. Although given for a single invariant, it applies equally to a system.

I give Mr Rogers' proof that any invariant of a *pure* reciprocant (the proof will not hold for impure ones) is a pure reciprocant; or rather I use his method to prove the analogous theorem that any invariant of an invariant is itself an invariant. It will be seen hereafter that this same proof applies to *pure* reciprocants with only trifling changes; but the proof as given by Mr Rogers requires some further considerations to be gone into for which we are not yet ripe.

Consider, for the sake of simplicity, the binary Quintic

$$(a, b, c, d, e, f \between x, y)^5,$$

and let I be any invariant of it (satisfied or unsatisfied); then

$$I = a_0 f^n + a_1 f^{n-1} + a_2 f^{n-2} + \ldots + a_n,$$

where $a_0, a_1, a_2, \ldots a_n$ do not contain f, but are functions of a, b, c, d, e alone.

Let the Protomorphs for our Quintic be denoted by A, B, C, D, E, F; then

$$F = a^2 f - 5abe + 2acd + 8b^2 d + 6bc^2.$$

Eliminating f from I by means of this equation, we have

$$Ia^{2n} = A_0 F^n + A_1 F^{n-1} + A_2 F^{n-2} + \ldots + A_n,$$

where $A_0, A_1, A_2, \ldots A_n$ are all of them invariants (not necessarily integral

* The theorem is, however, given in Vol. XI. p. 98 of the *Bulletin de la Société Mathématique de France*, in a paper by M. Perrin, which has only recently come under the lecturer's notice.

forms, but this is immaterial to the proof, for Ω annihilates fractional and integral invariants alike). For

$$\Omega(Ia^{2n}) = \Omega(A_0F^n + A_1F^{n-1} + \ldots + A_n),$$

and, in consequence of Ia^{2n} and F being invariants, so that, as regards Ω, F may be treated as if it were a constant, this becomes

$$0 = F^n\Omega A_0 + F^{n-1}\Omega A_1 + F^{n-2}\Omega A_2 + \ldots + \Omega A_n,$$

in which the coefficients of the several powers of F must be separately equated to zero. In other words, $A_0, A_1, A_2, \ldots A_n$ are all of them invariants. Now, any invariant of

$$A_0F^n + A_1F^{n-1} + A_2F^{n-2} + \ldots + A_n$$

is a function of A_0, A_1, A_2, ... A_n, and therefore an invariant.

(N.B.—We cannot assume that any function of general reciprocants is itself a reciprocant.)

Again, since $\quad A_0F^n + \ldots + A_n$, and $a_0f^n + \ldots + a_n$ are connected by the substitution

$$F = a^2f - 5abe + \ldots,$$

which is *linear* in respect to the letters F and f, any invariant of

$$A_0F^n + \ldots + A_n$$

is (to a factor *près*, that factor being a power of a which is itself an invariant) equal to the corresponding invariant of

$$a_0f^n + \ldots + a_n.$$

But every invariant of the former has been shown to be an invariant of the original quantic, and therefore every invariant of the latter is so also.

I add some examples in illustration of this theorem:

Ex. 1. Take the invariant of the Quintic

$$a^2f^2 - 10abef + 4acdf + 16b^2df - 12bc^2f + 16ace^2 + 9b^2e^2 - 12ad^2e - 76bcde + 48c^3e + 48bd^3 - 32c^2d^2.$$

The discriminant of this, considered as a quadratic in f, is

$$\begin{aligned} &a^2(16ace^2 + 9b^2e^2 - 12ad^2e - 76bcde + 48c^3e + 48bd^3 - 32c^2d^2) \\ &\qquad - (5abe - 2acd - 8b^2d + 6bc^2)^2 \\ &= 16a^3ce^2 - 16a^2b^2e^2 - 12a^3d^2e - 56a^2bcde + 48a^2c^3e + 80ab^3de - 60ab^2c^2e \\ &\qquad + 48a^2bd^3 - 36a^2c^2d^2 - 32ab^2cd^2 - 64b^4d^2 + 24abc^3d + 96b^3c^2d - 36b^2c^4. \end{aligned}$$

It will be found on trial that this is divisible by the invariant

$$4(ae - 4bd + 3c^2),$$

the quotient being

$$\begin{aligned} &4a^2ce - 4ab^2e - 3a^2d^2 + 2abcd + 4b^3d - 3b^2c^2 \\ &\qquad = 3a(ace - b^2e - ad^2 + 2bcd - c^3) + (ac - b^2)(ae - 4bd + 3c^2). \end{aligned}$$

Thus the discriminant of the quadratic in f, that is, of the invariant

$$a^2f^2 - 2f(5abe - 2acd + 8b^2d - 6bc^2) + \dots,$$

is shown to be an invariant. It will further illustrate the proof of the theorem if we remark that precisely the same invariant is obtained by eliminating f between the above form and the protomorph

$$a^2f - 5abe + 2acd + 8b^2d - 6bc^2.$$

Ex. 2. If we take the pure reciprocant

$$45a^3d^2 - 450a^2bcd + 400ab^3d + 192a^2c^3 + 165ab^2c^2 - 400b^4c,$$

which, from its similarity to the Discriminant of the Cubic, I have called the Quasi-Discriminant, and form *its* discriminant, when regarded as a quadratic in d, we find

$$45a^3(192a^2c^3 + 165ab^2c^2 - 400b^4c) - (225a^2bc - 200ab^3)^2.$$

If, in this expression, we write $P = 3ac - 5b^2$, so that $3ac = P + 5b^2$, it becomes

$$5 \,.\, 64a^2(P + 5b^2)^3 + 5 \,.\, 165a^2b^2(P + 5b^2)^2 - 15 \,.\, 400a^2b^4(P + 5b^2) \\ - 625a^2b^2(3P + 7b^2)^2.$$

On performing the calculation it will be found that all the terms involving b will disappear from this result, and there will remain the single term $320a^2P^3$, that is, $320a^2(3ac - 5b^2)^3$, which is a reciprocant.

LECTURE VIII.

In my last lecture the complete expression, both in terms of the modified and unmodified letters, was obtained for V, the annihilator for pure reciprocants assuming its existence and its form. These assumptions I shall now make good by proving, from first principles, the fundamental theorem that the satisfaction of the equation

$$VR = 0$$

is a necessary and sufficient condition in order that R may be a pure reciprocant.

It will be advantageous to use the modified system of letters, in which

$$t, a_0, a_1, a_2, \dots \text{ stand for } \frac{dy}{dx}, \frac{1}{1\,.\,2}\cdot\frac{d^2y}{dx^2}, \frac{1}{1\,.\,2\,.\,3}\cdot\frac{d^3y}{dx^3}, \frac{1}{1\,.\,2\,.\,3\,.\,4}\cdot\frac{d^4y}{dx^4}, \dots$$

$$\text{and} \quad \alpha_0, \alpha_1, \alpha_2, \dots \text{ for } \frac{1}{1\,.\,2}\cdot\frac{d^2x}{dy^2}, \frac{1}{1\,.\,2\,.\,3}\cdot\frac{d^3x}{dy^3}, \frac{1}{1\,.\,2\,.\,3\,.\,4}\cdot\frac{d^4x}{dy^4}, \dots$$

respectively. Let the variation due to the change of x into $x + \epsilon y$, where ϵ

is an infinitesimal number, be denoted by Δ. Obviously this change leaves the value of each of the quantities $\alpha_0, \alpha_1, \alpha_2, \ldots$ unaltered, and therefore

$$\Delta R(\alpha_0, \alpha_1, \alpha_2, \ldots) = 0,$$

whatever the nature of R may be. But when R is a pure reciprocant,

$$R(a_0, a_1, a_2, \ldots) = \pm t^{\mu} R(\alpha_0, \alpha_1, \alpha_2, \ldots),$$

whence it immediately follows that

$$\Delta t^{-\mu} R(a_0, a_1, a_2, \ldots) = 0^*.$$

Before proceeding to determine the values of

$$\Delta t,\ \Delta a_0,\ \Delta a_1,\ \Delta a_2, \ldots$$

it will be useful to remark that since

$$\frac{dy}{dx} = t,\ \frac{d^2y}{dx^2} = 1.2.a_0,\ \frac{d^3y}{dx^3} = 1.2.3.a_1, \ldots,$$

we have $$\frac{dt}{dx} = 2a_0,\ \frac{da_0}{dx} = 3a_1, \ldots,$$

and generally $$\frac{da_n}{dx} = (n+3)\,a_{n+1}.$$

Now let $[t]$ denote the augmented value of t, and in general let $[\ ]$ be used to signify that the augmented value of the quantity enclosed in it is to be taken. Then

$$[t] = \frac{dy}{d[x]} = \frac{dy}{d(x+\epsilon y)} = \frac{dy}{dx\left(1+\epsilon\dfrac{dy}{dx}\right)} = \frac{t}{1+\epsilon t} = t - \epsilon t^2;$$

so also $$2[a_0] = [2a_0] = \frac{d[t]}{d[x]} = \frac{d[t]}{d(x+\epsilon y)} = \frac{d[t]}{dx(1+\epsilon t)} = (1-\epsilon t)\frac{d[t]}{dx}$$

$$= (1-\epsilon t)\frac{d}{dx}(t-\epsilon t^2) = (1-\epsilon t)(2a_0 - 4\epsilon t a_0)$$

$$= 2a_0 - 6\epsilon t a_0;$$

that is $$[a_0] = a_0 - 3\epsilon t a_0.$$

Reasoning precisely similar to that which gave

$$2[a_0] = (1-\epsilon t)\frac{d}{dx}[t],$$

leads to the formula

$$(n+3)[a_{n+1}] = (1-\epsilon t)\frac{d}{dx}[a_n],$$

* It has been suggested by Mr J. Chevallier that the proof might be simplified by considering the variation $\Delta a_0^{-\frac{\mu}{3}} R(a_0, a_1, a_2, \ldots)$ instead of $\Delta t^{-\mu} R(a_0, a_1, a_2, \ldots)$.

from which the augmented values of $a_1, a_2, a_3, \ldots$ may be found by giving to n the values 0, 1, 2, ... in succession. Thus, writing $n=0$, we have

$$3[a_1] = (1-\epsilon t)\frac{d}{dx}[a_0] = (1-\epsilon t)\frac{d}{dx}(a_0 - 3\epsilon t a_0)$$

$$= (1-\epsilon t)(3a_1 - 9\epsilon t a_1 - 6\epsilon a_0^2) = 3a_1 - \epsilon(12ta_1 + 6a_0^2),$$

or $$[a_1] = a_1 - \epsilon(4ta_1 + 2a_0^2).$$

Similarly, when $n=1$,

$$4[a_2] = (1-\epsilon t)\frac{d}{dx}[a_1] = (1-\epsilon t)\frac{d}{dx}(a_1 - 4\epsilon t a_1 - 2\epsilon a_0^2)$$

$$= (1-\epsilon t)(4a_2 - 16\epsilon t a_2 - 20\epsilon a_0 a_1)$$

$$= 4a_2 - 20\epsilon t a_2 - 20\epsilon a_0 a_1,$$

and $$[a_2] = a_2 - 5\epsilon(ta_2 + a_0 a_1).$$

Again, $$5[a_3] = (1-\epsilon t)\frac{d}{dx}[a_2] = (1-\epsilon t)\frac{d}{dx}(a_2 - 5\epsilon t a_2 - 5\epsilon a_0 a_1)$$

$$= (1-\epsilon t)(5a_3 - 25\epsilon t a_3 - 30\epsilon a_0 a_2 - 15\epsilon a_1^2)$$

$$= 5a_3 - 30\epsilon t a_3 - 30\epsilon a_0 a_2 - 15\epsilon a_1^2,$$

so that $$[a_3] = a_3 - \epsilon(6ta_3 + 6a_0 a_2 + 3a_1^2).$$

In like manner we shall find

$$[a_4] = a_4 - 7\epsilon(ta_4 + a_0 a_3 + a_1 a_2).$$

These results may be written in a more symmetrical form; thus:

$$2[t] = 2t - 2\epsilon t^2,$$

$$2[a_0] = 2a_0 - 3\epsilon(ta_0 + a_0 t),$$

$$2[a_1] = 2a_1 - 4\epsilon(ta_1 + a_0^2 + a_1 t),$$

$$2[a_2] = 2a_2 - 5\epsilon(ta_2 + a_0 a_1 + a_1 a_0 + a_2 t),$$

$$2[a_3] = 2a_3 - 6\epsilon(ta_3 + a_0 a_2 + a_1^2 + a_2 a_0 + a_3 t),$$

$$2[a_4] = 2a_4 - 7\epsilon(ta_4 + a_0 a_3 + a_1 a_2 + a_2 a_1 + a_3 a_0 + a_4 t).$$

The general law

$$2[a_n] = 2a_n - (n+3)\epsilon(ta_n + a_0 a_{n-1} + \ldots + a_{n-1}a_0 + a_n t),$$

or, as it may also be written,

$$\Delta a_n = -\frac{n+3}{2}\epsilon(ta_n + a_0 a_{n-1} + \ldots + a_{n-1}a_0 + a_n t),$$

admits of an easy inductive proof.

Assuming the truth of the theorem for $[a_n]$, and writing for brevity in what follows,

$$S_n = ta_n + a_0 a_{n-1} + a_1 a_{n-2} + \ldots + a_{n-2}a_1 + a_{n-1}a_0 + a_n t,$$

we have $$[a_n] = a_n - \frac{n+3}{2}\epsilon S_n.$$

Now, $$\frac{dS_n}{dx} = (n+3)\,ta_{n+1} + 2a_0a_n$$
$$+ (n+2)\,a_0a_n + 3a_1a_{n-1}$$
$$+ (n+1)\,a_1a_{n-1} + 4a_2a_{n-2}$$
$$+ \ldots\ldots\ldots\ldots\ldots + \ldots\ldots\ldots$$
$$+ 4a_{n-2}a_2 + (n+1)\,a_{n-1}a_1$$
$$+ 3a_{n-1}a_1 + (n+2)\,a_na_0$$
$$+ 2a_na_0 + (n+3)\,a_{n+1}t$$
$$= (n+4)\,(ta_{n+1} + a_0a_n + a_1a_{n-1} + \ldots$$
$$+ a_{n-1}a_1 + a_na_0 + a_{n+1}t) - 2ta_{n+1}$$
$$= (n+4)\,S_{n+1} - 2ta_{n+1}.$$

Hence $$\frac{d}{dx}[a_n] = (n+3)\,a_{n+1} - \frac{n+3}{2}\,\epsilon\,\{(n+4)\,S_{n+1} - 2ta_{n+1}\}.$$

But, as we have already seen,
$$(n+3)\,[a_{n+1}] = (1-\epsilon t)\,\frac{d}{dx}[a_n]\,;$$
consequently,
$$[a_{n+1}] = (1-\epsilon t)\,a_{n+1} - \frac{n+4}{2}\,\epsilon S_{n+1} + \epsilon t a_{n+1} = a_{n+1} - \frac{n+4}{2}\,\epsilon S_{n+1}\,;$$
that is, the theorem holds for $[a_{n+1}]$ when it holds for $[a_n]$. But we know that it is true for the cases $n=0, 1, 2, 3, 4$, and therefore it is true universally.

Resuming the proof of the main theorem, it has been shown that
$$\Delta t^{-\mu}R\,(a_0,\, a_1,\, a_2,\, \ldots) = 0\,;$$
that is $$-\mu t^{-1}\Delta t + R^{-1}\Delta R = 0,$$
or $$-\mu R t^{-1}\Delta t + \frac{dR}{da_0}\Delta a_0 + \frac{dR}{da_1}\Delta a_1 + \frac{dR}{da_2}\Delta a_2 + \ldots = 0.$$

But $$\Delta t = -\epsilon t^2,$$
$$\Delta a_0 = -3\epsilon t a_0,$$
$$\Delta a_1 = -\epsilon\,(4ta_1 + 2a_0^2),$$
$$\Delta a_2 = -\epsilon\,(5ta_2 + 5a_0a_1),$$
$$\Delta a_3 = -\epsilon\,(6ta_3 + 6a_0a_2 + 3a_1^2),$$
$$\Delta a_4 = -\epsilon\,(7ta_4 + 7a_0a_3 + 7a_1a_2),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
and consequently
$$t\,(\mu - 3a_0\partial_{a_0} - 4a_1\partial_{a_1} - 5a_2\partial_{a_2} - 6a_3\partial_{a_3} - 7a_4\partial_{a_4} - \ldots)\,R$$
$$-\left\{4\left(\frac{a_0^2}{2}\right)\partial_{a_1} + 5\,(a_0a_1)\,\partial_{a_2} + 6\left(a_0a_2 + \frac{a_1^2}{2}\right)\partial_{a_3}\right.$$
$$\left. + 7\,(a_0a_3 + a_1a_2)\,\partial_{a_4} + \ldots\right\}R = 0.$$

This is equivalent to the two conditions

$$(3a_0\partial_{a_0} + 4a_1\partial_{a_1} + 5a_2\partial_{a_2} + \dots)\, R = \mu R$$

and

$$VR = 0,$$

where

$$V = 4\left(\frac{a_0^2}{2}\right)\partial_{a_1} + 5\,(a_0 a_1)\,\partial_{a_2} + 6\left(a_0 a_2 + \frac{a_1^2}{2}\right)\partial_{a_3} + 7\,(a_0 a_3 + a_1 a_2)\,\partial_{a_4} + \dots.$$

For greater simplicity I confine what I have to say to the only essential case, to which every other may be reduced, of a *homogeneous* pure reciprocant. The equation

$$(3a_0\partial_{a_0} + 4a_1\partial_{a_1} + 5a_2\partial_{a_2} + \dots)\, R = \mu R$$

shows that for every term $w + 3i$ is constant; that is, w is constant and therefore the function R is isobaric. This is also immediately deducible from the form of the relations between $a_0, a_1, a_2, \dots; \alpha_0, \alpha_1, \alpha_2, \dots$, and, what is important to notice, for future purposes,

$$F(a_0, a_1, a_2, \dots) - t^{\mu} F(\alpha_0, \alpha_1, \alpha_2, \dots),$$

when F is a homogeneous isobaric function, and $\mu = w + 3i$ is itself a homogeneous function of $(a_0, a_1, a_2, \dots)$, whose degree is the same as that of F.

The only condition affecting R, a function of $a_0, a_1, a_2, \dots$, supposed homogeneous and isobaric, is

$$VR = 0.$$

I shall now prove the converse, that if $R = F(a_0, a_1, a_2, \dots)$ (being homogeneous and isobaric) has V for its annihilator, then R is a pure reciprocant. Let D be the value of $F(a_0, a_1, a_2, \dots) - t^{\mu} F(\alpha_0, \alpha_1, \alpha_2, \dots)$ expressed as a function of $a_0, a_1, a_2, \dots$ alone. Then D will be a function of the same type as $F(a_0, a_1, a_2, \dots)$.

Suppose that

$$\Delta D = 0;$$

that is, that the variation of D due to the change of x into $x + \epsilon y$ vanishes in virtue of the equation $VR = 0$.

Let D become D' when y receives an arbitrary variation $y + \eta u$, where η is an infinitesimal constant and u an arbitrary function of x; then the variation of D' will vanish when x is changed into $x + \epsilon y + \epsilon\eta u$, and consequently when x is changed into $x + \epsilon y$ the variation of D' will also vanish. Hence

$$\Delta D' = 0,$$

and if we take the difference of the variations of D and D', we shall find

$$\Delta\left(u'' \frac{d}{da_0} D + u''' \frac{d}{da_1} D + u^{\text{IV}} \frac{d}{da_2} D + \dots\right) = 0.$$

Now, the arbitrary nature of the function u shows that we must have

$$\Delta \frac{d}{da_0} D = 0, \quad \Delta \frac{d}{da_1} D = 0, \quad \Delta \frac{d}{da_2} D = 0, \quad \dots$$

and if we reason on $\frac{d}{da_0}D$, $\frac{d}{da_1}D$, ... in the same way as we have on D, we see that the variation Δ of each of the second differential derivatives of D will also vanish; and, pursuing the same argument further, it will be evident that the Δ of any derivative of D, of any order whatever, with respect to $a_0, a_1, a_2, \ldots$ will vanish. Hence

$$D = 0;$$

for if this is not so we may, supposing D to be a function of degree i in the letters $a_0, a_1, a_2, \ldots$, take the Δ of each of the differential derivatives of D of the order $i-1$; each of these variations would vanish by what precedes; that is, the variation due to the change of x into $x + \epsilon y$ of each of the letters $a_0, a_1, a_2, \ldots$ contained in D would be identically zero, which is absurd. We see, therefore, that when $\Delta D = 0$ (that is, when R is annihilated by V), $D = 0$, or

$$F(a_0, a_1, a_2, \ldots) = t^\mu F(\alpha_0, \alpha_1, \alpha_2, \ldots),$$

which proves the converse proposition.

It will not fail to be noticed how much language, and as a consequence algebraical thought (for words are the tools of thought), is facilitated by the use of the concept of annihilation in lieu of that of equality as expressed by a partial differential equation.

It is somewhat to the point that in the recent two grand determinations of the order of precedence among the so-called fixed stars relative to our planet, as approximately represented by the intensities of the light from them which reaches the eye, the one is directed by the principle of annihilation, the other by that of equality. Prof. Pritchard's method essentially consists in determining what relative thicknesses of an interposed glass screen, effected by means of a sliding wedge of glass, will serve to extinguish the light of a star; that employed by Prof. Pickering depends on finding what degree of rotation of an interposed prism of Iceland spar (a Nicol Prism) will serve to bring to an equality the ordinary image of one star with the extraordinary one of another. As these intensities depend on the squared sines and cosines of this angle of rotation measured from the position of non-visibility of one of them, it follows that the tangent squared of the twist measures the relative intensities by this method.

Hereafter it will be shown that if F is a homogeneous isobaric function of

$$y, y', y'', y''', \ldots,$$

whose weights are reckoned as

$$-2, -1, 0, 1, \ldots$$

then, when x becomes $x + hy$, where h is any constant quantity, F becomes

$$(1 + ht)^{-\mu} e^{-\frac{hV_1}{1+ht}} F,$$

where $\quad t = y', \; V_1 = -t^2\partial_t + V, \text{ and } \mu = 3i + w,$

i being the degree and w the weight of F.

From this, by an obvious course of reasoning, could be deduced as a particular case the condition of $F(a_0, a_1, a_2, \ldots)$ remaining a factor of its altered self when *any* linear substitutions are impressed on x and y; namely, the necessary and sufficient condition is that F has V for its annihilator.

LECTURE IX.

The prerogative of a Pure Reciprocant is that it continues a factor of its altered self when the variables x and y are subjected to any linear substitution. Its form, like that of any other reciprocant, is of course persistent when the variables are interchanged; that is, when in the general substitution, in which y is changed into

$$fy + gx + h$$

and x into

$$f'y + g'x + h',$$

we give the particular values $h = 0$, $h' = 0$, $f = 0$, $g' = 0$, $f' = 1$, $g = 1$, to the constants. Stated geometrically, the theorem is that the evanescence of any pure reciprocant R indicates a property independent of transformation of axes in a plane. We suppose R to be homogeneous and isobaric in $a, b, c, \ldots$. (If it were not, the theorem could not hold, for either the change of y into κy or that of x into λx would destroy the form.)

The persistence, under any linear substitution, of the form of pure reciprocants may be easily established as follows:

By a *semi-substitution* understand one where one of the variables remains unaltered. There are two such semi-substitutions, namely, where x remains unaltered, and where y does.

(1) Let x remain unaltered and y become $fy + gx + h$; then $a, b, c, \ldots$ become $fa, fb, fc, \ldots$ respectively; and therefore

$$R(a, b, c, \ldots) \text{ becomes } f^i R(a, b, c, \ldots),$$

where i is the degree of R.

(2) Let y remain unchanged and x become $f'y + g'x + h'$. Then, instead of R, I look to its equal

$$qt^\mu R(\alpha, \beta, \gamma, \ldots)\ (q = \pm 1);$$

that is, to

$$q\tau^{-\mu} R(\alpha, \beta, \gamma, \ldots),$$

which becomes

$$q(f' + g'\tau)^{-\mu} g'^i R(\alpha, \beta, \gamma, \ldots).$$

Since R is a reciprocant, this is equal to

$$\frac{\tau^\mu}{(f' + g'\tau)^\mu} g'^i R(a, b, c, \ldots),$$

or, replacing τ by its equivalent $\dfrac{1}{t}$,

$$(f't + g')^{-\mu} g'^i R(a, b, c, \ldots).$$

Thus we see that the proposition is true for a semi-substitution of either kind. Consider now the complete substitution made by changing y into

$$fy + gx + h$$

and x into $$Fy + Gx + H.$$

If $f = 0$ and $G = 0$, then $\frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \ldots$ become $\frac{g}{F^2} \cdot \frac{d^2x}{dy^2}, \frac{g}{F^3} \cdot \frac{d^3x}{dy^3}, \ldots$; so that $R(a, b, c, \ldots)$ becomes $\frac{g^i}{F^{2i+w}} \cdot R(\alpha, \beta, \gamma, \ldots)$; and since this is equal to

$$\frac{g^i}{F^{2i+w}} \cdot qt^{-\mu} R(a, b, c, \ldots),$$

the proposition is true.

But if either of the two letters f, G (say f) is not zero, we may combine two semi-substitutions so as to obtain the complete substitution, in which y changes into

$$fy + gx + h,$$

and x changes into $$Fy + Gx + H.$$

(1) Substitute $y_1 (= fy + gx + h)$ for y, and $x_1 (= x)$ for x.

(2) Then substitute $y_2 (= y_1)$ for y_1, and $x_2 (= f'y_1 + g'x_1 + h')$ for x_1.

By the first of these semi-substitutions

$$R(a, b, c, \ldots)$$

takes up an extraneous factor f^i. By the second it acquires the factor

$$\left(f' \frac{dy_1}{dx_1} + g'\right)^{-\mu} g'^i, \text{ where } \frac{dy_1}{dx_1} = f\frac{dy}{dx} + g = ft + g.$$

Hence we see that the extraneous factor is a negative power of a linear function of t, which we shall presently particularize, though it is not essential to the present demonstration to do so.

It only remains to show how the combination of these two semi-substitutions can be made to give the complete one in question. We have

$$y_2 = y_1 = fy + gx + h$$

and
$$\begin{aligned} x_2 &= f'y_1 + g'x_1 + h' = f'(fy + gx + h) + g'x + h' \\ &= ff'y + (f'g + g')x + (f'h + h'). \end{aligned}$$

In order that this may be equal to $Fy + Gx + H$, we must be able to satisfy the equations

$$f' = \frac{F}{f}, \quad g' = G - \frac{gF}{f}, \quad h' = H - \frac{hF}{f},$$

which is always possible, since by hypothesis f is not zero. Similarly it may be shown that when f vanishes, but G does not, by substituting

(1) $x_1 (= Fy + Gx + H)$ for x, and $y_1 (= y)$ for y,

(2) $x_2 (= x_1)$ for x_1, and $y_2 (= f''y_1 + g''x_1 + h'')$ for y_1,

we may so determine f'', g'', h'' as to get the complete substitution as before.

In every case, therefore, any linear substitution impressed upon the variables x and y will leave $R(a, b, c, \dots)$ unaltered, barring the acquisition of an extraneous factor which is a negative power of a linear function of t.

Now, the first semi-substitution introduces, as we have seen, the constant factor

$$f^i;$$

the second introduces the factor

$$\left(f'\frac{dy_1}{dx_1}+g'\right)^{-\mu} g'^i,$$

where
$$\frac{dy_1}{dx_1}=ft+g.$$

The complete extraneous factor is the product of these two, and is therefore

$$f^i g'^i (ff't+f'g+g')^{-\mu}.$$

To express f' and g' in terms of the constants of the complete substitution we have

$$f'=\frac{F}{f},\quad g'=G-\frac{gF}{f}.$$

Writing these values for f' and g' in the expression just found, we obtain

$$(fG-gF)^i (Ft+G)^{-\mu},$$

which is the extraneous factor acquired by R when the complete substitution is made. For example, if x becomes

$$Fy+Gx+H,$$

and y becomes
$$fy+gx+h,$$

the altered value of a $\left(\text{that is, of } \frac{d^2y}{dx^2}\right)$ is

$$(fG-gF)(Ft+G)^{-3}a.$$

Corresponding to the simple interchange of the variables, we have

$$F=1,\ G=0,\ H=0;\ f=0,\ g=1,\ h=0,$$

so that
$$fG-gF=-1,$$

and the altered value of a is $\frac{d^2x}{dy^2}$, or

$$\alpha=-\frac{a}{t^3},$$

which is right. In this case the general value of the acquired extraneous factor

$$(fG-gF)^i (Ft+G)^{-\mu} \text{ becomes } (-)^i t^{-\mu},$$

thus showing, what we have already proved from other considerations, that the character of a pure reciprocant is odd or even according as its degree is odd or even.

We saw in the last lecture that *every* pure reciprocant necessarily satisfied the two conditions

$$(3a_0\partial_{a_0} + 4a_1\partial_{a_1} + 5a_2\partial_{a_2} + \ldots)\, R = \mu R$$

(where μ is the characteristic), and

$$VR = 0.$$

We also saw that $VR = 0$ was a sufficient as well as necessary condition that *any homogeneous function* R of a_0, a_1, a_2, ... should be a pure reciprocant. It will now be shown that every pure reciprocant is either homogeneous and isobaric, or else resoluble into a sum of homogeneous and isobaric reciprocants. Non-homogeneous mixed ones, it may be observed, are not so resoluble, so that the theorem only holds for pure reciprocants.

(1) Let us suppose that R (a pure reciprocant) is homogeneous in $a_0, a_1, a_2 \ldots$; then it must be isobaric also. For, if i is the degree of R, Euler's theorem shows that

$$(3a_0\partial_{a_0} + 3a_1\partial_{a_1} + 3a_2\partial_{a_2} + 3a_3\partial_{a_3} + \ldots)\, R = 3iR\,;$$

and since R is a pure reciprocant, the condition

$$(3a_0\partial_{a_0} + 4a_1\partial_{a_1} + 5a_2\partial_{a_2} + 6a_3\partial_{a_3} + \ldots)\, R = \mu R$$

is necessarily satisfied. Hence

$$(a_1\partial_{a_1} + 2a_2\partial_{a_2} + 3a_3\partial_{a_3} + \ldots)\, R = (\mu - 3i)\, R = \text{a constant multiple of } R,$$

which is the distinctive property of isobaric functions.

And, *vice versâ*, if R is homogeneous and isobaric of weight w and degree i, then

$$(3a_0\partial_{a_0} + 4a_1\partial_{a_1} + 5a_2\partial_{a_2} + \ldots)\, R = (w + 3i)\, R = \mu R.$$

Thus homogeneous pure reciprocants are also isobaric and their characteristic is $3i + w$. (This property is also true for mixed reciprocants, as we have previously shown.)

(2) Suppose that R is not homogeneous, but made up of the homogeneous parts

$$R_{,},\ R_{,,},\ R_{,,,},\ \ldots.$$

Then, since $$V(R_{,} + R_{,,} + R_{,,,} + \ldots) = 0$$

is satisfied identically, it is obvious that

$$VR_{,} + VR_{,,} + VR_{,,,} + \ldots = 0$$

must also be satisfied identically.

But since all the terms are of different degrees, the only way in which this can happen is by making $VR_{,}$, $VR_{,,}$, $VR_{,,,}$, ... separately vanish. Now, $R_{,}$, $R_{,,}$, $R_{,,,}$, ... are by hypothesis *homogeneous* functions of a_0, a_1, a_2, ..., and it has just been shown that each of them is annihilated by V, which has been shown to be a sufficient condition that any homogeneous function of a_0, a_1, a_2, ... may be a pure reciprocant. Thus each part $R_{,}$, $R_{,,}$, $R_{,,,}$, ... of R is a pure reciprocant.

Also, the condition

$$(3a_0\partial_{a_0} + 4a_1\partial_{a_1} + 5a_2\partial_{a_2} + \ldots) R = \mu R$$

shows that if i_1, w_1; i_2, w_2; i_3, w_3; ... are the deg. weights of $R_{,}$, $R_{,,}$, $R_{,,,}$, ..., we must have

$$3i_1 + w_1 = \mu,\ 3i_2 + w_2 = \mu,\ 3i_3 + w_3 = \mu,\ \ldots.$$

Thus non-homogeneous pure reciprocants are severable into parts each of which is a homogeneous and isobaric pure reciprocant, the characteristic of each part being equal to the same quantity μ, which is the characteristic of the whole.

I will now explain what information concerning the number of pure reciprocants of a given type is afforded by the equation $VR = 0$. Let

$$Aa_0^{\lambda_0}a_1^{\lambda_1}a_2^{\lambda_2}\ldots a_j^{\lambda_j}$$

be a term of a homogeneous isobaric function (with its full number of terms) of a_0, a_1, a_2, ... a_j, whose degree is i, extent j, and weight w, and which we will call R.

Then in the entire function there are as many terms as there are solutions in integers of the equations

$$\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3 + \ldots + \lambda_j = i,$$

$$\lambda_1 + 2\lambda_2 + 3\lambda_3 + \ldots + j\lambda_j = w.$$

In other words, the number of terms in R is equal to the number of ways in which w can be made up of i or fewer parts, none greater than j. This number will be denoted by

$$(w;\ i, j).$$

Since the function R is the sum of every possible term of the form

$$Aa_0^{\lambda_0}a_1^{\lambda_1}\ldots a_j^{\lambda_j},$$

each multiplied by an arbitrary constant, the number of these arbitrary constants is also

$$(w;\ i, j).$$

Now, suppose R to be a reciprocant; this imposes the condition

$$VR = 0.$$

Consider the effect produced by the operation of any term of

$$V = 4\left(\frac{a_0^2}{2}\right)\partial_{a_1} + 5a_0a_1\partial_{a_2} + 6\left(a_0a_2 + \frac{a_1^2}{2}\right)\partial_{a_3} + \ldots,$$

say $\left(a_0a_2 + \frac{a_1^2}{2}\right)\partial_{a_3}$ (rejecting the numerical coefficient 6).

Operating on R with ∂_{a_3} decreases its weight by 3 and its degree by 1 unit. The subsequent multiplication by $a_0a_2 + \frac{a_1^2}{2}$, on the other hand, increases the weight by 2 and the degree by 2 units. Hence the total effect of $\left(a_0a_2 + \frac{a_1^2}{2}\right)\partial_{a_3}$ is to increase the degree by 1 and to diminish the weight by 1 unit. The same is evidently true for any other term of V. Thus the total effect of V operating on the general homogeneous isobaric function R of weight w, degree i, extent j, is to change it into another homogeneous isobaric function whose weight, degree and extent are respectively $w-1$, $i+1, j$. Observe that the extent is not altered by the operation of V.

It is easily seen that the coefficients of VR are linear functions of the coefficients of R; for example, if

$$R = Aa_0^2a_3 + Ba_0a_1a_2 + Ca_1^3,$$

$$VR = a_0^3a_2(6A + 2B) + a_0^2a_1^2(3A + 5B + 6C).$$

Hence the condition $VR = 0$ gives us $(w-1;\ i+1, j)$ *linear* equations between the $(w;\ i, j)$ coefficients of R; so that, *assuming that these equations of condition are all independent*, after they have been satisfied the number of arbitrary constants remaining in R (that is, the number of linearly independent reciprocants of the type $w;\ i, j$) is equal to

$$(w;\ i, j) - (w-1;\ i+1, j),$$

when this difference is positive; but when it is zero or negative there are no reciprocants of the given type.

If, however, any r of the $(w-1;\ i+1, j)$ equations of condition should not be independent of the rest, these equations would be equivalent to $(w-1;\ i+1, j) - r$ *independent* conditions, and therefore the number of linearly independent reciprocants of the type $w;\ i, j$ would be

$$(w;\ i, j) - (w-1;\ i+1, j) + r.$$

It is therefore certain that this number *cannot be less* than

$$(w;\ i, j) - (w-1;\ i+1, j).$$

We shall assume provisionally that $r = 0$, or in other words that the above partition formula is exact, instead of merely giving an inferior limit. Though it would be unsafe to rely on its accuracy, no positive grounds for doubting its exactitude have been revealed by calculation.

Such attempts as I have hitherto made to demonstrate the theorem have proved infructuous, but it must be remembered that more than a quarter of a century elapsed between the promulgation of Cayley's analogous theorem and its final establishment by myself on a secure basis of demonstration.

LECTURE X.

I will commence this lecture with a proof of Capt. MacMahon's theorem that if R is any pure reciprocant and μ its characteristic (that is, its weight added to three times its degree),

$$\left(V^m \frac{d^m}{dx^m}\right) R = 1 \cdot 2 \cdot 3 \ldots m \{\mu(\mu+2)(\mu+4)\ldots(\mu+2m-2)\}(y'')^m R,$$

where y'' may be replaced by either $2a_0$ or a, according as the modified or unmodified system of letters is employed.

Instead of a pure reciprocant, let us consider any homogeneous isobaric function F of degree i and weight w; and $\left(\text{for the sake of simplicity writing } \partial_x \text{ for } \frac{d}{dx}\right)$ instead of the operator $V^m \partial_x{}^m$ let us consider $V^m \partial_x{}^n - \partial_x{}^n V^m$. We have identically

$$\begin{aligned}(V^m\partial_x{}^n - \partial_x{}^n V^m) F = {} & V^{m-1}(V\partial_x - \partial_x V)\partial_x{}^{n-1}F \\ & + V^{m-2}(V\partial_x - \partial_x V)\, V\partial_x{}^{n-1}F \\ & + V^{m-3}(V\partial_x - \partial_x V)\, V^2\partial_x{}^{n-1}F \\ & + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ & + V(V\partial_x - \partial_x V)\, V^{m-2}\partial_x{}^{n-1}F \\ & + \quad (V\partial_x - \partial_x V)\, V^{m-1}\partial_x{}^{n-1}F \\ & + \partial_x (V^m\partial_x{}^{n-1} - \partial_x{}^{n-1}V^m) F.\end{aligned}$$

Now, the operation of $(V\partial_x - \partial_x V)$ on any homogeneous isobaric function whose characteristic is μ_1 is equivalent, as we have seen in Lecture VII, to multiplication by $\mu_1 y''$; so that if the characteristics of

$$\partial_x{}^{n-1}F, \quad V\partial_x{}^{n-1}F, \quad V^2\partial_x{}^{n-1}F, \ \ldots \ V^{m-1}\partial_x{}^{n-1}F$$

are $\qquad \mu_1, \quad \mu_2, \quad \mu_3, \ \ldots \ \mu_m,$

it follows that

$$(V^m\partial_x{}^n - \partial_x{}^n V^m) = (\mu_1 + \mu_2 + \mu_3 + \ldots + \mu_m)\, y'' V^{m-1}\partial_x{}^{n-1}F + \partial_x (V^m \partial_x{}^{n-1} - \partial_x{}^{n-1} V^m) F.$$

Observe that

$$V^{m-1}(V\partial_x - \partial_x V)\,\partial_x{}^{n-1}F = V^{m-1}\mu_1 y''\partial_x{}^{n-1}F = \mu_1 y'' V^{m-1}\partial_x{}^{n-1}F,$$

where the transposition of the y'' is permissible because V does not act on it; but if y'' were preceded by ∂_x it could not be similarly transposed.

The numbers $\mu_1, \mu_2, \mu_3, \ldots$ form an arithmetical progression, for each operation of V increases the degree by unity and diminishes the weight by unity, so that

$$\mu_1 = 3i_1 + w_1 \ \text{ becomes } \ \mu_2 = 3(i_1+1) + (w_1 - 1) = \mu_1 + 2.$$

Similarly $\quad \mu_3 = \mu_1 + 4, \quad \mu_4 = \mu_1 + 6, \ldots \ \mu_m = \mu_1 + 2m - 2.$

The characteristic of F being

$$\mu = 3i + w, \text{ that of } \partial_x^{n-1}F \text{ is } \mu_1 = \mu + n - 1;$$

for each operation of ∂_x leaves the degree unaltered, but adds an unit to the weight; hence

$$\mu_1 + \mu_2 + \mu_3 + \ldots + \mu_m = m(\mu + m + n - 2);$$

so that

$$(V^m\partial_x^n - \partial_x^n V^m)F = m(\mu + m + n - 2)y'' V^{m-1}\partial_x^{n-1}F + \partial_x(V^m\partial_x^{n-1} - \partial_x^{n-1}V^m)F. \quad (1)$$

When $F = R$, a pure reciprocant, so that $VR = 0$, our formula becomes

$$V^m\partial_x^n R = m(\mu + m + n - 2)y'' V^{m-1}\partial_x^{n-1}R + \partial_x V^m\partial_x^{n-1}R. \quad (2)$$

Suppose that in (2) $m > n$, then $V^m\partial_x^n R = 0$. This is obviously true when $n = 0$, and when $n = 1$. When $n = 2$ we find

$$V^m\partial_x^2 R = m(\mu + m)y'' V^{m-1}\partial_x R + \partial_x V^m \partial_x R$$
$$= 0 \text{ if } m > 2.$$

Similarly the case $n = 3$, $m > 3$ can be made to depend on $n = 2$, $m > 2$, and in general each case depends on the one immediately preceding it. Next let $n = m$ in (2); then, remembering that $V^m\partial_x^{m-1}R = 0$, we have

$$V^m\partial_x^m R = m(\mu + 2m - 2)y'' V^{m-1}\partial_x^{m-1}R,$$

from which MacMahon's theorem that

$$V^m\partial_x^m R = 1 \cdot 2 \cdot 3 \ldots m\{\mu(\mu+2)(\mu+4)\ldots(\mu+2m-2)\}(y'')^m R$$

is an immediate consequence.

Another special case of Formula (1) is worthy of notice, namely, that in which we take $n = 1$, when we obtain the simple formula

$$(V^m\partial_x - \partial_x V^m)F = m(\mu + m - 1)y'' V^{m-1}F. \quad (3)$$

If in this we write a_n in the place of F, and (the modified system of letters being used) $2a_0$ for y'', μ becomes $3 + n$, and we have

$$(V^m\partial_x - \partial_x V^m)a_n = 2m(m + n + 2)a_0 V^{m-1}a_n,$$

or, as it may also be written,

$$\frac{V^m\partial_x a_n}{1 \cdot 2 \cdot 3 \ldots m} = \frac{\partial_x V^m a_n}{1 \cdot 2 \cdot 3 \ldots m} + \frac{2(m + n + 2)a_0 V^{m-1}a_n}{1 \cdot 2 \cdot 3 \ldots (m-1)}. \quad (4)$$

Mr Hammond remarks that this last formula may be used to prove the theorem

$$\alpha_n = -t^{-n-3}(e^{-\frac{V}{t}})a_n,$$

which was given without proof in Lecture II. Assuming that

$$\alpha_n = -t^{-n-3}a_n + t^{-n-4}Va_n - t^{-n-5}\frac{V^2a_n}{1 \cdot 2} + \ldots,$$

we have to prove that the theorem is also true when n is increased by unity. Differentiating both sides of the assumed identity with respect to x, we find

$$\partial_x \alpha_n = \partial_x \left(- t^{-n-3} a_n + t^{-n-4} V a_n - t^{-n-5} \frac{V^2 a_n}{1.2} + \dots \right)$$
$$= - t^{-n-3} \partial_x a_n + t^{-n-4} \{ \partial_x V a_n + 2(n+3) a_0 a_n \}$$
$$- t^{-n-5} \left\{ \frac{\partial_x V^2 a_n}{1.2} + 2(n+4) a_0 V a_n \right\}$$
$$+ \dots\dots\dots\dots ;$$

the general term being

$$(-)^{m+1} t^{-n-m-3} \left\{ \frac{\partial_x V^m a_n}{1.2.3\dots m} + \frac{2(m+n+2) a_0 V^{m-1} a_n}{1.2.3\dots(m-1)} \right\}$$

which, by means of (4), reduces to

$$(-)^{m+1} t^{-n-m-3} \frac{V^m \partial_x a_n}{1.2.3\dots m}.$$

Hence $$\partial_x \alpha_n = - t^{-n-3} \partial_x a_n + t^{-n-4} V \partial_x a_n - t^{-n-5} \frac{V^2 \partial_x a_n}{1.2} + \dots,$$

or, more concisely,

$$\partial_x \alpha_n = - t^{-n-3} (e^{-\frac{V}{t}}) \partial_x a_n.$$

But $\partial_x a_n = (n+3) a_{n+1}$, and $\partial_x \alpha_n = t \partial_y \alpha_n = (n+3) t \alpha_{n+1}$,

and therefore

$$(n+3) t \alpha_{n+1} = - (n+3) t^{-n-3} (e^{-\frac{V}{t}}) a_{n+1},$$

or

$$\alpha_{n+1} = - t^{-n-4} (e^{-\frac{V}{t}}) a_{n+1}.$$

The theorem is easily seen to be true, for $n = 0, 1, 2$, and is thus proved to be true universally.

I will now return to the point at which I left off in my previous lecture. We saw that the exactitude of the formula

$$(w; i, j) - (w-1; i+1, j)$$

for the number of pure reciprocants of the type $w; i, j$ could not be inferred with certainty unless we were able to prove that the $(w-1; i+1, j)$ linear equations between the coefficients of R, found by equating VR to zero, were all of them independent. A similar difficulty presents itself in the proof of the corresponding formula $(w; i, j) - (w-1; i, j)$ in the invariantive theory; but in that case I succeeded in making out a proof of the independence of the equations of condition founded on the fact that $\Omega^m O^m I$ is a numerical multiple of I, where I is any invariant, and Ω, O are the well-known operators

$$a_0 \partial_{a_1} + 2a_1 \partial_{a_2} + 3a_2 \partial_{a_3} + \dots + j a_{j-1} \partial_{a_j}$$
$$a_j \partial_{a_{j-1}} + 2a_{j-1} \partial_{a_{j-2}} + 3a_{j-2} \partial_{a_{j-3}} + \dots + j a_1 \partial_{a_0}.$$

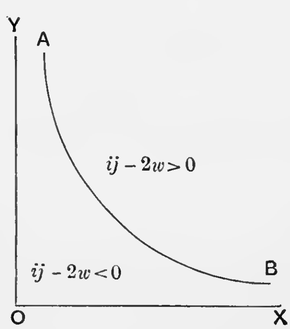
I have since discovered a second proof of the theorem for invariants which, though very interesting, is less simple than my first; but neither of these methods can be extended to the case of reciprocants.

It was suggested by Capt. MacMahon that the fact that $V^m \partial_x{}^m R$ is a numerical multiple of $a^m R$ ought to lead to a proof of the theorem for reciprocants similar to that obtained for invariants by my first method, alluded to above, but this I find is not the case; and indeed it is capable of being shown *a priori* that it cannot lead to a proof. One great distinction between the two theories, which is fatal to the success of the proposed method, is well worthy of notice.

If
$$(w; i, j) - (w-1; i, j) = > 0$$
(I shall sometimes call this positive), then
$$(w'; i, j) - (w'-1; i, j) = > 0$$
for all values of w' less than w; the condition that this difference, say $\Delta(w; i, j)$ shall be positive being simply that $ij - 2w$ is positive (that is, $ij - 2w = > 0$). This is not the case with the difference
$$(w; i, j) - (w-1; i+1, j),$$
say $E(w; i, j)$; it by no means follows that if this is positive for a given value of w (i, j being kept constant), it will be so for any inferior value of w.

We may illustrate geometrically the condition $ij - 2w = > 0$, which holds when $\Delta(w; i, j)$ is non-negative.

Let (i, j) be co-ordinates of a point in a plane and draw the positive branch of the rectangular hyperbola
$$ij - 2w = 0.$$



Then, $ij - 2w < 0$ for all points in the area $YOXBA$ between the curve and its asymptotes; but for points on the curve AB,
$$ij - 2w = 0,$$
and for all points of the infinite area on the side of AB remote from the origin,
$$ij - 2w > 0.$$

Thus, for all points which lie either on or beyond the curve AB, $\Delta(w; i, j)$ is non-negative, and for all points between the curve and the asymptotes $\Delta(w; i, j)$ is non-positive.

We have here considered w as constant and i, j as variable, but in the case where all three are variable we should have to consider the hyperbolic paraboloid
$$ij - 2w = 0,$$
of which the curve AB is a section, by the plane $w = \text{const.}$; and the condition

of $\Delta(w; i, j)$ being non-negative or non-positive depends on the variable point (i, j, w) lying in the one case on or beyond the surface, and in the other between the surface and the planes of reference.

The function of i, j, w, whose positive or negative sign determines in like manner that of $E(w; i, j)$, cannot be linear in w. What its form is, or whether it is an Algebraical or Transcendental function, no one at present can say. Indeed, except for the light shed on the subject by the Algebraical Theory of Invariants, it would have been exceedingly difficult (as I know from vain efforts made by myself and others in Baltimore) to prove the much simpler theorem that $\Delta(w; i, j)$ is positive (that is, non-negative) when $ij - 2w$ is so. It amounts to the assertion that the coefficient of $a^i x^w$ in the expansion of

$$\frac{1-x}{(1-a)(1-ax)(1-ax^2)\dots(1-ax^j)}$$

is always non-negative, provided that $ij - 2w$ is non-negative.

This is a theorem of great importance in the ordinary Theory of Invariants, and may be seen to be a consequence of the fact, which I have proved, that (using $[w; i, j]$ to denote a function of the type $w; i, j$ having its full number of arbitrary coefficients) there are no linear connections between the coefficients of $\Omega\,[w; i, j]$ when $ij - 2w = > 0$; but no one, as far as I know, has ever found a *direct* proof of it.

Viewing the connection between the two theories of Invariants and Reciprocants, I think it desirable to recapitulate with some improvements the proof, given in the *Phil. Mag.* for March, 1878, of the theorem that the number of linearly independent invariants of the type $w; i, j$ is exactly $\Delta(w; i, j)$ when this quantity is positive, and exactly zero when it is 0 or negative.

As regards reciprocants, at present we can only say that the number of linearly independent ones of the type $w; i, j$ is never less than $E(w; i, j)$, leaving to some gifted member of the class to prove or disprove that the first is always exactly equal to the second. The *exact* theorem to be proved in the theory of invariants is as follows:

If $ij - 2w = > 0$, the number of linearly independent invariants of the type $w; i, j$ is $\Delta(w; i, j)$.

If $ij - 2w < 0$, the number of such invariants is zero; that is, there are none. The proof is made to depend on the properties of

$$\Omega = a_0\partial_{a_1} + 2a_1\partial_{a_2} + 3a_2\partial_{a_3} + \dots + ja_{j-1}\partial_{a_j}$$

and of $$O = a_j\partial_{a_{j-1}} + 2a_{j-1}\partial_{a_{j-2}} + 3a_{j-2}\partial_{a_{j-3}} + \dots + ja_1\partial_{a_0}.$$

If U be any homogeneous isobaric function of degree i and weight w in the letters $a_0, a_1, a_2, \dots a_j$, it is easy to prove that

$$(\Omega O - O\Omega)\,U = (ij - 2w)\,U,$$

and consequently, if U is an invariant I, so that $\Omega I = 0$,

$$\Omega O I = (ij - 2w) I.$$

I call $ij - 2w$ the *excess* and denote it by η, and shall first show that if η is negative $I = 0$; that is, there exists no invariant with a negative excess. This will prove that when $\Delta(w; i, j)$ is negative, that is, when

$$(w-1; i, j) > (w; i, j),$$

the number of independent functions of the coefficients of $[w; i, j]$ which appear in $\Omega[w; i, j]$ is exactly equal to $(w; i, j)$, which is the number of the coefficients themselves. Clearly it cannot be greater; for, no matter what the number of linear functions of n quantities may be, only n at the utmost can be independent; there might be fewer, there cannot possibly be more. The complete theorem is that the number of independent coefficients in $\Omega[w; i, j]$ is the *subdominant* of two numbers: one the number of terms of the type $w; i, j$, the other the number of terms of the type $w-1; i, j$.

N.B. That one of two numbers which is not greater than the other is called the subdominant.

LECTURE XI.

We may write for the Annihilator of an Invariant

$$\Omega = a_0 \dot{a}_1 + 2a_1 \dot{a}_2 + 3a_2 \dot{a}_3 + \ldots + j a_{j-1} \dot{a}_j$$

and for its opposite

$$O = j a_1 \dot{a}_0 + (j-1) a_2 \dot{a}_1 + (j-2) a_3 \dot{a}_2 + \ldots + a_j \dot{a}_{j-1},$$

where the pointed letters $\dot{a}_0, \dot{a}_1, \dot{a}_2, \ldots \dot{a}_j$ stand for the partial differential operators

$$\partial_{a_0}, \partial_{a_1}, \partial_{a_2}, \ldots \partial_{a_j}.$$

Suppose Ω and O to operate on any function $U(a_0, a_1, a_2, \ldots a_j)$; then

$$\Omega O U = (\Omega . O + \Omega * O) U$$

and

$$O \Omega U = (O . \Omega + O * \Omega) U,$$

where the full stop between O and Ω signifies multiplication, and the asterisk operation on the unpointed letters only. Thus,

$$\Omega . O = O . \Omega,$$

and, consequently, $(\Omega O - O \Omega) U = (\Omega * O - O * \Omega) U.$

Now,

$$\Omega * O U = \{1 . j a_0 \dot{a}_0 + 2(j-1) a_1 \dot{a}_1 + 3(j-2) a_2 \dot{a}_2 + \ldots + j . 1 a_{j-1} \dot{a}_{j-1}\} U,$$

and

$$O * \Omega U = \{1 . j a_1 \dot{a}_1 + 2(j-1) a_2 \dot{a}_2 + \ldots + (j-1) 2 a_{j-1} \dot{a}_{j-1} + j . 1 a_j \dot{a}_j\} U,$$

whence we readily obtain

$$(\Omega O - O\Omega) U = j(a_0\dot{a}_0 + a_1\dot{a}_1 + a_2\dot{a}_2 + \ldots + a_j\dot{a}_j) U$$
$$- 2(a_1\dot{a}_1 + 2a_2\dot{a}_2 + 3a_3\dot{a}_3 + \ldots + ja_j\dot{a}_j) U.$$

Introducing the conditions of homogeneity and isobarism, namely,

$$(a_0\dot{a}_0 + a_1\dot{a}_1 + a_2\dot{a}_2 + \ldots + a_j\dot{a}_j) U = iU$$

and $$(a_1\dot{a}_1 + 2a_2\dot{a}_2 + 3a_3\dot{a}_3 + \ldots + ja_j\dot{a}_j) U = wU,$$

where i and w denote the degree and weight of U, supposed now to be a rational integral homogeneous and isobaric function (or, to avoid a tedious periphrasis, say a *gradient*), we see that if the complete type of the gradient U is $w; i, j$,

$$(\Omega O - O\Omega) U = (ij - 2w) U = \eta U,$$

where η is the excess.

Since the operation of O increases the weight of the operand by unity, but does not alter either its degree or its extent, it is clear that the type of $O^\theta U$ is $w + \theta; i, j$. The excess of $O^\theta U$ is therefore

$$ij - 2(w + \theta) = \eta - 2\theta,$$

and the theorem just proved shows that

$$(\Omega O - O\Omega) O^\theta U = (\eta - 2\theta) O^\theta U.$$

From this we pass on to prove that $\Omega O^q - O^q\Omega$, acting on any gradient as its objective, is equivalent to $q(\eta - q + 1) O^{q-1}$; that is, when q is any positive integer, we shall show that

$$(\Omega O^q - O^q\Omega) U = q(\eta - q + 1) O^{q-1} U.$$

The subsequent consideration of a special case of this formula, in which U is replaced by any invariant I, will enable us to prove that there can be no invariants for which the excess $ij - 2w$ is negative. Let

$$O^{q-\theta}\Omega O^\theta U = P_\theta U;$$

then $$O^{q-\theta-1}\Omega O^{\theta+1} U = P_{\theta+1} U,$$

and therefore $$(P_{\theta+1} - P_\theta) U = O^{q-\theta-1}(\Omega O - O\Omega) O^\theta U.$$

Substituting in this for

$$(\Omega O - O\Omega) O^\theta U \text{ its value } (\eta - 2\theta) O^\theta U,$$

we have $$(P_{\theta+1} - P_\theta) U = (\eta - 2\theta) O^{q-1} U.$$

Hence

$$(P_q - P_0) U = \{(P_1 - P_0) + (P_2 - P_1) + (P_3 - P_2) + \ldots + (P_q - P_{q-1})\} U$$
$$= \{\eta + (\eta - 2) + (\eta - 4) + \ldots + (\eta - 2q + 2)\} O^{q-1} U$$
$$= q(\eta - q + 1) O^{q-1} U.$$

But since $P_q = \Omega O^q$ and $P_0 = O^q\Omega$, this result may be written

$$(\Omega O^q - O^q\Omega) U = q(\eta - q + 1) O^{q-1} U.$$

If now $U = I$, an invariant, we have $\Omega U = 0$, and our formula becomes

$$\Omega O^q I = q(\eta - q + 1) O^{q-1} I.$$

Writing in succession $q = m, m-1, \ldots 1$, we obtain

$$m(\eta - m + 1) O^{m-1} I = \Omega O^m I$$
$$(m-1)(\eta - m + 2) O^{m-2} \;\; = \Omega O^{m-1} I$$
$$(m-2)(\eta - m + 3) O^{m-3} I = \Omega O^{m-2} I$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$1 \, . \, \eta I = \Omega O I.$$

By assigning to m a sufficiently large value we are able to make $O^m I$ vanish as well as ΩI; for, the type of I being w; i, j, that of $O^m I$ is $w + m$; i, j. But it is evident that no *gradient* can have a greater weight than ij, the product of its degree and extent, for each term is a product of i letters none of them having a weight greater than j. If, then, we suppose that $m = ij - w + 1$, the weight of $O^m I$ is

$$w + m = ij + 1.$$

Therefore $$O^m I = 0.$$

Again, $$\eta - m + 1 = ij - 2w - (ij - w + 1) + 1 = -w.$$

If, then, η is negative, every term in the series

$$m(\eta - m + 1), \; (m-1)(\eta - m + 2), \ldots 2(\eta - 1), \; 1 \, . \, \eta$$

is negative and can never vanish. Hence we have successively

$$O^{m-1} I = 0, \; O^{m-2} I = 0, \ldots I = 0;$$

that is, when $ij - 2w < 0$ no invariant of the type w; i, j exists.

Observe that the elenchus of the demonstration consists in the fact that the successive numerical factors $\eta - m + 1$, $\eta - m + 2$, $\eta - m + 3, \ldots \eta$ are all non-zero on account of η being negative; but if η were positive we should eventually come to a factor $\eta - \mu$ which would be zero, and we could not conclude from $(\mu + 1)(\eta - \mu) O^\mu I$ being zero that $O^\mu I = 0$. Since $\eta - (m-1)$ passes from $\eta - (ij - w)$ to η, that is, from $-w$ to η, it passes through zero when η is positive.

The second part of Cayley's completed theorem remains to be proved, namely, that when $ij - 2w => 0$, the number of linearly independent invariants of the type w; i, j is precisely equal to $\Delta(w; i, j)$; that is, to

$$(w; i, j) - (w - 1; i, j).$$

I show this by proving that if $D(w; i, j)$ is the number in question, keeping i and j constant and taking $w <= \frac{ij}{2}$,

$$D(w; i, j) + D(w-1; i, j) + D(w-2; i, j) + \ldots + D(0; i, j)$$

cannot be greater than

$$\Delta(w; i, j) + \Delta(w-1; i, j) + \Delta(w-2; i, j) + \ldots + \Delta(0; i, j),$$

and consequently, since we know that no single $D(w; i, j)$ can possibly be less than the corresponding $\Delta(w; i, j)$, it follows that

$$D(w; i, j) + D(w-1; i, j) + D(w-2; i, j) + \ldots + D(0; i, j)$$
$$= \Delta(w; i, j) + \Delta(w-1; i, j) + \Delta(w-2; i, j) + \ldots + \Delta(0; i, j);$$

and, furthermore, that each

$$D(w; i, j) = \Delta(w; i, j).$$

For if any D were greater than its corresponding Δ, some other D would have to be less, which is impossible.

This principle of reasoning may be illustrated by imagining a row of ballot-boxes and supposing it to be ascertained that no single box contains fewer white balls than black ones. If, then, there are not more white than black balls altogether, the total number of whites must be the same as that of the blacks. And since there are just as many whites as blacks distributed among the ballot-boxes, the number of white and black balls must be the same in each box; for otherwise some box must contain fewer whites than blacks, which is contrary to the hypothesis.

Observe that the sum of these Δ's is $(w; i, j)$; for

$$(w; i, j) - (w-1; i, j) + (w-1; i, j) - (w-2; i, j) + \ldots + (0; i, j) - (-1; i, j)$$
$$= (w; i, j) - (-1; i, j)$$

and
$$(-1; i, j) = 0,$$

since there is no way of composing -1 with parts 0, 1, 2, ... j. Hence what I have to show is that

$$D(w; i, j) + D(w-1; i, j) + \ldots + D(1; i, j) + D(0; i, j) = (w; i, j).$$

I want preliminarily to express $\Omega^q O^q I$ as a multiple of I*.

This can be done by a formula previously demonstrated, namely,

$$\Omega O^q I = q(\eta - q + 1) O^{q-1} I,$$

which gives

$$\Omega^2 O^q I = q(\eta - q + 1)\Omega O^{q-1} I = q(\eta - q + 1)(q-1)(\eta - q + 2) O^{q-2} I;$$

similarly

$$\Omega^3 O^q I = q(\eta - q + 1)(q-1)(\eta - q + 2)(q-2)(\eta - q + 3) O^{q-3} I;$$

and finally, changing the order of the numerical factors,

$$\Omega^q O^q I = 1 . 2 . 3 \ldots q \{\eta(\eta - 1)(\eta - 2) \ldots (\eta - q + 1)\} I.$$

This shows that $\Omega^q O^q I$ and *à fortiori* $O^q I$ can never vanish unless $\eta - q + 1$ becomes negative.

* The result of operating on I with O and Ω each q times, the two operations following each other according to any law of distribution whatever, will always be a numerical multiple of I; but the value of this multiple will differ for different laws of distribution.

Suppose now that I^q means an invariant of the type $w - q; i, j$; its excess is $ij - 2(w-q)$, and consequently $O^q I_q$ cannot vanish unless

$$ij - 2(w-q) - q + 1$$

becomes negative, which is impossible. For

$$ij - 2(w-q) - q + 1 = ij - 2w + q + 1,$$

and $ij - 2w = > 0$ by hypothesis.

By taking $O^q I_q$ as an *image*, so to say, of I_q we shall be able to obtain a limit to the number of I_q's by obtaining a limit to the number of their images. In fact, taking the *image* $O^q I_q$ of each of the $D(w-q; i, j)$ linearly independent invariants of the type $w-q; i, j$ (this is what is meant by the I_q's) and giving q all possible values from 0 to w inclusive, the total number of these images is obviously

$$D(w; i, j) + D(w-1; i, j) + \ldots + D(0; i, j).$$

Each of them will be a gradient of the weight $w - q + q$ (that is, of weight w), and will consist of terms of weight w, degree i, and extent j. The total number of such terms will be the number of ways of making up w with i of the numbers 0, 1, 2, 3, ... j, or with the usual notation $(w; i, j)$. If, then, it can be shown that none of these forms are linearly connected, then, inasmuch as they are all functions of the same $(w; i, j)$ arguments, it will follow that their total number cannot exceed $(w; i, j)$. That is, we shall have shown that

$$D(w; i, j) + D(w-1; i, j) + D(w-2; i, j) + \ldots + D(0; i, j)$$

cannot exceed

$$\Delta(w; i, j) + \Delta(w-1; i, j) + \Delta(w-2; i, j) + \ldots + \Delta(0; i, j),$$

and by the ballot-box principle, as already stated (inasmuch as no D is less than its corresponding Δ), it will follow that each D is the same as the corresponding Δ, and the theorem to be proved is established.

The proof of this independence is easy. For (1) suppose that there is any linear relation between the forms

$$O^q I_q,\ O^q I_q',\ O^q I_q'',\ \ldots,$$

for each of which the value of q is the same. Denoting these forms by

$$P_q,\ P_q',\ P_q'',$$

let the relation in question be

$$\lambda P_q + \lambda' P_q' + \lambda'' P_q'' + \ldots = 0.$$

Then $$\lambda \Omega^q P_q + \lambda' \Omega^q P_q' + \lambda'' \Omega^q P_q'' + \ldots = 0.$$

But each argument $\Omega^q P_q$ is of the form $\Omega^q O^q I_q$, and since this is equal to I_q multiplied by a number which does not vanish*, we have a linear relation between I_q, I_q', I_q'', ..., namely

$$\lambda I_q + \lambda' I_q' + \lambda'' I_q'' + \ldots = 0;$$

that is, the I_q's would not be linearly independent, contrary to hypothesis. Thus the *images* ($O^q I_q$, $O^q I_q'$, $O^q I_q''$ ...) belonging to invariants of the same type $w - q$; i, j cannot be linearly connected.

(2) I say that the images of invariants of different types cannot be linearly connected. For let q, q', q'', ... arranged in descending order of magnitude, be the different values of q in the images supposed to be linearly related. The result of operating with Ω^q on any image of the form $O^{q'} I_{q'}$ is to bring it to the form $\Omega^{q-q'} \Omega^{q'} O^{q'} I_{q'}$, which is a multiple of $\Omega^{q-q'} I_{q'}$, and therefore vanishes. But Ω^q, acting on any of the images $O^q I_q$, $O^q I_q'$, ..., will, as we have seen, bring back the multiple of I_q; thus the operation of Ω^q on the supposed relation will give a linear equation connecting $I_q, I_q', I_q'', \ldots$, and for the same reason as before this is impossible. Hence there can be no linear relation whatever between the images of the invariants whose types extend from w; i, j to 0; i, j, and the number of these images will accordingly be not greater than $(w; i, j)$, as was to be proved.

It is well worthy of notice that $D(w; i, j)$ may be zero, but obviously cannot be negative, as it denotes a number of things which may have any value from zero upwards. Hence follows a remarkable theorem in the pure theory of partitions which it would be extremely difficult to prove from first principles, namely, that the difference between the two partition numbers

$$(w; i, j) - (w-1; i, j)$$

can never be negative when $ij - 2w = > 0$. It may be zero, but cannot be less than zero. This explains what I said about the hyperbolic paraboloid $ij - 2w = 0$, where i, j, w are treated as co-ordinates of a point in space. We might call the value of $(w; i, j) - (w-1; i, j)$ the density of any point i, j, w, and the theorem may then be expressed by saying that at points within or upon the hyperbolic paraboloid the density can never be negative; for points outside this surface it can never be positive.

As regards the analogous formula in the Theory of Reciprocants

$$(w; i, j) - (w-1; i+1, j),$$

we do not know that any algebraical surface can be constructed which will enable us to discriminate between the cases in which this difference, say $E(w; i, j)$, is positive or negative. Should such a surface exist, its equation must contain w in a higher degree than the first. Supposing that the above

* In fact, remembering that the excess of the type $w-q$; i, j is $ij - 2(w-q) = \eta + 2q$, we find

$$\Omega^q O^q I_q = 1 \,.\, 2 \,.\, 3 \ldots q \{(\eta + 2q)(\eta + 2q - 1) \ldots (\eta + q + 1)\} I_q,$$

in which both η and q are *positive* integers.

formula represents the actual number of reciprocants, it will follow (and this is confirmed by experience) that there can be no reciprocants to a type of negative excess. For

$$\begin{aligned}&(w; i, j) - (w-1; i+1, j)\\ &= (w; i, j) - (w-1; i, j) - [(w-1; i+1, j) - (w-1; i, j)]\\ &= (w; i, j) - (w-1; i, j) - (w-i-2; i+1, j-1).\end{aligned}$$

But if $ij - 2w$ is negative, $(w; i, j) - (w-1; i, j)$ is zero or negative. Hence $(w; i, j) - (w-1; i+1, j)$ is non-positive.

For *satisfied* invariants (those ordinarily so called) $w = \frac{ij}{2}$, and the formula for their number becomes $\left(\frac{ij}{2}; i, j\right) - \left(\frac{ij}{2} - 1; i, j\right)$.

As these form a well-defined class apart, it would have seemed very natural to begin with them in endeavouring to establish the theorem, reserving the theory of unsatisfied invariants (sources of covariants) for future consideration. But to all appearance it would have been very difficult, if not impossible, to have succeeded in dealing with them alone.

This is another example of the law in Heuristic that the whole is easier of deglutition than its part.

LECTURE XII.

Before proceeding further with the development of the pure analytical theory of reciprocants, it may be useful to point out some instances of its relations and applications to geometrical questions.

Using $y_1, y_2, y_3, \ldots y_n$ to denote the successive derivatives of y with respect to x*, let the complete primitive of the differential equation

$$F(x, y, y_1, y_2, \ldots y_n) = 0$$

be
$$\phi(x, y, \lambda, \mu, \nu, \ldots) = 0.$$

We can in general so determine the n constants $\lambda, \mu, \nu, \ldots$ that the curve ϕ may pass through n given points, and if we take these to be consecutive points on the curve

$$\Phi(x, y) = 0,$$

ϕ and Φ will have a contact of the $(n-1)$th order at a given point of Φ. In order that the curves may have a contact of the nth order at a point

* In future $y_1, y_2, y_3, \ldots y_n$ will always have this meaning, the derivatives of x with respect to y will be denoted by $x_1, x_2, x_3, \ldots$, and whenever the letters $t, a, b, c, \ldots$ are used they will stand for $y_1, \frac{y_2}{1.2}, \frac{y_3}{1.2.3}, \frac{y_4}{1.2.3.4}, \ldots$ respectively.

whose abscissa is x, the ordinates of Φ and ϕ at that point and their 1st, 2nd, ... nth derivatives with respect to x must be the same for both curves. But at every point of ϕ its differential equation

$$F(x, y, y_1, y_2, \ldots y_n) = 0$$

has to be satisfied, and therefore the $x, y, y_1, y_2, \ldots y_n$ of any point on Φ, at which contact of the nth order with ϕ is possible, must also satisfy the same equation.

Now, suppose that for x and y we substitute given functions of them, X and Y; the curves ϕ and Φ become

$$\phi(X, Y, \lambda, \mu, \nu, \ldots) = 0 \text{ and } \Phi(X, Y) = 0.$$

Contact of the nth order with the transformed ϕ will therefore be possible at any point of the transformed Φ for which

$$F(X, Y, Y_1, Y_2, \ldots Y_n) = 0,$$

where $Y_1, Y_2, Y_3, \ldots Y_n$ are the derivatives of Y with respect to X.

But, unless the function F and the substitutions $X = f_1(x, y)$, $Y = f_2(x, y)$ are so related that the transformed differential equation

$$F(X, Y, Y_1, Y_2, \ldots Y_n) = 0$$

is identical with the untransformed one, the property marked by the contact of the transformed curves will not be identical with that marked by the contact of the untransformed ones.

For example, let $F = y_2$; then the relation between $\phi \equiv y + \lambda x + \mu = 0$ (the complete primitive of $y_2 = 0$) and an arbitrary curve Φ is that the constants λ and μ may be so chosen that the line $y + \lambda x + \mu = 0$ may have a contact of the second order at any point of Φ for which $y_2 = 0$; and the property marked is an inflexion on Φ. But if we make the substitution $X = x^2$, $Y = y^2$, so that the differential equation $y_2 = 0$ is transformed into $\left(\frac{d}{dx^2}\right)^2 y^2 = 0$ and its complete primitive into $y^2 + \lambda x^2 + \mu = 0$, it will still be possible so to choose λ and μ that $y^2 + \lambda x^2 + \mu = 0$ may have a contact of the second order at any point of an arbitrary curve for which $\left(\frac{d}{dx^2}\right)^2 y^2 = 0$, but the property marked, instead of being an inflexion, will be a *contact of the second order with a conic having a pair of conjugate diameters coincident with the co-ordinate axes.*

The property remains unaltered when the co-ordinate axes are interchanged, and therefore the differential equation $\left(\frac{d}{dx^2}\right)^2 y^2 = 0$ will be identical with $\left(\frac{d}{dy^2}\right)^2 x^2 = 0$, in which the variables x and y have changed places. The identity of the two differential equations is easily verified, for

$$\left(\frac{d}{dx^2}\right)^2 y^2 = \frac{1}{2x}\cdot\frac{d}{dx}\left(\frac{y}{x}\cdot\frac{dy}{dx}\right) = \frac{1}{2x}\cdot\left\{\frac{y}{x}\cdot\frac{d^2y}{dx^2}+\frac{1}{x}\left(\frac{dy}{dx}\right)^2-\frac{y}{x^2}\cdot\frac{dy}{dx}\right\}$$

$$= \frac{1}{2x^3}(xyy_2 + xy_1^2 - yy_1);$$

so that the differential equation may be written

$$xyy_2 + xy_1^2 - yy_1 = 0.$$

Interchanging x and y in this, we have

$$yxx_2 + yx_1^2 - xx_1 = 0,$$

in which, if we write $x_1 = \frac{dx}{dy} = \frac{1}{y_1}$, and $x_2 = \frac{d^2x}{dy^2} = -\frac{y_2}{y_1^3}$, it follows immediately that

$$yxx_2 + yx_1^2 - xx_1 = -\frac{1}{y_1^3}(xyy_2 + xy_1^2 - yy_1),$$

and the identity in question is established.

Such a form as the above, which merely acquires an extraneous factor when the variables are interchanged, might be called a reciprocant, if it were not convenient to restrict the use of the word to forms in which the variables x and y do not appear explicitly. With this limitation, the geometrical property indicated by the evanescence of a reciprocant will be independent of the position of the origin, but not in general independent of the directions of the co-ordinate axes. Thus, we may prove that the equation

$$2y_1y_3 - 3y_2^2 = 0$$

indicates the possibility of 4-point contact with a hyperbola whose asymptotes are *parallel to the co-ordinate axes.* To do this it is sufficient to show that its complete primitive is the equation to such a hyperbola.

Writing the equation in the form

$$\frac{y_3}{y_2} = \frac{3}{2}\cdot\frac{y_2}{y_1},$$

we see that its first integral is

$$\log y_2 = \frac{3}{2}\log y_1 + \text{const.};$$

or, when prepared for a second integration,

$$-\frac{1}{2}\cdot y_1^{-\frac{3}{2}}y_2 = \lambda.$$

Hence
$$y_1^{-\frac{1}{2}} = \lambda x + \mu,$$

$$y_1 = (\lambda x + \mu)^{-2},$$

and finally we obtain the complete primitive

$$\lambda(\nu - y) = (\lambda x + \mu)^{-1},$$

which proves the proposition.

With the notation previously explained, in which $y_1 = t$, $y_2 = 2a$, $y_3 = 6b$, the differential equation is $bt - a^2 = 0$. We have therefore proved that at all points of a general curve for which the Schwarzian $(bt - a^2)$ vanishes, 4-point contact with a hyperbola whose asymptotes are parallel to the co-ordinate axes is possible.

We now consider the important case in which the conditioning differential equation remains unchanged when the axes are orthogonally transformed, and is therefore found by equating to zero an orthogonal reciprocant. The simplest example of this class of equations is that which marks the points of maximum or minimum curvature on a curve. Since these points are points of 4-point contact with a circle, the conditioning differential equation will be that of the circle

$$(x + \lambda)^2 + (y + \mu)^2 + \nu = 0.$$

Differentiating this three times in succession, we have

$$x + \lambda + (y + \mu)\, t = 0,$$

$$1 + t^2 + 2a\,(y + \mu) = 0,$$

$$at + b\,(y + \mu) = 0.$$

Eliminating μ from the last two of these equations, y will disappear at the same time, and the condition for points of maximum or minimum curvature is found to be

$$2a^2t - b\,(1 + t^2) = 0.$$

In Salmon's *Higher Plane Curves* (2nd edition, p. 357) the "aberrancy of curvature" is given by the formula

$$\tan\delta = y_1 - \frac{(1 + y_1^2)\, y_3}{3y_2^2} = t - \frac{(1 + t^2)\, b}{2a^2}.$$

The above differential equation is therefore equivalent to $\delta = 0$.

If we differentiate the radius of curvature $\rho = \dfrac{(1 + y_1^2)^{\frac{3}{2}}}{y_2} = \dfrac{(1 + t^2)^{\frac{3}{2}}}{2a}$, we find

$$\frac{d\rho}{dx} = \frac{6a^2t\,(1 + t^2)^{\frac{1}{2}} - 3b\,(1 + t^2)^{\frac{3}{2}}}{2a^2} = 3\,(1 + t^2)^{\frac{1}{2}} \tan\delta = 3\tan\delta\,.\frac{ds}{dx}.$$

Hence it follows that

$$\tan\delta = \frac{1}{3}\,.\frac{d\rho}{ds}.$$

The conditioning equation for points at which $\dfrac{d\rho}{ds}$ or $\tan\delta$ is a maximum or minimum is $\dfrac{d^2\rho}{ds^2} = 0$; or the same condition may be expressed by

$$\frac{d\tan\delta}{dx} = 0.$$

Now

$$\frac{d\tan\delta}{dx} = \frac{d}{dx}\left\{t - \frac{b\,(1 + t^2)}{2a^2}\right\} = 2a - \frac{2c\,(1 + t^2)}{a^2} - \frac{2abt}{a^2} + \frac{3b^2\,(1 + t^2)}{a^3}$$

is an orthogonal reciprocant, for it can be expressed in terms of legitimate combinations of $1+t^2$, which is an orthogonal reciprocant of even character, with the three orthogonal reciprocants of odd character,

$$a,\ b(1+t^2)-2a^2t,\ c(1+t^2)-5abt+5a^3.$$

In fact, the above expression for $\dfrac{d\tan\delta}{dx}$, when multiplied by a^3 to clear of fractions, becomes

$$\begin{aligned}&2a^4-2a^2bt+3b^2(1+t^2)-2ac(1+t^2)\\&\quad=\frac{3}{1+t^2}\{b(1+t^2)-2a^2t\}^2+\frac{12a^4}{1+t^2}-2a\{c(1+t^2)-5abt+5a^3\},\end{aligned}$$

where the right-hand side is a linear function of orthogonal reciprocants of the same (even) character, so that the combination is legitimate.

Quantities such as $\rho,\ \dfrac{d\rho}{ds},\ \dfrac{d^2\rho}{ds^2},\ \ldots$, or $\rho,\ \dfrac{d\rho}{d\phi},\ \dfrac{d^2\rho}{d\phi^2},\ \ldots$, where $d\phi$ is the angle subtended by the arc ds at the centré of curvature, have values independent of the particular position of the co-ordinate axes (supposed rectangular), and consequently these values, expressed in terms of $t, a, b, c, \ldots$ will be absolute orthogonal reciprocants. A differential equation expressing the condition that any one of these quantities vanishes, or that any one of them has a maximum or minimum value, will also be independent of the position of the rectangular axes, and must therefore be expressible in the form of an orthogonal reciprocant equated to zero.

Mr Hammond remarks that, since the radii of curvature at corresponding points of a curve and its evolute are ρ and $\dfrac{d\rho}{d\phi}$, the radius of curvature of its nth evolute is $\dfrac{d^n\rho}{d\phi^n}$. The radius of curvature of the nth evolute of any nth involute of a circle is constant, and, consequently, the differential equation of an nth involute to a circle is

$$\frac{d^{n+1}\rho}{d\phi^{n+1}}=0.$$

Writing this in the form

$$\left(\frac{1+t^2}{a}\cdot\frac{d}{dx}\right)^{n+1}\cdot\frac{(1+t^2)^{\frac{3}{2}}}{a}=0,$$

to which it is easily reduced, since

$$\frac{d}{d\phi}=\rho\cdot\frac{d}{ds}=\frac{\rho}{(1+t^2)^{\frac{1}{2}}}\cdot\frac{d}{dx}=\frac{(1+t^2)}{2a}\cdot\frac{d}{dx},$$

we see by what precedes that the left-hand member of the differential equation is an orthogonal reciprocant.

As an example of the class of singularities which next presents itself for consideration, let us find the differential condition which holds at points of contact of the fourth order with a common parabola. This condition is expressible by the differential equation whose complete primitive is

$$(y+\kappa x)^2+2\lambda x+2\mu y+\nu=0.$$

Differentiating three times in succession, we obtain

$$(y+\kappa x)(t+\kappa)+\lambda+\mu t=0,$$
$$2a(y+\kappa x+\mu)+(t+\kappa)^2=0,$$
$$b(y+\kappa x+\mu)+a(t+\kappa)=0.$$

The arbitrary constants ν and λ do not appear in the last two of these equations, from which, if we eliminate μ, the variables x and y disappear at the same time, and we find

$$2a^2-b(t+\kappa)=0.$$

A final differentiation and elimination give

$$10ab-4c(t+\kappa)=0,$$
$$4ac-5b^2=0.$$

Points of 5-point contact with a parabola are therefore indicated by the evanescence of the pure reciprocant $4ac-5b^2$. And in general the differential equation $R=0$, where R is any pure reciprocant, indicates a property of a curve which may be called a descriptive singularity, since it is totally unaffected by the arbitrary choice of any two lines on the plane for the axes of co-ordinates. For it was proved in Lecture IX of the present course that if i be the degree and μ the characteristic of R, the substitution of $ly+mx+n$ for x and $l'y+m'x+n'$ for y changes R into $(l'm-lm')^i(lt+m)^{-\mu}R$, so that the differential equation $R=0$ and the geometrical property corresponding to it are left unchanged by the substitution.

Six-point contact with a cubical parabola is another example of a descriptive singularity. Its defining differential equation may be written in any of the following forms:

$$45y_2^3y_5^2-450y_2^2y_3y_4y_5+192y_2^2y_4^3+400y_2y_3^3y_5+165y_2y_3^2y_4^2-400y_3^4y_4=0,$$
$$125a^3d^2-750a^2bcd+256a^2c^3+500ab^3d+165ab^2c^2-300b^4c=0,$$
$$5(9y_2^2y_5-45y_2y_3y_4+40y_3^3)^2+64(3y_2y_4-5y_3^2)^3=0,$$
$$125(a^2d-3abc+2b^3)^2+4(4ac-5b^2)^3=0;$$

or, if we make $a^2d-3abc+2b^3=A$ and $ac-\frac{5}{4}b^2=M$, the equation may be put in the form

$$\left(\frac{A}{16}\right)^2+\left(\frac{M}{5}\right)^3=0.$$

In the theory of Binary Forms, when the numerical parameter κ in

$$(a^2d-3abc+2b^3)^2+\kappa(ac-b^2)^3$$

is so chosen that the highest powers of b cancel each other, the form divides by a^2 and gives the Discriminant of the Cubic

$$a^2d^2 - 6abcd + 4b^3d + 4ac^3 - 3b^2c^2.$$

In the parallel theory of Reciprocants the form

$$125A^2 + 256M^3$$

is divisible by a (instead of by a^2), giving

$$125a^3d^2 - 750a^2bcd + 500ab^3d + 256a^2c^3 + 165ab^2c^2 - 300b^4c,$$

which may be called the Quasi-Discriminant.

A complete discussion of the differential equation

$$A^2 + \kappa M^3 = 0$$

is reserved for the next ensuing lecture, in the course of which it will appear that the Quasi-Discriminant equated to zero is the differential equation of the cubical parabola.

LECTURE XIII.

We may integrate the general homogeneous equation in reciprocants extending to d, inclusive, as follows:

Calling $$ac - \frac{5}{4}b^2 = M \text{ and } a^2d - 3abc + 2b^3 = A,$$

the equation in question will be of the form

$$A^2 + \kappa M^3 = 0.$$

But if we write $$\beta = \Lambda\alpha^\lambda,$$

where β, α are general linear functions of the co-ordinates, say

$$y + mx + n, \; y + m'x + n',$$

we may eliminate the five constants m, n, m', n', Λ, and the result will evidently be a pure reciprocant extending to d, inclusive, and, being homogeneous and isobaric, can only be of the form

$$A^2 + \kappa M^3 = 0,$$

so that it remains only to determine κ in terms of λ, or, which is the same thing, λ in terms of κ.

The solution $\beta = \Lambda\alpha^\lambda$ implies $\alpha = \Lambda^{-\frac{1}{\lambda}}\beta^{\frac{1}{\lambda}}$. Hence the equation between M and A must be of the form

$$\theta\,\{(\lambda + p)(p\lambda + 1)\}^i M^3 + \{(\lambda + q)(q\lambda + 1)\}^j A^2 = 0,$$

where θ is a constant, for otherwise there would be more than one general solution to it. It only remains then to determine the values of p, q, θ, i, j, which may be affected by considering the particular solution $y = x^\lambda$.

When $\lambda = 2$, M and A both vanish, and if $\lambda = 2 + \epsilon$, where ϵ is an infinitesimal, M and A will each be of the same order as ϵ (that the first power of ϵ does not vanish in M or A may be easily verified). Hence $2 + q + \epsilon$ is of the order ϵ, and therefore $q = -2$ and $j = 1$.

When $\lambda = -1 + \epsilon$, M remains finite and A is of the order ϵ. Hence $p = 1$ and $i = 1$. Thus, the equation is

$$\theta (\lambda + 1)^2 M^3 + (\lambda - 2)(2\lambda - 1) A^2 = 0.$$

To find θ, let $\lambda = 3$ and $y = x^3$; then

$$a = 3x,\ b = 1,\ c = 0,\ d = 0,\ M = -\frac{5}{4},\ A = 2,$$

so that
$$-\theta \cdot \frac{5^3}{4} + 5 \cdot 4 = 0,\quad \theta = \frac{16}{25},$$

and finally
$$16 (\lambda + 1)^2 M^3 + 25 (2\lambda^2 - 5\lambda + 2) A^2 = 0$$

has for its integral
$$\beta = \Lambda \alpha^\lambda.$$

If $\lambda = \infty$, we may make

$$y = \left(1 + \frac{x}{\lambda}\right)^{\lambda l} = e^{lx},$$

and, consequently, $\beta = e^{l\alpha}$, which contains five independent arbitrary constants, will be the general integral.

For a parallel method of deducing the Integral of $A^8 + \kappa\Delta^3 = 0$, where Δ (our future $AC - B^2$) is the projective reciprocant whose letters go up to f, see Halphen's *Thèse sur les Invariants Différentiels*, Paris, 1878.

Mr Hammond has succeeded in deducing the equation between A and M from the primitive $\beta = \Lambda\alpha^\lambda$ by direct elimination, as shown in what follows. Possibly he, or some other algebraist, may eventually succeed in the more difficult task of obtaining the Differential Equation to $\gamma = \beta^\lambda \alpha^{1-\lambda}$ (that is, the linear relation between A^8 and Δ^3) by some similar direct process.

Differentiating the equation $\beta\alpha^{-\lambda} = \Lambda$ three times in succession, and observing that, since $\alpha = y + mx + n$ and $\beta = y + m'x + n'$,

$$\alpha'' = \beta'' = \frac{d^2y}{dx^2} = y_2,$$

we have
$$\alpha\beta' - \lambda\alpha'\beta = 0,$$
$$y_2 (\alpha - \lambda\beta) + (1 - \lambda)\, \alpha'\beta' = 0,$$
$$y_3 (\alpha - \lambda\beta) + y_2 \{(2 - \lambda)\, \alpha' + (1 - 2\lambda)\, \beta'\} = 0.$$

From the last two of these three equations we obtain, by eliminating $(\alpha - \lambda\beta)$,

$$y_3 (1 - \lambda)\, \alpha'\beta' - y_2^2 \{(2 - \lambda)\, \alpha' + (1 - 2\lambda)\, \beta'\} = 0;$$

or, writing

$$y_2 = 2a,\ y_3 = 6b,\ 2 - \lambda = 3q^2,\ 1 - 2\lambda = -3r^2,\ 1 - \lambda = q^2 - r^2,$$

and dividing by $\alpha'\beta'$, the equation assumes the form

$$\frac{b}{2a^2}(q^2-r^2)=\frac{q^2}{\beta'}-\frac{r^2}{\alpha'}.$$

Differentiating again, remembering that

$$\alpha''=\beta''=2a, \text{ and } \frac{da}{dx}=3b, \ \frac{db}{dx}=4c,$$

we find
$$\frac{4ac-6b^2}{4a^4}(q^2-r^2)=-\frac{q^2}{\beta'^2}+\frac{r^2}{\alpha'^2}.$$

The elimination of β' between this and the equation immediately preceding it gives

$$\frac{4ac-6b^2}{4a^4}(q^2-r^2)\,q^2+\left\{\frac{b}{2a^2}(q^2-r^2)+\frac{r^2}{\alpha'}\right\}^2-\frac{q^2r^2}{\alpha'^2}=0.$$

Writing in this $4ac-5b^2=4M$, we obtain by an easy reduction

$$4q^2M\alpha'^2=r^2\{2a^2-b\alpha'\}^2,$$

and, taking the square root of each side,

$$\alpha'(2q\sqrt{M}+rb)-2a^2r=0.$$

A final differentiation gives

$$\alpha'\left(\frac{qM'}{\sqrt{M}}+4cr\right)+2a(2q\sqrt{M}-5br)=0.$$

Finally, eliminating α', we obtain

$$(2q\sqrt{M}+rb)(2q\sqrt{M}-5rb)+ar\left(4cr+\frac{qM'}{\sqrt{M}}\right)=0.$$

Hence
$$4Mq^2+qr\left(\frac{aM'}{\sqrt{M}}-8b\sqrt{M}\right)+r^2(4ac-5b^2)=0;$$

or,
$$4(q^2+r^2)M^{\frac{3}{2}}+qr(aM'-8bM)=0.$$

Now
$$M'=\frac{dM}{dx}=\frac{d}{dx}\left(ac-\frac{5b^2}{4}\right)=5ad-7bc,$$

and, consequently,

$$aM'-8bM=a(5ad-7bc)-b(8ac-10b^2)=5(a^2d-3abc+2b^3)=5A;$$

so that we may write

$$4(q^2+r^2)M^{\frac{3}{2}}=-qr(aM'-8bM)=-5qrA;$$

or,
$$16(q^2+r^2)^2M^3-25q^2r^2A^2=0,$$

where
$$3q^2=2-\lambda \text{ and } -3r^2=1-2\lambda.$$

Replacing q^2 and r^2 by their expressions in terms of λ, the differential equation becomes

$$16(\lambda+1)^2M^3+25(2\lambda^2-5\lambda+2)A^2=0.$$

Some special cases may be noticed.

When $\lambda = 2$ or $\frac{1}{2}$, the equation reduces to $M = 0$, which is the differential equation of the common parabola previously obtained.

When $\lambda = 3$ or $\frac{1}{3}$, we obtain $256M^3 + 125A^2 = 0$ for the equation of the cubical parabola, where the expression on the left-hand side is the Quasi-Discriminant.

When $\lambda = -1$, we find $A = 0$ for the differential equation of the general conic.

When λ is an imaginary cube root of negative unity, so that $\lambda^2 - \lambda + 1 = 0$, we have

$$(\lambda + 1)^2 + (2\lambda^2 - 5\lambda + 2) = 0,$$

and the differential equation becomes

$$16M^3 - 25A^2 = 0.$$

We shall subsequently avail ourselves of this result in finding the complete primitive of the Halphenian Δ.

In the case where λ is infinite, from the complete primitive $\beta = e^{l\alpha}$ we first eliminate the exponential function and afterwards the arbitrary constant l.

Thus we find $$\beta' = l\alpha'\beta \text{ and } \frac{y_2}{\beta'} = \frac{y_2}{\alpha'} + \frac{\beta'}{\beta};$$

or, $$y_2\beta(\alpha' - \beta') - \alpha'\beta'^2 = 0.$$

Hence $$y_3\beta(\alpha' - \beta') - y_2\beta'(\alpha' + 2\beta') = 0.$$

The elimination of β gives

$$y_3\alpha'\beta' - y_2^2(\alpha' + 2\beta') = 0;$$

or, $$\frac{3b}{2a^2} = \frac{1}{\beta'} + \frac{2}{\alpha'}.$$

Comparing this with the equation previously obtained,

$$\frac{b}{2a^2}(q^2 - r^2) = \frac{q^2}{\beta'} - \frac{r^2}{\alpha'},$$

we see that $q^2 = 1$ and $r^2 = -2$. Substituting these values in the differential equation

$$16(q^2 + r^2)^2 M^3 - 25q^2r^2A^2 = 0,$$

it becomes $$8M^3 + 25A^2 = 0,$$

which is the differential equation corresponding to the complete primitive $\beta = e^{l\alpha}$.

We shall hereafter consider in detail the theory of that special class of pure reciprocants (M. Halphen's Differential Invariants) which retain their form when any homographic substitution is impressed on the variables; that is, when, instead of x and y, we write

$$\frac{lx + my + n}{l''x + m''y + n''} \text{ and } \frac{l'x + m'y + n'}{l''x + m''y + n''}.$$

Since perspective projection is the geometrical equivalent of homographic substitution, it follows from the definition of Differential Invariants that they are connected with the properties and relations of curves which remain unaffected by perspective projection. For this reason Differential Invariants are sometimes called Projective Reciprocants. Two reciprocants with which we are familiar belong to this important class. One of them, y_2 or a, vanishes at points of inflexion on the curve $y = f(x)$; the other,

$$9y_2^2y_5 - 45y_2y_3y_4 + 40y_3^3, \text{ or } a^2d - 3abc + 2b^3,$$

which, for reasons given below, we shall call the Mongian, vanishes at sextactic points; that is, at points where a conic can be drawn having 6-point contact with the given curve.

To illustrate the distinction between a projective and a merely descriptive singularity, consider for an instant the pure reciprocant $4ac - 5b^2$, which, as we have seen, vanishes at all points of a general curve where 5-point contact with a parabola is possible. Now, 5-point contact with a parabola is a descriptive but not a projective singularity; after projection the parabola becomes a general conic, and 5-point contact with it becomes 5-point contact with a general conic, which is not a singularity at all. But inflexions and sextactic points are indelible by projection, and thus belong to the class of projective singularities.

The differential equation to a conic was originally obtained by Monge in the form

$$9y_2^2y_5 - 45y_2y_3y_4 + 40y_3^3 = 0$$

(see Monge, "Sur les Équations différentielles des Courbes du Second Degré," *Corresp. sur l'École Polytech.*, Paris, II. 1809–13, pp. 51–54, and *Bulletin de la Soc. Philom.*, Paris, 1810, pp. 87, 88). At the end of the first chapter of his *Differential Equations*, Boole mentions this form of equation as due to Monge, but without any reference, and adds the remark: "But here our powers of geometrical interpretation fail, and results such as this can scarcely be otherwise useful than as a registry of integrable forms." The theory of Reciprocants, however, furnishes both a simple interpretation of the Mongian equation and an obvious method of integrating it.

To see that the differential equation of a conic is satisfied at the sextactic points of a given curve, we have only to remember that at such points the derivatives of y with respect to x, up to the fifth order, inclusive, are the same for the given curve as for a conic.

We proceed to show how the Mongian may be integrated. Writing in the above equation

$$y_2 = 2a,\ y_3 = 2.3b,\ y_4 = 2.3.4c,\ y_5 = 2.3.4.5d,$$

it becomes

$$a^2d - 3abc + 2b^3 = 0,$$

where it can hardly fail to be noticed that the left-hand member of the equation is an ordinary Invariant as well as a Reciprocant. It will be proved hereafter that all Differential Invariants possess this double nature.

Now, if $\mu = 3i + w$, where i is the degree and w the weight of any pure reciprocant R, the ordinary theory of eduction shows that

$$\frac{d}{dx}\left(\frac{R}{a^{\frac{\mu}{3}}}\right) = \frac{a\dfrac{dR}{dx} - \mu bR}{a^{\frac{\mu}{3}+1}}$$

is another pure reciprocant.

When we consider the letters $a, b, c, \ldots$ in any invariant I to mean $\frac{y_2}{2}, \frac{y_3}{2.3}, \frac{y_4}{2.3.4}, \ldots$ the parallel theory of generation for Invariants gives the corresponding theorem that if $\nu = 3i + 2w$, where i is the degree and w the weight of I,

$$\frac{d}{dx}\left(\frac{I}{a^{\frac{\nu}{3}}}\right) = \frac{a\dfrac{dI}{dx} - \nu bI}{a^{\frac{\nu}{3}+1}}$$

is also an invariant.

A strict proof of this theorem will subsequently be given. For present purposes it is sufficient to notice the easily verified special cases of the two theorems

$$\frac{d}{dx}\left(\frac{4ac - 5b^2}{a^{\frac{8}{3}}}\right) = \frac{20\,(a^2d - 3abc + 2b^3)}{a^{\frac{11}{3}}},$$

and

$$\frac{d}{dx}\left(\frac{ac - b^2}{a^{\frac{10}{3}}}\right) = \frac{5\,(a^2d - 3abc + 2b^3)}{a^{\frac{13}{3}}}.$$

It follows as an immediate consequence that the equation

$$a^2d - 3abc + 2b^3 = 0$$

admits of the two first integrals

$$a^{-\frac{8}{3}}(4ac - 5b^2) = \text{const.}$$

and

$$a^{-\frac{10}{3}}(ac - b^2) = \text{const.}$$

Now, $$a^{-\frac{8}{3}}(4ac - 5b^2) = \frac{d}{dx}(a^{-\frac{5}{3}}b) = -\frac{1}{2}\frac{d^2}{dx^2}(a^{-\frac{2}{3}});$$

so that the Mongian equation is equivalent to

$$\frac{d^3}{dx^3}(a^{-\frac{2}{3}}) = 0, \text{ or to } \frac{d^3}{dx^3}(y_2{}^{-\frac{2}{3}}) = 0.$$

We thus obtain an integral of the form

$$y_2{}^{-\frac{2}{3}} = l + 2mx + nx^2,$$

from which the complete primitive may be found by two easy integrations. Thus,

$$y_1 + p = \int \frac{dx}{(l + 2mx + nx^2)^{\frac{3}{2}}} = \frac{m + nx}{(ln - m^2)(l + 2mx + nx^2)^{\frac{1}{2}}}$$

gives
$$y + px + q = \frac{1}{ln - m^2}(l + 2mx + nx^2)^{\frac{1}{2}},$$

which is the equation of a general conic.

By first interchanging the variables x, y in the Mongian equation (whose form remains unaltered by this interchange, since $a^2d - 3abc + 2b^3$ is a reciprocant) and then integrating three times with respect to x, we should find another integral of the form

$$x_2^{-\frac{2}{3}} = l' + 2m'y + n'y^2.$$

The solution may be completed by two integrations, as in the former method.

Mr Hammond remarks that $\dfrac{2(ac - b^2)}{a^{\frac{10}{3}}} = \dfrac{d^2}{dt^2}(a^{\frac{2}{3}})$, where $t = y_1$. For, since

$$\frac{d}{dt} = \frac{dx}{dt} \cdot \frac{d}{dx} = \frac{1}{2a} \cdot \frac{d}{dx},$$

we have
$$\frac{d}{dt}(a^{\frac{2}{3}}) = \frac{1}{2a} \cdot \frac{2}{3} \cdot a^{-\frac{1}{3}} . 3b = \frac{b}{a^{\frac{4}{3}}},$$

and, consequently,

$$\frac{d^2}{dt^2}(a^{\frac{2}{3}}) = \frac{1}{2a} \cdot \frac{d}{dx}(a^{-\frac{4}{3}}b) = 2a^{-\frac{10}{3}}(ac - b^2).$$

Hence the integral $a^{-\frac{10}{3}}(ac - b^2) = \text{const.}$ previously obtained for the Mongian is equivalent to $\dfrac{d^2}{dt^2}(a^{\frac{2}{3}}) = \text{constant}$; that is, to $\dfrac{d^2}{dy_1^2}(y_2^{\frac{2}{3}}) = \text{const.}$ Thus we have another integral of the form

$$y_2^{\frac{2}{3}} = \lambda + 2\mu y_1 + \nu y_1^2,$$

from which it is also easy to pass to the complete primitive.

I add a few general remarks relating to the subject-matter of this and the preceding lecture. Instead of the cumbrous terms Projective Reciprocants or Differential Invariants, it may be better to use the single word Principiants to denominate that crowning class or order of Reciprocants which remain, to a factor *près*, unaltered for any homographic substitutions impressed on the variables. This is the *species princeps*. If we go back to the *species infima*, we see the beginning of life in the subject. In general Reciprocants, all that is affirmed is that there exist forms-functions of the derivatives of y in regard to x which (to a factor *près*) remain unaltered when the variables x and y are interchanged, so that $f(y_1, y_2, y_3, \ldots)$ becomes

$\phi(x_1, x_2, x_3, \ldots)$. The function ϕ only differs from f by the acquisition of an extraneous factor $(-)^\kappa y_1^\mu$; that is,

$$f(y_1, y_2, y_3, \ldots) = (-)^\kappa y_1^\mu f(x_1, x_2, x_3, \ldots).$$

A particular species of these general (mixed) reciprocants arises when $f(y_1, y_2, y_3, \ldots)$, differentiated in regard to y_1, gives a reciprocant. These are Orthogonal Reciprocants, and in them we see the first dawn of free continuous motion as distinguished from mere displacement (or mere interchange of axes). Orthogonal Reciprocants, when x, y are rectangular co-ordinates, remain unaltered (save as to a factor) when the orthogonal axes are moved continuously. A quarter of a revolution of course will reverse their original positions, so that we see the condition of mutual displacement is fulfilled. Thirdly, Reciprocants into whose form the first derivative y_1 does not enter are called Pure. Their form is invariable when the axes (now taken generally) undergo separate displacement (instead of turning round together) in a plane. Here there is a further development, so to say, of life in the subject.

Finally, in Principiants, a particular species of Pure Reciprocants, the invariance remains good, not merely for any position of the axes of reference, but for any homographic deformation of the plane in which they lie, so that the evanescence of a Principiant corresponds to some property of a curve not only intrinsic but indelible by projection, as, for example, an inflexion, or a double point, or a sextactic point, and so on.

It is clear from this review that the Theory as we have given it goes to the root of the subject, and that the word Reciprocant is rightly chosen as conveying the notion of a property which is common to the entire continuous series of forms bearing that name. All the links of this connected chain are thus comprehended under the general name of Reciprocants.

LECTURE XIV.

The remaining lectures of the course will be devoted to the theory of Pure and Projective Reciprocants. I shall first treat of the existence and properties of the Protomorphs of Invariants and Reciprocants, using the latter system of protomorphs to obtain all the fundamental forms of Reciprocants in the letters a, b, c, d, e. I shall then pass on to the theory of Projective Reciprocants, or Principiants, with its applications contained in M. Halphen's *Thèse pour obtenir le grade de docteur ès sciences* (Paris, Gauthier-Villars, 1878). It will be seen that M. Halphen's very ingenious methods become greatly simplified when his results are read by the light of an important discovery in the theory of Principiants recently made by myself and Mr Hammond working conjointly, arising out of a theorem put forward by one of my hearers. This theorem, on examination, we found was necessarily erroneous and would fail at the very first step of its application. But although the proposition stated was wrong, it contained an Idea which survives and may be incorporated in a valid and extremely important theorem, which I will endeavour to explain.

A Principiant, besides being an Invariant in the original letters $a, b, c, d, \ldots$ is also an Invariant in the letters $a, A, B, C, D, \ldots$ where each capital letter is itself a Reciprocant; and, conversely, every invariant in the capital letters $A, B, C, D, \ldots$ is a Principiant. The invariants in the capital letters form a system of protomorphs for Principiants, so that every Principiant is either some such invariant simply, or a rational integral function of such invariants provided by some power of a. Thus, for example, it will be proved that the Cubic Criterium (that is, the Principiant which gives, when equated to zero, the differential equation of a cubic curve) may be expressed as the quotient of

$$\frac{9}{64}A^5+\frac{5}{4}A(A^2D-3ABC+2B^3)-(ACE-AD^2-B^2E+2BCD-C^3)$$

by the fifth power of a.

The proof of this theorem is based upon the fact that we can form a series of terms beginning with the Mongian (namely, $a^2d-3abc+2b^3$), say $A, B, C, D, \ldots$ such that

$$\Omega A = 0,$$
$$\Omega B = A \times \frac{a}{2},$$
$$\Omega C = 2B \times \frac{a}{2},$$
$$\Omega D = 3C \times \frac{a}{2},$$
$$\cdots\cdots\cdots\cdots\cdots$$

where
$$\Omega = a\partial_b + 2b\partial_c + 3c\partial_d + \ldots,$$

coupled with the fact that every Principiant must be a function of the letters in such series and the small a.

Each consequent of the series $A, B, C, D, \ldots$ is, so to say, an Invariant relative to its antecedent; it becomes an actual Invariant when its antecedent vanishes.

In the theorem as originally proposed, each letter of the series was derived by the operation of an eductive generator upon the one which precedes. In the true theorem the scale of relation is between three and not two consecutive terms. Calling the letters $u_0, u_1, u_2, \ldots u_i$, we have

$$(i+7)\,u_{i+2} - Gu_{i+1} + (i+1)\,Mu_i = 0,$$

where G is the ordinary eductive generator,

$$4(ac-b^2)\partial_b+5(ad-bc)\partial_c+6(ae-bd)\partial_d+\ldots,$$

M is the first pure reciprocant after the monomial a, namely, $M=ac-\frac{5}{4}b^2$, $u_0=A=a^2d-3abc+2b^3$, and $6u_1=GA$.

But although, as I have said, the theorem in the form proposed was absolutely erroneous, its proposer has rendered an invaluable service to the theory by the mere suggestion of what turns out to be true, namely, that every Principiant is an Invariant in regard to a known series of Reciprocants considered as simple elements.

To this theorem there is a correlative one, for it will be shown that there exists a series of invariants $A_0, A_1, A_2, \ldots$, the first term of which, A_0, is the same as the Mongian A, each of the other terms of the series being a Reciprocant relative to the one that precedes it. In fact, we have

$$VA_0=0,$$
$$VA_1=-a^2A_0,$$
$$VA_2=-2a^2A_1,$$
$$\ldots\ldots\ldots\ldots$$
$$VA_n=-na^2A_{n-1},$$

where
$$V=4\left(\frac{a^2}{2}\right)\partial_b+5ab\partial_c+6\left(ac+\frac{b^2}{2}\right)\partial_d+\ldots,$$

and, as a consequence, every Principiant will be an Invariant in respect to these Invariants and the first small letter a.

Thus, speaking symbolically, we have not only

$$P=R+I$$

(a logical equation meaning that P has the same qualities as both R and I, or that a Principiant is both a Reciprocant and an Invariant), but also

$$P=IR \text{ and } P=II,$$

meaning that a Principiant is an Invariant of Reciprocantive elements, and an Invariant whose elements are themselves Invariants.

I may add that the invariantive elements $A_0, A_1, A_2, A_3, \ldots$ are defined by the equations

$$A_0=A,$$
$$A_1=B-\frac{b}{2}A,$$
$$A_2=C-2\left(\frac{b}{2}\right)B+\left(\frac{b}{2}\right)^2A,$$
$$A_3=D-3\left(\frac{b}{2}\right)C+3\left(\frac{b}{2}\right)^2B-\left(\frac{b}{2}\right)^3A,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$

so that any invariant in the reciprocantive elements $A, B, C, D, \ldots$ is equal to the corresponding invariant in $A_0, A_1, A_2, A_3, \ldots$. Thus,

$$A = A_0,$$
$$AC - B^2 = A_0A_2 - A_1^2,$$
$$A^2D - 3ABC + 2B^3 = A_0^2A_3 - 3A_0A_1A_2 + 2A_1^3,$$
$$AE - 4BD + 3C^2 = A_0A_4 - 4A_1A_3 + 3A_2^2,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

M. Halphen appears not to have noticed the Principiant $AE - 4BD + 3C^2$, which presents itself naturally when the theory is viewed from our present ground of vantage, but A, $AC - B^2$ and $A^2D - 3ABC + 2B^3$ occur in his *Thèse* in connection with the curve

$$\alpha = \beta^\lambda \gamma^{1-\lambda},$$

in which α, β, γ are any linear functions of $x, y, 1$.

When $\lambda = -1$ the differential equation of this curve (the conic $\alpha\beta = \gamma^2$) is $A = 0$, but it is

$$AC - B^2 = 0$$

when λ is a cube root of negative unity, and

$$A^2D - 3ABC + 2B^3 = 0$$

when λ has an arbitrary value.

Before making out an exhaustive table of all the irreducible forms of pure reciprocants in the letters a, b, c, d, e similar to, but not identical with, the corresponding table for invariants, it seems to me desirable to say something of Protomorphs in general; and this will be better understood if we devote a short space to the protomorphs of Invariants. The simplest forms of these are the following well-known ones of alternately the second and third degrees:

$$P_2 = ac - b^2,$$
$$P_3 = a^2d - 3abc + 2b^3,$$
$$P_4 = ae - 4bd + 3c^2,$$
$$P_5 = a^2f - 5abe + 2acd + 8b^2d - 6bc^2,$$
$$P_6 = ag - 6bf + 15ce - 10d^2,$$
$$P_7 = a^2h - 7abg + 9acf - 5ade + 12b^2f - 30bce + 20bd^2,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

The quadratic Protomorphs $P_2, P_4, P_6, \ldots$, are absolutely unique, for the number of invariants of the type $j; 2, j$ is $(j; 2, j) - (j-1; 2, j) = 1$ if j is even, and $= 0$ if j is odd. Their form is so well known that there is no need to dilate upon it here.

The cubic ones P_3, P_5, P_7, ..., may be derived from the quadratic ones by means of Cayley's generators, given early in the course, namely,

$$P = (ac - b^2)\partial_b + (ad - bc)\partial_c + (ae - bd)\partial_d + \dots,$$

$$Q = (ac - 2b^2)\partial_b + 2(ad - 2bc)\partial_c + 3(ae - 2bd)\partial_d + \dots.$$

Let us first use the P generator

$$P(ac - b^2) = a(ad - bc) - 2b(ac - b^2) = a^2d - 3abc + 2b^3,$$

$$P(ae - 4bd + 3c^2) = a(af - be) - 4b(ae - bd) + 6c(ad - bc) - 4d(ac - b^2)$$
$$= a^2f - 5abe + 2acd + 8b^2d - 6bc^2.$$

Similarly, we find

$$P(ag - 6bf + 15ce - 10d^2) = a^2h - 7abg + 9acf - 5ade + 12b^2f - 30bce + 20bd^2,$$

and so on.

Let I be any invariant whatever of the type w; i, j (satisfied or unsatisfied); then using the original forms of the generators P and Q as given by Cayley (see Lecture IV), we have

$$PI = a(b\partial_a + c\partial_b + d\partial_c + \dots)I - ibI,$$

$$QI = a(c\partial_b + 2d\partial_c + 3e\partial_d + \dots)I - 2wbI,$$

and, consequently,

$$(jP - Q)I = a\{jb\partial_a + (j-1)c\partial_b + (j-2)d\partial_c + \dots\}I - (ij - 2w)bI.$$

If in this formula we write

$$O = jb\partial_a + (j-1)c\partial_b + (j-2)d\partial_c + \dots,$$

it becomes

$$(jP - Q)I = aOI - (ij - 2w)bI,$$

which, when I is a satisfied invariant, so that $ij - 2w = 0$ and $OI = 0$, reduces to

$$(jP - Q)I = 0,$$

showing that the forms obtained by operating with either P or Q on any satisfied invariant are the same to a numerical factor *près*.

Now, each quadratic protomorph is a satisfied invariant (for when $w = j$ and $i = 2$, $ij - 2w = 0$), and therefore the cubic protomorphs found by operating on the quadratic ones with Q will only differ by a numerical factor from those already obtained by the operation of P. But we must not conclude from this that the cubic protomorphs are unique. Their number is in fact given by the formula

$$(j;\ 3,\ j) - (j-1;\ 3,\ j),$$

where it is obvious that

$$(j-1;\ 3,\ j) = (j-1;\ 3,\ j-1);$$

so that the above formula may be written

$$(j;\ 3,\ j) - (j-1;\ 3,\ j-1), \text{ or say } \Delta(j;\ 3,\ j).$$

Now, there is a simple rule for finding $(j\,;\,3,j)$; it is the nearest integer to $\frac{(j+3)^2}{12}$. From the following table, obtained by the use of this rule,

$j=$	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$(j;\,3,j)=$	2	3	4	5	7	8	10	12	14	16	19	21	24	27
$\Delta\,(j;\,3,j)=$		1		1		1		2		2		2		3

it may be seen that for any odd number $j => 9$ there are two or more forms of extent j equally entitled to rank as protomorphs. If l be the last letter which occurs in one of these forms, its first term will of course be a^2l; the difference between any two such forms will not involve the letter l, and will only extend to k, but will still be of the same (potential) extent as l.

The property of the protomorphs $a, P_2, P_3, P_4, \ldots$ is that every invariant is a rational integral function of them divided by some power of a, as appears from the fact that Q, any given rational integral function whatever of the letters $a, b, c, d, e, \ldots$, may obviously be expressed as a rational integral function of $a, b, P_2, P_3, P_4, \ldots$ divided by some power of a. Thus,

$$Q = a^{-m}\phi\,(a, b, P_2, P_3, P_4, \ldots).$$

Suppose Q to be an invariant I; then

$$Ia^m = \phi\,(a, b, P_2, P_3, P_4, \ldots),$$

and, consequently,

$$\Omega\,(Ia^m) = \frac{d\phi}{da}\,\Omega a + \frac{d\phi}{db}\,\Omega b + \frac{d\phi}{dP_2}\,\Omega P_2 + \frac{d\phi}{dP_3}\,\Omega P_3 + \ldots,$$

where Ω is the annihilator for invariants; so that

$$\Omega\,(Ia^m) = 0,\ \ \Omega a = 0,\ \ \Omega P_2 = 0,\ \ \Omega P_3 = 0,\ \ldots.$$

We have therefore

$$\frac{d\phi}{db}\,\Omega b = a\,\frac{d\phi}{db} = 0.$$

Hence ϕ does not contain b, but is a rational integral function of the protomorphs alone, and

$$I = a^{-m}\phi\,(a, P_2, P_3, P_4, \ldots).$$

I shall show how to obtain a similar scale of forms possessing like properties for pure reciprocants.

LECTURE XV.

A Protomorph may be defined as a form whose weight is equal to its actual extent, so that its type is $j\,;\,i, j$. The first protomorph is a, which corresponds to $j = 0$. For higher values of j it follows immediately from the definition that every protomorph will contain a term $a^{i-1}l$, in which the letter of highest extent appears only in the first degree multiplied by a power of the first letter. The existence of this term enables us to instantly recognize a protomorph. As in the case of invariants, it will be shown that every pure reciprocant is either a rational integral function of protomorphs or else such a function divided by some power of a. But first it will be better to prove *à priori* their existence and exhibit examples of them for the earlier values of j.

It was proved, in Lecture IX, that the number of pure reciprocants of the type $w; i, j$ is at least equal to

$$(w; i, j) - (w-1; i+1, j).$$

Now, obviously, the number of partitions of w into i parts not exceeding $w+\epsilon$ is the same as the number of partitions of w into i parts not exceeding w, so that

$$(w; i, w+\epsilon) = (w; i, w);$$

and since, by a well-known theorem, $(w; i, j) = (w; j, i)$, we see that

$$(w; w+\epsilon, j) = (w; j, w+\epsilon) = (w; j, w) = (w; w, j),$$

a result which follows more immediately from the consideration that the partitions of $w; w+\epsilon, j$ differ only from those of $w; w, j$ by ϵ columns of zeros, as we see in the annexed example:

3; 5, 3	3; 3, 3
30000	300
21000	210
11100	111

Hence, if $w = j$, and $i => j$, we have

$$(w; i, j) = (j; j, j)$$

and $$(w-1; i+1, j) = (j-1; j-1, j-1).$$

Thus, the number of pure reciprocants of the type $j; j, j$ is

$$(j; j, j) - (j-1; j-1, j-1),$$

in other words, the difference between the indefinite partitions of j and those of $j-1$. Expressed by means of generating functions, this difference is the coefficient of x^j in

$$\frac{1-x}{(1-x)(1-x^2)(1-x^3)\dots(1-x^j)}$$

= coefficient of x^j in the expansion of

$$\frac{1}{(1-x^2)(1-x^3)\dots(1-x^j)}.$$

This coefficient is a positive integer for all values of j (except $j=1$, when it is zero), which proves the existence of reciprocants of the type $j; j, j$ when j has any value except unity.

But we wish to prove the existence of one or more reciprocants of the type $j; j, j$ which actually contain a term of the form $a^{j-1}l$, where the letter l

is of extent j. The number of such forms is the difference between the number of pure reciprocants of the types $j; j, j$ and $j; j, j-1$.

Now, the number of linearly independent pure reciprocants of the type $j; j, j$ has just been shown to be

$$(j;\ j,\ j)-(j-1;\ j-1,\ j-1).$$

And, in like manner, that of the linearly independent reciprocants of the type $j; j, j-1$ is

$$(j;\ j,\ j-1)-(j-1;\ j+1,\ j-1)$$
$$=(j;\ j,\ j-1)-(j-1;\ j-1,\ j-1).$$

The difference between these two numbers is therefore

$$(j;\ j,\ j)-(j;\ j,\ j-1)=1.$$

For the only partition not common to the two types is $j\,.\,0^{j-1}$, made up of one j and $j-1$ zeros, which belongs to the first type, but not to the second. Hence reciprocants of the type $j; j, j$ contain one term which those of the type $j; j, j-1$ do not, and which can only be $a^{j-1}l$. This proves the existence of protomorphs.

In the latter part of the above proof we have assumed the truth of the theorem, which, however probable, is not demonstrated, that the number of reciprocants of the type $w; i, j$ is $(w;\ i,\ j)-(w-1;\ i+1,\ j)$ and *no more* [that concerns the subtrahend, namely, $(j;\ j,\ j-1)-(j-1;\ j-1,\ j-1)$].

We shall, however, have an independent method of arriving at Protomorphs by direct generation, just as we saw that all the cubic protomorphs to invariants were derivable by direct operation of generators from the quadratic ones.

The difference between the two cases is that the lowest degree of Invariantive Protomorphs fluctuates alternately between 2 and 3. For Reciprocantive Protomorphs the lowest degree corresponding to a given extent fluctuates, but has a tendency to rise, and goes on progressing until it exceeds any assignable number.

It is interesting to find what the degrees are for successive values of j. The calculations required are greatly facilitated by an extensive table of partitions given by Euler in 1750, and partly reproduced by Cayley in the *American Journal of Mathematics*, Vol. IV., Part III. In the table as presented by Cayley, the number in column j and line i means the number of ways of partitioning j into exactly i parts (zeros excluded). Hence, to find the number of ways of partitioning j into i parts or fewer, that is, to find $(j;\ i,\ \infty)$ or its equivalent $(j;\ i,\ j)$, we must add up the numbers in the 1st, 2nd, 3rd, ... ith lines of column j.

When these summations are made we obtain the subjoined table:

EXTENT $j=$

DEGREE $i=$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	1	2	2	3	3	4	4	5	5	6	6	7	7	8	8	9	9	10
3	1	1	2	3	4	5	7	8	10	12	14	16	19	21	24	27	30	33	37
4	1	1	2	3	5	6	9	11	15	18	23	27	34	39	47	54	64	72	84
5	1	1	2	3	5	7	10	13	18	23	30	37	47	57	70	84	101	119	141
6	1	1	2	3	5	7	11	14	20	26	35	44	58	71	90	110	136	163	199
7	1	1	2	3	5	7	11	15	21	28	38	49	65	82	105	131	164	201	248
8	1	1	2	3	5	7	11	15	22	29	40	52	70	89	116	146	186	230	288

The number of pure reciprocants of the type $j;\, i, j$ is

$$(j;\, i, j) - (j-1;\, i+1, j) = (j;\, i, j) - (j-1;\, i+1, j-1).$$

To find the minimum degree for protomorphs of extent j we have therefore only to see for what value of i any figure in the j column first becomes greater than the figure in the column to the left one place lower down. The fluctuations of the minimum degree are indicated by the dark irregularly waving line which runs through the table.

Accordingly, we find that the types of the protomorphs, omitting w, which is always equal to j, are as follows:

(2, 2), (3, 3), (3, 4), (4, 5), (3, 6), (4, 7), (4, 8), (5, 9), (5, 10), (5, 11), (5, 12), ...,

whereas for invariants they are

(2, 2), (3, 3), (2, 4), (3, 5), (2, 6), (3, 7), (2, 8), (3, 9), (2, 10), (3, 11), (2, 12), ...

Corresponding to the extents

2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, ...,

the lowest degrees of the Reciprocantive Protomorphs are

2, 3, 3, 4, 3, 4, 4, 5, 5, 5, 5, ...

Contrast this with the regularly fluctuating series

2, 3, 2, 3, 2, 3, 2, 3, 2, 3, 2, 3, ...,

which shows the minimum degrees of invariantive protomorphs for successive extents.

It may be proved, from known formulae in the theory of partitions, that as the extent increases the minimum degree of reciprocantive protomorphs increases (on the whole) and ultimately becomes infinite when the extent is so.

The apparent number of protomorphs to the several types is

$$\begin{matrix}(2,2), & (3,3), & (3,4), & (4,5), & (3,6), & (4,7), & (4,8), & (5,9), & (5,10), & (5,11), & (5,12), \dots \\ 1 & 1 & 1 & 1 & 1 & 1 & 2 & 3 & 4 & 2 & 3\end{matrix}$$

The explanation of this multiplicity is the same as that previously given for the case of invariants: the difference between any two protomorphs of a given type j; i, j will be a reciprocant (no longer a protomorph) of the type j; i, $j-1$.

For the only term containing the letter l (of extent j) will disappear from the result of subtraction; and, accordingly, the above numbers, each diminished by unity, will give the numbers of a set of reciprocants of the same degree-weight as the protomorphs, but of a smaller (actual) extent.

Assuming that the number of pure reciprocants of the type w; i, j is correctly given by the formula

$$(w;\ i,\ j) - (w-1;\ i+1,\ j),$$

Euler's great table of partitions, already referred to, enables us to carry on the determination of the minimum degree and multiplicity of protomorphs for all extents as far as 59.

If m is the multiplicity corresponding to the minimum degree i of a reciprocantive protomorph whose extent is j, we form without difficulty, using only the principles explained above, the following table:

$j=$	0	1	2	3	4	5	6	7	8	9	10	11
$i=$	1	–	2	3	3	4	3	4	4	5	5	5
$m=$	1	0	1	1	1	1	1	1	2	3	4	2

,

$j=$	12	13	14	15	16	17	18	19	20	21	22	23
$i=$	5	6	6	6	6	7	7	7	7	7	8	8
$m=$	3	6	8	5	5	15	18	12	12	2	40	32

,

$j=$	24	25	26	27	28	29	30	31	32	33	34	35
$i=$	8	8	8	9	9	9	9	10	10	10	10	10
$m=$	32	14	6	84	82	58	45	207	211	180	161	102

,

$j=$	36	37	38	39	40	41	42	43	44	45	46	47
$i=$	10	11	11	11	11	11	11	12	12	12	12	12
$m=$	45	482	469	391	320	167	13	1126	1064	881	687	337

,

$j=$	48	49	50	51	52	53	54	55	56	57	58	59
$i=$	13	13	13	13	13	13	13	14	14	14	14	14
$m=$	2829	2666	2492	2097	1643	892	26	6394	6017	5227	4266	2755

.

Notice the repetitions of i indicated by the series

$$1^1,\ 0^1,\ 2^1,\ 3^2,\ 4^1,\ 3^1,\ 4^2,\ 5^4,\ 6^4,\ 7^5,\ 8^5,\ 9^4,\ 10^6,\ 11^6,\ 12^5,\ 13^7,\ 14^{5+?}.$$

It will be observed that there is a general tendency of the number of equal values of i to increase, but that this is subject to occasional fluctuations. When $j=5$, $i=4$; but when $j=6$, $i=3$, so that the minimum value of i recedes. After this point is reached, i either advances or remains stationary, but never recedes.

In order actually to find the protomorphs, we may use the annihilator V. This was my original method of obtaining them; a shorter way, analogous to that used by Halphen for differential invariants (principiants), has been previously mentioned, but it will be instructive to begin with the method of indeterminate coefficients. In the first place we have the form a of weight 0, which is annihilated by

$$V = 2a^2\partial_b + 5ab\partial_c + (6ac+3b^2)\partial_d + (7ad+7bc)\partial_e + \dots.$$

For weight 1 there is no pure reciprocant. We could not make $R=\lambda a^{i-1}b$, for then $VR = 2\lambda a^{i+1}$, which cannot vanish unless $\lambda=0$ and consequently $R=0$.

To find the Protomorph of extent 2, assume $R=\lambda ac+\mu b^2$; then

$$VR = 4\mu a^2b + 5\lambda a^2b = (4\mu+5\lambda)\,a^2b.$$

Hence λ and μ are proportional to 4 and -5, and we may write

$$R = 4ac - 5b^2.$$

For extent 3, assuming $R=\lambda a^2d+\mu abc+\nu b^3$, we have

$$VR = 2\mu a^3c + 6\nu a^2b^2 + 5\mu a^2b^2 + 6\lambda a^3c + 3\lambda a^2b^2,$$

which vanishes when

$$2\mu+6\lambda=0,\ \ 6\nu+5\mu+3\lambda=0.$$

We may therefore write $\lambda=1$, $\mu=-3$, $\nu=2$, and thus obtain

$$R = a^2d - 3abc + 2b^3.$$

For extent 4 the table of minimum degrees indicates the existence of a protomorph of degree 3. To find its value we assume

$$R = \kappa a^2e + \lambda abd + \mu ac^2 + \nu b^2c.$$

Operating with V, we find

	a^3d	a^2bc	ab^3
$VR=$	2λ	4ν	.
	.	10μ	5ν
	.	6λ	3λ
	7κ	7κ	.

In order that VR may vanish, we must have

$$2\lambda+7\kappa=0,\ \ 4\nu+10\mu+6\lambda+7\kappa=0,\ \text{and}\ \ 5\nu+3\lambda=0.$$

To avoid fractions, let $\kappa=50$; then $\lambda=-175$, $\nu=105$, and $\mu=28$; thus,

$$R = 50a^2e - 175abd + 28ac^2 + 105b^2c;$$

whereas, the protomorph of extent 4 for Invariants is $ae-4bd+3c^2$. There is no reciprocant of degree 2 weight 4 to correspond to this.

LECTURE XVI.

By using the generator for pure reciprocants instead of the annihilator V, we readily obtain the protomorph of extent 5 and of the fourth degree whose existence is indicated in the previously given table of minimum degrees. We have only to operate on the protomorph of degree 3 and extent 4 with

$$G = 4(ac - b^2)\partial_b + 5(ad - bc)\partial_c + 6(ae - bd)\partial_d + 7(af - be)\partial_e + \ldots$$

Thus,

$$\begin{aligned} &G(50a^2e - 175abd + 28ac^2 + 105b^2c) \\ &= 4(ac - b^2)(-175ad + 210bc) \\ &+ 5(ad - bc)(56ac + 105b^2) \\ &+ 6(ae - bd)(-175ab) \\ &+ 7(af - be)(50a^2). \end{aligned}$$

Rejecting the numerical factor 35, which is common to all the terms in the result, and at the same time writing the terms themselves in reverse order, we have

$$\begin{aligned} 10a^2(af - be) - 30ab(ae - bd) + (ad - bc)(8ac + 15b^2) + 4(ac - b^2)(-5ad + 6bc) \\ = 10a^3f - 40a^2be - 12a^2cd + 65ab^2d + 16abc^2 - 39b^3c, \end{aligned}$$

which is the protomorph in question.

The form just found is irreducible, as indeed it ought to be, since the minimum degree for extent 5 is greater than that for extent 4 by unity, which exactly corresponds with the unit increase of degree due to the operation of G. But if we use G to generate a protomorph of extent 4 from that of extent 3, the resulting form will be reducible. In fact

$$\begin{aligned} &G(a^2d - 3abc + 2b^3) \\ &= 4(ac - b^2)(-3ac + 6b^2) + 5(ad - bc)(-3ab) + 6(ae - bd)a^2 \\ &= 3(2a^3e - 7a^2bd - 4a^2c^2 + 17ab^2c - 8b^4). \end{aligned}$$

If now we write

$$ac - \frac{5}{4}b^2 = M,$$

$$a^2d - 3abc + 2b^3 = A,$$

$$a^3e - \frac{7}{2}a^2bd - 2a^2c^2 + \frac{17}{2}ab^2c - 4b^4 = B,$$

we have shown that $$GA = 6B.$$

But

$$\begin{aligned} 50B + 128M^2 &= 25(2a^3e - 7a^2bd - 4a^2c^2 + 17ab^2c - 8b^4) + 8(4ac - 5b^2)^2 \\ &= a(50a^2e - 175abd + 28ac^2 + 105b^2c); \end{aligned}$$

so that B is reducible, being expressible as a rational integral function of a, M, and the previously obtained protomorph of degree 3 and extent 4.

The general theory of the generator G is contained in that of the differentiation of absolute reciprocants, in which, if $\mu = 3i + w$, where w is the weight and i the degree of any pure reciprocant R, we have

$$\frac{R}{a^{\frac{\mu}{3}}} = \pm \frac{R_1}{a_1^{\frac{\mu}{3}}},$$

and, consequently,

$$\frac{d}{dx}\left(\frac{R}{a^{\frac{\mu}{3}}}\right) = \pm \frac{dy}{dx} \cdot \frac{d}{dy}\left(\frac{R_1}{a_1^{\frac{\mu}{3}}}\right),$$

where R_1 and a_1 are what R and a become when x and y are interchanged. Hence

$$\frac{a\dfrac{dR}{dx} - \dfrac{\mu}{3} R \dfrac{da}{dx}}{a^{\frac{\mu}{3}+1}},$$

and therefore also the numerator of this fraction is a reciprocant.

Remembering that

$$\frac{da}{dx} = 3b, \quad \frac{db}{dx} = 4c, \quad \frac{dc}{dx} = 5d, \ \ldots,$$

the numerator may be written

$$a\frac{dR}{dx} - \mu b R = GR.$$

The ordinary expression for G is found by writing

$$a\frac{d}{dx} - \mu b = a\,(3b\partial_a + 4c\partial_b + 5d\partial_c + \ldots)$$
$$- b\,(3a\partial_a + 4b\partial_b + 5c\partial_c + \ldots).$$

If the actual extent of R is j, that of GR is $j+1$; for the operation of G introduces an additional letter. Both the weight and degree are also increased by unity. Thus, the type of R being w; i, j, that of GR is

$$w+1;\ i+1,\ j+1.$$

Suppose the weight of R to be equal to its actual extent; then R is a protomorph of the type j; i, j, and GR, whose type is $j+1$; $i+1$, $j+1$, is also a protomorph. This proves the existence of protomorphs for every possible extent. Starting with the form $4ac - 5b^2$ we obtain, by successive eduction, a series of protomorphs of the type j; j, j for which the general expression is

$$G^{j-2}(4ac - 5b^2),$$

where j has any of the values 2, 3, 4, ...

If R is a protomorph of minimum degree, GR (if irreducible) will also be a protomorph of minimum degree. Hence the minimum degree can never increase by more than one unit when the extent is increased by unity.

The second educt G^2R is always reducible; for

$$G^2R = \left\{ a\frac{d}{dx} - (\mu+4)b \right\} \left(a\frac{d}{dx} - \mu b \right) R$$

$$= \left\{ a^2 \frac{d^2}{dx^2} - (2\mu+1)ab\frac{d}{dx} - 4\mu ac + \mu(\mu+4)b^2 \right\} R.$$

Combining this with $M = ac - \frac{5}{4}b^2$, we have

$$5G^2R + 4\mu(\mu+4)MR = a\left\{ 5a\frac{d^2}{dx^2} - 5(2\mu+1)b\frac{d}{dx} + 4\mu(\mu-1)c \right\} R,$$

where the right-hand side is divisible by a, showing that the degree of G^2R is always depressible by unity. R being a protomorph of degree i and extent j,

$$\left\{ 5a\frac{d^2}{dx^2} - 5(2\mu+1)b\frac{d}{dx} + 4\mu(\mu-1)c \right\} R$$

is one of degree $i+1$ and extent $j+2$. Hence we may conclude that an increase in the minimum degree for protomorphs cannot be immediately followed by another increase; for, if this were possible, the minimum degree for extent $j+2$ would be $i+2$, instead of being $i+1$ at most.

This conclusion is in accordance with the sequence of the values of i in the table of minimum degrees, and as far as it goes confirms the exactitude of the formula $(w; i, j) - (w-1; i+1, j)$ for the number of pure reciprocants which was assumed in calculating the table.

The method previously employed to prove that every invariant is a rational integral function of protomorphs, or such function divided by a power of a, may be very easily extended to the case of reciprocants.

In the first place, it is obvious that every rational integral function of the letters $a, b, c, d, \ldots$ is by successive substitutions reducible to the form

$$a^{-\theta}\Phi(a, b, P_2, P_3, P_4, \ldots P_j),$$

where P_j means the protomorph of extent j.

Let any reciprocant R be put under this form; then

$$a^{\theta}R = \Phi(a, b, P_2, P_3, P_4, \ldots P_j),$$

and, consequently,

$$V(a^{\theta}R) = \frac{d\Phi}{da}Va + \frac{d\Phi}{db}Vb + \frac{d\Phi}{dP_2}VP_2 + \ldots + \frac{d\Phi}{dP_j}VP_j.$$

Now, V annihilates $R, a, P_2, P_3, \ldots P_j$, since these are all pure reciprocants. Hence the above identity reduces to $\frac{d\Phi}{db}Vb = 0$, from which (since Vb does not vanish) we conclude that Φ does not contain b explicitly. Thus,

$$a^{\theta}R = \Phi(a, P_2, P_3, P_4, \ldots P_j),$$

and the theorem is established for reciprocants.

The Protomorphs for Reciprocants as far as extent 8 are as follows:

$P_2 = 4ac - 5b^2,$

$P_3 = a^2d - 3abc + 2b^3,$

$P_4 = 50a^2e - 175abd + 28ac^2 + 105b^2c,$

$P_5 = 10a^3f - 40a^2be - 12a^2cd + 65ab^2d + 16abc^2 - 39b^3c,$

$P_6 = 14a^2g - 63abf - 1350ace + 1782b^2e + 1470ad^2 - 4158bcd + 2310c^3,$

$$P_7 = 7a^3h - 35a^2bg - 539a^2cf + 735ab^2f + 605a^2de + 306abce - 1485b^3e - 2135abd^2 + 1001ac^2d + 3465b^2cd - 1925bc^3,$$

$$P_8 = 420a^3i - 2310a^2bh - 25648a^2cg + 9240a^2df + 21780a^2e^2 + 36680ab^2g + 85386abcf - 191730abde - 59220ac^2e + 120540acd^2 - 126945b^3f + 252126b^2ce + 169260b^2d^2 - 419034bc^2d + 129360c^4.$$

The work necessary for obtaining the first four of these, P_2, P_3, P_4, P_5, has been fully set out. Since P_4 is of degree 3, its second educt, G^2P_4, is of degree 5 and its reduced second educt of degree 4. A linear combination of this with a form whose leading term is a^2ce becomes divisible by a and gives P_6; but as this requires the preliminary calculation of the form (a^2ce), it is simpler to find P_6 directly by the method of indeterminate coefficients, and thence by eduction to get P_7 and P_8. Thus (to a numerical factor *près*) P_7 is the educt and P_8 the reduced second educt of P_6. Beyond this point the calculation of protomorphs has not at present been carried.

Referring to the table which gives the minimum degree and multiplicity for a Protomorph of any extent, we see that the multiplicity exceeds unity when the extent $j = > 8$, and is exactly equal to 2 when $j = 8$, 11, or 21.

Hence the protomorphs as far as P_7 inclusive are unique; but there are two forms of extent 8 and degree 4, any linear combination of which (provided it contains the term a^3i) may be regarded as a protomorph. One of these forms is P_8, whose value is given above; the other is a linear combination of P_8 with a form, whose leading term is a^2cg, hereafter to be set forth.

The irreducible forms for extent 2 are a and P_2; every other form must be simply a power of P_2 multiplied by a power of a. We proceed to the calculation of all the Irreducible Forms for the extents 3 and 4 respectively. When $j = 3$, we may combine the protomorphs

$$4ac - 5b^2$$

and

$$a^2d - 3abc + 2b^3$$

with one another.

Adding 125 times the square of the latter to 4 times the cube of the former and dividing by a, there results the form

$$125a^3d^2 - 750a^2bcd + 500b^3d + 256a^2c^3 + 165ab^2c^2 - 300b^4c.$$

This form is analogous to the discriminant of the cubic, but is of a higher degree by one unit. Its type is 6; 5, 3, whereas that of the discriminant is 6; 4, 3.

In the case of invariants, we have to combine $ac - b^2$ with $a^2d - 3abc + 2b^3$. The square of the second, added to 4 times the cube of the first, gives

$$a^4d^2 - 6a^3bcd + 4a^2b^3d + 9a^2b^2c^2 - 12ab^4c + 4b^6 + 4a^3c^3 - 12a^2b^2c^2 + 12ab^4c - 4b^6.$$

Here the term $12ab^4c$ is nullified by $-12ab^4c$, so that the result contains a^2, the other factor being the discriminant

$$a^2d^2 - 6abcd + 4b^3d + 4ac^3 - 3b^2c^2,$$

which is of the type 6; 4, 3.

We may show *à priori*, assuming the problematical but highly probable formula $(w;\ i,\ j) - (w-1;\ i+1,\ j)$, that the type 6; 4, 3 does not belong to any reciprocant.

For, as seen in the partitionments set out below,

$(6;\ 4,\ 3) - (5;\ 5,\ 3) = 5 - 5 = 0$	
3 . 3	3 . 2
3 . 2 . 1	3 . 1 . 1
3 . 1 . 1 . 1	2 . 2 . 1
2 . 2 . 2	2 . 1 . 1 . 1
2 . 2 . 1 . 1	1 . 1 . 1 . 1 . 1

We can by no other means combine the protomorphs with one another or with the Quasi-Discriminant $(125a^3d^2 \ldots)$ so as to obtain additional fundamental forms. Every Rational Integral Pure Reciprocant of extent 3 is therefore necessarily a rational integral function of the four forms

deg. wt.		
1 . 0	$a,$	
2 . 2	$4M$	$= 4ac - 5b^2,$
3 . 3	A	$= a^2d - 3abc + 2b^3,$
5 . 6	(a^3d^2)	$= 125a^3d^2 - 750a^2bcd + 500b^3d + 256a^2c^3 + 165ab^2c^2 - 300b^4c.$

These are connected by a syzygy of degree-weight 6 . 6, namely

$$125A^2 + 256M^3 = a\,(a^3d^2),$$

analogous to the syzygy of the same degree-weight, in the Theory of the Binary Cubic, which connects the Discriminant with a and the Protomorphs of extent 2 and 3.

It will be clearly seen from an inspection of the fundamental forms that there is no law for the coefficients of Reciprocants akin to that of their algebraical sum being zero in Invariants.

LECTURE XVII.

The fundamental reciprocants for extent 3, given in the last lecture, agree with the irreducible invariants of a binary cubic both in number and type, with the single exception that the degree of the cubic discriminant is lower by unity than that of the reciprocant corresponding to it. When the extent is raised to 4, both the discriminant and its analogue cease to rank among the irreducible forms, the former being expressible as a rational integral function of invariants of lower degree, and the latter as a similar function of reciprocants. But the increase of extent introduces three additional reciprocants whose leading terms are a^2e, a^2ce and a^3e^2, whereas the additional invariants are only two in number and begin with ae and ace respectively.

The irreducible reciprocants of extent 4 are as follows:

deg.	wt.		
1	0	a,	
2	2	$4M$	$= 4ac - 5b^2$,
3	3	A	$= a^2d - 3abc + 2b^3$,
3	4	P_4	$= 50a^2e - 175abd + 28ac^2 + 105b^2c$*,
4	6	(a^2ce)	$= 800a^2ce - 1000ab^2e - 875a^2d^2 + 2450abcd - 1344ac^3 - 35b^2c^2$,
5	8	(a^3e^2)	$= 625a^3e^2 - 4375a^2bde - 49700a^2c^2e + 128625ab^2ce - 78750b^4e + 55125a^2cd^2 - 61250ab^2d^2 - 156800abc^2d + 183750b^3cd + 84868ac^4 - 102165b^2c^3$.

The similar list of invariants for the quartic is

deg.	wt.	
1	0	a,
2	2	$ac - b^2$,
3	3	$a^2d - 3abc + 2b^3$,
2	4	$ae - 4bd + 3c^2$,
3	6	$ace - b^2e - ad^2 + 2bcd - c^3$.

To obtain the fundamental forms of extent 4 we have to combine M, A and the Quasi-Discriminant

$$(a^3d^2) = 125a^3d^2 - 750a^2bcd + 500ab^3d + 256a^2c^3 + 165ab^2c^2 - 300b^4c$$

with the additional Protomorph

$$P_4 = 50a^2e - 175abd + 28ac^2 + 105b^2c$$

* P_4 is the protomorph of minimum degree; the other protomorph, B, which will be used when we treat of Principiants, is, when expressed in terms of the irreducible forms,

$$B = \frac{1}{50}(aP_4 - 128M^2).$$

in such a manner that the combination contains a factor a. The removal of this factor gives rise to a form of lower degree, and the process is repeated as often as possible.

Calling that portion of any form which does not contain a its residue, the residue of $4M$ is $-5b^2$, that of (a^3d^2) being $-300b^4c$, and that of P_4 being $105b^2c$. Thus

$$16MP_4 - 7(a^3d^2)$$

contains the factor a, and leads to (a^2ce) of the type 6; 4, 4, which is the analogue to the Catalecticant

$$\begin{vmatrix} a & b & c \\ b & c & d \\ c & d & e \end{vmatrix}.$$

The form (a^3d^2) now ceases to be a groundform (= irreducible form) and is replaced by the Quasi-Catalecticant (a^2ce), for

$$(a^3d^2) = \frac{16}{7} MP_4 - \frac{1}{7} a (a^2ce).$$

Similarly, the Cubic Discriminant, a groundform *quâ* the letters a, b, c, d, becomes reducible when a new letter, e, is introduced, and is then replaced by the Catalecticant.

We now come to an extra form which has no analogue in invariants. The residue of the Quasi-Catalecticant (a^2ce) is $-35b^2c^2$, and consequently

$$P_4^2 - 252M(a^2ce)$$

divides by a numerical multiple of a (as it happens by $4a$) and yields the form (a^3e^2), whose type is 8; 5, 4.

Here the deduction of new fundamental forms comes to an end on account of the appearance of e in the residue of (a^3e^2). It would have ended sooner but for the apparently accidental non-appearance of the term b^3d (of the same type 6; 4, 4 as b^2c^2) in the residue of (a^2ce). Had this term appeared, no combination could have been made leading to a new groundform after (a^2ce). We are able to show from *à priori* considerations that it cannot exist.

For the arguments in the annihilator V, up to ∂_e inclusive, are

$$a^2\partial_b,\ ab\partial_c,\ ac\partial_d,\ b^2\partial_d,\ ad\partial_e, \text{ and } bc\partial_e.$$

If, now, the term μb^3d were to form part of a Pure Reciprocant, $b^2\partial_d$ operating upon it would give μb^5; but every other portion of the operator would necessarily give terms containing one or other of the letters a, c. Since such terms cannot destroy μb^5, we must have $\mu b^5 = 0$. Hence the term in question is necessarily non-existent.

The method of combining the protomorphs which we have followed shows that the fundamental reciprocants of extent 4 are connected *inter se* by the two relations or syzygies

$$7(256M^3+125A^2)-16aMP_4+a^2(a^2ce)=0,$$
$$P_4^2-252M(a^2ce)-4a(a^3e^2)=0.$$

The invariants of the binary quartic are connected by only one syzygy, similar to the first of these; the second has no analogue in the theory of Invariants. It has been shown that the irreducible reciprocants of extent 3 are connected by the syzygy

$$256M^3+125A^2-a(a^3d^2)=0.$$

Substituting in this for the Quasi-Discriminant (a^3d^2) its value expressed in terms of the fundamental forms of extent 4, by means of the equation

$$16MP_4-7(a^3d^2)=a(a^2ce),$$

we obtain the first of the above syzygies. By a precisely similar substitution, the syzygy connecting the invariants of the quartic is derived from the one which connects the invariants of the cubic.

Every reciprocant of extent 4 is a rational integral function of the six fundamental forms given in the table; and, by means of the syzygies, powers, but not products, of A and P_4 can be removed from this function. For the first syzygy gives A^2 and the second gives P_4^2 as a rational integral function of the four remaining forms a, M, (a^2ce), and (a^3e^2). Hence every reciprocant of extent 4 is of one or other of the forms

$$\Phi,\ A\Phi,\ P_4\Phi,\ AP_4\Phi,$$

where Φ does not contain either A or P_4, but is a rational integral function of the other four fundamental forms.

Let the four forms which appear in Φ occur raised to the powers κ, λ, μ, ν, respectively, in one of its terms. Since the degree-weights of these four forms are

$$1.0,\quad 2.2,\quad 4.6 \text{ and } 5.8,$$

any such term may be represented by

$$a^\kappa(a^2x^2)^\lambda(a^4x^6)^\mu(a^5x^8)^\nu.$$

Thus the totality of the terms in Φ will be represented by

$$\Sigma\, a^\kappa(a^2x^2)^\lambda(a^4x^6)^\mu(a^5x^8)^\nu=\frac{1}{(1-a)(1-a^2x^2)(1-a^4x^6)(1-a^5x^8)}.$$

Now, A, P_4 and AP_4 have the degree-weights

$$3.3,\quad 3.4 \text{ and } 6.7,$$

and consequently the totality of terms in

$$\Phi,\ A\Phi,\ P_4\Phi \text{ and } AP_4\Phi$$

(that is, the totality of the pure reciprocants of extent 4) will be represented by

$$(1 + a^3x^3 + a^3x^4 + a^6x^7)\,\Sigma\, a^\kappa(a^2x^2)^\lambda(a^4x^6)^\mu(a^5x^8)^\nu$$
$$= \frac{1 + a^3x^3 + a^3x^4 + a^6x^7}{(1-a)(1-a^2x^2)(1-a^4x^6)(1-a^5x^8)}.$$

Hence the number of Pure Reciprocants of the type $w;\, i,\, 4$ is the coefficient of a^ix^w in the expansion of a fraction whose numerator is

$$1 + a^3x^3 + a^3x^4 + a^6x^7,$$

with the denominator

$$(1-a)(1-a^2x^2)(1-a^4x^6)(1-a^5x^8).$$

This fraction is called the Representative Form of the Generating Function, in contradistinction to the Crude Form, which is a fraction with the numerator

$$1 - a^{-1}x,$$

having for its denominator

$$(1-a)(1-ax)(1-ax^2)(1-ax^3)(1-ax^4).$$

The crude form expresses the fact that the number of pure reciprocants of the type

$$w;\, i,\, j$$

is
$$(w;\, i,\, j) - (w-1;\, i+1,\, j).$$

Its numerator is $1 - a^{-1}x$ for all extents; for the general case in which the extent is j, its denominator consists of the $j+1$ factors

$$(1-a)(1-ax)(1-ax^2)\ldots(1-ax^j).$$

The removal of the negative terms [corresponding to cases in which $(w;\, i,\, j) < (w-1;\; i+1,\, j)$] from the crude form would give either the representative form or one equivalent to it, according as the representative form is or is not in its lowest terms. In the parallel theory of Invariants the terms to be rejected are those for which $ij - 2w < 0$; but we do not at present know of any similar criterion for reciprocants, and are thus unable to pass directly from the crude to the representative form of their generating function.

Knowing both the crude and the representative form for reciprocants of extent 4, we may verify that the difference between these two forms of the generating function is omninegative. It will be found that

$$\frac{1 - a^{-1}x}{(1-a)(1-ax)(1-ax^2)(1-ax^3)(1-ax^4)}$$
$$= \frac{1 + a^3x^3 + a^3x^4 + a^6x^7}{(1-a)(1-a^2x^2)(1-a^4x^6)(1-a^5x^8)}$$
$$- \frac{1}{(1-ax^2)(1-ax^3)(1-ax^4)}\left(\frac{a^{-1}x + a^2x^5}{1-a^4x^6} + \frac{x^2 + a^2x^6}{1-a^5x^8}\right)$$
$$- \frac{1}{(1-ax^4)(1-a^4x^6)(1-a^5x^8)}\left(\frac{x + a^5x^{10}}{1-ax^2} + \frac{a^3x^5 + a^2x^7}{1-ax^3}\right).$$

Thus the crude form is seen to consist of an omnipositive part, equal to the representative form, and an omninegative part.

There is no difficulty in obtaining the representative form of the generating function for pure reciprocants of extents 2 and 3. In the one case every reciprocant is a rational integral function of two forms of degree-weight, 1.0 and 2.2 respectively. The generating function is therefore

$$\frac{1}{(1-a)(1-a^2x^2)}.$$

In the other case (that is, for extent 3) every pure reciprocant can be expressed as a rational integral function of four forms, of which the degree-weights are 1.0, 2.2, 3.3 and 5.6, no higher power than the first of the form 3.3 occurring in the function. Thus the representative form is

$$\frac{1+a^3x^3}{(1-a)(1-a^2x^2)(1-a^5x^6)}.$$

LECTURE XVIII.

The number of Pure Reciprocants of a given degree is finite; the number of Invariants of the same degree is infinite. Thus, for example, we have the well-known series of invariants

$$ac-b^2,\ ae-4bd+3c^2,\ \ldots,$$

all of degree 2, but of weights and extents proceeding to infinity. This may be proved from the theory of partitions (see *American Journal of Mathematics*, Vol. v., No. 1, "On Subinvariants," Excursus on Rational Fractions and Partitions). It will be seen in that article that if $N(w:i)$ is the number of ways in which w can be divided into i parts, and if P is the least common multiple of 2, 3, 4, ..., i, then $N(w:i)$ can be expressed under the form

$$F(w,i)+F'(w,i,p),$$

where p is the residue of w in respect of P.

Writing $$w+\frac{i(i+1)}{4}=\nu,$$

$F(w, i)$ is of the form

$$\frac{\nu^{i-1}}{2^2.3^2\ldots(i-1)^2.i}+\ldots,$$

all the succeeding indices of the powers of ν in $F(w, i)$ decreasing by 2, and their coefficients being transcendental functions of i which involve Bernoulli's Numbers.

In $F'(w, i, p)$ the highest index of ν is one unit less than the number of times that i is divisible by 2, that is, is $\frac{i-2}{2}$ or $\frac{i-3}{2}$, according as i is even or odd.

Thus, for the partitions of w into 3 parts, we have the formula

$$N(w:3)=\left\{\frac{\nu^2}{12}-\frac{7}{72}\right\}+\left\{\frac{1}{8}(-1)^{\nu+1}+\frac{1}{9}(\rho_1{}^\nu+\rho_2{}^\nu)\right\},$$

where $$\nu=w+\frac{1+2+3}{2}=w+3.$$

And, for the partitions of w into 4 parts,

$$N(w:4)=\left\{\frac{\nu^3}{144}-\frac{5\nu}{96}\right\}+\left\{\frac{1}{32}(-)^{\nu+1}\nu+\frac{1}{27}(\rho_1{}^{\nu+1}+\rho_2{}^{\nu+1}-\rho_1{}^{\nu-1}-\rho_2{}^{\nu-1})\right.$$
$$\left.-\frac{1}{32}(i_1{}^{\nu+1}+i_2{}^{\nu+1}-i_1{}^{\nu-1}-i_2{}^{\nu-1})\right\},$$

where $$\nu=w+\frac{1+2+3+4}{2}=w+5,$$

and ρ_1, ρ_2 are the roots of $\rho^2+\rho+1=0$,

i_1, i_2 „ „ $i^2+1=0$;

in other words, ρ_1 and ρ_2 are primitive cube roots, and i_1, i_2 primitive fourth roots of unity.

The principal term of $N(w:3)$, regarded as a function of w, is

$$\frac{w^2}{12}=\frac{w^2}{2^2\,.\,3},\text{ that of } N(w:4) \text{ being } \frac{w^3}{144}=\frac{w^3}{2^2\,.\,3^2\,.\,4}.$$

And in general the principal term of $N(w:i)$ is

$$\frac{w^{i-1}}{2^2\,.\,3^2\,.\,4^2\,\ldots\,(i-1)^2\,.\,i}.$$

Hence it follows, from a general algebraical principle, that for all values of w above a certain limit, which depends on the value of i and may be determined by the aid of partition tables, $(w;\,i,\,\infty)-(w-1;\,i+1,\,\infty)$ must become negative.

Ultimately, $\dfrac{(w-1;\ i+1,\ \infty)}{(w;\ i,\ \infty)}=\dfrac{w}{i(i+1)}$, which must eventually be greater than unity. This shows that beyond a certain value of w there can be no pure reciprocant, and consequently that the number of pure reciprocants of a given degree i is finite.

Mr Hammond remarks that the formulae for $N(w:3)$ and $N(w:4)$ may, by the substitution of trigonometrical expressions for the roots of unity, accompanied by some easy reductions, be transformed into

$$N(w:3)=\frac{\nu^2}{12}+\frac{1}{4}\sin^2\frac{\nu\pi}{2}-\frac{4}{9}\sin^2\frac{\nu\pi}{3},$$

and $$N(w:4)=\frac{\nu^3}{144}-\frac{\nu}{12}+\frac{\nu}{16}\sin^2\frac{\nu\pi}{2}+\frac{1}{8}\sin\frac{\nu\pi}{2}-\frac{2}{9\sqrt{3}}\sin\frac{\nu\pi}{3},$$

where, in the first formula, $\nu=w+3$, and in the second $\nu=w+5$. He also obtains the principal term of $N(w:i)$ from first principles as follows:

The partitions of w into i parts may be separated into two sets, the first containing at least one zero part in each of its partitions, the second consisting of partitions in which no zero part occurs.

Suppressing one zero part in each partition of the first set, we see that the number of partitions in which 0 occurs is $N(w:i-1)$. Diminishing each part by unity in those partitions which contain no zeros, their number is seen to be $N(w-i:i)$. The sum of these two numbers is $N(w:i)$, which is the total number of partitions, and consequently

$$N(w:i) = N(w:i-1) + N(w-i:i).$$

Let the principal term of $N(w:i-1)$ be αw^{i-2}, where α is independent of w, and write

$$w = ix,\ N(w:i) = u_x,\ N(w-i:i) = u_{x-1}.$$

Then $$u_x - u_{x-1} = \alpha w^{i-2} + \ldots = \alpha i^{i-2} x^{i-2} + \ldots.$$

Hence, by a simple summation, we find

$$u_x = \alpha i^{i-2}\{x^{i-2} + (x-1)^{i-2} + (x-2)^{i-2} + \ldots\} + \ldots.$$

But, since only the principal term of u_x is required, this summation may be replaced by an integration. Thus the principal term of u_x is

$$\alpha i^{i-2}\int x^{i-2}dx = \frac{\alpha i^{i-2}x^{i-1}}{i-1}.$$

Restoring $$w = ix \text{ and } N(w:i) = u_x,$$

we see that the principal term of $N(w:i)$ is $\dfrac{\alpha w^{i-1}}{(i-1)i}$. Thus the principal term of $N(w:i)$ is found from that of $N(w:i-1)$ by multiplying it by

$$\frac{w}{(i-1)i}.$$

When $i=3$, the principal term is $\dfrac{w^2}{2^2.3}$; it is therefore $\dfrac{w^3}{2^2.3^2.4}$ when $i=4$; and for the general case it is $\dfrac{w^{i-1}}{2^2.3^2.4^2\ldots(i-1)^2.i}$.

The value of $N(w:i)$ is given in line i and column w of the following table:

	1	2	3	4	5	6	7	8	9	10	11	12	13	14
2	1	2	2	3	3	4	4	5	5	6	6	7	7	8
3	1	2	3	4	5	7	8	10	12	14	16	19	21	24
4	1	2	3	5	6	9	11	15	18	23	27	34	39	47
5	1	2	3	5	7	10	13	18	23	30	37	47	57	70
6	1	2	3	5	7	11	14	20	26	35	44	58	71	90

From an inspection of the tabulated values of $N(w:i)$ we see that

$N(w:2) - N(w-1:3)$ is negative or zero when $w > 2$,
$N(w:3) - N(w-1:4)$ „ „ $w > 6$,
$N(w:4) - N(w-1:5)$ „ „ $w > 8$,
$N(w:5) - N(w-1:6)$ „ „ $w > 12$.

Hence for pure reciprocants of indefinite extent, whose degrees are

2, 3, 4, 5,

the highest possible weights are 2, 6, 8 and 12, respectively.

In like manner, from Euler's table, in his memoir "De Partitione Numerorum" (published in 1750), it will be found that

for degrees	2	3	4	5	6	7	8	9	10	11	12	13
the highest weights are	2	6	8	12	16	21	26	30	36	42	49	55

Further than this the table, which goes up to $w = 59$, will not enable us to proceed.

The actual number of pure reciprocants of degree i, weight w, and of indefinite extent, is seen in the following table, which gives the value of $N(w:i) - N(w-1:i+1)$ when positive, blank spaces being left in the table when this difference is zero or negative.

Degree $i=$ \ Weight $w=$	2	3	4	5	6	7	8	9	10	11	12	13	14
2	1												
3	1	1	1		1								
4	1	1	2	1	2	1	2						
5	1	1	2	2	3	2	4	3	4	2	3		

Thus, for degree 2, there is only one pure reciprocant, namely

$$(ac) = 4ac - 5b^2.$$

For degree 3 the table shows that, in addition to the compound form

$$a(ac) = a(4ac - 5b^2),$$

there are three others whose weights are 3, 4 and 6 respectively.

These are the three protomorphs,

$$(a^2d) = a^2d - 3abc + 2b^3,$$
$$(a^2e) = 50a^2e - 175abd + 28ac^2 + 105b^2c,$$
$$(a^2g) = 14a^2g - 63abf - 1350ace + 1782b^2e + 1470ad^2 - 4158bcd + 2310c^3.$$

With the above forms and a we are able to form the following compounds of degree 4:

$$a^2(ac),\quad a(a^2d),\quad (ac)^2,\quad a(a^2e),\quad a(a^2g),$$

whose weights are 2, 3, 4, 4, 6.

The forms of degree 4 and weights 5, 7, 8, and one of the forms of weight 6, cannot be similarly made up of forms of inferior degree, and are therefore groundforms. Three of them are the protomorphs (a^3f), (a^3h) and (a^3i) of weights 5, 7 and 8, whose values were given in Lecture XVI. The groundform of weight 6 is the Quasi-Catalecticant given in the last lecture. All the forms of degree 4 have thus been accounted for except one of the two forms of weight 8, which will be seen to be of extent 6, and to have a^2cg for its leading term.

We know from Euler's table that $N(8:4) - N(7:5) = 2$; that is,

$$(8\,;\,4,\,8) - (7\,;\,5,\,8) = 2.$$

Now, $(8\,;\,4,\,7) = N(8:4) - 1$, the omitted partition being $8\,.\,0\,.\,0\,.\,0$,

$(8\,;\,4,\,6) = N(8:4) - 2$, the partition $7\,.\,1\,.\,0\,.\,0$ being also left out,

$(8\,;\,4,\,5) = N(8:4) - 4$, $\left\{\begin{array}{l}\text{for } 6\,.\,2\,.\,0\,.\,0 \text{ and } 6\,.\,1\,.\,1\,.\,0 \text{ are excluded from} \\ (8\,;\,4,\,5), \text{but make their appearance in } (8\,;\,4,\,6).\end{array}\right.$

Similarly,

$$(7\,;\,5,\,7) = N(7:5),$$
$$(7\,;\,5,\,6) = N(7:5) - 1,$$
$$(7\,;\,5,\,5) = N(7:5) - 2.$$

We have, therefore,

$$(8\,;\,4,\,8) - (7\,;\,5,\,8) = 2,$$
$$(8\,;\,4,\,7) - (7\,;\,5,\,7) = 1,$$
$$(8\,;\,4,\,6) - (7\,;\,5,\,6) = 1,$$
$$(8\,;\,4,\,5) - (7\,;\,5,\,5) = 0.$$

Hence we may draw the following inferences:

(1) No pure reciprocant exists whose type is 8 ; 4, 5.

(2) The one whose type is 8 ; 4, 6 must contain the letter g.

(3) No fresh form is found by making the extent 7 instead of 6, so that there is no pure reciprocant of weight 8 and degree 4 whose *actual extent* is 7.

(4) There is a pure reciprocant (the Protomorph whose leading term is a^3i) whose actual extent is 8.

(5) This, with the one whose actual extent is 6, makes up the two given by $(8\,;\,4,\,8) - (7\,;\,5,\,8) = 2$.

LECTURE XIX.

The following is a complete list of the irreducible reciprocants of indefinite extent for the degrees 2, 3 and 4:

Deg. wt.	
2 . 2	(ac),
3 . 3	(a^2d),
3 . 4	(a^2e),
3 . 6	(a^2g),
4 . 5	(a^3f),
4 . 6	(a^2ce),
4 . 7	(a^3h),
4 . 8	(a^3i), (a^2cg).

The values of all of them except (a^2cg) have been given in previous lectures, and the method of obtaining them sufficiently indicated. Thus (ac), (a^2d), (a^2e), (a^3f), (a^2g), (a^3h) and (a^3i) are the Protomorphs of minimum degree P_2, P_3, P_4, P_5, P_6, P_7 and P_8, respectively; and (a^2ce) is the Quasi-Catalecticant whose value has been set forth in the table of irreducible forms of extent 4. It will be remembered that (a^2ce) was found by combining the Quasi-Discriminant (a^3d^2) with P_2P_4 linearly in such a manner that the combination, which is of the 5th degree, divides by a and gives (a^2ce) of the 4th degree. If we try to find (a^2cg) by a similar process, it will be necessary to rise as high as the 7th degree, and then to drop down by successive divisions by a to the fourth.

In fact, since to a numerical factor *près* the residues of

	$P_2,\ P_3,\ P_4,\ P_5$
are	$b^2,\ b^3,\ b^2c,\ b^3c$,
that of	P_3P_5 will be b^6c,
and that of	$P_2{}^2P_4$ will be b^6c.

Thus a linear combination of P_3P_5 and $P_2{}^2P_4$ will be divisible by a, and, taking account of the numerical coefficients, we shall find

$$26P_2{}^2P_4 + 875P_3P_5 \equiv 0 \text{ (mod. } a).$$

As a result of calculation, it will be seen that the above combination of the protomorphs divided by a,

$$\frac{1}{a}(26P_2{}^2P_4 + 875P_3P_5),$$

has (to a numerical factor *près*) the same residue as $P_4{}^2$.

Making a second combination and division by a, we find

$$7\left(\frac{26P_2^2P_4+875P_3P_5}{a}\right)-25P_4^2\equiv 0 \text{ (mod. } a) = aS, \text{ suppose.}$$

Then, by actual calculation, the residue of S is found to be

$$-262500b^4e+612500b^3cd-339080b^2c^3.$$

Two reductions have already been made in obtaining this form S of the 5th degree. A final combination of S with P_2P_6 and the form (a^3e^2), whose value was given in a former lecture, enables us to divide out once more by a and thus get the form (a^2cg) of the 4th degree.

It is the fact that P_2P_6 and (a^3e^2) have residues which are not the same to a numerical factor *près* which necessitates the long calculation above described. No linear combination of P_2P_6 and (a^3e^2) with one another is divisible by a, and it is necessary to find a third form S a linear combination of which with both P_2P_6 and (a^3e^2) will divide by a.

There is, however, another way of arriving at the form (a^2cg) by using the eductive generator

$$G=4(ac-b^2)\partial_b+5(ad-bc)\partial_c+6(ae-bd)\partial_d+\ldots.$$

Starting with the Quasi-Catalecticant

$$(a^2ce)=800a^2ce-1000ab^2e-875a^2d^2+2450abcd-1344ac^3-35b^2c^2,$$

and operating on it with G, we have

$$\begin{aligned}G(a^2ce)=&\,4(ac-b^2)(-2000abe+2450acd-70bc^2)\\&+5(ad-bc)(800a^2e+2450abd-4032ac^2-70b^2c)\\&+6(ae-bd)(-1750a^2d+2450abc)\\&+7(af-be)(800a^2c-1000ab^2).\end{aligned}$$

The terms of this expression contain the common numerical factor 10, which may be rejected; thus we have

$$G(a^2ce)=10(a^3cf),$$

where $(a^3cf)=560a^3cf-700a^2b^2f-650a^3de-290a^2bce+1500ab^3e$

$$+2275a^2bd^2-1036a^2c^2d-3710ab^2cd+1988abc^3+63b^3c^2.$$

This form (a^3cf) is the first educt of (a^2ce), and is irreducible (but, being of the fifth degree, does not appear in our list, which contains no forms of higher degree than the fourth). Operating on it with G, we obtain the educt of (a^3cf), which is the second educt of (a^2ce). This second educt will be of the 6th degree (its leading term will be a^4cg), but is reducible to the 5th when combined with

$$(4ac-5b^2)(a^2ce),$$

as we know from the general theorem concerning the reduction of second educts. We shall thus obtain a form (a^3cg), the reduced second educt of (a^2ce), of the 5th degree, and a final combination of (a^3cg) with one or both of the forms P_2P_6 and (a^3e^2) will enable us to divide once more by a and thus arrive at (a^2cg) of the 4th degree.

By either of these methods we obtain

$$\begin{aligned}(a^2cg) = 1176a^2cg - 8085a^2df + 7040a^2e^2 - 1470ab^2g + 18963abcf \\ - 16940abde - 27160ac^2e + 26460acd^2 - 9555b^3f \\ + 28098b^2ce + 12740b^2d^2 - 52822bc^2d + 21560c^4;\end{aligned}$$

but the second way, besides being more direct, gives us at the same time the value of the irreducible form (a^3cf).

Every Pure Reciprocant is an Invariant of a Binary Quantic whose coefficients A, B, C, D, ... are functions of the original elements a, b, c, d, ... such that

$$\begin{aligned} VA &= 0, \\ VB &= A, \\ VC &= 2B, \\ VD &= 3C, \\ &\ldots\ldots\ldots \end{aligned}$$

and conversely, every Invariant of this Binary Quantic, or of a system of such Binary Quantics, is a Pure Reciprocant.

This is a particular case of the more general theorem, due to Mr Hammond, that if Θ is the operator,

$$\phi_1(a)\partial_b + \phi_2(a, b)\partial_c + \phi_3(a, b, c)\partial_d + \ldots,$$

where ϕ_1, ϕ_2, ϕ_3, ... are arbitrary rational integral functions, and if

$$A, B, C, D, \ldots, A', B', C', D', \ldots, A'', B'', C'', \ldots$$

be any rational integral functions of the original letters a, b, c, ... which satisfy the conditions

$$\begin{array}{lll} \Theta A = 0, & \Theta A' = 0, & \Theta A'' = 0, \\ \Theta B = A, & \Theta B' = A', & \Theta B'' = A'', \\ \Theta C = 2B, & \Theta C' = 2B', & \Theta C'' = 2B'', \\ \Theta D = 3C, & \Theta D' = 3C', & \Theta D'' = 3C'', \\ \ldots\ldots\ldots & \ldots\ldots\ldots & \ldots\ldots\ldots \end{array}$$

then every invariant in respect to the elements

$$A, B, C, D, \ldots, A', B', C', D', \ldots, A'', B'', C'', D'', \ldots$$

is a rational integral solution of the equation

$$\Theta = 0.$$

Obviously, every rational integral solution of $\Theta = 0$ is an invariant in the above elements, so that the converse of the proposition is true. For the only conditions imposed upon A, A', A'', ... are that they shall be rational integral functions of $a, b, c, d, \ldots$ annihilated by Θ. Let

$$\Phi(A, B, C, D, \ldots, A', B', C', D', \ldots, A'', B'', C'', D'', \ldots)$$

be any invariant in the large letters. We have to show that

$$\Theta\Phi = 0.$$

Now,
$$\Theta\Phi = \frac{d\Phi}{dA}\Theta A + \frac{d\Phi}{dB}\Theta B + \frac{d\Phi}{dC}\Theta C + \ldots$$
$$+ \frac{d\Phi}{dA'}\Theta A' + \frac{d\Phi}{dB'}\Theta B' + \frac{d\Phi}{dC'}\Theta C' + \ldots$$
$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Hence, writing for ΘA, ΘB, ΘC, ..., their values given above, we have

$$\Theta\Phi = (A\partial_B + 2B\partial_C + 3C\partial_D + \ldots)\Phi$$
$$+ (A'\partial_{B'} + 2B'\partial_{C'} + 3C'\partial_{D'} + \ldots)\Phi$$
$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$= 0 \text{ (since } \Phi \text{ is an invariant)};$$

which proves the proposition.

Confining our attention to a single set of letters, the Binary Quantic

$$(A, B, C, \ldots J, K, L \between x, y)^n,$$

whose coefficients are formed from one another by the successive operation of Θ as above, may be called a Quasi-Covariant; and it will follow immediately from the Theory of Binary Forms that every Covariant of a Quasi-Covariant is itself a Quasi-Covariant, and that every Invariant of any Quasi-Covariant (or system of Quasi-Covariants) is an Invariant in respect to the letters $A, B, C, \ldots$, and therefore, by what precedes, a rational integral solution of $\Theta = 0$.

Writing the terms of

$$(A, B, C, \ldots J, K, L \between x, y)^n$$

in reverse order, we have

$$Ly^n + nKxy^{n-1} + \frac{n(n-1)}{1.2}Jxy^{n-2} + \ldots + Ax^n,$$

where
$$\Theta L = nK,\ \Theta K = (n-1)J,\ \ldots \Theta A = 0.$$

Thus the Quasi-Covariant may be written

$$Ly^n + \Theta Lxy^{n-1} + \frac{\Theta^2 L}{1.2}x^2y^{n-2} + \ldots + \frac{\Theta^n L}{1.2.3\ldots n}x^n = y^n\left(e^{\frac{x\Theta}{y}}\right)L,$$

where $\Theta^{n+1}L = 0$.

This is the general symbolic expression for a Quasi-Covariant. An example of a Quasi-Covariant has already been given in Lecture II. [p. 310, above], where it was stated, and afterwards proved [p. 360], that the reciprocal of the nth modified derivative could be put under the form

$$-t^{-n-3}\left(e^{-\frac{V}{t}}\right)a_n.$$

The numerator of this reciprocal expression, which may be called the reciprocal function, is

$$t^n\left(e^{-\frac{V}{t}}\right)a_n,$$

which is identical with the general expression

$$y^n\left(e^{\frac{x\Theta}{y}}\right)L,$$

if

$$x=-1,\ y=t,\ L=a_n \text{ and } \Theta=V.$$

Hence every Invariant of the reciprocal function is a Pure Reciprocant.

This property of the reciprocal function was discovered independently by Mr C. Leudesdorf, who published his results in the *Proceedings of the London Mathematical Society* (Vol. XVII. p. 208). Mr Hammond's results were given in two letters to me dated January 15th and January 20th, 1886, and were briefly alluded to by him at a meeting of the London Mathematical Society. They are here published for the first time.

Recalling the form of the operator

$$\Theta=\phi_1(a)\partial_b+\phi_2(a, b)\partial_c+\phi_3(a, b, c)\partial_d+\ldots,$$

where $\phi_1, \phi_2, \phi_3, \ldots$ are rational integral functions, we can form a Quasi-Covariant of extent j by a finite number of successive operations on a single letter of that extent.

To fix the ideas, take the letter d of extent 3, and operate on it with Θ; then

$$\Theta d=\phi_3(a, b, c).$$

Since $\phi_1, \phi_2, \phi_3, \ldots$ are by definition rational integral functions, we can, by operating a finite number of times with Θ, remove first c and then b from $\phi_3(a, b, c)$, and thus obtain

$$\Theta^n d=\text{funct. } a,$$

where n denotes a finite number of operations. Since $\Theta a=0$, we have

$$\Theta^{n+1}d=0.$$

In this manner we form the Quasi-Covariant of the nth order

$$y^n\left(e^{\frac{x\Theta}{y}}\right)d.$$

If $\phi_2, \phi_3, \phi_4, \ldots$ do not contain higher powers than the first of the last letter in each, the order of the above Quasi-Covariant will be the same as its extent. This is the case with the reciprocal function, which is a co-reciprocant (that is, a Quasi-Covariant relative to V).

Ex. $$y^2\left(e^{\frac{xV}{y}}\right)c = cy^2 + Vcxy + \frac{V^2c}{1.2}x^2 = cy^2 + 5abxy + 5a^3x^2.$$

The discriminant of this is the pure reciprocant

$$5a^2\left(ac - \frac{5b^2}{4}\right).$$

As an additional example, consider the pair of linear co-reciprocants

$$4a(4ac - 5b^2)x + (5ad - 7bc)y,$$

$$50a(a^2d - 3abc + 2b^3)x + (25abd - 32ac^2 + 5b^2c)y.$$

The resultant of this pair is

$$2a(125a^3d^2 - 750a^2bcd + 500ab^3d + 256a^2c^3 + 165ab^2c^2 - 300b^4c),$$

that is, is the Quasi-Discriminant multiplied by $2a$.

LECTURE XX

"Quintessenced into a finer substance."—*Drummond of Hawthornden.*

Before proceeding with the proper subject of this day's lecture, I should like to mention a geometrical theorem which has fallen in my way, and which, *inter alia*, gives an immediate proof of the existence of 27 straight lines on a general cubic surface. It is proved by means of a Lemma (itself of quasi-geometrical origin) which finds its principal application in an extension of Bring's or Tschirnhausen's method, and shows how any number of specified terms, reckoning from either end, can be taken away from any equation of a sufficiently high degree*.

Subjectively speaking, I was led to the Lemma by considering the question, closely connected with Differential Invariants, of the method of depriving a linear differential equation of several terms.

Let ϕ be a cubic and u a linear function in x, y, z, t, say

$$\phi = ax^3 + \ldots + fx^2y + \ldots,$$

$$u = lx + my + nz + pt.$$

Then, if ψ is a scroll which contains all the straight lines on $\phi + \lambda u^3$, when the parameter λ has any arbitrary numerical value from $+\infty$ to $-\infty$, I prove that

$$\psi = \phi^2 A + \phi u^3 B + u^6 C,$$

* I recover all Hamilton's results contained in his Report to the British Association, 1836, "On Jerrard's Method," in a much more clear and concise manner, and make important additions to his theory.

where ψ is of the degree 15 in the variables x, y, z, t,

.... 6 in the coefficients (l, m, n, p) of u,

.... 11 .... $(a, \ldots\ldots\ldots)$ of ϕ.

Or, more briefly, in	x	l	a
ψ is of degree	15	6	11, and consequently
C...............	9	0	11.

The intersections of ϕ with ψ are its intersections with u^6 and with C, of which the intersections with the arbitrary plane u^6 are clearly foreign to the question, but the cubic ϕ and the $9^c C$ intersect in 27 straight lines, which are the 27 ridges on ϕ.

C is identical with the covariant found by Clebsch and given in Salmon's *Geometry of Three Dimensions* at the end of the chapter on Cubic Surfaces. It may with propriety be called the Clebschian.

By giving the parameter λ (which occurs in $\phi + \lambda u^3$) an infinitesimal variation, it is easily proved that

$$B = -2EC, \quad A = E^2C, \quad E^3C = 0,$$

where E is the operator $l^3\partial_a + \ldots + 3l^2m\partial_f + \ldots$, which may be simply and completely defined by its property of changing the general cubic ϕ into $(lx + my + nz + pt)^3$.

The equation $E^3C = 0$ expresses a new property of the Clebschian: it shows that if a, f are the coefficients of x^3 and any other term in ϕ containing x^2, neither a^3 nor a^2f can occur in any one of the terms of C. Defining a principal term in ϕ as one which contains the cube of one of the variables, and a term adjacent to it as one which contains the square of the same variable, this is equivalent to saying that neither the cube of the coefficient of a principal term nor its square multiplied by the coefficient of any adjacent term can appear in any of the terms of C.

An interesting special case of the general theorem is when the arbitrary plane u is taken to be one of the planes of reference, say $u = x$. Then

$$l = 1, \quad m = 0, \quad n = 0, \quad p = 0,$$

and the operator E becomes simply $\dfrac{d}{da}$. Thus we learn that

$$\phi^2 \frac{d^2C}{da^2} - 2x^3\phi \frac{dC}{da} + x^6C$$

is a Scroll of the fifteenth order which contains all the Ridges on

$$\phi + \lambda x^3$$

for any arbitrary value of the parameter λ.

It also contains 6 times over the curve of intersection of $\phi = 0$ with $x = 0$.

I now propose to give the substance, with a brief commentary, of some very interesting letters I have recently received from Capt. MacMahon. I abstain from giving a proof of his results, as I am informed that he intends to do this himself at an early meeting of the London Mathematical Society.

Using V to signify the Reciprocant Annihilator and Ω the Annihilator of Invariants, we have studied the properties of

$$V\frac{d}{dx}-\frac{d}{dx}V$$

and those of

$$\Omega\frac{d}{dx}-\frac{d}{dx}\Omega.$$

These may be written in the form

$$\begin{vmatrix} V\dfrac{d}{dx} \\ V\dfrac{d}{dx} \end{vmatrix} \qquad \begin{vmatrix} \Omega\dfrac{d}{dx} \\ \Omega\dfrac{d}{dx} \end{vmatrix},$$

and may be called alternants to $V, \dfrac{d}{dx}$ and to $\Omega, \dfrac{d}{dx}$ respectively.

It has been shown in Lecture VII. [p. 341, above] that

$$V\frac{d}{dx}-\frac{d}{dx}V=2(3i+w)a.$$

The corresponding formula is

$$\Omega\frac{d}{dx}-\frac{d}{dx}\Omega=3i+2w,$$

as may be seen by writing $\kappa=0$, $\lambda=3$, $\mu=4$, $\nu=5$, ... in a more general formula given in Lecture V. [p. 329, above].

Observe that operating with the alternant to $\Omega, \dfrac{d}{dx}$ is equivalent to multiplication by a number, and that operating with the alternant to $V, \dfrac{d}{dx}$ merely introduces a numerical multiple of a as a factor. No such property exists for the Alternant

$$V\Omega-\Omega V,$$

but one much more extraordinary.

MacMahon has found that this alternant, which he calls J, is a generator to a Reciprocant and a generator to an Invariant; that is, it converts a Reciprocant into another Reciprocant, and an Invariant into another Invariant. As regards a Differential Invariant, which is at once an Invariant and a Reciprocant, it is an Annihilator. He shows, in fact, that

$$\Omega J-J\Omega=0$$

and

$$VJ-JV=0.$$

If, then, $\Omega R = 0$, it follows immediately that $\Omega(JR) = 0$; that is, if R is an invariant, JR is so too. And in like manner, if

$$VR = 0, \quad V(JR) = 0,$$

that is, if R is a reciprocant, so is JR.

Of course, if M is a Differential Invariant,

$$JM = V(\Omega M) - \Omega(VM) = 0.$$

Let me here give a caution which may be necessary: The fact that a form is annihilated by J is not sufficient to show that it is a Differential Invariant, though all Differential Invariants are necessarily annihilated by J. Forms exist which are subject to annihilation by

$$J = a^2\partial_c + 3ab\partial_d + \ldots,$$

but are, notwithstanding, *neither* invariants nor reciprocants.

Such a form is the monomial b, which is obviously annihilated by J. Another is $ad - 3bc$. For, since

$$a^2d - 3abc + 2b^3$$

is a Differential Invariant, we have

$$J(a^2d - 3abc + 2b^3) = 0.$$

But $$Jb^3 = 0 \quad \text{and} \quad Ja = 0;$$

therefore, also, $$aJ(ad - 3bc) = 0.$$

The general theorem is as follows, and is a most remarkable one: If we write

$$\begin{aligned} mP(m, \mu, v, n) = \mu a^m \partial_{a_n} &+ (\mu + v)\, m a^{m-1} b \partial_{a_{n+1}} \\ &+ (\mu + 2v)\left(m a^{m-1} c + \frac{m(m-1)}{2} a^{m-2} b^2\right) \partial_{a_{n+2}} \\ &+ (\mu + 3v)\left\{m a^{m-1} d + m(m-1) a^{m-2} bc \right. \\ &\left. + \frac{m(m-1)(m-2)}{6} a^{m-3} b^3\right\} \partial_{a_{n+3}} + \ldots, \end{aligned}$$

where the coefficients of the terms inside the brackets are the same as those of the corresponding terms in the expansion of $(a + b + c + \ldots)^m$, and where a_n stands for the nth letter of the series $a, b, c, d, \ldots$, then Capt. MacMahon establishes that *the alternant of any two P's is another P.*

A question here suggests itself naturally: What would be the alternant of three or more P's? For instance, would the alternant

$$\begin{vmatrix} P_1 & P_2 & P_3 \\ P_1 & P_2 & P_3 \\ P_1 & P_2 & P_3 \end{vmatrix} = P_1P_2P_3 - P_1P_3P_2 + P_2P_3P_1 - P_2P_1P_3 + P_3P_1P_2 - P_3P_2P_1$$

be another P?*

* In my Multiple Algebra investigations, which I hope some day to resume, I have made important use of similar Alternants, which, it may be noticed, do not vanish when their elements

Moreover, he obtains expressions for the parameters m, μ, v, n of the resulting P in terms of the parameters of its two components. He proves that if P_1, P_2 are the two components whose alternant is P, supposing

m_1, μ_1, v_1, n_1 to be the parameters of P_1,
m_2, μ_2, v_2, n_2 .... P_2,

then the parameters m, μ, v, n of their resultant P are given by the equations

$$m = m_1 + m_2 - 1,$$
$$\mu = (m_1 + m_2 - 1)\left\{\frac{\mu_2}{m_2}(\mu_1 + n_2 v_1) - \frac{\mu_1}{m_1}(\mu_2 + n_1 v_2)\right\},$$
$$v = (n_2 - n_1)\, v_1 v_2 - \frac{m_2 - 1}{m_1}\mu_1 v_2 + \frac{m_1 - 1}{m_2}\mu_2 v_1,$$
$$n = n_1 + n_2.$$

It will be seen that Ω and V are special forms of P. Thus,

$$\Omega = P(1, 1, 1, 1),$$
$$V = P(2, 4, 1, 1).$$

Now, if the second and third parameters are zero, every term of P vanishes, and MacMahon finds that in the following two cases the second and third parameters of the resultant above given vanish.

(1) Supposing $\frac{\mu_1}{m_1 v_1}$ to be an integer, this takes place when the two component systems of parameters are

$$m_1,\ \mu_1,\quad v_1,\quad n_1,$$
$$m_2,\ \mu_1 m_2,\ m_1 v_1,\ n_1 + \frac{\mu_1}{m_1 v_1}(m_2 - m_1).$$

(2) When they are

$$m_1,\ \mu_1,\quad v_1,\quad n_1,$$
$$m_2,\ n_1 m_2,\ m_1 - 1,\ \frac{\mu_1}{m_1 v_1}(m_2 - 1).$$

Now,
$$P(1, 1, 1, 1) = \Omega,$$
$$P(2, 4, 1, 1) = V,$$

and by the law of composition

$$J = \Omega V - V\Omega = P(2, 2, 1, 2).$$

Also, $\left.\begin{matrix} 2, 2, 1, 2 \\ 1, 1, 1, 1 \end{matrix}\right\}$ will be found to come under the first case;

and $\left.\begin{matrix} 2, 2, 1, 2 \\ 2, 4, 1, 1 \end{matrix}\right\}$ .... the second.

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are non-commutative. In this connection it is well worthy of observation that the P's (as indeed would be true of any operators linear in the differential inverses) obey the *associative* law.

It would be interesting to ascertain under what arithmetical conditions, if any, other than MacMahon's, *any* two linear operators of the same general form as his P's become commutative.

Perhaps it would also be worthy of inquiry whether the P theory might not admit of extension in some form to operators non-linear in the differential inverses, and whether to every such operator of degrees i and j in the letters and their differential inverses there is not correlated another in which i and j are interchanged.

Hence, $\Omega J - J\Omega = 0$ and $VJ - JV = 0$.

The above theorem is one of extraordinary beauty, and must play an important part in the future of Algebra.

In another letter Capt. MacMahon calls my attention to the fact that the operator called by me Cayley's generator P, in Lecture IV. of this course [p. 323, above], is a particular case of one of a much more general character given by him in the *Quarterly Mathematical Journal* (Vol. XX., p. 362).

He also states that every pure reciprocant, when multiplied by the needful power of a, is an invariant of the binary quantic

$$\{2.(2n+1)!\}\,a^{n+1} - n\,\{1!\,(2n+1)!\}\,a^{n-1}bt + \frac{n(n-1)}{1.2}\{2!\,(2n)!\}\left\{a^{n-2}c + \frac{n-2}{2}a^{n-3}b^2\right\}t^2$$

$$-\frac{n(n-1)(n-2)}{1.2.3}\{3!\,(2n-1)!\}\left\{a^{n-3}d + (n-3)\,a^{n-4}bc + \frac{(n-3)(n-4)}{1.2.3}a^{n-5}b^3\right\}t^3$$

$$+ \ldots\ldots\ldots$$

which I have written in the non-homogeneous form.

But this expression is (to a numerical factor *près*) identical with the numerator of $\dfrac{d^{n+2}x}{dy^{n+2}}$ when $t, a, b, \ldots$ are taken to be the modified differential derivatives $\dfrac{dy}{dx}, \dfrac{1}{2}\dfrac{d^2y}{dx^2}, \dfrac{1}{2.3}\dfrac{d^3y}{dx^3}, \ldots$. See my note on Burman's law for the Inversion of the Independent Variable [Vol. II. of this Reprint, p. 44].

The property that its invariants are pure reciprocants has already been proved in the lectures [above, p. 412].

LECTURE XXI.

I take blame to myself for not earlier communicating to the class the substance of a note of Mr Hammond's under date of January 20th, 1886, in which he makes an interesting application of the theorem that any invariant of the form

$$y^n\left(e^{\frac{x}{y}V}\right)F(a, b, c, \ldots),$$

in which the function F is subject to the condition

$$V^{n+1}F = 0,$$

or of any combination of such forms, is a pure reciprocant.

Forms such as the above, whose invariants are pure reciprocants, he calls *co-reciprocants.* It follows that any covariant of one or more co-reciprocants is itself a co-reciprocant, for any invariant of a covariant is an invariant.

Taking F to be a single letter b, c, d, he forms the functions

$$by + 2a^2x, \qquad (1)$$

$$cy^2 + 5abxy + 5a^3x^2, \qquad (2)$$

$$dy^3 + 3(2ac + b^2)xy^2 + 21a^2bx^2y + 14a^4x^3, \qquad (3)$$

in which

$$2a^2 = Vb,$$

$$5ab = Vc,\quad 5a^3 = \frac{V^2c}{1.2},$$

$$3(2ac + b^2) = Vd,\quad 21a^2b = \frac{V^2d}{1.2},\quad 14a^4 = \frac{V^3d}{1.2.3}.$$

On writing $y = t$, $x = -1$, it will be observed that these three forms are the numerators of

$$\frac{1}{3!}\frac{d^3x}{dy^3},\quad \frac{1}{4!}\frac{d^4x}{dy^4},\quad \frac{1}{5!}\frac{d^5x}{dy^5}.$$

The Jacobian of (1) and (2) is

$$(4ac - 5b^2)\,ay;$$

the coefficient of ay is the familiar pure reciprocant $4ac - 5b^2$.

The Jacobian of (1) and (3) is the determinant

$$\begin{vmatrix} b & 2a^2 \\ dy^2 + (4ac - 5b^2)xy & (2ac + b^2)y^2 \end{vmatrix},$$

which is divisible by y, giving the quotient

$$(2a^2d - 2abc - b^3)y + 2a^2(4ac - 5b^2)x. \qquad (4)$$

This is

$$y\left(e^{\frac{x}{y}V}\right)(2a^2d - 2abc - b^3),$$

the terms involving $\frac{x^2}{y}$, $\frac{x^3}{y^2}$, ... vanishing identically.

Looking at $2a^2d - 2abc - b^3$ as the anti-source to a Co-reciprocant*, we might at first sight expect that it would give rise to a co-reciprocant of the third order in x, y, whereas we see it is the anti-source of a linear co-reciprocant.

* What differentiates Reciprocants from Invariants is that we have no reverser to V as O is to Ω in the theory of Invariants, that is, no reverser which does not introduce an additional letter.

The coefficients of a covariant are obtained either from the source by continually operating with O, or from the anti-source by continually operating with Ω. But in the case of a co-reciprocant, we are only able to proceed in one direction (namely from the anti-source, or coefficient of the highest power of y, to the source), as we have only one operator, V, at our disposal.

We have $$V(2a^2d - 2abc - b^3) = 2a^2(4ac - 5b^2).$$
Combining this with
$$V(a^2d - 3abc + 2b^3) = 0 \quad \text{(the well-known Mongian),}$$
and dividing by a, he obtains
$$V(5ad - 7bc) = 4a(4ac - 5b^2).$$
Hence $$(5ad - 7bc)\,y + 4a(4ac - 5b^2)\,x \qquad (5)$$
is a co-reciprocant. It is in fact (4) reduced in degree.

The Jacobian of (5) and of $cy^2 + 5abxy + 5a^3x^2$, that is,
$$\begin{vmatrix} 5ad - 7bc & 4a(4ac - 5b^2) \\ 2cy + 5abx & 5aby + 10a^3x \end{vmatrix},$$
will divide by a, and gives the new linear co-reciprocant
$$(25abd - 32ac^2 + 5b^2c)\,y + 50a(a^2d - 3abc + 2b^3)\,x. \qquad (6)$$

The coefficient of y is of weight 4, but instead of giving rise to a co-reciprocant of the 4th order, we see that this again is the anti-source of a linear co-reciprocant.

The resultant of the two linear co-reciprocants (4) and (6) divided by a numerical multiple of a gives the well-known Quasi-Discriminant $125a^3d^2 + \ldots$, as was stated at the end of Lecture XIX [above, p. 413].

The noticeable fact is that (including $by + 2a^2x$) there exist 3 linear independent co-reciprocants of extent 3. Probably there are no more, but this requires proof.

The promised land of Differential Invariants or Projective Reciprocants is now in sight, and the remainder of the course will be devoted to its elucidation. Twenty lectures have been given on the underlying matter, and probably ten more, at least, will have to be expended on this higher portion of the theory.

One is surprised to reflect on the change which has come over the face of Algebra in the last quarter of a century. It is now possible to enlarge to an almost unlimited extent on any branch of it. These thirty lectures, embracing only a fragment of the theory of reciprocants, might be compared to an unfinished epic in thirty cantos. Does it not seem as if Algebra had attained to the character of a fine art, in which the workman has a free hand to develop his conceptions as in a musical theme or a subject for painting? Formerly it consisted almost exclusively of detached theorems, but now-a-days it has reached a point in which every properly developed algebraical composition, like a skilful landscape, is expected to suggest the notion of an infinite distance lying beyond the limits of the canvas.

It is quite conceivable that the results we have been investigating may be descended upon from a higher and more general point of view. Many circumstances point to such a consummation being probable. But man must creep before he can walk or run, and a house cannot be built downwards from the roof. I think the mere fact that our work enables us to simplify and extend the results obtained by so splendid a genius as M. Halphen, is sufficient to convey to us the assurance that we have not been beating the wind or chasing a phantom, but doing solid work. Let me instance one single point: M. Halphen has succeeded, by a prodigious effort of ingenuity, in obtaining the differential equation to a cubic curve with a given absolute invariant. His method involves the integration of a complicated differential equation. In the method which I employ the same result is obtained by a simple act of substitution in an exceedingly simple special form of Aronhold's S and T, capable of being executed in the course of a few minutes on half a sheet of paper, without performing any integration whatever. This will be seen to be a simple inference from the theorem invoked under three names, to which allusion has been made in a preceding lecture and the demonstration of which will shortly occupy our attention.

Before entering upon the theory of Differential Invariants, I think it desirable to bring forward the exceedingly valuable and interesting communication with which I have been favoured by M. Halphen establishing *à priori* the existence of *invariants* in general.

Sur l'Existence des Invariants.

(*Extracted from a Letter of M. Halphen to Professor Sylvester.*)

Dans des théories diverses on a rencontré des Invariants sans qu'on ait pénétré la cause générale de leur existence. C'est cette lacune qu'il s'agit ici de faire disparaître.

1. Soient $A, B, \ldots, L$ des quantités auxquelles on puisse attribuer des valeurs *ad libitum.*

Une *substitution* consiste à remplacer ces quantités $(A, B, \ldots, L)$ par d'autres $(a, b, \ldots, l)$.

Les substitutions, que l'on doit considérer ici, sont définies par des relations algébriques, de forme supposée donnée, mais contenant des *paramètres* arbitraires $p, q, \ldots$.

$$\left.\begin{array}{l} a=f(A, B, \ldots, L;\ p, q, \ldots) \\ b=f_1(A, B, \ldots, L;\ p, q, \ldots) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{array}\right\}. \qquad (1)$$

Soit maintenant une seconde substitution, de même espèce, mais avec d'autres paramètres $\pi, \chi, \ldots$, et donnant lieu à $(\alpha, \beta, \ldots, \lambda)$, en sorte qu'on ait

$$\left.\begin{array}{l} \alpha=f(A, B, \ldots, L;\ \pi, \chi, \ldots) \\ \beta=f_1(A, B, \ldots, L;\ \pi, \chi, \ldots) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{array}\right\}. \qquad (1\ \text{bis})$$

2. DÉFINITION. Les substitutions dont il s'agit forment un GROUPE, si, quels que soient les paramètres $p, q, \ldots, \pi, \chi, \ldots,$ ainsi que $A, B, \ldots, L,$ il existe des quantités $P, Q, \ldots$ vérifiant les égalités semblables

$$\left.\begin{array}{l}\alpha=f(a, b, \ldots, l;\ P, Q, \ldots)\\ \beta=f_1(a, b, \ldots, l;\ P, Q, \ldots)\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\end{array}\right\}. \qquad (1 \text{ ter})$$

Les invariants sont l'apanage exclusif des substitutions formant groupe. On va le montrer. Mais auparavant, pour éviter toute confusion, on doit faire une remarque sur la définition.

3. Dans les diverses théories où l'on a rencontré des Invariants, les substitutions forment groupe, en effet, suivant cette définition; mais il s'y rencontre encore une circonstance particulière de plus, c'est que les paramètres $P, Q, \ldots$ de la substitution composée (1 ter) dépendent uniquement des paramètres $p, q, \ldots, \pi, \chi, \ldots$ des substitutions composantes (1) et (1 bis). Cette propriété *n'est pas nécessaire* à l'existence des Invariants, et nous ne la supposerons pas ici. Il sera donc entendu que $P, Q, \ldots$ peuvent dépendre, non seulement de $p, q, \ldots, \pi, \chi, \ldots,$ mais aussi de $A, B, \ldots, L.$

EXEMPLES:

I. $\quad a = Ap^2, \quad b = Apq + Bp, \quad c = Aq^2 + 2Bq + C;$

$\alpha = A\pi^2, \quad \beta = A\pi\chi + B\pi, \quad \gamma = A\chi^2 + 2B\chi + C;$

$\alpha = aP^2, \quad \beta = aPQ + bP, \quad \gamma = aQ^2 + 2bQ + c;$

$$P = \frac{\pi}{p}, \qquad Q = \frac{\chi - q}{p}.$$

P et Q ne dépendent pas de $A, B, C.$

II. $\quad a = A^3p^2, \quad b = A^2pq + ABp, \quad c = Aq^2 + 2Bq + C;$

$\alpha = A^3\pi^2, \quad \beta = A^2\pi\chi + AB\pi, \quad \gamma = A\chi^2 + 2B\chi + C;$

$\alpha = a^3P^2, \quad \beta = a^2PQ + abP, \quad \gamma = aQ^2 + 2bQ + c;$

$$P = \frac{\pi}{A^3p^3}, \qquad Q = \frac{\chi - q}{Ap}.$$

P et Q dépendent de A.

Dans ces deux exemples, il y a un invariant absolu, $\dfrac{B^2 - AC}{A}$.

4. Dans la substitution (1) nous supposerons que le nombre des paramètres soit inférieur au nombre des quantités $A, B, \ldots, L.$

Soient ainsi $\quad m$ le nombre des paramètres $p, q, \ldots,$

n le nombre des quantités $A, B, \ldots, L,$

on suppose $\quad m < n.$

Cela étant, on peut éliminer les paramètres entre les équations (1), et il reste $(n-m)$ équations

$$\left.\begin{array}{l} F(a, b, \ldots, l;\ A, B, \ldots, L)=0 \\ F_1(a, b, \ldots, l;\ A, B, \ldots, L)=0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}\right\}. \qquad (2)$$

THÉORÈME: *Si les substitutions considérées forment* GROUPE, *les* $(n-m)$ *équations* (2) *peuvent être mises sous la forme*

$$\left.\begin{array}{l} \Phi\ (a, b, \ldots, l)=\Phi\ (A, B, \ldots, L) \\ \Phi_1(a, b, \ldots, l)=\Phi_1(A, B, \ldots, L) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}\right\}, \qquad (3)$$

en d'autres termes, *il y a* $(n-m)$ *invariants absolus.*

Réciproquement, s'il y a $(n-m)$ *invariants absolus* (distincts), *les substitutions forment groupe.*

5. DÉMONSTRATION. Prouvons d'abord la seconde partie, ou réciproque. Voici l'hypothèse: des équations (1), par élimination de $p, q, \ldots$ resultent les équations (3).

Par conséquent, $A, B, \ldots, L$ et $a, b, \ldots, l$ étant quelconques, mais satisfaisant aux équations (3), on peut déterminer $p, q,$ au moyen des équations (1).

Soient $A, B, \ldots, L, p, q, \ldots, \pi, \chi, \ldots$ pris arbitrairement, et $a, b, \ldots, l,$ $\alpha, \beta, \ldots, \lambda$ déterminés par (1) et (1 bis). Suivant l'hypothèse, on a

$$\Phi(a, b, \ldots, l)=\Phi(A, B, \ldots, L) \quad \text{et} \quad \Phi(\alpha, \beta, \ldots, \lambda)=\Phi(A, B, \ldots, L);$$

donc $$\Phi(a, b, \ldots, l)=\Phi(\alpha, \beta, \ldots, \lambda), \text{ etc.}$$

Donc on peut déterminer $P, Q, \ldots$ par les équations (1 ter), ce qu'il fallait démontrer.

Démontrons maintenant la première partie, ou théorème direct. Par hypothèse, $A, B, \ldots, L, p, q, \ldots, \pi, \chi, \ldots$ étant pris à volonté et $a, b, \ldots, l,$ $\alpha, \beta, \ldots, \lambda$ déterminés au moyen de (1) et (1 bis), il en résulte les relations (1 ter).

Des équations (1) résulte le système (2); de même, de (1 bis) et de (1 ter) resultent

$$\left.\begin{array}{l} F\ (\alpha, \beta, \ldots, \lambda;\ A, B, \ldots, L)=0 \\ F_1(\alpha, \beta, \ldots, \lambda;\ A, B, \ldots, L)=0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}\right\}, \qquad (2 \text{ bis})$$

$$\left.\begin{array}{l} F\ (\alpha, \beta, \ldots, \lambda;\ a, b, \ldots, l)\ \ =0 \\ F_1(\alpha, \beta, \ldots, \lambda;\ a, b, \ldots, l)\ \ =0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}\right\}. \qquad (2 \text{ ter})$$

Je dis que le système (2 ter) résulte de (2) et de (2 bis).

En effet, $a, b, \ldots, l$ et $\alpha, \beta, \ldots, \lambda$ n'étant definis que par (1) et (1 bis), le système (2 ter) résulte de (1) et de (1 bis) par l'élimination de $p, q, \ldots, \pi, \chi, \ldots$ et $A, B, \ldots, L$. Mais l'élimination de $p, q, \ldots$ remplace le système (1) par le système (2), celle de $\pi, \chi, \ldots$ remplace le système (1 bis) par (2 bis); donc (2 ter) résulte de l'élimination de $A, B, \ldots, L$ entre (2) et (2 bis).

Le système (2), (2 bis) est formé par $2(n-m)$ équations, et cependant l'élimination de n lettres $A, B, \ldots, L$, au lieu de donner $(n-2m)$ équations, en donne $(n-m)$, les équations (2 ter). Si donc on élimine seulement $(n-m)$ lettres $A, B, \ldots, G$, les m autres $H, \ldots, L$ disparaîtront d'elles-mêmes. Tirons $A, B, \ldots, G$ des équations (2), et nous aurons

$$A = \Psi\ (a, b, \ldots, l;\ H, \ldots, L),$$
$$B = \Psi_1(a, b, \ldots, l;\ H, \ldots, L),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

Tirons de même $A, B, \ldots, G$ des équations (2 bis), et nous aurons

$$A = \Psi\ (\alpha, \beta, \ldots, \lambda;\ H, \ldots, L),$$
$$B = \Psi_1(\alpha, \beta, \ldots, \lambda;\ H, \ldots, L),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

Le résultat de l'élimination est donc représenté par $(n-m)$ équations telles que

$$\left.\begin{array}{l} \Psi\ (a, b, \ldots, l;\ H, \ldots, L) = \Psi\ (\alpha, \beta, \ldots, \lambda;\ H, \ldots, L) \\ \Psi_1(a, b, \ldots, l;\ H, \ldots, L) = \Psi_1(\alpha, \beta, \ldots, \lambda;\ H, \ldots, L) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{array}\right\}, \qquad (4)$$

et l'on sait que $H, \ldots, L$ disparaissent, d'eux-mêmes, de ces équations.

En assignant donc à $H, \ldots, L$ des valeurs numériques à volonté, on voit donc bien que les équations résultantes, équivalentes à (2 ter), ont la forme

$$\Phi\ (a, b, \ldots, l) = \Phi\ (\alpha, \beta, \ldots, \lambda),$$
$$\Phi_1(a, b, \ldots, l) = \Phi_1(\alpha, \beta, \ldots, \lambda),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

C'est ce qu'il fallait démontrer.

6. Remarques. Si les équations (4) sont rationnelles, la disparition de $H, \ldots, L$ exige que Ψ ait la forme suivante

$$\Psi = \Phi(a, b, \ldots, l)\, \Theta(H, \ldots, L) + \theta(H, \ldots, L),$$

et de même pour Ψ_1, etc. Sous cette forme, on voit que Θ et θ disparaissent dans les équations (4), et l'invariant résultant est Φ.

Mais, si les équations (4) sont irrationnelles, la disparition de $H, \ldots, L$ peut n'être pas immédiate. En assignant à $H, \ldots, L$ des valeurs numériques à volonté, comme on l'a dit dans la démonstration, c'est-à-dire en considérant $H, \ldots, L$ comme des *constantes arbitraires,* on voit les invariants se présenter avec des constantes arbitraires. Ceci ne doit pas étonner, puisqu'il s'agit ici d'invariants *absolus*, que l'on peut effectivement modifier en leur ajoutant des constantes arbitraires ou en les multipliant par des constantes arbitraires, sans troubler la propriété d'invariance.

L'analyse employée dans la démonstration fournit un moyen régulier de former les invariants ; ce moyen consiste à éliminer les paramètres dans les équations (1), puis à résoudre par rapport à $(n-m)$ quantités $A, B, \ldots, G$. Mais, les substitutions formant groupe, on peut aussi résoudre par rapport à $a, b, \ldots, g$, en éliminant les paramètres.

EXEMPLE : $a = Ap^2,\ b = Apq + Bp,\ c = Aq^2 + 2Bq + C.$

En résolvant par rapport à c, c'est-à-dire en tirant p, q des deux premières, on obtient

$$c = A\left(\frac{b - Bp}{Ap}\right)^2 + 2B\frac{b - Bp}{Ap} + C = \frac{b^2}{Ap^2} + C - \frac{B^2}{A} = \frac{b^2}{a} + C - \frac{B^2}{A}.$$

Voici l'invariant $C - \dfrac{B^2}{A}$.

En résolvant par rapport à b, on trouve $b = \sqrt{a}\sqrt{\left(\dfrac{B^2 - AC}{A}\right)} + c$, ce qui donne l'invariant $\dfrac{B^2 - AC}{A} + c$, où c est une constante arbitraire.

LECTURE XXII.

E pur si muove.

The theory still moves on. We have now emerged from the narrows and are entering on the mid-ocean of Differential Invariants, or of Principiants, as I have called them. These, it will now be seen, are perfectly defined by their property of being at one and the same time invariants and pure reciprocants. In other words, if P be a Principiant, it has both Ω and V for its annihilators. Thus, for example, the Mongian

$$A = a^2d - 3abc + 2b^3$$

is necessarily a Principiant. For

$$\Omega A = (a\partial_b + 2b\partial_c + 3c\partial_d)(a^2d - 3abc + 2b^3) = 0,$$

and at the same time

$$VA = \{2a^2\partial_b + 5ab\partial_c + (6ac + 3b^2)\partial_d\}(a^2d - 3abc + 2b^3) = 0.$$

Among Pure Reciprocants, those only are entitled to rank as Principiants whose form is persistent (merely taking up an extraneous factor, but otherwise unchanged) under the most general homographic substitution (see

Lecture XIII. [pp. 379, 382 above]. We have therefore to show that such reciprocants and no others are subject to annihilation by Ω.

With this end in view, let us consider the effect of substituting $\frac{x}{1+hx}$ for x and $\frac{y}{1+hx}$ for y in any rational integral function of y and its derivatives with respect to x. Suppose that, in consequence of this substitution, the function

$$F(y, y_1, y_2, y_3, \ldots y_n)$$

becomes changed into

$$F_1(x, y, y_1, y_2, y_3, \ldots y_n);$$

then the transformed function will be

$$F(Y, Y_1, Y_2, Y_3, \ldots Y_n),$$

where $X = \frac{x}{1+hx}$, $Y = \frac{y}{1+hx}$, and $Y_1, Y_2, Y_3, \ldots Y_n$ are the successive derivatives of Y with respect to X.

If, for the moment, we agree to consider h as an infinitesimal (we shall afterwards give it a finite value), neglecting squares and higher powers of h, we may write

$$X = x - hx^2,$$
$$Y = y - hxy.$$

Hence, by n successive differentiations of Y with respect to X, neglecting squares of h whenever they occur, we deduce

$$Y_1 = y_1 + hxy_1 - hy,$$
$$Y_2 = y_2 + 3hxy_2,$$
$$Y_3 = y_3 + 5hxy_3 + 3hy_2,$$
$$Y_4 = y_4 + 7hxy_4 + 8hy_3,$$
$$Y_5 = y_5 + 9hxy_5 + 15hy_4,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$Y_{n-1} = y_{n-1} + (2n-3)\,hxy_{n-1} + (n-1)(n-3)\,hy_{n-2},$$
$$Y_n = y_n + (2n-1)\,hxy_n + n(n-2)\,hy_{n-1}.$$

The last of these, for instance, is obtained as follows:

We have $$Y_n = \frac{dY_{n-1}}{dX}.$$

But $$\frac{d}{dX} = \frac{1}{1-2hx}\cdot\frac{d}{dx} = (1+2hx)\frac{d}{dx},$$

and $$\frac{dY_{n-1}}{dx} = \frac{d}{dx}\{y_{n-1} + (2n-3)\,hxy_{n-1} + (n-1)(n-3)\,hy_{n-2}\}$$
$$= y_n + (2n-3)\,hxy_n + n(n-2)\,hy_{n-1}.$$

Consequently, $Y_n = (1+2hx)\dfrac{dY_{n-1}}{dx}$

$$= (1+2hx)\{y_n + (2n-3)\,hxy_n + n\,(n-2)\,hy_{n-1}\}$$
$$= y_n + (2n-1)\,hxy_n + n\,(n-2)\,hy_{n-1}.$$

On substituting the above values of $Y, Y_1, Y_2, \ldots Y_n$ in the transformed function, we find immediately

$$F(Y, Y_1, Y_2, \ldots Y_n) = (1 + hx\nu + h\Theta)\,F(y, y_1, y_2, \ldots y_n),$$

where ν and Θ are the partial differential operators

$$\nu = -y\partial_y + y_1\partial_{y_1} + 3y_2\partial_{y_2} + 5y_3\partial_{y_3} + 7y_4\partial_{y_4} + \ldots,$$
$$\Theta = -y\partial_{y_1} + 3y_2\partial_{y_3} + 8y_3\partial_{y_4} + 15y_4\partial_{y_5} + \ldots + n\,(n-2)\,y_{n-1}\partial_{y_n}.$$

Changing to our usual notation, we write

$$y_1 = t,\ y_2 = 2a,\ y_3 = 2\,.\,3b,\ y_4 = 2\,.\,3\,.\,4c, \ldots,$$

and then if F_1 is what F (a rational integral function of $a, b, c, \ldots$) becomes when we substitute $\dfrac{x}{1+hx}, \dfrac{y}{1+hx}$ for x, y (regarding h as *infinitesimal*), we have

$$F_1 = (1 + hx\nu + h\Theta)\,F,$$

where $\nu = -y\partial_y + t\partial_t + 3a\partial_a + 5b\partial_b + 7c\partial_c + 9d\partial_d + \ldots,$

and $\Theta = -y\partial_t + a\partial_b + 2b\partial_c + 3c\partial_d + 4d\partial_e + \ldots.$

In general ν is merely the partial differential operator written above; but when its subject, F, is homogeneous, of degree i, and isobaric, of weight w, in

the letters	y,	t,	a,	b,	c,	d,	... supposed to be
of degrees	1,	1,	1,	1,	1,	1,	...
and of weights	-2,	-1,	0,	1,	2,	3,	...,

its operation is equivalent to multiplication by the number $3i + 2w$. For in this case we have

$$y\partial_y + t\partial_t + a\partial_a + b\partial_b + c\partial_c + d\partial_d + \ldots = i,$$

and $$-2y\partial_y - t\partial_t + b\partial_b + 2c\partial_c + 3d\partial_d + \ldots = w;$$

so that we may regard ν as a number, simply writing

$$\nu = 3i + 2w$$

when we have occasion to do so.

We are now able to show that if F is a persistent form, we must necessarily have

$$\Theta F = 0.$$

For $$\frac{F_1}{F} = 1 + \nu hx + \frac{h\Theta F}{F};$$

and consequently, if F_1 is divisible by F (this is what is meant by saying that F is a persistent form), unless ΘF vanishes, $\frac{\Theta F}{F}$ must be a rational integral function of $y, t, a, b, c, \ldots$. But since the operation of Θ diminishes the weight by unity without altering the degree, $\frac{\Theta F}{F}$ must be of degree 0 and weight -1. The impossibility of the existence of such a function leads to the necessary conclusion that

$$\Theta F = 0.$$

Let us apply this result to the case of a pure reciprocant. We have

$$\Theta = -y\partial_t + a\partial_b + 2b\partial_c + 3c\partial_d + \ldots = -y\partial_t + \Omega.$$

Thus when F is a pure reciprocant, or indeed any function in which t does not appear, $y\partial_t F = 0$ and Θ reduces to Ω. We have therefore shown, in what precedes, that the condition

$$\Omega F = 0$$

is necessary to ensure the persistence of the form of F under a particular homographic substitution; *à fortiori*, this condition is also necessarily satisfied when the form of F is persistent under the most general homographic substitution $\left(\text{in which } x, y \text{ are changed into } \frac{lx + my + n}{l''x + m''y + n''}, \frac{l'x + m'y + n'}{l''x + m''y + n''}\right)$.

The satisfaction of $\Omega F = 0$ is of itself inadequate to ensure persistence under the general homographic substitution; the necessary and sufficient condition of pure reciprocants

$$VF = 0$$

must also be satisfied. This follows from the fact that the general linear substitution, for which all pure reciprocants are persistent, is merely a particular case of the most general homographic substitution.

It only remains to be proved that the two conditions $VF = 0$, $\Omega F = 0$, taken conjointly, are sufficient as well as necessary.

In what follows I use a method which may be termed that of composition of variations. Its nature and value will be better understood if I first apply it to the rigorous demonstration of the theorem that the substitution of $x + hy$ for x in the Quantic

$$(a, b, c, \ldots \between x, y)^n$$

changes any function whatever of its coefficients, say

$$F(a, b, c, \ldots), \text{ into } e^{h\Omega} F(a, b, c, \ldots).$$

This is not proved, but only verified up to terms of the second order of differentiation, in Salmon's *Modern Higher Algebra* (3rd ed. 1876, p. 59). Remembering that, whatever the order n of the Quantic may be, the changed values of the coefficients $a, b, c, d, \ldots$ are

$$\begin{aligned} a' &= a, \\ b' &= b + ah, \\ c' &= c + 2bh + ah^2, \\ d' &= d + 3ch + 3bh^2 + ah^3, \\ &\ldots\ldots\ldots\ldots\ldots\ldots, \end{aligned}$$

what we have to prove is that, for all values of h,

$$F(a', b', c', d', \ldots) = e^{h\Omega} F(a, b, c, d, \ldots).$$

In other words, if for brevity we write

$$F(a, b, c, \ldots) = F,$$

and

$$F(a', b', c', \ldots) = F_1,$$

it is required to show that

$$F_1 = F + h\Omega F + \frac{h^2}{1 \,.\, 2}\Omega^2 F + \frac{h^3}{1 \,.\, 2 \,.\, 3}\Omega^3 F + \ldots,$$

where

$$\Omega = a\partial_b + 2b\partial_c + 3c\partial_d + \ldots.$$

When h is infinitesimal, it is obvious that

$$F_1 = F + h\Omega F.$$

Hence, when h has a general value, we may assume

$$F_1 = F + h\Omega F + \frac{h^2}{1 \,.\, 2} P + \frac{h^3}{1 \,.\, 2 \,.\, 3} Q + \frac{h^4}{1 \,.\, 2 \,.\, 3 \,.\, 4} R + \ldots.$$

Let h be increased by the infinitesimal quantity ϵ; then, considering this increase as resulting from a second substitution similar to the first, we see that F_1 becomes

$$F_1 + \epsilon\Omega F_1.$$

But it also becomes

$$\begin{aligned} F + (h+\epsilon)\,\Omega F + \frac{(h+\epsilon)^2}{1 \,.\, 2} P + \frac{(h+\epsilon)^3}{1 \,.\, 2 \,.\, 3} Q + \ldots &= F_1 + \epsilon \frac{dF_1}{dh} \\ &= F_1 + \epsilon\left(\Omega F + hP + \frac{h^2}{1 \,.\, 2} Q + \frac{h^3}{1 \,.\, 2 \,.\, 3} R + \ldots\right). \end{aligned}$$

Equating this to $F_1 + \epsilon\Omega F_1$, we obtain

$$\Omega F_1 = \Omega F + hP + \frac{h^2}{1 \,.\, 2} Q + \frac{h^3}{1 \,.\, 2 \,.\, 3} R + \ldots.$$

But

$$\Omega F_1 = \Omega\left(F + h\Omega F + \frac{h^2}{1 \,.\, 2} P + \frac{h^3}{1 \,.\, 2 \,.\, 3} Q + \ldots\right).$$

The comparison of these two expressions gives

$$P = \Omega^2 F,$$
$$Q = \Omega P = \Omega^3 F,$$
$$R = \Omega Q = \Omega^4 F,$$
$$\ldots\ldots\ldots\ldots$$

Substituting these values in the assumed expansion for F_1, there results

$$F_1 = F + h\Omega F + \frac{h^2}{1.2}\Omega^2 F + \frac{h^3}{1.2.3}\Omega^3 F + \ldots,$$

which is the expanded form of

$$F_1 = e^{h\Omega} F.$$

A similar method of procedure will enable us to establish the corresponding but more elaborate formula

$$F_1 = (1 + hx)^{\nu} e^{\frac{h\Theta}{1+hx}} F,$$

in which F is any *homogeneous* and *isobaric* function* of degree i and weight w in y and its modified derivatives $(t, a, b, c, \ldots)$ with respect to x; the operator $\Theta = -y\partial_t + a\partial_b + 2b\partial_c + 3c\partial_d + \ldots$; the function F_1 is what F becomes in consequence of the substitution of $\frac{x}{1+hx}, \frac{y}{1+hx}$ for x, y; h is any finite quantity, and $\nu = 3i + 2w$.

Before giving the proof of this theorem, I will show that, upon the assumption of its truth, two inverse finite substitutions will, as they ought, nullify each other, leaving the function operated upon unaltered in form.

To avoid needless periphrasis, we call the substitution of $\frac{x}{1+hx}, \frac{y}{1+hx}$ for x, y the substitution h.

Either of the two substitutions, $h, -h$, reverses the effect of the other; for the substitution $-h$ turns

$$\frac{x}{1+hx} \text{ into } \frac{x}{1-hx} \div 1 + \frac{hx}{1-hx} = x,$$

and

$$\frac{y}{1+hx} \text{ into } \frac{y}{1-hx} \div 1 + \frac{hx}{1-hx} = y.$$

The two substitutions $h, -h$, performed successively on F, ought therefore to leave its value unaltered. But by hypothesis the substitution h converts F into F_1; consequently the substitution $-h$ performed on F_1 ought to change it back again into F.

* F need not be integral or even rational; whenever it is homogeneous or isobaric, ν will be a number.

It must be carefully observed that (since the operation of Θ decreases the weight by unity, leaving the degree unchanged) the weight of $\Theta^\kappa F$ is κ units lower than that of F, whilst the degree is the same for both.

Thus for F we have $3i + 2w = \nu$,

and for $\Theta^\kappa F$ $\quad 3i + 2(w - \kappa) = \nu - 2\kappa$.

Hence the substitution $-h$, which changes

$$F \text{ into } (1 - hx)^\nu e^{-\frac{h\Theta}{1-hx}} F,$$

also changes

$$\Theta F \quad ,, \quad (1 - hx)^{\nu-2} e^{-\frac{h\Theta}{1-hx}} \Theta F,$$

$$\Theta^2 F \quad ,, \quad (1 - hx)^{\nu-4} e^{-\frac{h\Theta}{1-hx}} \Theta^2 F,$$

....

and in general

$$\Theta^\kappa F \text{ into } (1 - hx)^{\nu-2\kappa} e^{-\frac{h\Theta}{1-hx}} \Theta^\kappa F.$$

Moreover, $1 + hx$ becomes $1 + \dfrac{hx}{1 - hx} = (1 - hx)^{-1}$, so that

$$(1 + hx)^{\nu-\kappa} \Theta^\kappa F \text{ becomes } (1 - hx)^{-(\nu-\kappa)} (1 - hx)^{\nu-2\kappa} e^{-\frac{h\Theta}{1-hx}} \Theta^\kappa F$$

$$= (1 - hx)^{-\kappa} e^{-\frac{h\Theta}{1-hx}} \Theta^\kappa F$$

$$= e^{-\frac{h\Theta}{1-hx}} (1 - hx)^{-\kappa} \Theta^\kappa F \text{ (since } \Theta \text{ does not act on } x).$$

Consequently, $(1 + hx)^\nu F$ becomes $e^{-\frac{h\Theta}{1-hx}} F$,

$$(1 + hx)^{\nu-1} \Theta F \quad ,, \quad e^{-\frac{h\Theta}{1-hx}} (1 - hx)^{-1} \Theta F,$$

$$(1 + hx)^{\nu-2} \Theta^2 F \quad ,, \quad e^{-\frac{h\Theta}{1-hx}} (1 - hx)^{-2} \Theta^2 F,$$

....

And since, by the formula to be verified,

$$F_1 = (1 + hx)^\nu F + h(1 + hx)^{\nu-1} \Theta F + \frac{h^2}{1 . 2} (1 + hx)^{\nu-2} \Theta^2 F + \ldots,$$

$$F_1 \text{ becomes } e^{-\frac{h\Theta}{1-hx}} \left\{1 + h(1 - hx)^{-1} \Theta + \frac{h^2}{1 . 2} (1 - hx)^{-2} \Theta^2 + \ldots\right\} F$$

$$= e^{-\frac{h\Theta}{1-hx}} e^{\frac{h\Theta}{1-hx}} F = F.$$

LECTURE XXIII.

We now proceed to show how the composition of variations can be made to furnish a strict proof of the formula

$$F_1 = (1 + hx)^\nu e^{\frac{h\Theta}{1+hx}} F,$$

which was set forth in the preceding lecture.

As before, calling the change of x, y into $\frac{x}{1+hx}$, $\frac{y}{1+hx}$, the substitution h, it is easy to see that the *product* of two substitutions, h, ϵ, is the substitution $h+\epsilon$. For

$$\frac{x}{1+hx} \div 1 + \epsilon \frac{x}{1+hx} = \frac{x}{1+(h+\epsilon)x},$$

$$\frac{y}{1+hx} \div 1 + \epsilon \frac{x}{1+hx} = \frac{y}{1+(h+\epsilon)x}.$$

This shows that if

	F_1 is what F becomes on making the substitution h,						
and	F_2	„	F_1	„	„	„	„ ϵ,
then	F_2	„	F	„	„	„	„ $h+\epsilon$.

Thus we can find two expressions for F_2, the comparison of which will enable us to assign the coefficients of all the powers of h in the expanded values of F_1.

The first two terms of this expansion were obtained, in the preceding lecture, by treating h as an infinitesimal. We may therefore write

$$F_1 = F + h(\nu x + \Theta) F + \frac{h^2}{1.2} N_2 + \frac{h^3}{1.2.3} N_3 + \dots.$$

Changing h into $h+\epsilon$, we deduce

$$F_2 = F + (h+\epsilon)(\nu x + \Theta) F + \frac{(h+\epsilon)^2}{1.2} N_2 + \frac{(h+\epsilon)^3}{1.2.3} N_3 + \dots.$$

For greater simplicity, let ϵ be an infinitesimal, and write

$$\frac{F_2 - F_1}{\epsilon} = \Delta F_1.$$

Then
$$\Delta F_1 = (\nu x + \Theta) F + h N_2 + \frac{h^2}{1.2} N_3 + \dots.$$

Now look at each term in the expansion of F_1 and find its increment (that is, its Δ) when x, y undergo the substitution ϵ. We thus obtain

$$\Delta F_1 = \Delta F + h\Delta(\nu x + \Theta) F + \frac{h^2}{1.2} \Delta N_2 + \frac{h^3}{1.2.3} \Delta N_3 + \dots.$$

Comparing these two values of ΔF_1, we find

$$N_2 = \Delta(\nu x + \Theta) F,$$
$$N_3 = \Delta N_2,$$
$$N_4 = \Delta N_3,$$
$$\cdots\cdots\cdots\cdots$$

and generally $$N_r = \Delta N_{r-1}.$$

These equations are sufficient to determine all the coefficients of F_1; it only remains to show how the operations Δ may be performed.

We have in fact

$$F_1 = F + h\Delta F + \frac{h^2}{1.2}\Delta^2 F + \frac{h^3}{1.2.3}\Delta^3 F + \ldots,$$

where $\Delta F = (\nu x + \Theta) F.$

But we must not from this rashly infer that

$$\Delta^n F = (\nu x + \Theta)^n F.$$

To do so would be tantamount to regarding ν as a constant number, whereas its value depends on the degree and weight of the subject of operation.

This will be clearly seen in the calculation which follows*. We first generalize the formula

$$\Delta F = (\nu x + \Theta) F$$

by making $\Theta^\kappa F$ the operand instead of F.

Then, since i is the degree and $w - \kappa$ the weight of $\Theta^\kappa F$, instead of

$$3i + 2w = \nu,$$

we have $$3i + 2(w - \kappa) = \nu - 2\kappa.$$

Thus, $$\Delta\Theta^\kappa F = \{(\nu - 2\kappa) x + \Theta\} \Theta^\kappa F.$$

Again, since $$\Delta x = \left(\frac{x}{1 + \epsilon x} - x\right) \div \epsilon = - x^2,$$

we find

$$\Delta x^\lambda \Theta^\kappa F = \lambda x^{\lambda-1} \Theta^\kappa F . \Delta x + x^\lambda \Delta\Theta^\kappa F = - \lambda x^{\lambda+1} \Theta^\kappa F + x^\lambda \{(\nu - 2\kappa) x + \Theta\} \Theta^\kappa F.$$

Hence we obtain the general formula

$$\Delta x^\lambda \Theta^\kappa F = x^\lambda \{(\nu - 2\kappa - \lambda) x + \Theta\} \Theta^\kappa F,$$

* If our sole object were to show that $\Theta F = 0$ is a sufficient as well as necessary condition of the persistence of F, we might dispense with all further calculation. Thus it is obvious that, since $\Delta F = (\nu x + \Theta) F$, $\Delta^n F$ must be of the form $(x, \Theta)^n F$; for the dependence of ν on the degree-weight of the operand will not affect the *form* of Δ^n, but only its numerical coefficients. Hence we conclude that F_1 is of the form $\phi(x, \Theta) F$; and remembering that $\Theta^2 F = 0$, $\Theta^3 F = 0$, ... whenever $\Theta F = 0$, it is at once seen that not only (as was shown in the last lecture) must ΘF vanish when F is persistent under the substitution h, but, conversely, that when $\Theta F = 0$, the altered value of F contains the original value as a factor (the other factor being in this case a function of x only); that is, F is persistent.

by means of which we calculate in succession the values of $\Delta^2 F$, $\Delta^3 F$, ... Thus,

$$\begin{aligned}\Delta^2 F &= \Delta(\nu x + \Theta) F \\ &= \nu \Delta x F + \Delta \Theta F \\ &= \nu x \{(\nu - 1) x + \Theta\} F + \{(\nu - 2) x + \Theta\} \Theta F \\ &= \{\nu(\nu - 1) x^2 + 2(\nu - 1) x\Theta + \Theta^2\} F.\end{aligned}$$

Hence

$$\begin{aligned}\Delta^3 F &= \nu(\nu - 1) \Delta x^2 F + 2(\nu - 1) \Delta x \Theta F + \Delta \Theta^2 F \\ &= \nu(\nu - 1) x^2 \{(\nu - 2) x + \Theta\} F + 2(\nu - 1) x \{(\nu - 3) x + \Theta\} \Theta F \\ &\qquad + \{(\nu - 4) x + \Theta\} \Theta^2 F \\ &= \{\nu(\nu - 1)(\nu - 2) x^3 + 3(\nu - 1)(\nu - 2) x^2\Theta + 3(\nu - 2) x\Theta^2 + \Theta^3\} F.\end{aligned}$$

If $[\nu]^n$ is used to denote $\nu(\nu - 1)(\nu - 2)\ldots$ to n factors ($[\nu]^1$ will of course mean ν), we have shown that

$$\begin{aligned}\Delta F &= ([\nu]^1 x + \Theta) F, \\ \Delta^2 F &= ([\nu]^2 x^2 + 2[\nu - 1]^1 x\Theta + \Theta^2) F, \\ \Delta^3 F &= ([\nu]^3 x^3 + 3[\nu - 1]^2 x^2\Theta + 3[\nu - 1]^1 x\Theta^2 + \Theta^3) F,\end{aligned}$$

and by induction it may be proved that in general

$$\Delta^n F = \left\{[\nu]^n x^n + n[\nu - 1]^{n-1} x^{n-1}\Theta + \frac{n(n-1)}{1.2}[\nu - 2]^{n-2} x^{n-2}\Theta^2 + \ldots + \Theta^n\right\} F.$$

That the last term of this expression is $\Theta^n F$ is sufficiently obvious; what we wish to prove is that, when m is any positive integer less than n, the term in $\Delta^n F$ which involves Θ^m will be

$$\frac{n(n-1)\ldots(n-m+1)}{1.2.3\ldots m}[\nu - m]^{n-m} x^{n-m}\Theta^m F.$$

To find the term involving Θ^m in $\Delta^{n+1} F$, we need only consider the operation of Δ on two consecutive terms of $\Delta^n F$; none of the remaining terms will affect the result. Suppose, then, that

$$\Delta^n F = \ldots + p x^{n-m}\Theta^m F + q x^{n-m+1}\Theta^{m-1} F + \ldots.$$

Operating with Δ, we find

$$\begin{aligned}\Delta^{n+1} F &= \ldots + p\Delta x^{n-m}\Theta^m F + q\Delta x^{n-m+1}\Theta^{m-1} F + \ldots \\ &= \ldots + p x^{n-m}\{(\nu - n - m) x + \Theta\}\Theta^m F \\ &\qquad + q x^{n-m+1}\{(\nu - n - m + 1) x + \Theta\}\Theta^{m-1} F + \ldots \\ &= \ldots + \{p(\nu - n - m) + q\} x^{n+1-m}\Theta^m F + \ldots.\end{aligned}$$

Now, assuming the general term of $\Delta^n F$ to be as written above, we have

$$p = \frac{n(n-1)\ldots(n-m+1)}{1.2.3\ldots m}[\nu - m]^{n-m},$$

$$q = \frac{n(n-1)\ldots(n-m+2)}{1.2.3\ldots(m-1)}[\nu - m + 1]^{n-m+1};$$

so that $$q = p\left\{\frac{m(\nu - m + 1)}{n - m + 1}\right\}.$$

Thus the general term of $\Delta^{n+1}F$ has for its numerical coefficient

$$p(\nu-n-m)+q=p\left\{\frac{m(\nu-m+1)+(\nu-n-m)(n-m+1)}{n-m+1}\right\}$$

$$=p\left\{\frac{(n+1)(\nu-n)}{n-m+1}\right\}=\frac{(n+1)n\ldots(n-m+2)}{1.2.3\ldots m}[\nu-m]^{n+1-m},$$

which shows that the numerical coefficients in $\Delta^{n+1}F$ obey the same law as those in $\Delta^n F$; and as this law is true for $n=1, 2, 3$, it is also true universally.

We have thus shown that the general term in $\Delta^n F$ is

$$\frac{n(n-1)\ldots(n-m+1)}{1.2.3\ldots m}[\nu-m]^{n-m}x^{n-m}\Theta^m F,$$

and, consequently, the corresponding general term in

$$\frac{h^n\Delta^n F}{1.2.3\ldots n}\text{ is }\frac{[\nu-m]^{n-m}}{1.2.3\ldots(n-m)}h^{n-m}x^{n-m}.\frac{h^m\Theta^m F}{1.2.3\ldots m}.$$

Now, as we have already seen,

$$F_1=\left(1+h\Delta+\frac{h^2}{1.2}\Delta^2+\frac{h^3}{1.2.3}\Delta^3+\ldots\right)F,$$

which, by merely expressing the symbolic factor as a series of powers of Θ, may be transformed into

$$F_1=\left(1+[\nu]^1hx+\frac{[\nu]^2}{1.2}h^2x^2+\frac{[\nu]^3}{1.2.3}h^3x^3+\ldots\right)F$$

$$+\left(1+[\nu-1]^1hx+\frac{[\nu-1]^2}{1.2}h^2x^2+\frac{[\nu-1]^3}{1.2.3}h^3x^3+\ldots\right)h\Theta F$$

$$+\left(1+[\nu-2]^1hx+\frac{[\nu-2]^2}{1.2}h^2x^2+\frac{[\nu-2]^3}{1.2.3}h^3x^3+\ldots\right)\frac{h^2\Theta^2F}{1.2}$$

$$+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

where, remembering that $[\nu]^n$ stands for $\nu(\nu-1)(\nu-2)\ldots$ to n factors, it is evident that the functions of x which multiply F, $h\Theta F$, $\frac{h^2}{1.2}\Theta^2F$, ... are all of them binomial expansions. Hence we immediately obtain

$$F_1=(1+hx)^\nu F+(1+hx)^{\nu-1}h\Theta F+(1+hx)^{\nu-2}\frac{h^2}{1.2}\Theta^2F+\ldots$$

$$=(1+hx)^\nu\left\{1+(1+hx)^{-1}h\Theta+(1+hx)^{-2}\frac{h^2\Theta^2F}{1.2}+\ldots\right\}F,$$

and finally, $$F_1=(1+hx)^\nu e^{\frac{h\Theta}{1+hx}}F.$$

Mr Hammond has remarked that, with a slight modification, the foregoing demonstration will serve to establish the analogous theorem, that

$$F_1=(1+ht)^{-\mu}e^{-\frac{hV_1}{1+ht}}F,$$

where, as before, F means any homogeneous and isobaric function of degree i and weight w in the letters $y, t, a, b, c, \ldots$; and F_1 is what F becomes when, leaving y unaltered, we change x into $x + hy$, where h is any finite quantity. Instead of the operator

$$\Theta = -y\partial_t + a\partial_b + 2b\partial_c + 3c\partial_d + \ldots = -y\partial_t + \Omega$$

we have $$-V_1 = yt\partial_y + t^2\partial_t - 2a^2\partial_b - 5ab\partial_c - \ldots = yt\partial_y + t^2\partial_t - V\,*;$$

and instead of $\nu = 3i + 2w$, a different number, $\mu = 3i + w$ (which I have called the characteristic), taken negatively.

If we suppose that

F_1 is what F becomes on changing x into $x + hy$,

and F_2 „ F „ „ „ x „ $x + \epsilon y$,

then F_2 „ F „ „ „ x „ $x + (h + \epsilon)\, y$.

Hence, if $$F_1 = F + hP + \frac{h^2}{1\,.\,2} Q + \frac{h^3}{1\,.\,2\,.\,3} R + \ldots,$$

we must have $$F_2 = F + (h + \epsilon)\, P + \frac{(h + \epsilon)^2}{1\,.\,2} Q + \frac{(h + \epsilon)^3}{1\,.\,2\,.\,3} R + \ldots$$

$$= F_1 + \epsilon \frac{dF_1}{dh} + \ldots.$$

Thus, if ϵ be regarded as infinitesimal, and we write

$$\frac{F_2 - F_1}{\epsilon} = \Delta F_1,$$

it follows that $$\Delta F_1 = P + hQ + \frac{h^2}{1\,.\,2} R + \ldots.$$

But, by the direct operation of Δ, we find

$$\Delta F_1 = \Delta F + h\Delta P + \frac{h^2}{1\,.\,2} \Delta Q + \ldots,$$

and, comparing these two values of ΔF_1,

$$P = \Delta F,$$
$$Q = \Delta P = \Delta^2 F,$$
$$R = \Delta Q = \Delta^3 F,$$
$$\ldots\ldots\ldots\ldots\ldots$$

Hence it follows that

$$F_1 = F + h\Delta F + \frac{h^2}{1\,.\,2} \Delta^2 F + \frac{h^3}{1\,.\,2\,.\,3} \Delta^3 F + \ldots.$$

* This theorem was stated without proof in Lecture VIII, where, through inadvertence, the term $yt\partial_y$ in the expression for V_1 was omitted [p. 352, above].

It remains to find the value of $\Delta^n F$. This can be effected by means of formulae given in Lecture VIII. [p. 350, above], where it is shown that

$$\begin{aligned}
\Delta x &= y,\\
\Delta y &= 0,\\
\Delta t &= -t^2,\\
\Delta a &= -3at,\\
\Delta b &= -4bt - 2a^2,\\
\Delta c &= -5ct - 5ab,\\
\Delta d &= -6dt - 6ac - 3b^2,\\
\Delta e &= -7et - 7ad - 7bc,\\
&\ldots\ldots\ldots\ldots
\end{aligned}$$

We now show that

$$\Delta F = -(\mu t + V_1) F,$$

where

$$V_1 = V - t^2\partial_t - yt\partial_y,$$

just as in the cognate theorem we had

$$\Delta F = (\nu x + \Theta) F.$$

Since F is a function of $y, t, a, b, c, \ldots$ without x, it is evident that

$$\begin{aligned}
\Delta F &= \frac{dF}{dy}\Delta y + \frac{dF}{dt}\Delta t + \ldots\\
&= -t(t\partial_t + 3a\partial_a + 4b\partial_b + 5c\partial_c + \ldots) F\\
&\quad - \{2a^2\partial_b + 5ab\partial_c + (6ac + 3b^2)\partial_d + \ldots\} F,
\end{aligned}$$

where the part of ΔF which is independent of t is $-VF$.

Now, $\quad y\partial_y + t\partial_t + a\partial_a + b\partial_b + c\partial_c + \ldots = i$

and $\quad -2y\partial_y - t\partial_t + b\partial_b + 2c\partial_c + \ldots = w;$

so that $\quad t\partial_t + 3a\partial_a + 4b\partial_b + 5c\partial_c + \ldots = 3i + w - y\partial_y - t\partial_t.$

Hence, writing $3i + w = \mu$,

$$\begin{aligned}
\Delta F &= -t(\mu - y\partial_y - t\partial_t) F - VF\\
&= -(\mu t + V_1) F,
\end{aligned}$$

where

$$V_1 = V - t^2\partial_t - yt\partial_y.$$

Observing that $V_1^\kappa F$ is of degree $i + \kappa$ and weight $w - \kappa$; since

$$3(i + \kappa) + (w - \kappa) = \mu + 2\kappa,$$

we see that

$$\Delta V_1^\kappa F = -\{(\mu + 2\kappa) t + V_1\} V_1^\kappa F.$$

Again,

$$\begin{aligned}
\Delta t^\lambda V_1^\kappa F &= \lambda t^{\lambda-1} V_1^\kappa F \,.\, \Delta t + t^\lambda \Delta V_1^\kappa F\\
&= -\lambda t^{\lambda+1} V_1^\kappa F - t^\lambda \{(\mu + 2\kappa) t + V_1\} V_1^\kappa F.
\end{aligned}$$

We thus obtain the formula

$$\Delta t^\lambda V_1^\kappa F = -t^\lambda \{(\mu + \lambda + 2\kappa) t + V_1\} V_1^\kappa F, \qquad (1)$$

analogous to the one previously employed,

$$\Delta x^\lambda \Theta^\kappa F = x^\lambda \{(\nu - 2\kappa - \lambda) x + \Theta\} \Theta^\kappa F. \qquad (2)$$

The remainder of the work will be step for step the same for this as for the previous theorem. In fact, by using (1) just as we used (2), we shall deduce

$$F_1 = (1 + ht)^{-\mu} e^{-\frac{hV_1}{1+ht}} F, \qquad (3)$$

just as we deduced the analogous formula

$$F_1 = (1 + hx)^{\nu} e^{\frac{h\Theta}{1+hx}} F. \qquad (4)$$

The reason of this is obvious: by interchanging x and t, μ and $-\nu$, Θ and $-V_1$, we interchange the formulae (1) and (2), (3) and (4).

It may be well to observe that if we use S_h to denote a substitution of such a nature that

$$S_\epsilon S_h = S_{h+\epsilon},$$

and if (regarding ϵ as an infinitesimal) we write

$$\frac{S_\epsilon - 1}{\epsilon} = \Delta,$$

then in general $$S_h F = e^{h\Delta} F.$$

The proof of this proposition is virtually contained in what precedes.

LECTURE XXIV.

Whenever a rational integral function of $x, y, t, a, b, c, \ldots$ is persistent in form under the general linear substitution, it cannot contain explicitly either x, y or t, but must be a function of the remaining letters $a, b, c, \ldots$ (the successive modified derivatives, beginning with the second, of y with respect to x) alone.

For if, keeping y unaltered, we change x into $x + \alpha$, where α is any arbitrary constant which may be regarded as an infinitesimal, the derivatives $t, a, b, c, \ldots$ are not affected by this change, and consequently the function

$$F = F(x, y, t, a, b, c, \ldots) \text{ becomes } F + \alpha \frac{dF}{dx},$$

which cannot be divisible by F unless $\frac{dF}{dx} = 0$.

(The alternative hypothesis of $\frac{dF}{dx}$ being divisible by F is inadmissible, because F is a rational integral function.)

Hence F cannot contain x explicitly; and if we write $y + \beta$ for y, keeping x unchanged, we see, in like manner, that F cannot contain y explicitly.

Again, if in the function

$$F = F(t, a, b, c, \ldots)$$

we change x, y into $x + \alpha$, $y + \beta_, x + \beta$, the effect of this substitution will be to increase t by the arbitrary constant $\beta_,$, without altering any of the remaining derivatives $a, b, c, \ldots$.

Hence, in order that the form of F may still be persistent, we must have $\frac{dF}{dt} = 0$; the reasoning being just the same as that by which $\frac{dF}{dx}$ was seen to vanish. Thus, F does not contain t explicitly. Moreover, the function

$$F = F(a, b, c, \ldots)$$

must be both homogeneous and isobaric.

For the substitution of $\alpha_, x + \alpha$, $\beta_{,,} y + \beta_, x + \beta$ for x, y, respectively, will multiply the letters

$$a \;,\quad b \;,\quad c \;,\quad d \;, \ldots$$

by $$\beta_{,,}\alpha_,^{-2},\quad \beta_{,,}\alpha_,^{-3},\quad \beta_{,,}\alpha_,^{-4},\quad \beta_{,,}\alpha_,^{-5},\ \ldots.$$

Each term of F will therefore be multiplied by a positive power of $\beta_{,,}$ and a negative power of $\alpha_,$.

Let one of the terms of F be $a^{\lambda_0} b^{\lambda_1} c^{\lambda_2} d^{\lambda_3} \ldots$. It will be multiplied by

$$\beta_{,,}^{\lambda_0+\lambda_1+\lambda_2+\lambda_3+\ldots}\,\alpha_,^{-(2\lambda_0+3\lambda_1+4\lambda_2+5\lambda_3+\ldots)}.$$

In order that F may retain its form, this multiplier must be the same for every term of F, no matter what arbitrary values are assigned to $\alpha_,$ and $\beta_{,,}$. This can only happen when, for all terms of the function F, we have

$$\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3 + \ldots = \text{const.}$$

and $$\lambda_1 + 2\lambda_2 + 3\lambda_3 + \ldots = \text{const.},$$

that is, when F is homogeneous and isobaric.

We have thus proved that among all the rational integral functions of $x, y, t, a, b, c, \ldots$ the only ones persistent under the substitution of $\alpha + \alpha_, x$, $\beta + \beta_, x + \beta_{,,} y$ for x, y, respectively, are such as simultaneously satisfy the conditions of not explicitly containing x, y or t, and of being homogeneous and isobaric in the remaining letters $a, b, c, \ldots$.

If F, any function satisfying these conditions, merely acquires an extraneous factor when, leaving y unaltered, we change x into $x + hy$, the form of F will be persistent under the general linear substitution. For both $\alpha + \alpha_, (x + hy)$ and $\beta + \beta_, (x + hy) + \beta_{,,} y$ are general linear functions of $x, y, 1$.

Now, the change of x into $x + hy$ converts (as was shown in the preceding lecture) F into

$$F_1 = (1 + ht)^{-\mu} e^{-\frac{hV_1}{1+ht}} F,$$

where $$V_1 = V - t^2\partial_t - yt\partial_y.$$

But, since neither y nor t occurs in F, we must have

$$\partial_y F = 0 \text{ and } \partial_t F = 0.$$

Consequently, $$V_1 F = VF, \quad V_1^2 F = V^2 F,$$

and so on. Hence

$$F_1 = (1+ht)^{-\mu} e^{-\frac{hV}{1+ht}} F$$

$$= (1+ht)^{-\mu} F - (1+ht)^{-\mu-1} hVF + (1+ht)^{-\mu-2} \frac{h^2 V^2}{1 \cdot 2} F - \dots .$$

Unless VF, V^2F, V^3F, ... all of them vanish, F_1 cannot contain F as a factor. If it could, VF, V^2F, ... would all have to be divisible by F. But this is impossible; for VF, a rational integral function of a, b, c, ... whose weight is $w-1$, cannot be divisible by F, a rational integral function of weight w.

We must therefore have

$$VF = 0$$

(which implies $V^2F = 0$, etc.) as the necessary and sufficient condition of the persistence of the form of F under the general linear substitution. In other words, F must be a pure reciprocant.

In order that F may also be persistent in form under the general homographic substitution, it must (besides being a pure reciprocant) be subject to annihilation by the operator

$$\Omega = a\partial_b + 2b\partial_c + 3c\partial_d + \dots .$$

For it was seen, in the preceding lecture, that the special homographic substitution in which $\frac{x}{1+hx}$, $\frac{y}{1+hx}$ are written instead of x, y, respectively, has the effect of changing any homogeneous and isobaric function F into F_1, where

$$F_1 = (1+hx)^{\nu} e^{\frac{h\Theta}{1+hx}} F,$$

$$\Theta = \Omega - y\partial_t.$$

When the letter t does not occur in F, we may write $\partial_t F = 0$, so that Θ becomes simply Ω, and the above formula becomes

$$F_1 = (1+hx)^{\nu} e^{\frac{h\Omega}{1+hx}} F.$$

Hence it follows immediately that, when F is a rational integral function of the letters a, b, c, ..., the condition $\Omega F = 0$ is sufficient as well as necessary to ensure the persistence of the form of F under the special homographic substitution we have employed.

But when F is a pure reciprocant it also satisfies the condition $VF = 0$, and it is the simultaneous satisfaction of $\Omega F = 0$ and $VF = 0$ that ensures the persistence of the form of F under the most general homographic substitution. This may be shown by combining the substitution $\frac{x}{1+hx}, \frac{y}{1+hx}$ (for which F is persistent when, and only when, $\Omega F=0$) with the general linear substitution (for which $VF=0$ is the necessary and sufficient condition of the persistence of the form of F), so as to obtain the most general homographic substitution. Thus the linear substitution

$$\left.\begin{array}{l} x = lx_, + my_, + n \\ y = l'x_, + m'y_, + n' \end{array}\right\},$$

when combined with

$$x_, = \frac{x_{,,}}{1+hx_{,,}}, \quad y_, = \frac{y_{,,}}{1+hx_{,,}},$$

gives the substitution

$$\left.\begin{array}{l} x = \dfrac{lx_{,,} + my_{,,} + n\,(1+hx_{,,})}{1+hx_{,,}} \\ y = \dfrac{l'x_{,,} + m'y_{,,} + n'\,(1+hx_{,,})}{1+hx_{,,}} \end{array}\right\},$$

in which both the numerators are general linear functions.

By combining the substitution just obtained with the linear substitution

$$x_{,,} = \lambda x_{,,,} + \mu y_{,,,} + \nu, \quad y_{,,} = y_{,,,},$$

the denominator of each fraction is changed into a general linear function, and thus, by combining the special homographic substitution $\frac{x}{1+hx}, \frac{y}{1+hx}$ with two linear substitutions, we arrive at the most general homographic substitution.

This proves that the necessary and sufficient condition of F being a *homographically persistent* form is the coexistence of the two conditions

$$VF = 0, \quad \Omega F = 0.$$

Thus a Projective Reciprocant, or Principiant, or Differential Invariant, combines the natures of a Pure Reciprocant and Invariant *in respect of the elements.*

Notice that every Pure Reciprocant is an Invariant of the Reciprocal Function (that is, the numerator of the expression for $\frac{d^n x}{dy^n}$ in terms of $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, ..., or what is the same in terms of the modified derivatives $t, a, b, \ldots$), but the elements of such invariants are *not* the original simple elements, but more or less complicated functions of them.

What has just been stated is obvious from the fact that all invariants of the "reciprocal function" have been shown to be pure reciprocants (*vide** Lecture XIX.). The ordinary protomorph invariants of this function will

[* above, p. 412.]

have for their leading term a power of a multiplied by a single letter. Consequently, by reasoning previously employed in these lectures, every pure reciprocant will be a rational function of invariants of the Reciprocal Function divided by some power of a. Thus, for example, the Reciprocal Function

$$14a^4 - 21a^2bt + 3(2ac + b^2)t^2 - dt^3 = (\alpha, \beta, \gamma, \delta \between 1, -t)^3$$

if $$\alpha = 14a^4,\ \beta = 7a^2b,\ \gamma = 2ac + b^2,\ \delta = d.$$

The two protomorph invariants of this reciprocal function are

$$\alpha\gamma - \beta^2 = 7a^4(4ac - 5b^2)$$

and $$\alpha^2\delta - 3\alpha\beta\gamma + 2\beta^3 = 196a^6(a^2d - 3abc + 2b^3).$$

All other pure reciprocants of extent 3 may be rationally expressed in terms of a and the two protomorphs $4ac - 5b^2$, $a^2d - 3abc + 2b^3$; that is, all pure reciprocants of extent 3 are invariants of the reciprocal function of extent 3.

The reasoning employed can be applied with equal facility to the general case of extent n.

Instead of $\dfrac{x}{1+hx}$, $\dfrac{y}{1+hx}$, let us consider the special homographic substitution $\dfrac{1}{x}$, $\dfrac{y}{x}$ employed by M. Halphen.

Writing $$X = \frac{1}{x} \text{ and } Y = \frac{y}{x},$$

let $Y_1, Y_2, Y_3, \ldots$ denote the successive derivatives of Y with respect to X, and $y_1, y_2, y_3, \ldots$ those of y with respect to x. Then

$$\begin{aligned}
Y &= x^{-1}y,\\
Y_1 &= -x\left(y_1 - \frac{1}{x}y\right),\\
Y_2 &= x^3y_2,\\
Y_3 &= -x^5\left(y_3 + \frac{3}{x}y_2\right),\\
Y_4 &= x^7\left(y_4 + \frac{8}{x}y_3 + \frac{12}{x^2}y_2\right),\\
Y_5 &= -x^9\left(y_5 + \frac{15}{x}y_4 + \frac{60}{x^2}y_3 + \frac{60}{x^3}y_2\right),\\
&\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}$$

Hence, if $a, b, c, d, \ldots$ are the successive modified derivatives (beginning with the second) of y with respect to x, and $a', b', c', d', \ldots$ the corresponding modified derivatives of Y with respect to X, it follows immediately that

$$a' = x^3 a,$$

$$b' = -x^5\left(b + \frac{1}{x}a\right),$$

$$c' = x^7\left(c + \frac{2}{x}b + \frac{1}{x^2}a\right),$$

$$d' = -x^9\left(d + \frac{3}{x}c + \frac{3}{x^2}b + \frac{1}{x^3}a\right),$$

....

Attributing the weights 0, 1, 2, 3, ... to the letters $a, b, c, d, \ldots$, it is very easily seen that if F is any homogeneous and isobaric function of degree i and weight w,

$$F(a', b', c', \ldots) = (-)^w x^{3i+2w} F\left(a,\, b + \frac{1}{x}a,\, c + \frac{2}{x}b + \frac{1}{x^2}a,\, \ldots\right).$$

But we proved (in Lecture XXII.) [above, p. 429] that for all values of h

$$F(a,\, b + ah,\, c + 2bh + ah^2,\, \ldots) = e^{h\Omega} F(a, b, c, \ldots).$$

Hence, making $h = \frac{1}{x}$, we obtain

$$F(a', b', c', d', \ldots) = (-)^w x^{3i+2w} e^{\frac{\Omega}{x}} F(a, b, c, \ldots),$$

which proves that the satisfaction of

$$\Omega F(a, b, c, \ldots) = 0$$

is the necessary and sufficient condition for the persistence of the form of F under the Halphenian substitution $\frac{1}{x}, \frac{y}{x}$.

Similarly we might prove that $F(y, t, a, b, c, \ldots)$, which contains y and t, but not x, is changed by the substitution $\frac{1}{x}, \frac{y}{x}$ into

$$(-)^w x^\nu e^{\frac{\Theta}{x}} F(y, t, a, b, c, \ldots),$$

where

$$\Theta = -y\partial_t + a\partial_b + 2b\partial_c + \ldots = \Omega - y\partial_t;$$

or we may deduce this result from the formula, demonstrated in the preceding lecture of this course,

$$F_1 = (1 + hx)^\nu e^{\frac{h\Theta}{1+hx}} F,$$

in which F_1 is what F becomes in consequence of the substitution $\frac{x}{1+hx}$, $\frac{y}{1+hx}$ impressed on the variables.

Let i be the degree and ω the weight measured by the sum of the orders of differentiation in each term of

$$F(y, t, a, b, c, \ldots).$$

If we measure the weight by the sum of the orders of differentiation of every term of F diminished by 2 units for each letter in the term, then

$$w = \omega - 2i \text{ and } 2\omega - i = 3i + 2w = \nu.$$

Let $\qquad F(y, t, a, b, c, \ldots)$ become $F'(y, t, a, b, c, \ldots)$,

when we change

$$x \text{ into } qx + p \text{ and } y \text{ into } ry;$$

then $\qquad F'(y, t, a, b, c, \ldots) = r^i q^{-\omega} F(y, t, a, b, c, \ldots).$

A further substitution $\dfrac{x}{1+hx}, \dfrac{y}{1+hx}$, impressed on the variables in F', will convert the original variables into

$$\frac{qx}{1+hx} + p \text{ and } \frac{ry}{1+hx},$$

that is, into $\qquad \dfrac{p(1+hx)+qx}{1+hx}$ and $\dfrac{ry}{1+hx}$.

The function F' is at the same time changed into

$$r^i q^{-\omega}(1+hx)^\nu e^{\frac{h\Theta}{1+hx}} F(y, t, a, b, c, \ldots).$$

If now, in the above, we write $p = h$, $q = -h^2$, $r = h$, we shall have changed the original variables x, y into $\dfrac{h}{1+hx}, \dfrac{hy}{1+hx}$, and the original function F into

$$h^i(-h^2)^{-\omega}(1+hx)^\nu e^{\frac{h\Theta}{1+hx}} F = (-)^\omega h^{i-2\omega}(1+hx)^\nu e^{\frac{h\Theta}{1+hx}} F = (-)^w \left(\frac{1+hx}{h}\right)^\nu e^{\frac{h\Theta}{1+hx}} F.$$

Let h become infinite; then $\dfrac{h}{1+hx}, \dfrac{hy}{1+hx}$ and $(-)^w\left(\dfrac{1+hx}{h}\right)^\nu e^{\frac{h\Theta}{1+hx}} F$ become $\dfrac{1}{x}, \dfrac{y}{x}$ and $(-)^w x^\nu e^{\frac{\Theta}{x}} F$, showing that the substitution $\dfrac{1}{x}, \dfrac{y}{x}$ changes F into $(-)^w x^\nu e^{\frac{\Theta}{x}} F$.

LECTURE XXV.

In a letter to me dated June 14th, 1886, M. Halphen calls forms which are persistent under the substitution $\frac{1}{x}, \frac{y}{x}$, *Invariants d'homologie.* He uses the letters

$$a_0,\ a_1,\ a_2,\ a_3,\ \ldots\ a_n,$$

to denote y and its successive modified derivatives with respect to x; and, supposing them to become

$$A_0,\ A_1,\ A_2,\ A_3,\ \ldots\ A_n,$$

in consequence of the substitution $\frac{1}{x}, \frac{y}{x}$, gives, in the briefest possible manner, two very ingenious proofs of the formula

$$A_n = (-)^n x^{2n-1}\left\{a_n + \frac{n-2}{1\,.\,x}a_{n-1} + \frac{(n-2)(n-3)}{1\,.\,2\,.\,x^2}a_{n-2} + \ldots\right\},$$

from which he deduces the theorem that the substitution in question changes any homogeneous and isobaric function f, of degree i and weight ω in

$$a_0,\ a_1,\ a_2,\ a_3,\ \ldots\ a_n,$$

into
$$F = (-)^\omega x^{2\omega - i} e^{\frac{\Theta}{x}} f,$$

where Θ is the partial differential operator

$$-a_0\partial_{a_1} + a_2\partial_{a_3} + 2a_3\partial_{a_4} + \ldots + (n-2)\,a_{n-1}\partial_{a_n}.$$

I give the two proofs mentioned above in M. Halphen's own words, adding occasional footnotes, and making slight changes in the literation of his formulae when it seems desirable to do so.

Soient
$$X = \frac{1}{x},\quad Y = \frac{y}{x}.$$

Par une formule connue (Schlömilch, Compendium II.)

$$\frac{d^n y}{dX^n} = (-1)^n x^{n+1}\frac{d^n}{dx^n}(x^{n-1}y)\,*$$

* An easy inductive proof of this may be obtained as follows:

Since $\frac{d}{dX} = -x^2\frac{d}{dx}$ we have $\frac{d^{\kappa+1}y}{dX^{\kappa+1}} = -x^2\frac{d}{dx}\left(\frac{d^\kappa y}{dX^\kappa}\right)$.

Hence, assuming the truth of the formula when $n=\kappa$, we find

$$\begin{aligned}\frac{d^{\kappa+1}y}{dX^{\kappa+1}} &= (-)^{\kappa+1}x^2\frac{d}{dx}\left\{x^{\kappa+1}\frac{d^\kappa}{dx^\kappa}(x^{\kappa-1}y)\right\}\\ &= (-)^{\kappa+1}x^2\left\{x^{\kappa+1}\frac{d^{\kappa+1}}{dx^{\kappa+1}}(x^{\kappa-1}y) + (\kappa+1)\,x^\kappa\frac{d^\kappa}{dx^\kappa}(x^{\kappa-1}y)\right\}\\ &= (-)^{\kappa+1}x^{\kappa+2}\left\{x\frac{d^{\kappa+1}}{dx^{\kappa+1}}(x^{\kappa-1}y) + (\kappa+1)\frac{d^\kappa}{dx^\kappa}(x^{\kappa-1}y)\right\}\\ &= (-)^{\kappa+1}x^{\kappa+2}\frac{d^{\kappa+1}}{dx^{\kappa+1}}(x^\kappa y).\end{aligned}$$

Thus, if the formula is true for $n=\kappa$, it will be equally so when $n=\kappa+1$. But it is obviously true when $n=1$ $\left(\text{when it becomes } \frac{dy}{dX} = -x^2\frac{dy}{dx}\right)$, and therefore holds universally.

et puisque $$Y = Xy,$$
il en resulte

$$\frac{d^nY}{dX^n} = X\frac{d^ny}{dX^n} + n\frac{d^{n-1}y}{dX^{n-1}} = (-1)^n x^n\left\{\frac{d^n}{dx^n}(x^{n-1}y) - n\frac{d^{n-1}}{dx^{n-1}}(x^{n-2}y)\right\}$$

$$= (-1)^n x^{2n-1}\left\{y_n + \frac{n(n-2)}{1.x}y_{n-1} + \frac{n(n-1)(n-2)(n-3)}{1.2.x^2}y_{n-2} + \dots\right\}^*.$$

Si l'on pose $$\frac{d^nY}{dX^n} = n!\,A_n,\quad y_n = n!\,a_n,$$

il vient

$$A_n = (-1)^n x^{2n-1}\left\{a_n + \frac{n-2}{1.x}a_{n-1} + \frac{(n-2)(n-3)}{1.2.x^2}a_{n-2} + \dots\right\}. \qquad \text{(I)}$$

Soit $$\Theta f = \Sigma\,(n-2)\,a_{n-1}\frac{\partial f}{\partial a_n}\,\dagger$$

on aura $\Theta a_n = (n-2)\,a_{n-1}$,

$\Theta^2 a_n = (n-2)(n-3)\,a_{n-2}$,

....

$$A_n = (-1)^n x^{2n-1}\left\{a_n + \frac{1}{1.x}\Theta a_n + \frac{1}{1.2.x^2}\Theta^2 a_n + \dots\right\}.$$

Par conséquent, pour une fonction contenant $a_0, a_1, a_2, \dots$, de degré i et de poids ω, à chaque terme, on aura

$$F = (-1)^\omega x^{2\omega-i}\left\{f + \frac{1}{1.x}\Theta f + \frac{1}{1.2.x^2}\Theta^2 f + \dots\right\}\ddagger. \qquad \text{C. Q. F. D.}$$

* For, expanding by Leibnitz's Theorem,

$$\frac{d^n}{dx^n}(x^{n-1}y) - n\frac{d^{n-1}}{dx^{n-1}}(x^{n-2}y) = x^{n-1}y_n + n(n-1)x^{n-2}y_{n-1} + \frac{n(n-1)}{1.2}(n-1)(n-2)x^{n-3}y_{n-2} + \dots$$
$$- n\{x^{n-2}y_{n-1} + (n-1)(n-2)x^{n-3}y_{n-2} + \dots\}$$
$$= x^{n-1}y_n + n(n-2)x^{n-2}y_{n-1} + \frac{n(n-1)(n-2)(n-3)}{1.2}x^{n-3}y_{n-2} + \dots.$$

† The summation extending to all positive integral values of n, from 1 to ∞, so that

$$\Theta = -a_0\partial_{a_1} + a_2\partial_{a_3} + 2a_3\partial_{a_4} + 3a_4\partial_{a_5} + \dots.$$

Remembering that Halphen's $a_0, a_1, a_2, a_3, \dots$ have the same meaning as our $y, t, a, b, \dots$, this operator is $-y\partial_t + a\partial_b + 2b\partial_c + 3c\partial_d + \dots$ identical with the Θ used in previous lectures.

‡ We may show without much difficulty that, when $\Theta_1, \Theta_2, \Theta_3, \dots$ are each of them equivalent to Θ, but Θ_1 acts on u only, Θ_2 on v, Θ_3 on w, and so on, $\Theta uvw \dots = (\Theta_1 + \Theta_2 + \Theta_3 + \dots)\,uvw \dots$. From this it can be deduced that $\Theta^\kappa uvw \dots = (\Theta_1 + \Theta_2 + \Theta_3 + \dots)^\kappa uvw \dots$, when κ is any positive integer. Now let the number of the functions $u, v, w, \dots$ be i, and suppose that

$$u = a_n,\quad v = a_p,\quad w = a_q,\ \dots;$$

suppose, also, that the weight $n + p + q + \dots = \omega$. Then

$$A_nA_pA_q \dots = (-)^\omega x^{2\omega-i}\left(e^{\frac{\Theta}{x}}a_n\right)\left(e^{\frac{\Theta}{x}}a_p\right)\left(e^{\frac{\Theta}{x}}a_q\right)\dots = (-)^\omega x^{2\omega-i}e^{\frac{1}{x}(\Theta_1+\Theta_2+\Theta_3+\dots)}a_na_pa_q\dots$$
$$= (-)^\omega x^{2\omega-i}e^{\frac{\Theta}{x}}a_na_pa_q\dots$$

(for by what precedes $\Theta_1 + \Theta_2 + \Theta_3 + \dots$ may be replaced by Θ). Taking $a_na_pa_q\dots$ and $A_nA_pA_q\dots$ to be corresponding terms of f and F, we see at once that

$$F = (-)^\omega x^{2\omega-i}e^{\frac{\Theta}{x}}f.$$

Autre Demonstration de la Formule (I)*.

Si l'on change X et x en $X+H$ et $x+h$, on a

$$h=-\frac{H}{X(X+H)}.$$

Maintenant la formule

$$y=a_0+ha_1+h^2a_2+\ldots+h^n a_n+\ldots$$

écrite *symboliquement*†

$$y=\frac{1}{1-ah}$$

devient
$$y=\frac{X(X+H)}{X^2+H(X+a)}.$$

D'ailleurs
$$Y=(X+H)\,y;$$

donc *symboliquement*

$$Y=\frac{X(X+H)^2}{X^2+H(X+a)}. \qquad \text{(II)}$$

Si l'on développe le second membre (II) suivant les puissances ascendants de H, le coefficient de H^n est A_n. Or ce développement est

$$Y=X\left\{1+\left(1-\frac{a}{X}\right)\frac{H}{X}+\left(\frac{a}{X}\right)^2\left(\frac{H}{X}\right)^2 + \ldots+(-1)^n\left(\frac{H}{X}\right)^n\left(1+\frac{a}{X}\right)^{n-2}\left(\frac{a}{X}\right)^2+\ldots\right\}$$

* If x becomes $x+h$ in consequence of the augmentation of X by an arbitrary quantity H, the increment of x will not be a constant, but will depend on X as well as on H. The value of h may be found at once by eliminating x between $X=\frac{1}{x}$ and $X+H=\frac{1}{x+h}$, when we obtain $X+H=\frac{X}{1+hX}$, and consequently $h=-\frac{H}{X(X+H)}$.

This increase of X also changes y and Y (functions of x and X, whose original values were a_0 and A_0 before the augmentation of X took place) into

$$y=a_0+ha_1+h^2a_2+\ldots+h^na_n+\ldots$$

and into
$$Y=A_0+HA_1+H^2A_2+\ldots+H^nA_n+\ldots.$$

These altered values of y and Y are the ones used in this second proof; the other letters retain their original signification.

† The word *symboliquement* indicates, whenever it is used, that powers of a are to be replaced by suffixes of corresponding value. For example, in the final result

$$A_n=(-)^n x^{2n-1}\left(a^n+\frac{n-2}{x}a^{n-1}+\ldots\right)$$

is to be replaced by
$$A_n=(-)^n x^{2n-1}\left(a_n+\frac{n-2}{x}a_{n-1}+\ldots\right).$$

In our notation the final result is $A_{n+2}=(-)^n x^{2n+3}\left(a,\ b,\ c,\ d,\ \ldots\middle)\!\!\middle(\frac{1}{x},\ 1\right)^n$.

donc *symboliquement*

$$A_n = (-1)^n \frac{1}{X^{n+1}} \left(1 + \frac{a}{X}\right)^{n-2} a^2 = (-1)^n x^{2n-1} \left(a + \frac{1}{x}\right)^{n-2} a^2$$

ce qui est justement la formule (I).

We may regard the coefficients $a, b, c, \ldots$ of the ordinary binary Quantic in u, v,

$$(a, b, c, \ldots \between u, v)^n,$$

as the successive modified derivatives, beginning with the second, of a new variable y with respect to another new variable x.

Any invariant I of this Quantic will then retain its form unaltered, or at most merely acquire an extraneous factor, if

(1) leaving x, y, v unaltered we change u into $u + \lambda v$,

(2) " u, v " " " x, y " $\dfrac{x}{1+hx}, \dfrac{y}{1+hx}$,

(3) " u, v " " " x, y " $\dfrac{1}{x}, \dfrac{y}{x}$,

where λ and h are arbitrary constants.

For we have seen that these three substitutions will severally convert any homogeneous and isobaric function F, of degree i and weight w in the letters $a, b, c, \ldots$, into

$$e^{\lambda\Omega} F, \ (1 + hx)^\nu e^{\frac{h\Omega}{1+hx}} F, \text{ and } (-)^w x^\nu e^{\frac{\Omega}{x}} F,$$

where, in each case, $\Omega = a\partial_b + 2b\partial_c + 3c\partial_d + \ldots$, and $\nu = 3i + 2w$. From our point of view an invariant is defined as a homogeneous and isobaric solution of the equation

$$\Omega I = 0.$$

Hence the above substitutions convert the invariant I into

$$I, \ (1 + hx)^\nu I, \text{ and } (-)^w x^\nu I, \text{ respectively.}$$

An *absolute invariant* with respect to any substitution is one which, disregarding its sign, remains unchanged in absolute value by that substitution. Thus, any invariant for which

$$\nu = 3i + 2w = 0$$

is an absolute invariant with respect to each of the three substitutions here considered.

An invariant is of odd or even character with respect to any substitution according as its sign is or is not changed by that substitution. Thus, invariants are of odd or even character with respect to the substitution $\dfrac{1}{x}, \dfrac{y}{x}$ according as their *weights* are odd or even.

This corresponds to the theorem that the character (with respect to the interchange of x and y) of a pure reciprocant is odd or even according as its degree is odd or even [p. 316, above].

From any two invariants for which ν has the same value we can form an absolute invariant (that is, one for which $\nu = 0$) by taking their ratio, and then by differentiating the absolute invariant thus formed obtain another invariant.

Suppose I_1 to be an invariant of degree i_1 and weight w_1,

I_2 " " " i_2 " w_2,

and let $$3i_1 + 2w_1 = \nu_1,\ 3i_2 + 2w_2 = \nu_2;$$

then the ν for $I_1^{\nu_2}$ is the same as that for $I_2^{\nu_1}$, and consequently $I_1^{\nu_2}I_2^{-\nu_1}$ is an absolute invariant.

We proceed to show that $\frac{d}{dx}(I_1^{\nu_2}I_2^{-\nu_1})$ is an invariant, though not an absolute one.

Using accents to denote differential derivation with respect to x, we have

$$\frac{d}{dx}(I_1^{\nu_2}I_2^{-\nu_1}) = I_1^{\nu_2-1}I_2^{-\nu_1-1}(\nu_2 I_1' I_2 - \nu_1 I_1 I_2').$$

If, then, we can prove that $\nu_2 I_1' I_2 - \nu_1 I_1 I_2'$ is an invariant, it will follow that $\frac{d}{dx}(I_1^{\nu_2}I_2^{-\nu_1})$ will be one also, and the proposition will be established. It may be very easily shown that this is the case by using Cayley's generators P and Q. For [p. 327, above], I being any invariant of degree i and weight w, PI and QI are also invariants where

$$P = a(b\partial_a + c\partial_b + d\partial_c + e\partial_d + \dots) - ib,$$

and $$Q = a(c\partial_b + 2d\partial_c + 3e\partial_d + \dots) - 2wb.$$

Hence $(3P + Q)I$ is an invariant.

Now, since $$3b\partial_a + 4c\partial_b + 5d\partial_c + \dots = \frac{d}{dx},$$

and $$3i + 2w = \nu,$$

$$(3P + Q)I = a(3b\partial_a + 4c\partial_b + 5d\partial_c + \dots)I - (3i + 2w)bI = aI' - \nu bI.$$

Consequently $aI_1' - \nu_1 bI_1$ and $aI_2' - \nu_2 bI_2$ are both of them invariants. Hence the combination

$$\nu_2 I_2(aI_1' - \nu_1 bI_1) - \nu_1 I_1(aI_2' - \nu_2 bI_2) = a(\nu_2 I_1' I_2 - \nu_1 I_1 I_2')$$

is also an invariant; that is

$$\nu_2 I_1' I_2 - \nu_1 I_1 I_2'$$

is one; which is the theorem to be demonstrated.

The invariant $aI' - \nu bI$, which we generated from I, is of degree $i + 1$ and weight $w + 1$; its ν is therefore the original ν increased by 5 units, three for the unit increase in the degree and two for the unit increase in the weight. Hence, on repeating the process of generation, we obtain the invariant

$$\left\{a \frac{d}{dx} - (\nu + 5)\, b\right\}(aI' - \nu bI) = a^2 I'' - 2(\nu + 1)\, abI' - 4\nu acI + \nu(\nu + 5)\, b^2 I.$$

By adding on the invariant $\nu(\nu + 5)(ac - b^2)\, I$ and dividing the sum by a, the above invariant is reduced to

$$aI'' - 2(\nu + 1)\, bI' + \nu(\nu + 1)\, cI,$$

which is an invariant of lower degree by unity than the unreduced form.

The results obtained above may be compared with the corresponding ones in the theory of reciprocants.

Thus to the invariants	correspond the reciprocants
I (deg. i, wt. w),	R (deg. i, wt. w),
$aI' - \nu bI$,	$aR' - \mu bR$,
$\nu_2 I_1' I_2 - \nu_1 I_1 I_2'$,	$\mu_2 R_1' R_2 - \mu_1 R_1 R_2'$,
$aI'' - 2(\nu + 1)\, bI' + \nu(\nu + 1)\, cI$,	$5aR'' - 5(2\mu + 1)\, bR' + 4\mu(\mu - 1)\, cR$,
where $\nu = 3i + 2w$,	where $\mu = 3i + w$.

Defining a *plenarily absolute* form to be one whose degree and weight are both zero ($i = 0$, $w = 0$), the theorem I shall now prove may be stated as follows:

By differentiating a plenarily absolute principiant we obtain another principiant.

Let P be any principiant of degree i and weight w. Then, by what precedes, since P is both an invariant and a reciprocant,

$$a \frac{dP}{dx} - \nu bP \text{ is an invariant,}$$

and

$$a \frac{dP}{dx} - \mu bP \text{ is a reciprocant.}$$

Hence, when $\nu = 0$ (that is, when $3i + 2w = 0$),

$$\frac{dP}{dx} \text{ is an invariant,}$$

and when $\mu = 0$ (that is, when $3i + w = 0$),

$$\frac{dP}{dx} \text{ is a reciprocant.}$$

When both $\mu = 0$ and $\nu = 0$ (which happens when $i = 0$, $w = 0$),

$$\frac{dP}{dx} \text{ is both a reciprocant and an invariant;}$$

that is,

$$\frac{dP}{dx} \text{ is a principiant.}$$

LECTURE XXVI.

In the theory of Invariants the annihilator Ω has two independent reversors any linear combination of which will also be a reversor. To each of these reversors there corresponds a generator for invariants. Thus Cayley's two generators

$$a(b\partial_a + c\partial_b + d\partial_c + e\partial_d + \ldots) - ib,$$
$$a(c\partial_b + 2d\partial_c + 3e\partial_d + \ldots) - 2wb,$$

correspond to the two reversors

$$b\partial_a + c\partial_b + d\partial_c + e\partial_d + \ldots,$$
$$c\partial_b + 2d\partial_c + 3e\partial_d + \ldots.$$

The only linear combination of these which does not increase the extent j as well as the weight of the operand is

$$O = jb\partial_a + (j-1)\,c\partial_b + (j-2)\,d\partial_c + \ldots.$$

It is convenient to take this for one of our reversors, and for the other

$$\frac{d}{dx} = 3b\partial_a + 4c\partial_b + 5d\partial_c + \ldots,$$

which is a reversor to V, the annihilator for reciprocants, as well as to Ω, the annihilator for invariants.

We saw in Lecture XI. [p. 364, above] that when F is any homogeneous and isobaric function of degree i and weight w in the $j+1$ letters $a, b, c, \ldots$

$$(\Omega O - O\Omega)\,F = (ij - 2w)\,F.$$

The method employed in proving this can also be applied to show that

$$\left(\Omega\frac{d}{dx} - \frac{d}{dx}\Omega\right)F = \nu F,$$

where $\nu = 3i + 2w$.

Corresponding to the reversors O and $\frac{d}{dx}$ we have the two generators for invariants

$$a\frac{d}{dx} - \nu b \text{ and } aO - (ij - 2w)\,b,$$

which are linear combinations of Cayley's generators.

Thus, if I be any invariant,

$$\left(a\frac{d}{dx} - \nu b\right)I \text{ and } \{aO - (ij - 2w)\,b\}\,I$$

are also invariants.

The operator $\frac{d}{dx}$ has, but O has not, analogous properties in the theory of Reciprocants; namely, $\frac{d}{dx}$ is a reversor to V and $a\frac{d}{dx} - \mu b$ is a generator for reciprocants. Thus, we have shown in previous lectures that

$$\left(V\frac{d}{dx} - \frac{d}{dx}V\right)F = 2\mu a F,$$

where F is any homogeneous and isobaric function, and $\mu = 3i + w$, and that if R is any pure reciprocant $\left(a\frac{d}{dx} - \mu b\right)R$ is one also.

Now, Mr Hammond has found that if

$$W = \frac{b}{a}\partial_a + \frac{2ac - b^2}{a^2}\partial_b + \frac{3a^2d - 3abc + b^3}{a^3}\partial_c + \ldots,$$

W is a reversor to V, and $a^2W - ib$ is a generator for pure reciprocants. In fact we have

$$\begin{aligned} VW - WV = {} & V\left(\frac{b}{a}\right)\partial_a \\ & + \left\{V\left(\frac{2ac - b^2}{a^2}\right) - W(2a^2)\right\}\partial_b \\ & + \left\{V\left(\frac{3a^2d - 3abc + b^3}{a^3}\right) - W(5ab)\right\}\partial_c \\ & + \ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

But, since

$$\begin{aligned} & V\left(\frac{b}{a}\right) = 2a, \\ & V\left(\frac{2ac - b^2}{a^2}\right) = 10b - 4b = 6b, \\ & V\left(\frac{3a^2d - 3abc + b^3}{a^3}\right) = \left(18c + 9\frac{b^2}{a}\right) - \left(15\frac{b^2}{a} + 6c\right) + 6\frac{b^2}{a} = 12c, \\ & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

and

$$\begin{aligned} & W(2a^2) = 4b, \\ & W(5ab) = 5\frac{b^2}{a} + 5\left(\frac{2ac - b^2}{a}\right) = 10c, \\ & \ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

it follows that

$$VW - WV = 2a\partial_a + 2b\partial_b + 2c\partial_c + \ldots = 2i.$$

Thus W is a reversor to V. Moreover, $a^2W - ib$ acting on any pure reciprocant generates another.

Let R be a pure reciprocant of degree i; then, by what precedes,

$$(VW - WV)R = 2iR.$$

But, since R is a pure reciprocant, $VR = 0$, and consequently $VWR = 2iR$.

Now, $V(a^2W - ib)R = a^2VWR - iRVb = a^2 . 2iR - iR . 2a^2 = 0$.

Hence $$(a^2W - ib)R$$

is a pure reciprocant; that is $a^2W - ib$

is a generator for pure reciprocants.

Mr Hammond shows that W is a reversor to V in the following manner:

Let
$$u = a_0 + a_1e^\theta + a_2e^{2\theta} + a_3e^{3\theta} + \ldots,$$
$$\phi(u) = A_0 + A_1e^\theta + A_2e^{2\theta} + A_3e^{3\theta} + \ldots,$$
$$\psi(u) = A_0' + A_1'e^\theta + A_2'e^{2\theta} + A_3'e^{3\theta} + \ldots,$$

and consider the operators
$$P = \lambda A_0\partial_{a_n} + (\lambda + \mu) A_1\partial_{a_{n+1}} + (\lambda + 2\mu) A_2\partial_{a_{n+2}} + \ldots,$$
$$Q = \lambda' A_0'\partial_{a_{n'}} + (\lambda' + \mu') A_1'\partial_{a_{n'+1}} + (\lambda' + 2\mu') A_2'\partial_{a_{n'+2}} + \ldots.$$

Regarding e^θ as an operative symbol defined by the equation
$$e^{\kappa\theta}[\partial_{a_0}] = \partial_{a_\kappa},$$

we may write
$$P = \{\lambda A_0e^{n\theta} + (\lambda + \mu) A_1e^{(n+1)\theta} + (\lambda + 2\mu) A_2e^{(n+2)\theta} + \ldots\}[\partial_{a_0}]$$
$$= e^{n\theta}\lambda(A_0 + A_1e^\theta + A_2e^{2\theta} + \ldots)[\partial_{a_0}]$$
$$+ e^{n\theta}\mu(A_1e^\theta + 2A_2e^{2\theta} + \ldots)[\partial_{a_0}]$$
$$= e^{n\theta}\left(\lambda + \mu\frac{d}{d\theta}\right)\phi(u)[\partial_{a_0}].$$

Similarly, $$Q = e^{n'\theta}\left(\lambda' + \mu'\frac{d}{d\theta}\right)\psi(u)[\partial_{a_0}].$$

Now, $$PQ - QP = \left\{Pe^{n'\theta}\left(\lambda' + \mu'\frac{d}{d\theta}\right)\psi(u) - Qe^{n\theta}\left(\lambda + \mu\frac{d}{d\theta}\right)\phi(u)\right\}[\partial_{a_0}]$$
$$= \left\{e^{n'\theta}\left(\lambda' + \mu'\frac{d}{d\theta}\right)P\psi(u) - e^{n\theta}\left(\lambda + \mu\frac{d}{d\theta}\right)Q\phi(u)\right\}[\partial_{a_0}].$$

For $$Q\phi(u) = QA_0 + e^\theta QA_1 + e^{2\theta}QA_2 + \ldots;$$

so that $$e^{n\theta}\frac{d}{d\theta}Q\phi(u) = e^{n\theta}(e^\theta QA_1 + 2e^{2\theta}QA_2 + \ldots)$$

and $$e^{n\theta}\frac{d}{d\theta}\phi(u) = e^{n\theta}(e^\theta A_1 + 2e^{2\theta}A_2 + \ldots);$$

so that $$Qe^{n\theta}\frac{d}{d\theta}\phi(u) = e^{n\theta}(e^\theta QA_1 + 2e^{2\theta}QA_2 + \ldots)$$
$$= e^{n\theta}\frac{d}{d\theta}Q\phi(u).$$

Similarly, $$Pe^{n'\theta}\frac{d}{d\theta}\psi(u) = e^{n'\theta}\frac{d}{d\theta}P\psi(u).$$

Moreover,

$$P\psi(u) = \psi'(u)\,Pu = \psi'(u)\,P(a_0 + a_1e^\theta + a_2e^{2\theta} + \ldots)$$
$$= \psi'(u)\,\{e^{n\theta}\lambda A_0 + e^{(n+1)\theta}(\lambda+\mu)\,A_1 + e^{(n+2)\theta}(\lambda+2\mu)\,A_2 + \ldots\}$$
$$= e^{n\theta}\psi'(u)\left(\lambda + \mu\frac{d}{d\theta}\right)\phi(u).$$

Similarly, $$Q\phi(u) = e^{n'\theta}\phi'(u)\left(\lambda' + \mu'\frac{d}{d\theta}\right)\psi(u).$$

Hence

$$PQ - QP = \left\{e^{n'\theta}\left(\lambda' + \mu'\frac{d}{d\theta}\right)e^{n\theta}\,\psi'(u)\left(\lambda + \mu\frac{d}{d\theta}\right)\phi(u)\right.$$
$$\left. - e^{n\theta}\left(\lambda + \mu\frac{d}{d\theta}\right)e^{n'\theta}\phi'(u)\left(\lambda' + \mu'\frac{d}{d\theta}\right)\psi(u)\right\}[\partial_{a_0}]$$
$$= e^{(n+n')\theta}\left\{\left(\lambda' + \mu'n + \mu'\frac{d}{d\theta}\right)\psi'(u)\left(\lambda + \mu\frac{d}{d\theta}\right)\phi(u)\right.$$
$$\left. - \left(\lambda + \mu n' + \mu\frac{d}{d\theta}\right)\phi'(u)\left(\lambda' + \mu'\frac{d}{d\theta}\right)\psi(u)\right\}[\partial_{a_0}].$$

If in this we write

$$\phi = \frac{u^2}{2},\qquad \lambda = 4,\ \ \mu = 1,\ \ n = 1,$$
$$\psi = \log u,\ \ \lambda' = 0,\ \ \mu' = 1,\ \ n' = -1,$$

we have

$$PQ - QP = \left\{\left(1 + \frac{d}{d\theta}\right)u^{-1}\left(4 + \frac{d}{d\theta}\right)\frac{u^2}{2} - \left(3 + \frac{d}{d\theta}\right)u\frac{d}{d\theta}\log u\right\}[\partial_{a_0}]$$
$$= \left\{\left(1 + \frac{d}{d\theta}\right)\left(2u + \frac{du}{d\theta}\right) - \left(3 + \frac{d}{d\theta}\right)\frac{du}{d\theta}\right\}[\partial_{a_0}]$$
$$= \left\{\left(1 + \frac{d}{d\theta}\right)\left(2 + \frac{d}{d\theta}\right) - \left(3 + \frac{d}{d\theta}\right)\frac{d}{d\theta}\right\}u\,[\partial_{a_0}]$$
$$= 2u\,[\partial_{a_0}].$$

Now, $$2u\,[\partial_{a_0}] = 2\,(a_0 + a_1e^\theta + a_2e^{2\theta} + \ldots)\,[\partial_{a_0}]$$
$$= 2\,(a_0\partial_{a_0} + a_1\partial_{a_1} + a_2\partial_{a_2} + \ldots).$$

Also $$P = 4A_0\partial_{a_1} + 5A_1\partial_{a_2} + 6A_2\partial_{a_3} + \ldots,$$
$$Q = A_1'\partial_{a_0} + 2A_2'\partial_{a_1} + 3A_3'\partial_{a_2} + \ldots,$$

where $$\frac{1}{2}(a_0 + a_1e^\theta + a_2e^{2\theta} + \ldots)^2 = A_0 + A_1e^\theta + A_2e^{2\theta} + \ldots$$

and $$\log(a_0 + a_1e^\theta + a_2e^{2\theta} + \ldots) = \log a_0 + A_1'e^\theta + A_2'e^{2\theta} + \ldots.$$

Equating coefficients, we have

$$A_0 = \frac{1}{2}a_0^2,\ \ A_1 = a_0a_1,\ \ A_2 = a_0a_2 + \frac{a_1^2}{2},\ \ldots$$
$$A_1' = \frac{a_1}{a_0},\ \ A_2' = \frac{2a_0a_2 - a_1^2}{2a_0^2},\ \ldots.$$

It is easily seen by expanding the logarithm that the general value of A_n' is $(-)^{n+1}\dfrac{S_n}{n}$ where S_n denotes the sum of the nth powers of the roots of

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_n.$$

Thus we have shown that if

$$P = 2a_0^2\partial_{a_1} + 5a_0a_1\partial_{a_2} + (6a_0a_2 + 3a_1^2)\,\partial_{a_3}$$

and
$$Q = \frac{a_1}{a_0}\partial_{a_0} + \frac{2a_0a_2 - a_1^2}{a_0^2}\partial_{a_1} + \frac{3a_0^2a_3 - 3a_0a_1a_2 + a_1^3}{a_0^3}\partial_{a_2} + \ldots,$$

then
$$PQ - QP = 2\,(a_0\partial_{a_0} + a_1\partial_{a_1} + a_2\partial_{a_2} + \ldots) = 2i.$$

The general formula obtained for $PQ - QP$ is an extension of a result of Capt. MacMahon's, who considers the case in which

$$\phi(u) = \frac{u^m}{m},\quad \psi(u) = \frac{u^{m'}}{m'}.$$

When $\phi(u)$ and $\psi(u)$ have these values, the general formula becomes

$$PQ - QP = e^{(n+n')\theta}\left\{\left(\lambda' + \mu'n + \mu'\frac{d}{d\theta}\right)\left(\frac{\lambda u^{m+m'-1}}{m} + \mu u^{m+m'-2}\frac{du}{d\theta}\right) - \ldots\ldots\ldots\ldots\right\}[\partial_{a_0}].$$

But
$$\left(\lambda' + \mu'n + \mu'\frac{d}{d\theta}\right)\left(\frac{\lambda}{m}u^{m+m'-1} + \mu u^{m+m'-2}\frac{du}{d\theta}\right)$$
$$= \left(\lambda' + \mu'n + \mu'\frac{d}{d\theta}\right)\left(\frac{\lambda}{m} + \frac{\mu}{m+m'-1}\frac{d}{d\theta}\right)u^{m+m'-1}.$$

Consequently

$$PQ - QP = e^{(n+n')\theta}\left\{\left(\lambda' + \mu'n + \mu'\frac{d}{d\theta}\right)\left(\frac{\lambda}{m} + \frac{\mu}{m+m'-1}\frac{d}{d\theta}\right) - \ldots\ldots\ldots\ldots\right\}u^{m+m'-1}[\partial_{a_0}].$$

In Capt. MacMahon's notation

$$P = (m, \lambda, \mu, n),\quad Q = (m', \lambda', \mu', n');$$

in our notation

$$P = e^{n\theta}\left(\lambda + \mu\frac{d}{d\theta}\right)\frac{u^m}{m}[\partial_{a_0}],$$

$$Q = e^{n'\theta}\left(\lambda' + \mu'\frac{d}{d\theta}\right)\frac{u^{m'}}{m'}[\partial_{a_0}].$$

If now we write

$$PQ - QP = e^{(n+n')\theta}\left(\lambda_1 + \mu_1\frac{d}{d\theta}\right)\frac{u^{m+m'-1}}{m+m'-1}[\partial_{a_0}],$$

which is equivalent to

$$PQ - QP = (m + m' - 1,\ \lambda_1,\ \mu_1,\ n + n'),$$

we have
$$\left(\lambda' + \mu' n + \mu' \frac{d}{d\theta}\right)\left\{\frac{\lambda}{m}(m+m'-1) + \mu \frac{d}{d\theta}\right\}$$
$$- \left(\lambda + \mu n' + \mu \frac{d}{d\theta}\right)\left\{\frac{\lambda'}{m'}(m+m'-1) + \mu' \frac{d}{d\theta}\right\} = \lambda_1 + \mu_1 \frac{d}{d\theta}.$$

Hence we obtain
$$\lambda_1 = (m+m'-1)\left\{\frac{\lambda}{m}(\lambda' + \mu' n) - \frac{\lambda'}{m'}(\lambda + \mu n')\right\},$$
$$\mu_1 = \mu\mu'(n-n') + \frac{\lambda\mu'}{m}(m'-1) - \frac{\lambda'\mu}{m'}(m-1).$$

This agrees with Capt. MacMahon's result, a statement of which was given in Lecture XX. [above, p. 417].

Let Q be a reversor to the operator $P = \lambda a^m \partial_b + (\ldots)\partial_c + (\ldots)\partial_d + \ldots$, and suppose that
$$(PQ - QP)F = \kappa a^{m-1}F,$$
where F is any homogeneous and isobaric function and κ some number depending on its degree and weight. Then $\lambda a Q - \kappa b$ will be the generator corresponding to Q. In other words, we have to prove that
$$P(\lambda a Q - \kappa b)F = 0 \text{ whenever } PF = 0.$$

Now, by hypothesis, $Pa = 0$, $Pb = \lambda a^m$, and when $PF = 0$,
$$PQF = \kappa a^{m-1}F.$$

Thus,
$$P(\lambda a Q - \kappa b)F = \lambda a PQF - \kappa F.Pb$$
$$= \lambda\kappa a^m F - \lambda\kappa a^m F = 0.$$

As an example, consider the case of the reversor $\frac{d}{dx}$ in the theory of reciprocants. Here
$$P = V,\ \lambda = 2,\ m = 2;$$

and since
$$\left(V\frac{d}{dx} - \frac{d}{dx}V\right)F = 2\mu a F,$$

we have $\kappa = 2\mu$. Hence the corresponding generator is $2\left(a\frac{d}{dx} - \mu b\right)$; or, disregarding the numerical factor 2, we may take $a\frac{d}{dx} - \mu b$ for the generator in question, which is usually denoted by the letter G.

We may also write G in the equivalent form
$$G = 4(ac - b^2)\partial_b + 5(ad - bc)\partial_c + 6(ae - bd)\partial_d + \ldots,$$
which it is sometimes more convenient to use.

I shall now show that
$$\Omega G - G\Omega = aw - b\Omega,$$
where w is the weight of the operand.

It is very easily seen that

$$\begin{aligned}\Omega(ac-b^2) &= 0,\\ \Omega(ad-bc) &= 2(ac-b^2),\\ \Omega(ae-bd) &= 3(ad-bc),\\ \Omega(af-be) &= 4(ae-bd),\\ &\dots\dots\end{aligned}$$

Hence it follows, by a direct and very simple calculation, that

$$\Omega G - G\Omega = 2(ac-b^2)\partial_c + 3(ad-bc)\partial_d + 4(ae-bd)\partial_e + \dots.$$

But, since $\quad b\partial_b + 2c\partial_c + 3d\partial_d + 4e\partial_e + \dots = w,$

and $\quad a\partial_b + 2b\partial_c + 3c\partial_d + 4d\partial_e + \dots = \Omega,$

$$aw - b\Omega = 2(ac-b^2)\partial_c + 3(ad-bc)\partial_d + 4(ae-bd)\partial_e + \dots.$$

Consequently $\quad \Omega G - G\Omega = aw - b\Omega.$

The use of this formula will be seen in a subsequent lecture.

We may also prove an analogous theorem relating to the invariant generator $a\dfrac{d}{dx} - \nu b$, which we shall call G'.

Let the operand be F, a homogeneous and isobaric function of degree i and weight w. Then VF is of degree $i+1$ and weight $w-1$; its ν is therefore

$$3(i+1) + 2(w-1) = \nu + 1.$$

Thus, $\quad (VG' - G'V)F = \left\{V\left(a\dfrac{d}{dx} - \nu b\right) - \left(a\dfrac{d}{dx} - \nu b - b\right)V\right\}F$

$$= a\left(V\frac{d}{dx} - \frac{d}{dx}V\right)F - \nu(Vb - bV)F + bVF.$$

But $\quad \left(V\dfrac{d}{dx} - \dfrac{d}{dx}V\right)F = 2\mu aF = 2(3i+w)aF,$

and $\quad VbF = bVF + 2a^2F.$

Consequently $\quad VG' - G'V = 2(3i+w)a^2F - 2\nu a^2F + bVF$

$$= 2(3i+w-\nu)a^2F + bVF$$

$$= -2wa^2F + bVF.$$

It is perhaps worthy of notice that if I is an invariant of weight w and R a pure reciprocant, also of weight w, then

$$\Omega GI = awI \text{ and } VG'R = -2a^2wR;$$

whereas $\quad \Omega G'I = 0 \text{ and } VGR = 0.$

LECTURE XXVII.

I should like to make a momentary pause in the development of the theory which now engages our attention and to revert to the proof of Cayley's theorem for the enumeration of linearly independent invariants contained in Lecture XI. and expressed by the formula $(w; i, j) - (w - 1; i, j)$.

Since that proof was written out I have endeavoured to obtain one that might be capable of being extended to the supposed analogous theorem, regarding pure reciprocants, expressed by the formula $(w; i, j)-(w-1; i+1, j)$, but all my efforts and those of another and most skilful algebraist in this direction have hitherto proved ineffectual.

In aiming at this object, however, I obtained a second proof of Cayley's theorem, less compendious than the previous one, and subject to the drawback that it assumes the law of Reciprocity, but which possesses the advantage over it of being more direct and of looking the question, so to say, more squarely in the face. The forms of thought employed in it seem to me too peculiar and precious to be consigned to oblivion. I am not one of those who look upon Analysis as only valuable for the positive results to which it leads, and who regard proofs as almost a superfluity, thinking it sufficient that mathematical formulae should be obtained, no matter how, and duly entered on a register.

I look upon Mathematics not merely as a language, an art, and a science, but also as a branch of Philosophy, and regard the forms of reasoning which it embodies and enshrines as among the most valuable possessions of the human mind. Add to this that it is scarcely possible that a well-reasoned mathematical proof shall not contain within itself subordinate theorems—germs of thought of intrinsic value and capable of extended application.

That such was the opinion of our High Pontiff is shown by the publication of his seven proofs of the Theorem of Reciprocity, a number to which subsequent researches have made almost annual additions (like so many continually augmenting asteroids in the Arithmetical Firmament) to such an extent that it would seem to be an interesting task for some one to undertake to form a corolla of these various proofs and to construct a reasoned bibliography, a *catalogue raisonnée*, of this one single theorem. For these reasons, I shall venture to put on record (*valeat quantum*) the following Second Proof of Cayley's Theorem.

The notation which I proceed to explain will be found very convenient. A rational integral homogeneous isobaric function will be called a *gradient;* its weight, degree, extent (extent meaning the number of letters after the first) will be denoted by w; i, j and spoken of as the *type* of the gradient. Either a single letter, such as ϕ, will be employed to denote a gradient, or else its type enclosed in a parenthesis thus $[w; i, j]$. The abbreviation $T\phi$ signifies the type of ϕ; thus, $T\phi = w; i, j$.

The number of terms in the most general gradient whose type is the same as that of ϕ will be spoken of as the *denumerant* of ϕ. The letter N will be used to denote such a denumerant; thus, $N\phi$ signifies the denumerant of ϕ.

In like manner, the letter Δ will be used to denote the number of linear relations between the coefficients of any gradient, whenever such relations exist. Hence $N\phi - \Delta\phi$ expresses the number of terms in ϕ whose coefficients are left arbitrary. Obviously, when ϕ is the most general gradient of its type, we have

$$\Delta\phi = 0.$$

We also use E to denote the $ij - 2w$, which may be called the *excess*, of the gradient of type $w; i, j$. Thus, if $T\phi = w; i, j$, we write $E\phi = ij - 2w$.

The operators which we shall employ, namely, Ω and Ω', are defined by the equations

$$\Omega \ = a_0\partial_{a_1} + a_1\partial_{a_2} + a_2\partial_{a_3} + \ldots,$$
$$\Omega' = \qquad\quad a_1\partial_{a_2} + a_2\partial_{a_3} + \ldots.$$

The first of these is of course an equivalent, but for present purposes more convenient, form of $a\partial_b + 2b\partial_c + 3c\partial_d + \ldots$, the ordinary invariant annihilator Ω $\left(\text{as will be evident on writing } a_0 = a,\ a_1 = \frac{b}{1},\ a_2 = \frac{c}{1\,.\,2},\ \ldots\right)$; the second of them, Ω', is merely Ω deprived of its first term.

We may now give the following enunciation of the theorem to be proved:

If ϕ is the most general gradient of its type, $\Omega\phi$ is also the most general gradient of its type whenever $E\phi$ is not negative. In other words, we shall prove that, subject to the condition stated above, $\Delta\Omega\phi = 0$ whenever $\Delta\phi = 0$. This is equivalent to Cayley's Theorem on the number of linearly independent invariants. For the number of forms of the same type as ϕ, and subject to annihilation by Ω, is

$$N\phi - N\Omega\phi + \Delta\Omega\phi;$$

and Cayley's Theorem states that the number of such forms is $N\phi - N\Omega\phi$, which will be the case when

$$\Delta\Omega\phi = 0.$$

The theorem of Reciprocity enables us to dispense with the discussion of those cases in which the extent j is greater than the degree i. For since [Vol. III. of this Reprint, p. 151] the number of linearly independent invariants for the type $w; j, i$ is the same as for the type $w; i, j$, we can substitute the first of these types for the second, using ψ, whose type is $w; j, i$, instead of ϕ, whose type is $w; i, j$. Thus we have

$$N\psi - N\Omega\psi + \Delta\Omega\psi = N\phi - N\Omega\phi + \Delta\Omega\phi.$$

But by Ferrers' proof of Euler's Theorem (*vide* "A Constructive Theory of Partitions" [p. 1, above]),

$$N\psi = N\phi \text{ and } N\Omega\psi = N\Omega\phi.$$

It obviously follows that

$$\Delta\Omega\psi = \Delta\Omega\phi.$$

Cases for which the extent is greater than the degree may therefore be made to depend on those for which the degree is greater than the extent. Hence Cayley's Theorem depends on the proof that $\Delta\Omega\phi = 0$ when $i => j$ and $ij => 2w$.

In the course of the demonstration, the following Lemma will be used:

If $\quad T\phi = w;\, i, j$ and $T\psi = ij - w;\, i, j$, then $N\phi = N\psi$.

The types of the two gradients we are now considering may be said to be *complementary*, and then the Lemma may be enunciated in words as follows:

The denumerants of two gradients are equal when the types of the gradients are complementary.

The proof consists in showing that to each term of the type $w;\, i, j$ there corresponds a term of the type $ij - w;\, i, j$. Let $a_0^{\lambda_0} a_1^{\lambda_1} a_2^{\lambda_2} \ldots a_j^{\lambda_j}$ be any term of the type $w;\, i, j$; then

$$w = \lambda_1 + 2\lambda_2 + 3\lambda_3 + \ldots + j\lambda_j$$

and

$$i = \lambda_0 + \lambda_1 + \lambda_2 + \lambda_3 + \ldots + \lambda_j.$$

Writing the suffixes of the letters $a_0, a_1, a_2, \ldots a_j$ in reverse order, everything else being kept unchanged, we obtain the term $a_j^{\lambda_0} a_{j-1}^{\lambda_1} a_{j-2}^{\lambda_2} \ldots a_0^{\lambda_j}$, whose weight we will call w'. Then

$$\begin{aligned} w' &= j\lambda_0 + (j-1)\lambda_1 + (j-2)\lambda_2 + \ldots + \lambda_{j-1} \\ &= j(\lambda_0 + \lambda_1 + \lambda_2 + \ldots + \lambda_j) - (\lambda_1 + 2\lambda_2 + 3\lambda_3 + \ldots + j\lambda_j) \\ &= ij - w. \end{aligned}$$

The degree of the transformed term is still i, and its extent is still j, while its weight has become $ij - w$; its type is therefore complementary to that of the original term. Hence to each term of any given type there corresponds a term of the complementary type, and consequently the total number of possible terms (that is, the Denumerant) for each type is the same.

By means of this Lemma it can be shown that $\Delta\Omega\phi = 0$ when $E\phi = -1$. Let

$$T\phi = w;\, i, j \text{ where } ij - 2w = -1;$$

then, since $T\Omega\phi = w = 1;\, i, j$, the types $T\phi$ and $T\Omega\phi$ are complementary (the sum of the weights being $w + w - 1 = ij$).

It follows from the Lemma that the Denumerants of ϕ and $\Omega\phi$ are equal. Hence

$$\Delta\Omega\phi = 0.$$

For if not, the number of independent terms in $\Omega\phi$ being less than the denumerant of $\Omega\phi$, will also be less than its equal, the denumerant of ϕ, and therefore there will be one or more invariants of the type $w; i, j$ for which the excess is negative. Since this is known to be impossible, we must have

$$\Delta\Omega\phi = 0.$$

We next prove that, in all cases for which $i = > w$, the number of linearly independent invariants of the type $w; i, j$ is correctly given by the formula

$$(w; i, j) - (w-1; i, j),$$

which is equivalent (as we showed at the beginning of Lecture XV.) to

$$(w; w, j) - (w-1; w, j),$$

or, what is the same thing, to the coefficient of $a^w x^w$ in the expansion of

$$F = \frac{1-x}{(1-a)(1-ax)(1-ax^2)(1-ax^3)\dots(1-ax^j)}.$$

Let the expansion of

$$G = \frac{1-x}{(1-ax)(1-ax^2)(1-ax^3)\dots(1-ax^j)}$$

be $$1 + (a-1)x + A_2x^2 + \dots + A_w x^w + \dots.$$

The expansion of F is obtained by multiplying that of G by the infinite geometrical series

$$1 + a + a^2 + a^3 + \dots.$$

But we only require the coefficient of $a^w x^w$ in the expansion of F, so that we need only retain the portion

$$A_w x^w (1 + a + a^2 + \dots + a^w)$$

of the above product instead of its complete expression.

It is of importance to notice here that A_w, which is independent of x, cannot contain any higher power of a than a^w. (That this is so will be evident from the constitution of the fraction G, for clearly no power of a in the expansion of G can be associated with a lower power of x.) Thus we see that

$$A_w = \alpha a^w + \beta a^{w-1} + \gamma a^{w-2} + \dots + \kappa a + \lambda,$$

and consequently

$$A_w x^w (1 + a + a^2 + \dots + a^w) = \dots + a^w x^w (\alpha + \beta + \gamma + \dots + \kappa + \lambda) + \dots.$$

Hence the coefficient of $a^w x^w$ in the expansion of F is

$$\alpha + \beta + \gamma + \dots + \kappa + \lambda,$$

which is the value assumed by A_w when in it we write $a = 1$. Call this value A_w', and let the value of G when $a = 1$ be denoted by G'. Then A_w' is the coefficient of x^w in

$$G' = \frac{1}{(1-x^2)(1-x^3)\dots(1-x^j)}.$$

Hence we see that, when $i = > w$, the value of $(w; i, j) - (w-1; i, j)$ is the total number of ways in which w can be made up of the parts 2, 3, ... j.

We have yet to show that this number is the same as that of the linearly independent invariants of the type $w; i, j$ when $i = > w$.

This follows from the known theorem that every invariant is either a rational integral function of the Protomorphs $a, P_2, P_3, \ldots P_j$ (meaning the invariant a and those of the second and third degrees alternately whose first terms are $ac, a^2d, ae, a^2f, \ldots$), or can be made so by multiplying it by a suitable power of a. Thus, if I be any invariant of degree i and weight w,

$$Ia^{w-i} = \Phi(a, P_2, P_3, \ldots P_j),$$

where Φ, which is of degree-weight $w.w$ when expressed in terms of $a, b, c, \ldots$, is rational and integral as regards the protomorphs.

When $i = > w$, writing

$$I = a^{i-w}\Phi(a, P_2, P_3, \ldots P_j),$$

Φ consists of a series of terms of the form $Aa^\theta P_2^\lambda P_3^\mu \ldots P_j^\rho$, *each with an arbitrary coefficient*, where, since

$$2\lambda + 3\mu + 4\nu + \ldots + j\rho = w,$$

the number of arbitrary constants in Φ is the total number of partitions of w into parts 2, 3, ... j. Hence the number of linearly independent invariants of the type $w; i, j$ is also this number of partitions, that is, by what precedes is $(w; i, j) - (w-1; i, j)$. This proves Cayley's theorem for cases in which $i = > w$.

But when $i < w$, the equation

$$Ia^{w-i} = \Phi(a, P_2, P_3, \ldots P_j)$$

shows that the coefficients of Φ are not all arbitrary, but must be so chosen that Φ may be divisible by a^{w-i}, and the reasoning employed in the case of $i = > w$ no longer holds.

It will be convenient at this point of the investigation to review the results we have hitherto obtained and to see what remains to be proved.

Cayley's Theorem has been demonstrated for cases in which the degree is not less than the weight. This will be expressed by saying that

$$\Delta\Omega\,[w; i, j] = 0 \text{ when } i = > w.$$

We have also proved that

$$\Delta\Omega\,[w; i, j] = 0 \text{ when } ij - 2w = -1.$$

The law of reciprocity has been expressed in the form

$$\Delta\Omega\,[w; i, j] = \Delta\Omega\,[w; j, i],$$

where $[w; i, j]$ denotes the most general gradient of the type $w; i, j$.

The theorem to be proved is that

$$\Delta\Omega\,[w;\,i,j]=0 \text{ when } ij-2w=>0\,;$$

but we may at once dismiss those cases in which $i=>w$, and (assuming the theorem to have been proved for Quantics of order inferior to j) those in which $i<j$, for these depend on the truth of the theorem for a Quantic of order i.

It remains, then, to prove that, when $ij-2w=>0$, $\Delta\Omega\,[w;\,i,j]=0$ for values of i inferior to w, but not inferior to j. This may be effected as follows:

Let ϕ be the most general gradient of the type $w;\,i+1,j$, and suppose

$$\phi=P+Qa+Ra^2+Sa^3,$$

where P, Q and R do not contain the letter a, though S may do so. Then, writing

$$\phi_1=Q+Ra+Sa^2,$$

ϕ_1 is the most general gradient of the type $w;\,i,j$.

Now, if $\Omega=a\partial_b+b\partial_c+c\partial_d+\ldots$, and $\Omega'=b\partial_c+c\partial_d+\ldots$, we have

$$\Omega\phi=\Omega'P+\left(\Omega'Q+\frac{dP}{db}\right)a+\left(\Omega'R+\frac{dQ}{db}\right)a^2+\left(\Omega S+\frac{dR}{db}\right)a^3, \qquad (1)$$

and $$\Omega\phi_1=\Omega'Q+\left(\Omega'R+\frac{dQ}{db}\right)a+\left(\Omega S+\frac{dR}{db}\right)a^2.$$

Confining our attention for the present to $\Omega\phi_1$, it is clear that if no linear relations exist among the coefficients of $\Omega'R$ (that is, if $\Delta\Omega'R=0$) the coefficients of $\Omega'Q$ are not connected with those of $\Omega'R+\dfrac{dQ}{db}$ by any linear relation. For the coefficient of each term of $\Omega'R+\dfrac{dQ}{db}$ is the sum of a single coefficient of Q and an independent linear function of the coefficients of R. Moreover, obviously the coefficients of $\Omega'Q$ are unconnected with those of $\Omega S+\dfrac{dR}{db}$.

If, then, the coefficients of $\Omega'Q$ are not related *inter se* (that is, if $\Delta\Omega'Q=0$), we have

$$\Delta\Omega\phi_1=\Delta\left\{\left(\Omega'R+\frac{dQ}{db}\right)a+\left(\Omega S+\frac{dR}{db}\right)a^2\right\}. \qquad (2)$$

Looking now to the expression (1) for $\Omega\phi$, we see immediately from (2) that any linear relation subsisting between the coefficients of $\Omega\phi_1$ will also subsist between those of $\Omega\phi$, and therefore that $\Delta\Omega\phi_1$ is not greater than $\Delta\Omega\phi$.

If, then, $\Delta\Omega\phi=0$, it follows that $\Delta\Omega\phi_1=0$, provided that both the supplementary conditions $\Delta\Omega'Q=0$ and $\Delta\Omega'R=0$ are also satisfied.

Now, since $\phi_1 = Q + Ra + Sa^2$ is the most general gradient of the type $w; i, j$,

Q will be the most general gradient of the type $w - i; i, j-1$,

and R „ „ „ „ „ $w - i + 1; i-1, j-1$,

when in Q and R we change $b, c, d, \ldots$ into $a, b, c, \ldots$. This change converts $\Omega' = b\partial_c + c\partial_d + \ldots$ into $\Omega = a\partial_b + b\partial_c + \ldots$. Hence the conditions $\Delta\Omega' Q = 0$ and $\Delta\Omega' R = 0$ are respectively equivalent to

$$\Delta\Omega\,[w - i; i, j-1] = 0 \text{ and } \Delta\Omega\,[w - i + 1; i-1, j-1] = 0.$$

Supposing these supplementary conditions to be satisfied, what we have proved is that when

$$\Delta\Omega\,[w; i+1, j] = 0 \text{ (that is, } \Delta\Omega\phi = 0),$$

then also $\quad \Delta\Omega\,[w; i, j] \quad = 0$ (that is, $\Delta\Omega\phi_1 = 0$).

Now,

$T\phi = w; i+1, j$, so that $E\phi = (i+1)j - 2w = (ij - 2w) + j$,

$TQ = w - i; i, j-1$, so that $EQ = i(j-1) - 2(w-i) = (ij-2w) + i$,

$TR = w - i + 1; i-1, j-1$, so that $ER = (i-1)(j-1) - 2(w-i+1)$

$$= (ij - 2w) + i - j - 1.$$

Thus, when $ij - 2w = > 0$ and $i = > j$,

$E\phi$ and EQ are both positive.

ER is in general $= > 0$, but in the special case where $ij - 2w = 0$ and $i = j$, we have $ER = -1$. Except in this case (which gives us no trouble, since we have seen that $\Delta\Omega R = 0$ in consequence of $ER = -1$), we have never to deal with a type of which the excess is negative.

Hence, if we assume Cayley's Theorem to have been proved for all extents up to $j - 1$ inclusive, we have

$$\Delta\Omega\,[w - i; i, j-1] = 0,$$

and $$\Delta\Omega\,[w - i + 1; i-1, j-1] = 0,$$

(that is, the two supplementary conditions are satisfied).

We wish to extend the theorem to the extent j.

Subject to the conditions $i = > j$ and $ij - 2w = > 0$, we have

$$\Delta\Omega\,[w; i, j] = 0 \text{ if } \Delta\Omega\,[w; i+1, j] = 0.$$

But we need consider no value of i greater than w, as we have proved that

$$\Delta\Omega\,[w; w, j] = 0 = \Delta\Omega\,[w; w+\kappa, j];$$

therefore $$\Delta\Omega\,[w; w-1, j] = 0,$$

$$\Delta\Omega\,[w; w-2, j] = 0,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\Delta\Omega\,[w; j, j] \quad = 0.$$

As previously shown, the theorem is true for all values of i inferior to j if it is true for all Quantics of inferior order. Thus the theorem is true for a Quantic of order j and for every value of i if it is true for all Quantics of order inferior to j. But it is true for the Quadric (where $j = 2$)*; therefore also for the Cubic ($j=3$); therefore also for the Quartic ($j=4$), and so universally. Hence the theorem to be proved is demonstrated.

LECTURE XXVIII.

We now resume the theory of Principiants and proceed to prove the important theorem that every Principiant is either simply an invariant in respect to a known series of pure reciprocants, which we call $A, B, C, D, \ldots$, or else becomes such an invariant when multiplied by a^{w-i}, where w is the weight and i the degree of the Principiant in question. Using the letter M to denote the pure reciprocant $ac - \frac{5}{4}b^2$, and G the ordinary eductive generator,

$$4(ac-b^2)\partial_b + 5(ad-bc)\partial_c + 6(ae-bd)\partial_d + 7(af-be)\partial_e + \ldots$$

(which, it will be remembered, is only another form of $a\frac{d}{dx} - \mu b$, with the advantage of the μ being suppressed, that is, only implicitly contained), we obtain in succession the values of $A, B, C, D, \ldots$ from the following equations:

$$5A = GM,$$
$$6B = GA,$$
$$7C = GB - MA,$$
$$8D = GC - 2MB,$$
$$9E = GD - 3MC,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

On performing the calculations indicated by these equations we shall find

$$A = a^2d - 3abc + 2b^3,$$

$$B = a^3e - 2a^2c^2 - \frac{7}{2}a^2bd + \frac{17}{2}ab^2c - 4b^4,$$

$$C = a^4f - 5a^3cd - 4a^3be + 13a^2bc^2 + \frac{45}{4}a^2b^2d - \frac{103}{4}ab^3c + \frac{19}{2}b^5,$$

$$D = a^5g - \frac{25}{8}a^4d^2 - 6a^4ce + 7a^3c^3 + \text{terms involving } b,$$

$$E = a^6h - \frac{15}{2}a^5de - 7a^5cf + 29a^4c^2d + \text{terms involving } b.$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

* When $j=2$ the condition $ij = > 2w$ becomes identical with $i = > w$; but we have already seen that the theorem is true whenever $i = > w$.

The fact that D is a pure reciprocant enables us to calculate the terms in E which are independent of b without a previous knowledge of the values of those terms in D which involve b. For, since

$$G = 4(ac - b^2)\partial_b + \dots \text{ and } V = 2a^2\partial_b + \dots,$$

$$a^2G - 2(ac - b^2)V \text{ does not contain } \partial_b.$$

Hence the operation of $a^2G - 2(ac - b^2)V$ on terms involving b cannot give rise to terms independent of b. But,

$$D \text{ being a pure reciprocant, } VD = 0;$$

so that
$$\{a^2G - 2(ac - b^2)V\}D = a^2GD,$$

and the terms of a^2GD which do not involve b are found by operating with

$$[a^2G - 2(ac - b^2)V]_{b=0}$$

on the terms of D which do not involve b.

If, now, we use $M_0, A_0, B_0, C_0, \dots$ to denote those portions of $M, A, B, C, \dots$ which are independent of b, and write

$$[a^2G - 2(ac - b^2)V]_{b=0} = a^2G_0,$$

we shall still have
$$9E_0 = G_0D_0 - 3M_0C_0;$$

and in general the law of successive derivation for $A_0, B_0, C_0, D_0, \dots$ is the same as that for $A, B, C, D, \dots$ except that G_0 takes the place of G.

We have

$$\begin{aligned} a^2G_0 &= [a^2G - 2(ac - b^2)V]_{b=0} \\ &= a^2(5ad\partial_c + 6ae\partial_d + 7af\partial_e + 8ag\partial_f + 9ah\partial_g + \dots) \\ &\quad - 2ac\{6ac\partial_d + 7ad\partial_e + (8ae + 4c^2)\partial_f + (9af + 9cd)\partial_g + \dots\}; \end{aligned}$$

so that

$$\begin{aligned} G_0 = 5ad\partial_c + 6(ae - 2c^2)\partial_d + 7(af - 2cd)\partial_e \\ + \frac{8}{a}(a^2g - 2ace - c^3)\partial_f + \frac{9}{a}(a^2h - 2acf - 2c^2d)\partial_g + \dots; \end{aligned}$$

and consequently (since $M_0 = ac$),

$$\begin{aligned} 5A_0 &= G_0M_0 &\text{gives } A_0 &= a^2d, \\ 6B_0 &= G_0A_0 &\text{,, } B_0 &= a^3e - 2a^2c^2, \\ 7C_0 &= G_0B_0 - M_0A_0 &\text{,, } C_0 &= a^4f - 5a^3cd, \\ 8D_0 &= G_0C_0 - 2M_0B_0 &\text{,, } D_0 &= a^5g - \frac{25}{8}a^4d^2 - 6a^4ce + 7a^3c^3, \\ 9E_0 &= G_0D_0 - 3M_0C_0 &\text{,, } E_0 &= a^6h - \frac{15}{2}a^5de - 7a^5cf + 29a^4c^2d, \\ &\dots\dots &\text{,, } &\dots\dots \end{aligned}$$

Thus, for example,

$$8D_0 = G_0(a^4f - 5a^3cd) - 2ac(a^3e - 2a^2c^2)$$
$$= -25a^4d^2 - 30a^3c(ae - 2c^2) + 8a^3(a^2g - 2ace - c^3) - 2ac(a^3e - 2a^2c^2);$$

whence
$$D_0 = a^5g - \frac{25}{8}a^4d^2 - 6a^4ce + 7a^3c^3.$$

Again,
$$9E_0 = G_0\left(a^5g - \frac{25}{8}a^4d^2 - 6a^4ce + 7a^3c^3\right) - 3ac(a^4f - 5a^3cd)$$
$$= 5ad(-6a^4e + 21a^3c^2) - \frac{75}{2}(ae - 2c^2)a^4d - 42(af - 2cd)a^4c$$
$$+ 9(a^2h - 2acf - 2c^2d)a^4 - 3ac(a^4f - 5a^3cd),$$

gives
$$E_0 = a^6h - \frac{15}{2}a^5de - 7a^5cf + 29a^4c^2d.$$

Similarly, from the known values of D_0 and E_0 we may deduce that of the next letter, F_0, and so on to any extent.

It may be noticed that each of the pure reciprocants $A, B, C, D, \ldots$ can be determined without ambiguity, by means of the annihilator V, when the portions of them, $A_0, B_0, C_0, D_0, \ldots$ independent of b are known.

For suppose R and R' to be two reciprocants, of weight w, for each of which the terms independent of b are the same. Then their difference is divisible by b. Let

$$R - R' = b\phi;\ \text{then}\ V(b\phi) = 0;\ \text{that is,}\ 2a^2\phi + bV\phi = 0.$$

Hence ϕ is divisible by b, and $R - R'$ is divisible by b^2; say $R - R' = b^2\psi$. Then

$$V(b^2\psi) = 4a^2b\psi + b^2V\psi = 0,$$

showing that ψ is divisible by b, and $R - R'$ by b^3.

By continually reasoning in this manner, we prove that $R - R'$ must be divisible by b^w; and then the remaining factor (being of weight 0) is necessarily of the form λa^θ, where λ and θ are numerical constants. Thus

$$R - R' = \lambda a^\theta b^w,\ \text{and consequently}\ V(\lambda a^\theta b^w) = 0.$$

This is impossible unless $\lambda = 0$, when the two reciprocants R, R' become equal, showing that there cannot be two different reciprocants for which the terms independent of b are the same. When, therefore, the terms which do not involve b of any pure reciprocant are known, the complete expression of that reciprocant can be determined without ambiguity.

Each reciprocant of the series $A, B, C, D, \ldots$ possesses the property of being, so to say, an Invariant relative to the one which precedes it, meaning that the operation of $\Omega = a\partial_b + 2b\partial_c + 3c\partial_d + \ldots$ on any letter gives (to a factor *près*) the one immediately preceding it. The first letter, A, is an Invariant in the ordinary sense. We can in fact show that

$$\Omega A = 0,$$

$$\Omega B = A \times \frac{a}{2},$$

$$\Omega C = 2B \times \frac{a}{2},$$

$$\Omega D = 3C \times \frac{a}{2},$$

$$\Omega E = 4D \times \frac{a}{2},$$

....

The proof depends on a formula established in Lecture XXVI. of this course [p. 457, above], namely

$$\Omega G - G\Omega = wa - b\Omega,$$

where G is the generator $4(ac - b^2)\partial_b + 5(ad - bc)\partial_c + \ldots$, and w is the weight of the operand.

Thus, observing that the weights of A, B, C, D, ... are 3, 4, 5, 6, ... respectively, we have

$$(\Omega G - G\Omega) A = (3a - b\Omega) A,$$

$$(\Omega G - G\Omega) B = (4a - b\Omega) B,$$

$$(\Omega G - G\Omega) C = (5a - b\Omega) C,$$

....

Now, since A is the well-known invariant $a^2d - 3abc + 2b^3$, we may write $\Omega A = 0$ in the first of these equations, which then reduces to

$$\Omega GA = 3aA.$$

But, since
$$6B = GA,$$
we have
$$6\Omega B = \Omega GA = 3aA.$$

Thus
$$\Omega B = A \times \frac{a}{2}.$$

Again, substituting for ΩB in the formula

$$(\Omega G - G\Omega) B = (4a - b\Omega) B,$$

we find
$$\Omega GB - G\left(\frac{aA}{2}\right) = 4aB - \frac{ab}{2} A,$$

where, since G (which is linear in ∂_b, ∂_c, ... and does not contain ∂_a) does not operate on a,

$$G\left(\frac{aA}{2}\right) = \frac{a}{2} GA = 3aB,$$

and consequently
$$\Omega GB + \frac{ab}{2} A = 7aB.$$

Now, $7C = GB - MA$;

so that $7\Omega C = \Omega GB - A\Omega M - M\Omega A.$

But, since $\Omega M = \Omega\left(ac - \frac{5b^2}{4}\right) = -\frac{ab}{2}$ and $\Omega A = 0$,

$$7\Omega C = \Omega GB + \frac{ab}{2}A = 7aB.$$

Thus $\Omega C = 2B \times \frac{a}{2}.$

We may, in exactly the same way, prove that

$$\Omega D = 3C \times \frac{a}{2},$$

$$\Omega E = 4D \times \frac{a}{2},$$

and so on to any extent.

In the following inductive proof it will be convenient to denote the letters

$$A, B, C, D, E, \ldots$$

by

$$u_0, u_1, u_2, u_3, u_4, \ldots,$$

and then the theorem to be proved is that

$$\Omega u_n = n u_{n-1} \times \frac{a}{2}.$$

When this notation is used, the law of successive derivation which defines the capital letters is expressed by the equation

$$(n+7)\, u_{n+2} - G u_{n+1} + (n+1)\, M u_n = 0, \qquad (1)$$

where G is the generator

$$4\,(ac - b^2)\,\partial_b + 5\,(ad - bc)\,\partial_c + \ldots, \text{ and } M = ac - \frac{5b^2}{4}.$$

Operating with Ω on the above equation, we obtain

$$(n+7)\,\Omega u_{n+2} - \Omega G u_{n+1} + (n+1)\,(M\Omega u_n + u_n \Omega M) = 0. \qquad (2)$$

Now, the weights of $u_0, u_1, u_2, \ldots$ are 3, 4, 5, ... respectively, and consequently the operation of

$$\Omega G - G\Omega = wa - b\Omega$$

on u_{n+1} (whose weight is $n+4$) gives

$$(\Omega G - G\Omega)\, u_{n+1} = (n+4)\, a u_{n+1} - b\Omega u_{n+1}.$$

Or, assuming that $\Omega u_\kappa = \kappa u_{\kappa-1} \times \frac{a}{2}$ for all values of κ as far as $n+1$ inclusive (it has previously been shown that $\Omega B = A \times \frac{a}{2}$ and $\Omega C = 2B \times \frac{a}{2}$, so that the theorem is true for $\kappa = 1$ and $\kappa = 2$),

$$\begin{aligned}\Omega G u_{n+1} &= G\Omega u_{n+1} + (n+4)\, a u_{n+1} - b\Omega u_{n+1}\\ &= (n+1)\, G\left(\frac{a}{2} u_n\right) + (n+4)\, a u_{n+1} - (n+1)\frac{ab}{2} u_n.\end{aligned}$$

But (remembering that G does not operate on a, so that $G.\frac{a}{2}u_n = \frac{a}{2}Gu_n$) we have, in virtue of equation (1),

$$G\left(\frac{a}{2}u_n\right) = \frac{a}{2}\{(n+6)u_{n+1} + nMu_{n-1}\}.$$

Hence it follows that

$$\Omega Gu_{n+1} = \frac{n+1}{2}a\{(n+6)u_{n+1} + nMu_{n-1}\} + (n+4)au_{n+1} - (n+1)\frac{ab}{2}u_n$$

$$= \frac{(n+2)(n+7)}{2}au_{n+1} + \frac{n(n+1)}{2}aMu_{n-1} - (n+1)\frac{ab}{2}u_n.$$

On substituting this in (2) we obtain

$$(n+7)\left\{\Omega u_{n+2} - (n+2)\frac{a}{2}u_{n+1}\right\}$$
$$+ (n+1)M\left\{\Omega u_n - n\frac{a}{2}u_{n-1}\right\}$$
$$+ (n+1)u_n\left\{\Omega M + \frac{ab}{2}\right\} = 0.$$

This reduces to $\qquad \Omega u_{n+2} = (n+2)\frac{a}{2}u_{n+1}.$

For, according to the assumption previously made in the course of the demonstration,

$$\Omega u_n = n\frac{a}{2}u_{n-1};$$

so that the second term vanishes; and the third term vanishes because

$$\Omega M = \Omega\left(ac - \frac{5b^2}{4}\right) = -\frac{ab}{2}.$$

We have therefore proved that if the theorem is true for Ωu_κ, when κ has any value up to $n+1$ inclusive, it is also true for Ωu_{n+2}. But the theorem holds for $\kappa = 1$, and for $\kappa = 2$. It therefore holds universally for any positive integer value of κ.

Recalling the known values of the reciprocants $M, A, B, C, D, \ldots$ we observe that their principal terms are $ac, a^2d, a^3e, a^4f, a^5g, \ldots$, where it is to be noticed that the most advanced of the small letters in the expression for any capital letter occurs only in the first degree multiplied by a power of a. In other words, $M, A, B, C, D, \ldots$ form a series of Protomorphs, and consequently every Pure Reciprocant can, as we have already seen (vide [p. 384, above]), be expressed as a function of $a, M, A, B, C, D, \ldots$ rational in all of them and integral in all except a.

But it is further to be noticed that whereas

a	is of degree	1	and weight	0,
M	„	2	„	2,
A	„	3	„	3,
B	„	4	„	4,

and in fact that every capital letter is of equal weight and degree.

From this it will follow that every Pure Reciprocant will be the product of a power of a into a function of the capital letters alone.

For let i be the degree and w the weight of any pure reciprocant expressed in terms of $a, M, A, B, C, \ldots$, and suppose one of its terms to be

$$a^\eta M^\theta A^\kappa B^\lambda C^\mu \ldots;$$

then
$$\eta + 2\theta + 3\kappa + 4\lambda + 5\mu + \ldots = i,$$
and
$$2\theta + 3\kappa + 4\lambda + 5\mu + \ldots = w.$$

Hence
$$\eta = i - w,$$
which is the same for every term of the pure reciprocant in question. Thus each term contains a^{i-w} as a factor, and the reciprocant is of the form

$$a^{i-w}\Phi(M, A, B, C, D, \ldots).$$

Let us now consider any Principiant P; since P is a pure reciprocant, we must have

$$P = a^{i-w}\Phi(M, A, B, C, D, \ldots).$$

But Principiants are subject to annihilation by Ω, and consequently $\Omega P = 0$, which gives

$$\frac{d\Phi}{dM}\Omega M + \frac{d\Phi}{dA}\Omega A + \frac{d\Phi}{dB}\Omega B + \frac{d\Phi}{dC}\Omega C + \ldots = 0.$$

On writing for $\qquad \Omega M, \quad \Omega A, \quad \Omega B, \quad \Omega C, \ldots$

their values $\qquad -b \times \frac{a}{2}, \quad 0, \quad A \times \frac{a}{2}, \quad 2B \times \frac{a}{2}, \ldots$

we obtain
$$\frac{a}{2}(-b\partial_M + A\partial_B + 2B\partial_C + 3C\partial_D + \ldots)\Phi = 0.$$

From this it would follow that Φ is an invariant in the two sets of letters

$$-b, M \text{ and } A, B, C, D, \ldots;$$

but it is easy to see that it is an invariant in the latter set exclusively. For M and $A, B, C, D, \ldots$ being all of them pure reciprocants,

$$\Phi \text{ and } \partial_M\Phi, \partial_B\Phi, \partial_C\Phi, \partial_D\Phi, \ldots,$$

which are functions of $M, A, B, C, \ldots$ exclusively, must also be pure reciprocants.

If, then, we operate with V on

$$(-b\partial_M + A\partial_B + 2B\partial_C + 3C\partial_D, \ldots)\,\Phi = 0,$$

we shall find $V(-b\partial_M)\,\Phi = 0$ (every other term being annihilated by V). Thus

$$V(b\partial_M)\,\Phi = (\partial_M\Phi)\,Vb = 2a^2\partial_M\Phi = 0,$$

and consequently $\partial_M\Phi = 0$. Hence

$$(A\partial_B + 2B\partial_C + 3C\partial_D + \ldots)\,\Phi = 0.$$

The equation $\partial_M\Phi = 0$ shows that M does not appear in the expression for any principiant in terms of the capital letters, while

$$(A\partial_B + 2B\partial_C + 3C\partial_D + \ldots)\,\Phi = 0$$

shows that Φ is an invariant in $A, B, C, D, \ldots$.

We have thus shown that every invariant of

$$(A, B, C, \ldots)(x, y)^j$$

is a principiant, and conversely that every principiant is an invariant of

$$(A, B, C, \ldots)(x, y)^j,$$

or such an invariant multiplied by a power of a.

LECTURE XXIX.

From the theorem that every Principiant is (to a power of *a près*) an Invariant in the reciprocantive elements $A, B, C, \ldots$ we readily deduce its correlative in which, everything else remaining unchanged, the *reciprocantive* elements $A, B, C, \ldots$ are replaced by a set of *invariantive* elements which we call $A_0, A_1, A_2, \ldots$. The equations connecting the new elements with the old ones are as follows:

$$A_0 = A,$$

$$A_1 = B - \left(\frac{b}{2}\right) A,$$

$$A_2 = C - 2\left(\frac{b}{2}\right) B + \left(\frac{b}{2}\right)^2 A,$$

$$A_3 = D - 3\left(\frac{b}{2}\right) C + 3\left(\frac{b}{2}\right)^2 B - \left(\frac{b}{2}\right)^3 A,$$

$$A_4 = E - 4\left(\frac{b}{2}\right) D + 6\left(\frac{b}{2}\right)^2 C - 4\left(\frac{b}{2}\right)^3 B + \left(\frac{b}{2}\right)^4 A,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

We have, in the first place, to prove that $A_0, A_1, A_2, \ldots$ are all of them invariants in the small letters $a, b, c, \ldots$. This is an immediate consequence of the identities

$$\Omega A = 0,$$

$$\Omega B = A \times \frac{a}{2},$$

$$\Omega C = 2B \times \frac{a}{2},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

established in the preceding Lecture, coupled with the fact that $\Omega b = a$. Thus

$$\Omega A_0 = \Omega A = 0,$$

$$\Omega A_1 = -\frac{b}{2}\Omega A + \left(\Omega B - A \times \frac{a}{2}\right) = 0,$$

$$\Omega A_2 = \left(\frac{b}{2}\right)^2 \Omega A - 2\left(\frac{b}{2}\right)\left(\Omega B - A \times \frac{a}{2}\right) + \left(\Omega C - 2B \times \frac{a}{2}\right) = 0;$$

and in general, writing the equation which gives A_n in the form

$$A_n = \left(-\frac{b}{2}\right)^n A + n\left(-\frac{b}{2}\right)^{n-1} B + \frac{n(n-1)}{1.2}\left(-\frac{b}{2}\right)^{n-2} C + \frac{n(n-1)(n-2)}{1.2.3}\left(-\frac{b}{2}\right)^{n-3} D + \ldots,$$

and operating on it with Ω, we find

$$\Omega A_n = \left(-\frac{b}{2}\right)^n \Omega A + n\left(-\frac{b}{2}\right)^{n-1}\left(\Omega B - A \times \frac{a}{2}\right) + \frac{n(n-1)}{1.2}\left(-\frac{b}{2}\right)^{n-2}\left(\Omega C - 2B \times \frac{a}{2}\right) + \frac{n(n-1)(n-2)}{1.2.3}\left(-\frac{b}{2}\right)^{n-3}\left(\Omega D - 3C \times \frac{a}{2}\right) + \ldots$$

$$= 0 \text{ (each term vanishing separately).}$$

We next observe that

$(A_0, A_1, A_2, \ldots)(x, y)^j$, being equal to $(A, B, C, \ldots)\left(x - \frac{b}{2}y, y\right)^j$,

is a linear transformation of $(A, B, C, \ldots)(x, y)^j$,

and that the determinant of the transformation $\begin{vmatrix} 1 & -\frac{b}{2} \\ 0 & 1 \end{vmatrix}$ is equal to unity.

Hence every invariant in $A_0, A_1, A_2, \ldots$ is equal to the corresponding invariant in $A, B, C, \ldots$, which proves the theorem in question.

Each of the invariantive elements A_0, A_1, A_2, ... is, so to say, a *reciprocant* relative to the one which immediately precedes it, just as in the cognate theorem each of the capital letters A, B, C, ... was an *invariant* relative to its antecedent. It is in fact easily seen that

$$VA_0 = 0,$$

$$VA_1 = -A_0a^2,$$

$$VA_2 = -2A_1a^2,$$

$$VA_3 = -3A_2a^2,$$

....

and in general $$VA_n = -nA_{n-1}a^2.$$

Thus, for example, if we operate with V on

$$A_3 = D - 3\left(\frac{b}{2}\right)C + 3\left(\frac{b}{2}\right)^2 B - \left(\frac{b}{2}\right)^3 A,$$

remembering that A, B, C, D are pure reciprocants, we shall find

$$VA_3 = -\frac{3}{2}\left\{C - 2\left(\frac{b}{2}\right)B + \left(\frac{b}{2}\right)^2 A\right\} VB.$$

But $$C - 2\left(\frac{b}{2}\right)B + \left(\frac{b}{2}\right)^2 A = A_2 \text{ and } Vb = 2a^2;$$

so that $$VA_3 = -3A_2a^2.$$

In like manner, operating with V on

$$A_n = (A, B, C, \ldots)\left(-\frac{b}{2}, 1\right)^n,$$

we obtain $$VA_n = -\frac{n}{2}(A, B, C, \ldots)\left(-\frac{b}{2}, 1\right)^{n-1} Vb$$

$$= -nA_{n-1}a^2.$$

This property enables us to give a proof (exactly similar to the proof of the cognate theorem in the preceding Lecture) of the theorem that every principiant is expressible as the product of an invariant in A_0, A_1, A_2, ... by a suitable power of a. We first observe that, using N to denote $ac - b^2$,

$$N, A_0, A_1, A_2, \ldots$$

form a series of invariantive protomorphs of equal degree and weight.

Hence it follows that any invariant of degree i and weight w can be expressed in the form

$$a^{i-w}\Phi(N, A_0, A_1, A_2, \ldots),$$

and consequently that every Principiant can be expressed in this form, provided only that

$$V\Phi = 0.$$

Substituting for VA_0, VA_1, VA_2, ... their values given above, and at the same time observing that

$$VN = V(ac - b^2) = 5a^2b - 4a^2b = a^2b,$$

we find $$V\Phi = a^2(b\partial_N - A_0\partial_{A_1} - 2A_1\partial_{A_2} - 3A_2\partial_{A_3} - \ldots)\Phi = 0.$$

Finally, we prove that Φ does not contain N, but is an invariant in A_0, A_1, A_2, ... alone, by operating with Ω on

$$(b\partial_N - A_0\partial_{A_1} - 2A_1\partial_{A_2} - 3A_2\partial_{A_3} - \ldots)\Phi = 0,$$

when it is easily seen that every term vanishes except the first, which gives

$$\Omega(b\partial_N\Phi) = \Omega b \times \partial_N\Phi = 0,$$

where, $\Omega b = a$ being different from zero, we must have $\partial_N\Phi = 0$.

The invariants N, A_0, A_1, A_2, ... obey a law of successive derivation similar to that which holds for the reciprocants M, A, B, C, ...

Starting with $N = ac - b^2$ and operating continually with

$$G' = a\frac{d}{dx} - (3i + 2w)b = (4ac - 5b^2)\partial_b + (5ad - 7bc)\partial_c + \ldots,$$

we shall find
$$\begin{aligned} G'N &= 5A_0, \\ G'A_0 &= 6A_1, \\ G'A_1 &= 7A_2 - NA_0, \\ G'A_2 &= 8A_3 - 2NA_1, \\ G'A_3 &= 9A_4 - 3NA_2, \\ &\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

and generally $$G'A_n = (n+6)A_{n+1} - nNA_{n-1}.$$

These equations are exactly analogous to

$$\begin{aligned} GM &= 5A, \\ GA &= 6B, \\ GB &= 7C + MA, \\ GC &= 8D + 2MB, \\ GD &= 9E + 3MC, \\ &\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

in which $M = ac - \frac{5}{4}b^2$, and GM, GA, GB, ... are the educts of M, A, B, ... obtained by operating with

$$G = a\frac{d}{dx} - (3i + w)b = 4(ac - b^2)\partial_b + 5(ad - bc)\partial_c + \ldots.$$

It should be noticed that the two generators G and G' are connected by the relation

$$G' = G - wb,$$

where w is the weight of the operand.

Also, that

$$Gb = 4(ac - b^2) = 4N, \text{ and } G'b = 4ac - 5b^2 = 4M.$$

We may easily verify that

$$G'N = 5A_0 = 5(a^2d - 3abc + 2b^3)$$

by operating with $G' = (4ac - 5b^2)\partial_b + (5ad - 7bc)\partial_c$ on $N = ac - b^2$.

To prove that $\quad G'A_0 = 6A_1,$

we operate on $\quad A_0 = A,$

for which the weight is 3, with

$$G' = G - 3b.$$

Thus $\quad G'A_0 = (G - 3b)A = 6B - 3bA = 6A_1.$

For by definition $\quad A_1 = B - \left(\frac{b}{2}\right)A.$

In general, to find $G'A_n$, we have by definition

$$A_n = (A, B, C, \ldots)\left(-\frac{b}{2}, 1\right)^n,$$

and, since the weight of A_n is $n+3$,

$$G'A_n = GA_n - (n+3)bA_n.$$

Now

$$GA_n = G(A, B, C, \ldots)\left(-\frac{b}{2}, 1\right)^n$$

$$= (GA, GB, GC, \ldots)\left(-\frac{b}{2}, 1\right)^n - \frac{n}{2}(A, B, C, \ldots)\left(-\frac{b}{2}, 1\right)^{n-1} Gb.$$

Substituting for GA, GB, GC, ... their known values, and remembering that $Gb = 4N$ and that $(A, B, C, \ldots)\left(-\frac{b}{2}, 1\right)^{n-1} = A_{n-1}$, we have

$$GA_n = (6B, 7C, 8D, \ldots)\left(-\frac{b}{2}, 1\right)^n$$

$$+ M(0, A, 2B, 3C, \ldots)\left(-\frac{b}{2}, 1\right)^n - 2nNA_{n-1}$$

$$= 6(B, C, D, \ldots)\left(-\frac{b}{2}, 1\right)^n + (0, C, 2D, 3E, \ldots)\left(-\frac{b}{2}, 1\right)^n$$

$$+ M(0, A, 2B, 3C, \ldots)\left(-\frac{b}{2}, 1\right)^n - 2nNA_{n-1}.$$

But $\quad (0, C, 2D, 3E, \ldots)\left(-\frac{b}{2}, 1\right)^n$

$$= nC\left(-\frac{b}{2}\right)^{n-1} + n(n-1)D\left(-\frac{b}{2}\right)^{n-2} + \frac{n(n-1)(n-2)}{1.2}E\left(-\frac{b}{2}\right)^{n-3} + \ldots$$

$$= n(C, D, E, \ldots)\left(-\frac{b}{2}, 1\right)^{n-1};$$

and similarly

$$(0, A, 2B, 3C, \ldots)\left(-\frac{b}{2}, 1\right)^n = n(A, B, C, \ldots)\left(-\frac{b}{2}, 1\right)^{n-1} = nA_{n-1}.$$

Hence

$$GA_n = 6(B, C, D, \ldots)\left(-\frac{b}{2}, 1\right)^n + n(C, D, E, \ldots)\left(-\frac{b}{2}, 1\right)^{n-1}$$
$$+ n(M - 2N)A_{n-1}.$$

Now let $U = (A, B, C, \ldots)(u, v)^n$;

then $\dfrac{dU}{du} = n(A, B, C, \ldots)(u, v)^{n-1}$,

and $\dfrac{dU}{dv} = n(B, C, D, \ldots)(u, v)^{n-1}$;

whence it follows that

$$U = (A, B, C, \ldots)(u, v)^n = u(A, B, C, \ldots)(u, v)^{n-1}$$
$$+ v(B, C, D, \ldots)(u, v)^{n-1}. \quad (1)$$

Similarly, we see that

$$(B, C, D, \ldots)(u, v)^n = u(B, C, D, \ldots)(u, v)^{n-1}$$
$$+ v(C, D, E, \ldots)(u, v)^{n-1}. \quad (2)$$

Writing $u = -\frac{b}{2}$ and $v = 1$ in the above equations, and remembering that

$$(A, B, C, \ldots)\left(-\frac{b}{2}, 1\right)^n = A_n,$$

we obtain immediately from (1)

$$(B, C, D, \ldots)\left(-\frac{b}{2}, 1\right)^{n-1} = A_n + \frac{b}{2}A_{n-1},$$

and then (2) gives

$$(C, D, E, \ldots)\left(-\frac{b}{2}, 1\right)^{n-1} = \left(A_{n+1} + \frac{b}{2}A_n\right) + \frac{b}{2}\left(A_n + \frac{b}{2}A_{n-1}\right)$$
$$= A_{n+1} + bA_n + \frac{b^2}{4}A_{n-1}.$$

But it has been shown that

$$GA_n = 6(B, C, D, \ldots)\left(-\frac{b}{2}, 1\right)^n + n(C, D, E, \ldots)\left(-\frac{b}{2}, 1\right)^{n-1}$$
$$+ n(M - 2N)A_{n-1}.$$

Hence, by substitution,

$$GA_n = 6\left(A_{n+1} + \frac{b}{2}A_n\right) + n\left(A_{n+1} + bA_n + \frac{b^2}{4}A_{n-1}\right) + n(M - 2N)A_{n-1}$$
$$= (n+6)A_{n+1} + (n+3)bA_n + n\left(M + \frac{b^2}{4} - 2N\right)A_{n-1}.$$

Now, $G'A_n = GA_n - (n+3)\,bA_n = (n+6)\,A_{n+1} + n\left(M + \frac{b^2}{4} - 2N\right)A_{n-1}$,

where $$M + \frac{b^2}{4} = ac - \frac{5}{4}b^2 + \frac{b^2}{4} = ac - b^2 = N.$$

Thus $$G'A_n = (n+6)\,A_{n+1} - nNA_{n-1},$$

which proves the law of successive derivation for the invariantive elements $A_0, A_1, A_2, \ldots$*.

We now proceed to explain the method of transforming a Principiant, given in terms of the small letters $a, b, c, \ldots$, into one expressed in terms of $a, A, B, C, \ldots$.

Remembering that the expressions for

$$A, B, C, D, E, \ldots$$

have for their most advanced small letters

$$d, e, f, g, h, \ldots,$$

and that, in each capital letter, the most advanced letter occurs only in the first degree, multiplied by a power of a, it follows, as an immediate consequence, that we may, by continually substituting for the most advanced letter, eliminate $d, e, f, g, h, \ldots$ from any rational integral function

$$\phi\,(a, b, c, d, e, f, g, h, \ldots)$$

and thus transform it into another function whose arguments are

$$a, b, c, A, B, C, D, E, \ldots$$

and which is rational in all its arguments, and integral in all of them, with the possible exception of the first argument, a.

But (see Lecture XXVIII.) [above, p. 471] the result of this elimination is known to be

$$a^{i-w}\,\Phi\,(A, B, C, D, E, \ldots)$$

in the case where ϕ is a Principiant of known degree i and weight w. Hence b and c must disappear spontaneously during the process of elimination.

This being so, we can give b and c any arbitrary values, without thereby affecting the result, and it will greatly simplify the work to take $b = 0$ and $c = 0$.

It is also permissible to take $a = 1$; for, although the factor a^{i-w} is thereby lost, it can always be restored in the final result because both i and

* The establishment of the scale of relation between the terms of the $A_0, A_1, A_2, \ldots$ series, and the above proof of it, is due exclusively to Mr Hammond.

w are known numbers. Now, if we write $a=1$, $b=0$, $c=0$ in the known expressions for $A, B, C, D, \ldots$, we shall find

$$A = d,$$
$$B = e,$$
$$C = f,$$
$$D = g - \frac{25}{8} d^2,$$
$$E = h - \frac{15}{2} de,$$
$$\cdots\cdots\cdots\cdots$$

Hence we have to eliminate $d, e, f, g, h, \ldots$ between the above equations and

$$P = \phi\,(1, 0, 0, d, e, f, g, h, \ldots),$$

where P stands for the given Principiant. In other words, we have to substitute for

$$\begin{array}{cccccccccc} a, & b, & c, & d, & e, & f, & g, & h, & \ldots \\ 1, & 0, & 0, & A, & B, & C, & D+\frac{25}{8}A^2, & E+\frac{15}{2}AB, & \ldots \end{array}$$

in

$$P = \phi\,(a, b, c, d, e, f, g, h, \ldots).$$

The result of this substitution will be

$$P = \Phi\,(A, B, C, D, E, \ldots),$$

where, to compensate for the factor lost by taking $a=1$, we must multiply Φ by a^{i-w}. As an easy example, consider the Principiant which Halphen calls Δ, and for which he obtains the expression

$$\begin{vmatrix} b & c & d & e & f \\ a & b & c & d & e \\ -a^2 & 0 & b^2 & 2bc & 2bd+c^2 \\ 0 & a^2 & 2ab & 2ac+b^2 & 2ad+2bc \\ 0 & 0 & a^2 & 3ab & 3b^2+3ac \end{vmatrix}.$$

Here the degree $i=8$ and the weight $w=8$; so that $i-w=0$, and no factor has to be restored. On making the substitutions spoken of, the determinant becomes

$$\begin{vmatrix} 0 & 0 & A & B & C \\ 1 & 0 & 0 & A & B \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2A \\ 0 & 0 & 1 & 0 & 0 \end{vmatrix},$$

which immediately reduces to $AC-B^2$ by striking out the first three columns and the last three rows.

Of this Principiant we shall have more to say hereafter.

LECTURE XXX.

The method of substituting large letters for small ones will be better understood if we employ it to obtain an expression of the form

$$a^{i-w}\Phi(M, A, B, C, D, E, \ldots)$$

for any pure reciprocant

$$\phi(a, b, c, d, e, f, g, h, \ldots)$$

of known degree i and weight w in the small letters.

The transformation is effected by substituting in ϕ for $c, d, e, f, g, h, \ldots$ their values (which are perfectly definite) in terms of $a, b, M, A, B, C, D, E, \ldots$ But since b does not appear in the final result, we are at liberty to give it any arbitrary value, and it will be convenient to take $b=0$, for then (see Lecture XXVIII.) [above, p. 465] we have

$$\begin{aligned}
M &= ac,\\
A &= a^2d,\\
B &= a^3e - 2a^2c^2,\\
C &= a^4f - 5a^3cd,\\
D &= a^5g - \frac{25}{8}a^4d^2 - 6a^4ce + 7a^3c^3,\\
E &= a^6h - \frac{15}{2}a^5de - 7a^5cf + 29a^4c^2d,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

There is an additional advantage in taking $b=0$, namely, that then the values of the *invariants* $N, A_0, A_1, A_2, \ldots$ (see their definition at the beginning of* Lecture XXIX.) exactly coincide with those of the *reciprocants* $M, A, B, C, \ldots$ set forth above. Hence, merely interchanging the capital letters, the same substitutions enable us to express any invariant in terms of $a, N, A_0, A_1, \ldots$, as well as any reciprocant in terms of $a, M, A, B, \ldots$.

The solution of the above equations will give $\frac{c}{a}, \frac{d}{a}, \frac{e}{a}, \ldots$ in terms of $\frac{M}{a^2}, \frac{A}{a^3}, \frac{B}{a^4}, \ldots$; but we can, without loss of generality, put $a=1$, when we shall find

$$\begin{aligned}
a &= 1,\\
b &= 0,\\
c &= M,\\
d &= A,\\
e &= B + 2M^2,\\
f &= C + 5MA,\\
g &= D + \frac{25}{8}A^2 + 6MB + 5M^3,\\
h &= E + \frac{15}{2}AB + 7MC + 6MA^2,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

[* p. 472, above.]

The substitution of these values in the pure reciprocant

$$\phi(a, b, c, d, e, f, g, h, \ldots)$$

will convert it into

$$\Phi(M, A, B, C, D, E, \ldots).$$

We have written $a=1$ for the sake of simplicity; but without doing this we have, since ϕ is homogeneous of degree i,

$$\phi(a, 0, c, d, e, \ldots) = a^i\phi\left(1,\ 0,\ \frac{c}{a},\ \frac{d}{a},\ \frac{e}{a},\ \ldots\right).$$

Hence, substituting for $\frac{c}{a}, \frac{d}{a}, \frac{e}{a}, \ldots$ in terms of $\frac{M}{a^2}, \frac{A}{a^3}, \frac{B}{a^4}, \ldots$,

$$\phi(a, 0, c, d, e, \ldots) = a^i\Phi\left(\frac{M}{a^2}, \frac{A}{a^3}, \frac{B}{a^4}, \ldots\right);$$

or, since $M, A, B, \ldots$ are of weights 2, 3, 4, ... and Φ is of weight w,

$$\phi(a, 0, c, d, e, \ldots) = a^{i-w}\Phi(M, A, B, \ldots).$$

Thus, in consequence of writing $a=1$, the factor a^{i-w} has been lost; but this factor can always be restored, both i and w being known numbers.

When ϕ is a Principiant, M will not appear in the final result, which will be identical with that obtained by the simpler substitutions of the preceding Lecture. If, for example, we substitute for

$$\begin{array}{cccccc} a, & b, & c, & d, & e, & f, \\ 1, & 0, & M, & A, & B+2M^2, & C+5MA, \end{array}$$

instead of

$$\begin{array}{cccccc} 1, & 0, & 0, & A, & B, & C, \end{array}$$

in the determinant expression for Halphen's Δ, previously given, it becomes

$$\begin{vmatrix} 0 & M & A & B+2M^2 & C+5MA \\ 1 & 0 & M & A & B+2M^2 \\ -1 & 0 & 0 & 0 & M^2 \\ 0 & 1 & 0 & 2M & 2A \\ 0 & 0 & 1 & 0 & 3M \end{vmatrix}.$$

Subtracting the 4th row multiplied by M from the first, the determinant reduces to

$$\begin{vmatrix} 0 & A & B & C+3MA \\ 1 & M & A & B+2M^2 \\ -1 & 0 & 0 & M^2 \\ 0 & 1 & 0 & 3M \end{vmatrix}.$$

Again, subtracting the 2nd column multiplied by $3M$ from the last, and reducing, the determinant becomes

$$\begin{vmatrix} 0, & B, & C \\ 1, & A, & B-M^2 \\ -1, & 0, & M^2 \end{vmatrix} = AC - B^2,$$

where M disappears, as it ought to do, because Δ is a Principiant.

In what follows we shall have frequent occasion to make use of the fact that if R_a is an absolute pure reciprocant, $\dfrac{dR_a}{a^{\frac{1}{3}}dx}$, which we know is a pure reciprocant, is also an absolute one.

This is very easily proved. For let R be any pure reciprocant, of degree i and weight w, which becomes R_a when made absolute by division by a power of a, then

$$R_a = \frac{R}{a^{\frac{\mu}{3}}}, \text{ where } \mu = 3i + w,$$

and, using G as usual to denote the generator for pure reciprocants,

$$\frac{dR_a}{dx} = \frac{GR}{a^{\frac{\mu}{3}+1}}.$$

Hence
$$\frac{dR_a}{a^{\frac{1}{3}}dx} = \frac{GR}{a^{\frac{\mu+4}{3}}},$$

which is an absolute pure reciprocant because GR, which is of degree $i+1$ and weight $w+1$, must be divided by $a^{\frac{\mu+4}{3}}$ in order to make it absolute. Thus, if M_a, A_a, B_a, C_a, ... are what M, A, B, C, ... become when each of them is made absolute by division by a power of a, we have

$$a^{-\frac{1}{3}}\frac{d}{dx}M_a = 5A_a,$$

$$a^{-\frac{1}{3}}\frac{d}{dx}A_a = 6B_a,$$

$$a^{-\frac{1}{3}}\frac{d}{dx}B_a = 7C_a + M_aA_a,$$

....

We shall use these results in deducing the complete primitive of the differential equation

$$AC - B^2 = 0$$

from that of the equation in pure reciprocants,

$$25A^2 - 16M^3 = 0.$$

This equation may be written in the form

$$25A_a^2 = 16M_a^3;$$

whence, by differentiation, we obtain

$$50A_a\left(a^{-\frac{1}{3}}\frac{d}{dx}A_a\right) = 48M_a^2\left(a^{-\frac{1}{3}}\frac{d}{dx}M_a\right),$$

which gives
$$50A_a \,.\, 6B_a = 48M_a^2 \,.\, 5A_a;$$

that is,
$$5B_a = 4M_a^2.$$

Differentiating this result, we find

$$5\,(7C_a + M_a A_a) = 40 M_a A_a;$$

which gives $$C_a = M_a A_a.$$

We now restore the non-absolute reciprocants M, A, B, C; that is, we write

$$5B = 4M^2 \text{ and } C = MA.$$

Hence $25\,(AC - B^2) = M\,(25A^2 - 16M^3) = 0$ (because $25A^2 = 16M^3$).

Now, the equation $AC - B^2 = 0$ remains unaltered by any homographic substitution, so that it will be satisfied not only by any solution of the equation in pure reciprocants $25A^2 - 16M^3 = 0$, but also by any homographic transformation of such solution. But it has been shown (in Lecture XIII., [p. 379, above]) that the complete primitive of $25A^2 - 16M^3 = 0$ is a linear transformation of $y = x^\lambda$, where $\lambda^2 - \lambda + 1 = 0$ (that is, where λ is a cube root of negative unity).

Consequently any homographic transformation of $y = x^\lambda$ is a solution of

$$AC - B^2 = 0.$$

Moreover, this is its complete primitive; for the highest letter, f, which occurs in $AC - B^2$, corresponds to the seventh order of differentiation, and if we write

$$y = \frac{Y}{Z}, \quad x = \frac{X}{Z},$$

where X, Y, Z are general linear functions of $x, y, 1$ (that is, if we make the most general homographic substitution), $y = x^\lambda$ becomes $Y = X^\lambda Z^{1-\lambda}$, which will be found to contain exactly 7 independent arbitrary constants. Thus the complete primitive of $AC - B^2 = 0$ is $Y = X^\lambda Z^{1-\lambda}$, where X, Y, Z are general linear functions of $x, y, 1$, and λ is a cube root of negative unity.

Observe that although any solution of $M = 0$ also makes $A, B, C, \ldots$ all vanish, and so satisfies $AC - B^2 = 0$, we cannot from this infer that a homographic transformation of the parabola $y = x^2$ will be the complete primitive of $AC - B^2 = 0$. For, though $YZ = X^2$ is a solution of $AC - B^2 = 0$, it only contains 5 independent arbitrary constants, and therefore cannot be its complete primitive. Neither can $YZ = X^2$ be obtained from the complete primitive by giving special values to the arbitrary constants. Hence $YZ = X^2$ is a singular solution of $AC - B^2 = 0$.

We may also deduce the differential equation of the curve $Y = X^\lambda Z^{1-\lambda}$, where λ has a general value, from the corresponding equation in pure reciprocants,

$$25\,(2\lambda^2 - 5\lambda + 2)\,A^2 + 16\,(\lambda + 1)^2\,M^3 = 0,$$

which has (see [p. 377, above]) for its complete primitive any linear transformation of the general parabola $y = x^\lambda$.

Writing for shortness

$$2\lambda^2 - 5\lambda + 2 = p \text{ and } (\lambda + 1)^2 = q,$$

and at the same time making both A and M absolute, the above equation becomes

$$25pA_a^2 + 16qM_a^3 = 0.$$

Hence, by differentiation, we obtain

$$50pA_a . 6B_a + 48qM_a^2 . 5A_a = 0,$$

which gives
$$5pB_a + 4qM_a^2 = 0.$$

After a second differentiation we find

$$5p(7C_a + M_aA_a) + 40qM_aA_a = 0;$$

that is,
$$7pC_a + (p + 8q)M_aA_a = 0.$$

We now replace the absolute reciprocants M_a, A_a, B_a, C_a by M, A, B, C, and thus write the original equation and its two differentials in the form

$$25pA^2 = -16qM^3,$$
$$5pB = -4qM^2,$$
$$7pC = -(p + 8q)MA.$$

Hence we find

$$5^2 . 7 . p^2(AC - B^2) = -25p(p + 8q)MA^2 - 16 . 7q^2M^4$$
$$= 16q(p + q)M^4,$$
$$5^6 . 7^3 . p^6(AC - B^2)^3 = 16^3q^3(p + q)^3M^{12},$$
$$5^8p^4A^8 = 16^4q^4M^{12},$$

and, eliminating M from the two last equations,

$$2^4 . 7^3 . p^2q(AC - B^2)^3 = 5^2(p + q)^3A^8.$$

Now restoring
$$p = 2\lambda^2 - 5\lambda + 2 = (\lambda - 2)(2\lambda - 1)$$
and
$$q = (\lambda + 1)^2,$$
we have
$$p + q = 3(\lambda^2 - \lambda + 1);$$
so that the final equation becomes

$$2^4 . 7^3(\lambda + 1)^2(\lambda - 2)^2(2\lambda - 1)^2(AC - B^2)^3 = 3^3 . 5^2(\lambda^2 - \lambda + 1)^3A^8.$$

The same reasoning as before will show that, for a general value of λ, the complete primitive of this equation is the general homographic transformation $Y = X^\lambda Z^{1-\lambda}$ of the curve $y = x^\lambda$.

There is, however, a special exceptional case in which the differential equation becomes

$$2^6 . 7^3(AC - B^2)^3 = 3^3 . 5^2A^8,$$

the corresponding value of the parameter λ being either 0, 1 or ∞, as may be seen by solving the equation

$$(\lambda + 1)^2(\lambda - 2)^2(2\lambda - 1)^2 = 4(\lambda^2 - \lambda + 1)^3.$$

In the case where $\lambda = 0$ or ∞ we can, in the same manner as before, show that the complete primitive is a homographic transformation of the curve $y = e^x$ by deducing the differential equation from the corresponding equation in pure reciprocants,

$$25A^2 + 8M^3 = 0,$$

whose complete primitive is (see Lecture XIII.) [p. 379 above] a linear transformation of $y = e^x$.

When $\lambda = 1$ the corresponding equation in pure reciprocants is

$$25A^2 - 64M^3 = 0,$$

whose complete primitive may be shown to be a linear transformation of $y = x \log x$. The reason why these two distinct equations in pure reciprocants lead to the same equation in principiants is that the two curves $y = e^x$ and $y = x \log x$ are *homographically* equivalent but not *linearly* transformable into one another. For we may write the equation $y = x \log x$ in the form $x = e^{\frac{y}{x}}$, which is a homographic transformation of $y = e^x$.

Besides the special case just considered, in which the complete primitive of the equation in Principiants is $\frac{Y}{Z} = e^{\frac{X}{Z}}$, we may notice that in which the parameter λ is either -1, 2, or $\frac{1}{2}$, the differential equation reducing to $A = 0$ simply, and its complete primitive $Y = X^\lambda Z^{1-\lambda}$ being the equation to a conic, as it should be. The case where $\lambda^2 - \lambda + 1 = 0$ and the differential equation reduces to $AC - B^2 = 0$ has been considered already. There remains the case in which $\lambda = 3$, when the complete primitive becomes $YZ^2 = X^3$ (the equation of the general cuspidal cubic) and the differential equation assumes the simple form

$$\left(\frac{AC - B^2}{3}\right)^3 = \left(\frac{A}{2}\right)^8,$$

which is therefore the differential equation of cuspidal cubics.

We shall hereafter show that in this case the Principiant

$$2^8 (AC - B^2)^3 - 3^3 A^8,$$

which is apparently of the 24th degree, loses a factor a^4 and so sinks to the 20th degree. It is, however, generally difficult to determine the power of a contained as a factor in a Principiant given in terms of the large letters.

The results obtained in the present Lecture agree with those of M. Halphen contained in his *Thèse sur les Invariants différentiels* (Paris, Gauthier-Villars, 1878), which contains a complete investigation of the properties of the Principiant $AC - B^2$, which he calls Δ. But our point of view is different from his. He obtains Δ in the form of a determinant from geometrical considerations. With him $\Delta=0$ is the differential equation which expresses the condition that, at a point x, y on any curve, a nodal cubic shall exist, having its node at x, y, and such that *one of its branches* shall have 8-point contact with the curve at that point. With us $AC-B^2$ is the simplest example, after the Mongian A, of an invariant in the capital letters A, B, C, ...

LECTURE XXXI.

We may include λ among the arbitrary constants in the primitive equation $Y=X^\lambda Z^{1-\lambda}$, which can also be written in the form

$$\lambda \log X - \log Y + (1-\lambda)\log Z = 0,$$

or (X, Y, Z being general linear functions of x, y, 1) in the equivalent form

$$\lambda \log (y+\alpha x+\beta) - \log (y+\alpha' x+\beta') + (1-\lambda)\log (y+\alpha'' x+\beta'') = \text{const.},$$

which evidently contains 8 independent arbitrary constants.

One of these will be made to disappear by differentiation, and thus we shall obtain a differential equation of the first order, containing 7 arbitrary constants, identical (when the constants are rearranged) with

$$(y-xt)(lx+my)+t(l'x+m'y+n')+l''x+m''y+n''=0,$$

which is known as Jacobi's Equation.

For, by differentiating the primitive equation, we obtain

$$\lambda(t+\alpha)(y+\alpha x+\beta)^{-1}-(t+\alpha')(y+\alpha' x+\beta')^{-1} \\ +(1-\lambda)(t+\alpha'')(y+\alpha'' x+\beta'')^{-1}=0,$$

which, when cleared of negative indices by multiplication, becomes

$$\lambda(y+\alpha' x+\beta')\{(y+\alpha'' x+\beta'')(t+\alpha)-(y+\alpha x+\beta)(t+\alpha'')\} \\ +(y+\alpha x+\beta)\{(y+\alpha' x+\beta')(t+\alpha'')-(y+\alpha'' x+\beta'')(t+\alpha')\}=0.$$

Writing this equation in the equivalent form

$$\lambda(y+\alpha' x+\beta')\{(\alpha-\alpha'')(y-xt)+(\beta''-\beta)t+(\alpha\beta''-\alpha''\beta)\} \\ +(y+\alpha x+\beta)\{(\alpha''-\alpha')(y-xt)+(\beta'-\beta'')t+(\alpha''\beta'-\alpha'\beta'')\}=0,$$

it is easily seen to be identical with Jacobi's equation given above.

The seven arbitrary constants which occur in Jacobi's equation are the mutual ratios of the eight coefficients l, m, l', m', n', l'', m'', n'', any one of which may have an arbitrarily chosen value assigned to it.

Taking $m=-1$, the equation may be written in the form

$$Pt+lxy-y^2+l''x+m''y+n''=0,$$

where $$P=l'x+m'y+n'-lx^2+xy.$$

In order to eliminate n'' and l'', we differentiate the above equation twice. The first differentiation gives

$$2aP + t(P' + lx - 2y + m'') + ly + l'' = 0,$$

where $P' = \frac{dP}{dx} = l' + m't - 2lx + y + xt$, and the second differentiation gives

$$6bP + 2a(2P' + lx - 2y + m'') + t(P'' + 2l - 2t) = 0.$$

Now, $P'' = \frac{dP'}{dx} = 2a(m' + x) + 2(t - l)$; so that, on substituting this value, the above equation becomes

$$3bP + aQ = 0, \tag{1}$$

where
$$Q = 2P' + lx - 2y + m'' + m't + xt$$
$$= 2l' + 3m't - 3lx + 3xt + m''.$$

Differentiating (1) we have

$$12cP + 3bP' + 3bQ + aQ' = 0,$$

where $Q' = 3(t - l) + 6a(x + m') = 3R + 6aS$, suppose.

Thus we have $\quad 4cP + bP' + bQ + aR + 2a^2S = 0. \quad (2)$

Differentiating this 4 times in succession, and at each step substituting for

$$P'', \quad Q', \quad R', \quad S',$$

their values $\quad 2R + 2aS, \quad 3R + 6aS, \quad 2a, \quad 1,$

we obtain 4 more equations, from which, combined with the 2 previously obtained, we can eliminate

$$P, \ P', \ Q, \ R, \ S.$$

Thus, differentiating (2), we find

$$20dP + 8cP' + b(2R + 2aS) + 4cQ + b(3R + 6aS)$$
$$+ 3bR + 2a^2 + 12abS + 2a^2 = 0;$$

that is, $\quad 5dP + 2cP' + cQ + 2bR + 5abS + a^2 = 0, \quad (3)$

and continuing the same process,

$$6eP + 3dP' + dQ + 3cR + (6ac + 3b^2)S + 3ab = 0, \tag{4}$$
$$7fP + 4eP' + eQ + 4dR + (7ad + 7bc)S + (4ac + 2b^2) = 0, \tag{5}$$
$$8gP + 5fP' + fQ + 5eR + (8ae + 8bd + 4c^2)S + (5ad + 5bc) = 0. \tag{6}$$

The result of elimination is

$$\begin{vmatrix} 3b & 0 & a & 0 & 0 & 0 \\ 4c & b & b & a & 2a^2 & 0 \\ 5d & 2c & c & 2b & 5ab & a^2 \\ 6e & 3d & d & 3c & 6ac + 3b^2 & 3ab \\ 7f & 4e & e & 4d & 7ad + 7bc & 4ac + 2b^2 \\ 8g & 5f & f & 5e & 8ae + 8bd + 4c^2 & 5ad + 5bc \end{vmatrix} = 0,$$

where the determinant equated to zero is a Principiant.

In his *Thèse sur les Invariants différentiels*, p. 42, M. Halphen states that this equation can be found by eliminating the constants from Jacobi's equation, but he does not set out the work. When in the above determinant twice the 3rd column is added to the second, it becomes exactly identical with the one given by Halphen, which he calls T.

We proceed to express the above result in terms of the capital letters, using the method explained in Lecture XXIX., and observing that the determinant is of degree 8 and of weight 12; so that in this case $i-w=8-12=-4$, showing that the final result has to be multiplied by a^{-4}.

Substituting in the determinant for

$$\begin{matrix} a & b & c & d & e & f & g \\ 1 & 0 & 0 & A & B & C & D+\frac{25}{8}A^2, \end{matrix}$$

it becomes

$$\begin{vmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 5A & 0 & 0 & 0 & 0 & 1 \\ 6B & 3A & A & 0 & 0 & 0 \\ 7C & 4B & B & 4A & 7A & 0 \\ 8D+25A^2 & 5C & C & 5B & 8B & 5A \end{vmatrix}.$$

Subtracting the last column multiplied by $5A$ from the first, and the 4th column multiplied by 2 from the 5th, and then striking out rows and columns, we obtain

$$\begin{vmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 6B & 3A & A & 0 & 0 & 0 \\ 7C & 4B & B & 4A & -A & 0 \\ 8D & 5C & C & 5B & -2B & 5A \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 6B & 3A & 0 & 0 & 0 \\ 7C & 4B & 4A & -A & 0 \\ 8D & 5C & 5B & -2B & 5A \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & 0 & 1 \\ 6B & 3A & 0 & 0 \\ 7C & 4B & -A & 0 \\ 8D & 5C & -2B & 5A \end{vmatrix} = \begin{vmatrix} 6B & 3A & 0 \\ 7C & 4B & A \\ 8D & 5C & 2B \end{vmatrix}$$

$$= 24(A^2D - 3ABC + 2B^3).$$

If, using Halphen's notation, we call the principiant now under consideration T, what we have proved is that

$$T = 24a^{-4}(A^2D - 3ABC + 2B^3),$$

and consequently that $A^2D - 3ABC + 2B^3$ is divisible by a^4.

The differential equation $T = 0$ corresponds, as we have seen, to the complete primitive $Y = X^\lambda Z^{1-\lambda}$, in which λ is counted as one of the arbitrary constants.

This result may be otherwise obtained. For we have shown in the preceding Lecture that the differential equation of the seventh order, from which all the arbitrary constants except λ have disappeared, has the form

$$(AC - B^2)^3 = \kappa A^8,$$

where κ depends solely on λ.

Writing this equation in the form

$$(AC - B^2)A^{-\frac{8}{3}} = \text{const.},$$

and differentiating with respect to x, we remove the remaining arbitrary constant, and thus obtain the differential equation of the 8th order free from all arbitrary constants, a result which, to a factor *près*, must coincide with

$$T = 0.$$

We proceed to show how this differentiation may be performed without introducing any of the small letters. In the first place, it is clear that since

$$G = 4(ac - b^2)\partial_b + 5(ad - bc)\partial_c + 6(ae - bd)\partial_d + \dots$$

does not contain ∂_a and is linear in the other differential reciprocals $\partial_b, \partial_c, \dots$,

$$Ga^\theta\Phi(A, B, C, \dots) = a^\theta G\Phi(A, B, C, \dots)$$
$$= a^\theta\left(\frac{d\Phi}{dA}GA + \frac{d\Phi}{dB}GB + \frac{d\Phi}{dC}GC + \dots\right).$$

And since we have

$$GA = 6B,$$
$$GB = 7C + MA,$$
$$GC = 8D + 2MB,$$
$$\dots\dots\dots\dots\dots\dots\dots$$

it follows immediately that

$$Ga^\theta\Phi(A, B, C, \dots) = a^\theta(6B\partial_A + 7C\partial_B + 8D\partial_C + \dots)\Phi$$
$$+ a^\theta M(A\partial_B + 2B\partial_C + 3C\partial_D + \dots)\Phi.$$

This is true for any function of the capital letters, whatever its nature may be; but when Φ is a principiant, it is also an invariant in the large letters; so that in this case we have

$$(A\partial_B + 2B\partial_C + 3C\partial_D + \dots)\Phi = 0$$

and

$$Ga^\theta\Phi = a^\theta(6B\partial_A + 7C\partial_B + 8D\partial_C + \dots)\Phi.$$

Now, the operation of G on a function of degree i and weight w is equivalent to that of $a\dfrac{d}{dx}-(3i+w)\,b$, or to that of $a\dfrac{d}{dx}$, when both $i=0$ and $w=0$ (which happens in the case of a plenarily absolute form). Hence, if we suppose Φ to be a plenarily absolute principiant, $G\Phi$ is also a principiant, though not a plenarily absolute one.

For a is a principiant, and $\dfrac{d\Phi}{dx}$ is a principiant; therefore $a\dfrac{d\Phi}{dx}$ or $G\Phi$ is one also*. Thus

$$6B\partial_A+7C\partial_B+8D\partial_C+\ldots,$$

acting on any plenarily absolute principiant, generates another principiant, but not a plenarily absolute one.

We now resume the consideration of the equation

$$(AC-B^2)\,A^{-\frac{8}{3}}=\text{const.}$$

Differentiating and multiplying by a, we have

$$a\frac{d}{dx}\,\{(AC-B^2)\,A^{-\frac{8}{3}}\}=0.$$

Hence, by what precedes,

$$(6B\partial_A+7C\partial_B+8D\partial_C)\,\{(AC-B^2)\,A^{-\frac{8}{3}}\}=0;$$

or, using Θ to denote the operator,

$$6B\partial_A+7C\partial_B+8D\partial_C+\ldots,$$

$$A^{-\frac{8}{3}}\,\Theta\,(AC-B^2)-\frac{8}{3}\,A^{-\frac{11}{3}}\,(AC-B^2)\,\Theta A=0;$$

or, observing that $\Theta A=6B$,

$$A\Theta\,(AC-B^2)-16B\,(AC-B^2)=0.$$

This gives $\quad A\,(6BC-14BC+8AD)-16B\,(AC-B^2)=0;$

or finally $\quad A^2D-3ABC+2B^3=0.$

We may find a generator for principiants expressed in terms of the large letters similar to the expression for the reciprocant generator G in terms of

* See the concluding paragraph of Lecture XXV. [p. 450 above], where it was shown that P, being a principiant (of degree i and weight w), $a\dfrac{dP}{dx}-(3i+w)\,bP$ is a reciprocant, and $a\dfrac{dP}{dx}-(3i+2w)\,bP$ an invariant. This proves, what we omitted to mention there, that P being a *zero-weight* principiant,

$$GP=\left(a\frac{d}{dx}-3ib\right)P \text{ is a principiant.}$$

It may here be remarked that a principiant of degree i and of *zero weight* is equal to the corresponding plenarily absolute principiant (which is a function of the large letters only) multiplied by the factor a^i, on which the operator G does not act.

the small letters. For let P be any principiant, of weight w, which, when reduced to zero weight by division by $A^{\frac{w}{3}}$, becomes $PA^{-\frac{w}{3}}$; then

$$\Theta(PA^{-\frac{w}{3}})$$

is a principiant. But

$$\Theta(PA^{-\frac{w}{3}}) = A^{-\frac{w}{3}-1}(A\Theta - 2wB)P,$$

where, remembering that $A^{-\frac{w}{3}-1}$ is a principiant, $(A\Theta - 2wB)P$ is one also.

Now, the weights of $A, B, C, D, \ldots$

being $3, 4, 5, 6, \ldots,$

we may write $w = 3A\partial_A + 4B\partial_B + 5C\partial_C + 6D\partial_D + \ldots,$

and consequently

$$\begin{aligned} A\Theta - 2wB = A\,&(6B\partial_A + 7C\partial_B + 8D\partial_C + 9E\partial_D + \ldots) \\ &- 2B(3A\partial_A + 4B\partial_B + 5C\partial_C + 6D\partial_D + \ldots) \\ &= (7AC - 8B^2)\partial_B + (8AD - 10BC)\partial_C + (9AE - 12BD)\partial_D + \ldots, \end{aligned}$$

which is the generator in question.

As an easy example of its use, suppose it to operate on $AC - B^2$; then

$$\begin{aligned} &\{(7AC - 8B^2)\partial_B + (8AD - 10BC)\partial_C\}(AC - B^2) \\ &= -2B(7AC - 8B^2) + A(8AD - 10BC) \\ &= 8(A^2D - 3ABC + 2B^3). \end{aligned}$$

The generator just obtained,

$$(7AC - 8B^2)\partial_B + (8AD - 10BC)\partial_C + (9AE - 12BD)\partial_D + \ldots,$$

is a linear combination of Cayley's two generators (given in Lecture IV., [p. 327, above]), which, when we write $A, B, C, \ldots$ instead of the corresponding small letters, become

$$(AC - B^2)\partial_B + (AD - BC)\partial_C + (AE - BD)\partial_D + \ldots$$

and $$(AC - 2B^2)\partial_B + (2AD - 4BC)\partial_C + (3AE - 6BD)\partial_D + \ldots.$$

Thus we shall obtain the principiant generator by adding the second of Cayley's generators to six times the first. Either of Cayley's generators acting on a principiant would of course give an invariant in the large letters (that is, a principiant), but the combination we have used has special relation to the theory of the generation of principiants by differentiation.

LECTURE XXXII.

I will now pass on to the consideration of the Principiant which, when equated to zero, gives the Differential Equation to the most general Algebraic Curve of any order.

The Differential Equation to a Conic (see the reference given [p. 380, above]) was obtained by Monge in the first decade of this century. This was followed by the determination, in 1868, by Mr Samuel Roberts, of the Differential Equation to the general Cubic (see Vol. X. p. 47 of Mathematical Questions and Solutions from the *Educational Times*). I do not consider that any substantial advance was made upon this by Mr Muir, in the *Philosophical Magazine* for February, 1886, except that he sets out explicitly the quantities to be eliminated in obtaining the final result. These may, of course, be collected from the processes indicated by Mr Roberts, but are not set forth by him. In speaking of the history of this part of the subject, I pass over M. Halphen's process for obtaining the Differential Equation to a Conic. It is very ingenious, like everything that proceeds from his pen, but, being founded on the solution of a quadratic equation, does not admit of being extended to forms of a higher degree, and consequently, viewed in the light of subsequent experience, must be regarded as faulty in point of method.

Let the Differential Equation to a curve of any order, when written in its simplest form, containing no extraneous factor, be $\chi = 0$. It is convenient to give χ a single name; I call it the Criterion. The integral of the Criterion to a curve of order n must contain as many arbitrary constants as there are ratios between the coefficients of a curve of the nth order. The number of these ratios being $\frac{n^2+3n+2}{2} - 1$, the order of the Criterion ought to be $\frac{n^2+3n}{2}$.

It must be independent of Perspective Projection, because projection does not affect the order of a curve. Hence it is a Principiant, and as such ought not (when y is regarded as the dependent and x as the independent variable) to contain either x, y or $\frac{dy}{dx}$ (see Lecture XXIV. [p. 438, above]).

Let $U = 0$ be an algebraical equation of the nth order between x, y. I write symbolically

$$U = (p + qx + y)^n = u^n,$$

where the different powers and products of p, q, 1 which occur in the expansion of u^n are considered as representing the different coefficients in U; so that, for example, if $n = 3$ the coefficients of

$$y^3,\ 3y^2x,\ 3y^2,\ 3yx^2,\ 6yx,\ 3y,\ x^3,\ 3x^2,\ 3x,\ 1$$

are represented by

$$1,\quad q,\quad p,\quad q^2,\quad pq,\ p^2,\ q^3,\ pq^2,\ p^2q,\ p^3.$$

The number of terms in U is

$$1 + 2 + 3 + \ldots + (n+1) = \frac{(n+1)(n+2)}{2}.$$

The number of these containing y is

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}.$$

To obtain the Differential Equation we equate to zero the Differential Derivatives of U of all orders from $n+1$ to $\frac{1}{2}(n^2+3n)$ inclusive, and from the $\frac{1}{2}(n^2+n)$ equations thus formed eliminate the $\frac{1}{2}(n^2+n)$ coefficients of the terms in U containing y.

All the coefficients of pure powers of x will obviously disappear under differentiation; for no power of x higher than x^n occurs in U, and no differential derivative of U of lower order than $n+1$ is taken.

We thus find a differential equation of the order $\frac{1}{2}(n^2+3n)$, free from all the $\frac{1}{2}(n^2+3n+2)$ coefficients of U. This equation might conceivably contain x, y and *all* the successive differential derivatives of y with respect to x. But we know *à priori* that it ought not to contain either x, y or $\frac{dy}{dx}$; and in fact we shall be able so to conduct the elimination that x, y and $\frac{dy}{dx}$ appear only in the quantities to be eliminated and not in the final result.

Treating $u = p + qx + y$ as an ordinary algebraical quantity, we have, by Taylor's theorem,

$$\frac{1}{1.2.3\ldots r}\cdot\frac{d^r u^n}{dx^r} = \text{co. } h^r \text{ in } \left(u + u_1 h + u_2\frac{h^2}{1.2} + u_3\frac{h^3}{1.2.3} + \ldots\right)^n,$$

where $u_1, u_2, u_3, \ldots$ are the successive differential derivatives of u with respect to x. And this result will remain true when for u^n we write U, meaning thereby that $\frac{1}{1.2.3\ldots r}\cdot\frac{d^r U}{dx^r}$ will be the quantitative interpretation of the function of $u, u_1, u_2, \ldots$ which multiplies h^r in the expansion of

$$\left(u + u_1 h + u_2\frac{h^2}{1.2} + \ldots\right)^n,$$

subject to the condition that this function shall be *linear* in the coefficients of U. This condition can be fulfilled in only one way, so that there is no ambiguity in such interpretation. Hence the equations obtained by equating to zero the successive differential derivatives of U of all orders from $n+1$ to $\frac{1}{2}(n^2+3n)$ inclusive may be written under the form

$$\text{co. } h^r \text{ in } \left(u + u_1 h + u_2 \frac{h^2}{1.2} + u_3 \frac{h^3}{1.2.3} + \dots\right)^n = 0,$$

where $$r = n+1,\ n+2,\ n+3,\ \dots\ \tfrac{1}{2}(n^2+3n).$$

Now, using $y_1, y_2, y_3, \dots$ to denote the successive differential derivatives of y with respect to x, we have

$$u_1 = q + y_1, \quad u_2 = y_2, \quad u_3 = y_3, \dots,$$

and, in general, $u_i = y_i$ when i is any positive integer greater than 1. Thus

$$\text{co. } h^r \text{ in } \left(u + u_1 h + y_2 \frac{h^2}{1.2} + y_3 \frac{h^3}{1.2.3} + \dots\right)^n = 0;$$

or, employing the usual modified derivatives $a, b, c, \dots$,

$$\text{co. } h^r \text{ in } (u + u_1 h + ah^2 + bh^3 + ch^4 + \dots)^n = 0.$$

Writing now $$Q = ah^2 + bh^3 + ch^4 + \dots,$$

and expanding $(u + u_1 h + Q)^n$ in ascending powers of Q, we have

$$\text{co. } h^r \text{ in } \left\{(u+u_1h)^n + n(u+u_1h)^{n-1}Q + \frac{n(n-1)}{1.2}(u+u_1h)^{n-2}Q^2 + \dots\right\} = 0,$$

where, remembering that $r > n$, the value of co. h^r in $(u+u_1h)^n$ is zero; so that, omitting this term, we may write

$$\text{co. } h^r \text{ in } \left\{n(u+u_1h)^{n-1}Q + \frac{n(n-1)}{1.2}(u+u_1h)^{n-2}Q^2 + \dots + Q^n\right\} = 0.$$

The quantities to be eliminated will now be combinations of the various powers of u, u_1 and 1. Their number will be the same as that of the terms in $(u, u_1, 1)^{n-1}$, which is $\frac{1}{2}(n^2+n)$, the same number as that of the equations between which the elimination is to be performed.

We now use $(m.\mu)$ to denote the coefficient of h^m in Q^μ (which, since

$$Q = ah^2 + bh^3 + ch^4 + \dots,$$

will be independent of the combinations of u and u_1 to be eliminated), and in writing out the $\frac{1}{2}(n^2+n)$ equations which result from making the coefficients of $h^{n+1}, h^{n+2}, \dots h^{\frac{n^2+3n}{2}}$ in

$$n(u+u_1h)^{n-1}Q + \frac{n(n-1)}{1.2}(u+u_1h)^{n-2}Q^2 + \dots + Q^n$$

vanish, we arrange their terms according to ascending values of m and μ. Thus, making the coefficient of h^{n+1} vanish, we find

$$nu_1^{n-1}(2.1)+n(n-1)\,u_1^{n-2}u\,(3.1)+\frac{n(n-1)}{1.2}\,u_1^{n-2}(3.2)+\ldots+(n+1.n)=0,$$

and similarly, making the coefficient of h^{n+2} vanish,

$$nu_1^{n-1}(3.1)+n(n-1)\,u_1^{n-2}u\,(4.1)+\frac{n(n-1)}{1.2}\,u_1^{n-2}(4.2)+\ldots+(n+2.n)=0.$$

So in general the equation obtained by making the coefficient of $h^{n+\kappa}$ vanish consists of a series of numerical multiples (which are independent of the value of κ) of $u_1^{n-\theta}u^{\theta-\eta}(\theta+\kappa,\eta)$ where η has all values from 1 to θ inclusive, and θ all values from 1 to n inclusive. Hence, by elimination, we find

$$\begin{vmatrix}
(2.1) & (3.1) & (3.2) & (4.1) & (4.2) & (4.3) & (5.1) & (5.2) & (5.3) & (5.4)\ldots \\
(3.1) & (4.1) & (4.2) & (5.1) & (5.2) & (5.3) & (6.1) & (6.2) & (6.3) & (6.4)\ldots \\
(4.1) & (5.1) & (5.2) & (6.1) & (6.2) & (6.3) & (7.1) & (7.2) & (7.3) & (7.4)\ldots \\
(5.1) & (6.1) & (6.2) & (7.1) & (7.2) & (7.3) & (8.1) & (8.2) & (8.3) & (8.4)\ldots \\
(6.1) & (7.1) & (7.2) & (8.1) & (8.2) & (8.3) & (9.1) & (9.2) & (9.3) & (9.4)\ldots \\
(7.1) & (8.1) & (8.2) & (9.1) & (9.2) & (9.3) & (10.1) & (10.2) & (10.3) & (10.4)\ldots \\
(8.1) & (9.1) & (9.2) & (10.1) & (10.2) & (10.3) & (11.1) & (11.2) & (11.3) & (11.4)\ldots \\
(9.1) & (10.1) & (10.2) & (11.1) & (11.2) & (11.3) & (12.1) & (12.2) & (12.3) & (12.4)\ldots \\
(10.1) & (11.1) & (11.2) & (12.1) & (12.2) & (12.3) & (13.1) & (13.2) & (13.3) & (13.4)\ldots \\
(11.1) & (12.1) & (12.2) & (13.1) & (13.2) & (13.3) & (14.1) & (14.2) & (14.3) & (14.4)\ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots
\end{vmatrix}=0,$$

where the determinant on the left-hand side, consisting of $\frac{1}{2}(n^2+n)$ rows and columns, is the Criterion of the curve of the nth order.

Thus in the case of the Cubic Criterion, which we shall specially consider, we have $n=3$, and the elimination of $3u_1^2$, $6u_1u$, $3u_1$, $3u^2$, $3u$ and 1 between the six equations

$$\begin{aligned}
3u_1^2(2.1)+6u_1u\,(3.1)+3u_1(3.2)+3u^2(4.1)+3u\,(4.2)+(4.3)&=0,\\
3u_1^2(3.1)+6u_1u\,(4.1)+3u_1(4.2)+3u^2(5.1)+3u\,(5.2)+(5.3)&=0,\\
3u_1^2(4.1)+6u_1u\,(5.1)+3u_1(5.2)+3u^2(6.1)+3u\,(6.2)+(6.3)&=0,\\
3u_1^2(5.1)+6u_1u\,(6.1)+3u_1(6.2)+3u^2(7.1)+3u\,(7.2)+(7.3)&=0,\\
3u_1^2(6.1)+6u_1u\,(7.1)+3u_1(7.2)+3u^2(8.1)+3u\,(8.2)+(8.3)&=0,\\
3u_1^2(7.1)+6u_1u\,(8.1)+3u_1(8.2)+3u^2(9.1)+3u\,(9.2)+(9.3)&=0,
\end{aligned}$$

gives the Cubic Criterion in the form of the determinant

$$\begin{vmatrix}
(2.1) & (3.1) & (3.2) & (4.1) & (4.2) & (4.3) \\
(3.1) & (4.1) & (4.2) & (5.1) & (5.2) & (5.3) \\
(4.1) & (5.1) & (5.2) & (6.1) & (6.2) & (6.3) \\
(5.1) & (6.1) & (6.2) & (7.1) & (7.2) & (7.3) \\
(6.1) & (7.1) & (7.2) & (8.1) & (8.2) & (8.3) \\
(7.1) & (8.1) & (8.2) & (9.1) & (9.2) & (9.3)
\end{vmatrix}.$$

Remembering that

$$(m\,.\,\mu) = \text{co. } h^m \text{ in } (ah^2 + bh^3 + ch^4 + \dots)^\mu,$$

it is easy to express the Criterion explicitly in terms of $a, b, c, \dots$.

Thus, since

$$(ah^2 + bh^3 + ch^4 + \dots)^2 = a^2h^4 + 2abh^5 + (2ac + b^2)h^6 + (2ad + 2bc)h^7 + (2ae + 2bd + c^2)h^8 + (2af + 2be + 2cd)h^9 + \dots$$

and

$$(ah^2 + bh^3 + ch^4 + \dots)^3 = a^3h^6 + 3a^2bh^7 + (3a^2c + 3ab^2)h^8 + (3a^2d + 6abc + b^3)h^9 + \dots,$$

the Cubic Criterion may be written in the form

$$\begin{vmatrix} a & b & 0 & c & a^2 & 0 \\ b & c & a^2 & d & 2ab & 0 \\ c & d & 2ab & e & 2ac + b^2 & a^3 \\ d & e & 2ac + b^2 & f & 2ad + 2bc & 3a^2b \\ e & f & 2ad + 2bc & g & 2ae + 2bd + c^2 & 3a^2c + 3ab^2 \\ f & g & 2ae + 2bd + c^2 & h & 2af + 2be + 2cd & 3a^2d + 6abc + b^3 \end{vmatrix}$$

in which it was originally obtained by Mr Roberts.

M. Halphen has remarked that the minor of h in the Cubic Criterion is the Principiant which he calls Δ (our $AC - B^2$) multiplied by a (see p. 50 of his *Thèse*).

We proceed to determine the degree and weight of the Criterion of the curve of the nth order. These are the same as the degree and weight of its diagonal

$$(2\,.\,1)(4\,.\,1)(5\,.\,2)(7\,.\,1)(8\,.\,2)(9\,.\,3)(11\,.\,1)(12\,.\,2)(13\,.\,3)(14\,.\,4)\dots,$$

which consists of $\frac{1}{2}(n^2 + n)$ factors, separable into n groups,

$$(2\,.\,1),\ (4\,.\,1)(5\,.\,2),\ (7\,.\,1)(8\,.\,2)(9\,.\,3),\ (11\,.\,1)(12\,.\,2)(13\,.\,3)(14\,.\,4), \dots$$

containing 1, 2, 3, 4, ... n factors respectively. Now,

$$\begin{aligned}(m\,.\,\mu) &= \text{co. } h^m \text{ in } (ah^2 + bh^3 + ch^4 + \dots)^\mu \\ &= \text{co. } h^{m-2\mu} \text{ in } (a + bh + ch^2 + \dots)^\mu,\end{aligned}$$

and consequently $(m\,.\,\mu)$ is of degree μ and weight $m - 2\mu$. Hence the degree of the Criterion (found by adding together the second numbers of the duads which occur in the diagonal) is

$$\begin{aligned}1 + (1 + 2) + (1 + 2 + 3) + (1 + 2 + 3 + 4) + \dots + (1 + 2 + 3 + \dots + n) \\ = 1 + 3 + 6 + 10 + \dots + \frac{n^2 + n}{2} \\ = \frac{n(n+1)(n+2)}{6}.\end{aligned}$$

To find the weight of the Criterion, we begin by arranging the factors of its diagonal according to their weight. This is done by writing each group of factors in reverse order, so that the diagonal is written thus:

$$(2.1)(5.2)(4.1)(9.3)(8.2)(7.1)(14.4)(13.3)(12.2)(11.1)\ldots.$$

The weights of the factors are now seen to be $0, 1, 2, 3, \ldots \frac{n^2+n}{2}-1$; there being $\frac{1}{2}(n^2+n)$ factors in the diagonal, one of them of zero weight. Hence the weight of the Criterion is

$$1+2+3+\ldots+\left(\frac{n^2+n}{2}-1\right)$$

$$=\frac{\left(\frac{n^2+n}{2}-1\right)\frac{n^2+n}{2}}{2}=\frac{(n-1)n(n+1)(n+2)}{8}.$$

If, in the above formulae, we make $n=2$, we shall find that the degree is 4 and the weight 3, whereas the Mongian $a^2d-3abc+2b^3$ (which is the Criterion of the second order) is of degree 3 and weight 3.

To account for this discrepancy, observe that in this case

$$\begin{vmatrix} (2.1) & (3.1) & (3.2) \\ (3.1) & (4.1) & (4.2) \\ (4.1) & (5.1) & (5.2) \end{vmatrix} = \begin{vmatrix} a & b & 0 \\ b & c & a^2 \\ c & d & 2ab \end{vmatrix},$$

which is divisible by a, the other factor being the Mongian, as may easily be verified. This is the only case in which the determinant expression for the Criterion contains an irrelevant factor.

To express the Cubic Criterion in terms of a, A, B, C, D, E, we first remark that its degree is $\frac{3.4.5}{6}=10$, and its weight $\frac{2.3.4.5}{8}=15$. Thus the Cubic Criterion is expressible as the product of $a^{-5}\,(\overset{\text{deg.}}{10}-\overset{\text{wt.}}{15}=-5)$ into a function of the capital letters, which we determine by the usual method of substituting for

$$\begin{matrix} a, & b, & c, & d, & e, & f, & g, & h, \\ 1, & 0, & 0, & A, & B, & C, & D+\frac{25}{8}A^2, & E+\frac{15}{2}AB. \end{matrix}$$

When these substitutions are made, the Cubic Criterion becomes

$$\begin{vmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & A & 0 & 0 \\ 0 & A & 0 & B & 0 & 1 \\ A & B & 0 & C & 2A & 0 \\ B & C & 2A & D+\frac{25}{8}A^2 & 2B & 0 \\ C & D+\frac{25}{8}A^2 & 2B & E+\frac{15}{2}AB & 2C & 3A \end{vmatrix}.$$

Subtracting the first column of this determinant from the fifth and reducing, we obtain

$$\begin{vmatrix} 0 & 1 & A & 0 & 0 \\ A & 0 & B & 0 & 1 \\ B & 0 & C & A & 0 \\ C & 2A & D+\frac{25}{8}A^2 & B & 0 \\ D+\frac{25}{8}A^2 & 2B & E+\frac{15}{2}AB & C & 3A \end{vmatrix}.$$

Again, subtracting the second column multiplied by A from the third and reducing, there results

$$-\begin{vmatrix} A & B & 0 & 1 \\ B & C & A & 0 \\ C & D+\frac{9}{8}A^2 & B & 0 \\ D+\frac{25}{8}A^2 & E+\frac{11}{2}AB & C & 3A \end{vmatrix},$$

which, after subtracting the first row multiplied by $3A$ from the last and reducing, becomes

$$\begin{vmatrix} B & C & A \\ C & D+\frac{9}{8}A^2 & B \\ D+\frac{1}{8}A^2 & E+\frac{5}{2}AB & C \end{vmatrix}$$

$$= B\left(CD+\frac{9}{8}A^2C-BE-\frac{5}{2}AB^2\right)+C\left(BD+\frac{1}{8}A^2B-C^2\right)$$

$$+A\left(CE+\frac{5}{2}ABC-D^2-\frac{5}{4}A^2D-\frac{9}{64}A^4\right)$$

$$=(ACE-B^2E-AD^2+2BCD-C^3)-\frac{5}{4}A\,(A^2D-3ABC+2B^3)-\frac{9}{64}A^5.$$

This expression, which is of degree-weight 15 . 15, instead of 10 . 15, must be divided by a^5 to give the correct value of the Cubic Criterion.

LECTURE XXXIII.

In this Lecture it is proposed to investigate the differential equation of a cubic curve having a given absolute invariant $\frac{S^3}{T^2}$.

Since the value of $\frac{S^3}{T^2}$ is the same for any homographic transformation of the cubic as for the original curve, the differential equation in question must be of the form

$$\text{Plenarily absolute principiant} = \frac{S^3}{T^2}.$$

This equation is (as we see at once by differentiating it) the integral of another of the form

$$\text{Principiant} = 0,$$

which is satisfied, independently of the value of the absolute invariant, at all points on a perfectly general cubic.

Now, the differential equation of the general cubic is of the 9th order, and when expressed in terms of $A, B, C, \ldots$ contains no letter beyond E. Hence the integral of this equation, which we are in search of, will be of the 8th order and will contain no capital letter beyond D.

When no letters beyond D are involved, all plenarily absolute principiants are functions of the two fundamental, or protomorphic, ones,

$$\frac{AC - B^2}{A^{\frac{8}{3}}}, \quad \frac{A^2D - 3ABC + 2B^3}{A^4}.$$

Thus the differential equation of a cubic with a given absolute invariant is of the form

$$F\left(\frac{AC - B^2}{A^{\frac{8}{3}}}, \quad \frac{A^2D - 3ABC + 2B^3}{A^4}\right) = \frac{S^3}{T^2}.$$

M. Halphen actually integrates the differential equation of the general cubic, which he shows (on p. 52 of his *Thèse sur les Invariants Différentiels*) may be put under the form

$$\xi\zeta d\xi + \left\{\zeta - \frac{3}{8}(\xi + 3)(\xi + 27)\right\} d\zeta = 0,$$

where, in our notation,

$$\xi = \frac{24(A^2D - 3ABC + 2B^3)}{A^4}, \quad \zeta = \frac{288\,(AC - B^2)^3}{A^8}.$$

The integral of this equation, which M. Halphen obtains partly from geometrical considerations, involves an arbitrary parameter depending on $\frac{S^3}{T^2}$. His result is as follows:

$$R^2 = hQ^3,$$

where

$$2^9 R = 2^9 \zeta^3 + 2^6 . 3 [(\xi - 3^2)^2 + 2^4 . 3^4] \zeta^2 + 2^3 . 3 (\xi + 3^3)^3 (\xi - 3^2 . 5) \zeta + (\xi + 3^3)^6,$$

$$2^6 Q = 2^6 \zeta^2 + 2^4 (\xi + 3^3)(\xi - 3^2 . 5) \zeta + (\xi + 3^3)^4,$$

and $$T^2 - 64hS^3 = 0.$$

(Two misprints, which are here corrected, occur in the expression for R as given on p. 54 of the *Thèse.*)

In this result the invariant S differs in sign from the invariant usually denoted by that letter. Thus the discriminant is $T^2 - 64S^3$ instead of $T^2 + 64S^3$.

When $h = 1$ the discriminant vanishes and the differential equation becomes

$$R^2 - Q^3 = 0.$$

This is divisible by a numerical multiple of ζ^3; in fact,

$$R^2 = Q^3 + 2^3 . 3^5 \zeta^3 P,$$

where $$2^6 P \equiv (2^3 \zeta + \xi^2 - 2 . 3^3 \xi - 3^5)^2 + 2^6 . 3 \xi^3 = 0$$

is the differential equation of a nodal cubic, previously obtained by Halphen.

It is from a knowledge of the fact that $P = 0$ and another algebraic relation between ξ and ζ, which he finds by trial to be $Q = 0$, constitute two particular integrals of the differential equation to the general cubic, that he arrives, not by any regular method but by repeated strokes of penetrative genius, at the general integral

$$R^2 = hQ^3.$$

In establishing the relation $T^2 - 64hS^3 = 0$ he supposes that, by means of the equation to the cubic and its differentials as far as the 8th order inclusive, the coefficients of the cubic have been expressed in terms of the variables x, y and the derivatives of y with respect to x up to the 8th order, and that the values thus obtained for the coefficients have been substituted in Aronhold's S and T.

The abbreviations introduced by the use of our notation enable us to actually perform this calculation, which would otherwise be impracticable in consequence of the enormous amount of labour required; and we shall use this method to obtain the plenarily absolute principiant which, equated to $\frac{S^3}{T^2}$, gives the differential equation to a cubic with a known absolute invariant.

Using the symbolic notation explained in Lecture XXXII. [above, p. 492], the equation of the cubic and its first eight differentials are

$$u^3 = 0,$$
$$u^2u_1 = 0,$$
$$2uu_1^2 + u^2u_2 = 0,$$
$$2u_1^3 + 6uu_1u_2 + u^2u_3 = 0,$$
$$3u_1^2(2.1) + 6u_1u(3.1) + 3u_1(3.2) + 3u^2(4.1) + 3u(4.2) + (4.3) = 0,$$
$$3u_1^2(3.1) + 6u_1u(4.1) + 3u_1(4.2) + 3u^2(5.1) + 3u(5.2) + (5.3) = 0,$$
$$3u_1^2(4.1) + 6u_1u(5.1) + 3u_1(5.2) + 3u^2(6.1) + 3u(6.2) + (6.3) = 0,$$
$$3u_1^2(5.1) + 6u_1u(6.1) + 3u_1(6.2) + 3u^2(7.1) + 3u(7.2) + (7.3) = 0,$$
$$3u_1^2(6.1) + 6u_1u(7.1) + 3u_1(7.2) + 3u^2(8.1) + 3u(8.2) + (8.3) = 0,$$

where $\quad u = p + qx + y, \quad u_1 = q + t, \quad u_2 = 2a, \quad u_3 = 6b;$

as usual, $\quad t = \frac{dy}{dx}, \quad a = \frac{1}{2}\cdot\frac{d^2y}{dx^2}, \quad b = \frac{1}{6}\cdot\frac{d^3y}{dx^3}, \ldots;$

$(m.\mu)$ denotes the coefficient of h^m in $(ah^2 + bh^3 + ch^4 + \ldots)^\mu$; and if, as in Salmon's *Higher Plane Curves* (2nd edit., p. 187), the equation of the cubic is taken to be

$$r + 3a_0x + 3a_1y + 3b_0x^2 + 6b_1xy + 3b_2y^2 + c_0x^3 + 3c_1x^2y + 3c_2xy^2 + c_3y^3 = 0,$$

then, in the above equations, the symbols

$$p^3;\ p^2q,\ p^2,\ pq^2,\ pq,\ p,\ q^3,\ q^2,\ q,\ 1$$

stand for $\quad r,\ a_0,\ a_1,\ b_0,\ b_1,\ b_2,\ c_0,\ c_1,\ c_2,\ c_3.$

These nine equations are sufficient to determine the values of the coefficients of the cubic which have to be substituted in $\frac{S^3}{T^2}$ in order to obtain our differential equation, which will be, as we have seen, of the form

$$F\left(\frac{AC - B^2}{A^{\frac{8}{3}}},\ \frac{A^2D - 3ABC + 2B^3}{A^4}\right) = \frac{S^3}{T^2}.$$

Since this equation *contains nothing which involves x, y, or t*, these letters must have disappeared spontaneously in the process of forming it, and consequently we may, at any stage of the work, *give x, y, and t any arbitrary values without thereby affecting the result.* Let, then,

$$x = 0,\ y = 0,\ t = 0, \text{ so that } u = p,\ u_1 = q,\ u_2 = 2a,\ u_3 = 6b,$$

and the first four equations become

$$u^3 = p^3 = r = 0,$$
$$u^2u_1 = p^2q = a_0 = 0,$$
$$\frac{1}{2}(2uu_1^2 + u^2u_2) = pq^2 + p^2a = b_0 + a_1a = 0,$$
$$\frac{1}{2}(2u_1^3 + 6uu_1u_2 + u^2u_3) = q^3 + 6pqa + 3p^2b = c_0 + 6b_1a + 3a_1b = 0.$$

Writing in the last five equations

$$\begin{aligned} u_1^2 = q^2 &= c_1, \\ u_1 u = pq &= b_1, \\ u_1 = q &= c_2, \\ u^2 = p^2 &= a_1, \\ u = p &= b_2, \\ 1 &= c_3, \end{aligned}$$

we have

$$\begin{aligned} &3c_1(2.1) + 6b_1(3.1) + 3c_2(3.2) + 3a_1(4.1) + 3b_2(4.2) + c_3(4.3) = 0, \\ &3c_1(3.1) + 6b_1(4.1) + 3c_2(4.2) + 3a_1(5.1) + 3b_2(5.2) + c_3(5.3) = 0, \\ &3c_1(4.1) + 6b_1(5.1) + 3c_2(5.2) + 3a_1(6.1) + 3b_2(6.2) + c_3(6.3) = 0, \\ &3c_1(5.1) + 6b_1(6.1) + 3c_2(6.2) + 3a_1(7.1) + 3b_2(7.2) + c_3(7.3) = 0, \\ &3c_1(6.1) + 6b_1(7.1) + 3c_2(7.2) + 3a_1(8.1) + 3b_2(8.2) + c_3(8.3) = 0^*. \end{aligned}$$

Substituting in $\frac{S^3}{T^2}$ for r, a_0, b_0, c_0 their values given by the equations

$$r = 0, \quad a_0 = 0, \quad b_0 + a_1 a = 0, \quad c_0 + 6b_1 a + 3a_1 b = 0,$$

and for the mutual ratios of a_1, b_1, b_2, c_1, c_2, c_3 their values found by solving the last five equations, we obtain the differential equation required.

Referring to Salmon's *Higher Plane Curves*, p. 188, we see that, when $r = 0$,

$$\begin{aligned} S &= (c^2a^2) + (cb^2a) - (b^2)^2, \\ T &= 4(c^3a^3) - 3(c^2b^2a^2) - 12(b^2)(cb^2a) + 8(b^2)^3, \end{aligned}$$

where (c^2a^2), (cb^2a), ... are functions of a_0, a_1, b_0, b_1, b_2, c_0, c_1, c_2, c_3, which, when $a_0 = 0$, become

$$\begin{aligned} (c^2a^2) &= (c_0c_2 - c_1^2)\,a_1^2, \\ (cb^2a) &= (b_0^2c_3 - 3b_0b_1c_2 + b_0b_2c_1 + 2b_1^2c_1 - b_1b_2c_0)\,a_1, \\ (b^2) &= b_0b_2 - b_1^2, \\ (c^3a^3) &= (c_0^2c_3 - 3c_0c_1c_2 + 2c_1^3)\,a_1^3, \\ (c^2b^2a^2) &= (c_0^2b_2^2 - 4c_0c_1b_1b_2 - 2c_0c_2b_0b_2 - 4c_0c_2b_1^2 + 8c_0c_3b_0b_1 \\ &\quad + 8c_1^2b_1^2 + 4c_1^2b_0b_2 - 12c_1c_2b_0b_1 - 8c_1c_3b_0^2 + 9c_2^2b_0^2)\,a_1^2. \end{aligned}$$

We have now reached a point at which the work will be greatly facilitated by the introduction of the capital letters A, B, C, D. This is usually done by writing for

$$\begin{matrix} a, & b, & c, & d, & e, & f, & g, \\ 1, & 0, & 0, & A, & B, & C, & D + \frac{25}{8}A^2. \end{matrix}$$

* These equations are only set out for the sake of distinctness; when our abbreviations are introduced, only two terms survive in the first three, and only three terms in the last two of these five equations.

But in the present instance we may make a further simplification by writing

$$A = 1, \quad B = 0, \quad C = C_1, \quad D = D_1,$$

for the only effect of this will be to make the final result take the form

$$F(C_1, D_1) = \frac{S^3}{T^2}$$

instead of $$F\left(\frac{AC - B^2}{A^{\frac{8}{3}}}, \ \frac{A^2D - 3ABC + 2B^3}{A^4}\right) = \frac{S^3}{T^2}.$$

The form of the function will not be affected by writing in it $A = 1$, $B = 0$, and the letters A, B can be restored at pleasure by making

$$C_1 = \frac{AC - B^2}{A^{\frac{8}{3}}}, \quad D_1 = \frac{A^2D - 3ABC + 2B^3}{A^4}.$$

Hence we may write for

$$a, \ b, \ c, \ d, \ e, \ f, \ g,$$

$$1, \ 0, \ 0, \ 1, \ 0, \ C_1, \ D_1 + \frac{25}{8}.$$

Instead of the coefficient of

$$h^m \text{ in } (ah^2 + bh^3 + ch^4 + \dots)^\mu,$$

$(m \,.\, \mu)$ will now signify

$$\text{co. } h^m \text{ in } \left\{h^2 + h^5 + C_1 h^7 + \left(D_1 + \frac{25}{8}\right) h^8\right\}^\mu.$$

Thus we have

$$\begin{array}{lll}
(2\,.\,1) = 1 & & \\
(3\,.\,1) = 0 & (3\,.\,2) = 0 & \\
(4\,.\,1) = 0 & (4\,.\,2) = 1 & (4\,.\,3) = 0, \\
(5\,.\,1) = 1 & (5\,.\,2) = 0 & (5\,.\,3) = 0, \\
(6\,.\,1) = 0 & (6\,.\,2) = 0 & (6\,.\,3) = 1, \\
(7\,.\,1) = C_1 & (7\,.\,2) = 2 & (7\,.\,3) = 0, \\
(8\,.\,1) = D_1 + \frac{25}{8} & (8\,.\,2) = 0 & (8\,.\,3) = 0.
\end{array}$$

Hence the equations which give $a_1, b_1, b_2, c_1, c_2, c_3$ become

$$c_1 + b_2 = 0,$$

$$c_2 + a_1 = 0,$$

$$6b_1 + c_3 = 0,$$

$$c_1 + a_1 C_1 + 2b_2 = 0,$$

$$2b_1 C_1 + 2c_2 + a_1\left(D_1 + \frac{25}{8}\right) = 0.$$

From the first four of these, coupled with the equations

$$b_0 + a_1 = 0, \quad c_0 + 6b_1 = 0,$$

obtained by making $a=1$ and $b=0$ in the original equations which give b_0, c_0, we find

$$\begin{aligned} c_0 &= \quad c_3 = -6b_1, \\ c_1 &= -b_2 = -C_1^2, \\ c_2 &= \quad b_0 = -a_1 = C_1, \end{aligned}$$

by assuming $a_1 = -C_1$ (which we are at liberty to do since any one of the coefficients may be chosen arbitrarily).

The last equation then gives

$$b_1 = \frac{D_1}{2} + \frac{9}{16}.$$

Substituting these values in the previously given expressions for (c^2a^2), $(cb^2a), \ldots$ we have

$$\begin{aligned} (c^2a^2) &= -(6b_1 + C_1^3)\, C_1^3, \\ (cb^2a) &= -(4b_1^2 - 9b_1 - C_1^3)\, C_1^3, \\ (b^2) &= C_1^3 - b_1^2, \\ (c^3a^3) &= (216b_1^3 + 18b_1C_1^3 + 2C_1^6)\, C_1^3, \\ (c^2b^2a^2) &= (312b_1^3 + 20b_1^2C_1^3 - 24b_1C_1^3 + 9C_1^3 + 4C_1^6)\, C_1^3. \end{aligned}$$

Hence
$$\begin{aligned} S &= (c^2a^2) + (cb^2a) - (b^2)^2 \\ &= -C_1^6 + 3b_1C_1^3 - 2b_1^2C_1^3 - b_1^4, \end{aligned}$$

and
$$\begin{aligned} T &= 4\,(c^3a^3) - 3\,(c^2b^2a^2) - 12\,(b^2)\,(cb^2a) + 8\,(b^2)^3 \\ &= -8C_1^9 - 3\,(8b_1^2 - 12b_1 + 9)\, C_1^6 - 12b_1^3\,(2b_1 - 3)\, C_1^3 - 8b_1^6. \end{aligned}$$

To express S and T in terms of A, B, C, D, we write

$$C_1 = \frac{AC - B^2}{A^{\frac{8}{3}}}, \quad b_1 = \frac{D_1}{2} + \frac{9}{16} = \frac{A^2D - 3ABC + 2B^3}{2A^4} + \frac{9}{16},$$

or, if we use Halphen's notation in which

$$\zeta = \frac{288\,(AC - B^2)^3}{A^8}, \quad \xi = \frac{24\,(A^2D - 3ABC + 2B^3)}{A^4},$$

we have
$$2^5 \,.\, 3^2 C_1^3 = \zeta, \quad 2^4 \,.\, 3b_1 = \xi + 3^3,$$

and consequently,

$$2^3 \,.\, 3\,(2b_1 - 3) = \xi - 3^2 \,.\, 5,$$

$$2^5 \,.\, 3^2\,(8b_1^2 - 12b_1 + 9) = (2^4 \,.\, 3b_1 - 2^2 \,.\, 3^2)^2 + 2^4 \,.\, 3^4 = (\xi - 3^2)^2 + 2^4 \,.\, 3^4.$$

Hence

$$\begin{aligned} -2^{16} \,.\, 3^4 S &= 2^{16} \,.\, 3^4 C_1^6 + 2^{16} \,.\, 3^4 b_1\,(2b_1 - 3)\, C_1^3 + 2^{16} \,.\, 3^4 b_1^4 \\ &= 2^6\zeta^2 + 2^4\,(\xi + 3^3)\,(\xi - 3^2 \,.\, 5)\,\zeta + (\xi + 3^3)^4, \\ -2^{21} \,.\, 3^6 T &= 2^{24} \,.\, 3^6 C_1^9 + 2^{21} \,.\, 3^7\,(8b_1^2 - 12b_1 + 9)\, C_1^6 \\ &\qquad + 2^{23} \,.\, 3^7 b_1^3\,(2b_1 - 3)\, C_1^3 + 2^{24} \,.\, 3^6 b_1^6 \\ &= 2^9\zeta^3 + 2^6 \,.\, 3\,[(\xi - 3^2)^2 + 2^4 \,.\, 3^4]\,\zeta^2 \\ &\qquad + 2^3 \,.\, 3\,(\xi + 3^3)^3\,(\xi - 3^2 \,.\, 5)\,\zeta + (\xi + 3^3)^6, \end{aligned}$$

where the expressions on the right-hand side are 2^6Q and 2^9R in Halphen's notation. Thus

$$-2^{10}.3^4S=Q,\quad -2^{12}.3^6T=R;$$

so that

$$\frac{Q^3}{R^2}=-\frac{2^{30}.3^{12}S^3}{2^{24}.3^{12}T^2}=-\frac{64S^3}{T^2}.$$

This result agrees exactly with Halphen's, if we remember that his S is taken with a different sign from ours.

Since

$$b_1=\frac{D_1}{2}+\frac{9}{16}=\frac{A^2D-3ABC+2B^3}{2A^4}+\frac{3^2}{2^4},$$

we may write

$$\Phi=2^4A^4b_1=2^3(A^2D-3ABC+2B^3)+3^2A^4,$$

and in like manner

$$\Psi=A^8C_1^3=(AC-B^2)^3.$$

Now

$$2^8A^8(b_1^2+C_1^3)=\Phi^2+2^8\Psi,$$

which is divisible by A^2. Hence if

$$\Phi^2+2^8\Psi=A^2\Theta,$$

we have

$$\begin{aligned}\Theta&=2^8A^6(b_1^2+C_1^3)\\&=2^6(A^2D^2-6ABCD+4AC^3+4B^3D-3B^2C^2)\\&\qquad+2^4.3^2A^2(A^2D-3ABC+2B^3)+3^4A^6.\end{aligned}$$

The equations which give S and T in terms of b_1 and C_1 may be written

$$-S=(b_1^2+C_1^3)^2-3b_1C_1^3,$$

$$-T=2^3(b_1^2+C_1^3)^3-2^2.3^2(b_1^2+C_1^3)b_1C_1^3+3^3C_1^6,$$

and consequently,

$$-2^{16}A^{12}S=\Theta^2-2^{12}.3\Phi\Psi,$$

$$-2^{21}A^{18}T=\Theta^3-2^{11}.3^2\Theta\Phi\Psi+2^{21}.3^3A^2\Psi^2,$$

where Θ, Φ, Ψ are the rational integral principiants

$$\begin{aligned}\Theta&=2^6(A^2D^2-6ABCD+4AC^3+4B^3D-3B^2C^2)\\&\qquad+2^4.3^2A^2(A^2D-3ABC+2B^3)+3^4A^6,\\\Phi&=2^3(A^2D-3ABC+2B^3)+3^2A^4,\\\Psi&=(AC-B^2)^3,\end{aligned}$$

which, as we have seen, are connected by the relation

$$\Phi^2+2^8\Psi=A^2\Theta.$$

The differential equation of cubics with a given absolute invariant is

$$\frac{(\Theta^2-2^{12}.3\Phi\Psi)^3}{(\Theta^3-2^{11}.3^2\Theta\Phi\Psi+2^{21}.3^3A^2\Psi^2)^2}=-\frac{2^6S^3}{T^2},$$

or, as it may also be written,

$$(\Theta^2-2^{12}.3\Phi\Psi)^3T^2+2^6S^3(\Theta^3-2^{11}.3^2\Theta\Phi\Psi+2^{21}.3^3A^2\Psi^2)^2=0.$$

For a nodal cubic, the discriminant $T^2 + 2^6S^3$ vanishes. Hence the differential equation of a nodal cubic is

$$(\Theta^3 - 2^{11}.3^2\Theta\Phi\Psi + 2^{21}.3^3A^2\Psi^2)^2 - (\Theta^2 - 2^{12}.3\Phi\Psi)^3 = 0.$$

When expanded, and divided by $2^{22}.3^3\Psi^2$, this reduces to

$$A^2\Theta^3 - \Theta^2\Phi^2 - 2^{11}.3^2A^2\Theta\Phi\Psi + 2^{14}\Phi^3\Psi + 2^{20}.3^3A^4\Psi^2 = 0,$$

which (since $A^2\Theta - \Phi^2 = 2^8\Psi$) divides out by $2^8\Psi$, giving

$$\Theta^2 - 2^3.3^2A^2\Theta\Phi + 2^6\Phi^3 + 2^{12}.3^3A^4\Psi = 0,$$

or, what is the same thing,

$$\Theta^2 - 2^3.3^2A^2\Theta\Phi + 2^6\Phi^3 + 2^4.3^3A^4(A^2\Theta - \Phi^2) = 0.$$

This may also be written in the form

$$(\Theta - 2^2.3^2A^2\Phi + 2^3.3^3A^6)^2 + 2^6(\Phi - 3^2A^4)^3 = 0,$$

or, replacing Θ and Φ by their values in terms of A, B, C, D,

$$\{2^6(A^2D^2 - 6ABCD + 4AC^3 + 4B^3D + 3B^2C^2)$$
$$- 2^4.3^2A^2(A^2D - 3ABC + 2B^3) - 3^3A^6\}^2 + 2^{15}(A^2D - 3ABC + 2B^3)^3 = 0.$$

For a cubic whose invariant S vanishes, the differential equation is

$$\Theta^2 - 2^{12}.3\Phi\Psi = 0,$$

and for a cubic whose invariant T vanishes,

$$\Theta^3 - 2^{11}.3^2\Theta\Phi\Psi + 2^{21}.3^3A^2\Psi^2 = 0.$$

For the cuspidal cubic, both S and T vanish, so that the algebraic equation of the cuspidal cubic is a particular solution of each of these equations. We can, however, replace the system

$$\Theta^2 - 2^{12}.3\Phi\Psi = 0, \quad (1)$$

$$\Theta^3 - 2^{11}.3^2\Theta\Phi\Psi + 2^{21}.3^3A^2\Psi^2 = 0, \quad (2)$$

by another pair of equations, for one of which the cuspidal cubic is a particular solution, and for the other the complete primitive.

Multiplying the first equation by Θ and subtracting the second from it, we have, after dividing by $2^{11}.3\Psi$,

$$\Theta\Phi - 2^{10}.3^2A^2\Psi = 0. \quad (3)$$

From (1) and (3) we obtain

$$\Theta^2\Phi^2 = 2^{12}.3\Phi^3\Psi = 2^{20}.3^4A^4\Psi^2.$$

Hence $$\Phi^3 = 2^8.3^3A^4\Psi. \quad (4)$$

But $$A^2\Theta = \Phi^2 + 2^8\Psi,$$

so that $$A^2\Theta\Phi = \Phi^3 + 2^8\Phi\Psi.$$

Substituting in this the values of $\Theta\Phi$ and Φ^3 found from (3) and (4) and dividing by Ψ, we have

$$2^{10}.3^2A^4 = 2^8.3^3A^4 + 2^8\Phi,$$

which gives $$\Phi = 3^2A^4. \quad (5)$$

Substituting this value of Φ in (4) and rejecting the factor 3^3A^4, we obtain

$$3^3A^8 = 2^8\Psi;$$

that is

$$\left(\frac{A}{2}\right)^8 = \left(\frac{AC-B^2}{3}\right)^3.$$

In the course of the work we have only rejected powers of Ψ (that is of $AC-B^2$) and of A, of which neither corresponds to the cuspidal cubic.

Since $\Phi = 3^2A^4$, it follows that $A^2D - 3ABC + 2B^3 = 0$. The equation to the cuspidal cubic above obtained is a particular solution of this, its complete primitive being (see Lecture XXXI. [above, p. 486]), $Y = X^\lambda Z^{1-\lambda}$, where λ is an arbitrary constant.

LECTURE XXXIV.

The preceding 33 lectures contain the substance of the lectures on Reciprocants actually delivered, entire or in abstract, in the course of three terms, to a class at the University of Oxford.

A good deal of material remains over which the lecturer has lacked leisure or energy to throw into form, which he hopes to be able to recover and annex to what has gone before as supplemental matter in the convenient form of lectures numbered on from those which have already appeared.

The one that follows is entirely due to Mr Hammond, who has rendered invaluable aid in compiling, and in many cases bettering, the lectures previously published.

It constitutes probably the most difficult problem in elimination which has been effected up to the present time. J. J. S.

The problem in question is to obtain the differential equation corresponding to the complete primitive

$$(l'x + m'y + n') = (lx + my + n)^\lambda (l''x + m''y + n'')^{1-\lambda}$$

(say $Y = X^\lambda Z^{1-\lambda}$) by the process of eliminating all the arbitrary constants except λ.

The eliminations to be performed become greatly simplified by aid of the following Lemma. If X be any linear function of x and y, and M_a the absolute pure reciprocant corresponding to M; then

$$X_3 - 4M_aX_1 = 0,$$

where

$$\frac{dX}{dx} = a^{\frac{1}{3}}X_1,\quad \frac{dX_1}{dx} = a^{\frac{1}{3}}X_2,\quad \frac{dX_2}{dx} = a^{\frac{1}{3}}X_3.$$

For if we suppose

$$X = lx + my + n,$$

two successive differentiations give

$$a^{\frac{1}{3}}X_1 = l + mt$$

and

$$a^{\frac{2}{3}}X_2 + a^{-\frac{2}{3}}bX_1 = 2ma.$$

Writing the second of these equations in the form

$$a^{-\frac{1}{3}}X_2 + a^{-\frac{5}{3}}bX_1 = 2m,$$

and differentiating again, we find

$$X_3 - a^{-\frac{4}{3}}bX_2 + a^{-\frac{4}{3}}bX_2 + (4ac - 5b^2)\, a^{-\frac{8}{3}}X_1 = 0,$$

or, since $4M_a = (4ac - 5b^2)\, a^{-\frac{8}{3}}$,

$$X_3 + 4M_a X_1 = 0.$$

N.B.—Throughout the following work all letters with numerical suffixes are to be considered as derived from the corresponding unsuffixed letters in the same way as, in what precedes, X_1, X_2, and X_3 are derived from X; namely by successive differentiations, each of which is accompanied by a division by $a^{\frac{1}{3}}$.

Writing the equation

$$Y = X^{\lambda} Z^{1-\lambda}$$

(in which X, Y, Z denote any three linear functions of x, y) in the form

$$\log Y = \lambda \log X + (1 - \lambda) \log Z,$$

we obtain by differentiation and division by $a^{\frac{1}{3}}$,

$$\frac{Y_1}{Y} = \lambda \frac{X_1}{X} + (1 - \lambda) \frac{Z_1}{Z}. \qquad (1)$$

Let now

$$X_1 = uX,$$
$$Y_1 = vY,$$
$$Z_1 = wZ,$$

so that (1) takes the form

$$v = \lambda u + (1 - \lambda)\, w,$$

and consequently

$$v_1 = \lambda u_1 + (1 - \lambda)\, w_1,$$
$$v_2 = \lambda u_2 + (1 - \lambda)\, w_2.$$

By means of the Lemma it can be shown that

$$u^3 + 3uu_1 + u_2 + 4M_a u = 0, \qquad (2)$$
$$v^3 + 3vv_1 + v_2 + 4M_a v = 0, \qquad (3)$$
$$w^3 + 3ww_1 + w_2 + 4M_a w = 0. \qquad (4)$$

For, since $X_1 = Xu$,

we have $X_2 = X_1 u + X u_1 = X(u^2 + u_1)$

and $X_3 = X_2 u + 2X_1 u_1 + X u_2 = X(u^3 + 3uu_1 + u_2)$.

Substituting these values for X_1 and X_3 in

$$X_3 + 4M_a X_1 = 0,$$

we obtain $u^3 + 3uu_1 + u_2 + 4M_a u = 0,$

which proves equation (2). The equations (3) and (4) connecting v, v_1, v_2 and w, w_1, w_2 are similarly established. We now write

$$\left.\begin{array}{r} u+v+w=3\omega \\ u-w=3z \end{array}\right\}.$$

These, combined with $v=\lambda u+(1-\lambda)w$,

give

$$\left.\begin{array}{l} u=\omega-(\lambda-2)z \\ v=\omega-(1-2\lambda)z \\ w=\omega-(\lambda+1)z \end{array}\right\},$$

which, when operated on by $a^{-\frac{1}{3}}\dfrac{d}{dx}$ twice in succession, yield

$$\left.\begin{array}{l} u_1=\omega_1-(\lambda-2)z_1 \\ v_1=\omega_1-(1-2\lambda)z_1 \\ w_1=\omega_1-(\lambda+1)z_1 \end{array}\right\}, \qquad \left.\begin{array}{l} u_2=\omega_2-(\lambda-2)z_2 \\ v_2=\omega_2-(1-2\lambda)z_2 \\ w_2=\omega_2-(\lambda+1)z_2 \end{array}\right\}.$$

When expressed in terms of $\omega, \omega_1, \omega_2$ and z, z_1, z_2, equations (2), (3), and (4) become transformed into

$$P-(\lambda-2)Q+(\lambda-2)^2R-(\lambda-2)^3z^3=0, \tag{5}$$

$$P-(1-2\lambda)Q+(1-2\lambda)^2R-(1-2\lambda)^3z^3=0, \tag{6}$$

$$P-(\lambda+1)Q+(\lambda+1)^2R-(\lambda+1)^3z^3=0, \tag{7}$$

where, for the sake of brevity, we write

$$\omega^3+3\omega\omega_1+\omega_2+4M_a\omega=P,$$

$$3\omega^2z+3\omega z_1+3\omega_1z+z_2+4M_az=Q,$$

$$3\omega z^2+3zz_1=R.$$

In order to simplify (5), (6), and (7), we multiply the first of them by λ, the second by -1, and the third by $1-\lambda$, and take their sum, which is obviously independent of P, and from which it is easily seen that the terms containing Q and z^3 will also disappear. For

$$\lambda(\lambda-2)-(1-2\lambda)+(1-\lambda)(\lambda+1)=0,$$

and

$$\lambda(\lambda-2)^3-(1-2\lambda)^3+(1-\lambda)(\lambda+1)^3=0.$$

We are thus left with

$$\{\lambda(\lambda-2)^2-(1-2\lambda)^2+(1-\lambda)(\lambda+1)^2\}R=0,$$

which, on restoring the value of R and reducing, becomes

$$\lambda(\lambda-1)z(\omega z+z_1)=0.$$

Now the values of u, v, w, which are equal to $\dfrac{X_1}{X}, \dfrac{Y_1}{Y}, \dfrac{Z_1}{Z}$ respectively, being distinct from each other, z cannot vanish; for $z=0$ would imply $u=v=w$. Hence, considering λ to have any finite numerical value except 1 or 0, we may write

$$\omega z+z_1=0$$

in equations (5), (6), (7), which will then become

$$P-(\lambda-2)\ (3\omega_1 z+z_2+4M_a z)-(\lambda-2)^3 z^3 = 0, \tag{8}$$

$$P-(1-2\lambda)(3\omega_1 z+z_2+4M_a z)-(1-2\lambda)^3 z^3=0, \tag{9}$$

$$P-(\lambda+1)\ (3\omega_1 z+z_2+4M_a z)-(\lambda+1)^3 z^3 = 0. \tag{10}$$

Adding these together, we find

$$3P=\{(\lambda-2)^3+(1-2\lambda)^3+(\lambda+1)^3\}\,z^3$$

$$=3(\lambda-2)(1-2\lambda)(\lambda+1)\,z^3.$$

Restoring the value of P, and writing for shortness

$$(\lambda-2)(\lambda+1)(2\lambda-1)=p,$$

there results $$\omega^3+3\omega\omega_1+\omega_2+4M_a\omega+pz^3=0.$$

From any pair of the equations (8), (9), (10) we obtain by subtraction

$$3\omega_1 z+z_2+4M_a z+3(\lambda^2-\lambda+1)\,z^3=0.$$

Thus, for example, subtracting (10) from (8), we have

$$3(3\omega_1 z+z_2+4M_a z)=\{(\lambda-2)^3-(\lambda+1)^3\}\,z^3=-9(\lambda^2-\lambda+1)\,z^3.$$

Collecting our results, we see that equations (5), (6), (7) may be replaced by

$$\omega^3+3\omega\omega_1+\omega_2+4M_a\omega+pz^3=0, \tag{11}$$

$$3\omega_1 z+z_2+4M_a z+3qz^3=0, \tag{12}$$

$$\omega z+z_1=0, \tag{13}$$

where $$p=(\lambda-2)(\lambda+1)(2\lambda-1),$$

and $$q=\lambda^2-\lambda+1.$$

Differentiating (13), we obtain

$$\omega_1 z+\omega z_1+z_2=0.$$

Subtracting this from (12) and adding (13) multiplied by ω, the result divides by z, and we find

$$\omega^2+2\omega_1+4M_a+3qz^2=0, \tag{14}$$

which, when multiplied by ω and subtracted from (11), reduces it to

$$\omega\omega_1+\omega_2+pz^3-3qz^2\omega=0. \tag{15}$$

Now it has been shown in Lecture XXX. [above, p. 482] that

$$a^{-\frac{1}{3}}\frac{d}{dx}M_a=5A_a,$$

$$a^{-\frac{1}{3}}\frac{d}{dx}A_a=6B_a,$$

$$a^{-\frac{1}{3}}\frac{d}{dx}B_a=7C_a+M_aA_a,$$

whence it follows that (14) gives on differentiation

$$\omega\omega_1+\omega_2+10A_a+3qzz_1=0.$$

Combining this with (15) we have

$$10A_a = pz^3 - 3qz(\omega z + z_1),$$

or, finally, since $\omega z + z_1 = 0$,

$$10A_a = pz^3.$$

Differentiating this, we have

$$20B_a = pz^2 z_1 = -pz^3\omega;$$

that is

$$2B_a + A_a\omega = 0, \tag{16}$$

whence, by differentiation,

$$14C_a + 2M_a A_a + 6B_a\omega + A_a\omega_1 = 0.$$

Subtracting (14) multiplied by A_a from the double of this, we have

$$28C_a - A_a\omega^2 + 12B_a\omega - 3qz^2 A_a = 0.$$

Substituting in this for ω its value $-\frac{2B_a}{A_a}$, found from (16), there results

$$28(A_a C_a - B_a^2) = 3qz^2 A_a^2.$$

But it has been shown that

$$10A_a = pz^3.$$

Hence the elimination of z gives

$$28^3 p^2 (A_a C_a - B_a^2)^3 = 3^3 q^3 p^2 z^6 A_a^6 = 10^2 3^3 q^3 A_a^8.$$

Or restoring for p and q their values in terms of λ, and replacing the absolute reciprocants A_a, B_a, C_a by the non-absolute ones A, B, C (which is effected by merely multiplying throughout by a power of a), we have

$$2^4 . 7^3 (\lambda - 2)^2 (\lambda + 1)^2 (2\lambda - 1)^2 (AC - B^2)^3 = 3^3 . 5^2 (\lambda^2 - \lambda + 1)^3 A^8. \tag{17}$$

For other methods of obtaining this differential equation see Halphen's *Thèse sur les Invariants Différentiels*, p. 30, and Lecture XXX. of the present course. It corresponds *in general* (that is unless $\lambda = 0, 1, \infty$) to the complete primitive

$$Y = X^\lambda Z^{1-\lambda}.$$

When $\lambda = 0, 1, \infty$, the differential equation (17) becomes

$$28^3 (AC - B^2)^3 = 3^3 . 5^2 A^8, \tag{18}$$

which corresponds to the complete primitive

$$Y = Xe^{\frac{Z}{X}}. \tag{19}$$

This case has been discussed in the *Thèse* and in Lecture XXX. [above, p. 480].

We may obtain (18) from (19) by a method of elimination similar to that employed in deducing (17) from its complete primitive. Thus the first differential of (19) may be written

$$\frac{Y_1}{Y} = \frac{X_1}{X} + \frac{Z_1 X - Z X_1}{X^2},$$

which becomes

$$v = u + 3z$$

when we assume $X_1 = Xu$, $Y_1 = Yv$, $Z_1 = Zu + 3Xz$.

By means of the Lemma we obtain

$$u^3 + 3uu_1 + u_2 + 4M_a u = 0, \tag{20}$$

$$v^3 + 3vv_1 + v_2 + 4M_a v = 0, \tag{21}$$

$$3u^2 z + 3u_1 z + 3uz_1 + z_2 + 4M_a z = 0. \tag{22}$$

The first two of these are identical with (2) and (3) previously given; the third is found as follows. Since

$$Z_1 = Zu + 3Xz,$$

$$Z_2 = Z_1 u + Zu_1 + 3X_1 z + 3Xz_1$$

$$= Z(u^2 + u_1) + 3X(2uz + z_1).$$

Hence

$$Z_3 = Z_1(u^2 + u_1) + Z(2uu_1 + u_2) + 3X_1(2uz + z_1) + 3X(2u_1 z + 2uz_1 + z_2)$$

$$= Z(u^3 + 3uu_1 + u_2) + 3X(3u^2 z + 3u_1 z + 3uz_1 + z_2).$$

Thus we have

$$Z_3 + 4M_a Z_1 = Z(u^3 + 3uu_1 + u_2 + 4M_a u) + 3X(3u^2 z + 3u_1 z + 3uz_1 + z_2 + 4M_a z).$$

But $Z_3 + 4M_a Z_1 = 0$, and $u^3 + 3uu_1 + u_2 + 4M_a u = 0$, which shows that

$$3u^2 z + 3u_1 z + 3uz_1 + z_2 + 4M_a z = 0.$$

Equations (20), (21), and (22), of which we have just proved the last, are merely convenient expressions of the fact that X, Y, Z are linear functions of x, y. We combine them with the first, second, and third differentials of the primitive equation (19) by writing

$$\left.\begin{aligned} v &= u + 3z \\ v_1 &= u_1 + 3z_1 \\ v_2 &= u_2 + 3z_2 \end{aligned}\right\}.$$

When this is done (21) becomes

$$(u^3 + 3uu_1 + u_2 + 4M_a u) + 3(3u^2 z + 3uz_1 + 3u_1 z + z_2 + 4M_a z)$$
$$+ 27z(uz + z^2 + z_1) = 0,$$

which, in consequence of the identities (20) and (22), reduces to

$$(u + z)z + z_1 = 0.$$

Let now $u = \omega - z$ (so that $\omega z + z_1 = 0$). Substituting in (20) and (22) we find

$$\omega^3 + 3\omega\omega_1 + \omega_2 + 4M_a\omega - 3(\omega - z)(\omega z + z_1) - z^3 - 3\omega_1 z - z_2 - 4M_a z = 0,$$

and
$$(3\omega - 6z)(\omega z + z_1) + 3z^3 + 3\omega_1 z + z_2 + 4M_a z = 0$$

respectively. Adding both equations together, and remembering that

$$\omega z + z_1 = 0,$$

we obtain
$$\omega^3 + 3\omega\omega_1 + \omega_2 + 4M_a\omega + 2z^3 = 0, \tag{23}$$

$$3\omega_1 z + z_2 + 4M_a z + 3z^3 = 0, \tag{24}$$

which, combined with
$$\omega z + z_1 = 0, \tag{25}$$

replace the system (20), (21), (22).

Comparing these equations with (11), (12), (13), we see that the two sets are identical if we make $\lambda = 0$, when p becomes 2 and $q = 1$. Hence, by performing exactly the same work as in the previous case, we shall find

$$5A_a = z^3 \qquad \text{(instead of } 10A_a = pz^3\text{)}$$

and $$28(A_aC_a - B_a^2) = 3z^2A_a^2 \text{ (instead of } 3qz^2A_a^2\text{)}.$$

And, finally, eliminating z between this pair of equations, at the same time replacing the absolute reciprocants A_a, B_a, C_a by the corresponding non-absolute ones A, B, C, we have

$$28^3(AC - B^2)^3 = 3^3 . 5^2A^8,$$

which is what (17) becomes when λ has any of the values 0, 1, or ∞.

43.

SUR LES RÉCIPROCANTS PURS IRRÉDUCTIBLES DU QUATRIÈME ORDRE.

[*Comptes Rendus*, CII. (1886), pp. 152, 153.]

DANS une Note précédente*, nous avons voulu donner le système de réciprocants irréductibles par rapport aux lettres a, b, c, d, e.

Malheureusement une erreur de calcul s'est glissée dans la détermination de la forme numérotée (5) [p. 248, above], et conséquemment la forme (6) qui, d'après notre méthode de calcul, dépend en partie de la forme (5) est aussi erronée. L'erreur est grave, car, en conséquence, un terme contenant b^3d se trouve dans cette dernière forme qui ne doit pas y paraître; cela empêcherait une combinaison ultérieure linéaire de cette forme avec le carré de la forme (4), qui donne naissance à une nouvelle forme irréductible.

Dans la forme (5) donnée, au lieu de $1585ab^2c^2$ on doit lire $1485ab^2c^2$, et, au lieu de $-18000b^4c$, on doit lire $-3600b^4c$. Ainsi corrigée, la forme, en divisant par 9, devient

$$45a^3d^2 - 450a^2bcd + 192a^2c^3 + 165ab^2c^2 + 400ab^3d - 400b^4c,$$

et, en combinant celle-ci linéairement avec le produit de (2) et (4), on obtient, en divisant par a, pour la forme (6),

$$240a^2ce - 400ab^2e - 315a^2d^2 + 1470abcd - 1008ac^3 - 35b^2c^2.$$

Sans aucun calcul arithmétique, on aurait dû prévoir que l'argument b^3d ne doit pas paraître là-dedans; car le terme qui contient $b^2\delta_d$ dans V, opérant sur b^3d, donne b^5, et évidemment aucune autre partie de V, opérant sur un terme quelconque de la forme commençant par a^2ce, ne peut donner ce même argument.

En combinant linéairement le produit de cette forme par la forme $ac - b^2$ avec le carré de (4) [p. 248, above], on obtient, en divisant par a, une nouvelle forme irréductible (7). C'est M. Hammond qui m'a averti de mon erreur de calcul et qui a calculé lui-même cette nouvelle forme dont il a vérifié l'exactitude par le moyen de l'équation différentielle partielle. On peut donc accepter avec pleine confiance pour (7) la forme

$$\begin{aligned}25a^3e^2 - 350a^2bde - 4970a^2c^2e + 17150ab^2ce + 6615a^2cd^2\\ - 9800ab^2d^2 - 31360abc^2d + 21217ac^4 - 14000b^4e\\ + 49000b^3cd - 34055b^2c^3.\end{aligned}$$

Avec ces conventions le système complet de *Grundformen*, pour le système de lettres a, b, c, d, e, sera constitué par les formes (1), (2), (3), (4), (6), (7).

[* Above, p. 242.]

44.

SUR UNE EXTENSION DU THÉORÈME RELATIF AU NOMBRE D'INVARIANTS ASYZYGÉTIQUES D'UN TYPE DONNÉ À UNE CLASSE DE FORMES ANALOGUES.

[*Comptes Rendus*, CII. (1886), pp. 1430—1435.]

[Cf. p. 459, above.]

NOUS employons toujours aujourd'hui le mot *invariant* pour désigner les sous-invariants et les invariants (ainsi ordinairement nommés) sans distinction.

Le type d'un invariant est l'ensemble de trois éléments, le poids, le degré et l'étendue, que nous désignerons ordinairement par les lettres w, i, j, et nous nous servons de cet ensemble entre parenthèses $(w:i,\ j)$ pour signifier le nombre de manières de composer w avec i des chiffres 0, 1, 2, ..., j ou bien, ce qui revient au même, avec j des chiffres 0, 1, 2, ..., i.

Il est quelquefois utile d'ajouter à ces trois éléments un autre dont il est fonction, à savoir l'*excès* qu'on prend égal à $ij - 2w$.

Quand on considère un invariant comme source d'un covariant, l'excès coïncide avec l'ordre dans les variables de ce dernier.

Le théorème connu, dont nous parlons dans le titre de cette Note, se divise en deux parties:

(1) Il n'existe aucun invariant dont l'excès du type soit négatif;

(2) Quand l'excès est positif, le nombre des invariants asyzygétiques du type $w:i, j$ est $(w:i, j) - (w-1:i, j)$ qu'on peut représenter par $\Delta(w:i, j)$.

Évidemment, ces résultats peuvent être étendus au cas des formes rationnelles et entières qui sont *anéanties* par l'opérateur

$$\lambda_1 a_0 \delta_{a_1} + \lambda_2 a_1 \delta_{a_2} + \ldots + \lambda_j a_{j-1} \delta_{a_j},$$

pourvu qu'aucun des λ ne soit nul; car alors, en remplaçant les a par des multiples numériques convenables, l'anéantisseur peut être changé dans la forme $a_0 \delta_a + 2a_1 \delta_{a_2} + \ldots + j a_{j-1} \delta_{a_j}$.

Quand tous les λ dans l'opérateur sont pris égaux à l'unité, on peut donner aux formes qu'il anéantit le nom de *binariants*.

De même, on peut considérer un anéantisseur

$$a_0\delta_{a_k} + a_1\delta_{a_{k+1}} + \ldots + a_{j-k}\delta_{a_j},$$

et donner aux formes qu'il anéantit le nom de *binariants de raison* k*; en particulier, quand $k = 2$, on peut les nommer *transbinariants*. C'est sur les transbinariants pour lesquels l'étendue j est un nombre PAIR que nous allons démontrer un théorème analogue à celui que nous avons énoncé sur les binariants ordinaires.

Si nous considérons les binariants de raison k, voici comment on pourrait procéder pour trouver toutes les formes du type $(w:i,j)$:

On prendra la forme la plus générale de ce type qui contiendra $(w:i,j)$ constantes disponibles. On opérera sur elle avec l'anéantisseur $a_0\delta_{a_k} + \ldots$, ce qui donnera une forme du type $(w-k:i,j)$ dont les $(w-k:i,j)$ coefficients seront des fonctions linéaires de ceux de la forme primitive, et l'on égalera à zéro tous ces coefficients. Ainsi l'on pourrait être porté à croire que, pourvu que le nombre des coefficients de la forme primitive excède le nombre de coefficients de la dérivée, la différence de ces deux nombres doit être le nombre de binariants de raison k asyzygétiques. Mais tout ce qu'on peut légitimement conclure dans ce cas, c'est que ce dernier nombre ne peut pas être moindre que cette différence; car les équations dont on a parlé ne sont pas nécessairement indépendantes. Cette précaution n'est nullement surérogatoire; un seul exemple suffira à le démontrer. Prenons $k = 2$ et cherchons le nombre des transbinariants du type $(6:2, 5)$.

On a

$(6:2, 5) = 3$, car 6 peut être composé avec $5+1$, $4+2$, $3+3$,

$(4:2, 5) = 3$, car 4 „ $4+0$, $3+1$, $2+2$.

Donc $(6:2, 5) - (4:2, 5) = 0$.

Cependant le nombre des transbinariants du type donné n'est pas zéro, mais 1; car, évidemment, $2bf - d^2$ est anéanti par l'opérateur

$$a\delta_c + b\delta_d + c\delta_e + d\delta_f.$$

On voit donc que c'est un théorème bien réel et nullement nugatoire, qui énonce que, pour le cas où j est un nombre pair, le nombre des transbinariants du type $(w:i, j)$ est égal exactement à $(w:i,j) - (w-2:i,j)$ quand cette différence n'est pas négative. On peut ajouter que cette différence est négative seulement dans le cas où l'*excès* du type est négatif et qu'alors (comme on va le démontrer) il n'y a pas de binariants de ce type.

Si l'on a $$\Theta = a_0\delta_{a_2} + a_1\delta_{a_3} + \ldots + a_{2\eta-2}\delta_{a_{2\eta}},$$

on peut écrire $$\Theta = \theta_1 + \theta_2,$$

* Le théorème de Brioschi montre qu'un binariant de raison k est une fonction de

$$s_1,\ s_2,\ \ldots,\ s_{k-1},\ s_{k+1},\ \ldots,\ s_j,\ s_0$$

étant la somme des puissances $\theta^{\text{ièmes}}$ des racines de l'équation $a_0x^j + a_1x^{j-1} + \ldots + a_j = 0$.

en posant $\theta_1 = a_0\delta_{a_2} + a_2\delta_{a_4} + a_4\delta_{a_6} + \ldots + a_{2\eta-2}\delta_{a_{2\eta}},$

$$\theta_2 = a_1\delta_{a_3} + a_3\delta_{a_5} + \ldots\ldots\ldots + a_{2\eta-3}\delta_{a_{2\eta-1}}.$$

En faisant $t = t_1 + t_2,$

avec $t_1 = 1 \,.\, \eta a_2\delta_{a_0} + 2(\eta-1)\,a_4\delta_{a_2} + 3(\eta-2)\,a_6\delta_{a_4} + \ldots + \eta \,.\, 1 \,.\, a_{2\eta}\delta_{a_{2\eta-2}},$

$$t_2 = 1(\eta-1)\,a_3\delta_{a_1} + 2(\eta-2)\,a_5\delta_{a_3} + \ldots + (\eta-1)\,1 \,.\, a_{2\eta-1}\delta_{a_{2\eta-3}},$$

on trouvera

$$\theta_1 t_1 - t_1\theta_1 = \eta a_0\delta_{a_0} + (\eta-2)\,a_2\delta_{a_2} + \ldots - (\eta-2)\,a_{2\eta-2}\delta_{a_{2\eta-2}} - \eta a_{2\eta}\delta_{a_{2\eta}},$$

$$\theta_2 t_2 - t_2\theta_2 = (\eta-1)\,a_1\delta_{a_1} + (\eta-3)\,a_3\delta_{a_3} + \ldots - (\eta-1)\,a_{2\eta-1}\delta_{a_{2\eta-2}}.$$

Donc, si I est une fonction homogène et isobarique dans les lettres a du type $w\,;\,i, j$, on aura

$$(\Theta T - T\Theta)\,I = [\eta a_0\delta_{a_0} + (\eta-1)\,a_1\delta_{a_1} + \ldots - (\eta-1)\,a_{2\eta-1}\delta_{a_{2\eta-1}} - \eta a_{2\eta}\delta_{a_{2\eta}}]\,I$$
$$= (i\eta - j)\,I = \frac{2\eta i - 2w}{2}\,I\,;$$

car on remarquera que ni l'un ni l'autre θ n'agit sur l'un ou l'autre t, et que ni l'un ni l'autre t n'agit sur l'un ou l'autre θ.

Le coefficient de I, on le remarquera, est la moitié de l'*excès* au type $w : i,\ 2\eta$.

Il est bon d'observer qu'il n'est pas possible d'obtenir un résultat semblable dans le cas où j est impair, c'est-à-dire qu'on ne peut pas trouver, dans ce cas, une forme T telle que le résultat de l'opération $(\Theta T - T\Theta)$ sur une forme homogène et isobarique soit équivalent au produit de cette forme par une fonction quelconque de $w\,;\,i, j$.

Avec l'aide de la formule ci-dessus, suivant la même marche que nous avons prise pour les invariants dans le *Philosophical Magazine* * (mars 1878), on parvient à des résultats tout à fait semblables.

En appelant ϵ la moitié de l'excès et en supposant que I est un transbinariant, on trouve

$$\epsilon I = \Theta T I$$

et, plus généralement, $\mu T^{q-1} I = \Theta T^q I,$

où $\mu = q(\epsilon - q + 1)$.

Or il est évident que, puisque l'effet de T est d'augmenter (par deux unités) le poids de la forme sur laquelle il agit sans en changer le degré ni l'étendue, et que le poids d'une forme homogène et isobarique ne peut pas excéder le produit du degré par l'étendue, en prenant q suffisamment grand, on aura

$$TI = 0,$$

et, à plus forte raison, $\Theta T I = 0.$

On trouvera donc successivement $T^{q-1}I = 0$, $T^{q-n}I = 0, \ldots, TI = 0, I = 0$, pourvu que le μ ne devienne pas nul dans le cours de cette déduction: ceci

[* Vol. III. of this Reprint, p. 117.]

ne peut pas arriver quand ϵ est négatif, car on trouvera que les valeurs de μ, dans ce cas, resteront toujours négatives.

Cela démontre qu'un transbinariant, dont le type a un excès négatif, ne peut pas être autre que zéro, c'est-à-dire n'a pas d'existence actuelle quand l'excès est non négatif; en désignant par $E(w:i,j)$ le nombre

$$(w:i,j)-(w-2:i,j),$$

et par $D(w:i,j)$ le nombre de transbinariants du type $(w:i,j)$, on prouve que $D(w:i,j)=E(w:i,j)$ de la manière suivante.

En remarquant que, pour w négatif, $E(w:i,j)=0$, on trouve immédiatement

$$\sum_{q=\infty}^{q=0} E(w-2q:i,j)=(w:i,j),$$

et, puisque chaque D est au moins égal au E correspondant, on a

$$\sum_{q=\infty}^{q=0} D(w-2q:i,j) \geqq (w:i,j).$$

Or on peut démontrer facilement que, si $ij-2w$ est non négatif, en appelant $I_{w:i,j}$ un transbinariant du type $(w:i,j)$, $\Theta^q T^q I_{w-2q:i,j}$ sera égal à un multiple numérique de $I_{w-2q:i,j}$ différent de zéro pour toutes les valeurs de q qu'on a besoin de considérer.

Or, dans l'ensemble des transbinariants asyzygétiques, dont le type est $w-2q:i,j$, on peut substituer à chacun, pour ainsi dire, son *image* $T^q I_{w-2q:i,j}$. Le nombre de ces images sera

$$\sum_{q=\infty}^{q=0} D(w-2q:i,j).$$

De plus, chaque image sera du même type $(w:i,j)$.

On démontre facilement qu'il ne peut pas exister entre ces images une relation linéaire; car, dans le cas contraire, en opérant sur l'équation qui les lie ensemble avec une puissance convenable de Θ, on tomberait sur une équation linéaire entre les transbinariants asyzygétiques eux-mêmes. Donc, évidemment, le nombre des images ne peut pas excéder la valeur de $(w:i,j)$. Donc

$$\sum_{q=\infty}^{q=0} D(w-2q;i,j)$$

n'est ni plus grand ni plus petit que $\sum_{q=\infty}^{q=0} E(w-2q;i,j)$; il lui est donc égal, et conséquemment, puisque aucun D ne peut être moins que le E qui lui correspond pour chaque valeur de q,

$$D(w-2q;i,j)=E(w-2q;i,j);$$

car si un D quelconque était plus grand que le E qui lui correspond, un autre D serait nécessairement plus petit, ce qui est inadmissible.

On aura donc $$D(w:i,j) = E(w:i,j),$$
pourvu que $ij - 2w$ ne soit pas négatif. C.Q.F.D.

On démontre facilement les mêmes théorèmes pour des formes anéantissables par une *somme* d'opérateurs

$$a_0\delta_{a_2} + \ldots + a_{j-2}\delta_{a_j},$$
$$a_0'\delta_{a_2'} + \ldots + a'_{j'-2}\delta_{a'_{j'}},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$

En supposant que chaque j soit pair et en regardant $w:i,j:i',j',\ldots$ comme leur type, on parvient à cette conclusion qu'aucun transbinariant d'un tel type n'existe dans le cas où $ij + i'j' + \ldots - 2w$ est négatif et que, quand cette quantité n'est pas négative, le nombre des transbinariants asyzygétiques est égal à $(w:i,j:i',j':\ldots) - (w-2:i,j:i',j';\ldots)$, où $(w:i,j:i',j':\ldots)$ désigne le nombre de manières de composer w avec i des chiffres $0, 1, 2, \ldots$, combinés avec i' des chiffres $0, 1, 2, \ldots, j'$, etc.

Il est utile de remarquer que les formes et les syzygies fondamentales des intégrales de l'équation

$$(a_0\delta_{a_2} + a_1\delta_{a_3} + \ldots + a_{\eta-2}\delta_{a_{2\eta}})I = 0$$

sont des mêmes types que les invariants et les syzygies fondamentales d'un système formé avec deux *quantics* d'ordres η et $\eta - 1$ respectivement; ce qui donne un moyen facile de vérifier la formule que nous avons démontrée pour le nombre de transbinariants asyzygétiques d'un type donné. Il va sans dire que nous n'avons pas négligé de nous servir de cette méthode pour vérifier la justesse de nos conclusions.

45.

NOTE SUR LES INVARIANTS DIFFÉRENTIELS.

[*Comptes Rendus*, CII. (1886), pp. 31—34.]

EN affirmant, dans notre Lettre à M. Hermite (dont un Extrait a paru dans les *Comptes rendus*), que les invariants différentiels de M. Halphen sont identiques avec nos réciprocants purs, nous sommes allé trop loin; nous aurions dû dire qu'ils sont identiques avec la classe spéciale de ces derniers que nous avons nommés *réciprocants projectifs*; en effet, en prenant pour *éléments*

$$\frac{1}{1.2}\frac{d^2y}{dx^2}, \quad \frac{1}{1.2.3}\frac{d^3y}{dx^3}, \quad \frac{1}{1.2.3.4}\frac{d^4y}{dx^4}, \quad \ldots,$$

regardés comme quantités algébriques, lesquelles on peut nommer (selon l'usage quand on parle de formes binaires) $a, b, c, d, \ldots$, un invariant différentiel possède la propriété vraiment étonnante d'être en même temps un réciprocant et un sous-invariant ordinaire.

En accommodant la valeur de V à cette notation nouvelle, il devient

$$4aa\delta_b + 5(ab + ba)\delta_c + 6(ac + bb + ca)\delta_d, \quad \ldots;$$

et, en posant
$$a\delta_b + 2b\delta_c + 3c\delta_d + \ldots = \Omega,$$

un invariant différentiel I satisfait en même temps aux deux équations partielles différentielles

$$V.I = 0, \quad \Omega.I = 0.$$

Voici comment on peut établir le fait que $\Omega.I = 0$.

En commençant avec les trois premiers invariants différentiels, c'est-à-dire a, $a^2d - 3abc + 2b^3$, et le Δ de M. Halphen (dans sa thèse immortelle), on sait que les deux premiers, et l'on vérifie sans trop de peine que le troisième sont tous les trois des sous-invariants.

De plus, on sait que, en commençant avec ces trois invariants que nous nommerons I_0, I_1, I_2, on peut former une suite indéfinie de formes protomorphiques

$$I_0,\quad I_1,\quad I_2,\quad I_3,\quad \ldots,\quad I_p,\quad \ldots,$$

dont tous les autres seront des fonctions rationnelles.

Pour obtenir cette suite, on n'a qu'à former une fonction J de $I_0, I_1, \ldots, I_p, \ldots$, dont le degré et le poids soient tous deux zéro ; en opérant alors sur J (considéré comme fonction des dérivées de y par rapport à x) avec δ_x, on obtient I_{p+1}.

Si donc on peut démontrer que $\Omega\delta_x J = \delta_x \Omega J$, il s'ensuivra que I_{p+1} sera un sous-invariant, pourvu que I_p en soit un, et le théorème en question sera démontré.

Or remarquons en premier lieu que, à cause de la valeur zéro du degré et du poids de J, la quantité

$$(\lambda a\delta_a + \mu b\delta_b + \nu c\delta_c + \ldots)\, J$$

sera nulle si λ, μ, ν, ... forment une progression arithmétique quelconque ; et, en second lieu, que (par rapport à une fonction de dérivées de J par rapport à x), $\delta_x = 3b\delta_a + 4c\delta_b + 5d\delta_c + \ldots$ identiquement.

Conséquemment

$$(\Omega\delta_x - \delta_x\Omega)\, J = [(3a\delta_a + 8b\delta_b + 15c\delta_c + \ldots) - (3b\delta_b + 8c\delta_c + \ldots)]\, J$$
$$= (3a\delta_a + 5b\delta_b + 7c\delta_c + \ldots)\, J = 0,$$

ce qu'il fallait démontrer.

M. Halphen, à qui j'avais communiqué ce résultat, en a trouvé une tout autre démonstration qu'il m'autorise à communiquer à l'Académie. Elle possède sur la mienne l'avantage d'aller plus au fond de la question, en faisant voir que l'équation $\Omega \,.\, I = 0$ équivaut à dire que, en se servant de x, y, z au lieu de $x, y, 1$, un invariant différentiel peut subir le changement entre eux de x et z. Or, puisque $V \,.\, I = 0$ signifie qu'on peut imposer des substitutions linéaires quelconques sur x et y, il s'ensuit, en combinant les deux équations, que la même chose aura lieu quand x, y, z subissent tous les trois des substitutions linéaires quelconques. Voici la démonstration très élégante de M. Halphen :

" Si l'on fait le changement de variables

$$X = \frac{1}{x},\qquad Y = \frac{y}{x},$$

et qu'on écrive

$$\frac{dy}{dx} = y',\quad \frac{d^2y}{dx^2} = y'',\quad \ldots,\quad \frac{d^ny}{dx^n} = y^{(n)},\quad \ldots,$$

on a $Y = + x^{-1} y,$

$$\frac{dY}{dX} = - x^{+1}\left(y' - \frac{1}{x} y\right),$$

$$\frac{d^2Y}{dX^2} = + x^3 y'',$$

$$\frac{d^3Y}{dX^3} = - x^5\left(y''' + \frac{3}{x} y''\right),$$

$$\frac{d^4Y}{dX^4} = + x^7\left(y^{\text{IV}} + \frac{8}{x} y''' + \frac{12}{x^2} y''\right),$$

$$\frac{d^5Y}{dX^5} = - x^9\left(y^{\text{V}} + \frac{15}{x} y^{\text{IV}} + \frac{60}{x^2} y''' + \frac{60}{x^3} y''\right),$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

$$\frac{d^nY}{dX^n} = (-1)^n x^{2n-1}\left[y^{(n)} + \frac{n(n-2)}{x} y^{(n-1)} + \frac{\alpha}{x^2} y^{(n-2)} + \frac{\beta}{x^3} y^{(n-3)} + \ldots\right].$$

"Posant $\qquad \dfrac{d^nY}{dX^n} = n' A_n, \quad \dfrac{d^ny}{dx^n} = n' a_n, \quad \dfrac{1}{x} = \epsilon,$

on a $\qquad A_n = (-1)^n x^{2n-1}[a_n + (n-2)\epsilon a_{n-1} + \alpha' \epsilon^2 a_{n-2} + \ldots].$

"Soit une fonction $f(A_0, A_1, \ldots, A_n)$ dont tous les termes soient de poids et de degré constants p, δ; en supposant ϵ infiniment petit, on aura

$$f(A_0, A_1, \ldots, A_n) = (-1)^p x^{2p-\delta}\left\{f(a_0, a_1, \ldots, a_n) + \epsilon\left[-a_0\frac{\partial f}{\partial a_1} + 2a_2\frac{\partial f}{\partial a_3} + 2a_3\frac{\partial f}{\partial a_4} + \ldots + (n-2)\, a_{n-1}\frac{\partial f}{\partial a_n}\right]\right\}.$$

"Donc, pour que f soit invariant pour la substitution considérée, il faut qu'on ait

$$a_2\frac{\partial f}{\partial a_3} + 2a_3\frac{\partial f}{\partial a_4} + 3a_4\frac{\partial f}{\partial a_5} + \ldots + (n-2)\, a_{n-1}\frac{\partial f}{\partial a_n} = a_0\frac{\partial f}{\partial a_1}.$$

"En particulier, si f ne contient pas a_1, ce qui est le cas des *réciprocants purs*, on aura

$$a_2\frac{\partial f}{\partial a_3} + 2a_3\frac{\partial f}{\partial a_4} + \ldots + (n-2)\, a_{n-1}\frac{\partial f}{\partial a_n} = 0. \qquad \text{C.Q.F.D.''}$$

Ainsi, l'on voit qu'un invariant différentiel est en même temps réciprocant et sous-invariant; ce n'est nullement un mélange ou une combinaison de deux choses différentes, mais plutôt, pour ainsi dire, une personnalité seule et indivisible douée de deux natures tout à fait distinctes.

Afin de compléter la théorie, il faut démontrer la réciproque, c'est-à-dire que toute forme douée de ces deux natures est un réciprocant projectif. M. Halphen effectue cela en trouvant le développement complet de sa série et en faisant voir que, quand le coefficient de la première puissance de ϵ disparaît, la même chose aura lieu pour tous les coefficients suivants. Voici notre méthode, à nous de l'effectuer.

Soit H une forme rationnelle et entière dont le terme principal (c'est-à-dire celui qui contient la plus haute puissance du terme le plus avancé) est Gh^i. On suppose que le théorème à démontrer est vrai jusqu'à la lettre g incluse, et que $VA = 0$, $\Omega H = 0$ sans que H soit projectif.

Alors évidemment $VG = 0$, $\Omega G = 0$ et G, par hypothèse, sera projectif. Soit H' une puissance d'un protomorphe pour laquelle le terme principal est $G'h^i$, alors, si $H_1 = G'H - GH'$, G, G', H' sont projectifs, mais H non projectif; donc, H_1 (qui, comme H, est anéanti par V et par Ω) sera non projectif: de plus, dans H_1 le degré du terme principal en h est abaissé. De la même manière on peut construire H_2, H_3, ... jusqu'à ce qu'on parvienne à une forme* qui ne contient pas h, laquelle possédera les mêmes caractères que H, ce qui est impossible par hypothèse. Donc, si le théorème à démontrer est vrai pour un nombre quelconque donné de lettres, il sera vrai universellement: mais il est évidemment vrai pour la fonction a qui est le seul réciprocant à une lettre. Donc, si $VI = 0$ et $RI = 0$, I est un réciprocant projectif, c'est-à-dire un invariant différentiel. Ce qui était à démontrer.

* Cette forme sera, en effet, le résultant de H et de la première puissance du protomorphe. Nous avons jugé inutile de dire dans le texte que G', comme G, sera anéanti par V et par Ω et conséquemment, par hypothèse, sera lui aussi projectif.

46.

SUR L'ÉQUATION DIFFÉRENTIELLE D'UNE COURBE D'ORDRE QUELCONQUE.

[*Comptes Rendus,* CIII. (1886), pp. 408—411.]

[Also, above, p. 492.]

On peut obtenir une solution directe et universelle de ce problème: *Trouver l'équation différentielle d'une courbe de l'ordre n,* en représentant la fonction de l'équation (avec l'unité pour terme constant), soit U ou $(x, y, 1)^n$, sous la forme symbolique u^n, où $u = a + bx + y$. Alors, en mettant $\left(\frac{d}{dx}\right)^n y = y_r$, on aura
$$\frac{du}{dx} = b + y_1, \quad \frac{d^{i+1}u}{dx^{i+1}} = y_{i+1} \text{*}.$$

Égalons à zéro les dérivées de u^n des degrés $n+1, n+2, \ldots, \frac{(n+1)(n+2)}{2}$; il en résultera $\frac{n^2+n}{2}$ équations entre lesquelles on peut éliminer le même nombre de coefficients, c'est-à-dire tous les coefficients en U, sauf ceux qui ne contiennent nulle puissance de y, lesquels ne paraîtraient pas dans les équations dont nous parlons.

Pour obtenir le déterminant qui correspond à ce système d'équations, remarquons que le théorème de Taylor donne immédiatement†
$$\frac{1}{\Pi r}\partial_x{}^r u^n = \text{co}_r\left(u + u'h + u''\frac{h^2}{1\,.\,2} + u'''\frac{h^3}{1\,.\,2\,.\,3} + \ldots\right)^n$$
$$= \text{co}_r\left((u+u'h)^n + n\,.\,(u+u'h)^{n-1}V + n\,.\,\frac{n-1}{2}(u+u'h)^{n-2}V^2 + \ldots\right),$$
où l'on peut prendre
$$V = y_2\frac{h^2}{1\,.\,2} + y_3\frac{h^3}{1\,.\,2\,.\,3} + y_4\frac{h^4}{1\,.\,2\,.\,3\,.\,4} + \ldots,$$
ce qui suffit à résoudre le problème.

* On remarquera qu'avec cette notation toute fonction entière de u et $\partial_x u$ représentera sans ambiguïté une quantité algébrique ordinaire, pourvu que l'on sache *a priori* qu'elle doit être linéaire dans les coefficients de u^n. C'est pourquoi dans le texte on est libre d'exprimer toute dérivée différentielle de U comme fonction de u et u'.

† Par co_r on sous-entend les mots "le coefficient de h^r dans."

Pour cela, on considère toutes les dérivées de U comme fonctions linéaires des termes qui paraissent dans le développement de $(u, u', 1)^{n-1}$ *.

Alors, en représentant par $m.\mu$ le coefficient de h^m dans

$$\left(\frac{y^2}{1.2}h^2+\frac{y^3}{1.2.3}h^3+\ldots\right)^\mu,$$

on trouvera, sans calcul algébrique aucun, que la $q^{\text{ième}}$ ligne du déterminant cherché peut être prise sous la forme

$$(1+q).1\ (2+q).1\ (2+q).2\ (3+q).1\ (3+q).2\ (3+q).3\ \ldots$$
$$(n+q).1\ (n+q).2\ \ldots\ (n+q).n.$$

Par exemple, prenons le cas de $n=4$; le déterminant

2.1	3.1	3.2	4.1	4.2	4.3	5.1	5.2	5.3	5.4
3.1	4.1	4.2	5.1	5.2	5.3	6.1	6.2	6.3	6.4
4.1	5.1	5.2	6.1	6.2	6.3	7.1	7.2	7.3	7.4
5.1	6.1	6.2	7.1	7.2	7.3	8.1	8.2	8.3	8.4
6.1	7.1	7.2	8.1	8.2	8.3	9.1	9.2	9.3	9.4
7.1	8.1	8.2	9.1	9.2	9.3	10.1	10.2	10.3	10.4
8.1	9.1	9.2	10.1	10.2	10.3	11.1	11.2	11.3	11.4
9.1	10.1	10.2	11.1	11.2	11.3	12.1	12.2	12.3	12.4
10.1	11.1	11.2	12.1	12.2	12.3	13.1	13.2	13.3	13.4
11.1	12.1	12.2	13.1	13.2	13.3	14.1	14.2	14.3	14.4

sera le premier membre de l'équation différentielle (disons le critérium différentiel) d'une courbe du quatrième degré.

Si l'on se borne aux termes contenus dans les six premières lignes et colonnes, on aura le critérium pour la cubique, et, en se bornant aux termes contenus dans les trois premières lignes et colonnes, celui pour la conique, ou plutôt ce critérium multiplié par 2.1, ce qui constitue un cas exceptionnel.

2.1 lui-même, c'est-à-dire $\frac{\partial_x^2 y}{2}$, est naturellement le critérium pour la ligne droite. On remarquera que 3.2, 4.3, 5.3, 5.4, 6.4, 7.4 sont des combinaisons pour ainsi dire fictives, qui ont pour valeur zéro†. De même, en général, il y aura toujours des termes nuls dans les $(n-1)$ premières lignes du critérium de la courbe de degré n; au-dessous de la $(n-1)^{\text{ième}}$ ligne, toutes les places seront remplies par des combinaisons qui correspondent à des non-zéros.

Quand $n=3$, en substituant pour $\frac{y''}{1.2}, \frac{y'''}{1.2.3}, \frac{y^{\text{IV}}}{1.2.3.4}, \ldots$ les lettres $a, b, c, \ldots$, on retombe sur la formule trouvée pour la cubique par M. Samuel

* Ou plutôt les termes avec leurs coefficients numériques de $(u, u', 1)^n$, en omettant les $(n+1)$ termes du degré n.

† Évidemment $m.\mu$ est zéro quand $m<2\mu$.

Roberts (voir *Mathematical Questions from the Educational Times*, t. x. p. 47)*, c'est-à-dire la même matrice que celle donnée par M. Roberts, mais avec ses colonnes autrement présentées.

On voit immédiatement que le degré du critérium pour une courbe du $n^{\text{ième}}$ ordre sera $\frac{n(n+1)(n+2)}{6}$ et, par un calcul facile, que son poids sera $\frac{(n-1)n(n+1)(n+2)}{8}+\frac{n(n+1)(n+2)}{3}$†. Ce dernier nombre suppose que le poids de $d_x^i y$ est compté comme i. Dans le calcul des réciprocants, on le compte toujours comme étant $i-2$ et, en faisant cette réduction, le poids devient tout simplement $\frac{(n-1)n(n+1)(n+2)}{8}$.

M. Halphen nous a appris que les formules qu'il a données dans son Mémoire intitulé: *Recherches des points d'une courbe algébrique plane*, etc. (*Journal de Mathématiques*, 3e série, t. II. pp. 373, 374 et 400; 1876) fournissent un moyen pour calculer le degré et le poids du critérium $n^{\text{ième}}$ et conduisent aux mêmes résultats que ceux donnés ci-dessus. Dans le cas de la conique, le déterminant, comme nous l'avons dit, se divise par y'', de sorte que son poids-degré s'abaisse et, au lieu d'être 3 . 4, devient 3 . 3; en effet, c'est la forme bien connue $a^2d-3abc+2b^3$, trouvée par Monge.

* Ce travail a été cité et reproduit dans le *Philosophical Magazine* de février 1886, par M. Muir, qui y construit pour ainsi dire le tableau du calcul dont M. Roberts avait déjà fait le procès-verbal.

† Car le degré sera la somme de n termes de la série $1+3+6+\ldots$, c'est-à-dire $\frac{n(n+1)(n+2)}{6}$, et le poids, moins deux fois le degré, la somme de n termes de la série

$$0+(2+1)+(5+4+3)+(9+8+7+6)+\ldots$$

ou bien de $\frac{n^2+n-2}{2}$ termes de la progression naturelle $1+2+3+4+5+\ldots$, c'est-à-dire

$$\frac{n^2+n-2}{2}\,\frac{n^2+n}{4}.$$

47.

SUR UNE EXTENSION D'UN THÉORÈME DE CLEBSCH RELATIF AUX COURBES DU QUATRIÈME DEGRÉ.

[*Comptes Rendus*, CII. (1886), pp. 1532—1534.]

En appliquant un terme quelconque du développement de

$$(\delta_x, \delta_y, \delta_z, \ldots)^\eta$$

au quantic $(x, y, z, \ldots)^{2\eta}$, on obtient autant de fonctions de degré η qu'il y a de termes dans chaque fonction. L'ensemble de leurs coefficients peut donc être regardé comme la matrice d'un déterminant auquel nous donnerons le même nom de *catalecticant*, dont on fait usage dans le cas des formes binaires.

On voit très aisément que la matrice catalectique, pour une puissance d'une fonction linéaire de variables, possède cette propriété que chaque déterminant mineur du second ordre qu'elle contient s'évanouit. Conséquemment, deux colonnes quelconques d'une telle matrice, associées à d'autres colonnes arbitraires, en nombre suffisant pour former une matrice carrée nouvelle, feront s'évanouir le déterminant de cette dernière.

Or la matrice catalectique d'une somme de puissances de fonctions linéaires des mêmes variables est la somme des matrices qui appartiennent à chacune prise séparément; donc, comme conséquence immédiate de cette propriété dont nous avons parlé, si le nombre de ces matrices est moindre que l'ordre de chacune, le déterminant de leur somme s'évanouira, car il pourra être résolu dans une somme de déterminants dont chacun aura la valeur zéro *.

* S'il y a n matrices, chacune de l'ordre N (de sorte que N est le nombre des colonnes dans chaque matrice), on associera à volonté la première colonne d'une quelconque des n matrices avec la seconde, avec la troisième, etc. colonne, prises ou dans la même ou dans aucune autre matrice, en sorte que le nombre des nouvelles matrices partielles sera n^N. Il est évident que, N étant par hypothèse plus grand que n, deux colonnes *au moins* de chaque matrice ainsi formée appartiendront à une même matrice fondamentale, c'est-à-dire à la matrice catalectique d'une puissance d'une fonction linéaire des variables. Voilà la raison pour laquelle chacun des n^N déterminants partiels est égal à zéro.

(1) Prenons deux variables. Le catalecticant sera de l'ordre $\eta+1$; on retrouve ainsi cette règle bien connue, et qui ne contient rien d'exceptionnel ni de paradoxal : pour qu'une forme binaire d'ordre 2η soit équivalente à la somme de η puissances de fonctions linéaires, il faut que le catalecticant de la forme soit nul.

(2) Prenons trois variables et faisons $\eta=2$: l'ordre du déterminant catalectique de $(ax+by+cz)^4$ étant 6, le catalecticant de

$$\sum_{\theta=5}^{\theta=1}(a_\theta x+b_\theta y+c_\theta z)^4=0.$$

Cela donne le théorème de Clebsch, à savoir que le premier membre de l'équation d'une courbe du quatrième degré n'est pas, en général, exprimable en une somme de cinq puissances de fonctions linéaires des variables.

(3) Prenons cinq variables, en faisant encore $\eta=2$. L'ordre du déterminant catalectique $(ax+by+cz+dt+eu)^4$ étant 15, le catalecticant de

$$\sum_{\theta=14}^{\theta=1}(a_\theta x+b_\theta y+c_\theta z+d_\theta t+e_\theta u)^4$$

s'évanouit.

Or $5\times 14=70$, ce qui est justement le nombre $\dfrac{5.6.7.8}{1.2.3.4}$ des coefficients de $(x, y, z, t, u)^4$.

On arrive ainsi à cette conclusion nouvelle, et un peu paradoxale, que l'équation d'une hypersurface du quatrième degré, bien que contenant le même nombre de constantes que la somme de 14 puissances biquadratiques de fonctions linéaires des variables, ne peut pas en général être exprimée comme une telle somme ; car, pour que cela fût possible, il faudrait que le catalecticant de l'hypersurface s'évanouît.

(4) Prenons encore $\eta=2$, et considérons la somme de 9 puissances quatrièmes de fonctions linéaires de x, y, z, t. Le catalecticant de cette somme sera de l'ordre 10 et, conséquemment, zéro.

Donc le premier membre de l'équation d'une surface du quatrième degré qui ne contient que 35 constantes ne peut pas en général être mis sous la forme d'une somme de 9 puissances de fonctions linéaires des variables, quoique cette somme contienne 36 constantes disponibles.

Ce résultat pour les surfaces est, on le voit, un peu plus *paradoxal*, en apparence, que le théorème de Clebsch, sur les courbes du quatrième degré, quoiqu'en effet il n'y ait aucun paradoxe, ni dans l'un ni dans l'autre de ces théorèmes, pour ceux qui sont convaincus qu'on ne doit jamais se fier, sans contrôle, aux conclusions apparentes, fournies par la comparaison numérique de constantes.

48.

ON THE DIFFERENTIAL EQUATION TO A CURVE OF ANY ORDER.

[*Nature,* XXXIV. (1886), pp. 365, 366.]

To Mr Samuel Roberts (see Reprint of *Educational Times,* X. p. 47) is due the credit of having been the first to show that a direct method of elimination properly conducted leads to the differential equation for a cubic curve; but he has not attempted to obtain the general formula for a curve of any order. By aid of a very simple idea explained in a paper intended to appear in the *Comptes Rendus* of the Institute, I find* without calculation the general form of this equation. The left-hand member of it may be conveniently termed the differential *criterion* to the curve. One single matrix will then serve to express the criteria for all curves whose order does not exceed any prescribed number. For instance, suppose we wish to have the criteria for the orders 1, 2, 3, 4:—

Let $m\cdot\mu$ be used in general to denote the coefficient of h^m in

$$\left(\frac{1}{1.2}y''h^2+\frac{1}{1.2.3}y'''h^3+\frac{1}{1.2.3.4}y''''h^4+\dots\right)^\mu.$$

Write down the matrix—

2·1	3·1	3·2	4·1	4·2	4·3	5·1	5·2	5·3	5·4
3·1	4·1	4·2	5·1	5·2	5·3	6·1	6·2	6·3	6·4
4·1	5·1	5·2	6·1	6·2	6·3	7·1	7·2	7·3	7·4
5·1	6·1	6·2	7·1	7·2	7·3	8·1	8·2	8·3	8·4
6·1	7·1	7·2	8·1	8·2	8·3	9·1	9·2	9·3	9·4
7·1	8·1	8·2	9·1	9·2	9·3	10·1	10·2	10·3	10·4
8·1	9·1	9·2	10·1	10·2	10·3	11·1	11·2	11·3	11·4
9·1	10·1	10·2	11·1	11·2	11·3	12·1	12·2	12·3	12·4
10·1	11·1	11·2	12·1	12·2	12·3	13·1	13·2	13·3	13·4
11·1	12·1	12·2	13·1	13·2	13·3	14·1	14·2	14·3	14·4

[* Cf. pp. 492, 524 above.]

The determinant of the entire matrix, which is of the tenth order, is the criterion for a quartic curve. The determinant of the minor of the sixth order, comprised within the first six lines and columns, is the criterion for a cubic. The determinant of the third order, comprised within the first three lines and columns (subject to a remark about to be made) will furnish the criterion for a conic, and the apex of the matrix is the criterion for the straight line. By adding on five more lines and columns, according to an obvious law, the matrix may be extended so as to give the criterion for a quintic; then six more lines and columns a sextic, and so on as far as may be required.

The remark to be made concerning the determinant of the third order referred to is that it contains the irrelevant factor 2·1, that is, $\frac{y''}{2}$, so that the criterion for a conic (Monge's) is this determinant divested of such factor. It is *certain* that the next determinant is indecomposable, and is therefore the criterion for a cubic. There is no reason that I know of to suppose that any other determinant except that one which corresponds to the conic, is decomposable into factors. If this is made out, then, observing that the single term which is the criterion for the right line is indecomposable, we have another example of what may be called, in Babbage's words, a miraculous exception to a general law.

A well-known similar case of such miraculous exception I had occasion many years ago to notice in connection with the criteria for determining the number of real and imaginary roots in an algebraical equation. Such criteria may, with one single exception, be expressed by means of invariants. The case of exception is the biquadratic equation, for which it is impossible to assign an invariantive criterion that shall serve to distinguish between the cases of all the roots being real and all imaginary.

It is proper to notice that it follows, from the definition of the symbol $m \cdot \mu$, that its value is zero whenever m is less than 2μ. Thus, in the matrix written out above, the symbols 3·2, 4·3, 5·3, 5·4, 6·4, 7·4 may be replaced by zeros.

The above general result for a curve of any order is actually obtained by a far less expenditure of thought and labour than was employed by Monge, Halphen, and others to obtain it for the trifling case of a conic. I touch a secret spring, and the doors of the cabinet fly wide open*.

* Adopting the convention for degree and weight of a differential coefficient usual in the theory of reciprocants the deg : weight of the differential criterion of the nth order will be easily found to be

$$\frac{n \, . \, n+1 \, . \, n+2}{6} : \frac{n-1 \, . \, n \, . \, n+1 \, . \, n+2}{8}$$

except that for $n=2$ it is 3 : 3 instead of 4 : 3.

49.

ON THE SO-CALLED TSCHIRNHAUSEN TRANSFORMATION.

[*Crelle's Journal*, C. (1887), pp. 465—486.]

EXACTLY one hundred years ago, E. S. Bring (Dissertation, University of Lund, 1786. Meletemata quaedam mathematica circa transformationem aequationum algebraicarum) gave the method to which the name of Tschirnhausen by a common consent in error is now usually attached*. Sometimes but more rarely the method is attributed to Jerrard who came much later into the field. This is especially the case in England; Hamilton for instance in his "Report on Jerrard's method" published exactly 50 years ago in the

* The expression $$P_\theta - L_{n-1} Q_{n-1-\theta} + M_n R_{n-2-\theta}$$
where L, M are given entire functions in x of degrees $n-1$, n,
P, Q, R ,, disposable ,, ,, ,, ,, θ, $n-1-\theta$, $n-2-\theta$,
may be made identically zero by solving $2n-1-\theta$ homogeneous linear equations between the $2n-\theta$ disposable constants contained collectively in P, Q, R, and when this is done we have

$$\frac{P_\theta}{Q_{n-1-\theta}} \equiv L_{n-1} \text{ [mod. } M_n].$$

Hence it follows that the Tschirnhausen substitution has a one-to-one correspondence with any fractional substitution containing the requisite number of disposable constants: so for instance in the case of a *quintic* the Bring substitution

$$lx^4 + mx^3 + nx^2 + px + q$$

is only *another name* for the general *quadratic substitution* $\frac{ax^2+bx+c}{dx^2+ex+f}$.

This change of form in the substitution, supposed to be generalised, is interesting for the reason that it completes the analogy between the Tschirnhausen method of simplifying an algebraical equation and Combescure's method of simplifying a linear differential equation. Sir James Cockle appears to have arrived at the same result as M. Combescure in a paper on Linear Differential Equations. (*Quarterly Journal of Mathematics*, Aug. 1864.)

This method involves two quadratures, the integration of a differential equation of the second order, and substitutions impressed simultaneously upon *the two variables*.

The quadratures and solution of an equation of the second order are, of course, analogous to the solution of two simple and one quadratic algebraical equation; the substitutions impressed on the two variables run parallel to the two integral substitutions to be performed upon the two variables of the algebraical equation put under the form of a quantic which are equivalent to a fractional substitution performed upon the single variable of a non-homogeneous form.

Reports of the British Association makes hardly any mention of any other author but Jerrard in connexion with the subject.

In the following memoir I propose to present Hamilton's process under what appears to me to be a clearer and more easily intelligible form, to extend his numerical results and to establish the principles of a more general method than that to which he has confined himself.

But previously to entering upon this part of my work I think it may be well to call attention to a circumstance connected with the so-called Tschirnhausen transformation, as bearing upon the character of the transformed equation to which it leads, which hitherto appears to have escaped observation, and which is of particular interest as regards the application of the method to the equation of the 5th degree when it is reduced to the form

$$y^5 + By + C = 0,$$

for I shall be able to show in that case that in general the coefficients which remain (notwithstanding the large element of indeterminateness of which the method admits) cannot be made real when more than one of the roots of the original equation is real; this remark will be found to apply whether the method be used under its original form or under the modified form employed so advantageously by Hermite.

In order to make out this proposition it will be useful to give a somewhat more extended statement of the Law of Inertia (Trägheitsgesetz) for quadratic forms than that originally presented by me in the memoir: "On a theory of the syzygetic relations of two rational integral functions comprising an application to the theory of Sturm's functions and that of the greatest algebraical common measure" (*Phil. Trans.* for 1853)*.

Let us suppose a quadratic function of $m + n$ letters, either independent or connected by linear relations which in the latter case reduce the number of independent quantities to $\mu + \nu$.

Let the function be supposed to be expressed

(1) by the sum of m positive and n negative squares,

(2) by the sum of μ positive and ν negative squares

of *real* linear functions of the variables.

Then I affirm the impossibility of either of the two inequalities

$$\mu > m; \quad \nu > n.$$

(1) I say that the conjunction of the inequalities $m > \mu$, $\nu > n$ is impossible.

For suppose the two expressions of the same quadratic function to be

$$a_1^2 + a_2^2 + \ldots + a_m^2 - b_1^2 - b_2^2 - \ldots - b_n^2$$

and $$\alpha_1^2 + \alpha_2^2 + \ldots + \alpha_\mu^2 - \beta_1^2 - \beta_2^2 - \ldots - \beta_\nu^2.$$

[* Vol. I. of this Reprint, p. 511.]

Then $$a_1^2 + a_2^2 + \dots + a_m^2 + \beta_1^2 + \beta_2^2 + \dots + \beta_\nu^2$$
$$= b_1^2 + b_2^2 + \dots + b_n^2 + \alpha_1^2 + \alpha_2^2 + \dots + \alpha_\mu^2.$$

By hypothesis $$\mu + n < \mu + \nu,$$
$$\mu + n < m + \nu.$$

By virtue of the first inequality it must be possible to establish $\mu + n$ relations between the $\mu + \nu$ independent variables.

Consequently we may equate each square on the right-hand side of the equation to some distinct square on the other side, and then by virtue of the second inequality some squares will remain over on the left-hand side of the equation whose sum will be identically zero. Which is impossible. Hence the inequalities $m > \mu$, $\nu > n$ cannot exist simultaneously. In like manner it follows that $n > \nu$, $\mu > m$ cannot exist simultaneously.

Now the only suppositions of combined relations of greater and less that can connect m, n; μ, ν are the following:

$$m < \mu,\ n < \nu;\quad m < \mu,\ n = \nu;\quad m < \mu,\ n > \nu;$$
$$m = \mu,\ n < \nu;\quad m = \mu,\ n = \nu;\quad m = \mu,\ n > \nu;$$
$$m > \mu,\ n < \nu;\quad m > \mu,\ n = \nu;\quad m > \mu,\ n > \nu.$$

Of these 9 suppositions the 1st, 2nd, and 4th are excluded by the condition $m + n =$ or $> \mu + \nu$, and the 3rd and 7th by virtue of what has just been proved. Hence the only hypotheses admissible are the four contained in the negative statements:

$$\mu \text{ not } > m \text{ and } \nu \text{ not } > n. \qquad \text{Q.E.D}$$

Although the only application which I shall have to make of this Lemma is to the case where $m + n = \mu + \nu + 1$, I have thought that it is of sufficient interest in itself and collaterally in the logical process of its proof to deserve setting out in full.

Suppose now that we have the equation $f(x) = (x, 1)^n = 0$ where all the coefficients in f are supposed to be real, and that we write in conformity with the ordinary so-called Tschirnhausen process:

$$y = u_1 x + u_2 x^2 + \dots + u_{n-1} x^{n-1} - S,$$

where $$nS = u_1 \Sigma x + u_2 \Sigma x^2 + \dots + u_{n-1} \Sigma x^{n-1}$$

so that the transformed equation will be of the form:

$$y^n + B_2 y^{n-2} + B_3 y^{n-3} + \dots + B_n = 0,$$

where B_i is a quantic of degree i in the letters $u_1, u_2, \dots u_{n-1}$. Let us consider the projective character of the quadratic function B_2. This character is determined by the nature of the succession of algebraical signs in the sum of positive and negative squares to which B_2 regarded as a function of the $n - 1$ letters u may be reduced by *real* linear transformations.

Since
$$y_1 + y_2 + \dots + y_n = 0,$$
$$-2B_2 = -\Sigma 2y_1y_2 = \Sigma y^2,$$
so that it is the character of Σy^2 which determines the projective character of B_2. The number of real values of y is the same as of x. Hence if f has i pairs of imaginary roots, Σy^2 will be the sum of $n-i$ positive and i negative squares of real linear functions of $u_1, u_2, \dots u_{n-1}$.

Consequently, by virtue of the lemma above proved, there is only one element of uncertainty as to the character of Σy^2, that is, it must we know *à priori*, when reduced to a sum of $n-1$ positive and negative squares of linear functions of $u_1, u_2, \dots u_{n-1}$, contain either i or $i-1$ negative squares. This uncertainty may be removed by means of a second lemma, namely, that the discriminant of B_2 is a numerical multiplier of the discriminant of f.

When two of the roots of f are equal, two of the values of y become equal so that Σy^2 becomes reducible to a sum of $n-2$ instead of a sum of $n-1$ squares.

Hence the former contains the latter as a factor: moreover it is obvious from the form of each value of y that its discriminant regarded as a function of the n roots of f will be of the degree $2\{1+2+\dots+(n-1)\}$, that is, $n(n-1)$ which is the same as that of the squared product of the differences of the roots of f. Hence B_2 is a *numerical* multiplier of such squared product. To find the value of the multiplier, I observe that in general it follows from known algebraical principles that if F is a sum of the squares of n linear functions of $n-1$ variables the discriminant of F may be found as follows. Form an oblong matrix with the coefficients of the several linear functions. The determinant represented by what Cauchy would have called the square of this matrix, but which is more correctly to be called the product of this matrix by its transverse, will be the discriminant in question, or which is the same thing this discriminant is the sum of the squares of all the *complete* minors that are contained in the oblong matrix.

In the case before us if we make $f = x^n - 1$* it will easily be seen that

* When $f=x^n-1$ the value of S (the mean of the values of y) is obviously zero. Suppose now by way of illustration that $n=5$, then calling the imaginary 5th roots of unity $\rho_1, \rho_2, \rho_3, \rho_4$, one of the complete minors referred to in the text will be the determinant of the matrix

$$\begin{matrix} \rho_1 & \rho_2 & \rho_3 & \rho_4 \\ \rho_1^2 & \rho_2^2 & \rho_3^2 & \rho_4^2 \\ \rho_1^3 & \rho_2^3 & \rho_3^3 & \rho_4^3 \\ \rho_1^4 & \rho_2^4 & \rho_3^4 & \rho_4^4, \end{matrix}$$

and when the columns of this matrix are divided respectively by $\rho_1^\theta, \rho_2^\theta, \rho_3^\theta, \rho_4^\theta$, $[\theta=1, 2, 3, 4]$, which will leave the value of the determinant unaltered, the determinant of the matrix so modified will represent in succession each of the other 4 minors.

The value of the one above written, paying no attention to the algebraical sign, is by a well known theorem the product of the differences of $\rho_1, \rho_2, \rho_3, \rho_4$, that is, inasmuch as
$$(1-\rho_1)(1-\rho_2)(1-\rho_3)(1-\rho_4) = 5$$

the n minors in question, paying no regard to algebraical sign, become all equal, and each will be the product of the differences of the roots of $x^n - 1$ when the root 1 is excluded, or which is the same thing will be the product of the differences of all the roots (not excluding 1) divided by n.

Hence the sum of the n squared minors will be the nth part of the square of the products of the differences of the roots of $x^n - 1$. Consequently in general the discriminant of Σy^2 is the nth part of the product of the squares of the differences of the roots of the function f, and therefore by the process of reduction of $-\Sigma y^2$ to a sum of $n-1$ squares it is the *positive* sign always which will undergo the diminution of a unit, the number of negative signs remaining unaltered.

Hence when there are no imaginary roots in f, $-B_2$ will have all its signs positive; but when there are i pairs of imaginary roots in f, i of the signs in $-B_2$ will be negative, and thus the character of B_2, or of the quadratic *contour* (that is, curve, surface, hypersurface, etc.) represented by $B_2 = 0$ is completely determined when the number of real and imaginary roots in f is given.

If we suppose $n = 5$ we see that according as the number of real roots in f is 5, 3, or 1, the signs of $-B_2$ regarded as a sum of positive and negative squares of real linear functions of 4 letters will be:

$$\begin{array}{cccc} + & + & + & + \\ + & + & + & - \\ + & + & - & - \end{array}$$

In the first case the contour B_2 is completely imaginary, and it is not only not possible to apply the Bring-Tschirnhausen method so as to make simultaneously $B_2 = 0$, $B_3 = 0$ by real quantities u_1, u_2, u_3, u_4, but it is also the case that such values of u_1, u_2, u_3, u_4 do not exist. This indeed is evident *à priori*, from the fact that the equation

$$y^5 + B_4 y + B_5 = 0$$

must have at least two imaginary roots and therefore the equation in x would have at least two imaginary roots if the quantities u_1, u_2, u_3, u_4 were all real and unequal; whereas all the roots of that equation are supposed to be real.

In the second case the intersection of the contours B_2, B_3 may be real or imaginary: but even if it be real the method will not serve to determine any

it is the 5th part of the product of the differences of 1, ρ_1, ρ_2, ρ_3, ρ_4, and consequently the sum of the squares of the 5 minors is 5 times the 25th part of the squared product of the differences of the 5 roots. Here $\frac{5}{25}$ represents the general numerical multiplier $\frac{n}{n^2}$, that is, $\frac{1}{n}$.

single point in such section, because no *real* right line can be drawn to B_2 at any point which shall lie on the surface.

In the 3rd case at each point of B_2 two real right lines can be drawn each of which will intersect B_3 in one real point at least, and accordingly there will be a duplex-infinity of systems of real values of the u's which will make $B_2 = 0$, $B_3 = 0$ capable of being found by solving only a quadratic and a cubic equation in succession, and any one of such systems will lead to an equation of the form

$$y^5 + B_4 y + B_5 = 0,$$

where B_4, B_5 (which it is hardly necessary to notice become respectively $\frac{1}{4}\Sigma y^4$, $-\frac{1}{5}\Sigma y^5$) will each be real.

The B_2 found by Hermite's method may be obtained from the B_2 above given by a real linear substitution impressed on the letters u_1, u_2, u_3, u_4, and consequently the same conclusions continue to apply, that is, the coefficient of y and the constant will not in general be real unless four of the roots of the equation in x are imaginary*.

I will now proceed to the principal object of this paper, namely, the elucidation and extension of the method, contained in Hamilton's report, for determining the least number of letters which must be contained in one or more equations in order that they may admit of being solved by means of equations whose degrees are subject to satisfy certain prescribed conditions.

Before proceeding to the Lemma upon which all that follows is based, it will be useful to give one or two definitions.

1. Let S be a system of homogeneous equations in an indefinite number of variables $x, y, \ldots$, and let $x = a$, $y = b$, ... satisfy all the equations. I call $a, b, \ldots$ a solution of S.

2. If $a, b, \ldots$ is a given solution of S, I call the equation obtained by operating upon any of those in S with $(a\partial_x + b\partial_y + \ldots)^q$ where q has any integer value whatever not excluding zero, an emanant of such equation in respect to the solution $a, b, \ldots$, and the new system S_1 which contains all the emanants of all the equations in S an emanant to S in respect to the given solution.

* Hamilton remarks (*Report of British Association*, 1836, p. 307) that "the coefficients of the new or transformed equation will *often* be imaginary even when the coefficients of the original equation are real." Apparently he was not aware that the criterion for determining when this is so, depends solely on the intrinsic character of the equation to be transformed.

It should have been noticed before that when two of the roots in the given quintic are equal the quadratic surface represented by the coefficient of y^3 in the transformed equation becomes a cone and the reasoning employed in the text falls to the ground. But inasmuch as in this case two of the values of y become equal, we know *à priori* that the equation in y must be reducible to a form with real coefficients, namely,

$$y^5 - 5y + 4 = 0.$$

3. Let $a, b, \ldots, a_1, b_1, \ldots$, be any two solutions of a system of equations. I call

$$a + \lambda a_1, \quad b + \lambda b_1, \quad c + \lambda c_1, \quad \ldots$$

an alliance of the two solutions.

We may now state the following Sub-lemma. The alliance of any solution of a system S with a solution of S_1, its emanant in respect to the first named solution, is a solution of S_1.

For let $a, b, \ldots$ be the solution of S which gives rise to S_1. Then, calling

$$a\partial_x + b\partial_y + \ldots = E,$$

the general form of the equations in S_1 is $E^q\Phi(x, y, \ldots)$, and, supposing Φ to be of n dimensions,

$$E^q\Phi(x + \lambda a, \quad y + \lambda b, \quad \ldots)$$
$$= E^q\Phi + \lambda E^{q+1}\Phi + \tfrac{1}{2}\lambda^2 E^{q+2}\Phi + \ldots + \lambda^{n-1}E^{q+n-1}\Phi.$$

Hence the effect of substituting $x + \lambda a$, $y + \lambda b$, ... for $x, y, \ldots$ in S_1 is merely to effect upon it a linear transformation, and consequently the alliance of the solution $a, b, c, \ldots$ with any solution of S_1 will be a solution of that system. If now we find a solution $a_2, b_2, \ldots$ of S_1 and form an emanant S_2 of it in respect to that solution, it will follow from the sub-lemma that an alliance of the solutions $a, b, \ldots; a_1, b_1, \ldots$; with any solution $a_2, b_2, \ldots$ of S_2, that is, the solution $a + \lambda a_1 + \mu a_2$, $b + \lambda b_1 + \mu b_2$, ... will be a solution of S_2, and so in general. This I call the Lemma.

An ordinary solution of a system of equations may be called a point solution, an alliance in which 1, 2, 3, ... parameters enter a line, a plane, a hyperplane, ... solution.

It will of course be observed that any solution of an emanant to a system is *à fortiori* a solution of the primitive which as observed forms a portion of its emanant.

If S be a system of $q_i, q_{i-1}, \ldots q_1$ equations of degrees $i, i-1, \ldots 1$ respectively in the variables, it is obvious the 1st emanant will consist of

$$q_i, \quad q_i + q_{i-1}, \quad q_i + q_{i-1} + q_{i-2}, \quad \ldots \quad q_i + q_{i-1} + \ldots + q_1$$

equations of the degrees $i, i-1, i-2, \ldots 1$ respectively, and more generally in the rth emanant the number of equations of the degree

i	will be	q_i
$i-1$	" "	$rq_i + q_{i-1}$
$i-2$	" "	$\frac{1}{2}r(r+1)q_i + (r-1)q_{i-1} + q_{i-2}$
....	....	....
....	....	....
1	" "	$[r+i-1]_i q_i + [r+i-2]_{i-1} q_{i-1} + \ldots + q_1$,

where in general $[\theta]_i$ is used to denote $\dfrac{\theta(\theta-1)\ldots(\theta-i+1)}{1 \cdot 2 \ldots i}$.

The question now arises as to what must be the number of variables in a system S in order that its rth emanant S_r may admit of a general solution. If the total number of equations in S_r be called N, it might at first sight be supposed that the number of variables, or letters as I prefer to call them, in S must have $N+1$ as an inferior limit: but the case is not so—the least number of variables required will be r greater than this, that is, $N+r+1$.

Thus, for example, suppose we consider a first emanant S_1; then if a_1, b_1, c_1, ... is a solution we know that $a_1+\lambda a$, $b_1+\lambda b$, $c_1+\lambda c$, ... is also a solution whatever λ may be. Hence making $\lambda = -\frac{a_1}{a}$ and remembering that the equations are homogeneous we see that zero associated with any system of independent minors of the matrix

$$\begin{matrix} a & b & c & \dots, \\ a_1 & b_1 & c_1 & \dots \end{matrix}$$

will constitute a solution, as for instance 0; $ab_1 - ba_1$; $ac_1 - ca_1$; ...*. Hence the number of independent quantities in S_1 will be 1 less than the number of letters in S.

* As an illustration suppose Φ is a quantic of degree n in $(n+2)$ letters representing what may be termed a *contour*, the analogue in general space of a curve in 2-dimensional or a surface in 3-dimensional space. If we take all the successive emanants of Φ in respect to a point upon it $a, b, c, \dots$ the n resulting functions [Φ included] being functions of the $n+1$ minors to the matrix [$(n+2)$ places in length]

$$\begin{vmatrix} a & b & c & \dots \\ x & y & z & \dots \end{vmatrix}$$

the contours which they represent will intersect in a faisceau of right lines—showing that on a contour of the nth degree in $(n+1)$-dimensional space $1 . 2 . 3 \dots n$ right lines lying in the contour will pass through every point thereof, a fact we are familiar with in the case of a quadric surface where $n=2$. We might with equal propriety and more convenience say that Πn straight lines may be drawn upon and at every point of an n-fold contour of the nth *order*.

As I have already referred in this footnote to right lines drawn on contours I venture upon a slight digression connected with this conception. If we have a cubic twofold contour (an ordinary cubic surface) expressed as a quantic in x, y, z, t, we see that on writing x, y as linear functions of z, t and substituting their values in Φ in order to make the result, a cubic function of z, t vanish, we have to satisfy 4 equations between the 4 coefficients of substitution, which at once shows that a finite number of right lines may be drawn upon such contour of which the number we see at once cannot exceed 3^4 and which we know aliunde is 3^3.

It would seem then that for a contour in n letters of the degree $2n-5$ (unless there is some lurking fallacy in the counting of the constants) we ought in like manner to be able, by expressing $n-2$ of the letters as linear functions of the two remaining ones, to make the result vanish by solving $2n-4$ non-homogeneous equations of the degree $2n-5$ between the like number of coefficients of substitution, and as if upon such a contour we must be able to draw a definite number of straight lines of which the number, supposing that there is no latent fallacy of constant-counting, would be not greater and in all probability less than $(2n-5)^{2n-4}$, in fact $(2n-5)^{2n-5}$.

Also it may be shown that, as by Bedetti's theorem we know that every twofold contour (an ordinary surface) is cut by its linear polar (its tangent plane) in respect to a point upon it, in a curve having a double point thereat, so a contour of the 3rd order will be cut by its linear and quadratic polars in respect to any point upon it in a curve having a sextuple point thereat, and so in general an n-fold contour will be cut by $n-1$ consecutive polars (starting from the tangential

Similarly for the system S_r; r zeros associated with any independent system of complete minors of the matrix

$$\begin{matrix} a & b & c & \dots \\ a_1 & b_1 & c_1 & \dots \\ a_2 & b_2 & c_2 & \dots \\ \dots & \dots & \dots & \dots \\ a_r & b_r & c_r & \dots \end{matrix}$$

may be taken as the variables, and consequently it is $N-r-1$ and not $N-1$ which has for its inferior limit the number of equations in S_r. We may restore to the variables their independence by associating with the equations in S_r r additional perfectly arbitrary linear functions and there is sometimes a convenience in substituting in place of the rth emanant as it stands such emanant augmented by r arbitrary linear functions, which may be called the *completed* emanant.

For the purpose of greater clearness of exposition there will be an advantage in ignoring in the first instance all considerations based upon any other alliance except of the 1st order, that is, involving only one arbitrary parameter.

Suppose a system of equations S_1 consisting of a system S and one equation more Q. If we are in possession of a linear solution of S, that is, a solution

$$x = a_1 + \lambda a, \quad y = b_1 + \lambda b, \quad \dots$$

by substituting these values in Q, λ may be found by solving an equation whose degree is that of Q, and thus a point (or ordinary) solution of S_1 will have been found.

Let us now consider the question of a linear solution of S containing $q_i, q_{i-1}, \dots q_1$ equations of degree $i, i-1, \dots 1$ respectively. This we shall call of the type $[q_i, q_{i-1}, \dots q_1]$. Let

$a, b, \dots$ be any point solution of S,

and $a_1, b_1, \dots$ any point solution of ES,

homaloid as the first of them) in respect to any point upon it in a curve having thereat a point of multiplicity $1 . 2 . 3 \dots n$.

It may be well here to notice that a uni-parametrical solution of $\Phi = 0$ corresponds to drawing a straight line upon the contour represented by Φ, and in like manner a bi-parametric solution corresponds to drawing a plane upon the contour, a tri-parametric solution to drawing a hyper-plane upon the contour, and so in general. This is why I call such solutions linear, planar, hyperplanar, etc.

So again in this connexion it may be remarked that upon a quadratic contour in trans-hyper-space 6 planes lying on the contour pass through every point and in like manner upon a quadratic contour in $2n$ letters, $1 . 2 . 3 \dots n$ n-fold homaloids may be drawn upon the contour through every point thereon.

the *completed* emanant of S in regard to $a, b, \ldots$. This will be of the type

$$q_i;\; q_i+q_{i-1};\; \ldots;\; q_i+q_{i-1}+\ldots+q_2;\; q_i+q_{i-1}+\ldots+q_1+1.$$

The alliance of these two point solutions will be a linear solution of S. Again, the number of variables required for the point solution of ES need not exceed the number required for the linear solution of the system to which S is reduced by the abstraction of one equation of the degree i. Hence if we use $[p, q, r, \ldots \eta, \theta]$ to denote the number of variables sufficient for the solution of a system of p equations of degree i, q of degree $i-1$, etc. (i being the number of indices $p, q, \ldots \theta$) we obtain the formula of reduction

$$[p, q, r, \ldots \theta] = [p-1, p+q, p+q+r, \ldots p+q+r+\ldots+\theta]+1^*.$$

Continuing this process of reduction until the first index is reduced to zero a very easy calculation leads to the *formula of obliteration*

$$[p, q, r, s, \ldots \theta] = p + [q_1, r_1, s_1, \ldots \theta_1]\dagger,$$

where

$$q_1 = \frac{p(p+1)}{2} + q,$$

$$r_1 = \frac{p(p+1)(2p+1)}{6} + pq + r,$$

$$s_1 = \frac{p(p+1)(p+2)(3p+1)}{24} + \frac{p(p+1)}{2}q + pr + s,$$

....

$$\theta_1 = \frac{p(p+1)\ldots(p+i-1)(ip+1)}{1.2.3\ldots(i+1)} + \frac{p(p+1)\ldots(p+i-2)}{1.2.3\ldots(i-1)}q + \ldots + \theta.$$

In applying the process of reduction in the way indicated, the system S will have been replaced by two systems which we may call a diminished S and a diminished emanant of S, that is, S and ES each deprived of an equation (not necessarily the same in both) of degree i.

In like manner each of these will give rise to two systems, namely, a diminished self-system and a diminished emanant-system; but as the object is to obtain a formula of reduction for the number of letters required to obtain a linear solution of S, and as this number is greater for an emanant of any system than for the system itself, it was sufficient to follow the main-stream of deduction, in which the first alone is taken account of, in order to arrive at the required formula. In doing so, 2^p independent equations of the degree i will have been set apart each of which will have to be solved in its proper turn.

In the formula of *obliteration* the index in the first place has disappeared. Repeating the process we shall come to

$$p + q_1 + [r_2, s_2, \ldots \theta_2]$$

* In this and all subsequent formulae of reduction or obliteration the sign "=" is to be understood to mean "not greater than."

† Compare Hamilton, *Report of British Association*, 1836, p. 335, formula 244, and p. 345, formula 320.

where $r_2, s_2, \ldots \theta_2$ are derived from $q_1, r_1, s_1, \ldots \theta_1$ in the same way as $q_1, r_1, s_1, \ldots \theta_1$ from $p, q, r, s, \ldots \theta$ except that i will be replaced by $i-1$; and thus pursuing the same process we shall arrive at

$$[p + q_1 + r_2 + \ldots + \eta_{i-1} + \theta_i]$$

or say $[\sigma]$. The number of variables required for a solution involving one arbitrary parameter of σ homogeneous linear equations being $\sigma + 2$, this latter will be the number sufficient for S to admit of a linear solution without giving occasion to solve any equation of a degree exceeding i, and also without having occasion to solve any simultaneous system of equations other than linear ones.

Suppose a system of equations of the respective degrees 1, 2, 3, ... i and a single equation of the degree $i + 1$.

The type of the former will be 1, 1, 1, ... 1 to i places,
and of the latter 1, 0, 0, 0, ... 0 „ $i+1$ „ .

By the rule which has been established the number of letters required for the linear solution of the latter will be *one* more than for the former.

Hence the determination of the Tschirnhausen question of finding what the degree of an equation must be in order that i consecutive terms following immediately after the first term in the transformed equation, conjoined with any more advanced term, may admit of a solution of minimum weight, contains a determination of the number of variables required to ensure the possibility of obtaining a linear solution by a system of equations of minimum weight of a single equation of degree $i+1$; for the latter number will be the former increased by a unit*. The first form of the question is the more simple in itself; but as the other is more immediately connected with the object in which the theory originated, I prefer to put it in the latter form.

We may apply the obliteration formula to the indefinite type and obtain the annexed Table.

Triangle of Obliteration.

1	1	1	1	1	1	1	......
	2	3	4	5	6	7	......
		6	15	29	49	76	......
			36	210	804	2449	......
				876	24570	401134	......
					408696	246382080	......
						83762796636	......
							......
							....
							...

* For example, to take away the 2nd, 3rd, and another term the degree required is 5: and to obtain a linear solution of a cubic the number of variables required is 6.

To take away the 2nd, 3rd, 4th, and another term, employing a solution of *the lowest weight*, 11 variables are required; in order to obtain a solution, of lowest weight, of a single function of the fourth degree, 12 variables are required, and so on.

The degree of the equation sufficient to allow

$$2,\quad 3,\quad 4,\quad 5,\quad 6,\quad 7,\quad \ldots$$

consecutive terms following the first to be removed by a solution of *minimum weight* of the auxiliary equations, will be the continued sum of

$$1,\quad 2,\quad 6,\quad 36,\quad 876,\quad 408\,696,\quad 83\,762\,796\,636,\quad \ldots$$

each increased by 2, that is,

$$3,\quad 5,\quad 11,\quad 47,\quad 923,\quad 409\,619,\quad 83\,763\,206\,255,\quad \ldots$$

These numbers up to 923 agree with those found by Hamilton (*Report*, p. 346), the two last have been calculated here probably for the first time.

It would be too arduous a task to seek to give a much further extension to the table inasmuch as each successive term in the series 1, 2, 6, 36, ... is a fraction converging to $\frac{1}{2}$ of the square of the preceding term. This becomes obvious from inspection of the series formed by dividing each number in the above series by the square of the one before it; we thus obtain the fractions:

$$\frac{4}{1},\quad \frac{6}{4},\quad \frac{36}{36},\quad \frac{876}{1296},\quad \frac{408696}{767376},\quad \frac{83762796636}{167032420416},$$

which are continually diminishing.

But if we call two successive and infinitely distant rows of the Triangle of Obliteration

$$\begin{matrix} a & b & \ldots \\ & B & \ldots, \end{matrix}$$

$$B = \frac{a^2 + a}{2} + b.$$

Hence $\frac{B}{a^2}$ converges to $\frac{1}{2} + \frac{b}{a^2}$ which is always greater than $\frac{1}{2}$. Moreover $\frac{b}{a^2}$, calculated for the successive values as far as the table extends, will be seen to be a continually decreasing fraction and assuming (what awaits exact proof) that it eventually vanishes, $\frac{B}{a^2}$ must converge to $\frac{1}{2}$.

The successive values of $\frac{b}{a^2}$ for the different rows are

$$\frac{3}{4},\quad \frac{15}{36},\quad \frac{210}{1296},\quad \frac{24570}{767376},\quad \frac{246382080}{167032420416}.$$

Inverting these fractions the values, to the nearest integer, become 1, 2, 6, 31, 678, so that there can be no doubt of the truth of the law that the asymptotic value of the square of each term divided by the square of its antecedent is $\frac{1}{2}$.

Moreover the numbers last found themselves obviously obey a parallel law to that of the original series which raises a presumption that it may be possible to obtain an exact expression for the general term in the original series or even in the Obliteration Table in its entirety. But be that as it may, as evidently the asymptotic law is equally true for the sums of the terms in the first diagonal as for the terms themselves, we arrive at the interesting fact that if $\Phi(i)$ is the minimum degree of an equation from which i consecutive terms immediately following the first can be removed, $2\Phi(i+1)$ converges to a ratio of equality with $\Phi(i)^2$ when i increases indefinitely.

The minimum number of letters thus found is we see a minimum, at all events in this sense that the *method employed* to obtain a solution is inapplicable if that number of letters be reduced. In the words of Jerrard as quoted by Hamilton (*Report*, pp. 326, 327) "to discover $m-1$ ratios of m disposable quantities,

$$a_1, \quad a_2, \quad \ldots \quad a_m$$

which shall satisfy a given system of h_1 rational and integral and homogeneous equations of the first degree

$$A'=0, \quad A''=0, \quad \ldots \quad A^{(h_1)}=0,$$

h_2 such equations of the second degree

$$B'=0, \quad B''=0, \quad \ldots \quad B^{(h_2)}=0,$$

h_3 of the third degree

$$C'=0, \quad C''=0, \quad \ldots \quad C^{(h_3)}=0,$$

and so on, as far as h_t equations of the tth degree

$$T'=0, \quad T''=0, \quad \ldots \quad T^{(h_t)}=0$$

without being obliged, in any part of the process, to introduce any elevation of degree by elimination."

But this definition may be superseded by another in which only the intrinsic character of the result arrived at is in question, and not the particular method pursued to reach it.

Let us agree to consider all equations of the same degree to have the same weight and that this weight is infinitely greater than that of an equation of any lower degree. The weight of a system of equations to be regarded as the sum of the weights of the equations which it contains.

We may, extending but not altering the meaning previously attached to the word "solution," call the *ensemble* of the equations to be solved in order to obtain any solution of the given system a solution thereof. If now a system of equations is given in number and in the degree of each, and each equation is supposed to be the most general of its kind, but the number of variables in the system is left disposable, it is easy to see that the above process, when it is practicable, leads to a solution of the lowest weight, so that no increase in the number of letters will have any effect in diminishing the weight of the solution, whatever may be the process employed to obtain it. Thus the numbers given by the linear method are *minima* in regard to solutions of the *lowest weight.*

We may however suppose another and more natural condition attached to the solution to be obtained; let n be the highest degree of any equation in a given general system proposed for solution; we know that it is impossible to avoid the solution of one or more equations of the nth degree. We may therefore propose to ourselves the problem of determining what is the least number of letters necessary in order that no equation in the solution shall be of a degree exceeding n. The minimum thus obtained will in general be inferior to the minimum required for obtaining a solution of the lowest weight, and to arrive at it in any particular case it becomes necessary to make use of the Lemma in its general form which introduces the notion of alliances above the first order. Hamilton has not touched upon this part of the subject except in a single case which it was impossible to overlook: namely, where he considers the problem of taking away four consecutive terms from the general equation of the tenth or any higher degree.

The process we have seen leads to the conclusion that as many letters are required as are needed for the solution of two quadratics and seven linear equations. The solution of one biquadratic equation in the application of the process being indispensable, he felt the absurdity (if I may use the word) of stickling at the introduction of one biquadratic more, the use of which has the effect of lowering the minimum from 11 to 10. See *Report of British Association*, 1836, p. 326.

The linear method however or theory of solutions of lowest weight enjoys this prerogative that the reduction formulae are of a purely algebraical kind, whereas when the other condition above referred to is introduced, questions of numerical equality and inequality have to be considered and the theory ceases to be strictly algebraical. In what follows therefore I shall confine myself to the only case of any particular interest, namely, that which arises from the original problem of removing any given number of consecutive terms (immediately following the first) from an algebraical equation.

We may accept as the general condition to be observed that the degree of no equation appearing in the solution of a system of equations shall exceed the highest degree which must perforce figure in such solution, that is, the highest degree in the system of equations to be solved. In the case then of n equations of the successive degrees $1, 2, 3, \dots i$ the condition will be that no equation in the solution shall be of a higher degree than i.

Thus, for example, if we look back to the easy case of a quaternary succession of such terms to be removed, we find that the problem reduces itself to finding the number of letters required to obtain a line-solution of the system whose type is 1, 1, 1, and that again to finding the number of letters required to obtain a line-solution to its augmented emanant 2, 4, that is, a system of 2 quadratic and 4 linear solutions, that is, a point solution of the completed emanant to this system which will be of the type 2, 7. The condition imposed here is that no equation shall appear of a higher degree than a biquadratic. Consequently subject to this condition the number of letters required to solve a system of one linear, one quadratic, and one cubic equation, is that sufficient for the plane-solution of a system of 7 linear equations, that is, 10, which is less by 1 than the number required in order to obtain a solution of the same system which shall be of the lowest weight.

It might at first sight be supposed that in general the introduction of solutions involving 2 or more parameters would lead to a very considerable reduction of the numbers found in the obliteration table; this however is not the case, the reduction in the values obtained by this extended method bears in general a very small ratio to the number reduced. This is a consequence of the following rule:—

In passing from the point solution of a system to a solution of any kind with a reduced type, the reduction is effected by *segregating* a certain number i of the given equations and obtaining a solution of the remainder which shall contain i arbitrary parameters.

Now it will be found that the *literant* (by which I mean the number of letters sufficient for the solution) will never be diminished by any other kind of segregation than what may be termed an *external segregation**.

* Imagine the type of a set of equations to be represented by a broad ribbon, in which each group of equations of the same degree is represented by a band of a distinct colour occupying as many units of space as there are units in the group. The legitimate process of segregation will then consist in dividing the band into two, obeying the same conditions as the original one, and the rule of "external segregation" amounts to saying that this separation must be effected by a single straight cut so that no middle portion is to be cut out.

According to this (which is a perfectly natural) representation the rule of external segregation may in the language of logic be described as the rule of the *excluded middle*. Thus, for example, suppose we wish to find the smallest number of variables required for the solution of a system of equations of which the type is 1, 1, 1, 0 without solving an equation beyond the 8th degree. The number required may be made equal to (cf. p. 547)

$$.\,[1, 1, 0] \text{ or to } :[1, 0, 0].$$

But $$.\,[1, 1, 0]=[1, 2, 3]=:[3],$$

and $$:[1, 0, 0]=[1, 2, 5]=:[5].$$

Thus the simultaneous segregation of the equations of the 4th and 2nd degrees *contrary to the rule* not only raises the weight of the solution but also increases the number of variables required in the given system in order that the solution may be possible.

As a consequence of this rule it may easily be seen (in the problem of determining the

Let $f, g, \ldots k, l$ be the type of the system of equations segregated, this will have no effect in diminishing the literant unless $f, g, \ldots k$ are the initial numbers of the type of the given system, in such case I call the segregation *external.*

Thus in starting with a system of the type 1, 1, ... 1, 1 the first act of segregation must consist in setting apart the equation of the highest degree and finding a line-solution of the system thus reduced. Suppose, to fix the ideas, that the highest degree is 6 and that we have arrived in the course of the deduction at a system of linear, quadratic, and cubic equations denoted by the type m, n, p.

So far as regards observance of the limit 6 for the highest degree in any substituted system, it would be permissible to segregate one cubic and one quadratic, but according to the rule of external segregation this will not be profitable (it will in general be quite the reverse unless $m = 1$) and so in general.

Let us now proceed to obtain the literant required for the point-solution of a sequence of i equations of all degrees from 1 to i subject to the condition that no auxiliary system shall contain an equation of degree higher than i for the values $i = 5, 6, 7, 8$ which is as far as the table of obliteration extends. The rule teaches that this is the same as the literant of a line-solution of a system of $i-1$ equations whose degrees extend from 1 to $i-1$.

It will be useful in what follows to obtain a general formula for the plane-literant of a system of i quadratics denoted by the type $i, 0$.

Let us signify by a symbol consisting of a type preceded by q points the literant to the form of solution containing q parameters of the system to which the type refers.

Then calling the plane-literant for $[i, 0]$ v_i, we have by virtue of the Lemma

$$v_i = :[i, 0] = [i-2, 2i+2] \quad = v_{i-2} + 2i + 2,$$
$$v_1 = :[1, 0] = .[1, 2] \qquad = .[4] = 6,$$
$$v_2 = :[2, 0] = .[2, 3] = .[1, 6] = [8] = 9.$$

Hence by integrating $v_i - v_{i-2} = 2i + 2$ we shall easily obtain:

$$v_{2q} \;\; = 2q^2 + 4q + 3,$$
$$v_{2q-1} = 2q^2 + 2q + 2.$$

In treating of the literant to [1, 1, 1, 1, 1, 1, 1, 1] it will be convenient to find

minimum degree of the equation required for taking away i consecutive terms without any equation in the solution exceeding the ith degree) that the occasion can never arise in the act of segregation to take account of any other numerical equalities and inequalities than one or the other of the two following

$$q^i = \text{ or } < n, \quad q^i (q-1)^j = \text{ or } < n.$$

the value of $\vdots\,[i, 0, 0]$ the general expression of which rid of exponentials will give rise to 3 cases.

Not being desirous of encumbering this memoir with formulae, and as we shall only have occasion to consider a single case of these formulae, I adjourn the calculation until we know what the form is of i in regard to 3 in the case to be calculated, and shall obtain the value of $\vdots\,[i, 0, 0]$ for that case alone.

I will now consider in succession the *literants* denoted by

$$.[1, 1, 1, 1] \quad .[1, 1, 1, 1, 1] \quad .[1, 1, 1, 1, 1, 1] \quad .[1, 1, 1, 1, 1, 1, 1]$$

subject to the conditions of the solution containing no equation of a degree higher than the 5th, 6th, 7th, 8th respectively

$$.[1, 1, 1, 1] = .[2, 3, 5] = .[1, 5, 11] = .[6, 18]$$
$$= :[4, \ 25] = 25 + 2\,.\,2^2 + 4\,.\,2 + 3 = 44.$$

This is the literant for the solution of minimum highest degree and is 3 units less than 47, the literant for the solution of lowest weight.

It will be observed that $.[6, 18]$ has been expressed in the course of the deduction by $:[4, 25]$ instead of $.[5, 25]$. In fact $.[6, 18] = [6, 25]$ and this latter according as we segregate 1 or 2 of the quadratics is expressible by $.[5, 25]$ or by $:[4, 25]$.

The expression $.[6, 18]$ might have been obtained immediately from the triangle of obliteration

1	1	1	1	...
	2	3	4	...
		6	15	...
			.	...

by simply substituting $1 + 2 + 15$ for 18. (It is worth noticing that in the table of obliteration after the 2nd line every initial number in any line ends with 6 and after the 3rd line every second number in each line ends with 0.)

So in like manner observing that $1 + 2 + 6 + 210 = 219$, we have

$$.[1, 1, 1, 1, 1] = .[36, 219]$$

which must have been *led up to* from

$$[1, 36, 219].$$

Hence $\quad .[1, 1, 1, 1, 1] = .[1, 35, 182] = [1, 36, 219] = :[35, 219]$

$$= 219 + 2\,.\,18^2 + 2\,.\,18 + 2 = 905$$

which is 18 units less than the corresponding literant of lowest weight 923. Similarly observing that

$$1 + 2 + 6 + 36 + 24\,570 = 24\,615,$$

$$.[1, 1, 1, 1, 1, 1] = :[875, 24\,615] = 24\,615 + 2\,(438)^2 + 2\,(438) + 2 = 409\,181$$

which is 438 less than the corresponding literant of lowest weight 409 619. In like manner calling

$$246\,382\,080 + 876 + 36 + 6 + 2 + 1 = 246\,383\,175 = s,$$

$$.[1, 1, 1, 1, 1, 1, 1] = s + :[408\,695, 0] = [408\,695, 0] + t = \vdots[408\,692, 0] + t$$

where $$t = s + 2 \times 408\,695 + 2 = 247\,200\,567.$$

Here $$408\,695 \equiv 2 \;[\text{mod. } 3].$$

But in general
$$\vdots[3q+2, 0] = \vdots[3q-1, 0] + 9q + 9$$
$$= \vdots[2, 0] + 9\{(q+1) + q + (q-1) + \dots + 2\}$$
$$= \vdots[2, 0] + \frac{9(q^2+3q)}{2} = \frac{9q^2+27q+24}{2} *$$
$$= \frac{(3q+2)^2 + 5(3q+2)}{2} + 5.$$

Therefore
$$.[1, 1, 1, 1, 1, 1, 1] = t + 5 + (204\,346)(408\,697)$$
$$= 247\,200\,572 + 83\,515\,597\,162$$
$$= 83\,762\,797\,734.$$

This number is the minimum degree of equation which admits of 8 of its terms being removed without solving any equation above the 8th degree in the same sense as 5 is the minimum degree of equation from which 3 terms can be removed without solving an equation above the 3rd degree.

The Hamiltonian numbers corresponding to the solutions of lowest weight, have been found to be

3, 5, 11, 47, 923, 409 619, 83 763 206 255

the reduced numbers due to the introduction of planar and hyperplanar solutions

3, 5, 10, 44, 905, 409 181, 83 762 797 734,

the differences are 1, 3, 18, 438, 408 521.

The ratio of these last numbers to the numbers above them constituting a rapidly decreasing series, it is obvious that the "asymptotic law" will remain good for the second as well as for the first line of numbers: so that if $\phi(i)$ expresses the minimum degree of an equation from which i terms can be abstracted without solving an equation above the ith degree, $\frac{2\phi(i+1)}{\phi(i)^2}$ will continually decrease towards and finally (when i is infinite) coincide with unity.

I have already defined the weight of a solution. According to analogy (as, for example, in the case of a given symmetric function $\Sigma a^\alpha . b^\beta . c^\gamma \dots$) the degree of the equation of highest degree in a solution may be termed its *order*.

* For $\vdots[2, 0] = [2, 9] = :[9] = 12.$

Thus then the two first series of numbers which have been given express the first of them the literant of the solution of lowest *weight*, the second the literant of the solution of lowest *order*. The numbers in the first series up to 923 and in the second series up to 10 appear in Hamilton's Report, all the others are here presented (it is believed) for the first time.

A solution is of course to be understood to mean a *non-simultaneous* but *not independent* system of equations from which a solution of a given system of equations may be derived. The equations in the solution-system form an arborescence or a ramification of consecutive systems, meaning thereby that the solution of any one of them depends upon a successive process of substitution of values of variables deduced from equations which precede it in such ramification. Some of the simpler of these arborescences I propose to *delineate* graphically in a subsequent communication.

Invited to participate in the centenary number of the leading Mathematical Journal in the world, it occurred to me that compatibly with my feeble means no more suitable contribution could be made than one which at the same time celebrates the centenary of the discovery due to the long and persistently ignored author of the method which it is the object of this memoir to elucidate and extend. I offer it (an aloe-flower of 100 years' growth) as a tardy Bessarabian "satisfaction to the Manes of" Bring.

50.

SUR UNE DÉCOUVERTE DE M. JAMES HAMMOND RELATIVE À UNE CERTAINE SÉRIE DE NOMBRES QUI FIGURENT DANS LA THÉORIE DE LA TRANSFORMATION TSCHIRNHAUSEN.

[*Comptes Rendus*, CIV. (1887), pp. 1228—1231.]

On peut se proposer le problème suivant:

Étant donné un quantic, le faire disparaître en exprimant chaque variable comme une fonction linéaire et homogène de deux variables.

Si le nombre des variables dans le quantic est suffisamment grand, quel que soit son degré n, ce problème peut s'effectuer au moyen d'un système auxiliaire d'équations, tel que pour résoudre le système on n'aura jamais occasion de résoudre une équation d'un degré supérieur à n.

En nommant N le nombre minimum des variables nécessaire pour que cela soit possible, cette question se présente: *trouver la valeur de N pour une valeur donnée de n.*

Par exemple, pour $n = 2$, on voit bien que N est 4.

Pour $n = 3$, on peut démontrer que N est 6; pour $n = 4$, $N = 11$, etc.

Mais on peut imposer une condition plus rigoureuse sur le caractère du système auxiliaire d'équations qui aura l'effet d'augmenter la valeur minimum N. On peut exiger que le type du système auxiliaire d'équations sera *le plus simple possible* ou, comme je préfère le dire, sera d'*un poids minimum.* Le poids d'une équation dépend seulement de son degré i et peut être pris égal à ρ^i, où ρ est une constante indéfiniment grande. De plus, le poids d'un système d'équations peut être défini comme étant la somme des poids des équations individuelles qu'il contient.

On a ainsi un criterium exact pour déterminer lequel des deux systèmes a son poids inférieur à celui d'un autre; le terme *poids minimum* devient exempt de toute ambiguïté, et l'on comprend ce que veut dire le système d'équations le plus simple d'un nombre quelconque de tels systèmes.

Avec la première définition de N, ses valeurs successives seront

3, 4, 6, 11, 45, 906, 409182, 83762797735, ...

En imposant la condition la plus rigoureuse, on obtient la série moins transcendante

3, 4, 6, 12, 48, 924, 409620, 83763206256, ...

que je nommerai E_0, E_1, E_2, E_3, ...

En diminuant ces derniers chiffres de l'unité, on trouve la série de nombres

2, 3, 5, 11, 47, 923, 409619, 83763206255, ...,

dont les six premiers ont été calculés par Hamilton (voir *Report of 6th Meeting of British Association*, pp. 346—7, 1837).

Hamilton a, en effet, montré que le degré d'une équation algébrique, étant pris successivement égal à 2, 3, 5, 11, 47, ..., on peut, par la méthode dite *de Tschirnhausen*, la transformer dans une autre où 1, 2, 3, 4, 5, ... termes consécutifs, après le premier, manquent, sans avoir occasion de résoudre aucune équation au-dessus des degrés 1, 2, 3, 4, 5, ... respectivement.

J'ajoute que le système d'équations auxiliaires, auquel on parvient par la méthode qu'il emploie, sera *du type le plus simple possible.* Si, pour ôter i termes consécutifs, on voulait se borner à la seule condition de n'avoir pas à résoudre une équation au-dessus du degré i, alors, au lieu des nombres 2, 3, 5, 11, 47, ..., on aurait les nombres plus transcendants 2, 3, 5, 10, 44, ... C'est la série 2, 3, 5, 11, 47, ... que je nomme les *nombres de Hamilton*, et que je désigne par H_0, H_1, H_2, H_3, H_4, ... Pour les obtenir (ou plutôt leurs différences) par la méthode de Hamilton, on a besoin de construire un triangle de chiffres (*voir* mon Mémoire dans le *Journal de Kronecker*, t. C. p. 477 [above, p. 541]).

Mon collaborateur, M. James Hammond, a trouvé un très beau théorème pour déduire les N immédiatement et successivement les uns des autres, sans introduire de nombres étrangers.

En se servant de $\beta_r(q)$ pour représenter $\dfrac{q(q-1)\dots(q-r+1)}{1.2\dots r}$, il a trouvé la formule vraiment remarquable

$$H_i = 2 + \beta_2(H_{i-1}) - \beta_3(H_{i-2}) + \beta_4(H_{i-3}) - \dots.$$

A ce théorème, j'ajoute comme corollaire une formule qui se rapporte à la série de nombres E (qui ne sont autre chose que les nombres H, augmentés chacun de l'unité), qui est bonne pour toutes les valeurs de r supérieures à l'unité,

$$\beta_0(E_r) - \beta_1(E_{r-1}) + \beta_2(E_{r-2}) - \dots + (-)^r \beta_r(E_0) = 0,$$

c'est-à-dire $\quad E_{r-1} = 1 + \beta_2(E_{r-2}) - \beta_3(E_{r-3}) + \dots + (-)^r \beta_r(E_0).$

Par exemple, $1-\frac{4}{1}+\frac{3.2}{1.2}=0,$

$$1-\frac{6}{1}+\frac{4.3}{1.2}-\frac{3.2.1}{1.2.3}=0,$$

$$1-\frac{12}{1}+\frac{6.5}{1.2}-\frac{4.3.2}{1.2.3}=0,$$

$$1-\frac{48}{1}+\frac{12.11}{1.2}-\frac{6.5.4}{1.2.3}+\frac{4.3.2.1}{1.2.3.4}=0,$$

$$1-\frac{924}{1}+\frac{48.47}{1.2}-\frac{12.11.10}{1.2.3}+\frac{6.5.4.3}{1.2.3.4}=0.$$

C'est par la méthode de fonctions génératrices que M. Hammond a réussi à établir cette échelle de relation entre les nombres de Hamilton, lequel évidemment n'avait pas le moindre soupçon de l'existence d'une échelle pareille.

Si l'on prend les différences des nombres de Hamilton, on obtient la série 1, 2, 6, 36, 876, ..., qu'on peut nommer h_1, h_2, h_3, h_4, h_5, ... On savait déjà par démonstration que $h_{i+1}\div h_i^2$ est plus grand que $\frac{1}{2}$ pour toute valeur finie de i et avec certitude morale que ce rapport devient $\frac{1}{2}$ quand i est infini. Avec la formule de M. Hammond, on peut donner une démonstration rigoureuse de ce dernier fait et en même temps établir ce nouveau théorème: *$H_{i+1}\div H_i^2$ est plus petit que $\frac{1}{2}$ pour toute valeur de i finie et plus grande que l'unité, et égal à $\frac{1}{2}$ quand i est infini.*

51.

ON HAMILTON'S NUMBERS.

By J. J. Sylvester and James Hammond.

[*Philosophical Transactions of the Royal Society of London*, CLXXVIII. (1887), pp. 285—312; CLXXIX. (1888), pp. 65—71.]

Introduction.

In the year 1786 Erland Samuel Bring, Professor at the University of Lund in Sweden, showed how by an extension of the method of Tschirnhausen it was possible to deprive the general algebraical equation of the 5th degree of three of its terms without solving an equation higher than the 3rd degree. By a well-understood, however singular, academical fiction, this discovery was ascribed by him to one of his own pupils, a certain Sven Gustaf Sommelius, and embodied in a thesis humbly submitted to himself for approval by that pupil, as a preliminary to his obtaining his degree of Doctor of Philosophy in the University*. The process for effecting this reduction seems to have been overlooked or forgotten, and was subsequently rediscovered many years later by Mr Jerrard. In a memoir contained in the *Report of the British Association*, for 1836, Sir William Hamilton showed that Mr Jerrard was mistaken in supposing that the method was adequate to taking away more than three terms of the equation of the 5th degree, but supplemented this somewhat unnecessary refutation of a result known *à priori* to be impossible, by an extremely valuable discussion of a question raised by Mr Jerrard as to the number of variables required in order that any system of equations of given degrees in those variables shall

* Bring's "Reduction of the Quintic Equation" was republished by the Rev. Robert Harley, F.R.S., in the *Quarterly Journal of Pure and Applied Mathematics*, vol. VI. 1864, p. 45. The full title of the Lund Thesis, as given by Mr Harley (see *Quart. Journ. of Math.*, pp. 44, 45) is as follows: "B. cum D. Meletemata quaedam mathematica circa transformationem aequationum algebraicarum, quae consent. Ampliss. Facult. Philos. in Regia Academia Carolina Praeside D. Erland Sam. Bring, Hist. Profess. Reg. & Ord. publico Eruditorum Examini modeste subjicit Sven Gustaf Sommelius, Stipendiarius Regius & Palmcrentzianus Lundensis. Die XIV Decemb., MDCCLXXXVI, L.H.Q.S.—Lundae, typis Berlingianis."

admit of being satisfied without solving any equation of a degree higher than the highest of the given degrees.

In the year 1886 the senior author of this memoir showed in a paper* in Kronecker's (better known as *Crelle's*) *Journal* that the trinomial equation of the 5th degree, upon which by Bring's method the general equation of that degree can be made to depend, has necessarily imaginary coefficients except in the case where four of the roots of the original equation are imaginary, and also pointed out a method of obtaining the absolute minimum degree M of an equation from which any given number of specified terms can be taken away subject to the condition of not having to solve any equation of a degree higher than M†. The numbers furnished by Hamilton's method, it is to be observed, are not minima unless a more stringent condition than this is substituted, namely, that the system of equations which have to be resolved in order to take away the proposed terms shall be the simplest possible, that is, of the lowest possible weight and not merely of the lowest order; in the memoir in *Crelle,* above referred to, the author has explained in what sense the words weight and order are here employed. He has given the name of Hamilton's Numbers to these relative minima (minima, that is, in regard to weight) for the case where the terms to be taken away from the equation occupy consecutive places in it, beginning with the second.

Mr James Hammond has quite recently discovered by the method of generating functions a very simple formula of reduction, or scale of relation, whereby any one of these numbers may be expressed in terms of those that precede it: his investigation will be found in the second section of this paper, and constitutes its most valuable portion. The principal results obtained by its senior author, consequential in great measure to Mr Hammond's remarkable and unexpected discovery, refer to the proof of a theorem left undemonstrated in the memoir in *Crelle* above referred to, and the establishment of certain other asymptotic laws to which Hamilton's Numbers and their differences are subject, by a mixed kind of reasoning, in the main apodictic, but in part also founded on observation‡. It thus

[* Above, p. 531.]

† For instance, an equation of not lower than the 905th degree may be transformed into another of that degree, in which the 2nd, 3rd, 4th, 5th, 6th, 7th, terms are all wanting, by means of the successive solution of a ramificatory system of equations, of no one of which the degree exceeds 6, whereas by the Jerrard-Hamiltonian method this transformation could not be effected for the general equation of degree lower than the 6th Hamiltonian Number, namely, 923. So for the analogous removal of 5 consecutive terms the inferior limit of degree of the equation to be transformed would be 47 by the one method, but 44 (the lowest possible) by the other. In the case of 4 consecutive terms Hamilton could not avoid being aware that 11, the 4th number which I have named after him, might be replaced by 10, as the lowest possible inferior limit of the equation to be transformed.

‡ In the 3rd section, communicated to the Society after the 1st and 2nd had gone to press, the empirical element is entirely eliminated, and the results reduced to apodictic certainty.

became necessary to calculate out the 10th Hamiltonian Number, which contains 43 places of figures. The highest number calculated by Hamilton (the 6th) was the number 923, which comes third in order after 5 (the Bring Number), 11 and 47 being the two intervening numbers. It is to be hoped that some one will be found willing to undertake the labour (considerable, but not overwhelming) of calculating some further numbers in the scale.

The theory has been "a plant of slow growth." The Lund Thesis, of December 1786 (a matter of a couple of pages), Hamilton's Report of 1836, with the tract of Mr Jerrard therein referred to, and the memoir in *Crelle*, of December 1886, constitute, as far as we are aware, the complete bibliography of the subject up to the present date.

§ 1. *On the Asymptotic Laws of the Numbers of Hamilton and their Differences.*

Consider the following Table:—

1	0	0	0	0	0	0	0	......
	1	1	1	1	1	1	1	......
		2	3	4	5	6	7	......
			6	15	29	49	76	......
				36	210	804	2449	......
					876	24570	401134	......
						408696	246382080	......
							83762796636	......
								......
								......

Any line of figures, say $p, q, r, s, t, \dots \theta$, in the Table being given, to form the subsequent line $q_1, r_1, s_1, t_1, \dots \theta_1$, we write

$$q_1 = \frac{p(p+1)}{1.2} + q,$$

$$r_1 = \frac{p(p+1)(2p+1)}{1.2.3} + pq + r,$$

$$s_1 = \frac{p(p+1)(p+2)(3p+1)}{1.2.3.4} + \frac{p(p+1)}{1.2} q + pr + s,$$

$$t_1 = \frac{p(p+1)(p+2)(p+3)(4p+1)}{1.2.3.4.5} + \frac{p(p+1)(p+2)}{1.2.3} q + \frac{p(p+1)}{1.2} r + ps + t,$$

....

$$\theta_1 = \frac{p(p+1)\dots(p+i-1)(ip+1)}{1.2.3\dots(i+1)} + \frac{p(p+1)\dots(p+i-2)}{1.2.3\dots(i-1)} q + \dots + \theta.$$

If we call the nth term of the mth line $[m, n]$, the general law of deduction may be expressed by the formula

$$[m+1, n] = -B_{n+1}([m, 1]-1) + \sum_{i=0}^{i=n} [m, n+1-i]\, B_i [m, 1],$$

where $B_i k$ means the coefficient of z^i in $(1-z)^{-k}$.

The negative term $-B_{n+1}([m, 1]-1)$, it may be noticed, arises from decomposing the first term of $[m+1, n]$, as given by the original formulæ, into two parts, of which it is one. Thus, for example,

$$\frac{p(p+1)(p+2)(p+3)(4p+1)}{1.2.3.4.5}$$

is changed into

$$-\frac{(p-1)p(p+1)(p+2)(p+3)}{1.2.3.4.5} + \frac{p(p+1)(p+2)(p+3)}{1.2.3.4} p.$$

The numbers in the hypothenuse of this infinite triangle, namely,

1, 1, 2, 6, 36, 876, 408696, 83762796636, 35081259062070955 91916,

6153473687096578758445014683368786661634996, ...,

are what I call the Hamiltonian Differences, or Hypothenusal Numbers*; and their continued sums augmented by unity, namely,

2, 3, 5, 11, 47, 923, 409619, 83763206255, 3508125906290858798171,

6153473687096578758448522809275077520433167, ...,

are what I call the Hamiltonian Numbers. The two latter of these have been calculated by means of Mr Hammond's formula, presently to be mentioned, and the corresponding Hypothenusal Numbers deduced from them by simple subtraction. Their connection with the theory of the Tschirnhausen Transformation will be found fully explained in my memoir on the subject in Vol. C. of *Crelle*. My present object is to speak of the numbers as they stand, without reference to their origin or application†.

* The other numbers of the "triangle," whose properties it may be some day desirable to investigate, may be termed co-hypothenusal numbers of *order* measured by their horizontal distance from the hypothenuse—their vertical distance below the top line denoting their *rank*. In the sequel the development is given of the half of a hypothenusal number (of the first order) in a descending series of powers (with fractional indices) of the half of its antecedent, the coefficients in the principal part of such series being (not, as might have been the case, functions of the rank, but) absolute constants. These may be termed the hypothenusal constants. The values of the first four of them are shown to be 1, $\frac{4}{3}$, $\frac{11}{18}$, $\frac{10}{81}$.

† The reader will be disappointed who seeks in Hamilton's Report any systematic deduction of the numbers which I have called after his name. He treats therein the more general question of finding the number of letters sufficient for satisfying any system of equations of given degrees by means of a certain prescribed uniform process whereby the necessity is obviated of solving any equation of a higher degree than the highest one of the given equations, and among, and mixed up with, other examples considers systems of equations of degrees 1, 2, 3; 1, 2, 3, 4;

The question arises as to whether it is possible to deduce the Hamiltonian Differences, or to deduce the Hamiltonian Numbers, directly in a continued chain from one another without the use of any intermediate numbers. Mr James Hammond has shown that it is possible, and has made the remarkable discovery that it is the Numbers of Hamilton, and not the Hypothenusal Numbers, which are subject to a very simple scale of relation. These being found, of course the Differences become known. This is contrary to what one would have expected. *À priori*, one would have anticipated that the determination of the Hypothenusal Numbers would have preceded that of their sums.

I leave Mr Hammond to give his own account of his mode of obtaining the wonderful formula of reduction, which, by a slight modification, I find, may be expressed as follows:—Using E_i to denote the $(i+1)$th Hamiltonian Number augmented by unity, so that $E_0=3$, $E_1=4$, $E_2=6$, $E_3=12$, $E_4=48,\ldots$;

1, 2, 3, 4, 5; 1, 2, 3, 4, 5, 6; for which the minimum numbers of letters required to make such process possible (when the equations are homogeneous) are 5, 11, 47, 923, respectively. Accordingly he has no occasion to employ the infinitely developable Triangle which gives unity and cohesion to the problem which deals with an indefinite number of equations of all consecutive degrees from 1 upwards. This triangle, which plays an important part in the systematic treatment of the problem, first appears in my memoir on the subject in the 100th volume of *Crelle*.

It is proper also again to notice that what I call the Numbers of Hamilton (at all events those subsequent to the number 5) are not the smallest numbers requisite for fulfilling the condition above specified. Smaller numbers will serve to satisfy that condition taken alone; but when such smaller numbers are substituted for Hamilton's the resolving equations will be less simple, inasmuch as they will contain a greater number of equations of the higher degrees than when the larger Hamiltonian numbers are employed. This distinction will be found fully explained in the memoir cited, and the smallest numbers substitutable for Hamilton's are there actually determined for r equations of degrees extending from 1 to r for all values of r up to 8 inclusive.

I have added nothing (for there is nothing to be added) to the fundamental formula of Hamilton expressed by the equation

$$[\lambda, \mu, \nu, \ldots \pi]=1+[\lambda-1, \lambda+\mu, \lambda+\mu+\nu, \ldots, \lambda+\mu+\nu+\ldots+\pi],$$

where, supposing the letters $\lambda, \mu, \nu, \ldots \pi$, to be i in number, $[\lambda, \mu, \nu, \ldots \pi]$ means the number of letters required in order that it may be possible to satisfy, according to the process employed by Hamilton (in conformity with a certain stipulation of Jerrard), a system of λ equations of degree i, μ equations of degree $i-1$, ν equations of degree $i-2$, ..., π equations of the degree 1, without solving any single equation of a degree higher than i. This formula, applied λ times successively, will have the effect of abolishing λ and causing $[\lambda, \mu, \nu, \ldots \pi]$ to depend on $[\mu', \nu', \ldots \pi']$, where $\mu', \nu', \ldots \pi'$ are connected with $\lambda, \mu, \nu, \ldots \pi$ by means of the formulæ given at the commencement of the present paper, but where instead of the letters $\lambda, \mu, \nu, \ldots$ I have used the letters $p, q, r, \ldots$.

It is presumable that the *reduced* Hamiltonian numbers would be found much less amenable to algebraical treatment than the Hamiltonian numbers proper; for numerical equalities and inequalities have to be taken account of, in determining them, which have no place in the determination of the latter numbers. Hamilton, as already stated, expressly alludes to the reduction of 11 to 10, but with that exception has avoided the general question of finding the *absolutely* lowest number of letters required in order that a system of equations (expressed in terms of those letters) of given degrees may admit of being satisfied without the necessity arising to solve any equation of a higher degree than the highest of the given ones.

and $\beta_i m$ to signify the coefficient of t^i in $(1+t)^m$; then, for any value of i greater than unity,

$$\beta_0 E_i - \beta_1 E_{i-1} + \beta_2 E_{i-2} - \beta_3 E_{i-3} + \ldots + (-)^i \beta_i E_0 = 0.$$

Or in other words, writing $\beta_0 E_i = 1$, $\beta_1 E_{i-1} = E_{i-1}$, and replacing $i-1$ by i,

$$E_i = 1 + \beta_2 E_{i-1} - \beta_3 E_{i-2} + \ldots + (-)^{i+1} \beta_{i+1} E_0$$

for all values of i greater than zero.

This is eminently a practical formula, as all the numerical calculations made use of to obtain any E are available for finding the E which follows. Dispensing with the symbol β, we may deduce all the values of E successively from those that go before by means of the *equivalence*

$$S = (1-t)^{E_0} + t(1-t)^{E_1} + t^2(1-t)^{E_2} + \ldots \equiv 1 - 2t,$$

which, by equating the powers of t on the two sides of the equivalence, gives

$$E_0 = 3,$$

$$E_1 = 1 + \frac{3 \cdot 2}{1 \cdot 2} = 4,$$

$$E_2 = 1 + \frac{4 \cdot 3}{1 \cdot 2} - \frac{3 \cdot 2 \cdot 1}{1 \cdot 2 \cdot 3} = 6,$$

$$E_3 = 1 + \frac{6 \cdot 5}{1 \cdot 2} - \frac{4 \cdot 3 \cdot 2}{1 \cdot 2 \cdot 3} + \frac{3 \cdot 2 \cdot 1 \cdot 0}{1 \cdot 2 \cdot 3 \cdot 4} = 12,$$

and so on.

I use the term equivalence and its symbol in order to convey the necessary caution that the relation indicated is not one of quantitative equality; for, although the series on the left-hand side of the symbol converges for all positive values of t less than 2, it is never equal to the expression on the right-hand side except when $t = 0$. Thus, for example, when t is unity the two terms of the equivalence are 0 and -1, and when $t = \frac{1}{2}$ they are

$$2^{-E_0} + 2^{-E_1 - 1} + 2^{-E_2 - 2} + \ldots \text{ and } 0, \text{ respectively;}$$

and for all values of t within the limits of convergence the value of the left-hand side is in excess of the value of the right-hand side of the equivalence by a finite quantity which decreases continuously as t decreases from 2 to 0, and which vanishes when $t = 0$*.

In a word, the *generating equation* is not an equation in the usual sense of the term. Conceiving each term of the series S to be expanded in ascending powers of t, and like powers of t to be placed in columns under and above

* Of the truth of the statement that the excess never changes sign, and continually decreases, I have scarcely a doubt, but it requires proof. Mr Hammond remarks that

$$(1-t)^{E_0} + t(1-t)^{E_1} + t^2(1-t)^{E_2} + \ldots + t^n(1-t)^{E_n} = (1-2t) + t^2(1-t)^{E_n - 2} F_n(t) - t^{n+1}(1-t)^{E_n - 1},$$

where $F_n(t)$ is positive for all positive values of t. Probably a proof of the point in question might be deduced from this expression, but I have not thought it necessary to investigate the matter.

each other, the double sum may be taken as a vertical sum of line-sums or as a horizontal sum of column-sums, and, although for licit values of t each sum has a finite value, the two finite values are not identical, just as a double definite integral may undergo a change of value when the order of its integrations is reversed *.

I have noticed [see above, p. 542] that the value of any Hamiltonian Difference divided by the square of the preceding one was always greater than $\frac{1}{2}$, and stated as morally certain, but "awaiting exact proof," that this ratio ultimately becomes $\frac{1}{2}$. By aid of Mr Hammond's formula for the numbers, I shall now be able to supply this proof, and at the same time to show that the ratio of a Hamiltonian *Number* to the square of its antecedent (which, of course, converges to the same asymptotic value $\frac{1}{2}$) is always *less* than that limit †.

We must in the first place prove that in the series

$$\beta_2 E_{i-1} - \beta_3 E_{i-2} + \beta_4 E_{i-3} - \beta_5 E_{i-4} + \dots$$

the absolute value of each term is greater than that of the one which follows it.

In proving this, I shall avail myself of the property of the Hypothenusal Numbers disclosed in the process of forming the triangle given at the outset of the memoir, namely, that $E_i - E_{i-1}$ is greater than $(E_{i-1} - E_{i-2})^2/2$.

Let us suppose that the law to be established holds good for a certain value of i. For the sake of brevity, I denote $E_i, E_{i-1}, E_{i-2}, E_{i-3}, \dots$ by $N, P, Q, R, \dots$.

We have then

$$P - 1 = \frac{Q(Q-1)}{2} - \frac{R(R-1)(R-2)}{2.3} + \frac{S(S-1)(S-2)(S-3)}{2.3.4} - \dots$$

$$N - 1 = \frac{P(P-1)}{2} - \frac{Q(Q-1)(Q-2)}{2.3} + \frac{R(R-1)(R-2)(R-3)}{2.3.4} - \frac{S(S-1)(S-2)(S-3)(S-4)}{2.3.4.5} + \dots.$$

* Professor Cayley has brought under my notice a not altogether dissimilar, but perhaps less striking, phenomenon, pointed out by Cauchy, that, although the series

$$1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \dots$$

is convergent, its square

$$u_0^2 + (2u_0u_1) + (2u_0u_2 + u_1^2) + \dots,$$

that is,

$$1 - \sqrt{2} + \left(\frac{2}{\sqrt{3}} + \tfrac{1}{2}\right) - \dots$$

is divergent.

† The fortunate circumstance of the two ratios in question being always respectively less and greater than the common asymptotic value of each of them enables us to find the value of the constant in the expression c^{2^x}, which is asymptotically equivalent to the half of the xth Hamiltonian or Hypothenusal Number by a method exactly analogous to that of exhaustions for finding the Archimedian constant correct to any required number of decimal places. See end of this section [p. 566, below].

If, then, the law to be proved is true for all the consecutive terms of the upper series it will obviously be true for the second series, *abstraction being made of its first term*, provided that no antecedent is less than its consequent in the series

$$\frac{Q-2}{3},\ \frac{R-3}{4},\ \frac{S-4}{5},\ \ldots,$$

which is true *à fortiori* if

$$\frac{Q}{3},\ \frac{R}{4},\ \frac{S}{5},\ \ldots$$

continually decrease, as is obviously the case, inasmuch as

$$Q,\ R,\ S,\ \ldots$$

form a descending series.

In order, then, to establish the necessary chain of induction, it only remains to show that

$$3P(P-1)-Q(Q-1)(Q-2)$$

is positive.

Now $(P-Q)-\frac{(Q-R)^2}{2}$, and *à fortiori* $P-\frac{(Q-R)^2}{2}$,

is positive for a reason previously given.

And, if in the series 3, 4, 6, 12, 48, 924, ... we make exclusion of the first three terms, we have always

$$R = \text{ or } < \frac{Q}{4},$$

and consequently $P > \frac{9Q^2}{32}$ *.

And, since under the same condition $(P-1)/(Q-1) > 4$,

$3P(P-1)-Q^2(Q-1)$, and *à fortiori* $3P(P-1)-Q(Q-1)(Q-2)$,

is positive if $12P-Q^2$ is positive, which is the case, since $P > 9Q^2/32$.

Hence, since the theorem to be proved is true for the several series

$$(1)\quad \frac{4.3}{1.2}-\frac{3.2.1}{1.2.3},$$

$$(2)\quad \frac{6.5}{1.2}-\frac{4.3.2}{1.2.3},$$

$$(3)\quad \frac{12.11}{1.2}-\frac{6.5.4}{1.2.3}+\frac{4.3.2.1}{1.2.3.4},$$

$$(4)\quad \frac{48.47}{1.2}-\frac{12.11.10}{1.2.3}+\frac{6.5.4.3}{1.2.3.4},$$

* The proof that the ratio of each term of the series 4, 6, 12, 48, 924, ... to its antecedent continually increases is too easy and too tedious to be worth setting forth in the text.

it will be true universally; for in all the succeeding series the term we have called R will be higher than the term 6 in the scale 3, 4, 6, 12, 48, ...

Hence $$P-1 = \text{ or } < \tfrac{1}{2}(Q^2-Q).$$

For the initial values of Q, P, (namely, 3, 4)

$$P-1 = \tfrac{1}{2}(Q^2-Q).$$

(When P represents any term beyond the first it is very easy to prove, but too tedious to set out the proof, that the sum of all the terms after the first in the series equated to $P-1$ will be less than -2; so that, except in the case stated, $P < \frac{1}{2}(Q^2-Q)$.)

For the series 12, 48, 924, ... we have seen that $P > 9Q^2/32$.

Hence, for the series 48, 924, ...,

$$Q > \frac{9R^2}{32} \text{ or } R < \sqrt{\frac{32Q}{9}}.$$

But $$P > \frac{Q^2-Q}{2} - \frac{R(R-1)(R-2)}{6}$$

$$> \frac{Q^2-Q}{2} - \frac{R^3}{6}.$$

Hence $$P > \frac{Q^2-Q}{2} - \frac{64\sqrt{2}}{81}Q^{\frac{3}{2}}, \text{ and } P < \frac{Q^2-Q}{2}.$$

Hence, when P, Q, are at an infinite distance from the origin,

$$\frac{P}{Q^2} = \tfrac{1}{2}.$$

Hence, also, $$\frac{P-Q}{(Q-R)^2} \text{ ultimately} = \frac{P}{Q^2} = \tfrac{1}{2},$$

which proves the theorem left over for "exact proof" in the memoir referred to.

It is convenient to deal with the halves of the *sharpened** Numbers of Hamilton, which may be called the reduced Hamiltonian Numbers, and denoted by h with a subscript, or, when required, by p, q, r, ... (the halves of P, Q, R, ... respectively).

We have then

$$2p < \frac{4q^2-2q}{2},$$

or $$p < q^2 - \frac{q}{2},$$

$$p > q^2 - \frac{q}{2} - \tfrac{128}{81}q^{\frac{3}{2}}.$$

* Numbers increased by unity may conveniently be denominated sharpened numbers, and numbers diminished by unity flattened numbers.

We may find a closer superior limit to p in terms of q as follows—

$$P-1 = \text{or} < \frac{Q^2-Q}{2} - \frac{R(R-1)(R-2)}{6} + \frac{S(S-1)(S-2)(S-3)}{24},$$

in which inequality it may be shown by inspection up to a certain point, and after that by demonstration, the tedium of writing out or reading which I spare my readers and myself, that P may be substituted for its flattened value $P-1$.

We have then $$P < \frac{Q^2-Q}{2} - \frac{R^3-3R^2}{6} + \frac{S^4}{24}.$$

Let us suppose that S, R, are not lower in the scale of the E's than 12, 48, respectively; so that P is not lower than E_6, which is 409620.

Then, as we have previously shown,

$$Q^2 < \tfrac{32}{9}P, \quad R^2 < \tfrac{32}{9}Q, \quad S^2 < \tfrac{32}{9}R.$$

Moreover, we have

$$P < \tfrac{1}{2}(Q^2-Q), \text{ whence it follows that } Q^2 > 2P + Q,$$

and, *à fortiori*, $$Q^2 > 2P.$$

Similarly $$R^2 > 2Q,$$

and $$S^2 > 2R.$$

Now $$P < \frac{Q^2-Q}{2} - \frac{R^3}{6} + \frac{R^2}{2} + \frac{S^4}{24}$$

$$< \frac{Q^2-Q}{2} - \tfrac{1}{6}(2Q)^{\frac{3}{2}} + \tfrac{1}{2}(\tfrac{32}{9}Q) + \tfrac{1}{24}(\tfrac{32}{9}R)^2$$

$$< \frac{Q^2}{2} - \frac{\sqrt{2}}{3}Q^{\frac{3}{2}} + \tfrac{23}{18}Q + \tfrac{1}{24}(\tfrac{32}{9})^3 Q,$$

that is, $$P < \tfrac{1}{2}Q^2 - \frac{\sqrt{2}}{3}Q^{\frac{3}{2}} + \tfrac{13781}{4374}Q.$$

This result, expressed in terms of the reduced numbers p, q, takes the form

$$p < q^2 - \tfrac{2}{3}q^{\frac{3}{2}} + \tfrac{13781}{4374}q,$$

and we have previously shown that

$$p > q^2 - \tfrac{128}{81}q^{\frac{3}{2}} - \frac{q}{2},$$

at all events when P is not lower in the scale than E_6.

The fraction $\frac{128}{81}$ arises from our having substituted for R^3 the inferior value $(\frac{32}{9}Q)^{\frac{3}{2}}$; but, the higher we advance P in the scale, the nearer R^2 approaches to $2Q$, and is ultimately in a ratio of equality with it. But, if we had written $(2Q)^{\frac{3}{2}}$ for R^3, the coefficient, which now stands at $-\frac{128}{81}$, would have been $-\frac{2}{3}$. In like manner, as P and Q are travelled on in the scale, R^2 and S^4 become indefinitely near to $2Q$ and $(2R)^2$, that is, $8Q$, so that the coefficient of Q in the superior limit approximates indefinitely near to

$$-\tfrac{1}{2}+1+\tfrac{1}{3}, \text{ that is, } \tfrac{5}{6},$$

and the two limits of p which have been obtained become

$$q^2-\tfrac{2}{3}q^{\frac{3}{2}}+(\tfrac{5}{6}+\epsilon)q,$$
$$q^2-(\tfrac{2}{3}+\eta)q^{\frac{3}{2}}-\tfrac{1}{2}q,$$

where ultimately ϵ and η are infinitesimals*.

Hence it follows that the ultimate value of

$$(p-q^2)\div q^{\frac{3}{2}} \text{ is } -\tfrac{2}{3},$$

that is, $$\frac{2E_i-E^2_{i-1}}{E^{\frac{3}{2}}_{i-1}}=-\sqrt{\tfrac{8}{9}} \text{ when } i=\infty.$$

Let λ, μ, ν, ... represent the halves of the Hypothenusal Numbers in the triangle given at the commencement of the paper, that is, the differences of the numbers which we have called p, q, r, ...

Since $$p=q^2-\tfrac{2}{3}q^{\frac{3}{2}} \text{ and } q=r^2-\tfrac{2}{3}r^{\frac{3}{2}},$$

$$p-q=q^2-\tfrac{2}{3}q^{\frac{3}{2}}-q, \text{ and } q-r=r^2-\tfrac{2}{3}r^{\frac{3}{2}}-r.$$

Obviously, therefore, as a first approximation when λ, μ, are very advanced terms in the hypothenuse,

$$\lambda=\mu^2.$$

Let us write $$\lambda=\mu^2+\kappa\mu^{\alpha}$$

for a second approximation.

Then $$q^2-\tfrac{2}{3}q^{\frac{3}{2}}-q=(r^2-\tfrac{2}{3}r^{\frac{3}{2}}-r)^2+\kappa(r^2-\tfrac{2}{3}r^{\frac{3}{2}}-r)^{\alpha},$$

or, *neglecting* terms of lower dimensions than r^3,

$$(r^2-\tfrac{2}{3}r^{\frac{3}{2}})^2-\tfrac{2}{3}r^3\left(1-\frac{1}{r^{\frac{1}{2}}}+\frac{1}{6r}-\dots\right)=(r^2-\tfrac{2}{3}r^{\frac{3}{2}}-r)^2+\kappa r^{2\alpha}.$$

Therefore $$-\tfrac{2}{3}r^3=-2r^3+\kappa r^{2\alpha}.$$

Consequently $$\alpha=\tfrac{3}{2} \text{ and } \kappa=\tfrac{4}{3}.$$

Thus, then, for the consecutive Hypothenusal Numbers λ, μ,

$$\lambda=\mu^2+\tfrac{4}{3}\mu^{\frac{3}{2}}+\dots.$$

Let $$\lambda=\mu^2+\tfrac{4}{3}\mu^{\frac{3}{2}}+\theta\mu,$$

or say $$\eta_{x+1}=\eta_x{}^2+\tfrac{4}{3}\eta_x{}^{\frac{3}{2}}+\rho_x\eta_x,$$

where η_x is the xth term in the series $\frac{1}{2}$, 1, 3, 18, ...

* As a matter of fact, it will be found that, as soon as q and p attain the values 6, 24, $q^2-\frac{2}{3}q^{\frac{3}{2}}$ may be taken as a superior limit. It may be noticed also, to prevent a wrong inference being drawn from the above expressions, that, as will hereafter appear, η is an infinitesimal of the order $1/q^{\frac{1}{4}}$, when q is infinite.

The successive values of ρ_x and their differences are given in the annexed Table.

x	η_x	ρ_x	$\Delta\rho_x$
1	·5	·55719096	
2	1	·66666666	+·10947570
3	3	·69059893	+·02393227
4	18	·67647909	–·01411984
5	438	·64334761	–·03313148
6	204348	·61769722	–·02565039
7	41881398318	·61139243	–·00630479
8	1754062953103547795958	·61111171	–·00028072

The decimal figures following those given in ρ_8, required for ulterior purposes, being 5795.

An examination of the column of differences for $x = 5$, 6, 7, 8, shows that the ratios of each to the rest go on decreasing somewhat faster than their squares: this makes it almost certain that $\rho_8 - \rho_9$ will be between the 400th and 500th part of ·000280, and that accordingly the value of ρ_9 will be ·6111111, &c. I believe it is beyond all moral doubt that the ultimate value of ρ is exactly $\frac{11}{18}$; and, indeed, it was the conviction I entertained of this being its true value, when I had calculated ρ_7, that led me to undertake the very considerable labour of ascertaining the 10th Hamiltonian Number in order to deduce from it the value of ρ_8. This being taken for granted*, we may proceed to ascertain a further term in the asymptotic value of η_{x+1} expressed as a function of η_x.

For, calling $\rho_x - \frac{11}{18} = \delta_x$ and $\sqrt[4]{\eta_x} = q_x$,

we have

$$\delta_6 = \cdot 00658611,$$
$$\delta_7 = \cdot 00028132,$$
$$\delta_8 = \cdot 0000006047,$$

$$\left.\begin{aligned} q_6 &= 21, \\ q_7 &= 452, \\ q_8 &= 204649, \end{aligned}\right\} \text{neglecting decimals.}$$

Thus

$$(\delta q)_6 = \cdot 1383,$$
$$(\delta q)_7 = \cdot 1272,$$
$$(\delta q)_8 = \cdot 12375.$$

The value of $(\delta q)_6 - (\delta q)_7$ being ·0111,

and of $(\delta q)_7 - (\delta q)_8$ „ ·0035,

* It is reduced to *certainty* in the supplemental 3rd section.

we may feel tolerably certain, from the Law of Squares, that $(\delta q)_8 - (\delta q)_9$ will be somewhere in the neighbourhood of the tenth part of ·0035, and accordingly that $(\delta q)_9$ is about ·1234, so that the probable value of $(\delta q)_\infty$ is ·1234 ....

Thus we have found

$$\eta_{x+1} = \eta_x^2 + \tfrac{4}{3}\eta_x^{\frac{3}{2}} + \tfrac{11}{18}\eta_x + [\quad]\eta_x^{\frac{3}{4}} + \ldots,$$

the only moral doubt being as to the degree of closeness of propinquity of the coefficient of $\eta_x^{\frac{3}{4}}$ to the decimal ·1234 ...*.

For the benefit of those who may wish to carry on the work, I give the following numerical results which have been employed in the preceding arithmetical determinations:—

$$\frac{E_8(E_8-1)}{1.2} = 61534736871945297028957640011158846858717 06$$

$$\frac{E_7(E_7-1)(E_7-2)}{1.2.3} = 979509444484142161376072006375 20$$

$$\frac{E_6(E_6-1)(E_6-2)(E_6-3)}{1.2.3.4} = 117302430235229583844 5$$

$$\frac{E_5(E_5-1)(E_5-2)(E_5-3)(E_5-4)}{1.2.3.4.5} = 5552272910184$$

$$\frac{E_4(E_4-1)(E_4-2)(E_4-3)(E_4-4)(E_4-5)}{1.2.3.4.5.6} = 12271512$$

$$\frac{E_3(E_3-1)(E_3-2)(E_3-3)(E_3-4)(E_3-5)(E_3-6)}{1.2.3.4.5.6.7} = 792$$

$$\eta_5 \div \eta_4 = 24{\cdot}33333333\ \ldots$$

$$\eta_6 \div \eta_5 = 466{\cdot}54794520\ \ldots$$

$$\eta_7 \div \eta_6 = 204951{\cdot}34925714\ \ldots$$

$$\eta_8 \div \eta_7 = 41881671184{\cdot}54776412\ \ldots$$

$$\eta_9 \div \eta_8 = 1754062953159389842293{\cdot}346657805\ \ldots$$

$$\sqrt{\eta_4} = 4{\cdot}24264068\ \ldots$$

$$\sqrt{\eta_5} = 20{\cdot}92844819\ \ldots$$

$$\sqrt{\eta_6} = 452{\cdot}04866994\ \ldots$$

$$\sqrt{\eta_7} = 204649{\cdot}45227877\ \ldots$$

$$\sqrt{\eta_8} = 41881534751{\cdot}051659567667\ \ldots$$

Finally, it is interesting to find the asymptotic values of h_x and η_x (the halves of the sharpened Hamiltonian and of the Hypothenusal Numbers), which are ultimately in a ratio of equality to each other, in terms of x.

* The exact value of the coefficient of $\eta_x^{\frac{3}{4}}$, left blank in the text, is proved in section 3 to be $\frac{10}{81}$, that is, the recurring decimal $\cdot\dot{1}2345679\dot{0}$.

Obviously each of these is ultimately in a ratio of equality with M^{2^x}, where M is a constant to be determined.

Let $$M = 10^{2^a} \text{ and } u_x = 10^{2^{x+a}}.$$

Then, for finite values of x, remembering that (in the preceding notation)

$$p < q^2 \text{ and } \lambda > \mu^2,$$

u_x must be intermediate between the corresponding terms of the two series

$$\eta = \tfrac{1}{2},\ 1,\ 3,\ 18,\ 438,\ 204348,\ 41881398318,\ \ldots,$$
$$h = 2,\ 3,\ 6,\ 24,\ 462,\ 204810,\ 41881603128,\ \ldots.$$

By means of this formula, writing for u_x corresponding values of η and h, and retaining so much of the two corresponding determinations of a as is common to both, we can find a precisely to any desired number of places of decimals, as shown in the following Table, in which 18 and 24 are taken as the terms of place zero in the respective series:

$u_x =$ 18,	438,	204348,	41881398318,
$a =$ ·32,	·401,	·4088,	·4089863 ...
$u_x =$ 24,	463,	204810,	41881603128,
$a =$ ·46,	·413,	·4090,	·4089866 ....

Hence, if we now change the origin, taking $\frac{1}{2}$ and 2 as the zero terms, we have approximately

$$M^{2^{x+3}} = 10^{2^{x+a}}$$

and $$8 \log M = 2^{\cdot 408986},$$

which gives $$M = 1{\cdot}4654433\ \ldots\text{*}.$$

As a verification, since $2^3 = 8$, $(1{\cdot}46544)^8$ should lie between 18 and 24; and, as a matter of fact, a rough calculation gives

$$(1{\cdot}46544)^2 = 2{\cdot}1473\ \ldots,$$
$$(2{\cdot}1473)^2 = 4{\cdot}608\ \ldots,$$
$$(4{\cdot}608)^2 = 21{\cdot}234\ \ldots,$$

which is about midway between the two limits.—J. J. S.

§ 2. *Proof of the Formula for the Successive Determination of each in turn of Hamilton's Numbers from its Antecedents.*

Let
$$1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + \ldots = F_0(x),$$
$$2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + 7x^6 + \ldots = F_1(x),$$
$$6x^2 + 15x^3 + 29x^4 + 49x^5 + 76x^6 + \ldots = F_2(x),$$
$$36x^3 + 210x^4 + 804x^5 + 2449x^6 + \ldots = F_3(x),$$
....

* See Note 1, p. 578 [below].

where the coefficients of the various powers of x are the numbers set out in the triangular Table at the commencement of this paper.

If, in general, we write

$$F_n(x) = a_n x^n + b_n x^{n+1} + c_n x^{n+2} + d_n x^{n+3} + \ldots,$$

the coefficients of $F_{n+1}(x)$, expressed in terms of those of $F_n(x)$, are as follows:—

$$a_{n+1} = b_n + \frac{a_n(a_n+1)}{1\,.\,2}$$

$$b_{n+1} = c_n + a_n b_n + \frac{a_n(a_n+1)(2a_n+1)}{1\,.\,2\,.\,3}$$

$$c_{n+1} = d_n + a_n c_n + \frac{a_n(a_n+1)}{1\,.\,2} b_n + \frac{a_n(a_n+1)(a_n+2)(3a_n+1)}{1\,.\,2\,.\,3\,.\,4}$$

....

Now $(1-x)^{-a_n} = 1 + a_n x + \frac{a_n(a_n+1)}{1\,.\,2} x^2 + \frac{a_n(a_n+1)(a_n+2)}{1\,.\,2\,.\,3} x^3 + \ldots,$

when multiplied by

$$F_n(x) = a_n x^n + b_n x^{n+1} + c_n x^{n+2} + d_n x^{n+3} + \ldots,$$

gives

$$\begin{aligned}(1-x)^{-a_n} F_n(x) = a_n x^n &+ b_n x^{n+1} + c_n x^{n+2} + d_n x^{n+3} + \ldots \\ &+ a_n^2 x^{n+1} + a_n b_n x^{n+2} + a_n c_n x^{n+3} + \ldots \\ &+ \frac{a_n^2(a_n+1)}{1\,.\,2} x^{n+2} + \frac{a_n(a_n+1)}{1\,.\,2} b_n x^{n+3} + \ldots \\ &+ \frac{a_n^2(a_n+1)(a_n+2)}{1\,.\,2\,.\,3} x^{n+3} + \ldots \\ &+ \ldots.\end{aligned}$$

Comparing this with

$$\begin{aligned}F_{n+1}(x) = b_n x^{n+1} &+ c_n x^{n+2} + d_n x^{n+3} + \ldots \\ &+ \frac{a_n(a_n+1)}{1\,.\,2} x^{n+1} + a_n b_n x^{n+2} + a_n c_n x^{n+3} + \ldots \\ &+ \frac{a_n(a_n+1)(2a_n+1)}{1\,.\,2\,.\,3} x^{n+2} + \frac{a_n(a_n+1)}{1\,.\,2} b_n x^{n+3} + \ldots \\ &+ \frac{a_n(a_n+1)(a_n+2)(3a_n+1)}{1\,.\,2\,.\,3\,.\,4} x^{n+3} + \ldots \\ &+ \ldots,\end{aligned}$$

we see that the difference of the two expressions is

$$a_n x^n + \frac{(a_n-1)a_n}{1\,.\,2} x^{n+1} + \frac{(a_n-1)a_n(a_n+1)}{1\,.\,2\,.\,3} x^{n+2} + \frac{(a_n-1)a_n(a_n+1)(a_n+2)}{1\,.\,2\,.\,3\,.\,4} x^{n+3} + \ldots,$$

which is equal to $\quad x^{n-1}(1-x)^{-(a_n-1)} - x^{n-1}(1-x).$

Thus $F_{n+1}(x) = (1-x)^{-a_n} F_n(x) - x^{n-1}(1-x)^{-a_n+1} + x^{n-1}(1-x)$*.

Multiplying this equation by $(1-x)^{s_{n+1}}$, where

$$s_{n+1} = a_0 + a_1 + a_2 + \ldots + a_{n-1} + a_n,$$

we obtain

$$(1-x)^{s_{n+1}} F_{n+1}(x) = (1-x)^{s_n} F_n(x) + x^{n-1}(1-x)^{s_{n+1}+1} - x^{n-1}(1-x)^{s_n+1},$$

which gives, when we write successively $n-1$, $n-2$, $n-3$, ... 0 in the place of n,

$$(1-x)^{s_n} F_n(x) = (1-x)^{s_{n-1}} F_{n-1}(x) + x^{n-2}(1-x)^{s_n+1} - x^{n-2}(1-x)^{s_{n-1}+1};$$

$$(1-x)^{s_{n-1}} F_{n-1}(x) = (1-x)^{s_{n-2}} F_{n-2}(x) + x^{n-3}(1-x)^{s_{n-1}+1} - x^{n-3}(1-x)^{s_{n-2}+1};$$

....

$$(1-x)^{s_1} F_1(x) = (1-x)^{s_0} F_0(x) + x^{-1}(1-x)^{s_1+1} - x^{-1}(1-x)^{s_0+1}.$$

Hence, by addition of these n equations, we find

$$(1-x)^{s_n} F_n(x) = (1-x)^{s_0} F_0(x) + x^{n-2}(1-x)^{s_n+1} - x^{-1}(1-x)^{s_0+1}$$
$$+ x^{n-3}(1-x)^{s_{n-1}+2} + x^{n-4}(1-x)^{s_{n-2}+2} + \ldots + x^{-1}(1-x)^{s_1+2},$$

where it has been assumed that it is possible to assign to s_0 (previously undefined) such a value as will make the last of the above n equations, namely,

$$(1-x)^{s_1} F_1(x) = (1-x)^{s_0} F_0(x) + x^{-1}(1-x)^{s_1+1} - x^{-1}(1-x)^{s_0+1},$$

identically true. That this can be done is obvious; for, if in that equation we write for $F_1(x)$, $F_0(x)$, and s_1 their values, namely,

$$F_1(x) = (1-x)^{-2} - 1, \quad F_0(x) = (1-x)^{-1}, \text{ and } s_1 = a_0 = 1,$$

then, on making $s_0 = 0$, the equation becomes

$$(1-x)^{-1} - (1-x) = (1-x)^{-1} + x^{-1}(1-x)(1-x-1).$$

Thus the general value of $F_n(x)$ is given by the equation

$$(1-x)^{s_n} F_n(x) = (1-x)^{-1} + x^{n-2}(1-x)^{s_n+1} - x^{-1}(1-x)$$
$$+ x^{n-3}(1-x)^{s_{n-1}+2} + x^{n-4}(1-x)^{s_{n-2}+2} + \ldots + x^{-1}(1-x)^{s_1+2},$$

which is equivalent to

$$(1-x)^{s_n} F_n(x) - (1-x)^{-1} + x^{-1}(1-x) - x^{n-1}(1-x)^{s_n+1}$$
$$= x^{n-2}(1-x)^{s_n+2} + x^{n-3}(1-x)^{s_{n-1}+2} + x^{n-4}(1-x)^{s_{n-2}+2} + \ldots + x^{-1}(1-x)^{s_1+2},$$

where, a_0, a_1, a_2, a_3, ... being the Hypothenusal Numbers 1, 2, 6, 36, ... we have

$$s_1 = a_0 = 1,$$
$$s_2 = a_0 + a_1 = 3,$$
$$s_3 = a_0 + a_1 + a_2 = 9,$$
$$\ldots\ldots\ldots\ldots;$$

that is, the successive values of $s_n + 2$ are the Hamiltonian Numbers 3, 5, 11, 47 ....

* See Note 2, p. 578 [below].

Now $F_n(x) = a_n x^n + \ldots$, so that the coefficient of x^n in $(1-x)^{s_n} F_n(x)$ is the same as the coefficient of x^n in $F_n(x)$, namely, a_n. Consequently, equating coefficients of x^n on each side of the equation just obtained, we find

$$a_n - 1 + (s_n + 1) = \frac{(s_n+2)(s_n+1)}{1\,.\,2} - \frac{(s_{n-1}+2)(s_{n-1}+1)\,s_{n-1}}{1\,.\,2\,.\,3}$$
$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$+ (-)^{n+1} \frac{(s_1+2)(s_1+1)\ldots(s_1+2-n)}{1\,.\,2\,.\,3\ldots(n+1)}.$$

Remembering that $a_n + s_n = s_{n+1}$,

if we call the Hamiltonian Number $s_n + 2$, H_n, the above relation may be written thus:

$$H_{n+1} - 2 = \frac{H_n(H_n-1)}{1\,.\,2} - \frac{H_{n-1}(H_{n-1}-1)(H_{n-1}-2)}{1\,.\,2\,.\,3}$$
$$+ \frac{H_{n-2}(H_{n-2}-1)(H_{n-2}-2)(H_{n-2}-3)}{1\,.\,2\,.\,3\,.\,4}$$
$$- \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$+ (-)^{n+1} \frac{H_1(H_1-1)(H_1-2)\ldots(H_1-n)}{1\,.\,2\,.\,3\ldots(n+1)}.$$

To obtain Professor Sylvester's modification of this formula given in the preceding portion of this memoir, we multiply the equation from which it was obtained by $1-x$ before proceeding to equate coefficients. Thus we have to equate coefficients of x^n on both sides of

$$(1-x)^{s_n+1} F_n(x) - 1 + x^{-1}(1-x)^2 - x^{n-1}(1-x)^{s_n+2}$$
$$= x^{n-2}(1-x)^{s_n+3} + x^{n-3}(1-x)^{s_{n-1}+3} + x^{n-4}(1-x)^{s_{n-2}+3} + \ldots + x^{-1}(1-x)^{s_1+3}.$$

Or, writing $s_n + 3 = E_n$,

we equate coefficients on both sides of

$$(1-x)^{E_n-2} F_n(x) - 1 + x^{-1}(1-x)^2 - x^{n-1}(1-x)^{E_n-1}$$
$$= x^{n-2}(1-x)^{E_n} + x^{n-3}(1-x)^{E_{n-1}} + x^{n-4}(1-x)^{E_{n-2}} + \ldots + x^{-1}(1-x)^{E_1}.$$

This equation is easily transformed into

$$(1-x)^{E_0} + x(1-x)^{E_1} + x^2(1-x)^{E_2} + \ldots + x^n(1-x)^{E_n}$$
$$= 1 - 2x + x^2(1-x)^{E_n-2} F_n(x) - x^{n+1}(1-x)^{E_n-1},$$

from which, as Professor Sylvester has pointed out in this memoir, by equating coefficients of all powers of x from 0 to n, we can obtain the successive values of E_n.

The general formula

$$1 - E_{n-1} + \frac{E_{n-2}(E_{n-2}-1)}{1\,.\,2} - \ldots + (-)^n \frac{E_0(E_0-1)\ldots(E_0-n+1)}{1\,.\,2\ldots n} = 0$$

arises from equating the coefficients of x^n.—J. H.*

* See Note 3, p. 578 [below].

§ 3. *Sequel to the Asymptotic Theory contained in* § 1.

The relation $$p = q^2 - \tfrac{2}{3}q^{\frac{3}{2}}, \text{ etc.}$$
previously obtained supplies only the two first terms of the remarkable asymptotic development

$$\frac{q^2 - p}{q} = \tfrac{2}{3}\left(q^{\frac{1}{2}} + q^{\frac{1}{4}} + q^{\frac{1}{8}} + \ldots + q^{(\frac{1}{2})^i}\right) + \Xi q,$$

where i is any assigned integer and Ξ is of a lower order of magnitude than the lowest power of q in the series which precedes it. This may be easily established as follows :—

By the scale of relation proved in the preceding section we have

$$p = q^2 - \tfrac{2}{3}r^3 + \frac{s^4}{3} + \ldots$$

$$= q^2 - \tfrac{2}{3}r^3 + \text{terms whose maximum order is that of } r^2.$$

Let, now, $$p = q^2 - \tfrac{2}{3}q^{\frac{3}{2}} - \tfrac{2}{3}hq^{\alpha} - \tfrac{2}{3}kq^{\beta} - \tfrac{2}{3}lq^{\gamma} \ldots;$$

therefore $$q = r^2 - \tfrac{2}{3}r^{\frac{3}{2}} - \tfrac{2}{3}hr^{\alpha} - \tfrac{2}{3}kr^{\beta} - \tfrac{2}{3}lr^{\gamma} \ldots$$

and $$p = q^2 - \tfrac{2}{3}r^3\left(1 - r^{-\frac{1}{2}} - hr^{\alpha-2} - kr^{\beta-2} - lr^{\gamma-2} \ldots\right) + \ldots$$
$$- \tfrac{2}{3}hr^{2\alpha} - \tfrac{2}{3}kr^{2\beta} - \tfrac{2}{3}lr^{2\gamma} \ldots$$
$$= q^2 - \tfrac{2}{3}r^3 + \tfrac{2}{3}\left(r^{\frac{5}{2}} + hr^{\alpha+1} + kr^{\beta+1} + lr^{\gamma+1} + \ldots\right) + \ldots$$
$$- \tfrac{2}{3}hr^{2\alpha} - \tfrac{2}{3}kr^{2\beta} - \tfrac{2}{3}lr^{2\gamma} - \ldots\ldots.$$

Therefore $$h = 1, \quad k = 1, \quad l = 1, \quad m = 1, \ldots$$
$$2\alpha = \tfrac{5}{2}, \quad 2\beta = 1 + \alpha, \quad 2\gamma = 1 + \beta, \quad 2\delta = 1 + \gamma, \ldots$$

that is, $$\alpha = \tfrac{5}{4}, \quad \beta = \tfrac{9}{8}, \quad \gamma = \tfrac{17}{16}, \quad \delta = \tfrac{33}{32}, \ldots$$

and thus $$p = q^2 - \tfrac{2}{3}q\left(q^{\frac{1}{2}} + q^{\frac{1}{4}} + q^{\frac{1}{8}} + q^{\frac{1}{16}} + \ldots + q^{(\frac{1}{2})^i}\right) + \Xi q,$$

as was to be shown*.

* This theorem may be rigorously demonstrated, and reduced to a more precise analytical form, as follows :—

For the sake of brevity, we may call $-p/q + q$ the relative deficiency of p, and denote it by Δ.

First it may be noticed that, if in the equation

$$F(q) = \Sigma_0^\infty \left(q^{2^{-s}} - q^{-2^{-s}}\right)$$

we write $\log q = k$,

$$F(q) = 2\left(k + \frac{k^3}{1.2.3.7} + \frac{k^5}{1.2.3.4.5.31} + \frac{k^7}{1.2.3.4.5.6.7.127} + \ldots\right),$$

which is always convergent.

Moreover, the value of $F(q)$ may be calculated for any given value of q within close limits. For, if we call U the right-hand branch of the series in q, beginning with $z - z^{-1}$, the terms of U will easily be seen to lie between those of two geometrical series of which $z - z^{-1}$ is the first term, and of one of which $\frac{1}{2}$, and of the other $(z^{\frac{1}{2}} + z^{-\frac{1}{2}})^{-1}$, is the common ratio.

Hence U is intermediate between $2(z^2 - 1)/z$ and $(z^2 - 1)(z + 1)/z(z - z^{\frac{1}{2}} + 1)$.

It is interesting to notice that the formula *apparently* remains arithmetically true for *finite* values of p and q, provided that q is not less than

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The difference between these limits, it may be parenthetically observed, is

$$(z-z^{-1})\frac{(z^{\frac{1}{4}}-z^{-\frac{1}{4}})^2}{z^{\frac{1}{2}}-1+z^{-\frac{1}{2}}},$$

which, when z is nearly unity (the limit to which $q^{(\frac{1}{2})^s}$ converges), is nearly equal to $\frac{1}{16}(z-z^{-1})^3$; that is, if $z=1+\tau$, the difference between the limits (for τ small) is very near to $\tau^3/2$.

Now on p. 575 (*post*) it is shown that $\surd q-r+s-\frac{2}{3}\surd r=\epsilon$, and that, when the rank of q is taken indefinitely great, ϵ converges to $\frac{1}{4}$. Hence ϵ always lies between finite limits.

For, in general, x being any one of a series of increasing numbers, and $\psi(x)$ a function of x which is always finite for finite values of x, but ultimately converges to c, by taking for x a value of L sufficiently great, we make the series of terms for $x>L$ intermediate between $c+\delta$ and $c-\delta$, where δ is any assigned positive quantity; and consequently, if μ, ν, are the greatest and least values of $\psi(x)$ when x does not exceed L, the greater of the two values, $c+\delta$, μ, and the lesser of the two, $c-\delta$, ν, will be superior and inferior limits to the value of $\psi(x)$ for all values of x.

Hence, writing

$$\begin{aligned}
&\surd(p)-q+r-\tfrac{2}{3}\surd(q)=\epsilon_1,\\
&\surd(q)-r+s-\tfrac{2}{3}\surd(r)=\epsilon_2,\\
&\surd(r)-s+t-\tfrac{2}{3}\surd(s)=\epsilon_3,\\
&\dots\dots\dots\dots\dots\dots\dots\dots\\
&\surd(6)-3+2-\tfrac{2}{3}\surd(3)=\epsilon_{x-1},
\end{aligned}$$

we obtain, by summation,

$$\surd(p)-q+\tfrac{1}{3}\{\surd(q)+\surd(r)+\surd(s)+\dots+\surd(6)\}=\Sigma\epsilon-2+\tfrac{2}{3}\surd(3),$$

and, consequently, $\surd(p)-q+\frac{1}{3}\{\surd(q)+\surd(r)+\surd(s)+\dots+\surd(6)\}=\rho x$,

where ρ is always a finite quantity lying between determinable limits. But again (p. 573)—

$$p=\{q-\theta\surd(q)\}^2,$$

where θ (whose ultimate value is $\frac{1}{3}$) is always a proper fraction. Hence

$$\frac{q^2-p}{q}=2\{q-\surd(p)\}-\theta^2.$$

Hence, from what has been shown above,

$$\frac{q^2-p}{q}=\tfrac{2}{3}\{\surd(q)+\surd(r)+\surd(s)+\dots+\surd(6)\}-2\rho' x.$$

In this equation we may write

$$\left.\begin{aligned}
\surd(r)&=q^{\frac{1}{4}}+k_1\\
\surd(s)&=q^{\frac{1}{8}}+k_2\\
\surd(t)&=q^{\frac{1}{16}}+k_3\\
&\dots\dots\dots\dots
\end{aligned}\right\}(x-3 \text{ equations}),$$

where k_1, k_2, k_3, ... are all of them finite (and, as a matter of fact, of no consequence for our immediate object, positive proper fractions). For, ultimately,

$$k_1=\surd(r)-q^{\frac{1}{4}}=\frac{r-q^{\frac{1}{2}}}{r^{\frac{1}{2}}+q^{\frac{1}{4}}}=\frac{\theta_1 r^{\frac{1}{2}}}{r^{\frac{1}{2}}+q^{\frac{1}{4}}}=\tfrac{1}{2}\theta_1=\tfrac{1}{6}\ \text{(see p. 574)},$$

and consequently the finiteness of each k is a direct inference from the general principle previously applied in the case of the ϵ's.

Applying this result to the equation previously given, it follows that

$$\begin{aligned}
&q^{\frac{1}{2}}+q^{\frac{1}{4}}+\dots+q^{(\frac{1}{2})^{x-2}}=\tfrac{3}{2}\Delta-vx \text{ (where } v \text{ is finite)}\\
&=F(q)+(q^{-\frac{1}{2}}+q^{-\frac{1}{4}}+\dots+q^{-(\frac{1}{2})^{x-2}})-\{(z-z^{-1})+(z^{\frac{1}{2}}-z^{-\frac{1}{2}})+(z^{\frac{1}{4}}-z^{-\frac{1}{4}})+\dots\},
\end{aligned}$$

where z lies between 1 and 2.

The series of negative powers of q is obviously less than x, and the z-series, which follows it, is less than the finite quantity $2(z-1/z)$, that is, $<2(2-\frac{1}{2})$. Hence $\frac{3}{2}\Delta=F(q)+\Theta x$, where Θ is

24, when we replace each term in the formula by its integer portion, and in the series on the right stop at the term immediately preceding the first term for which

$$Eq^{(\frac{1}{2})^i} = 1.$$

Thus, when $\qquad p = 462$ and $q = 24$,

we have
$$E\left(\frac{q^2 - p}{q}\right) = E\left(\frac{576 - 462}{24}\right) = E\left(\frac{114}{24}\right) = 4,$$

and
$$E\left\{\tfrac{2}{3}\left(Eq^{\frac{1}{2}} + Eq^{\frac{1}{4}}\right)\right\} = E\left\{\tfrac{2}{3}(4 + 2)\right\} = 4.$$

So also, when $\qquad p = 41881603128, \quad q = 204810,$

$$E\left(\frac{q^2 - p}{q}\right) = 319,$$

and $E\left\{\tfrac{2}{3}\left(Eq^{\frac{1}{2}} + Eq^{\frac{1}{4}} + Eq^{\frac{1}{4}} + Eq^{\frac{1}{8}}\right)\right\} = E\left\{\tfrac{2}{3}(452 + 21 + 4 + 2)\right\} = E\left(\tfrac{958}{3}\right) = 319.$

But, if we had included the term $Eq^{\frac{1}{16}}$, the result would have been

$$E\left\{\tfrac{2}{3}(452 + 21 + 4 + 2 + 1)\right\} = 320.$$

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a number lying between fixed limits, and x, the rank of q, is of the same order of magnitude as $\log\log q$. This equation contains as a consequence the asymptotic theorem to be proved; for, using i to denote any positive integer,

$$\tfrac{3}{2}\Delta - \Sigma_1{}^i q^{(\frac{1}{2})^i} = F(q) - \Sigma_1{}^i q^{(\frac{1}{2})^i} - \Theta x = q^{(\frac{1}{2})^{i+1}} + \sum_{s=i+2}^{s=\infty} \left(q^{(\frac{1}{2})^s} - q^{-(\frac{1}{2})^s}\right) - \Sigma_0{}^i 1/q^{(\frac{1}{2})^{i+1}} - \Theta x.$$

Hence, remembering that x is of the same order of magnitude as $\log\log q$, and that

$$\sum_{s=i+2}^{s=\infty} \left(q^{(\frac{1}{2})^s} - q^{-(\frac{1}{2})^s}\right) < 2\left(q^{(\frac{1}{2})^{i+2}} - q^{-(\frac{1}{2})^{i+2}}\right),$$

which is of a lower order of magnitude than $q^{(\frac{1}{2})^{i+1}}$, it follows that $\frac{3}{2}\Delta - \Sigma_1{}^i q^{(\frac{1}{2})^i}$ for all values of i is ultimately in a ratio of equality with $q^{(\frac{1}{2})^{i+1}}$, which is the theorem to be proved.

We have thought it desirable to obtain the formula $\frac{3}{2}\Delta = Fq + \Theta x$ for its own sake, but, so far as regards the proof in question, that might be obtained more expeditiously from the expression given for $3\Delta/2 - \upsilon x$ without introducing the series Fq.

It is easy to ascertain the ultimate value to which Θ converges. In the first place, the series of fractions $1/q^{\frac{1}{2}} + 1/q^{\frac{1}{4}} + 1/q^{\frac{1}{8}} + \ldots$ to $x - 2$ terms (where x is the rank of q) may be shown to be always finite, and consequently, when divided by x, converges to zero.

For we know that $(p - q) > (q - r)^2 > (r - s)^4 \ldots > (6 - 3)^{2^{x-2}}$. Hence the last term of the series $q^{\frac{1}{2}}, q^{\frac{1}{4}}, q^{\frac{1}{8}} \ldots$ (namely, $q^{(\frac{1}{2})^{x-2}}$) > 3. Hence the finite series $1/q^{\frac{1}{2}} + 1/q^{\frac{1}{4}} + 1/q^{\frac{1}{8}} + \ldots$ for a double *à fortiori* reason is less than the infinite geometrical series $\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \ldots < \frac{1}{2}$.

In fact, from § 1 (p. 566) it may easily be shown that the last term of the series

$$q^{\frac{1}{2}}, q^{\frac{1}{4}}, q^{\frac{1}{8}} \ldots > M^4 > (1{\cdot}465)^4 > 4{\cdot}608,$$

so that the sum is really less than $\dfrac{1}{3{\cdot}608}$.

Hence, retracing the steps by which Θ has been obtained, and observing that ρ' differs from ρ by a finite multiple of $1/x$, we have ultimately $\Theta = \upsilon = k - 3\rho' = k - 3\rho = k - 3\epsilon = \frac{1}{6} - \frac{3}{4} = -\frac{7}{12}$. If, then (using u_x to denote the half of the sharpened xth Hamiltonian number), we write $u_x - 1/u_x = v_x$, and understand by $G(t - 1/t)$ the infinite series

$$\left(t^{\frac{1}{2}} - t^{-\frac{1}{2}}\right) + \left(t^{\frac{1}{4}} - t^{-\frac{1}{4}}\right) + \left(t^{\frac{1}{8}} - t^{-\frac{1}{8}}\right) + \ldots,$$

it is easily seen that the *principal* part of $\sqrt{(v_{x+1})}$, regarded as a function of v_x and x, is

$$v_x - \tfrac{1}{3}Gv_x + \tfrac{7}{36}x.$$

Again, when

$$p = 307673684354828937922426140463753876021 6584,$$
$$q = 175406295314542939 9086,$$

$$E\left(\frac{q^2 - p}{q}\right) = 27921159919,$$

and $E\left\{\frac{2}{3}\left(Eq^{\frac{1}{2}} + Eq^{\frac{1}{4}} + Eq^{\frac{1}{8}} + Eq^{\frac{1}{16}} + Eq^{\frac{1}{32}} + Eq^{\frac{1}{64}}\right)\right\}$

$$= E\left\{\tfrac{2}{3}(41881534751 + 204649 + 452 + 21 + 4 + 2)\right\} = 27921159919^*.$$

We will now proceed to consider afresh the asymptotic development of any *Hypothenusal* Number $p - q$ in terms of its antecedent $q - r$, and to reduce to apodictic certainty results which in the first section were partly obtained by observation. It has already been shown in that section that

$$p > q^2 - \tfrac{128}{81} q^{\frac{3}{2}} - \frac{q}{2}$$

when p is not lower than 204810 in the scale 2, 3, 6, 24, 462, 204810, ..., that is, when q is not less than 462.

Hence $$p > q^2 - 2q^{\frac{3}{2}} + q + \left(\tfrac{34}{81}q^{\frac{3}{2}} - \tfrac{3}{2}q\right),$$
or, since $\frac{34}{81}q^{\frac{3}{2}} - \frac{3}{2}q$ is a positive quantity,

$$p > (q - \sqrt{q})^2,$$

at all events when $q =$ or > 462.

It will be found also on trial that this formula remains true for all the values of q inferior to 462.

Thus
$$462 > (24 - \sqrt{24})^2,$$
$$24 > (6 - \sqrt{6})^2,$$
$$6 > (3 - \sqrt{3})^2,$$
$$3 > (2 - \sqrt{2})^2.$$

Hence, universally, $p > (q - \sqrt{q})^2$†.

But we know that $p < q^2$.

We may therefore write $p = (q - \theta\sqrt{q})^2$,
where θ is some quantity between 0 and 1.

Similarly,
$$q = (r - \theta_1\sqrt{r})^2,$$
$$r = (s - \theta_2\sqrt{s})^2,$$
$$\dots\dots\dots\dots\dots\dots$$

where θ_1, θ_2, ... are also positive fractions.

* The authors must be understood merely to affirm the *possibility* of the theorem being true, and to offer no opinion on the strength of the presumption raised that it is so.

† Had this inequality been true only for values of q sufficiently great, it would have been enough for the purposes of the text.

When p and q become infinite,

$$\frac{q^2-p}{q^{\frac{3}{2}}}=\tfrac{2}{3}=2\theta.$$

Hence the ultimate value of θ is $\frac{1}{3}$. Similarly, θ_1, θ_2, ... all of them converge to the value $\frac{1}{3}$.

This agrees with the result previously demonstrated (p. 563), and is the starting point of all that follows.

We know that letters $p, q, r, s, \ldots$, being used to denote the halves of the augmented Hamiltonian Numbers, they are connected by the scale of relation

$$p=\tfrac{1}{2}+\frac{q\,(2q-1)}{2}-\frac{r\,(2r-1)\,(2r-2)}{2\,.\,3}+S-T,$$

where
$$S=\frac{s\,(2s-1)\,(2s-2)\,(2s-3)}{2\,.\,3\,.\,4}$$

and T stands for the remaining terms, involving

$$t,\ u,\ v,\ \ldots.$$

Considering $\qquad q,\ r,\ s,\ t,\ \ldots$

to be of the order $\qquad 1,\ \frac{1}{2},\ \frac{1}{4},\ \frac{1}{8},\ \ldots,$

we may reject the term $\frac{1}{2}$, which is of zero order, and write

$$p=q^2-\tfrac{2}{3}r^3\,;\ \ -\frac{q}{2}+r^2-\frac{r}{3}+S-T.$$

Hence, rejecting terms of order less than $\frac{3}{2}$ (which have, however, to be retained in obtaining the subsequent approximations),

$$\left.\begin{array}{r}(p-q)\\-(q-r)^2\end{array}\right\}=\left\{\begin{array}{ll}q^2-\tfrac{2}{3}r^3\,; & -\tfrac{3}{2}q+r^2-\dfrac{r}{3}+S-T\\-q^2+2qr\,; & -r^2\end{array}\right.$$
$$=(2qr-\tfrac{2}{3}r^3)\,;$$

that is,
$$(p-q)-(q-r)^2=\tfrac{4}{3}q^{\frac{3}{2}}$$

when q is infinite.

Again, writing for S its expanded value, namely,

$$\frac{s^4}{3}-s^3+\tfrac{11}{12}s^2-\frac{s}{4},$$

we have

$$\left.\begin{array}{r}(p-q)\\-\ \ (q-r)^2\\-\tfrac{4}{3}\,(q-r)^{\frac{3}{2}}\end{array}\right\}=\left\{\begin{array}{lll}2qr-\tfrac{2}{3}r^3-\tfrac{4}{3}q^{\frac{3}{2}} & \text{Order} & \tfrac{3}{2}\,;\\+\,2q^{\frac{1}{2}}r-\tfrac{3}{2}q+\tfrac{1}{3}s^4 & \ \ \ \text{,,} & 1\,;\\-\tfrac{1}{2}q^{-\frac{1}{2}}r^2-\dfrac{r}{3}-s^3+\tfrac{11}{12}s^2-\tfrac{1}{4}s-T & \ \ \ \text{,,} & <1,\end{array}\right.$$

rejecting the terms $q^{-\frac{3}{2}}r^3$, $q^{-\frac{5}{2}}r^4$, ... in the expansion of $(q-r)^{\frac{3}{2}}$ because the order of none of them is superior to zero.

We now write $$q = (r - \theta_1 \sqrt{r})^2,$$
so that
$$2qr - \tfrac{2}{3}r^3 - \tfrac{4}{3}q^{\frac{3}{2}} = (2r^3 - 4\theta_1 r^{\frac{5}{2}} + 2\theta_1^2 r^2) - \tfrac{2}{3}r^3 - (\tfrac{4}{3}r^3 - 4\theta_1 r^{\frac{5}{2}} + 4\theta_1^2 r^2 - \tfrac{4}{3}\theta_1^3 r^{\frac{3}{2}})$$
$$= -2\theta_1^2 r^2 + \tfrac{4}{3}\theta_1^3 r^{\frac{3}{2}}.$$

Hence
$$\left.\begin{array}{r} (p-q) \\ -\ (q-r)^2 \\ -\tfrac{4}{3}(q-r)^{\frac{3}{2}} \end{array}\right\} = \begin{cases} -2\theta_1^2 r^2 + 2q^{\frac{1}{2}} r - \tfrac{3}{2}q + \tfrac{1}{3}s^4 & \text{Order } 1; \\ +\tfrac{4}{3}\theta_1^3 r^{\frac{3}{2}} - s^3 & \quad ,, \quad \tfrac{3}{4}; \\ -\tfrac{1}{2}q^{-\frac{1}{2}} r^2 - \dfrac{r}{3} + \tfrac{11}{12}s^2 - \tfrac{1}{4}s - T & \quad ,, \quad < \tfrac{3}{4}. \end{cases}$$

Since $$q = r^2 = s^4 \text{ (ultimately)},$$
the terms of Order 1 (which are the only ones with which we have to do at present) are ultimately equal to
$$(-2\theta_1^2 + 2 - \tfrac{3}{2} + \tfrac{1}{3})\,q\,;$$
or, giving θ_1 its ultimate value $\frac{1}{3}$, to $\frac{11}{18}q$, or to the same order of approximation to $\frac{11}{18}(q-r)$.

Hence, ultimately,
$$(p-q) = (q-r)^2 + \tfrac{4}{3}(q-r)^{\frac{3}{2}} + \tfrac{11}{18}(q-r)\text{*}.$$

We use this result to obtain a closer approximation to $\sqrt{q}$ than $r - \theta_1\sqrt{r}$, and to find the relation between the general values of θ_1 and θ_2.

Thus, assuming $$\sqrt{(q-r)} = r - s + \tfrac{2}{3}\sqrt{(r-s)} + k,$$
we have, ultimately,
$$q - r = (r-s)^2 + \tfrac{4}{3}(r-s)^{\frac{3}{2}} + (\tfrac{4}{9} + 2k)(r-s)$$
$$= (r-s)^2 + \tfrac{4}{3}(r-s)^{\frac{3}{2}} + \tfrac{11}{18}(r-s).$$

Consequently, as r becomes indefinitely great, k converges to the value $\frac{1}{2}(\frac{11}{18} - \frac{4}{9}) = \frac{1}{12}$.

Now $$\sqrt{(q-r)} = \sqrt{(q)} - \tfrac{1}{2}\frac{r}{\sqrt{q}} \ldots = \sqrt{(q)} - \tfrac{1}{2} \text{ ultimately};$$
and similarly $$\sqrt{(r-s)} = \sqrt{(r)} - \tfrac{1}{2} \text{ ultimately}.$$

Hence, ultimately,
$$\sqrt{(q)} = r - s + \tfrac{2}{3}\sqrt{(r)} + \tfrac{1}{12} + \tfrac{1}{2} - \tfrac{1}{3} = r - s + \tfrac{2}{3}\sqrt{(r)} + \tfrac{1}{4}.$$

We may therefore write
$$\sqrt{(q)} = r - s + \tfrac{2}{3}\sqrt{(r)} + \epsilon \text{ (where ultimately } \epsilon = \tfrac{1}{4}).$$

But $$\sqrt{(q)} = r - \theta_1\sqrt{r},$$
and therefore $$\theta_1\sqrt{(r)} = s - \tfrac{2}{3}\sqrt{(r)} - \epsilon.$$

* As previously obtained by observation in § 1 (p. 563). It will, of course, be understood that in the above and similar passages the sign = is to be interpreted to mean "is in a ratio of equality with."

Moreover $\sqrt{(r)} = s - \theta_2 \sqrt{s}$,

whence it follows that

$$\theta_1 \sqrt{(r)} = \tfrac{1}{3}s + \tfrac{2}{3}\theta_2 \sqrt{(s)} - \epsilon \text{ (where } \epsilon = \tfrac{1}{4} \text{ ultimately).}$$

Resuming the development of $(p-q)$ in terms of $(q-r)$, we have

$$\left.\begin{array}{r} (p-q) \\ -\quad (q-r)^2 \\ -\ \tfrac{4}{3}(q-r)^{\frac{3}{2}} \\ -\tfrac{11}{18}(q-r) \end{array}\right\} = \left\{\begin{array}{lr} -2\theta_1 r^2 + 2q^{\frac{1}{2}}r - \tfrac{19}{9}q + \dfrac{s^4}{3} & \text{Order } 1, \\ +\tfrac{4}{3}\theta_1{}^3 r^{\frac{3}{2}} - s^3 & ,,\quad \tfrac{3}{4}, \\ & \\ -\tfrac{1}{2}q^{-\frac{1}{2}}r^2 + \tfrac{5}{18}r + \tfrac{11}{12}s^2 - \dfrac{s}{4} - T & ,,\quad < \tfrac{3}{4}. \end{array}\right.$$

The terms of order inferior to $\frac{3}{4}$ are of no value for present purposes, and are only retained for the benefit of those who may wish to carry on the work.

To reduce the terms of Order 1, we write, in succession,

$$q = (r - \theta_1 \sqrt{r})^2,$$
$$\theta_1 \sqrt{(r)} = \tfrac{1}{3}s + \tfrac{2}{3}\theta_2 \sqrt{(s)} - \epsilon,$$
$$r = (s - \theta_2 \sqrt{s})^2.$$

Thus $\dfrac{s^4}{3} - 2\theta_1{}^2 r^2 + 2q^{\frac{1}{2}}r - \tfrac{19}{9}q$

$$= \frac{s^4}{3} - 2\theta_1{}^2 r^2 + 2r^2 - \tfrac{19}{9}r^2;\ -2\theta_1 r^{\frac{3}{2}} + \tfrac{38}{9}\theta_1 r^{\frac{3}{2}};\ -\tfrac{19}{9}\theta_1{}^2 r$$

$$= \frac{s^4}{3} - \frac{r^2}{9} - 2\theta_1{}^2 r^2;\ +\tfrac{20}{9}\theta_1 r^{\frac{3}{2}};\ -\tfrac{19}{9}\theta_1{}^2 r$$

$$= \frac{s^4}{3} - \frac{r^2}{9} - 2r\left(\frac{s}{3} + \tfrac{2}{3}\theta_2\sqrt{s}\right)^2;\ +4\epsilon r\left(\frac{s}{3} + \tfrac{2}{3}\theta_2\sqrt{s}\right) + \tfrac{20}{9}\theta_1 r^{\frac{3}{2}};\ -2\epsilon^2 r - \tfrac{19}{9}\theta_1{}^2 r$$

$$= \frac{s^4}{3} - \tfrac{1}{9}(s^4 - 4\theta_2 s^{\frac{7}{2}} + 6\theta_2{}^2 s^3 - 4\theta_2{}^3 s^{\frac{5}{2}} + \theta_2{}^4 s^2) - \tfrac{2}{9}s^2(s^2 + 4\theta_2 s^{\frac{3}{2}} + 4\theta_2{}^2 s)$$
$$+\tfrac{4}{9}\theta_2 s^{\frac{3}{2}}(s^2 + 4\theta_2 s^{\frac{3}{2}} + 4\theta_2{}^2 s) - \tfrac{2}{9}\theta_2{}^2 s(s^2 + 4\theta_2 s^{\frac{3}{2}} + 4\theta_2{}^2 s);$$
$$+\tfrac{4}{3}\epsilon rs + \tfrac{20}{9}\theta_1 r^{\frac{3}{2}};\ +\tfrac{8}{3}\epsilon\theta_2 r\sqrt{(s)} - 2\epsilon^2 r - \tfrac{19}{9}\theta_1{}^2 r$$
$$= \tfrac{4}{3}\epsilon rs + \tfrac{20}{9}\theta_1 r^{\frac{3}{2}} \qquad \text{Order } \tfrac{3}{4},$$
$$+\tfrac{4}{3}\theta_2{}^3 s^{\frac{5}{2}} + \tfrac{8}{3}\epsilon\theta_2 r\sqrt{(s)} - \theta_2{}^4 s^2 - 2\epsilon^2 r - \tfrac{19}{9}\theta_1{}^2 r \qquad ,,\ < \tfrac{3}{4}.$$

Hence

$$\left.\begin{array}{r} (p-q) \\ -\quad (q-r)^2 \\ -\ \tfrac{4}{3}(q-r)^{\frac{3}{2}} \\ -\tfrac{11}{18}(q-r) \end{array}\right\} = \left\{\begin{array}{lr} \tfrac{4}{3}\theta_1{}^3 r^{\frac{3}{2}} - s^3 + \tfrac{4}{3}\epsilon rs + \tfrac{20}{9}\theta_1 r^{\frac{3}{2}} & \text{Order } \tfrac{3}{4}, \\ & \\ -T + \tfrac{4}{3}\theta_2{}^3 s^{\frac{5}{2}} + \tfrac{8}{3}\epsilon\theta_2 r s^{\frac{1}{2}} & ,,\quad \tfrac{5}{8}, \\ -\tfrac{1}{2}q^{-\frac{1}{2}}r^2 + \tfrac{5}{18}r + \tfrac{11}{12}s^2 - \dfrac{s}{4} - \theta_2{}^4 s^2 - 2\epsilon^2 r - \tfrac{19}{9}\theta_1{}^2 r & ,,\quad < \tfrac{5}{8}. \end{array}\right.$$

Here the terms of Order $\frac{3}{4}$ are ultimately equal to

$$\left(\tfrac{4}{3}\theta_1^3 - 1 + \tfrac{4}{3}\epsilon + \tfrac{20}{9}\theta_1\right) q^{\frac{3}{4}},$$

which, when θ_1 and ϵ receive their ultimate values, $\frac{1}{3}$ and $\frac{1}{4}$, becomes

$$\left(\tfrac{4}{81} - 1 + \tfrac{1}{3} + \tfrac{20}{27}\right) q^{\frac{3}{4}} = \tfrac{10}{81} q^{\frac{3}{4}} *.$$

From this it follows immediately that (rejecting terms of an order of magnitude inferior to that of $(q-r)^{\frac{3}{4}}$)

$$p - q = (q-r)^2 + \tfrac{4}{3}(q-r)^{\frac{3}{2}} + \tfrac{11}{18}(q-r) + \tfrac{10}{81}(q-r)^{\frac{3}{4}}.$$

The law of the indices in the complete development is easily deduced from the relation

$$p = \tfrac{1}{2} + \frac{q(2q-1)}{2} - \frac{r(2r-1)(2r-2)}{2.3} + \frac{s(2s-1)(2s-2)(2s-3)}{2.3.4} - \ldots.$$

The terms carrying the arguments

$$q^2,\ q,\ r^3,\ r^2,\ r,\ s^4,\ s^3,\ s^2,\ s,\ t^5, \ldots$$

furnish the indices $\quad 2,\ 1,\ \frac{3}{2},\ 1,\ \frac{1}{2},\ 1,\ \frac{3}{4},\ \frac{1}{2},\ \frac{1}{4},\ \frac{5}{8}, \ldots,$

which, arranged in order of magnitude, become

$$2,\ \tfrac{3}{2},\ 1,\ \tfrac{3}{4},\ \tfrac{5}{8},\ \tfrac{1}{2},\ \tfrac{3}{8},\ \tfrac{5}{16},\ \tfrac{1}{4}, \ldots.$$

Thus, calling $p-q$ and $q-r$ y and x respectively, the expansion for y in terms of x will be of the form

$$y = \Sigma A x^{\frac{2m+1}{2^n}},$$

where n has all values from 0 to ∞, and $2m+1$ does not exceed $n+2$, that is, m has all positive values from 0 to $n/2$ or $\frac{1}{2}(n+1)$, according as n is even or odd.

But, besides this expressed portion of the development of a Hypothenusal Number, say η_{x+1}, as a function of its antecedent, η_x, there will be another portion, consisting of terms with zero and negative indices of η_x having functions of x for their coefficients, which observation is incompetent to reveal, and with the nature of which we are at present unacquainted. The study of Hamilton's Numbers, far from being exhausted, has, in leaving our hands, little more than reached its first stage, and it is believed will furnish a plentiful aftermath to those who may feel hereafter inclined to pursue to the end the thorny path we have here contented ourselves with indicating, which lies so remote from the beaten track of research, and offers an example and suggestion of infinite series (as far as we are aware) wholly unlike any which have previously engaged the attention of mathematicians.

J. J. S. and J. H.

* Agreeing closely with what had been previously found by observation in § 1 (p. 563).

NOTE 1, p. 566.

It is easy to see that, if δM and $\delta\alpha$ are corresponding errors in the values of M and α respectively,

$$\delta M = (M \log_e M \log_e 2)\, \delta\alpha = (\cdot 38822 \ldots)\, \delta\alpha$$

(since $M = 1{\cdot}46544 \ldots$, $\log_e M = \cdot 38220 \ldots$, and $\log_e 2 = \cdot 69314 \ldots$).

Hence, $\delta\alpha$ being intermediate between ·0000003 and ·0000006,

δM lies between ·000000116 and ·000000233.

The value of M (the base of the Hamiltonian Numbers) is thus found to be 1·465443 ..., correct to the last figure inclusive.—J. J. S.

NOTE 2, p. 568.

This equation may be obtained more simply from the *fundamental* formula of Hamilton (middle of above note). It follows from the law of derivation there given that, if we write ${}^1F_n = (1-x)^{-1} F_n - x^n$, and, in general, ${}^{j+1}F_n = (1-x)^{-1}\, {}^jF_n - x^n$, then $F_{n+1} = {}^{a_n}F_n$; and, consequently,

$$F_{n+1} - (1-x)^{-a_n} F_n = -x^n \{1 + (1-x)^{-1} + (1-x)^{-2} + \ldots + (1-x)^{-a_n+1}\}$$

$$= x^{n-1} \{(1-x) - (1-x)^{-a_n+1}\}.\text{—J. J. S.}$$

NOTE 3, p. 569.

It is curious to notice the sort of affinity which exists between a form of writing the scale of relation for Bernoulli's Numbers and that given at p. 569 for Hamilton's.

If we write

$G_0 = 1,\ G_1 = -1,\ G_2 = (-4) B_1,\ G_3 = 0,\ G_4 = (-4)^2 B_2,\ G_5 = 0,\ G_6 = (-4)^3 B_3, \ldots$

then, using β_κ in the same sense as at p. 558, we shall find the scale of relation between the B's (Bernoulli's Numbers) is given by the equation

$$\sum_{\kappa=0}^{\kappa=i} (-)^\kappa \beta_\kappa i \,.\, G_{i-\kappa} = 0, \textit{ provided } i \textit{ is odd.}$$

On striking out the i which intervenes between β_κ and $G_{i-\kappa}$, so as to make the former operate on the latter, the equation becomes that given at p. 569 for the E's, the sharpened numbers of Hamilton.—J. J. S.

§ 4. *Continuation, to an infinite number of terms, of the Asymptotic Development for Hypothenusal Numbers.*

"This was sometime a paradox, but now the time gives it proof."
Hamlet, Act III., Scene 1.

In the third section of this paper [above, p. 575] it was stated, on what is now seen to be insufficient evidence, that the asymptotic development of $p-q$, the half of any Hypothenusal Number, could be expressed as a series of powers of $q-r$, the half of its antecedent, in which the indices followed the sequence

$$2, \tfrac{3}{2}, 1, \tfrac{3}{4}, \tfrac{5}{8}, \tfrac{1}{2}, \ldots$$

It was there shown that, when quantities of an order of magnitude inferior to that of $(q-r)^{\frac{3}{4}}$ are neglected,

$$p-q=(q-r)^2+\tfrac{4}{3}(q-r)^{\frac{3}{2}}+\tfrac{11}{18}(q-r)+\tfrac{10}{81}(q-r)^{\frac{3}{4}};$$

but, on attempting to carry this development further, it was found that, though the next term came out $\frac{88}{1215}(q-r)^{\frac{5}{8}}$, there was an infinite series of terms interposed between this one and $(q-r)^{\frac{1}{2}}$, namely, as proved in the present section, between $(q-r)^{\frac{3}{4}}$ and $(q-r)^{\frac{1}{2}}$ there lies an infinite series of terms whose indices are

$$\tfrac{5}{8}, \tfrac{9}{16}, \tfrac{17}{32}, \tfrac{33}{64}, \tfrac{65}{128}, \ldots,$$

and whose coefficients form a geometrical series of which the first term is $\frac{88}{1215}$ and the common ratio $\frac{2}{3}$.

We shall assume the law of the indices (which, it may be remarked, is identical with that given in the introduction to this paper as originally printed in the *Proceedings**, but subsequently altered in the *Transactions*) and write

$$\begin{aligned} p-q=&(q-r)^2+\tfrac{4}{3}(q-r)^{\frac{3}{2}}+\tfrac{11}{18}(q-r)+\tfrac{10}{81}(q-r)^{\frac{3}{4}} \\ &+\tfrac{2^3}{3^3}A\,(q-r)^{\frac{5}{8}}+\tfrac{2^4}{3^4}B\,(q-r)^{\frac{9}{16}}+\tfrac{2^5}{3^5}C\,(q-r)^{\frac{17}{32}} \\ &+\tfrac{2^6}{3^6}D\,(q-r)^{\frac{33}{64}}+\tfrac{2^7}{3^7}E\,(q-r)^{\frac{65}{128}}+\text{\&c., ad inf.} \\ &+\Theta\,\dagger. \end{aligned} \qquad (1)$$

The law of the coefficients will then be established by proving that

$$A=B=C=D=E=\ldots=\tfrac{11}{45}.$$

If there were any terms, of an order superior to that of $(q-r)^{\frac{1}{2}}$, whose indices did not obey the assumed law, any such term would make its presence felt in the course of the work; for, in the process we shall employ, the coefficient of each term has to be determined before that of any subsequent

[* See footnote, p. 584, below.]

† In the text above Θ represents some unknown function, the asymptotic value of whose ratio to $(q-r)^{\frac{1}{2}}$ is not infinite.

term can be found. It was in this way that the existence of terms between $(q-r)^{\frac{5}{8}}$ and $(q-r)^{\frac{1}{2}}$ was made manifest in the unsuccessful attempt to calculate the coefficient of $(q-r)^{\frac{1}{2}}$. It thus appears that the assumed law of the indices is the true one.

It will be remembered that $p, q, r, \ldots$, are the halves of the sharpened Hamiltonian Numbers $E_{n+1}, E_n, E_{n-1}, \ldots$, and that consequently the relation

$$E_{n+1} = 1 + \frac{E_n(E_n-1)}{1.2} - \frac{E_{n-1}(E_{n-1}-1)(E_{n-1}-2)}{1.2.3} + \ldots$$

may be written in the form

$$p = \tfrac{1}{2} + \frac{q(2q-1)}{2} - \frac{r(2r-1)(2r-2)}{2.3} + \frac{s(2s-1)(2s-2)(2s-3)}{2.3.4}$$
$$- \frac{t(2t-1)(2t-2)(2t-3)(2t-4)}{2.3.4.5} + \frac{u(2u-1)(2u-2)(2u-3)(2u-4)(2u-5)}{2.3.4.5.6}$$
$$- \ldots \qquad (2)$$

The comparison of this value of p with that given by (1) furnishes an equation which, after several reductions have been made, in which special attention must be paid to the order of the quantities under consideration, ultimately leads to the determination of the values $A, B, C, \ldots$, in succession.

Taking unity to represent the order of q, the orders of

$$p,\ q,\ r,\ s,\ t,\ u,\ v,\ w,\ \ldots$$

will be

$$2,\ 1,\ \tfrac{1}{2},\ \tfrac{1}{4},\ \tfrac{1}{8},\ \tfrac{1}{16},\ \tfrac{1}{32},\ \tfrac{1}{64},\ \ldots$$

Hence, after expanding each of the binomials on the right-hand side of (1) and arranging the terms in descending order, retaining only terms for which the order is superior to $\frac{1}{2}$, we shall find

Order	2	$p = q^2$
„	$\frac{3}{2}$	$-2qr + \frac{4}{3}q^{\frac{3}{2}}$
„	1	$+r^2 - 2q^{\frac{1}{2}}r + \frac{29}{18}q$
„	$\frac{3}{4}$	$+\frac{10}{81}q^{\frac{3}{4}}$
„	$\frac{5}{8}$	$+\frac{2^3}{3^3}A\,q^{\frac{5}{8}}$
„	$\frac{9}{16}$	$+\frac{2^4}{3^4}B\,q^{\frac{9}{16}}$
„	$\frac{17}{32}$	$+\frac{2^5}{3^5}C\,q^{\frac{17}{32}}$
„	$\frac{33}{64}$	$+\frac{2^6}{3^6}D\,q^{\frac{33}{64}}$
„	$\frac{65}{128}$	$+\frac{2^7}{3^7}E\,q^{\frac{65}{128}} + \ldots.$ (3)

Again, retaining only those terms of (2) whose order is superior to $\frac{1}{2}$, we have

$$p = q^2;\ -\tfrac{2}{3}r^3;\ -\tfrac{1}{2}q + r^2 + \tfrac{1}{3}s^4;\ -s^3;\ -\tfrac{2}{15}t^5 \qquad (4)$$

$$\text{Order} \qquad 2;\quad \tfrac{2}{2};\qquad 1\qquad;\ \tfrac{3}{4};\quad \tfrac{5}{8}\ .$$

From (3) and (4) we obtain by subtraction

$$
\begin{array}{lll}
\text{Order } \frac{3}{2} & 0 = \frac{2}{3}r^3 - 2qr + \frac{4}{3}q^{\frac{3}{2}} & \\
\quad ,, \quad 1 & \quad - \frac{1}{3}s^4 - 2q^{\frac{1}{2}}r + \frac{19}{9}q & \\
\quad ,, \quad \frac{3}{4} & \quad + s^3 + \frac{10}{81}q^{\frac{3}{4}} & \\
\quad ,, \quad \frac{5}{8} & \quad + \frac{2}{15}t^5 + \frac{2^3}{3^3}A\,q^{\frac{5}{8}} & \\
\quad ,, \quad \frac{9}{16} & \quad + \frac{2^4}{3^4}B\,q^{\frac{9}{16}} & \\
\quad ,, \quad \frac{17}{32} & \quad + \frac{2^5}{3^5}C\,q^{\frac{17}{32}} & \\
\quad ,, \quad \frac{33}{64} & \quad + \frac{2^6}{3^6}D\,q^{\frac{33}{64}} & \\
\quad ,, \quad \frac{65}{128} & \quad + \frac{2^7}{3^7}E\,q^{\frac{65}{128}} + \ldots & (5)
\end{array}
$$

Changing $p, q, r, \ldots$ into $q, r, s, \ldots$ respectively, equation (4) becomes

$$q = r^2 - \tfrac{2}{3}s^3 - \tfrac{1}{2}r + s^2 + \tfrac{1}{3}t^4 - t^3 - \tfrac{2}{15}u^5,$$

so that, if we assume $q = r^2(1-\alpha)$, the order of α will be the same as that of $r^{-2}s^3$, namely, $-\frac{2}{2} + \frac{3}{4} = -\frac{1}{4}$.

Hence, if we substitute $r^2(1-\alpha)$ for q in (5), neglecting in the result quantities of the order $\frac{1}{2}$, we shall find

$$
\begin{aligned}
&\tfrac{2}{3}r^3 - 2qr + \tfrac{4}{3}q^{\frac{3}{2}} - \tfrac{1}{3}s^4 - 2q^{\frac{1}{2}}r + \tfrac{19}{9}q \\
&= \tfrac{2}{3}r^3 - 2r^3(1-\alpha) + \tfrac{4}{3}r^3(1 - \tfrac{3}{2}\alpha + \tfrac{3}{8}\alpha^2 + \tfrac{1}{16}\alpha^3) \\
&\quad - \tfrac{1}{3}s^4 - 2r^2(1 - \tfrac{1}{2}\alpha) + \tfrac{19}{9}r^2(1-\alpha) \\
&= \tfrac{1}{2}r^3\alpha^2 + \tfrac{1}{12}r^3\alpha^3 - \tfrac{1}{3}s^4 + \tfrac{1}{9}r^2 - \tfrac{10}{9}r^2\alpha\,;
\end{aligned}
$$

while at the same time, since the order of $r^{\frac{3}{2}}\alpha$ does not exceed $\frac{1}{2}$, we have

$$q^{\frac{3}{4}} = r^{\frac{3}{2}}(1-\alpha)^{\frac{3}{4}} = r^{\frac{3}{2}},$$

and in like manner $\quad q^{\frac{5}{8}} = r^{\frac{5}{4}},\ q^{\frac{9}{16}} = r^{\frac{9}{8}}$, and so on.

Thus equation (5) becomes

$$
\begin{array}{lll}
\text{Order } 1 & 0 = \frac{1}{2}r^3\alpha^2 - \frac{1}{3}s^4 + \frac{1}{9}r^2 & \\
\quad ,, \quad \frac{3}{4} & \quad + \frac{1}{12}r^3\alpha^3 - \frac{10}{9}r^2\alpha + s^3 + \frac{10}{81}r^{\frac{3}{2}} & \\
\quad ,, \quad \frac{5}{8} & \quad + \frac{2}{15}t^5 + \frac{2^3}{3^3}A\,r^{\frac{5}{4}} & \\
\quad ,, \quad \frac{9}{16} & \quad + \frac{2^4}{3^4}B\,r^{\frac{9}{8}} & \\
\quad ,, \quad \frac{17}{32} & \quad + \frac{2^5}{3^5}C\,r^{\frac{17}{16}} & \\
\quad ,, \quad \frac{33}{64} & \quad + \frac{2^6}{3^6}D\,r^{\frac{33}{32}} & \\
\quad ,, \quad \frac{65}{128} & \quad + \frac{2^7}{3^7}E\,r^{\frac{65}{64}} + \ldots & (6)
\end{array}
$$

where $\quad \alpha = \frac{2}{3}r^{-2}s^3\,;\ + \frac{1}{2}r^{-1} - r^{-2}s^2 - \frac{1}{3}r^{-2}t^4\,;\ + r^{-2}t^3\,;\ + \frac{2}{15}r^{-2}u^5,$

order $\quad -\frac{1}{4}\ ;\qquad -\frac{1}{2}\qquad ;\ -\frac{5}{8}\ ;\ -\frac{11}{16}$.

Let $\quad \alpha = \frac{2}{3}r^{-2}s^3(1+\alpha')$

then $\quad \alpha' = \frac{3}{2}s^{-3}(\frac{1}{2}r - s^2 - \frac{1}{3}t^4 + t^3 + \frac{2}{15}u^5)$

where terms as far as, but not beyond, $-\frac{7}{16}$ (which is the order of $s^{-3}u^5$) have been retained.

Now p consists of terms whose orders are $2,\ \frac{3}{2},\ 1,\ \frac{3}{4},\ \frac{5}{8},\ \frac{1}{2}, \ldots$

q „ „ „ $1,\ \frac{3}{4},\ \frac{1}{2},\ \frac{3}{8},\ \frac{5}{16},\ \frac{1}{4}, \ldots$

α „ „ „ $-\frac{1}{4}, -\frac{1}{2}, -\frac{5}{8}, -\frac{11}{16}, -\frac{3}{4}, \ldots$

α' „ „ „ $-\frac{1}{4}, -\frac{3}{8}, -\frac{7}{16}, -\frac{1}{2}, \ldots.$

Thus the order of α' is $-\frac{1}{4}$, and in the above expression all terms of α' superior to $-\frac{1}{2}$ have been retained, and consequently (rejecting the square of α' whose order is $-\frac{1}{2}$) in the first line of (6) we may write

$$\begin{aligned}\tfrac{1}{2} r^3\alpha^2 &= \tfrac{2}{9} r^{-1}s^6(1+2\alpha')\\ &= \tfrac{2}{9} r^{-1}s^6 + \tfrac{2}{3} r^{-1}s^3\left(\tfrac{1}{2} r - s^2 - \tfrac{1}{3} t^4 + t^3 + \tfrac{2}{15} u^5\right)\\ &= \tfrac{2}{9} r^{-1}s^6 + \tfrac{1}{3} s^3 - \tfrac{2}{3} r^{-1}s^5 - \tfrac{2}{9} r^{-1}s^3t^4 + \tfrac{2}{3} r^{-1}s^3t^3 + \tfrac{4}{45} r^{-1}s^3u^5.\end{aligned}$$

In the second line of (6) we may reject the whole of α', since its order is $-\frac{1}{4}$, and write

$$\begin{aligned}&\tfrac{1}{12} r^3\alpha^3 - \tfrac{10}{9} r^2\alpha + s^3\\ &= \tfrac{2}{81} r^{-3}s^9 + \tfrac{7}{27} s^3.\end{aligned}$$

After substituting their values for the terms in (6) which contain α, and at the same time dividing throughout by $\frac{2}{3}$, we shall obtain

$$\begin{array}{lll}
\text{Order} & 1 & 0 = \tfrac{1}{3} r^{-1}s^6 - \tfrac{1}{2} s^4 + \tfrac{1}{6} r^2\\
\quad „ & \tfrac{3}{4} & \quad + \tfrac{1}{27} r^{-3}s^9 - r^{-1}s^5 - \tfrac{1}{3} r^{-1}s^3t^4 + \tfrac{8}{9} s^3 + \tfrac{5}{27} r^{\frac{3}{2}}\\
\quad „ & \tfrac{5}{8} & \quad + r^{-1}s^3t^3 + \tfrac{1}{5} t^5 + \tfrac{2^2}{3^2} A r^{\frac{5}{4}}\\
\quad „ & \tfrac{9}{16} & \qquad\qquad + \tfrac{2}{15} r^{-1}s^3u^5 + \tfrac{2^3}{3^3} B r^{\frac{9}{8}}\\
\quad „ & \tfrac{17}{32} & \qquad\qquad\qquad + \tfrac{2^4}{3^4} C r^{\frac{17}{16}}\\
\quad „ & \tfrac{33}{64} & \qquad\qquad\qquad + \tfrac{2^5}{3^5} D r^{\frac{33}{32}}\\
\quad „ & \tfrac{65}{128} & \qquad\qquad\qquad + \tfrac{2^6}{3^6} E r^{\frac{65}{64}} + \ldots
\end{array} \qquad (7)$$

We now write

$$r = s^2(1-\beta) \text{ and } \beta = \tfrac{2}{3} s^{-2} t^3 (1+\beta')$$

where, observing that the values of β and β' can be immediately deduced from those of α and α' by changing $r, s, t, \ldots$ into $s, t, u, \ldots$, it is evident that β and β' are both of the order $-\frac{1}{8}$; for α and α' are both of the order $-\frac{1}{4}$. Thus (neglecting quantities whose order is equal to, or less than, $\frac{1}{2}$) we have

$$\begin{aligned}&\tfrac{1}{3} r^{-1}s^6 - \tfrac{1}{2} s^4 + \tfrac{1}{6} r^2\\ &= \tfrac{1}{3} s^4(1+\beta+\beta^2+\beta^3) - \tfrac{1}{2} s^4 + \tfrac{1}{6} s^4(1-2\beta+\beta^2) = \tfrac{1}{2} s^4\beta^2 + \tfrac{1}{3} s^4\beta^3\\ &= \tfrac{2}{9} t^6(1+2\beta') + \tfrac{8}{81} s^{-2}t^9\\ &= \tfrac{2}{9} t^6 + \tfrac{2}{3} t^3\left(\tfrac{1}{2} s - t^2 - \tfrac{1}{3} u^4 + u^3 + \tfrac{2}{15} v^5\right) + \tfrac{8}{81} s^{-2}t^9\\ &= \tfrac{2}{9} t^6;\ + \tfrac{1}{3} st^3 - \tfrac{2}{3} t^5 - \tfrac{2}{9} t^3u^4 + \tfrac{8}{81} s^{-2}t^9;\ + \tfrac{2}{3} t^3u^3;\ + \tfrac{4}{45} t^3v^5.\end{aligned}$$

Order $\frac{3}{4}$; $\frac{5}{8}$; $\frac{9}{16}$; $\frac{17}{32}$.

$$\begin{aligned}
&\tfrac{1}{27} r^{-3} s^9 - r^{-1} s^5 - \tfrac{1}{3} r^{-1} s^3 t^4 + \tfrac{8}{9} s^3 + \tfrac{5}{27} r^{\frac{3}{2}} \\
&= \tfrac{1}{27} s^3 (1+3\beta) - s^3 (1+\beta) - \tfrac{1}{3} st^4 (1+\beta) + \tfrac{8}{9} s^3 + \tfrac{5}{27} s^3 (1 - \tfrac{3}{2}\beta) \\
&= \tfrac{1}{9} s^3 - \tfrac{7}{6} s^3 \beta - \tfrac{1}{3} st^4 (1+\beta) \\
&= \tfrac{1}{9} s^3 - \tfrac{1}{3} st^4 ; \; - \tfrac{7}{9} st^3 - \tfrac{2}{9} s^{-1} t^7 .
\end{aligned}$$

Order $\quad \tfrac{3}{4} \quad ; \quad \tfrac{5}{8} \quad .$

$$r^{-1} s^3 t^3 + \tfrac{1}{5} t^5 + \tfrac{2^2}{3^2} A r^{\frac{5}{4}} = st^3 + \tfrac{1}{5} t^5 + \tfrac{2^2}{3^2} A s^{\frac{5}{2}},$$

$$\tfrac{2}{15} r^{-1} s^3 u^5 + \tfrac{2^3}{3^3} B r^{\frac{9}{8}} = \tfrac{2}{15} su^5 + \tfrac{2^3}{3^3} B s^{\frac{9}{4}},$$

and so on.

Hence (7) becomes

$$\begin{array}{lll}
\text{Order } \tfrac{3}{4} & 0 = \tfrac{2}{9} t^6 - \tfrac{1}{3} st^4 + \tfrac{1}{9} s^3 & \\
\quad ,, \quad \tfrac{5}{8} & \quad + \tfrac{8}{81} s^{-2} t^9 - \tfrac{7}{15} t^5 - \tfrac{2}{9} t^3 u^4 + \tfrac{5}{9} st^3 - \tfrac{2}{9} s^{-1} t^7 + \tfrac{2^2}{3^2} A s^{\frac{5}{2}} & \\
\quad ,, \quad \tfrac{9}{16} & \quad + \tfrac{2}{3} t^3 u^3 + \tfrac{2}{15} su^5 + \tfrac{2^3}{3^3} B s^{\frac{9}{4}} & \\
\quad ,, \quad \tfrac{17}{32} & \qquad + \tfrac{4}{45} t^3 v^5 + \tfrac{2^4}{3^4} C s^{\frac{17}{8}} & \\
\quad ,, \quad \tfrac{33}{64} & \qquad\quad + \tfrac{2^5}{3^5} D s^{\frac{33}{16}} & \\
\quad ,, \quad \tfrac{65}{128} & \qquad\quad + \tfrac{2^6}{3^6} E s^{\frac{65}{32}} + \dots & (8)
\end{array}$$

Dividing this throughout by $\tfrac{2}{3}s$, and then writing

$$s = t^2 (1-\gamma) \text{ and } \gamma = \tfrac{2}{3} t^{-2} u^3 (1+\gamma'),$$

we obtain in exactly the same manner as before, merely altering the letters in the previous work,

$$\begin{aligned}
&\tfrac{1}{3} s^{-1} t^6 - \tfrac{1}{2} t^4 + \tfrac{1}{6} s^2 \\
&= \tfrac{2}{9} u^6 ; \; + \tfrac{1}{3} tu^3 - \tfrac{2}{3} u^5 - \tfrac{2}{9} u^3 v^4 + \tfrac{8}{81} t^{-2} u^9 ; \; + \tfrac{2}{3} u^3 v^3 ; \; + \tfrac{4}{45} u^3 w^5 .
\end{aligned}$$

Order $\quad \tfrac{3}{8} \quad ; \quad \tfrac{5}{16} \quad ; \quad \tfrac{9}{32} \quad ; \quad \tfrac{17}{64}$

where quantities of the order $\tfrac{1}{4}$, or less, are now neglected.

Similarly $\tfrac{4}{27} s^{-3} t^9 - \tfrac{7}{10} s^{-1} t^5 - \tfrac{1}{3} s^{-1} t^3 u^4 + \tfrac{5}{6} t^3 - \tfrac{1}{3} s^{-2} t^7 + \tfrac{2}{3} A s^{\frac{3}{2}}$

$$\begin{aligned}
&= \tfrac{4}{27} t^3 (1+3\gamma) - \tfrac{7}{10} t^3 (1+\gamma) - \tfrac{1}{3} tu^4 (1+\gamma) + \tfrac{5}{6} t^3 - \tfrac{1}{3} t^3 (1+2\gamma) + \tfrac{2}{3} A t^3 (1 - \tfrac{3}{2}\gamma) \\
&= (\tfrac{2}{3} A - \tfrac{7}{135}) t^3 - \tfrac{1}{3} tu^4 - (A + \tfrac{83}{90}) t^3 \gamma - \tfrac{1}{3} tu^4 \gamma \\
&= (\tfrac{2}{3} A - \tfrac{7}{135}) t^3 - \tfrac{1}{3} tu^4 ; \; - (\tfrac{2}{3} A + \tfrac{83}{135}) tu^3 - \tfrac{2}{9} t^{-1} u^7 .
\end{aligned}$$

Order $\quad \tfrac{3}{8} \quad ; \quad \tfrac{5}{16} \quad .$

$$s^{-1} t^3 u^3 + \tfrac{1}{5} u^5 + \tfrac{2^2}{3^2} B s^{\frac{5}{4}} = tu^3 + \tfrac{1}{5} u^5 + \tfrac{2^2}{3^2} B t^{\frac{5}{2}}$$

$$\tfrac{2}{15} s^{-1} t^3 v^5 + \tfrac{2^3}{3^3} C s^{\frac{9}{8}} = \tfrac{2}{15} tv^5 + \tfrac{2^3}{3^3} C t^{\frac{9}{4}},$$

and so on.

Thus (8) becomes

$$\begin{array}{ll}
\text{Order } \tfrac{3}{8} & 0 = \tfrac{2}{9} u^6 - \tfrac{1}{3} tu^4 + (\tfrac{2}{3} A - \tfrac{7}{135}) t^3 \\
\quad ,, \quad \tfrac{5}{16} & \quad + \tfrac{8}{81} t^{-2} u^9 - \tfrac{7}{15} u^5 - \tfrac{2}{9} u^3 v^4 + (\tfrac{97}{135} - \tfrac{2}{3} A) tu^3 - \tfrac{2}{9} t^{-1} u^7 + \tfrac{2^2}{3^2} B t^{\frac{5}{2}} \\
\quad ,, \quad \tfrac{9}{32} & \quad + \tfrac{2}{3} u^3 v^3 + \tfrac{2}{15} tv^5 + \tfrac{2^3}{3^3} C t^{\frac{9}{4}} \\
\quad ,, \quad \tfrac{17}{64} & \qquad + \tfrac{4}{45} u^3 w^5 + \tfrac{2^4}{3^4} D t^{\frac{17}{8}} \\
\quad ,, \quad \tfrac{33}{128} & \qquad\quad + \tfrac{2^5}{3^5} E t^{\frac{33}{16}} + \dots
\end{array}$$

Now the terms of the highest order in this equation must vanish when we write $t=u^2$, and therefore $\frac{2}{9}-\frac{1}{3}+\frac{2}{3}A-\frac{7}{135}=0$, which gives $A=\frac{11}{45}$. Substituting this value for A, we find

$$\begin{array}{lll} \text{Order} & \frac{3}{8} & 0=\frac{2}{9}u^6-\frac{1}{3}tu^4+\frac{1}{9}t^3 \\ \text{,,} & \frac{5}{16} & \quad +\frac{8}{81}t^{-2}u^9-\frac{7}{15}u^5-\frac{2}{9}u^3v^4+\frac{5}{9}tu^3-\frac{2}{9}t^{-1}u^7+\frac{2^2}{3^2}Bt^{\frac{5}{2}} \\ \text{,,} & \frac{9}{32} & \quad +\frac{2}{3}u^3v^3+\frac{2}{15}tv^5+\frac{2^3}{3^3}Ct^{\frac{9}{4}} \\ \text{,,} & \frac{17}{64} & \qquad +\frac{4}{45}u^3w^5+\frac{2^4}{3^4}Dt^{\frac{17}{8}} \\ \text{,,} & \frac{33}{128} & \qquad\quad +\frac{2^5}{3^5}Et^{\frac{33}{16}}+\dots, \end{array}$$

which is a mere repetition of equation (8), with all the letters moved forward one place. Hence it is evident that, if we treat this equation as we treated (8), we shall find $B=\frac{11}{45}$, arriving, at the same time, at another equation which will be merely a repetition of (8), with all its letters moved forward two places; and this process can be continued as long as we please.

Thus we arrive at the result—

$$A=B=C=D=E=\dots=\tfrac{11}{45},$$

and the asymptotic development for Hypothenusal Numbers

$$p-q=(q-r)^2+\tfrac{4}{3}(q-r)^{\frac{3}{2}}+\tfrac{11}{18}(q-r)+\tfrac{10}{81}(q-r)^{\frac{3}{4}}$$
$$+\tfrac{11}{45}(q-r)^{\frac{1}{2}}\left\{\tfrac{2^3}{3^3}(q-r)^{\frac{1}{8}}+\tfrac{2^4}{3^4}(q-r)^{\frac{1}{16}}+\tfrac{2^5}{3^5}(q-r)^{\frac{1}{32}}+\dots\right\}$$

is established.

Comparing this with the corresponding formula for Hamiltonian Numbers,

$$p=q^2-\tfrac{2}{3}q\,(q^{\frac{1}{2}}+q^{\frac{1}{4}}+q^{\frac{1}{8}}+q^{\frac{1}{16}}+\dots+q^{(\frac{1}{2})^i})+\Xi q,$$

given at the beginning of the third section [p. 570], it will be noticed that each of the two developments begins with an irregular portion consisting respectively of four and one terms, followed by a regular series. In the one case the regular portion is $\frac{11}{45}(q-r)^{\frac{1}{2}}$, multiplied by a series whose general term is $\frac{2^n}{3^n}(q-r)^{(\frac{1}{2})^n}$; in the other it consists of a series of terms of the form $q^{(\frac{1}{2})^n}$ multiplied by $-\frac{2}{3}q$.

[To p. 579, footnote*. The reference is to *Proceedings of the Royal Society*, Vol. 42 (1887), pp. 470, 471, where is printed an Abstract identical with the Introduction to this paper (pp. 553–555 above) save for the insertion after the word "scale" (p. 555 above) of the words "in order to establish or disprove conclusively the presumptive law of the asymptotic branch of the series connecting any two consecutive semi-differences η_x, η_{x+1} of the Hamiltonian Numbers, viz.: —

$$\eta_{x+1}-\eta_x{}^2=\eta_x{}^{\frac{1}{2}}\sum_{r=0}^{\infty}c_r\eta_x{}^{(\frac{1}{2})^r}."$$

There is also a paper, *Proceedings of the Royal Society*, Vol. 44 (1888), pp. 99—101, containing what is here given on p. 579 and the first half of p. 580.]

52.

SUR LES NOMBRES DITS DE HAMILTON.

[*Compte Rendu de l'Assoc. Française (Toulouse),* 1887, pp. 164—168.]

CONSIDÉRONS ce tableau formé en bas par un procédé qui à peu près s'explique de soi-même :

1	0	0	0	0	0	0	....
	1	1	1	1	1	1	....
		2	3	4	5	6	....
		1	5	9	14	20	....
			6	15	29	49	...
			5	21	50	89	...
			4	26	76	175	...
			3	30	106	231	...
			2	33	139	420	...
			1	35	174	594	...
				36	210	804	...
				.	.	.	...
				.	.	.	...

Ce tableau peut être étendu indéfiniment.

On voit qu'il se divise en étages et que les nombres initiaux des premières lignes de ces étages sont :

$$1, 1, 2, 6, 36.$$

En les additionnant et en ajoutant l'unité aux sommes, on obtient les nombres 2, 3, 5, 11, 47 ....

Ces nombres sont ce que j'appelle les nombres de Hamilton qui a trouvé les nombres 11, 47, et encore le nombre qui vient après 47, c'est-à-dire 923, dans un rapport qu'il a publié dans les *Reports of the British Association* 1836, sur la méthode de Jerrard pour réduire les équations du cinquième degré, méthode qui remonte, en effet, à Bring, professeur à Lund, qui l'a publié dans un opuscule en 1786 qui restait inconnu ou oublié. De même qu'on peut ôter 3 termes d'une équation dont le degré est au moins 5 sans résoudre aucune équation d'un degré supérieur à 3, de même aussi on peut ôter 4 termes d'une équation dont le degré est au moins 11 sans résoudre des équations d'un degré supérieur à 4; 5 termes d'une équation dont le degré est au moins 47 sans résoudre des équations d'un degré supérieur à 5 et ainsi de suite.

Mais il est nécessaire d'avertir ici que la même chose aura lieu pour des équations de degrés moindres, en général, que ceux fournis par les nombres de Hamilton. En effet, au lieu de 11, 47, 923 ... on peut substituer 10, 44, 905 ...: mais le système d'équations résolvantes deviendra plus compliqué quand on fait cette diminution du degré minimum. Ainsi, par exemple, il est bien vrai que pour ôter 4 termes à une équation du degré 10, le système d'équations à résoudre ne contiendra nulle équation d'un degré supérieur à 4: mais il y aura 3 équations de ce degré à résoudre tandis que quand l'équation donnée est du degré 11 ou plus haut que 11, on n'aura à résoudre (en combinaison bien entendu avec des équations cubiques quadratiques et linéaires) qu'une seule équation biquadratique au lieu de trois: et ainsi en général.

Pour trouver les nombres de Hamilton, mon coadjuteur, M. Hammond a trouvé une échelle de relation d'une simplicité merveilleuse.

On peut former avec les lignes successives du tableau les fonctions

$1 + 0x + 0x^2 + 0x^3 + \ 0x^4 \dots$	disons	F_0 (qui en effet est l'unité).
$x + \ x^2 + \ x^3 + \ \ x^4 \dots$	„	F_1
$2x^2 + 3x^3 + \ 4x^4 \dots$	„	F_2
$x^2 + 5x^3 + \ 9x^4 \dots$	„	${}^1F_2$
$6x^3 + 15x^4 \dots$	„	${}^2F_2 = F_3$
$5x^3 + 21x^4 \dots$	„	${}^1F_3$
$4x^3 + 26x^4 \dots$	„	${}^2F_3$
$3x^3 + 30x^4 \dots$	„	${}^3F_3$
$2x^3 + 35x^4 \dots$	„	${}^4F_3$
$x^3 + 35x^5 \dots$	„	${}^5F_3$
$36x^5 \dots$	„	${}^6F_3 = F_4$

et ainsi de suite.

Donnons à 1, 1, 2, 6, 36 ... les noms $a_0, a_1, a_2, a_3, a_4 \dots$ alors il est facile à voir qu'en général ${}^nF_n = F_{n+1}$; mais aussi on voit que

$$ {}^{i+1}F_n = (1-x)^{-1}\, {}^iF_n - x^n. $$

Donc

$$ F_{n+1} - (1-x)^{-a_n} . F_n = -x^n \{1 + (1-x)^{-1} + (1-x)^{-2} + \dots + (1-x)^{-a_n+1}\} $$
$$ = x^{n-1} \{(1-x) - (1-x)^{-a_n+1}\}. $$

Faisons $a_0 + a_1 + a_2 + \dots + a_n = S_{n+1}$ alors en multipliant l'équation par $(1-x)^{S_{n+1}}$, on obtient:

$$ (1-x)^{S_{n+1}} . F_{n+1} - (1-x)^{S_n} . F_n = x^n (1-x)^{S_{n+1}+1} - x^n (1-x)^{S_{n+1}}. $$

Cette équation qui existe pour toutes les valeurs S_n jusqu'à S_1 exclusif reste vraie comme identité même pour S_0 si on met $S_0 = 0$. Alors en donnant à n toutes les valeurs depuis $n-1$ jusqu'à 0 inclusivement et en faisant la sommation des équations ainsi formées, on obtient facilement:

$$(1-x)^{S_n} F_n - 1 + x^{-1}(1-x) - x^{n-1}(1-x)^{S_n+1}$$
$$= x^{n-2}(1-x)^{S_n+2} + x^{n-3}(1-x)^{S_{n-1}+2} + x^{n-n}(1-x)^{S_{n-n}+2} + \ldots.$$

Si dans cette équation on compare les coefficients de x^n en se rappelant que le coefficient de x^n en F_n est a_n, c'est-à-dire $S_{n+1} - S_n$, et que $S_n + 1$ est le nombre n^{me} de M. Hamilton, de sorte que $S_n + 2$ que je nommerai E_n est ce nombre augmenté de l'unité, on trouve:

$$E_{n+1} = 1 + E_n . \frac{E_n - 1}{2} - \frac{E_{n-1}(E_{n-1}-1)(E_{n-1}-2)}{2.3} + \ldots$$

formule de relation entre les nombres de Hamilton qu'on peut écrire sous la forme symétrique

$$1 - (E_n)_1 + (E_{n-1})_2 - (E_{n-2})_3 \ldots = 0.$$

En augmentant les nombres de Hamilton de l'unité, on obtient pour E les valeurs successives

3, 4, 6, 12, 48, 924

qu'on trouve très facilement par la formule de la relation donnée.

Ainsi par exemple:

$$\frac{3.2}{2} = 4 - 1 = 3$$
$$\frac{4.3}{2} - \frac{3.2.1}{1.2.3} = 6 - 1 = 5$$
$$\frac{6.5}{2} - \frac{4.3.2}{1.2.3} = 12 - 1 = 11$$
$$\frac{12.11}{2} - \frac{6.5.4}{1.2.3} + \frac{4.3.2.1}{1.2.3.4} = 48 - 1 = 47$$
$$\frac{48.47}{2} - \frac{12.11.10}{1.2.3} + \frac{6.5.4.3}{1.2.3.4} = 924 - 1 = 923$$
$$\frac{924.923}{2} - \frac{48.67.44}{2.3} + \frac{12.11.10.9}{1.2.3.4} - \frac{6.5.4.3.2}{1.2.3.4.5} = 409620 - 1 = 409619.$$

Les nombres de Hamilton ainsi calculés sont:

2, 3, 5, 11, 47, 923, 409619, 83763206255 ...

où comme première approximation asymptotique on peut remarquer que si u_x est le nombre de rang x, $u_{2+1} \div u_2^2$ devient de plus en plus près de δ mais toujours moindre que l'unité quand x croît indéfiniment.

Telle est la formule bien remarquable trouvée par M. Hammond, dont j'ai un peu simplifié et abrégé la démonstration.

Un travail sur les nombres de Hamilton, fait par M. Hammond et moi-même va prochainement paraître dans les *Philosophical Transactions* [above, p. 553].

53.

NOTE ON A PROPOSED ADDITION TO THE VOCABULARY OF ORDINARY ARITHMETIC*.

[*Nature*, XXXVII. (1888), pp. 152, 153.]

THE total number of distinct primes which divide a given number I call its Manifoldness or Multiplicity.

A number whose Manifoldness is n I call an n-fold number. It may also be called an n-ary number, and for $n = 1, 2, 3, 4, \ldots$ a unitary (or primary), a binary, a ternary, a quaternary,... number. Its prime divisors I call the *elements* of a number; the highest powers of these elements which divide a number its *components*; the degrees of these powers its *indices*; so that the indices of a number are the totality of the indices of its several components. Thus, we may say, a prime is a one-fold number whose index is unity.

So, too, we may say that all the components but one of an odd perfect number must have even indices, and that the excepted one must have its base and index each of them congruous to 1 to modulus 4.

Again, a remarkable theorem of Euler, contained in a memoir relating to the Divisors of Numbers (*Opuscula Minora*, II. p. 514), may be expressed by saying that *every even perfect number is a two-fold number, one of whose* components *is a prime, and such that when augmented by unity it becomes a power of* 2, *and double the other component*†.

* Perhaps I may without immodesty lay claim to the appellation of the Mathematical Adam, as I believe that I have given more names (passed into general circulation) to the creatures of the mathematical reason than all the other mathematicians of the age combined.

† It may be well to recall that a perfect number is one which is the half of the sum of its divisors. The converse of the theorem in the text, namely that $2^n(2^{n+1}-1)$, when $2^{n+1}-1$ is a prime, is a perfect number, is enunciated and proved by Euclid in the 36th (the last) proposition of the 9th Book of the "Elements," the second factor being expressed by him as the sum of a geometric series whose first term is unity and the common ratio 2. In Isaac Barrow's English translation, published in 1660, the enunciation is as follows: "If from a unite be taken how many numbers soever 1, A, B, C, D, in double proportion continually, until the whole

Euler's function $\phi(n)$, which means the number of numbers not exceeding n and prime to it, I call the *totient* of n; and in the new nomenclature we may enunciate that *the totient of a number is equal to the product of that number multiplied by the several excesses of unity above the reciprocals of its elements.* The numbers prime to a number and less than it, I call its *totitives.*

Thus we may express Wilson's generalized theorem by saying that any number is contained as a factor in the product of its totitives increased by unity if it is the number 4, or a prime, or the double of a prime, and diminished by unity in every other case.

I am in the habit of representing the totient of n by the symbol τn, τ (taken from the initial of the word it denotes) being a less hackneyed letter than Euler's ϕ, which has no claim to preference over any other letter of the Greek alphabet, but rather the reverse.

It is easy to prove that the half of any perfect number must exceed in magnitude its totient.

Hence, since $\frac{3}{2}.\frac{5}{4}$ is less than 2, it follows that no odd two-fold perfect number exists.

---

added together E be a prime number; and if this whole E multiplying the last produce a number F, that which is produced F shall be a perfect number."

The direct theorem that every even perfect number is of the above form could probably only have been proved with extreme difficulty, if at all, by the resources of Greek Arithmetic. Euler's proof is not very easy to follow in his own words, but is substantially as follows:

Suppose P (an even perfect number) $=2^nA$. Then, using in general $\int X$ to denote the sum of the divisors of X,

$$2=\frac{\int P}{P}=\frac{\int 2^n . \int A}{2^nA}=\frac{2^{n+1}-1}{2^n}.\frac{\int A}{A}.$$

Hence $$\frac{\int A}{A}=\frac{2^{n+1}}{2^{n+1}-1}, \text{ say}=\frac{Q+1}{Q}.$$

Hence $A=\mu Q$, and $\int A=1+\mu+Q+\mu Q+ \ldots$ (if μ be supposed >1). Hence unless $\mu=1$ and at the same time Q is a prime

$$\int A>\mu(Q+1),$$

that is $\frac{\int A}{A}$ is greater than itself.

Hence an even number P cannot be a perfect number if it is not of the form $2^n(2^{n+1}-1)$, where $2^{n+1}-1$ is a prime, which of course implies that $n+1$ must itself be a prime.

It is remarkable that Euler makes no reference to Euclid in proving his own theorem. It must always stand to the credit of the Greek geometers that they succeeded in discovering a class of perfect numbers which in all probability are the only numbers which are perfect. Reference is made to *so-called* perfect numbers in Plato's "Republic," H, 546 B, and also by Aristotle, Probl. I E 3 and "Metaph." A 5. Mr Margoliouth has pointed out to me that Muhamad Al-Sharastani, in his *Book of Religious and Philosophical Sects*, Careton, 1856, p. 267 of the Arabic text, assigns reasons for regarding all the numbers up to 10 inclusive as perfect numbers, which he attributes to Pythagoras, but which are purely fanciful and entitled to no more serious consideration than the late Dr Cummings's ingenious speculations on the number of the Beast. My particular attention was called to perfect numbers by a letter from Mr Christie, dated from "Carlton, Selby," containing some inquiries relative to the subject.

Similarly, the fact of $\frac{3}{2} \cdot \frac{7}{6} \cdot \frac{11}{10}$ being less than 2 is sufficient to show that 3, 5 must be the two least elements of any three-fold perfect number; furthermore, $\frac{3}{2} \cdot \frac{5}{4} \cdot \frac{17}{16}$ being less than 2, shows that 11 or 13 must be the third element of any such number if it exists*—each of which hypotheses admits of an easy disproof. But to disprove the existence of a four-fold perfect number by my actual method makes a somewhat long and intricate, but still highly interesting, investigation of a multitude of special cases. I hope, *numine favente,* sooner or later to discover a general principle which may serve as a key to a universal proof of the non-existence of any other than the Euclidean perfect numbers, for a prolonged meditation on the subject has satisfied me that the existence of any one such—its escape, so to say, from the complex web of conditions which hem it in on all sides—would be little short of a miracle. Thus then there seems every reason to believe that Euclid's perfect numbers are the only perfect numbers which exist!

In the higher theory of congruences (see Serret's *Cours d'Algèbre Supérieure*) there is frequent occasion to speak of "a number n which does not contain any prime factor other than those which are contained in another number M."

In the new nomenclature n would be defined as *a number whose elements are all of them elements of M.*

As τN is used to denote the totient of N, so we may use μN to denote its multiplicity, and then a well-known theorem in congruences may be expressed as follows.

The number of solutions of the congruence

$$x^2 - 1 \equiv 0 \pmod{P}$$

is $2^{\mu P}$ if P is odd,

$2^{\mu P - 1}$ if P is the double of an odd number,

$2^{\mu P}$ if P is the quadruple of an odd number,

and $2^{\mu P + 1}$ in every other case.

In the memoir above referred to, Euler says that no one has demonstrated whether or not any odd perfect numbers exist. I have found a method for determining what (if any) odd perfect numbers exist of any specified order of manifoldness. Thus, for example, I have proved that there exist no perfect odd numbers of the 1st, 2nd, 3rd, or 4th orders of manifold-

* 3, 5, 7 can never co-exist as elements in *any* perfect number as shown by the fact that $\frac{1+3+3^2}{9} \cdot \frac{1+5}{5} \cdot \frac{1+7+49}{49}$, that is $\frac{26}{15}\left(1+\frac{1}{7}+\frac{1}{49}\right)$, is *greater* than 2. Thus we see that no perfect number can be a multiple of 105. So again the fact that $\frac{5}{4} \cdot \frac{7}{6} \cdot \frac{11}{10} \cdot \frac{13}{12} \cdot \frac{17}{16} \cdot \frac{19}{18}$ is *less* than 2 is sufficient to prove that any odd perfect number of multiplicity less than 7 must be divisible by 3.

ness, or in other words, no odd primary, binary, ternary, or quaternary number can be a perfect number. Had any such existed, my method must infallibly have dragged each of them to light*.

In connection with the theory of perfect numbers I have found it useful to denote $p^i - 1$ when p and i are left general as the Fermatian function, and when p and i have specific values as the ith Fermatian of p. In such case p may be called the base, and i the index of the Fermatian.

Then we may express Fermat's theorem by saying [cf. p. 625 below] that *either the Fermatian itself whose index is one unit below a given prime or else its base must be divisible by that prime*†.

It is also convenient to speak of a Fermatian divided by the excess of its base above unity as a Reduced Fermatian and of that excess itself as the Reducing Factor.

The spirit of my actual method of disproving the existence of odd perfect numbers consists in showing that an n-fold perfect number must have more than n elements, which is absurd. The chief instruments of the investigation are the two inequalities to which the elements of any perfect number must be subject and the properties of the prime divisors of a Reduced Fermatian with an odd prime index.

* I have, since the above was in print, extended the proof to quinary numbers, and anticipate no difficulty in doing so for numbers of higher degrees of multiplicity, so that it is to be hoped that the way is now paved towards obtaining a general proof of this *palmary* theorem.

† So too we may state the important theorem that if *an element of a Fermatian is its index the component which has that index for its base must be its square.*

54.

ON CERTAIN INEQUALITIES RELATING TO PRIME NUMBERS.

[*Nature*, XXXVIII. (1888), pp. 259—262.]

I SHALL begin with a method of proving that the number of prime numbers is infinite, which is not new, but which it is worth while to recall as an introduction to a similar method, by series, which will subsequently be employed in order to prove that the number of primes of the form $4n+3$, as also of the form $6n+5$, is infinite.

It is obvious that the reciprocal of the product

$$\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\left(1-\frac{1}{p_3}\right)\dots\left(1-\frac{1}{p_{N.p}}\right)$$

(where p_i means the ith in the natural succession of primes, and $p_{N.p}$ means the highest prime number not exceeding N)* will be equal to

$$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\dots+\frac{1}{N}+R,$$

and therefore greater than $\log N$ (R consisting exclusively of positive terms).

Hence $$\left(1+\frac{1}{p_1}\right)\left(1+\frac{1}{p_2}\right)\dots\left(1+\frac{1}{p_{N.p}}\right)>M\log N,$$

where $$M=\left(1-\frac{1}{{p_1}^2}\right)\left(1-\frac{1}{{p_2}^2}\right)\dots\left(1-\frac{1}{{p_{N.p}}^2}\right),$$

and is therefore greater than $\frac{2}{\pi}$.

Hence the number of terms in the product must increase indefinitely with N.

By taking the logarithms of both sides we obtain the inequality

$$S_1-\frac{1}{2}S_2+\frac{1}{3}S_3-\frac{1}{4}S_4+\dots>\log\log N+\log M,$$

* $N.p$ itself of course denotes in the above notation the number of primes (p) not exceeding N.

where in general S_i means the sum of inverse ith powers of all the primes not exceeding N; and accordingly is finite, except when $i = 1$, for any value of N. We have therefore

$$S_1 > \log \log N + \text{Const.}$$

The actual value of S_1 is observed to differ only by a limited quantity from the second logarithm of N, but I am not aware whether this has ever been strictly proved.

Legendre has found that for large values of N

$$\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{5}\right) \cdots \left(1 - \frac{1}{p_{N.p}}\right) = \frac{1{\cdot}104}{\log N}.$$

Consequently

$$\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_{N.p}}\right) = \frac{{\cdot}552}{\log N}.$$

This would show that the value of our R bears a finite ratio to $\log N$; calling it $\theta \log N$ we obtain, according to Legendre's formula,

$$\frac{1}{1+\theta} = {\cdot}552, \text{ which gives } \theta = {\cdot}811,$$

so that the nebulous matter, so to say, in the expansion of the reciprocal of the product of the differences between unity and the reciprocals of all the primes not exceeding a given number, stands in the relation of about 4 to 5 to the condensed portion consisting of the reciprocals of the natural numbers.

I will now proceed to establish similar inequalities relating to prime numbers of the respective forms $4n + 3$ and $6n + 5$.

Beginning with the case $4n + 3$, I shall use q_j to signify the jth in the natural succession of primes of the form $4n + 3$, and $q_{N.q}$ to signify the highest q not exceeding N, $N.q$ itself signifying the number of q's not exceeding N.

Let us first, without any reference to convergence, consider the product obtained by the usual mode of multiplication of the infinite series

$$S = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots \textit{ ad inf.}$$

by the product

$$\frac{1}{1 - \frac{1}{2}} \cdot \frac{1 + \frac{1}{q_1}}{1 - \frac{1}{q_1}} \cdot \frac{1 + \frac{1}{q_2}}{1 - \frac{1}{q_2}} \cdot \frac{1 + \frac{1}{q_3}}{1 - \frac{1}{q_3}} \ldots \textit{ ad inf.}$$

It is clear that the effect of the multiplication of S by the numerator of the above product will be to deprive the series S of all its negative terms. Then the effect of dividing by the denominator of the product, with the exception of the factor $1-\frac{1}{2}$, will be to restore all the obliterated terms, but with the sign + instead of −. Lastly, the effect of multiplying by the reciprocal of $(1-\frac{1}{2})$ will be to supply the even numbers that were wanting in the denominators of the terms of S, and we shall thus get the indefinite series

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\dots \textit{ad inf.}$$

Call now

$$Q_N=\frac{1}{1-\frac{1}{2}}\cdot\frac{1+\frac{1}{q_1}}{1-\frac{1}{q_1}}\cdot\frac{1+\frac{1}{q_2}}{1-\frac{1}{q_2}}\cdots\frac{1+\frac{1}{q_{N.q}}}{1-\frac{1}{q_{N.q}}}.$$

Q_N, which is finite when N is finite, may be expanded into an infinite aggregate of positive terms, found by multiplying together the series

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\dots$$

$$1+\frac{2}{q_1}+\frac{2}{q_1^2}+\frac{2}{q_1^3}+\dots$$

$$1+\frac{2}{q_2}+\frac{2}{q_2^2}+\frac{2}{q_2^3}+\dots$$

....

....

$$1+\frac{2}{q_{N.q}}+\frac{2}{q_{N.q}^2}+\frac{2}{q_{N.q}^3}+\dots.$$

Let $$S_N=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\dots\pm\frac{1}{N},$$

then from what has been said it is obvious that we may write

$$Q_NS_N=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\dots+\frac{1}{N}+V-R,$$

where V and R may be constructed according to the following rule: Let the denominator of any term in the aggregate Q_N be called t, and let θ be the smallest odd number which, multiplied by t, makes $t\theta$ greater than N; then if θ is of the form $4n+1$ it will contribute to V a portion represented by the product of the term by some portion of the series S_N of the form

$$\frac{1}{\theta}-\frac{1}{\theta+2}+\frac{1}{\theta+4}-\dots$$

and if θ is of the form $4n+3$ it will contribute to $-R$ a portion equal to the term multiplied by a series of the form

$$-\frac{1}{\theta}+\frac{1}{\theta+2}-\frac{1}{\theta+4}+\dots.$$

Hence R is made up of the sum of products of portions of the aggregate Q_N multiplied respectively by the series

$$\frac{1}{3}-\frac{1}{5}+\frac{1}{7}-\frac{1}{9}+\frac{1}{11}-\frac{1}{13}+\dots$$

$$\frac{1}{7}-\frac{1}{9}+\frac{1}{11}-\frac{1}{13}+\dots$$

$$\frac{1}{11}-\frac{1}{13}+\dots$$

of which the greatest is obviously the first, whose value is $1-S_N$.

Consequently R must be less than the total aggregate Q_N multiplied by $1-S_N$.

Therefore

$$Q_N S_N + Q_N(1-S_N) > 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\dots+\frac{1}{N} > \log N,$$

that is, $$Q_N > \log N,$$

from which it follows that when N increases indefinitely the number of factors in Q_N also increases indefinitely, and there must therefore be an infinite number of primes of the form $4n+3$.

Denoting by M_N the quantity

$$\left(1-\frac{1}{q_1^2}\right)\left(1-\frac{1}{q_2^2}\right)\dots\left(1-\frac{1}{q_{N.q}{}^2}\right)$$

we obtain the inequality

$$\left(1+\frac{1}{q_1}\right)^2\left(1+\frac{1}{q_2}\right)^2\dots\left(1+\frac{1}{q_{N.q}}\right)^2 > \frac{1}{2}M_N \log N,$$

and taking the logarithms of both sides

$$\Sigma_1-\frac{1}{2}\Sigma_2+\frac{1}{3}\Sigma_3-\dots > \frac{1}{2}\log\log N+\frac{1}{2}\log M_N-\frac{1}{2}\log 2,$$

where in general Σ_i denotes the sum of the ith powers of the reciprocals of all prime numbers of the form $4n+3$ not surpassing N.

Hence it follows that $\Sigma_1 > \frac{1}{2}\log\log N$.

If we could determine the ultimate ratio of the sum of those terms of Q_N whose denominators are greater than N to the total aggregate, and should find that μ, the limiting value of this ratio, is not unity, then the method employed to find an inferior limit would enable us also to find a superior limit to Q_N; for we should have $V < \mu Q_N$ added to the sum of portions of what remains of the aggregate when μQ_N is taken from it multiplied respectively by the several series

$$\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\frac{1}{13}-\frac{1}{15}+\ldots \textit{ ad inf.}$$

$$\frac{1}{9}-\frac{1}{11}+\frac{1}{13}-\frac{1}{15}+\ldots \textit{ ad inf.}$$

$$\frac{1}{13}-\frac{1}{15}+\ldots \textit{ ad inf.}$$

the total value of the sum of which products would evidently be less than

$$(1-\mu)\left(S-1+\frac{1}{3}\right)Q_N.$$

Hence the total value of V would be less than

$$\mu Q_N S+(1-\mu)\,Q_N\left(S-\frac{2}{3}\right),$$

that is, less than $$Q_N S-\frac{2}{3}(1-\mu)\,Q_N,$$

and consequently we should have

$$\frac{2}{3}(1-\mu)\,Q_N<\log N,$$

that is $$Q_N<\frac{3}{2(1-\mu)}\log N.$$

From which we may draw the important conclusion that if μ is less than 1, that is, if when N is infinite the portion of the aggregate $S_N Q_N$ comprising the terms whose denominators exceed N does not become infinitely greater than the remaining portion, the sum of the reciprocals of all the prime numbers of the form $4n+3$ not exceeding N would differ by a limited quantity from half the second logarithm of N.

A precisely similar treatment may be applied to prime numbers of the form $6n+5$. We begin with making

$$S_N=1-\frac{1}{5}+\frac{1}{7}-\frac{1}{11}+\frac{1}{13}-\frac{1}{17}+\frac{1}{19}-\ldots.$$

We write

$$Q_N=\frac{1}{1-\frac{1}{2}}\cdot\frac{1}{1-\frac{1}{3}}\cdot\frac{1+\frac{1}{r_1}}{1-\frac{1}{r_1}}\cdot\frac{1+\frac{1}{r_2}}{1-\frac{1}{r_2}}\cdots\frac{1+\frac{1}{r_{N.r}}}{1-\frac{1}{r_{N.r}}}.$$

We make $$Q_N S_N=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots+\frac{1}{N}+V-R.$$

We prove as before that $$R<(1-S)\,Q_N,$$

and thus obtain $$Q_N>\log N,$$

and then putting $M_N = \left(1 - \frac{1}{r_1^2}\right)\left(1 - \frac{1}{r_2^2}\right) \ldots \left(1 - \frac{1}{r_{N.r}^2}\right)$,

and finally noticing that $\frac{1}{1-\frac{1}{2}} \cdot \frac{1}{1-\frac{1}{3}} = 3$,

we obtain $\left(1 + \frac{1}{r_1}\right)^2 \left(1 + \frac{1}{r_2}\right)^2 \ldots \left(1 + \frac{1}{r_{N.r}}\right)^2 > \frac{1}{3} M_N \log N.$

Taking the logarithms of both sides of the equation, we find

$$\Theta_1 - \frac{1}{2}\Theta_2 + \frac{1}{3}\Theta_3 - \ldots > \frac{1}{2}\log\log N + \frac{1}{2}\log M_N - \frac{1}{2}\log 3,$$

where Θ_i means the sum of ith powers of the reciprocals of all the prime numbers, not exceeding N, of the form $6n + 5$.

Either from this equation or from the one from which it is derived it at once follows that the number of primes of the form $6n + 5$ is greater than any assignable limit.

Parallel to what has been shown in the preceding case, if it could be ascertained that the sum of the terms of the aggregate Q_N whose denominators do not exceed N bears a ratio which becomes indefinitely small to the total aggregate, it would follow by strict demonstration that the sum of the reciprocals of the primes of the form $6n + 5$ inferior to N would always differ by a limited quantity from the half of the second logarithm of N.

It is perhaps worthy of remark that the infinitude of primes of the forms $4n + 3$ and $6n + 5$ may be regarded as a simple rider to Euclid's proof (Book IX., Prop. 20) of the infinitude of the number of primes in general.

The point of this is somewhat blunted in the way in which it is presented in our ordinary text-books on arithmetic and algebra.

What Euclid gives is something more than this*: his statement is, "There are more prime numbers than any proposed multitude (πλῆθος) of prime numbers"; which he establishes by giving a formula for finding at least one more than any proposed number. He does not say, as our text-book writers do, "if possible let $A, B, \ldots C$ be *all* the prime numbers," &c., but simply that if $A, B, \ldots C$ are *any* proposed prime numbers, one or more additional ones may be found by adding unity to their product which will either itself be a prime number, or contain at least one additional prime; which is all that can correctly be said, inasmuch as the augmented product may be the power of a prime.

* Whereas the English elementary book writers content themselves with showing that to suppose the number of primes finite involves an absurdity, Euclid shows how from any given prime or primes to generate an infinite succession of primes.

Thus from one prime number arbitrarily chosen, a progression may be instituted in which one new prime number at least is gained at each step, and so an indefinite number may be found by Euclid's formula: for example, 17 gives birth to 2 and 3; 2, 3, 17 to 103; 2, 3, 17, 103 to 7, 19, 79; and so on.

We may vary Euclid's mode of generation and avoid the transcendental process of decomposing a number into its prime factors by using the more general formula, $a, b, \ldots c+1$, where $a, b, \ldots c$, are any numbers relatively prime to each other; for this formula will obviously be a prime number or contain one or more distinct factors relatively prime to $a, b, \ldots c$.

The effect of this process will be to generate a continued series of numbers all of which remain prime to each other: if we form the progression

$$a,\ a+1,\ a^2+a+1,\ a(a+1)(a^2+a+1)+1, \ldots$$

and call these successive numbers

$$u_1,\ u_2,\ u_3,\ u_4, \ldots$$

we shall obviously have $$u_{x+1} = u_x^2 - u_x + 1.$$

It follows at once from Euclid's point of view that no primes contained in any term up to u_x can appear in u_{x+1}, so that all the terms must be relatively prime to each other. The same consequence follows *a posteriori* from the scale of relation above given; for, as I had occasion to observe in the *Comptes Rendus* for April 1888 [see p. 620, below], if dealing only with rational integer polynomials,

$$\phi(x) = (x-a) f(x) + a,$$

then, whatever value, c, we give to x, no two forms $\phi^i(c)$, $\phi^j(c)$ can have any common measure not contained in a: in this case $\phi(x) = (x-1)x+1$; so that $\phi^i(c)$ and $\phi^j(c)$ must be relative primes for all values of i and j*.

It is worthy of remark that all the primes, other than 3, implicitly obtained by this process will be of the form $6i+1$.

Euclid's own process, or the modified and less transcendental one, may be applied in like manner to obtain a continual succession of primes of the form $4n+3$ and $6n+5$.

As regards the former, we may use the formula

$$2 \,.\, a \,.\, b \ldots c + 1$$

(where $a, b, \ldots c$ are any "proposed" primes of the form $4n+3$), which will necessarily be of the form $4n+3$, and must therefore contain *one* factor at least of that form.

* Another theorem of a similar kind is that, whatever integer polynomial $\phi(x)$ may be, if i, j have for their greatest common measure k, then $\phi^k[\phi(0)]$ will be the greatest common measure of $\phi^i[\phi(0)]$, $\phi^j[\phi(0)]$.

As regards the latter, we may employ the formula

$$3 \,.\, a \,.\, b \ldots c + 2$$

(where $a, b, \ldots c$ are each of the form $6n + 5$), which will necessarily itself be, and therefore contain *one* factor at least, of that form.

The scale of relation in the first of these cases will be, as before,

$$u_{x+1} = u_x^2 - u_x + 1\,;$$

so that each term in the progression, abstracting 3, will be of the form $4i + 3$ and $6j + 1$ conjointly, and consequently of the form $12n + 7$; as for example,

$$3, \; 7, \; 43, \; 1807, \ldots.$$

In the latter case the scale of relation is

$$u_{x+1} = u_x^2 - 2u_x + 2,$$

which is of the form $(u_x - 2)\, u_x + 2$. It is obvious that in each progression at each step one new prime will be generated, and thus the number of ascertained primes of the given form go on indefinitely increasing, as also might be deduced *a posteriori* by aid of the general formula above referred to from the scale of relation applicable to each. Each term in the second case (the term 3, if it appears, excepted) will be simultaneously of the form $6i - 1$ and $4j + 1$, and consequently of the form $12n + 5$, as in the example 5, 17, 257, 65537, ...

The same simple considerations *cease* to apply to the genesis of primes of the forms $4n + 1$, $6n + 1$. We may indeed apply to them the formulae

$$(2 \,.\, a \,.\, b \ldots c)^2 + 1 \quad \text{and} \quad 3\,(a \,.\, b \ldots c)^2 + 1$$

respectively, but then we have to draw upon the theory of quadratic forms in order to learn that their divisors are of the form $4n + 1$ and $6n + 1$ respectively.

Of course the difference in their favour is that in their case *all* the divisors locked up in the successive terms of the two progressions respectively are of the prescribed form; whereas in the other two progressions, whose theory admits of so much simpler treatment, we can only be assured of the presence of *one* such factor in each of the several terms.

Euler has given the values of two infinite products, without any evidence of their truth except such as according to the lax method of dealing with series without regard to the laws of convergence prevalent in his day, and still held in honour in Cambridge down to the times of Peacock, De Morgan, and Herschel inclusive (and this long after Abel had justly denounced the use of divergent series as a crime against reason), was erroneously supposed to amount to a proof, from which the same consequences may be derived as shown in the foregoing pages, and something more besides*. These two theorems are

$$(1) \qquad \frac{3}{3+1}\cdot\frac{5}{5-1}\cdot\frac{7}{7+1}\cdot\frac{11}{11+1}\cdot\frac{13}{13-1}\cdots=\frac{\pi}{4}$$

(where, corresponding to the primes 3, 7, 11, ... of the form $4n+3$, the factors of the product on the left are

$$\frac{3}{3+1},\ \frac{7}{7+1},\ \frac{11}{11+1},\ \cdots$$

all of them with the sign + in the denominator; while the fractions corresponding to primes of the form $4n+1$ have the − sign in their denominators).

$$(2) \qquad \frac{5}{5+1}\cdot\frac{7}{7-1}\cdot\frac{11}{11+1}\cdot\frac{13}{13-1}\cdot\frac{17}{17+1}\cdots=\frac{\pi}{2}\sqrt{3}$$

where, as in the previous product, the sign in the denominator of each fraction depends on the form of the prime to which it corresponds (being + for primes of the form $6n-1$, and − for primes of the form $6n+1$).

Dr J. P. Gram (*Mémoires de l'Académie Royale de Copenhague*, 6me série, Vol. II. p. 191) refers to a paper by Mertens ("Ein Beitrag zur analytischen Zahlentheorie," *Borchardt's Journal*, Bd 78), as one in which the truth of the first of the two theorems is demonstrated—"fuldstœndigt Bevis af Mertens" are Gram's words†.

* It follows from the first of these theorems that with the understanding that no denominator is to exceed n (an indefinitely great number),

$$\left(1+\frac{1}{3}\right)\left(1+\frac{1}{7}\right)\left(1+\frac{1}{11}\right)\left(1+\frac{1}{19}\right)\cdots$$

bears a finite ratio to

$$\left(1+\frac{1}{5}\right)\left(1+\frac{1}{13}\right)\left(1+\frac{1}{17}\right)\cdots$$

so that as their *product* is known to be infinite, each of these two partial products must be separately infinite; in like manner from Euler's second theorem a similar conclusion may be inferred in regard to each of the two products

$$\left(1+\frac{1}{5}\right)\left(1+\frac{1}{11}\right)\left(1+\frac{1}{17}\right)\left(1+\frac{1}{23}\right)\left(1+\frac{1}{29}\right)\left(1+\frac{1}{41}\right)\cdots$$

and

$$\left(1+\frac{1}{7}\right)\left(1+\frac{1}{13}\right)\left(1+\frac{1}{19}\right)\left(1+\frac{1}{31}\right)\cdots.$$

† It always seems to me absurd to speak of a complete proof, or of a theorem being rigorously demonstrated. An incomplete proof is no proof, and a mathematical truth not rigorously demonstrated is not demonstrated at all. I do not mean to deny that there are mathematical truths, morally certain, which defy and will probably to the end of time continue to defy proof, as, for example, that every indecomposable integer polynomial function must represent an infinitude of primes. I have sometimes thought that the profound mystery which envelops our conceptions relative to prime numbers depends upon the limitation of our faculties in regard to time, which like space may be in its essence poly-dimensional, and that this and such sort of truths would become self-evident to a being whose mode of perception is according to *superficially* as distinguished from our own limitation to *linearly* extended time.

Assuming this to be the case, we shall easily find when N is indefinitely great, so that S_N becomes $\frac{\pi}{4}$,

$$Q_N S_N = \frac{1}{\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\ldots\left(1-\frac{1}{N}\right)},$$

which, according to Legendre's empirical law (Legendre, *Théorie des Nombres*, 3rd edition, Vol. II. p. 67, Art. 397), is equal to $\frac{2\log N}{K}$, where $K = 1{\cdot}104$; and as we have written $Q_N S_N = \log N + (V - R)$, we may deduce, upon the above assumptions,

$$V - R = \left(\frac{2}{K} - 1\right)\log N = 0{\cdot}811 \ldots \log N.$$

R, we know, is demonstrably less than $\left(1 - \frac{\pi}{4}\right)\log N$, consequently V must be less than $(0{\cdot}812 + 0{\cdot}215)\log N$, that is, less than $1{\cdot}027 \log N$, and *a fortiori* the portion of the omnipositive aggregate Q_N, which consists of terms whose denominators exceed N, when N is indefinitely great, cannot be less than $\frac{4}{\pi}\left(1 - \frac{\pi}{4}\right)\log N$, that is, $0{\cdot}273 \log N$.

Before concluding, let me add a word on Legendre's empirical formula for the value of

$$\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)\ldots\left(1-\frac{1}{p_{N.p}}\right),$$

referred to in the early part of this article.

If N is any odd number, the condition of its being a prime number is that when divided by any odd prime less than its own square root, it shall not leave a remainder zero. Now if N (an unknown odd number) is divided by p, its remainder is equally likely to be 0, 1, 2, 3, ... or $(p-1)$. Hence the chance that it is not divisible by p is $\left(1 - \frac{1}{p}\right)$, and, if we were at liberty to regard the like thing happening or not for any two values of p within the stated limit as independent events, the expectation of N being a prime number would be represented by

$$\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)\left(1-\frac{1}{7}\right)\left(1-\frac{1}{11}\right)\ldots\left(1-\frac{1}{p_{N^{\frac{1}{2}}.p}}\right),$$

which, according to the formula referred to, for infinitely large values of N is equal to $\frac{1{\cdot}104}{\log N^{\frac{1}{2}}}$. It is rather more convenient to regard N as entirely unknown instead of being given as odd, on which supposition the chance of its being a prime would be $\frac{1{\cdot}104}{2\log N^{\frac{1}{2}}}$ or $\frac{1{\cdot}104}{\log N}$.

Hence for very large values of N the sum of the logarithms of all the primes inferior to N might be expected to be something like $(1{\cdot}104)\,N$. This does not contravene Tchebycheff's formula (Serret, *Cours d'Algèbre Supérieure*, 4me ed., Vol. II. p. 233), which gives for the limits of this sum AN and BN, where $A = 0{\cdot}921292$, and $B = \frac{6A}{5} = 1{\cdot}10555$; but does contravene the narrower limits given by my advance upon Tchebycheff's method [see Vol. III. of this Reprint, p. 530], according to which for A, B, we may write A_1, B_1, where

$$A_1 = 0{\cdot}921423,\ B_1 = 1{\cdot}076577\,*.$$

That the method of probabilities *may* sometimes be successfully applied to questions concerning prime numbers I have shown reason for believing in the two tables published by me [above, p. 101] in the *Philosophical Magazine* for 1883†.

* Namely $A_1 = \frac{51072}{50999}A$, and $B_1 = \frac{59595}{50999}A$, the values of which are incorrectly stated in the memoir. Strange to say, Dr Gram, in his prize essay, previously quoted, on the number of prime numbers under a given limit, has omitted all reference to this paper in his bibliographical summary of the subject, which is only to be accounted for by its having escaped his notice; a narrowing of the asymptotic limits assigned to the sum of the logarithms of the prime numbers series being the most notable fact in the history of the subject since the publication of Tchebycheff's memoir. Subjectively, this paper has a peculiar claim upon the regard of its author, for it was his meditation upon the two simultaneous difference-equations which occur in it that formed the starting-point, or incunabulum, of that new and boundless world of thought to which he has given the name of Universal Algebra. But, apart from this, that the superior limit given by Tchebycheff as 1·1055 should be brought down by a more stringent solution of his own inequalities to only 1·076577—in other words, that the excess above the probable mean value (unity) should be reduced to little more than ⅔rds of its original amount—is in itself a surprising fact. Perhaps the numerous (or innumerable) misprints and arithmetical miscalculations which disfigure the paper may help to account for the singular neglect which it has experienced. It will be noticed that the mean of the limits of Tchebycheff is 1·01342, the mean of the new limits being 0·99900. The excess in the one case above and the defect in the other below the probable true mean are respectively 0·01342 and 0·00100.

† A principle precisely similar to that employed above if applied to determining the number of reduced proper fractions whose denominators do not exceed a given number n, leads to a correct result. The expectation of two numbers being prime to each other will be the product of the expectations of their not being each divisible by any the same prime number. But the probability of one of them being divisible by i is $\frac{1}{i}$, and therefore of two of them being not each divisible by i is $\frac{1}{i^2}$. Hence the probability of their having no common factor is

$$\left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right)\left(1-\frac{1}{49}\right)\left(1-\frac{1}{121}\right)\ldots \textit{ad inf.}, \text{ that is, is } \frac{6}{\pi^2}.$$

If, then, we take two sets of numbers, each limited to n, the probable number of relatively prime combinations of each of one set with each of the other should be $\frac{6n^2}{\pi^2}$, and the number of reduced proper fractions whose denominators do not exceed n should be the half of this or $\frac{3n^2}{\pi^2}$. I believe M. Césaro has claimed the prior publication of this mode of reasoning, to which he is heartily welcome. The number of these fractions is the same thing as the sum of the *totients* of all numbers not exceeding n. In the *Philosophical Magazine* for 1883 (Vol. xv. p. 251), a table of these sums of totients has been published by me for all values of n not exceeding 500, and [above, p. 101] in the same year (Vol. xvi. p. 231) the table was extended to values of n not exceeding 1000. In every case without any exception the estimated value of this totient sum is found to be intermediate between

$$\frac{3n^2}{\pi^2} \text{ and } \frac{3(n+1)^2}{\pi^2}.$$

Calling the totient sum to n, $T(n)$, I stated the exact equation

$$T(n)+T\left(\frac{n}{2}\right)+T\left(\frac{n}{3}\right)+T\left(\frac{n}{4}\right)+\ldots=\frac{n^2+n}{2},$$

from which it is capable of proof, without making any assumption as to the form of Tn, that its asymptotic value is $\frac{3n^2}{\pi^2}$. The functional equation itself is merely an integration (so to say) of the well-known theorem that any number is equal to the sum of the totients of its several divisors. The introduction to these tables will be found very suggestive, and besides contains an interesting bibliography of the subject of Farey series (*suites de Farey*), comprising, among other writers upon it, the names of Cauchy, Glaisher, and Sir G. Airy, the last-named as author of a paper on toothed wheels, published, I believe, in the "Selected Papers" of the Institute of Mechanical Engineers. The last word on the subject, as far as I am aware, forms one of the *interludes*, or rather the *postscript*, to my "Constructive Theory of Partitions," published in the *American Journal of Mathematics* [above, p. 55].

55.

SUR LES NOMBRES PARFAITS.

[*Comptes Rendus*, CVI. (1888), pp. 403—405.]*

EXISTE-T-IL des nombres parfaits impairs? C'est une question qui reste indécise.

Dans un article intéressant de M. Servais, paru dans le journal *Mathesis* en octobre 1887, on trouve cette proposition qu'*un nombre parfait* (*s'il y en a*) *qui ne contient que trois facteurs premiers distincts est nécessairement divisible par* 3 *et* 5. Je vais démontrer ici qu'un tel nombre n'existe pas, au moyen d'un genre de raisonnement qui m'a fourni aussi une démonstration de ce théorème qu'*il n'existe pas de nombre parfait qui contienne moins de six facteurs premiers distincts.*

On voit facilement que la somme de la série géométrique

$$1 + c + c^2 + \ldots + c^i,$$

où c est impair, sera elle-même paire quand i est impair; de plus, quand i est pair, cette somme sera toujours paire, mais impairement paire seulement dans le cas où $c \equiv i \equiv 1 \pmod 4$.

Donc, si un nombre parfait impair est de la forme $p^i q^j r^k \ldots$, ($p, q, r, \ldots$ étant des nombres premiers distincts), tous les indices $i, j, k, \ldots$ doivent être pairs à l'exception d'un seul, soit i, lequel, de même que sa base p, sera congru à 1 par rapport au module 4; car on doit avoir

$$\int p^i \int q^j \int r^k \ldots = 2p^i q^j r^k \ldots,$$

$\int x^i$ représentant $1 + x + \ldots + x^i$, c'est-à-dire $\dfrac{x^{i+1} - 1}{x - 1}$.

Ainsi, on voit qu'un nombre parfait impair (si un tel nombre existe) sera de la forme

$$M^2 (4q + 1)^{4k+1},$$

$4q + 1$ étant un nombre premier qui ne divise pas M.

[* See also below, p. 615.]

Comme corollaire, on peut déduire qu'aucun nombre parfait impair ne peut être divisible par 105; en effet, soit un tel nombre

$$3^{2i}5^x7^{2k}\ldots,$$

on aura $$\frac{\int 3^{2i}\int 5^x\int 7^{2k}}{3^{2i}5^x7^{2k}} \gtreqqless \left(1+\frac{1}{3}+\frac{1}{3^2}\right)\left(1+\frac{1}{5}\right)\left(1+\frac{1}{7}+\frac{1}{7^2}\right);$$

c'est-à-dire $\gtreqqless \dfrac{2\,.\,13\,.\,19}{5\,.\,49}$, c'est-à-dire $\dfrac{494}{245}$; qui est plus grand que 2.

Remarquons qu'en général, si $p^iq^jr^k\ldots$ est un nombre parfait, il fait que $\dfrac{p^{i+1}}{p^i(p-1)}\dfrac{q^{j+1}}{q^j(q-1)}\ldots$, c'est-à-dire $\dfrac{p}{p-1}\dfrac{q}{q-1}\dfrac{r}{r-1}\ldots$, soit plus grand que 2.

Ainsi, à moins que le plus petit des éléments p, q, r, ... ne soit plus grand que 3, on doit avoir

$$\frac{5}{4}\frac{7}{6}\frac{11}{10}\frac{13}{12}\frac{17}{16}\frac{19}{18}\ldots > 2;$$

mais en ne dépassant pas 19, ce produit est moindre que 1,94963. Conséquemment le nombre des éléments, dans ce cas, doit être 7, au moins. Puisque $1{,}95\times\left(1+\dfrac{1}{40}\right)<2$, on voit immédiatement que, si un nombre parfait à 7 éléments parmi lesquels 3 ne figurent pas existe, le septième élément ne pourrait pas dépasser 37.

Passons au cas de 3 éléments 3, q, r d'un nombre parfait impair. Puisque $\dfrac{3}{2}\dfrac{7}{6}\dfrac{11}{10}=\dfrac{231}{120}<2$, on voit que $3^i7^j11^k$, et à plus forte raison $3^ip^jq^k$, où p, q sont des nombres quelconques autres que 3 ou 5, ne peut être un nombre parfait.

Supposons donc que 3, 5, q sont les éléments d'un nombre parfait: puisque $\dfrac{3}{2}\dfrac{5}{4}\dfrac{17}{16}=\dfrac{255}{128}<2$, on voit que q ne peut être ni 17, ni un nombre quelconque plus grand que 17. Donc $q=11$ ou $q=13$; car nous avons vu que 3, 5, 7 ne peuvent jamais se trouver réunis comme éléments d'un nombre parfait quelconque.

(1) Soient 3, 5, 13 les éléments. L'indice de 13 ne peut pas être impair, car alors le nombre $\int 13^{2i+1}=\dfrac{13^{2i+2}-1}{13-1}$ contiendrait le facteur 7, et 7 devrait être un des éléments. Il s'ensuit que $(3^{2i+1}-1)(13^{2j+1}-1)$ devrait contenir 5; mais, par rapport au module 5, une puissance impaire quelconque de 3 ou 13 est congrue à 3 ou à 2. Donc la combinaison 3, 5, 13 est inadmissible.

(2) Soient 3, 5, 11 les éléments.

L'indice de 5 doit être de la forme $4j+1$; mais, si $j>0$,

$$\int 5^{4j+1} = \frac{5^{4j+2}-1}{5-1}$$

contiendra les trois nombres impairs premiers entre eux*

$$\frac{5^{2j+1}-1}{5-1}, \quad \frac{5^{2j+1}+1}{5+1}, \quad \frac{5+1}{2}.$$

Conséquemment, il y aura au moins trois autres éléments en plus de 5, ce qui est inadmissible : donc le nombre sera de la forme $3^{2i}5\ 11^{2k}$.

Donc $(1+5)(11^{2k+1}-1)$ doit contenir 9, ce qui est impossible ; car $11^{2k+1} \equiv 2 \pmod 3$.

Ainsi, on voit qu'un nombre impair avec 3 éléments seulement ne peut exister.

Quant aux nombres parfaits pairs, Euclide a démontré que $2^n \int 2^n$, c'est-à-dire $2^n(2^{n+1}-1)$, est un nombre parfait pourvu que $2^{n+1}-1$ soit un nombre premier. Mais on doit à Euler la seule preuve que je connaisse de la proposition réciproque qu'*il n'existe pas de nombres pairs parfaits autres que ceux d'Euclide.*

[* See below, p. 615.]

56.

SUR UNE CLASSE SPÉCIALE DES DIVISEURS DE LA SOMME D'UNE SÉRIE GÉOMÉTRIQUE.

[*Comptes Rendus*, CVI. (1888), pp. 446—450.]

En l'honneur du grand et surprenant Fermat, dont j'ai vu avec une émotion indicible gravés sur le buste au musée de Toulouse les mots qui lui étaient adressés par Blaise Pascal : "Au plus grand homme de l'Europe," je me propose de nommer la fonction fondamentale de la haute Arithmétique $\Theta^M - 1$ le *fermatien* à la *base* Θ et à l'*indice* M.

De plus, je nommerai la fonction $\dfrac{\Theta^M - 1}{\Theta - 1}$, qui n'est autre chose que la somme d'une série géométrique dont la raison est un entier, le *fermatien réduit.* M (bien entendu) est un entier positif quelconque, mais Θ un entier positif ou *négatif.*

Les nombres premiers qui divisent un nombre quelconque, je les nomme ses *éléments.*

On sait, d'après Euler, que tout diviseur d'un fermatien sera de la forme $x\mu + 1$, où μ est M ou bien un diviseur quelconque de M. Parmi ces diviseurs, il y a une classe toute spéciale qui correspond aux cas de $\mu = 1$ et de $\mu = -1$. Le caractère spécial de ces diviseurs du fermatien, c'est qu'ils doivent nécessairement être (comme on verra immédiatement) en même temps diviseurs de son indice. Je remarque préalablement que, $\Theta^{p^a} - 1$ (où p est un nombre premier) étant, par rapport au module p, congru à $\Theta - 1$, afin que ce fermatien contienne p, il faut que $\Theta - 1$ le contienne.

(1) Soit $M = p$ un nombre premier *impair* : je dis que le fermatien réduit $\dfrac{\Theta^p - 1}{\Theta - 1}$ contiendra p, mais non pas p^2. Car, en mettant $\Theta = kp + 1$, on voit que le fermatien réduit $\dfrac{\Theta^p - 1}{\Theta - 1}$, envisagé comme la somme d'une série géométrique, sera congru par rapport au module p^2 à $p + k\dfrac{p^2 - p}{2}p$, c'est-à-dire à p.

(2) Soit M la puissance d'un nombre premier impair p^a. En supposant toujours que $\Theta - 1$ contient p, $\Theta^x - 1$ le contiendra.

Conséquemment, puisque $\dfrac{\Theta^{p^a}-1}{\Theta-1} = \dfrac{\Theta^{p^a}-1}{\Theta^{p^{a-1}}-1}\,\dfrac{\Theta^{p^{a-1}}-1}{\Theta^{p^{a-2}}-1}\cdots\dfrac{\Theta^{p}-1}{\Theta-1}$, il suit comme conséquence de ce qui précède que $\dfrac{\Theta^{p^a}-1}{\Theta-1}$ sera divisible par p^a, mais non pas par p^{a+1}.

(3) Soit $M = Np^a$, où N est premier à p; on a

$$\frac{\Theta^{Np^a}-1}{\Theta-1} = \frac{\Theta^{Np^a}-1}{\Theta^N-1}\,\frac{\Theta^N-1}{\Theta-1};$$

le premier facteur peut être envisagé comme fonction de Θ^N et par le cas précédent sera divisible par p^a, mais non pas par p^{a+1}. Le second facteur, envisagé comme la somme d'une série géométrique, sera congru à N par rapport à p (quel que soit N pair ou impair) et conséquemment ne contiendra pas p. Donc $\dfrac{\Theta^{Np^a}-1}{\Theta-1}$ sera divisible par p^a, mais non par p^{a+1}.

Ainsi, si p est un élément quelconque impair de $\Theta - 1$ et p^a la plus haute puissance de p contenu dans M, le fermatien réduit $\dfrac{\Theta^M-1}{\Theta-1}$ contiendra p^a, mais ne contiendra pas p^{a+1} et, comme conséquence particulière, ne contiendra nul élément de $\Theta - 1$ qui n'est pas un diviseur de M.

On peut aussi supposer que $\Theta - 1$ contient chaque élément de M, et l'on obtient le théorème suivant:

Un fermatien réduit à indice impair, dont le dénominateur est divisible par chaque élément de son indice, sera lui-même divisible par cet indice, et de plus le quotient qui résulte de la division de l'une de ces quantités par l'autre sera premier relatif à l'indice.

C'est dans les recherches sur la possibilité de l'existence de nombres parfaits autres que ceux d'Euclide que se rencontre cette théorie des fermatiens réduits qui y joue un rôle indispensable. Comme exemple de son utilité, je vais faire voir qu'un nombre de la forme $3N \pm 1$ à 7 éléments ne peut pas être un nombre parfait.

Remarquons que, si g est un des nombres gaussiens 3, 5, 17, 257,..., c'est-à-dire un nombre premier de la forme $2^n + 1$, g ne peut pas diviser un fermatien réduit à indice impair s'il ne divise pas le dénominateur; car, afin que cela eût lieu, $g - 1$ par le théorème déjà cité d'Euler devrait contenir un facteur impair.

Donc un tel fermatien réduit sera de la forme $\dfrac{(gx+1)^{\mu g}-1}{(gx+1)-1}$.

Or nous avons vu, dans la Note précédente [p. 604, above], qu'un nombre $3N \pm 1$ à 6 éléments ne peut pas être un nombre parfait, et que, si un tel nombre à 7 éléments est un nombre parfait, le plus grand d'entre eux ne peut pas excéder 37.

Il est facile de voir que ce nombre doit contenir 5, parce que

$$\frac{7}{6} \cdot \frac{11}{10} \cdot \frac{13}{12} \cdot \frac{17}{16} \cdot \frac{19}{18} \cdot \frac{23}{22} \cdot \frac{29}{28} < 2;$$

en effet, ce produit est moindre que 1,69.

Soit donc, s'il est possible, $3N \pm 1$ un nombre parfait à 7 éléments.

Les nombres premiers de la forme $4x+1$ pas plus grands que 37 sont 13, 17, 29, 37. Mais 17 ne peut pas être l'élément exceptionnel de $3N \pm 1$ parce que la somme des diviseurs du component* qui répond à 17 sera la somme d'un nombre pair de termes de la série $1 + 17 + 17^2 + 17^3 + \ldots$, laquelle nécessairement contient 3. La même chose est évidemment vraie pour un nombre quelconque, comme $2q$, qui est de la forme $12x + 5$.

Donc le component exceptionnel aura pour élément ou 13 ou 37; mais ni $13^2 - 1$ ni $37^2 - 1$ ne contient 5. Il faut donc que la somme des diviseurs du component ou à l'élément 11 ou sinon à l'élément 31 soit respectivement de la forme $\frac{11^{5\mu} - 1}{11 - 1}$ ou $\frac{31^{5\nu} - 1}{31 - 1}$, car 11 et 31 sont les seuls nombres pas plus grands que 37 de la forme $5x + 1$. Conséquemment tous les diviseurs d'une au moins des deux quantités $\frac{11^5 - 1}{11 - 1}$ ou $\frac{31^5 - 1}{31 - 1}$ seront compris parmi les éléments de $3N \pm 1$.

Selon notre théorème, les diviseurs ni de l'un ni de l'autre de ces deux fonctions ne peuvent contenir 5 et conséquemment par le théorème d'Euler seront de la forme $10x + 1$.

Or, puisque 11 n'est pas un résidu quadratique de 31, $11^5 - 1$ ne peut pas contenir 31; donc les diviseurs de $\frac{11^5 - 1}{11 - 1}$ sont compris parmi les nombres 41, 61, 71, 101, ...

$\frac{31^5 - 1}{31 - 1}$ contiendra 11, mais ne peut pas être une puissance de 11, car au module 11^2

$$4^5 (31^5 - 1) \equiv 3^5 - 4^5 \equiv 1 - 4^3 \equiv -1023,$$

c'est-à-dire $-11 . 93$,

de sorte que $31^5 - 1$ n'est pas divisible même par 11^2.

Donc les diviseurs de $\frac{31^5 - 1}{31 - 1}$ sont aussi compris parmi les nombres 41, 61, 71, 101, ...

* La plus haute puissance d'un élément d'un nombre qu'il contient se nomme un *component* de ce nombre.

Conséquemment il y aura au moins un élément du nombre parfait $3N \pm 1$ qui n'est pas moindre que 41; cette conclusion est contradictoire à l'existence de la limite supérieure 37 à la grandeur des éléments. Donc on peut affirmer en toute sûreté qu'un nombre non divisible par 3 qui contient moins que 8 facteurs premiers distincts ne peut pas être un nombre parfait.

Il y a une méthode un peu plus expéditive pour parvenir au résultat dernièrement acquis; mais, tout de même, supprimer la première méthode serait un procédé mal avisé, puisque son principe est applicable à d'autres cas où celui dont je vais faire usage se trouverait en défaut; par exemple en combinant les deux méthodes, c'est-à-dire en tenant compte en même temps des conséquences de la présence de 17 quand il figure comme élément, et de la présence de l'élément 5 dans le cas où 17 manque, je crois avoir démontré qu'un entier $3N \pm 1$ à 8 éléments ne peut pas être un nombre parfait.

Remarquons que, puisque le produit suivant, à 7 termes, où 17 manque dans le numérateur, $\frac{5}{4}\frac{7}{6}\frac{11}{10}\frac{13}{12}\frac{19}{18}\frac{23}{22}\frac{29}{28}$, est moindre que 1,988, un nombre parfait à 7 éléments non divisible par 3 ne peut pas exister sans l'élément 17. Supposons qu'un tel nombre existe. Soit η un de ses éléments (autre que 17). La somme des diviseurs du *component* qui y correspond sera de la forme $\frac{\eta^{2i+2}-1}{\eta-1}$ si η est un élément ordinaire, et de la forme $\frac{(\eta^2)^{2j+1}-1}{\eta^2-1}(\eta+1)$ si η est l'élément exceptionnel.

Dans l'un et dans l'autre cas, cette somme ne peut contenir 17 que sous la condition que η^2-1 soit divisible par 17.

Donc, puisque le produit des sommes des diviseurs des components d'un nombre parfait doit contenir tous ses éléments, il existe au moins un élément η tel que η^2-1 contient 17, c'est-à-dire il y a un élément qui est un nombre premier compris dans l'une ou l'autre des formules $17x+1$, $17x-1$; mais le plus petit nombre premier contenu dans ces formules est 67*. Ainsi, puisque

$$\frac{5}{4}\frac{7}{6}\frac{11}{10}\frac{13}{12}\frac{17}{16}\frac{19}{18}\frac{67}{66} < (1,95)\left(1+\frac{1}{66}\right) < 1,98,$$

l'existence d'un nombre parfait $3N \pm 1$ à 7 éléments est impossible.

* On pourrait facilement prouver (s'il était nécessaire pour les besoins de la démonstration du théorème) que η doit être un nombre premier de la forme $17x+1$ ou un nombre premier en même temps de la forme $17x-1$ et $12y+1$, c'est-à-dire de la forme $204x+169$, et ainsi il y aurait au moins un élément plus grand que 103.

57.

SUR L'IMPOSSIBILITÉ DE L'EXISTENCE D'UN NOMBRE PARFAIT IMPAIR QUI NE CONTIENT PAS AU MOINS 5 DIVISEURS PREMIERS DISTINCTS.

[*Comptes Rendus*, CVI. (1888), pp. 522—526.]

NOUS avons vu, dans une Note précédente, qu'un nombre parfait impair avec moins de 7 facteurs doit être divisible par 3, et aussi que nul nombre parfait ne peut être divisible par 105. Ajoutons que, puisque

$$\frac{3}{2}\cdot\frac{11}{10}\cdot\frac{13}{12}\cdot\frac{17}{16}=\frac{151\frac{15}{16}}{80}<2$$

et que, en changeant 11, 13, 17 pour d'autres éléments, on ne peut diminuer ce produit qu'en empiétant sur les chiffres 5 ou 7, il s'ensuit que l'élément 3 doit être associé ou avec 7 ou avec 5 dans un nombre parfait à quatre éléments, s'il y en a.

Supposons donc qu'un tel nombre N existe.

(1) Soient 3 et 7 deux de ses éléments. Le troisième élément en ordre de grandeur ne peut pas excéder 13; car

$$\frac{3}{2}\cdot\frac{7}{6}\cdot\frac{17}{16}\cdot\frac{19}{18}=\frac{119}{64}\left(1+\frac{1}{18}\right)<\frac{126}{64}<2.$$

(α) Soit 11 le troisième élément; puisque

$$\frac{3}{2}\cdot\frac{7}{6}\cdot\frac{11}{10}\cdot\frac{29}{28}=\frac{77}{40}\left(1+\frac{1}{28}\right)<2,$$

on voit que le quatrième élément ne peut être qu'un des nombres 13, 17, 19, 23.

Mais, parmi les éléments, un au moins doit être de la forme $4x+1$.

De plus, nous avons vu dans une Note précédente que *nul* nombre parfait ne peut contenir l'élément 17 sans contenir en même temps un élément pas plus petit que 67. Donc les quatre éléments seront 3, 7, 11, 13.

Le diviseur-somme* à 7 ne peut pas contenir le facteur algébrique 7^9-1, car alors $\frac{1}{3}\cdot\frac{7^3-1}{7-1}$, $\frac{1}{3}\cdot\frac{7^9-1}{7^3-1}$ seront diviseurs de cette somme premiers entre eux, à 3 et à 7, et en plus ne contenant pas 13 parce que 13 n'est ni une fonction unilinéaire† de q ni diviseur de 7^3-1. Ainsi sur cette supposition il y aurait au moins cinq éléments distincts. Donc le diviseur-somme à 7 ne peut pas contenir 9, mais le component à 3 contient nécessairement 3^2; conséquemment, puisque le diviseur-somme à 11 (élément ordinaire et non pas de la forme $3x+1$) ne peut pas contenir 3, le diviseur-somme à 13 contiendra un facteur algébrique de la forme $\frac{13^3-1}{13-1}$ qui est égal à $169+13+1$. Donc 61 sera un élément en plus de 3, 7, 11, 13 qui est contraire à l'hypothèse.

1. (β) Soit 13 le troisième élément.

Puisque $\frac{3}{2}\cdot\frac{7}{6}\cdot\frac{13}{12}\cdot\frac{23}{22}=\frac{91}{48}\left(1+\frac{1}{22}\right)<2$, le quatrième élément sera necessairement moins que 23, et le système des éléments sera 3, 7, 13, 19, car 17 est exclus.

Les diviseurs-sommes, ni à 13 ni à 19, ne peuvent pas contenir 3; parce qu'ils contiendraient nécessairement les facteurs $\frac{13^3-1}{13-1}$ et $\frac{19^3-1}{19-1}$, et ainsi $\frac{1+13+13^2}{3}$, c'est-à-dire 61, et $\frac{1+19+19^2}{3}$, c'est-à-dire 127.

Donc le diviseur-somme à 7 doit contenir algébriquement les facteurs $\frac{1}{3}\cdot\frac{7^9-1}{7^3-1}$, $\frac{1}{3}\cdot\frac{7^3-1}{7-1}$; ce dernier est égal à 19; le premier sera nécessairement premier à 3, 7, 19 et, pour la raison déjà donnée, à 13.

Il est donc démontré que 7 ne peut pas être un élément de N.

(2) Supposons que 3 et 5 sont deux de ses éléments.

2. A. Soit 5 l'élément exceptionnel.

2. A (α). Si l'indice à l'élément 3 est 2, alors, puisque $1+3+3^2=13$, on aura les éléments 3, 5, 13; donc le diviseur-somme à 13 doit contenir 3, et, conséquemment, contiendra algébriquement le facteur $\frac{13^2+13+1}{3}$, c'est-à-dire 61.

Ainsi on aura les éléments 3, 5, 13, 61.

Mais $\frac{1+3+3^2}{9}\cdot\frac{1+5}{5}\cdot\frac{13}{12}\cdot\frac{61}{60}<2$, ce qui est inadmissible.

* Si p est un élément et p^i un *component* d'un nombre N, on nomme p^i le *component* à p, et $\frac{p^{i+1}-1}{p-1}$ le *diviseur-somme* à p.

† Il est très commode, dans ce genre de recherches, de se servir de la phrase "fonction unilinéaire de x" pour signifier $kx+1$.

2. A (β). On peut donc supposer l'indice du component à 3 au moins 4.

Soient 3, 5, p les trois éléments ; l'indice du diviseur-somme à p ne peut pas être 9, car alors on aurait en plus de 3, 5, p deux autres éléments au moins premiers entre eux et à 3, 5, p.

Soit q le quatrième élément ; la même chose sera vraie du diviseur-somme à q.

Donc le produit des diviseurs-sommes à 3, 5, p, q ne peut pas contenir une plus haute puissance de 3 que 3^3 ; mais elle doit contenir au moins 3^4.

Ainsi l'hypothèse que 5 est l'élément exceptionnel est inadmissible.

2. B. Passons à l'hypothèse que 5 est un élément ordinaire.

Remarquons que $\frac{3}{2}\cdot\frac{5}{4}\cdot\frac{31}{30}\cdot\frac{37}{36} < 1{,}992 < 2$.

Conséquemment, il y aura au moins un élément, disons p, qui n'excède pas 29 : je dis que p ne peut pas être contenu dans le diviseur-somme de 5 ; car, si cela avait lieu, l'indice de cette somme serait nécessairement un diviseur impair de l'excès au-dessus de l'unité de quelque nombre premier inférieur à 31, c'est-à-dire 3, 5, 7, 9 ou 11, dont les quatre derniers correspondent respectivement aux nombres premiers 11, 29, 19 et 23.

Il ne peut pas être 3, car $\frac{5^3-1}{5-1}=31$; ni 5, car $\frac{5^5-1}{5-1} = 11 \,.\, 71$ (et l'on aurait une combinaison d'éléments 3, 5, 11, 71 ; laquelle est inadmissible, parce que 5 est, par hypothèse, non exceptionnel, et les autres éléments sont de la forme $4x+3$).

Il ne peut pas être 7, car on trouve facilement que 5^7-1 ne contient pas 29 ni 9 ; car, quoiqu'il soit vrai que (5 étant résidu quadratique de 19) 5^9-1 contient 19, il contient en même temps 5^3-1, et l'on aurait la combinaison 3, 5, 19, 31, qui est défendue par la même raison que l'est 3, 5, 11, 71.

Reste seulement 11, mais $5^{11}-1$ ne peut pas contenir 23, parce que 5 n'est pas résidu quadratique de 23.

Ainsi l'élément 5 ne peut pas *engendrer* (au moyen du diviseur-somme qui lui répond) un élément qui n'est pas en dehors de la limite 29.

Le diviseur-somme à un tel élément (s'il est 11 et seulement dans ce cas-là) peut contenir 5, mais non pas 5^2 ; car, s'il contenait 5^2, on aurait au moins deux diviseurs de cette somme premiers entre eux et à 3, 5, 11.

Remarquons que le component à l'élément exceptionnel ne peut pas être une puissance (à exponent $4j+1$) d'un nombre ; car, si $j>0$, $q^{4j+2}-1$ contiendrait nécessairement deux facteurs premiers distincts en addition à 3, 5 et p ; donc $j=0$; ainsi l'on voit que $q+1$ doit contenir au moins les puissances de 3 et 5 contenues en $3^2 . 5^2$, qui ne sont pas contenues dans le diviseur-somme de l'autre élément indéterminé, lequel on montre facilement ne pouvoir contenir que 3 ou 5 et non pas 3^2, $3 . 5$, ou 5^2; car, sur la première ou la dernière de ces trois hypothèses, le nombre des éléments serait plus grand que 4, et sur l'hypothèse qui reste plus grand même que 5. Donc l'élément exceptionnel augmenté par l'unité sera de la forme ou $2k . 3^2 . 5 - 1$ ou $2k . 3 . 5^2 - 1$: conséquemment sa valeur doit excéder 89; cela prouve que le p dont nous avons parlé n'est pas l'élément exceptionnel.

Soit q cet élément, on aura

$$q = 30\lambda - 1.$$

Or le diviseur-somme à 5 ne contient ni 3 ni p.

On aura donc forcément

$$\frac{5^x - 1}{5 - 1} = q = 30\lambda - 1,$$

c'est-à-dire

$$5^x - 120\lambda + 3 = 0,$$

ce qui est impossible.

Cela démontre que l'hypothèse 2. B est inadmissible, et finalement le résultat est acquis qu'il n'existe pas de nombres parfaits impairs qui soient divisibles par moins de 5 facteurs premiers; car ce théorème, pour les cas d'une multiplicité 3, 2, 1, a déjà été démontré.

Ajoutons quelques mots sur les nombres parfaits à cinq éléments.

Ici, puisque

$$\frac{3}{2} \cdot \frac{11}{10} \cdot \frac{13}{12} \cdot \frac{17}{16} \cdot \frac{23}{22} < 1{,}986,$$

mais

$$\frac{3}{2} \cdot \frac{11}{10} \cdot \frac{13}{12} \cdot \frac{17}{16} \cdot \frac{19}{18} > 2{,}004.$$

On voit qu'un nombre parfait à cinq éléments, où 5 et 7 manquent, ne peut avoir pour ces éléments que les chiffres 3, 11, 13, 17, 19.

Mais 17 (un nombre cyclotomique de Gauss) ne peut pas exister sans un élément satellite de la forme $17k \pm 1$. Donc un nombre parfait à cinq éléments, s'il existe, aura nécessairement ou les éléments 3, 5 ou les éléments 3, 7.

J'ai réussi à démontrer l'impossibilité de l'une et de l'autre de ces hypothèses; mais la preuve est trop longue pour être insérée ici.

58.

SUR LES NOMBRES PARFAITS.

[*Comptes Rendus*, CVI. (1888), pp. 641, 642; *Mathesis*, VIII. (1888), pp. 57—61.]

DANS la démonstration de l'impossibilité qu'un nombre à 3 éléments soit un nombre parfait, qui a paru dans les *Comptes rendus* du 6 février dernier, il y a une petite omission que M. Mansion a eu la bonté de me signaler. Il est dit [p. 606, above], que les nombres $\frac{5^{2j+1}-1}{5-1}$, $\frac{5^{2j+1}+1}{5+1}$, $\frac{5+1}{2}$ sont premiers entre eux.

Cela n'est pas vrai si $2j+1$ contient 3, mais, dans ce cas-là, $5^{2j+1}+1$ contiendra 5^3+1 qui contient 7: conséquemment, on aura les quatre éléments 3, 5, 7, 11. Donc la démonstration reste bonne.

M. Sylvester vient de publier [p. 604, above], dans les *Comptes Rendus de l'Académie des Sciences de Paris* (séance du 6 février 1888, t. CVI. pp. 403—405), une importante contribution à l'étude des nombres parfaits, à l'occasion de remarques de notre collaborateur M. Servais (*Mathesis*, t. VII. pp. 228—230).

Nous sommes heureux de reproduire ici les considérations développées par l'illustre géomètre anglais, comme complément des articles publiés à ce sujet dans *Mathesis* (t. VI. pp. 100—101, 145—148, 178, 248—250, et t. VII. pp. 228—230, 245—246).

La notation $c \equiv i \equiv 1 \pmod 4$ est équivalente à la notation plus explicite:

$$c = i + \mathfrak{M}\,4 = 1 + \mathfrak{M}\,4$$

et se prononce: *c est congru à i et à* 1, *suivant le module* 4.

Nous ajoutons quelques notes à l'article un peu bref de M. Sylvester pour en faciliter l'intelligence*.
P. MANSION.

Existe-t-il des nombres parfaits impairs? C'est une question qui reste indécise.

* Dans les nos. des *C. R.* du 13 et du 20 février, M. Sylvester a publié de nouvelles recherches sur les nombres parfaits dont nous ne pouvons, faute d'espace, que signaler plus bas, les conclusions en note. Il s'est aussi occupé des nombres parfaits dans les nos. de *Nature*, du 15 et du 22 décembre 1887, et dans l'*Educational Times* du 1er mars 1888.

Dans un article intéressant de M. Servais, paru dans le journal *Mathesis*, en octobre 1887, on trouve cette proposition qu'*un nombre parfait impair (s'il y en a) qui ne contient que trois facteurs premiers distincts est nécessairement divisible par* 3 *et* 5. Je vais démontrer ici qu'un tel nombre n'existe pas, au moyen d'un genre de raisonnement qui m'a fourni aussi une démonstration de ce théorème qu'*il n'existe pas de nombre parfait impair qui contienne moins de six facteurs premiers distincts.*

On voit facilement que la somme de la série géométrique

$$1 + c + c^2 + \ldots + c^i$$

où c est impair, sera elle-même paire quand i est impair; de plus, quand i est pair, cette somme sera toujours impaire, mais impairement paire seulement dans le cas où $c \equiv i \equiv 1 \pmod 4$.

Donc, si un nombre parfait impair est de la forme $p^i q^j r^k \ldots$ ($p, q, r, \ldots$ étant des nombres premiers distincts), tous les indices $i, j, k, \ldots$ doivent être pairs à l'exception d'un seul, soit i, lequel, de même que sa base p, sera congru à 1 par rapport au module 4; car on doit avoir

$$\int p^i \int q^j \int r^k \ldots = 2p^i q^j r^k \ldots,$$

$\int x^i$ représentant $1 + x + \ldots + x^i$, c'est-à-dire $\dfrac{x^{i+1} - 1}{x - 1}$.

Ainsi, on voit qu'un nombre parfait impair (si un tel nombre existe) sera de la forme $M^2 (4q + 1)^{4k+1}$, $4q + 1$ étant un nombre premier qui ne divise pas M*.

Comme corollaire, on peut déduire qu'aucun nombre parfait impair ne peut être divisible par 105. En effet, soit un tel nombre $3^{2i} 5^{2j} 7^{2k} \ldots$; on aura

$$\frac{\int 3^{2i} \int 5^{2j} \int 7^{2k}}{3^{2i} 5^{2j} 7^{2k}} \geqq \left(1 + \frac{1}{3} + \frac{1}{3^2}\right)\left(1 + \frac{1}{5}\right)\left(1 + \frac{1}{7} + \frac{1}{7^2}\right);$$

c'est-à-dire $\geqq \dfrac{2 \,.\, 13 \,.\, 19}{5 \,.\, 49}$; c'est-à-dire $\dfrac{494}{245}$, qui est plus grand que 2.

Remarquons qu'en général, si $p^i q^j r^k \ldots$ est un nombre parfait, il faut que

$$\frac{p^{i+1}}{p^i (p-1)} \frac{q^{j+1}}{q^j (q-1)} \ldots \text{ c'est-à-dire } \frac{p}{p-1} \frac{q}{q-1} \frac{r}{r-1} \ldots$$

soit plus grand que 2†.

* Théorème démontré aussi, en 1886, par M. Stern, dans *Mathesis*, t. VI. pp. 248—250, mais que l'on trouve également au no. 109, du chapitre III, de l'opuscule d'Euler: *Tractatus de numerorum doctrina*, publié dans les *Commentationes arithmeticae collectae* (voir t. II. pp. 514—515).

Il en résulte que, si 3, 7, ou 11, etc. entrent comme facteur dans un nombre parfait impair, ils y entrent avec un exposant pair, car ils sont de la forme $(4p + 3)$.

† Voir, par exemple, l'article de M. Servais, p. 230. D'après la définition des nombres parfaits, on a

$$\frac{\int p^i \,.\, \int q^j \,.\, \int r^k}{p^i q^j r^k} = 2,$$

Ainsi, à moins que le plus petit des éléments p, q, r ... ne soit pas plus grand que 3, on doit avoir

$$\frac{5}{4}\frac{7}{6}\frac{11}{10}\frac{13}{12}\frac{17}{16}\frac{19}{18}\ldots > 2;$$

mais en ne dépassant pas 19, ce produit est moindre que 1,94963. Conséquemment le nombre des éléments, dans ce cas, doit être 7 au moins. Puisque

$$1{,}95 \times \left(1 + \frac{1}{40}\right) < 2,$$

on voit immédiatement que, si un nombre parfait à 7 éléments parmi lesquels 3 ne figure pas, existe, le septième élément ne pourrait pas dépasser 37*.

Passons au cas de 3 éléments 3, q, r d'un nombre parfait impair. Puisque

$$\frac{3}{2}\frac{7}{6}\frac{11}{10} = \frac{231}{120} < 2,$$

on voit que $3^i . 7^j . 11^k$, et à plus forte raison $3^i p^j q^k$, où p, q sont des nombres quelconques autres que 3 ou 5, ne peut être un nombre parfait.

Supposons donc que 3, 5, q sont les éléments d'un nombre parfait; puisque

$$\frac{3}{2}\frac{5}{4}\frac{17}{16} = \frac{255}{128} < 2,$$

on voit que q ne peut être ni 17, ni un nombre quelconque plus grand que 17. Donc $q = 11$ ou $q = 13$; car nous avons vu que 3, 5, 7 ne peuvent jamais se trouver réunis comme éléments d'un nombre parfait quelconque.

(1) Soient 3, 5, 13 les éléments. L'indice de 13 ne peut pas être impair, car alors le nombre

$$\int 13^{2i+1} = \frac{13^{2i+2} - 1}{13 - 1}$$

---

ou encore $$\frac{p^{i+1} - 1}{p^i (p-1)} \cdot \frac{q^{j+1} - 1}{q^j (q-1)} \cdot \frac{r^{k+1} - 1}{r^k (r-1)} = 2.$$

On déduit aisément de là (1) que $(q^{j+1} - 1)(r^{k+1} - 1)$ doit être divisible par p. (2) En supprimant (-1) dans les numérateurs,

$$\frac{p}{p-1} \cdot \frac{q}{q-1} \cdot \frac{r}{r-1} > 2.$$

* Dans les *C. R.* du 13 février, M. Sylvester a prouvé qu'il ne peut y avoir de nombre parfait premier avec 3, ayant même 7 ou 8 éléments. Il se sert pour arriver à ce résultat de propriétés (déduites du théorème de Fermat) des expressions $\theta^n - 1$, $[(\theta^n - 1) : (\theta - 1)]$; il nomme ces expressions *fermatien* de base θ et d'*indice* n, et *fermatien réduit* en l'honneur du grand géomètre de Toulouse. Il rappelle, à ce propos, les mots adressés à celui-ce par Pascal : "Au plus grand homme de l'Europe," mots gravés sur le buste de Fermat au musée de Toulouse. La citation exacte de Pascal est : "Quoique vous soyez celui *de toute l'Europe* que je tiens pour *le plus grand géomètre*, etc." (Lettre du 10 août 1660).

contiendrait le facteur 7, et 7 devrait être un des éléments*. Il s'ensuit que $(3^{2i+1}-1)(13^{2j+1}-1)$ devrait contenir 5 ; mais, par rapport au module 5, une puissance impaire quelconque de 3 ou 13 est congrue à 3 ou à 2. Donc la combinaison 3, 5, 13 est inadmissible.

(2) Soient 3, 5, 11† les éléments. L'indice de 5 doit être de la forme $4j+1$; mais, si $j>0$,

$$\int 5^{4j+1} = \frac{5^{4j+2}-1}{5-1}$$

contiendra les trois nombres impairs premiers entre eux‡

$$\frac{5^{2j+1}-1}{5-1},\quad \frac{5^{2j+1}+1}{5+1},\quad \frac{5+1}{2}$$

[pourvu que $2j+1$ ne soit pas divisible par 3 ; dans ce cas, $5^{2j+1}+1$ contiendrait $5^3+1=18.7$, de sorte que 7 serait un élément].

Conséquemment, il y aura au moins trois autres éléments en plus de 5, ce qui est inadmissible ; donc le nombre sera de la forme $3^{2i}.5.11^{2k}$.

Donc $(1+5)(11^{2k+1}-1)$ doit contenir 9, ce qui est impossible ; car $11^{2k+1} \equiv 2 \pmod 3$.

Ainsi, on voit qu'un nombre parfait impair avec 3 éléments seulement ne peut exister§.

Quant aux nombres parfaits pairs, Euclide a démontré que $2^n \int 2^n$, c'est-à-dire $2^n(2^{n+1}-1)$ est un nombre parfait pourvu que $2^{n+1}-1$ soit un nombre premier. Mais on doit à Euler la seule preuve‖ que je connaisse de la proposition réciproque qu'*il n'existe pas de nombres pairs parfaits autres que ceux d'Euclide.*

NOTE. On peut encore établir le (2) comme il suit. Le nombre $\frac{5^{2j+1}-1}{5-1}$ introduit dans le premier membre de l'égalité hypothétique

$$\frac{3^{2i+1}-1}{3-1}\cdot\frac{5^{4j+2}-1}{5-1}\cdot\frac{11^{2k+1}-1}{11-1} = 2.3^{2i}.5^{4j+1}.11^{2k}$$

* $13^{2i+2}-1$ est divisible par $13^2-1=168=7\times 24$.

† 3 et 11 ont des exposants pairs (voir la première note, p. [616]).

‡ Les nombres $5^{2j+1}-1$, $5^{2j+1}+1$ n'ont d'autre diviseur commun que leur différence 2; ensuite on a

$$5^{2j+1} = \mathfrak{M}\,3+2,$$

donc $5^{2j+1}-1$ et $\frac{1}{4}(5^{2j+1}-1)$ ne sont pas divisibles par $\frac{5+1}{2}=3$. Mais $\frac{5^{2j+1}+1}{5+1}$ n'est pas toujours premier avec 3 ; en effet, $5^{2j+1}+1$ est un multiple de 9 plus 6, 0 ou 3 suivant que $2j+1$ est de la forme $3p+1$, $3p$, ou $3p+2$. Les lignes entre crochets manquent dans les *C. R.* ; elles nous ont été obligeamment communiquées par l'auteur, pour compléter la démonstration, dans le cas où 5^{2j+1} est divisible par 9. (Voir aussi la note à la suite de l'article.)

§ Dans les *C. R.* du 20 février, M. Sylvester démontre qu'il n'y a pas nombre parfait impair avec quatre éléments et annonce qu'il a prouvé qu'il n'en existe pas même avec cinq éléments.

‖ *Commentationes arithm. coll.*, p. 514, no. 107 (cité par M. Sylvester, *Nature*, 15 déc. 1887, p. 152). Voir une autre démonstration due à M. Lucas, dans *Mathesis*, t. VI. pp. 146—147.

au moins un facteur différent de 3, 5, 11. En effet, ce nombre $\frac{5^{2j+1}-1}{5-1}$ s'il n'est pas divisible par 11, introduit un autre facteur que 3, 5, 11, puisqu'il est premier avec 3, 5, 11. D'autre part, s'il est divisible par 11, il est aussi divisible par 71 ; car on a

$$5-1=4, \qquad 5^{5(2p+1)-4}-1=\mathfrak{M}\,11+4,$$

$$5^3-1=\mathfrak{M}\,11+3, \qquad 5^{5(2p+1)-2}-1=\mathfrak{M}\,11+3,$$

$$5^5-1=4\,.\,11\,.\,71, \qquad 5^{5(2p+1)}-1=\mathfrak{M}\,(5^5-1)=\mathfrak{M}\,71,$$

$$5^7-1=\mathfrak{M}\,11+2, \qquad 5^{5(2p+1)+2}-1=\mathfrak{M}\,11+2,$$

$$5^9-1=\mathfrak{M}\,11+8, \qquad 5^{5(2p+1)+4}-1=\mathfrak{M}\,11+8.$$

P. MANSION.

59.

PREUVE ÉLÉMENTAIRE DU THÉORÈME DE DIRICHLET SUR LES PROGRESSIONS ARITHMÉTIQUES DANS LES CAS OÙ LA *RAISON* EST 8 OU 12.

[*Comptes Rendus*, CVI. (1888), pp. 1278—1281, 1385—1386.]

LE principe (ou pour ainsi dire le moment intellectuel) dont nous nous servons est le suivant :

Pour démontrer que le nombre de nombres premiers d'une forme donnée est infini, cherchons à construire une progression infinie d'entiers relativement premiers entre eux, et dont chacun contiendra un nombre premier (au moins) de la forme donnée.

Dans ce qui suit, f signifie une forme fonctionnelle rationnelle entière et ne contenant que des coefficients rationnels.

LEMME I.—*Si* $u_{x+1} = fu_x$ *et si* $ff0 = f0$, *alors,* r *et* s *étant deux entiers quelconques, le plus grand diviseur commun à* u_r *et* u_s *sera un diviseur de* $f0$.

Car évidemment $u_{r+\epsilon} \equiv ff \dots f0$ (c'est-à-dire $f^\epsilon 0$) [mod u_r]. Mais $f^\epsilon 0$, par hypothèse, $= f0$.

Conséquemment, tout diviseur de u_r et u_s sera un diviseur de $f0$.

LEMME II.—*Si* $u_{x+1} = fu_x$ *et si, de plus,* $u_1 = f0$, *le plus grand diviseur commun de* u_r *et* u_s *sera* u_t, *où* t *est le plus grand diviseur commun de* r *et* s.

(1) On aura évidemment

$$u_{s+\epsilon} \equiv u_\epsilon \quad (\text{mod } u_s).$$

Conséquemment u_t sera un diviseur de $u_{2t}, u_{3t}, \dots, u_{mt}$ quel que soit m.

(2) Écrivons un *schéma* pareil à celui qui s'applique à la recherche du plus grand diviseur de r et s, c'est-à-dire

$$r - hs = v, \quad s - kv = w, \quad \dots, \quad z - ly = t, \quad y - mt = 0\,;$$

alors, en vertu de ce qui précède, u_t sera un diviseur de u_r et u_s, et tout diviseur de u_r et de u_s sera un diviseur de u_t.

Donc, si t est le plus grand diviseur commun à r et s, u_t sera le plus grand diviseur commun à u_r et u_s, ce qui était à démontrer. Il s'ensuit que, si r est premier relativement à s, u_r et u_s auront u_1 pour leur plus grand diviseur commun.

Je vais faire l'application de ce principe : (A) aux progressions arithmétiques à la raison 8, (B) à la raison 12.

A. 1. *Cas de* $8x+3$.—Écrivons

$$u_1 = 1, \quad u_2 = 2u_1^2 + 1 = 3, \quad u_3 = 2u_2^2 + 1 = 19, \quad \ldots.$$

On démontre facilement que tout u est de la forme $8m+3$, et l'on sait que les facteurs premiers de tout u sont de la forme $8n+1$ ou $8n+3$.

Conséquemment, tout u contiendra au moins un facteur de la forme $8m+3$, et tout terme de la progression infinie

$$u_3, \quad u_5, \quad u_7, \quad u_{11}, \quad u_{13}, \quad \ldots$$

contiendra un facteur premier de la forme voulue.

De plus, en vertu du second lemme, tous ces facteurs seront distincts l'un de l'autre; car sinon u_r et u_s, où r est premier à s, auraient un facteur commun autre que u_1.

On pourrait prendre une série plus générale en écrivant u_1 égal à un produit d'un nombre quelconque de nombres premiers dont aucun n'est de la forme $8m+3$, tellement combinés que $u_1 \equiv 1$ [mod 8]; le résultat restera acquis que chaque terme de la progression des u contiendra un facteur premier de la forme $8x+3$, et que tous ces facteurs seront distincts entre eux.

A. 2. *Cas de* $8x+7$.—Écrivons

$$u_1 = 1, \quad u_2 = 2(u_1+1)^2 - 1 = 7, \quad u_3 = 2(u_2+1)^2 - 1 = 127, \quad \ldots.$$

Tout $u \equiv 7$ [mod 8] : chaque diviseur premier de tout u sera de la forme $8m+1$ ou $8m+7$. Donc il entrera dans chaque terme de la progression

$$u_2, \quad u_3, \quad u_5, \quad u_7, \quad \ldots$$

un facteur de la forme $8x+7$, et de plus, en vertu du second lemme (puisque $f0 = 1$), tous ces facteurs seront distincts.

A. 3. *Cas de* $8x+1$.—Écrivons

$$u_1 = 1, \quad u_2 = u_1^4 + 1 = 2, \quad u_3 = u_2^4 + 1 = 17, \quad \ldots.$$

Tous les facteurs de chaque u, à l'exception de 2, seront de la forme $8x+1$, et, en vertu du second lemme u_3, u_5, u_7, u_{11}, u_{13}, u_{17}, seront premiers entre eux.

A. 4. *Cas de* $8x+5$.—Écrivons

$$u_1 = 1, \quad u_2 = u_1^2 + 1 = 2, \quad u_3 = u_2^2 + 1 = 5,$$
$$u_4 = u_3^2 + 1 = 26, \quad u_5 = u_4^2 + 1 = 677, \quad \ldots.$$

Chaque u_{2i+1} sera de la forme $8m+5$, et chaque diviseur premier sera ou de la forme $8n+1$ ou $8n+5$, de sorte qu'il s'en trouvera un au moins de la forme $8x+5$. Donc par le second lemme la progression

$$u_3, \quad u_5, \quad u_7, \quad u_{11}, \quad u_{13}, \quad \ldots$$

contiendra un nombre infini de nombres premiers distincts de cette forme.

B. 1. *Cas de* $12x+5$.—On démontre facilement par induction que chaque terme de rang pair de la progression précédente au delà du second sera de la forme $2(24n+13)$, et chaque terme de rang impair au delà du premier de la forme $24n+5$.

Les diviseurs premiers de chaque u seront de l'une ou l'autre des six formes $24x+1$, 5, 19, 17, 13, 21.

Supposons qu'il n'existe aucun facteur premier de la forme $24x+17$ ni de la forme $24x+5$. Alors les résidus des facteurs (par rapport à 12) appartiendront au groupe 1, 9, 13, 21. Mais on voit facilement que ce groupe est un groupe fermé : car toutes ces combinaisons binaires ne font que reproduire ces mêmes nombres.

Conséquemment, tout terme de rang impair contiendra nécessairement un facteur ou de la forme $24x+5$ ou de la forme $24x+17$, et ainsi, en vertu du second lemme, on voit que la progression déjà écrite contiendra un nombre infini de nombres premiers de la forme $12n+5$.

B. 2. *Cas de* $12x+7$.—Écrivons

$$u_1=7,\quad u_2=u_1^2-u_1+1=43,\quad u_3=u_2^2-u_2+1=1807,\quad \ldots.$$

Les diviseurs premiers de chaque u seront de la forme $12n+1$ ou $12n+7$ et u lui-même de la forme $12m+7$. Donc, en vertu du premier lemme, la suite $u_1, u_2, u_3, u_4, \ldots$ contiendra un nombre infini de nombres premiers de la forme $12x+7$*.

B. 3. *Cas de* $12x+11$.—Écrivons

$$u_1=-1,\quad u_2=3u_1^2-1=2,\quad u_3=3u_2^2-1=11,$$
$$u_4=3u_3^2-1=362,\quad \ldots.$$

Tous les u de rang impair seront de la forme $12m+11$, de sorte que leurs diviseurs premiers étant, ou de la forme $12x+1$ ou $12x+11$, il y aura un nombre infini de nombres premiers distincts contenus dans les termes de la progression

$$u_3,\quad u_5,\quad u_7,\quad u_{11},\quad \ldots.$$

B. 4. *Cas de* $12x+1$.—Écrivons

$$u_1=\theta^4-\theta^2+1,\quad u_2=u_1^4-u_1^2+1,\quad u_3=u_2^4-u_2^2+1,\quad \ldots.$$

Chaque u, selon la loi cyclotomique, ne contiendra que des facteurs de la forme $12x+1$ et, en vertu du premier lemme, $u_1, u_2, u_3, u_4, u_5, \ldots$ seront tous

* Par un procédé analogue à celui que nous avons appliqué à la progression dont nous nous sommes servis dans les cas A. 4 et B. 1 ; on peut démontrer avec l'aide de la progression 7, 43, 1807, ..., donnée plus haut, que le nombre de nombres premiers dans la double progression arithmétique à raison 30,

7, 13, 37, 43, 67, 73, ...,

contient un nombre infini de nombres premiers : à plus forte raison cette conclusion s'applique à la double progression à raison 5

2, 3, 7, 8, 12, 13, ...

premiers entre eux: donc cette progression contiendra un nombre infini de facteurs de la forme $12x+1$.

L'application du principe général énoncé au commencement n'est nullement astreinte aux progressions de la forme $\phi\theta$, $\phi\phi\theta$, $\phi\phi\phi\theta$, ... C'est ce que j'ai montré au Congrès scientifique d'Oran.

Au Congrès scientifique d'Oran nous avons indiqué :

(1) Une démonstration instantanée du théorème de Dirichlet pour le cas $Ax+1$, quel que soit A, en nous servant des fonctions cyclotomiques de l'espèce ordinaire en u, en prenant pour les indices successifs A, $2A$, $3A$, ... et en donnant à u une valeur quelconque. Ces fonctions cyclotomiques sont les facteurs irréductibles des fermatiens. Par exemple, en prenant 3 pour la base des fonctions cyclotomiques, et en ôtant de chaque cyclotome dont l'indice est une puissance de 2 le *facteur singulier* 2, on obtient la progression 2, 2, 13, 5, 121, 7, 1093, ..., dont tous les termes, en omettant le second, sont premiers entre eux, et où le terme à l'indice i (le second excepté) ne contient d'autres facteurs premiers que ceux de la forme $ix+1$. Conséquemment, en se bornant aux $i^{\text{ème}}$, $(2i)^{\text{ème}}$, $(3i)^{\text{ème}}$, $(4i)^{\text{ème}}$, ... termes, et en décomposant chacun de ces termes dans un produit de facteurs premiers distincts, la totalité de ces facteurs fournira un nombre infini de nombres premiers de la forme $ix+1$;

(2) Une démonstration beaucoup plus cachée pour le cas $Ax-1$, quand A est une puissance d'un nombre premier, au moyen des fonctions cyclotomiques qui se déduisent des fonctions dont nous avons parlé en les divisant par une puissance convenable de u, en exprimant le quotient comme fonction de $u+\frac{1}{u}$, disons v, et en attribuant à v une valeur constante dont la forme par rapport au module A ou bien à un multiple de A (capable de grandir indéfiniment) dépend de la forme du nombre premier dont A est une puissance, par rapport au module 8.

Plus récemment, nous avons étendu la même démonstration aux cas où A est une combinaison de puissances de 2, 3, 5, 7, de sorte qu'il nous paraît peu douteux que les propriétés cyclotomiques donnent le moyen de prouver le théorème de Dirichlet aussi bien pour le cas de $Ax-1$, comme pour le cas de $Ax+1$, quelle que soit la forme de A. Il nous semble donc qu'il y a quelque lieu d'espérer que le principe général (qu'on peut nommer constructif ou cosmothétique) peut servir à donner une démonstration pour le cas le plus général du théorème de Dirichlet. En addition à la méthode ici donnée et celle fournie par la théorie cyclotomique pour obtenir des progressions infinies de nombres relativement premiers entre eux, on peut se servir comme troisième méthode des *cumulants* (les numérateurs et dénominateurs de fractions continues) et sans doute d'une infinité d'autres espèces de fonctions. Toute la difficulté consiste à trouver la *forme* de progression convenable à chaque cas donné.

En ce qui regarde la théorie générale des diviseurs des fonctions cyclotomiques de toute espèce, nous renvoyons à notre article, intitulé : *Excursus A : On the divisors of cyclotomic functions* [Vol. III. of this Reprint, p. 317]; et en ce qui regarde la propriété des nombres cyclotomiques de la première et seconde espèce, privés de leur *facteur singulier*, d'être relativement premiers entre eux, à un article paru dans le journal *Nature* [see pp. 591, 625 of this Volume] du mois de mars de cette année*.

* Le cas de $12x+5$ (page [622] de la Note précédente) est mal expliqué. Afin de démontrer le théorème de Dirichlet pour ce cas il suffit de remarquer que chaque terme de rang impair (après le premier) dans la progression 1, 2, 5, 26, 677, ... est de la forme $12m+5$, et chacun de ses facteurs *premiers* de la forme $4x+1$, c'est-à-dire de la forme $12x+1$ ou $12x+5$; conséquemment il contiendra au moins un facteur premier de la forme $12x+5$.

60.

ON THE DIVISORS OF THE SUM OF A GEOMETRICAL SERIES WHOSE FIRST TERM IS UNITY AND COMMON RATIO ANY POSITIVE OR NEGATIVE INTEGER.

[*Nature,* XXXVII. (1888), pp. 417, 418.]

"Nein! Wir sind Dichter*."

—*Kronecker in Berlin.*

A REDUCED Fermatian†, $\frac{r^p-1}{r-1}$, is obviously only another name for the sum of a geometrical series whose first term is unity and common ratio an integer, r.

If p is a prime number, it is easily seen that the above reduced Fermatian will not be divisible by p, unless $r-1$ is so, in which case (unless p is 2) it will be divisible by p, but not by p^2.

This is the theorem which I meant to express [p. 591, above] in the footnote to the second column of this journal for December 15, 1887, p. 153, but by an oversight, committed in the act of committing the idea to paper, the expression there given to it is erroneous.

Following up this simple and almost self-evident theorem, I have been led to a theory of the divisors of a reduced Fermatian, and consequently of the Fermatian itself, which very far transcends in completeness the condition

* Such were the pregnant words recently uttered by the youngest of the splendid triumvirate of Berlin, when challenged to declare if he still held the opinion advanced in his early inaugural thesis (to the effect that mathematics consists exclusively in the setting out of self-evident truths,—in fact, amounts to no more than showing that two and two make four), and maintained unflinchingly by him in the face of the elegant raillery of the late M. Duhamel at a dinner in Paris, where his interrogator—the writer of these lines—was present. This doctoral thesis ought to be capable of being found in the archives of the University (I believe) of Breslau.

† The word Fermatian, formed in analogy with the words Hessian, Jacobian, Pfaffian, Bezoutiant, Cayleyan, is derived from the name of Fermat, to whom it owes its existence among recognized algebraical forms.

in which the subject was left by Euler (see Legendre's *Theory of Numbers*, 3rd edition, vol. I. chap. 2, § 5, pp. 223—27, of Maser's literal translation, Leipzig, 1886)*, and must, I think, in many particulars be here stated for the first time. This theory was called for to overcome certain difficulties which beset my phantom-chase in the chimerical region haunted by those doubtful or supposititious entities called odd perfect numbers. Whoever shall succeed in demonstrating their absolute non-existence will have solved a *problem of the ages* comparable in difficulty to that which previously to the labours of Hermite and Lindemann (whom I am wont to call the Vanquisher of PI, a prouder title in my eyes than if he had been the conqueror at Solferino or Sadowa) environed the subject of the quadrature of the circle. Lambert had proved that the Ludolphian† number could not be a fraction nor the square root of a fraction. Lindemann within the last few years, standing on the shoulders of Hermite, has succeeded in showing that it cannot be the root of any algebraical equation with rational coefficients (see Weierstrass' abridgment of Lindemann's method, *Sitzungsberichte der A. D. W. Berlin*, Dec. 3, 1885).

It had already been shown by M. Servais (*Mathesis*, Liège, October 1887), that no one-fold integer or two-fold odd integer could be a perfect number, of which the proof is extremely simple. The proof for three-fold and four-fold numbers will be seen in articles of mine in the course of publication in the *Comptes Rendus* [above, pp. 604—619], and I have been able also to extend the proof to five-fold numbers. I have also proved that no odd number not divisible by 3 containing less than eight elements can be a perfect number, and see my way to extending the proof to the case of nine elements.

How little had previously been done in this direction is obvious from the fact that, in the paper by M. Servais referred to, the non-existence of three-fold perfect numbers is still considered as problematical; for it contains a "Theorem" that if such form of perfect number exists it must be divisible by fifteen: the ascertained fact, as we must know, being that this hypothetical

* I find, not without surprise, that some of the theorems here produced, including the one contained in the corrected footnote, have been previously stated by myself in a portion of a paper "On certain Ternary Cubic Form Equations," entitled "Excursus A—On the divisors of Cyclotomic Functions" [Vol. III. of this Reprint, p. 317] the contents and almost the existence of which I had forgotten: but the mode of presentation of the theory is different, and I think clearer and more compact here than in the preceding paper; the concluding theorem (which is the important one for the theory of perfect numbers) and the propositions immediately leading up to it in this, are undoubtedly not contained in the previous paper.

I need hardly add that the term *cyclotomic* function is employed to designate the core or primitive factor of a Fermatian, because the resolution into factors of such function, whose index is a given number, is virtually the same problem as to divide a circle into that number of equal parts.

† So the Germans wisely name π, after Ludolph van Ceulen, best known to us by his second name, as the calculator of π up to thirty-six places of decimals.

theorem is the first step in the *reductio ad absurdum* proof of the non-existence of perfect numbers of this sort (see *Nature*, December 15, 1887, p. 153, written before I knew of M. Servais' paper, and recent numbers of the *Comptes Rendus*).

But after this digression it is time to return to the subject of the numerical divisors of a reduced Fermatian.

We know that it can be separated algebraically into as many irreducible functions as there are divisors in the index (unity not counting as a divisor, but a number being counted as a divisor of itself), so that if the components of the index be a^α, b^β, c^γ, ... the number of such functions augmented by unity is

$$(\alpha+1)(\beta+1)(\gamma+1)\ldots.$$

All but one of these algebraical divisors, with the exception of a single one, will also be a divisor of some other reduced Fermatian with a lower index: that one, the core so to say (or, as it is more commonly called, the irreducible primitive factor), I call a cyclotomic function of the base, or, taken absolutely, a cyclotome whose index is the index of the Fermatian in which it is contained.

It is obvious that the whole infinite number of such cyclotomes form a single infinite complex. Now it is of high importance in the inquiry into the existability of perfect numbers to ascertain under what circumstances the divisors of the same reduced Fermatian, that is, cyclotomes of different indices to the same base, can have any, and what, numerical factor in common. For this purpose I distinguish such divisors into superior or external and inferior or internal divisors, the former being greater, and the latter less, than the index.

As regards the superior divisors, the rule is that any one such cannot be other than a unilinear function of the index (I call $kx+1$ a unilinear function of x, and k the unilinear coefficient) and that a prime number which is a unilinear function of the index will be a divisor of the cyclotome when the base in regard to the index as modulus is congruous to a power of an integer whose exponent is equal to the unilinear coefficient.

As regards the inferior divisors, the case stands thus. If the index is a prime, or the power of a prime, such index will be itself a divisor. If the index is not a prime, or power of a prime, then the only possible internal divisor is the largest element contained in the index, and such element will not be a divisor unless it is a unilinear function of the product of the highest powers of all the other elements contained in the index.

It must be understood that such internal divisor in either case only appears in the first power; its square cannot be a divisor of the cyclotome.

It is easy to prove the important theorem that no two cyclotomes to the same base can have any the same *external* divisor*.

We thus arrive at a result of great importance for the investigation into the existence or otherwise of perfect odd numbers, which (it being borne in mind that in this theorem the divisors of a number include the number itself, but *not* unity) may be expressed as follows:

The sum of a geometrical series whose first term is unity and common ratio any positive or negative integer other than $+1$ *or* -1 *must contain at least as many distinct prime divisors as the number of its terms contains divisors of all kinds; except when the common ratio is* -2 *or* 2, *and the number of terms is*

* The proof of this valuable theorem is extremely simple. It rests on the following principles:

(1) That any number which is a common measure to two cyclotomes to the same base must divide the Fermatian to that base whose index is their greatest common measure. This theorem needs only to be stated for the proof to become apparent.

(2) That any cyclotome is contained in the quotient of a Fermatian of the same index by another Fermatian whose index is an aliquot part of the former one. The truth of this will become apparent on considering the form of the linear factors of a cyclotome.

Suppose now that any prime number, k, is a common measure to two cyclotomes whose indices are PQ, PR respectively, where Q is prime to R, and whose common base is Θ. Then k must measure $\Theta^P - 1$ and also $\dfrac{\Theta^{PQ}-1}{\Theta^P - 1}$; it will therefore measure Q, and similarly it will measure R; therefore $k=1$ [unless $Q=1$ or $R=1$; for suppose $Q=1$, then $\dfrac{\Theta^{PQ}-1}{\Theta^P-1}$ is unity, and no longer contains the *core* of $\Theta^{PQ}-1$]. Hence k being contained in R can only be an internal factor to one of the cyclotomes (namely, the one whose index is the greater of the two). (See footnote at end.)

The other theorem preceding this one in the text, and already given in the "Excursus," may be proved as follows:

Let k, any non-unilinear function of P, the index of a cyclotome χ, be a divisor thereto. Then, by Euler's law, there exists some number, μ, such that k divides $x^{\frac{P}{\mu}}-1$, but the cyclotome is contained algebraically in $\dfrac{x^P-1}{x^{\frac{P}{\mu}}-1}$; hence k must be contained in μ, and therefore in P. Also, k will be a divisor of $x^{\frac{P}{k}}-1$ and of $\dfrac{x^P-1}{x^{\frac{P}{k}}-1}$, which contain $x^{\frac{P}{\mu}}-1$ and χ respectively; consequently, if k is odd, k^2 will not be a divisor of $\dfrac{x^P-1}{x^{\frac{P}{k}}-1}$, and *a fortiori* not of χ. (A proof may easily be given applicable to the case of $k=2$.)

Again, let $P=Qk^i$, where Q does not contain k. Then, by Fermat's theorem, $x^{k^i} \equiv x$ [mod. k] and therefore k divides x^Q-1; but it is prime to Q. Hence, by what has been shown, k must be an external divisor of this function, and consequently a unilinear function of Q. Thus, it is seen that a cyclotome can have only one internal divisor, for this divisor, as has been shown, must be an element of the index, and a unilinear function of the product of the highest powers of all the other elements which are contained in the index.

For an extension of this law to "cyclotomes of the second order and conjugate species," see the "Excursus," where I find the words *extrinsic* and *intrinsic* are used instead of *external* and *internal.*

even in the first case, and 6 *or a multiple of* 6 *in the other, in which cases the number of prime divisors may be one less than in the general case*.*

In the theory of odd perfect numbers, the fact that, in every geometrical series which has to be considered, the common ratio (which is an element of the supposed perfect number) is necessarily odd prevents the exceptional case from ever arising.

The establishment of these laws concerning the divisors and mutual relations of cyclotomes, so far as they are new, has taken its origin in the felt necessity of proving a purely negative and seemingly barren theorem, namely the non-existence of certain classes of those probably altogether imaginary entities called odd perfect numbers: the moral is obvious, that every genuine effort to arrive at a secure basis even of a negative proposition, whether the object of the pursuit is attained or not, and however unimportant such truth, if it were established, may appear in itself, is not to be regarded as a mere gymnastic effort of the intellect, but is almost certain to bring about the discovery of solid and positive knowledge that might otherwise have remained hidden†.

* A reduced Fermatian obviously may be resolved into as many cyclotomes, less one, as its index contains divisors (unity and the number itself as usual counting among the divisors). But, barring the internal divisors, all these cyclotomes to a given base have been proved to be prime to one another, and, consequently, there must be at least as many distinct prime divisors as there are cyclotomes, except in the very special case where the base and index are such that one at least of the cyclotomes becomes equal to its internal divisor or to unity. It may easily be shown that this case only happens when the base is -2 and the index any even number, or when the base is $+2$ and the index divisible by 6; and that in either of these cases there is only a single unit lost in the inferior limit to the number of the elements in the reduced Fermatian.

† Since receiving the revise, I have noticed that it is easy to prove that the algebraical resultant of two cyclotomes to the same base is unity, except when their indices are respectively of the forms $Q(kQ+1)^h$ and $Q(kQ+1)^i$, where $(kQ+1)$ is a prime number, and Q any number (unity *not* excluded), in which case the resultant is $kQ+1$. This theorem supplies the *raison raisonnée* of the proposition proved otherwise in the first part of the long footnote.

61.

NOTE ON CERTAIN DIFFERENCE EQUATIONS WHICH POSSESS AN UNIQUE INTEGRAL.

[*Messenger of Mathematics*, XVIII. (1888–9), pp. 113—122.]

FOR greater simplicity suppose in what follows that a difference equation is expressed in terms of the arguments

$$u_x, u_{x+1}, \dots u_{x+i}.$$

I shall call u_{x+i} the highest and u_x the lowest argument respectively, or collectively the extreme or principal arguments, and the degrees in which they enter into the equation the upper and lower or extreme or principal degrees. It is these partial degrees rather than the total degree of the entire equation which determine the essential character of the solution.

If m is the upper degree and $u_0, u_1, \dots u_{i-1}$ be given it is obvious that for any value of x higher than $(i-1)$, u_x will have m^{x-i+1} values, and consequently in general there will be an infinite number of integrals whether complete or of a given order of deficiency (the deficiency being estimated by the number of relations connecting the initial values $u_0, u_1, \dots u_{i-1}$); but it may be, and is in some cases, possible to assign an integral which shall have m^{x-i+1} values, and in such case there can exist no other; such an integral may be called an unique or exhaustive one, and the equations which possess such integrals may be termed uni-solutional.

As the simplest example of such, suppose

$$u^m{}_{x+1} - u_x{}^n = 0,$$

where m and n are integers.

If we write
$$u_x = \alpha^{\left(\frac{n}{m}\right)^x}$$

we have
$$u_{x+1} = \left(\alpha^{\left(\frac{n}{m}\right)^x}\right)^{\frac{n}{m}}$$

or
$$u^m{}_{x+1} = u_x{}^n.$$

Here $u_x = \alpha^{\left(\frac{n}{m}\right)^x}$ is the one and sole complete integral of the equation; for it possesses m^x values so that there can be no other integrals whatever.

Let us now seek to form difference uni-solutional equations of the 2nd order.

To this end let $u_x = C(\alpha^{2^x} - \beta^{2^x})$, where $\alpha\beta = 1$.

Then calling $\alpha^{2^x} = P$ and $\beta^{2^x} = Q$, $PQ = 1$,

$$u_x = C(P - Q),$$
$$u_{x+1} = C(P^2 - Q^2),$$
$$u_{x+2} = C(P^4 - Q^4).$$

Hence $$\frac{u_{x+1}}{u_x} = P + Q, \quad \frac{u_{x+2}}{u_{x+1}} = P^2 + Q^2 = (P+Q)^2 - 2,$$

and $$\frac{u_{x+2}}{u_{x+1}} = \left(\frac{u_{x+1}}{u_x}\right)^2 - 2.$$

Hence the equation

$$u_x^2 u_{x+2} - u^3_{x+1} + 2u_x^2 u_{x+1} = 0$$

has for its complete integral $u_x = C(\alpha^{2^x} - \alpha^{-2^x})$, and there can be no other because when u_0, u_1 are given u_x is absolutely determined.

But furthermore we may invert the above equation by interchanging u_x and u_{x+2}, which gives the equation

$$(u_x + 2u_{x+1})\, u^2_{x+2} - u^3_{x+1} = 0,$$

of which the solution will obviously be $u_x = C\left(P - \frac{1}{P}\right)$, where $P = \alpha^{(\frac{1}{2})^x}$.

Suppose u_0, u_1 to be given; then

$$C\left(\alpha - \frac{1}{\alpha}\right) = u_0, \quad C\left(\alpha^{\frac{1}{2}} - \frac{1}{\alpha^{\frac{1}{2}}}\right) = u_1,$$

and calling $$\frac{u_0}{u_1} = 2r, \quad \alpha^{\frac{1}{2}} + \frac{1}{\alpha^{\frac{1}{2}}} = 2r, \quad \alpha^{\frac{1}{2}} - \frac{1}{\alpha^{\frac{1}{2}}} = 2\sqrt{(r^2-1)},$$

$$C = \frac{u_0}{4r\sqrt{(r^2-1)}}.$$

Hence

$$u_x = \frac{u_0}{4r\sqrt{(r^2-1)}}\left[\{r + \sqrt{(r^2-1)}\}^{(\frac{1}{2})^{x-1}} - \{r - \sqrt{(r^2-1)}\}^{(\frac{1}{2})^{x-1}}\right],$$

has exactly 2^{x-1} values, for the change of $\sqrt{(r^2-1)}$ into $-\sqrt{(r^2-1)}$ changes simultaneously the signs of the numerator and denominator of this fraction. But by the general principle u_x ought to have 2^{x-1} values in terms of u_0, u_1. Hence the above integral is *exhaustive.*

Suppose now we were to write

$$u_x = C(\alpha^{2^x} + \beta^{2^x}) \text{ with } \alpha\beta = 1;$$

for brevity sake call $u_x = f,\ u_{x+1} = g,\ u_{x+2} = h,$ then

$$C(P+Q) = f,$$
$$C(P^2+Q^2) = g,$$
$$C(P^4+Q^4) = h,$$
$$PQ = 1.$$

Hence
$$f^2 = Cg + 2C^2,$$
$$g^2 = Ch + 2C^2,$$
$$C = \frac{f^2-g^2}{g-h},$$
$$f^2 = \frac{f^2-g^2}{g-h} \cdot \frac{2f^2-g^2-gh}{g-h},$$

or
$$f^2g^2 - 2f^2gh + f^2h^2 = 2f^4 - 3f^2g^2 + g^4 - f^2gh + g^3h,$$

or
$$f^2h^2 - (g^3 + f^2g)h - g^4 + 4f^2g^2 - 2f^4 = 0,$$

or
$$u_x^2 u_{x+2}^2 - (u_{x+1}^3 + u_x^2 u_{x+1}) u_{x+2} - u_{x+1}^4 + 4u_x^2 u_{x+1}^2 - 2u_x^4 = 0,$$

of which the correlative equation is
$$-2u_{x+2}^4 + (4u_{x+1}^2 - u_{x+1}u_x + u_x^2) u_{x+2}^2 - u_{x+1}^3 u_x - u_{x+1}^4 = 0.$$

A complete solution of the former of these will therefore be
$$u_x = C(\alpha^{2^x} + \beta^{2^x}),$$

and of the latter
$$u_x = C(\alpha^{(\frac{1}{2})^x} + \beta^{(\frac{1}{2})^x}),$$

but neither of these will be an *exhaustive* solution, for in the one the most general value of u_x ought to be a 2^{x-1}-valued function and in the latter a 4^{x-1}-valued function, whereas the actual value is only one-valued in the one case and 2^{x-1}-valued in the other.

Suppose again we write
$$u_x = C(\alpha^{3^x} - \beta^{3^x}), \text{ where } \alpha\beta = 1, \text{ as before,}$$

say
$$u_x = C(P-Q), \text{ where } PQ = 1.$$

Then with the same notation as before
$$C(P-Q) = f,$$
$$C(P^3-Q^3) = g,$$
$$C(P^9-Q^9) = h,$$
$$\frac{g}{f} - 1 = P^2 + Q^2,\quad \frac{h}{g} - 1 = P^6 + Q^6,$$
$$\frac{h}{g} - 1 = \left(\frac{g}{f} - 1\right)^3 - 3\left(\frac{g}{f} - 1\right),$$

or
$$\frac{h}{g} = \left(\frac{g}{f}\right)^3 - 3\left(\frac{g}{f}\right)^2 + 3,$$
$$f^3h - 3f^3g = g^4 - 3g^3f,$$
$$\frac{h-3g}{g-3f} = \frac{g^3}{f^3}.$$

Whence it follows that the integrals of

$$\frac{u_{x+2} - 3u_{x+1}}{u_{x+1} - 3u_x} - \frac{u^3_{x+1}}{u_x^3} = 0,$$

and of

$$\frac{u^3_{x+2}}{u^3_{x+1}} - \frac{3u_{x+2} - u_{x+1}}{3u_{x+1} - u_x} = 0,$$

are respectively

$$u_x = C(\alpha^{3^x} - \alpha^{-3^x}),$$

and

$$u_x = C(\alpha^{(\frac{1}{3})^x} - \alpha^{-(\frac{1}{3})^x}),$$

with the understanding that $\alpha^{-\frac{1}{3}} . \alpha^{\frac{1}{3}} = 1$.

These integrals are evidently *exhaustive*.

By writing $\sqrt{(-1)}\,\alpha$, $-\sqrt{(-1)}\,\alpha^{-1}$ for α, α^{-1} respectively, f, g, h become increased in the ratio of $\sqrt{(-1)}$, $-\sqrt{(-1)}$, $\sqrt{(-1)}$, respectively.

Hence the equations

$$\frac{u_{x+2} + 3u_{x+1}}{u_{x+1} + 3u_x} - \frac{u^3_{x+1}}{u_x^3} = 0,$$

and

$$\frac{u^3_{x+2}}{u^3_{x+1}} - \frac{3u_{x+2} + u_{x+1}}{3u_{x+1} + u_x} = 0,$$

have for their solutions

$$u_x = C(\alpha^{3^x} + \alpha^{-3^x}) \text{ and } u_x = C(\alpha^{(\frac{1}{3})^x} + \alpha^{-(\frac{1}{3})^x}).$$

Hitherto we have been dealing with *homogeneous* uni-solutional equations. It is easy, however, to form non-homogeneous ones by an obvious process. For, if we write

$$u_x = a_1^{m^x} + a_2^{m^x} + \dots + a_i^{m^x} \text{ (m being an integer)},$$

by eliminating between

$$f_0 = \Sigma a,\ f_1 = \Sigma a^m,\ f_2 = \Sigma a^{m^2},\ \dots f_i = \Sigma a^{m^i},$$

we shall obtain a relation between the f's of the first degree in f_i and of the degree m^i in f_0, corresponding to which there will be a difference equation of the ith order in which the upper extreme degree is unity and the lower one m^i, of which the integral will be the value of u_x above written, and by interchanging $u_x, u_{x+1}, \dots u_{x+i}$ respectively with $u_{x+i}, u_{x+i-1}, \dots u_x$, another in which the lower degree is unity and the upper one m^i, of which the integral will be

$$u_x = a_1^{\left(\frac{1}{m}\right)^x} + a_2^{\left(\frac{1}{m}\right)^x} + \dots + a_i^{\left(\frac{1}{m}\right)^x},$$

each of which equations will evidently be uni-solutional.

Or, again, if instead of the a's being independent we make their product equal to unity we shall obtain uni-solutional equations of the $(i-1)$th instead of the ith order.

Thus, for example, let

$$u_x = a^{2^x} + b^{2^x} + c^{2^x} \text{ with the condition } abc = 1.$$

Then writing $u_x = f$, $u_{x+1} = g$, $u_{x+2} = h$,

$$f = A + B + C,\ g = A^2 + B^2 + C^2,\ h = A^4 + B^4 + C^4,$$

$$f^2 - g = 2(AB + AC + BC),$$

$$2(g^2 - h) = 4(A^2B^2 + A^2C^2 + B^2C^2)$$

$$= (f^2 - g)^2 - 8f.$$

Hence we obtain the uni-solutional equations

$$2u_{x+2} - u^2_{x+1} - 2u_{x+1}u_x^2 + u_x^4 - 8u_x = 0,$$

$$u^4_{x+2} - 2u_{x+1}u^2_{x+2} - 8u_{x+2} - u^2_{x+1} + 2u_x = 0,$$

of which the integrals are known and are exhaustive.

We may in a similar manner obtain uni-solutional *simultaneous* difference equations.

Thus let $\qquad u_x = C(\alpha^{3^x} - \beta^{3^x}),\ v_x = C'(\alpha^{3^x} + \beta^{3^x}),$

and call $\qquad u_x, u_{x+1}, u_{x+2}$ as before $f, g, h,$

and $\qquad v_x, v_{x+1}, v_{x+2}$ $\qquad l, m, n.$

Then

$$\frac{g}{f} = P^2 + PQ + Q^2, \quad \frac{m}{l} = P^2 - PQ + Q^2,$$

$$\frac{h}{g} = P^6 + P^3Q^3 + Q^6, \quad \frac{n}{m} = P^6 - P^3Q^3 + Q^6.$$

Hence

$$\frac{h}{g} - \frac{n}{m} = \tfrac{1}{4}\left(\frac{g}{f} - \frac{m}{l}\right)^3,$$

$$\frac{h}{g} + \frac{n}{m} = 2(P^6 + Q^6)$$

$$= 2(P^2 + Q^2)(P^4 - P^2Q^2 + Q^4)$$

$$= 2(P^2 + Q^2)\{(P^2 + Q^2)^2 - 3P^2Q^2\}$$

$$= \tfrac{1}{4}\left(\frac{g}{f} + \frac{m}{l}\right)\left\{\left(\frac{g}{f} + \frac{m}{l}\right)^2 - 3\left(\frac{g}{f} - \frac{m}{l}\right)^2\right\}$$

$$= -\tfrac{1}{4}\left(\frac{g}{f} + \frac{m}{l}\right)\left(2\frac{g^2}{f^2} - 8\frac{g}{f}\cdot\frac{m}{l} + 2\frac{m^2}{l^2}\right).$$

Hence

$$\frac{h}{g} = \tfrac{1}{8}\left(-\frac{g^3}{f^3} + 3\frac{g^2}{f^2}\cdot\frac{m}{l} + 9\frac{g}{f}\cdot\frac{m^2}{l^2} - 3\frac{m^3}{l^3}\right),$$

$$\frac{n}{m} = \tfrac{1}{8}\left(-3\frac{g^3}{f^3} + 9\frac{g^2}{f^2}\cdot\frac{m}{l} + 3\frac{g}{f}\cdot\frac{m^2}{l^2} - \frac{m^3}{l^3}\right).$$

Obviously, when u_0, u_1; v_0, v_1 are given, each u_x and v_x deduced from the above system of equations has only one value, so that their exhaustive integrals will be

$$u_x = C(\alpha^{3^x} - \beta^{3^x}),\ v_x = C'(\alpha^{3^x} + \beta^{3^x}).$$

The related system found by interchanging f with h and l with n will be

$$\frac{f}{g}=\tfrac{1}{8}\left(-\frac{g^3}{h^3}+3\frac{g^2}{h^2}\cdot\frac{m}{n}+9\frac{g}{h}\cdot\frac{m^2}{n^2}-3\frac{m^3}{n^3}\right),$$

$$\frac{l}{m}=\tfrac{1}{8}\left(-3\frac{g^3}{h^3}+9\frac{g^2}{h^2}\cdot\frac{m}{n}+3\frac{g}{h}\cdot\frac{m^2}{n^2}-\frac{m^3}{n^3}\right).$$

When f, g; l, m are given the system $\frac{1}{h}, \frac{1}{n}$ may be found by solving an equation of the 9th degree. Hence, when u_0, u_1; v_0, v_1 are given, u_2, v_2 will have 9; u_3, v_3, 81, and in general u_x, v_x will have $3^{2(x-1)}$ values which will correspond to the $3^{x-1}.3^{x-1}$ values of u_x, v_x.

The apparent number of values of each of these is $(3^x)^2$, which, however, must be reducible to $3^{x-1}.3^{x-1}$ when expressed in terms of the two initial values of u and of v, similarly to what was noticed at the outset on the reduction of the apparent multiplicity 2^x to a multiplicity 2^{x-1}.

In fact, we write

$$u_x=C\left(\alpha^{(\frac{1}{3})^x}-\beta^{(\frac{1}{3})^x}\right),\quad v_x=C'\left(\alpha^{(\frac{1}{3})^x}+\beta^{(\frac{1}{3})^x}\right),$$

$$u_0=C(\alpha-\beta),\ u_1=C(\alpha^{\frac{1}{3}}-\beta^{\frac{1}{3}});\ v_0=C'(\alpha+\beta),\ v_1=C'(\alpha^{\frac{1}{3}}+\beta^{\frac{1}{3}}),$$

$$\alpha^{\frac{2}{3}}+\alpha^{\frac{1}{3}}\beta^{\frac{1}{3}}+\beta^{\frac{2}{3}}=\frac{u_0}{u_1},\quad \alpha^{\frac{2}{3}}-\alpha^{\frac{1}{3}}\beta^{\frac{1}{3}}+\beta^{\frac{2}{3}}=\frac{v_0}{v_1},$$

$$\alpha^{\frac{1}{3}}\beta^{\frac{1}{3}}=\tfrac{1}{2}\left(\frac{u_0}{u_1}-\frac{v_0}{v_1}\right),$$

$$\alpha^{\frac{2}{3}}+\beta^{\frac{2}{3}}=\tfrac{1}{2}\left(\frac{u_0}{u_1}+\frac{v_0}{v_1}\right),$$

$$\alpha^{\frac{1}{3}}-\beta^{\frac{1}{3}}=\sqrt{\left\{\tfrac{1}{2}\left(\frac{3v_0}{v_1}-\frac{u_0}{u_1}\right)\right\}},$$

$$\alpha^{\frac{1}{3}}+\beta^{\frac{1}{3}}=\sqrt{\left\{\tfrac{1}{2}\left(\frac{3u_0}{u_1}-\frac{v_0}{v_1}\right)\right\}},$$

$$C=\frac{u_1}{\sqrt{\left\{\tfrac{1}{2}\left(\frac{3v_0}{v_1}-\frac{u_0}{u_1}\right)\right\}}},\quad C'=\frac{v_1}{\sqrt{\left\{\tfrac{1}{2}\left(\frac{3u_0}{u_1}-\frac{v_0}{v_1}\right)\right\}}},$$

$$\alpha^{\frac{1}{3}}=\sqrt{\left\{\tfrac{1}{8}\left(\frac{3u_0}{u_1}-\frac{v_0}{v_1}\right)\right\}}+\sqrt{\left\{\tfrac{1}{8}\left(\frac{3v_0}{v_1}-\frac{u_0}{u_1}\right)\right\}},$$

$$\beta^{\frac{1}{3}}=\sqrt{\left\{\tfrac{1}{8}\left(\frac{3u_0}{u_1}-\frac{v_0}{v_1}\right)\right\}}-\sqrt{\left\{\tfrac{1}{8}\left(\frac{3v_0}{v_1}-\frac{u_0}{u_1}\right)\right\}},$$

and thus for the final values of u_x and v_x, we find

$$u_x = \frac{u_1}{\sqrt{\left\{\frac{1}{2}\left(\frac{3v_0}{v_1} - \frac{u_0}{u_1}\right)\right\}}} \times \left\{\left[\sqrt{\left\{\frac{1}{8}\left(\frac{3u_0}{u_1} - \frac{v_0}{v_1}\right)\right\}} + \sqrt{\left\{\frac{1}{8}\left(\frac{3v_0}{v_1} - \frac{u_0}{u_1}\right)\right\}}\right]^{(\frac{1}{3})^{x-1}} - \left[\sqrt{\left\{\frac{1}{8}\left(\frac{3u_0}{u_1} - \frac{v_0}{v_1}\right)\right\}} - \sqrt{\left\{\frac{1}{8}\left(\frac{3v_0}{v_1} - \frac{u_0}{u_1}\right)\right\}}\right]^{(\frac{1}{3})^{x-1}}\right\},$$

$$v_x = \frac{v_1}{\sqrt{\left\{\frac{1}{2}\left(\frac{3u_0}{u_1} - \frac{v_0}{v_1}\right)\right\}}} \times \left\{\left[\sqrt{\left\{\frac{1}{8}\left(\frac{3u_0}{u_1} - \frac{v_0}{v_1}\right)\right\}} + \sqrt{\left\{\frac{1}{8}\left(\frac{3v_0}{v_1} - \frac{u_0}{u_1}\right)\right\}}\right]^{(\frac{1}{3})^{x-1}} + \left[\sqrt{\left\{\frac{1}{8}\left(\frac{3u_0}{u_1} - \frac{v_0}{v_1}\right)\right\}} - \sqrt{\left\{\frac{1}{8}\left(\frac{3v_0}{v_1} - \frac{u_0}{u_1}\right)\right\}}\right]^{(\frac{1}{3})^{x-1}}\right\},$$

each of which is unaffected by a change in the signs of the square roots, so that u_x and v_x are seen to be 3^{x-1}-valued functions, and (u_x, v_x) a 9^{x-1}-valued system, as should be the case for an exhaustive solution of the last written difference equations.

Let us tentatively go a step further in the same direction and suppose that we are given

$$u_x = C(\alpha^{5^x} - \beta^{5^x}), \quad v_x = C'(\alpha^{5^x} + \beta^{5^x}),$$

and use $f, g, h; l, m, n$ in the same way as before, and furthermore, write

$$\tfrac{1}{2}\left(\frac{g}{f} + \frac{m}{l}\right) = L, \quad \tfrac{1}{2}\left(\frac{h}{g} + \frac{n}{m}\right) = N,$$

$$\tfrac{1}{2}\left(\frac{g}{f} - \frac{m}{l}\right) = M, \quad \tfrac{1}{2}\left(\frac{h}{g} - \frac{n}{m}\right) = P,$$

we shall find

$$L = A^4 + A^2B^2 + B^4, \quad N = A^{20} + A^{10}B^{10} + B^{20},$$

$$M = AB(A^2 + B^2), \quad P = A^5B^5(A^{10} + B^{10}),$$

(where $A = \alpha^{5^x}$ and $B = \beta^{5^x}$).

Let $$A^2 + B^2 = \lambda, \quad AB = \mu.$$

Then $$L = \lambda^2 - \mu^2, \quad M = \lambda\mu,$$

and it will be seen that

$$N = (\lambda^5 - 5\lambda^3\mu^2 + 5\lambda\mu^4)^2 - \mu^{10},$$

$$P = \lambda^5\mu^5 - 5\lambda\mu^7(\lambda^2 - \mu^2).$$

For $$A^6 + B^6 = \lambda^3 - 3\lambda\mu^2,$$

and consequently

$$\lambda^5 = A^{10} + B^{10} + 5A^2B^2(A^6 + B^6) + 10A^4B^4(A^2 + B^2)$$

$$= A^{10} + B^{10} + 5\mu^2(\lambda^3 - 3\lambda\mu^2) + 10\lambda\mu^4,$$

that is $$A^{10} + B^{10} = \lambda^5 - 5\lambda^3\mu^2 + 5\lambda\mu^4.$$

The above values of N and P (remembering that $AB = \mu$) are found by substituting the expression just obtained for $A^{10} + B^{10}$ in

$$N = A^{20} + A^{10}B^{10} + B^{20},$$

$$P = A^5B^5(A^{10} + B^{10}).$$

From $$P = \lambda^5\mu^5 - 5\lambda\mu^7(\lambda^2 - \mu^2),$$

(remembering that $\lambda\mu = M$, $\lambda^2 - \mu^2 = L$), we obtain

$$P = M^5 - 5LM\mu^6.$$

Hence $$\left.\begin{aligned} \frac{M^5 - P}{5LM^4} &= \frac{\mu^3}{\lambda^3} \\ \frac{L}{M} &= \frac{\lambda}{\mu} - \frac{\mu}{\lambda} \end{aligned}\right\}.$$

From these equations we obtain by elimination

$$\left(\frac{P}{M}\right)^2 + (2M^4 + 15L^2M^2 + 5L^4)\frac{P}{M} + M^4(M^4 - 10L^2M^2 + 5L^4) = 0. \quad (1)$$

Similarly by an elimination into the details of which it is unnecessary to enter we obtain

$$3LMP + (L^2 + M^2)N = L(L^2 - 2M^2)(L^2 - M^2)^2, \quad (2)$$

which gives a *linear* relation between N and P.

Equations (1) and (2) form a non-uni-solutional system of which (as also of its inverse) we are in possession of one complete integral, and I have some grounds for suspecting that it may be possible to obtain from this a second (so-called *indirect*) integral, but am unable for the present to pursue the subject further.

The preceding investigation originated in my attention happening to be called to Vieta's well known theorem for approximating to the *Archimedean* constant (π) by means of an indefinite product of cosines of continually bisected angles. The implied connection of ideas will become apparent when one considers that any one of such cosines may be expressed as a sum of two binary exponentials with $\frac{1}{2}$ for the first index, and that thus Vieta's theorem (although presumably obtained by him as a very simple consequence of the method of exhaustions) in its essence depends on the integrability of a uni-solutional difference equation of the 2nd order of the form treated of at the outset of this paper.

62.

SUR LA RÉDUCTION BIORTHOGONALE D'UNE FORME LINÉO-LINÉAIRE À SA FORME CANONIQUE.

[*Comptes Rendus*, CVIII. (1889), pp. 651—653.]

SOIT F une fonction linéo-linéaire des deux séries de lettres

$$x_1, x_2, \ldots, x_n; \quad \xi_1, \xi_2, \ldots, \xi_n;$$

alors F contiendra n^2 termes. En assujettissant les x et les ξ respectivement à deux substitutions orthogonales indépendantes, on introduit dans la transformée $n^2 - n$ quantités arbitraires, de sorte que, en leur donnant des valeurs convenables, on doit pouvoir faire disparaître ce nombre de termes en ne conservant que les n paires dont les arguments seront (par exemple)

$$x_1\xi_1, x_2\xi_2, \ldots, x_n\xi_n.$$

On peut nommer les multiples de ces arguments les *multiplicateurs canoniques*; je vais donner la règle pour les déterminer, et en même temps pour trouver les deux substitutions orthogonales simultanées qui amènent la forme canonique. La marche à suivre sera parfaitement analogue à celle qui s'applique à la réduction d'une forme quadrique à n lettres à sa forme canonique au moyen d'une seule substitution orthogonale ; mais on remarquera, *a priori*, une distinction essentielle entre les deux questions. Pour le cas d'une seule quadrique, les multiplicateurs canoniques sont absolument déterminés; mais, pour le cas actuel, il est évident que chacun de ces multiplicateurs peut changer son signe, de sorte que ce sont les carrés de ces multiplicateurs qui doivent se présenter dans le résultat.

Il sera utile de rappeler quelques faits élémentaires sur les matrices. Le carré d'une matrice est la matrice qui se produit par la multiplication des lignes par les colonnes; il sera une matrice non symétrique dont les *racines latentes* seront les carrés des racines latentes d'une matrice donnée. Au contraire, le produit d'une matrice par son transverse donnera (selon l'ordre de la multiplication) lieu à deux matrices symétriques qu'on obtient par la multiplication des lignes par des lignes ou bien par celle des colonnes par les colonnes; ces matrices seront distinctes, mais posséderont les mêmes racines latentes, c'est-à-dire en affectant tous les termes dans la diagonale de symétrie de l'un ou de l'autre avec la même addition, soit $-\lambda$, le déterminant d'une matrice ainsi affectée sera le même pour l'un comme pour l'autre*.

En différentiant F par rapport aux x et aux ξ, on obtient deux matrices, dont l'une sera la transverse de l'autre, que je nommerai les matrices déterminatives. Avec l'aide de ces matrices on obtient une solution complète du problème voulu.

(1) Pour déterminer les multiplicateurs canoniques:

Je dis que les racines latentes de leur produit seront les carrés des multiplicateurs canoniques.

Il peut arriver qu'un de ces multiplicateurs soit zéro; alors le dernier terme de l'équation aux racines latentes, qui n'est autre chose que le carré du déterminant d'une matrice déterminative, s'évanouit; et l'on voit que le cas de la disparition d'un des n termes dans la réduite canonique est indiqué par l'évanouissement du déterminant de la matrice déterminative.

(2) Pour trouver les deux substitutions orthogonales canoniques:

Prenons une des deux matrices symétriques affectées de $-\lambda$ dans chaque terme de sa diagonale; en supprimant une quelconque de ses lignes, les n premiers mineurs de la matrice diminuée qui restent divisés chacun par la racine carrée de la somme de leurs carrés (fonctions de λ), en donnant à λ successivement les valeurs des n racines latentes, fourniront les n^2 termes d'une des substitutions orthogonales, et de même on obtient l'autre substitution orthogonale en agissant semblablement *pas à pas* sur l'autre matrice affectée: ainsi le problème de la réduction voulue est complètement résolu.

Prenons, par exemple,

$$F = 8x\xi - x\eta - 4y\xi + 7y\eta.$$

* Toutes ces racines latentes seront non seulement réelles (comme elles doivent l'être à cause de la forme symétrique de la matrice), mais aussi positives; car, en substituant λ à $-\lambda$, les coefficients de l'équation latente (en commençant avec le dernier) sont, respectivement, le carré du déterminant complet, la somme des carrés des premiers mineurs, des seconds mineurs, etc., de la matrice déterminative (le premier coefficient étant l'unité et le second la somme des carrés des coefficients de la forme bilinéaire). Chacune de ces sommes sera un invariant biorthogonal, et le déterminant de la matrice déterminative lui-même sera un invariant gauche de la forme bilinéaire.

Ajoutons que les deux matrices qui sont les carrés cauchiens de cette matrice, envisagées comme discriminants, fourniront deux quadriques (dont chacune contiendra un seul des deux systèmes donnés de lettres) qui seront des covariants orthogonaux simultanés de la fonction bilinéaire donnée.

(1) Pour trouver les multiplicateurs canoniques:

On prend la matrice déterminative dans ses deux formes

$$\begin{matrix} 8\,; & -1 \\ -4\,; & 7 \end{matrix}, \qquad \begin{matrix} 8\,; & -4 \\ -1\,; & 7 \end{matrix},$$

dont les produits affectés seront

$$\begin{matrix} 65-\lambda\,; & -39 \\ -39\,; & 65-\lambda \end{matrix}, \qquad \begin{matrix} 80-\lambda\,; & -36 \\ -36\,; & 50-\lambda \end{matrix}.$$

Ainsi, en se servant de l'un ou de l'autre, on obtient

$$\lambda^2 - 130\lambda + 2704 = 0,$$

dont les racines sont 26 et 104, de sorte que $\sqrt{26}$ et $2\sqrt{26}$ seront les multiplicateurs canoniques.

(2) Pour trouver les substitutions, on assigne ses deux valeurs à

$$39 : 65 - \lambda, \quad \text{c'est-à-dire} \quad 39 : 39 \ \text{et} \ 39 : -39$$

et à

$$36 : 80 - \lambda, \quad \text{c'est-à-dire} \quad 36 : 54 \ \text{et} \ 36 : -24.$$

Ainsi l'on aura, pour les deux matrices de substitution,

$$\begin{matrix} \dfrac{1}{\sqrt{2}}\,; & \dfrac{1}{\sqrt{2}} \\ -\dfrac{1}{\sqrt{2}}\,; & \dfrac{1}{\sqrt{2}} \end{matrix} \quad \text{et} \quad \begin{matrix} \dfrac{2}{\sqrt{13}}\,; & \dfrac{3}{\sqrt{13}} \\ -\dfrac{3}{\sqrt{13}}\,; & \dfrac{2}{\sqrt{13}} \end{matrix},$$

et, en effet, on vérifie facilement que

$$\sqrt{26}\left(\frac{x}{\sqrt{2}}+\frac{y}{\sqrt{2}}\right)\left(\frac{2\xi}{\sqrt{13}}+\frac{3\eta}{\sqrt{13}}\right)+2\sqrt{26}\left(-\frac{x}{\sqrt{2}}+\frac{y}{\sqrt{2}}\right)\left(-\frac{3\xi}{\sqrt{13}}+\frac{2\eta}{\sqrt{13}}\right)$$
$$= 8x\xi - x\eta - 4y\xi + 7y\eta.$$

Si l'on donne les deux matrices symétriques ayant les mêmes racines latentes qui doivent représenter respectivement les deux produits *cauchiens* d'une matrice de l'ordre n par elle-même, on verra facilement que le problème de trouver cette dernière matrice a été virtuellement résolu plus haut, et que, comme le problème de trouver la véritable racine carrée d'une seule matrice générale donnée, il admet 2^n solutions.

63.

SUR LA CORRESPONDANCE COMPLÈTE ENTRE LES FRACTIONS CONTINUES QUI EXPRIMENT LES DEUX RACINES D'UNE ÉQUATION QUADRATIQUE DONT LES COEFFICIENTS SONT DES NOMBRES RATIONNELS.

[*Comptes Rendus*, CVIII. (1889), pp. 1037—1041.]

SI $u_i = \lambda_i u_{i-1} + u_{i-2}$ et $u_{-1} = 0$, $u_0 = 1$, on peut appeler u_i un cumulant dont la succession $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_i$ est le type; désignons-le par t.

Alors on peut représenter

Par ${}^{\backprime}t$ la succession............ $\lambda_2, \lambda_3, \ldots, \lambda_i$
Par t' „ ....$\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_{i-1}$
Par ${}^{\backprime}t'$ „ .... $\lambda_2, \lambda_3, \ldots, \lambda_{i-1}$.

De plus, on peut représenter par θt la réunion du type θ suivi par le type t; par $\theta 0 t$ ce que devient θt quand on intercale un zéro entre la succession θ et la succession t; par $\theta(0t)^i$ la succession θ suivie par la succession $0t$ répétée i fois; et par $t(0\theta)^i \tau$ ce que devient $t\tau$ quand on intercale 0θ i fois entre le t et le τ.

T étant un type quelconque, on peut désigner par $[T]$ le cumulant dont T est le type.

Ainsi, si les éléments en T sont regardés comme les quotients partiels d'une fraction continue, et que, suivant la notation de l'immortel Lejeune-Dirichlet, on représente par (T) la dernière convergente à cette fraction, on aura

$$(T) = [T] \div [{}^{\backprime}T].$$

Désignons par $\underline{\theta}$ ce que devient θ quand on renverse l'ordre, et par $\bar{\theta}$ ce qu'il devient quand on change le signe de chacun de ses éléments. Posons

$$T_i = \underline{\theta} t (0t)^i \bar{\theta};$$

j'ai trouvé et démontré le lemme suivant*:

* Pour établir cette proposition, on n'a besoin que de se servir des *deux* identités suivantes. Si $T = t\theta$,

$$[T] = [t][\theta] + [t'][{}^{\backprime}\theta].$$

Les rapports des trois quantités $[T_i] : [`T_i] - [T_i'] : [`T_i']$ *sont indépendants de* $\underline{i}$*; c'est-à-dire sont les mêmes que les rapports de*

$$[\underline{\theta}t\bar{\theta}] : [`\underline{\theta}t\bar{\theta}] - [\underline{\theta}t\bar{\theta}'] : [`\underline{\theta}t\bar{\theta}'].$$

Avec l'aide de ce théorème et de l'équation qui exprime une propriété bien connue des convergentes successives de fractions continues, savoir

$$[T][`T'] - [`T][T'] = \pm 1,$$

on établit facilement le théorème suivant :

On peut écrire et d'une seule manière les deux racines d'une équation quadratique simultanément sous les formes

$$(\theta t (0t)^\infty), \quad -(\bar{\theta}\underline{t}(0\underline{t})^\infty),$$

où tous les éléments de θ*, sauf le dernier (qui peut être zéro), et tous les éléments de* t *sont positifs.*

Comme un simple corollaire de ce théorème de correspondance, en appliquant à la seconde forme la méthode donnée par Dirichlet pour *régulariser* une succession de quotients partiels dont quelques-uns au commencement sont négatifs, on voit que les périodes des deux fractions convergentes contiendront les mêmes éléments, mais en ordre inverse.

Un exemple fera mieux comprendre la portée du théorème.

Prenons l'équation

$$23x^2 - 68x + 50 = 0,$$

dont les racines sont

$$\frac{34 + \sqrt{6}}{23}, \quad \frac{34 - \sqrt{6}}{23}.$$

On trouve, pour le développement de ces deux quantités, les fractions périodiques en fractions continues

$$(1, 2, 1, 2 ; 4, 2 ; 4, 2 ; 4, 2 ; \ldots)$$

et

$$(1, 1, 1, 2 ; 2, 4 ; 2, 4 ; 2, 4 ; \ldots)$$

respectivement.

Si $T = t\theta\tau$, $\qquad [T] = [t][\theta][\tau] + [t'][`\theta][\tau] + [t][\theta'][`\tau] + [t'][`\theta'][`\tau].$

On peut cependant ajouter que, de même, si $T = t\theta\tau\omega$,

$$[T] = [t][\theta][\tau][\omega] + [t'][`\theta][\tau][\omega] + [t][\theta'][`\tau][\omega] + [t][\theta][\tau'][`\omega]$$
$$+ [t'][`\theta'][`\tau][\omega] + [t'][`\theta][\tau'][`\omega] + [t][\theta'][`\tau'][`\omega] + [t'][`\theta'][`\tau'][`\omega],$$

où l'on remarquera que les trois premiers produits de la deuxième ligne sont composés de deux (le premier et le dernier) de formes analogues, et d'un troisième d'une forme différente, et ainsi, en général, si le nombre des types partiels t, θ, τ, ... est $\underline{i}$, on aura 2^{i-1} produits de cumulants partiels et de leurs dérivés simples et doubles ; car il y aura $(i-1)$ intervalles entre les i types sur lesquels on doit faire tomber dans chaque manière possible 1, 2, 3, ... $(i-1)$ paires d'accents. Quand les types partiels deviennent monomiaux, les termes avec les accents doubles dans la somme des produits deviennent zéros, et l'on retrouve la règle connue pour exprimer un cumulant comme somme des produits des agrégats de ses éléments, en élisant ou en traitant comme unités des paires et combinaisons de paires d'éléments consécutifs.

Or, en écrivant

$$\theta = 1,\ 2, \quad t = 1,\ 2,\ 3,$$

on aura

$$(\theta t\,(0t)^\infty) = (1,\ 2,\ 1,\ 2,\ 3,\ 0,\ 1,\ 2,\ 3,\ 0,\ 1,\ 2,\ 3,\ 0,\ 1,\ 2, \ldots)$$
$$= (1,\ 2,\ 1,\ 2; \quad 4,\ 2; \quad 4,\ 2; \quad 4,\ 2; \ldots),$$

ce qui répond à la première racine.

On aura aussi

$$(\bar{\theta}\underline{t}\,(0\underline{t})^\infty) = (-1,\ -2,\ 3,\ 2,\ 1,\ 0,\ 3,\ 2,\ 1,\ 0,\ 3, \ldots)$$
$$= (-1,\ -2,\ 3;\ 2, \quad 4;\ 2, \quad 4; \ldots),$$

laquelle convergente, *régularisée* selon les règles de Dirichlet*, peut être remplacée par

$$(-2,\ 1,\ 0,\ 1,\ 2;\ 2,\ 4;\ 2,\ 4; \ldots),$$

c'est-à-dire

$$(-2,\ 2,\ 2;\ 2,\ 4;\ 2,\ 4; \ldots),$$

ce qui, selon les mêmes règles, équivaut à

$$-(1,\ 1,\ 1,\ 2;\ 2,\ 4;\ 2,\ 4; \ldots),$$

laquelle est la valeur prise négativement de la seconde racine.

Terminons par l'exemple très simple

$$x^2 - 10x - 1 = 0,$$

dont les deux racines sont $5 + \sqrt{26}$, $5 - \sqrt{26}$, qui équivalent aux fractions continues

$$(10,\ 10,\ 10, \ldots), \quad -(0,\ 10,\ 10,\ 10, \ldots).$$

Faisons $\theta = 9,\ 0, \quad t = 1,\ 9.$

Alors $(\theta t\,(0t)^\infty)$ devient

$$(9,\ 0;\ 1,\ 9;\ 0,\ 1,\ 9;\ 0,\ 1,\ 9; \ldots),$$

c'est-à-dire

$$(10;\ 10;\ 10; \ldots),$$

la première racine; et $(\bar{\theta}\underline{t}\,(0\underline{t})^\infty)$ devient

$$(-9,\ 0;\ 9,\ 1;\ 0,\ 9,\ 1;\ 0,\ 9,\ 1; \ldots),$$

ce qui équivaut à

$$(0,\ 10,\ 10, \ldots),$$

laquelle est la valeur prise négativement de la seconde racine.

On comprendra que dans les formules pour une racine et la négative de l'autre, rien n'empêche que le θ disparaisse et qu'ainsi les formules deviennent

$$(t\,(0t)^\infty), \quad (\underline{t}\,(0\underline{t})^\infty)$$

respectivement.

* *Vorlesungen über Zahlentheorie*, § 80; 1871.

Dans le cas où les deux racines sont égales, mais de signes contraires, non seulement le $\underline{\theta}$ disparaît, mais aussi le $\underline{t}$ devient symétrique : ainsi l'on retrouve la forme applicable à l'équation $Ax^2 - \beta = 0$, pour lequel cas la racine positive peut être mise sous la forme

$$(abc, \ldots, cba, 0, abc, \ldots, cba, 0, abc),$$

c'est-à-dire

$$(a_j bc, \ldots, cb, 2a_j bc, cb, 2a_j).$$

On peut encore simplifier un peu les expressions pour x et x' (où x et x' sont les racines de la même équation quadratique) en écrivant

$$x = (\theta\,(t,\,0)^\infty), \quad x' = -\,(\overline{\theta}\,(\underline{t},\,0)^\infty),$$

formule vraiment surprenante par sa simplicité et sa symétrie.

64.

SUR LA REPRÉSENTATION DES FRACTIONS CONTINUES QUI EXPRIMENT LES DEUX RACINES D'UNE ÉQUATION QUADRATIQUE.

[*Comptes Rendus*, CVIII. (1889), pp. 1084—1086.]

NOUS avons donné dans une Note précédente [p. 644, above], pour les deux racines x et x' d'une équation quadratique à coefficients entiers, les formules jumelles

$$x = (t\,(\tau 0)^{\infty}), \quad -x' = (\bar{t}\,(\underline{\tau}0)^{\infty}).$$

Mais ces formules admettent encore une simplification importante au moyen des considérations suivantes.

Un type peut être nommé *omni-positif* ou *omni-négatif* quand tous ses éléments sont positifs pour un des cas et tous négatifs pour l'autre : il sera nommé *homonyme* quand il est *ou* omni-positif *ou* omni-négatif sans spécifier lequel des deux il est.

Le zéro sera regardé comme un *nombre* (non pas neutre, mais) *amphibolique*, c'est-à-dire qui est en même temps positif et négatif, de sorte qu'un type omni-positif ou omni-négatif ne cesse pas d'être homonyme en y ajoutant ou y entremêlant un ou plusieurs zéros.

De plus, on remarquera que $(\bar{T}) = -(T)$.

Alors la théorie, atteignant son dernier terme de simplicité et de généralité, donne lieu à l'énoncé suivant :

En supposant que t est un type homonyme quelconque et τ un autre, et que x, x' sont les deux racines d'une équation quadratique à coefficients entiers, on aura toujours

$$x = (t\tau^{\infty}), \quad x' = (t0\,(\bar{\tau})^{\infty})$$

avec la faculté à t de disparaître.

Ainsi, par exemple, en supposant que t disparaisse et que τ devienne monomial et égal à a, si

$$x = (a,\ a,\ a,\ \ldots,\ \textit{ad infinitum}),$$

on aura

$$x' = (0,\ -a,\ -a,\ \ldots,\ \textit{ad infinitum}),$$

c'est-à-dire $$x' = -(0, a, a, a, \ldots, \textit{ad infinitum});$$

de sorte que $$x' = -\frac{1}{x}\,*.$$

On remarquera que les types $t\tau^{\infty}$, $t0\underline{\bar{\tau}}^{\infty}$ sont mutuellement inverses l'un de l'autre, car $(t00\underline{\underline{\bar{\bar{\tau}}}}^{\infty}) = (t\tau^{\infty})$.

Nous nous sommes déjà servi† dans nos conférences, tenues à King's College London en 1859, sur la détermination du nombre de solutions en nombres entiers d'un système d'équations numériques‡, avec grand avantage de cette idée d'une série de quantités omni-positive, omni-négative ou homonyme et de la conception du caractère du zéro comme appartenant aux deux catégories des quantités positives et négatives à la fois.

Dans une Note à suivre, nous nous proposons de faire connaître la connexion§ remarquable qui subsiste entre les racines de l'équation

$$ax^2 + 2bx + c = 0$$

et les développements en fractions continues des fractions ordinaires $\dfrac{p \pm bq}{aq}$, où p, q sont les nombres de Pell qui appartiennent au déterminant $b^2 - ac$, et, si nous ne nous sommes pas trompé, nous espérons fonder là-dessus une règle pour l'extraction simultanée des deux racines de l'équation au moyen de ces deux développements.

* Et, en général, quand $x = -\dfrac{1}{x'}$, on aura

$$x = ((\theta)^{\infty}),$$

où θ est un type symétrique, ce qui est le théorème de Gallois (*Journal de Liouville*, t. II. p. 385).

De même, si $x = ((\theta 0)^{\infty})$ (θ étant symétrique) et ainsi $\underline{\theta} = \theta$, on aura

$$-x' = ((00\theta 0)^{\infty}) = ((\theta 0)^{\infty}) = x,$$

de sorte que $((\theta 0)^{\infty})$ est la forme générale de la fraction continue qui exprime la racine carrée d'une quantité rationnelle quelconque.

[† See Vol. II. of this Reprint, p. 122.]

‡ Inédites jusqu'à ce jour, mais qui doivent paraître prochainement dans l'*American Journal of Mathematics*. C'est dans nos recherches sur ce sujet que nous avons rencontré et discuté la théorie géométrique de dispositions de points dans un plan et dans l'espace que notre éminent Confrère M. Halphen a retrouvée indépendamment depuis et à laquelle il a donné le nom de *théorie d'aspects*. C'est en réduisant la détermination du nombre de solutions en nombres entiers d'un système de 3 équations à dépendre d'un agrégat de pareilles déterminations pour des systèmes de 2 équations que cette théorie s'est forcément mise en évidence pour les points dans un plan. De même, en faisant dépendre le problème pour un système de 4 de celui de systèmes de 3 équations, on est amené à une théorie semblable pour l'espace ; bien entendu, l'œil regardé comme un seul point dans la théorie pour le plan devient linéaire, ou, ce qui revient à la même chose, un système de deux points, pour l'espace.

§ Pour l'établir, nous nous servons encore de notre théorème de l'immutabilité des rapports de $[`T'] : [T'] - [`T] : [T]$ quand $T = t\tau\,(0\tau)^i\,\underline{\bar{t}}$ pour toute valeur positive et entière de i.

65.

SUR LA VALEUR D'UNE FRACTION CONTINUE FINIE ET PUREMENT PÉRIODIQUE.

[*Comptes Rendus*, CVIII. (1889), pp. 1195—1198.]

On sait que la valeur de la fraction purement périodique infinie (t^∞), où t est un type (c'est-à-dire une succession) d'éléments quelconques, est la racine positive de l'équation

$$[`t]\,x^2 - ([t] - [`t'])\,x - [t'] = 0. \qquad (1)$$

Cela conduit naturellement à la question de trouver la valeur de la fraction continue analogue périodique mais finie (t^n).

Avec l'aide de notre formule donnée dans une Note précédente, qui sert à exprimer un cumulant à un type composé de i types partiels comme une somme de 2^{i-1} produits des i cumulants partiels et leurs dérivées simples et doubles, on peut résoudre cette question sans aucune difficulté.

On a
$$(t^n) = \frac{[t^n]}{[`t^n]} = \frac{[t^n]}{[`tt^{n-1}]}.$$

Soient
$$[t^n] = u_n, \quad [`tt^{n-1}] = `tv_n,$$

on trouve que v_n sera une fonction entière et l'on établit, au moyen de la formule citée, entre u_n et v_n les équations aux différences

$$u_n - au_{n-1} - Bu_{n-2} = cBv_{n-2}, \quad v_{n-1} - cv_{n-2} = u_{n-2},$$

où
$$a = [t], \quad B = [`t]\,[t'], \quad c = [`t'].$$

Donc
$$Bv_{n-1} = u_n - au_{n-1},$$
$$av_n + (B - ac)\,v_{n-1} = u_n = v_{n+1} - cv_n,$$
$$v_{n+1} - (a + c)\,v_n + (-)^{\mu-1}\,v_{n-1} = 0$$

[car $B - ac = (-)^{\mu-1}$, μ étant le nombre d'éléments en t].

Conséquemment, par un principe bien connu, v_n et u_n seront les coefficients de k^n dans le développement d'une fraction de la forme

$$\frac{A + Bk}{1 - (a + c)\,k - \epsilon k^2},$$

où $\epsilon = (-)^\mu$, A et B étant convenablement déterminés pour l'un et pour l'autre cas.

Or
$$u_0 = 1, \quad u_1 = a,$$
$$v_0 = 0, \quad v_1 = 1.$$

Donc u_n est le coefficient de k^n en $\dfrac{1 - ck}{1 - (a+c)k - \epsilon k^2}$ et v_n le coefficient de k^n en $\dfrac{k}{1-(a+c)k - \epsilon k^2}$, de sorte que, si l'on écrit

$$\Phi_n(x) = x^n + (n-1)\epsilon x^{n-2} + \frac{(n-2)(n-3)}{2}\epsilon^2 x^{n-4} + \ldots$$

jusqu'au premier terme qui devient zéro, on aura

$$v_n = \Phi_{n-1}(a+c)$$

et
$$u_n = \Phi_n(a+c) - c\Phi_{n-1}(a+c).$$

Ainsi l'on voit que
$$(t^n) = \frac{(\Phi_n - [{}^{\backprime}t']\,\Phi_{n-1})(a+c)}{[{}^{\backprime}t]\,\Phi_{n-1}(a+c)}.$$

On peut aussi exprimer u_n et v_n au moyen des racines de l'équation

$$m^2 - ([t] + [{}^{\backprime}t'])\,m - \epsilon = 0,$$

dont on remarquera que le déterminant $\frac{1}{4}([t] + [{}^{\backprime}t'])^2 + \epsilon$ est le même que celui de l'équation (1), puisque

$$\tfrac{1}{4}([t] - [{}^{\backprime}t'])^2 + [{}^{\backprime}t]\,[t'] = \tfrac{1}{4}([t] + [{}^{\backprime}t'])^2 + \epsilon\,;$$

car, en supposant que ρ et σ sont les deux racines, on aura

$$\frac{u_n}{v_n} = \frac{A\rho^n - B\sigma^n}{\rho^n - \sigma^n},$$

où A, B sont des quantités connues; et, en supposant que $\rho^2 = > \sigma^2$, on aura $\dfrac{u_\infty}{v_\infty} = A$ et $(t^\infty) = \dfrac{A}{{}^{\backprime}t}$, laquelle valeur on identifiera facilement avec la racine positive de l'équation

$$[{}^{\backprime}t]\,x^2 - ([t] - [{}^{\backprime}t'])\,x - [t'] = 0.$$

Si l'on suppose que les éléments de t sont m en nombre et tous identiques avec l'unité, on aura

$$[{}^{\backprime}t] = [1^{m-1}], \quad [{}^{\backprime}t^n] = [1^{mn-1}],$$

et l'on obtient la formule peut-être nouvelle

$$\frac{\Phi_{mn-1}(1)}{\Phi_{m-1}(1)} = \Phi_{n-1}(\Psi_m),$$

où $\Psi_m = \Phi_m(1) + \Phi_{m-2}(1)$.

Si l'on suppose que m est impair, ϵ sera positif et Ψ_m prendra la forme

$$1 + m + m\frac{m-3}{2} + m\frac{(m-4)(m-5)}{2.3} + \ldots,$$

en s'arrêtant au premier terme qui devient zéro.

Cette formule donne naissance à un corollaire intéressant. Supposons que la somme de deux termes séparés par un seul dans la série *phyllotactique* 1, 2, 3, 5, 8, 13, 21, ... est un nombre premier p. Soit $m, m-2$ l'ordre de ces deux termes; alors je dis que le quotient du nombre de l'ordre $mi-1$ par celui de l'ordre $m-1$ (nombre toujours entier) par rapport au module p sera congru à l'unité si i est impair et à zéro si i est pair; de plus, dans ce dernier cas où $i = 2j$, le quotient de ce quotient divisé par p sera congru à $(-)^j (j+1)$ par rapport au même module p.

On pourrait tirer sans doute d'autres théorèmes analogues, mais apparemment moins simples, au moyen de l'équation

$$[t^n] = \Phi_n [t] - [t'] \Phi_{n-1} [t].$$

C'est une chose qu'on n'avait nul droit (*a priori*) d'attendre que le quotient $[`t^n] \div [`t]$, au lieu d'être une fonction rationnelle et entière de quatre quantités $[t]$, $[`t]$, $[t']$, $[`t']$ ou (ce qui est équivalent) rationnelle et fractionnelle de $[t]$, $[`t]$, $[t']$, est en effet une fonction rationnelle et entière d'une seule quantité, savoir de $[t] + [`t']$, c'est-à-dire est un nombre *phyllotactique* affecté ou paramétrique, nom qu'on peut convenablement donner à la valeur de $[x^n]$, où x est monomial et entier, $[1^n]$ prenant alors le nom de *nombre phyllotactique simple* ou *unitaire.*

66.

A NEW PROOF THAT A GENERAL QUADRIC MAY BE REDUCED TO ITS CANONICAL FORM (THAT IS, A LINEAR FUNCTION OF SQUARES) BY MEANS OF A REAL ORTHOGONAL SUBSTITUTION.

[*Messenger of Mathematics,* XIX. (1890), pp. 1—5.]

ALL the proofs that I am acquainted with (and their name is legion) of the possibility of depriving a quadric, in three or more variables, of its mixed terms by a real orthogonal transformation are made to depend on the theorem that the "*latent roots*" of any symmetrical matrix are all real.

By the latent roots is understood the roots of the determinant expressed by tacking on a variable $-\lambda$ to each term in the diagonal of symmetry to such matrix.

I shall show that the same conclusion may be established *à priori* by purely algebraical ratiocination and without constructing any equation, by the method of cumulative variation. The proof I employ is inductive: that is, if the theorem is true for two or any number of variables I prove that it will be true for one more.

To illustrate the method let us begin with two variables. Consider the form $ax^2 + 2hxy + by^2$.

If in any such form $b = a$, then by an obvious orthogonal transformation, namely, writing $\frac{x+y}{\sqrt{(2)}}$ and $\frac{x-y}{\sqrt{(2)}}$ for x and y, the form becomes

$$a(x^2 + y^2) + h(x^2 - y^2),$$

or

$$(a+h)x^2 + (a-h)y^2.$$

Now in general on imposing on x, y any orthogonal infinitesimal substitution, so that

$$x \text{ becomes } x + \epsilon y,$$

$$y \quad " \quad y - \epsilon x,$$

h in the new form becomes $h+(a-b)\epsilon$, or say $\delta h=(a-b)\epsilon$, and

$$\tfrac{1}{2}\delta(h^2)=(a-b)h\epsilon;$$

the variations of a and b need not be set forth.

Let an infinite succession of such transformations be instituted; then either a and b become equal and the orthogonal substitution above referred to reduces the quadric to its canonical form, in which case this one combined with the preceding infinite series of such substitutions may be compounded into a single substitution, or else by giving ϵ the sign of $(b-a)$ the variation of h^2 may at each step be made negative so that h^2 continually decreases, unless h vanishes. If h does not vanish it must have a minimum value, and this minimum value may be diminished, which involves a contradiction: hence, in the infinite series of substitutions supposed, either a and b become equal or h vanishes, and in either case the quadric is reduced or reducible to its canonical form.

Let us now take the case of three variables x, y, z.

Obviously, by the preceding case, we may make the term involving xy disappear and commence with the initial form

$$ax^2+by^2+2fxz+2gyz+cz^2.$$

If f or g become zero the quadric may be canonified by virtue of the preceding case.

Again, if $b=a$, by imposing on x, y the orthogonal substitution

$$\frac{g}{\sqrt{(f^2+g^2)}}x+\frac{f}{\sqrt{(f^2+g^2)}}y$$
$$-\frac{f}{\sqrt{(f^2+g^2)}}x+\frac{g}{\sqrt{(f^2+g^2)}}y,$$

the term involving xz will disappear and the final result is the same as if f were zero.

Let us now introduce the infinitesimal orthogonal substitution which changes

$$\begin{array}{lll} x & \text{into} & x+\epsilon y+\eta z, \\ y & ,, & -\epsilon x+y+\theta z, \\ z & ,, & -\eta x-\theta y+z, \end{array}$$

where ϵ, η, θ are supposed to be of the same order of magnitude so that only first powers of them have to be considered.

Then
$$\delta f=(a-c)\eta-g\epsilon,$$
$$\delta g=(b-c)\theta+f\epsilon,$$

also the coefficient of $2xy$ becomes $(a-b)\epsilon-f\theta-g\eta$.

Now whatever η, θ may be, we may determine ϵ in terms of η, θ so that this may be made to vanish, and the initial form of the quadric will be maintained, provided that b is not equal to a.

Hence instituting an infinite series of these infinitesimal substitutions, provided we do not reach a stage where a and b become equal, we may maintain the original form keeping η, θ arbitrary, and shall have

$$\tfrac{1}{2}\delta(f^2+g^2)=(a-c)f\eta+(b-c)g\theta.$$

Suppose a and b to be unequal; therefore $(a-c)$, $(b-c)$ do not vanish simultaneously, and consequently we may make $\delta(f^2+g^2)$ negative unless at least one of the two quantities f, g vanishes.

If neither of them vanishes f^2+g^2 may be made continually to decrease and will have a minimum other than zero, which involves a contradiction.

Hence the infinite series of infinitesimal orthogonal substitutions may be so conducted that either $a-b$ or one at least of the letters f, g shall become zero; and then two additional orthogonal substitutions at most will serve to reduce the Quadric immediately to its canonical form.

I shall go one step further to the case of four variables x, y, z, t and then the course of the induction will become manifest. We may, by virtue of what has been shown, take as our quadric

$$ax^2+by^2+cz^2+2fxt+2gyt+2hzt+dt^2.$$

Here, if any one of the mixed terms disappears, the quadric is immediately reducible by the preceding case, and if any two of the grouped pure coefficients a, b, c become equal (as for instance a, b), then by an orthogonal transformation one of the mixed terms (f or g in the case supposed) may be got rid of; so that this supposition merges in the preceding one.

Impose on x, y, z, t an infinitesimal orthogonal substitution, writing

$$\begin{array}{rl} x+\epsilon y+\theta z+\lambda t & \text{for } x,\\ -\epsilon x+y+\eta z+\mu t & \quad ,, \quad y,\\ -\theta x-\eta y+z+\nu t & \quad ,, \quad z,\\ -\lambda x-\mu y-\nu z+t & \quad ,, \quad t.\end{array}$$

Then
$$\begin{aligned}\delta f&=(a-d)\lambda-g\epsilon-h\theta,\\ \delta g&=(b-d)\mu+f\epsilon-h\eta,\\ \delta h&=(c-d)\nu+f\theta+g\eta.\end{aligned}$$

Also the coefficients of $2xy$, $2xz$, $2yz$ respectively become

$$\begin{gathered}(a-b)\epsilon-f\mu-g\lambda,\\ (a-c)\theta-f\nu-h\lambda,\\ (b-c)\eta-g\nu-h\mu.\end{gathered}$$

Suppose that no two of the grouped pure coefficients a, b, c are equal; then ϵ, θ, η can be, and are to be, expressed in terms of λ, μ, ν so as to make these three expressions vanish; that being done the initial form of the Quadric is maintained throughout the series of substitutions and we may write

$$f\delta f+g\delta g+h\delta h=(a-d)f\lambda+(b-d)g\mu+(c-d)h\nu.$$

Of the three quantities λ, μ, ν it is sufficient for the purpose of the argument to retain any two as λ, μ and to suppose $\nu = 0$.

Then, since we suppose that a and b are not equal,

$$(a-d) f\lambda + (b-d) g\mu$$

(where λ, μ are arbitrary) can always be made negative unless f, g are none of them zero; so that if a and b never become equal *nor* f or g vanish $f^2 + g^2 + h^2$ cannot have any minimum value other than zero, which involves a contradiction; hence in the course of the series of infinitesimal transformations either a and b must become equal, or f or g or both of them vanish. If f and g vanish simultaneously or even if one only of them vanish, then *one* succeeding substitution, and if a and b become equal *two* succeeding substitutions, will effect the reduction to the canonical form. This proves the theorem for four variables.

The method is obviously extendible to any number of variables; in the case just considered it is seen that in the infinitesimal orthogonal matrix of substitution for the *exceptional* line or column (that which relates to the *excepted* variable the t) it is not necessary to employ more than two arbitrary infinitesimals and a like remark applies to the general case, so that if there are n variables, whilst $\frac{1}{2}(n^2 - n)$ is the number of infinitesimals that would appear in the complete matrix, $\frac{1}{2}(n^2 - 3n + 6)$, that is $\frac{1}{2}\{(n-1)(n-2)\} + 2$, are sufficient for the purpose of the demonstration.

Thus then without recourse to any theorem of Equations it is proved that any Quadric may be reduced by a real orthogonal substitution to its canonical form *.

* I have applied the same method to prove that by two real independent orthogonal substitutions operated on

$$x_1, x_2, \ldots x_n;\ y_1, y_2, \ldots y_n$$

the general lineo-linear Quantic in the x's and y's (with real coefficients) may be reduced to the canonical form $\Sigma x_i y_i$, and have sent for insertion in the *Comptes Rendus* of the Institute a Note in which I give the rule for effecting this reduction [above, p. 638].

It may be sufficient here to mention that if U is the given lineo-linear Quantic, its n canonical multipliers are the square roots of the n canonical multipliers of the Quadric $\Sigma\left(\frac{dU}{dy}\right)^2$, or if we please of $\Sigma\left(\frac{dU}{dx}\right)^2$, which it may easily be shown *à posteriori* are necessarily omni-positive; and I need hardly add that although these two Quadrics are different, their canonical multipliers are the same.

67.

ON THE REDUCTION OF A BILINEAR QUANTIC OF THE n^{TH} ORDER TO THE FORM OF A SUM OF n PRODUCTS BY A DOUBLE ORTHOGONAL SUBSTITUTION*.

[*Messenger of Mathematics*, XIX. (1890), pp. 42—46.]

A HOMOGENEOUS lineo-linear function in two sets of variables

$$x, y, \dots z; \ u, v, \dots w$$

will contain n^2 terms: two independent orthogonal substitutions performed on the two sets will introduce twice $\frac{1}{2}n(n-1)$ disposable constants, and by a suitable choice of these, n^2-n terms of the transformed function may be made to vanish so as to leave a sum of products of the new $x, y, \dots z$ paired with the new $u, v, \dots w$: it will of course be found in general impossible to obliterate *any* arbitrarily chosen (n^2-n) terms in the transformed function; since if in the n remaining products one letter of one set were combined with more than one of the other set, this would (by means of a further superimposed orthogonal substitution) be equivalent to taking away more than (n^2-n) terms by means of only (n^2-n) disposable constants. It is very easy to effect the transformation indicated by a method very analogous to that of reducing a quadric in n variables by an orthogonal substitution to its canonical form, and to show *à posteriori* that the substitutions are always real in this case as in the other, when the original coefficients are real; but it will, I think (although not necessary), be found interesting and instructive to prove *à priori* the latter assertion by a similar method to that applied to Quadrics in the last number of the *Messenger*. I will begin then with this proof, reserving the complete solution of the problem to the end of the article. The leading idea in this as in the preceding article is to regard a finite orthogonal substitution as the product of an infinite number of infinitesimal ones.

For $$axu + \alpha xv + \beta yu + byv.$$

Let $x, y; u, v$ become $x+\epsilon y, -\epsilon x + y; u+\lambda v, -\lambda u + v$ respectively, then

$$\delta\alpha = a\lambda - b\epsilon, \quad \delta\beta = a\epsilon - b\lambda,$$
$$\alpha\delta\alpha + \beta\delta\beta = (a\alpha - b\beta)\lambda + (a\beta - b\alpha)\epsilon.$$

[* Cf. p. 638 above.]

Hence $\alpha^2 + \beta^2$ may be made to decrease unless $a = 0$, $b = 0$, or $\alpha = 0$, $\beta = 0$, or $\frac{a}{b} = \frac{\alpha}{\beta} = \pm 1$, in which case since

$$(a+\alpha)(x+y)(u+v) + (a-\alpha)(x-y)(u-v) = 2a(xu+yv) + 2\alpha(xv+yu),$$
$$(a-\alpha)(x+y)(u-v) + (a+\alpha)(x-y)(u+v) = 2a(xu-yv) + 2\alpha(xv-yu),$$

the form is immediately canonizable.

Hence in the infinite succession of infinitesimal orthogonal substitutions (equivalent to a single one) either a and b or α and β must vanish simultaneously, on which supposition the form is canonical or else it is reducible to the canonical form by a second finite orthogonal substitution.

Let us now proceed to the case of a *ternary* bilinear form in x, y, z; u, v, w.

I suppose by the previous case the form to be deprived of two terms, and that we have to deal with the form

$$axu + byv + fxw + guz + hyw + kvz + czw.$$

Lemma. If $f = 0$, $g = 0$, or $h = 0$, $k = 0$ the above form is reducible by the previous case. Also if $a^2 = b^2$ and $f = 0$, $h = 0$, or $g = 0$, $k = 0$, or $a^2 = b^2$ and $\left(\frac{f}{h}\right)^2 = \left(\frac{g}{k}\right)^2$ the form is reducible to the previous case by a single additional finite orthogonal transformation.

For the sake of brevity I leave the proof to my readers.

Introducing now two infinitesimal orthogonal substitutions with parameters ϵ, η, θ; λ, μ, ν*, we obtain the variations

$$\delta f = a\mu - h\epsilon - c\eta, \quad \delta h = b\nu + f\epsilon - c\theta,$$
$$\delta g = a\eta - k\lambda - c\mu, \quad \delta k = b\theta + g\lambda - c\nu,$$

also in order to keep the coefficients of xv, yu at *null*, we must have

$$a\lambda - b\epsilon - f\nu - k\eta = 0,$$
$$-b\lambda + a\epsilon - g\theta - h\mu = 0.$$

From the previous equations we obtain

$$f\delta f + g\delta g + h\delta h + k\delta k = (af - cg)\mu + (bh - ck)\nu + (ag - cf)\eta + (bk - ch)\theta.$$

(1) Suppose $a^2 - b^2$ *not* zero; then μ, ν, η, θ will be independent and their coefficients cannot all become zero unless $f^2 = g^2$ and $h^2 = k^2$, or else $f = 0$ and $g = 0$, or $h = 0$ and $k = 0$, on either of which suppositions the form becomes canonizable by virtue of the Lemma.

(2) Let $a^2 = b^2$. Then we must have

$$f\nu + k\eta \pm (g\theta + h\mu) = 0,$$

which I shall satisfy by making $f\nu \pm g\theta = 0$, $k\eta \pm h\mu = 0$.

* The positive values of the parameters in each system are supposed to belong to the upper, and the negative values to the lower half of each orthogonal matrix.

Hence

$$\Sigma f\delta f = \{(af - cg)k \mp (ag - cf)h\}\rho + \{(ah - ck)g \mp (ak - ch)f\}\tau,$$

ρ, τ being two arbitrary infinitesimals.

Therefore $\Sigma f\delta f$ may be made negative unless the multipliers of ρ and τ are both zero, in which case by addition or subtraction we obtain $fk = gh$; consequently two out of the four variables f, g, h, k are zero, or else $\frac{f}{h} = \frac{g}{k}$, and on either of these suppositions the transformed function may be canonized by virtue of what has been proved in the case of two biliteral sets, or may by a finite orthogonal substitution be brought to a form so canonizable.

Hence it is clear that either f, g, h, k may all be made to vanish, or else we must pass through a form known to be canonizable. This is the proof for a bilinear function of triliteral sets, which may be easily extended to a bilinear function of n-literal sets.

I will now give the method for effecting the reduction which is thus proved to be always capable of being effected by real substitutions.

Let $\Sigma a_{r,s} x_r y_s$ be the given bilinear function B.

Then $\Sigma \left(\frac{dB}{dy_s}\right)^2$, which is an orthogonal invariant of B quâ the y's, is a Quadratic function of the x's, which will have an orthogonal substitute quâ the x's of the form $\Sigma[\lambda_r x_r^2]$.

If then B is reducible by a double orthogonal substitution to the form $\Sigma[\theta_r x_r y_s]$, we must have $\Sigma[\theta_r x_r]^2$ orthogonally equivalent to $\Sigma[\lambda_r x_r^2]$, and this can only be the case when the θ's are respectively (in any order) the squares of the λ's.

The θ's I call the Canonical Multipliers to B.

This gives rise to the following rule:

Form the Matrix $[m]$.

$$\begin{array}{l} a_{1,1}, \quad a_{2,1}, \ldots a_{n,1}, \\ a_{1,2}, \quad a_{2,2}, \ldots a_{n,2}, \\ \ldots\ldots\ldots\ldots\ldots\ldots, \\ a_{1,n}, \quad a_{2,n}, \ldots a_{n,n}. \end{array}$$

From this derive a Matrix $[M]$, a *false* square of $[m]$, obtained by multiplying each line in it by *all* the lines (according to Cauchy's rule, in fact, for the multiplication of *Determinants*). Then the latent roots of $[M]$ are the squares of the Canonical Multipliers to B.

But if instead of $\Sigma \left(\frac{dB}{dy_s}\right)^2$ we take $\Sigma \left(\frac{dB}{dx_r}\right)^2$ and deal with it in like manner, we shall obtain a matrix $[n]$, such that $[m]$ and $[n]$ are transverse to each other, the lines and columns of the one being the columns and lines of the other: the Cauchian Square of $[n]$ will give rise to a matrix $[N]$ different from $[M]$ but having the same latent roots: in fact the coefficients of the equation to the latent roots alike of $[m]$ and of $[n]$ with the signs in the alternate places changed will be unity, the sum of the squares of all the terms in $[m]$ or $[n]$, the sum of the squares of the minors of the 2nd, 3rd, ... orders in $[m]$ or $[n]$; and finally the last coefficient will be the square of the determinant to $[m]$ or $[n]$: so that we shall obtain as we ought the same set of canonical multipliers whichever matrix $[M]$ or $[N]$ we employ; but in order to obtain the substitutions which must be impressed on the x set and the y set to arrive at the Canonical form in which only n products appear we shall want both $[M]$ and $[N]$. Let me, however, pause for a moment to call attention to the interesting fact that the sum of the squares of the coefficients in B by virtue of being a coefficient of the latent function to $[M]$ or $[N]$ is necessarily a bi-orthogonal invariant to B; so, too, all the other coefficients in this function are such invariants: and among them the last, which is the square of the determinant to $[m]$ or $[n]$. Thus then this determinant (which may be termed the discriminant) is an invariant alike for the two theories; namely the better known one in which the x set and the y set are subjected to the same general substitution, and the one here considered where these sets are subjected to two independent orthogonal substitutions.

In either theory the vanishing of the discriminant is the signal of the Canonical form becoming short of one term.

It is also proper to notice that the latent roots of $[M]$ or $[N]$, which by virtue of $[M]$ and $[N]$ being symmetrical matrices are necessarily real, are for these particular forms of $[M]$ and $[N]$ *positive* as well as real since the coefficients with the alternate signs changed are all positive, being the sums of squares of real numbers.

To complete the solution it remains to find the two canonizing orthogonal matrices, but these are known by the ordinary theory for quadrics: thus the x substitution will be that which canonizes $[M]$ and the y substitution that which canonizes $[N]$.

Conversely, if $[M]$ and $[N]$ are supposed given, we shall know the linear functions of the x's which substituted for $x_1, x_2, \ldots x_n$ and the linear function of the y's which substituted for $y_1, y_2, \ldots y_n$, such that $\Sigma\lambda_1^{\frac{1}{2}}x_1y_1$ shall be identical with B, the λ's being the latent roots common to $[M]$ and $[N]$. There will be 2^n systems of values represented by $\lambda_1^{\frac{1}{2}}, \lambda_2^{\frac{1}{2}}, \ldots \lambda_n^{\frac{1}{2}}$: thus then 2^n matrices transverse to one another can be found such that their *false* squares shall be respectively identical with any two given symmetrical matrices having the same latent roots, and we are thus enabled indirectly, through the theory of bi-orthogonal canonization, to obtain the solution of a problem which intrinsically has or seems to have nothing to do with orthogonal or other transformation.

It is worthy of observation that this problem of finding the so-to-say *false* square root common to two given symmetrical matrices having the same latent equation, admits of precisely the same number (2^n) solutions as the problem of finding the true square root of one general matrix. For if $[M]$ be any given matrix of order n and $[1]$ represents the unit matrix of that order, namely the matrix all of whose terms are zeros except those in the principal diagonal which are units, we know by virtue of a general theorem that calling $\lambda_1, \lambda_2, \ldots \lambda_n$ its n latent roots, each true square root of $[M]$ is represented by

$$\Sigma\lambda_1^{\frac{1}{2}} \frac{([M]-\lambda_2[1])([M]-\lambda_3[1])\ldots([M]-\lambda_n[1])}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)\ldots(\lambda_1-\lambda_n)}.$$

68.

ON AN ARITHMETICAL THEOREM IN PERIODIC CONTINUED FRACTIONS.

[*Messenger of Mathematics*, XIX. (1890), pp. 63—67.]

THE well-known form of continued fraction for the square root of N, an integer, is

$$(a\,;\ b,\ c,\ d, \ldots,\ d,\ c,\ b,\ 2a\,;\ b,\ c,\ d, \ldots,\ d,\ c,\ b,\ 2a\,;\ \text{indefinitely continued})$$

which, if we denote the type $a, b, c, d, \ldots, d, c, b, a$ by t, may be written under the more convenient form

$$(t,\ 0,\ t,\ 0,\ t,\ 0, \ldots\ \textit{ad inf.}).$$

If now we use $[t]$ to signify the cumulant of which t is the type, and $[{}^{\backprime}t]$, $[t']$, $[{}^{\backprime}t']$ respectively, the cumulants of the types got by cutting off a from either end and from both ends of t, it is easily shown that *whatever* numbers $a, b, c, \ldots$ represent, the value of the continued fraction $\{(t,\ 0)^{\infty}\}$ is $\sqrt{\dfrac{[t]}{[{}^{\backprime}t']}}$, so that if $\{(t,\ 0)^{\infty}\}$ represents the square root of an integer, $[t]$ must be divisible by $[{}^{\backprime}t']$.

At first sight one would imagine that it would be a difficult matter to give a rule for determining whether such condition is fulfilled or not by any assigned value of the symmetrical type t, but Mr C. E. Bickmore, of New College, Oxford, has noticed that the case is quite otherwise, for that if we put t under the form a, τ, a, then, in order that $\{(a,\ \tau,\ a,\ 0)^{\infty}\}$ may satisfy the requirement of being the square root of an integer, the sufficient and necessary condition is the equivalence

$$2a \equiv (-)^{\mu}\,[\tau']\,[{}^{\backprime}\tau']\ (\text{mod. } [\tau]),$$

where μ is the number of elements in τ.

Consequently τ may be taken quite arbitrarily, and then an infinite number of values be assigned to a, except in the case where $[\tau]$ is even, and at the same time $[\tau']$ and $[{}^{\backprime}\tau']$ are each of them odd.

The proof in my notation is as follows:

Since $t = a, \tau, a$, we have $`t' = \tau$, and consequently $\frac{[t]}{[`t']}$ will be an integer if

$$[a, \tau, a] \equiv 0 \text{ (mod. } [\tau]).$$

Expanding and remembering that $[`\tau] = [\tau']$ (the type τ being symmetrical), we obtain

$$a^2 [\tau] + 2a [\tau'] + [`\tau'] \equiv 0 \text{ (mod. } [\tau]).$$

Hence $$2a [\tau'] + [`\tau'] \equiv 0 \text{ (mod. } [\tau]), \quad (1)$$

and $$2a [\tau']^2 + [\tau'] [`\tau'] \equiv 0 \text{ (mod. } [\tau]). \quad (2)$$

But $$[\tau']^2 - [\tau] [`\tau'] = (-1)^{\mu+1},$$

so that $$[\tau']^2 \equiv (-1)^{\mu+1} \text{ (mod. } [\tau]),$$

and therefore (2) becomes

$$2a \equiv (-)^{\mu} [\tau'] [`\tau'] \text{ (mod. } [\tau]), \quad (3)$$

which is thus shown to be a necessary condition.

It is also a sufficient condition, for multiplying (3) by $[\tau']$ we have

$$2a [\tau'] \equiv (-)^{\mu} [\tau']^2 [`\tau'] \text{ (mod. } [\tau]),$$

or, since $$[\tau']^2 \equiv (-)^{\mu+1} \text{ (mod. } [\tau]),$$

$$2a [\tau'] \equiv - [`\tau'] \text{ (mod. } [\tau]),$$

which is the same as (1).

Suppose now that $`\tau'$ is given and that we wish to ascertain if $a$ can be found of such a value that the congruence (3) shall be soluble. This will obviously be the case if $[\tau]$ is odd. It will also be the case if $[\tau]$ is even, provided $[`\tau']$ is also even, and only in that case; for, when $[\tau]$ is even, then by virtue of the equation

$$[`\tau'] [\tau] - [\tau']^2 = \pm 1,$$

$[\tau']$ must be odd.

We have, therefore, to find under what circumstances $[`\tau']$ will be odd and $[\tau]$ even; in all other cases but these the congruence (3) will be soluble, and then the most general value of $a$ will be any term in an arithmetical series of which the common difference is $[\tau]$, unless $[\tau]$ and $[`\tau']$ are both of them even, in which case the common difference will be $\frac{1}{2} [\tau]$.

I proceed now to give a rule for determining the possible and impossible cases of the solution of (3), to explain the grounds of which the following statement will suffice.

(1) The value of a cumulant is not affected by striking out any even number of consecutive zeros from its type.

(2) The *parity* (that is the character *quâ* the modulus 2) of any cumulant will not be affected if we strike out three consecutive odd terms, whether

they occur in the middle or at either extremity. For if t, τ be any two types, the cumulant

$$[t, 1, 1, 1, \tau] = 3[t][\tau] + 2[t'][\tau] + 2[t][`\tau] + [t'][`\tau]$$
$$\equiv [t][\tau] + [t'][`\tau] \pmod{2},$$

that is

$$\equiv [t, \tau] \pmod{2}.$$

Also

$$[1, 1, 1, t] = [t, 1, 1, 1] = 3[t] + 2[`t] \equiv [t] \pmod{2}.$$

(3) The value of any cumulant in the type of which 1, 0, 1 occurs anywhere is the same as if 2 is substituted for 1, 0, 1; and therefore its parity is not affected if the units on each side of the 0 are omitted.

In what precedes in Nos. (1), (2), (3) the result, to modulus 2, is obviously unaffected if for 0 we write any even and for 1 any odd number.

In order then to determine the parity of $[`\tau']$ and of $[\tau]$ we may proceed as follows:

Let τ be any assigned symmetrical type, $`\tau'$ will then represent the type divested of its two equal terminals.

Rules—(1) for each even number in $`\tau'$ write 0, and for each odd number, 1;

(2) elide any even number of consecutive *zeros*, and any number divisible by 3 of consecutive *units*;

(3) elide any pair of units lying on each side of a zero;

(4) repeat these processes as often as possible;

then, I say, eventually we must arrive at one or other of the six following irreducible types, namely

( ); 0; 1; 1, 1; 0, 1, 0; 0, 1, 1, 0*,

where ( ) means absolute vacuity; accordingly $`\tau'$ may be said to be affected with one or the other of these six *characters*.

If now the reduced form of $`\tau'$ is 0; 1, 1; 0, 1, 0, $[`\tau']$ is even, and the congruence (3) will be soluble. In the other three cases $[`\tau']$ is odd, but $[\tau]$ will also be odd unless its terminal elements are odd in the case where the reduced form of $`\tau'$ is ( ), and even for the reduced forms 1, and 0, 1, 1, 0.

In the following exhaustive table the second column indicates the evenness or oddness of the terminals of τ denoted by e and u respectively.

The third and fourth columns indicate the evenness or oddness (denoted as above) of $[`\tau']$ and $[\tau]$, along with the character of $`\tau'$ in the third column. In the fifth column the answer is given as to the determining congruence

* Except for the symmetrical form of τ there would be two additional (virtually undistinguishable) reduced forms 0, 1 and 1, 0.

being soluble or insoluble, denoted by s and i respectively; and the last column shows whether the common difference of the arithmetical series of the values of either terminal, in the case of solubility, is equal to the modulus $[\tau]$ or its moiety.

Cases	Terminals	$'\tau'$		$[\tau]$	Sol. or Insol.	C. D.
1	e	( )	u	u	s	$[\tau]$
2	u	( )	u	e	i	
3	e	1	u	e	i	
4	u	1	u	u	s	$[\tau]$
5	e	0, 1, 1, 0	u	e	i	
6	u	0, 1, 1, 0	u	u	s	$[\tau]$
7	e	0	e	e	s	$\frac{1}{2}[\tau]$
8	u	0	e	e	s	$\frac{1}{2}[\tau]$
9	e	1, 1	e	u	s	$[\tau]$
10	u	1, 1	e	u	s	$[\tau]$
11	e	0, 1, 0	e	u	s	$[\tau]$
12	u	0, 1, 0	e	u	s	$[\tau]$

The following examples are given to prevent the possibility of misapprehension in the application of the Algorithm.

(α) Let

$$\tau = 1, 9, 1, 1, 1, 2, 1, 7, 4, 2, 2, 2, 4, 7, 1, 2, 1, 1, 1, 9, 1.$$

Then
$$\begin{aligned} '\tau' &\equiv 1, 1, 1, 1, 0, 1, 1, 0, 0, 0, 0, 0, 1, 1, 0, 1, 1, 1, 1 \\ &\equiv 0,\ 1,\ 0, 1,\ 0 \\ &\equiv 0,\ 0,\ 0 \\ &\equiv 0. \end{aligned}$$

This corresponds to case (8), which is a soluble one, and accordingly we have from Degen's Table

$$\{(15, \tau, 15, 0)^\infty\} = \sqrt{(251)},$$

15 being the first term of an arithmetical series whose common difference is $\frac{1}{2}[\tau]$.

(β) Let $\tau = 2, 3, 1, 2, 4, 1, 6, 6, 1, 4, 2, 1, 3, 2.$

Then
$$\begin{aligned} '\tau' &\equiv 1, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 1 \\ &\equiv 1, 1,\ 1,\ 1,\ 1, 1 \\ &\equiv (\ ). \end{aligned}$$

This corresponds to the soluble case (1), and accordingly we find from Degen's Table $\{(10, \tau, 10, 0)^\infty\} = \sqrt{(109)}$; 10 being the first term of an arithmetical series whose common difference is $[\tau]$.

69.

ON A FUNICULAR SOLUTION OF BUFFON'S "PROBLEM OF THE NEEDLE" IN ITS MOST GENERAL FORM.

[*Acta Mathematica*, XIV. (1890–1), pp. 185—205.]

"...quaintly made of cords."
(*Two Gentlemen of Verona*, Act III. Sc. 1.)

THE founder of the theory of Local Probability appears to have been Buffon (better known as a Naturalist, but who began his career as a Mathematician). Among a few other questions of a similar kind, which he proposed in his *Essai d'Arithmétique Morale,* the one which has obtained the greatest notoriety is the celebrated one which goes by the name of the *Problème de l'Aiguille,* the purport of which is as follows.

On an area of indefinite extent (say a planked floor) a number of parallel straight lines are ruled at equal distances, upon which a needle, not long enough to cross more than one of the parallels at the same time, is thrown down: the probability is required of its falling in such a position as to be intersected by one of the parallels.

An easier question of the same kind, which Buffon treats before the other, is when a circle is used instead of the needle. This latter question he solves by simple geometrical considerations too obvious to need recapitulation; to obtain a solution of the former he, and after him Laplace, had recourse to a process of integration.

In a question given in the late Mr Todhunter's *Integral Calculus* (1st edition, 1857, p. 268) the solution of the problem is correctly stated for an ellipse, whose major axis is less than the distance between two consecutive parallels, instead of for a circle or straight line: this important step in the development of the theory is, I am informed, currently attributed to the late Mr Leslie Ellis, of the University of Cambridge.

In the year 1860, Lamé proposed to give a course of lectures on the subject at the Sorbonne, and, apparently without knowledge of the result contained in Todhunter's treatise, reproduced the solution for the ellipse and for any equilateral polygon. In the same year M. Emile Barbier, whose lamented decease occurred in the course of the present year and who had attended Lamé's lectures, discovered and published in *Liouville's Journal* for that year a universal solution for an undivided plane contour of any form whatever.

The subsequent history I am not able to trace further than to state that in Czuber's *Geometrische Wahrscheinlichkeiten* (Leipzig, 1884) Barbier's solution is extended to the case of any two rigidly connected convex figures (in a plane)*. I propose to give here the finishing stroke to the theory as regards plane figures by extending it to any number of them, rigidly connected and of any forms, in the same plane. It is always to be understood, in what precedes as in what follows, that the greatest diameter of the figure, or system of figures, is less than the distance between two consecutive parallels.

Barbier's principle (see Czuber, pp. 117, 125) leads at once to the conclusion that the probability of any figure (subject to the restriction above stated) intersecting the system of parallels is to certainty as the length of a cord stretched round the figure is to the circumference of a circle touched by two adjoining parallels†. This circumference (with a view to simplicity of expression) we shall adopt as the unit of length in all subsequent formulae.

By the disjunctive probability of a set of figures I shall understand the probability of *one or more* of them intersecting one of the parallels: by the conjunctive probability of the same, the probability of *all* of them intersecting one of the parallels.

I start from Barbier's theorem that for a single figure the probability of intersection is measured by the length of a stretched string passing round it: this, it should be observed, is universally true whether the contour be curvilinear or rectilinear or mixtilinear, composed of a single line straight or curved or of any number of such—a theorem almost unexampled for its generality. The disjunctive probability for any number of figures $A, B, C, \ldots, H$ I shall for the present denote by $A:B:C:\ldots:H$, the conjunctive by $A.B.C\ldots H$.

Let there be $n+1$ figures given, let p_i be the sum of the conjunctive and ϖ_i of the disjunctive probabilities for these figures taken i and i together; so that ϖ_1 and p_1 are identical, and ϖ_{n+1}, p_{n+1} are monomial quantities. Then by a universal theorem of *logic* we have the reciprocal formulae

$$\varpi_{n+1} = \sum_{i=1}^{i=n+1} (-)^{i+1} p_i, \tag{1}$$

$$p_{n+1} = \sum_{i=1}^{i=n+1} (-)^{i+1} \varpi_i. \tag{2}$$

* See *Postscriptum*, p. [679, below].

† The case of a straight line (the original question of the *needle*) may be made to fall under this rule: for the line, as Barbier has observed, may be regarded as an indefinitely narrow ellipse or other oval.

Let us now suppose that we have obtained expressions for the disjunctive and conjunctive probabilities of any number not exceeding n figures of any kind: we may extend these to the case of $n+1$ figures as follows.

(1) When the $n+1$ figures are so situated that it is impossible for all of them to be cut by the same straight line, we have $p_{n+1}=0$ so that ϖ_{n+1} can be found immediately in terms of $p_1, p_2, \ldots, p_n$ by using formula (1), or in terms of $\varpi_1, \varpi_2, \ldots, \varpi_n$ by using (2); that is ϖ_{n+1} can be found in terms of known quantities; for by hypothesis all the terms of p_i or of ϖ_i are known when i is any number not exceeding n.

(2) When all the $n+1$ figures are capable of being cut by the same straight line, let XY be some straight line which cuts them all and call the figures taken in the order in which they are cut by XY

$$A_1, A_2, A_3, \ldots, A_{n+1}\text{*}.$$

Let a stretched string be made to wind round these $n+1$ contours passing alternately from one side of XY to the other, as in Fig. 1, and crossing itself

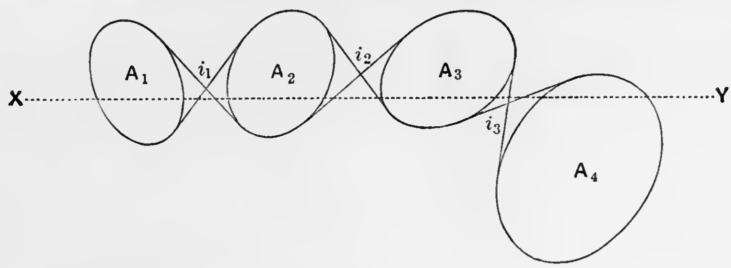

Fig. 1.

in the n points $i_1, i_2, \ldots, i_n$ lying between A_1, A_2; A_2, A_3; ... A_n, A_{n+1} respectively. Let us call the figures enclosed by the successive $n+1$ *loops* of the winding string

$$B_1, B_2, B_3, \ldots, B_{n+1}.$$

It is obvious that any straight line which cuts all these loops will cut all the given figures, and *vice versâ*.

Hence $$A_1 . A_2 . A_3 \ldots A_{n+1} = B_1 . B_2 . B_3 \ldots B_{n+1}.$$

Let P_i, Π_i represent what p_i, ϖ_i become when for the figures A we substitute the loops B, so that

$$\Pi_{n+1} = \sum_{i=1}^{i=n+1} (-)^{i+1} P_i,$$

$$P_{n+1} = \sum_{i=1}^{i=n+1} (-)^{i+1} \Pi_i,$$

and $$P_{n+1} = p_{n+1}.$$

* It may be well to draw at once attention to the fact that different systems of straight lines do not necessarily cut the figures $A_1, A_2, A_3, \ldots$ in the same order; as, for example, if three circles touch, or so nearly touch one another that each blocks the channel between the other two, straight lines may be drawn whose intersections with *any* one of the three shall be intermediate to their intersections with the other two.

Π_{n+1} is known by Barbier's rule, because the loops taken together form a single figure, in fact

$$\Pi_{n+1} = L,$$

where L is the length of the uncrossed string stretched round the system of figures B, which is no other than that stretched round the given figures A. Also, by hypothesis, Π_i is known for all values of i not exceeding n. We therefore know p_{n+1} which is the same as P_{n+1}. Hence ϖ_{n+1} is known from (1): thus then p_{n+1} and ϖ_{n+1} are both known, so that when the conjunctive and disjunctive probabilities are known in general for n figures they become known for $n+1$ figures; but when $n=1$, p_1 and ϖ_1 are equal to one another and to the length of a given stretched string. Hence, by the usual process of induction, we may conclude that the conjunctive and disjunctive probabilities for any number of figures can always be expressed as a linear function with positive and negative integer coefficients, or in a word as a Diophantine linear function, of a finite number of lengths of certain stretched strings.

When there are only *two* figures A_1, A_2 we pass a stretched string between them crossing itself in i (see Fig. 2): then using $(A_1 \times A_2)$ to

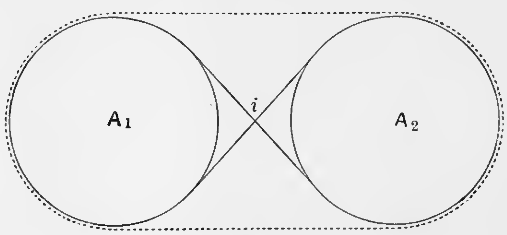

Fig. 2.

denote the length of this string, and (A_1A_2) to denote the length of the uncrossed string (indicated by dots in the figure) stretched round A_1, A_2 we have

$$\Pi_2 = (A_1 \times A_2) - P_2$$

and

$$\varpi_2 = (A_1) + (A_2) - p_2$$

(where (A_1), (A_2) denote the lengths of the separate bands round A_1, A_2 respectively).

But

$$\Pi_2 = (A_1A_2),$$

and consequently

$$p_2 = P_2 = (A_1 \times A_2) - (A_1A_2),$$

$$\varpi_2 = (A_1) + (A_2) + (A_1A_2) - (A_1 \times A_2).$$

We will now proceed to consider in detail the application of the inductive method to the case of *three* figures for which, since each of these may be replaced by a convex band passing round it, we may if we please for greater graphical simplicity substitute three convexes (that is contours which any secant must intersect in exactly two points). Many cases requiring separate discussion will arise, but one important consequence, rising to the dignity of a principle, which holds good whatever may be the number of figures, governs them all; namely that the final result for either probability is a linear homogeneous function of lengths of stretched bands drawn in various ways round the given figures and depending for their course on the forms and disposition of these figures exclusively, *wholly uninfluenced* by the presence of any points external to them. Lines drawn from the pointed ends, or apices, of the loops enclosing them do it is true make their appearance in the computations but, either coalesce into portions of the bands referred to, or else, entering in pairs with opposite algebraical signs, disappear from the final result. As a consequence, if for the sake of illustration we suppose the figures to be any closed *curves* without singular points, the probability, disjunctive or conjunctive, to be ascertained is a function exclusively of the complete system of lengths of double tangents that can be drawn between the curves and of the arcs into which they are severally divided by their points of contact with those tangents.

We have for all the cases of three figures

$$\varpi_3 = p_1 - p_2 + p_3$$

where $$p_1 = (A_1) + (A_2) + (A_3)$$

and $$p_2 = (A_2 \times A_3) - (A_2A_3) + (A_3 \times A_1) - (A_3A_1) + (A_1 \times A_2) - (A_1A_2).$$

Thus $$\varpi_3 - p_3 = (A_1) + (A_2) + (A_3) + (A_2A_3) + (A_3A_1) + (A_1A_2) - (A_2 \times A_3) - (A_3 \times A_1) - (A_1 \times A_2). \quad (3)$$

Similarly $$\Pi_3 - P_3 = (B_1) + (B_2) + (B_3) + (B_2B_3) + (B_3B_1) + (B_1B_2) - (B_2 \times B_3) - (B_3 \times B_1) - (B_1 \times B_2),$$

where B_1, B_2, B_3 are the loops of the string which passes round the figures A_1, A_2, A_3 and crosses itself at i and j, as shown in Fig. 3. But $P_3 = p_3$,

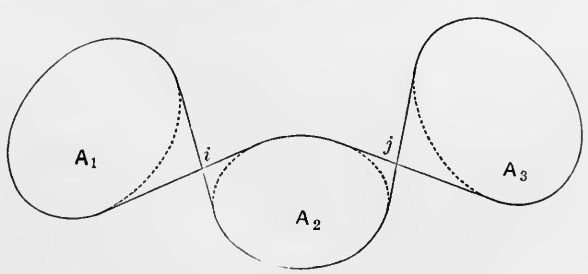

Fig. 3.

and Π_3 is the length of an uncrossed band stretched round the entire system of figures A_1, A_2, A_3 (which will be expressed in symbols by writing

$$\Pi_3 = (A_1A_2A_3).)$$

Hence $p_3 = (A_1 A_2 A_3) + (B_2 \times B_3) + (B_3 \times B_1)$
$$+ (B_1 \times B_2) - (B_1) - (B_2) - (B_3) - (B_2 B_3) - (B_3 B_1) - (B_1 B_2).$$

Moreover $$(B_1 \times B_2) = (B_1) + (B_2)$$

and $$(B_2 \times B_3) = (B_2) + (B_3),$$

because B_1, B_2 and B_2, B_3 are pairs of consecutive loops. And whenever the three given figures are capable of being cut by a straight line in the order A_1, A_2, A_3 (that is except in the case $p_3 = 0$, which is separately considered)

$$(B_3 B_1) = (A_3 A_1),$$

because both the crossing points, i and j, of the looped string necessarily fall inside the uncrossed band round A_1, A_3. Thus the value of p_3 is given by the equation

$$p_3 = (A_1 A_2 A_3) - (A_1 A_3) + (B_3 \times B_1) + (B_2) - (B_2 B_3) - (B_1 B_2) \qquad (4)$$

which, for immediate purposes, we shall find convenient to write under the form

$$p_3 = (A_1 A_2 A_3) - (A_1 A_3) + (B_2 \times B_3) - (B_2 B_3) + (B_3 \times B_1) - (B_1 B_2) - (B_3). \quad (5)$$

We shall apply the formula to the two classes which between them comprise all the cases of three figures, namely

Class A. One of the figures, which we call A_2, lies either wholly or partially inside the crossed band round the other two.

Class B. Each figure lies entirely outside the crossed band round the other two.

In Class A we recognize three species, namely

Aa. The figure A_2 does not cut either of the crossed strings *ab*, *cd* of the band looped round A_1, A_3 (Fig. 4), but lies wholly in the same loop as one of them, which we call A_1.

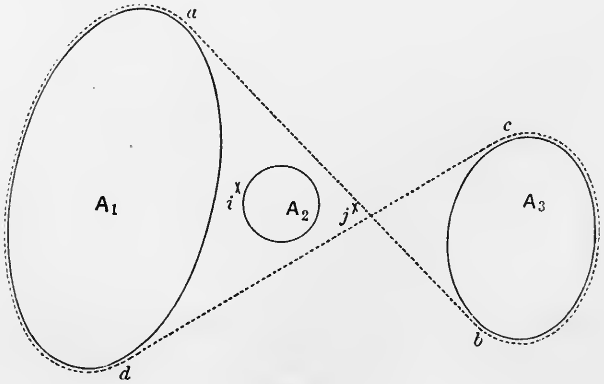

Fig. 4.

Ab. The figure A_2 cuts one, but not both, of the crossed strings *ab, cd* (Fig. 5), and part of it lies in the same loop as A_1.

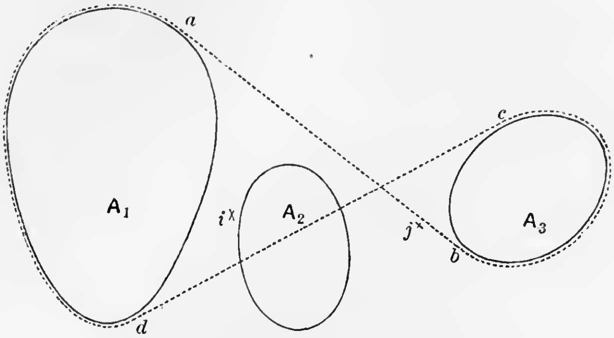

Fig. 5.

Ac. The figure A_2 cuts both the crossed strings *ab, cd* (Figs. 6 and 7) and part of it lies in the same loop as A_1.

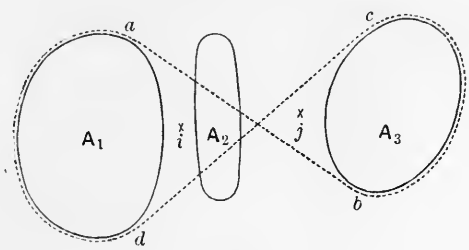

Fig. 6.

To avoid complicating these figures (4, 5, 6, 7) the band (looped round A_1, A_2, A_3 as shown in Fig. 3) which crosses itself at i, j is not given, but the position of each crossing point is marked by a small cross. It should be

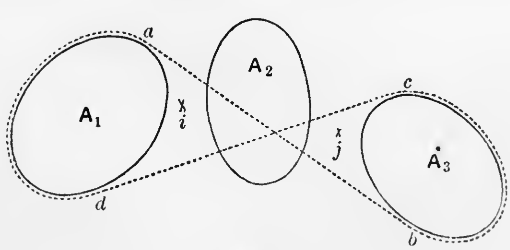

Fig. 7.

observed that in Fig. 5 (species Ab) j lies outside the crossed band round A_1, A_3; in Fig. 4 (species Aa) i and j lie in the same loop, and in Figs. 6, 7 (species Ac) i and j lie in opposite loops of the crossed band round A_1, A_3.

The discussion of species Aa is very simple; for it is clear that the conjunctive probability is

$$p_3 = (A_2 \times A_3) - (A_2 A_3)$$

since it is obviously impossible for a straight line to cut A_2 and A_3 without cutting A_1. Substituting this value for p_3 in formula (3) we obtain the disjunctive probability

$$\varpi_3 = (A_1) + (A_2) + (A_3) + (A_1 A_2) + (A_1 A_3) - (A_1 \times A_2) - (A_1 \times A_3).$$

The remaining two species belonging to class A may be discussed simultaneously; for we have in all the cases (see Fig. 8), using e, f to denote the

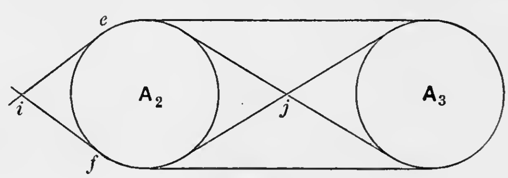

Fig. 8.

points of contact with the figure A_2 of the strings which cross at the point i (between A_1 and A_2),

$$(B_2 \times B_3) = (A_2 \times A_3) + fi + ie - ef,$$
$$(B_2 B_3) = (A_2 A_3) + fi + ie - ef,$$

so that $$(B_2 \times B_3) - (B_2 B_3) = (A_2 \times A_3) - (A_2 A_3).$$

Hence, for all the species of class A, formula (5) becomes

$$p_3 = (A_1 A_2 A_3) - (A_1 A_3) + (A_2 \times A_3) - (A_2 A_3) + (B_1 \times B_3) - (B_1 B_2) - (B_3).$$

In reducing the last three terms of this expression to a form which involves the lengths of bands round the A's, a slight difference arises between species Ab (in which, see Fig. 5, the point j and the figure A_1 are on the same side of the string ab) and species Ac (in which j and A_1 are on opposite sides of the string ab, see Figs. 6 and 7).

Thus, for species Ac, the crossed band round B_1, B_3 will not encounter either of the points i, j, but will be identical with the crossed band (*abcda*, Figs. 6 and 7) round A_1, A_3; that is

$$(B_3 \times B_1) = (A_3 \times A_1).$$

Moreover, a moment's reflexion will show that the uncrossed band round B_1, B_2 will combine with the loop B_3 so as to form a single band: in fact we have

$$(B_1 B_2) + (B_3) = D,$$

where D is the crossed band round A_1, A_3 with the loop which contains A_1 distended until it also contains A_2.

But in species Ab (see Fig. 9), let the points of contact with A_3 of the strings which cross at j (between A_2, A_3) be g, h; and let a string jk, in contact with A_1 at k, be stretched from j to the figure A_1: then

$$(B_1 \times B_3) = (A_1 \times A_3) + gj + jk + ka - ab - bg,$$

and

$$(B_1 B_2) + (B_3) = D + gj + jk + ka - ab - bg,$$

where D is the band ($abgchjlmna$), derived from the crossed band ($abgcdna$) round A_1, A_3 by distending the loop which contains A_1 until it also contains A_2.

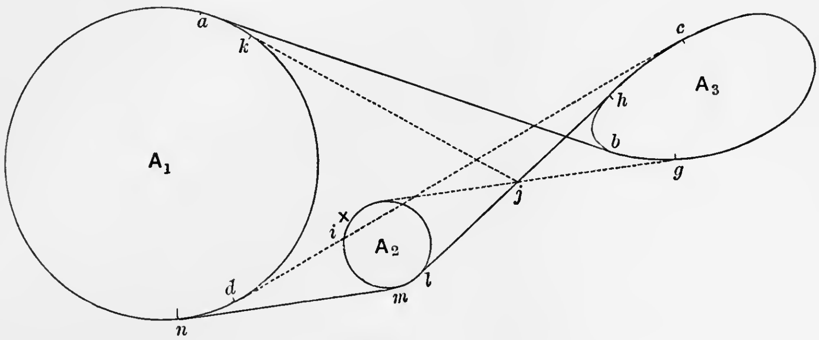

Fig. 9.

Hence $$(B_1 \times B_3) - (B_1 B_2) - (B_3) = (A_1 \times A_3) - D,$$

and the general formula for the conjunctive probability (for class A) becomes

$$p_3 = (A_1 A_2 A_3) + (A_1 \times A_3) + (A_2 \times A_3) - (A_1 A_3) - (A_2 A_3) - D. \qquad (6)$$

Combining this with formula (3), which belongs to all cases of three figures, we obtain

$$\varpi_3 = (A_1) + (A_2) + (A_3) + (A_1 A_2) + (A_1 A_2 A_3) - (A_1 \times A_2) - D.$$

The species Aa, Ab, Ac are distinguishable from one another by the difference in shape of the band D belonging to each. Thus in Aa the band

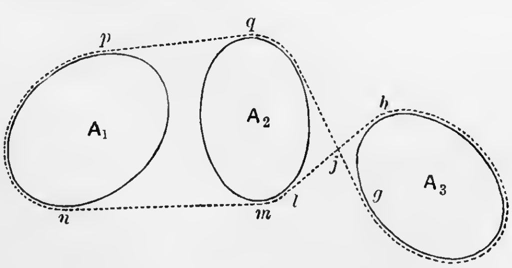

Fig. 10.

D is not distended at all, but is simply $(A_1 \times A_3)$; in Ab the loop containing A_1 is distended on one side only; and in Ac is distended on both sides (see Figs. 10 and 11). This difference in shape will be denoted by writing D_1

for D in the general formula when the species is Ab, and D_2 for D when the species is Ac.

The dotted bands (*pqjghjlmnp*) of Fig. 10, and (*abhlmna*) of Fig. 11 are what the dotted bands of Fig. 7 (species Ac) and Fig. 5 (species Ab) become, when the former is doubly and the latter singly distended.

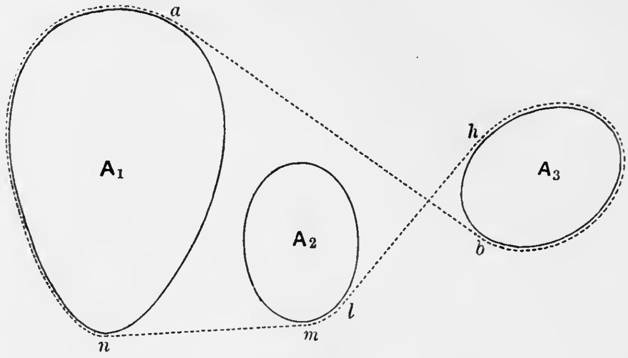

Fig. 11.

Varieties of the species in class A (namely one variety for Aa, two for Ab, and three for Ac, making 6 cases in all) occur when we consider the situation of the figure A_2 with respect to the uncrossed band round A_1, A_3. In all cases where A_2 lies wholly inside this band we have $(A_1A_2A_3) = (A_1A_3)$, so that in all such cases the general formula (6), which gives the conjunctive probability, becomes

$$p_3 = (A_1 X A_3) + (A_2 X A_3) - (A_2A_3) - D.$$

Aa. We have $D = (A_1 X A_3)$

so that $p_3 = (A_2 X A_3) - (A_2A_3)$

(the same as the result previously obtained from *à priori* considerations).

Ab. 1. The figure A_2 lies wholly within the uncrossed band round A_1, A_3

$$p_3 = (A_1 X A_3) + (A_2 X A_3) - (A_2A_3) - D_1.$$

Ab. 2. The figure A_2 cuts the uncrossed band round A_1, A_3

$$p_3 = (A_1A_2A_3) + (A_1 X A_3) + (A_2 X A_3) - (A_1A_3) - (A_2A_3) - D_1.$$

Ac. 1. The figure A_2 lies wholly within the uncrossed band round A_1, A_3.

Ac. 2. The figure A_2 cuts only one string of the uncrossed band round A_1, A_3. In these two cases the formulae which give p_3 are the same as in the corresponding varieties of Ab, except that D_2 takes the place of D_1.

Ac. 3. The figure A_2 cuts both strings of the uncrossed band round A_1, A_3. In this case the formula for the conjunctive probability

$$p_3 = (A_1A_2A_3) + (A_1 \times A_3) + (A_2 \times A_3) - (A_1A_3) - (A_2A_3) - D_2$$

becomes greatly simplified; for (see Fig. 12)

$$D_2 - (A_1A_2A_3) = rsjgu + vljht - rt - vu = (A_2 \times A_3) - (A_2A_3)$$

so that

$$p_3 = (A_1 \times A_3) - (A_1A_3),$$

which is evidently true, since every straight line which cuts both A_1 and A_3 must also (in this case) cut A_2.

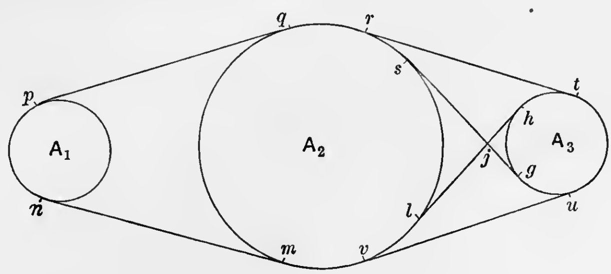

Fig. 12.

We have now enumerated all the six cases of Class A, and given in each case the formula for the conjunctive probability (from which, by means of formula (3), the disjunctive probability may be determined immediately). We proceed to the discussion of Class B.

In Class B (that is in the class where each figure lies entirely outside the crossed band round the other two) we recognize four species, and in one of them two varieties, making five cases in all. The enumeration is as follows.

Ba. There is one definite order of succession in which the three figures can be cut by a system of straight lines. There are two varieties of this species, namely

Ba. 1. The middle figure (A_2, see Fig. 13) lies wholly inside the uncrossed band round the other two. The small crosses in this figure, as in others, indicate the positions of the points i, j where the string looped round A_1, A_2, A_3 (see Fig. 3) crosses itself.

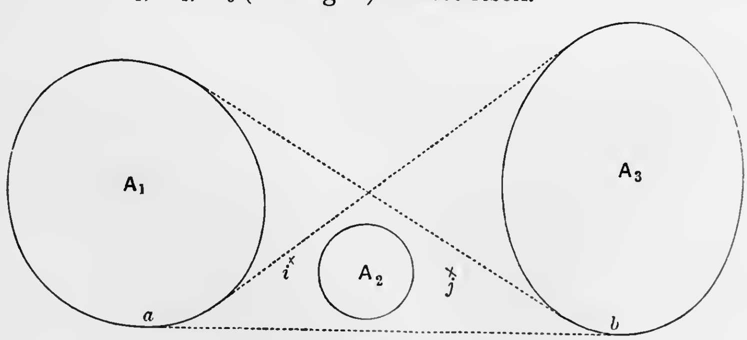

Fig. 13.

Ba. 2. The middle figure cuts the uncrossed band round the other two as shown in Fig. 14. In this, as in the preceding case, both i and j lie outside the crossed, but inside the uncrossed, band round A_1, A_3*.

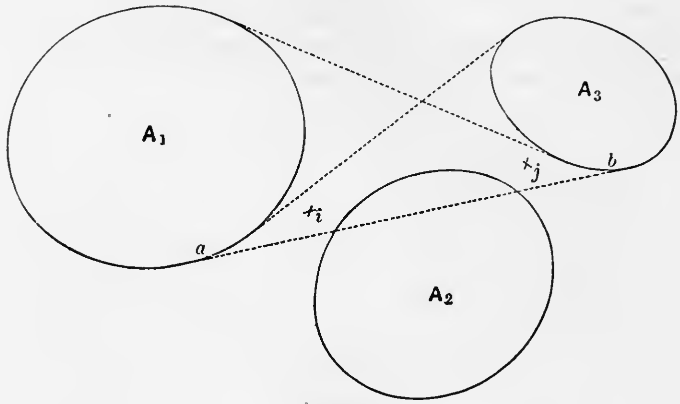

Fig. 14.

Bb. The figures may be cut in two different orders by two distinct systems of straight lines (see Fig. 15). One system of straight lines cuts the figures in the order A_1, A_2, A_3; the other system cuts them in the order A_3, A_1, A_2.

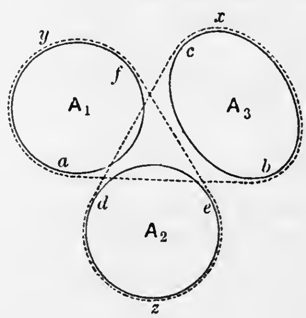

Fig. 15.

Bc. The figures may be cut by three distinct systems of straight lines (Fig. 16).

Bd. The three figures cannot all be cut by any straight line (Fig. 17).

In all cases with the exception of Bd, which will be treated separately, we have (see formula (4) *ante* [p. 668])

$$p_3 = (A_1A_2A_3) - (A_1A_3) + (B_1 \times B_3) + (B_2) - (B_2B_3) - (B_1B_2).$$

* This circumstance enables us to discuss Ba. 1 and Ba. 2 simultaneously.

In Ba (see Fig. 18) we have

$$(B_2 B_3) = (A_2 A_3) + hi + ik - kc - cd - dh,$$

$$(B_1 B_2) = (A_1 A_2) + mj + jn - nf - fe - em,$$

$$(B_1 \times B_3) = (B_1) + (B_3) + ik - kc - cr - rj + jn - nf - fp - pi.$$

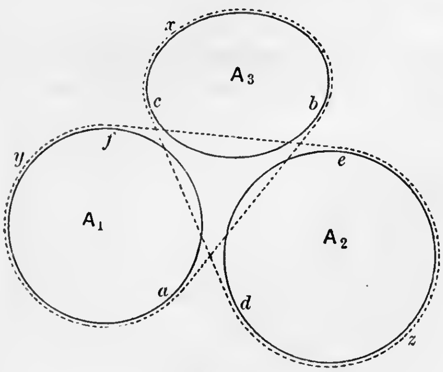

Fig. 16.

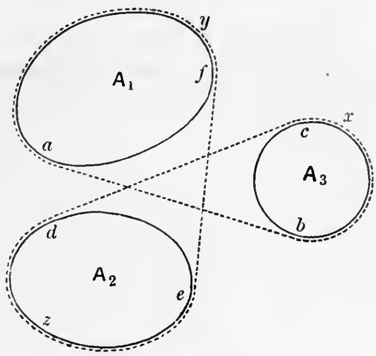

Fig. 17.

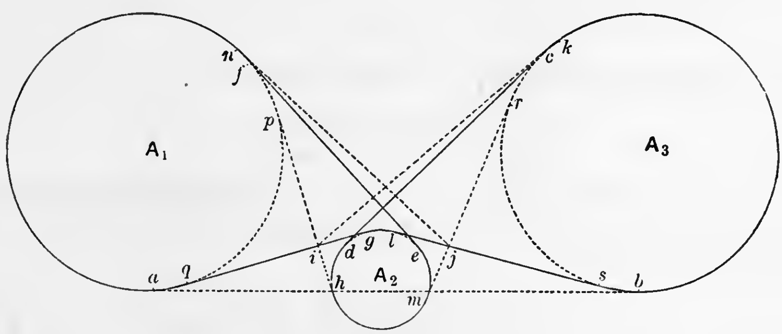

Fig. 18.

Substituting these values in the general expression for p_3, we obtain

$$p_3 = (A_1 A_2 A_3) - (A_2 A_3) - (A_3 A_1) - (A_1 A_2)$$
$$+ (B_1) + (B_2) + (B_3) - mr - rc + cd + dh - hp - pf + fe + em$$

where the term $-mr$ comes from $-mj - rj$, and the term $-hp$ comes from $-hi - pi$; the other terms involving the points i, j or the points of contact k, n of tangents drawn from them to the original figures disappear in pairs. The terms

$$(B_1) + (B_2) + (B_3) - mr - rc + cd + dh - hp - pf + fe + em$$

will be seen to coalesce into a single band (whose course is marked in Fig. 18 by the continuous line *aqigljsbkcdhmefna*, all other lines in the figure being dotted). This band we shall call Δ_1.

Fig. 18 is drawn for the case Ba. 2, but the investigation of case Ba. 1 is precisely the same as that of Ba. 2. In both cases we find

$$p_3 = (A_1A_2A_3) - (A_2A_3) - (A_3A_1) - (A_1A_2) + \Delta_1$$

for the conjunctive probability, and consequently

$$\varpi_3 = (A_1) + (A_2) + (A_3) + (A_1A_2A_3) - (A_2 \times A_3) - (A_3 \times A_1) - (A_1 \times A_2) + \Delta_1$$

gives the disjunctive probability in both cases.

The band Δ_1 for the case Ba. 1 is shown by the continuous line of Fig. 19, that is Δ_1 is the band *atqglsvbxcdwuefya*: its course is precisely the same as that of the Δ_1 for the case Ba. 2.

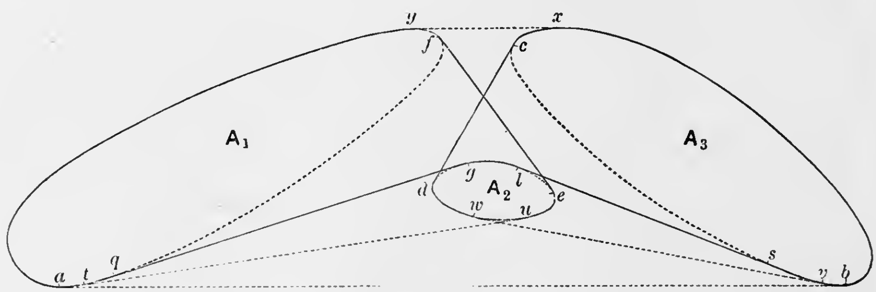

Fig. 19.

The difference between the two cases is this: in Ba. 1 we have

$$(A_1A_2A_3) = (A_1A_3)$$

so that
$$p_3 = \Delta_1 - (A_1A_2) - (A_2A_3)^*,$$

whereas in Ba. 2 (and in all the cases to be subsequently considered) the terms $(A_2A_3) + (A_3A_1) + (A_1A_2) - (A_1A_2A_3)$ coalesce into a single band which we shall call Δ, so that

$$p_3 = \Delta_1 - \Delta.$$

The course of the band Δ is marked by the letters *abkcdhmefna* in Fig. 18. The band Δ_1 may be derived from Δ by supposing its rectilinear portion *ab* to be pressed inwards by the figure A_2 so as to occupy the position *aqglsb*.

The investigation of the case Bb proceeds on exactly the same lines as that of Ba. 2; we start from the same general formula and, by performing precisely similar work, obtain the result

$$p_3 = \Delta_2 - \Delta,$$

where (see Fig. 15) Δ is the band *abxcdzefya* whose course is indicated by dots, and Δ_2 is the band derived from Δ by supposing *two* of its rectilinear portions *ab, cd* to be pressed inwards by the figures A_1 and A_2.

* By an easy rearrangement of the bands the value of p_3 for this case may be expressed as the difference of the two bands, *atuelgdwvbxya* and *atqgleuwdglsvbxya* (see Fig. 19), derived from the uncrossed band *abxya* round A_1, A_3 by *twisting* its rectilinear portion *ab* right round A_2 in opposite directions.

In the case Bc (Fig. 16) the work is simplified by observing that each of the figures A_1, A_2, A_3 blocks the channel between the other two (that is, no straight line can pass between any two of them without cutting the third). Hence every straight line which cuts the uncrossed band round all the figures must cut one or more of them; that is

$$\varpi_3 = (A_1 A_2 A_3)$$

and consequently formula (3) gives

$$p_3 = (A_1 A_2 A_3) - (A_2 A_3) - (A_3 A_1) - (A_1 A_2)$$
$$+ (A_2 \times A_3) + (A_3 \times A_1) + (A_1 \times A_2) - (A_1) - (A_2) - (A_3).$$

Now it is easily seen that

$$(A_2 A_3) + (A_3 A_1) + (A_1 A_2) - (A_1 A_2 A_3) = \Delta$$

and $$(A_2 \times A_3) + (A_3 \times A_1) + (A_1 \times A_2) - (A_1) - (A_2) - (A_3) = \Delta_3$$

where Δ is the band *abxcdzefya* (shown by the dotted line in Fig. 16) and Δ_3 is what Δ becomes when its rectilinear portions *ab, cd, ef* are pressed inwards by the figures A_1, A_2, A_3.

Thus $$p_3 = \Delta_3 - \Delta.$$

The sole remaining case of three figures is Bd (Fig. 17), the case in which no straight line can possibly cut all three figures. In it we have obviously

$$p_3 = 0,$$

and therefore

$$\varpi_3 = (A_1) + (A_2) + (A_3) + (A_2 A_3) + (A_3 A_1) + (A_1 A_2)$$
$$- (A_2 \times A_3) - (A_3 \times A_1) - (A_1 \times A_2).$$

This case forms no exception to the general rule for finding the conjunctive probability in cases belonging to class B.

We have $$\Delta = abxcdzefya$$

(that is, Δ is the dotted band of Fig. 17), and since this band is not pressed inwards by any of the figures the conjunctive probability according to the rule would be $\Delta - \Delta = 0$, which is right.

Having thus pointed out the general method of procedure, and illustrated it by treating in detail the case of three figures, it does not seem desirable to pursue the subject further in this direction for the present; but, before concluding, it may be worth while to notice that, in the general case of n limited right lines, the probabilities with which we have to do become Diophantine linear functions of the sides of the complete $2n$-gonal figure of which the n pairs of extremities of the lines are the angles. There will be a group of such linear functions depending on the mutual disposition of the n lines, but the number of formulae in any such group will be much greater than in the case of n general figures: for, when we pass from these to indefinitely narrow ovals, the portion of a definite band (appearing in any formula), partially surrounding any one of such ovals, may, according to the mutual disposition of their major axes, have in common with it an infinitesimal arc in some cases, in others an arc (to an infinitesimal *près*) equal to a circumference, and again in others to a semicircumference of the oval; which latter is ultimately the same as the length of the line whose double the complete circumference represents.

By way of illustration let us consider the question of two needles or limited straight lines rigidly connected. Neglecting the limiting cases, where one of the lines terminates in the other, there will remain three hypotheses:

A. The lines intersect.

B. The lines tend to intersect in a point external to each of them.

C. One of the lines tends towards a point lying within the other.

Let p_2 denote the chance of both the needles AB, CD being cut by one of the parallels, ϖ_2 the chance of one or other of them being cut: then we have the general formulae applicable to all cases

$$\varpi_2 = 2AB + 2CD - p_2,$$

p_2 = difference between the crossed and uncrossed bands round AB, CD.

A. When the lines intersect

$$\varpi_2 = AD + DB + BC + CA,$$
$$p_2 = 2AB + 2CD - AD - DB - BC - CA.$$

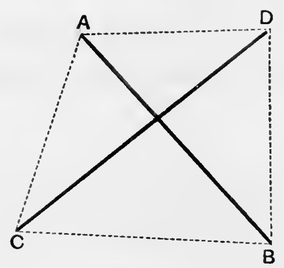

Fig. 20.

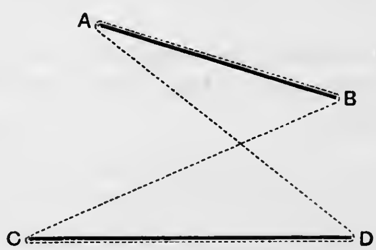

Fig. 21.

B. When the lines tend to intersect in a point external to each of them

$$p_2 = (AB + BC + CD + DA) - (AB + BD + DC + CA)$$
$$= BC - CA + AD - DB^*,$$
$$\varpi_2 = 2AB + 2CD - BC + CA - AD + DB.$$

* Imagine a string passing from B to C, from C to A, from A to D, and from D to B. This string cannot be kept tight unless fastened by pins at A, B, C, D. Inserting the necessary pins and tightening the string, we agree to consider the consecutive portions of the string as alternately positive and negative.

On these suppositions p_2 is the algebraical length of the band $BCADB$ stretched round the pins. The method of representation by means of pinned bands may be extended to the case of two (or any number of) general figures.

C. When one of the lines tends towards a point lying within the other

$$p_2 = (2AB + BC + CD + DB) - (AC + CD + DA)$$
$$= 2AB + BC - CA - AD + DB,$$
$$\varpi_2 = 2CD - BC + CA + AD - DB.$$

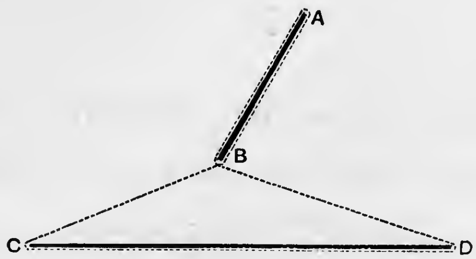

Fig. 22.

The complexity of cases for three right lines is such as would require a separate study even to obtain a perfect enumeration of them; consequently I shall leave it to others to pursue the subject further whether as regards principles or details. I will only add that the ascertainment of the general law that the formulae contain no other arguments than lengths of tight endless bands variously drawn round the given contours appears to me, a distinct step achieved in the prosecution of this extensive theory, and one that is far from being obvious *à priori*. Buffon's problem of the needle, it will be seen, has now expanded into a problem of n needles rigidly connected, which may be treated as a corollary to that of n *entirely separate* general contours, the mode of solution of which, it is believed, has been sufficiently indicated in the investigations which form the subject of this memoir.

Postscriptum. Since the above was set up in print my attention has been called to the fact that the extension of Barbier's theorem referred to on p. [664] is due to Prof. Crofton and is given by him in his celebrated paper on the *Theory of Local Probability* contained in the *Philosophical Transactions* for 1868. Strange to say, no reference to this, so far as I can find, is made in Czuber's treatise. It is the more singular that I should have overlooked the fact inasmuch as it was an outcome of conversations with myself, when Prof. Crofton was serving under me in the Royal Military Academy at Woolwich, that he was put upon the track of investigations in local probability in which he has since earned for himself so great and well merited celebrity. It may be added that Prof. Crofton seems to have written in entire ignorance of Barbier's discovery as he makes no allusion to it in his paper.

It is indeed a romantic incident in mathematical history that Buffon's problem of the needle should have led up (as is undoubtedly the case) to Crofton's new and striking theorems in the integral calculus reproduced in Bertrand's *Calcul intégral*.

70.

SUR LE RAPPORT DE LA CIRCONFÉRENCE AU DIAMÈTRE.

[*Comptes Rendus*, CXI. (1890), pp. 778—780.]

[See p. 682, below ; footnote.]

En étudiant la preuve de Lambert, du théorème que π ne peut pas être la racine carrée d'un nombre entier, je crois avoir trouvé le moyen d'en faire l'extension au théorème de Lindemann, c'est-à-dire que π ne peut pas être la racine d'une équation rationnelle. Par exemple, supposons que π soit une racine de l'équation

$$Ax^2 + Bx + C = 0,$$

ou en mettant $Ax = \rho$, que $A\pi$ soit une racine de

$$\rho^2 + B\rho + AC = 0 ;$$

prenons un nombre entier K, tel que $K(B - A\pi)$ soit de la forme

$$2m\pi + (1 - \theta)\frac{\pi}{2},$$

θ étant < 1 ; en mettant $K\rho = R$, nous aurons l'équation

$$R^2 + DR + E = 0, \qquad (1)$$

dont $KA\pi$ sera une racine et l'autre une quantité dont la tangente sera positive, η.

Considérons la fraction continue

$$S = 3 - \frac{R^2}{5 -}\,\frac{R^2}{7 -}\cdots,$$

en mettant $R = KA\pi$, on aura

$$S = 0 ;$$

en mettant $R = \eta$, on aura

$$S' = \eta.$$

Or, prenons un nombre ν tel que $2\nu > R^2$ et considérons les deux fractions continues

$$S_\nu = \frac{R^2}{2\nu + 1 -}\,\frac{R^2}{2\nu + 3 -}\,\frac{R^2}{2\nu + 5}\cdots,$$

$$S'_\nu = \frac{R'^2}{2\nu + 1 -}\,\frac{R'^2}{2\nu + 3 -}\,\frac{R'^2}{2\nu + 5}\cdots,$$

R, R' étant les deux racines de l'équation quadratique (1)

$$S_\nu = \frac{B}{A}, \qquad S_{\nu+1} = \frac{C}{B}, \qquad S_{\nu+2} = \frac{D}{C}, \qquad \ldots,$$

$A, B, C, D, \ldots$ étant des fonctions linéaires avec des coefficients entiers de R, et l'on aura

$$S'_\nu = \frac{B' - B'_1\eta}{A' - A'_1\eta}, \qquad S'_{\nu+1} = \frac{C' - C'_1\eta}{B' - B'_1\eta}, \qquad \ldots,$$

A', B', C' étant les mêmes fonctions de R' que le sont A, B, C de R.

Or, on peut démontrer que $A', B', C', \ldots$ seront des nombres positifs, et $\frac{A'}{A'_1}, \frac{B'}{B'_1}, \frac{C'}{C'_1} \ldots$ chacune $> \eta$.

De plus, toutes les fractions $\frac{B' - B'_1\eta}{A' - A'_1\eta}$ seront des quantités positives et moindres que l'unité.

Mais $\frac{B'}{A'} - \frac{B' - B'_1\eta}{A' - A'_1\eta} = \frac{R'^2\eta}{A'^2\left(1 - \frac{A'_1}{A'}\eta\right)}$, dont le dénominateur sera nécessairement positif.

Donc la quantité positive $\frac{B'}{A'}$ égale une fraction positive diminuée d'une autre fraction positive.

Donc $\frac{B'}{A'}$, et les quantités semblables, $\frac{C'}{B'}, \frac{D'}{C'}, \ldots$, seront toutes des fractions positives et moindres que l'unité.

Donc $\frac{BB'}{AA'}, \frac{CC'}{BB'}, \frac{DD'}{CC'}, \ldots$ seront des fractions possédant ce même caractère.

Mais tous ces *produits* AA', BB', CC' seront des *nombres entiers*, ce qui est impossible.

Je crois pouvoir faire une démonstration tout à fait semblable pour établir que π ne peut pas être la racine d'une équation d'un degré quelconque dont toutes les racines sont réelles. Pour le cas d'équations avec des racines imaginaires, il y aura quelque chose de plus à faire pour achever la démonstration ; mais j'ai lieu de croire qu'avec l'aide de la théorie des modules de quantités imaginaires il n'y aura pas de grosses difficultés à vaincre. Enfin j'ajoute que deux quantités réelles ou imaginaires, dont l'une est la tangente ou le logarithme népérien de l'autre, ne peuvent être toutes les deux fonctions algébriques des racines de la même équation irréductible, à coefficients entiers.

71.

PREUVE QUE π NE PEUT PAS ÊTRE RACINE D'UNE ÉQUATION ALGÉBRIQUE À COEFFICIENTS ENTIERS*.

[*Comptes Rendus*, CXI. (1890), pp. 866—871.]

LEMME. *Soit*

$$J = \cfrac{\epsilon m}{n + \cfrac{\epsilon' m'}{n' + \cfrac{\epsilon'' m''}{n'' + \ddots}}},$$

où $\epsilon^2 = \epsilon'^2 = \epsilon''^2 = \ldots = 1$; $n, n', n'', \ldots$ *sont des nombres réels positifs et plus grands que l'unité;* $m, m', m'', \ldots,$ *des nombres réels ou complexes, et où chaque quotient partiel est assujetti à la condition que* $n - 1$ *est plus grand que le module de* m.

Alors je dis que le module de J sera moindre que l'unité.

Supposons que ces conditions soient satisfaites par $\frac{m}{n}$, $\frac{m_{\prime}}{n_{\prime}}$.

Soit $$m = \alpha + i\beta.$$

Par hypothèse $$n - 1 > \sqrt{(\alpha^2 + \beta^2)}.$$

Servons-nous de $M(x)$ pour signifier le module de x, alors

$$M\left(\frac{m}{n}\right) = \frac{M(m)}{n} < \frac{n-1}{n} < 1,$$

de sorte que, si $\frac{m}{n} = \alpha_{\prime} + i\beta_{\prime}$, $\alpha_{\prime}^2 + \beta_{\prime}^2 < 1$ et, à plus forte raison, $\alpha_{\prime}^2 < 1$,

$$M\left(\frac{m_{\prime}}{n_{\prime} + \frac{m}{n}}\right) = \frac{M(m_{\prime})}{M(n_{\prime} + \alpha_{\prime} + i\beta_{\prime})} = \frac{M(m_{\prime})}{\sqrt{\{(n_{\prime} + \alpha_{\prime})^2 + \beta_{\prime}^2\}}} < \frac{M(m_{\prime})}{n_{\prime} - 1},$$

* Cette Note doit être substituée à la Note de l'auteur qui a été insérée, par suite d'un malentendu, dans les *Comptes rendus* du 24 novembre dernier. La Note précédente, qui ne traitait que le cas le plus restreint du théorème du texte, est affectée d'inexactitudes qui la rendent de nulle valeur.

car $(n_{,}+\alpha_{,})^2$, quand $\alpha_{,}$ est compris entre les limites 1, -1, est plus grand que $(n_{,}-1)^2$.

Donc, par hypothèse,

$$M\left(\frac{m_{,}}{n_{,}+\dfrac{m}{n}}\right)<1,$$

et, évidemment, par le même raisonnement, on trouve successivement

$$M\left(\frac{m}{n}\right),\quad M\left(\frac{m_{,}}{n_{,}+\dfrac{m}{n}}\right),\quad M\left(\frac{m_{,,}}{n_{,,}+\dfrac{m_{,}}{n_{,}+\dfrac{m}{n}}}\right),\quad \ldots$$

ou, ce qui revient à la même chose, toutes les quantités

$$M\left(\frac{\epsilon m}{n}\right),\quad M\left(\frac{\epsilon_{,}m_{,}}{n_{,}+\dfrac{\epsilon m}{n}}\right),\quad M\left(\frac{\epsilon_{,,}m_{,,}}{n_{,,}+\dfrac{\epsilon_{,}m_{,}}{n_{,}+\dfrac{\epsilon m}{n}}}\right),\quad \ldots$$

seront moindres que l'unité*.

Nous allons démontrer, à l'aide de ce lemme, que, si θ est une racine d'une équation *irréductible* à coefficients entiers, tang θ ne peut pas être rationnel ou même une fonction rationnelle à coefficients rationnels de θ.

Supposons que $A\theta^n+B\theta^{n-1}+\ldots+L=0$ et que tang θ soit une fonction rationnelle de θ. On peut supposer que $A=1$, car, si nous écrivons $\theta'=A\theta$†, alors l'équation pour θ' peut s'exprimer semblablement à celle pour θ, mais avec le premier coefficient égal à l'unité. De plus, si l'on peut démontrer que tang θ' ne peut pas être une fonction rationnelle de θ', alors, puisque $\theta'=A\theta$, et conséquemment tang θ', est une fonction rationnelle de tang θ, il s'ensuivra que, si tang θ est une fonction rationnelle de θ, tang θ' sera une fonction rationnelle de θ', ce qui est contraire à la supposition faite‡.

* Ce lemme peut être envisagé comme une application de la proposition 8, III d'Euclide. En prenant O le centre d'un cercle à rayon unité et N un point extérieur à ce cercle, Euclide y enseigne que le segment de ON, compris entre N et le contour convexe, sera moindre que toute autre ligne droite menée de N au cercle: à plus forte raison il sera moindre que la distance de N à un point quelconque d'un cercle intérieur au premier. Voir la Note au bas de la page [685, below] pour une addition qu'on doit faire à ce lemme.

† Voir le scolie pour le cas plus général où les coefficients de l'équation en θ sont des nombres complexes [p. 686, below].

‡ L'illustre Legendre aurait, il me semble, dû faire une transformation analogue dans sa présentation célèbre de la preuve de Lambert de son théorème (Note IV, *Éléments de Géométrie*). Pour avoir négligé cette précaution, la succession infinie de quantités toujours décroissantes qu'il trouve par le moyen du lemme de Lambert ne forme pas nécessairement une succession de nombres entiers, mais de tels nombres divisés par des puissances toujours croissantes de A, le dénominateur de θ, supposé rationnel, exprimé comme fraction vulgaire réduite, ce qui n'est nullement impossible.

Donc, nous pouvons supposer que l'équation en θ soit de la forme

$$\theta^n + B\theta^{n-1} + \ldots + L = 0.$$

Évidemment on peut aussi supposer que l'équation en θ soit irréductible.

Écrivons $\theta \tang \theta = \tau(\theta)$, de sorte que

$$\tau(\theta) = \cfrac{\theta^2}{1 - \cfrac{\theta^2}{3 - \cfrac{\theta^2}{5 - \ddots}}};$$

on trouvera

$$\cfrac{\theta^2}{3 - \cfrac{\theta^2}{5 - \ddots}} = \frac{\tau(\theta) - \theta^2}{\tau(\theta)},$$

$$\cfrac{\theta^2}{5 - \cfrac{\theta^2}{7 - \ddots}} = \frac{\tau(\theta)(3 - \theta^2) - 3\theta^2}{\tau(\theta) - \theta^2},$$

$$\cfrac{\theta^2}{7 - \cfrac{\theta^2}{9 - \ddots}} = \frac{\tau(\theta)(15 - 6\theta^2) - 15\theta^2 + \theta^4}{\tau(\theta)(3 - \theta^2) - 3\theta^2},$$

et, en nommant

$$\cfrac{\theta^2}{2r + 1 - \cfrac{\theta^2}{2r + 3 - \ddots}} = \Theta_r(\theta),$$

$$\Theta_r(\theta) = \frac{A_{r+1}(\theta)\,\tau(\theta) - B_{r+1}(\theta)}{A_r(\theta)\,\tau(\theta) - B_r(\theta)},$$

$$\Theta_{r+1}(\theta) = \frac{A_{r+2}(\theta)\,\tau(\theta) - B_{r+2}(\theta)}{A_{r+1}(\theta)\,\tau(\theta) - B_{r+1}(\theta)}.$$

....

....

Soit $\Theta_{r,i}(\theta)$ ce que devient $\Theta_r(\theta)$ quand on substitue θ_i pour θ dans la valeur de $\tau(\theta)$. Si, pour une certaine racine θ_i de l'équation supposée en θ, $\tau_{r,i}(\theta) = \tau_r(\theta_i)$, alors $\tau_{r,i}(\theta)$ en vertu du lemme aura un module moindre que l'unité; sinon, ce module deviendra éventuellement et restera, pour une certaine valeur r, et pour toute valeur supérieure, au-dessous d'une certaine limite, parce que dans ce cas $\Theta_{r,i}(\theta)$ différera et continuera à différer par une quantité aussi petite qu'on veut de $\dfrac{A_{r+1}(\theta_i)}{A_r(\theta_i)}$ (dont le module a une limite supérieure dépendant de la grandeur de θ_i) quand r est pris suffisamment grand. Cela sera développé au long dans une Communication ultérieure.

Supposons que N soit le plus grand des modules carrés des n racines,

$\theta_1, \theta_2, \theta_3, \ldots, \theta_n$ les n racines de l'équation proposée en θ. Prenons $2r > N$; alors, en vertu du lemme* et à cause du principe énoncé plus haut, on aura éventuellement (en prenant $2r - N$ suffisamment grand) le produit des modules de $\Theta_r(\theta_1), \Theta_r(\theta_2), \ldots, \Theta_r(\theta_n)$ moindre que l'unité pour une certaine valeur de r et toute valeur de r supérieure à celle-ci.

Or, remarquons que, à cause de la valeur l'*unité* du coefficient de θ^n dans l'équation en θ, tous les $A(\theta)$ et les $B(\theta)$ seront des fonctions linéaires et *entières* de $\theta, \theta^2, \ldots, \theta^{n-1}$, car si $\mu > n-1$, θ^μ devient une fonction linéaire et entière de $\theta, \theta^2, \ldots, \theta^{n-1}$.

Ainsi, en supposant que k soit un nombre tel qui rende $k\tau(\theta)$ une fonction linéaire entière de $\theta, \theta^2, \ldots, \theta^{n-1}$, pour toute valeur de r,

$$k[A_r(\theta)\,\tau(\theta) - B_r(\theta)]$$

sera une fonction rationnelle et entière de θ; or, en vertu de ce qui a été dit, le produit des modules de

$$\Theta_\mu(\theta_1), \quad \Theta_\mu(\theta_2), \quad \ldots, \quad \Theta_\mu(\theta_n)$$

sera moindre que l'unité quand μ est plus grand que le nombre que nous avons nommé r. Mais le produit des modules de n quantités est le module de leur produit; donc

$$\begin{array}{l} k^n \Pi\,[A_r(\theta)\,\tau(\theta) - B_r(\theta)], \\ k^n \Pi\,[A_{r+1}(\theta)\,\tau(\theta) - B_{r+1}(\theta)], \\ k^n \Pi\,[A_{r+2}(\theta)\,\tau(\theta) - B_{r+2}(\theta)], \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}$$

formeront une succession infinie de nombres entiers décroissants, ce qui est impossible†.

Ainsi $\tau(\theta)$ et conséquemment tang θ ne peut pas être une fonction rationnelle de θ quand θ est racine d'une équation à coefficients entiers.

Si nous supposons que tang θ soit une quantité rationnelle pure et simple, cela ne fait nul changement dans notre raisonnement; ainsi, puisque tang π $\left(\text{ou bien si l'on veut tang}\,\frac{\pi}{4}\right)$ est rationnel, π ne peut pas être la racine d'une équation algébrique à coefficients entiers.

Je démontre par un procédé à peu près pareil à ce qui précède, la proposition inverse, c'est-à-dire que, si tang θ est racine d'une équation algébrique, alors θ ne peut pas être une fonction rationnelle à coefficients rationnels de tang θ. Or, dans cette théorie, il n'y a nulle distinction entre les quantités réelles et complexes, de sorte que $\sqrt{(-1)}$ compte comme quantité entière. Donc tang $\sqrt{(-1)}$, et conséquemment e, base des logarithmes népériens (qui

* On doit sous-entendre par le *lemme* la proposition ainsi nommée au commencement de cette Note, mais avec l'addition essentielle, facilement prouvée, que quand les n croissent continuellement et les m restent constants, alors, en commençant avec un r suffisamment grand, le module de J deviendra une quantité aussi petite que l'on veut.

† Voir le scolie [p. 686, below] pour le cas plus général où l'équation en θ a des coefficients complexes.

en est une fonction algébrique) ne peut pas être racine d'une équation algébrique à coefficients entiers. En réunissant les deux procédés applicables à ces deux cas, on parvient à démontrer un théorème plus général, à savoir :

*Si une fonction trigonométrique quelconque et son amplitude sont liées ensemble par une équation algébrique à coefficients entiers, ni l'une ni l'autre ne peut satisfaire à une équation algébrique à coefficients entiers, et comme cas particulier compris dans ce théorème, une fonction trigonométrique et son amplitude ne peuvent pas être l'une une racine d'une équation algébrique à coefficients entiers et l'autre aussi une racine d'une telle équation**.

Il y a un théorème un peu plus général, au moins en apparence, qu'on peut démontrer par un raisonnement tout à fait semblable.

Nommons une quantité qui est racine d'une équation algébrique irréductible à coefficients entiers, simples ou complexes, *quantité équationnelle,* et les racines de la même équation algébrique irréductible à coefficients entiers, *quantités équationnelles associées*; de plus, nommons une quantité qui est racine d'une équation dont les coefficients sont fonctions rationnelles d'un nombre quelconque d'autres quantités données *fonction équationnelle* de ces quantités ; alors on peut affirmer qu'une fonction trigonométrique et son amplitude ne peuvent pas être, toutes les deux, fonctions équationnelles d'un même système de quantités équationnelles associées. Cette proposition donne lieu de soupçonner qu'au moyen de formules propres aux fonctions elliptiques on pourrait démontrer qu'une fonction elliptique, son amplitude et son paramètre ne peuvent pas être, tous les trois, fonctions équationnelles d'un même système de quantités équationnelles associées.

Scolie. On ne doit nullement exclure le cas où θ serait proposé comme racine d'une équation à coefficients entiers, mais complexes.

Dans ce cas, si le coefficient du premier terme en cette équation est $\alpha + i\beta$, alors afin de pouvoir réduire l'équation à sa forme canonique où ce coefficient est l'unité, sans que le tangent du nouveau θ cesse d'être fonction rationnelle de tang θ, il faut écrire $\theta' = (\alpha^2 + \beta^2)\,\theta$.

On remarquera aussi que les produits [p. 685, above]

$$k^n \Pi\,[A_r(\theta)\,\tau(\theta) - B_r(\theta)], \qquad k^n \Pi\,[A_{r+1}(\theta)\,\tau(\theta) - B_{r+1}(\theta)], \qquad \ldots,$$

au lieu d'être entiers et réels, deviendront quantités complexes, mais entières, dont les *modules* vont à l'infini en décroissant; de sorte que la démonstration donnée, pour le cas où les coefficients de l'équation en θ sont des nombres ordinaires, reste bonne pour le cas général.

* Ainsi on peut affirmer qu'une fonction trigonométrique et son amplitude, ou bien un nombre et son logarithme, ne peuvent pas être tous les deux racines de deux équations algébriques quelconques à coefficients entiers. Par exemple, $\cos(\cos\lambda\pi)$ ne peut pas être un nombre algébrique de Kronecker, quand λ est rationnel, car son amplitude $\cos\lambda\pi$ est un tel nombre. De même $e^{\sqrt[i]{\lambda} + \sqrt[j]{\mu} + \sqrt[k]{\nu} + \cdots}$ ne peut pas être racine d'une équation algébrique à coefficients entiers.

72.

ON ARITHMETICAL SERIES.

[*Messenger of Mathematics*, XXI. (1892), pp. 1—19, 87—120.]

THE first part of this article relates to the prime numbers (or so to say latent primes) contained as factors of the terms of given arithmetical series; the second part will deal with the actual or, say, visible primes included among such terms. Both investigations repose alike upon certain elementary theorems concerning the "index-sums" (relative to any given prime) of arithmetical series, whether simple and continuous as in the case of series ordinarily so called or compound and interstitial as such before named series become when subjected to certain periodic and uniform interruptions.

PART I.

§ 1. *Preliminary Notions.*

Consider any given sequence

$$m+1,\ m+2,\ m+3,\ \ldots,\ m+n,$$

in relation to any given prime number q.

Let r be the sum of the indices of the highest powers of q which are contained in the several terms of the natural sequence

$$1,\ 2,\ 3,\ \ldots,\ n,$$

s the sum of the indices of the highest powers of q contained in the given sequence.

Then it is almost immediately obvious that $s =$ or $> r$, that is $s > r-1$.

For the index-sum of the natural sequence will be represented by

$$r = E\left(\frac{n}{q}\right) + E\left(\frac{n}{q^2}\right) + E\left(\frac{n}{q^3}\right) + \ldots,$$

and the index-sum of the given sequence by

$$\begin{aligned} s = E\left(\frac{m+n}{q}\right) + E\left(\frac{m+n}{q^2}\right) + E\left(\frac{m+n}{q^3}\right) + \ldots \\ -E\left(\frac{m}{q}\right) - E\left(\frac{m}{q^2}\right) - E\left(\frac{m}{q^3}\right) - \ldots \end{aligned}$$

and this is at least equal to

$$E\left(\frac{n}{q}\right) + E\left(\frac{n}{q^2}\right) + E\left(\frac{n}{q^3}\right) + \ldots,$$

that is $s =$ or $> r$.

But there is another and more important theorem, less immediately obvious, and more germane to the subject-matter of the following section, which I proceed to explain.

Suppose $\sigma_1, \sigma_2, \sigma_3, \ldots, \sigma_n$ to be the several exponents of the highest powers of q which are contained in

$$x+1,\ x+2,\ x+3,\ \ldots,\ x+n,$$

and let σ be one of these n exponents which is not less than any other of them.

Call any term in the sequence

$$x+1,\ x+2,\ x+3,\ \ldots,\ x+n$$

which contains q^σ, say P, a principal q-term.

On one side of P the terms are less, on the other greater than P; in lieu of any term substitute the difference between it and P, then I say that the q-index of such altered term will be the same as when it was unaltered.

For let the principal term, or the chosen principal term if there are more than one, be λq^σ, and let μq^ρ be any other term.

If $\rho < \sigma$, $\lambda q^\sigma \sim \mu q^\rho$ will obviously have ρ for its q-index; also if $\rho = \sigma$ the same will be true, that is supposing $\mu q^\rho - \lambda q^\rho$ to be positive, ρ will be its q-index: for if we write $\lambda = aq + b$ and $\mu = cq + d$, where $b < q$ and $d < q$, a and c must be equal, since otherwise between λq^ρ and μq^ρ there would be a term $(a+1)\,q\,.\,q^\rho$ containing a higher power of q than the principal term: hence $\mu - \lambda = d - b$ and does not contain q. In like manner if $\lambda q^\rho - \mu q^\rho$ is positive, ρ is its q-index for the same reason as before.

Hence the index-sum, *quâ* any prime q, of the two sequences

$$m+1,\ m+2,\ \ldots,\ P-1;\quad P+1,\ P+2,\ \ldots,\ m+n-1,\ m+n$$

is the same as the sum of the index-sums of

$$1,\ 2,\ 3,\ \ldots,\ P-m-1,$$
$$1,\ 2,\ \ldots,\ m+n-P.$$

Call the sum of these two index-sums s', then

$$s' = E\left(\frac{P-m-1}{q}\right) + E\left(\frac{P-m-1}{q^2}\right) + E\left(\frac{P-m-1}{q^3}\right) + \ldots$$
$$+ E\left(\frac{m+n-P}{q}\right) + E\left(\frac{m+n-P}{q^2}\right) + E\left(\frac{m+n-P}{q^3}\right) + \ldots$$

and this is

$$= \text{or} < E\left(\frac{n-1}{q}\right) + E\left(\frac{n-1}{q^2}\right) + E\left(\frac{n-1}{q^3}\right) + \ldots$$
$$= \text{or} < E\left(\frac{n}{q}\right) + E\left(\frac{n}{q^2}\right) + E\left(\frac{n}{q^3}\right) + \ldots$$
$$= \text{or} < r.$$

Hence $s' =$ or $< r$. But the original index-sum of the sequence is diminished by σ on account of P being omitted.

Hence $s - \sigma$ or $s' =$ or $< r$.

Thus we have $s > r - 1$, $s - \sigma < r + 1$.

But this is not all: we may for certain relative values of m, n, and q (without regard to the situation of the principal term) establish the inequality $s - \sigma < r$.

I premise the obviously true statement that if $f + g < h$, then

$$f + E\left(\frac{f}{q}\right) + E\left(\frac{f}{q^2}\right) + \ldots + g + E\left(\frac{g}{q}\right) + E\left(\frac{g}{q^2}\right) + \ldots$$

$$< h + E\left(\frac{h}{q}\right) + E\left(\frac{h}{q^2}\right) + \ldots.$$

Let now h be the number of terms in the natural sequence from 1 to n which contain q.

Then in the given sequence the number will be

$$h + E\left(\frac{m+n}{q}\right) - E\left(\frac{m}{q}\right) - E\left(\frac{n}{q}\right), \text{ say } h + e,$$

and the sum of the number of terms divisible by q in the partial sequences on each side of P will be $h + e - 1$, where $e = 1$ or 0; let the respective numbers be f, g. Then $f + g = h - 1 + e$, where $e = 0$ or 1, and, using the same notation as before,

$$s - \sigma = f + E\left(\frac{f}{q}\right) + E\left(\frac{f}{q^2}\right) + \ldots$$

$$+ g + E\left(\frac{g}{q}\right) + E\left(\frac{g}{q^2}\right) + \ldots,$$

and

$$r = h + E\left(\frac{h}{q}\right) + E\left(\frac{h}{q^2}\right) + \ldots.$$

Hence if $e = 0$, $s - \sigma < r$,

if $e = 1$, $s - \sigma < r + 1$,

the former inequality subsisting whenever

$$E\left(\frac{m+n}{q}\right) - E\left(\frac{m}{q}\right) - E\left(\frac{n}{q}\right) = 0.$$

If for example $m = n$, then $s - \sigma < r$ when

$$E\left(\frac{2n}{q}\right) - 2E\left(\frac{n}{q}\right) = 0.$$

which it is easily seen happens whenever $E\left(\frac{2n}{q}\right)$ is an even number.

§ 2. *Proof that* $(m+1)(m+2)\dots(m+n)$ *when* $m > n-1$ *contains a prime not contained in* $1 . 2 . 3 \dots n$*.

The universal condition independent of the relation between m, n, q, above found, namely, $s - \sigma =$ or $< r$ will be found sufficient to establish the theorem which constitutes the object of this section and which is as follows:—

"If the first term of a sequence is greater than the number of terms in it, then one term at least must be a prime or a multiple of a prime greater than that number."

When the first term exceeds by unity the number of terms, the sequence takes the form $m+1$, $m+2$, ..., $2m-1$, and since no term in this sequence can be a multiple of m, the theorem for such case is tantamount to affirming that one term at least is a prime number which is in accord with and an easy inference from the well-known "postulate of Bertrand," that between m and $2m-2$ there must always be included some prime number when $m > \frac{7}{2}$.

Suppose if possible that $m+1$, $m+2$, ..., $m+n$ contains no other primes than such as are not greater than n, and which therefore divide some of the numbers from 1 to n.

Let q be any such prime, and P_q a principal term of the sequence

$$m+1,\ m+2,\ \dots,\ m+n, \quad \text{quâ } q.$$

Then, by virtue of the proposition above established,

$$\frac{(m+1)(m+2)\dots(m+n)}{P_q}$$

will contain no higher power of q than does $1 . 2 . 3 \dots n$, and consequently if P be the least common multiple of the principal terms in respect to the several primes, say ν in number (unity not being reckoned one of them), none greater than n, we may infer that

$$\frac{(m+1)(m+2)\dots(m+n)}{P}$$

will be wholly contained in, and therefore not greater than $1 . 2 . 3 \dots n$, if the sequence $m+1$, $m+2$, ..., $m+n$ contains no prime or multiple of a prime greater than n. To fix the ideas let us agree to consider that term in the sequence which contains the highest power of q, and is the greatest of all that do the same (if there be more than one), *the* principal q-term. The least common multiple cannot be greater than the product of the principal terms which are *distinct* from each other, and since even if they are all distinct, their number cannot exceed ν (the number of primes other than

* It will readily be seen that, if this theorem is true, for n any prime, it will be so *à fortiori* when n is a composite number.

unity less than $n+1$), it follows that P cannot be greater than the product of the *highest* ν terms in the given sequence. Hence we may infer that unless

$$(m+1)(m+2)\dots(m+n-\nu)$$

is less than $1.2.3\dots n$, some prime greater than n must divide one term at least of the sequence

$$m+1,\ m+2,\ \dots,\ m+n.$$

We might go further and say that unless $1.2.3\dots n$ is greater than

$$(m+1)(m+2)\dots(m+n-\nu)D,$$

where
$$D=\Pi q^{1+E\left(\frac{m}{q}\right)+E\left(\frac{n}{q}\right)-E\left(\frac{m+n}{q}\right)},$$

(q being made successively each of the ν primes between 2 and n inclusive and Π being used in the ordinary sense of indicating products), this same conclusion must obtain.

Conversely the theorem is true when either of these inequalities is denied. The denial of the first of them, which is sufficient for the object in view, is implied in the inequality

$$(m+1)(m+2)\dots(m+n-\nu)>1.2.3\dots n,$$

which, since ν depends only on n, may be written under the form

$$F(m,n)>1.2.3\dots n.$$

This will be referred to hereafter, in this section, as the *fundamental inequality**.

Since $F(m,n)$ increases with m, the theorem if true for m must be true for any greater value of m, when n remains constant.

From this it will be seen at once that the theorem must be true when m has any value exceeding n^2 and $n>7$.

For when $n=8$ the number of primes in the range from 1 to 8 is 4 and is equal to $\frac{1}{2}n$: but as n increases the number of new primes being less than the number of odd numbers must be less than $\frac{1}{2}n$.

Hence if $n>7$ and $m>n^2$,

$$F(m,n)>m^{n-\nu}>(n^2)^{\frac{1}{2}n}>n^n>1.2.3\dots n.$$

This result enables us to prove that the theorem is true when

$$13<n<3000.$$

The theorem it will be borne in mind is true if some prime number occurs in the sequence $m+1, m+2, \dots, m+n$, or in other words if the above sequence does not consist exclusively of composite numbers. But

* The subsistence of the fundamental inequality for any given value of n implies for that value of n the truth of the theorem to be established: but the converse does not necessarily hold. The theorem may be true when the fundamental inequality is *not* satisfied.

Dr Glaisher has found* that the highest sequence of composite numbers within the first 9000000 contains only 153 terms, namely, the sequence 4652354 to 4652506 (both inclusive). Hence if the theorem is not true when $n < 3000$, in which case $n^2 + n < 9000000$, we must have $n =$ or < 153, and there ought to be a sequence of n composite numbers in which the first term is less than $(153)^2$ which is 23409. But the longest sequence of composite numbers under 23409 is that which extends from 19610 to 19660 containing 51 terms, the square of 51 is 2601 and the longest sequence under this number is that which extends from 1328 to 1360 comprising 33 terms. The square of 33 is 1089, the longest sequence below which is from 888 to 906 comprising 19 terms: the square of 19 is 361, the longest sequence below which stretches from 114 to 126 comprising 13 terms. Hence the theorem is true for all values of n not greater than 3000 and not less than 13.

It is easy to show that the theorem is true for all values of n not greater than 13.

(1) Suppose $n = 13$, which gives $\nu = 6$.

The theorem must be true when m is taken so great that

$$(m+1)(m+2)(m+3)(m+4)(m+5)(m+6)(m+7) > 1.2.3.4.5.6.7.8.9.10.11.12.13,$$

which is easily seen to be satisfied when $m =$ or > 100.

But there is no sequence of 13 composite numbers till we come to the sequence 114 to 126, so that when $m < 100$ the theorem must be true as well as when $m =$ or > 100.

(2) Suppose $n = 11$, for which value of n, $\nu = 5$.

The theorem is true if

$$(m+1)(m+2)(m+3)(m+4)(m+5)(m+6) > 1.2.3.4.5.6.7.8.9.10.11,$$

which is obviously satisfied as before when $m = 100$, but there is no sequence of 11 which precedes the sequence before named from 114 to 126. Hence the theorem is true generally for $n = 11$.

When $n = 7$, $\nu = 4$ and the theorem is true for all values of m which make

$$(m+1)(m+2)(m+3) > 1.2.3.4.5.6.7, \text{ that is, } > 5040,$$

which is obviously the case if $m =$ or > 20, but there is no sequence of 7 composite numbers till we come to 89 to 97. Hence the theorem is proved for $n = 7$.

When $n = 5$, $\nu = 3$ and the condition of the theorem is satisfied if

$$(m+1)(m+2) > 2.3.4.5, \text{ that is, } > 120,$$

* See table at the end of this section.

as is the case if $m =$ or > 10, but the first composite sequence of 5 terms is 24 to 28. In like manner when $n = 3$, $\nu = 2$ and the theorem is true when $m + 1 =$ or $> 1.2.3$, that is, $m =$ or > 5, but 8, 9, 10 is the first composite sequence of 3 terms. Similarly, when $n = 2$, $\nu = 1$ and the condition $m + 1 =$ or > 2 is necessarily satisfied since $m =$ or $> n$ by hypothesis.

Finally, the theorem is obviously true when $n = 1$, because $m + 1$, whatever m may be, contains a factor greater than 1.

Being true for the prime numbers not exceeding 13, the slightest consideration will serve to prove that, as previously remarked in a footnote, it must be true *à fortiori* for all the composite numbers between them. Hence the theorem is verified for all values of n not greater than 3000, and it only remains to establish it for values of n exceeding that limit.

To prove it for this case we must begin with finding a superior limit to ν, when $n > 3000$, under the convenient form of a multiple of $\frac{n}{\log n}$.

If we multiply together the first 9 prime numbers from 2 to 23 and divide their product by that of the natural numbers up to 9 increased in the ratio of 1 to 2^9, the quotient will be found to exceed unity; and since the following primes are all more than twice the corresponding natural numbers, if we denote by $p_1, p_2, p_3, \ldots$, the prime numbers 2, 3, 5, ..., we must have

$$p_1 . p_2 . p_3 \ldots p_\nu > 2^\nu (1 . 2 . 3 \ldots \nu),$$

(provided that $\nu > 22$, as is the case if $n =$ or > 89),

or $$\log (1 . 2 . 3 \ldots \nu) + (\log 2)\, \nu < \log (p_1 . p_2 . p_3 \ldots p_\nu).$$

But by Stirling's theorem (Serret, *Cours d'Alg. Sup.*, ed. 4, vol. II. p. 226),

$$\nu \log \nu - \nu - \tfrac{1}{2} \log \nu + \tfrac{1}{2} \log 2\pi < \log (1 . 2 . 3 \ldots \nu),$$

and by Tchebycheff's theorem (Serret, vol. II. p. 236)*,

$$\log (p_1 . p_2 . p_3 \ldots p_\nu) < n',$$

where $n' = \frac{6}{5} A n + \frac{5}{4 \log 6} (\log n)^2 + \frac{5}{2} \log n + 2$, and $A = \cdot 921292 \ldots$.

Hence $(\log \nu)(\nu - \tfrac{1}{2}) - (1 - \log 2)(\nu - \tfrac{1}{2}) + (\tfrac{1}{2} \log 2\pi - \tfrac{1}{2} \log \tfrac{1}{2} e) < n'$,

and *à fortiori* $$\log (\nu - \tfrac{1}{2})(\nu - \tfrac{1}{2}) - (\log \tfrac{1}{2} e)(\nu - \tfrac{1}{2}) < n',$$

or $$\frac{2}{e} (\nu - \tfrac{1}{2}) \log \left\{ \frac{2}{e} (\nu - \tfrac{1}{2}) \right\} < \frac{2}{e} n'.$$

Hence, if we write $$\mu \log \mu = \frac{2}{e} n' = n_1$$

we shall have $$\nu - \tfrac{1}{2} < \tfrac{1}{2} e \mu.$$

* For greater simplicity I have left out the term $-An^{\frac{1}{2}}$, and thereby *increased* the superior limit.

But
$$\mu = \frac{n_1}{\log \mu},$$
and therefore
$$\log \mu = \log n_1 - \log\log \mu = \log n_1 - \log(\log n_1 - \log\log \mu) > \log n_1 - \log\log n_1.$$

Hence
$$\mu < \frac{n_1}{\log n_1 - \log\log n_1}$$
$$< \frac{2}{e}\frac{n'}{\log n' - \log\log n' + \log\frac{2}{e}}.$$

and
$$\nu < \tfrac{1}{2} + \frac{n'}{\log n' - \log\log n' - (1 - \log 2)}\text{*}.$$

Hence, observing that $\frac{1}{u}$, $\frac{\log u}{u}$, $\frac{(\log u)^2}{u}$, $\frac{\log\log u}{\log u}$ *all* decrease as the denominators increase (provided as regards the second of these fractions that $u > e$, as regards the third that $u > e^2$, and as regards the fourth that $u > e^e$), we may find a superior limit to ν in the case before us, where $n > 3000$, by writing in the numerator of $\nu - \frac{1}{2}$,
$$\frac{(\log 3000)^2}{3000}n,\quad \frac{\log 3000}{3000}n,\quad \frac{2}{3000}n,$$
for
$$(\log n)^2,\qquad \log n,\qquad 2,$$
and in its denominator, first, $\log n - \log\log n$ for $\log n' - \log\log n'$, and then
$$\frac{\log\log 3000}{\log 3000}\log n \quad\text{and}\quad \frac{1-\log 2}{\log 3000}\log n,$$
for
$$\log\log n \qquad\text{and}\qquad 1 - \log 2 \text{ respectively.}$$

Making the calculations it will be found that we shall get
$$\nu - \tfrac{1}{2} < 1{\cdot}606\frac{n}{\log n}.$$

With the aid of this limit it will now be easy to prove the truth of the theorem when $n =$ or > 3000.

Let us suppose $n =$ or > 3000.

(1) Suppose $m < 2n$, then $m + n > \frac{3}{2}m$ and the theorem will be proved for this case, if it can be shown that in the range of numbers from m to $\frac{3}{2}m$, there is at least one prime number when $m =$ or > 3000.

* From this it will be seen that the asymptotic ratio of ν to $\frac{n}{\log n}$ is less than the asymptotic ratio which any superior limit to the sum of the logarithms of the primes not exceeding n bears to n: this perhaps is a new result, at all events it is not to be found in Serret nor indeed is it wanted for Tchebycheff's proof of the famous postulate which Serret has so lucidly expounded. The correlative theorem that the asymptotic ratio of ν to $\frac{n}{\log n}$ is always *greater* than the asymptotic ratio which any *inferior* limit to the sum aforesaid bears to n is of course an obvious and familiar fact.

This will be the case (Serret, vol. II. p. 239), if (on that supposition) $\frac{5}{6}\cdot\frac{3}{2}n - n$, that is, if

$$\frac{n}{4} > 2\sqrt{(\tfrac{3}{2}n)} + \frac{25\,(\log\frac{3}{2}n)^2}{16A\log 6} + \frac{125}{24A}(\log\tfrac{3}{2}n) + \frac{25}{6A},$$

where $A = \cdot 92129202\ldots.$

But when $n = 3000$, it will be found that the terms on the second side of the inequality are respectively less than

$$134\cdot 1641,\quad 66\cdot 9773,\quad 47\cdot 5546,\quad 4\cdot 5227,$$

whose sum is less than 750.

Hence, the inequality is satisfied, and accordingly the theorem is true when $m < 2n$ and n is equal to or *greater* than 3000; for when n satisfies that condition the derivative in respect to n of the right-hand side of the above inequality will be always less than $\frac{1}{4}$.

(2) Suppose $m =$ or $> 2n$, then it is only necessary to prove that

$$\log(2n+1)(2n+2)\ldots(3n-\nu) > \log(1.2.3\ldots n),$$

or, what is the same thing, that

$$\log\{1.2.3.4\ldots(3n-\nu)\} > \log(1.2.3\ldots n) + \log(1.2.3\ldots 2n),$$

ν being the number of primes not greater than n, and n being at least 3000.

Call the two sides of the inequality P and Q. Then (Serret, vol. II. p. 226)

$$P > \log\sqrt{(2\pi)} + (3n-\nu)\log(3n-\nu) - (3n-\nu) - \tfrac{1}{2}\log(3n-\nu)$$

$$> \log\sqrt{(2\pi)} + (3n-\nu)\log 3n + (3n-\nu)\log\left(1-\frac{\nu}{3n}\right) - 3n + \nu - \tfrac{1}{2}\log 3n$$

$$> \log\sqrt{(2\pi)} + 3(\log n)\,n + (3\log 3 - 3)\,n - (\log n)\,\nu$$
$$+ (1-\log 3)\,\nu - \tfrac{1}{2}\log 3 - \tfrac{1}{2}\log n - \nu,$$

for $-(3n-\nu)\log\left(1-\dfrac{\nu}{3n}\right) = \nu\left\{1 - \frac{1}{2}\left(\dfrac{\nu}{3n}\right) - \frac{1}{6}\left(\dfrac{\nu}{3n}\right)^2 - \frac{1}{12}\left(\dfrac{\nu}{3n}\right)^3 - \ldots\right\} < \nu.$

On the other hand,

$$Q < \log\sqrt{(2\pi)} + n\log n - n + \tfrac{1}{2}\log n + \tfrac{1}{12}$$
$$+ \log\sqrt{(2\pi)} + 2n\log 2n - 2n + \tfrac{1}{2}\log 2n + \tfrac{1}{12}$$
$$< \{2\log\sqrt{(2\pi)} + \tfrac{1}{2}\log 2 + \tfrac{1}{6}\} + 3(\log n)\,n + (2\log 2 - 3)\,n + \log n.$$

Hence

$$P - Q > (3\log 3 - 2\log 2)\,n - (\log n)\,\nu - \tfrac{3}{2}\log n - (\log 3)\,\nu - \{\tfrac{1}{2}\log(12\pi) + \tfrac{1}{6}\}$$
$$> (3\log 3 - 2\log 2)\,n - \log n\,(\nu - \tfrac{1}{2})$$
$$- 2\log n - \log 3\,(\nu - \tfrac{1}{2}) - \{\tfrac{1}{2}\log(36\pi) + \tfrac{1}{6}\}$$

where $\nu - \frac{1}{2} < 1\cdot 606\,\dfrac{n}{\log n}$.

But $$3 \log 3 - 2 \log 2 = 1{\cdot}9095415 > 1{\cdot}909.$$

Hence*

$$P - Q > ({\cdot}303)\, n - (1{\cdot}606 \log 3) \frac{n}{\log n} - 2 \log n - \{\tfrac{1}{2} \log (36\pi) + \tfrac{1}{6}\},$$

say $P - Q > f(n) > 0$ when $n = 3000$.

Also the derivative with respect to n of $(\log n) f(n)$ being

$$({\cdot}303)(1 + \log n) - 1{\cdot}606 \log 3 - \frac{4 \log n}{n} - \frac{\tfrac{1}{2} \log (36\pi) + \tfrac{1}{6}}{n},$$

$P - Q$ will increase as n increases and will remain positive for all values of n superior to 3000.

Hence the theorem is true, whatever m may be, when $n =$ or > 3000, and since it has been proved previously for the case of $n < 3000$, it is true universally.

I subjoin the valuable table, kindly communicated to me by Dr Glaisher, referred to in the text above.

Table of Increasing Sequences of Composite Numbers interposed between Consecutive Primes included in the first nine million numbers.

Limits to sequence	Number of terms
7 to 11	3
23 „ 29	5
89 „ 97	7
113 „ 127	13
523 „ 541	17
887 „ 907	19
1129 „ 1151	21
1327 „ 1361	33
9551 „ 9587	35
15683 „ 15727	43
19609 „ 19661	51
31397 „ 31469	71
155921 „ 156007	85
373261 „ 373373	111
492113 „ 492227	113
1349533 „ 1349651	117
1357201 „ 1357333	131
2010733 „ 2010881	147
4652353 „ 4652507	153

* It will now be seen why I take separately the two cases of m greater and m less than $2n$. If we were to take *simpliciter* $m =$ or $> n$ and were to attempt to prove

$$\log \{1 \,.\, 2 \,.\, 3 \ldots (2n - \nu)\} > 2 \log (1 \,.\, 2 \,.\, 3 \ldots n)$$

the inferior limit to the difference between these two quantities would then have for its principal term, not $(3 \log 3 - 2 \log 2 - 1{\cdot}606)\, n$ but $(2 \log 2 - 1{\cdot}606)\, n$, which would be *negative*.

Of course there is no special reason except of convenience (in dealing with an integer instead of a fraction) for making $2n$ the dividing point between the two suppositions separately considered in the text; κn where κ as far as regards the second inequality does not fall short of some

The table is to be understood as follows. The lowest sequence of as many as 3 consecutive composite numbers is that included between 7 and 11: the lowest of as many as 5 is that included between 23 and 29, of as many as 7 that included between 89 and 97; between 13 and 17 there is a break—this indicates that the lowest sequence of as many as 15, or as many as 17 first occurs in the sequence of 17 interposed between 523, 541. Similarly the break between 21 and 33 indicates that the lowest sequence containing 23 or 25 or 27 or 29 or 31 or 33 terms first occurs in the sequence of 33 composite numbers interposed between the primes 1327, 1361.

It is also necessary to add that in the first nine million numbers there is no succession of *more* than 153 consecutive composite numbers.

§ 3. *Relating to irreducible arithmetical series in general**.

Let P be a principal term quâ q in any *irreducible* arithmetical series whose common difference is i, N any other term greater or less than P, and D their difference. If q is not prime to i, no term in the series will be divisible by q.

Just as in the case of a natural sequence when there is only one principal term in the series it may be shown that the index of D quâ q will be the same as that of N; when there is more than one principal term it appears by the same reasoning as before that the index of N cannot be greater than that of D: (it will not now necessarily be equal unless q is greater than the common difference i).

The index-sum quâ q is zero when q has a common measure with i, and we may therefore consider only the case where q is relatively prime to i:

certain limit, would have served as well: this inferior limit to κ would be some quantity a little greater (how much exactly would have to be found by trial) than the quantity θ which makes $\theta \log \theta - (\theta - 1) \log (\theta - 1)$ equal to the coefficient of $\frac{n}{\log n}$ in the superior limit to ν. As regards the first inequality κ would have to be a quantity somewhat less (how much less to be found by trial) than the quantity η which makes $\frac{\eta+1}{\eta} = \frac{6}{5}$, that is, $\eta = 5$. This is on the supposition made throughout of using Tchebycheff's own limits, but if we use the more general, but less compact, limits indicated in my paper in vol. IV. of the *American Journal of Mathematics*‡, any fraction not less than $\frac{6}{5}$ and not so great as $\frac{59595}{51072}$ would take the place of $\frac{6}{5}$, and the extreme value of η would be $\frac{51072}{8523}$, which is a trifle under 6. By a judicious choice of the value given to κ, a value of n could be found considerably less than 3000, which would satisfy both inequalities, and this in the absence of Dr Glaisher's table would have been a matter of some practical importance, but is of next to none when we have that table to draw upon. How low down in the scale of number, n may be taken, without interruption of the existence of the fundamental inequality for the minimum value of n in the case treated of in this section, it has not been necessary for the purpose in hand to ascertain. That it holds good for all values of n above a certain limit follows from the fact that $2 \log 2$ is greater than the coefficient of the leading term in the superior functional limit to the sum of the logarithms of the primes not greater than n.

* An irreducible arithmetical series is one whose terms are prime to their common difference.

[‡ Vol. III. of this Reprint, p. 530.]

on this supposition, by virtue of what has been stated above, the index-sum quâ q of the series whose first term is $m+i$, and number of terms n, will be equal to or less than

$$E\left(\frac{P-m-i}{iq}\right)+E\left(\frac{P-m-i}{iq^2}\right)+E\left(\frac{P-m-i}{iq^3}\right)+\dots$$
$$+E\left(\frac{m+ni-P}{iq}\right)+E\left(\frac{m+ni-P}{iq^2}\right)+E\left(\frac{m+ni-P}{iq^3}\right)+\dots;$$

and therefore *à fortiori*

$$< \text{or} = E\left(\frac{(n-1)\,i}{iq}\right)+E\left(\frac{(n-1)\,i}{iq^2}\right)+E\left(\frac{(n-1)\,i}{iq^3}\right)+\dots$$
$$< \text{or} = E\left(\frac{n}{q}\right)+E\left(\frac{n}{q^2}\right)+E\left(\frac{n}{q^3}\right)+\dots,$$

that is, not greater than the index-sum of 2, 3, ..., n quâ q.

Consequently, by the same reasoning as that employed in the last section, the theorem now to be proved, namely, that if m (prime to i) = or > n, then $(m+i)(m+2i)\dots(m+ni)$ must contain some one or more prime numbers greater than n, must be true whenever

$$(m+i)(m+2i)(m+3i)\dots\{m+(n-\nu_1)\,i\} > 1.2.3\dots n \qquad (\Theta)^*$$

where ν_1 is the number of prime numbers not exceeding n, and not contained in i, and *à fortiori* when for ν_1, we substitute, as for the present we shall do, ν the *entire* number of primes not greater than n. This I term the *fundamental inequality* for the general case now under consideration.

Suppose n = or > 3000. The logarithm of the first side of the fundamental inequality when we write ν for ν_1 is obviously greater than the ith part of the logarithm of

$$(m+1)(m+2)\dots(m+i)(m+i+1)\dots\{m+(n-\nu)\,i\};$$

and the inequality (subject to certain suppositions) to be established will be satisfied, if on the same suppositions,

$$\frac{1}{i}\log\,[1.2.3\dots\{m+(n-\nu)\,i\}] > \log\,(1.2.3\dots n)+\frac{1}{i}\log\,(1.2.3\dots m).$$

Suppose $m=n$, and make

$$\log\,[1.2.3\dots\{(i+1)\,n-i\nu\}]=T,$$
$$(i+1)\log\,(1.2.3\dots n)=U,$$
$$F(n,i)=T-U.$$

* If it had been necessary the condition in the text might have been stated in the more stringent form that *some aliquot part* of the factorial of n (namely, this factorial divested of all powers of prime numbers contained in i) would have to be greater than

$$(m+i)(m+2i)\dots\{m+(n-\nu_1)\,i\}$$

if the theorem were not true for any specified values of m, n, i.

It will be noticed that when i is relatively prime to n, ν_1 is less than ν so that $n-\nu_1>n-\nu$: some use will be made of the formula in the text when dealing with certain small values of n and $m-n$ towards the end of the section.

Then $T > \log(2\pi) + \{(i+1)n - i\nu\} \log\{(i+1)n - i\nu\}$

$$- \{(i+1)n - i\nu\} - \tfrac{1}{2}\log\{(i+1)n - i\nu\},$$

$$U < (i+1)\log\sqrt{(2\pi)} + (i+1)n\log n - (i+1)n + \tfrac{1}{2}(i+1)\log n + \tfrac{1}{12}(i+1).$$

Hence $F(n, i) > -i\log\sqrt{(2\pi)} + \{(i+1)n - i\nu\}\log\{(i+1)n\}$

$$+ \{(i+1)n - i\nu\}\log\left\{1 - \frac{i\nu}{(i+1)n}\right\}$$

$$+ i\nu - (i+1)n\log n - \tfrac{1}{2}\log\{(i+1)n - i\nu\} - \tfrac{1}{2}(i+1)\log n - \tfrac{1}{12}(i+1)$$

$$> \{(i+1)\log(i+1)\}n - i\log\{(i+1)n\}\nu - \tfrac{1}{2}\log\{(i+1)n\} - \tfrac{1}{2}(i+1)\log n$$

$$- \tfrac{1}{2}i\log(2\pi) - \tfrac{1}{12}(i+1),$$

that is $> \{(i+1)\log(i+1)\}n - i\log\{(i+1)n\}\nu - \tfrac{1}{2}(i+2)\log n - \tfrac{1}{2}\log(i+1)$

$$- \tfrac{1}{2}i\log(2\pi) - \tfrac{1}{12}(i+1) \qquad \text{(H)},$$

so that when $n > 3000$ and consequently $\nu < \tfrac{1}{2} + (1{\cdot}606)\dfrac{n}{\log n}$, the inequality to be established will be true *à fortiori* if

$$F(n, i) > \left\{(i+1)\log(i+1) - (1{\cdot}606)\,i\left[1 + \frac{\log(i+1)}{\log n}\right]\right\}n - (i+1)\log n$$

$$- [\tfrac{1}{2}(i+1)\log(i+1) + \tfrac{1}{2}\{i\log(2\pi)\} + \tfrac{1}{12}(i+1)].$$

When $i = 1$ or 2 or 3 the coefficient of n is negative; consequently the limit to ν before found is no longer applicable to bring out the desired result.

The case of $i = 1$ has been already disposed of; that of $i = 2$ may be disposed of, as I shall show, in a similar manner; when $i = 3$, I shall raise the limit n from 3000 to 8100 of which the logarithm is so near to 9 that it may, for the purpose of the proof in hand, be regarded as equal to 9 without introducing any error in the inequality to be established, as the error involved will only affect the result in a figure beyond the 4th or 5th place of decimals, whereas the inequality in question depends on figures in the first decimal place. When this is done the theorem will be in effect demonstrated for the case of $i = 3$ and $n > 8100$. For all values of n not greater than 8100 I shall be able to show that the fundamental inequality (Θ) is satisfied by employing the actual value of ν_1 or ν instead of a limiting value of the latter.

Thus the fundamental inequality will be shown to subsist for all values of n when $i = 3$ and $m = n$, and *à fortiori* therefore for all values of m and i *not less* than n and 3 respectively.

Case of $i = 2$.

Suppose $n =$ or > 3000, and take separately the cases $m <$ or $= 2n$, $m > 2n$.

(1) Let m be not greater than $2n$ so that $m + 2n$ is greater than $2m - 1$.

By hypothesis m must be odd, and by Bertrand's Postulate

$$m+2,\ m+3,\ m+4,\ \ldots,\ 2m,$$

and therefore $$m+2,\ m+4,\ m+6,\ \ldots,\ (2m-1)$$

(seeing that the interpolated terms are all even) must contain a prime, and thus the first case is disposed of.

(2) Since the fundamental inequality has been shown to be satisfied when $n > 3000$, $m > 2n$, $i = 1$, it will *à fortiori* be so when $n > 3000$, $m > 2n$, $i = 2$.

Hence the theorem is established for $i = 2$ when $n > 3000$. Finally as regards values of n inferior to 3000, the reasoning employed for the case of $i = 1$ applies *à fortiori* to the case of $i = 2$.

To see this let us recall the first step of the reasoning applicable to the supposition of $i = 1$.

Because in the first nine million numbers there is no sequence of 3000 composite numbers, from Dr Glaisher's Table of Sequences (taken in conjunction with the fact that when $m > n^2$, the theorem has been proved to be true whatever n may be), we were able to infer that it must be true when n does not exceed 153: in the present case, if the theorem were not true when $3000 > n > 153$, there would be a sequence of 153 composite odd numbers and therefore of over 305 composite consecutive numbers in the first 9000000 numbers, whereas there are not more than 153, and so we may proceed step by step till we arrive at the conclusion that the theorem must be true when $n > 13$; and when $n = 13, 11, 7, 5, 3, 2, 1$ a like method of disproof (but briefer) will apply as for the case of $i = 1$.

Case of $i =$ *or* > 3.

Let $n =$ or > 8100. Then we may without ultimate error write

$$\nu - \tfrac{1}{2} < \frac{1\cdot1056 + \dfrac{5}{4\log 6}\dfrac{81}{8100} + \tfrac{5}{2}\dfrac{9}{8100} + \dfrac{2}{8100}}{1 - \dfrac{\log 9}{9} - \dfrac{1-\log 2}{9}}\frac{n}{\log n} < 1\cdot546\frac{n}{\log n},$$

and accordingly

$$F(n, 3) > \left\{4\log 4 - (3 \times 1\cdot546)\left(1 + \frac{\log 4}{9}\right)\right\} n$$
$$- 4\log n - (2\log 4 + \tfrac{3}{2}\log 2\pi + \tfrac{1}{3})$$

and $$F(8100, 3) > (5\cdot545 - 5\cdot352)(8100) - 36 - 5\cdot863 > 0.$$

Hence the Fundamental Inequality is satisfied when $n =$ or > 8100.

To prove that it is satisfied for values inferior to 8100, observe that by virtue of the formula (H) it will be so, *ex abundantiâ*, for all values of n not less than $'n$ and not greater than n', provided that, calling n'_ν the number of primes not exceeding n',

$$(5{\cdot}545)\,'n - 3\log(4n')\,n'_\nu - \tfrac{5}{2}\log n' - C > 0,$$

where $$C = \tfrac{1}{3} + \log 2 + \tfrac{3}{2}\log(2\pi) = 3{\cdot}783.$$

On trial it will be found that the above inequality is satisfied when we successively substitute for $'n$, n', and for n'_ν (found from any Table for the enumeration of primes) the values given in the annexed table:

n'	n'_ν	$'n$
8100	1018	5725
5724	753	4096
4095	564	2967
2966	427	2172
2171	326	1604
1603	252	1200
1199	196	903
902	154	687
686	124	535
534	99	415
414	80	325
324	66	260
259	55	210
209	46	171
170	39	141
140	34	111
110	29	99
98	25	84
83	23	76
75	21	68
67	19	62
61	18	57
56	16	50
49	15	46
45	14	42
41	13	39
38	12	36
35	11	32
31	11	31
30	10	30
29	10	29

The fundamental theorem is therefore established when $i > 2$ for all values of n down to 29 inclusive.

It remains to consider the case where n is any prime number less than 29.

Calling μ the difference between n and the number of primes (exclusive of 1) not greater than n, to

$$n = 2,\ 3,\ 11,\ 17,\ 23$$

will correspond

$$\mu = 1,\ 1,\ \ 6,\ 10,\ 14$$

and for each combination of these corresponding numbers it will be found that

$$1.2.3 \ldots n = \text{or} < (n+3)(n+6) \ldots (n+3\mu).$$

Hence the theorem is proved for these values of n, whatever n may be, when $i =$ or > 3. To

$$n = 13, \quad n = 19$$

corresponds

$$\mu = 7, \quad \mu = 11,$$

and for these combinations of n and μ it will be found that

$$1.2.3 \ldots n < (n+4)(n+7) \ldots (n+1+3\mu),$$

so that the theorem is true for

$$n = 13, \ 19,$$

except in the case where

$$m = 13, \ 19.$$

That it is true in these excepted cases follows from inspection of the series,

$$16, \ 19, \ 22, \ 25, \ \&c.,$$

$$22, \ 25, \ 28, \ 31, \ \&c.,$$

where $19 > 13$, $31 > 19$: or it might be proved, but more cumbrously, by the same method as that applied below to the only two values of n remaining to be considered, namely

$$n = 5, \quad n = 7,$$

for which we have respectively

$$\mu = 2, \quad \mu = 3.$$

If $n = 5$ and i has no common measure with $2.3.4.5$, i must be not less than 7, but $1.2.3.4.5 < 12.19$.

On the other hand, if i has a common measure with $2.3.4.5$, then what we have called ν_1, in formula (Θ), is less than ν, so that $n - \nu_1 > 2$, but

$$1.2.3.4.5 < 8.11.14.$$

These two inequalities combined serve to prove that, whatever i may be, the inequality (Θ) is satisfied, and the theorem is consequently proved for $n = 5$.

So again, when $n = 7$, if i has no common measure with $2.3.4.5.6.7$ it must be 11 at least. In that case the inequality $2.3.4.5.6.7 < 18.29.40$, and in the contrary case the inequality $2.3.4.5.6.7 < 10.13.16.19$ serves to prove the theorem.

When $n = 1$ the truth of the theorem is obvious: hence combining the results obtained in this and the preceding section, it will be seen we have proved that whatever n and whatever i may be, provided that m is relatively prime to i and not less than n, the product

$$(m+i)(m+2i) \ldots (m+ni)$$

must contain some prime number by which $2.3 \ldots n$ is not divisible, and the wearisome proof is thus brought to a close. It will not surprise the author of it, if his work should sooner or later be superseded by one of a less piece-meal character—but he has sought in vain for any more compendious proof. He has not thought it necessary to produce the figures or refer in detail to the calculations giving the numerical results inserted in various places in the text: had he done so the number of pages, already exceeding what he had any previous idea of, would probably have been more than doubled*.

PART II†.

Explicit Primes.

In this part I shall consider the asymptotic limits to the number of primes of certain *irreducible* linear forms $mz + r$ comprised between a number x and a given fractional multiple thereof kx, the method of investigation being such that the asymptotic limits determined will be unaffected by the value of r, and will be the same for all values of m which

* The author was wandering in an endless maze in his attempts at a general proof of his theorem, until in an auspicious hour when taking a walk on the Banbury road (which leads out of Oxford) the Law of Ademption flashed upon his brain : meaning thereby the law (the nerve, so to say, of the preceding investigation) that *if all the terms of a natural arithmetical series be increased by the same quantity so as to form a second such series, no prime number can enter in a higher power as a factor of the product of the terms of this latter series, when a suitable term has been* taken away *from it, than the highest power in which it enters as a factor into the product of the terms of the original series.*

In Part II. I shall be able to apply the same method to demonstrate a theorem showing that it is always possible to split up an infinite arithmetical series, if not in the general case, at least for certain values of the common difference, into an infinite number of successive finite and determinable segments such that one or more primes shall be found in each such segment : a theorem which is, so to say, Dirichlet's theorem on arithmetical progressions cut up into slices.

The whole matter is thus made to rest on an elementary fundamental *equality* (Tchebycheff's) which, with the aid of an application of Stirling's theorem, leads (as the former has so admirably shown) *inter alia* to a superior limit to the sum of the logarithms of the primes not exceeding a given number, from which as has been seen in § 2, a superior limit may be deduced to the number of such primes. With the aid of this last limit together with an elementary fundamental *inequality* and a *renewed* application of Stirling's theorem, all my results are made to flow. Thus a theorem of pure form is brought to depend on considerations of greater and less, or as we may express it, Quality is made to stoop its neck to the levelling yoke of Quantity.

Long and vain were my previous efforts to make the desired results hinge upon the properties of transposed Eratosthenes' scales: now we may hope to reverse the process and compel these scales to reveal the secret of their laws under the new light shed upon them by the successful application of the Quantitative method.

† I ought to have stated that the theorem contained in section 2 of Part I. originally appeared in the form of a question (No. 10951) in the *Educational Times* for April of this year.

have the same totient. The simplest case, and the foundation of all that follows, is that in which $k=0$ and $m=2$: this will form the subject of the ensuing chapter which may be regarded as a supplement to Tchebycheff's celebrated memoir of 1850*, and as superseding my article thereon in vol. IV. of the *Amer. Math. Journ.* [Vol. III. of this Reprint, p. 530].

CHAPTER I.

ON THE ASYMPTOTIC LIMITS TO THE NUMBER OF PRIMES INFERIOR TO A GIVEN NUMBER.

§ 1. *Crude determination of the asymptotic limits.*

Call the sum of the logarithms of primes not exceeding x (any real positive quantity) the prime-number-logarithmic sum, or more briefly the prime-log-sum to x, and the sum of such sums to x and all its positive integer roots the prime-log-sum-sum, which in Serret is called $\psi(x)$.

Then it follows from elementary arithmetical principles that the sum of this sum-sum to x and all its aliquot parts, that is

$$\psi(x)+\psi\left(\frac{x}{2}\right)+\psi\left(\frac{x}{3}\right)+\ldots,$$

which we may call the natural series of sum-sums and denote by $T(x)$, is identical with the logarithm of the factorial of the highest integer not exceeding x, and accordingly from Stirling's theorem may be shown to have for its asymptotic limit $x\log x - x$, the superior and inferior limits being this quantity with a residue which, as well for the one as for the other, is a known linear function of $\log x$. Serret, vol. II. p. 226.

If now we take two sets of positive integers,

$$p, p', p'', \ldots;\ q, q', q'', \ldots,$$

together forming what may be termed a *harmonic scheme,* meaning thereby that the sum of the reciprocals of the numbers in the two sets is the same, and extend the T series over x divided by the respective numbers in each set and take the difference between the two sums thus obtained, there will result a new series of the form

$$\sum_{n=1}^{n=\infty} f(n)\,\psi\left(\frac{x}{n}\right),$$

of which the asymptotic limit will be x multiplied by

$$\Sigma\frac{\log p}{p}-\Sigma\frac{\log q}{q},$$

and the value of $f(n)$ will be

$$\Sigma\frac{n}{p:}-\Sigma\frac{n}{q:},$$

* Published in the *St Petersburg Transactions* for 1854.

where, in general, $\frac{n}{t:}$ means 1 or 0 according as n does or does not contain t, or in other words the "denumerant" of the equation $ty = n$.

I shall call the p's and q's the *stigmata* of the scheme:

$$\Sigma \frac{\log p}{p} - \Sigma \frac{\log q}{q}$$

the stigmatic multiplier, and the new series in $\psi(x)$ a stigmatic series of sum-sums (obtained, it will be noticed, by a four-fold process of summation—namely, two infinite and two finite summations).

It is possible, in general (as will hereafter appear), to deduce from the asymptotic value of a stigmatic series of sum-sums, superior and inferior asymptotic limits to the sum-sum itself. The *asymptotic* limits to the simple sum will then be the same as those last named (Serret, vol. II. p. 236, formulae (8) and (9)*) and will be multiples of x: dividing these respectively by $\log x$, we obtain superior and inferior asymptotic limits to the number of primes not exceeding x (*Messenger*, May 1891, p. 9, footnote [above, p. 694]).

It is obviously simplest always to take unity as one of the stigmata; those employed by Tchebycheff are 1, 30; 2, 3, 5; this *scheme* as I term it leads to the relation

$$\begin{aligned}
&\psi\left(\frac{x}{1}\right) - \psi\left(\frac{x}{6}\right) + \psi\left(\frac{x}{7}\right) - \psi\left(\frac{x}{10}\right) + \psi\left(\frac{x}{11}\right) - \psi\left(\frac{x}{12}\right) \\
&+ \psi\left(\frac{x}{13}\right) - \psi\left(\frac{x}{15}\right) + \psi\left(\frac{x}{17}\right) - \psi\left(\frac{x}{18}\right) + \psi\left(\frac{x}{19}\right) - \psi\left(\frac{x}{20}\right) \\
&+ \psi\left(\frac{x}{23}\right) - \psi\left(\frac{x}{24}\right) + \psi\left(\frac{x}{29}\right) - \psi\left(\frac{x}{30}\right) \\
&+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&= \left(\tfrac{1}{2}\log 2 + \tfrac{1}{3}\log 3 + \tfrac{1}{5}\log 5 - \tfrac{1}{30}\log 30\right)x + \dagger,
\end{aligned}$$

the series extending to infinity but consisting of repetitions (with a difference) of the above period, obtained by adding for the second period 30, for the third period 60, for the fourth period 90, and so on, to each denominator in the period set out. We may call this a period of 30 terms in which the coefficients are $+1$, 0, or -1. So, in general, whatever the stigmata may be, the stigmatic series will consist of periods of terms in each of which the total number of terms will be the least common multiple of the stigmata.

* The fourth edition, 1879, of Serret's *Cours d'Algèbre Supérieure* is referred to here and throughout the paper.

† The + is used to denote that a quantity is omitted of inferior *order* of magnitude to x. The strict interpretation of the "relation" is that the sum of the stigmatic series less the stigmatic multiplier into x is intermediate to two known linear functions of $\log x$.

Thus, for example, the schemes 1; 2, 2 and 1, 6; 2, 3, 3 would give rise to the relations

$$\psi(x) - \psi\left(\frac{x}{2}\right) + \psi\left(\frac{x}{3}\right) - \psi\left(\frac{x}{4}\right) + \psi\left(\frac{x}{5}\right) - \psi\left(\frac{x}{6}\right) + \dots$$
$$= (\tfrac{1}{2}\log 2 + \tfrac{1}{2}\log 2)\,x + = (\log 2)\,x + \dots,$$

$$\psi(x) - \psi\left(\frac{x}{3}\right) + \psi\left(\frac{x}{5}\right) - \psi\left(\frac{x}{6}\right) + \psi\left(\frac{x}{7}\right) - \psi\left(\frac{x}{9}\right) + \psi\left(\frac{x}{11}\right) - \psi\left(\frac{x}{12}\right) + \dots$$
$$= (\tfrac{1}{2}\log 2 + \tfrac{2}{3}\log 3 - \tfrac{1}{6}\log 6)\,x + = (\tfrac{1}{3}\log 2 + \tfrac{1}{2}\log 3)\,x + \dots$$

of which the periods are 2 and 6 respectively.

The three schemes above given, whose keys, so to say, are 2, 3, 5 respectively (these being the highest prime numbers contained in the stigmata), possess the property that their effective coefficients are alternately plus and minus 1, and, in consequence thereof, we may *immediately* deduce from them asymptotic limits superior and inferior to the logarithmic sum-sum $\psi(x)$.

Thus, calling the stigmatic multipliers in the three cases

$$St_2,\ St_3,\ St_5,$$

we obtain as limits to the coefficient of x in $\psi(x)$,

	St_2	and	$2St_2$	from	the	first,
	St_3	,,	$\frac{3}{2}St_3$	,,	,,	second,
and	St_5	,,	$\frac{6}{5}St_5$	,,	,,	third scheme.

(Compare Serret, pp. 233, 234, where the A is the present St_5.)

The three pairs of limits will thus be

·6931472 : 1·3862944,
·7803552 : 1·1705328,
·9212920 : 1·1055504,

which are in regular order of closer and closer propinquity to unity on each side of it*.

The question then arises can no further schemes be discovered which will enable us to bring the asymptotic coefficients still nearer to this empirical limit†?

* Mr Hammond has noticed that the harmonic scheme 1, 12; 2, 3, 4 will also give rise to a stigmatic series in which the effective terms are alternately positive and negative units, namely,

$$\psi(x) - \psi\left(\frac{x}{4}\right) + \psi\left(\frac{x}{5}\right) - \psi\left(\frac{x}{6}\right) + \psi\left(\frac{x}{7}\right) - \psi\left(\frac{x}{8}\right) + \psi\left(\frac{x}{11}\right) - \psi\left(\frac{x}{12}\right) + \psi\left(\frac{x}{13}\right) - \psi\left(\frac{x}{16}\right) + \dots,$$

the stigmatic multiplier corresponding to which, say St_{12}, is ·8522758 ..., and therefore will furnish the asymptotic coefficients St_{12} and $\frac{4}{3}St_{12}$, that is, ·8522758 ... and 1·1363687 ....

† The true asymptotic limit to the number of primes below x being according to Legendre's empirical rule $\frac{x}{\log x}$, the asymptotic value of $\psi(x)$ should presumably be x.

It would, I believe, be perfectly futile to seek for stigmatic schemes, involving higher prime numbers than 5, that should give rise to stigmatic series of sum-sums in which the successive coefficients should be alternately positive and negative unity, as in the above instances, but this although a sufficient is not a necessary condition in order that limits to a sum-sum may be capable of being extracted from the known limits to the sum of a series of such sum-sums.

This will be most easily explained by actually exhibiting a new scheme which is effective to the end in view, and showing why it is so.

Such a scheme is 1, 6, 70; 2, 3, 5, 7, 210, which, it will be observed, satisfies the necessary *harmonic* condition: for we have

$$1 + \tfrac{1}{6} + \tfrac{1}{70} = \frac{210 + 35 + 3}{210} = \tfrac{248}{210},$$

$$\tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{5} + \tfrac{1}{7} + \tfrac{1}{210} = \frac{105 + 70 + 42 + 30 + 1}{210} = \tfrac{248}{210}.$$

The *stigmatic multiplier* is here

$$\tfrac{1}{2}\log 2 + \tfrac{1}{3}\log 3 + \tfrac{1}{5}\log 5 + \tfrac{1}{7}\log 7 + \tfrac{1}{210}\log 210 - \tfrac{1}{6}\log 6 - \tfrac{1}{70}\log 70 = \cdot 9787955,$$

which I shall call D.

The stigmatic series arranged in sets in two different ways then becomes as a first arrangement

$$\psi(x) - \psi\left(\frac{x}{10}\right):$$

$$+\psi\left(\frac{x}{11}\right) + \psi\left(\frac{x}{13}\right) - \psi\left(\frac{x}{14}\right) - \psi\left(\frac{x}{15}\right); \ +\psi\left(\frac{x}{17}\right) + \psi\left(\frac{x}{19}\right)$$

$$-\psi\left(\frac{x}{20}\right) - \psi\left(\frac{x}{21}\right); \ +\psi\left(\frac{x}{23}\right) - \psi\left(\frac{x}{28}\right); \ +\psi\left(\frac{x}{29}\right) - \psi\left(\frac{x}{30}\right);$$

$$+\psi\left(\frac{x}{31}\right) - \psi\left(\frac{x}{35}\right); \ +\psi\left(\frac{x}{37}\right) - \psi\left(\frac{x}{40}\right); \ +\psi\left(\frac{x}{41}\right) - \psi\left(\frac{x}{42}\right);$$

$$+\psi\left(\frac{x}{43}\right) - \psi\left(\frac{x}{45}\right); \ +\psi\left(\frac{x}{47}\right) - \psi\left(\frac{x}{50}\right); \ +\psi\left(\frac{x}{53}\right) - \psi\left(\frac{x}{56}\right);$$

$$+\psi\left(\frac{x}{59}\right) - \psi\left(\frac{x}{60}\right); \ +\psi\left(\frac{x}{61}\right) - \psi\left(\frac{x}{63}\right); \ +\psi\left(\frac{x}{67}\right) - \psi\left(\frac{x}{70}\right);$$

$$+\psi\left(\frac{x}{71}\right) + \psi\left(\frac{x}{73}\right) - \psi\left(\frac{x}{75}\right) + \psi\left(\frac{x}{79}\right) - \psi\left(\frac{x}{80}\right) + \psi\left(\frac{x}{83}\right)$$

$$-\psi\left(\frac{x}{84}\right) + \psi\left(\frac{x}{89}\right) - \psi\left(\frac{x}{90}\right) + \psi\left(\frac{x}{97}\right) - \psi\left(\frac{x}{98}\right) - \psi\left(\frac{x}{100}\right);$$

$$+\psi\left(\frac{x}{101}\right) + \psi\left(\frac{x}{103}\right) - \psi\left(\frac{x}{105}\right) - \psi\left(\frac{x}{105}\right); \ +\psi\left(\frac{x}{107}\right)$$

$$+\psi\left(\frac{x}{109}\right)-\psi\left(\frac{x}{110}\right)-\psi\left(\frac{x}{112}\right);\ +\psi\left(\frac{x}{113}\right)-\psi\left(\frac{x}{120}\right);$$

$$+\psi\left(\frac{x}{121}\right)-\psi\left(\frac{x}{126}\right);\ +\psi\left(\frac{x}{127}\right)-\psi\left(\frac{x}{130}\right);\ +\psi\left(\frac{x}{131}\right)$$

$$-\psi\left(\frac{x}{135}\right);\ +\psi\left(\frac{x}{137}\right)+\psi\left(\frac{x}{139}\right)-\psi\left(\frac{x}{140}\right)+\psi\left(\frac{x}{143}\right)$$

$$-\psi\left(\frac{x}{147}\right)+\psi\left(\frac{x}{149}\right)-\psi\left(\frac{x}{150}\right)+\psi\left(\frac{x}{151}\right)-\psi\left(\frac{x}{154}\right)$$

$$+\psi\left(\frac{x}{157}\right)-\psi\left(\frac{x}{160}\right)+\psi\left(\frac{x}{163}\right)-\psi\left(\frac{x}{165}\right)+\psi\left(\frac{x}{167}\right)$$

$$-\psi\left(\frac{x}{168}\right)+\psi\left(\frac{x}{169}\right)-\psi\left(\frac{x}{170}\right)+\psi\left(\frac{x}{173}\right)-\psi\left(\frac{x}{175}\right)$$

$$+\psi\left(\frac{x}{179}\right)-\psi\left(\frac{x}{180}\right)+\psi\left(\frac{x}{181}\right)-\psi\left(\frac{x}{182}\right)+\psi\left(\frac{x}{187}\right)$$

$$-\psi\left(\frac{x}{189}\right)-\psi\left(\frac{x}{190}\right);\ +\psi\left(\frac{x}{191}\right)+\psi\left(\frac{x}{193}\right)-\psi\left(\frac{x}{195}\right)$$

$$-\psi\left(\frac{x}{196}\right);\ +\psi\left(\frac{x}{197}\right)+\psi\left(\frac{x}{199}\right)-\psi\left(\frac{x}{200}\right)+\psi\left(\frac{x}{209}\right)$$

$$-\psi\left(\frac{x}{210}\right)-\psi\left(\frac{x}{210}\right);\ +\psi\left(\frac{x}{211}\right)-\psi\left(\frac{x}{220}\right);$$

....

....

the correlative arrangement being

$$\psi(x)-\psi\left(\frac{x}{10}\right)+\psi\left(\frac{x}{11}\right)+\psi\left(\frac{x}{13}\right):$$

$$-\psi\left(\frac{x}{14}\right)-\psi\left(\frac{x}{15}\right)+\psi\left(\frac{x}{17}\right)+\psi\left(\frac{x}{19}\right);\ -\psi\left(\frac{x}{20}\right)-\psi\left(\frac{x}{21}\right)$$

$$+\psi\left(\frac{x}{23}\right)-\psi\left(\frac{x}{28}\right)+\psi\left(\frac{x}{29}\right)-\psi\left(\frac{x}{30}\right)+\psi\left(\frac{x}{31}\right)-\psi\left(\frac{x}{35}\right)$$

$$+\psi\left(\frac{x}{37}\right)-\psi\left(\frac{x}{40}\right)+\psi\left(\frac{x}{41}\right)-\psi\left(\frac{x}{42}\right)+\psi\left(\frac{x}{43}\right)-\psi\left(\frac{x}{45}\right)$$

$$+\psi\left(\frac{x}{47}\right)-\psi\left(\frac{x}{50}\right)+\psi\left(\frac{x}{53}\right)-\psi\left(\frac{x}{56}\right)+\psi\left(\frac{x}{59}\right)-\psi\left(\frac{x}{60}\right)$$

$$+\psi\left(\frac{x}{61}\right)-\psi\left(\frac{x}{63}\right)+\psi\left(\frac{x}{67}\right)-\psi\left(\frac{x}{70}\right)+\psi\left(\frac{x}{71}\right)+\psi\left(\frac{x}{73}\right);$$

$$-\psi\left(\frac{x}{75}\right)+\psi\left(\frac{x}{79}\right);\ -\psi\left(\frac{x}{80}\right)+\psi\left(\frac{x}{83}\right);\ -\psi\left(\frac{x}{84}\right)+\psi\left(\frac{x}{89}\right);$$

$$-\psi\left(\frac{x}{90}\right)+\psi\left(\frac{x}{97}\right);\ -\psi\left(\frac{x}{98}\right)-\psi\left(\frac{x}{100}\right)+\psi\left(\frac{x}{101}\right)$$

$$+\psi\left(\frac{x}{103}\right);\ -\psi\left(\frac{x}{105}\right)-\psi\left(\frac{x}{105}\right)+\psi\left(\frac{x}{107}\right)+\psi\left(\frac{x}{109}\right);$$

$$-\psi\left(\frac{x}{110}\right)-\psi\left(\frac{x}{112}\right)+\psi\left(\frac{x}{113}\right)-\psi\left(\frac{x}{120}\right)+\psi\left(\frac{x}{121}\right)$$

$$-\psi\left(\frac{x}{126}\right)+\psi\left(\frac{x}{127}\right)-\psi\left(\frac{x}{130}\right)+\psi\left(\frac{x}{131}\right)-\psi\left(\frac{x}{135}\right)$$

$$+\psi\left(\frac{x}{137}\right)+\psi\left(\frac{x}{139}\right);\ -\psi\left(\frac{x}{140}\right)+\psi\left(\frac{x}{143}\right);\ -\psi\left(\frac{x}{147}\right)$$

$$+\psi\left(\frac{x}{149}\right);\ -\psi\left(\frac{x}{150}\right)+\psi\left(\frac{x}{151}\right);\ -\psi\left(\frac{x}{154}\right)+\psi\left(\frac{x}{157}\right);$$

$$-\psi\left(\frac{x}{160}\right)+\psi\left(\frac{x}{163}\right);\ -\psi\left(\frac{x}{165}\right)+\psi\left(\frac{x}{167}\right);\ -\psi\left(\frac{x}{168}\right)$$

$$+\psi\left(\frac{x}{169}\right);\ -\psi\left(\frac{x}{170}\right)+\psi\left(\frac{x}{173}\right);\ -\psi\left(\frac{x}{175}\right)+\psi\left(\frac{x}{179}\right);$$

$$-\psi\left(\frac{x}{180}\right)+\psi\left(\frac{x}{181}\right);\ -\psi\left(\frac{x}{182}\right)+\psi\left(\frac{x}{187}\right);\ -\psi\left(\frac{x}{189}\right)$$

$$-\psi\left(\frac{x}{190}\right)+\psi\left(\frac{x}{191}\right)+\psi\left(\frac{x}{193}\right);\ -\psi\left(\frac{x}{195}\right)-\psi\left(\frac{x}{196}\right)$$

$$+\psi\left(\frac{x}{197}\right)+\psi\left(\frac{x}{199}\right);\ -\psi\left(\frac{x}{200}\right)+\psi\left(\frac{x}{209}\right);\ -\psi\left(\frac{x}{210}\right)$$

$$-\psi\left(\frac{x}{210}\right)+\psi\left(\frac{x}{211}\right)-\psi\left(\frac{x}{220}\right)+\psi\left(\frac{x}{221}\right)+\psi\left(\frac{x}{223}\right):$$

....

....*.

The terms in each arrangement, it will be seen, are separated by marks of punctuation into groups: omitting the first group in either of them, which may be called the outstanding group, in each of the others the sum of the coefficients is zero.

Moreover, the sum of the coefficients from the beginning of each group is always homonymous in sign, that is, will be non-negative in the first and non-positive in the second arrangement: the consequence of this is that all the terms of such groups may be resolved into pairs, whose sum will be necessarily positive in the one and negative in the other.

Thus, for example, in the first arrangement the last but one of the groups may be resolved into the pairs

$$\psi\left(\frac{x}{197}\right)-\psi\left(\frac{x}{200}\right);\ \psi\left(\frac{x}{199}\right)-\psi\left(\frac{x}{210}\right);\ \psi\left(\frac{x}{209}\right)-\psi\left(\frac{x}{210}\right).$$

* Each of these arrangements is to be regarded as made up of the outstanding group and an infinite succession of periodic groups. In the text we have set out the outstanding group and the first period, the other periods will be formed from this one by adding to each denominator in it successive multiples of 210.

each of which is equal to zero or a positive quantity. So the eighth group of the second arrangement is resoluble into the pairs

$$-\psi\left(\frac{x}{98}\right)+\psi\left(\frac{x}{101}\right);\ -\psi\left(\frac{x}{100}\right)+\psi\left(\frac{x}{103}\right),$$

each of which is zero or a negative quantity.

It may be as well to notice in this place that the sum of the coefficients, reckoning from the first term of the outstanding group to the term whose denominator is n, is

$$\sum_{t=0}^{t=n}\Sigma\left(\frac{t}{p:}-\frac{t}{q:}\right),$$

which by virtue of the obvious identity,

$$\sum_{t=0}^{t=n}\left(\frac{t}{i:}\right)=E\left(\frac{n}{i}\right),$$

is equal to $$\Sigma\left\{E\left(\frac{n}{p}\right)-E\left(\frac{n}{q}\right)\right\}.$$

This formula supplies an easy and valuable test for ascertaining the correctness of the determination of the coefficients up to any given term in the series.

These observations may be extended to any harmonic scheme whatever: for it will be observed that

$$\Sigma\left\{E\left(\frac{n}{p}\right)-E\left(\frac{n}{q}\right)\right\}$$

is a periodic quantity, and therefore possesses both a maximum and a minimum; whence it is easy to see that, by taking the outstanding group of terms sufficiently extensive, all the remaining terms in either kind of arrangement may be separated into groups similar to those above set out; namely, such that the *complete* sum of the coefficients in each group from its first to its end term is zero and up to any intermediate term is *homonymous,* that is, always positive in one and always negative in the other arrangement*.

* For example, from the harmonic scheme 1, 15; 2, 3, 5, 30, we may derive a stigmatic series under the two forms of arrangement

$$\psi(x)-\psi\left(\frac{x}{6}\right):+\psi\left(\frac{x}{7}\right)-\psi\left(\frac{x}{10}\right);\ +\psi\left(\frac{x}{11}\right)-\psi\left(\frac{x}{12}\right);\ +\psi\left(\frac{x}{13}\right)+\psi\left(\frac{x}{17}\right)-\psi\left(\frac{x}{18}\right)$$
$$+\psi\left(\frac{x}{19}\right)-\psi\left(\frac{x}{20}\right)+\psi\left(\frac{x}{23}\right)-\psi\left(\frac{x}{24}\right)+\psi\left(\frac{x}{29}\right)-\psi\left(\frac{x}{30}\right)-\psi\left(\frac{x}{30}\right);\ +\psi\left(\frac{x}{31}\right)-\psi\left(\frac{x}{36}\right):\ \&c.,$$

$$\psi(x)-\psi\left(\frac{x}{6}\right)+\psi\left(\frac{x}{7}\right)-\psi\left(\frac{x}{10}\right)+\psi\left(\frac{x}{11}\right)-\psi\left(\frac{x}{12}\right)+\psi\left(\frac{x}{13}\right)+\psi\left(\frac{x}{17}\right):\ -\psi\left(\frac{x}{18}\right)$$
$$+\psi\left(\frac{x}{19}\right);\ -\psi\left(\frac{x}{20}\right)+\psi\left(\frac{x}{23}\right);\ -\psi\left(\frac{x}{24}\right)+\psi\left(\frac{x}{29}\right);\ -\psi\left(\frac{x}{30}\right)-\psi\left(\frac{x}{30}\right)+\psi\left(\frac{x}{31}\right)$$
$$-\psi\left(\frac{x}{36}\right)+\psi\left(\frac{x}{37}\right)-\psi\left(\frac{x}{40}\right)+\psi\left(\frac{x}{41}\right)-\psi\left(\frac{x}{42}\right)+\psi\left(\frac{x}{43}\right)+\psi\left(\frac{x}{47}\right):\ \&c.$$

In the above arrangements the groups are separated by semicolons and the period is marked out by the colons. In this instance it will be observed that minimum and maximum values of

The consequence of this is that the outstanding group in the first arrangement will always be less, and in the second arrangement always greater, than a function of which the principal, or, as we may call it, the asymptotic term, is the product of x by the stigmatic multiplier, say (St), the complete function being in each case of the form $(St)\,x$ associated with a known linear function of $\log x$. (Compare Serret, vol. II. p. 232.)

The importance of this observation will become apparent in a subsequent section.

In the case before us (that is, for the scheme in the key of 7) confining our attention to the principal term of either limit, the first arrangement leads immediately (Serret, p. 234) to the superior asymptotic limit $\frac{10}{9}Dx$.

As regards the inferior limit, we have

$$\psi(x) + \psi\left(\frac{x}{13}\right) > Dx,$$

$$\psi(x) > Dx - \tfrac{1}{13} \cdot \tfrac{10}{9} Dx > \tfrac{107}{117} Dx^*.$$

Substituting for D its value ·9787955, we obtain the asymptotic limits 1·0873505 and ·8951370.

The corresponding values got from the Tchebycheffian scheme (1, 30; 2, 3, 5) being 1·1055504 and ·9212920, which are the $\frac{6}{5}A$ and A of Serret.

We know *aliunde* that the true asymptotic values are each of them presumably unity. The superior value above obtained by the new scheme is thus seen to be better, and the inferior value worse than those given by Tchebycheff's scheme. But these values correspond to what may be termed the *crude* determination of the limits which the schemes are capable of affording. The contraction of these asymptotic limits by a method of continual successive approximation will form the subject of the following section†.

$E(n) + E\left(\frac{n}{15}\right) - E\left(\frac{n}{2}\right) - E\left(\frac{n}{3}\right) - E\left(\frac{n}{5}\right) - E\left(\frac{n}{30}\right)$ are 0 and 2, and accordingly in the first arrangement the outstanding group has to be continued until the sum of the coefficients of the terms which it contains is 0, and in the second until such sum is 2.

Writing $Q = \frac{1}{2}\log 2 + \frac{1}{3}\log 3 + \frac{1}{5}\log 5 + \frac{1}{30}\log 30 - \frac{1}{15}\log 15 = ·96750\ldots,$

we may deduce from the above, the asymptotic coefficients $\frac{6}{5}Q$ and $Q - \frac{1}{17}\cdot\frac{6}{5}Q$; that is, 1·1610 ... and ·8992....

* Compare the determination of the limits for the harmonic scheme 1; 2, 3, 6 (*American Journal of Mathematics*, vol. IV. pp. 243, 244 [Vol. III. of this Reprint, p. 542]).

† By the method about to be explained, it should be noticed, we may not merely improve upon the results obtained by the *crude* method from certain harmonic schemes (which form a very restricted class) but may also obtain limits to $\psi(x) \div x$ from harmonic schemes which without its aid would be absolutely sterile (see p. [715]).

§ 2. *On a method of obtaining continually contracting asymptotic limits to* $\dfrac{\psi(x)}{x}$.

To fix the ideas let us consider the scheme (1, 30; 2, 3, 5) which leads to the stigmatic series

$$(1)-(6)+(7)-(10)+(11)-(12)+(13)-(15)+(17)-(18)+(19)$$
$$-(20)+(23)-(24)+(29)-(30)+(31)\ldots,$$

in which for brevity (n) is used to denote $\psi\left(\dfrac{x}{n}\right)$.

The sum of this series is, we know, intermediate between

$$Dx+R(\log x) \quad \text{and} \quad D_1x+R_1(\log x),$$

where $\qquad D=\cdot 9212920\ldots, \quad D_1=1\cdot 1055504\ldots=\tfrac{6}{5}D,$

and R, R_1 signify two known quantities which for uniformity may both be regarded as quadratic functions of $\log x$ (in the first of which the coefficient of $(\log x)^2$ is zero). (Serret, pp. 233, 235.)

Omitting every pair of consecutive terms $-(m)+(\mu)$ in which $\dfrac{\mu}{m}<\tfrac{6}{5}$, and using $[\psi(x)]$ to signify the asymptotic value of $\psi(x)$, we find

$$[\psi(x)]>Dx+\left[\psi\left(\frac{x}{24}\right)\right]-\left[\psi\left(\frac{x}{29}\right)\right]>Dx+D\frac{x}{24}-D_1\frac{x}{29},$$

say $\qquad >D'x.$

Similarly, omitting every consecutive pair of terms $(m)-(\mu)$ in which $\dfrac{\mu}{m}<\tfrac{6}{5}$, we find

$$[\psi(x)]<Dx+D_1\frac{x}{6}-D\frac{x}{7}+D_1\frac{x}{10},$$

say $\qquad <D_1'x.$

If instead of $[\psi(x)]$ we had deduced limits to $\psi(x)$ in the manner indicated above, we should have found

$$\psi(x)>D'x+R'(\log x), \quad \psi(x)<D_1'x+R_1'(\log x);$$

the added terms being each of them quadratic functions of $\log x$.

Repeating this process we shall obtain

$$[\psi(x)]>D''x, \quad [\psi(x)]<D_1''x,$$

where $\qquad D''=D+\tfrac{1}{24}D'-\tfrac{1}{29}D_1', \quad D_1''=D+\tfrac{1}{6}D_1'-\tfrac{1}{7}D'+\tfrac{1}{10}D_1'.$

Similarly we may write

$$[\psi(x)]>D'''x, \quad [\psi(x)]<D_1'''x,$$

where $\qquad D'''=D+\tfrac{1}{24}D''-\tfrac{1}{29}D_1'', \quad D_1'''=D+\tfrac{1}{6}D_1''-\tfrac{1}{7}D''+\tfrac{1}{10}D_1'',$

and so on.

If then we write for $D,\ D',\ D'',\ \ldots,\ v_0,\ v_1,\ v_2,\ \ldots,$
and for $D_1,\ D_1',\ D_1'',\ \ldots,\ u_0,\ u_1,\ u_2,\ \ldots,$
we shall find in general

$$[\psi(x)] > v_i x, \quad [\psi(x)] < u_i x;$$

where

$$v_{i+1} = D + \frac{v_i}{24} - \frac{u_i}{29},$$

$$u_{i+1} = D + (\tfrac{1}{6} + \tfrac{1}{10})\, u_i - \tfrac{1}{7} v_i;$$

the complete statement of the inequalities being

$$\psi(x) > v_i x + R^{(i)}(\log x), \quad \psi(x) < u_i x + R_1^{(i)}(\log x),$$

where it is to be noticed that the supplemental terms always remain *quadratic* functions of $\log x$.

(The result thus obtained differs in this particular from that stated by me in the *Amer. Math. Jour.* (vol. IV. p. 241)*; the process therein employed giving as supplemental terms rational integral functions of continually rising degrees of $\log x$. I am indebted to Mr Hammond for drawing my attention to this simple but important circumstance which had strangely escaped my attention previously.) To integrate the equations in u, v we have only to write

$$v_i = V_i + F, \qquad u_i = U_i + E,$$

$$F(1 - \tfrac{1}{24}) + \tfrac{1}{29} E = D, \quad V_i = C_1 \rho_1^i + C_2 \rho_2^i,$$

$$\tfrac{1}{7} F + (1 - \tfrac{1}{6} - \tfrac{1}{10})\, E = D, \quad U_i = K_1 \rho_1^i + K_2 \rho_2^i;$$

and to take for $\rho_1,\ \rho_2$ the two roots of the equation

$$\begin{vmatrix} \rho - \tfrac{1}{24}, & \tfrac{1}{29} \\ \tfrac{1}{7}\ , & \rho - \tfrac{1}{6} - \tfrac{1}{10} \end{vmatrix} = \rho^2 - (\tfrac{1}{6} + \tfrac{1}{10} + \tfrac{1}{24})\,\rho + \tfrac{1}{24}(\tfrac{1}{6} + \tfrac{1}{10}) - \tfrac{1}{203} = 0,$$

that is

$$\rho^2 - \tfrac{37}{120}\rho + \tfrac{113}{18270} = 0.$$

The roots of this equation being each less than 1, on making $i = \infty$ we obtain $v_\infty = F,\ u_\infty = E$, where E, F are deduced from the two algebraic equations

$$\tfrac{23}{24} F + \tfrac{1}{29} E = D,$$

$$\tfrac{1}{7} F + \tfrac{11}{15} E = D.$$

This gives

$$\frac{E}{F} = (\tfrac{23}{24} - \tfrac{1}{7}) \div (\tfrac{11}{15} - \tfrac{1}{29}) = \frac{137 \times 145}{304 \times 56} = \frac{19865}{17024} = q$$

(compare *Amer. Math. Jour.*, vol. IV. p. 242),

$$E = \tfrac{59595}{50999} D = 1{\cdot}0765779 \ldots,$$

$$F = \tfrac{51072}{50999} D = {\cdot}9226107 \ldots;$$

whence we may infer that $\psi(x)$ may be made intermediate between two

[* See Vol. III. of this Reprint, p. 539.]

known functions $u_i x + r(\log x)$, $v_i x + s(\log x)$, where u_i, v_i may be brought *indefinitely near* to the numbers

$$1{\cdot}0765779\ldots,\quad {\cdot}9226107\ldots;$$

and the supplemental terms are quadratic functions of $\log x$ depending upon the value of i that may be employed. We may, therefore (subject to an obvious interpretation), treat E and F as asymptotic limits to $\frac{\psi(x)}{x}$.*

If we examine the ratio of the denominators m, μ of any pair of consecutive terms throughout the entire infinite series, whether of the form $(m)-(\mu)$ or $-(m)+(\mu)$, we shall find that $\frac{\mu}{m}$ is always less than q (namely $1{\cdot}16688\ldots$), except in the case of the pairs that have been retained in forming the equations between E and F, from which we may infer that if any of the discarded pairs had been retained we should have obtained values of E and F respectively greater and less than those above set forth.

If, on the other hand, q had turned out to be so much less than $\frac{6}{5}$ as to cause $\frac{\mu}{m}$ in any rejected pair to be greater than q, in such case in order to obtain a value of E the least, and of F the greatest, capable of being extracted from the given scheme, it would have been necessary to take account of every such pair and perform the calculations afresh, thereby obtaining a new value of q (say q') less than the former one; we should then have had to continue the process of examining the rejected pairs and reinstating those (if any) whose denominators furnished a ratio $\frac{\mu}{m}$ greater than q', thereby obtaining a still smaller value q''. Repeating these operations *toties quoties* we should at last arrive at a value of q superior to every ratio $\frac{\mu}{m}$ throughout the entire stigmatic series; the corresponding values of the asymptotic limits would then be the best capable of being deduced from the given scheme.

Per contra had we retained at the start any of the discarded pairs of terms, we should have found for q a value greater than the value of $\frac{\mu}{m}$ in some of the terms retained, which would be a sure indication that the retention of those terms had led to a greater value of q than was necessary; those pairs would then have to be omitted; the q calculated from the reformed equations would be diminished by so doing and the resulting values of E, F

* For the complete analytical determination of the limits to $\psi(x)$ see § 3 of this chapter.

By making i sufficiently great u_i, v_i may be brought indefinitely near to E, F: furthermore, when the superior and inferior limits of $\psi(x) \div x$ are expressed as functions of x and i of the form mentioned in the text, these limits may, by taking x sufficiently great, be brought indefinitely near to u_i, v_i, and therefore to E, F, which I therefore speak of throughout as asymptotic limits to $\psi(x) \div x$. But more strictly the optimistic limits actually arrived at are E' as little as we please greater than E, and F' as little as we please less than F.

would be the best attainable, provided that care was taken at the outset that no rejected pair gave a larger value to $\frac{\mu}{m}$ than any pair that had been retained.

In the case we have considered initial asymptotic limits (namely D and D_1) to $\frac{\psi(x)}{x}$ were obtained from the scheme itself, but it will not always be possible to do this when we are dealing with any harmonic scheme.

Thus, for example, from the fact that the minor arrangement of the stigmatic series corresponding to the scheme [1, 6, 10, 210, 231, 1155; 2, 3, 5, 7, 11, 105] has (1) + (13) for its outstanding group [see p. 718], we may deduce that $\psi(x) + \psi\left(\frac{x}{13}\right)$ has Nx for its inferior asymptotic limit, but are unable from this arrangement to obtain an initial inferior asymptotic limit to $\psi(x)$ itself, and still less shall we be able to obtain an initial superior asymptotic limit to $\psi(x)$ from the major arrangement of the same stigmatic series. It is therefore important to notice that the final asymptotic limits arrived at by the method explained in this section, depend only on the stigmatic multiplier and the coefficients of the stigmatic series, being quite independent of the *initial* values employed, so that in the general case we may start from any given asymptotic limits to $\frac{\psi(x)}{x}$, *however obtained*, without thereby producing any effect in the final result. The limits $u_0 = 2\log 2$ and $v_0 = \log 2$ obtained from the scheme [1; 2, 2] will do as well as any others for our initial asymptotic limits to $\frac{\psi(x)}{x}$, and we may, by substituting these limits in the retained portion of the stigmatic series, arrive at new limits u_1, v_1 which in their turn will give rise to fresh limits u_2, v_2 and so on. We shall in this way obtain a pair of difference equations (connecting u_{i+1}, v_{i+1} with u_i, v_i) which will be of the same form as those previously given [p. 713], and it is to be noticed that in the solutions of these equations, namely

$$u_i = C\rho^i + C_1\rho_1^i + E, \quad v_i = K\rho^i + K_1\rho_1^i + F,$$

only the values of C, C_1, K, K_1 will depend on the initial values of u, v; so that, provided the roots of the quadratic in ρ (which are always real) are each less than unity, we may, by taking i sufficiently great, make u_i and v_i approach as near as we please to E and F respectively; that is as near as we please to two quantities whose values depend solely on the stigmatic series employed.

The positive and negative divergences from unity of the E and F previously found are respectively

$$\cdot 0765779\ldots, \quad \cdot 0773893\ldots;$$

these divergences as found by Tchebycheff being

$$\cdot 1055504\ldots, \quad \cdot 0787080\ldots,$$

which is already an important gain; but by varying the scheme we shall obtain still better results.

Let us apply the method of indefinite successive approximation to the scheme in the key of 7 treated of in the preceding section, namely [1, 6, 70; 2, 3, 5, 7, 210], for which the stigmatic multiplier (the D of p. [707]), namely

$$\tfrac{1}{2}\log 2 + \tfrac{1}{3}\log 3 + \tfrac{1}{5}\log 5 + \tfrac{1}{7}\log 7 + \tfrac{1}{210}\log 210 - \tfrac{1}{6}\log 6 - \tfrac{1}{70}\log 70$$

is ·9787955....

Preliminary calculations having served to satisfy me that the asymptotic ratio $\frac{E}{F}$ (the q) for this system was not likely to differ much from 1·10, which may be called the *regulator*, I form the corresponding equations for E and F by retaining only those pairs $(m)-(\mu)$ in the stigmatic series for which $\frac{\mu}{m}$ is greater than 1·10.

As previously explained no *error* can result whatever regulator we employ; the worst that can happen will be that the result will not be the best attainable from the scheme, and such imperfection can be ascertained by means of the method previously explained; the result, if the best possible, will prove itself to be so, and, if not the best, will indicate whether the regulator (or criterion of retention) has been taken too small or too great.

Let us examine separately the two arrangements set out in the previous section, the first being employed to obtain by successive approximations the superior, and the second the inferior, limit.

Consider 1° the periodic part of the first arrangement: in the group $(11)+(13)-(14)-(15)$, the pair $(13)-(14)$ being rejected, $(11)-(15)$ remains. Similarly, in the following group $(19)-(20)$ being rejected, $(17)-(21)$ remains; in the third and fourth groups $(23)-(28)$ and $(31)-(35)$ are to be retained. In the following group, all the consecutive pairs from (73) to (98) both inclusive are to be rejected, leaving $(71)-(100)$ available. (The corresponding pair to this in the next period, namely $(281)-(310)$, gives $\frac{310}{281}$, which is less than the assumed regulator.) All the groups in the first period, following $-(100)$, will have to be rejected until we come to the group beginning with (137), which leads to the available pair $(137)-(190)$: in the next period all the ratios will be too small with the exception of $(347)-(400)$ which must be retained, but the term corresponding to this in the third period, namely $(557)-(610)$, will have to be neglected.

Hence, in approximating to the superior limit, we may write

$$u_{i+1} = M + \left(\tfrac{1}{10} + \tfrac{1}{15} + \tfrac{1}{21} + \tfrac{1}{28} + \tfrac{1}{35} + \tfrac{1}{100} + \tfrac{1}{190} + \tfrac{1}{400}\right) u_i$$
$$- \left(\tfrac{1}{11} + \tfrac{1}{17} + \tfrac{1}{23} + \tfrac{1}{31} + \tfrac{1}{71} + \tfrac{1}{137} + \tfrac{1}{347}\right) v_i.$$

2°. In the second arrangement, the first group in the periodic part being $-(14)-(15)+(17)+(19)$, and $\frac{17}{15}$ (and *à fortiori* $\frac{19}{14}$) exceeding the regulator, all these terms are to be preserved.

In addition to these, we shall find in the first period the available couples $-(20)+(73)$ and $-(110)+(139)$, and in the second period $-(230)+(283)$; no other couples will be available, and accordingly, we shall have

$$v_{i+1} = M + \left(\tfrac{1}{10} + \tfrac{1}{14} + \tfrac{1}{15} + \tfrac{1}{20} + \tfrac{1}{110} + \tfrac{1}{230}\right) v_i$$
$$- \left(\tfrac{1}{11} + \tfrac{1}{13} + \tfrac{1}{17} + \tfrac{1}{19} + \tfrac{1}{73} + \tfrac{1}{139} + \tfrac{1}{283}\right) u_i.$$

If then we write a, b for the coefficients of u_i, $-v_i$ in the first, and c, d for the coefficients of v_i, $-u_i$ in the second of the above equations, and make $u_i = U_i + E$, $v_i = V_i + F$, we shall obtain

$$u_i = C\rho^i + C_1\rho_1{}^i + E,$$
$$v_i = K\rho^i + K_1\rho_1{}^i + F,$$

where ρ, ρ_1 are the roots of the equation

$$\begin{vmatrix} \rho - a, & b \\ d \;\;, & \rho - c \end{vmatrix} = 0,$$

that is
$$\rho^2 - (a+c)\rho + (ac - bd) = 0,$$

and E, F are subject to the equations

$$(1-a)E + bF = M,$$
$$dE + (1-c)F = M,$$

which give

$$E = \frac{1-b-c}{(1-a)(1-c)-bd} M, \quad F = \frac{1-a-d}{(1-a)(1-c)-bd} M.$$

On performing the calculations, we shall find

$$a = {\cdot}29633\ldots, \quad b = {\cdot}24973\ldots,$$
$$c = {\cdot}30153\ldots, \quad d = {\cdot}30371\ldots,$$
$$1-b-c = {\cdot}44873\ldots, \quad 1-a-d = {\cdot}39995\ldots,$$
$$ac = {\cdot}08935\ldots, \quad bd = {\cdot}07584\ldots,$$
$$a+c = {\cdot}59786\ldots, \quad (1-a)(1-c)-bd = {\cdot}41563\ldots,$$

ρ, ρ_1 will therefore be the roots of

$$\rho^2 - {\cdot}59786\rho + {\cdot}01350 = 0,$$

which are each less than unity.

Also
$$E = 1{\cdot}0567265\ldots, \quad F = {\cdot}9418543\ldots,$$

$$q = \frac{1-b-c}{1-a-d} = 1{\cdot}12196\ldots.$$

This last number being *greater* than the assumed regulator 1·10, and *less* than any of the retained ratios "$\left[\frac{\mu}{m}\right]$," it follows that no better limits than E, F can be extracted from the scheme [1, 6, 70; 2, 3, 5, 7, 210]; or (as we may phrase it) E, F are the optimistic asymptotic limits to that scheme.

Obviously, there is no reason to suppose that these are the closest asymptotic limits that can be obtained from the infinite choice of schemes at our disposal: on the contrary, there is every reason to suppose that these limits may by schemes in higher and higher keys be brought to coincide as nearly as may be desired to each other and to unity.

We shall presently obtain by aid of a new scheme a better result than the E, F of the preceding investigation. But first it should be observed that instead of forming the difference equations in u, v from the two arrangements, say the major and minor, of one and the same stigmatic series (the former meaning the one used to find the superior and the latter the inferior asymptotic limit), we may take these two arrangements, if we please, from two distinct series corresponding to two different schemes.

I have had calculated, from beginning to end, the value of the coefficient of each term in the stigmatic series of sum-sums corresponding to the first natural period, containing 2310 terms of the scheme (1, 6, 10, 210, 231, 1155; 2, 3, 5, 7, 11, 105), the stigmatic multiplier to which, namely

$$\tfrac{1}{2}\log 2 + \tfrac{1}{3}\log 3 + \tfrac{1}{5}\log 5 + \tfrac{1}{7}\log 7 + \tfrac{1}{11}\log 11 + \tfrac{1}{105}\log 105$$
$$- \tfrac{1}{6}\log 6 - \tfrac{1}{10}\log 10 - \tfrac{1}{210}\log 210 - \tfrac{1}{231}\log 231 - \tfrac{1}{1155}\log 1155,$$

is ·9909532 ... (say N).

This stigmatic series, though too long for printing at full in the restricted space of this Journal, is given later on in a condensed tabular form (see Table A, p. 721). I will proceed to describe its essential features and the use made of it to bring the asymptotic limits closer together. The maximum and minimum sums of its coefficients are 2 and -2: the first terms being $(1)+(13)-(14)-(15)$, the maximum is first reached at the second term; so that the outstanding group in the minor arrangement will be $(1)+(13)$. But the minimum sum, -2, is not reached before the term whose argument is (616). The outstanding group in the major arrangement will therefore contain a very great number of terms, and there might be some trouble in handling the groups, so as to secure the greatest possible advantage. For this reason, I have thought it sufficient for the present to combine the major arrangement of the scheme [1, 6, 70; 2, 3, 5, 7, 210] with the minor one of the scheme [1, 6, 10, 210, 231, 1155; 2, 3, 5, 7, 11, 105].

Maintaining the regulator still at the same value as before, namely 1·10, the major arrangement will remain unaltered from what it was in the preceding case. In the minor arrangement there will be found to exist the following 17 available pairs, all of which, except the last, belong to the first period (the last one belonging to the second period), namely

(14) – (19), (15) – (17), (21) – (31), (33) – (41), (44) – (53), (63) – (73), (84) – (97), (105) – (241), (110) – (131), (195) – (223), (315) – (481), (525) – (703), (735) – (943), (945) – (1231), (1484) – (1693), (1694) – (2323), (4004) – (4633).

We may accordingly write

$$u_{i+1} = M + au_i - bv_i,$$
$$v_{i+1} = N + \gamma v_i - \delta u_i,$$

where

$$a = \frac{1}{10} + \frac{1}{15} + \frac{1}{21} + \frac{1}{28} + \frac{1}{35} + \frac{1}{100} + \frac{1}{190} + \frac{1}{400},$$

$$b = \frac{1}{11} + \frac{1}{17} + \frac{1}{23} + \frac{1}{31} + \frac{1}{71} + \frac{1}{137} + \frac{1}{347},$$

$$\gamma = \frac{1}{14} + \frac{1}{15} + \frac{1}{21} + \frac{1}{33} + \frac{1}{44} + \frac{1}{63} + \frac{1}{84} + \frac{1}{105} + \frac{1}{110} + \frac{1}{195} + \frac{1}{315} + \frac{1}{525} + \frac{1}{735} + \frac{1}{945} + \frac{1}{1484} + \frac{1}{1694} + \frac{1}{4004},$$

$$\delta = \frac{1}{13} + \frac{1}{17} + \frac{1}{19} + \frac{1}{31} + \frac{1}{41} + \frac{1}{53} + \frac{1}{73} + \frac{1}{97} + \frac{1}{131} + \frac{1}{223} + \frac{1}{241} + \frac{1}{481} + \frac{1}{703} + \frac{1}{943} + \frac{1}{1231} + \frac{1}{1693} + \frac{1}{2323} + \frac{1}{4633},$$

from which, writing

$$(1 - a)E + bF = M,$$
$$\delta E + (1 - \gamma)F = N,$$

we shall find

$$u_i = C\rho^i + C_1\rho_1{}^i + E,$$
$$v_i = K\rho^i + K_1\rho_1{}^i + F,$$

where ρ, ρ_1 are the roots of

$$\begin{vmatrix} \rho - a, & b \\ \delta\ , & \rho - \gamma \end{vmatrix} = 0,$$

that is

$$\rho^2 - (a + \gamma)\rho + a\gamma - b\delta = 0.$$

The values of a, b; γ, δ are respectively

·2963346 ..., ·2497346 ...; ·2992774 ..., ·3107808 ...,

from which we see that ρ, ρ_1 being each less than unity the values of u_∞, v_∞ will be E, F, where

$$E = \frac{(1 - \gamma)M - bN}{(1 - a)(1 - \gamma) - b\delta},$$

$$F = \frac{(1 - a)N - \delta M}{(1 - a)(1 - \gamma) - b\delta},$$

and on performing the calculation it will be found that

$$E = 1{\cdot}0551851\ldots,\quad F = {\cdot}9461974.$$

Also $$q = \frac{E}{F} = 1{\cdot}11518\ldots,$$

which being greater than the assumed regulator, but less than any of the retained ratios $\frac{\mu}{m}$, the results thus obtained are *optimistic*, that is no better can be found without having recourse to some other harmonic scheme.

The advance made upon the determination of the asymptotic limits beyond what was known previously is already remarkable. Tchebycheff's asymptotic numbers stood at

$$1{\cdot}1055504\ldots,$$

$${\cdot}9212920\ldots,$$

corresponding to a divergence from unity

·1055504 ... in excess,

and ·0787080 ... in defect;

by the combined effect of scheme variation and successive substitution we have succeeded in reducing these divergences to

·0551851 ... in excess,

and ·0538026 ... in defect;

in which it will be noticed that the divergence for the superior limit is only a little more than half the original one.

The mean of the two limits, it will be seen, is now less than

$$1{\cdot}0007.$$

The annexed table, in which for brevity $\bar{c}$ is written for $-c$, gives in a condensed form the stigmatic series to the scheme [1, 6, 10, 210, 231, 1155; 2, 3, 5, 7, 11, 105].

The coefficients, for all the terms $\psi\left(\frac{x}{m}\right)$ from $m=1$ to $m=1155$ (the half modulus), are written down in regular batches of 10. The coefficients for the ensuing terms up to 2309 can be got from these by the formula $c_{1155+t} = c_{1155-t}$, the term following will have the coefficient zero; the rest of the infinite series is then known from the formula $c_{t+2310i} = c_t$.

TABLE A.

The coefficients of the first 1155 *terms of the stigmatic series to* [1, 6, 10, 210, 231, 1155; 2, 3, 5, 7, 11, 105]*.

1000000000	$001\bar{1}\bar{1}01010$	$\bar{1}\bar{1}10000\bar{1}10$	$10\bar{1}0\bar{1}01000$
$1\bar{1}1\bar{1}\bar{1}01000$	$0010\bar{1}\bar{1}0010$	$10\bar{1}00\bar{1}100\bar{1}$	$1010\bar{1}0\bar{1}010$
$001\bar{1}000\bar{1}10$	$0000001\bar{1}\bar{1}0$	$1010\bar{3}01011\bar{1}$	$0\bar{1}10000000$
$00000\bar{1}1000$	$1\bar{1}00\bar{1}01011\bar{1}$	$000000\bar{1}010$	$100\bar{2}001000$
$0010\bar{2}01\bar{1}10$	$0010\bar{1}\bar{1}0010$	$1\bar{1}000000\bar{1}0$	$1010\bar{1}\bar{1}1\bar{1}10$
$000000000\bar{1}$	$100000000\bar{1}$	$101\bar{1}\bar{1}01010$	$\bar{1}010000\bar{1}10$
$1\bar{1}00\bar{1}01000$	$1\bar{1}00\bar{1}01000$	$001\bar{1}0\bar{1}0010$	$10\bar{1}0\bar{1}0100\bar{1}$
$1010\bar{1}\bar{1}0010$	$001\bar{1}00\bar{1}010$	$0000001\bar{2}00$	$1010\bar{3}01000$
$0\bar{1}1000000\bar{1}$	$10000\bar{1}1000$	$0000\bar{1}0101\bar{1}$	$0\bar{1}1000\bar{1}010$
$10\bar{1}\bar{1}001000$	$001\bar{1}\bar{1}01\bar{1}10$	$0010\bar{2}00010$	$1\bar{1}000\bar{1}10\bar{1}0$
$1010\bar{1}\bar{1}0010$	$0000000\bar{1}1\bar{1}$	$10000000\bar{1}0$	$101\bar{1}\bar{1}0101\bar{1}$
$\bar{1}010000\bar{1}10$	$0000\bar{1}01000$	$1\bar{1}10\bar{1}01000$	$00000\bar{1}0010$
$10\bar{1}\bar{1}00100\bar{1}$	$1010\bar{2}00010$	$001\bar{1}0\bar{1}0010$	$0000000\bar{1}00$
$1010\bar{3}01\bar{1}10$	$0\bar{1}100000\bar{1}0$	$10000\bar{1}100\bar{1}$	$1000\bar{1}0101\bar{1}$
$\bar{1}01000\bar{1}010$	$1\bar{1}0\bar{1}001000$	$0000\bar{1}01\bar{1}10$	$001\bar{1}\bar{1}00010$
$1\bar{1}00\bar{1}010\bar{1}0$	$1010\bar{1}\bar{2}1010$	$000000\bar{1}01\bar{1}$	$1000000\bar{1}00$
$101\bar{1}\bar{1}01000$	$\bar{1}010000\bar{1}1\bar{1}$	$1000\bar{1}01000$	$0\bar{1}10\bar{1}01000$
$0\bar{1}100\bar{1}0010$	$10\bar{1}000100\bar{1}$	$101\bar{1}\bar{1}00010$	$001\bar{1}\bar{1}00010$
$00000\bar{1}1\bar{1}00$	$1010\bar{3}00010$	$0\bar{1}10000\bar{1}00$	$10000\bar{1}10\bar{1}0$
$1000\bar{1}0101\bar{2}$	$001000\bar{1}010$	$000\bar{1}001000$	$0\bar{1}10\bar{1}01\bar{1}10$
$0000\bar{1}00010$	$1\bar{1}0\bar{1}0010\bar{1}0$	$1010\bar{2}\bar{1}1010$	$00000\bar{1}001\bar{1}$
$100000\bar{1}000$	$101\bar{1}\bar{1}01\bar{1}10$	$\bar{1}010000\bar{1}00$	$1000\bar{1}0100\bar{1}$
$1\bar{1}10\bar{1}01000$	$\bar{1}0100\bar{1}0010$	$1\bar{1}\bar{1}000100\bar{1}$	$1000\bar{1}00010$
$001\bar{1}000010$	$0000\bar{1}01\bar{1}00$	$1010\bar{3}\bar{1}1010$	$0\bar{1}1000\bar{1}000$
$10000\bar{1}1\bar{1}00$	$1000\bar{1}0100\bar{1}$	$001000\bar{1}01\bar{1}$	$100\bar{1}001000$
$\bar{1}010\bar{1}01\bar{1}10$	$0\bar{1}10\bar{1}00010$	$1\bar{1}\bar{1}00010\bar{1}0$	$101\bar{1}\bar{1}\bar{1}1010$
$0000\bar{1}0001\bar{1}$	$10000\bar{1}0000$	$101\bar{1}\bar{1}00010$	$\bar{1}010000\bar{2}10$
$1000\bar{1}010\bar{1}0$	$1\bar{1}10\bar{1}0100\bar{1}$	$00100\bar{1}0010$	$00\bar{1}000100\bar{1}$
$1\bar{1}10\bar{1}00010$	$000\bar{1}000010$	$000\bar{1}001\bar{1}00$	$1010\bar{2}$†.

* This table is to be read off in lines. The first three lines set out in full (omitting the *null* terms) will mean

$$\psi(x)+\psi\left(\frac{x}{13}\right)-\psi\left(\frac{x}{14}\right)-\psi\left(\frac{x}{15}\right)+\psi\left(\frac{x}{17}\right)+\psi\left(\frac{x}{19}\right)-\psi\left(\frac{x}{21}\right)-\psi\left(\frac{x}{22}\right)$$
$$+\psi\left(\frac{x}{23}\right)-\psi\left(\frac{x}{28}\right)+\psi\left(\frac{x}{29}\right)+\psi\left(\frac{x}{31}\right)-\psi\left(\frac{x}{33}\right)-\psi\left(\frac{x}{35}\right)+\psi\left(\frac{x}{37}\right)$$
$$+\psi\left(\frac{x}{41}\right)-\psi\left(\frac{x}{42}\right)+\psi\left(\frac{x}{43}\right)-\psi\left(\frac{x}{44}\right)-\psi\left(\frac{x}{45}\right)+\psi\left(\frac{x}{47}\right)+\psi\left(\frac{x}{53}\right)$$
$$-\psi\left(\frac{x}{55}\right)-\psi\left(\frac{x}{56}\right)+\psi\left(\frac{x}{59}\right)+\psi\left(\frac{x}{61}\right)-\psi\left(\frac{x}{63}\right)-\psi\left(\frac{x}{66}\right)+\psi\left(\frac{x}{67}\right)$$
$$-\psi\left(\frac{x}{70}\right)+\psi\left(\frac{x}{71}\right)+\psi\left(\frac{x}{73}\right)-\psi\left(\frac{x}{75}\right)-\psi\left(\frac{x}{77}\right)+\psi\left(\frac{x}{79}\right)$$
$$+\psi\left(\frac{x}{83}\right)-\psi\left(\frac{x}{85}\right)-\psi\left(\frac{x}{88}\right)+\psi\left(\frac{x}{89}\right)+\psi\left(\frac{x}{97}\right)-\psi\left(\frac{x}{98}\right)-\psi\left(\frac{x}{99}\right)+\psi\left(\frac{x}{101}\right)$$
$$+\psi\left(\frac{x}{103}\right)-3\psi\left(\frac{x}{105}\right)+\psi\left(\frac{x}{107}\right)+\psi\left(\frac{x}{109}\right)-\psi\left(\frac{x}{110}\right)-\psi\left(\frac{x}{112}\right)+\psi\left(\frac{x}{113}\right).$$

† By actual summation it will be found as stated above [p. 718] that the sum reckoned from the beginning of the positive and negative integers in this table always lies between 2 and −2 (both inclusive).

If we confine our attention exclusively to the outstanding group of the Major Arrangement, which extends to the 616th term inclusive, *without taking advantage* of any of the other groups, we shall find, on making $E = 1{\cdot}0551851$, $F = {\cdot}9461974$, and N (the stigmatic multiplier) $= {\cdot}9909532$,

$$\frac{[\psi(x)]}{x} < N + \left(\frac{1}{15} + \frac{1}{22} + \frac{1}{28} + \frac{1}{35} + \frac{1}{45} + \frac{1}{56} + \frac{1}{66} + \frac{1}{77} + \frac{1}{88} + \frac{1}{99} + \frac{1}{105} + \frac{1}{126} + \frac{1}{525} + \frac{1}{616}\right) E$$
$$- \left(\frac{1}{17} + \frac{1}{23} + \frac{1}{29} + \frac{1}{37} + \frac{1}{47} + \frac{1}{59} + \frac{1}{71} + \frac{1}{79} + \frac{1}{89} + \frac{1}{113} + \frac{1}{227}\right) F$$
$$< 1{\cdot}0542390 \ldots$$ which is inferior in value to E.

This is enough to assure us that a better result than the one last found would be obtained by using the above scheme to furnish the major as well as the minor arrangement, instead of combining it, as we have done, with the scheme [1, 6, 70; 2, 3, 5, 7, 210].

Mr Hammond has been good enough to work out for me in the annexed scholium the *complete* approximation to the limits to $\psi(x)$ given by the original scheme of Tchebycheff [1, 30; 2, 3, 5]: this approximation preserves precisely the same form as that obtained by the crude method, and, although it lies a little out of the track which I had marked out for myself in this paper, will, I think, besides being possibly valuable for future purposes in a more or less remote future, serve as an example to clear up any obscurity that may have pervaded the previous exposition of the purely asymptotic portion of these limits*.

§ 3. *Scholium. Containing an example of the complete ith approximation to the limits to the prime-log-sum-sum to x.*

Using S to denote the stigmatic series

$$\psi(x) - \psi\left(\frac{x}{6}\right) + \psi\left(\frac{x}{7}\right) - \psi\left(\frac{x}{10}\right) + \cdots,$$

we have the inequalities

$$\left.\begin{array}{l} S > Ax - \frac{5}{2}\log x - 1 \\ S < Ax + \frac{5}{2}\log x \end{array}\right\} \text{(Serret, p. 233),}$$

which, as explained in the preceding section, may be replaced by

$$\psi(x) > Ax - \tfrac{5}{2}\log x - 1 + \psi\left(\frac{x}{24}\right) - \psi\left(\frac{x}{29}\right) \quad (1),$$

$$\psi(x) < Ax + \tfrac{5}{2}\log x + \psi\left(\frac{x}{6}\right) - \psi\left(\frac{x}{7}\right) + \psi\left(\frac{x}{10}\right) \quad (2).$$

* In the paragraph [last but one] of p. [709] in the preceding number, a theorem (too simple to require a formal proof) is tacitly assumed which virtually amounts to saying:

If an equal number of black and white beads be strung upon a wire, in such a way that on telling them all, from left to right, more white than black ones are never told off, then the whole number of beads, as they stand, may be sorted into pairs, in each of which a black bead lies to the left of a white one.

If now we assume

$$\psi(x) > p_i Ax + q_i (\log x)^2 + r_i (\log x) + s_i \qquad (3),$$
$$\psi(x) < t_i Ax + u_i (\log x)^2 + v_i (\log x) + w_i \qquad (4),$$

we obtain, by combining these inequalities with (1),

$$\begin{aligned}\psi(x) > Ax &\qquad\qquad\qquad\qquad - \tfrac{5}{2}\log x \qquad\qquad - 1\\ &+ \tfrac{1}{24} p_i Ax + q_i (\log x - \log 24)^2 + r_i (\log x - \log 24) + s_i\\ &- \tfrac{1}{29} t_i Ax - u_i (\log x - \log 29)^2 - v_i (\log x - \log 29) - w_i.\end{aligned}$$

Say $\psi(x) > p_{i+1} Ax + q_{i+1} (\log x)^2 + r_{i+1} (\log x) + s_{i+1}$,

where

$$p_{i+1} = \tfrac{1}{24} p_i - \tfrac{1}{29} t_i + 1,$$
$$q_{i+1} = q_i - u_i,$$
$$r_{i+1} = r_i - v_i + 2u_i \log 29 - 2q_i \log 24 - \tfrac{5}{2},$$
$$s_{i+i} = s_i - w_i + q_i (\log 24)^2 - u_i (\log 29)^2 - r_i \log 24 + v_i \log 29 - 1.$$

Similarly, combining (3) and (4) with (2), we find

$$\begin{aligned}\psi(x) < Ax &\qquad\qquad\qquad\qquad + \tfrac{5}{2}\log x\\ &+ \tfrac{1}{6} t_i Ax + u_i (\log x - \log 6)^2 + v_i (\log x - \log 6) + w_i\\ &- \tfrac{1}{7} p_i Ax - q_i (\log x - \log 7)^2 - r_i (\log x - \log 7) - s_i\\ &+ \tfrac{1}{10} t_i Ax + u_i (\log x - \log 10)^2 + v_i (\log x - \log 10) + w_i.\end{aligned}$$

Say $\psi(x) < t_{i+1} Ax + u_{i+1} (\log x)^2 + v_{i+1} (\log x) + w_{i+1}$,

where

$$t_{i+1} = \tfrac{4}{15} t_i - \tfrac{1}{7} p_i + 1,$$
$$u_{i+1} = 2u_i - q_i,$$
$$v_{i+1} = 2v_i - r_i + 2q_i \log 7 - 2u_i \log 60 + \tfrac{5}{2},$$
$$w_{i+1} = 2w_i - s_i - q_i (\log 7)^2 + u_i \{(\log 6)^2 + (\log 10)^2\} + r_i \log 7 - v_i \log 60.$$

These, together with the four given above, constitute a set of eight difference equations for the determination of p_i, q_i, r_i, s_i, t_i, u_i, v_i, w_i. Their initial values are furnished by the inequalities

$$\left.\begin{aligned}&\psi(x) > Ax - \tfrac{5}{2}\log x - 1\\ &\psi(x) < \tfrac{6}{5} Ax + \frac{5}{4\log 6} (\log x)^2 + \tfrac{5}{4}\log x + 1\end{aligned}\right\} \text{(Serret, p. 236)},$$

which give

$$p_0 = 1,\ q_0 = 0,\ r_0 = -\tfrac{5}{2},\ s_0 = -1,$$
$$t_0 = \tfrac{6}{5},\ u_0 = \frac{5}{4\log 6},\ v_0 = \tfrac{5}{4},\ w_0 = 1.$$

The values of p_i, t_i will be found to be

$$p_i = \frac{1}{50999}\left\{51072 - 36\tfrac{1}{2}(\rho^i + \rho_1^i) - 47\tfrac{211}{2320}\left(\frac{\rho^i - \rho_1^i}{\rho - \rho_1}\right)\right\},$$
$$t_i = \frac{1}{50999}\left\{59595 + 801\tfrac{9}{10}(\rho^i + \rho_1^i) + 190\tfrac{2397}{2800}\left(\frac{\rho^i - \rho_1^i}{\rho - \rho_1}\right)\right\},$$

where ρ, ρ_1 are the roots of the equation

$$(\rho - \tfrac{4}{15})(\rho - \tfrac{1}{24}) = \frac{1}{203},$$

and it is easy to verify that these values (which agree with the general ones, involving arbitrary constants, obtained in the preceding section) satisfy the initial conditions

$$p_0 = 1,\ \ p_1 = \tfrac{1}{24}p_0 - \tfrac{1}{29}t_0 + 1 = 1\tfrac{1}{3480},$$
$$t_0 = \tfrac{6}{5},\ \ t_1 = \tfrac{4}{15}t_0 - \tfrac{1}{7}p_0 + 1 = 1\tfrac{31}{175}.$$

The values of q_i and u_i, obtained from the equations

$$q_{i+1} = q_i - u_i, \quad u_{i+1} = 2u_i - q_i,$$

with the initial conditions

$$q_0 = 0,\ \ u_0 = \frac{5}{4\log 6},$$

are

$$q_i = -\frac{5}{4\log 6}\left(\frac{\alpha^i - \alpha^{-i}}{\alpha - \alpha^{-1}}\right),$$

$$u_i = \frac{5}{8\log 6}\left(\alpha^i + \alpha^{-i} + \frac{\alpha^i - \alpha^{-i}}{\alpha - \alpha^{-1}}\right),$$

where α, α^{-1} are the roots of the equation

$$\alpha^2 - 3\alpha + 1 = 0.$$

The values of r_i, s_i, v_i, w_i are linear functions of q_i, u_i whose coefficients are linear functions of i in the case of r_i, v_i and quadratic functions of i in the case of s_i, w_i.

Thus we find, when the constants are properly determined,

$$r_i = -(2\log 6 + \lambda i)\,u_i + \{\kappa - \lambda - 2\log 29 + \log 6 - (\kappa + \lambda)\,i\}\,q_i,$$
$$v_i = \ \ (3\log 6 - \kappa i)\,u_i + (2\log 10 + \lambda - 2\kappa - \lambda i)\,q_i - \tfrac{5}{2},$$

where

$$\kappa = \tfrac{2}{5}\log\left(\frac{24^3 \,.\, 60^2}{7 \,.\, 29}\right), \quad \lambda = \tfrac{2}{5}\log\left(\frac{24^4 \,.\, 60}{7^3 \,.\, 29^3}\right).$$

The substitution of these values of r_i and v_i in the equations for determining s_i and w_i, will give a pair of equations of the form

$$s_{i+1} = s_i - w_i \ + (a + bi)\,q_i + (c + di)\,u_i - (1 + \tfrac{5}{2}\log 29),$$
$$w_{i+1} = 2w_i - s_i + (e + fi)\,q_i + (g + hi)\,u_i - \tfrac{5}{2}\log 60,$$

where a, b, c, d, e, f, g, h are known constants, and q_i, u_i are known linear functions of α^i, α^{-i}.

For example, the value of a is

$$(\log 24)^2 - (\kappa - \lambda - 2\log 29 + \log 6)\log 24 + (2\log 10 + \lambda - 2\kappa)\log 29.$$

From these equations we should obtain a result of the form

$$s_i = Q_1\alpha^i + R_1\alpha^{-i} + C_1,$$
$$w_i = Q_2\alpha^i + R_2\alpha^{-i} + C_2,$$

in which C_1, C_2 are constants and Q_1, Q_2, R_1, R_2 quadratic functions of i, but the complete determination of these would occupy too much space to be given here.

Sequel to Part II., Chapter I. § 2.

Since § 2 of this chapter was sent to press I have had asymptotic limits to $\psi(x) \div x$ computed by means of a scheme whose stigmata contain simply and in combination all the prime numbers up to 13 inclusive. The numerical results obtained on the one hand and on the other the process employed to determine *à priori* (so as to save the labour of working out the 30030 terms of a complete period) the minimum and maximum values (-1 and 4) of the sum of the coefficients of any number of consecutive terms (the first included) in the stigmatic series proper to the scheme, appear to me too noteworthy to be consigned to oblivion.

This calculation differs from those that precede it in the circumstance that it does not attempt to give the *optimistic* limits which the scheme will afford, notwithstanding which the limits actually obtained will be found to be each of them materially closer to unity than the optimistic limits furnished by any of the preceding schemes.

The scheme I adopt is [1, 6, 10, 14, 105; 2, 3, 5, 7, 11, 13, 385, 1001], which satisfies the necessary condition that the sums of the reciprocals of the numbers on the two sides of the semicolon are equal to one another.

The first thing to be done is to discover the maximum and minimum values of

$$S_n = E\left(\frac{n}{1}\right) + E\left(\frac{n}{6}\right) + E\left(\frac{n}{10}\right) + E\left(\frac{n}{14}\right) + E\left(\frac{n}{105}\right)$$
$$- E\left(\frac{n}{2}\right) - E\left(\frac{n}{3}\right) - E\left(\frac{n}{5}\right) - E\left(\frac{n}{7}\right) - E\left(\frac{n}{11}\right)$$
$$- E\left(\frac{n}{13}\right) - E\left(\frac{n}{385}\right) - E\left(\frac{n}{1001}\right).$$

On taking n equal to 66, it will be found that the value of S_n is -1: I shall proceed to show that this is the minimum, in other words that $-S_n$ cannot be so great as 2.

Denote the fractional part of any quantity x by $F(x)$: if $-S_n$ is not less than 2, then it may be shown that *à fortiori*

$$F\left(\frac{n}{6}\right) + F\left(\frac{n}{10}\right) + F\left(\frac{n}{14}\right) - F\left(\frac{n}{2}\right) - F\left(\frac{n}{3}\right) - F\left(\frac{n}{5}\right) - F\left(\frac{n}{7}\right) + F\left(\frac{n}{105}\right),$$

say $Q(n) + F\left(\frac{n}{105}\right)$ must not be less than 2, and therefore $Q(n)$ must be greater than 1 : now it is not difficult to show that $Q(n)$ is only greater than unity when

$$n = 106 + 210\kappa \quad \text{or} \quad n = 136 + 210\kappa$$

(κ being a positive integer). But corresponding to these two values it will be found that

$$Q(106) + F\left(\frac{106}{105}\right) = \tfrac{1}{3} + \tfrac{2}{5} + \tfrac{3}{7} + \tfrac{1}{105},$$

$$Q(136) + F\left(\frac{136}{105}\right) = \tfrac{1}{3} + \tfrac{2}{5} + \tfrac{2}{7} + \tfrac{31}{105},$$

so that on either supposition $Q(n) + F\left(\frac{n}{105}\right)$ is less than 2.

Hence the minimum value of S_n is -1, and consequently, since the stigmatic excess is here $8-5$, the maximum value, as appears from the footnote below, will be $8-5+1$, that is 4*. (By the stigmatic excess for any scheme I mean the number of stigmata in the right-hand less the number of those in the left-hand set. This excess is obviously equal to the coefficient, with its sign changed, of $\psi\left(\frac{x}{\mu}\right)$ in the stigmatic series, where μ is any common multiple of the stigmata.)

It will be found, on summing up the numbers in Table B, that S_n first attains the value 4 when $n = 1891$, and the value -1 when $n = 66$.

For the inferior limit the outstanding group consists of all the terms up to 1891 inclusive, and for the superior limit all the terms up to 66 inclusive. But in obtaining this limit advantage has been taken of the next three groups, which end with 78, 418, and 2068 respectively. Thus the extreme limit of the following table is 2068, instead of being 30030 (that is 2.3.5.7.11.13) which is the number of terms in a complete period. It contains the coefficients of the first 2068 terms of the stigmatic series for the scheme [1, 6, 10, 14, 105; 2, 3, 5, 7, 11, 13, 385, 1001] written down in horizontal order in regular batches of ten, as was done in Table A for the

* If we call c_n the coefficient of $\psi\left(\frac{x}{n}\right)$ and S_n the sum of such coefficients up to c_n inclusive (regarding c_0 and S_0 as zero), and take μ the least common multiple of the stigmata, we have, obviously,

$$S_\mu = 0, \quad c_n = c_{\mu-n}, \quad \text{and } (S_n + S_{\mu-1-n}) - (S_{n-1} + S_{\mu-n}) = c_n - c_{\mu-n} = 0.$$

Consequently, $S_n + S_{\mu-1-n} = S_0 + S_{\mu-1} = -c_\mu = \eta$ (the stigmatic excess).

This is a valuable formula of verification, and moreover gives a rule for finding either the maximum or minimum coefficient-sum when the other sum is given; for if S_n has the maximum value, $S_{\mu-1-n} = \eta - S_n$; if this is not the minimum let S_n' be less than $\eta - S_n$, then $S'_{\mu-1-n}$ will be greater than S_n, contrary to hypothesis. Hence the minimum value of a coefficient-sum may be found by subtracting the maximum from the stigmatic excess and *vice versâ*.

(I may perhaps be allowed to add that this theorem suggests a generalization of itself, which I think it is safe to anticipate may be formally deduced from it, namely :

If $a_1, a_2, \ldots, a_n$; $\alpha_1, \alpha_2, \ldots, \alpha_\nu$ *be any given positive quantities (integer or fractional, rational or irrational) such that* $\Sigma a = \Sigma \alpha$, *and if* $-m$, M *be the least and greatest values that* $\Sigma E(ax) - \Sigma E(\alpha x)$ *can assume when* x *is any positive quantity whatever, then* $M - m = \nu - n$.)

scheme [1, 6, 10, 210, 231, 1155; 2, 3, 5, 7, 11, 105] with the unimportant difference that (for typographical convenience) negative coefficients are indicated by dots instead of by bars placed over them.

Table B.

The coefficients of the first 2068 *terms of the stigmatic series to* [1, 6, 10, 14, 105; 2, 3, 5, 7, 11, 13, 385, 1001].

1000000000	0000i01010	ii100i0010	10i0i010i0
101ii01000	0i10i00010	10i0ii1000	1010i0ii10
0010000i10	i0000010i0	101ii0101i	001000i000
000000100i	1i00i01010	00i000i010	100i0i1000
00102̇01000	0010ii0010	1i000000i0	10102̇01i10
0000000i01	100000000i	0010i01010	2̇01i000010
1i00i00000	1000i0100i	001i000010	102̇0i01000
1010i2̇0010	001000i000	0000001i00	1i10i01000
0010i0000i	1000001i00	0000i01010	ii1000i010
10ii001000	001ii00010	00103̇0001i	10000i10i0
1000i00010	00000i0i11	100000002̇0	1010i0101i
ii10000010	00002̇01000	1i10i01i00	0000000010
00ii001000	101i2̇00010	00100ii010	000000000i
1010i01i10	00000000i0	10000i100i	1000i01000
i01000i010	12̇00001000	00002̇01010	001ii00i10
1000i010i0	0010ii1010	000i00i011	100000ii00
1010i0100i	i01000001i	10i0i01000	0010ii1000
0i10000000	102̇0001000	1i1ii00010	00102̇00010
00000i1i00	1010i00010	i010000i00	100i0010i0
1000i00012̇	001000i01i	0000001000	0i00i01010
0000ii0010	100i00102̇0	10102̇01010	0i000i0011
1000i0i000	1010i012̇10	i010000000	0000i0100i
101ii01000	i01000i010	1ii000100i	1000i00010
000i000010	0000ii1000	1010ii1000	001000i000
1i00001i00	10002̇01000	001000ii1i	1000001000
3̇010i01010	0i1ii00010	10i00000i0	101ii0101i
0000i00011	10i00i0000	1010ii0010	i010000i00
1000i010i0	1i10i0100i	0010i00010	00i0001i00
1i10i00010	i000000010	0002̇001000	10103̇00010
00100i000i	1000000000	10i0i01i10	00100ii000
10000010ii	0010i01010	ii10i00010	1i00i010i0
1000i01i10	000i000011	0000i00000	101iii1010
i010002̇010	1000i01i0i	1010i010i0	000000001i
10i00i1000	0010i00000	0i10000010	0ii0001000
101i2̇01010	0010i00i00	10000i1000	0000i00010
001i00ii10	10000000i0	0010i01012̇	0010i00010
00100010i0	1i10ii1010	00i0000001	100i000000
1i102̇01010	i010ii0010	1000i00i00	1010i01i00
i010000000	10ii001002̇	1010i0i010	i01000001i
0i00001000	10i0i01010	001i0i0000	1000i010i0

1000ii1010	0i10002̇010	1000i01i00	0010i01i00
0010i0001i	0000001010i0	001ii01010	0i0000i011
10i000000i	101ii01010	i000i00010	1000i2̇1000
1010i000i0	0010000i10	1ii00010i0	10102̇0001i
0010000i10	i000001000	0i10i01010	000i000000
100i000000	10002̇0101i	00100ii010	10i0000000
0010ii1i10	0010i000i0	1000001 0ii	1i10i01010
i000i00011	1i00000i00	1000i01010	2̇01i000010
100i3̇01000	1010ii0000	001000i01i	10i0001i00
1000i00000	00100i001i	0000001 0i0	0010i01010
3̇010000000	10i0i01000	100ii01i10	0010i0i010
00000i1000	001ii00010	0010i0ii	

In Tables I and II below, in addition to pairs of numbers $-(\eta)+(\eta+\theta)$, meaning

$$-\psi\left(\frac{x}{\eta}\right)+\psi\left(\frac{x}{\eta+\theta}\right),$$

and $+(\eta)-(\eta+\theta)$ meaning

$$+\psi\left(\frac{x}{\eta}\right)-\psi\left(\frac{x}{\eta+\theta}\right),$$

there will be found the unpaired numbers (15) and (66) in the one and (19), (229) and (1891) in the other; to understand how these are got, it should be observed that S_n (the sum of the first n numbers in Table B) first becomes 0 when $n=15$, first becomes -1 when $n=66$ and first becomes 2, 3, 4 when $n=19$, 229, 1891 respectively*.

TABLE I.

				+(15)
−	(17)	+	(22)	
−	(19)	+	(21)	
−	(23)	+	(26)	
−	(29)	+	(35)	
−	(41)	+	(45)	
−	(47)	+	(52)	
−	(59)	+	(65)	+(66)
−	(67)	+	(78)	
−	(79)	+	(418)	
−	(107)	+	(135)	
−	(210)	+	(275)	
−	(289)	+	(385)	
−	(419)	+	(2068)	
−	(521)	+	(585)	
−	(629)	+	(795)	
−	(839)	+	(936)	
−	(1049)	+	(1144)	
−	(1717)	+	(1925)	

TABLE II.

+	(15)	−	(17)	−	(19)
+	(21)	−	(31)		
+	(26)	−	(29)		
+	(33)	−	(43)		
+	(44)	−	(61)		
+	(63)	−	(73)		
+	(65)	−	(71)		
+	(75)	−	(103)	−	(229)
+	(242)	−	(271)		
+	(285)	−	(323)		
+	(385)	−	(421)		
+	(385)	−	(439)		
+	(440)	−	(493)		
+	(494)	−	(571)		
+	(770)	−	(841)		
+	(1155)	−	(1273)	−	(1891)

* Call Σ the sum of the infinite series given by Table B: it may then easily be verified that

$$\{\psi(x)-\Sigma\}-\left\{\psi\left(\frac{x}{15}\right)+\psi\left(\frac{x}{66}\right)\right\}$$

The reasoning employed in dealing with previous schemes serves to show that superior and inferior asymptotic limits to $\psi(x) \div x$, which we shall call E_1, F_1 in order to distinguish them from the corresponding optimistic limits (E, F), may be found from the equations

$$\left.\begin{aligned} E_1 &= M + aE_1 - bF_1 \\ F_1 &= M + cF_1 - dE_1 \end{aligned}\right\},$$

where a is the sum of the reciprocals of the numbers occurring in Table I with the sign +

b " " " " " —

c " " " in Table II " +

d " " " " " —

and M is the stigmatic multiplier,

namely
$$a = \frac{1}{15} + \frac{1}{21} + \frac{1}{22} + \ldots + \frac{1}{2068} = \cdot 33352 \ldots,$$

$$b = \frac{1}{17} + \frac{1}{19} + \frac{1}{23} + \ldots + \frac{1}{1717} = \cdot 30580 \ldots,$$

$$c = \frac{1}{15} + \frac{1}{21} + \frac{1}{26} + \ldots + \frac{1}{1155} = \cdot 26966 \ldots,$$

$$d = \frac{1}{17} + \frac{1}{19} + \frac{1}{29} + \ldots + \frac{1}{1891} = \cdot 27742 \ldots *,$$

---

may be resolved into term-pairs of the form

$$-\psi\left(\frac{x}{\eta}\right) + \psi\left(\frac{x}{\eta+\theta}\right)$$

that shall contain among them all those in Table I, and

$$\{\psi(x) - \Sigma\} + \left\{\psi\left(\frac{x}{19}\right) + \psi\left(\frac{x}{229}\right) + \psi\left(\frac{x}{1891}\right)\right\}$$

into term-pairs of the form $+\psi\left(\frac{x}{\eta}\right) - \psi\left(\frac{x}{\eta+\theta}\right)$ that shall contain among them all those in Table II above.

The maximum value of S_n is here 4: if it had been 2, then instead of 3 unpaired positive terms appended to $\{\psi(x) - \Sigma\}$ there would have been but 1. This is what happens for the scheme [1, 15; 2, 3, 5, 30] given in the footnote on p. [710]: and accordingly, we see that $\{\psi(x) - \Sigma\} + \psi\left(\frac{x}{17}\right)$, for that scheme, is resoluble into paired terms of the form

$$+\psi\left(\frac{x}{\eta}\right) - \psi\left(\frac{x}{\eta+\theta}\right).$$

So again, the minimum being 0 (instead of -1), there will be but 1 unpaired negative term to append to $\{\psi(x) - \Sigma\}$, and accordingly, we see that $\{\psi(x) - \Sigma\} - \psi\left(\frac{x}{6}\right)$ in that scheme is resoluble into term-pairs of the form $-\psi\left(\frac{x}{\eta}\right) + \psi\left(\frac{x}{\eta+\theta}\right)$.

* The above values of a, b, c, d give $a + c = \cdot 603 \ldots$ and $ac - bd = \cdot 005 \ldots$, and consequently the roots of the "characteristic" equation $\rho^2 - (a+c)\rho + (ac - bd) = 0$ satisfy the necessary condition of being each less than unity in absolute value.

and $$M=\frac{1}{2}\log 2+\frac{1}{3}\log 3+\frac{1}{5}\log 5+\frac{1}{7}\log 7$$
$$+\frac{1}{11}\log 11+\frac{1}{13}\log 13+\frac{1}{385}\log 385+\frac{1}{1001}\log 1001$$
$$-\frac{1}{6}\log 6-\frac{1}{10}\log 10-\frac{1}{14}\log 14-\frac{1}{105}\log 105=\cdot 98859\ldots.$$

Hence $$E_1=\frac{(1-c-b)M}{(1-a)(1-c)-bd}=1\cdot 04423\ldots,$$
$$F_1=\frac{(1-a-d)M}{(1-a)(1-c)-db}=\cdot 95695\ldots,$$

(so that the mean of E_1 and F_1 is less than ·0006), and $\frac{E_1}{F_1}=1\cdot 09120\ldots$ *.

Thus then (see footnote to p. [694]) by taking x sufficiently great, the number of primes not exceeding x, multiplied by $\log x$ and divided by x, may always be made to lie between the numbers

1·04423 ... and ·95695 ...,

the divergences of which from unity are

·04423 ... and ·04304 ... (as against

Tchebycheff's ·10555 ... and ·07807 ...).

These divergences, there is little doubt, would become even more nearly equal than they are, if anyone should feel inclined to undertake the very laborious task of extracting the *optimistic* values (E, F) from the scheme employed.

In order to understand this necessarily abbreviated sketch of a method more easy to think out and apply than to find language to express, I must not conceal that a careful study of the several schemes given, and of the principles embodied in the calculations relating to them, is a *sine quâ non.* It may somewhat lighten the burden thrown upon the reader, if I add a few words concerning one or two points, perhaps inadequately explained in what precedes.

Let μ be the least common multiple of the stigmata of any given harmonic scheme and S_n the sum of the coefficients of

$$\psi(x),\quad \psi\left(\frac{x}{2}\right),\quad \psi\left(\frac{x}{3}\right),\ldots\psi\left(\frac{x}{n}\right)$$

* In Tables I and II above, the ratio $\frac{\eta+\theta}{\eta}$ is greater than 1·09120 ... for every pair of terms, except $-(1049)+(1144)$ in Table I. In the case of this pair, we have $\frac{1144}{1049}=1\cdot 0905\ldots$, which shows that the exclusion of it from that table would have led to asymptotic limits better (but very slightly so) than those arrived at in the text.

in the corresponding stigmatic series. Then from the formula of [p. 710] combined with the equation which connects the stigmata, it follows that

$$S_\mu = 0, \quad S_{n+\mu} = S_n.$$

Hence an infinite number of values of n will give S_n its greatest value; the difference of these values will be of the form $k\mu - \mu'$ where μ' may, and in general will, besides zero have various other values less than μ, thus giving rise to the collections of terms called *groups* (see p. [709]) of which the period of μ terms will be composed. The same will be true when we substitute the word *least* for *greatest.*

If now i be taken *any* number such that S_i has its greatest value it may be shown that the sum of all the terms in the stigmatic series subsequent to the one containing $\psi\left(\frac{x}{i}\right)$ will be *negative* or zero, and similarly when S_i has its least value such sum will be *positive* or zero*; consequently when i is properly determined we can find immediately a superior limit in the one case and an inferior limit in the other, to the sum of the first i terms of the series.

I will conclude this portion of the subject with the remark that from the values of E_1 and F_1 it is easy to infer that if μ is equal to or less than $(\cdot 95695 \ldots) k - (1\cdot 04423 \ldots)$, and x exceeds a certain ascertainable number whose value depends on k and μ, then between x and kx there will be found more than $\mu \frac{x}{\log x}$ primes†.

* The reason of this is that the sum of all the terms beyond the ith may be separated into partial sums, each containing μ terms, which ultimately vanish. If now

$$\gamma_1 (k\mu + i + 1) + \gamma_2 (k\mu + i + 2) + \ldots + \gamma_\mu (k\mu + i + \mu)$$

be one of them, then $\gamma_1 + \gamma_2 + \ldots + \gamma_t$ will be zero when $t = \mu$, and will have a constant algebraical sign (or else be zero) when $t < \mu$; from which it follows (see footnote p. [722] where, be it observed, a coefficient $+\lambda$ or $-\lambda$ is supposed to be represented by a *sequence* of λ black or λ white beads) that each partial sum may be decomposed into an aggregate of quantities of the form $+(\eta) - (\eta + \theta)$ or $-(\eta) + (\eta + \theta)$ according as the first coefficient in each such sum is positive or negative, and will therefore, if not zero, have the same algebraical sign as that coefficient has, namely $-$ or $+$ according as S_i has its greatest or least value.

† In order that μ may be positive (which ensures the existence of *some* primes between x and kx, when x exceeds a certain limit) it is only necessary to take $k > 1\cdot 09120 \ldots$ (which differs very little from $\frac{12}{11}$), whereas if we limited ourselves to the results of the oft-quoted memoir of 1850 [see p. 704, above], we could not prove the existence of prime numbers between x and kx, for a given value of x, however great, unless k exceeds $\frac{6}{5}$.

73.

NOTE ON A NINE SCHOOLGIRLS PROBLEM.

[*Messenger of Mathematics*, XXII. (1893), pp. 159, 160.]

THIS is a parallel question to the well-known one of fifteen schoolgirls extended to the supposition of their walking for *one* week, three and three together, so that in any the same *day* no two, and at the end of the *week* no three, taking four walks a day, shall have walked more than once together.

Let us understand by the development of the array

$$\begin{array}{ccc} a, & b, & c, \\ d, & e, & f, \\ g, & h, & k, \end{array}$$

the four arrangements

$$\begin{array}{ccc} (abc, & def, & ghk), \\ (adg, & beh, & cfk), \\ (aek, & bfg, & cdh), \\ (afh, & bdk, & ceg), \end{array}$$

(corresponding, in fact, to the four sets of three lines through the nine inflexions of a cubic).

If we suppose the nine girls to walk out four times a day, the same two never being together more than once in the same day, and that at the week's end each has been associated with every pair of the remaining eight, the above will serve to represent one day's walks. To find the other six, I first form the three following pairs of subsidiary arrays, by circular motion performed successively on the three columns of the primitive array, namely

$$\begin{array}{ccccccc} g, & b, & c, & \quad & d, & b, & c, \\ a, & e, & f, & & g, & e, & f, \\ d, & h, & k, & & a, & h, & k, \end{array}$$

$$\begin{array}{ccccccc} a, & h, & c, & \quad & a, & e, & c, \\ d, & b, & f, & & d, & h, & f, \\ g, & e, & k, & & g, & b, & k, \end{array}$$

$$\begin{array}{ccccccc} a, & b, & k, & \quad & a, & b, & f, \\ d, & e, & c, & & d, & e, & k, \\ g, & h, & f, & & g, & h, & c. \end{array}$$

Then making any similarly placed line (I have taken the first) in each of the above six groups circulate in one direction as regards the three on the left, and in the opposite direction as regards the three on the right, we obtain six new arrays: these together with the original one furnish the following table:

			a,	*b*,	*c*,			
			d,	*e*,	*f*,			
			g,	*h*,	*k*,			
c,	*g*,	*b*,				*b*,	*c*,	*d*,
a,	*e*,	*f*,				*g*,	*e*,	*f*,
d,	*h*,	*k*,				*a*,	*h*,	*k*,
c,	*a*,	*h*,				*e*,	*c*,	*a*,
d,	*b*,	*f*,				*d*,	*h*,	*f*,
g,	*e*,	*k*,				*g*,	*b*,	*k*,
k,	*a*,	*b*,				*b*,	*f*,	*a*,
d,	*e*,	*c*,				*d*,	*e*,	*k*,
g,	*h*,	*f*,				*g*,	*h*,	*c*.

When the seven arrays in the above table are developed according to the rule previously given, the triads thus arising will be found to be all distinct or, which is the same thing, will comprise among them the whole of the eighty-four ternary combinations of the nine symbols. We have therefore in this table a solution of the proposed problem.

Of course the general problem, when n is any odd multiple of 3, is to construct sets of $\frac{1}{2}(n-1)$ synthemes, each containing $\frac{1}{3}n$ triads with no element in common, and to distribute the whole number of triads into $(n-2)$ such sets.

This problem I solved very many years ago, but I believe have nowhere published, for the case where n is any power of 3, by a method of compound rhythmical displacement strictly analogous to (but of course more intricate than) the one here exhibited.

74.

ON THE GOLDBACH-EULER THEOREM REGARDING PRIME NUMBERS.

[*Nature*, LV. (1896–7), pp. 196, 197; 269.]

In the published correspondence of Euler there is a note from him to Goldbach, or, the other way, from Goldbach to Euler, in which a very wonderful theorem is stated which has never been proved by Euler or any one else, which I hope I may be able to do by an entirely original method that I have applied with perfect success to the problem of partitions and to the more general problem of denumeration, that is, to determine the number of solutions in positive integers of any number of linear equations with any number of variables. In applying this method I saw that the possibility of its success depended on the theorem named being true in a stricter sense than that used by its authors, of whom Euler verified but without proving the theorem by innumerable examples. As given by him, the theorem is this: *every even number* may be broken up in one or more ways into two primes.

My stricter theorem consists in adding the words "where, if $2n$ is the given number, one of the primes will be greater than $\frac{n}{2}$, and the other less than $\frac{3n}{2}$." This theorem I have verified by innumerable examples. Such primes as these may be called mid-primes, and the other integers between 1 and $2n-1$ extreme primes in regard to the range $1, 2, 3 \ldots, 2n-1$.

I have found that with the exception of the number 10, Euler's theorem is true for the resolution of $2n$ into two *extreme* primes; but this I do not propose to consider at present, my theorem being that every even number $2n$ may be resolved into the sum of two mid-primes of the range

$$(1, 2, 3 \ldots, 2n-1).$$

As, for example

$$
\begin{array}{llll}
4 = 2 + 2 & 6 = 3 + 3 & 8 = 3 + 5 & 10 = 3 + 7 \\
12 = 5 + 7 & 14 = 7 + 7 & 16 = 5 + 11 & \\
18 = 5 + 13 = & 7 + 11 & 20 = 7 + 13 & \\
40 = 11 + 29 = & 17 + 23 & 50 = 13 + 37 = 19 + 31 & \\
100 = 29 + 71 = & 41 + 59 & & \\
200 = 61 + 139 = & 73 + 127 = \&c. & & \\
500 = 127 + 373 = & 193 + 307 = \&c. & & \\
1000 = 257 + 743 = \&c. & & &
\end{array}
$$

And so on.

My method of investigation is as follows. I prove that the number of ways of solving the equation $x + y = 2n$, where x and y are two mid-primes to the range $2n - 1$, that is twice the number* of ways of breaking up $2n$ into two mid-primes + zero or unity, according as n is a composite or a prime number, is exactly equal to the coefficient of x^{2n} in the series

$$\left(\frac{1}{1 - x^p} + \frac{1}{1 - x^q} + \dots + \frac{1}{1 - x^l}\right)^2$$

where $p, q, \dots, l$ are the mid-primes in question. This coefficient, we know *à priori*, is always a positive integer, and therefore if we can show that the coefficient in question is not zero, my theorem is proved, and as a consequence the narrower one of Goldbach and Euler. By means of my general method of expressing any rational algebraical fraction, say ϕx, as a residue, by taking the distinct roots of the denominator, say ρ, and writing the variable equal to $\rho\epsilon^t$, and taking the residue with changed sign of $\Sigma\rho^{-n}\,\epsilon^{-nt}\phi(\rho\epsilon^t)$, we can find the coefficient of x^n or (if we please to say so) of x^{2n} in the above square, and obtain a superior and an inferior limit to the same in terms of $p, q, \dots, l$; and if, as I *expect* (or rather, I should say, *hope*) may be the case, these two limits do not include zero between them, the theorems (mine, and therefore *ex abundantia* Euler's) will be apodictically established.

The two limits in question will be algebraic functions of $p, q, \dots, l$, whereas the *absolute* value of the coefficient included within these limits would require a knowledge of the residues of each of these numbers in respect to every other as a modulus, and of $2n$ in respect of each of them. In a word, the limits will be algebraical, but the quantity limited is an algebraical function of the mid-primes $p, q, r, \dots, l$.

Postscript. The shortest way of stating my refinement on the Goldbach-Euler theorem is as follows:—"It is always possible to find two primes

* This number may be shown to be of the order $\frac{n}{(\log n)^2}$, and a very fair approximate value of it is $\frac{\mu^2}{n}$ where μ is the number of mid-primes corresponding to the frangible number $2n$.

differing by less than any given number whose sum is equal to twice that number."

Another more instructive and slightly more stringent statement of the new theorem is as follows. Any number n being given, it is possible to find two primes whose sum is $2n$, and whose difference is less than n, $n-1$, $n-2$, $n-3$, according as n divided by 4 leaves the remainders 1, 0, -1, -2 respectively.

Major MacMahon, to whom and to the Council of the Mathematical Society of London I owe my renewed interest in this subject, informs me that in a very old paper in the *Philosophical Magazine* I stated that I was in possession of "a subtle method, which I had communicated to Prof. Cayley," of finding the number of solutions in positive integers of any number of linear equations in any number of variables. This method (never printed) must have been in essence identical with that which within the last month I have discovered and shall, I hope, shortly publish.

I have verified the new law for all the even numbers from 2 to 1000, but will not encumber the pages of *Nature* with the details. The approximate formula hazarded for the number of resolutions of $2n$ into two primes, namely $\frac{\mu^2}{n}$, where μ is the number of mid-primes, does not always come near to the true value. I have reasons for thinking that when n is sufficiently great, $\frac{\mu^2}{2n}$ may possibly be an inferior limit. The generating function

$$\left[\Sigma \frac{1}{1-x^p}\right]^2$$

is subject to a singular correction when the partible number $2n$ is the double of a prime. In this case, since the development to be squared is

$$\mu + x^n + x^{2n} + \ldots + x^p + x^{2p} + \ldots + \&c.,$$

the coefficient of x^{2n} will contain 2μ, arising from the combination of 0 with $2n$, which is foreign to the question, and accordingly the result given by the generating function would be too great by 2μ.

This may be provided against by always rejecting the centre of the mid-range from the number of mid-primes. The formula will then in all cases give twice the number of ways of breaking up $2n$ into two unequal primes. Another method would be to take as the generating function not the square of the sum, but the product of the fractions $\frac{1}{1-x^p}$ (without casting out n when it is a prime), but this method would be inordinately more difficult to work with in computing series involving the roots of unity than the one chosen, which is in itself a felicitous invention*. Whether the method turns out successful or not, it at the very least gives an analytical expression for the number of ways of conjoining the mid-primes to make up $2n$ without trial, which in itself is a somewhat surprising result. Having lost my preliminary calculations, it may be some little time before I shall be able to say whether the method does or does not contain a proof of the new theorem; but that this can be ascertained, there is no manner of doubt. This is the first serious attempt to deal with Euler's theorem, or to bring the question into line with the general theory of partitions.

It is proper to regard the range 1 to $2n-1$ as consisting of two complementary flank regions, two lateral mid-prime regions, and a region reduced to a single term in the middle, as for example,

$$1,\ 2,\ 3 : 4,\ 5 : 6 : 7,\ 8 : 9,\ 10,\ 11.$$

Or, again, $\quad 1,\ 2,\ 3 : 4,\ 5,\ 6 : 7 : 8,\ 9,\ 10 : 11,\ 12,\ 13.$

And the question of $2n$ being resoluble into 2 primes breaks up into three, namely, whether $2n$ can be composed with two flank primes, two lateral mid-primes, or with the number in the central region repeated.

* For the generating function we may take any power greater than 2, instead of the square, and the coefficient of x^{2n} will then be the number of couples making up $2n$ multiplied by $(r^2-r)\,\mu^{r-1}$, which can be calculated by the same method as for the square, but is more difficult and must give rise to numerous theorems of great interest, arising from the multiform representation of the same quantity.

75.

ON THE NUMBER OF PROPER VULGAR FRACTIONS IN THEIR LOWEST TERMS THAT CAN BE FORMED WITH INTEGERS NOT GREATER THAN A GIVEN NUMBER.

[*Messenger of Mathematics*, XXVII. (1898), pp. 1—5.]

A SLIGHT reflexion will show that the number of such fractions $\left(\frac{1}{1} \text{ counting as one of them}\right)$ with the limit n is the sum of the totients of all the numbers from 1 to n.

Let us use Ej as usual to denote the integer part of j, τEj to denote the totient (or number of numbers not exceeding and prime) to Ej, and JEj to denote the sum of such totients for all numbers from 1 to j. Then we may establish the following exact equation given by the author of this article, but without proof and with some slight inaccuracy, in the *Phil. Mag.* for April, 1883 [p. 102, above]. The equation is

$$JEj + JE\left(\tfrac{1}{2}j\right) + JE\left(\tfrac{1}{3}j\right) + \text{etc.},$$

or, more shortly,

$$\sum_1^\infty JE\frac{j}{i} = \tfrac{1}{2}\left\{(Ej)^2 + (Ej)\right\}. \qquad (1)$$

The proof is as follows. Remarking that $E(j-1) = Ej - 1$, the right-hand side of equation (1), when j is reduced to $j-1$ obviously suffers a diminution equal to Ej.

On the left-hand side of the equation any term $JE\frac{j}{i}$ remains unaltered, when for j is written $(j-1)$, unless Ej is divisible by i, in which case the term undergoes a diminution JEj. Thus for example $J\frac{100}{11} - J\frac{99}{11} = 0$, but $J\frac{100}{5} - J\frac{99}{5} = J(20)$. And, as in the case supposed, $\frac{Ej}{i}$ is a factor of Ej, the total diminution, when $j-1$ replaces j, will be the sum of the totients of the factors of Ej, which by a known theorem equals Ej. Hence equation (1) is satisfied for j if it is satisfied for $j-1$, and as it is true when $Ej=1$ it is true for all values of j, as was to be proved. From equation (1) it follows that JEj is of the order $(Ej)^2$, and making

$$JEj = \tfrac{1}{2}\mu(Ej)^2 + \epsilon j,$$

where ϵj is zero when $j=\infty$, we obtain

$$\mu(1+\tfrac{1}{4}+\tfrac{1}{9}+\tfrac{1}{16}+\ldots)=1,$$

or $$\mu=\frac{6}{\pi^2}, \text{ or approximately } Jj=\frac{3j^2}{\pi^2}.$$

In the tables in the *Phil. Mag.* for April and September, 1883 *, the value of Jj is computed up to $j=1{,}000$ and compared with the mean value $\frac{3}{\pi^2}j^2$. From this table it appears that Jj is always intermediate between $\frac{3}{\pi^2}j^2$ and $\frac{3}{\pi^2}(j+1)^2$, and much nearer to their mean, which to an insignificant fraction *près* is the same as $\frac{3}{\pi^2}(j^2+j)$, than it is to either extreme. The first, at least, of these statements ought to be susceptible of proof.

As a matter of philosophical interest as embodying a principle applicable to other cases, I will show how I originally found the value $\frac{3}{\pi^2}j^2$ for the number of proper vulgar fractions in their lowest terms that can be formed by means of the first integers.

It is obvious that the probability of any unknown number being divisible by a prime number i is $\frac{1}{i}$, and of any two numbers, being each so divisible, is $\frac{1}{i^2}$, so that the probability of two unknown numbers being each *not* divisible either by 2, 3, 5, 7, n, or any other prime, will be

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{5^2}\right)\left(1-\frac{1}{7^2}\right)\left(1-\frac{1}{n^2}\right)\ldots,$$

which we know is equal to the sum of the reciprocal of the squares of the natural numbers, that is, is equal to $\frac{6}{\pi^2}$. Hence the number of fractions in their lowest terms that can be got by combining each of j integers with each of i others, found *roughly* by adding together the probable expectation of any such combination consisting of two relative primes, will be $\frac{6}{\pi^2}j^2$, and the number of *proper* fractions in their lowest terms so capable of being formed will be the half of this or $\frac{3j^2}{\pi^2}$. It appears incidentally from this

[* p. 103, above.]

that the average or mean value of the totient of any number is $\frac{3}{\pi^2}$ into, or rather more than, $\frac{3}{10}$ths of that number.

In like manner, if we define a mid-prime to the number $2n$ to be one which is greater than $\frac{1}{2}n$ and less than $\frac{3}{2}n$, the *range* of numbers amongst which such primes are to be found will, to a unit *près*, be n. Let us call the number of such mid-primes μ. Then the probability of any number and its complement in respect to $2n$ being each of them primes will be $\frac{\mu^2}{n^2}$. If now we seek the number of solutions of the equation in prime numbers $x + y = 2n$, which will be an even or an odd number, according as n is a composite number or a prime, we may suppose a row of n white balls and n black balls, each series being marked with all the numbers from 1 to n inclusive. It follows from what has been said that the sum of the expectation of x being inscribed on any one of the white balls being itself a prime, and its complement $2n - x$ upon one of the black balls being so likewise, will be $n \cdot \frac{\mu^2}{n^2}$, that is $\frac{\mu^2}{n}$,* and as the same will be true when x is a figure on a black ball and $2n - x$ on a white, the total value of the expectation of the equation in *primes* $x + y = 2n$ being satisfied will be the double of this, or $\frac{2\mu^2}{n}$. I have had tables constructed for determining the number of the solutions of this equation (x and y being primes) from $2n = 2$ up to $2n = 500$.

Call the number of solutions for any value of n, $\theta\frac{\mu^2}{n}$; on taking the average value of θ for all values of $2n$ on the 1st, 2nd, 3rd, 4th, 5th, centuries respectively, it will be found that

$$\begin{aligned} \tfrac{1}{2}\theta &= \cdot 96344 \\ &= \cdot 99349 \\ &= 1\cdot 00603 \\ &= \cdot 98281 \\ &= \cdot 99764, \end{aligned}$$

of which the sum is 4·94341 and the average is ·98868, agreeing with wonderful nearness to the rough estimate of the number of solutions being $\frac{2\mu^2}{n}$.

* μ is of the order of, and ultimately in a ratio of equality with, $\frac{n}{\log n}$, in the sense that, however small ϵ be taken, a limit $L\epsilon$ can be found such that for all values of n beyond it, μ will be limited on the two sides by $(1 \pm \epsilon)\frac{n}{\log n}$; this follows demonstrably from a known theorem proved within the last few years, and as a consequence we see that the number of solutions in "mid-primes" of the equation $x + y = 2n$ will necessarily be of the same order as $\frac{n}{(\log n)^2}$ and *presumably* in a ratio of equality with it in the sense explained above, but this, of course, awaits demonstration.

I ought not, however, to suppress the fact that, from another point of view, this number might be expected to eventuate as $\frac{\mu^2}{n}$ instead of $\frac{2\mu^2}{n}$.

In equation (1) we may write $F(j)$ for the sum of the totients of all the numbers not exceeding j, and it then takes the form

$$\phi j = \tfrac{1}{2}\{Ej + (Ej)^2\} = Fj + F(\tfrac{1}{2}j) + F(\tfrac{1}{3}j) + F(\tfrac{1}{4}j) + \text{etc.},$$

which, by the well-known formula of reversion (see *Phil. Mag.*, December, 1884*), gives

$$Fj = \phi j - \phi(\tfrac{1}{2}j) - \phi(\tfrac{1}{3}j) - \phi(\tfrac{1}{5}j) + \phi(\tfrac{1}{6}j) - \text{etc.}$$

Thus for example the number of terms in a Farey series with 17 as a limit should be equal to

$$\tfrac{1}{2}(17 - 8 - 5 - 3 + 2 - 2 + 1 - 1 - 1 + 1 + 1 - 1)$$
$$+ \tfrac{1}{2}(289 - 64 - 25 - 9 + 4 - 4 + 1 - 1 - 1 + 1 + 1 - 1)$$

that is $\tfrac{1}{2}(1) + \tfrac{1}{2}(191)$ or 96, which is right†.

* I do not know whether the annexed important case of reversion has been noticed or not: i being greater than unity, let σ_i denote the sum of the *negative* ith powers of the prime numbers 2, 3, 5, 7, etc., and s_i the *logarithm* of the sum of the negative ith powers of the natural numbers 1, 2, 3, 4, etc. (which, when i is an even integer, is a known quantity), then it is easily shown that

$$s_i = \sigma_i + \tfrac{1}{2}\sigma_{2i} + \tfrac{1}{3}\sigma_{3i} + \tfrac{1}{4}\sigma_{4i} + \tfrac{1}{5}\sigma_{5i} + \text{etc.},$$

and therefore by reversion

$$\sigma_i = s_i - \tfrac{1}{2}s_{2i} - \tfrac{1}{3}s_{3i} - \tfrac{1}{5}s_{5i} + \tfrac{1}{6}s_{6i} - \tfrac{1}{7}s_{7i} + \tfrac{1}{10}s_{10i} + \text{etc.}$$

A very general case for reversion arises when $\phi i = \Sigma \frac{1}{n^2}\phi(n^s . i)$. In this last application of the formula $r=1$, $s=1$; in the case considered in the text relating to Farey series $r=0$, $s=-1$.

† And so in general, since by a well-known theorem

$$Ej - E(\tfrac{1}{2}j) - E(\tfrac{1}{3}j) + E(\tfrac{1}{6}j) + \text{etc.}$$

is always equal to unity, so that

$$(Ej)^2 - 2JEj + 1 = E(\tfrac{1}{2}j)^2 + E(\tfrac{1}{3}j)^2 - E(\tfrac{1}{6}j)^2 + \text{etc.},$$

we have always

$$2JEj - 1 = (Ej)^2 - E(\tfrac{1}{2}j)^2 - E(\tfrac{1}{3}j)^2 + E(\tfrac{1}{6}j)^2 + \text{etc.}$$

a very convenient, and, I believe, new formula for calculating the number of fractions in their lowest terms where neither numerator nor denominator exceeds j.

To this E theorem there exists a pendant which may be called the H theorem, namely let Hx mean the nearest integer (when there is one) to x, but when x is midway between two integers Hx is to denote the first integer above x; let $p, q, r, \ldots$ be the primes not exceeding the integer n, and call

$$H_n = n - \Sigma H\frac{n}{p} + \Sigma H\frac{n}{pq} - \Sigma H\frac{n}{pqr} + \text{etc.};$$

then H_n will be the number of primes greater than n and less than $2n$, so that H_n is always greater than zero; and if $\epsilon(x)$ means unity or zero according as x is a prime or not, we shall always have

$$H_n - H_{n-1} = \epsilon(2n-1) - \epsilon(n).$$

I do not know whether this theorem has been previously noticed. It may be obtained by the Eratosthenes sieve process applied to the progression $n+1, n+2, n+3, \ldots, 2n$, replacing therein every prime number by unity.

If not already known, it may be worth while to register the two following additional theorems concerning $E_1 n$ and $H_1 n$, by which I mean what E_n and H_n become when the even prime 2 does not count among the primes p, q, r, which are less than n, namely

$$E_1 n = E\left(\frac{n}{2}\right) - \Sigma E\,\frac{n}{2p} + \Sigma E\,\frac{n}{2pq} + \text{etc.} = E\left(\frac{\log n}{\log 2}\right),$$

$$H_1 n = H\,\frac{n}{2} - \Sigma H\,\frac{n}{2p} + \Sigma H\,\frac{n}{2pq} + \text{etc.} = 1.$$

This paper was sent by Professor Sylvester to the editor on Feb. 12th, 1897, with a letter in which he wrote "I could subsequently send you the valuable table referred to in the text, giving the number of solutions of the equation $x + y = 2n$ in prime numbers for all values of n up to 500." In subsequent letters he made several slight additions to the paper. He corrected the proof sheets about the end of the month, and then added the first footnote and the last paragraph of the third note. His death took place on March 15th.

INDEX TO PROFESSOR SYLVESTER'S CONTRIBUTIONS TO "MATHEMATICAL QUESTIONS...FROM THE *EDUCATIONAL TIMES*."

In the following index, prepared by Mr C. C. Scott, of the Library of St John's College, Cambridge, the date is that of the volume of the "Mathematical Questions...from the *Educational Times*," the page is that of the volume, the number is that of the question, and the word 'Solution' indicates that a solution of a question is given by Professor Sylvester. It will be noticed that in some cases a question occurs several times, in different volumes (as indicated by the number of the question being the same).

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