COMMENTARIES
ON THE
PRINCIPIA
OF
SIR ISAAC NEWTON.

COMMENTARIES

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PRINCIPIA OF SIR ISAAC NEWTON,

RESPECTING HIS THEORY

THAT THE FORCES OF THE GRAVITATION OF THE PLANETS

ARE INVERSELY AS THE SQUARES OF THEIR

MEAN DISTANCES FROM THE SUN:

WHICH THEORY IS CALLED IN QUESTION IN THESE COMMENTARIES.

BY THE AUTHOR OF

" A NEW THEORY OF GRAVITATION," &c.

" A short access has been made to much knowledge, at which Sir Isaac Newton arrived through arduous and circuitous paths. Yet we look with peculiar reverence on the Principia."—Mr. Macaulay's Essays.

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TO THE

RIGHT HONOURABLE LORD BROUGHAM,

&c. &c. &c.

My Lord,

In dedicating these Commentaries to your Lordship, it is due from me to state that they have not been seen by your Lordship, and that I am not aware that you have ever given any opinion contrary to the doctrines of the Principia. I offer my Commentaries on the Principia to your Lordship's notice and protection, as a tribute to the zeal and ardour with which your Lordship has ever promoted the interests of science; but more particularly as a tribute of respect and admiration of your Lordship's great and acknowledged proficiency in these abstruse and recondite investigations.

I have the honour to be,

My Lord,

Your Lordship's very obedient Servant,

THE AUTHOR.

Sunbury Park,
Middlesex, June 4, 1846.

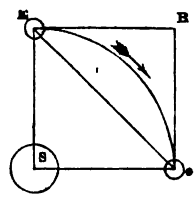
CHAPTER I.

INTRODUCTION.

In a work published by me in 1844, entitled, "A New Theory of Gravitation," the main proposition which I advanced was this: viz., that the forces of the gravitation of the planets towards the sun are inversely as the *square-roots* (that is, in the *sub-duplicate* ratio,) of their mean distances from the sun; a proposition very much at variance with the received doctrine of Sir Isaac Newton, viz., that the forces of the gravitation of the planets towards the sun are inversely as the *squares* (that is, in the duplicate ratio) of their mean distances from the sun. To exhibit the difference between these two theories, one instance may suffice :—According to the Newtonian Theory, if one of two planets were at four times the distance of the other from the sun, the force of gravitation of the nearer planet would be sixteen times that of the more distant one (that is, the *square* of four times); but according to the new theory, the force of gravitation of the nearer planet would be only twice that of the other (that is, the *square-root* of four times).

The investigations which led me to the adoption of this new theory did not originate in any preconceived doubt of the truth of the received theory of Newton; on the contrary, I began those investigations under the impression that the received theory was true, being part of that great system of gravitation, the discovery of which had been made by Newton. But Kepler had, about forty-seven years prior to that discovery, discovered his famous analogy that the squares of the periodic times of the planets are as the cubes of their mean distances from the sun; whence it had been inferred by Newton (Coroll. 6 to Proposition iv.—Principia) that the mean *velocities* of the planets are inversely as the square-roots of their mean distances from the sun; an inference adopted by La Place in his "System of the World," vol. i. p. 236, but neither he nor Newton has given any demonstration of it: (in commenting on Coroll. 6, to Prop. iv.—Principia, I propose to demonstrate this inverse analogy of the velocities to the distances from Kepler's Analogy.) Taking the estimated distances of the planets to be true, and their orbits to be circular (which they are, nearly), I easily computed their respective velocities, and by comparing the velocities of the planets with their distances from the sun, I found that the velocities are (according to the inference from Kepler's Analogy) inversely as the square-roots (that is, in the sub-duplicate ratio) of the distances. In 1842 I published the results of this investigation in another work, entitled, "A New Analogy for discovering the the Distances of the Planets from the Sun," &c. Thus, by showing that the analogy derived from Kepler's is conformable to the observed phenomena, both analogies became more fully established.

It subsequently occurred to me that the gravitating forces of the planets must be as their velocities; and therefore (by equality of ratios) inversely as the *square-roots* of their mean distances from the sun; instead of being (according to the Newtonian doctrine) inversely as the *squares* of their mean distances from the sun. For (by the Scholium to Proposition iv.—Principia) the centrifugal and centripetal forces of each planet are equal to each other; and the revolution of each planet round the sun is the resultant of the composition of these two equal forces. This resultant, if we consider the orbit as circular, is a given quantity, with regard to all the planets and the satellites; for their periodic times are known by observation, and the distances of the planets from the sun, and of the satellites from their primaries, have been determined by Kepler's Analogy, and from these data we may determine the centrifugal and centripetal forces in the following manner:—



Taking the earth, for instance, at any point E of its orbit considered circular, to be at the distance of 96,000,000 miles from S, the centre of the sun, she would in $\frac{365}{4} = 91{\cdot}25$ days, arrive at the point *e*, after describing one quarter of her orbit. Now the earth would in the same time have arrived at the point *e*, if at the point E she had been simultaneously impressed with the two equal forces represented by the right lines E B, and E S, at right-angles to each other, of which forces one, that is, E B, would (if acting alone) cause her to move from E to B, in 91·25 days, and the other force E S, equal to E B, would (if acting alone) occasion her to move from E to S, in the same time; and since the centrifugal force is represented by the line E B, the gravitating or centripetal force is equal to the force represented by the line E S, that is, to 96,000,000 miles in 91·25 days; or to $\frac{96{\cdot}000{,}000}{91{\cdot}25} = 1{,}052{,}055$ miles per diem, nearly.

In this manner I determined the force of the sun's gravity on the earth, and in like manner I determined the gravitating forces of the other known primary planets in the solar system. I then proceeded to compare these forces with their mean distances from the sun, with a view to discover whether the forces were as the velocities, that is, inversely as the square-roots of the distances, or, according to Newton's Analogy, inversely as the squares of the distances; and I found by actual computation that the forces are inversely as the *square-roots* (not as the squares) of the mean distances. In the "New Theory of Gravitation," I have given in several tables the gravitating forces of the planets and satellites as I determined them, and have fully detailed the method and process of my calculations.

Sir Isaac Newton founded his theory (that the gravitating forces are inversely as the *squares* of the distances) on the hypothesis, that the force of gravity emanates from the sun as rays from a centre, in which case his theory would probably hold good. But his hypothesis was not collected from ascertained phenomena; but seems to have been assumed, by attributing to gravity the properties of light.

On the other hand, taking the phenomena to be as I have endeavoured to prove, viz., that the forces are inversely as the square-roots of the distances (or, in other words, that the distances are inversely in the duplicate ratio of the forces), I proceeded to investigate in what manner the centripetal and centrifugal forces operate upon the solar system, so as to occasion this analogy between the gravitating forces of the planets and their mean distances from the sun. In this investigation I proceeded upon the basis of the centripetal and centrifugal forces of each planet being equal to each other, which equality I considered necessary to retain the planet in its orbit. I conceived that the *force* of the sun's attraction upon the planets emanates from his whole *hemisphere,* and not from his centre only; and therefore that the forces of gravitation would be represented by *cones,* of which the sun's hemisphere would be the common base, and the vertices of which cones would be those points where the respective shadows of the planets terminate. I considered this hypothesis as being more consistent with probability, at least, than the other hypothesis, which ascribes the emanation to the sun's centre only; for such I understand Newton's hypothesis to be. Taking the force of the sun's gravitation on the planets as emanating from his whole hemisphere, it follows that because cones on the same base are to each other as their altitudes (12 Euclid xiv.), the forces of gravitation would be inversely as the distances nearly; that is, rejecting the parts of the cones lying beyond the centres of the planets. But the *action* of these forces would be *directly* as the distances; and the full effect of gravitation on the planets would be in the compound ratio of the forces and their action. For the action of the greater force on the nearer planet would be *more oblique* than the action of the less force on the urther planet, and the distances express the *direct proportion* of their respective action.

In the "New Theory," I have demonstrated that taking D=the greater distance, and d = the less distance, the full effect of gravitation on the nearer planet would be as $D \times \surd d$ and on the further planet as $d \times \surd D$, and taking F = the full effect of gravitation on the nearer planet, and f = the full effect of gravitation on the further planet, $F : f :: d \times D^{\frac{1}{2}} : D \times d^{\frac{1}{2}}$, whence (multiplying extremes and means) $F \times D \times d^{\frac{1}{2}} = f \times d \times D^{\frac{1}{2}}$, and (dividing by $\surd D$ and $\surd d$) $\frac{F \times D}{\surd D} = \frac{f \times d}{\surd d}$. Wherefore $F \times D^{\frac{1}{2}} = f \times d^{\frac{1}{2}}$ *i. e.* $F : f :: d^{\frac{1}{2}} : D^{\frac{1}{2}}$, which is the expression of the analogy enunciated in the "New Theory."

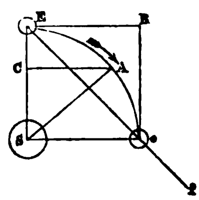
Besides the demonstrations given in the "New Theory," I have shown that the forces of the sun's gravitation on the planets, if calculated according to Newton's theory, would be much less than their centrifugal forces; although, according to his theory, these forces ought to be equal.

Now, although I conceive that the demonstrations contained in the New Theory fully establish that theory, and consequently negative the theory advanced by Newton, with which it is inconsistent, yet it seems proper not to leave the question to rest on that footing only; for, supposing Newton's theory to fail, it follows that there must be some fallacy in his demonstrations of it, and therefore a full investigation of the subject requires that his demonstrations should be examined, and the supposed fallacy pointed out, which I propose to undertake in the following Commentaries,—a difficult task, indeed, yet it may well interest the student to pursue the train of reasoning of that great philosopher. But before we enter upon this examination, it will be proper to determine the nature and quality of the centrifugal and centripetal forces; by the combination of which the planets perform their revolutions round the sun, and the satellites their revolutions round their primaries.

1st. The centrifugal force originates in one *projectile* impulse, instantaneously communicated to the planet, and impelling it with an uniform motion in, or nearly in, the direction of a tangent to some part of its orbit, and in which direction it would (by the first and second laws of motion) continue to move onward in a right line until, by some other force communicated to it, that motion was disturbed. This centrifugal force resembles in its effects the momentary impulse by which an arrow is shot from a bow, or a ball from a gun.

2ndly. The centripetal force, or gravity, is not a force originating in a momentary impulse, like the centrifugal force, but is a continually acting force, generating a succession of small impulses, always in the direction of a right line from the planet, towards the centre of the sun, in every part of the planet's orbit.

Now, a planet's orbit, considered as circular, being the resultant of the composition of the centrifugal and centripetal forces, it is obvious that these two forces must be equal to each other, from the following considerations.



We have already seen that by the composition of the two equal forces E B, E S, simultaneously applied, the body would move from E to *e* in the diagonal E *e*, in the same time that the force E B, acting alone, would carry it to B, or the force E S, acting alone, would carry it to S.

But it is obvious, that beyond the point *e*, the body would continue to move onward in the direction *e f*, that is, of the diagonal E *e*, produced.

Let E B, represent the centrifugal force, acting in the direction of the right line, E B, which is a tangent to the orbit at E, and instead of the impulsive force, E S, equal to E B, let a centripetal force be applied simultaneously with the centrifugal force at E, such centripetal force generating a succession of small impulses; and let the body impelled by both these forces move from E, to *e*, it is obvious that the sum of the successive centripetal impulses accruing between E and *e*, is equal to E S, which is equal to E B, the centrifugal force; that is, the centripetal force is equal to the centrifugal force. It is also obvious, that the result arising from the composition of the centripetal and centrifugal forces, at the end of one quarter of the planet's periodic time, will be similar at the end of the second, third, and fourth quarters; so that such a composition of these forces must produce a perpetual circular orbit.

It follows that the *versed sine* of any arc represents the sum of the centripetal impulses accruing whilst that arc is described; thus S E, is the versed sine (as S *e*, is the sine) of the arc E *e*, which versed sine, as we have seen, is equal to the sum of the centripetal impulses accruing between E, and *e*. So C E is the versed sine of the arc E A, and represents the sum of the centripetal forces accruing between E and A. Hence the centripetal force accruing in one quarter of the planet's orbit is the versed sine of an angle of 90°; hence it is equal to the planet's mean distance from the sun. But Newton, in the Principia, treats these versed sines as representing the centripetal forces at different points of the orbit. Considering the orbit as circular, it is obvious that the centripetal force must be the same in every point of the orbit. But if the orbit be elliptical, the planet will have different degrees of velocity in different points of its orbit, and the joint effect of the centripetal and centrifugal forces will be greater in one point than in another, owing to those forces acting more or less obliquely in different points of the ellipse; and it is obviously the drift of Newton's reasoning, to treat the versed sines as the measures of the centripetal forces in different points of an elliptical orbit. But this consideration is foreign to my purpose, for I am not considering what are the comparative centripetal forces of any one planet in different parts of an elliptical orbit (for these vary, from the continually *differing composition* of the centripetal and centrifugal forces, as well as from the variation of distance from the sun, and probably more so from the former cause); but I am considering only the difference between the centripetal forces of two or more, or all the planets, as occasioned by the differences of their mean distances from the sun, without regard to the consideration whether their orbits are elliptical or circular, considering that the mean distance of an elliptical orbit is on the same footing as the radius of a circular one.

In an elliptical orbit, indeed, the planet is sometimes nearer to the sun, and sometimes further from it; and the centripetal forces of the planet, in its nearer or further distance from the sun, must necessarily be in the ratio of the centripetal forces of two planets, revolving in circular orbits, one at the nearer, and the other at the further, distance from the sun. But, in an elliptical orbit, the centripetal and centrifugal forces are sometimes (that is, when the planet is approaching its perihelion) acting in augmentation of each other, and at other times (as in approaching its aphelion), in diminution one of the other; and this augmentation or diminution of the two forces is quite independent of the variation of gravitation, in respect of distance. But it is obvious, that the centripetal forces of any one planet revolving in an elliptical orbit round the sun, vary according to the concurrence of these causes, or as both causes conjunctly, in every point in its orbit; that is, first, according as the two forces are in augmentation or diminution of each other; and, secondly, accordingly to the planet's distance, for the time being, from the sun. And, therefore, if, by means of the versed sines, or otherwise, it could be determined what is the velocity or combined force acting upon the planet in different points of its elliptical orbit, the differences of velocity or force in those points, being owing to both these causes together, cannot be ascribed to one of them only, that is, to difference of distance, for this would be to ascribe to one of two causes the effect produced by both. The effect of difference of distance being involved with the effect produced by varying combinations of the two forces, cannot be ascertained without separating one cause from the other. But the effect of difference of distance is much more easily, because separately, ascertainable, from the phenomena of two or more planets; for the differences of their mean velocities or centripetal forces cannot be ascribed to any other cause than their being situated at different distances from the sun.

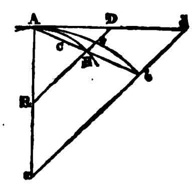
It follows from these considerations, that in the Divine work of creation it was necessary to communicate to each planet that precise centrifugal impulse which would balance its own distinct centripetal force, or its attraction towards the sun; that is, such an impulse as, acting alone, would urge the planet in a direction at right angles to a right line between the sun and the planet, in whatever position the planet might be when created, with this additional nicety of adjustment, namely, that the centrifugal force must be equal to the sum of the centripetal impulses, at the moment when the planet would have completed one quarter of its orbit,—that is, the centrifugal force must be such as, acting by itself, would move the planet in the direction aforesaid, a space equal to its distance from the sun, in one-fourth of its periodic time. At least, such must have been the case, had the orbits of the planets been perfectly circular. That their orbits are not quite circular, but only nearly so, is owing to small inequalities between the two forces at the moment of the planets' projection, so as to cause them to describe ellipses of small eccentricity, and thereby to produce greater variety of seasons, or, in some cases, to mollify them; as is the case with the earth, the inhabitable parts of which are situate principally in its northern hemisphere; and the earth being in aphelion about Midsummer, and in perihelion about the 1st of January, it is obvious that the intensity of heat in summer, and of cold in winter, is mollified in our northern latitudes, by means of the eccentricity of the earth's orbit.

I may add, in concluding this introduction, that it follows from what has been observed, that the original initiatory centrifugal impulse given to each planet, was different from the one given to each of the others, that impulse being proportioned to the force and action, combined, of each planet's gravitation to the sun; but that the centripetal forces of the gravitation of the planets are uniform and equal forces in their nature, varying only with regard to their respective distances from their common centre or focus, the sun; that these centrifugal and centripetal forces precisely balance each other, without loss or gain to either; and that each of them is incapable of increase, diminution, or change, and will for ever continue so, except by the intervention of some power capable of disturbing the equilibrium.

CHAPTER II.

COMMENTARIES ON THE FIRST SECTION, BOOK I. OF THE PRINCIPIA, TREATING OF THE METHOD OF FIRST AND LAST RATIOS OF QUANTITIES; THAT IS, OF THE RATIOS OF NASCENT AND EVANESCENT QUANTITIES.

THE sixth lemma is enunciated and demonstrated in the Principia as follows :—



"If any arc, A C B, given in position, is subtended by its chord, A B, and in any point, A, in the middle of the continued curvature, is touched by a right line, A D, produced both ways; then if the points A and B approach one another and meet, I say the angle B A D, contained between the chord and the tangent, will be diminished *in infinitum*, and ultimately will vanish.

"For if that angle does not vanish, the

arc A C B will contain, with the tangent A D, an angle equal to a rectilinear angle; and therefore the curvature at the point A will not be continued, which is against the supposition."

Comment. Now this diagram does not show in what manner the points A and B approach each other, and meet; however, the most obvious and simple manner of accomplishing this approach and meeting of the two points A and B, is to suppose that at the point R, as a fixed and immoveable centre, the radius R B moves forward towards the other radius R A, until it meets and coincides with it; so that the point B would, in approaching and at length meeting the point A, describe the arc B C A. Now it is obvious that, on this supposition, when the point B approaches A, the angle B A D, contained between the chord and the tangent, will be diminished *in infinitum,* and ultimately will vanish; that is, when the point B arrives at A.

But the reasoning of this demonstration is by no means correct. The condition of the construction, viz. that A shall be a point in the middle of the continued curvature, is unnecessary, as is also the other condition, that the right line A D is produced both ways. Indeed, the proposition requires no demonstration, for two points, when met together, cannot contain an angle. But I notice this lemma chiefly because the same construction is adopted in the seventh lemma.

The seventh lemma is enunciated and demonstrated as follows.

"The same things being supposed, I say that the ultimate ratio of the arc, chord, and tangent, any one to any other (*ad invicem*), is the ratio of equality.

"For while the point B approaches towards the point A, consider always A B and A D as produced to the remote points *b* and *d*, and parallel to the secant B D draw *b d*; and let the arc A *c b* be always similar to the arc A C B. Then, supposing the points A and B to coincide, the angle *d* A *b* will vanish, by the preceding lemma; and therefore the right lines A *b*, A *d* (which are always finite), and the intermediate arc

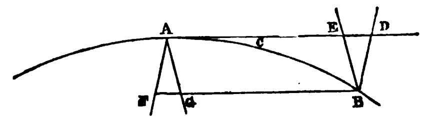
A *c* *b* will coincide, and become equal among themselves. Wherefore, the right lines A B, A D, and the intermediate arc A C B (which are always proportional to the former), will vanish, and ultimately acquire the ratio of equality. Q. E. D."

Comment. This proposition is obvious, and scarcely needs demonstration; for when the arc A C B is reduced to the point A, and the chord A B and the tangent A D are also reduced to the same point A, they are all equal, because each is=o.

But Newton's demonstration deserves notice. "While the point B approaches towards the point A," says he. This is new information, and confirms our first supposition, viz. that B moves towards A, and that A is at rest. But when he proceeds to say, "Consider always A B and A D as produced to the remote points, *b* and *d*," &c., this consideration is not requisite to the demonstration, nor is it consistent with the original construction. For while the point B approaches towards the point A, by what supposition can that approach occasion the right lines A B and A D to be produced to the remote points *b* and *d*? The large figure *r* A *d*, and the large arc and chord which it contains, do no more than exhibit, on a larger scale, the smaller figure R A D and its arc and chord; and Newton's demonstration amounts to no more than this, viz. that since the angle *d* A *b* on the larger scale would vanish, *ergo*, the angle D A B, on the smaller scale, would also vanish, and similar consequences would ensue in each case; which is perfectly true, but leaves unexplained how the approach of B to A can elongate the right lines A B, A D. The right line A R is not stated to be elongated in the demonstration, and yet it gets to *r* in the diagram, and very necessarily, for how otherwise could the triangle *d* A *b* be formed?

The first corollary to lemma seven is enunciated and demonstrated as follows:—

"*Cor.* 1. Whence, if through B we



draw B F, parallel to the tangent, always cutting any right line, A F, passing through

A in F, this line B F, will be *ultimately* in the ratio of equality with the evanescent arc A C B; because, completing the parallelogram A F B D, it is always in a ratio of equality with A D."

Comment. This corollary is equally true with the lemma from which it is deduced, and is not much better demonstrated; for it should have been made part of the construction, that as B approaches A, the right lines F B and B D continually approach their parallels A D and A F, in which case this corollary would have been very well demonstrated, for if F B be always equal to A D, in every part of their approach to each other, it is quite certain that when B coincides with A, whereby A D becomes equal to the evanescent arc A C B, F B will also be equal to the evanescent arc, A C B. But the converse of this reasoning will by no means hold good, viz. that because the evanescent quantities become equal at the vanishing point, *ergo*, they are always equal, that is, at any time during their approach; for they are only approaching to equality, and are therefore unequal before they arrive at that point; and yet we shall find that Newton's theory depends altogether on that converse reasoning.

"*Cor.* 2. And if through B and A more right lines are drawn, as B E, B D, A F, A G, cutting the tangent A D and its parallel B F; the ultimate ratio of all the abscissas A D, A E, B F, B G, and of the chord and arc A B, any one to any other will be the ratio of equality."

Comment. This corollary is equally true with the former one, but it advances a much stronger and better proposition, viz. that at the vanishing point, A E, the less, becomes equal to A D, the greater, and B G, the less, becomes equal to B F, the greater; which manifestly cannot be the case, before the points B and C coincide.

"*Cor.* 3. And therefore in all our reasoning about ultimate ratios, we may freely use any one of those lines for any other."

Comment. True, but not in our reasoning, when these lines represent existing quantities; and yet we shall see that in Prop. 4, Principia, Newton applies this lemma to existing quantities. We shall find, in our comment on Prop. 4, in the next chapter, that Newton professes to demonstrate that proposition by the aid of this lemma; and since that proposition, if established, would support Newton's theory, that the centripetal forces are inversely as the squares of the distances, this seventh lemma would seem to be the main foundation of that theory, which it was the principal object of the Principia to substantiate. Whether this seventh lemma serves to support Prop. 4 or not, we shall show in the proper place, in the next chapter.

Lemma 8 is enunciated and demonstrated in the Principia as follows :—

"Lemma 8. If the right lines A R, B R, (diagram to lemma 6) with the arc A C B, the chord A B, and the tangent A D, constitute three triangles, R A B, R A C B, R A D, and the points A and B approach and meet, I say that the ultimate form of these evanescent triangles is that of similitude, and their ultimate ratio that of equality.

"For while the point B approaches towards the point A, consider always A B, A D, A R, as produced to the remote points *b d* and *r*, and *r b d* as drawn parallel to R D, and let the arc A *c b* be always similar to the arc A C B. Then, supposing the points A and B to coincide, the angle *b* A *d* will vanish; and therefore the three triangles *r* A *b*, *r* A *c b*, *r* A *d*, (which are always finite,) will coincide, and on that account become both similar and equal. And, therefore, the triangles R A B, R A C B, R A D, which are always similar and proportional to them, will ultimately become both similar and equal among themselves. Q. E. D."

Comment. That the ultimate form of these triangles is that of similitude and their ultimate ratio that of equality is true, because since they continually approach to similitude and equality *ad infinitum* as B approaches A, they must become similar and equal when B coincides with A. But neither the construction nor the diagram illustrate the proposition; for according to the construction, by what motion or flux of the line R B, by which B approaches A and the triangle R A B becomes evanescent, would the triangle *r* A *d* be described?

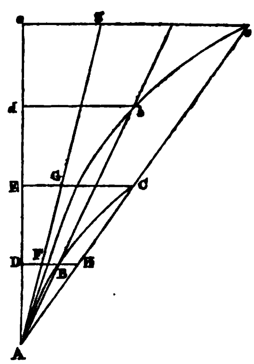
"*Cor*. And hence, in all our reasonings about ultimate ratios, we may indifferently use any one of these triangles for any other."

Comment. True, but only in the case of all the triangles being evanescent; for in no other case could the less triangle R A B be equal to the greater triangle R A D.

Lemma 9. This lemma is enunciated and demonstrated in the Principia as follows:—

"If a right line, A E, and a curve, A B C, both given in position, cut each other in a given angle, A, and to that right line in another given angle, B D and C E are ordinately applied, meeting the curve in B C; and the points B and C meet together in the point A; I say that the areas of the triangles A B D, A C E, will be ultimately one to the other in the duplicate ratio of the sides.

"For while the points B C approach to-



wards the point A, suppose always A D to be produced to the remote points *d* and *e*, so as A *d*, A *e* may be proportional to A D, A E; and the ordinates *d b*, *e c*, to be drawn parallel to the ordinates D B and E C, and meeting A B and A C, produced in *b* and *c*. Let the curve A *b c* be similar to the curve A B C, and draw the right-line A *g* so as to touch both curves in A, and cut the ordinates D B, E C, *d b*, *e c*, in F, G, *f*, *g*. Then supposing the length A *e* to remain the same, let the points B and C meet in the point A; and the angle *c* A *g* vanishing, the curvilineal areas, A *b d*, A *c e*, will coincide with the rectilineal areas A *f d*, A *g e*; and therefore (by lemma 5) will be one to the other in the duplicate ratio of their sides A *d*, A *e*; but the areas A B D, A C E are always proportional to these areas, and so the sides A D, A E are to these sides, and therefore the areas A B D, A C E are ultimately one to the other in the duplicate ratio of the sides A D, A E. Q. E. D."

Comment. The hypothesis in this lemma is very peculiar; for the points B and C are supposed to approach the point A, in the curvilinear paths B A and C A, with velocities proportional to the lengths of those curves, so that they may meet or coincide simultaneously in the point A; and with them the ordinates move parallel to each other until they coincide in the point A; but as the points B and C move towards A, the right lines B A and C A not only diminish in length, but they also vary their position or direction every instant. Now it does not in this case, as in the former ones, involve any inconsistency, that while the ordinates D B and C E approach towards A, the right line A D may be produced to the remote points *d* and *e*, so that A *d*, A *e*, may be proportional to A D, A E, and the ordinates *d b*, *e c*, may be drawn as directed in the demonstration. We may therefore admit this construction. But there seems to be no necessity for drawing the curve A *b* *c*; for the right line A *g*, drawn so as to touch the curve A B C, would cut all the ordinates in the points F, G, *f*, *g*. Then, supposing the length A *e* to remain the same, the angle *c* A *g* would vanish when the points B and C meet in A.

Now it is obvious that the triangles D A B and E A C are dissimilar, as also the triangles *d* A *b* and *e* A *c*, because they have not a common angle at A; hence these triangles are not in the duplicate ratio of their sides D A, E A, *d* A, *e* A, although those sides are homologous. It is also obvious that the triangles A D B and A *d b* are similar, as also are the triangles A E C and A *e c*; wherefore they are respectively in the duplicate ratio of their homologous sides; that is, A D B : A *d b* :: A D *qu.* : A *d qu.* and A E C : A *e c* :: A E *qu.* : A *e qu.*

Now when the angle *c* A *g* vanishes, it might be said that the rectilinear areas, or triangles, A *b d*, A *c e*, will coincide with (or rather will be reduced to) the rectilinear areas, or triangles, A *f d*, A *g e*; and because those remaining triangles are obviously similar (that is, A *f d* is similar to A *g e*), *ergo*, the parts being similar, the wholes are similar, that is, the whole A *b d* is similar to the whole A *c e*. This seems to be the detailed reasoning of Newton's demonstration, which in the text omits some of the connecting links in the process. It follows, if we admit the dissimilar triangles A *d b*, A *e c*, become similar, by the vanishing of the evanescent triangle A *g c*, that the rectilinear areas, A *d b*, A *e c*, will be in the duplicate ratio of the sides A *d*, A *e*, and that the areas A B D, A E C, similar to the areas A *d b*, A *e c*, would ultimately be one to the other in the duplicate ratio of the sides A D, A E.

This demonstration, however, proves no more than that if from two dissimilar triangles the dissimilar parts be taken, the remaining parts will be similar. Besides, in the enunciation it is said that the triangles A B D, A C E will be *ultimately* one to the other in the duplicate ratio of the sides; that is, when the triangles are wholly vanished; which may be admitted;

because when they become = o, the triangles become both equal and similar, and therefore by 19 Euclid, book vi., they are to one another in the duplicate ratio of their homologous sides A D, A E. But the demonstration here purports to show that the triangles become similar to each other before they wholly vanish (that is, when the parts comprised in the triangle *g* A *e* remain); which only amounts to this, that two dissimilar triangles will become, or may be made, similar to each other, by taking away their dissimilar parts; which seems a truism; but the question occurs, what useful inference can be drawn from it?

The introduction of the comparison between the curvilinear and rectilinear areas in this demonstration is irrelevant, and only serves to divert the mind from the main point in question.

The ninth lemma might be shortly demonstrated as follows:—since the two triangles (by lemma 8) ultimately become similar, and consequently the approximating sides homologous, and (6 Euclid 19) because similar triangles are to one another in the duplicate ratio of their homologous sides; the evanescent triangles are in the duplicate ratio of the homologous sides; and this seems the proper demonstration of this lemma. But the demonstration only proves the proposition to be true in the particular case when the two triangles A B D and A C E have vanished.

Indeed, it might be admitted, without any demonstration, that as the points C and B approach to A, the triangles B A D and C A E continually approach towards similarity, and therefore when they coincide in A, they become altogether similar, and are to each other in the duplicate ratio of their homologous sides. But the question recurs, what useful inference can be drawn from this lemma, which amounts to no more than that O is in the duplicate ratio of O? So that the objection to this lemma is that it is useless, not that it is false. How different is this from the case of an evanescent fluxion, such as $a\dot{x} - 2x\dot{x} = 0$ (1 Simpson's Fluxions, page 15); whence we deduce the important inference, $a\dot{x} =$

$2x\dot{x}$, and consequently $x = \frac{1}{2}a$; that is, the unknown quantity x is found (by means of the evanescence of the fluxion) to be equal to half of a, the known quantity. But we shall find, in the next chapter, that Newton ascribes the properties of nascent and evanescent quantities to the quantities *after* their beginning, and *before* their evanescence.

The tenth lemma is enunciated and demonstrated as follows:—

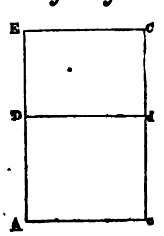
"The spaces which a body describes by any finite force urging it, whether that force is determined and immutable, or is *continually* augmented, or *continually* diminished, are, in the very beginning of the motion, one to the other in the duplicate ratio of the times.

"Let the times be represented by the lines A D, A E, (see the last diagram,) and the velocities generated in those times by the ordinates D B, E C. The spaces described with these velocities will be as the areas A B D, A C E, described by those ordinates, that is, at the very beginning of the motion (by lemma 9), in the duplicate ratio of the times A D, A E.—Q. E. D."

Comment. The proposition advanced, in effect, in this tenth lemma, is that when a motion generated by a force is either, 1st, a constant motion; or, 2ndly, an uniformly accelerated motion; or, 3rdly, an uniformly retarded motion; the spaces described are, in the beginning of the motion, in the duplicate ratio of the times.

First *case.* Let the motion generated be a constant and uniform motion.

Now, in this case, the times may be represented as Newton requires, by the lines A D and A E; but since (by the hypothesis) the velocity, that is, the motion, is uniform or constant, it cannot be represented by the two ordinates D B, E C, (for these ordinates are different and unequal,) but only by one of them. Let this uniform velocity be represented by the greater ordinate E C, and let A E be the double of A D; that is, let the time represented by A E, be twice the time represented by A D: in this case, the rectangle A E C *e*, will represent the space described in the double time A E,

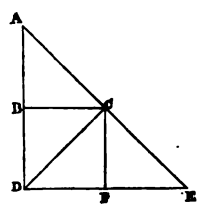


with the uniform velocity E C; and the rectangle A D *d e* will represent the space described in the single time A D, with that uniform velocity. But parallelograms of the same altitude are as their bases (6 Euclid 1); wherefore the parallelogram A E C *e*, is to the parallelogram A D *d e*, as the base A E, to the base A D; that is, in the simple and not in the *duplicate* ratio of the times, in this first case of an uniform motion. Now, even upon the principle laid down in the corollaries to lemmas 7 and 8 of ultimate ratios, the ratio of the spaces could not be said to be the duplicate of that of the times, in the very beginning of this uniform motion, unless, at the instant afterwards, there would subsist a very close approximation to that duplicate ratio, which would become less and less close, as the motion proceeded. But in this case, the ratio of the spaces to the times is always simple in every part of the motion, however near the beginning, or distant from it; there is, therefore, no such approximation; the want of which leaves the tenth lemma without foundation in the first case.

This lemma, which does not hold good in the case of an uniform motion, is, nevertheless, true in the case of an uniformly accelerated motion, and is true inversely of an uniformly retarded motion; that is, it is true, both in the very beginning of the motion, and during the whole of its continuance. But Newton's demonstration seems rather to disprove his lemma in both these cases. For, taking the *times* to be represented by the lines A D and A E, and the velocities by the ordinates D B and E C, and the spaces described with these velocities to be as the rectilinear areas A B D and A C E, it is obvious that the spaces described will *not* be in the duplicate ratio of the times; for the rectilinear areas A B D and A C E, which represent the spaces, are not in the duplicate ratio of A D and A E, which represent the times; because these areas are dissimilar, the angle C A E being greater than the angle B A D, and the angle A B D being greater than the angle A C E. However, we may easily demonstrate the truth of this lemma, in the two cases of uniformly accelerated and retarded motions.

Second *case* ; viz., of an uniformly accelerated motion.

If a motion be uniformly accelerated, then, whatever be its velocity at the end of one portion of time, its velocity will become doubled at the end of twice that time. Let the right line A B, represent the first time, at the end of which the velocity B C, has accrued; and let the line A D=2 A B, represent twice the first time, at the end of which second time, the double velocity D E=2 B C, has accrued.



It is obvious that the area of the whole triangle A D E, is four times the area of the triangle A B C, which is the duplicate ratio of A D, to A B; and trebling the time, the space would become nine-fold; and quadrupling the time, the spaces would be as 16 to 1; and so on in the duplicate ratio of the times, in this case of an uniformly accelerated motion.

And so the third *case*, (viz. that of an uniformly retarded motion,) might be demonstrated, merely by inverting this process; only it is obvious, in this third case, that the ratio is inverse.

I beg to remind the learned reader that the second case is that of the gravitation of heavy bodies, falling freely from a state of rest; and that the third case is that of heavy bodies projected upwards, perpendicularly to the plane of the horizon. See a "New Treatise on Mechanics." Whittaker and Co., 1841.

Now in the second case, that of an uniformly accelerated motion, we have seen that in every part of the motion, the spaces are in the duplicate ratio of the times, and that this duplicate ratio exists without variation or any tendency to vary; there is no approximation to any other ratio at the nascent point of the motion, and there cannot be a point of evanescence, because the accelerated motion will be at its maximum whenever it stops. Therefore, to affirm that in this second case the spaces are in the duplicate ratio of the times in the very beginning of the motion, is only attributing to the motion at its nascent point the properties which belong to it ever afterwards, and brings forward no new property. The tenth lemma, therefore, as applicable to the second case, seems to be a mere truism without affording any useful inference.

The third case, on the other hand, seems to imply a solecism; for a nascent point means a point = o; but this third case supposes a diminishing motion, that is, a motion growing continually less and less than o; which is absurd.

The eleventh, which is the concluding lemma in this section, does not affect the question in controversy, and therefore does not demand any comment.

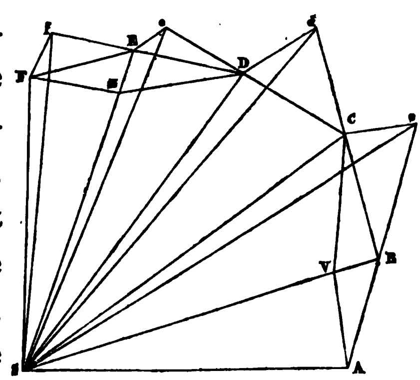
On the whole, Newton does not seem to have formed any complete scheme or system in which the properties ascribed to nascent and evanescent quantities can be applied to practical purposes. The seventh and eighth lemmas show that whatever quantities are unequal or dissimilar while they are in esse, they become equal and similar when evanescent; which is called their ultimate ratio, because they continually approximate to this ratio as they approach the point of evanescence. The ninth lemma was intended to demonstrate in effect that the *ultimate* ratio of dissimilar triangles is the duplicate ratio of their sides, and the tenth lemma to demonstrate the same of the *first* ratio. From the seventh and eighth lemmas another step seems to be gained towards the formation of a system, by enabling us in our reasonings of ultimate ratios to substitute any quantity denoted by a right line for any other quantity denoted by another right line, and any one triangle for any other, as stated in the last corollaries to those lemmas; which seems to be a very bold, not to say a startling, deduction. Hence we might have inferred (but which is not the case) that a more complete method or system of ultimate ratios was designed to be constructed as a *norma*, or instrument for demonstrating the propositions in the Principia, such as the method of Fluxions. At least we might have expected that Newton would have explained in what manner the properties of evanescent quantities are connected with those of existing quantities, or how they could illustrate the latter. But we had no reason to infer that the properties of *evanescent* quantities were intended without any such explanation to be applied to *existing* quantities; as we shall find to be the case in Proposition 4, where the seventh lemma forms one link in the chain of demonstration, upon which Newton's Theory of the force of gravitation altogether depends.

CHAPTER III.

COMMENTARIES ON THE SECOND SECTION OF THE FIRST BOOK OF THE PRINCIPIA, TREATING OF THE INVENTION OF CENTRIPETAL FORCES.

THE first Proposition in the Principia is enunciated and demonstrated as follows:—

"The areas which revolving bodies describe by radii drawn to an immoveable centre of force lie in the same immoveable planes, and are proportional to the times in which they are described.



"For suppose the time to be divided into equal parts, and in the first part of that time let the body, by its innate force, describe the right line A B. In the second part of that time, the same would (by law 1) if not hindered, proceed directly to *c*, along the line B *c*, equal to A B; so that by the radii A S, B S, *c* S, drawn to the centre, the equal areas A B S, B S *c*, would be described. But when the body is arrived at B, suppose that a centripetal force acts at once with a great impulse, and turning aside the body from the right line B *c*, compels it afterwards to continue its motion along the right line B C. Draw *c* C parallel to B S, meeting B C in C; and at the end of the second part of the time, the body (by Cor. 1 of the laws) will be found in C, in the same plane with the triangle A S B. Join S C, and because S B and C *c*, are parallel, the triangle S B C will be equal to the triangle S B *c*, and therefore also to the triangle S A B. By the like argument, if the centripetal force acts successively in C, D, E, &c., and makes the body in each single particle of time to describe the right lines C D, D E, E F, &c. they will all lie in the same plane; and the triangle S C D will be equal to the triangle S B C, and S D E to S C D, and S E F to S D E. And therefore, in equal times, equal areas are described in one immoveable plane: and by composition any sums, S A D S, S A F S, of those areas, are one to the other as the times in which they are described. Now let the number of those triangles be augmented, and their breadth diminished in infinitum; and (by Cor. 4, lem. 3) their ultimate perimeter A D F, will be a curve line; and therefore the centripetal force by which the body is perpetually drawn back from the tangent of this curve will act continually; and any described areas S A D S, S A F S, which are always proportional to the times of description, will, in this case also, be proportional to those times. Q. E. D."

Comment. In this admirable proposition Newton demonstrates Kepler's other analogy, viz. that the planets describe equal areas in equal times.

As this first proposition includes the two cases of a circular and an elliptical orbit, it was requisite in strictness to demonstrate each case; and, adopting Newton's demonstration, we must consider the figure A B C D E F S, as representing an approximation to a segment either of a circle or of an ellipse: which we shall not find it easy to do when we come to examine the Corollaries deduced by Newton from this proposition.

"*Cor.* 1. The velocity of a body attracted towards an immoveable centre, in spaces void of resistance, is reciprocally as the perpendicular let fall from that centre on the right line that touches the orbit. For the velocities in those places A, B, C, D, E, are as the bases A B, B C, C D, D E, E F, of equal triangles; and these bases are reciprocally as the perpendiculars let fall upon them."

Comment. This Corollary does not seem to apply to the case of a circular orbit; for in that case the velocity is the same in every point of the orbit; and the radii being all equal, the velocities are *directly* as the radii. But it is true with respect to an elliptical orbit; for the bases A B, B C, C D, D E, and E F, being supposed indefinitely small, may be taken to represent the velocities at the points A, B, C, D, and E; and the triangles being all equal to each other, it is obvious that the bases of these equal triangles are inversely as their altitudes; which, I presume, is what is meant by saying that they are reciprocally as the perpendiculars let fall upon them. For, according to the figure, a perpendicular from the point S, could not be let fall scarcely upon any of the bases. I suspect, however, that Newton means that the bases are reciprocally as the radii or distances A S, B S, C S, D S, E S, and F S, which may probably be true, but ought to have been demonstrated, and without demonstration ought not to be taken for granted. But in whatever sense we construe this Corollary we are carefully to distinguish the analogy between the velocities of one and the same planet moving in an elliptical orbit, and its distances from the sun in different parts of its orbit, from the analogy between the velocities and distances from the sun of different planets, which latter analogy, according to Newton himself (as we shall see), is that their velocities are reciprocally (not as their mean distances but) as the *square-roots* of their mean distances from the sun.

Corollaries 2 and 3 to Proposition 1, do not affect the point in question.

"*Cor.* 4. The forces by which bodies, in spaces void of resistance, are drawn back from rectilinear motions, and turned into curvilinear orbits, are one to another as the versed sines of arcs described in equal times; which versed sines tend to the centre of force, and bisect the chords when those arcs are diminished to infinity. For such versed sines are the halves of the diagonals mentioned in Corollary 3."

Comment. That the centripetal forces of the planets are to one another as the versed sines of arcs described in equal times, is true; and we shall demonstrate this part of Corollary 4, in our commentaries on the fourth Proposition of the Principia.

"Proposition 4. Theorem 4. The centripetal forces of bodies, which, by equable motions, describe different circles, tend to the centres of the same circles; and are one to the other as the squares of the arcs described in equal times, applied to the radii of the circles.

"These forces tend to the centres of the circles (by Prop. ii. and cor. 2, Prop. i.), and are one to another as the versed sines of the least arcs described in equal times (by cor. 4, Prop. i.); *that is*[1], *as the squares of the same arcs applied to the diameters of the circles* (*by lem.* 7)*;* and therefore, since those arcs are as arcs described in any equal times, and the diameters are as the radii, the forces will be as the squares of any arcs described in the same time applied to the radii of the circles. Q. E. D."

[1] I consider the part in italics to be the main fallacy of the reasoning in the Principia; Corollary i. does no more than exhibit Proposition 4 in another form; and the part in italics of Corollary i. is therefore only a different version of that fallacy.

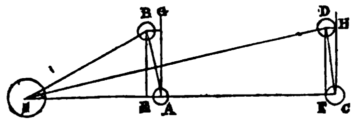
"*Cor.* 1. Therefore, since those arcs are as the velocities of the bodies, *the centripetal forces are in a ratio compounded of the duplicate ratio of the velocities directly, and of the simple ratio of the radii inversely.*"

Comment. The second branch of Proposition 4—"and are one to the other as the squares of the arcs described in equal times, *applied to the radii of the circles*"—is explained by this first Corollary, as meaning the ratio compounded of the duplicate ratio of the arcs directly, *and of the simple ratio of* the radii inversely; a meaning very obscurely indicated in the enunciation.

The first branch of the Proposition (viz., that the centripetal force *tends* to the centre), is true, although it does not thence follow that the whole of that force *emanates* from the centre only; it is obvious that the tendency would be to the centre, although the emanation proceeded from the whole hemisphere.

This tendency of gravity to the centre most probably gave rise to the notion, that the force of gravity emanates from the centre only, as rays of light; and therefore that gravity possesses the same property that light is known to possess, namely, that its intensity is inversely as the *squares* of the distances. This notion (which is either a gratuitous assumption or an illogical inference) is evidently the basis of Newton's theory, and is the principle which pervades the Principia.

The Proposition advanced in the second branch of this demonstration is, that the forces are as the versed sines of the *least* arcs described in equal times; which Proposition, although limited (unnecessarily) to least arcs, is true of all arcs, and is indeed enunciated generally of all arcs, in the fourth Corollary to Proposition I. It may be demonstrated as follows:—



Let A be the nearer, and C the further planet from the sun, and in the same time that A moves to B, let C move to D: draw B E, perpendicular upon S A, and D F, perpendicular upon S C, and G A, H C, parallel to B E, D F; also draw B G parallel to A E, and D H parallel to F C. It is obvious that A E represents the whole centripetal force, which urged the nearer planet during that time, and that C F represents the whole centripetal force which urged the further planet during the same time; wherefore the centripetal forces of the planets are as A E to C F, which are the versed sines of the arcs A B, C D. It is obvious, therefore, that the forces are as the versed sines, not merely of very small arcs, but of all arcs whatever, described by any two or more planets, in equal times.

Now this analogy between the forces and the versed sines of the arcs is absolute, and is totally independent of, and unaffected by, the difference of the distances of the planets from the sun; for the versed sines E A and F C represent the centripetal forces, whatever may be the lengths of the two arcs A B and D C, simultaneously described, or whatever may be the magnitudes of the angles which those arcs measure.

Newton then proceeds to infer from

Lemma 7, that the versed sines are as the squares of the same arcs applied to the diameters of the circles, that is, as he explains it, in a ratio compounded of the duplicate ratio of the arcs directly and of the simple ratio of the diameters inversely. Now the Proposition advanced in the seventh Lemma is this, that the *ultimate* ratio of the arc, chord, and tangent, any one to any other, is the ratio of equality; and in the third Corollary to that Lemma it is laid down, that in all our reasoning about ultimate ratios we may freely use any one of those lines for any other, in which lines the versed sines are included. But this is predicated *there* only of *evanescent* quantities, but *here* it is attempted to be applied to *existing* quantities of any magnitude; and even allowing it to be so applied, then by the seventh Lemma the versed sines would be equal, and therefore the forces would be equal in all cases; which is absurd, because this Proposition assumes, in accordance with the fact, that the centripetal forces of the planets are unequal, and this fourth Proposition is advanced for the purpose of demonstrating the ratio or analogy of these unequal forces.

It follows, therefore, that the inference from Lemma 7, that the versed sines are in a ratio compounded of the duplicate ratio of the arcs directly, and of the simple ratio of the diameters inversely, not being demonstrated by, or deducible from, that Lemma; but, on the other hand, being, in the instance just noticed, inconsistent with it, the enunciation of that ratio stands wholly unvouched, and is merely a gratuitous assumption; and as this assumed ratio would, if it were well founded, sustain Newton's Theory, that the centripetal forces are inversely as the squares of the distances (as we shall show in the sequel), it follows, that Proposition 4 is a petitio principis, or a taking for granted of the whole principle upon which the theory in question depends. Here, therefore, we discover the fallacy of which we have been in search; namely, the want of demonstration of Proposition 4, which is the basis of Newton's Theory. Had he demonstrated this

Proposition, or had it been capable of demonstration, it would have better deserved the character of *dignity* than Proposition 11, in the Principia, upon which he bestows it; although that Proposition involves, comparatively speaking, a very narrow and confined principle, namely, one which regulates the motions of any one planet in an elliptical orbit; which is much less important than the analogy of the centripetal forces of the planets between one another.

By adopting the following symbols, we may express algebraically the Proposition advanced in Corol. 1 to this Proposition :—

Let F, represent the centripetal force of the nearer of any two planets;

V, its velocity,

D, its mean distance from the sun;

and let *f*, *v*, and *d*, represent the centripetal force, velocity, and distance of the further planet; then, according to Corol. 1,

$$F : f :: V^2 \times \frac{1}{D} : v^2 \times \frac{1}{d} :: \frac{V^2}{D} : \frac{v^2}{d}$$

We shall show in our commentary on the sixth Corollary to this Proposition, that this analogy is altogether erroneous; and, on the contrary, that

$$F : f :: V : v; \text{ and that } F : f :: \frac{1}{\sqrt{D}} : \frac{1}{\sqrt{d}}$$

"*Cor.* 2. And since the periodic times are in a ratio compounded of the ratio of the radii directly, and the ratio of the velocities inversely, the centripetal forces are in a ratio compounded of the ratio of the radii directly, and the duplicate ratio of the periodic times inversely."

Comment. The first branch of this Corollary is expressed algebraically thus:—

$$T : t :: D \times \frac{1}{V} : d \times \frac{1}{v} :: \frac{D}{V} : \frac{d}{v};$$

and correctly expresses the analogy of the periodic times to the distances and velocities of the planets. It is indeed an important Proposition, and was well deserving of being demonstrated, which is not done in the Principia. It may be demonstrated as follows:—

By Kepler's Analogy,

$$T^2 : t^2 :: D^3 : d^3;$$

and dividing the last two terms by D and *d*,

$$T^2 : t^2 :: \frac{D^3}{d} : \frac{d^3}{D} =:: D^3 \times \frac{1}{d} : d^3 \times \frac{1}{D};$$

and extracting the square-root;

$$T : t :: D \times \frac{1}{\sqrt{d}} : d \times \frac{1}{\sqrt{D}};$$

and (by the "New Analogy," demonstrated in our commentary on Corollary 6)

$$V : v :: d^{\frac{1}{2}} : D^{\frac{1}{2}},$$

and by substituting V and v

$$T : t :: D \times \frac{1}{V} : d \times \frac{1}{v} :: \frac{D}{V} : \frac{d}{v};$$

Q. E. D.

This true analogy Newton engrafts upon that expressed in the first Corol. to Prop. 4, and deduces from the two together the main proposition (that is, the one secondly) advanced in Corol. 2, which is expressed algebraically thus:—

$$F : f :: D \times \frac{1}{T^2} : d \times \frac{1}{t^2} :: \frac{D}{T^2} : \frac{d}{t^2}$$

which analogy is correctly deduced from the demonstrated analogy, $T : t :: \frac{D}{V} : \frac{d}{v}$, and from the analogy, $F : f :: \frac{V^2}{D} : \frac{v^2}{d}$, enunciated in the first Corol.; for, *by inversion*, the demonstrated analogy becomes $\frac{1}{T} : \frac{1}{t} :: \frac{V}{D} : \frac{v}{d}$, and, by squaring all the terms, it becomes—

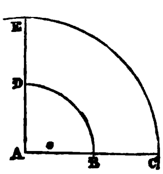
$\frac{1}{T^2} : \frac{1}{t^2} :: \frac{V^2}{D^2} : \frac{v^2}{d^2}$, and by dividing each of the antecedents by $\frac{1}{D}$, and each of the consequents by $\frac{1}{d}$, the analogy becomes—

$\frac{D}{T^2} : \frac{d}{t^2} :: \frac{V^2}{D} : \frac{v^2}{d}$, which, by the first Corollary, is the ratio of the forces; that is—

$F : f :: \frac{D}{T^2} : \frac{d}{t^2}$.

But it is obvious that this last analogy must fail if Proposition 4 cannot be supported.

"*Cor.* 3. Whence if the periodic times are equal, and the velocities therefore as the radii, the centripetal forces will be also as the radii."



Comment. This Corollary is obvious; for let the distance or radius A C be double the distance or radius A B; then since the periodic times are equal, while the nearer planet describes the arc B D, the further planet will describe the arc C E; let these arcs be each a quarter of their respective circumferences, and because circumferences of circles are as their radii, the quarters of the circumferences are as the radii; and are also as the velocities, for they are the measures of the velocities; and are likewise as the centripetal forces, as demonstrated in the "Exposition." "*And the contrary*," here means *and conversely*. This hypothetical Corollary is not an inference from Proposition 4, but is independent of it, as appears from the above demonstration.

"*Cor.* 4. If the periodic times and the velocities are both in the subduplicate ratio of the radii, the centripetal forces will be equal among themselves, and the contrary."

Comment. Subduplicate ratio being the ratio of the square-roots of quantities; this Corollary is expressed algebraically as follows:—if $T : t :: \sqrt{D} : \sqrt{d}$, and if $V : v :: \sqrt{D} : \sqrt{d}$; then $F = f$. We may easily demonstrate this Corollary according to Newton's Theory as follows:—

$$\text{By Corol. 1. } F : f :: \frac{V^2}{D} : \frac{v^2}{d}; \quad . \quad . \quad . \quad A.$$

$$\text{and by Corol. 2. } F : f :: \frac{D}{T^2} : \frac{d}{t^2}; \quad . \quad . \quad . \quad B.$$

And by the hypothesis—

$$\text{T} : t :: \text{D}^{\frac{1}{2}} : d^{\frac{1}{2}};$$

$$\text{and } \text{V} : v :: \text{D}^{\frac{1}{2}} : d^{\frac{1}{2}};$$

$$\text{wherefore } \text{V}^2 : v^2 : \text{D} : d,$$

and substituting V^2, v^2 for D, d in expression A

$$\text{F} : f :: \frac{\text{V}^2}{\text{V}^2} : \frac{v^2}{v^2} :: 1 : 1;$$

$$\text{wherefore } \text{F} = f;$$

$$\text{Again, because } \text{T} : t :: \sqrt{\text{D}} : \sqrt{d},$$

$$\text{T}^2 : t^2 :: \text{D} : d,$$

and substituting D, d, for T^2, t^2 in expression B,

$$\text{F} : f :: \frac{\text{D}}{\text{D}} : \frac{d}{d} :: 1 : 1;$$

$$\text{wherefore } \text{F} = f.$$

The rationale of this Corollary is this :—the ratio of the times is inverse to that of the velocities, and therefore the ratios of the times and velocities cannot both be the same with that of any one other ratio, *e. g.* the subduplicate ratio of the radii, except in the case when the times are equal to each other, and the velocities are also equal to each other, and consequently when the square roots of the radii (that is, the radii themselves) are equal to each other.

This Corollary is obviously an hypothesis totally inconsistent with the phenomena of the planetary motions.

"*Cor.* 5. If the periodic times are as the radii, and therefore the velocities equal, the centripetal forces will be reciprocally as the radii; and the contrary."

Comment. This Corollary may be expressed algebraically thus: if $T : t :: D : d$; then $V = v$, and $F : f : \frac{1}{D} : \frac{1}{d}$; and may be demonstrated according to Newton's theory as follows,—let the radius of the nearer planet be half that of the further one, the periodic time of the nearer one will, by the hypothesis, be half that of the further one; and because the circumferences of circles are as their radii, the orbit of the nearer planet will be equal to half that of the further one; and in the time that the nearer planet describes its orbit the further planet will describe half of its orbit, that is, they will describe equal spaces in equal times; hence, their velocities are equal, that is, $V = v$, and $V^2 = v^2$, and by the first Corollary $F : f :: \frac{V^2}{D} : \frac{v^2}{d}$; and dividing the last two terms by the equals V^2, v^2, $F : f :: \frac{1}{D} : \frac{1}{d}$ Q. E. D.

This Corollary is another hypothesis, and is also inconsistent with the planetary motions.

"*Cor.* 6. If the periodic times are in the sesquiplicate ratio of the radii, and therefore the velocities reciprocally in the subduplicate ratio of the radii, the centripetal forces will be in the duplicate ratio of the radii inversely; and the contrary."

Comment. In this Corollary Kepler's analogy is put hypothetically, which is expressed algebraically thus:—$T : t :: D^{\frac{3}{2}} : d^{\frac{3}{2}}$, (for sesquiplicate ratio means the ratio of the square roots of the cubes of quantities), from which the inferences are—

$$\text{1st.} \quad V : v :: \frac{1}{D^{\frac{1}{2}}} : \frac{1}{d^{\frac{1}{2}}}, \text{ and}$$

$$\text{2ndly,} \quad F : f :: \frac{1}{D^2} : \frac{1}{d^2};$$

but neither of these analogies is demonstrated in the Principia, but they are both left for the reader to infer them, or to demonstrate them for himself.

The first inference may be demonstrated from Kepler's analogy, as follows:—

$$T : t :: D^{\frac{3}{2}} : d^{\frac{3}{2}};$$

And since the velocity of every planet is the circumference of its orbit divided by its periodic time, and the circumference is as the diameter, that is, as the radius, or mean distance from the sun,

$$V : v :: \frac{D}{T} : \frac{d}{t};$$

and substituting $D^{\frac{3}{2}} : d^{\frac{3}{2}}$, for $T : t$,

$$V : v :: \frac{D}{D^{\frac{3}{2}}} : \frac{d}{d^{\frac{3}{2}}};$$

but $D^{\frac{3}{2}} = D^{1\frac{1}{2}} = D \times D^{\frac{1}{2}}$ and $d^{\frac{3}{2}} = d^{1\frac{1}{2}} = d \times d^{\frac{1}{2}}$, and substituting these equals, $V : v :: \frac{D}{D \times D^{\frac{1}{2}}} : \frac{d}{d \times d_{\frac{1}{2}}} :: \frac{1}{\sqrt{D}} : \frac{1}{\sqrt{d}}$.

Q. E. D.

Thus we derive from Kepler's Analogy, by direct demonstration, the other important analogy, that the velocities of the planets are inversely as the square roots of their mean distances from the sun; which analogy is strictly conformable to the ascertained phenomena of these celestial bodies, as we have amply shown in the "New Analogy."

The second proposition advanced in Corollary 6, viz., $F : f :: \frac{1}{D^2} : \frac{1}{d^2}$, is the theory which it appears to have been the main scope and object of the Principia to establish; and if that Corollary would hold good, this theory might be demonstrated from it, and from the first part of this sixth Corollary, (which, as we have seen, is deduced from Kepler's *good* analogy), as follows:—

$$\text{By Cor. 1, } \mathrm{F} : f :: \frac{\mathrm{V}^2}{\mathrm{D}} : \frac{v^2}{d}; \quad . \ . \ . \quad \mathrm{A}.$$

$$\text{and by Cor. 6, } \mathrm{V} : v :: \frac{1}{\sqrt{\mathrm{D}}} : \frac{1}{\sqrt{d}};$$

wherefore, squaring all the terms,

$$\mathrm{V}^2 : v^2 :: \frac{1}{\mathrm{D}} : \frac{1}{d};$$

and substituting $\frac{1}{\mathrm{D}} : \frac{1}{d}$, for $\mathrm{V}^2 : v^2$, in the expression A,

$$\mathrm{F} : f :: \frac{\left(\frac{1}{\mathrm{D}}\right)}{\mathrm{D}} : \frac{\left(\frac{1}{d}\right)}{d} :: \frac{1}{\mathrm{D}^2} : \frac{1}{d^2};$$

which is Newton's theory, thus deduced from the *fallacious* analogy assumed in Prop. 4, combined with the *true* analogy, $\mathrm{V} : v :: \frac{1}{\sqrt{\mathrm{D}}} : \frac{1}{\sqrt{d}}$; which is deduced from Kepler's *good* Analogy. It is obvious that the fallacy of *part* of the premises vitiates the theory thus deduced from the bad and good together.

Now it is obvious, that if Newton assumed, *à priori*, his long preconceived theory to be true, he might have deduced analytically from it combined with Kepler's *true* Analogy, the analogy enunciated in Cor. 1 to Prop. 4, namely, $F : f :: \frac{V^2}{D} : \frac{v^2}{d}$; as follows:—

$F : f :: \frac{1}{D^2} : \frac{1}{d^2}$, by Newton's preconceived Theory;

and $V : v :: \frac{1}{\sqrt{D}} : \frac{1}{\sqrt{d}}$; as deduced from Kepler's Analogy;

and squaring all the terms,

$$V^2 : v^2 :: \frac{1}{D} : \frac{1}{d},$$

$$\text{and } F : f :: \frac{1}{D} \times \frac{1}{D} : \frac{1}{d} \times \frac{1}{d},$$

and substituting V^2 for $\frac{1}{D}$ and v^2 for $\frac{1}{d}$,

$$F : f :: \frac{1}{D} \times V^2 : \frac{1}{d} \times v^2$$

$$\text{that is } F : f :: \frac{V^2}{D} : \frac{v^2}{d};$$

Q. E. D.

which is the expression of Cor. 1, Prop. 4.

By adopting this method of assuming the truth of the theory, and then deducing from it and from the above analogies the analogy enunciated in the first Corollary, it is obvious that the theory might, *vice versâ*, be demonstrated from the first Corollary, if that Corollary could be demonstrated *aliunde*.

By this analytical method of deducing the consequences of his preconceived theory, he could discover what he had to demonstrate, in order to substantiate a ground-work for his theory; for whatever consequences would necessarily follow from the assumed theory, then, by inverting the process, the assumed theory would become demonstrated from its own consequences; only he was forced to look *elsewhere* for his demonstration of these consequences, thus become antecedents. Newton's method, therefore, reduced him to the necessity of demonstrating *in limine* the *consequences* of his theory, to serve as the *foundation* of it. His ingenuity led him to adopt the system of nascent and evanescent quantities, as being all similar in form, equal in magnitude, and promiscuously convertible one with the other. This system has two faults; each fatal to it; first, that the properties of nascent and evanescent quantities do not belong to *existing* quantities, to which he nevertheless applies them; and secondly, (if the first objection could be overcome), that they are indefinite or universal. In Corollary 3 to Lemma 7, Newton says, " In all our reasoning about ultimate ratios, we may freely use any one of these lines for any other." But if we use any one line, do we exclude the others? Again, in the Corollary to Lemma 8, he says, " In all our reasonings about ultimate ratios, we may indifferently use any one of these triangles for any other." If we use one particular triangle, is that one so used, therefore appropriated so as to exclude the rest, any one of which might have been used instead of the one which was used?

However, these nascent and evanescent quantities would answer Newton's purpose, by the short process of transferring and attributing their properties to the *existing* quantities of which he was treating. He accordingly adopted them as the means of demonstrating the *consequences* of his theory. That he employed this inverted method there can be no doubt. For how could Newton have known that it was necessary for him to demonstrate the analogy that the versed sines are in a ratio compounded of the duplicate ratio of the arcs directly, and of the simple ratio of the radii inversely, as the basis of his theory, unless he had previously deduced that analogy analytically from his theory?

And how does he demonstrate this deduced analogy? Not by geometrical reasoning; not by reference to any proposition in which that analogy had been established; but merely by a reference to Lemma 7, which affirms no more than this; "that the ultimate ratio of the arc, chord, and tangent, any one to any other, is the ratio of equality." (See *ante*, page 22.) The proposition advanced by this lemma, is true as to the ultimate ratio of the arc, chord, and tangent, but is obviously false if predicated of them prior to evanescence. But the difficulty is to discover any connexion whatever between Lemma 7, and the analogy which it is cited to support. Newton's commentators, in general, state (what I think cannot be meant in commendation), that in his demonstrations he omits intermediate steps; which is vulgarly, though expressively, called jumping to his conclusion. The main stress of the argument is on this point. Lemma 7 does not support the analogy in question. Its connexion with the analogy is not attempted to be shown, and, in point of fact, it has no connexion with it whatever.

I forbear to make any comment on the remaining Corollaries to Prop. 4, they being only further deductions from the same theory.

The scholium which follows these Corollaries, deserves the greatest consideration.

"*Scholium.* The case of the sixth Corollary obtains in the celestial bodies (as Sir Christopher Wren, Dr. Hooke, and Dr. Halley have severally observed): and therefore, in what follows, I intend to treat more at large of those things which relate to centripetal force decreasing in a duplicate ratio of the distances from the centres."

Comment. It seems as remarkable that Newton should here ascribe the discovery of Kepler's Analogy (which is that of the sixth Corollary) to Wren, Hooke, and Halley, or one of them, as it was that he should in the sixth Corollary treat Kepler's Analogy as an hypothesis only; for he could not be ignorant that Kepler, and not one of the others, had discovered that analogy, and he must have considered the analogy to be true, because he has adopted it throughout the Principia.

The analogy that the velocities are inversely as the square-roots of the distances has been substantively determined by actual computation, from the phenomena of the planetary motions (see a "New Analogy"); and we may demonstrate Kepler's Analogy from it as follows:—

Demonstration of Kepler's analogy, namely,

$$T^2 : t^2 :: D^3 : d^3.$$

We have already seen that

$$V : v :: \frac{D}{T} : \frac{d}{t};$$

and by the "New Analogy,"

$$V : v :: \frac{1}{D^{\frac{1}{2}}} : \frac{1}{d^{\frac{1}{2}}}; \therefore \frac{D}{T} : \frac{d}{t} :: \frac{1}{D^{\frac{1}{2}}} : \frac{1}{d^{\frac{1}{2}}};$$

$$\text{and } \frac{D^2}{T^2} : \frac{d^2}{t^2} : \frac{1}{D} : \frac{1}{d},$$

and dividing by D^2 and by d^2,

$$\frac{1}{T^2} : \frac{1}{t^2} :: \frac{1}{D^3} : \frac{1}{d^3},$$

that is, $T^2 : t^2 :: D^3 : d^3$.

Q. E. D.

Newton concludes the scholium to Proposition 4 as follows :—

"The preceding proposition may likewise be demonstrated in this manner. In any circle suppose a polygon to be inscribed of any number of sides. And if a body, moved with a given velocity along the sides of the polygon, is reflected from the circle at the several angular points, *the force*, with which at every reflexion it strikes the circle, *will be as its velocity*: and therefore the sum of the forces, in a given time, will be as that velocity and the number of reflexions conjunctly; that is, (if the species of the polygon be given,) as the length described in that given time, and increased or diminished in the ratio of the same length to the radius of the circle; that is, as the square of that length applied to the radius; and therefore the polygon, by having its sides diminished *in infinitum*, coincides with the circle, as the square of the arc described in a given time applied to the radius."

Comment. Here Newton alleges that the force is as the velocity (that is, $F : f :: V : v$); and taking this important truth as the basis of a new argument, he proceeds to demonstrate from it Proposition 4 afresh; thus giving a double demonstration, or a confirmation of the first one. But before we follow him into the second demonstration, let us see whether this new allegation here broached for the first time in the Principia squares with the main doctrine advanced in the sixth Corollary (viz. $F : f :: \frac{1}{D^2} : \frac{1}{d^2}$); and we shall find that this new allegation and the said doctrine are totally inconsistent with each other; as will appear as follows:—for by the hypothesis of Corollary 6, $V : v :: \frac{1}{D^{\frac{1}{2}}} : \frac{1}{d^{\frac{1}{2}}}$; and by the scholium $F : f :: V : v$, wherefore by equality of ratios, $F : f :: \frac{1}{\sqrt{D}} : \frac{1}{\sqrt{d}}$. But by Corollary 6, $F : f :: \frac{1}{D^2} : \frac{1}{d^2}$; wherefore by equality of ratios $\frac{1}{D^{\frac{1}{2}}} : \frac{1}{d^{\frac{1}{2}}} :: \frac{1}{D^2} : \frac{1}{d^2}$; which could only be the case where the distances were equal, that is, when $D=d$; but the distances of the planets from the sun are not equal, wherefore this conclusion, (which affirms in effect that the subduplicate ratio of the distances is equal to their duplicate ratio,) is absurd. It follows, therefore, that the two analogies advanced by Newton, viz. $F : f :: \frac{1}{D^2} : \frac{1}{d^2}$ and $F : f :: V : v$, cannot both be true; and if the former analogy should still be deemed to stand, this second demonstration must be held to fail.

But the analogy $F : f :: V : v$ is true, by Newton's second law of motion, that "the alteration of motion is ever *proportional* to the motive force impressed" (which includes the change from a state of rest to that of motion); and unless this second law of motion is denied, the other analogy $F : f :: \frac{1}{D^2} : \frac{1}{d^2}$, which is inconsistent with it, must fail; and yet this latter analogy is that of which Newton says, "In what follows," (that is, in the remainder of the Principia,) "I intend to treat more at large of those things which relate to centripetal force decreasing in a duplicate ratio of the distances from the centres;" thus in the preceding part of the Principia he had laboured to establish this analogy, and in the remainder he proposed to enlarge upon it, which shows that this analogy was the main scope and object of that celebrated work. This, no doubt, was the analogy which Dr. Halley is said to have attempted to demonstrate in vain, and in which attempt Newton also failed for many years.

But let us next examine in what manner Newton proceeds to demonstrate from this true analogy, $F : f :: V : v$, his other analogy, $F : f :: \frac{V^2}{D} : \frac{v^2}{d}$; Corol. 1. Prop. 4. "The force," says he, "with which the body at every reflexion strikes the circle, will be as its velocity." True, and the motion being circular, the force at each point of reflexion will be equal; but it is not, therefore, or for any other reason, true, that "the sum of the forces in a given time will be as that velocity and the number of reflexions conjunctly." For let the given time be one quarter of the planet's periodic time, then, as we have seen in our commentary on Prop. 4, the sum of the forces will be equal to the versed sine of an arc of 90°, that is, equal to radius, or the planet's distance from the sun, and may be thus expressed, $F = \frac{D}{\left(\frac{T}{4}\right)}$. (See an "Exposition of the Nature, Force, Action, &c. of Gravitation on the Planets." Whittaker and Co., 1842.) But Newton says, that the sum of the forces in this time will be as the velocity and the number of reflexions conjunctly; that is, as he explains it, as the length of the arc, increased or diminished in the ratio of the same length to the radius of the circle. Now the number of reflexions seems here put for the length of the arc, and therefore it would seem that the sum of the forces is as the length of the arc conjunctly with the length of the arc; that is, if s denotes the arc, $F \propto s^2$. But Newton obviously does not mean this; for in this stage of the demonstration, he goes back to his original position, $F \propto S$, that is, the force varies as the arc, or as the velocity. So that the step in the argument, that "the sum of the forces in a given time will be as that velocity and the number of reflexions conjunctly," is an advance from which he retreats back immediately; and not without good reason; for it advances, I conceive, that the forces are as the squares of the velocities, whereas his major proposition was, that the forces are as the velocities.

But whatever be the meaning of this part of the argument, since it is no sooner advanced than abandoned, it is needless to remark further upon it; though it might afford scope for comment on the looseness or negligence with which the Principia was drawn up. For at the next step, we find that the sum of the forces is "as the length described in that given time, and increased or diminished in the ratio of the same length to the radius of the circle." Now the meaning of this is, that the sum of the forces is as the arc, or as the velocity; the "length" being the arc: but with this condition, that the arc is to be increased or diminished in the ratio of the same arc to the radius of the circle, which qualification only asserts (what is perfectly true) that the length of the arc will be greater or less, in proportion to the greater or less distance of the planet from the sun. From this qualification, however, Newton infers, that because the length of the arc may be determined by the radius, therefore the force is as the square of that length applied to the radius; or (as explained in Cor. 1) in a ratio compounded of the duplicate ratio of the velocities directly, and of the simple ratio of the radii inversely; *i. e.* $F \propto \frac{V^2}{D}$, which is the same ratio as in Cor. 1, which amounts, on the whole, to this, that the forces are as the velocities, that is, as the squares of the velocities divided by the radii or distances. But this second inference is a mere *petitio principis*, and is inconsistent with the first inference, viz., $F : f :: V : v$; as we have already shown, in our commentary on this branch of the scholium. That it is a *petitio principis* is evident, for there is no ground assigned for it, but that the length of the arc may be determined by the radius; which is no more a reason for this particular ratio, than it would be for any other.

In the concluding sentence of the scholium, it is truly stated that the centrifugal force is equal to the centripetal force. "This," says Newton, "is the centrifugal force, with which the body impels the circle; and to which the contrary force, wherewith the circle continually repels the body towards the centre, is equal."

Now I have shown, in Chapter I. of these Commentaries, that the centrifugal force is equal to the planet's mean distance from the sun, divided by $\frac{1}{4}$ of the periodic time $=\frac{D}{\frac{1}{4}T}$; and consequently the centripetal force is also $=\frac{D}{\frac{1}{4}T}$; let, therefore, F, f denote the centripetal forces of any two planets, then by the scholium, $F : f :: \frac{D}{\frac{1}{4}T} : \frac{d}{\frac{1}{4}t} :: \frac{D}{T} : \frac{d}{t}$; but

$\frac{D}{T}$, $\frac{d}{t}$ (as we have seen) represent the velocities; that is, $V : v :: \frac{D}{T} : \frac{d}{t}$; whence, by equality of ratios, $F : f :: V : v$; and by Corol. 6, $V : v :: \frac{1}{\sqrt{D}} : \frac{1}{\sqrt{d}}$; wherefore, $F : f :: \frac{1}{\sqrt{D}} : \frac{1}{\sqrt{d}}$; contradictory to Newton's theory, viz., $F : f :: \frac{1}{D^2} : \frac{1}{d^2}$.

I may observe, that the equality of the centrifugal and centripetal forces is a matter of so great importance, as to have deserved an enlarged consideration, and also a demonstration at the hands of Newton, instead of being enunciated in a single sentence at the end of this scholium, and never noticed afterwards.

The remainder of the Principia, so far as respects Newton's theory, consists of Propositions founded upon it, which it is not requisite for me to examine; for I have already commented on all that Newton has advanced in support of his theory, and therefore these Commentaries will not be affected by the consideration whether or not his subsequent Propositions are or are not correctly deduced from his theory.

To conclude these Commentaries: we shall find that Newton has given us other means of putting his theory to the test, in the following passage, in his "System of the World":—

"That the circumterrestrial force decreases in the duplicate proportion of the distances, I infer thus:

"The mean distance of the moon from the centre of the earth is, in semidiameters of the earth, according to Ptolemy, Kepler in his Ephemerides, Bullialdus, Hevelius, and Ricciolus, 59; according to Flamstead, $59\frac{1}{3}$; according to Tycho, $56\frac{1}{2}$; to Vendelin, 60; to Copernicus, $60\frac{1}{3}$; to Kircher, $62\frac{1}{2}$.

"But Tycho, and all that follow his tables of refraction, making the refractions of the sun and moon (altogether against the nature of light) to exceed those of the fixed stars, and that by about 4 or 5 minutes in the horizon, did thereby augment the horizontal parallax of the moon, by about the like number of minutes; that is, by about the twelfth or fifteenth part of the whole parallax. Correct this error, and the distance will become 60 or 61 semidiameters of the earth, nearly agreeing with what others have determined.

"Let us then assume the mean distance of the moon 60 semidiameters of the earth, and its periodic time in respect of the fixed stars, 27 days, 7 hours, 43′, as astronomers have determined it. And (by Corol. 6, Prop. 4) a body revolved in our air, near the surface of the earth supposed at rest, by means of a centripetal force, which should be to the same force at the distance of the moon in the *reciprocal duplicate proportion of the distances from the centre of the earth, that is, as* 3600 *to* 1, would (secluding the resistance of the air) complete a revolution in 1 hour, 24′, 27″."

Now, this last-mentioned periodic time is correct, or very nearly so; but so far from its being an inference from Newton's theory, it is inconsistent with it, and will be found to confirm our new theory: it is an inference derived solely from Kepler's Analogy.

In the first place, it is obvious that the hypothetical periodic time of the body revolving round the earth in an orbit near the earth's surface, could not be found by means of Newton's theory alone; for Newton's theory does not state the relation between the *forces* and the *times*, but only between the *forces* and the *distances*; but Kepler's Analogy gives the *true* relation between the distances and the times, without the aid of Newton's theory; and consequently, since the distances in this hypothetical case are given, as also the moon's periodic time, we may find, as Newton doubtless found, from these three data, the fourth quantity, that is, the periodic time of the other revolving body, as follows:—

By Kepler—

$$60^3 : 1^3 :: 655{\cdot}6^2 : \frac{655{\cdot}6^2}{60^3},$$

where 60^3 denotes the cube of the moon's distance in semidiameters of the earth;

1 denotes the cube of the distance of the body revolving near the earth's surface; $655 \cdot 6^2$ denotes the square of the moon's periodic time in hours, and consequently the fourth term $\frac{655 \cdot 6^2}{60^3}$ will express the square of the periodic time of the other revolving body; conformably to the formula of Kepler's Analogy,

$$D^3 : d^3 : : T^2 : t^2 ;$$

Now, by actual computation, this Rule of Three problem becomes—

$$216{,}000 : 1 : : 429811 \cdot 36 : \frac{429811 \cdot 36}{216{,}000} = 1 \cdot 989, \&c.$$

and $\sqrt{1 \cdot 989}$, &c.$=1 \cdot 41=1$ hour, 24′, 36″, which is sufficiently near to Newton's computation (1 hour, 24′, 27″) to show that both computations are based on Kepler's Analogy; the difference between them being only 9″. Newton's conclusion is therefore deduced, not from the sixth Corollary (which it purports to be), but from Kepler's Analogy, which is the hypothesis upon which the sixth Corollary is stated (but not shown) to be founded. That Newton's conclusion is just, we can only know because it is correctly deduced from Kepler's true Analogy; but that just conclusion has no tendency to show that another conclusion (viz. the sixth Corollary), drawn from the same premises, is true also.

But, in the second place, this periodic time thus found, is inconsistent with Newton's theory; according to which, the centripetal force of the body revolving near the earth's surface, would be 3600 times the centripetal force of the moon, that is, as the squares of the distances; but, according to our new theory, it would be as the square-roots of the distances, that is, as $\sqrt{60}$ to $\sqrt{1}$, or as 7·74 to 1, nearly. Now, since from the principles before mentioned, and recognized by Newton himself, we can determine these centripetal forces, we shall thereby, as before premised, have a test for ascertaining the validity of the two conflicting theories.

For we have already shown that the mean distance divided by one-fourth of the periodic time represents the centrifugal force, and (because the centrifugal force is equal to the centripetal force, Schol. to Prop. 4) also represents the centripetal force in terms of the time. Hence we may determine the centripetal forces of the moon and the other revolving body, and therefore the ratio of these forces, and their relation to their distances from the centre of the earth, as follows :—

The moon's mean distance from the earth is estimated at 240,000 miles, nearly; and its periodic time, 655·6 hours, nearly; one-fourth of which is=163·4, and $\frac{240,000}{163 \cdot 4}=1468$ miles *per horam* is the moon's centripetal force, very nearly.

By the hypothesis, the distance of the other revolving body from the centre of the earth is 4000 miles, nearly; and its periodic time is found to be 1 hour, 24′ 27″, that is, 5067 seconds; one-fourth of which is 1267 seconds, nearly; which is at the rate of 11,365 miles *per horam*.

Consequently, the ratio of the centripetal force of the moon to that of the other revolving body, is as 1468 to 11,365; and since the analogy between the forces and the times (whatever it may be) is inverse, if we take the number 1 as the third term in this proportion, as being one semidiameter of the earth, being the distance of the nearer body from the earth's centre, the fourth term will be found by the operations of the Rule of Three, and will either be 3600, or 7·74, according as Newton's theory or the new theory is the true one.

Thus—1468 : 11365 : : 1 : $\left(\frac{11365}{1468}=7{\cdot}74, \&c.\right)$

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1468)11365(7·74
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      10890
      10276
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        6140
        5872
        ----
         268
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that is, the periodic times, in this case, are inversely as the square-roots of the distances.

On the whole, therefore, we conclude that the centripetal or gravitating forces of the planets are inversely in the sub-duplicate (not the duplicate) ratio of their distances from the sun; and the defect in the reasoning in support of Newton's theory appears to consist of the assumption in the demonstration of Prop. 4, that the versed sines of arcs are as the squares of the arcs directly, and as the diameters inversely; an assumption which, as we have shown, might have been demonstrated from Newton's theory, taking it to be established, as well as his theory from the assumption, if admitted; but the assumption which he has taken as the basis of his theory, is itself without foundation.

From "the peculiar reverence" with which we are apt to regard the works of this great philosopher, many will scarcely believe that he could err in any particular. But if we admit the fallibility of human reason, from which even the ablest and wisest men have not considered themselves exempt, it will appear more wonderful, rather that Newton should have made only one mistake, than that he should have made that one. The system of gravitation was his own unaided discovery, independent of experience and instruction; not a deduction from facts recognized in the University, but the assignment of a great cause for a phenomenon disregarded because common—the falling of an apple in an orchard. In order to establish this cause as universal, it was necessary for him to adopt one of the three systems—the Pythagorean, the Ptolemaic, or the Tychonic, to which to apply his theory. He found it agreed with the Pythagorean system only, which he therefore made choice of; and he rejected the other two systems, which were inconsistent with it.

There remained only the two analogies which Kepler had established in the early part of the seventeenth century, with which Newton had to apply or compare his theory of gravitation. In the first Proposition in the Principia, he has admirably demonstrated Kepler's Analogy, that the planets describe equal areas in equal times. It is impossible to extol in too high terms the merit of this demonstration, whether on the score of ingenuity, judgment, or accuracy. But it does not appear that Newton ever attempted to demonstrate Kepler's other Analogy, that the squares of the periodic times of the planets are as the cubes of their mean distances from the sun. On the contrary, he assumes it hypothetically, in the sixth Corollary to Proposition 4, and rests its foundation on the authority of Wren, Hooke, and Halley, without even alluding to Kepler himself, who had discovered it. Kepler did not attempt to discover the principles upon which that analogy is founded; nor was it ever attempted to be demonstrated, that I am aware of, until I undertook the demonstration, which I published in 1842, in my work, intituled, "An Exposition of the Nature, Force, Action, and other Properties of Gravitation," &c.

Newton's conjecture, that gravitation emanates as rays of light from a centre, led him to attribute to gravity the properties of light; according to which analogy, the force of gravity would be inversely in the duplicate ratio of the distance. But the emanation of light is very different from that of gravity: the one is absolute and independent; the other only relative. Light would be diffused from the sun, and pervade the whole *expansum*, if there were no bodies to receive his rays. The planets, as they revolve, receive new light in every part of their orbits, because they at all times intercept the rays of light which are shot forth from the sun in all directions in the spaces through which the planets revolve; and which intercepted rays would otherwise pervade the *expansum*, and (as most of them probably do) become dissipated by an infinite diffusion. But the attraction of gravitation cannot exist unless there are at least two bodies mutually attracted to each other; there is no emanation of attraction into the void expanse; and therefore the analogy of gravitation to light, which is the basis of Newton's theory of the *force* of gravity, fails, and the theory with it.

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