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ABSTRACT

In infinite dimensional commodity spaces whose positive cone has an empty norm interior the core equivalence theorem fails. To this end we introduce a new condition called commodity pair desirability which enables us to preserve the core equivalence theorem. This condition is related to the assumption of an extremely desirable commodity or uniform properness and if the norm interior of the positive cone of commodity space is nonempty and preferences are monotone then it is automatically satisfied.

1. INTRODUCTION

The purpose of this paper is to study the core-Walras equivalence in economies with a continuum of agents and with an infinite dimensional commodity space.

It may be useful to discuss briefly the general importance of infinite dimensional commodity spaces in economics. As others have noted (e.g., Court (1941), Debreu (1954), Gabszewicz (1991), Bewley (1970) and Peleg-Yaari (1970)), infinite dimensional commodity spaces arise quite naturally in economics. In particular, an infinite dimensional commodity space may be desirable in problems involving an infinite time horizon, uncertainty about the possibly infinite number of states of nature of the world, or infinite varieties of commodity characteristics. For instance, the Lebesgue space L_∞ of bounded measurable functions on a measure space considered by Bewley (1970), Gabszewicz (1991) and Mertens (1991) is useful in modeling uncertainty or an infinite time horizon. The space L_2 of square-integrable functions on a measure space is useful in modeling the trading of long-lived securities over time. Finally, the space $M(\Omega)$ of measures on a compact metric space considered by Mas-Colell (1975), is useful in modeling differentiated commodities.

In this paper, we study core-Walras equivalence results for perfectly competitive economies with an infinite dimensional commodity space which is general enough to include all of the spaces that have been found most useful in equilibrium analysis. In particular, we cover all the Lebesgue spaces L_p, $(1 \le p \le \infty)$, the space of measures, $M(\Omega)$ and the space of continuous functions on a compact metric space $C(X)$.

It turns out that in infinite dimensional commodity space whose positive cone has a non-empty (norm) interior one can obtain core-Walras equivalence results under quite mild assumptions. However, in infinite dimensional commodity spaces whose positive cone has an empty (norm) interior, as Rustichini-Yannelis (1991) showed, even under quite strong assumptions on preferences and endowments, core-Walras equivalence fails. In particular, the above authors showed, that even when preferences are strictly convex, monotone, and weak* continuous and initial endowments are strictly positive, core-Walras equivalence fails to hold. It is important to note that this failure results despite the fact that these assumptions are much stronger than the standard assumptions which guarantee equivalence in the Aumann (1964) finite dimensional commodity space setting.

Our main objective is to obtain core-Walras equivalence for infinite dimensional commodity spaces (in particular, Banach lattices) whose positive cone may have an empty (norm) interior and are general enough to cover the space L_p $(1 \leq p \leq \infty)$ and $M(\Omega)$. In view of the Rustichini-Yannelis counterexample to the core-Walras equivalence in spaces whose positive cone has an empty interior, we introduce a new condition on preferences called commodity pair desirability. In essence, this assumption is a strengthening of the assumption of an extremely desirable commodity used in Yannelis-Zame (1986), which in turn is related to the condition of uniform properness in Mas-Colell (1986) (see also Chichilnisky-Kalman (1980)). All of these assumptions are essentially bounds on the marginal rate of substitution, and in practice turn out to be quite weak. For example, all three of these assumptions are automatically satisfied whenever preferences are monotone and the positive cone of the commodity space has a non-empty (norm) interior. Hence, this assumption is implicit in the infinite dimensional work of Gabszewicz (1991), Mertens (1991), and Bewley (1973), and is automatically satisfied in the finite dimensional work of Aumann (1964) and Schmeidler-Hildenbrand (Hildenbrand (1972), (1974)).[1] We also wish to note that in addition to the commodity pair desirability assumption, the lattice structure of the commodity space will play a crucial role in our analysis.

The remainder of the paper is organized as follows: Section 2 contains notation, definitions and some results on Banach lattices and the integration of correspondences. The economic model is outlined in Section 3 where we also prove a core-Walras equivalence theorem for an ordered separable Banach space of commodities, whose positive cone has a non-empty (norm) interior. The central assumption of the paper, commodity pair desirability, is introduced in Section 4. In Section 5 we prove a core-Walras equivalence result for a commodity space which can be any arbitrary separable Banach lattice, provided that the assumption of commodity pair desirability holds. Finally, some concluding remarks are given in Section 6.

2. PRELIMINARIES

2.1 <u>Notation</u>

R^ℓ denotes the ℓ-fold Cartesian product of the set of real numbers R.

intA denotes the interior of the set A.

2^A denotes the set of all nonempty subsets of the set A.

$\emptyset$ denotes the empty set.

/ denotes the set theoretic subtraction.

dist denotes distance.

If $A \subset X$ where X is a Banach space, $c\ell A$ denotes the norm closure of A.

If X is a Banach space its dual is the space X^* of all continuous linear functionals on X.

If $q \in X^*$ and $y \in X$ the value of q at y is denoted by $q \cdot y$.

2.2 Definitions

We begin by collecting some useful notions on Banach lattices (a more detailed exposition may be found in Aliprantis-Burkinshaw (1978, 1985)), which will be needed in the sequel.

A normed vector square is a real vector space E equipped with a norm $\|\cdot\|: E \to [0,\infty)$ satisfying:

(i) $\|x\| \geq 0$ for all x in E, and $\|x\| = 0$ if and only if $x = 0$;

(ii) $\|\alpha x\| = |\alpha| \, \|x\|$ for all x in E and all α in R;

(iii) $\|x+y\| \leq \|x\| + \|y\|$ for all x,y in E.

A Banach space is a normed vector space for which the metric induced by the norm is complete.

If E is a Banach space, then its dual space E^* is the set of continuous linear functionals on E. The dual space E^* is itself a Banach space, when equipped with the norm

$$\|\phi\| = sup\{|\phi(x)| : x \in E, \|x\| \leq 1\}.$$

A <u>Banach lattice</u> is a Banach space L endowed with a partial order $\leq$ (i.e., $\leq$ is a reflexive, antisymmetric, transitive relation) satisfying:

(1) $x \leq y$ implies $x + z \leq y + z$ (for all $x, y, z \in L$);

(2) $x \leq y$ implies $tx \leq ty$ (for all $x, y \in L$, all real numbers $t \geq 0$);

(3) every pair of elements $x, y \in L$ has a supremum (least upper bound) $x \vee y$ and an infimum (greatest lower bound) $x \wedge y$;

(4) $|x| \leq |y|$ implies $\|x\| \leq \|y\|$ (for all $x, y \in L$).

Here we have written, as $|x| = x^+ + x^-$ where $x^+ = x \vee 0$, $x^- = (-x) \vee 0$; we call x^+, x^- the <u>positive</u> and <u>negative parts</u> of x respectively and $|x|$ the <u>absolute value</u> of x. Note that $x = x^+ - x^-$, and that $x^+ \wedge x^- = 0$. We say that $x \in L$ is <u>positive</u> if $x \geq 0$; we write L_+ (or L^+) for the set of all positive elements of L and refer to L_+ (or L^+) as the <u>positive cone</u> of L.

We will say that an element x of L is <u>strictly positive</u> (and write $x >> 0$) if $\phi(x) > 0$ whenever ϕ is a positive non-zero element of L_+^*. Strictly positive elements are usually called <u>quasi-interior</u> to L_+. Note that if the positive cone L_+ of L has a non-empty (norm) interior, then the set of strictly positive elements coincides with the interior of L_+. However, many Banach lattices contain strictly positive elements even though the positive cone L_+ has an empty interior (see Aliprantis-Burkinshaw (1985, p. 259)). We will now give basic examples of separable Banach lattices.

(i) the Euclidean space R^N;

(ii) the space ℓ_p $(1 \le p < \infty)$ of real sequences (a_n) for which the norm $\|(a_n)\|_p = (\Sigma\ |a_n|^p)^{1/p}$ is finite;

(iii) the space $L_p(\Omega,\mathbb{R},\mu)$ $(1 \le p < \infty)$ of equivalence classes of measurable function f on the separable measure space $(\Omega,\mathbb{R},\mu)$ for which the norm $\|f\|_p = (\int_\Omega |f|^p d\mu)^{1/p}$ is finite;

(iv) the space C(K) of continuous, real-valued functions on the compact Hausdorff space K (with supremum norm);

A basic property of Banach lattices which will play a crucial role in the sequel, is the Riesz Decomposition Property.

<u>Riesz Decomposition Property</u>: Let L be a Banach lattice and let x, $y_1,\ldots,y_n$ be positive elements of L such that $0 \le x \le \sum_{i=1}^{n} y_i$. Then there are positive elements $x_1,\ldots,x_n$ in L such that $\sum_{i=1}^{n} x_i = x$ and $0 \le x_i \le y_i$ for each i.

We now define some measure theoretic notions as well as the concepts of a Bochner integrable function and the integral of a correspondence.

Let X, Y be sets. The <u>graph</u> of the correspondence $\phi : X \to 2^Y$ is denoted by $G_\phi = \{(x,y) \in X \times Y : y \in \phi(x)\}$. Let (T,τ,μ) be a finite measure space (i.e., μ is a real-valued, non-negative countably additive measure defined on a σ-field τ of subsets of T such that $\mu(T) < \infty$), and X a Banach space. The correspondence $\phi : T \to 2^X$ is said to have a <u>measurable graph</u> if $G_\phi \in \tau \otimes \mathfrak{B}(X)$, where $\mathfrak{B}(X)$ denotes the Borel σ-algebra on X and $\otimes$ denotes the product σ-algebra. A function $f : T \to X$ is called <u>simple</u> if there exist $x_1, x_2, \ldots, x_n$ in X and $a_1, a_2, \ldots, a_n$ in τ such that $f = \sum_{i=1}^{n} x_i\chi_{a_i}$ where $\chi_{a_i}(t) = 1$ if $t \in a_i$

and $\chi_{a_i}(t) = 0$ if $t \notin a_i$. A function $f : T \to X$ is said to be μ-<u>measurable</u> if there exists a sequence of simple functions $f_n : T \to X$ such that $\lim_{n\to\infty} \|f_n(t) - f(t)\| = 0$ μ - a.e. A μ-measurable function $f : T \to X$ is said to be <u>Bochner integrable</u> if there exists a sequence of simple functions $\{f_n : n = 1,2,...\}$ such that

$$\lim_{\mu\to\infty} \int_T \|f_n(t) - f(t)\| d\mu(t) = 0.$$

In this case we define for each $E \in \tau$ the integral to be $\int_E f(t)d\mu(t) = \lim_{n\to\infty} \int_E f_n(t)d\mu(t)$. It can be easily shown (see Diestel-Uhl (1977, p. 45)) that if $f : T \to X$ is a μ-measurable function then f is Bochner integrable if and only if $\int_T \|f(t)\| d\mu(t) < \infty$. We denote by $L_1(\mu,X)$ the space of equivalence classes of X-valued Bochner integrable functions $x : T \to X$ normed by $\|x\| = \int_T \|x(t)\| d\mu(t)$. Note that one can easily show that if (T,τ,μ) is atomless, the subset of simple functions given by $B = \{x : T \to X,\ x = \sum_{i=1}^{m} x_i \chi_{T_i},\ \mu(T_i) = \frac{1}{m}\}$ is norm-dense in $L_1(\mu,X)$. Moreover, we denote by S_ϕ the <u>set of all X-valued Bochner integrable selections from the correspondence</u> $\phi : T \to 2^X$, i.e.

$$S_\phi = \{x \in L_1(\mu,X) : x(t) \in \phi(t)\ \mu - a.e.\}.$$

The <u>integral of the correspondence</u> $\phi : T \to 2^X$ is defined as:

$$\int_T \phi(t)\, d\mu(t) = \{\int_T x(t)\, d\mu(t) : x \in S_\phi\}$$

In the sequel we will denote the above integral by

$$\int \phi \text{ or } \int_T \phi.$$

2.3 Lemmata

If (T,τ,μ) is atomless and $X = R^{\ell}$, it follows from Lyapunov's Theorem that the integral of the correspondence $\phi : T \to 2^X$, i.e., $\int\phi$, is convex. However, this result is false in infinite dimensional spaces (see for instance Yannelis (1991)). Nevertheless, it can be easily deduced (see for instance Hiai-Umegaki (1977), Khan (1985) or Yannelis (1991)) from the approximate version of Lyapunov's Theorem in infinite dimensional spaces that the norm closure of $\int\phi$, i.e., $c\ell\int\phi$, is convex. More formally the following Lemma is true.

<u>Lemma 2.1</u>: Let (T,τ,μ) be a finite atomless measure space, X be a Banach space and $\phi : T \to 2^X$ be a correspondence. Then $c\ell\int\phi$ is convex.

We will also need the following result whose proof follows from the measurable selection theorem and can be found in Hiai-Umegaki (1977, Theorem 2.2, p. 156).

<u>Lemma 2.2</u>: Let (T,τ,μ) be a finite measure space, X be a separable Banach space, and $\phi : T \to 2^X$ be a correspondence having a measurable graph. Suppose that $\int\phi \neq \emptyset$. Then for every $p \in X^*$ we have that

$$\inf_{z\in\int\phi} p \cdot z = \int \inf_{y\in\phi(\cdot)} p \cdot y.$$

It should be noted that Lemma 2.2 has been proved in Hildenbrand (1974, Proposition 6, p. 63) for $X = R^{\ell}$. However, by recalling that the Aumann measurable selection theorem holds in separable metric spaces, one can easily see that Hildenbrand's argument remains true in separable

Banach spaces. In fact, it is even true in arbitrary Hausdorff separable and metrizable linear topological spaces.

With all these preliminaries out of the way, we can now turn to our model.

3. THE MODEL AND A PRELIMINARY THEOREM

Denote by E the commodity space. Throughout this section the commodity space E will be an ordered Banach space.

An economy $\mathscr{E}$ is a quadruple $[(T,\tau,\mu), X, \succ_t, e]$ where

(1) (T,τ,μ) is a measure space of agents,

(2) $X : T \to 2^E$ is the consumption correspondence,

(3) $\succ_t \subset X(t) \times X(t)$ is the preference relation of agent t, and

(4) $e : T \to E$ is the initial endowment, where e is Bochner integrable and $e(t) \in X(t)$ for all $t \in T$.

An allocation for the economy $\mathscr{E}$ is a Bochner integrable function $x : T \to E_+$. An allocation x is said to be feasible if $\int_T x(t)d\mu(t) = \int_T e(t)d\mu(t)$. A coalition S is an element of τ such that $\mu(S) > 0$. The coalition S can improve upon the allocation x if there exists an allocation g such that

(i) $g(t) \succ_t x(t)$ μ - a.e. in S, and

(ii) $\int_S g(t)d\mu(t) = \int_S e(t)d\mu(t)$.

The set of all feasible allocations for the economy $\mathscr{E}$ that no coalition can improve upon is called the core of the economy $\mathscr{E}$ and it is denoted by $\mathcal{C}(\mathscr{E})$.

An allocation x and a price $p \in E_+^*/\{0\}$ are said to be a competitive equilibrium (or a Walras equilibrium) for the economy $\mathscr{E}$, if

(i) $x(t)$ is a maximal element for $\succ_t$ in the budget set $\{y \in X(t) : p \cdot y \leq p \cdot e(t)\}$ μ - a.e., and

(ii) $\int_T x(t)d\mu(t) = \int_T e(t)d\mu(t)$.

We denote by $W(\mathscr{E})$ the set of all competitive equilibria for the economy $\mathscr{E}$.

We begin by stating some assumptions needed for the proof of our core-Walras equivalence result.

(a.0) E is an ordered separable Banach space whose positive cone E_+ has a non-empty norm interior, i.e., $\text{int}E_+ \neq \emptyset$.

(a.1) (<u>Perfect Competition</u>): (T,τ,μ) is a finite atomless measure space.

(a.2) $X(t) = E_+$ for all $t \in T$.

(a.3) (<u>Resource availability</u>): The aggregate initial endowment $\int_T e(t)d\mu(t)$ is strictly positive, i.e., $\int e >> 0$.

(a.4) (<u>Continuity</u>): For each $x \in E_+$ the set $\{y \in E_+ : y \succ_t x\}$ is norm open in E_+ for all $t \in T$,

(a.5) $\succ_t$ is irreflexive and transitive for all $t \in T$.

(a.6) (<u>Measurability</u>): The set $\{(t,y) \in T \times E_+ : y \succ_t x\}$ belongs to $\tau \otimes \mathfrak{B}(E_+)$.

(a.7) (<u>Monotonicity</u>): If $x \in E_+$ and $v \in E_+/\{0\}$, then $x + v \succ_t x$ for all $t \in T$.

Theorem 3.1 below is taken from Rustichini-Yannelis (1991). Since the second part of the proof of Theorem 3.1 is the same as that of Theorem 5.1, we will provide the argument for the sake of completeness. It should be noted, that in view of Remark 3.1 (see below), Theorem 3.1 is more general than Theorem 4.1 of Rustichini-Yannelis (1991). In particular, E need not be separable.

<u>Theorem 3.1</u>: Under assumption (a.0) - (a.7), $\mathcal{C}(\mathscr{E}) = W(\mathscr{E})$.

<u>Proof</u>: The fact that $W(\mathscr{E}) \subset \mathcal{C}(\mathscr{E})$ is well known, and therefore its proof is not repeated here. We begin the proof by assuming that the allocation x is an element of the core of $\mathscr{E}$. We wish to show that for some price p, the pair (x,p) is a competitive equilibrium for $\mathscr{E}$.

To this end, define the correspondence $\phi : T \to 2^{E_+}$ by

(3.0) $$\phi(t) = \{z \in E_+ : z \succ_t x(t)\} \cup \{e(t)\}.$$

We claim that:

(3.1) $$cl\left(\int_T \phi - \int_T e\right) \cap int\, E_- = \emptyset,$$

or equivalently,[2]

(3.2) $$\left(\int_T \phi - \int_T e\right) \cap int\, E_- = \emptyset.$$

Suppose otherwise, i.e.,

$$\left(\int_T \phi - \int_T e\right) \cap int\, E_- \neq \emptyset$$

then there exists $v \in int\, E_+$ such that

(3.3) $$\int e - v \in \int \phi.$$

It follows from (3.3) that there exists a function $y : T \to E_+$ such that

(3.4) $$\int_T y = \int_T e - v,$$

and $y(t) \in \phi(t)$ μ - a.e.

Let

$$S = \{t : y(t) \succ_t x(t)\}, \text{ and}$$

$$S' = \{t : y(t) = e(t)\}.$$

Since $\int y \neq \int e$ we have that $\mu(S) > 0$. Define $\tilde{y} : S \to E_+$ by $\tilde{y}(t) = y(t) + \dfrac{v}{\mu(S)}$ for all $t \in S$. By monotonicity (assumption (a.7)) $\tilde{y}(t) \succ_t y(t)$. Since $y(t) \succ_t x(t)$ for all $t \in S$, by transitivity (assumption (a.5)) $\tilde{y}(t) \succ_t x(t)$ for all $t \in S$. Moreover, it can be easily seen that $\tilde{y}(\cdot)$ is feasible for the coalition S, i.e.

$$\int_S \tilde{y} = \int_S y + v = \int_T y - \int_{S'} e + v$$

$$= \int_T e - \int_{S'} e = \int_S e, \ (recall (3.4)).$$

Therefore, we have found an allocation $\tilde{y}(\cdot)$ which is feasible for the coalition S and is also preferred to the allocation x, which in turn was assumed to be in the core of $\mathscr{E}$, a blatant contradiction. The above contradiction establishes the validity of (3.1).

We are now in a position to separate the set $c\ell(\int\phi - \int e) = c\ell\int\phi - \int e$ from int E_-. Clearly the set int E_- is convex and non-empty as well. Observe first that by the definition of $\phi(\cdot)$, 0 is an element of $\int\phi - \int e$ and this shows that $c\ell\int\phi - \int e$ is non-empty. Since, (T,τ,μ) is atomless (assumption (a.1)) by Lemma 2.1 $c\ell\int\phi$ is convex. Thus, by Theorem 9.10 in Aliprantis-Burkinshaw (1985, p. 136) there exists a continuous linear functional $p \in E^*/\{0\}$, $p \geq 0$ such that

(3.5) $$p \cdot y \geq p \cdot \int e \text{ for all } y \in \int \phi.$$

Since by assumption (a.6), $\succ_t$ has a measurable graph, so does $\phi(\cdot)$, i.e., $G_\phi \in \tau \otimes \mathcal{B}(E_+)$. Therefore, it follows from Lemma 2.2 that

(3.6) $$\inf_{y \in \int \phi} p \cdot y = \int \inf_{z \in \phi(\cdot)} p \cdot z \geq \int p \cdot e.$$

It follows from (3.6) that

(3.7) $$\mu - a.e.\ p \cdot z \geq p \cdot e(t) \text{ for all } z \succ_t x(t).$$

To see this, suppose that for $z \in \phi(\cdot)$, $p \cdot z < p \cdot e(t)$ for all $t \in S$, $\mu(S) > 0$.

Define the function $\tilde{z} : T \to E_+$ by

$$\tilde{z}(t) = \begin{cases} z(t) & \text{if } t \in S \\ e(t) & \text{if } t \notin S. \end{cases}$$

Obviously, $\tilde{z} \in \phi(\cdot)$. Moreover,

$$\int_T p \cdot \tilde{z} = \int_S p \cdot z + \int_{T/S} p \cdot e$$

$$< \int_S p \cdot e + \int_{T/S} p \cdot e = \int p \cdot e,$$

a contradiction of (3.6).

We now show that μ - a.e. $p \cdot x(t) = p \cdot e(t)$. First note that it follows directly from (3.7) that $p \cdot x(t) \geq p \cdot e(t)$ μ - a.e. If now $p \cdot x(t) > p \cdot e(t)$ for all $t \in S$, $\mu(S) > 0$ then,

$$p \cdot \int_T x = p \cdot \int_{T/S} x + p \cdot \int_S x$$

$$> p \cdot \int_{T/S} e + p \cdot \int_S e = p \cdot \int_T e,$$

contradicting $\int_T x = \int_T e$, since $p \geq 0$, $p \neq 0$.

To complete the proof we must show that $x(t)$ is maximal in the budget set $\{z \in E_+ : p \cdot e(t)\}$ μ - a.e. The argument is now routine. Since $\int_T e$ is strictly positive (assumption (a.3)) it follows that $\mu(\{t : p \cdot e(t)\}) > 0$, for if $p \cdot e(t) = 0$ μ - a.e. then $p \cdot \int e = 0$ contradicting the fact that $\int e$ is strictly positive since $p \geq 0$, $p \neq 0$.

Thus, we can safely pick an agent t with positive income, i.e., $p \cdot e(t) > 0$. Since $p \cdot e(t) > 0$ there exists an allocation x' such that $p \cdot x' < p \cdot e(t)$. Let y be such that $p \cdot y \leq p \cdot e(t)$ and let $y(\lambda) = \lambda x' + (1-\lambda)y$ for $\lambda \in (0,1)$. Then for any $\lambda \in (0,1)$, $p \cdot y(\lambda) < p \cdot e(t)$ and by (3.7) $y(\lambda) \nsucc_t x(t)$. It follows from the norm continuity of $\succ_t$ (assumption (a.4)) that $y \nsucc_t x(t)$. This proves that $x(t)$ is maximal in the budget set of agent t, i.e., $\{w : p \cdot w \leq p \cdot e(t)\}$. This, together with the monotonicity of preferences (assumption (a.7)) implies that prices are strictly positive, i.e., $p >> 0$. Indeed, if there exists $v \in E_+/\{0\}$ such that $p \cdot v = 0$ then $p \cdot (x(t) + v) = p \cdot e(t)$ and by monotonicity $x(t) + v \succ_t x(t)$ contradicting the maximality of $x(t)$ in the budget set.

Thus $p >> 0$ and $x(t)$ is maximal in the budget set whenever $p \cdot e(t) > 0$. Consider now an agent t with zero income, i.e., $p \cdot e(t) = 0$. Since $p >> 0$ his/her budget set $\{z : p \cdot z = 0\}$ consists of zero only, and moreover, $p \cdot x(t) = p \cdot e(t) = 0$. Hence, $x(t) = 0$ for almost all $t \in T$, with $p \cdot e(t) = 0$; i.e., zero in this case is the maximal element in the budget set. Consequently, (p,x) is a competitive equilibrium for $\mathscr{E}$, and this completes the proof of Theorem 3.1.

<u>Remark 3.1</u>: It is possible to relax the assumption that E is separable. The argument (which is given to us by one of the referees) proceeds as follows:

As before we can obtain (3.5), i.e., there exists $p \in E^*/\{0\}$, $p \geq 0$ such that

$$p \cdot y \geq p \cdot \int e \text{ for all } y \in \int \phi.$$

We now show that $p \cdot x(t) = p \cdot e(t)$ μ-a.e. Let $S \subset T$, $\mu(S) > 0$, $\varepsilon > 0$ and $v \in E_{++}$. Define $x' : T \to E$ by

$$x'(t) = \begin{cases} x(t) + \varepsilon v & \text{if } t \in S \\ e(t) & \text{if } t \notin S. \end{cases} \tag{3.8}$$

Then $x' \in S_\phi$ for all $S \subset T$. Hence,

$$p \cdot [\int_S x + \varepsilon v \, \mu(S) + \int_{T/S} e] > p \cdot e$$

and rearranging we have that $\int_S p \cdot x \geq \int_S p \cdot e$ for any $S \subset T$ since $\varepsilon > 0$ is arbitrary. Thus, it follows (since S is arbitrary) that $p \cdot x(t) \geq p \cdot e(t)$ μ-a.e. Since x is feasible, i.e., $\int x = \int e$ we must have that $\int p \cdot x = \int p \cdot e$ and therefore $p \cdot x(t) = p \cdot e(t)$ μ-a.e. Now (3.7) follows by replacing in (3.8) $x + \varepsilon v$ by $z \in \phi(\cdot)$. We then have that $\int_S p \cdot z + \int_{t \in T/S} p \cdot e \geq \int p \cdot e$. Rearranging we obtain $\int_S p \cdot z \geq \int p \cdot e$, $z \in S_\phi$ and we can conclude (since S is arbitrary) that μ-a.e., $p \cdot z \geq p \cdot e(t)$ for all $z \succ_t x(t)$. The rest of the proof is now identical with the one outlined above in the proof of Theorem 3.1.

Note that in the above step we avoided Lemma 2.2 which requires the use of the measurable selection theorem (recall that for Lemma 2.2, E needs to be separable) and as a consequence we do not need to assume that E is separable. In fact, one does not even need to assume that $\succ_t$ has a measurable graph (recall (a.6)).

It turns out that if one drops the assumption that the positive cone of the space E has a non-empty interior, then the above Theorem fails. A counterexample to that effect can be found in Rustichini-Yannelis (1991). In order to remedy this difficulty we introduce the assumption of commodity pair desirability.

4. EXTREMELY DESIRABLE COMMODITIES AND COMMODITY PAIR DESIRABILITY

For notational convenience, below we drop the subscript t on the preference relation $\succ$. We begin by defining the notion of an extremely desirable commodity. Let E be a Banach lattice and denote its positive cone (which may have an empty norm interior) by E_+. Let $v \in E_+$, $v \neq 0$ and U be an open neighborhood of zero. We say that $v \in E_+$ is an <u>extremely desirable commodity</u> if there exists U such that for each $x \in E_+$ we have that $x + \alpha v - z \succ x$ whenever $\alpha > 0$, $z \leq \alpha v$ and $z \in \alpha U$. In other words, v is extremely desirable if an agent would prefer to trade any commodity bundle z for an additional increment of the commodity bundle v, provided that the size of z is sufficiently small compared to the increment of v. The above notion has a natural geometric interpretation. In particular, let $v \in E_+$, $v \neq 0$, U be an open neighborhood and define the open cone C as follows:

$$C = \{\alpha v - z : \alpha > 0,\ z \in E,\ z \in \alpha U\}.$$

The bundle v is said to be an extremely desirable commodity, if for each $x \in E_+$, we have $y \succ x$ whenever y is an element of $(C + x) \cap E_+$. Note this implies that v is an extremely desirable commodity if for each

$x \in E_+$ we have that $((-C + x) \cap E_+) \cap \{y : y \succ x\} = \emptyset$, or equivalently $-C \cap \{y - x \in E_+ : y \succ x\} = \emptyset$. The latter property is precisely the assumption we need for the core-Walras equivalence if we consider L_p as a commodity space $1 \le p < \infty$ [see Rustichini-Yannelis (1991)].

Recall that if the preference relation $\succ$ is monotone and $\mathrm{int}E_+ \neq \emptyset$, then the assumption of an extremely desirable commodity is automatically satisfied [see for instance Yannelis-Zame (1984)].

We now turn to a strengthening of the above assumption.

A pair $(x,y) \in E_+ \times E_+$ is said to be a <u>desirable commodity pair</u> if for every $z \in E_+$ we have $z + x - y \succ_t z$ whenever $y \le x + z$ for each $t \in T$. The pair $(x,y) \in E \times E$ is said to have the <u>splitting property</u> if for any m-tuple $(s_1,\ldots,s_m) \in E \times \ldots \times E$ such that $\sum_{i=1}^{m} s_i = (x-y)^-$ there exists an m-tuple $(d_1,\ldots,d_m) \in E \times \ldots \times E$ such that $\sum_{i=1}^{m} d_i = (x-y)^+$ and the pair (d_i,s_i) is a desirable commodity pair.

We are now ready to define our key notion.

<u>Definition 4.1</u>: <u>Commodity pair desirability</u> obtains if there exists a $v \in E_+$, $v \neq 0$ and a neighborhood U of zero such that any commodity pair (w,u) of the form $w = \alpha v$, $\alpha > 0$ and $u \in \alpha U$ has the splitting property.

Let us discuss briefly the intuitive meaning of the commodity pair desirability condition. It may be considered as an extension to a multiperson setting of the idea of desirable commodity pair for the case of a single player decision. In this last case a pair (x,y) is desirable if the player is always willing to trade the bundle y in exchange for a bundle x; that is to accept the gain $(x-y)^+$ in exchange for the loss $(x-y)^-$. In the case of the splitting property we can imagine that the offer of exchanging $(x-y)^-$ for $(x-y)^+$ is presented to a group of m players. The splitting property that we introduce asks that no matter how the reduction in the consumption bundle (given by $(x-y)^-$) is allocated over the members of the group, they can always find a way of allocating the surplus ($(x-y)^+$) and make everyone better off.

Obviously the above concept is a substitutability condition which roughly speaking says that an agent would accept a sufficiently small amount of the commodity bundle s_i if he/she would be compensated by consuming more of the desirable commodity bundle d_i.[3]

A couple of comments are in order. First notice that for $m = 1$ in the above definition we have that for any $z \in E_+$, $z + (w-u)^+ - (w-u)^- = z - w-u \succ z$ whenever $u \leq z + w$, $w = \alpha v$, $\alpha > 0$, and $u \in \alpha U$, i.e., for $m = 1$ we are reduced to the assumption of an extremely desirable commodity.

Moreover, it is easy to show that if $\text{int}E_+ \neq \emptyset$ and the preference relation $\succ$ is monotone then the condition of commodity pair desirability is automatically satisfied. Specifically, let $v \in \text{int}E_+$ and U be such that $v + U \subset E_+$, then for any pair (w,u) with $w = \alpha v$, $\alpha > 0$, $u \in \alpha U$ we have $(w-u) \in E_+$ so $(w-u)^- = 0$ and therefore by monotonicity for any $z \in E_+$, $z + w-u \succ z$.

5. CORE-WALRAS EQUIVALENCE IN BANACH LATTICES WHOSE POSITIVE CONE HAS AN EMPTY INTERIOR

In this section we state and prove our main result, i.e., a core-Walras equivalence theorem for a commodity space which can be any arbitrary separable Banach lattice whose positive cone may have an empty norm interior. We begin by stating the following assumptions:

(a.0′) E is any separable Banach lattice.

(a.8) (Commodity pair desirability): There exists $v \in E_+/\{0\}$ and an open neighborhood U of zero such that any commodity pair (w,u) of the form $w = \alpha v$, $\alpha > 0$ and $u \in \alpha U$, has the splitting property.

We are now ready to state and prove the following result:

<u>Theorem 5.1</u>: Under assumption (a.0′), (a.1) - (a.8), $\mathcal{C}(\mathscr{E}) = W(\mathscr{E})$.

<u>Proof</u>: It can be easily shown that $W(\mathscr{E}) \subset \mathcal{C}(\mathscr{E})$. Hence, we will show that if $x \in \mathcal{C}(\mathscr{E})$ then for some price p, the pair (x,p) is a competitive equilibrium for $\mathscr{E}$. Define the correspondence $\phi : T \to 2^{E_+}$ by

(5.1) $$\phi(t) = \{z \in E_+ : z \succ_t x(t)\} \cup \{e(t)\}.$$

Let $C = \bigcup_{\alpha>0} \alpha(v - U)$ where $v \in E_+/\{0\}$ and U given as in (a.8). We claim that:

(5.2) $$c\ell(\int\phi - \int e) \cap -C = \emptyset$$

or equivalently

(5.3) $$(\int\phi - \int e) \cap -C = \emptyset.$$

Since -C is open it suffices to show that for any $y \in {}_{\phi}$ there exists a sequence $\{(\bar{y}^k, \bar{e}^k) : k=1,2,\ldots\}$ such that $\bar{y}^k$ converges in the $L_1(\mu,E)$ norm to y, $\int \bar{e}^k \to \int e$, and

(5.4)
$$\int_T \bar{y}^k - \int_T \bar{e}^k \notin - C.$$

Let S = {t : $y(t) \succ_t x(t)$}, S´ = T/S. Without loss of generality we may assume that $\mu(S) > 0$ (for if $\mu(S) = 0$ then $y(t) = e(t)$ μ - a.e. which implies that $\int y - \int e = \notin - C$. Consequently (5.3) holds.). In the argument below y and e are restricted to S. Moreover denote by μ_S the restriction of μ to S.

Since $y : S \to E_+$ is Bochner integrable and $\succ_t$ is norm continuous (assumption (a.4)) there exist $y^k_1, \ldots, y^k_{m_k}$ in E_+ and T^k_1, $T^k_2, \ldots, T^k_{m_k}$ in τ such that y^k converges in the $L_1(\mu_S, E)$ norm to y, and

(5.5)
$$y^k = \sum_{i=1}^{m_k} y^i_k \chi_{T^k_i}$$

(5.6)
$$y^k_i \succ_t x(t) \textit{ for all } t \in T^k_i \textit{ and all } i,\ i=1,\ldots,m_k, \textit{ and}$$

(5.7)
$$\mu_S(T^k_i) = \xi,\ i=1,\ldots,m_k.$$

Let $e^k = \sum_{i=1}^{m_k} (\int_{T^k_i} e(t)d\mu(t))\ \chi_{T^k_i}$.

In order to establish (5.4) we first show that

(5.8)
$$\int_S y^k - \int_S e^k \notin C.$$

Suppose that (5.8) is false, then

$$\sum_{i=1}^{m_k} y^k_i\xi - \sum_{i=1}^{m_k} e^k_i\xi \in - \alpha(v + U), \textit{ and therefore}$$

(5.9) $$\sum_{i=1}^{m_k} y_i^k + w - u = \sum_{i=1}^{m_k} e_i^k,$$

where $x = \frac{\alpha}{\xi} v$, $u \in \frac{\alpha}{\xi} U$.

Since $\sum_{i=1}^{m_k} e_i^k \geq 0$, it follows from (5.9) that

(5.10) $$(w - u)^- \leq \sum_{i=1}^{m_k} y_i^k.$$

Applying the Riesz Decomposition Property to (5.10) we can find $(s_1, \ldots, s_{m_k}) \in E_+ \times \ldots \times E_+$ such that

(5.11) $$\sum_{i=1}^{m_k} s_i = (w - u)^- \; 0 \leq s_i \leq y_i^k \textit{ for all } i.$$

It follows from the assumption of commodity pair desirability that there exists an m_k-tuple $(d_1, \ldots, d_{m_k}) \in E_+ \times \ldots \times E_+$, such that $\sum_{i=1}^{m_k} d_i = (w - u)^+$ and

(5.12) $$\tilde{y}_i^k = y_i^k + d_i - s_i \succ_t y_i^k \textit{ for all } t \in T_i^k \textit{ and for all } i.$$

Note that since $y_i^k \geq s_i$ it follows that $\tilde{y}_i^k \in E^+$. Moreover, since $y_i^k \succ_t x(t)$ for all $t \in T_i^k$ and all i, and $\tilde{y}_i^k \succ_t y_i^k$ for all $t \in T_i^k$ and all i, by transitivity of $\succ_t$ we have that $\tilde{y}_i^k \succ_t x(t)$ for all $t \in T_i^k$ and all i. Also,

$$\sum_{i=1}^{m} \tilde{y}_i^k \xi = \sum_{i=1}^{m_k} e_i^k \xi = \int e,$$

Define $\tilde{y}^k = \sum_{i=1}^{m_k} \tilde{y}_i^k \chi_{T_i^k}$. Notice that $\int_S \tilde{y}^k = \int_S e$. Therefore, we have found an allocation $\tilde{y}^k(\cdot)$ feasible for the coalition S and preferred to $x(\cdot)$ which in turn was assumed to be in the core of $\mathscr{E}$, a contradiction. Hence, (5.8) holds.

We are now ready to construct the sequence $\{(\bar{y}^k, \bar{e}^k) : k=1,2,\ldots\}$. In particular, define $\bar{y}^k : T \to E_+$ by

$$\bar{y}^k(t) = \begin{cases} y^k(t) & if\ t \in S \\ y(t) & if\ t \notin S. \end{cases}$$

Similarly define $\bar{e}^k : T \to E_+$ by

$$\bar{e}^k(t) = \begin{cases} e^k(t) & if\ t \in S \\ e(t) & if\ t \notin S. \end{cases}$$

Note that $\int_T \bar{y}^k - \int_T \bar{e}^k \notin -C$ and therefore (5.4) holds. We can now separate the convex nonempty set $c\ell \int \phi - \int e$ from the convex nonempty set $-C$. Proceeding as in the proof of Theorem 3.1 one can now complete the proof.

6. CONCLUDING REMARKS

Remark 6.1: Since in Banach lattices whose positive cone has a nonempty (norm) interior and preferences are monotone, assumption (a.8) is automatically satisfied, it follows that in such spaces Theorem 3.1 becomes a corollary of Theorem 5.1. However, since in Theorem 3.1 the commodity space is any arbitrary ordered Banach space (i.e., no lattice structure is required) we cannot derive Theorem 3.1 as a corollary of Theorem 5.1.

Remark 6.2: Notice that in Theorem 5.1 the commodity space was assumed to be a Banach lattice. However, in Theorem 3.1 we only needed the commodity space to be an ordered Banach space, i.e., no lattice structure was required. The reason we needed the lattice structure in Theorem 5.1 was to apply the Riesz Decomposition Property which in turn has a natural economic meaning as was indicated in Yannelis-Zame (1986, p. 89). However, we do not know whether or not one can dispose with the lattice structure in Theorem 5.1. Aliprantis-Burkinshaw (1991) have shown that the lattice structure is a necessary and sufficient condition for the existence of Edgeworth equilibria in economies with finitely many agents.

<u>Remark 6.3</u>: Using the notion of commodity pair desirability one can easily prove the second welfare economics theorem for economies with an atomless measure space of agents and with a commodity space which can be any arbitrary separable Banach lattice.

<u>Remark 6.4</u>: It is of interest to know whether or not under the assumption of Theorem 3.1 we have core-Walras existence as well. It is known, for instance, that this is not true if $X = R^n$ (see Aumann (1966, Section 8, p. 17)); this is also the case here. For instance, under (a.0) - (1.7) a competitive equilibrium may not exist. However, by replacing (a.2) by

(a.2′) $X(t) = K$ for all $t \in T$, where K is a weakly compact convex non-empty subset of E_+,

and adding the assumption:

(a.11) for all $t \in T$, and for all $x(t) \in X(t)$ the set $\{y : y \succ_t x(t)\}$ is convex and the set $\{y : x(t) \succ_t y\}$ is norm open, one can conclude, by virtue of the main theorem in Khan-Yannelis (1986) that both sets, i.e., $\mathcal{C}(\mathcal{E})$ and $W(\mathcal{E})$ are non-empty. Notice that in Khan-Yannelis it is assumed that preferences are convex (see also Bewley (1991)), an assumption which is dispensible in the finite dimensional case. In fact as Aumann (1964) showed, the Lyapunov convexity theorem convexifies the aggregate demand set. However, in infinite dimensional spaces, the Lyapunov convexity result fails. It is also worth mentioning that in addition to the failure of Lyapunov's theorem, Fatou's Lemma fails in infinite dimensional spaces as well (see Yannelis (1988) or Rustichini (1989)). Hence, there is no exact analogue to Schmeidler's version of Fatou's Lemma in infinite dimensions. However, approximate versions of Fatou's Lemma have been obtained by Khan-Majumdar (1986) and Yannelis (1988). Moreover, with additional assumptions exact versions of Fatou's Lemma in infinite dimensional spaces can be obtained as well (see Rustichini (1989) and Yannelis (1988)). Nevertheless, the latter versions of the Fatou Lemma are not sufficient to prove the existence of Walrasian equilibrium. For more information on this problem see Rustichini-Yannelis (1991a), who have showed that in economies with "many more" agents than commodities, there is a nice convexifying effect on aggregation and the Fatou Lemma still holds. As a consequence of this, the theorem on the existence of a Walrasian equilibrium is still true.

FOOTNOTES

[1]Hildenbrand (1972, p. 85), attributes the proof to Schmeidler.

[2]This is so since intE_ is an open set. In particular if A and B are subsets of any topological space an B is open, then it can be easily seen that $A \cap B = \emptyset$ if and only if $c\ell A \cap B = \emptyset$.

[3]Note that since we have allowed splitting for $(w - u)^-$ and $(w - u)^+$, we may think of i (i=1,2,...,m) as agents and the splitting property as a kind of redistribution. Hence, the notion of commodity pair desirability is a "coalitional-type" of uniform properness.

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