Florence Chase

Mamie Hooper

The numerator shows the number of parts taken.

The denominator shows the number of parts into which the whole is divided.

13)2

66

5000

2
2
4 (2)
2
8 (3)
2
16 (4)
2
32 (5)
2
64 (6)
2
128 (7)
2
256 (8)
2
512 (9)
2
1024

m + h

Eg + 0

THE

COMMON SCHOOL ALGEBRA.

BY

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BOSTON:
TAGGARD & THOMPSON,
29 CORNHILL,
1867.

STEREOTYPED AT THE
BOSTON TYPE AND STEREOTYPE FOUNDRY.

PREFACE.

THE great difficulty, in the study of Algebra, is to attain a clear comprehension of the earliest steps. The first principles should, therefore, be communicated to the learner gradually, and in the most simple and intelligible manner.

Experience proves that these principles are most successfully taught by means of easy problems. But even when this mode is pursued, a majority of pupils find trouble in expressing algebraically the conditions of the problems. The author has, therefore, placed at the commencement of his work a series of introductory exercises, designed to familiarize the learner with representing quantities and performing the simplest algebraic processes, also to prepare him for putting problems into equations.

These introductory exercises, which were written about three years since, were shown to several excellent teachers, and received their approbation. They were subsequently used in two of the Boston schools, and with such success, that the author was solicited by a number of gentlemen, who were acquainted with his "Elements of Algebra," and who knew his plan in the present work, to prepare a treatise for common schools.

An attempt has been made to render the science as easily attainable as possible, without prejudice to the main result; not to save the learner the trouble of thinking and reasoning, but to teach him to think and reason; not merely to supply a series of simple exercises, but to insure a good knowledge of the subject. To what extent the writer has attained his object, is left to intelligent instructors, school-committees, and others, to determine.

Teachers and pupils will observe that, to represent multiplication, the full point is generally used in this work rather than the sign $\times$. But to distinguish the sign of multiplication from the period used as a decimal point, the latter is elevated by inverting the type, while the former is larger, and placed down even with the lower extremities of the figures or letters, between which it stands.

THOMAS SHERWIN

CONTENTS.

COMMON SCHOOL ALGEBRA.

PRELIMINARY EXERCISES.

Art. **1.** Algebra has been called "Universal Arithmetic," and is principally distinguished from common arithmetic by this, that in algebra calculations are performed by means of letters and signs; letters being used to represent quantities, and signs to indicate operations, or to stand for certain words.

The sign $+$, called *plus*, which signifies *more*, represents *addition*. Thus, $6+4$ represents the addition of 6 and 4, or indicates the *sum* of these numbers.

The sign $-$, called *minus*, which signifies *less*, represents *subtraction*, and is placed immediately before the quantity to be subtracted. Thus, $8-3$ represents the subtraction of 3 from 8, or indicates the *difference* of these numbers.

Multiplication is represented by a full point, or the sign $\times$, placed between the quantities to be multiplied. Thus, 7.3, or 7×3, represents the multiplication of 7 by 3, or indicates the *product* of these numbers.

Division is represented in the form of a fraction, the dividend being placed over the divisor; also by the sign : or $\div$. Thus, each of the expressions, $\frac{3}{7}$, $3:7$, and $3 \div 7$, represents the division of 3 by 7, or indicates the *quotient* arising from that division When the sign : or

$\div$ is used, the quantity preceding the sign is the dividend, and that following it, the divisor.

Instead of the words "equal to," "equals," or others of similar meaning, we use the sign $=$, which is called the sign of *equality*. Thus, $5+3=8$, is read, "5 plus 3 equals 8."

Instead of the words "greater than," or "less than," we use the sign $>$ or $<$, called the sign of *inequality*. Thus, $6>4$ is read, "6 greater than 4," and $3<5$ is read, "3 less than 5;" the open end of the sign being turned towards the greater quantity.

The sign $\therefore$ is used instead of the word "therefore" or "consequently."

Art. 2. An axiom is a self-evident truth; the following are of this nature.

Axioms.

1. If the same quantity, or equal quantities, be *added* to equal quantities, the *sums* will be equal.

2. If the same quantity, or equal quantities, be *subtracted* from equal quantities, the *remainders* will be equal.

3. If equal quantities be *multiplied* by the same quantity, or by equal quantities, the *products* will be equal.

4. If equal quantities be *divided* by the same quantity, or by equal quantities, the *quotients* will be equal.

5. If the same quantity be both *added to* and *subtracted from* another, the value of the latter will not be changed.

6. If a quantity be both *multiplied* and *divided* by another, its value will not be changed.

7. Two quantities, each of which is equal to a third, are equal to each other.

8. The whole of a quantity is greater than a part of it.

9. The whole of a quantity is equal to the sun of all its parts.

ART. **3.** 1. Two boys, A and B, had together 40 apples; but B had 3 times as many as A. How many had each?

It is manifest that, if we knew the number A had, by multiplying by 3 we should find the number B had. The number that A had may, therefore, be called the *unknown quantity*. From the conditions of the question, we know that once the unknown quantity and 3 times that quantity must make 40. But for conciseness, and in order to avoid the repetition of the words "unknown quantity," we may use a symbol to represent that quantity. The symbols used in algebra to represent unknown quantities, are the last letters of the alphabet, as x, y, z; the first letters of the alphabet, when used, commonly represent known quantities.

Let x represent, in the question, the number of apples A had. Then $3x$ will represent the number B had. Hence,

$x+3x=40$. Combining x and $3x$, we have

$4x=40$; and $1x$ or x must be $\frac{1}{4}$ of 40, or

$x=10$, the number A had,

$3x=30$, the number B had.

2. The united ages of three brothers, A, B, and C, amount to 120 years. B is twice as old as A, and C's age is equal to the sum of the ages of his two brothers. Required the age of each.

Let x represent A's age; then B's will be represented by $2x$, and C's by $x+2x$, or $3x$. Hence,

$x+2x+3x=120$. Combining, we have

$6x=120$;

$\therefore x=20$ years, A's age.

$2x = 40$ years, B's age; and
$3x = 60$ years, C's age.

We have seen, in the two preceding questions, the use which may be made of letters to represent quantities whose value is to be found. We shall now give some exercises in representing quantities, and in performing very simple operations.

ART. 4. 1. If x represent the price of an apple in cents, what will represent the price of 2 apples? Of 3, 4, 5, 6? *Ans.* $2x$, $3x$, $4x$, $5x$, $6x$.

2. If one pear cost x cents, what will 4, 6, 9, 11 pears cost?

3. If x represent the number of miles a man can travel in one day, what will represent the number of miles he can travel in 3, 5, 7, 10, 15, 30 days?

4. If one yard of cloth cost x dollars, what will represent the price of 2, 7, 9, 18, 20 yards?

5. If $2x$ dollars represent the price of one barrel of flour, what will represent the price of 2, 3, 4, 5, 6, 7, 8, 9, 10 barrels?

6. If $3x$ represent the number of shillings a man earns in a day, what will represent the number of shillings he earns in 2, 5, 6, 8, 10, 12 days?

7. If $4x$ cents represent the value of one pound of coffee, what will represent the value of 3, 6, 12, 13 pounds?

8. If $5x$ represent the weight of one bag of coffee in pounds, what will represent the weight of 2, 3, 4, 7, 9, 15 bags?

Remark. The learner will perceive that the preceding are examples of multiplication. Thus, $3x$ is the product of 3 and x, and $12x$ is the product of x and 12, of $2x$ and 6, or of $3x$ and 4.

9. How much is 5 times x? *Ans.* $5x$. How much is x times 5? *Ans.* $5x$.

10. If one sheep cost 9 s., what will represent the price of a number x of sheep? Of $2x$, $3x$, $4x$, $5x$, $6x$ sheep?

11. If one man earn 10 s. per day, what will represent the daily wages of x, $2x$, $3x$, $5x$, $7x$, $11x$ men?

12. If x represent the daily wages of one man in shillings, what will represent the earnings of 2 men in 3 days, of 3 men in 3 days? of 4, 5, 6 men in the same time?

13. If x represent the daily wages of one man, what will represent the wages of 2 men for 5, 6, 7, 8, 9, 10 days?

14. If x represent the daily wages of one man, what will represent the wages of 2 men for 3 days? of 3 men for 4 days? of 4 men for 5 days? of 6 men for 7 days? of 7 men for 8 days?

15. If x represent, in shillings, the price of one yard of cloth, what will be the price of 1 piece, 3 yds. long? of 2 pieces, each 4 yds. long? of 5 pieces, each 6 yds. long?

16. If one man earn \$2 per day, what will represent the wages of 2, 3, 4, 5, 6, 7 men for $3x$ days? And what will represent the wages of 3, 5, 7, 11, 13 men for $5x$ days?

17. What is 7 times $2x$? 8 times $3x$? 10 times $5x$? 12 times $6x$?

18. What is x times 4 s.? $2x$ times 5 s.? $3x$ times 6 s.? $7x$ times 9 s.?

19. If x represent the price of a cow, and an ox is worth twice as much as a cow, what will represent the price of 5 oxen?

20. If a yard of black cloth cost twice as much as a yd. of white, and a yd. of blue cost 3 times as much as a

yd. of black, what will represent the price of 4 yds. of black, and what the price of 5 yds. of blue, the price of a yd. of white being represented by x?

Art. 5. The addition of quantities, as has been already stated, is expressed by means of the sign $+$ Thus, $3+4+6$, means that 3, 4, and 6 are added together, and is read, "3 plus 4 plus 6."

1. If x represent the price of an apple, and $2x$ that of a pear, what will represent the price of an apple and a pear together? Ans. $x+2x$.

2. If $3x$ represent the price of a cow, and $4x$ that of an ox, what will represent the price of both?

3. If x, $2x$, and $3x$ represent the respective ages of 3 men, what will represent the sum of their ages?

4. If $3x$, $4x$, $5x$, and $6x$ represent the respective lengths of 4 pieces of cloth, what will represent the length of the whole?

5. If apples cost 2 cents each, and pears 3 cents each, what will represent the whole cost of x apples and x pears?

6. If corn cost 4 s., rye 6 s., and wheat 8 s. per bushel, what will represent the sum of the prices of x bushels of each?

7. What will represent the entire price of $3x$ bushels of rye, at 5 s. a bushel, and $6x$ bushels of wheat, at 9 s. a bushel?

8. If x represent A's money, and B have 3 times and C 4 times as much as A, what will represent the amount of their money?

9. If x, $2x$, and $3x$ represent the respective daily wages of A, B, and C, what will represent the amount of their wages, A working 9, B 6, and C 5 days?

10. A man has 3 sons, whose ages are such that the 2d is twice as old as the youngest, and the eldest twice as

old as the 2d. What will represent the sum of their ages, that of the youngest being represented by x?

11. A has twice as much money as B, and B 3 times as much as C. What will represent the amount of their money, if x represent C's?

12. A is of a certain age, represented by x, B twice as old, C 3 times as old as B, and D as old as B and C both. What will represent the united ages of all four?

13. If John's money be represented by x, and Joseph's by y, what will represent the amount of their money?

14. If x represent A's age, and y B's, what will represent the sum of 3 times A's and 7 times B's?

15. If $3x$ represent the price of a barrel of beer, and $2y$ the price of a barrel of cider, what will represent the whole cost of 7 barrels of beer and 5 of cider?

16. If x represent A's age, and y B's, and C be twice as old as A, and D 3 times as old as B, what will represent the united ages of all?

17. If one town be $3x$ miles north of Boston, and another $4x$ miles south of Boston, what will represent the distance of these towns asunder?

18. Two men start, at the same time, from the same place, and travel in opposite directions. If one travel x miles and the other y miles per hour, what will represent their distance apart at the end of 2, 3, 5, 7 hours respectively?

Art. 6. 1. If $4x$ represent the price of 2 yds. of cloth, what will represent the price of one yd.?

Ans. $2x$.

2. If $10x$ represent the price of 2 oxen, what will represent that of one?

3. If $15x$ represent the worth of 5 barrels of flour, what will represent that of 1, 2, 3, 4 barrels?

Ans. $3x$, $6x$, $9x$, $12x$

4. If $20x$ represent the price of 10 peaches, what will represent the price of 1, 2, 3, 4, 5, 6, 7, 8, 9 peaches?

5. When $12x$ represents the worth of 4 yds. of cloth, what will represent that of 1, 2, 3, 5, 6, 7, 8, 9, 10, 12 yds.?

6. When $9x$ represents the number of miles a man travels in 3 days, what will represent the number of miles he travels in 5, 7, 9, 11, 20 days?

7. If $6x$ represent the price of 3 eggs, what will represent that of 9, 12, 13, 15, 17 eggs?

8. If $27x$ represent the number of miles a ship sails in 9 hours, what will represent the distance sailed over in 12, 15, 17, 24, 48 hours?

9. If $48x$ represent the distance a vessel sails in a day, or 24 hours, what will represent the distance she sails in $1\frac{1}{2}$, 2, $2\frac{1}{2}$, 3, $3\frac{1}{4}$ days?

10. If $4x$ represent the price of a bushel of corn, what will represent that of $\frac{1}{2}$ a bushel? of $1\frac{1}{2}$, $2\frac{1}{2}$, $3\frac{1}{2}$ bushels?

11. If $2x$ represent the price of a bushel of wheat, and wheat be worth twice as much as corn, what will represent the price of 3, 5, 7, 9, 10, 12 bushels of corn?

12. If $2x$ represent the price of corn per bushel, and wheat be worth 3 times as much as corn, and rye $\frac{1}{2}$ as much as wheat, what will represent the price of 1, 2, 3, 4, 5, 6, 7 bushels of rye?

13. If $6x$ represent the price of a barrel of flour, and a barrel of sugar be worth 2 barrels of flour, what will represent the price of 1, 3, 5, 7, 8, 9 barrels of rice, supposing rice worth $\frac{1}{3}$ as much per barrel as sugar?

14. If $4x$ represent the price of 2 barrels of cider, and $6x$ that of 3 barrels of beer, what will represent the sum of the prices of 1 barrel of each? of 4 barrels of beer and 2 of cider? of 6 barrels of beer and 5 of cider? of 9 barrels of beer and 14 of cider?

15. If $10x$ represent the weight of 5 boxes of sugar, and $4x$ that of 2 boxes of raisins, what will represent the whole weight of 3 boxes of sugar and 5 of raisins? 8 of sugar and 7 of raisins? 17 of sugar and 20 of raisins?

16. When $4x$ represents the price of corn, and $8x$ the price of beans, per bushel, what will represent the entire price of 3 pecks of corn and 6 pecks of beans? 9 pecks of corn and 7 pecks of beans?

17. Supposing that $6x$ represents the distance one man travels in 3 days, and $8y$ the distance another travels in 4 days; if they start at the same time, from the same place, and travel in opposite directions, what will represent their distance apart at the end of 1 day? of 2, 3, 4, 5 days?

X

ART. **7.** 1. If x represent A's age, and B be 2 years older, what will represent B's age? *Ans.* $x+2$.

2. If x represent A's age in years, and B be 9 months older, what will represent his age in months?

Ans. $12x+9$.

3. If x represent the distance, in miles, of a certain town from N. York, what will represent the distance from that city of another town, 12 miles more remote?

4. If $2x$ shillings be the price of corn per bushel, and wheat be worth 4 s. per bushel more than corn, what will represent the price of a bushel of wheat?

5. If x represent A's age in years, what will represent B's age, and what C's; B being 2 years older than A, and C 3 years older than B?

Ans. B's, $x+2$; and C's, $x+2+3$, or $x+5$.

6. A man has 4 sons, each of whom is 3 years older than his next younger brother. What will represent the ages of the 3 elder, if x represent the age of the youngest in years?

7. If black cloth cost 4 s a yard more than white, and

blue 5 s. a yard more than black, what will represent the price of a yard of the black and blue respectively, x representing the number of shillings given for a yard of white?

8. A started from Boston 2 hours before B, and 3 hours before C, and they all arrived at Philadelphia at the same time. What will represent the time B and C respectively were on the road, x representing the number of hours in which A performed the journey?

9. A's horse is worth \$5 more than B's, and B's is worth one eagle more than C's. What will represent, in dollars, the price of A's and B's respectively, if x represent that of C's?

10. A is worth \$100 more than twice what B is worth. What will represent A's estate, if x represent B's, in dollars?

11. A is 4 years more than twice as old as B, and B is 3 times as old as C. What will represent the age of A and B respectively, if x represent the number of years in C's?

12. A can earn $\frac{1}{2}$ as much as B, and C 2 s. more than twice as much as B, in a day. What will represent B's and C's daily wages, if x represent the number of shillings A earns per day?

13. A man has a number x of swine, 5 more than 3 times as many cows as swine; and his number of sheep is equal to that of his swine and cows together. What will represent the number of cows and sheep respectively? and what will represent the aggregate number of his whole stock?

14. What will represent 4 more than twice $3x$? 7 more than 4 times $3x$? 10 more than 3 times $5x$?

15. What will represent 3 more than $\frac{1}{2}$ of $4x$? 10 more than $\frac{1}{3}$ of $9x$? 7 more than $\frac{1}{5}$ of $20x$?

16. What will represent the sum of x and 2? of $3x$ and 4? of $6x$ and 9? of $7x$ and 10?

17. A is 2 years older than B, and B 4 years older than C. What will represent the sum of their ages, C's being represented by x?

18. B has \$4 more than A, and \$5 less than C. What will represent the number of dollars B and C have, respectively, that which A has being x?

19. A begins trade with x dollars, and gains \$100; B begins with twice as much money as A, and gains \$150; C begins with as much as A and B both, and gains \$70. What will represent the amount of their property, including stock and gain?

ART. 8. 1. If x represent the number of dollars A has, and B have \$2 less, what will represent B's money?

Ans. $x - 2$.

2. If x represent the number of cows a farmer has, what will represent the number of his cows after he shall have sold 10?

3. If x shillings represent the price of a bushel of wheat, and rye be 4 s. a bushel cheaper, what will represent the price of a bushel of rye?

4. A drover, having 100 sheep, sold a number of them represented by x. What will represent the number he had remaining?

Ans. $100 - x$.

5. Of 150 acres of land, a part is cultivated, and the rest is woodland. If x represent the number of acres cultivated, what will represent the number of acres of woodland?

6. The sum of two numbers being 20, what will represent the second, if x represent the first?

7. A man has 50 coins, consisting of dollars and sovereigns. If x represent the number of dollars, what will represent that of the sovereigns?

8. A walked a number of miles, represented by x; B

walked twice as far wanting 4 miles. What will represent the distance B walked?

9. A man having a certain number, x, of dollars, doubled his money, and afterwards lost $12. What will represent the money he then had?

10. A draper, having 50 yards of cloth, sold a number of yards, represented by x. What will represent the number of yards remaining?

11. A grocer, having 500 lbs. of coffee, sold 3 bags, each containing x lbs. What will represent the number of pounds remaining?

12. A has x, and B, y dollars. If A give B $5, what will represent the number of dollars each has then?

Ans. A, $x - 5$, and B, $y + 5$.

13. If A have $3x$, and B, $2y$ dollars, what will represent their money, after B shall have given A $10?

14. A farmer has 100 sheep in one pasture and 75 in another. What will represent the number in each flock, after he shall have taken $2x$ sheep from the larger and put them with the smaller?

15. A and B have each 50 cows. A sells a number, x, and B sells 3 times as many. What will represent the number each has left?

16. A poulterer has x turkeys and y geese. After selling 4 turkeys for 15 geese, what will represent the number of each he then has?

17. If a woman be x years old, and her husband twice as old as she, at the time of marriage, what will represent their respective ages 5 years before marriage?

X

ART. 9. According to Ax. 5, if the same quantity be both added to, and subtracted from another, that other will not be changed in value. Thus, if 2 be added to 8, the sum will be 10. Now, if 2 be subtracted from 10, the remainder will be 8, the same as at first.

If these operations are represented merely, the result will be expressed thus, $8 + 2 - 2$, in which $+2$ and -2 destroy or cancel each other, and leave 8.

But if more is added to a quantity than is subtracted from it, that quantity is increased by the excess of what is added above what is subtracted. Thus, $12 + 7 - 6$ is the same as $12 + 1$, or 13, because 1 more is added to 12 than is subtracted from it.

On the other hand, if more is subtracted from a quantity than is added to it, that quantity is diminished by the excess of what is subtracted above what is added. Thus, $15 - 7 + 3$, or $15 + 3 - 7$, is the same as $15 - 4$, or 11.

In like manner, $x + 5 - 3$, is the same as $x + 2$; and $x - 8 + 5$, is the same as $x - 3$.

1. What is the same as $x - 4 + 4$? as $x + 6 - 6$? as $5 - 5 + x$? as $-7 + 7 + 2x$?

2. What is the same as $x + 9 - 3$? as $x - 9 + 3$? as $2x - 7 + 3$? as $3 - 7 + 2x$? as $5x - 5 + 7$? as $5x + 7 - 9$?

3. What is the same as $4x - 6 + 3$? as $7x + 10 - 4$? as $12x - 12 + 7$?

The several parts of an algebraic quantity, connected together by the signs $+$ and $-$, are called *terms*. Thus, in $7x - 10 + 5x + 3$, $7x$, -10, $5x$, and 3 are the separate terms.

When a quantity contains many terms, all those consisting of numbers only may be reduced to one single term, and those which are alike with regard to the letters, to another. But it must be remembered, that when a term has no sign before it, it is supposed to have the sign $+$. Thus, $x + 2x + 5x$ is the same as $8x$; $4x + 2 + 3x + 7$ is the same as $7x + 9$; $10x - 2 + 3x - 5$ is the same as $13x - 7$. In the last example, $10x$ and $3x$ are added, whilst 2 and 5 are both subtracted, which is the same as subtracting their sum, 7

In a similar manner, $10x + 7 + 3x - 5$ is the same as $13x + 2$; $12x - 8 + 4x + 3$ is $16x - 5$; and $12x + 3 - 3x - 8$ is $9x - 5$.

Again, $4x + 10 + 3x - 7 + 8x - 3 - 4x + 6 - 3x - 4$ is $8x + 2$; $3x - 7 + 4x + 2 - x + 1 + 5x - 3 - 5x$ is $6x - 7$; $12 - 3x + 7 + 2x - 3 - 5x$ is $16 - 6x$; and $7 + 3x - 4 + 2x - x + 2$ is $5 + 4x$.

In the last four examples, we combine all the terms containing x, and preceded by the sign $+$, expressed or implied; then combine all those containing x, and preceded by the sign $-$, and take the difference of the two sums, giving it the sign of the greater sum. With the numbers we proceed in the same way.

4. What is the same as $x + 2 + 3x - 4 + 5x + 7$? as $4x - 2 + 2x + 7 + 3x - 9 + x + 10$? as $11x - 6 + 3x + 7 - 6x - 4$? as $7 + 3x - 6 + 4x + 10 - 13x$? as $5x + 7 - 10x + 3 - 4x - 2 + x + 5$?

Let the answers to the following questions be reduced in the manner shown above.

5. A is x years old; B is twice as old as A, and three years more; and C's age is 2 years less than the sum of A's and B's. What will represent C's age?

Ans. $x + 2x + 3 - 2$, or $3x + 1$.

6. A has x dollars, and B has twice as many. A gains \$20, and B loses \$5. What will then represent the sum of A's and B's money?

Ans. $x + 20 + 2x - 5$, or $3x + 15$.

7. A has \$50, and B \$30. A loses x dollars, and B gains 4 times as much money as A loses. What will then represent the amount of their money?

Ans. $50 - x + 30 + 4x$, or $80 + 3x$.

Observe that, in the three preceding examples, the addition of quantities containing several terms is performed by writing them after each other, without altering the signs; after which the result is reduced. That the signs should not be changed, may easily be shown by figures. Thus, $12-5$ and $17-3$, when added, give $12-5+17-3$, which is the same as $29-8$ or 21; for $12-5$ is 7, and $17-3$ is 14, and the sum of 7 and 14 is 21.

8. An army consists of x officers, 6 more than 3 times as many cavalry, and 30 less than 10 times as many infantry as officers. What will represent the number of men in the army?

9. A is 10 years younger, and C 20 years older, than B. What will represent the sum of their ages, if x represent the number of years in B's age?

10. Four towns are in a straight line, and in the order of the letters A, B, C, and D. The distance from A to B is 20 miles more, and the distance from C to D is 30 miles less, than the distance from B to C. What will represent the whole distance from A to D, if x represent the number of miles from B to C?

11. Three men owed x guineas. A could pay the whole debt wanting 10 guineas, B could pay it wanting 20 guineas, and C could pay it and have 15 guineas left. What will represent the number of guineas they all had?

12. A man has lived x years in France, 5 years more than twice as long in England, and 15 years less than 3 times as long in America. These being his only places of residence, what will represent his age?

ART. **10.** When a quantity containing several terms is to be multiplied, each term must be multiplied, and the signs remain unchanged, except under particular circumstances, which will be hereafter explained.

Thus, 3 times $2+5$ is $6+15$; for $2+5$ is 7, and 3 times 7 is 21, the same as $6+15$. In like manner, 5 times $x+10$ is $5x+50$.

Again, 4 times $9-3$ is $36-12$; for $9-3$ is 6, and 4 times 6 is 24, the same as $36-12$. In like manner, 7 times $3x-4$ is $21x-28$, and 9 times $7-2x$ is $63-18x$.

1. A, having x dollars, gains \$3, and afterwards doubles his money. What will represent what he then has?

Ans. $2x+6$.

2. A man, having $3x$ sheep, sells 5 of them, and afterwards triples his stock. What will then represent the number of his sheep? *Ans.* $9x-15$.

3. The ages of three brothers are as follows: the age of the eldest is 4 years more than twice that of the second, and that of the second is 2 years more than 3 times that of the youngest. What will represent the sum of their ages, the youngest being x years old?

4. Two men, A and B, have together \$100. At the end of a year, A has \$5 more than double what he had at first, and B has \$20 less than 3 times what he had at first. What will represent the amount of what both had then, if x represent the number of dollars A had at first?

5. What is 5 times the quantity represented by $8x+5$? 7 times $20-3x$? 9 times $3x+y-5$?

6. A has x dollars, and B has \$5 more than twice as much. B gives A \$7, after which each doubles the money he then has. What will then represent the money each has? what the amount of their money?

7. A lends B \$5, and doubles the money he has left; after which B doubles his money, and then repays A. What will then represent the stock of each, and what their joint stock, if A had $4x$ dollars at first, and B half as much?

8. A man paid away \$3 each morning, and doubled the remainder of his money during the day. If x represent the number of dollars he had at first, what will represent his money at the end of the 5th day?

9. A merchant doubles his stock each year, wanting £1,000. What will represent his stock at the end of the 4th year, if he begins trade with x pounds?

10. A has £100, and B £75. A gives B x pounds of his money, after which A doubles his money, and B triples his. What will then represent the whole of their money?

ART. **11.** When a quantity containing several terms is to be divided, each term must be divided, and the signs remain unchanged, except under particular circumstances, to be hereafter explained.

Thus, half of $10+4$ is $5+2$; for $10+4$ is 14, half of which is 7, the same as $5+2$. Again, $\frac{1}{3}$ of $15-6$ is $5-2$; for $15-6$ is 9, $\frac{1}{3}$ of which is 3, the same as $5-2$.

In like manner, $\frac{1}{5}$ of $20+10x$ is $4+2x$; $\frac{1}{7}$ of $21x-14$ is $3x-2$; $\frac{1}{9}$ of $27-18x$ is $3-2x$; and $\frac{1}{4}$ of $100-20x$ is $25-5x$.

1. B is 4 years older than A, and C's age is half of the sum of A's and B's. If x represent the number of years in A's age, what will represent C's?

2. B is 4 years older than A, and 7 years younger than C, and they have a sister whose age is $\frac{1}{3}$ of the sum of their ages. What will represent the sister's age, if x represent A's in years? What will represent hers, if x represent C's?

3. A man, after having doubled his money, lost \$6; he afterwards gained half as much as he then had. What will represent his last gain, if x represent the number of dollars he had at first?

4. A and B begin trade with equal stocks. A loses £50, and B gains £100; after which each gains $\frac{1}{2}$ the amount of what both have. What will represent what each then has, if each had x pounds at first?

5. A horse cost $12 more than 3 times as much as a cow, and an ox $\frac{2}{3}$ as much as the horse. What will represent the price of the ox, if the cow cost x dollars?

Ans. $2x + 8$.

6. A man receives for a year's wages $12x + 60$ dollars. What will represent his wages for a month? what for 7 months?

7. A man paid for a piece of blue cloth 5 s. more, and for a piece of white, 20 s. less, than he did for a piece of black. What will represent the average price per piece, if a piece of black cost x shillings?

8. A man has 6 sons, each being 6 years older than his next younger brother. What will represent their average age, if the youngest is x years old?

Art. **12.** When division cannot be exactly performed, it is expressed in the form of a fraction. Thus, $\frac{1}{3}$ of 2 is $\frac{2}{3}$, $\frac{1}{4}$ of 5 is $\frac{5}{4}$. In like manner, $\frac{1}{2}$ of the quantity x is $\frac{x}{2}$; $\frac{1}{5}$ of $2x$ is $\frac{2x}{5}$, the divisor always being placed under the dividend.

1. A had a number of dollars, represented by x, and B had half as much money as A. What will represent B's money?

Ans. $\frac{x}{2}$.

2. In a certain school, $\frac{1}{2}$ of the scholars learn to read, $\frac{1}{3}$ learn arithmetic, and $\frac{1}{8}$ learn algebra. What will represent the number in each of these classes respectively, if x represent the whole number of scholars?

3. A farmer bought x sheep, and half as many cows. What will represent the whole number of both?

Ans. $x+\frac{x}{2}$.

Remark. When fractions are represented as added or subtracted, the sign $+$ or $-$ should be placed even with the line separating numerator and denominator.

4. A boy, having x cents, doubled his money, and then lost $\frac{1}{3}$ of what he had. What will represent the number of cents he lost?

5. A's age is 5 years more than $\frac{1}{2}$ of B's. What will represent the sum of their ages, if x represent B's age in years?

6. Two men engaged in trade with equal stocks. The first gained \$5 more than $\frac{1}{3}$ of his stock, and the second gained \$10 less than $\frac{1}{2}$ his stock. What will represent the sum of their gains, if the stock of each was x dollars?

7. A woman, at the time of her marriage, was $\frac{3}{4}$ as old as her husband. The husband being x years old at the time of marriage, what will represent his wife's age 5 years before? what 10 years after?

ART. **13.** The division of a quantity consisting of several terms, is likewise represented by putting the divisor under the whole of the terms. Thus, $\frac{1}{2}$ of $3+4$ is $\frac{3+4}{2}$; $\frac{1}{5}$ of $7-3$ is $\frac{7-3}{5}$. In like manner, $\frac{1}{3}$ of $x+2$ is $\frac{x+2}{3}$; $\frac{1}{7}$ of $2x-4$ is $\frac{2x-4}{7}$.

1. B had 3 sheep less than twice the number A had, and C had $\frac{1}{5}$ as many as B The number A had being x, what will represent the number C had?

Ans. $\frac{2x-3}{5}$.

2. What will represent $\frac{1}{2}$ as much as $3x-4$?

3. What will represent $\frac{1}{3}$ as much as $10 - x$?

4. What will represent $\frac{1}{4}$ as much as $x + y - 10$?

5. What will represent $\frac{1}{5}$ as much as $x - 2y + 5$?

6. A is 5 years older than B. What will represen $\frac{1}{2}$ of the sum of their ages, B being x years old?

7. A and B had the same number of eggs. A gave B 5, after which each broke $\frac{1}{2}$ of what he then had. What will represent the number each broke, if each had x eggs at first?

8. A man, having x dollars, doubled his money, then gave away \$20, and finally lost $\frac{1}{3}$ of what he then had What will represent the number of dollars lost?

9. If a number, x, be increased by 2, and the sum be divided by 3, and the quotient be increased by 7, what will represent the result? *Ans.* $\frac{x+2}{3} + 7$.

10. A man, having x dollars, gained \$100, and then paid away $\frac{1}{5}$ of what he had, wanting \$6. What will represent the number of dollars paid away?

11. If a number, represented by x, be multiplied by 3, and 20 be added to the product, this result be divided by 7, and the quotient be increased by 9, what will represent the final result?

12. A had x sheep, B twice as many and 11 more, and C 35 more than $\frac{1}{5}$ as many as B. What will represent the number C had?

13. A man, having x dollars, borrowed \$25, and then lent $\frac{1}{7}$ of what he had, wanting \$7. What will represent the number of dollars he lent?

14. What is 7 less than $\frac{1}{20}$ of $x + 5$?

15. What is 25 more than $\frac{1}{13}$ of $x - 20$?

16. What is $\frac{1}{3}$ of $x + 2$ added to $\frac{1}{4}$ of $x - 3$?

Ans. $\frac{x+2}{3} + \frac{x-3}{4}$.

17. What is $\frac{1}{6}$ of $x+y$ added to $\frac{1}{5}$ of $x-7$?

18. Double a number represented by x, increase the product by 4, divide this result by 7, and increase the quotient by 33. What will represent the final result?

19. A had 3 times as much money as B; A lost $10, and B $5. What will then represent $\frac{1}{3}$ of A's money added to $\frac{1}{5}$ of B's, if B had x dollars at first?

20. A and B had each x dollars. A lost $35, and B $25; after which each borrowed $\frac{1}{3}$ as much money as the other had left. What will represent the number of dollars each had then?

ART. **14.** 1. If x represent the age of A in years, and B is $\frac{1}{2}$ as old, what else will represent A's age, if he is 20 years older than B?

Ans. $\frac{x}{2}+20$.

2. There is a fish, whose head is 5 inches long, whose tail is as long as his head and $\frac{1}{2}$ of the length of his body, and whose body is as long as his head and tail. If x represent the length of the body, what other expression will also represent the length of the body?

3. B is 10 years older than A, and 10 years younger than C. Moreover, A's age is $\frac{1}{3}$ of the sum of B's and C's. Let x years be A's age, and find another expression for the same.

4. One half of a school learn to read, $\frac{1}{4}$ learn to write, and the remainder, 10, learn arithmetic. Let x represent the whole number of scholars, and find another expression for the same.

5. In a mixture of tin, copper, and zinc, $\frac{1}{3}$ of the whole is tin, $\frac{1}{4}$ of the whole is copper, and there are 5 lbs. of zinc. Let x represent the number of pounds in the whole, and find another expression for the same.

6. A man has 30 coins in his purse, viz. eagles, dol-

lars, and half-dollars. He has twice as many dollars as eagles, and 10 more half-dollars than dollars. If x represent the number of eagles, what expression will represent the whole number of coins, and consequently be equal to 30?

7. B is 10 years older than A, and 3 times A's age is equal to twice B's. If x represent A's age in years, what two expressions will be equal? *Ans.* $3x$ and $2x + 20$.

8. Two men are of the same age; but if one were 18 years younger, and the other 10 years older, 3 times the age of the former would be the same as twice the age of the latter. What two expressions will be equal, if x represent the age of each in years?

9. Two flocks of sheep are equal in numbers; but if 20 sheep be transferred from one to the other, one flock will then contain 3 times as many as the other. Find two expressions which shall be equal after this change.

10. Half of a man's life was spent in Europe, $\frac{1}{3}$ of it in Asia, and the remainder, which was 10 years, in America. Find two expressions for his age.

11. A woman's age is $\frac{2}{3}$ of her husband's, and the difference of their ages is 9 years. Find two expressions for the man's age.

12. A man, having 60 dollars, lost a certain number of them, after which he found that 3 times what he lost was the same as twice what he had remaining. Find two equal expressions.

13. A farmer has twice as many sheep as cows; he buys 60 more sheep and 5 more cows, and finds that he has 4 times as many sheep as cows. Find two expressions for the number of his sheep after this purchase.

SECTION I.

EQUATIONS OF THE FIRST DEGREE, HAVING UNKNOWN TERMS ONLY IN THE FIRST MEMBER, AND ENTIRELY KNOWN TERMS IN THE SECOND.

ART. **15.** 1. Two boys have, together, 30 cents; but the elder has twice as many cents as the younger. How many cents has each?

Let x represent the number of cents the younger had; then $2x$ will represent the number the elder had.

Hence, $x + 2x = 30$. Reduce the terms containing x,

$3x = 30$. Since $3x$ is equal to 30, $1x$ or x must be $\frac{1}{3}$ as much; therefore,

$x = 10$ cents, the number the younger had, and

$2x = 20$ cents, the number the elder had.

2. A farmer bought a cow and a calf for $33. For the cow he gave 10 times as much as for the calf. Required the price of each.

Let x represent the price of the calf in dollars; then $10x$ will represent that of the cow.

Hence, $x + 10x = 33$. Reducing the terms containing x,

$11x = 33$; $1x$ or x will be $\frac{1}{11}$ as much; $\therefore$

$x = \$3$, price of the calf, and

$10x = \$30$, price of the cow.

Problems in algebra can be proved, as well as those in arithmetic. The proof of the last consists in adding the price of the cow to that of the calf, and ascertaining that the sum is $33. Let the learner prove the correctness of his answers as he advances.

An *equation* is a representation of the equality of quantities. Thus, $x + 10x = 33$ is an equation.

A *member* or *side* of an equation, signifies the whole of the quantities on the same side of the sign $=$; the *first member* being on the left, and the *second member* on the right hand side of this sign.

An equation of the *first degree* is one, in which, after it is freed from fractions, the unknown quantities are neither multiplied by themselves nor by each other.

Terms are the separate parts of an algebraic expression affected by the signs $+$ and $-$. Those terms which have no sign prefixed to them, are supposed to have the sign $+$; and a term is said to be affected by a sign, when it is immediately preceded by that sign, either expressed or understood.

When the first term of a member of an equation, or of any algebraic quantity, is affected by the sign $+$, it is usual to omit writing the sign before that term; but the sign $-$ must *always* be written before any term affected by it. Moreover, terms affected by the sign $+$ are called *positive*; those affected by the sign $-$ are called *negative* quantities.

The equation $x + 10x = 33$ consists of three terms, two in the first member and one in the second; and each of these terms is affected by the sign $+$, although that sign is written before only one of them.

A *coefficient* of a letter, or of any algebraic quantity, is a number placed immediately before it, or separated from it only by the sign of multiplication; and the coefficient shows how many times the letter or quantity is taken. Thus, in the expressions, $5x$, $5 . x$, $5 \times x$, the coefficient of x is 5. When a letter or quantity is written without any coefficient, it is supposed to have 1 for its coefficient; thus, x is the same as $1x$. Letters, as will be seen hereafter, may be used as coefficients.

The process by which an equation is formed from the conditions of a question, is called *putting the question into an equation;* and the process by which the value of the unknown quantity is found from the equation, is called *solving the equation.*

3. A man bought a certain number of pounds of coffee, at 10 cents per pound, and the same number of pounds of sugar, at 8 cents per pound, and the whole amounted to \$1·62. Required the quantity of each.

Let x represent the number of pounds of each; x lbs. of coffee, at 10 cents per lb., will cost $10x$ cents, and x lbs. of sugar, at 8 cents per pound, will cost $8x$ cents.

Hence, $10x + 8x = 162$. Reducing the first member,

$$18x = 162, \text{ and}$$

$$x = 9 \text{ lbs. of each, } Ans.$$

4. A and B hired a house for \$300, of which A was to pay twice as much as B. How much was each to pay?

5. A man is twice as old as his son, and his son twice as old as his daughter. Required the age of each, their united ages being 77 years.

6. A farmer sold equal quantities of wheat and rye, and received for both 150 shillings. Required the number of bushels of each, the wheat being sold at 8 s. and the rye at 7 s. per bushel.

7. A drover sold 6 oxen and 4 cows for \$320, receiving twice as much for an ox as for a cow. Required the price of a cow, also that of an ox.

8. In a company of 160 individuals, there were 3 times as many ladies as gentlemen, and 4 times as many children as there were ladies. Required the number of each.

9. Two travellers set out, at the same time, from two towns, 120 miles apart, and travel towards each other till they meet. How long would they be upon the road, if one goes 5 and the other 7 miles per hour?

10. A market-man sold 10 apples, 12 peaches, and 7

melons, for \$2·64, selling a peach at twice the price of an apple, and a melon at 7 times the price of a peach. At what price did he sell one of each kind?

11. A man bought 7 bushels of potatoes, 5 of corn, and 9 of wheat, for £7 19 s., giving twice as much a bushel for corn as for potatoes, and twice as much a bushel for wheat as for corn. How many shillings did he give a bushel for each?

12. A and B traded in company, and gained \$240. B put in twice as much stock as A. What is each man's share of the gain?

Remark. It is evident that, since B furnished twice as much stock as A, he should have twice as much of the gain.

13. A gentleman distributed 20 shillings among 3 beggars. To the first he gave twice as much as to the second, and to the second 3 times as much as to the third. How many shillings did he give to each?

14. A man bequeathed an estate of \$16,000 to his two sons and three daughters, directing that the daughters should all share alike, that the younger son should have twice as much as one daughter, and that the elder son should have as much as all the daughters. Required the share of each.

15. Three men, A, B, and C, built 670 rods of fence. A built 7, B 5, and C 4 rods a day. A wrought 3 times as many days as B, and B wrought 5 times as many days as C. Required the number of days each wrought.

16. A draper bought 16 pieces of cloth; 3 were white, 4 black, and 9 blue. A piece of black cost twice, and a piece of blue 3 times, as much as a piece of white. Required the price of a piece of each, the cost of the whole being \$152.

SECTION II.

EQUATIONS OF THE FIRST DEGREE, HAVING UNKNOWN TERMS IN ONE MEMBER ONLY, AND KNOWN TERMS IN BOTH.

ART. **16.** 1. The sum of the ages of A and B is 50 years, and B is 10 years older than A. Required the age of each.

Let x represent A's age in years; then $x+10$ will represent B's age. Hence,

$x+x+10=50$. Reducing the first member,

$2x+10=50$.

Since the two members of the equation are equal, we may now subtract 10 from each member, and the remainders will be equal, (Ax. 2.) First representing this subtraction, we have

$2x+10-10=50-10$. Now reducing, that is, performing the subtraction represented,

$2x=40$; from which

$$\left.\begin{array}{r} x=20 \text{ years, A's age;} \\ \text{and } x+10=20+10=30 \text{ years, B's age;} \end{array}\right\} Ans$$

2. A farmer had 6 more than twice as many sheep as cows, and the number of his sheep and cows together was 66. Required the number of each.

Let x represent the number of cows;

then $2x+6$ must be the number of sheep. Hence,

$x+2x+6=66$. Reduce the first member,

$3x+6=66$; subtract 6 from each member,

$3x+6-6=66-6$; reduce both members,

$3x=60$, and

$$\left.\begin{array}{r} x=20 \text{ cows,} \\ 2x+6=46 \text{ sheep,} \end{array}\right\} Ans.$$

Remark. In the two preceding questions, we see that, after reducing the first equation, we subtracted from both members the known term, which was in the same member with the unknown quantity. Let the learner solve the following problems in a similar manner, first representing the subtraction, and afterwards reducing.

3. A and B hired a pasture for \$75, of which A paid \$15 more than B. How much did each pay?

4. A man and his son could earn together, in one day, \$2·50, but the man earned 5 s. more than his son. How much could each earn?

5. A laborer wrought 10 days, having the assistance of his boy 4 days, and received for the wages of both \$12 but the man earned, in a day, \$0·50 more than the boy. Required the daily wages of each.

6. A is 2 years older than B, and B is 3 years older than C. Required the age of each, the sum of their ages being 68 years.

7. A grocer gave \$53 for 5 barrels of flour and 4 barrels of rice, paying \$2 a barrel more for the rice than for the flour. Required the price of a barrel of each.

8. Divide \$83 between A, B, and C, so that B shall have \$7 more than A, and C \$9 more than B.

9. In a certain town there are 10 more Irish than English, and 30 more French than Irish. Required the number of each class, there being 710 persons in all.

10. A man paid £3 4 s. for 4 bushels of corn and 5 bushels of wheat, giving 2 s. more a bushel for wheat than for corn. Required the price of a bushel of each.

If, in solving the equations arising from the preceding questions, the learner had, after representing the subtraction in both members, reduced only that member in which the unknown quantity was found, he would have perceived that the resulting equation might have been obtained by carrying the known term, which was on the same side as the unknown quantity, to the other member, and changing its sign from + to —. Thus, in the second question, we had the equation $3x+6=66$. If we represent the subtraction of 6 from each member, the equation becomes $3x+6-6=66-6$. Reducing the first member of this, we have $3x=66-6$, which might have been obtained from $3x+6=66$, merely by transferring 6 from the first member to the second, and changing its sign from + to —.

In like manner, if we had the equation $3x=x+20$, by representing the subtraction of x from each member, and then reducing the second member of the result, we should have $3x-x=20$, which might have been obtained from $3x=x+20$, by transferring x, supposed to have the sign +, from the second member to the first, and changing its sign from + to —.

Art. **17.** Removing a term from one member of an equation to the other, is called *transposing* that term, or *transposition*. *Hence, in an equation, any term affected by the sign + may be transposed, if this sign be changed to* —; for this is subtracting the same quantity from each member. (Ax. 2.)

Art. **18.** 1. Two men, A and B, hired a farm for \$450, of which A paid \$50 more than B. Required the rent paid by each.

Let x be the number of dollars A paid;
then $x-50$ will be the number B paid. We have, then,

$x+x-50=450$. Reducing the first member,

$2x-50=450$; adding 50 to each member,

$2x-50+50=450+50$; reducing, (Ax. 5,)

$2x=500$; and

$$\left.\begin{aligned} x &= \$250, \text{ what A paid;} \\ x-50 &= \$200, \text{ what B paid;} \end{aligned}\right\} Ans$$

2. B is 10 years older than C, and A is 15 years older than B. Required the age of each, the sum of their ages being 95 years.

Let x represent A's age in years;
then $x - 15$ will represent B's, and $x - 15 - 10$ will represent C's.

Hence, $x + x - 15 + x - 15 - 10 = 95$.

Reducing the first member,

$3x - 40 = 95$; adding 40 to each member,

$3x - 40 + 40 = 95 + 40$; reducing, (Ax. 5,)

$3x = 135$; and

$$\left.\begin{array}{r} x = 45 \text{ years, A's age;} \\ x - 15 = 30 \text{ years, B's age;} \\ x - 15 - 10 = 20 \text{ years, C's age;} \end{array}\right\} \textit{Ans.}$$

Remark 1st. It will be perceived, in the reduction of the first equation of this question, that subtracting several quantities separately, is equivalent to subtracting their sum at once.

Remark 2d. Most of the preceding questions in this section may be performed in the same way as the last two have been. Let the learner solve the following questions in a similar manner.

3. A man performed a journey of 70 miles in 3 days, travelling 4 miles less the 2d day than he did the 1st, and 3 miles less the 3d day than he did the 2d. Required the number of miles he went each day.

4. A had 3 times as much money as B. After A had lost \$20 and B \$15, it was found that they had, together, \$85. How much money had each at first?

5. When oats were 3 s. a bushel cheaper than corn, a stage owner paid \$12·50 for 7 bushels of oats and 9 bushels of corn. Required the price of each per bushel.

6. A trader exchanged 20 bushels of rye and 13 bushels of corn for 12 bushels of wheat, estimated at 8 s. per

bushel, and £5 9 s. in money. Required the estimated price of the rye and corn per bushel, corn being worth 2 s. less per bushel than rye.

7. The difference of 2 numbers is 5; and 7 times the greater, together with 10 times the less, makes 171. Required the numbers.

8. There is a pole consisting of two parts, the upper being 6 feet less than 3 times the length of the lower. Required the length of each part, the whole pole being 30 feet long.

9. A and B have equal sums of money. A doubles his, and B loses £50, after which they both together have £250. How much had each at first?

If, in solving the equations arising from the preceding questions in this article, the learner had, after representing the addition to both members, reduced only that member in which the unknown quantity was found, he would have perceived, that the resulting equation might have been obtained by carrying the known term, which was on the same side as the unknown quantity, to the other member, and changing its sign from $-$ to $+$.

Thus, in the first question, we had $2x - 50 = 450$. If we represent the addition of 50 to both members, the equation becomes $2x - 50 + 50 = 450 + 50$. Reducing the first member of this, we have $2x = 450 + 50$, which might have been obtained from $2x - 50 = 450$, merely by transferring -50 from the first member to the second, and changing its sign from $-$ to $+$.

In like manner, if we had the equation $7x = 81 - 2x$, by representing the addition of $2x$ to each member, and then reducing the second member of the result, we should have $7x + 2x = 81$, which might have been obtained from $7x = 81 - 2x$, by transferring $-2x$ from the second member to the first, and changing its sign from $-$ to $+$

ART. **19.** *Hence, any term affected by the sign —, may be transposed from one member of an equation to the other, if this sign be changed to +*; for this is adding the same quantity to each member.

ART. **20.** Combining the preceding principle with that given in Art. 17, we have the following

GENERAL RULE FOR TRANSPOSITION.

Any term may be transposed from one member of an equation to the other, provided its sign be changed from + to —, or from — to +.

ART. **21.** 1. B was 10 years older than A, C was 3 times as old as A wanting 20 years, D was 30 years older than B, and the sum of their ages was 150 years. Required the age of each.

Let x be A's age in years; then B's, C's, and D's will be, respectively, $x+10$, $3x-20$, and $x+10+30$, or $x+40$. Hence,

$x+x+10+3x-20+x+40=150$. Reduce the first member,

$6x+30=150$; transpose 30,

$6x=150-30$; reduce the second member,

$6x=120$; $\therefore$

$$\left.\begin{aligned} x &= 20 \text{ years, A's age;} \\ x+10 &= 30 \text{ years, B's age;} \\ 3x-20 &= 40 \text{ years, C's age:} \\ x+40 &= 60 \text{ years, D's age;} \end{aligned}\right\} \textit{Ans.}$$

2. Four men commenced trade with equal stocks. The first doubled his stock and £10 more; the second doubled his and £20 pounds more; the third tripled his stock wanting £100; and the fourth quadrupled his, wanting £250; after which they all, together, had £780. What was the stock of each, at first?

Let x represent the number of pounds each had at first. Then, $2x+10$, $2x+20$, $3x-100$, and $4x-250$, will represent their respective stocks after they had made their gains. Hence,

$$2x+10+2x+20+3x-100+4x-250=780.$$

Reduce the first member,

$$11x-320=780; \text{ transpose } -320,$$

$$11x=780+320; \text{ reduce,}$$

$$11x=1100; \therefore$$

$$x=£100, \text{ each man's stock at first, } \textit{Ans.}$$

3. A man walked 91 miles in 3 days. He went 12 miles more the 2d day than he did the 1st, and 20 miles less the 3d day than he did the 2d. Required the extent of each day's journey.

4. In a certain manufactory there are 10 more than 3 times as many boys as men, and 40 less than 4 times as many girls as boys. Required the number of each, there being 138 in the whole.

5. A farmer has a certain number of plum-trees, 10 less than twice as many peach-trees, 20 more pear-trees than peach-trees, and twice as many apple-trees as of all others. Required the number of each kind, the whole number of trees being 300.

6. In a school of 155 scholars, a certain number learn geometry, twice as many learn algebra, and 20 less than twice as many learn arithmetic as algebra. These three classes constituting the whole school, it is required to find the number in each class.

7. B has \$200 more than A, and C has \$300 less than twice as much as B, and they have, together, \$1900. Required each man's money.

8. Four towns are situated in a straight line, and in the order of the letters A, B, C, and D. The distance from B to C is 10 miles more than that from A to B; and the

distance from C to D is 20 miles less than twice that from B to C. Required the distances from A to B, from B to C, and from C to D, the entire distance from A to D being 210 miles.

9. A drover bought a certain number of oxen, at \$30 each; 5 more cows than oxen, at \$20 each; and 50 less than 5 times as many sheep as cows, at \$5 each. The whole cost him \$725. Required the number of each.

10. A draper bought 9 yards of broadcloth, 12 of cassimere, and 15 of silk. A yard of broadcloth cost \$5 more, and a yard of silk \$1 less, than a yard of cassimere, and the whole cost him \$102. Required the price of each per yard.

SECTION III.

EQUATIONS HAVING BOTH KNOWN AND UNKNOWN TERMS IN EACH MEMBER.

ART. **22.** 1. A certain man's age is such, that if he were 10 years older, he would be twice as old as he would be if he were 10 years younger. Required his age.

Let x be his age in years; then $x + 10$ must be double $x - 10$. Hence, $2x - 20 = x + 10$.

In solving an equation having one unknown quantity, we wish to get all the unknown terms into one member by themselves, and it is generally best to get the unknown into the first member, and the known into the second.

In the equation given above, transpose the -20,

$2x = x + 10 + 20$; transpose x,

$2x - x = 10 + 20$; reduce,

$x = 30$ years, *Ans.*

Let us solve the same equation written with its members reversed.

$$x + 10 = 2x - 20. \quad \text{Transpose } 10,$$
$$x = 2x - 20 - 10; \quad \text{transpose } 2x,$$
$$x - 2x = -20 - 10; \quad \text{reduce,}$$
$$-x = -30; \quad \text{transpose both members,}$$
$$30 = x, \text{ which is the same as}$$
$$x = 30.$$

This last equation, $x = 30$, might have been obtained from $-x = -30$, merely by changing the signs of both members.

Or we might have changed the signs of all the terms in the equation $x - 2x = -20 - 10$, which would then have become $-x + 2x = 20 + 10$, or $2x - x = 20 + 10$, the same as in the first solution after transposition. Hence,

In any equation, the signs of all the terms may be changed; for this is equivalent to transposing all the terms.

This change of signs should always be made, whenever the first member becomes minus. The student must recollect, however, to change the signs of *all* the terms, otherwise great errors will ensue.

2. A has twice as much money as B; but if B had £20 more, and A £10 less, B would have 3 times as much as A. Required the money of each.

Suppose B had x pounds; then A would have $2x$ pounds. If B had £20 more, he would have $x + 20$; and if A had £10 less, he would have $2x - 10$. Hence,

$$x + 20 = 6x - 30. \quad \text{Transpose } 20,$$
$$x = 6x - 30 - 20; \quad \text{transpose } 6x,$$
$$x - 6x = -30 - 20; \quad \text{reduce,}$$
$$-5x = -50; \quad \text{change the signs,}$$
$$5x = 50; \quad \therefore$$
$$\left.\begin{array}{l} x = \text{£}10, \text{ B's money;} \\ 2x = \text{£}20, \text{ A's money;} \end{array}\right\} \textit{Ans.}$$

If the first equation had been reversed, both members would, after transposition and reduction, have been affected by the sign +, without changing the signs.

3. A man bought 3 horses. For the first he gave $30 less, and for the third $20 more, than for the second; moreover, the price of the third was double that of the first. Required the price of each.

4. Two numbers differ by 20; and 10 more than 4 times the less is equal to 30 added to twice the greater. Required the numbers.

5. A man has 4 sons, each of whom is 3 years older than his next younger brother; and the sum of the ages of the younger two is equal to the age of the eldest. Required their ages.

6. A and B have equal sums of money; and if A gives B $50, B will have 3 times as much as A has left. How much money has each?

7. A carpenter, wanting a stick of timber of a certain length, had two sticks on hand, one of which was 6 feet too short, and the other 14 feet too long. Now, 3 times the length of the shorter was equal to twice that of the longer. What length of timber did he require?

8. The ages of two boys differ by 4 years, and 3 times the age of the elder is the same as 5 times that of the younger. Required their ages.

9. A gentleman bought a horse and chaise. For the chaise he gave $50 more than for the horse; and 5 times the cost of the chaise was the same as 6 times that of the horse. Required the price of each.

10. Two towns, 60 miles from each other, are in the same direction from Boston, on the same straight road, and one is 5 times as far from Boston as the other. Required the distance of each from Boston.

11. A grocer has two kinds of tea, one of which is

worth 3 s. per pound more than the other; moreover, 4 pounds of the one are worth as much as 7 of the other. Required the worth of a pound of each.

12. A boy, distributing some peaches among his companions, found that he wanted 3 in order to give them 7 apiece; he therefore gave them 6 apiece, and had 1 peach left What was the number of his companions, and how many peaches had he?

13. A boy was sent to market for a certain number of pounds of meat. If he bought beef at 10 cents a pound, he would have \$0·80 left; but if he bought mutton at 8 cents a pound, he would have \$1·04 left. How many pounds of meat was he to purchase, and how much money had he with him?

14. A farmer, having a certain number of young fruit trees, wished to set them in rows, with a certain number in a row; but he lacked 6 trees in order to make 7 rows, and if he set out 6 rows, he would have 2 trees remaining. How many trees did he wish to place in each row? and how many trees had he?

15. B has \$10 more than A, and \$5 less than C; moreover, 3 times what A and B together have, is equal to twice what B and C together have. How much money has each?

16. A farmer sold his barley for 1 s. less, and his rye for 2 s. more, per bushel, than he did his corn; and he found that 2 bushels of barley and 5 bushels of corn came to as much as 5 bushels of rye. Required the price of each, per bushel.

17. Separate 50 into two parts, so that 3 times the less shall be equal to twice the greater.

Let x represent the less; then $50 - x$ will be the greater.

18. A had \$60, and B \$20. B borrowed a certain sum

of A, after which he had in all twice as much as A had left. How much did he borrow of A?

19. When corn was worth 3 s. a bushel more than oats, a man gave 9 bushels of oats and 6 s. in money for 6 bushels of corn. What were the estimated prices of the oats and corn per bushel?

20. Two gentlemen, each having $20 in his pocket, contributed to a public charity, one giving twice as much as the other; and it was found that one had remaining 3 times as much money as the other. How much did each contribute?

SECTION IV.

EQUATIONS CONTAINING FRACTIONAL PARTS OF SINGLE TERMS.

ART. **23.** 1. Five sixths of a ton of hay cost $15. What was the price of a ton?

Let x represent the price of a ton, in dollars; then $\frac{5}{6}$ of a ton will cost $\frac{5}{6}$ of x. Five sixths of x is represented thus, $\frac{5}{6}x$, or thus, $\frac{5x}{6}$; the latter form is the most usual. This may be read, either "five sixths of x," "$5x$ sixths," "a sixth of $5x$," or "$5x$ divided by 6." Hence,

$$\frac{5x}{6} = 15.$$

Now, we may divide both members by 5, (Ax. 4;) and as a fraction is divided by dividing the numerator, we have $\frac{x}{6} = 3$. Since $\frac{1}{6}$ of x is equal to 3, the whole of x will be 6 times as much; that is,

$$x = \$18, \textit{Ans.}$$

To solve the equation another way, let us resume $\frac{5x}{6}=15$. We may first multiply both members by 6 and as a fraction is multiplied by dividing its denominator we have

$$\frac{5x}{1}=90;\ \text{that is, } 5x=90;\ \therefore$$
$$x=\$18,\ \textit{Ans.}$$

The latter mode of solution is generally to be preferred.

2. In a certain village, $\frac{1}{2}$ of the people are English, $\frac{1}{4}$ Irish, $\frac{1}{8}$ French, and the remainder, 50 in number, are Germans. What is the whole population of the village?

Let x be the number of inhabitants. Then,

$$x=\frac{x}{2}+\frac{x}{4}+\frac{x}{8}+50.\quad \text{Multiply by 2,}$$
$$2x=x+\frac{x}{2}+\frac{x}{4}+100;\ \text{multiply this by 2,}$$
$$4x=2x+x+\frac{x}{2}+200;\ \text{multiply this by 2,}$$
$$8x=4x+2x+x+400;\ \text{reduce,}$$
$$8x=7x+400;\ \text{transpose } 7x,$$
$$8x-7x=400;\ \text{reduce,}$$
$$x=400,\ \textit{Ans.}$$

The process, in this question, would have been shorter, if we had multiplied the first equation by 8 at once, since we should then have obtained

$$8x=\frac{8x}{2}+\frac{8x}{4}+\frac{8x}{8}+400;\ \text{that is.}$$

$8x=4x+2x+x+400$; but it is best to remove one denominator at a time, until the learner has become quite familiar with the process.

3. The sum of $\frac{2}{3}$, $\frac{3}{4}$, $\frac{5}{6}$, and $\frac{7}{8}$ of a certain number is 150 Required the number.

Let x represent the number. Then,

$$\frac{2x}{3}+\frac{3x}{4}+\frac{5x}{6}+\frac{7x}{8}=150.$$

We are now to multiply so as to remove successively, the denominators. But it is sometimes convenient to represent the multiplication of certain numbers in an equation. Multiply the equation by 3.

$$2x+\frac{9x}{4}+\frac{5x}{2}+\frac{21x}{8}=3.150; \text{ multiply by } 4,$$

$$8x+9x+10x+\frac{21x}{2}=3.4.150; \text{ multiply by } 2,$$

$$16x+18x+20x+21x=3.4.2.150; \text{ reduce,}$$

$$75x=3.4.2.150.$$

Now, since dividing one factor divides a product, in the second member we have only to divide 150 by 75. Dividing both members, we have

$$x=3.4.2.2=48, \textit{ Ans.}$$

4. Four fifths of a number, increased by 2, is the same as $\frac{7}{8}$ of the number diminished by 4. Required the number.

5. The cost of $\frac{7}{8}$ of an acre of land being $35, what will an acre cost at the same rate?

6. Two numbers are to each other in the ratio of 9 to 7, and their sum is 112. Required the numbers.

Remark. The ratio of 9 to 7 means that the greater is $\frac{9}{7}$ of the less, or that the less is $\frac{7}{9}$ of the greater.

7. A man's age is to that of his wife as 3 to 2, and his age exceeds hers by 10 years. Required their ages.

8. A grocer bought some tea, at $0·50 per pound, and $\frac{2}{5}$ as much coffee, at $0·12 per pound, and paid for both $21·92. How many pounds of each did he buy?

9. In a mixture of gold and copper, 1 ounce more than $\frac{8}{9}$ of the whole was gold, and $\frac{1}{12}$ of the whole was copper. Required the weight of the whole, and that of each ingredient.

10. In a mixture of copper, tin, and zinc, 5 oz. more than $\frac{3}{8}$ of the whole was copper, 2 oz. less than $\frac{1}{3}$ of the whole was tin, and 1 oz. less than $\frac{1}{4}$ of the whole was zinc.

Required the weight of the entire mixture, and that of each ingredient.

11. A man had passed $\frac{1}{6}$ of his life in Germany, 12 years more than $\frac{1}{5}$ of it in England, and 3 years less than $\frac{1}{3}$ of it in France. Required his age.

12. A trader bought a quantity of oats, at 2 s. a bushel, and twice as much corn, at 5 s. a bushel. He afterwards sold, at the same prices he gave, $\frac{1}{5}$ of his oats and $\frac{3}{4}$ of his corn for 79 s. How much of each did he buy?

13. If I multiply a certain number by 5, divide the product by 7, and diminish the quotient by 3, I shall obtain $\frac{1}{2}$ the original number. Required that number.

14. A man's age, at the time of his marriage, was to that of his wife as 3 to 2; but after they had been married 10 years, 3 times his age was equal to 4 times hers. Required their ages at the time of marriage.

15. When a barrel of apples was worth $\frac{2}{5}$ as much as a barrel of flour, a farmer gave 6 barrels of apples and $3 in money for 3 barrels of flour. Required the estimated price of each, per barrel.

16. Two boys set out, at the same time, and from the same place, to run to a certain goal. One could run only $\frac{7}{8}$ as fast as the other, and was 5 rods short of the goal at the time his companion had reached it. Required the distance which they proposed to run.

17. What number is that, $\frac{2}{3}$ of which, increased by 6, will be the same as $\frac{11}{12}$ of it diminished by 3?

SECTION V.

EQUATIONS CONTAINING FRACTIONAL PARTS OF QUANTITIES CONSISTING OF SEVERAL TERMS.

ART. **24.** 1. A is 20 years older than B, and B's age is equal to $\frac{7}{11}$ of A's. Required the age of each.

Let x represent B's age in years; then $x+20$ will represent A's. One eleventh of A's is $\frac{x+20}{11}$, and $\frac{7}{11}$ is 7 times as much; that is, $\frac{7x+140}{11}$. Hence,

$$x=\frac{7x+140}{11}. \quad \text{Multiply by 11,}$$

$$11x=7x+140; \text{ transpose } 7x,$$

$$11x-7x=140; \text{ reduce,}$$

$$4x=140; \therefore$$

$$\left.\begin{aligned} x&=35 \text{ years, B's age,} \\ x+20&=55 \text{ years, A's age;} \end{aligned}\right\} Ans.$$

2. A had twice as much money as B. A gained \$500, and B \$100; then $\frac{2}{3}$ of B's money was equal to $\frac{4}{15}$ of A's. How much had each at first?

Let x be the number of dollars B had; then A would have $2x$ dollars. After gaining, B would have $x+100$, and A $2x+500$; then $\frac{2}{3}$ of $x+100$ must be equal to $\frac{4}{15}$ of $2x+500$. Hence,

$$\frac{2x+200}{3}=\frac{8x+2000}{15}. \quad \text{Multiply by 3,}$$

$$2x+200=\frac{8x+2000}{5}; \text{ multiply by 5,}$$

$$10x+1000=8x+2000; \text{ transpose,}$$

$$10x-8x=2000-1000; \text{ reduce,}$$

$$2x=1000; \therefore$$

$$\left.\begin{aligned} x&=\$500, \text{ B's money;} \\ 2x&=\$1000, \text{ A's money;} \end{aligned}\right\} Ans.$$

3. A man paid for a horse and chaise \$300, and $\frac{4}{5}$ of the price of the horse was the same as $\frac{5}{7}$ the price of the chaise increased by \$28. Required the price of each.

Suppose the horse cost x dollars; then $300 - x$ will be the price of the chaise. Hence,

$$\frac{4x}{5} = \frac{1500 - 5x}{7} + 28. \quad \text{Multiply by 5,}$$

$$4x = \frac{7500 - 25x}{7} + 140; \quad \text{multiply by 7,}$$

$$28x = 7500 - 25x + 980; \quad \text{transpose } -25x,$$

$$28x + 25x = 7500 + 980; \quad \text{reduce,}$$

$$53x = 8480; \therefore$$

$$\left.\begin{aligned} x &= \$160, \text{ price of the horse;} \\ 300 - x &= \$140, \text{ price of the chaise;} \end{aligned}\right\} Ans.$$

4. A is 15 years older than B, and $\frac{1}{3}$ of A's age is the same as $\frac{1}{2}$ of B's. Required their ages.

5. What number is that to which if 6 be added, and from which if 5 be subtracted, $\frac{4}{7}$ of the sum shall exceed $\frac{3}{5}$ of the difference by 5?

6. What number is that from which if 10 be subtracted, and to which if 11 be added, $\frac{4}{5}$ of the difference shall be the same as $\frac{1}{3}$ of the sum?

7. A drover, having a certain number of oxen, and twice as many sheep, bought 5 oxen and sold 2 sheep, after which he found that $\frac{3}{5}$ of his number of oxen was equal to $\frac{1}{2}$ of his number of sheep. How many of each had he at first?

8. A man agreed to work a year for 50 bushels of corn and \$75 in money; but, on account of sickness, he wrought only 10 months, and received the 50 bushels of corn and \$54$\frac{1}{6}$ in money. What was the estimated price of the corn per bushel?

9. A man's age, 5 years before his marriage, was $\frac{2}{3}$ of his age 5 years after his marriage. Required his age at the time he was married.

10. A and B had equal sums of money. A lent B $5, and had left $\frac{9}{11}$ as much as B then had. How much money had each at first?

11. Separate 68 into two parts, such that $\frac{2}{3}$ of the greater shall be the same as $\frac{3}{4}$ of the less.

12. The joint stock of two partners, whose particular shares differed by £300, was to the greater as 20 to 13. Required the stock of each partner.

13. A and B began trade, A with twice as much capital as B. A doubled his stock wanting £300, and B doubled his stock and £50 over; after which A's stock was to B's as 14 to 11. Required the stock with which each began.

14. If I multiply a certain number by 3, subtract 10 from the product, and take $\frac{3}{5}$ of the remainder, the result will be once and a half of the original number. Required that number.

15. The united ages of a man and his wife amount to 104 years, and $\frac{3}{4}$ of the man's age is the same as $\frac{7}{8}$ of his wife's. Required the age of each.

16. A man had two horses, and a chaise which was worth $200. The first horse and chaise were together worth twice as much as the second horse; the second horse and chaise together were worth twice and a half as much as the first horse. Required the value of each horse.

17. B could walk 5 miles more in a day than A, and $\frac{3}{5}$ of what A could walk in 9 days, was 8 miles more than B could walk in 4 days. How many miles could each walk daily?

18. A farmer had a certain number of turkeys and twice as many geese. He bought 5 turkeys and sold 10 geese, when he found that $\frac{4}{5}$ of his number of turkeys exceeded half his number of geese by 1. How many of each had he at first?

SECTION VI.

EQUATIONS OF THE FIRST DEGREE, WHICH REQUIRE THE SUBTRACTION OF QUANTITIES CONTAINING NEGATIVE TERMS.

ART. **25.** 1. Separate 30 into two parts, such that the less subtracted from 5 times the greater, shall leave a remainder of 66.

If we knew that the greater part was 16, the less would be $30 - 16$, or 14. We then have to subtract $30 - 16$ or 14, from 5 times 16, or 80. If we subtract 14 from 80, the remainder is 66, as required.

Now, without reducing $30 - 16$, we will subtract it from 80. First subtract 30 from 80; the remainder is $80 - 30$, or 50. But we have subtracted too much by 16, because we were to subtract $30 - 16$; our remainder is, therefore, too small by 16; hence we must add 16 to $80 - 30$, which gives $80 - 30 + 16$, or 66, for the true remainder.

Now, to perform the question, let x be the greater part; then $30 - x$ will be the less. We have to subtract $30 - x$ from $5x$.

Subtracting 30 from $5x$, we have $5x - 30$; but we have subtracted too much by x, because we were to subtract $30 - x$; the remainder is, therefore, too small by x. Hence, we must add x to $5x - 30$, which gives $5x - 30 + x$, for the true remainder.

We have, then, according to the conditions of the question,

$5x - 30 + x = 66$. This equation solved gives

$$\left.\begin{array}{r} x = 16, \text{ the greater part;} \\ 30 - x = 14, \text{ the less part} \end{array}\right\} Ans$$

2. A had twice as much money as B. A lost \$20, and B gained \$36. Then B's money, subtracted from \$120, would leave the same remainder as A's, subtracted from \$114. How much money had each at first?

Let x represent B's money; then $2x$ will represent A's. After A had lost \$20, he would have left $2x-20$; and B, having gained \$36, would have $x+36$. Subtracting B's money from \$120, the remainder is $120-x-36$; subtracting A's from \$114, the remainder is $114-2x+20$. Hence,

$$120-x-36=114-2x+20. \quad \text{Transpose,}$$
$$2x-x=114+20-120+36; \text{ reduce,}$$
$$\left.\begin{array}{r} x=\$50, \text{ B's money;} \\ 2x=\$100, \text{ A's money;} \end{array}\right\} \textit{Ans.}$$

In the first question, the subtraction of $30-x$ from $5x$ gave $5x-30+x$; that is, we changed the sign of each term of $30-x$, and then wrote it after $5x$. In like manner, in the second question, the subtraction of $x+36$ from 114, gave $114-x-36$, and the subtraction of $2x-20$ from 120, gave $120-2x+20$.

ART. **26.** It follows from the preceding operations and explanations, *that any quantity is subtracted by changing the signs of all its terms, and writing it after the quantity from which it is to be subtracted.*

1. A man, having 50 sheep, bought a certain number more, after which he sold twice as many as he had bought, wanting 10, and found that he had 35 left. How many did he buy? and how many did he sell?

2. Separate 60 into two parts, such that the less subtracted from twice the greater, shall leave the same remainder as the greater subtracted from 5 times the less.

3. A, B, and C had together £120. B had £140 less than 3 times as much as A; and if B's money were sub-

tracted from A's, the difference would be C's. How much money had each?

4. Separate 75 into two parts, such that if the greater be subtracted from 60, and the less from 40, the former remainder shall be 4 times the latter.

5. A is 3 times as old as B; but if A were 15 and B 5 years younger, the excess of A's age above B's would be $\frac{3}{2}$ of B's actual age. Required the age of each.

6. A farmer had a certain number of sheep, each of which brought him two lambs. He then sold all the old ones except 10, and found the number of sheep and lambs remaining was $2\frac{1}{4}$ times his original number of sheep. Required the number of sheep he had at first.

7. A had 3 times as much money as B. A lost \$50, and B gained \$50; then $\frac{3}{10}$ of B's money subtracted from A's would leave \$7 more than $\frac{1}{2}$ what A had at first. How much had each at first?

Let x represent the number of dollars B had; then $3x$ will represent what A had. A having lost \$50, and B having gained \$50, A would have $3x - 50$, and B, $x + 50$. Then, according to the conditions,

$$3x - 50 - \frac{3x + 150}{10} = \frac{3x}{2} + 7. \quad \text{Multiply by 10,}$$

$$30x - 500 - 3x - 150 = 15x + 70; \text{ reduce,}$$

$$27x - 650 = 15x + 70; \text{ transpose,}$$

$$27x - 15x = 70 + 650; \text{ reduce,}$$

$$12x = 720; \therefore$$

$$\left.\begin{aligned} x &= \$60, \text{ B's money;} \\ 3x &= \$180, \text{ A's money;} \end{aligned}\right\} \textit{Ans.}$$

In the first equation, $\frac{3x + 150}{10}$ is represented as subtracted; and when we multiply by 10, 10 times that fraction, that is, the numerator, is to be subtracted. But since each of the terms $3x$ and 150, has the sign $+$, the sign of each must be changed to —, which gives the result as exhibited in the second equation.

Observe that when the sign + or — is placed before a fraction, it affects the whole fraction, and does not belong to any one term of the numerator. Thus, $-\frac{3x+150}{10}$ is the same as $-\frac{+3x+150}{10}$.

8. A gentleman has two horses and one chaise. The first horse is worth \$50 more, and the second \$50 less, than the chaise. If $\frac{3}{5}$ of the value of the first horse be subtracted from that of the chaise, the remainder will be the same as if $\frac{7}{3}$ of the value of the second horse be subtracted from twice that of the chaise. Required the value of each horse and of the chaise.

Let x be the price of the chaise; then $x+50$ will be that of the first horse, and $x-50$ that of the second. Hence, from the conditions of the question,

$$x-\frac{3x+150}{5}=2x-\frac{7x-350}{3}. \quad \text{Multiply by 5,}$$

$$5x-3x-150=10x-\frac{35x-1750}{3}; \text{ multiply by 3,}$$

$$15x-9x-450=30x-35x+1750; \text{ reduce,}$$

$$6x-450=-5x+1750; \text{ transpose and reduce,}$$

$$11x=2200, \therefore$$

$$\left.\begin{aligned} x&=\$200, \text{ value of the chaise;} \\ x+50&=\$250, \text{ value of the 1st horse;} \\ x-50&=\$150, \text{ value of the 2d horse;} \end{aligned}\right\} \textit{Ans.}$$

In the second member of the 2d equation, $\frac{35x-1750}{3}$ is represented as subtracted; but $35x$ is supposed to have the sign +, and 1750 has the sign —; hence, when we remove the denominator, 3, we subtract by changing these signs, which gives the result in equation 3d.

From the solution of the two preceding questions, we see that, when a fraction, having several terms in the numerator, is represented as subtracted, on removing the denominator, we change the signs of all the terms of the numerator. If, however, the fraction is preceded by the sign $+$, no change is to be made in the signs of the numerator, on removing the denominator.

9. If from a certain number I subtract 10, and then subtract $\frac{2}{5}$ of this remainder from the original number, the last remainder will exceed $\frac{1}{2}$ of the original number by 6. Required the number.

10. A grocer had two casks full of wine, one containing twice as much as the other. From the smaller leaked out 13, and from the larger 46, gallons. He then drew from the smaller $\frac{3}{8}$ as many gallons as remained in the larger, and from the larger $\frac{4}{5}$ as many gallons as remained in the smaller; after which there remained in the larger 1 gallon more than in the smaller. How many gallons did each cask hold?

11. A and B are of the same age; but if A were 10 and B 5 years younger, $\frac{3}{7}$ of B's age subtracted from A's, would leave the same remainder as if $\frac{7}{8}$ of A's were subtracted from B's. Required their ages.

12. A man, having a lease for 100 years, on being asked how much of it had transpired, said, that $\frac{5}{8}$ of the time past subtracted from the time to come, would leave the same remainder, as if $\frac{1}{12}$ of the time to come were subtracted from the time past. How many years of the lease had transpired?

13. There is a certain number such, that if 25 be subtracted from it, and if it be subtracted from 125, $\frac{3}{5}$ of the first remainder subtracted from $\frac{2}{3}$ of the second, will leave 10 more than $\frac{1}{2}$ of the original number. Required that number.

SECTION VII.

MULTIPLICATION OF MONOMIALS

Art. 27. In *pure* algebra, letters are generally used to represent known as well as unknown quantities. We shall now treat of operations upon quantities purely algebraical.

It is to be observed, that the addition, subtraction, multiplication, &c. of algebraic quantities cannot, strictly speaking, be performed, in the same sense as they are in arithmetic, but are, in general, only represented; these representations, however, receive the same names as the actual operations in arithmetic.

A *monomial* is a quantity consisting of a single term; as, a, bd, or $\frac{x}{y}$, (Art. 15.)

A *binomial* is a quantity consisting of two terms; as, $a + m$, or $x - y$.

A *trinomial* is a quantity consisting of three terms; as, $a + b - c$.

Polynomial is a general name applied to any quantity containing several terms.

Art. 28. The product of monomials is expressed by writing them after each other, either with or without the sign of multiplication; thus, $a \cdot b$, $a \times b$, or $a\,b$, signifies that a and b are multiplied together. The last form, viz. $a\,b$, is generally used. In like manner, the product of a, b, c, and d, is $a\,b\,c\,d$.

The order of the letters in a product is unimportant; thus $a\,b$ is the same as $b\,a$. This will be manifest, if numbers are put instead of the letters. Suppose $a = 4$, and $b = 5$; then $ab = 4 . 5 = 20$, and $ba = 5 . 4 = 20$.

Hence, each of the expressions, abc, acb, bca, bac, cba, and cab, is the product of a, b, and c. For the sake of uniformity, however, the letters of a product are usually written in alphabetical order.

If $3ab$ and $5mn$ were to be multiplied together, we might write the quantities thus, $3ab5mn$; but, since the order of the factors is unimportant, the numerical factors may be placed next to each other; thus, $3 . 5abmn$. Now, performing the multiplication of 3 by 5, we have $15abmn$.

But it would be erroneous to write $35abmn$ as the product of $3ab$ and $5mn$, because the value of a figure depends on its place with regard to other figures. If we would *represent* the multiplication of the figures also, they must be separated either by letters or by the sign of multiplication; as, $3ab5mn$, $3 . 5abmn$, or $3 \times 5abmn$.

In like manner, the product of $2am$, $3xy$, and $5cd$, is $30acdmxy$; that of 7, $2b$, and $4d$, is $56bd$.

We infer, from the preceding examples, that the product of several monomials consists of the product of the coefficients and all the letters of the several quantities.

1. Multiply $2cd$ by $3ax$.
2. Multiply $3pq$ by 7.
3. Multiply $4a$ by $3b$.
4. Multiply $17c$ by $2mn$.
5. Multiply $5ax$ by $11y$.
6. Multiply $7pq$ by $4rs$.
7. Multiply $9gh$ by $11x$.
8. Multiply 20 by $3xy$.
9. Multiply $13mx$ by $7z$.
10. Multiply $2ef$ by $5x$.
11. Multiply 25 by $8abc$.
12. Multiply mm by mmm.

In the last example, the product is, according to what has been shown above, *m m m m m*. But when the same letter occurs several times as a factor in any quantity, instead of writing that letter so many times, it is usual to write it once only, and place a small figure, a little elevated, at the right, to show how many times that letter is a factor. Thus, instead of *m m m m m*, we write m^5. In the same manner, we write a^6, instead of *a a a a a a*.

In all cases, a product contains all the factors of both multiplicand and multiplier. In the 12th example, *m* is twice a factor in the multiplicand, and three times in the multiplier; the product, therefore, must contain it five times as a factor; that is, m^2 multiplied by m^3, gives m^5 The product of $2a^2b^3$ and $3a^4b^4$ is $6a^6b^7$, because each letter must be found as many times a factor in the product as it is in both multiplicand and multiplier.

ART. **29.** The small figure, placed at the right of a letter, is called the *index* or *exponent* of that letter, and affects that letter only, after which it is immediately placed. *An exponent, then, shows how many times a quantity is used as a factor.*

Quantities written with exponents are called *powers* of those quantities; thus, m^2 is called the *second power* of *m*, m^3 the *third power*, m^5 the *fifth power*, &c.; and when a quantity is written without any exponent, it is supposed to have 1 for its exponent, and is called the *first power* of that quantity; thus, *m*, which is the same as m^1, is called the first power of *m*.

Sometimes the second power is called the *square*, and the third power the *cube*, of a quantity names which, though derived from geometry, are, for the sake of conciseness, very convenient in algebra.

Thus, m^2 is read *m square*, m^3 is read *m cube*.

Figures are also frequently written with exponents; thus,

$$3^1 = 3;$$
$$3^2 = 3.3 = 9;$$
$$3^3 = 3.3.3 = 27;$$
$$3^4 = 3.3.3.3 = 81.$$

Exponents must be carefully distinguished from coefficients; for $4a$ and a^4 are very different in their value. Suppose a to be 5; then $4a = 20$, and $a^4 = 5.5.5.5 = 625$.

ART. **30.** From what precedes, we derive the following

RULE FOR THE MULTIPLICATION OF MONOMIALS.

Multiply the coefficients together, and write after their product all the different letters of the several quantities, giving to each letter an exponent equal to the sum of its exponents in all the quantities.

1. Multiply $3a^2m$ by $4a^3m^5$.

In this question, the product of the coefficients is 12; the sum of the exponents of a is 5, and the sum of the exponents of m is 6. The answer, therefore, is $12a^5m^6$.

2. Multiply a^2b^2 by $3ab^3$.

In this example, the product of the coefficients is 1.3 or 3; the sum of the exponents of a is 3, that of the exponents of b is 5; the answer, therefore, is $3a^3b^5$.

3. Multiply $7ab$ by $2ac$.
4. " $13c^2$ by $4c^3d$.
5. " $11a^3m^3$ by $3a^4m$.
6. " $25x$ by $4x^5y$.
7. " $2c^7$ by c^5d.
8. " $8a^2b^3c^4$ by $9ab^3c$.
9. " m^5 by $3a^2m^3$.

10. Multiply $17 x^2$ by $5 x y^2$.
11. " $a^4 d^3$ by $a d c$.
12. " $3 a b^2 c$ by $4 a^3 b^2 c^2$.
13. " $30 p q^3$ by $p^5 q$.
14. " $7 m^2 n^3$ by $a m^7$.
15. Find the product of $3 a$, $4 d$, and $7 a^2 d$.
16. " " " " $5 m^2$, $3 m$, and $6 c m^3$.
17. " " " " $10 x$, $3 y^2$, and $4 x y$.
18. " " " " 2, $7 a$, and a^4.
19. " " " " 5, $6 m^3$, and $2 m^2 n^2$.
20. " " " " p^3, $2 p$, and $3 p^5 q$.
21. " " " " a^4, $a b^2$, and $a^3 b c$.
22. " " " " $x y^2 z^3$, $x^2 y^2 z^2$, and x^3.
23. " " " " $2 p x$, $2 p^2 x^2$, and $2 p^3 x^3$.
24. " " " " $3 b$, $4 c$, and $3 b c^2 d^3$.
25. " " " " z^3, $5 x z$, and $2 x y z$.
26. " " " " $40 m^5 n$, $2 m^3$, and $x y z$.

ART. **31.** What is the value of $3 a^3 b^2$, if $a = 2$ and $b = 5$?

Putting these numbers instead of the letters, we have

$$3 a^3 b^2 = 3 . 2^3 . 5^2 = 3 . 2 . 2 . 2 . 5 . 5 = 600, \textit{ Ans.}$$

Let the learner find the value of the following algebraical expressions, supposing $a = 1$, $b = 2$, $c = 3$, and $d = 5$.

1. $6 a^2 b$.
2. $4 a^3 b^2$.
3. $9 a b c^2$.
4. $2 c^3 d$.
5. $7 a b d^2$.
6. $5 a^5 b c^2$.
7. $8 a^6 b c d$.
8. $a^4 b^3 c^2$.
9. $21 c^2 d^3$.
10. $5 b^2 c d$.
11. $13 b^4 d$.
12. $15 a^2 b^2 c^2 d^2$.

SECTION VIII.

REDUCTION OF SIMILAR TERMS.

ART. **32.** *Similar terms are those which are entirely alike with regard to the letters and exponents.*

N. B. Similarity of terms is wholly independent of the algebraic signs + and —, the numerical coefficients, and the order of the letters. Thus, $a^2 b$ and $5 a^2 b$ are similar; so also are $3 x^2 y^3$ and $5 y^3 x^2$.

Any polynomial which contains similar terms, may always be reduced to a smaller number of terms. Of this the student has already had numerous examples in the reduction of equations.

Let us take the polynomial $2 a^2 b^3 + 4 m^3 n^3 + 7 a^2 b^3 - m^3 n^3$. The two terms $2 a^2 b^3$ and $7 a^2 b^3$ are similar, and it is evident that 7 times a quantity added to twice the same quantity, make 9 times that quantity; therefore, instead of $2 a^2 b^3 + 7 a^2 b^3$, we may write $9 a^2 b^3$. Again, $4 m^3 n^3$ and $- m^3 n^3$ are similar, and 4 times a quantity diminished by once the same quantity, leaves 3 times that quantity; hence, instead of $4 m^3 n^3 - m^3 n^3$, we may put $3 m^3 n^3$. By these reductions the polynomial $2 a^2 b^3 + 4 m^3 n^3 + 7 a^2 b^3 - m^3 n^3$ becomes $9 a^2 b^3 + 3 m^3 n^3$.

As a second example, take the polynomial $5 b^2 c^2 - 2 m n^2 + 17 b^2 c^2 - 10 m n^2$. Here, $5 b^2 c^2 + 17 b^2 c^2 = 22 b^2 c^2$, and $- 2 m n^2 - 10 m n^2 = - 12 m n^2$; so that the given quantitv, when reduced, becomes $22 b^2 c^2 - 12 m n^2$.

For a third example, take $10 a^2 b^3 + 3 a^3 b^2 - 6 a^2 b^3 + 3 a^2 b^3 - 7 a^3 b^2 - 5 a^2 b^3 + 6 a^3 b^2 - a^3 b^2$. First unite the terms of one kind, having the sign +; $10 a^2 b^3 + 3 a^2 b^3 = 13 a^2 b^3$; now unite the terms of the same kind, having the sign —; $-6a^2b^3-5a^2b^3=-11a^2b^3$. hence, $10a^2b^3+3a^2b^3-6a^2b^3-5a^2b^3=13a^2b^3-11a^2b^3=2a^2b^3$. In like manner, $3a^3b^2+6a^3b^2=9a^3b^2$, and $-7a^3b^2-a^3b^2=-8a^3b^2$; hence, instead of these four terms, we have $9a^3b^2-8a^3b^2$, which is the same as a^3b^2 The given quantity, therefore, reduced, becomes $2a^2b^3+a^3b^2$.

Whenever the total of the negative terms exceeds that of the similar positive terms, we take the difference of the two amounts, giving this difference the sign —.

Suppose we have to reduce $b+3x^2y^3+7x^2y^3-10x^2y^3-5x^2y^3$. Uniting the similar positive and negative terms separately, we have $b+10x^2y^3-15x^2y^3$; here we have 10 times x^2y^3 added, and 15 times the same quantity subtracted, which is the same as subtracting 5 times that quantity; hence, instead of $10x^2y^3-15x^2y^3$, we may put $-5x^2y^3$, and the original quantity becomes $b-5x^2y^3$. We see that this last term might have been obtained by subtracting 10 from 15, and putting before the remainder the sign —, and after it the common letters with their proper exponents.

Art. **33.** From the preceding examples we derive the following

RULE FOR THE REDUCTION OF SIMILAR TERMS.

Unite all the similar terms of one kind affected with the sign +, by adding their coefficients, and writing the sum before the common literal quantity; unite, in like manner, the similar quantities of the same kind affected with the sign —; then take the difference between these two sums, and give to the result the sign of the greater quantity.

Remark. To prevent errors, it is advisable to mark the terms as they are reduced.

Let the learner reduce the following quantities

1. $a + 3a + 5a + 6b + b + 2a.$
2. $10x - 9y + 17x - 6x - 2x + 3y + 10y.$
3. $3ax + 4bx - 6ax + 11bx - 2ax + 3bx.$
4. $m + 10xy - 9x^2 + 3xy - 8x^2 - 5r^2$
5. $3x^2y + 6xy^2 - 3x^2y + 5xy^2 + 3m^2x.$
6. $m^2 + 10a^2b^3 + 24a^4b + 2a^2b^3 - 8a^2b^3 - 4a^4b - 4a^2b^3 - 20a^4b.$
7. $8abc + 10x^2y - 4abc - 10xy^2 + 3abc - 4x^2y + 12y^2x + abc.$
8. $12a^2b^3c + 10ab^3c - 6a^2b^3c + 8ab^3c - 2a^3bc + 4m - 20ab^3c.$
9. $11x^2y^3 - 12x^2y + 15xy^2 - 15x^2y^3 + abc + 6xy^2 - 4x^2y - 16xy^2 - 2abc.$
10. $x^2 + 3x^3 - 4x^2 + 9x^3 + 6x^2y^2 - 10x^2 - 9x^3 + 5x^2y^2 + 15x^3 - 17x^2.$
11. $3mn - 6xy^2 + 6xz - 13xy^2 - 10xz - 2xy^2 + 4xy^2 + 2xz - 11y^2x.$
12. $17p^2q^2 + 3abc - 2pq - 20p^2q^2 - 3abc + 6pq + 21p^2q^2 - 14pq + 17.$
13. $120 + m^2n^3 - 6xy + 15m^2n^2 - 24xy + 5 - m^3n - 16 + 4m^3n.$
14. $7x^2y + 3 + 4x - 10x^2y - 17 + 3x - 14x^2y + 5 + 2x + 9 + 13x^2y + xy^2.$

SECTION IX.

ADDITION.

ART. **34.** The addition of *positive* monomials, as has already been seen, consists in writing them after each other, putting the sign $+$ before each except that placed first, which is understood to be affected with the sign $+$. Thus, to add a, b, c, and d, we write $a+b+c+d$.

ART. **35.** The addition of the two polynomials, $a+b$ and $c+d+e$, in which all the terms are affected with the sign $+$, gives $a+b+c+d+e$; for adding these two quantities is the same as adding the separate terms of which they are composed. As an example in numbers, let us add $10+3$ and $6+9$. Now, $10+3$ is 13, and $6+9$ is 15, and the sum of 13 and 15 is 28; also $10+3+6+9$ is 28.

But if some terms of the quantities to be added have the sign —, these terms retain that sign in the sum. This is easily seen in the case of numbers. Let us add $9-3$ to 13; $9-3$ is 6, and 6 added to 13 gives 19. But if we first add 9 to 13, which is expressed thus, $13+9$, the sum is too great by 3, because we were to add only $9-3$, or 6; we must, therefore, subtract 3 from $13+9$, which gives for the true sum $13+9-3$, or 19.

Let us now add $a-b$ to m. Adding a to m, we have $m+a$; this result is too great by b, because we were to add only $a-b$, which is less than a by b; therefore, after having added a, we must subtract b, which gives for the correct result $m+a-b$.

We see, in the foregoing examples, that in adding we merely write the quantities after each other, without any change in the signs.

As another example, let it be required to add the polynomials $a^2 + 3ab + 3b^2 - c^2$, and $5a^2 - 7ab + 10b^2 - 6c^2$. The sum is $a^2 + 3ab + 3b^2 - c^2 + 5a^2 - 7ab + 10b^2 - 6c^2$; but this result contains similar terms, and may be reduced; it then becomes $6a^2 - 4ab + 13b^2 - 7c^2$, which is the sum in its simplest form.

ART. **36.** From what precedes, we have the following

RULE FOR THE ADDITION OF ALGEBRAIC QUANTITIES.

Write the several quantities one after another, giving to each term its original sign, and then reduce the similar terms.

Remember that those terms which have no sign, are supposed to have the sign $+$.

1. Add $3a$, $7b$, $6c$, and $4a + 2b$.
2. Add $2a^2$, $4m^2 + 3mn$, and $7a^2 + 6mn$.
3. Add $4a$, $3a - 2c$, and $5m + 7c$.
4. Add $a^2 + 2ab + b^2$, and $3a^2 - 2ab + 4b^2$.
5. Add $3x^2 - 15xy + 11y^2$, $15xy$, and $20x^2 - 13y^2 + m^2$.
6. Add $4c^2 + 5m^2$, $3m^2 - 2c^2$, $15c^2 - 10m^2$, and $x^2 + 3y^2 - c^2$.
7. Add $-20 + 3x^2$, $17x^2 + 5y^2$, $45 - 7x^2$, and $x^2 - 3y^2$.
8. Add $-12a^2x - 20ax^2 + 8x^3$, and $24a^3 + 40ax^2 - 16a^2x$.
9. Add $a^6 + a^4c^2 + a^2c^4$, and $c^6 - a^4c^2 - a^2c^4$.
10. Add $x^3 + 3ax^2 + 4a^2x + 2$, and $4x^3 - 3ax^2 + 10a^2x - 1$.

11. Add $8a^2b - 5ab^2 - 8abc + 4bc^2$, and $6ab^2 - 2a^2b - abc - 4bc^2$.

12. Add $6ab + 2ac - 3bc$, $4bd - 7ab$, and $6bc + 5bd - 3ac$.

13. Add $10 + x^2$, $30 - 3y^2$, $4a^2b^2 + 3c^2$, $11x^2 + 7y^2$, and $23 - a^2b^2$.

14. Add $17a^2b^2 - 10ab^2c + 5abc^2$, $45a^2b^2 - 10abc^2$, and $25 + 10ab^2c$.

15. Add $a^3pm + apm^3$, $6 + 3ap^2m - 4apm^3$, $7a^3pm + a^3mp + 6amp^2$, and $10a^3mp - 8amp^2 - 12a^3mp$.

16. Add $a^2b^2 - 25 + x^2$, $3a^2b^2 + y^2 - 4x^2$, $75 - 9a^2b^2 + 3y^2$, and $45x^2y^2 + 3x^2 - 9$.

SECTION X.

SUBTRACTION.

Art. **37.** We have already seen that the subtraction of a positive monomial consists in giving it the sign —, and writing it either after or before the quantity from which it is to be subtracted. Thus, to subtract b from a, we write $a - b$, or $-b + a$.

If we have to subtract a polynomial in which all the terms are affected with the sign +, it is plain that each of the terms must be subtracted, that is, the sign of each term must be changed to —.

As an example in figures, if it is required to subtract $7 + 3$ from 18, we must subtract both 7 and 3; thus, $18 - 7 - 3$, which reduced becomes 8. This result is correct, because $7 + 3$ is 10, and 10 taken from 18 leaves 8.

In like manner, to subtract $m + n$ from a, we write $a - m - n$, changing the signs of both m and n.

If some of the terms in the quantity to be subtracted have the sign —, the signs of those terms must be changed to +.

As an example in numbers, let us subtract $7 - 4$ from 12; $7 - 4$ is 3, and 3 subtracted from 12 leaves 9. Now, to subtract $7 - 4$ without reducing it, if we first subtract 7, which is expressed thus, $12 - 7$, we subtract too much by 4, and the remainder, $12 - 7$, or 5, is too small by 4; consequently, after having subtracted 7, we must add 4, which gives $12 - 7 + 4$, or 9, for the true remainder.

In like manner, $a - b$ subtracted from c, gives $c - a + b$. For, if a be subtracted from c, we have $c - a$; but since a is greater than $a - b$ by b, we have subtracted too much by b, and the remainder, $c - a$, is too small by b; we must therefore add b to $c - a$, and we have $c - a + b$ for the true result.

For another example, let us subtract $a^2 - 2ab + 3m^2$ from $5a^2 + 4ab - 2m^2$. As in the previous examples, it may be shown that the terms in the quantity to be subtracted must have their signs changed. Making this change in the signs, and then writing the quantities after each other, we have, for the remainder, $5a^2 + 4ab - 2m^2 - a^2 + 2ab - 3m^2$. This remainder is correct, but it may be reduced, and it then becomes $4a^2 + 6ab - 5m^2$, which is the remainder in its simplest form.

Art. **38.** From what precedes, we derive the following

RULE FOR THE SUBTRACTION OF ALGEBRAIC QUANTITIES.

Change the signs of all the terms in the quantity to be subtracted, from + to —, or from — to +, and write it *after that from which it is to be subtracted; then reduce similar terms*

1. From $11\,a^2 b - 6\,a\,b^2$, subtract $4\,a^2 b - 7\,a\,b^2 + m^2$

Changing the signs of all the terms in the latter quantity, and writing the result after the former, we have $11\,a^2 b - 6\,a\,b^2 - 4\,a^2 b + 7\,a\,b^2 - m^2$, which, reduced, becomes $7\,a^2 b + a\,b^2 - m^2$, *Ans.*

2. From $a^2 + 2\,a\,b + b^2$, subtract $a^2 - 2\,a\,b + b^2$.
3. From $3\,m^2 + 9\,m\,c + 2\,c^2$, subtract $3\,m^2 - 9\,m\,c + 2\,c^2$.
4. From $6\,a^2 b^2 + 17\,a\,b\,c$, subtract $3\,a^2 b^2 - 5\,a\,b\,c$.
5. From $25 + x$, subtract $25 - 3\,x$.
6. From $x^2 + 27$, subtract $x^2 - 27$.
7. From $10\,a^2 b^2 c - 13\,m\,n + 4\,m^2$, subtract $-15\,m\,n + 6\,a^2 b^2 c - 3\,m^2$.
8. From $8\,a\,b - 2\,c\,d + 5\,a\,c - 7\,a\,d$, subtract $3\,a\,b + 4\,c\,d + 5\,a\,c$.
9. From $3\,b\,d + 2\,a + m$, subtract $2\,b\,d - 3\,a - b$.
10. From $7\,a^2 b^2 c + 5\,a\,b^2 c^2 - 9\,a\,b\,c + m$, subtract $2\,a^2 b^2 c - 4\,a\,b^2 c^2 - 8\,a\,b\,c - 2\,m$.

ART. **39.** The subtraction of a polynomial is *indicated* by enclosing it within *brackets*, or a *parenthesis*, and placing the sign — before the whole. Thus, $3\,m - [a^2 - 7\,a\,b + m]$, or $3\,m - (a^2 - 7\,a\,b + m)$, indicates the subtraction of $a^2 - 7\,a\,b + m$ from $3\,m$. Performing this subtraction, recollecting that a^2 is affected with the sign $+$, we have $2\,m - a^2 + 7\,a\,b$.

Let the learner perform the subtraction indicated in the following examples.

1. $3\,a^2 + b - (a^2 - 10\,b + c^2)$.
2. $4\,m^2 - n^2 - (7\,a\,b - 6\,m^2 + 3\,n^2)$.
3. $a^2 + 2\,a\,c + c^2 - (a^2 - 2\,a\,c + c^2)$.

4. $5abc + 14m^2 - (10abc - 3m^2 - 3cd)$.
5. $-7ax^2y^2 + 3x^2y - (15x^2y - 10x^2y^2 - 5abc)$.
6. $a^2 + 6ax + 10ax^2 - (a^2 - 6ax - 10ax^2)$.
7. $m^4 - (a^2 - 2m^4)$.
8. $7xz - 3xz^2 + 3z^3 - (3xz^2 - 7xz + 10z^3)$.
9. $4x^2y^2 - 4xy^3 - 1 - (2x^2y^2 - 4xy^3 + 5)$.
10 $30ab - 15ac^2 + 15m^2x - (15ac^2 + 30ab - 15m^2x)$.

ART. **40.** It is often useful to reverse the process in the last article, and place part of a polynomial within a parenthesis, preceded by the sign —. This may be done without altering the value of the polynomial, provided the signs of all the terms placed within the parenthesis are changed. Thus, $a - m + n$ may be written $a - (m - n)$; for if the subtraction indicated in the latter expression be performed, we obtain $a - m + n$.

Let the student throw all, except the first term, of each of the following polynomials into a parenthesis, preceded by the sign —.

1. $m^2 - c + a$.
2. $4a^2 + b - 3c^2$.
3. $150 - 2x^2 + 4y - 7ab$.
4. $6x^2y^2 + 2ax + 3by - 10z^2$.
5. $a^2m^2 - 10 + 6xy$.
6. $3abc + mx - 3y + 10$.
7. $4x^2y^3 - 6xy^3 + abc - m^2x$.
8. $20p^2q - 56 + x^2$.
9. $a^2b^2c^2 + 10 - a^3bc^2 + abc$.

SECTION XI.

MULTIPLICATION OF POLYNOMIALS.

ART. **41.** Let it be required to multiply $7+3$ by 5. Since $7+3$ is 10, the product must be $5 \cdot 10$, or 50. Now, to multiply without reducing the multiplicand, it is evident that we must multiply both 7 and 3 by 5, and add the products.

Operation.

$$\begin{array}{r} 7+3 \\ 5 \\ \hline \text{Product} = 35+15=50. \end{array}$$

In like manner, to multiply $a+b$ by c, we must multiply both a and b by c, and add the products.

Operation.

$$\begin{array}{r} a+b \\ c \\ \hline \text{Product} = a\,c+b\,c. \end{array}$$

But if any term of the multiplicand has the sign —, the monomial multiplier being affected with the sign +, the corresponding product must have the sign —.

As an example in numbers, let us multiply $9-5$ by 3. Since $9-5$ is 4, the product must be $3 \cdot 4$, or 12. But if we first multiply 9 by 3, the product 27 is too great by 3 times 5, or 15; we must, therefore, multiply 5 by 3, and subtract the product 15 from 27, which gives $27-15$, or 12.

Operation.

$$\begin{array}{r} 9-5. \\ 3 \\ \hline \text{Product} = 27-15=12. \end{array}$$

In a similar manner, to multiply $a - b$ by c, if we first multiply a by c, the product $a c$ is too great by c times b; we must, therefore, multiply b by c, and subtract the product $b c$ from $a c$, which gives $a c - b c$ for the correct result.

Operation.

$$\begin{array}{r} a - b \\ c \\ \hline \text{Product} = a c - b c. \end{array}$$

Hence we see that $-b$, multiplied by $+c$, gives $-b c$

If both of the quantities whose product is sought are polynomials, the whole of the multiplicand must be multiplied by each term of the multiplier.

Let it be required to multiply $7+2$ by $4+3$. Since $7+2$ is 9, and $4+3$ is 7, the product must be $7 \cdot 9$, or 63. To produce this, $7+2$ must first be multiplied by 4, which gives $28+8$; then $7+2$ must be multiplied by 3, which gives $21+6$; the sum of these products is $28+8+21+6$, or 63.

Operation.

$$\begin{array}{l} 7+2 \\ 4+3 \\ \hline \text{Product} = 28+8+21+6 = 63. \end{array}$$

In a similar manner, to multiply $a+b$ by $c+d$, we first multiply $a+b$ by c, which gives $a c + b c$; then multiply $a+b$ by d, which gives $a d + b d$; now, adding these products, we have $a c + b c + a d + b d$ for the entire product.

Operation.

$$\begin{array}{l} a+b \\ c+d \\ \hline \text{Product} = a c + b c + a d + b d. \end{array}$$

Now, let it be required to multiply $6-2$ by $5-3$. Since $6-2$ is 4, and $5-3$ is 2, the product must be 4.2, or 8. If we first multiply $6-2$ by 5, we have $30-10$, or 20; this is too great by 3 times $6-2$, which is $18-6$, or 12; we must, therefore, subtract $18-6$ from $30-10$, which gives $30-10-18+6$, or 8.

Operation.

$$\begin{array}{r} 6-2 \\ 5-3 \\ \hline \end{array}$$

$$\text{Product} = 30-10-18+6=8.$$

In like manner, to multiply $a-b$ by $c-d$, we first multiply $a-b$ by c, and the product, as has already been seen, is $ac-bc$; this is too great by d times $a-b$, which is $ad-bd$; this last product must therefore be subtracted from $ac-bc$, which gives $ac-bc-ad+bd$, for the true product of $a-b$ by $c-d$

Operation.

$$\begin{array}{l} a-b \\ c-d \\ \hline \end{array}$$

$$\text{Product} = ac-bc-ad+bd.$$

On examining the preceding product, we see that $-ad$ was produced by multiplying $+a$ by $-d$; hence, if a term having the sign $+$ be multiplied by a term having the sign $-$, the corresponding product must have the sign $-$. Also, $+bd$ was produced by multiplying $-b$ by $-d$; hence, if a term having the sign $-$ be multiplied by another term having the sign $-$, the product must have the sign $+$; in other words, if two negative terms are multiplied together, the product must be positive.

ART. **42.** From the preceding analysis, we derive the following

RULE FOR THE MULTIPLICATION OF POLYNOMIALS.

1. *Multiply all the terms of the multiplicand by each term of the multiplier separately, according to the rule for the multiplication of monomials.*

2. *With regard to the signs, observe, that if the two terms to be multiplied together have the same sign, either both +, or both —, the corresponding product must have the sign +; but if one term has the sign +, and the other the sign —, the corresponding product must have the sign —*

3. *Add together the several partial products, reducing terms which are similar.*

1 Multiply $4ax + 7c^2y$
by $2ax + 5c^2y$

$$\left.\begin{array}{l} 8a^2x^2 + 14ac^2xy \\ \qquad\qquad + 20ac^2xy + 35c^4y^2 \end{array}\right\} \text{Partial products.}$$

$$8a^2x^2 + 34ac^2xy + 35c^4y^2;$$ the entire product reduced.

To facilitate reduction, it is advisable to place the similar terms of the partial products under each other.

2. Multiply
$2a^2b + 3bc - c^2$ by
$a^2b - 5bc$

$$\left.\begin{array}{l} 2a^4b^2 + 3a^2b^2c - a^2bc^2 \\ \qquad - 5a^2b^2c - 15b^2c^2 + 5bc^3 \end{array}\right\} \text{Partial products.}$$

$$2a^4b^2 - 2a^2b^2c - a^2bc^2 - 15b^2c^2 + 5bc^3.$$ Result.

3. Multiply
$x^4 + x^3y + x^2y^2 + xy^3 + y^4$, by
$x - y$

$$\left.\begin{array}{l} x^5 + x^4y + x^3y^2 + x^2y^3 + xy^4 \\ \quad - x^4y - x^3y^2 - x^2y^3 - xy^4 - y^5 \end{array}\right\} \text{Partial products.}$$

$$x^5 - y^5.$$ Result.

4. Multiply $7ab + 3cd$ by $2mn + 4cx$.
5. Multiply $15xy + 7x^2 - 4m$ by $3xy - 2x^2$.
6. Multiply $3a^2 - 2ab + 5b^2$ by $3a - 5b$.
7. Multiply $a^2 + ab + b^2$ by $a - b$.
8. Multiply $x^4 - y^4$ by $x^4 + y^4$.
9. Multiply $2x^2 - 3xy + 6$ by $3x^2 + 3xy - 5$.
10. Multiply $3a - 2b + 2c$ by $2a - 4b + 5c$.

ART. **43.** The multiplication of polynomials is *indicated* by putting the quantities under a vinculum, or within parentheses, and writing them after each other, with, in the former case, and either with or without the sign of multiplication between them, in the latter. It is most usual to omit the sign of multiplication. Thus, each of the expressions, $\overline{x+y} \times \overline{x-y}$, $\overline{x+y} \,.\, \overline{x-y}$, $(x+y) \times (x-y)$, $(x+y) \,.\, (x-y)$, and $(x+y)\,(x-y)$, indicates the multiplication of $x+y$ by $x-y$.

The parenthesis is generally preferable to the vinculum; but the learner must be careful to include the whole of each polynomial within the parenthesis; for $(a+b)\,x-y$ signifies that $a+b$ is multiplied by x only, and that y is subtracted from the product.

The expression $(a+b)\,(c-d)\,(x+y)$ indicates that the first polynomial is to be multiplied by the second, and that product by the third.

Let the student perform the operations indicated in the following examples.

1. $(m^2 + n^2)\,(m^2 - n^2)$.
2. $(a^2 + 2ab + b^2)\,(a - b)$.
3. $(4a + 5b^2 - 6cd^2)\,(5a - 7b^2)$.
4. $(3a^2 - 5ab + 2)\,(4a^2 + 10ab - 6)$.
5. $(x+y)\,(x-y)\,(x^2+y^2)$.
6. $(a^2b - 3cd)\,(a^2 + xy) + (3a^2b + 3cd)\,(a^2 - 3xy)$.

In this last example, the product of the third and fourth polynomials is to be added to the product of the first and second.

7. $(3m^2+3mn)(3x+2y)+(5m^2+4ax)(2x-y)$.

8. $(17x^2-2y)(a^2+2b)+(25x^2+3y)(3a^2-5b)$.

9. $(6xy+3az)(2x^2-5az)-(2xy-mx)(2x^2-4mx)$.

In the 9th example, the product of the last two quantities is to be subtracted from that of the first two.

10. $(3ab+cd)(m^2x+ny)-(3m^2x-4ny)\,2ab$.

11 $(a^5+5a^4x+2y^2)\,b^2c-(a^4+4a^3x)(ab^2c-my^2)$.

12. $(5a^2-3ab+4b^2)(6a-5b)-3a(a^2+ab)$.

ART. **44.** The following examples deserve particular attention, on account of the use which will hereafter be made of the results.

The sum of the two quantities, a and b, is $a+b$, and their difference is $a-b$. Let us multiply this sum bv this difference.

Operation.

$$\begin{array}{l} a+b \\ a-b \\ \hline a^2+ab \\ \quad\;\; -ab-b^2 \\ \hline a^2-b^2. \end{array}$$

This product is the difference between the second power of a, and the second power of b. Hence,

The sum of two quantities multiplied by their difference gives the difference of the second powers of those quantities

According to this principle, $(8+3)(8-3)=64-9=55$, and $(3a+x)(3a-x)=9a^2-x^2$.

Let the learner give the results of the following examples, without actually multiplying.

1. $(x+y)\,(x-y)$.
2. $(m+n)\,(m-n)$.
3. $(3\,a-4\,x)\,(3\,a+4\,x)$.
4. $(4\,a^2+3\,y)\,(4\,a^2-3\,y)$.
5. $(10+2\,a)\,(10-2\,a)$.
6. $(6\,a^2\,b-3\,m)\,(6\,a^2\,b+3\,m)$.

ART. **45.** When a polynomial is multiplied by itself, the product is called the *second power*, and when multiplied twice by itself, the product is called the *third power* of that polynomial.

Find the second power of $a+b$.

Operation.

$$\begin{array}{l} a+b \\ a+b \\ \hline a^2+a\,b \\ \quad\ +a\,b+b^2 \\ \hline a^2+2\,a\,b+b^2 = \text{2d power of } a+b. \end{array}$$

Hence the second power of the sum of two quantities contains the second power of the first quantity, plus twice the product of the first by the second, plus the second power of the second.

Find the second power of $a-b$.

Operation.

$$\begin{array}{l} a-b \\ a-b \\ \hline a^2-a\,b \\ \quad\ -a\,b+b^2 \\ \hline a^2-2\,a\,b+b^2 = \text{2d power of } a-b. \end{array}$$

This result differs from the second power of $a+b$ only in the sign of $2\,a\,b$, which is in this case negative.

By means of the two principles in this and the preceding article, write the second powers of the following quantities.

1. $x+y$.
2. $x-y$.
3. $m+n$.
4. $m-n$.
5. $2\,x+3\,y$.
6. $7\,m-x$.
7. $3\,x+5\,y$.
8. $a^2+3\,b^2$.
9. $10-3\,x^2$.
10. $3\,x\,y+2\,m^2$.

Art. 46. If the second power of $a+b$ is multiplied by $a+b$, the product will be the third power of $a+b$, which is

$$a^3+3\,a^2\,b+3\,a\,b^2+b^3.$$

Hence the third power of the sum of two quantities contains the third power of the first quantity, plus three times the second power of the first into the second, plus three times the first into the second power of the second, plus the third power of the second.

In like manner, the third power of $a-b$ will be found to be

$$a^3-3\,a^2\,b+3\,a\,b^2-b^3,$$

which differs from that of $a+b$, in having its second and fourth terms negative.

Write the third powers of the following quantities.

1. $x+y$.
2. $x-y$.
3. $m+n$.
4. $m-n$
5. $2\,a+m$.
6. $4\,x-y$.
7. $3\,m+2\,x$.
8. $2\,m^2-3\,y^2$

SECTION XII.

DIVISION OF MONOMIALS.

ART. **47.** Division being the reverse of multiplication, the object is to find, for the quotient, a quantity, which, when multiplied by the divisor, shall produce the dividend: in other words, having a product and one of its factors, the object is to find the other factor.

Since, then, the product of the divisor and quotient must reproduce the dividend, the coefficient of the quotient must be such, that, when multiplied by the coefficient of the divisor, it shall produce that of the dividend; and the exponent of any letter in the quotient must be such, that, when added to the exponent of the same letter in the divisor, it shall give the exponent of that letter in the dividend. Also, it is manifest that the quotient must contain those letters of the dividend which are not found in the divisor.

Divide $m a$ by m, or find $\frac{1}{m}$ of $m a$. The answer is a; because, if m be multiplied by a, the product is $m a$.

Divide $a x$	by a.	Ans. x.
Divide $3 a b$	by $3 b$.	Ans. a.
Divide $6 a m x$	by $2 a$.	Ans. $3 m x$.
Divide a, or $1 a$	by 1.	Ans. a.
Divide a, or $1 a$	by a.	Ans. 1.
Divide $a b c$	by $a b c$.	Ans. 1.
Divide $7 a b x$	by $a x$.	Ans. $7 b$.
Divide $15 a m$	by $5 m$.	Ans. $3 a$.

We readily see the correctness of the preceding answers, because, in each case, the product of the divisor and quotient gives the dividend.

Divide a^5 by a^2. The quotient is a^3, because $a^2 . a^3 = a^5$.

Divide $7\,a^2\,b^5\,c$ by $7\,a\,b^3$. The quotient is $a\,b^2\,c$, because $7\,a\,b^3 . a\,b^2\,c = 7\,a^2\,b^5\,c$.

We found, in the multiplication of monomials, that, when the same letter occurred in both multiplicand and multiplier, the exponents were added, to obtain the exponent of that letter in the product. On the other hand, in dividing, as we see in the last two examples, the exponent of any letter in the divisor is subtracted from the exponent of the same letter in the dividend, in order to obtain the exponent of that letter in the quotient. Thus, a^5, divided by a^2, gives $a^{5-2} = a^3$.

ART. **48.** From what precedes, we deduce the following

RULE FOR DIVIDING ONE MONOMIAL BY ANOTHER.

1. *Divide the coefficient of the dividend by the coefficient of the divisor.*
2. *Strike out from the dividend the letters common to it and the divisor, when they have the same exponents in both; but if the exponents of any letter are different, subtract its exponent in the divisor from its exponent in the dividend, and write the letter in the quotient with the remainder for an exponent.*
3. *Write also in the quotient, with their respective exponents, the letters of the dividend not found in the divisor.*

Remark. If, however, the divisor and dividend are, in any case, alike, the quotient is 1.

1. Divide $6\,a^2\,b^3$ by $2\,a\,b$. Ans. $3\,a\,b^2$.
2 Divide $25\,a^4$ by $5\,a^2$.

3. Divide $16x^5y$ by $2x^2y$.
4. Divide $17am^2x^3$ by ax.
5. Divide $21x^3y^4z$ by $3xz$.
6. Divide a^2x^3 by a^2x^3.
7. Divide $35x^7y^4$ by $7xy^3$.
8. Divide $350m^7y^6$ by $10m^3y^3$
9. Divide $64a^3m^5x$ by a^2m^3x.
10. Divide $33x^9y^7z^6$ by $11xy^2z^3$.
11. Divide $150am^3x^4$ by $30ax^2$.
12. Divide $17ax$ by ax.
13. Divide $18a^2b^3c^2x$ by $6ab^2c$.
14. Divide $28mxp^2$ by $4mxp$.
15. Divide $333a^7xy^4z$ by $9axy^3$.
16. Divide $111p^5x^2y^5$ by $3p^2y$.
17. Divide $99a^4$ by $33a$.
18. Divide $17am^4x^4$ by ax^4.
19. Divide $3a^5b^4c^3x^7$ by $3ab^2c^3$.
20. Divide $75m^{10}y^9$ by $25m^7y^6$.
21. Divide $125x^8y^7z^6$ by $5x^3y^2z$.
22. Divide $3a^2bc$ by abc.
23. Divide $100m^{10}xy^5$ by $4m^3xy^2$.

SECTION XIII.

DIVISION OF POLYNOMIALS.

ART. **49.** If $a+m-c$ be multiplied by d, the product will be $ad+dm-cd$; consequently, if $ad+dm-cd$ be divided by d, the quotient will be $a+m-c$. This quotient is obtained by dividing each term of the dividend, $ad+dm-cd$, by the divisor d. In order, therefore, that a polynomial may be divisible by a monomial, each term of the dividend must be divisible by the divisor. The rule for the signs of the partial quotients may be determined as follows, if we recollect that, in all cases, the product of the divisor and quotient must give the dividend.

If $+am$ be divided by $+a$, the quotient must be $+m$, because the product of $+a$ and $+m$ is $+am$.

If $+am$ be divided by $-a$, the quotient must be $-m$, because the product of $-a$ and $-m$ is $+am$.

If $-am$ be divided by $+a$, the quotient must be $-m$, because the product of $+a$ and $-m$ is $-am$.

Finally, if $-am$ be divided by $-a$, the quotient must be $+m$, because the product of $-a$ and $+m$ is $-am$.

We perceive, therefore, that when two terms, whose quotient is sought, have the same sign, whether both $+$ or both $-$, the quotient must have the sign $+$; but when the signs of the two terms are different, the quotient must have the sign $-$.

ART. **50.** From what precedes, we derive the following

RULE FOR THE DIVISION OF A POLYNOMIAL BY A MONOMIAL.

1. *Divide each term of the dividend by the divisor, according to the rule for dividing one monomial by another; the partial quotients, taken together, will form the entire quotient.*

2. *With regard to the signs, observe that when the term of the dividend and the divisor have the same sign, the corresponding term of the quotient has the sign $+$; but when the term of the dividend and the divisor have dif-*

ferent signs, the corresponding term of the quotient has the sign —.

1 Divide $a m + m^2 x$ by m. Ans $a + m x$.
2. Divide $3 x^2 y + 6 x$ by $3 x$.
3. Divide $15 m y^2 - 30 y^3$ by $5 y^2$.
4. Divide $45 m^4 - 15 m^3 y + 5 m$ by $5 m$.
5. Divide $10 d x y + 16 d^2 + 4 d y^2$ by $2 d$.
6. Divide $11 x^2 y^2 + 33 x y^3 + 22 x y$ by $11 x y$.
7. Divide $34 a m^3 + 51 a m^2 n - 17 a^2 m^2$ by $17 a m$.
8. Divide $49 x - 63 x y^2 - 56 x^3$ by $7 x$.
9. Divide $-40 x^2 + 50 x y - 30 x y^2$ by $-5 x$.
10. Divide $10 a b c - 16 a^2 b^2 - 20 a^3 b$ by $-2 a b$.
11. Divide $-16 m x^2 + 32 m^2 x - 24 m^3 x^2$ by $8 m x$.
12. Divide $-45 a^2 x^3 + 3 a^3 x^2 - 60 a^4 x^2$ by $-3 a^2 x^2$.

Art. 51. When both dividend and divisor are polynomials, the process of dividing will be easily understood, if we observe the formation of a product in multiplication.

Let us multiply $3 a^2 + 2 a b$ by $4 a + 3 b$.

Operation.

$$
\begin{array}{l}
3 a^2 + 2 a b \\
\;\; 4 a + 3 b \\
\hline
12 a^3 + 8 a^2 b \\
\qquad\;\; + 9 a^2 b + 6 a b^2 \\
\hline
12 a^3 + 17 a^2 b + 6 a b^2.
\end{array}
$$

If this product be divided by the multiplicand, the quotient will be the multiplier; or if it be divided by the multiplier, the quotient will be the multiplicand.

Since, in multiplying, the entire multiplicand is multiplied by each term of the multiplier, the product, if no reduction took place, would contain a number of terms equal to the product obtained by multiplying the number of terms in the multiplicand by the number of terms in the multiplier. Thus, in the preceding example, there are 2 terms in the multiplicand, and 2 in the multiplier, and, if $8\,a^2 b$ and $9\,a^2 b$ had not been united, there would have been $2 . 2$ or 4 terms in the product. In like manner, if one factor had 5 terms and the other 3, the product, without reduction, would contain $5 . 3$ or 15 terms.

But generally, by reduction, some terms of the product are united, and others are cancelled and disappear. There are, however, two terms of the product which can neither be united with others, nor disappear. These are, 1st, the product of the term containing the *highest* power of any letter in the multiplicand by the term containing the highest power of the same letter in the multiplier; 2d, the product of the two terms containing the *lowest* powers of the same letter.

Since the dividend is to be considered as the product of the divisor and quotient, it is plain that if the term containing the highest power of any letter in the dividend be divided by the term containing the highest power of the same letter in the divisor, the result will be the term containing the highest power of that letter in the quotient.

Let us now take the product of the preceding multipli cation for a dividend, and the multiplicand for a divisor, and see how we can obtain the multiplier, considered as a quotient.

Operation.

Dividend.	Divisor.
$12\,a^3 + 17\,a^2 b + 6\,a b^2$	$3\,a^2 + 2\,a b$
$12\,a^3 + 8\,a^2 b$	$4\,a + 3\,b$. Quotient.
$9\,a^2 b + 6\,a b^2$	
$9\,a^2 b + 6\,a b^2$	
0.	

According to what has been said above, if we divide $12\,a^3$ by $3\,a^2$, we shall obtain the term of the quotient with the highest power of a. This division gives $4\,a$ for the first term of the quotient. Then, since the entire dividend is produced by multiplying the whole divisor by each term of the quotient, if we multiply the whole divisor by $4\,a$, this first term of the quotient, the product will be a part of the dividend. This product is $12\,a^3 + 8\,a^2\,b$, which we subtract from the dividend, and find for a remainder $9\,a^2\,b + 6\,a\,b^2$.

This remainder is to be considered as a new dividend, and as produced by multiplying the divisor by the remaining part of the quotient; and if the term containing the highest power of a in this new dividend be divided by the term containing the highest power of a in the divisor, the result must be a new term of the quotient.

The division of $9\,a^2\,b$ by $3\,a^2$ gives $+3\,b$, which we write as the second term of the quotient. We now multiply the whole divisor by $3\,b$, and the product must be the whole or a part of the new dividend. This product is $9\,a^2\,b + 6\,a\,b^2$, which, subtracted from the new dividend, leaves no remainder. The entire quotient, therefore, is $4\,a + 3\,b$.

Since we always divide the term containing the highest power of some letter in the dividend by the term containing the highest power of the same letter in the divisor, it is convenient to write the quantities so that the former of these terms shall stand first in the dividend, and the latter first in the divisor. This object will be attained by *arranging* both dividend and divisor according to the powers of the same letter.

Remark. A polynomial is said to be *arranged* according to the powers of a particular letter, when, in the successive terms, the powers of that letter increase or diminish from left to right. In the example of division just given, the polynomials were arranged according to the decreasing powers of a. The same arrangement should be preserved in each partial dividend, as was made at first in the entire dividend.

The rule for the signs of the partial quotients is manifestly the same as that given for the division of a polynomial by a monomial.

ART. **52.** From the preceding explanations, we deduce the following

RULE FOR DIVIDING ONE POLYNOMIAL BY ANOTHER.

1. *Arrange the dividend and divisor according to the powers of the same letter, beginning with the highest.*

2. *Divide the first term of the dividend by the first term of the divisor, and place the result as the first term of the quotient; recollecting, that if both terms have the same sign, the partial quotient must have the sign +, but if they have different signs, the partial quotient must have the sign —.*

3. *Multiply the whole divisor by this term of the quotient, subtract the product from the dividend, and the remainder will form a new dividend.*

4. *Divide the first term of the new dividend by the first term of the divisor, and the result will form the second term of the quotient; multiply the whole divisor by this second term of the quotient, and subtract the product from the second dividend. The remainder will form a new dividend from which another term of the quotient may be found.*

These operations are to be repeated, until all the terms of the original dividend are exhausted.

As an example, let us divide $50\,a^3\,b^2 - 41\,a^4\,b + 20\,a^5 - 33\,a^2\,b^3 + 10\,a\,b^4$ by $5\,a\,b^2 + 5\,a^3 - 4\,a^2\,b$.

Operation.

$$
\begin{array}{l|l}
20a^5 - 41a^4b + 50a^3b^2 - 33a^2b^3 + 10ab^4 & 5a^3 - 4a^2b + 5ab^2 \\
\underline{20a^5 - 16a^4b + 20a^3b^2} & \overline{4a^2 - 5ab + 2b^2} \\
\quad -25a^4b + 30a^3b^2 - 33a^2b^3 + 10ab^4 & \\
\quad \underline{-25a^4b + 20a^3b^2 - 25a^2b^3} & \\
\qquad 10a^3b^2 - 8a^2b^3 + 10ab^4 & \\
\qquad \underline{10a^3b^2 - 8a^2b^3 + 10ab^4} & \\
\qquad\qquad 0. &
\end{array}
$$

First, arranging the two quantities according to the powers of a, we place the divisor on the right of the dividend, separating it from the dividend by some mark, and draw a line below to separate it from the quotient.

We now divide $20\,a^5$ by $5\,a^3$, and have for a quotient $+4\,a^2$, which we write as the first term of the quotient Multiplying the whole divisor by $4\,a^2$, we place the product, $20\,a^5 - 16\,a^4b + 20\,a^3\,b^2$, under the dividend, and subtract it from the dividend. The subtraction is performed by changing the signs of the terms in the quantity to be subtracted, considering it as written after the dividend, and then reducing. Thus, changing the signs, we have $-20\,a^5 + 16\,a^4\,b - 20\,a^3\,b^2$; then $+20\,a^5$ and $-20\,a^5$ cancel, $-41\,a^4\,b$ and $+16\,a^4\,b$ make $-25\,a^4\,b$, and $+50\,a^3\,b^2$ and $-20\,a^3\,b^2$ make $+30\,a^3\,b^2$; bringing down the remaining terms of the dividend, we have for a remainder $-25\,a^4\,b + 30\,a^3\,b^2 - 33\,a^2\,b^3 + 10\,a\,b^4$, which we regard as a new dividend.

We now divide $-25\,a^4\,b$ by $5\,a^3$, and obtain $-5\,a\,b$, which we write as the second term of the quotient. Multiplying the whole divisor by $-5\,a\,b$, we subtract the product, $-25\,a^4\,b + 20\,a^3\,b^2 - 25\,a^2\,b^3$, from the second dividend, and find for a remainder $10\,a^3\,b^2 - 8\,a^2\,b^3 + 10\,a\,b^4$, which forms the third dividend.

Dividing $10\,a^3\,b^2$ by $5\,a^3$, we obtain $+2\,b^2$, which we write as the third term of the quotient. Multiplying and subtracting as before, we have no remainder. The entire quotient, therefore, is $4\,a^2 - 5\,a\,b + 2\,b^2$, the correctness of which may be shown by multiplying it by the divisor, and finding that the product is the same as the dividend

As another example, divide $x^4 - y^4$ by $x - y$.

Operation.

$$
\begin{array}{l|l}
x^4 - y^4 & x - y \\
x^4 - x^3 y & x^3 + x^2 y + x y^2 + y^3 \\
\hline
\quad + x^3 y - y^4 & \\
\quad + x^3 y - x^2 y^2 & \\
\hline
\qquad + x^2 y^2 - y^4 & \\
\qquad + x^2 y^2 - x y^3 & \\
\hline
\qquad\quad + x y^3 - y^4 & \\
\qquad\quad + x y^3 - y^4 & \\
\hline
\qquad\qquad 0. &
\end{array}
$$

In this example several terms are produced in the process, which are not found in the dividend. These terms disappear when the divisor and quotient are multiplied together.

1. Divide $a^3\,b^2 + b\,c^2\,d + a\,c^2\,d + a^2\,b^3$ by $a + b$.
2. Divide $9\,a^2\,b^4 + 27\,a^3\,b^2\,c^2 + 18\,a^4\,c^4$ by $3\,a\,b^2 + 3\,a^2\,c^2$.
3. Divide $6\,b^4 + 12\,b^2\,x + 6\,x^2$ by $b^2 + x$.
4. Divide $6\,a^3 + 5\,b^3 + 23\,a^2\,b + 22\,a\,b^2$ by $2\,a + 5\,b$.
5. Divide $x^5 - 5\,x^4\,y + 10\,x^3\,y^2 - 10\,x^2\,y^3 + 5\,x\,y^4 - y^5$ by $x^2 - 2\,x\,y + y^2$.
6. Divide $16\,x^4 - 25\,y^4$ by $4\,x^2 + 5\,y^2$.
7. Divide $a^6 - 16\,a^3\,x^3 + 64\,x^6$ by $a^2 - 4\,a\,x + 4\,x^2$.

8. Divide $6ax^2 - 11abx + 3ab^2$ by $3x - b$

9. Divide $10a^4 + 51a^2b^2 - 48a^3b + 4ab^3 - 15b^4$ by $4ab - 5a^2 + 3b^2$.

10. Divide $3x^4 + 4x^3y - 4x^2 - 4x^2y^2 + 16xy - 15$ by $2xy + x^2 - 3$.

Art. **53.** The following principles in division will often be useful, and are found demonstrated in works designed for the more advanced student.

The difference between similar powers of two quantities, the exponent of the powers being integral and positive, is divisible by the difference of those quantities.

Thus, $x - y$, $x^2 - y^2$, $x^3 - y^3$, $x^4 - y^4$, &c., are each divisible by $x - y$.

Also, the difference of similar even powers, and the sum of similar odd powers of two quantities, are each divisible by the sum of those quantities.

Thus, $a^2 - b^2$, $a^4 - b^4$, $a^6 - b^6$, &c.; also, $a + b$, $a^3 + b^3$, $a^5 + b^5$, &c., are each divisible by $a + b$.

1. Divide $x^5 - y^5$ by $x - y$.
2. Divide $x^4 - y^4$ by $x + y$.
3. Divide $x^3 + m^3$ by $x + m$.
4. Divide $1 - m^6$ by $1 + m$.
5. Divide $m^5 + n^5$ by $m + n$.

Art. **54.** When a product is represented in its factors, it is manifestly divided by dividing one of its factors. Thus, to divide $5 . 9$ by 3, we have to divide the 9 only, and the quotient is $5 . 3$, or 15. In like manner, to divide $15 . 16$ by 12, or $3 . 4$, we divide 15 by 3 and 16 by 4, and the quotient is $5 . 4$, or 20.

Also, to divide $(m^2 - n^2)(x + y)$ by $m + n$, we divide the factor $m^2 - n^2$, and the quotient is $(m - n)(x + y)$; likewise $15m^3(a^2 + 2ab + b^2)(x^2 - y^2)$ divided by

$3m(a+b)(x-y)$, gives $5m^2(a+b)(x+y)$, to obtain which we divide $15m^3$ by $3m$, $a^2+2ab+b^2$ by $a+b$ and x^2-y^2 by $x-y$.

1. Divide $m^3(x+y)$ by m^2. Ans. $m(x+y)$.
2. Divide $a^2(m+n)$ by $m+n$. Ans. a^2.
3. Divide $4x^5(a+b)$ by $2x^3$.
4. Divide $15a(m^2+n^2)$ by $5a$.
5. Divide $21m^2(x^2-y^2)$ by $3(x+y)$.
6. Divide $3(m^3-1)(a-b)$ by $(m-1)(a-b)$.
7. Divide $16m^4(a^2-2ab+b^2)(c^2-d^2)$ by $4m^3(a-b)(c+d)$.
8. Divide $6(m^3+1)(a^3-1)$ by $2(m+1)(a-1)$

SECTION XIV.

MULTIPLICATION OF FRACTIONS BY INTEGRAL QUANTITIES

ART. **55.** Fractions have the same meaning in algebra that they have in arithmetic. Thus, $\frac{m}{n}$ signifies that a unit is divided into n equal parts, and that m of those parts are taken; or it expresses division, and signifies that m is divided into n equal parts.

1. How much is 3 times $\frac{2}{7}$? Ans. $\frac{6}{7}$.
2. How much is 5 times $\frac{m}{n}$? Ans. $\frac{5m}{n}$.
3. How much is c times $\frac{a}{b}$? Ans. $\frac{ac}{b}$.
4. What is $\frac{7}{8}$ of 5? $\frac{1}{8}$ of 5 is $\frac{5}{8}$, and $\frac{7}{8}$ of 5 is $\frac{35}{8}$, Ans
5. What is $\frac{3}{5}$ of a? $\frac{1}{5}$ of a is $\frac{a}{5}$, and $\frac{3}{5}$ of a is $\frac{3a}{5}$, Ans

6. What is the $\frac{m}{n}$ part of a? $\frac{1}{n}$ of a is $\frac{a}{n}$, and $\frac{m}{n}$ of a is $\frac{am}{n}$, Ans.

In the first three of the preceding questions, the object was to multiply a fraction by a whole number; and in the last three, to find a fractional part of a whole number, that is, to multiply a whole number by a fraction; and we perceive that in both cases we multiplied the numerator and the whole number together.

Hence, to multiply a fraction by an integral quantity, or an integral quantity by a fraction, multiply the numerator by the integral quantity, and write the product over the denominator.

1. Multiply $\frac{a}{b}$ by m. Ans. $\frac{am}{b}$.
2. Multiply $\frac{x+y}{b}$ by c. Ans. $\frac{cx+cy}{b}$.
3. Multiply $\frac{m}{x+y}$ by a.
4. Multiply $\frac{ax}{b+c}$ by m.
5. Multiply $\frac{m+n}{xy}$ by b.
6. Multiply $\frac{b}{c}$ by x^2+y^2.
7. Multiply $\frac{a+b}{c}$ by $a-b$.
8. Multiply $\frac{x+y}{b+c}$ by m^2-n^2.
9. What is the $\frac{a}{b}$ part of $2x+3y$?
10. What is the $\frac{a+b}{m}$ part of $2c-x$?
11. Multiply $\frac{2c-d}{x+y}$ by $2c+d$.

12. Multiply $\frac{4x^2y+3xy^2}{2m-3n}$ by $3x-2y$.

13. Multiply $\frac{m}{6}$ by 3.

The fraction $\frac{m}{6}$ means that m is divided into 6 equal parts. If we divide the denominator by 3, which gives $\frac{m}{2}$, m is then divided into $\frac{1}{3}$ as many parts as it was before; consequently, the parts are 3 times as great as they were before; that is, 3 times $\frac{m}{6}$ is $\frac{m}{2}$.

14. Multiply $\frac{a}{mn}$ by m.

The fraction $\frac{a}{mn}$ means that a is divided into mn equal parts. If we divide the denominator by m, which gives $\frac{a}{n}$, a is then divided into $\frac{1}{m}$ as many parts as it previously was; the parts, therefore, are m times as great as they were before; that is, m times $\frac{a}{mn}$ is $\frac{a}{n}$.

Hence, to multiply a fraction and an integral quantity together, divide the denominator by the integral quantity, if possible.

ART. **56.** Combining this rule with the preceding, we have a

GENERAL RULE TO MULTIPLY A FRACTION AND AN INTEGRAL QUANTITY TOGETHER.

Divide the denominator by the integral quantity, if possible; if not, multiply the numerator by the integral quantity.

The learner may perform the following examples by dividing the denominator.

1. Multiply $\frac{3x}{a^2 m}$ by a.

2. Multiply $\frac{a+3b}{4x^2y^3}$ by $2x^2y$.

3. Multiply $\frac{am+cn}{15x^3z^5}$ by $5x^2z^2$.

4. Multiply $\frac{ax^2y}{x^2-y^2}$ by $x-y$.

5. Multiply $\frac{4x-17y}{a^2+2ab+b^2}$ by $a+b$.

6. Multiply $\frac{m^4+3x^3}{x^3+y^3}$ by $x+y$.

7. Multiply a^2+ab+b^2 by $\frac{m^4}{a^3-b^3}$.

8. Multiply x^2+y^2 by $\frac{3ax}{5(x^4-y^4)}$.

9. Multiply $3(m+n)(a-b)$ by $\frac{4b^2+3xy}{15(m^2-n^2)(a^3-b^3)}$.

10. Multiply $4y^2(x+y)$ by $\frac{(3a+b)(m+n)}{12y^4(x^3+y^3)(a+b)}$.

11. Multiply $\frac{a}{b}$ by b.

Dividing the denominator, we have $\frac{a}{1}$, or a.

12. Multiply $\frac{a+b}{m^2 n}$ by $m^2 n$.

Dividing the denominator by $m^2 n$, we have $\frac{a+b}{1}$, or $a+b$. Hence,

If a fraction be multiplied by a quantity equal to its denominator, the product will be its numerator.

13 Multiply $\frac{3a}{4b}$ by $4b$.

14. Multiply $\frac{a+b}{a x^2}$ by $a x^2$.

15. Multiply $\frac{a x^2}{a+b}$ by $a+b$.

16. Multiply $\frac{2 m^2+n}{3 a (x+y)}$ by $3 a (x+y)$.

17. Multiply $\frac{a^2 x^2 - 3 y^2}{m^2 - n^2}$ by $(m+n)(m-n)$.

18. Multiply $\frac{4 a b c - x y}{3 a x (c-d)}$ by $3 a c x - 3 a d x$

19. Multiply $\frac{5 m^2}{2 x y (x+y)}$ by $2 x^2 y + 2 x y^2$.

Some of the following examples can be performed by dividing the denominator; in others, the numerator must be multiplied.

20. Multiply $\frac{3 a^2 b^2}{4 x^2 y^3}$ by $2 x^2 y$.

21. Multiply $\frac{2 a^2}{4 m - 3 n}$ by $a+b$.

22. Multiply $\frac{a x - x^2}{4 (m^4 - n^4)}$ by $2 (m^2 + n^2)$.

23. Multiply $3 (x^2 - y^2)$ by $\frac{a b x^2}{12 (x^4 - y^4)}$.

24. Multiply $a b m^2$ by $\frac{a + 2 b}{x+y}$.

25. Multiply $x+y$ by $\frac{m^4}{a x + a y}$.

26. Multiply $a+b$ by $\frac{a+b}{3 x y}$.

27. Multiply $x^2 + y^2$ by $\frac{x^2 - y^2}{a+b}$.

28. Multiply $x^2 - 2 x y + y^2$ by $\frac{a x + m y}{3 (x^3 - 3 x^2 y + 3 x y^2 - y^3)}$

29. Multiply $\frac{3 (a-x)}{y^4}$ by $4 (a+x)$.

30. Multiply $3 (x+y)$ by $\frac{11 a x^4}{9 (x^3 + y^3)}$.

SECTION XV

DIVISION OF FRACTIONS BY INTEGRAL QUANTITIES.

Art. 57. Divide $\frac{4}{5}$ by 2, or find $\frac{1}{2}$ of $\frac{4}{5}$. Ans. $\frac{2}{5}$.

Divide $\frac{6b}{7}$ by 3, or find $\frac{1}{3}$ of $\frac{6b}{7}$. Ans. $\frac{2b}{7}$.

Divide $\frac{ab}{m}$ by a, or find $\frac{1}{a}$ of $\frac{ab}{m}$. Ans. $\frac{b}{m}$.

These answers may be shown to be correct by the fact, that the quotient multiplied by the divisor produces the dividend.

Hence, to divide a fraction by an integral quantity, divide the numerator by the integral quantity, if possible.

1. Divide $\frac{6m^2}{bc}$ by $3m$. Ans. $\frac{2m}{bc}$.
2. Divide $\frac{15x^4y^3}{a+b}$ by $5x^2y$.
3. Divide $\frac{25a^2b^3c}{4a+3b}$ by $5a^2c$.
4. Divide $\frac{am+a^2}{5c}$ by a.
5. Divide $\frac{3x^2+6xy}{a+b}$ by $3x$.
6. Divide $\frac{3a^2b+9a^3b^2-21a^4}{5m+4n}$ by $3a^2$.
7. Divide $\frac{a^3-b^3}{x+y}$ by $a-b$.
8. Divide $\frac{5(x^3+y^3)}{6(a-b)}$ by $x+y$.
9. Divide $\frac{16a^2+32ax+16x2}{15(x-y)}$ by $a^2+2ax+x^2$.

10. Divide $\frac{12x^3+29x^2+14x}{ab+7b^2}$ by $4x+7$

11. Divide $\frac{m}{n}$ by a

In this example, the numerator cannot be divided by the divisor. But we have seen that a fraction is multiplied by dividing the denominator; on the other hand, a fraction is divided by multiplying the denominator.

The fraction $\frac{m}{n}$ means that m is divided into n equal parts; and if the denominator be multiplied by 5, for example, m will be divided into 5 times as many parts as it was before; consequently, the parts will be $\frac{1}{5}$ as great as before, that is, $\frac{1}{5}$ of $\frac{m}{n}$ is $\frac{m}{5n}$. In like manner, if the denominator of $\frac{m}{n}$ be multiplied by a, m will be divided into a times as many parts as it was before, and the parts will be $\frac{1}{a}$ as great as before; that is, $\frac{1}{a}$ of $\frac{m}{n}$ is $\frac{m}{an}$.

Hence, to divide a fraction by an integral quantity, multiply the denominator by the integral quantity.

ART. **58.** Combining this rule with the preceding, we have the following

GENERAL RULE FOR DIVIDING A FRACTION BY AN INTEGRAL QUANTITY.

Divide the numerator, if it can be done, if not, multiply the denominator, by the integral quantity.

1. Divide $\frac{4a^2c^3}{m+n}$ by $2ac^2$. Ans. $\frac{2ac}{m+n}$.

2. Divide $\frac{25x^2y^5}{3bc}$ by $6ab$. Ans. $\frac{25x^2y^5}{18ab^2c}$.

3. Divide $\frac{15\,m^3\,x^2\,y}{4\,b-c}$ by $5\,m^3\,y$.

4. Divide $\frac{6\,m^4}{x-y}$ by $x+y$.

5. Divide $\frac{3\,(a+b)}{7\,m\,n}$ by $a+x$.

6. Divide $\frac{5\,(a^2+2\,a\,b+b^2)}{9\,(x+y)}$ by $a+b$.

7. Divide $\frac{x^3-y^3}{a+m}$ by $x-y$.

8. Divide $\frac{4\,(x^3+y^3)}{m^2-n}$ by $2\,(x+y)$.

9. Divide $\frac{17\,a\,b\,c}{3\,(x^2-y^2)}$ by x^2+y^2.

10. Divide $\frac{m^4-n^4}{a+b}$ by $m+n$.

11. Divide $\frac{(a^2-b^2)\,(x^2-y^2)}{a\,b\,c}$ by $(a+b)\,(x-y)$.

12. Divide $\frac{a\,x^2+b\,m^3}{x^3-y^3}$ by $5\,(x^3+y^3)$.

SECTION XVI.

FACTORS, OR DIVISORS OF ALGEBRAIC QUANTITIES.

ART. **59.** A *prime* quantity is one that can be divided by no entire and rational quantity, except itself and unity. Thus, a, b, and $a+m$ are prime quantities.

Two quantities are *prime with regard to each other*, when no quantity, except unity, will divide them both without a remainder. Thus, $a\,b$ and $c\,d$, although neither of them is a prime quantity, are prime with regard to each other.

Remark. Although, algebraically considered, we call a, b, and c, prime quantities, they are strictly speaking such, only when they represent prime numbers.

We have frequent occasion to separate quantities into their prime factors. In a monomial, this operation is attended with no difficulty. We have only to find, according to the method usually given in arithmetic, the prime factors of the coëfficient, and to represent them as multiplied together and followed by the several letters, each written as many times as it is a factor. Thus, $18\,a^2\,m^3 = 3 \,.\, 3 \,.\, 2\, a\,a\,m\,m\,m$. In this example, the different prime factors are 3, 2, a and m; 3 is contained twice, 2 once, a twice, and m three times, as a factor.

The same quantity may be expressed in its factors thus: $2 \,.\, 3^2\, a^2\, m^3$, in which the exponents show how many times each quantity enters as a factor.

When a quantity is the product of a monomial and a prime polynomial, in order to separate it into factors, it is only necessary to divide it by the greatest monomial that will exactly divide all the terms, and to place the divisor, separated into prime factors, before the quotient, the latter being included in a parenthesis.

Thus, $a\,m + a\,n = a\,(m + n)$, in which the factors are a and $m + n$. In like manner, $50\,a^2\,b^2 + 25\,a\,b^3 = 5^2\,a\,b^2 \times (2\,a + b)$, the factors of which are 5, a, b, and $2\,a + b$.

Let the learner separate the following quantities into prime factors.

1. $16\,a^4\,b$.
2. $40\,x^2\,y^2$.
3. $120\,m\,y$.
4. $15\,x^2\,y^3$.
5. $225\,m\,y$.
6. $54\,x^3\,y^2$.
7. $3\,a\,x + 7\,a\,y$. Ans. $a\,(3\,x + 7\,y)$.

8. $2a^3 + 6a^2 m$.
9. $25m^3 - 5m^2 n$
10 $54a^3 b^2 c - 27ab^2 c^2$
11. $81m^2 xy + 27m^2 pq - 5m^3 y$.
12. $44abc - 88a^2 b + 22a^3 x$.
13. $3m^2 x^2 + 6m^2 y^2 - 3m^2$
14. $30a^3 + 25a^2 b + 5a^2$

Art. 60. When a quantity is the product of several polynomials, the process of finding its factors becomes more difficult; but in many cases some of the factors may be easily ascertained

1. Any *power* of a polynomial may evidently be separated into as many factors, each equal to that polynomial, as there are units in the exponent of the power. Thus, $x^2 + 2xy + y^2 = (x + y)^2 = (x + y)(x + y)$; and $x^3 + 3x^2 y + 3xy^2 + y^3 = (x + y)^3 = (x + y)(x + y)(x + y)$.

2. The *difference* between the *second* powers of two quantities can be separated into two factors, one of which is the sum and the other the difference of those quantities. Thus, $x^2 - y^2 = (x + y)(x - y)$; also, $x^4 - y^4 = (x^2 + y^2)(x^2 - y^2) = (x^2 + y^2)(x + y)(x - y)$.

3. The *difference* between *similar* powers of two quantities can be separated into at least two factors, one of which is the difference of those quantities. Thus, $x - y$, $x^2 - y^2$, $x^3 - y^3$, &c., are each divisible by $x - y$.

4. The *difference* between similar *even* powers of two quantities, the powers being above the second, can always be separated into at least three factors, one of which is the *sum*, and another the difference, of the quantities. Thus, $m^4 - n^4 = (m^2 + n^2)(m^2 - n^2) = (m^2 + n^2)(m + n)(m - n)$.

5. The *sum* of similar *odd* powers of two quantities can be separated into two factors, one of which is the

sum of the quantities. Thus, $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$.

The quantity $x^6 - y^6$ can be separated into four factors. Thus, $x^6 - y^6 = (x^3 + y^3)(x^3 - y^3) = (x + y)(x^2 - xy + y^2)(x - y)(x^2 + xy + y^2)$.

Let the learner separate the following quantities into prime factors.

1. $6a^2 - 6b^2$. Ans. $2.3(a + b)(a - b)$.
2. $3x^2 - 6xy + 3y^2$. Ans. $3(x - y)(x - y)$.
3. $x^2 + 2x + 1$.
4. $4x^2 + 8x + 4$.
5. $x^4 - y^4$.
6. $9x^3 + 9y^3$.
7. $15(x^6 - y^6)$.
8. $25m^8 - 25y^8$.
9. $12m^5 + 12n^5$.
10. $6x^3 - 18x^2y + 18xy^2 - 6y^3$.

Remark. Any power, also any root, of 1 is 1.

11. $9m^7 + 9$.
12. $18x^8 - 18$.

SECTION XVII.

SIMPLIFICATION OF FRACTIONS.

ART. **61.** Both numerator and denominator of a fraction may be multiplied by the same quantity, without changing the value of the fraction; for multiplying the numerator multiplies the fraction, and multiplying the denominator divides the fraction; but when a quantity is multiplied, and the result is divided by the multiplier, the value of that quantity remains unchanged.

Also, both numerator and denominator of a fraction may be divided by the same quantity, without changing the value of the fraction; for, dividing the numerator divides the fraction, and dividing the denominator multiplies the fraction; but if a quantity is divided, and the result is multiplied by the divisor, the value of that quantity remains unchanged.

ART. **62.** From the principle last stated, we deduce the following

RULE FOR SIMPLIFYING A FRACTION.

Divide the numerator and denominator by all the factors common to both.

Simplify the following fractions.

1. $\frac{15\,a\,b\,c}{25\,a^2\,b^2}$. Ans. $\frac{3\,c}{5\,a\,b}$.

2. $\frac{12\,x^2\,y^3}{14\,x\,y^5}$.

3. $\frac{13\,a^4\,b^3}{39\,a\,x^5}$.

4. $\frac{45\,m\,x^2\,y^3}{85\,m^3\,x\,y^2}$.

5. $\frac{9\,a\,b\,c}{57\,a^2\,m^4}$.

6. $\frac{155\,a\,x\,y}{465\,a^2\,x\,y^4}$.

7. $\frac{21\,a}{147\,a^2\,b^2}$.

8. $\frac{450\,x\,y^2}{900\,x^3\,y}$.

9. $\frac{33\,a\,b\,c^2\,d^3}{99\,a^2\,b^2\,c^2\,d}$.

10. $\frac{3\,a\,b + 9\,a^2}{6\,a\,m}$.

11. $\frac{15\,a^2\,b^2 - 45\,a^3}{9\,a^2\,m^3}$.

12. $\frac{7\,x^2\,y^3 + 14\,m\,x^2\,y}{21\,m\,x^2\,y^2}$.

13. $\frac{m^2\,n + 6\,m\,n}{3\,m^4 + m^3}$.

In the following examples, some of the common factors are polynomials; but they can be easily discovered by separating the numerators and denominators into factors.

14. $\frac{3\,x^2 - 3\,y^2}{6\,x - 6\,y}$. This is the same as $\frac{3\,(x+y)\,(x-y)}{3\,.\,2\,(x-y)}$.

Suppressing now the common factors 3, and $x - y$, we have for the result $\frac{x+y}{2}$.

15. $\frac{x^3 - y^3}{3x - 3y}$.

16 $\frac{a^2 - b^2}{a^2 - 2ab + b^2}$.

17. $\frac{9(x^2 - y^2)}{27(x+y)}$.

18. $\frac{12(x^3 + y^3)}{30x^2 + 30xy}$.

19. $\frac{35a^2b^3(x-y)}{45ab^2(x^3 - y^3)}$.

20. $\frac{12x^2 - 12y^2}{16(x^4 - y^4)}$.

21. $\frac{5a^2 + 10ax + 5x^2}{8a^3 + 8a^2x}$.

ART. **63.** When the division of one integral quantity by another cannot be exactly performed, it is expressed in the form of a fraction, the divisor being placed under the dividend. The fraction should then be simplified.

1. Divide $3ab^2c$ by $6bc^2$.

Expressing the division, we have $\frac{3ab^2c}{6bc^2}$, which reduced, becomes $\frac{ab}{2c}$, Ans.

2. Divide $12am^2$ by $15a^2m^3$.
3. Divide $13x^4y$ by $39x^2y^3$.
4. Divide $22mx^2$ by $33m^2x$.
5. Divide $45abx^3$ by $50a^2b^3x$.
6. Divide $54b^2c^2x$ by $33b^5cx^3$.
7. Divide $3abc + 3abm$ by $15a^2b^2x$.
8. Divide $5x^2 + 5y^2$ by $20(x^4 - y^4)$.
9. Divide $3(a+b)$ by $15(a^2 + 2ab + b^2)$.
10. Divide $6a^2(x+y)$ by $14a^3(x^3 + y^3)$.

SECTION XVIII.

MULTIPLICATION OF FRACTIONS BY FRACTIONS.

ART. **64.** Find $\frac{3}{5}$ of $\frac{7}{8}$; that is, multiply $\frac{7}{8}$ by $\frac{3}{5}$.

According to the rule for the division of fractions by integers, $\frac{1}{5}$ of $\frac{7}{8}$ is $\frac{7}{40}$, and, according to the rule for the multiplication of fractions by integers, $\frac{3}{5}$ of $\frac{7}{8}$ is $\frac{21}{40}$, Ans.

Find the $\frac{a}{b}$ of $\frac{c}{d}$; that is, find the product of $\frac{a}{b}$ by $\frac{c}{d}$.

$\frac{1}{b}$ part of $\frac{c}{d}$ is $\frac{c}{b\,d}$, and $\frac{a}{b}$ of $\frac{c}{d}$ is $\frac{a\,c}{b\,d}$, Ans.

In like manner, $\frac{a}{b} \cdot \frac{m}{n} \cdot \frac{x}{y} = \frac{a\,m\,x}{b\,n\,y}$.

Hence we have the following

RULE FOR MULTIPLYING FRACTIONS BY FRACTIONS.

Multiply all the numerators together for a new numerator, and all the denominators together for a new denominator.

Remark. As the resulting fractions should be simplified, it is best to represent the operation, then strike out the common factors, previous to the actual performance of the multiplication. Thus, $\frac{2\,a\,m}{3\,x\,y} \cdot \frac{6\,(a+b)}{4\,m^2} = \frac{2 \cdot 6\,a\,m\,(a+b)}{3 \cdot 4\,m^2\,x\,y}$ $= \frac{a\,(a+b)}{m\,x\,y}$.

1. Multiply $\frac{3\,a\,b}{4\,x\,y}$ by $\frac{2\,a\,x^2}{9\,m}$.

2. Multiply $\frac{5\,a\,x^2}{4\,b\,y^4}$ by $\frac{16\,b^3}{25\,a^3\,x}$.

3. Multiply $\frac{2\,x+4\,y}{m^2}$ by $\frac{3\,a\,m}{4\,b}$.

4. Multiply $\frac{3x+4}{5}$ by $\frac{10\,a\,m^2}{b+c}$.

5. Multiply $\frac{7a+xy}{4x^2}$ by $\frac{12\,a^2}{x+y}$.

6. Multiply $\frac{x^2-y^2}{4\,b^2}$ by $\frac{5\,b^3}{x+y}$.

7. Multiply $\frac{3\,a^2-4\,a}{7(a+b)}$ by $\frac{14(a^2-b^2)}{3\,a^2}$.

8. Find the product of $\frac{2\,a}{b}$, $\frac{3\,b^2}{4\,c\,d}$, and $\frac{5\,x\,y}{7\,m^2}$.

9. Find the product of $\frac{64\,a^2\,b\,x}{9\,m^2}$, $\frac{2(x-y)}{5\,b\,x}$, and $\frac{15}{a^2(x^2-y^2)}$.

10. Find the product of $\frac{39\,x^2}{14\,b\,c}$, $\frac{a^2-2\,a\,b+b^2}{13}$, and $\frac{3\,b^2\,c}{4(a-b)}$.

SECTION XIX.

LEAST COMMON MULTIPLE

Art. 65. When one quantity is divisible by another, the former is called a *multiple* of the latter. Thus, 10 is a multiple of 5; it is also a multiple of 2.

A *common multiple* of two or more quantities is one which is divisible by them all; and the least common multiple is the least quantity divisible by them all. Thus, 24 is a common multiple of 6 and 4, but 12 is the least common multiple of these numbers.

Let it be required to find the least common multiple of $8\,a^2\,m$ and $6\,a\,m^3$.

It is manifest that this multiple must contain all the factors of $8\,a^2\,m$ and $6\,a\,m^3$. Separating these quantities into their prime factors, we have $8\,a^2\,m = 2^3\,a^2\,m$, and $6\,a\,m^3 = 2\,.\,3\,a\,m^3$. The different prime factors are 2, 3, a, and m, each of which must be contained in the multiple required, as many times as it is found in either of the given quantities; that is, 2 must be contained three times, 3 once, a twice, and m three times, as a factor. The least common multiple is, therefore, $2^3\,.\,3\,a^2\,m^3$, or $24\,a^2\,m^3$

ART. **66.** Hence we have the following

RULE FOR FINDING THE LEAST COMMON MULTIPLE OF SEVERAL QUANTITIES.

First, separate the quantities into their prime factors; then unite in one product all these different factors, each raised to the highest power found in either of the given quantities.

Find the least common multiple in each of the following examples.

1. $4\,a^2$, $10\,a\,b^3$. Ans. $2^2\,.\,5\,a^2\,b^3 = 20\,a^2\,b^3$
2. $6\,m^4\,x$, $8\,m\,y^2$.
3. $4\,x^3\,y$, $2\,x\,y^2$, $9\,m^3$.
4. 25, $15\,m^2$, $45\,x^3\,m$.
5. $3\,x\,y$, $15\,x^3\,y^2$, $3\,(a+b)$.
6. $11\,p^2\,q^3$, $33\,p\,y$, $22\,p\,q^2$.
7. $7\,(a+b)$, $14\,(a^2+2\,a\,b+b^2)$
8. $2\,a\,b+4\,b^2$, and $14\,a^2\,b^2$.
9. $18\,(x-y)$, $9\,(x^2-y^2)$.

SECTION XX.

ADDITION AND SUBTRACTION OF FRACTIONS. COMMON DENOMINATOR.

Art. **67.** The addition and subtraction of fractions are represented by writing them after each other with the sign + and — between them, thus, $\frac{a}{b}+\frac{x}{y}-\frac{m}{n}$, care being taken to place the signs even with the line which separates the numerator from the denominator.

But when the denominators are alike, we may perform the addition or subtraction upon the numerators, placing the result over the common denominator. Thus, $\frac{2}{7}+\frac{3}{7}=\frac{2+3}{7}=\frac{5}{7}$, and $\frac{a}{m}+\frac{b}{m}=\frac{a+b}{m}$; also, $\frac{5}{9}-\frac{2}{9}=\frac{5-2}{9}=\frac{3}{9}$, or $\frac{1}{3}$, and $\frac{a}{m}-\frac{b}{m}=\frac{a-b}{m}$.

Suppose it were required to add $\frac{a}{m}$ and $\frac{b}{n}$. In this case the denominators are different; but if the numerator and denominator of the first fraction be multiplied by n, and the numerator and denominator of the second fraction be multiplied by m, the denominators of the fractions will be made alike, while the value of the fractions remain unchanged. The first fraction then becomes $\frac{an}{mn}$, and the second becomes $\frac{bm}{mn}$, by the addition of which we have $\frac{an+bm}{mn}$. Also, the difference of the same fractions is $\frac{an-bm}{mn}$.

Let it be required to add $\frac{a}{b}$, $\frac{c}{m}$, and $\frac{x}{y}$. First, we reduce the fractions to a common denominator. If the numerator and denominator of each fraction be multiplied by the denominators of both the others, which does not change the value of the fractions, they become $\frac{amy}{bmy}$, $\frac{bcy}{bmy}$, and $\frac{bmx}{bmy}$, the sum of which is $\frac{amy+bcy+bmx}{bmy}$, Ans.

Hence we derive a

RULE FOR THE ADDITION AND SUBTRACTION OF FRACTIONS.

Reduce them to a common denominator, then add the numerators, or subtract one from the other, placing the result over the common denominator.

ART. **68.** From the preceding examples, we derive also the following

RULE FOR REDUCING FRACTIONS TO A COMMON DENOMINATOR.

Multiply all the denominators together for a common denominator, and multiply each numerator by all the denominators except its own, in order to obtain the numerators.

The results obtained by this rule are correct, but not always the simplest.

Let us reduce $\frac{x}{3a^2}$, $\frac{3y}{4ab}$, and $\frac{5m}{6b^3c}$ to a common denominator. By taking the product of all the denominators, we shall obtain a common denominator considerably greater than is necessary. In this case, the least common denominator will be, as in arithmetic, the least common multiple of the given denominators. The least common multiple of $3a^2$, $4ab$, and $6b^3c$ is $3.2^2a^2b^3c$, or $12a^2b^3c$, which is the least common denominator sought. To produce $12a^2b^3c$, the first denominator is multiplied by $4b^3c$, the second by $3ab^2c$, and the third by $2a^2$; these are, therefore, the quantities by which the respective numerators are to be multiplied, and are evidently obtained by dividing the common denominator by each of the given denominators separately. This multiplication being performed, and the results placed over the common denominator, the fractions become $\frac{4b^3cx}{12a^2b^3c}$, $\frac{9ab^2cy}{12a^2b^3c}$, and $\frac{10a^2m}{12a^2b^3c}$.

ART. **69.** Hence we have the following

RULE TO REDUCE FRACTIONS TO THE LEAST COMMON DENOMINATOR.

Find the least common multiple of all the given denominators, and this will be the least common denominator; then divide the common denominator by each of the given denominators, and multiply the numerators by the respective quotients, placing each of these products over the common denominator.

Remark. Fractions must be simplified before applying this rule.

1 Reduce $\frac{a}{b}$, $\frac{c}{d}$, and $\frac{x}{y}$ to a common denominator.

Ans. $\frac{ady}{bdy}$, $\frac{bcy}{bdy}$, and $\frac{bdx}{bdy}$.

2. Reduce $\frac{4}{7ab}$, $\frac{3x}{14a^2}$, and $\frac{11y}{21b^3}$ to the least common denominator.

Ans. $\frac{24ab^2}{42a^2b^3}$, $\frac{9b^3x}{42a^2b^3}$, and $\frac{22a^2y}{42a^2b^3}$.

3. Reduce $\frac{8a^2}{7m^3}$, and $\frac{13xy}{21ab}$ to the least common denominator.

4. Reduce $\frac{5ab}{6m^2x}$, $\frac{4xy}{9m^3}$, and $\frac{8bc}{21mx^2}$ to the least common denominator.

5. Add $\frac{x}{2y}$ and $\frac{am}{bc}$.

6. Add $\frac{abc}{3m^2}$ and $\frac{2a}{9mx^2}$.

7. Add $\frac{11}{75x^2}$ and $\frac{13a^2}{25x^3y^2}$.

8. Add $\frac{bc}{4ax}$ and $\frac{5xy}{8a^2x^3}$.

9. Add $\frac{a-b}{27}$ and $\frac{a+b}{81}$.

10. Add $\frac{1}{1+a}$ and $\frac{1}{1-a}$.

11. Add $\frac{7}{x^2-y^2}$ and $\frac{3}{x-y}$.

12. From $\frac{2ax}{3}$ subtract $\frac{3my}{8}$.

13. From $\frac{4xy}{7a}$ subtract $\frac{9by}{10a^2}$.

14. From $\frac{1}{1-x}$ subtract $\frac{1}{1+x}$.

15. From $\frac{4x}{7a}$ subtract $\frac{3a-x}{21a^2}$.

16. From $\frac{m^2}{a^2-b^2}$ subtract $\frac{xy}{a+b}$.

17. From $\frac{3x^2}{4m^3n}$ subtract $\frac{7bc}{m^2+mn}$.

18. Reduce $2\frac{3}{4}$, that is, $2+\frac{3}{4}$, to the form of a fraction

Since 4 fourths make 1, $2=\frac{8}{4}$; and $\frac{8}{4}+\frac{3}{4}=\frac{11}{4}$, Ans

In like manner, $7-\frac{3}{8}=\frac{56}{8}-\frac{3}{8}=\frac{53}{8}$.

19. Add b and $\frac{c}{d}$.

Reducing b to a fraction having d for a denominator we have $b=\frac{bd}{d}$; and $\frac{bd}{d}+\frac{c}{d}=\frac{bd+c}{d}$, Ans.

20. Reduce $3a+\frac{4c}{7d}$ to a fraction.

21. Reduce $7m^2-\frac{3x}{4y}$ to a fraction.

22. Reduce $3x-\frac{3x-4y}{7}$ to a fraction. Ans $\frac{18x+4y}{7}$

23. Add $a+b$ and $\frac{a-b}{4}$.

24. From $\frac{a^2+b^2}{a-b}$ subtract $x-a$.

25. Reduce $\frac{9x+4y}{7b}+9xy$ to a fraction.

26. Reduce $12a^2-\frac{4x-17a^2x}{3a}$ to a fraction.

27. From $\frac{x^2+y^2}{x-y}$ subtract $m-1$.

28. Reduce $\frac{7ab+3c^2}{4c}+2c$ to a fraction.

SECTION XXI.

DIVISION OF INTEGRAL AND FRACTIONAL QUANTITIES BY FRACTIONS.

Art. 70. How many times is $\frac{5}{7}$ contained in 9?

Since in 9 there are $\frac{63}{7}$, $\frac{1}{7}$ is contained 63 times in 9, and $\frac{5}{7}$ is contained $\frac{1}{5}$ as many times, that is, $\frac{63}{5}$; in other words, 9 divided by $\frac{5}{7}$ gives $\frac{63}{5}$ for a quotient. The result is the same as the product of 9 by $\frac{7}{5}$.

How many times is $\frac{4}{5}$ contained in a?

Since in a number a of units there are $5a$ fifths, $\frac{1}{5}$ is contained $5a$ times in a, and $\frac{4}{5}$ is contained $\frac{1}{4}$ as many times, that is, $\frac{5\,a}{4}$ times; in other words, a divided by $\frac{4}{5}$ gives $\frac{5\,a}{4}$ for a quotient. This result is the same as the product of a by $\frac{5}{4}$.

How many times is $\frac{m}{n}$ contained in a?

Since in a number a of units there are $\frac{n\,a}{n}$, $\frac{1}{n}$ is contained $n\,a$ times in a, and $\frac{m}{n}$ is contained $\frac{1}{m}$ as many times, that is, $\frac{n\,a}{m}$ times; or a divided by $\frac{m}{n}$ gives $\frac{n\,a}{m}$ for a quotient. This is the same as $a \cdot \frac{n}{m}$.

How many times is $\frac{4}{7}$ contained in $\frac{9}{11}$?

Reducing the fractions to a common denominator, we have $\frac{4}{7} = \frac{44}{77}$, and $\frac{9}{11} = \frac{63}{77}$. But $\frac{44}{77}$ is contained in $\frac{63}{77}$ as many times as 44 is contained in 63, which is $\frac{63}{44}$; or $\frac{9}{11}$ divided by $\frac{4}{7}$ gives $\frac{63}{44}$ for a quotient. This result is the same as $\frac{9}{11} \cdot \frac{7}{4}$.

How many times is $\frac{a}{b}$ contained in $\frac{m}{n}$?

Reducing the fractions to a common denominator, we have $\frac{a}{b} = \frac{a\,n}{b\,n}$, and $\frac{m}{n} = \frac{b\,m}{b\,n}$. But $\frac{a\,n}{b\,n}$ is contained in $\frac{b\,m}{b\,n}$ as many times as $a\,n$ is contained in $b\,m$, that is, $\frac{b\,m}{a\,n}$ times; or $\frac{m}{n}$ divided by $\frac{a}{b}$ gives $\frac{b\,m}{a\,n}$ for a quotient. The result is the same as $\frac{m}{n} \cdot \frac{b}{a}$.

From the solution of the preceding questions we derive the following

RULE FOR DIVIDING ANY QUANTITY BY A FRACTION.

Invert the divisor, and then proceed as in multiplication

Remark. When we wish to find what part one quantity is of another, we make the quantity called the part the dividend, and the other the divisor; also, when we wish to find the ratio of one quantity to another, we make the quantity mentioned first the dividend, and the other the diviso-

Perform the following questions, recollecting to simplify, as directed in Article 62. Thus, dividing $\frac{4m}{7n}$ by $\frac{6am}{5bc}$, we have $\frac{4.5bcm}{6.7amn}=\frac{2.5bc}{3.7an}=\frac{10bc}{21an}$.

1. Divide a by $\frac{4}{7}$.

2. Divide my by $\frac{2a}{b}$.

3. Divide $3m^2y^2$ by $\frac{4my}{3x}$.

4. Divide $5a^2-4b^2$ by $\frac{3mn}{7x^2}$.

5. Divide $3(a+b)$ by $\frac{6a^2}{5xy}$.

6. Divide $4abc^2$ by $\frac{8a^2}{9mx+3y}$.

7. Divide $11a^2-22x^2$ by $\frac{33x}{4y}$.

8. Divide $\frac{x}{y}$ by $\frac{3}{7}$.

9. Divide $\frac{3x}{4y^2}$ by $\frac{6x^2}{5ab}$.

10 Divide $\frac{a+b}{xy}$ by $\frac{3x}{y}$.

11. Divide $\frac{4x^2y^2}{3m}$ by $\frac{2xy}{9m^2}$.

12. Divide $\frac{5a^2x^2}{9m^3y^3}$ by $\frac{10ax}{27my^2}$.

13. m is what part of $\frac{a}{b}$?

14. $\frac{m}{m+n}$ is what pa.t of $\frac{x}{y}$?

15. $\frac{x+y}{8ab}$ is what part of $\frac{3m}{4n}$?

16 What is the ratio of $\frac{5(a+b)}{6my}$ to $\frac{10(a+b)}{3m^2y^3}$?

17. What is the ratio of $\frac{3(x^2-y^2)}{m+n}$ to $\frac{6(x-y)}{7x^3}$?

18 What is the ratio of $\frac{11(a^3-b^3)}{7(x+y)}$ to $\frac{22(a-b)}{21(x^2-y^2)}$?

SECTION XXII.

EQUATIONS OF THE FIRST DEGREE CONTAINING TWO UNKNOWN QUANTITIES.

ART. **71.** When a question involves several independent unknown quantities, in order that we may be able to determine them, there must be given as many conditions, and, consequently, we must be able to form as many independent equations, as there are unknown quantities.

1. A man gave 27 s. for 3 bushels of corn and 4 bushels of oats; and, at the same rate, he gave 50 s. for 7 bushels of corn and 5 bushels of oats. What was the price of each per bushel?

Let x (shill.) = the price of corn, and y (shill.) = the orice of oats, per bushel. Then,

(1) $3x+4y=27$; and } Multiply the 1st by 7, and the
(2) $7x+5y=50$. } 2d by 3,

(3) $21x+28y=189$; }
(4) $21x+15y=150$. } Subtract the 4th from the 3d,

$$21x+28y-21x-15y=189-150;\text{ reduce,}$$

$$13y=39 \therefore y=3\text{ s.}$$

Substitute 3 for y in the 1st,

$$3x+12=27;\therefore 3x=27-12, \therefore 3x=15 \therefore x=5\text{ s.}$$

Ans. Corn, 5 s., rye, 3 s. per bushel.

We might have multiplied the 1st equation by 5, and the 2d by 4, and then have subtracted one of the resulting equations from the other. We should thus have obtained an equation without y, and from this we could have found the value of x. This value of x, substituted in one of the first two equations, would have given the value of y.

2. The sum of two numbers is 20, and if twice the less be subtracted from 3 times the greater, the remainder will be 25. Required the numbers.

Let $x=$ the greater, and $y=$ the less. Then,

(1) $x+y=20$; }
(2) $3x-2y=25$. } Multiply the 1st by 3,

(3) $3x+3y=60$; subtract the 2d from the 3d,

$$3x+3y-3x+2y=60-25;\text{ reduce,}$$

$$5y=35 \therefore y=7.$$

Substitute 7 for y in the 1st.

$$x+7=20, \therefore x=13.$$

Ans. 13 the greater, and 7 the less.

This question may be performed in another way, as follows.

The first two equations are

(1) $x+y=20$; }
(2) $3x-2y=25$. } Multiply the 1st by 2,

(3) $2x + 2y = 40$; add the 2d and 3d,

$$3x - 2y + 2x + 2y = 25 + 40; \text{ reduce,}$$

$$5x = 65, \therefore x = 13.$$

Substituting 13 for x in the 1st,

$$13 + y = 20, \therefore y = 7.$$

3. A farmer sold 6 barrels of apples and 8 barrels of pears for \$22; also, at the same rate, 9 barrels of apples and 24 of pears for \$57. Required the price of each per barrel.

Let x (dollars) = the price of apples, and y (dollars) = the price of pears per barrel. Then,

(1) $6x + 8y = 22$;
(2) $9x + 24y = 57$. Multiply the 1st by 3,

(3) $18x + 24y = 66$; subtract the 2d from the 3d,

$9x = 9, \therefore x = \$1$, price of apples per barrel.

Substitute 1 for x in the 1st,

$6 + 8y = 22, \therefore y = \$2$, price of pears per barrel.

Since 18 is the least common multiple of 6 and 9, if we had multiplied the 1st by 3, and the 2d by 2, we should have obtained two equations in which the coefficients of x would have been alike. Then, after subtraction, we should have found the value of y, which, by substitution, would have given that of x.

But we will solve this problem in another way, which is sometimes a convenient one. Resuming the first two equations,

(1) $6x + 8y = 22$;
(2) $9x + 24y = 57$. Divide the 1st by 2, and the 2d by 3,

(3) $3x + 4y = 11$;
(4) $3x + 8y = 19$. Subtract the 3d from the 4th,

$4y = 8 \therefore y = \$2$, price of pears per barrel.

Substituting 2 for y in the 3d,

$3x + 8 = 11 \therefore x = \$1$, price of apples per barrel

4. Two numbers are such, that if $\frac{1}{2}$ of the first be increased by 10, the sum will be equal to $\frac{2}{3}$ of the second; and, if $\frac{4}{5}$ of the second be diminished by 10, the remainder will be equal to $\frac{7}{10}$ of the first. Required the numbers.

Let $x=$ the 1st, and $y=$ the 2d. Then,

$$\left.\begin{array}{ll}(1) & \frac{x}{2}+10=\frac{2y}{3}; \\ (2) & \frac{4y}{5}-10=\frac{7x}{10}.\end{array}\right\} \text{Clearing the equations of fractions,}$$

$$\left.\begin{array}{ll}(3) & 3x+60=4y; \\ (4) & 8y-100=7x.\end{array}\right\}$$

In the 3d and 4th, transpose the unknown terms into the 1st members, and the known into the 2d,

$$\left.\begin{array}{ll}(5) & 3x-4y=-60; \\ (6) & 8y-7x=100.\end{array}\right\} \text{Multiply the 5th by 2,}$$

(7) $6x-8y=-120.$ Add the 6th and 7th,

$$8y-7x+6x-8y=100-120; \text{ reduce,}$$

$-x=-20$ ∴ changing the signs, $x=20$, the 1st number.

Substitute 20 for x in the 5th,

$60-4y=-60$, or $4y-60=60$ ∴ $y=30$, the 2d number.

We perceive that, in the preceding problems, the conditions of each gave rise to two independent equations, which may be called the *original equations.* The succeeding equations were deduced from, or were mere modifications of, the original equations.

From the two original equations containing two unknown quantities, one was obtained containing only one unknown quantity. The process by which this is done is called *elimination.* It is eliminating that unknown quantity which the new equation does not contain. For example, in the 1st question of this Article, we eliminated x.

and obtained an equation containing no unknown quantity except y.

We perceive, from what precedes, that, when the unknown quantity to be eliminated is found in corresponding members in both equations, that is, in the first members of both, or in the second members of both, the elimination can be effected according to the following rule.

FIRST METHOD OF ELIMINATION.

RULE.

Multiply or divide the equations, if necessary, so as to make the coefficients of the quantity to be eliminated alike in the two equations; then subtract one of the resulting equations from the other, if the signs of the terms containing this quantity are alike in both equations; but add them together, if the signs are different.

Previously to applying this rule, it is advisable to free the equations of fractions, if they contain any; and to transpose all the unknown terms into the first members, and the known terms into the second. Moreover, if, in either equation, there are several terms containing the unknown quantity to be eliminated, these terms should all be reduced to one.

The coefficients of any letter in the two equations will be made alike, if, after the equations are prepared as prescribed above, each equation be multiplied by the coefficient of that letter in the other; or, if each equation be multiplied by the number by which the coefficient of that letter in this equation must be multiplied, in order to produce the *least common multiple* of the two coefficients of the letter to be eliminated.

Thus, in the 3d example, the least common multiple of 6 and 9, the coefficients of x in the 1st and 2d equations, is 18, which may be produced by multiplying 6 by 3, or 9 by 2. If, then, the 1st equation be multiplied by 3, and the 2d by 2, the coefficients of x will be alike in the two resulting equations.

5. Four bushels of wheat and 3 bushels of corn cost \$11; and, at the same rate, 5 bushels of wheat and 6 of corn cost \$16. Required the price of a bushel of each.

6. Twice A's money and 5 times B's make \$165; also 6 times A's money and 7 times B's make \$295. How much money has each?

7. A draper bought 2 pieces of cloth for £2 15 s., one at 4 s. per yard, and the other at 5 s. He sold the whole at an advance of 2 s. a yard, and thereby gained £1 4 s. How many yards were there in each piece?

8. Four cows and 1 sheep cost \$82; also 20 sheep and 1 cow cost \$60. Required the cost of a cow and that of a sheep.

9. A farmer gave 10 bushels of corn and \$2·50 for 7 bushels of wheat; he also gave 15 bushels of corn for 5 bushels of wheat and \$4·50. What was the estimated worth of a bushel of each?

10. The sum of $\frac{1}{3}$ of A's age and $\frac{1}{4}$ of B's is 40 years; and if $\frac{5}{12}$ of A's age be subtracted from B's, the remainder will be 55 years. Required the age of each.

11. Two drovers, A and B, counting their sheep, found that, if A had 10 more and B 10 less, their flocks would be equal; but if B had 10 more and A 10 less, A would have only $\frac{3}{7}$ as many as B. How many sheep had each?

12. A man wrought 6 days, having his son with him 4 days, and received for their joint labor \$8; he afterwards wrought 8 days, having his son with him 6 days, and received \$11. Required the daily wages of each.

ART. **72.** 1 The sum of two numbers is 13, and their difference is 3. Required these numbers

Let $x =$ the greater, and $y =$ the less. Then,

(1) $x + y = 13$, }
(2) $x - y = 3$. } Transpose y in both 1st and 2d,

(3) $x = 13 - y$;

(4) $x = 3 + y$.

Since the 2d members of the 3d and 4th are each equal to x, they are equal to each other (Ax. 7); $\therefore$

$3 + y = 13 - y$; $\therefore 2y = 10$, and $y = 5$, the less number.

Substitute 5 for y in the 4th,

$x = 3 + 5 = 8$, the greater number.

2. If 3 yards of linen and 4 yards of cotton cost \$2·85, and, at the same rate, 5 yards of linen and 7 yards of cotton cost \$4·80, what is the price of each per yard?

Let x (cents) $=$ the price of linen, and y (cents) $=$ that of cotton, per yard. Then,

(1) $3x + 4y = 285$; } Transpose $4y$ in 1st, and divide
(2) $5x + 7y = 480$. } by 3,

(3) $x = \frac{285 - 4y}{3}$. Transpose $7y$ in 2d, and divide by 5.

(4) $x = \frac{480 - 7y}{5}$.

Put the values of x in 3d and 4th equal to each other,

$\frac{285 - 4y}{3} = \frac{480 - 7y}{5}$. Multiply by 3 and 5, or 15,

$1425 - 20y = 1440 - 21y$. Transpose and reduce,

$y = 15$ cents, the price of a yard of cotton.

Substitute 15 for y in the 3d,

$x = \frac{285 - 60}{3} = 75$ cents, price of a yard of linen.

3. What fraction is that, from the numerator of which if 1 be subtracted, the value of the fraction will be $\frac{3}{5}$; but if 4 be added to the denominator, the value of the fraction will be $\frac{1}{2}$?

Let $x =$ the numerator, and $y =$ the denominator. Then,

(1) $\frac{x-1}{y} = \frac{3}{5}$;

(2) $\frac{x}{y+4} = \frac{1}{2}$. Multiply the 1st by y,

(3) $x - 1 = \frac{3y}{5}$; transpose -1,

(4) $x = \frac{3y}{5} + 1$. Multiply the 2d by $y + 4$,

(5) $x = \frac{y+4}{2}$. Put the values of x in 4th and 5th equal,

$\frac{3y}{5} + 1 = \frac{y+4}{2}$. Multiply by 10,

$6y + 10 = 5y + 20$. Transpose and reduce,

$y = 10$, the denominator.

Substitute 10 for y in the 5th

$x = \frac{10+4}{2} = 7$, the numerator.

The fraction sought, therefore, is $\frac{7}{10}$.

The solution of the three preceding questions has been effected as follows, viz.; we first found the value of x from each of the original equations, as if y were known; that is, we found from each equation an expression for x, consisting of y, and known numbers; then, by equalizing these two values of x, we obtained an equation without x, from which we determined the value of y. We might have eliminated y in a similar manner, and found an equation without that letter. Hence we have a

SECOND METHOD OF ELIMINATION.

RULE.

Find the value of one of the unknown quantities, from each of the equations, as if the other unknown quantity were determined; then form a new equation by putting these two values equal to each other.

Observe, however, that the unknown quantity itself must not be contained in any expression for its value.

Let the learner solve the following problems according to this second method.

4. A grocer paid $18 for 4 barrels of beer and 3 barrels of cider; and, at the same rate, he paid $27 for 5 barrels of beer and 6 barrels of cider. Required the price of each per barrel.

5. A carpenter received for 5 days' labor of himself, and 3 days' labor of his journeyman, $14·50; but he himself earned $8 more in 7 days than his journeyman did in 4 days. Required the daily wages of each.

6. Says A to B, "$\frac{3}{8}$ of my money, and $\frac{4}{5}$ of yours, make $55; and $\frac{1}{4}$ of my money, increased by $10, is equal to $\frac{1}{2}$ of yours diminished by $5." How much money has each?

7. If A gives B $10 of his money, they will hav equal sums; but if B gives A $10 of his money, he will then have only $\frac{3}{7}$ as much as A. Required the money of each.

8. A market-man bought eggs at 2 for a cent, also some at 8 for 5 cents, giving for the whole 50 cents, and sold them all at a cent apiece, gaining on the whole 40 cents. How many of each kind did he buy?

9. There is a fraction such that, if its numerator be increased by 1, the value of the fraction will be $\frac{1}{2}$; but if the denominator be increased by 3, the value of the fraction will be $\frac{1}{3}$. Required the fraction.

10. I can buy 4 pounds of beef, and 6 pounds of mutton, for 76 cents, and I find that 8 pounds of beef are worth 8 cents more than 12 pounds of mutton. Required the price of each per pound.

11. A grocer mixes tea at 3 s. with tea at 5 s. per lb., and finds the whole mixture worth £3 11 s.; but 3 times the number of lbs. of the first kind is 1 lb. more than

twice the number of lbs. of the second kind. How many lbs. of each did the mixture contain?

Art. 73. 1. Two pairs of boots, and 1 pair of shoes, cost \$12; and 1 pair of boots, and two pairs of shoes, cost \$9. Required the price of each per pair.

Let x (dolls.) = the price of a pair of boots, and y (dolls.) = that of a pair of shoes. Then,

(1) $2x+y=12$;
(2) $x+2y=9$. } Transpose y in 1st, and divide by 2

(3) $x=\frac{12-y}{2}$.

Now, since $\frac{12-y}{2}$ is the value of x, this value may be substituted instead of x in the 2d equation.

We then have

$$\frac{12-y}{2}+2y=9. \therefore 12-y+4y=18 \therefore$$

$3y=6$, and $y=\$2$, price of a pair of shoes.

Substitute 2 for y in 3d,

$x=\frac{12-2}{2}=\$5$, price of a pair of boots.

2. There are 2 numbers, such that 3 times the less, and twice the greater, make 120; and if 5 times the less be subtracted from 4 times the greater, the remainder will be 20. Required the numbers.

Let x = the greater, and y = the less. Then,

(1) $2x+3y=120$;
(2) $4x-5y=20$. } From the 1st we have

(3) $y=\frac{120-2x}{3}$. Multiply this value of y by 5, and substitute the result $\frac{600-10x}{3}$, instead of $5y$ in the 2d. But since $5y$ in the 2d is subtracted, this value of $5y$, when it is substituted, must also be subtracted. Making this substitution, we have

$4x \quad \frac{600-10x}{3}=20.$ Clearing this of fractions, transposing, reducing, and dividing,

$x=30$, the greater.

Substituting 30 for x in the 3d,

$$y=\frac{120-60}{3}=20, \text{ the less.}$$

From the solution of the two preceding questions, we derive a

THIRD METHOD OF ELIMINATION.

RULE.

Find, from one of the equations, the value of the quantity to be eliminated, as if the other unknown quantity were determined, and substitute this value in the other equation, instead of the unknown quantity itself.

Let the following questions be performed according to the third method of elimination.

3. A boy bought 3 pears, and 2 peaches, for 18 cents; his companion bought, at the same prices, 7 pears, and 5 peaches, for 44 cents. Required the price of a peach, also that of a pear.

4. If 7 lbs. of butter and 6 lbs. of sugar cost $1·53, and 10 lbs. of butter and 12 lbs. of sugar cost $2·46, what is the price of each per lb.?

5. A banker, having two drawers containing money, found that if he transferred $100 from the first drawer to the second, the former would contain $\frac{5}{6}$ as much money as the latter; but if he transferred only $50 from the first to the second, they would then contain equal sums. How much money was there in each?

6. Two boys, talking of their ages, the elder says to the younger, "3 years ago, my age was to yours as 4 to 3; but 3 years hence, if we live, my age will be to yours as 6 to 5." Required their ages at the time.

7. A laborer agreed to reap 6 acres of wheat and 5 acres of rye for $28; but after he had reaped 4 acres of the wheat, and 3 acres of the rye, he was taken sick, and received for the work he had done $18. Required the price of reaping an acre of each.

8. A jockey says, "$\frac{1}{2}$ of the worth of my horse, and $\frac{2}{3}$ of the worth of my saddle, make $60; also, $\frac{3}{10}$ of the worth of my horse, and $\frac{4}{5}$ of the worth of my saddle, make $42." Required the estimated value of each

9. A fishing-rod consists of two parts, such that the lower part, added to $\frac{1}{4}$ of the upper part, makes 20 feet; moreover, 5 times the lower part, added to 3 times the upper part, exceeds twice the whole length of the rod by 65 feet. Required the length of each part.

10. In a certain school, $\frac{7}{8}$ of the number of boys exceeds $\frac{3}{4}$ of the number of girls by 19; and $\frac{1}{2}$ of the number of girls, together with $\frac{3}{4}$ of the number of boys, makes 62. Required the number of each.

11. After A had gained $100, and B had lost $50, they had equal sums of money; but if A had lost $50, and B had gained $50, B would have had twice as much money as A. How much money had each?

Art. **74.** The following problems are intended to exercise the learner in the three different modes of elimination. Sometimes one mode, sometimes another, will be most convenient. It is advisable, however, that the pupil perform each question of this article in the three different ways, in order to acquire skill in eliminating.

The sum of two numbers is to their difference as 4 is to 1; moreover, the sum of twice the greater, and 3 times the less, is 190. Required the numbers.

2. The sum of two numbers is 12, and their difference is 2. Required these numbers.

3. A grocer, having two casks of wine, drew 9 gallons from the greater and 6 gallons from the less, and found the number of gallons remaining in the greater to the number remaining in the less as 9 is to 5. He then puts 6 gallons of water into the greater, and 5 into the less, and finds the number of gallons of liquor in the greater to the number in the less as 12 to 7. How many gallons of wine were there at first in each?

4. A's money and $\frac{1}{6}$ of B's make $30; and B's money, with $\frac{1}{5}$ of A's, makes $35. How much has each?

5. Three men and 4 boys earn in a day £1 6 s.; also, 5 men and 7 boys earn in a day £2 4 s. Required the daily wages of a man and a boy respectively.

6. There are two numbers, such that, if $\frac{3}{7}$ of the greater be subtracted from the less, the remainder will be 16; but if $\frac{7}{8}$ of the less be added to the greater, the sum will be 91. Required the numbers.

7. I hired a horse for a journey of 20 miles, and a chaise for a journey of 15 miles, for $2·65; but, changing my first intention, I rode in the chaise only 10 miles, and went a distance of 25 miles with the horse, and paid for both $2·70. Required the price of the horse and chaise, respectively, per mile.

8. A farmer found that 3 horses and 4 cows would, during the winter, consume 12 tons of hay; and that it required $20\frac{1}{2}$ tons to keep 5 horses and 7 cows the same time. Required the quantity of hay eaten by a cow and a horse, respectively, during the winter.

9. The sum of the distances passed over by two locomotives, the first running 6 hours, and the second 7 hours, is 290 miles; but the first goes 35 miles more in 3 hours than the second does in 2 hours. Required the distance each goes per hour.

10. After A had lent B $5, he had $\frac{1}{2}$ as much money

as B; but if B had lent A $5, each would have had the same sum. How much money had each?

11. Ten boxes of raisins and 3 barrels of flour weigh 850 lbs.; also, 12 boxes of raisins and 7 barrels of flour weigh 1700 lbs. Required the weight of a box of raisins, also that of a barrel of flour.

12. What fraction is that, to the numerator of which if 1 be added, the value of the fraction will be $\frac{2}{3}$, but if 3 be subtracted from the denominator, the value of the fraction will be $\frac{5}{6}$?

SECTION XXIII.

EQUATIONS OF THE FIRST DEGREE CONTAINING THREE UNKNOWN QUANTITIES.

ART. **75.** 1. A boy bought an apple, an orange, and a peach, for 6 cents; 2 apples, 3 oranges, and 4 peaches, for 19 cents; 5 apples, 2 oranges, and 7 peaches, for 25 cents. What did he pay for one of each?

Let $x =$ the price of an apple, $y =$ that of an orange, and $z =$ that of a peach, in cents. Then,

$$\left.\begin{array}{ll}(1) & x + y + z = 6; \\ (2) & 2x + 3y + 4z = 19; \\ (3) & 5x + 2y + 7z = 25.\end{array}\right\}$$

In this problem, we have three independent equations, containing three unknown quantities. Our first step, in the solution, is to obtain from these original equations two others, which shall contain only two unknown quantities. Let us eliminate x, that is, find two equations which shall not contain x.

First method. Bring down the 2d, and multiply the 1st by 2, so as to make the coefficient of x the same as it is in 2d,

(2) $2x+3y+4z+19$;

(4) $2x+2y+2z=12$. Subtract 4th from 2d,

(5) $y+2z=7$ { Multiply 1st by 5, and bring down 3d,

(6) $5x+5y+5z=30$;

(3) $5x+2y+7z=25$. Subtract 3d from 6th,

(7) $3y-2z=5$. } Bring down 5th,

(5) $y+2z=7$. }

The 5th and 7th contain only the two unknown quantities y and z. These equations may, therefore, be solved as we solved similar equations in the preceding section. Since the coefficients of z are alike in the 5th and 7th, by adding these equations, we have

$4y=12$; $\therefore y=3$ cents, price of an orange.

Substitute 3 for y in the 5th, which contains only y and z,

$3+2z=7$, $\therefore z=2$ cents, price of a peach.

Substitute 3 for y, and 2 for z, in the 1st,

$x+3+2=6$, $\therefore x=1$ cent, price of an apple.

Second method. Resume the original equations,

$$\left.\begin{array}{ll}(1) & x+y+z=6;\\ (2) & 2x+3y+4z=19;\\ (3) & 5x+2y+7z=25.\end{array}\right\}$$

Find the value of x from each of the equations, as if y and z were known.

$$\left.\begin{array}{ll}(4) & x=6-y-z;\\ (5) & x=\dfrac{19-3y-4z}{2};\\ (6) & x=\dfrac{25-2y-7z}{5}.\end{array}\right\}$$

Put equal to each other the values of x in the 4th and 5th; also the values of x in the 4th and 6th,

$$\left.\begin{array}{ll}(7) & 6-y-z=\frac{19-3y-4z}{2};\\ (8) & 6-y-z=\frac{25-2y-7z}{5}.\end{array}\right\}$$

Clear the 7th and 8th of fractions; then transpose the terms containing y into the first members, all the other terms into the second, and reduce,

$$\left.\begin{array}{lrl}(9) & y= & 7-2z;\\ (10) & -3y= & -5-2z.\end{array}\right\}$$ Change the signs in 10th, and find the value of y,

$$(11) \qquad y=\frac{5+2z}{3}.$$ Equalize the values of y in 9th and 11th,

$$\frac{5+2z}{3}=7-2z, \therefore 5+2z=21-6z, \therefore$$

$z=2$ cents, price of a peach.

Substitute 2 for z in 9th,

$y=7-4=3$ cents, price of an orange.

Substitute 3 for y, and 2 for z, in 4th,

$x=6-3-2=1$ cent, price of an apple.

Third method. Take the original equations,

$$\left.\begin{array}{ll}(1) & x+y+z=6;\\ (2) & 2x+3y+4z=19;\\ (3) & 5x+2y+7z=25.\end{array}\right\}$$ Find the value of x from the 1st, as if y and z were known.

(4) $x=6-y-z$. Substitute this value in 2d and 3d,

$$\left.\begin{array}{ll}(5) & 12-2y-2z+3y+4z=19;\\ (6) & 30-5y-5z+2y+7z=25.\end{array}\right\}$$ Transpose the known terms, and reduce in the 5th and 6th,

$$\left.\begin{array}{lrl}(7) & y+2z= & 7;\\ (8) & -3y+2z= & -5.\end{array}\right\}$$ Find the value of z in 7th,

(9) $z=\frac{7-y}{2}$. Substitute this value in 8th,

$$-3y+7-y=-5, \therefore -4y=-12, \therefore 4y=12, \text{ and}$$

$y=3$ cents, price of an orange.

Substitute 3 for y in 9th,

$$z = \frac{7-3}{2} = 2 \text{ cents, price of a peach.}$$

Substitute 3 for y, and 2 for z, in 4th,

$$x = 6 - 3 - 2 = 1 \text{ cent, price of an apple.}$$

2. A farmer found that the number of his sheep exceeded by 20 that of his cows and horses together; that his horses and $\frac{1}{4}$ of his cows equalled in number $\frac{1}{5}$ of his sheep; and that $\frac{1}{8}$ of his cows, $\frac{1}{2}$ of his horses, and $\frac{1}{5}$ of his sheep, made 12. Required the number of each.

Suppose he had x sheep, y cows, and z horses. Then

$$\left.\begin{array}{ll} (1) & x = y + z + 20; \\ (2) & z + \frac{y}{4} = \frac{x}{5}; \\ (3) & \frac{x}{5} + \frac{y}{8} + \frac{z}{2} = 12. \end{array}\right\}$$

First method Remove the denominators, and transpose the unknown terms into the first members.

$$\left.\begin{array}{lr} (4) & x - y - z = 20; \\ (5) & -4x + 5y + 20z = 0; \\ (6) & 8x + 5y + 20z = 480. \end{array}\right\}$$

Let us eliminate y. Multiply 4th by 5,

(7) $5x - 5y - 5z = 100$. Add 5th and 7th, also 6th and 7th,

$$\left.\begin{array}{ll} (8) & x + 15z = 100; \\ (9) & 13x + 15z = 580. \end{array}\right\}$$

Since, in the 8th and 9th, the coefficients of z are alike, by subtracting the 8th from the 9th, we have

$$12x = 480, \therefore x = 40 \text{ sheep.}$$

Substitute 40 for x in 8th,

$$40 + 15z = 100, \therefore z = 4 \text{ horses.}$$

Substitute 40 for x, and 4 for z, in 4th,

$$40 - y - 4 = 20, \therefore y = 16 \text{ cows.}$$

Second method. Resume the original equations, inverting the 2d,

$$\left.\begin{aligned}&(1)\ x = y + z + 20;\\&(2)\ \frac{x}{5} = z + \frac{y}{4};\\&(3)\ \frac{x}{5} + \frac{y}{8} + \frac{z}{2} = 12.\end{aligned}\right\}$$

Let us eliminate x. Bring down the 1st,

(1) $x = y + z + 20$. From 2d we have

(4) $x = \frac{20z + 5y}{4}$; from 3d

(5) $x = \frac{480 - 5y - 20z}{8}$.

Equalizing the values of x in 1st and 4th, also the values in 1st and 5th,

$$\left.\begin{aligned}&(6)\ \frac{20z + 5y}{4} = y + z + 20;\\&(7)\ y + z + 20 = \frac{480 - 5y - 20z}{8}.\end{aligned}\right\}$$

The 6th and 7th do not contain x. From these eliminate y. Multiply, transpose, and deduce the value of y from each. The 6th gives

(8) $y = 80 - 16z$. From 7th,

(9) $y = \frac{320 - 28z}{13}$. Equalizing the values of y in 8th and 9th,

$$\frac{320 - 28z}{13} = 80 - 16z,$$ from which we have

$$z = 4 \text{ horses.}$$

Substitute 4 for z in 8th

$$y = 80 - 64 = 16 \text{ cows.}$$

Substitute 16 for y, and 4 for z, in 1st,

$$x = 16 + 4 + 20 = 40 \text{ sheep.}$$

Third method. Bring down 1st, and clear 2d and 3d from fractions,

$$\left.\begin{aligned}&(1)\ x = y + z + 20;\\&(4)\ 4x = 20z + 5y;\\&(5)\ 8x + 5y + 20z = 480.\end{aligned}\right\}$$

Let us eliminate x. Substitute, in 4th and 5th, the value of x given in 1st,

(6) $4y+4z+80=20z+5y$;
(7) $8y+8z+160+5y+20z=480$. Transpose and reduce in 6th and 7th,

(8) $y+16z=80$;
(9) $13y+28z=320$.

Now eliminate y from 8th and 9th. The 8th gives
(10) $y=80-16z$. Substitute this value of y in 9th,

$$1040-208z+28z=320, \therefore -180z=-720,$$
$$\therefore 180z=720, \therefore$$
$$z=4 \text{ horses.}$$

Substitute 4 for z in 10th,

$$y=80-64=16 \text{ cows.}$$

Substitute 16 for y, and 4 for z, in 1st,

$$x=16+4+20=40 \text{ sheep.}$$

3. The ages of 3 children are as follows. Four times A's age and 3 times B's make 27 years; 3 times A's added to C's make 15 years; and 4 times B's with twice C's make 32 years. Required the age of each.

Let x, y, and z (years), represent the respective ages of A, B, and C. Then,

(1) $4x+3y=27$;
(2) $3x+z=15$;
(3) $4y+2z=32$.

In this example, each equation does not contain all the unknown quantities; nor is that necessary in order that the solution may be possible. Let us eliminate z by the first method. But since the 1st does not contain z, we have to eliminate that letter from the 2d and 3d only Divide 3d by 2,

(4) $2y+z=16$. Subtract 2d from 4th,
(5) $2y-3x=1$. Bring down 1st,
(1) $4x+3y=27$.

Let us now eliminate y from 5th and 1st. Multiply 5th by 3, and 1st by 2,

$$\left.\begin{array}{ll}(6) & 6y-9x=\ \ 3; \\ (7) & 8x+6y=54.\end{array}\right\}$$ Subtract 6th from 7th,

$$17x=51, \therefore x=3 \text{ years, A's age.}$$

Substitute 3 for x in 1st,

$$12+3y=27, \therefore y=5 \text{ years, B's age.}$$

Substitute 3 for x in 2d,

$$9+z=15, \therefore z=6 \text{ years, B's age.}$$

Let the learner solve this question according to the second and third methods

From what precedes, it is evident that the three modes of elimination, given in the last section, may be extended to any number of equations, provided the number of unknown quantities does not exceed the number of equations.

The *first method* is applied to *several* equations by operating upon these equations taken two and two.

In applying the *second method* to several equations, find, from each equation that contains it, the value of the unknown quantity to be eliminated, then put any two of these expressions for its value equal to each other.

To extend the *third method*, we must, after having found, from one of the equations, the value of the unknown quantity to be eliminated, substitute this value in every other equation that contains this unknown quantity.

If a question involves four unknown quantities, and gives rise to four independent equations, we first deduce from them three equations with only three unknown quantities; we then proceed with these three equations, as we have done with the preceding equations in this section.

If either of the equations does not contain the unknown quantity to be eliminated, that equation may be put aside to be placed in the next set of equations, viz., those which contain one less unknown quantity.

Either method of elimination may be used, but the first will generally be found the most convenient, because it does not give rise to fractions. The pupil is advised, however, to perform each problem in the three ways, in order to acquire skill and be able to judge which will be best in any particular case. It is not necessary that the same mode of elimination be pursued throughout the solution of a question, but either may be resorted to whenever it shall seem the most convenient.

Let the learner find the values of x, y, and z, in the following sets of equations.

4. $\left.\begin{aligned} x+y&=13; \\ x+z&=14; \\ y+z&=15. \end{aligned}\right\}$

5. $\left.\begin{aligned} 2x+3y+4z&=29; \\ 3x+2y+5z&=32; \\ 4x+3y+2z&=25. \end{aligned}\right\}$

6. $\left.\begin{aligned} x+\ y&=90-\ z; \\ 2x+40&=3y+20; \\ x+20&=2z+\ 5. \end{aligned}\right\}$

7. $\left.\begin{aligned} \tfrac{1}{2}x+\tfrac{1}{3}y+\tfrac{1}{4}z&=62; \\ \tfrac{1}{3}x+\tfrac{1}{4}y+\tfrac{1}{5}z&=47; \\ \tfrac{1}{4}x+\tfrac{1}{5}y+\tfrac{1}{6}z&=38. \end{aligned}\right\}$

3 ways.

SECTION XXIV.

SUBSTITUTION OF NUMBERS IN ALGEBRAIC QUANTITIES.

ART. **76.** Let the learner find the numerical value of the following expressions, when $a=4$, $b=3$, $c=2$, and $d=5$.

1. abc. Ans. $4.3.2=24$.
2. a^2b. Ans. $4^2.3=16.3=48$.
3. $b^2.c^3$. Ans. $3^2.2^3=9.8=72$.

4. $a\,b\,d$.
5. $a^2\,d^2$.
6. $a\,b\,c^2\,d$.
7. $a\,b\,c^3\,d$.
8. $b^2\,d^3$.
9. $\frac{a}{c}$. Ans. $\frac{4}{2} = 2$.
10. $\frac{a^2\,b}{c\,d}$.
11. $\frac{c^2\,d^3}{a}$.
12. $\frac{a\,b\,c}{d}$.
13. $\frac{a^2\,c^2}{b^3}$.
14. $\frac{b^3}{a\,c}$.
15. $\frac{d^2\,c}{a^3}$.
16. $a + b + c$. Ans. 9.
17. $a^2 + b + d$.
18. $a\,b + c\,d$.
19. $a\,b\,c + d$.
20. $a^2\,b^2 + c^3\,d$.
21. $a\,b + c - d$.
22. $a\,b\,c^2 - d^2$.
23. $a^2 - c + d$.
24. $2\,a + 3\,c - 2\,b$.
25. $5\,a - 2\,b^2 + c^3$.
26. $3\,a^2 + 4\,b^2 - c - 2\,d$.
27. $(a + b)\,c$.
28. $a\,c\,(b + d)$.
29. $(a + b)\,(c + d)$.
30. $(a + b)\,(d - c)$.
31. $(2\,a - b)\,(3\,c + 2\,d)$.
32. $(b^2 + a^2)\,(d - c)$.
33. $(3\,a^2 - 2\,b^2)\,(4\,d - 3\,c)$.
34. $(a^2 - b^2)\,(c^2 - d^2)$.
35. $a^2\,(3\,b - c + 2\,d)$.
36. $(b^2 - a^2)\,(d^2 - c^2)$.
37. $3\,c^2\,(a + b - d)$.
38. $2\,b^2\,(a + c^2 + d)$.
39. $6\,a\,(a^2 - b + c)$.
40. $a\,(2\,b^2 - c^2 + d^3)$.

Find the value of the following expressions, when $a = 5$, $b = 3$, $m = 2$, and $n = 0$.

41. $a + b - m + n$.
42. $3\,a - 5\,n + 4\,m$.
43. $a\,b + m\,n$.
44. $6\,m\,a - 5\,a\,b\,n$.
45. $m\,n + 3\,a\,b\,n$.
46. $4\,a\,b\,c\,n - 5\,a\,b^2$.
47 $\frac{a + b}{m - n}$.
48 $\frac{3\,a - b}{m^2}$.
49. $\frac{3\,m\,n}{2\,b}$.
50. $\frac{3\,(a - b)\,(m - n)}{m}$.
51. $\frac{(3\,b - 2\,a)\,m^2}{2\,m - 3\,n}$.
52. $\frac{2\,(a^2 - b^2)\,(a^2 + b^2)}{3\,m^2 - n^2}$.

Substitute numbers in the following equivalent expressions, showing their identity, whatever numbers are put instead of the letters; provided, however, that the same value be given to the same letter in the two members of an equation. An *identical* equation is one in which the two members are exactly alike; they may differ in *form*, but both can be reduced to the same form.

53. $(a+c)\,b=a\,b+b\,c.$

In this example, let $a=1$, $b=2$, and $c=3$. Then the first member becomes $(1+3)\,2=4.2=8$. The second member gives $1.2+2.3=2+6=8$. The results, we perceive, are alike.

54. $a\,m-b\,m=m\,(a-b).$

55. $(a-m)\,(a+m)=a^2-m^2.$

56. $(a+m)\,(a+m)=a^2+2\,a\,m+m^2.$

57. $(a-m)\,(a-m)=a^2-2\,a\,m+m^2.$

58. $\frac{a+b}{2}+\frac{a-b}{2}=a.$

59. $\frac{a+m}{2}-\frac{a-m}{2}=m.$

60. $m+\frac{b}{c}=\frac{m\,c+b}{c}.$

61. $\frac{a^3-1}{a-1}=a^2+a+1.$

62. $\frac{m^4-1}{m^2-1}=m^2+1.$

63. $\frac{a^3+m^3}{a+m}=a^2-a\,m+m^2.$

64. $(a+m)\,(a+n)=a^2+a\,(m+n)+m\,n.$

65 $\frac{b}{a+b}+\frac{a-b}{b}=\frac{a^2}{a\,b+b^2}.$

66. $\frac{a+m}{a-m}+\frac{a-m}{a+m}=\frac{2\,(a^2+m^2)}{a^2-m^2}.$

67. $\frac{a^2-3a+2}{a^2-10a+9}=\frac{a-2}{a-9}$.

68. $\frac{(n+1)(n+2)}{2}=\frac{n(n+1)}{2}+n+1$.

SECTION XXV.

LITERAL EQUATIONS.

Art. 77. Find the values of x in the following equations.

1. $\frac{a+mx}{b}=x+3a$. Multiply by b,

$a+mx=bx+3ab$. Transpose a and bx,

$mx-bx=3ab-a$. Separate the first member into factors, one of which is x,

$$(m-b)x=3ab-a.$$

In this equation, x is taken $m-b$ times, that is, the factor $m-b$ is the coefficient of x. Divide both members by this coefficient,

$$x=\frac{3ab-a}{m-b}.$$

2. $\frac{3c+ax-bx}{a+m}=2x-b$. Multiply by $a+m$,

$$3c+ax-bx=2ax-ab+2mx-bm.$$

Transpose the terms containing x into the first member, the others into the second, and reduce,

$-ax-bx-2mx=-ab-bm-3c$. Change the signs,

$ax+bx+2mx=ab+bm+3c$. Separate the first member into factors, one of which is x.

$$(a+b+2m)x=ab+bm+3c. \quad \text{Hence}$$

$$x=\frac{ab+bm+3c}{a+b+2m}.$$

3. $\frac{2x-a}{b}=\frac{bc-cx}{a}$.

4. $\frac{am-bx}{2b+c}=3ax-bm$.

5. $\frac{2ax-m^2}{b}=\frac{3c^2-4x}{a+2m}$.

6. $\frac{m^2x+bc}{2b-3x}=\frac{a+4c}{5}$.

7. $\frac{a^2x}{b+c}-bx=\frac{3m-5}{2m}$.

8. $\frac{3b-4a}{6-3x}=\frac{5a-4c^2}{5-3x}$.

SECTION XXVI.

GENERALIZATION.

Art. 78. In *pure algebra,* letters are used to represent known as well as unknown quantities. It enables us to conduct operations with greater facility, and to form rules. The results of purely algebraical solutions of problems are called *general formulæ.* The design of this section is to familiarize the learner with such solutions.

1. A and B have together \$270, but B has twice as much money as A. Required the money of each.

In this problem it is required to divide \$270 into two parts, one of which shall be double the other.

Let x dollars be A's share; then $2x$ dollars will be the share of B. Hence,

$x+2x=270$. This equation gives

$$\left.\begin{aligned} x &= \$90, \text{ A's share;} \\ 2x &= \$180, \text{ B's share;} \end{aligned}\right\} \textit{Ans.}$$

Now, suppose that, instead of \$270, A and B have any number a of dollars, of which B has twice as much as A.

In this case, the problem is to divide the number a into two parts, one of which shall be twice as great as the other.

Representing the shares as before, we have

$$x+2x=a$$

This equation gives

$$\left.\begin{aligned} x&=\frac{a}{3},\ \text{A's share;} \\ 2x&=\frac{2a}{3},\ \text{B's share;} \end{aligned}\right\} \textit{Ans.}\quad \text{General formulæ.}$$

The general formulæ show us that one part is $\frac{1}{3}$ and the other $\frac{2}{3}$ of the number to be divided, without reference to the particular value of that number.

If, in the general formulæ, we now put \$270 instead of a, we have

$$\left.\begin{aligned} x&=\frac{270}{3}=\$90,\ \text{A's share;} \\ 2x&=\frac{2.270}{3}=2.90=\$180,\ \text{B's share;} \end{aligned}\right\} \text{Particular answers.}$$

Let the learner substitute other numbers for a in the general formulæ, and find the particular answers. Any number divisible by 3 will give whole numbers for the answers.

2. The sum of the ages of two brothers is 76 years, and the difference of their ages is 16 years. Required the age of each.

In this problem we are to separate 76 into two parts, such that the difference of those parts shall be 16. Instead of 76, let us suppose that the number to be separated into parts is indefinite, and that it is represented by x; also, that d represents the difference of the parts, that is, the excess of the greater part above the less.

Let $x=$ the less part; then

$x+d=$ the greater part. Hence,

$x+x+d=a$. Reduce and transpose d,

$$2x = a - d; \text{ divide by } 2,$$

$$x = \frac{a}{2} - \frac{d}{2} = \frac{a-d}{2}, \textit{ the less part. } \text{(Art. 67.)}$$

To obtain the greater part, we add d to the less, and we have

$x + d = \frac{a}{2} - \frac{d}{2} + d.$ Change d in the 2d member to halves,

$$x + d = \frac{a}{2} - \frac{d}{2} + \frac{2d}{2}; \text{ reduce,}$$

$$x + d = \frac{a}{2} + \frac{d}{2} = \frac{a+d}{2}, \textit{ the greater part.}$$

By examining the general formulæ for the two parts, and recollecting that a and d may stand for any numbers, provided d is less than a, we see that they furnish the following rule for separating a number into two parts, whose difference is given.

The less part is found by subtracting half of the difference of the parts from half of their sum, that is, from half of the number to be separated; or, by subtracting the difference of the parts from their sum, and dividing the remainder by 2.

The greater part is found by adding half of the difference to half of the sum of the parts; or, by adding the difference to the sum of the parts, and dividing the amount by 2.

Let the learner separate the following numbers into parts, either by means of the formulæ, or by following the rule.

	Numbers to be separated.	*Differences of the Parts.*
3	50	10.
4.	30	16.
5.	100	20.
6.	35	5

	Numbers to be separated.	Differences of the Parts.
7.	106	50.
8.	50	5.
9.	33	8.

10. A man bought corn at a shillings, rye at b shillings, and wheat at c shillings, per bushel, and the whole amounted to d shillings. Required the number of bushels of each he bought, if he bought the same number of bushels of each.

Let $x =$ the number of bushels of each. Then,

$$a x + b x + c x = d.$$

Separating the 1st member into factors, one of which is x, (Art. 59.)

$(a + b + c) x = d$; dividing by $a + b + c$, the coefficient of x,

$$x = \frac{d}{a+b+c}, \text{ the number of bushels of each.}$$

This general formula may be translated into the following rule.

Divide the price of the whole by the sum of the prices of a bushel of each sort; the quotient will be the number of bushels of each.

If, in the formula, we substitute 5, 6, 7, and 180, for a, b, c, and d, respectively, we have

$$x = \frac{180}{5+6+7} = \frac{180}{18} = 10 \text{ bushels, particular answer.}$$

This rule may be extended to any number of articles; only we must change "bushel" and "number of bushels" into other corresponding expressions, as the case may require.

11. How many apples at 1 cent, pears at 2 cents, peaches at 3 cents, and oranges at 4 cents, apiece, of each an equal number, can I buy for $5?

12. In a certain manufactory there are employed b

times as many boys as men, and c times as many girls as boys. Required the number of each, the whole number of individuals being a.

$$\text{Let } x = \text{the number of men;}$$
$$\text{then } b\,x = \text{the number of boys,}$$
$$\text{and } b\,c\,x = \text{the number of girls. Hence,}$$
$$x + b\,x + b\,c\,x = a.$$

Here x is taken $1 + b + b\,c$ times; therefore, separating the first member into factors, one of which is x,

$$(1 + b + b\,c)\,x = a. \quad \text{Divide by } 1 + b + b\,c,$$

$$x = \frac{a}{1 + b + b\,c}, \text{ number of men;}$$

$$b\,x = \frac{a}{1 + b + b\,c} \times b, \text{ number of boys;}$$

$$b\,c\,x = \frac{a}{1 + b + b\,c} \times b\,c, \text{ number of girls.}$$

If, in these formulæ, we substitute 81 for a, 2 for b, and 3 for c, we have

$$\frac{81}{1+2+6} = \frac{81}{9} = 9 \text{ men; } 9 \,.\, 2 = 18 \text{ boys, and } 9 \,.\, 2 \,.\, 3 = 54 \text{ girls.}$$

13. What will be the particular answers in the last question, if a is 144, b is 5, and c is 6?

14. Two men, A and B, engaged to dig a well; A could dig it alone in a days, and B could dig it alone in b days. In what time could they both together dig it?

Let $x =$ the number of days required. Then, as A could dig the whole in a days, in 1 day he would dig $\frac{1}{a}$ of it; and, as B could dig the whole in b days, in 1 day he would dig $\frac{1}{b}$ of it. Both would, therefore, dig $\frac{1}{a} + \frac{1}{b}$ of it in 1 day; and in x days they would dig $\frac{x}{a} + \frac{x}{b}$ of it. But we have supposed that in x days they would dig the whole. Hence,

$$\frac{x}{a}+\frac{x}{b}=1, \text{ well.}$$

Multiply by a and b,

$$b\,x+a\,x=a\,b; \text{ or,}$$

$$(b+a)\,x=a\,b; \text{ divide by } b+a,$$

$$x=\frac{a\,b}{b+a}, \text{ Ans.}$$

If a is 6, and b is 7, we have

$$x=\frac{6.7}{7+6}=\tfrac{42}{13}=3\tfrac{3}{13} \text{ days, particular answer.}$$

From the general formula in this question, we derive the following rule for any case in which there are but two workmen.

Divide the product of the numbers expressing the times in which each would perform the work, by the sum of those numbers. The quotient will be the time in which both together will perform it.

Let the two following questions be performed by the preceding rule.

15. A man could do a piece of work in 6 days, and a boy could do the same in 10 days. In what time would they both together do it?

16. A man alone would consume a barrel of flour in 6 months, and the same quantity would last his wife 8 months. How long would a barrel supply both?

17. Let it be required to find a rule for dividing the gain or loss in partnership, or, as it is commonly called, *the rule of fellowship.* Let us first take a particular example.

Three men, A, B, and C, traded together, and gained \$300. A put into the partnership \$7 as often as B put in \$6, and as often as C put in \$2. Required each man's share of the gain.

Let $x =$ A's share. B put in $\frac{6}{7}$, and C put in $\frac{2}{7}$ as much stock as A. Therefore, B must have $\frac{6}{7}$, and C $\frac{2}{7}$ as much gain as A. Consequently, $\frac{6x}{7} =$ B's share, and $\frac{2x}{7}$ $=$ C's share. Hence,

$$x + \frac{6x}{7} + \frac{2x}{7} = 300.$$ This equation gives

$$x = \$140, \text{ A's share;}$$

$$\frac{6x}{7} = \frac{6 \cdot 140}{7} = 6 \cdot 20 = \$120, \text{ B's share,}$$

$$\frac{2x}{7} = \frac{2 \cdot 140}{7} = 2 \cdot 20 = \$40, \text{ C's share.}$$

To generalize this question, suppose that A put in m dollars as often as B put in n, and as often as C put in p dollars; and that they gained a dollars. Then B put in $\frac{n}{m}$, and C $\frac{p}{m}$ as much as A; they must, therefore, have, respectively, $\frac{n}{m}$ and $\frac{p}{m}$ as much gain as A.

Let $x =$ A's gain; then

$\frac{nx}{m} =$ B's gain, and

$\frac{px}{m} =$ C's gain. Hence,

$$x + \frac{nx}{m} + \frac{px}{m} = a.$$ Multiply by m,

$$mx + nx + px = ma; \text{ or,}$$

$$(m + n + p)\,x = ma;$$ divide by $m + n + p$,

$$x = \frac{ma}{m+n+p}, \text{ or } x = m \times \frac{a}{m+n+p}, \text{ A's share.}$$

B's share is $\frac{n}{m}$ of A's; $\frac{1}{m}$ of $m \times \frac{a}{m+n+p}$ is $\frac{a}{m+n+p}$, and $\frac{n}{m}$ of it is n times as much; hence, $\frac{nx}{m} = n \times \frac{a}{m+n+p}$, B's share.

C s share being $\frac{p}{m}$ as much as A's, we have, in like manner,

$$\frac{p\,x}{m} = p \times \frac{a}{m+n+p},\ \text{C's share};$$

If we recollect that m, n, and p are the proportions of stock furnished respectively by A, B, and C, we shall perceive that the whole gain a is divided by $m+n+p$, the sum of the proportions of stock furnished by all the partners, and that this quotient is multiplied by m, A's proportion, by n, B's proportion, and by p, C's proportion of the stock, to obtain their respective shares of the gain. Hence, since a may represent the loss as well as the gain, for finding each partner's share of gain or loss, we have the following rule.

Divide the whole gain or loss by the sum of the proportions of the stock, and multiply the quotient by each partner's proportion.

This rule applies, whatever be the number of partners.

18. James and White trade together, the former putting into the partnership \$800, and the latter \$600. They gain \$700. Find by the formulæ, or by the rule, each partner's share of the gain.

Remark. If the learner use the formulæ, in this case, since there are but two partners, p must be zero. Moreover, when the amounts actually put in are given, the simplest proportions of the stocks will be found by dividing these amounts by the greatest number that will divide them all without a remainder. Thus, 800 and 600 are both divisible by 200, and the quotients, 4 and 3, are the simplest proportions of the stock.

19. What would be each man's gain, if A furnished \$400, B \$300, and C \$200, the whole gain \$450?

20. What would each man lose, if four partners furnished, respectively, \$700, \$600, \$400, and \$200, the whole loss being \$380?

21. A put in \$500 for 4 months, B put in \$400 for 6 months, and C put in \$300 for 7 months. They gained \$325. Required each man's gain.

Remark. When the stocks are employed unequal times, it is manifest, that each partner's stock, or his proportion of the stock, must be multiplied by the number expressing the time which it is in trade, and that then the proportions of these products must be used.

ART. **79.** Generally, known quantities are represented by the first, and unknown quantities by the last letters of the alphabet. But it is frequently convenient to use the initial letters of the names of quantities, whether known or unknown.

In the following questions of simple interest, let p be the principal, r the rate, t the time, i the interest, and a the amount. It must be remembered that r is supposed to be a fraction, as ·06, ·05, &c., according as the rate is 6 per cent., 5 per cent., &c., and that the time is expressed in years and fractions of a year.

1. What is the simple interest of p dollars, for t years, at r per cent.?

Since the principal multiplied by the rate gives the interest for a year, we have

$$r\,p = \text{the interest for 1 year; and}$$
$$t\,r\,p = \text{the interest for } t \text{ years. Hence,}$$
$$i = t\,r\,p.$$

From this formula we deduce the following rule.

To find the interest, the principal, time, and rate being known, multiply the principal, time, and rate together.

2. Required the interest on \$247·50, for 5 years, at 6 per cent.

3. The equation $trp = i$ contains four quantities, any three of which being known, the fourth may be found. For example, to find p when the other quantities are given. Take

$trp = i$; divide both members by tr,

$$p = \frac{i}{tr}.$$

This formula gives the following rule.

To find the principal, when the interest, time, and rate are known, divide the interest by the product of the time and rate.

4. The interest being \$90·10, the time 4 years, and the rate 5 per cent., required the principal, according to the rule.

Let the learner find, from $trp = i$, formulæ for the time and the rate, translate these formulæ into rules, and perform by the rules the particular examples subjoined.

5. The interest, rate, and principal given, find the time.

6. The interest being \$54, rate 6 per cent., principal \$150, what is the time?

7. The interest, rate, and principal given, find the rate.

8. The interest being \$33·705, time 5 years, and principal \$96·30, what is the rate?

9. Required the amount of p dollars, for t years, at the rate r, simple interest.

Since the amount is the sum of principal and interest, trp being the interest, as before, we have

$$a = p + trp; \text{ or, } a = p(1 + tr).$$

Hence the following rule.

To find the amount, when the principal, time, and rate *are known, multiply the time and rate together, add* 1 *to the product, and multiply this sum by the principal.*

10. The principal being $630·20, the rate $4\frac{1}{2}$ per cent., and the time 7 years and 6 months, what is the amount?

11. Let us find the value of p from the equation, $p+trp=a$. Separate the first member into factors,

$$(1+tr)p=a; \text{ divide by } 1+tr,$$

$$p=\frac{a}{1+tr}.$$

Hence,

To find the principal, when the amount, time, and rate are given, multiply the time and rate together, add 1 *to the product, and divide the amount by the sum.*

12. The amount being $124, the rate 6 per cent., and the time 4 years, what is the principal?

Remark. The principal, in this case, is called the *present worth* of the amount.

13. Required the present worth of $400, due in 3 years and 3 months, at 5 per cent.

14. From the same equation, let us find the formula for r.

$$p+trp=a; \text{ transpose } p,$$

$$trp=a-p; \text{ divide by } tp,$$

$$r=\frac{a-p}{tp}. \text{ Hence,}$$

To find the rate, when the amount, time, and principal are given, divide the difference of the amount and principal by the product of the time and principal.

Remark. This rule is virtually the same as that given by problem 7th, since the difference of the amount and principal is the interest.

15. The amount being $368·875, time 3 years, and principal $325, required the rate.

16. In a similar manner, and from the same equation, let the learner find the formula for t, convert it into a rule, and perform by the rule, the subjoined particular example.

17. The amount being $1012·50, principal $750, and rate 5 per cent., required the time.

ART. **80.** 1. Separate a into two parts, such that one shall be m times the other.

2. A and B have together a dollars, of which B has m times as much as A. How many dollars has each?

3. Separate a into two parts, such that the second shall be the $\frac{m}{n}$ part of the first.

4. Separate a into three parts, so that the second shall be m times, and the third n times the first.

5. What number is that whose mth part, added to its nth part, makes the number a?

6. What number is that whose mth part exceeds its nth part by a?

7. Separate a into two parts, so that the mth part of one, added to the nth part of the other, shall make the number a.

Let $x =$ one part; the other will be $a - x$.

8. After paying away $\frac{1}{a}$ and $\frac{1}{b}$ of my money, I had c dollars left. How many dollars had I at first?

9. A man and his boy together could do a piece of work in a days, and the man could do it alone in b days. Required the number of days in which the boy could do it alone.

10 The heirs to an estate received a dollars each;

but if there had been b less heirs, they would have received c dollars each. Required the number of heirs.

11. A man has 4 sons, each of whom is a years older than his next younger brother, and the eldest is m times as old as the youngest. Required their ages.

12. A father gave his children m oranges apiece, and had a oranges left; but in order to give them n oranges apiece, he wanted b oranges more. How many children had he?

13. The sum of two numbers is a, and $\frac{1}{m}$ of the greater added to $\frac{1}{n}$ of the less makes b. Required the numbers.

Remark. This and the two following questions may be performed by means of two unknown quantities.

14. One ox and m cows cost a dollars; but one cow and n oxen cost b dollars. Required the price of a cow, and that of an ox.

15. There are two numbers, such that the first with $\frac{1}{m}$ of the second makes a; and the second with $\frac{1}{n}$ of the first also makes a. Required the numbers.

SECTION XXVII.

EXTRACTION OF THE SECOND ROOTS OF NUMBERS.

Art. 81. 1. Some market women, counting their eggs, found that each had 12 times as many eggs as there were women, and that they all together had 300. Required the number of women.

Let $x =$ the number of women;
then $12x =$ the number of eggs each had,
and $12x.x$, or $12x^2 =$ the whole number of eggs. Hence,

$$12x^2 = 300. \quad \text{Divide by 12,}$$

$$x^2, \text{ or } x.x = 25.$$

We see that x must be a number which, multiplied by itself, shall produce 25; and we know that $5.5 = 25$ Hence,

$$x = 5 \text{ women, } Ans.$$

The equation $12x^2 = 300$ is called an equation of the *second degree*, or a *quadratic equation*, because it contains the second power or square of the unknown quantity.

In general, an equation of the second degree is such as, when reduced to its simplest form, contains at least one term in which there are two, but no term in which there are more than two unknown factors.

When an equation with one unknown quantity contains the second power only of that quantity, it is called a *pure* equation of the second degree, or a *pure* quadratic equation. The equation given above is of this kind.

The first power of a quantity, with reference to the second, is called the *second root* or *square root*, and finding the first power when the second is given, is called *extracting the second* or *square root*. The second root of a quantity, then, is such as, being multiplied by itself, will produce that quantity.

The second powers of the first nine figures are as follows.

$$\begin{cases} 1, 2, 3, \ 4, \ 5, \ 6, \ 7, \ 8, \ 9. & \text{Roots.} \\ 1, 4, 9, 16, 25, 36, 49, 64, 81. & \text{Powers.} \end{cases}$$

Hence, when a number consists of only one figure, its second power cannot contain more than two figures. The least number consisting of two figures or places is

10, the second power of which is 100, consisting of three figures, or rather of three places.

It is to be observed, that, when a number has zeros at the right, its second power is terminated by twice as many zeros; that is, the second power has, at the right, twice as many zeros as the root. And, in general, the product of two or more numbers, each having zeros at the right, has, at its right, as many zeros as all the factors. For example, $(30)^2 = 900$, $(500)^2 = 250000$, &c.; $(400)^2 \times 30 = 4800000$, $(2000)^2 . 300 = 1200000000$, &c.

ART. **82.** In order to find a mode of extracting the second root of a number consisting of more than two figures, let us examine the second power of $a + b$, which is $a^2 + 2ab + b^2$, in which a may represent the tens, and b the units of the root. Let $a = 2$ (tens) or 20, and $b = 3$; then $a + b = 23$; $a^2 = 400$, $2ab = 2.20.3 = 120$, and $b^2 = 9$; consequently, $a^2 + 2ab + b^2 = 400 + 120 + 9 = 529$. Hence,

When a number contains units and tens, its second power contains the second power of the tens, plus twice the product of the tens by the units, plus the second power of the units.

Let us now, by a reverse operation, deduce the root from the power.

Operation.

5′29 (23. Root.
4
———
12′9 (4. Divisor.

Since the number contains hundreds, its root must necessarily contain tens. Our first object, then, is to find the tens of the root. But since the second power of the tens of the root cannot have any significant figure below hundreds, it must be found in the 5 (hundreds). We separate, therefore, the last two figures from the 5 by an accent.

Now the second power of 2 (tens) is 4 (hundreds), and that of 3 (tens) is 9 (hundreds). Therefore the greatest second power of tens found in 5 (hundreds) is 4 (hundreds), and the root of 4 (hundreds) is 2 (tens).

We place 2 at the right of the given number as the tens of the root, separating it from that number by a line, and subtract the second power 4 (hundreds), corresponding to a^2, from the 5 (hundreds).

To the right of the remainder 1 (hundred), we bring down the two figures, previously cut off, and have 129, which we call a dividend. This number corresponds to $2ab + b^2$. It contains twice the product of the tens by the units, plus the second power of the units. If it contained $2ab$ only, or exactly twice the product of the tens and units, we should find the units exactly by dividing by twice the tens, for $2ab$ divided by $2a$ gives b. As it is, if we divide by twice the tens, disregarding the remainder, we shall obtain the units of the root exactly, or a number a little too great.

Twice 2 (tens) are 4 (tens), by which we are to divide. But since tens multiplied by units can have no significant figure below tens, if we take 4 merely as our divisor, we must reject the right-hand figure 9 of the dividend; that is, since the divisor is considered as ten times less than its true value, that the quotient may not be changed, we must make the dividend also ten times less than its true value. Hence, before dividing, we separate, by an accent, the 9 from 12.

The divisor 4 is contained 3 times in 12; we place 3 in the root, at the right of the 2, and obtain for the entire root 23. The second power of 23 is 529, showing that 23 is the exact root sought.

ART. **83.** In the preceding example, after having found the unit figure, we raised the whole root to the second power, in order to ascertain its correctness. We shall now show how the correctness of the second or any succeeding figure of the root may be ascertained, without raising the whole to the second power.

Let us extract the second root of 2401.

Operation.

```
24'01 (49.   Root.
16
——
 80'1 (89
 801
 ———
  0.
```

Reasoning as in the preceding example, we find the greatest second power of tens contained in 24 (hundreds), to be 16 (hundreds), the root of which is 4 (tens). We put 4 as the first figure of the root, subtract its second power from 24, bring down, to the right of the remainder, the two figures cut off, and have 801 for a dividend. This corresponds to $2ab + b^2$, which is the same as $(2a + b)b$. Dividing 80 by 8, twice the tens of the root, we have 10 for the quotient; but since the unit figure cannot exceed 9, we put 9 in the root, at the right of 4, and have 49 for the entire root.

Now, in order to determine whether 9 is the proper unit figure, we observe that our divisor 8 corresponds to $2a$, and 9 is the presumed value of b; hence $80 + 9$ or 89 corresponds to $2a + b$; we therefore put 9 at the right of the divisor, and multiply 89 by 9; the product 801 answers to $(2a+b)b$, which, subtracted from the dividend, leaves no remainder. Therefore 49 is the correct root.

Let the learner find the second roots of the following numbers, according to the method just explained.

1. 625.
2. 1089.
3. 2809.
4. 1936.
5. 225.
6. 5476.
7. 6241.
8. 9801.

ART. 84. Let it now be required to extract the root of 105625.

Since $(10)^2=100$, $(100)^2=10000$, $(1000)^2=1000000$, it follows that the second power of any whole number greater than 10 and less than 100, that is, consisting of two figures, must be greater than 100 and less than 10000, that is, it must consist of three or four figures; also, the second power of any number between 100 and 1000, that is, consisting of three figures, must contain five or six figures. Therefore the number of figures in the root of any proposed whole number may be found, by beginning at the right and separating that number into parts or periods of two figures each. The left-hand period may contain either one or two figures. There will be as many figures in the root as there are periods in the power.

Separating 105625 into periods, thus, 10′56′25, we see that its root must consist of three figures, viz., hundreds, tens, and units.

Let a represent the hundreds, b the tens, and c the units of the root; then $a+b+c$ will represent the root.

The second power of $a+b+c$ is $a^2+2ab+b^2+2ac+2bc+c^2$. This may be put in either of the following forms, viz., $a^2+2ab+b^2+2(a+b)c+c^2$, or $a^2+(2a+b)b+[2(a+b)+c]c$. The first of these forms shows that the second power of a number consisting of hundreds, tens, and units, contains the second power of the hundreds, plus twice the product of the hundreds and tens, plus twice the sum of the hundreds and tens multiplied by the units, plus the second power of the units.

Let us now proceed to extract the root of 105625.

Operation.

$$
\begin{array}{ll}
10'56'25(\underline{325.} & \text{Root.} \\
\underline{9} & = a^2. \\
15'6(\underline{62} & = 2a + b. \\
\underline{124} & = (2a + b)\,b. \\
\quad 322'5(\underline{645} & = 2(a + b) + c. \\
\quad \underline{3225} & = [2(a + b) + c]\,c. \\
\quad 0. &
\end{array}
$$

After separating the number into periods, we first seek the second power of the hundreds of the root, which must be found in the 10, (that is, 100000); the greatest second power of hundreds in this part is 9, (90000), the root of which is 3, (300).

Putting 3 as the first figure, or hundreds, of the root, we subtract its second power 9 from 10; to the right of the remainder 1 we bring down the second period 56, and have 156, (15600), which we regard as a dividend.

This dividend contains $2ab + b^2$, or twice the product of the hundreds and tens, plus the second power of the tens; it also contains the hundreds which are produced by multiplying the sum of the hundreds and tens by the units.

We now wish to obtain b, the tens of the root; and, for this purpose, we divide by twice the hundreds, corresponding to $2a$. But since the product of hundreds by tens has three zeros at the right, this product can have no significant figure below the fourth place from the right. We therefore reject the 6, separating it by an accent, and divide 15,(15000), by 6,(600), twice the hundreds of the root.

The quotient is 2, (20), which we put as the second figure or tens of the root; we also place it at the right of the divisor. The divisor thus increased is 62, (620), and answers to $2a+b$, which we multiply by 6 or b, and have 124,(12400), answering to $(2a+b)b$.

We subtract 124 from the dividend 156, and have for a remainder 32, to the right of which we annex the last period 25, and regard the result 3225 as a new dividend.

This dividend corresponds to $2(a+b)c+c^2$, or $[2(a+b)+c]c$; that is, it contains twice the sum of the hundreds and tens multiplied by the units, plus the second power of the units. To find the units of the root, therefore, we must divide by twice the sum of the hundreds and tens, that is, by twice the whole root, so far as it has been already ascertained.

But since hundreds and tens, multiplied by units, must have one zero at the right, this product can have no significant figure below the second from the right; we therefore reject the figure 5, separating it by an accent. Twice the hundreds and tens make 64,(640), corresponding to $2(a+b)$, which is contained in 322 (3220) five times.

We put 5, which is the presumed value of c, as the next figure of the root, also at the right of the divisor. The divisor thus increased answers to $2(a+b)+c$, which multiplied by 5, or c, gives 3225, corresponding to $[2(a+b)+c]c$; this subtracted from the dividend leaves no remainder. Hence 325 is the root required.

Of the two formulæ, already given, of the second power of $a+b+c$, viz., $a^2+2ab+2(a+b)c+c^2$, and $a^2+(2a+b)b+[2(a+b)+c]c$, the former shows, that, after the first figure has been found, each succeeding figure is to be sought by dividing by twice the whole of the root previously found; and the latter shows that, in each case, the quotient is to be placed at the right of the divisor, and that the divisor, thus increased, is to be multiplied by the quotient.

Moreover, from the rank of the figures, it is evident that twice the root already found can produce no significant figure below the second from the right in each dividend.

Formulæ might be given for the second power of four or more figures; but from what has been already shown, the mode of proceeding, in all cases, will be sufficiently manifest.

We exhibit below the process of extracting a root consisting of five figures.

Operation.

```
28'11'22'64'41 ( 53021.   Root.
25
 31'1 ( 103
 309
    2226'4 ( 10602
    21204
       10604'1 ( 106041
       106041
          0.
```

In this example we find that the second divisor 106 is not contained in the dividend 222, the right-hand figure being rejected; this shows that there are no hundreds in required root. In such a case, we place a zero in the root, also at the right of the divisor, and annex the suc ceeding period to form a dividend.

Art. 85. From the foregoing examples and explanations we deduce the following

RULE FOR EXTRACTING THE SECOND ROOTS OF NUMBERS.

1. *Begin at the right, and, by means of accents or other marks, separate the number into periods of two figures each. The left-hand period may contain one or two figures.*

2. *Find the greatest second power in the left-hand period, place its root at the right of the proposed number, separating it by a line, and subtract the second power from the left-hand period.*

3. *To the right of the remainder bring down the next period to form a dividend. Double the root already found for a divisor. Seek how many times the divisor is contained in the dividend, the right-hand figure being rejected. Place the quotient in the root, at the right of the figure previously found, and also at the right of the divisor. Multiply the divisor, thus increased, by the last figure of the root, and subtract the product from the whole dividend.*

4. *Repeat the process in part third of the rule, until all the periods have been brought down.*

Remark 1*st*. If the dividend will not contain the divisor, the right-hand figure of the former being rejected, place a zero in the root, also at the right of the divisor, and bring down the next period.

Remark 2*d*. We may observe, that, if the last figure of the preceding divisor be doubled, the root will be doubled; for that divisor contains twice the whole root found, with the exception of the figure last obtained.

Remark 3*d*. To find the root of a product, as will be shown hereafter, we take the root of each factor and multiply these roots together. Thus, the root of $81 . 225$ is $9 \; 15 = 135$.

Extract the roots of the subjoined numbers.

1. 676.	8. 266256.	15. 121.144.
2. 1369.	9. 289444.	16. 81.256.
3. 2304.	10. 628849.	17. 400.361.
4. 10816.	11. 12759184	18. 16.25.9.
5. 22201.	12. 8590761.	19. 4.64.144.
6. 26569.	13. 9616201.	20. 676.441.
7. 66049.	14. 81036004.	

ART. **86.** Comparatively few numbers are exact second powers, and the roots of such as are not perfect powers cannot be found exactly, either in whole numbers or fractions. Thus, the second root of 5 is between 2 and 3; but no number can be obtained, which, multiplied by itself, will produce exactly 5. Such a root may, however, be approximated to any degree of accuracy.

All whole numbers and all definite fractions are called *commensurable* or *rational*, because they have a common measure with unity, or their ratio to unity can be exactly obtained. But the root of a number which is not a perfect power, is called *incommensurable* or *irrational*, because it has no common measure with unity, or its ratio to unity cannot be exactly found. Such roots, or rather expressions of them, are called also *surds*.

The second root of a quantity is denoted either by the exponent $\frac{1}{2}$, or by the sign $\sqrt{\ }$, called the *radical sign* Thus, $4^{\frac{1}{2}}$ or $\sqrt{4} = 2$, and $3^{\frac{1}{2}}$ or $\sqrt{3}$ means the second root of 3.

The second root of a negative quantity is called *imaginary*, because no quantity, either positive or negative, can, when multiplied by itself, produce a negative quantity. Thus, $(-4)^{\frac{1}{2}}$ or $\sqrt{-4}$ is an imaginary quantity. An imaginary result to a problem generally indicates absurdity in the conditions of that problem.

SECTION XXVIII.

SECOND ROOTS OF FRACTIONS, AND EXTRACTION OF SECOND ROOTS BY APPROXIMATION.

ART. 87. A fraction is raised to the second power by raising both numerator and denominator to that power, this being equivalent to multiplying the fraction by itself. Thus, $\left(\frac{2}{3}\right)^2 = \frac{2}{3} \cdot \frac{2}{3} = \frac{4}{9}$; and $\left(\frac{m}{n}\right)^2 = \frac{m}{n} \cdot \frac{m}{n} = \frac{m^2}{n^2}$.

Consequently, the second root of a fraction is found by extracting the root of both numerator and denominator. Thus, the second root of $\frac{9}{16}$ is $\frac{3}{4}$; that of $\frac{a^2}{b^2}$ is $\frac{a}{b}$.

Extract the second roots of the following fractions.

1. $\frac{25}{49}$.	5. $\frac{961}{1089}$.
2. $\frac{64}{81}$.	6. $\frac{1225}{1369}$.
3. $\frac{484}{529}$.	7. $\frac{576}{841}$.
4. $\frac{625}{729}$.	8. $\frac{1444}{1681}$.

ART. 88. But if either the numerator or denominator is not an exact second power, we can find only an approximate root of the fraction. Thus, the root of $\frac{15}{16}$ is between $\frac{3}{4}$ and $\frac{4}{4}$, or 1, the latter being nearer the true root than the former.

We can always make the denominator of a fraction a perfect second power, by multiplying both numerator and denominator by the denominator. This does not change the value, but only the form of the fraction. For example $\frac{5}{8} = \frac{5 . 8}{8 . 8} = \frac{40}{64}$, the approximate root of which is $\frac{6}{8} +$.

Remark. The sign $+$, placed after an approximate root, denotes that it is less, and the sign —, that it is greater, than the true root.

If a greater degree of accuracy is required, we may, after preparing the fraction as above, multiply both numerator and denominator of the result by any second power, and then extract the root. Thus, to find the root of $\frac{7}{9}$; after changing it to $\frac{63}{81}$, we may multiply the numerator and denominator of $\frac{63}{81}$ by 225, the second power of 15; this gives $\frac{14175}{81 \,.\, 225}$, the approximate root of which is $\frac{119}{9 \,.\, 15}=\frac{119}{135}+$.

Art. 89. The roots of whole numbers, which are not exact second powers, may be approximated in a similar manner. For example, to find the root of 3, accurate to within $\frac{1}{15}$, we convert it into a fraction having the second power of 15 for its denominator. Thus, $3=\frac{675}{225}$, the root of which is $\frac{26}{15}-$.

But the most convenient number for a denominator is the second power of 10, 100, or 1000, &c.; that is, we convert the number into 100ths, 10000ths, or 1000000ths, &c., and the root will be in decimals.

Thus. $3=\frac{300}{100}=\frac{30000}{10000}=\frac{3000000}{1000000}$; that is, $3=3{\cdot}00 =3{\cdot}0000=3{\cdot}000000$. The approximate root of the first is $\frac{17}{10}+=1{\cdot}7+$; that of the second is $\frac{173}{100}+=1{\cdot}73+$; that of the third, $\frac{1732}{1000}+=1{\cdot}732+$.

It is evident that there must be twice as many decimals in the power as we wish to find in the root; for the second power of 10ths produces 100ths, the second power of 100ths produces 10000ths, &c. Hence two zeros must be annexed to the number for each additional decimal in the root. Nor need the zeros be all written at once, but we may annex two zeros to each remainder, in the same manner as we bring down successive periods.

Let us, in this way, extract the second root of 5.

Operation.

```
5 ( 2·236 +.   Root.
4
10′0 ( 42
 84
 160′0 ( 443
 1329
  2710′0 ( 4466
  26796
    304.
```

The operation may be continued to any desirable extent.

When the given number contains decimals, the process of finding the root is the same; and any fraction may be changed to decimals and the root be found in the same way, care being taken, in both cases, to make the number of decimals even, and to point off half as many figures for decimals in the root, as there are in the power, including the zeros annexed.

In separating a number containing decimals into periods, it is best to begin at the decimal point, and separate the decimals by proceeding towards the right, and the integral numbers by proceeding towards the left.

Let the roots of the following numbers be found in decimals, each root containing three decimal figures.

1. 7.	6. $\frac{3}{7}$.
2. 24.	7. $\frac{5}{12}$.
3. 23.	8. $3\frac{5}{8}$.
4. 25·72.	9. $105\frac{7}{8}$.
5. 31·2.	10. $10\frac{3}{7}$.

SECTION XXIX.

QUESTIONS PRODUCING PURE EQUATIONS OF THE SECOND DEGREE

ART. **90.** 1. Two numbers are to each other as 2 to 3, and their product is 96. Required the numbers.

2. What number is that whose $\frac{1}{3}$ part multiplied by its $\frac{1}{4}$ part produces 48?

3. The length of a house is to its breadth as 10 to 9, and it covers 1440 square feet of land. Required the length and breadth.

4. What number is that to which if 5 be added, and from which if 5 be subtracted, the product of the sum and difference shall be 39?

5. A man bought a farm, giving $\frac{1}{5}$ as many dollars per acre as there were acres in the farm, and the whole amounted to $2000. Required the number of acres and the price per acre.

6. Two numbers are to each other as 8 to 5, and the difference of their second powers is 156. Required these numbers.

7. A gentleman has a rectangular piece of land 25 rods long and 9 rods wide, which he exchanges for a square piece of the same area. Required the length of one side of the square.

8. The sides of two square floors are to each other as 7 to 8, and it requires 15 square yards more of carpeting to cover the larger, than it does to cover the smaller. Required the length of one side of each floor.

9. A farmer bought two equal pieces of land, giving for the whole $1800. For one he gave $10 less, and for

the other $10 more, per acre, than there were acres in each piece. Required the number of acres in each.

10. The product of two numbers is 500, and the greater divided by the less gives 5. What are the numbers?

11. An acre contains 160 square rods. Required the length of one side of a square containing an acre.

12. What is the length of a square piece of land in which there are 5 acres?

13. A cistern having a square bottom, and being 4 feet deep, contains 600 gallons. Required the length of one side of the bottom, a gallon wine measure being 231 cubic inches.

SECTION XXX.

AFFECTED EQUATIONS OF THE SECOND DEGREE.

Art. **91.** Equations of the second degree which we have hitherto considered, contained the second power, but no other power, of the unknown quantity. But an equation of the second degree, in its most general sense, contains three kinds of terms, viz., one in which there are *two* unknown factors; another in which there is but *one* unknown factor; and a third composed wholly of *known quantities*.

Equations of this description, when they contain only one unknown quantity, are called *affected equations of the second degree*, or *affected quadratic equations*.

1. The length of a rectangular piece of land exceeds its breadth by 6 rods, and the piece contains 112 square rods. Required the dimensions.

Let x (rods) $=$ the breadth; then
$$x+6=\text{the length, and}$$
$$x^2+6x=\text{the area. Hence,}$$
$$x^2+6x=112.$$

In order to solve this equation, we compare the first member of it with the second power of $x+a$, which is $x^2+2ax+a^2$. We perceive that there are two terms in each, which correspond to each other, and we shall express this correspondency by the sign of equality. Thus,
$$x^2=x^2,$$
$$2ax=6x. \quad \text{Hence, by division,}$$
$$2a=6$$
$$a=3 \therefore$$
$$a^2=9.$$

Since 9 corresponds to a^2, if we add 9 to each member of $x^2+6x=112$, we have
$$x^2+6x+9=121,$$
the first member of which corresponds to $x^2+2ax+a^2$, and is a perfect second power.

We now extract the root of each member. The root of the first member is $x+3$, for $(x+3)(x+3)=x^2+6x+9$, and this root corresponds to $x+a$; the root of the second member is 11. Hence,
$$x+3=11, \therefore x=8 \text{ rods, breadth,}$$
$$x+6=14 \text{ rods, length.}$$

We remark that every positive quantity has two second roots, one positive, and the other negative. For the second power of $+a$, and the second power of $-a$, are each $+a^2$. Therefore, the second root of a^2 is either $+a$, or $-a$. In like manner, the second root of 121 is either $+11$, or -11.

Now, since, in an equation similar to $x+3=11$, the value of x is not ascertained, until the known term has been transposed from the first to the second member, it may happen that the negative as well as the positive value of the second member will answer the conditions of the question.

In all cases, therefore, of affected equations of the second degree, we prefix to the root of the second member the double sign $\pm$. which is read "*plus or minus*."

Giving this sign to 11, we have

$$x+3=\pm 11, \text{ and}$$
$$x=\pm 11-3.$$

Calling 11 positive, we have $x=8$;
calling it negative, we have $x=-14$.

The former value only of x will satisfy the conditions of the question. The latter value will, however, satisfy the original equation; for,

$$(-14)^2+6(-14)=196-84=112.$$

2. What number is that whose second power increased by 20 is equal to 12 times the number itself?

Let $x=$ the number. Then,
$x^2+20=12x$. By transposition,
$x^2-12x=-20$.

In this equation $12x$ is negative, and in order to render the first member a complete second power, we compare it with the second power of $x-a$, which is $x^2-2ax+a^2$. This comparison gives

$$x^2=x^2,$$
$$-2ax=-12x; \therefore$$
$$-2a=-12,$$
$$-a=-6,$$
$$a^2=36.$$

Adding 36 to each member of $x^2-12x=-20$, we have

$$x^2-12x+36=-20+36=16.$$

The first member now corresponds to $x^2-2ax+a^2$, and is a perfect second power. The next step is to extract the root of each member. The root of the first member is $x-6$, for $(x-6)(x-6)=x^2-12x+36$ This root corresponds to $x-a$; the root of the second member is ± 4. Hence,

$$x-6=\pm 4. \text{ Transposing,}$$
$$x=6\pm 4, \therefore$$
$$x=10, \text{ or } x=2.$$

Both values of x have the sign $+$, and, therefore, both answer the conditions of the question. Indeed, $(10)^2+20=12.10=120$, and $(2)^2+20=12.2=24$. Hence we see the propriety of giving the double sign to the root of the second member.

Art. **92.** Any affected equation of the second degree may be reduced to the form of $x^2+px=q$, in which p and q may stand for any known quantities, positive or negative.

An affected equation may, manifestly, be reduced to this form in the following manner, x being the unknown quantity.

1. *Clear the equation of fractions, if necessary; transpose all the terms containing x^2 and x into the first member, and the known terms into the second; reduce the terms which contain x^2 to one term, and those which contain x to another; also, reduce the known quantities in the second member to a single term.*

2. *If the term containing x^2 is not positive, make it so by changing the signs of all the terms.*

3. *If the coefficient of x^2 is not* 1, *divide all the terms by that coefficient.*

1. Let it be required to solve the following equation, viz.

$$\frac{6}{x+1}+\frac{2}{x}=3.$$

C earing the equation of fractions by multiplication

$$6x + 2x + 2 = 3x^2 + 3x.\quad \text{Transpose,}$$
$$-3x^2 - 3x + 6x + 2x = -2;\ \text{reduce}$$
$$-3x^2 + 5x = -2;\ \text{change the signs,}$$
$$3x^2 - 5x = 2;\ \text{divide by 3,}$$
$$x^2 - \tfrac{5}{3}x = \tfrac{2}{3}.$$

The equation is now reduced to the form of $x^2 + px = q$, and we have $p = -\frac{5}{3}$, and $q = \frac{2}{3}$.

Comparing the first member of $x^2 - \frac{5}{3}x = \frac{2}{3}$ with $x^2 - 2ax + a^2$, we have

$$x^2 = x^2;$$
$$-2ax = -\tfrac{5}{3}x;$$
$$-2a = -\tfrac{5}{3};$$
$$-a = -\tfrac{5}{6};$$
$$a^2 = \tfrac{25}{36}.$$

Adding $\frac{25}{36}$ to each member of $x^2 - \frac{5}{3}x = \frac{2}{3}$, we have

$$x^2 - \tfrac{5}{3}x + \tfrac{25}{36} = \tfrac{2}{3} + \tfrac{25}{36};$$

reducing the second member to a single fraction,

$$x^2 - \tfrac{5}{3}x + \tfrac{25}{36} = \tfrac{49}{36}.$$

Extracting the root of each member, we have for the root of the first $x - \frac{5}{6}$, for $(x - \frac{5}{6})(x - \frac{5}{6}) = x^2 - \frac{5}{3}x + \frac{25}{36}$, and for the root of the second $\pm \frac{7}{6}$. Hence,

$$x - \tfrac{5}{6} = \pm \tfrac{7}{6}.\quad \text{Transpose } -\tfrac{5}{6},$$
$$x = \tfrac{5}{6} \pm \tfrac{7}{6}, \therefore$$
$$x = \tfrac{12}{6} = 2,\ \text{or}\ x = -\tfrac{2}{6} = -\tfrac{1}{3}.$$

2. Since every affected equation of the second degree may be reduced to the form of $x^2 + px = q$, let us solve this general equation. Comparing the first member of it with $x^2 + 2ax + a^2$, we have

$$x^2 = x^2;$$
$$2ax = px;$$
$$2a = p;$$
$$a = \tfrac{1}{2}p;$$
$$a^2 = \tfrac{1}{4}p^2.$$

Adding $\frac{1}{4}p^2$ to each member of $x^2 + px = q$, we have

$$x^2+px+\tfrac{1}{4}p^2=q+\tfrac{1}{4}p^2.$$

Extracting the root of the first member, and expressing that of the second, (Art. 86),

$$x+\tfrac{1}{2}p=\pm(q+\tfrac{1}{4}p^2)^{\frac{1}{2}};\text{ hence,}$$

$$x=-\tfrac{1}{2}p\pm(q+\tfrac{1}{4}p^2)^{\frac{1}{2}}.$$

Remark. It is evident that the root of the second member must be indicated merely, until definite values are assigned to p and q. The second root of a quantity is properly indicated by the exponent $\frac{1}{2}$. For the product of the second root of a quantity by itself, must produce that quantity; and, according to the rule for the exponents in multiplication (Art. 30), $a^{\frac{1}{2}}\,.\,a^{\frac{1}{2}}=a^1$ or a; therefore, $a^{\frac{1}{2}}$ is the second root of a. For a similar reason, $(q+\frac{1}{4}p^2)^{\frac{1}{2}}$ represents the second root of $q+\frac{1}{4}p^2$ The radical sign may be used by those who prefer it Thus, $\sqrt{q+\frac{1}{4}p^2}$ likewise denotes the root of $q+\frac{1}{4}p^2$.

ART. **93.** From the solution of the preceding genera equation, we derive a

RULE FOR SOLVING AFFECTED EQUATIONS OF THE SECOND DEGREE

1. *Reduce the equation to the form of* $x^2+px=q$.

2. *Make the first member a perfect second power, by adding to both members the second power of half the coefficient of* x *(or of the first power of the unknown quantity).*

3. *Extract the root of each member. The root of the first member will consist of two terms, the first of which is* x*, or the unknown quantity, and the second is half the coefficient previously found, having the same sign as that coefficient, and the root of the second member must have the double sign* $\pm$.

4. *Transpose the known term from the first to the second member and reduce, and the value of x will be found.*

Since p and q may be either positive or negative, the general equation admits of four forms, differing only with regard to the signs of p and q, that of $\frac{1}{4}p^2$ being always positive. These forms, and the consequent values of x, are as follows.

(1) $x^2+px=q$; whence, $x=-\frac{1}{2}p\pm(q+\frac{1}{4}p^2)^{\frac{1}{2}}$.

(2) $x^2-px=q$; whence, $x=+\frac{1}{2}p\pm(q+\frac{1}{4}p^2)^{\frac{1}{2}}$.

(3) $x^2+px=-q$; whence, $x=-\frac{1}{2}p\pm(\frac{1}{4}p^2-q)^{\frac{1}{2}}$.

(4) $x^2-px=-q$; whence, $x=+\frac{1}{2}p\pm(\frac{1}{4}p^2-q)^{\frac{1}{2}}$.

ART. **94.** 1. Solve the following equation; viz.

$$\frac{x}{x+1}+\frac{x+1}{x}=\frac{13}{6}.$$

Clear the equation of fractions, and reduce,

$$12x^2+12x+6=13x^2+13x;\ \text{transpose,}$$
$$12x^2+12x-13x^2-13x=-6;\ \text{reduce,}$$
$$-x^2-x=-6;\ \text{change the signs,}$$
$$x^2+x=6.$$

The equation is now in the form of $x^2+px=q$. The coefficient of x being 1, we are to add to each member the second power of $\frac{1}{2}$, which is $\frac{1}{4}$. We then have

$$x^2+x+\tfrac{1}{4}=6+\tfrac{1}{4}=\tfrac{25}{4}.\ \text{Extracting the root,}$$
$$x+\tfrac{1}{2}=\pm\tfrac{5}{2}\ \therefore$$
$$x=-\tfrac{1}{2}\pm\tfrac{5}{2}=2,\ \text{or}\ -3.$$

2. What number is that whose second power, increased by $3\frac{1}{4}$, is equal to 7 times that number?

Let $x=$ the number. Then,

$$x^2+3\tfrac{1}{4}=7x.\ \text{By transposition,}$$
$$x^2-7x=-3\tfrac{1}{4}.$$

In this equation it is not necessary to remove the denominator 4, the equation being now in the general form. Half of the coefficient of x is $-\frac{7}{2}$, the second power of which, added to both members, gives

$$x^2 - 7x + \tfrac{49}{4} = -3\tfrac{1}{4} + \tfrac{49}{4} = \tfrac{36}{4}. \text{ Taking the root,}$$

$$x - \tfrac{7}{2} = \pm \tfrac{6}{2} \therefore$$

$$x = \tfrac{7}{2} \pm \tfrac{6}{2} = 6\tfrac{1}{2}, \text{ or } \tfrac{1}{2}.$$

3. Separate the number 10 into two parts, such that their product shall be 30.

Let x be one part; then $10 - x$ will be the other. Hence,

$$10x - x^2 = 30. \text{ Changing the signs,}$$

$$x^2 - 10x = -30. \text{ Completing the second power,}$$

$$x^2 - 10x + 25 = -30 + 25 = -5. \text{ Taking the root,}$$

$$x - 5 = \pm\sqrt{-5} \therefore x = 5 \pm \sqrt{-5}.$$

Since the second root of a negative quantity is *imaginary*, this problem is impossible; indeed, the greatest quantity that can result from the product of the two parts of a number, is $\frac{1}{4}$ of the second power of that number.

4. The length of a rectangular field exceeds its breadth by 8 rods, and the field contains 180 square rods. Required the dimensions.

5. A draper sold a piece of cloth at \$4 less per yard than there were yards in the piece, and the price of the whole was \$60. Required the number of yards and the price per yard.

6. There are two square rooms, the side of one being 1 yard longer than the side of the other. To carpet both rooms requires 85 yards of carpeting. How long is one side of each room?

7. What number is such, that if you subtract it from 10, and multiply the remainder by the number itself, the remainder shall be 21?

8. A farmer bought a certain number of sheep for \$80; but if he had bought 5 more for the same sum, the

price of one sheep would have been $\frac{4}{5}$ of a dollar less. How many sheep did he purchase? 20

9. What two numbers are those whose sum is 29 and whose product is 100? 25 - 4

10. A man built two pieces of wall, one of which was 2 rods longer than the other. He received as many shillings per rod for each as there were rods in the longer, and the whole amounted to $44. Required the length of each piece.

11. What number is that to which if 3 times its second root be added, the sum will be 40?

Let $x^2 =$ the number; then, $x^2 + 3x = 40$.

12. There are two numbers, such, that if the less be subtracted from 3 times the greater, the remainder will be 35; and if 4 times the greater be divided by 3 times the less, plus 1, the quotient will be the less number. Required the numbers.

The solution of this question requires two unknown quantities.

Let $x =$ the greater, and $y =$ the less. Then,

$$\left.\begin{array}{ll}(1) & 3x - y = 35; \\ (2) & \dfrac{4x}{3y+1} = y.\end{array}\right\}$$

Remove the denominator in 2d,

$$(3) \quad 4x = 3y^2 + y \therefore$$

$$(4) \quad x = \frac{3y^2 + y}{4}.$$

From the 1st we have

$$(5) \quad x = \frac{35 + y}{3}.$$

Making an equation with the values of x in the 4th and 5th,

$$\frac{3y^2 + y}{4} = \frac{35 + y}{3}.$$

Solving this equation, we have, successively,

$$9y^2 + 3y = 140 + 4y;$$
$$9y^2 - y = 140;$$
$$y^2 - \tfrac{1}{9}y = \tfrac{140}{9};$$
$$y^2 - \tfrac{1}{9}y + \tfrac{1}{324} = \tfrac{140}{9} + \tfrac{1}{324} = \tfrac{5041}{324};$$
$$y - \tfrac{1}{18} = \pm\tfrac{71}{18};$$
$$y = \tfrac{1}{18} \pm \tfrac{71}{18} = 4, \text{ or } -3\tfrac{8}{9}.$$

Taking 4 as the value of y, and substituting it in 5th.

$$x = \frac{35 + 4}{3} = 13.$$

13. Separate 100 into two such parts, that the sum of their second roots shall be 14.

Let x^2 and y^2 be the parts.

14. The sum of two numbers is 24, and the sum of their second powers is 306. Required the numbers.

15. The greater of two numbers divided by the second power of the less, gives 2 for a quotient, and $\frac{1}{3}$ of the difference of the numbers is 5. What are these numbers?

16. The less of two numbers added to twice the greater, makes 22, and half of their product, increased by the second power of the less, makes 60. Required the numbers.

17. Two numbers are such, that twice the second power of the greater and 3 times the second power of the less make 147, and 3 times the greater, diminished by twice the less, make 8. What are the numbers?

SECTION XXXI.

EXTRACTION OF THE THIRD ROOTS OF NUMBERS.

ART. **95.** The product of a number multiplied twice by itself, is called the *third power* or *cube* of that number. Thus, $27 = 3 . 3 . 3$ is the third power of 3, and $a^3 = a . a . a$ is the third power of a.

The first power of a quantity, with reference to the third, is called the *third root* or *cube root*. For example, 3 is the third root of 27, and a is the third root of a^3; and the process of deducing the first power from the third, is called *extracting the third* or *cube root*.

The third powers of the first nine integral numbers are as follows.

Roots.	1,	2,	3,	4,	5,	6,	7,	8,	9.
Third powers.	1,	8,	27,	64,	125,	216,	343,	512,	729.

The second line contains the third powers, of which the corresponding numbers in the first line are the third roots. We perceive, moreover, that, of integral numbers consisting of one, two, or three figures, there are only nine which are perfect third powers. The roots of such numbers as are not exact third powers cannot be obtained exactly, although they may, as we shall see hereafter, be approximated to any degree of accuracy. Thus, the third root of 29 is between 3 and 4, 3 being nearer than 4 to the true root.

The third power of a number having zeros on the right, contains three times as many zeros as that number. Thus, $(10)^3 = 1000$, $(100)^3 = 1000000$, $(1000)^3 = 1000000000$, &c. Consequently, the third power of a number between 10 and 100, that is, of a number containing two figures, must be greater than 1000, and less than 1000000; in other words, it cannot consist of less than four nor of more than six figures. Also, the third power of a number between 100 and 1000, that is, of a number consisting of three figures, must be greater than 1000000 and less than 1000000000; in other words, it cannot consist of less than seven nor more than nine figures. In like manner, when the root contains four figures, the power must contain either ten, eleven, or twelve figures.

Hence, we may readily ascertain the number of figures in the root of any number, by commencing at the right and separating the number into periods of three figures each. The left-hand period may consist of one, two, or three figures. There will be as many figures in the root as there are periods. This separation may be denoted by accents or other marks.

If a number is a perfect third power, and contains no more than three figures, its root may be found by inspection or trial. When the number consists of more than three figures, its root must also be found by trial, but a rule may be obtained which will greatly facilitate the operation.

Art. **96.** We now proceed to investigate the mode of extracting the root of a number consisting of more than three figures, that of 17576, for example, which is the third power of 26.

Let a represent the tens, and b the units of the root; then $a+b=20+6=26$. The third power of $a+b$ is $a^3+3a^2b+3ab^2+b^3$. Putting 20 instead of a, and 6 instead of b, we have

$$a^3=(20)^3=8000,$$

$$3a^2b=3.(20)^2.6=7200,$$

$$3ab^2 = 3 \,.\, 20 \,.\, 6^2 = 2160,$$
$$b^3 = \quad 6^3 \quad = 216.$$

Therefore, $a^3 + 3a^2b + 3ab^2 + b^3 = 8000 + 7200 + 2160 + 216 = 17576$. Hence,

The third power of a number consisting of tens and units, contains the third power of the tens, plus three times the second power of the tens into the units, plus three times the tens into the second power of the units, plus the third power of the units.

Now, supposing the root of 17576 unknown, let us trace the process of finding it.

Operation.

```
          17'576 ( 26.   Root.
           8            = a³.
           ----
           9576 ( 12    = 3 a².
(26)³ =   17576
          -----
              0.
```

Separating the given number into periods, we perceive that the root must consist of two figures, tens and units. As the third power of tens has three zeros at the right, it can have no significant figure below thousands. The third power of the tens sought, must therefore be found in the 17, (17000). The third power of 2 (tens) or 20 is 8, (8000), and the third power of 3 (tens) or 30 is 27, (27000). Hence, the greatest third power of tens contained in 17, (17000) is 8, (8000), the root of which is 2 (tens) or 20. We put 2 (tens) in the root, at the right of the given number, and subtract its third power 8, (8000), from 17576. The remainder is 9576.

As we have subtracted the part corresponding to a^3, the remainder corresponds to $3a^2b + 3ab^2 + b^3$. Our next step is to find the units of the root, corresponding to b

If our remainder contained $3a^2b$ only, or exactly three times the second power of the tens into the units, we should evidently find b or the units, by dividing this remainder by $3a^2$, that is, by three times the second power of the tens already found; for $3a^2b$ divided by $3a^2$ gives b. Nevertheless, if we divide by three times the second power of the tens, neglecting the remainder, we shall find the units exactly, or a number a little greater than the units.

Three times the second power of 2 (tens) is 12,(1200), which we place, as a divisor, at the right of 9576, considered as a dividend. The number 1200 is contained 7 times in 9576, or, what is the same thing, 12 is contained 7 times in 95. We put 7 in the root, at the right of the 2, and raise 27 to the third power. But the third power of 27 is greater than 17576; therefore, 7 is too great for the unit figure of the root. We next try 6, and find $(26)^3 = 17576$; therefore, 26 is the true root.

Let the learner extract the third roots of the following numbers, by a process similar to the preceding.

1. 5832.	3. 13824.	5. 50653.
2. 3375.	4. 32768.	6. 91125.

ART. **97.** Let us now extract the third root of a number consisting of more than six figures; for example, that of 14706125.

Operation.

14'706'125 (245. Root.
8 . . . . . .
1st Dividend = 67 . . . . . (12 . . . . = 1st Divisor.
$(24)^3 = 13824$ . . .
2d Dividend = 8821 . . (1728 . . = 2d Divisor.
$(245)^3 = 14706125$
0.

Remark. The points are used in the preceding operation, to show the rank of the figures placed before them. Zeros properly occupy the place of these points. But no marks need be used, provided the figures are placed according to their proper rank.

Having separated the number into periods, we see that the root must consist of three figures, viz., hundreds, tens, and units. Let a represent the hundreds, b the tens, and c the units, of the root.

By multiplication we find $(a+b+c)^3 = a^3 + 3\,a^2\,b + 3\,a\,b^2 + b^3 + 3\,a^2\,c + 6\,a\,b\,c + 3\,b^2\,c + 3\,a\,c^2 + 3\,b\,c^2 + c^3$. This may be put into the following form,

$$a^3 + 3\,a^2\,b + 3\,a\,b^2 + b^3 + 3\,(a+b)^2\,c + 3\,(a+b)\,c^2 + c^3; \text{ for,}$$

$$3\,a^2\,c + 6\,a\,b\,c + 3\,b^2\,c = 3\,(a^2 + 2\,a\,b + b^2)\,c = 3\,(a+b)^2\,c \text{ (Art. 59);}$$

$$\text{and } 3\,a\,c^2 + 3\,b\,c^2 = 3\,(a+b)\,c^2.$$

Our first step is to find the hundreds of the root, the third power of which is terminated by six zeros, and must therefore be found in the 14 (millions). The greatest third power of hundreds found in 14 (millions), is 8 (millions), the root of which is 2 (hundreds). We therefore place 2, corresponding to a, as the first figure or hundreds of the root, and subtract 8 (millions), corresponding to a^3, from the given number. The remainder 6706125 corresponds to $3\,a^2\,b + 3\,a\,b^2 + b^3$, &c.

Our next step is to find b, the tens of the root. The formula $3\,a^2\,b + 3\,a\,b^2 + b^3$, &c., shows that by dividing by $3\,a^2$, that is, by three times the second power of the hundreds of the root, we shall obtain either the tens of the root, or a number somewhat too great.

But three times the second power of hundreds multiplied by tens, must be terminated by five zeros, and can, herefore, contain no significant figure below the sixth place from the right. Consequently, it is sufficient to subtract the 8 (millions) from 14 (millions), the first period, and to bring down, at the right of the remainder, the sixth figure, that is, the first figure of the next period, in order to form a dividend.

Dividing this dividend 67, (6700000), by 12, (120000), three times the second power of 2 (hundreds), we have 5 for a quotient. But if we put 5, as the tens of the root, at the right of the 2 (hundreds), and raise 25 (tens) to the third power, the result will be too great. We next try 4, and find the third power of 24 (tens) to be 13824 (thousands), corresponding to $a^3 + 3a^2b + 3ab^2 + b^3$, which we subtract from the given number, and have for a remainder 882125.

This remainder corresponds to $3(a+b)^2c + 3(a+b)c^2 + c^3$; and in order to find c, the units of the root, we must evidently divide by $3(a+b)^2$, that is, by three times the second power of 24 (tens), already found, which is 1728 (hundreds). But three times the second power of tens into units, must be terminated by two zeros, and can, therefore, contain no significant figure below the third place from the right. It is sufficient, therefore, to subtract the third power of 24 (tens) from the first two periods, and to bring down, at the right of the remainder, the first figure of the next period to form a dividend.

This dividend 8821 (hundreds) divided by 1728 (hundreds), three times the second power of 24 (tens), gives for a quotient 5, which we place as the third or unit figure of the root. Raising 245 to the third power, we have 14706125, which shows that 245 is the true root.

ART. **98.** Hitherto, after having found a new figure of the root, we have raised the whole root, so far as ascertained to the third power, and subtracted the result from as many of the left-hand periods, as there were figures already found in the root. But, regard being paid to the local value of the figures, the process of extracting the root may be considerably shortened. To show the manner in which this is effected, we shall again extract the root of 14706125.

Operation

14'706'125 (245. Root.
8 . . . . . . $= a^3$.
67'06 . . (12 . . . $= 3a^2$.
24 . . . $= 3ab$.
16 . . $= b^2$.
1456 . . $= 3a^2 + 3ab + b^2$.
4 . $= b$.
5824 . . . $= 3a^2b + 3ab^2 + b^3$.
8821'25 (1728 . . $= 3(a + b)^2$.
360 . $= 3(a + b)c$.
25 $= c^2$.
176425 $= 3(a + b)^2 + 3(a + b)c + c^2$.
5 $= c$.
882125 $= 3(a + b)^2c + 3(a + b)c^2 + c^3$
0.

The process is the same as in the preceding Article, until we have found the second figure of the root; except that we bring down the whole of the second period, at the right of the first remainder, separating the two right-hand figures by an accent. Previously to finding b, the tens of the root, we subtracted a^3 from the first period, and our remainder with the succeeding period annexed, 6706, contains $b(3a^2 + 3ab + b^2)$, together with the thousands arising from the rest of the formula. Having found $b =$ 4 (tens), we are to ascertain the value of $b(3a^2 + 3ab$

$+ b^2$), and subtract it from the dividend, including the two figures cut off.

Our divisor is $3\,a^2 = 12, (120000)$, and $3\,a\,b = 3 \,.\, 2 \,.\, 4$, or rather $3 \,.\, 200 \,.\, 40 = 24000$, is three times the product of the figure last found by the preceding figure of the root. But since b is of the order of units next below a, the number corresponding to $3\,a\,b$ will contain a significant figure one degree lower than is found in $3\,a^2$. Therefore, $24, = 3\,a\,b$, is to be put under $12, = 3\,a^2$, one place farther to the right. We next find $b^2, = (4)^2 = 16$, or rather $(40)^2 = 1600$, and since it contains a significant figure one degree lower than $3\,a\,b$, we put 16 under $24, = 3\,a\,b$, one place farther to the right than this last.

Adding these three numbers, as the figures now stand, we have 1456 (hundreds), $= 3\,a^2 + 3\,a\,b + b^2$, which we multiply by 4 (tens), $= b$, and obtain 5824 (thousands), $= 3\,a^2\,b + 3\,a\,b^2 + b^3$. We subtract this product, 5824, from the dividend, including the two figures rejected in the division, bring down the next period to the right of the remainder, and have $882125 = 3\,(a + b)^2\,c + 3\,(a + b)\,c^2 + c^3 = [3\,(a + b)^2 + 3\,(a + b)\,c + c^2]\,c$.

Rejecting the two right-hand figures of 882125, we take the rest as a dividend, and divide by 1728 (hundreds), $= 3\,(a + b)^2$, that is, by three times the second power of the hundreds and tens of the root. The division gives $5, = c$, which we place as the units of the root. We now wish to find the value of $[3\,(a + b)^2 + 3\,(a + b)\,c + c^2]\,c$, and subtract it from 882125.

Our divisor is $3\,(a + b)^2, = 1728$ (hundreds), and $3\,(a + b)\,c = 3 \,.\, 24 \,.\, 5 = 360$, or rather $3 \,.\, 240 \,.\, 5 = 3600$, is the product of three times the figure last found by the preceding figures of the root; and as this product would, if the last figure of it did not happen to be zero, contain a significant figure one degree below the value of

$3(a+b)^2$, we put 360 under 1728, one place farther to the right. We now put $25, = c^2$, which contains a significant figure still one degree lower, under 360, one place still farther to the right.

Adding these numbers, as the figures now stand, we have $176425 = 3(a+b)^2 + 3(a+b)c + c^2$, which we multiply by $5, = c$, and have $882125 = 3(a+b)^2 c + 3(a+b)c^2 + c^3$. This subtracted from the last dividend, including the rejected figures, leaves no remainder. Hence, the work is complete.

ART. **99.** From the preceding analysis we deduce the following

RULE FOR EXTRACTING THE THIRD ROOTS OF NUMBERS.

1. *Commencing at the right, separate the number, by means of accents, into periods of three figures each; the left-hand period may contain one, two, or three figures.*

2. *Find the greatest third power in the left-hand period, place its root at the right, and subtract the power from that period.*

3. *To the right of the remainder bring down the next period, separating the two right-hand figures by an accent; those to the left of the accent will form a dividend. For a divisor take three times the second power of that part of the root already found. Divide the dividend by the divisor, and put the quotient as the second figure of the root.*

4. *Take three times the product of the figure last found by the preceding part of the root, and place it under the divisor, one place farther to the right; under which, one place farther to the right, place the second power of the figure of the root last found. Add together*

the divisor and the numbers placed under it, as the figures stand, and multiply the sum by the figure of the root last found. Subtract this product from the dividend, including the two rejected figures.

5. *To the right of the remainder bring down the next period, forming a new dividend, in the same manner as the first was formed. Take for a divisor three times the second power of the whole root so far as found; divide and place the quotient as the next figure of the root.*

6. *Find three times the product of the last figure by the whole of the preceding part of the root, and put it under the divisor, one place farther to the right; under this, one place farther to the right, put the second power of the last figure of the root found. Add the divisor and the numbers placed under it, as the figures stand, multiply the sum by the last figure of the root found, and subtract the product from the dividend, including the rejected figures.*

7. *Repeat the operations stated in the 5th and 6th parts of the rule, until the given number is exhausted.*

Remark 1*st.* Whenever the divisor is not contained in the dividend, or the figures to the left of the two rejected, put a zero in the root, and bring down the next period, separating the two right-hand figures; the divisor for finding the next figure of the root will then be the same as before, except with the annexation of two zeros.

Remark 2*d.* Whenever the number to be subtracted exceeds that from which it is to be taken, diminish the last figure found in the root, until a number is obtained which can be subtracted.

1. What is the third root of 525557943?

Operation.

```
525′557′943 ( 807.   Root.
512
 135′57 ( 192
 135579′43 ( 19200
               1680
                 49
            1936849
                  7
 13557943
     0.
```

Extract the third roots of the following numbers

2. 1815848.
3. 3652264.
4. 21024576.
5. 35937.
6. 18609625.
7. 66430125.
8. 147197952.
9. 167284151.
10. 491169069.
11. 1967221277.

SECTION XXXII.

THIRD ROOTS OF FRACTIONS — AND THE EXTRACTION OF THIRD ROOTS BY APPROXIMATION.

Art. **100.** Since fractions are multiplied together by taking the product of their numerators for a new numerator, and that of their denominators for a new denominator, the third power of a fraction is found by raising both numerator and denominator to the third power. Thus, $\left(\frac{3}{7}\right)^3 = \frac{27}{343}$, and $\left(\frac{a}{b}\right)^3 = \frac{a^3}{b^3}$.

Conversely, the third root of a fraction is found by extracting the third root of both numerator and denominator. For example, the third root of $\frac{8}{27}$ is $\frac{2}{3}$, that of $\frac{m^3}{n^3}$ is $\frac{m}{n}$.

Find the third roots of the following fractions.

1. $\frac{125}{1331}$.
2. $\frac{343}{1728}$.
3. $\frac{9261}{12167}$.
4. $\frac{3375}{4913}$.
5. $\frac{19683}{24389}$.
6. $1\frac{469}{1728}$.

Art. 101. But if either numerator or denominator is not an exact third power, we can obtain only an approximate root. We can, however, always render the denominator a perfect third power, without changing the value of the fraction. This is done by multiplying both numerator and denominator by the second power of the denominator. Thus, $\frac{3}{5}=\frac{3\,.\,5^2}{5\,.\,5^2}=\frac{3\,.\,25}{5^3}=\frac{75}{125}$; the nearest integral root of the numerator of which is 4, and the root of the denominator is 5. Therefore, $\frac{4}{5}+$ is the approximate root of $\frac{3}{5}$.

A nearer approximation may be made, if, after multiplying both numerator and denominator of the fraction by the second power of the denominator, we multiply both numerator and denominator of the result by any third power. Thus, after converting $\frac{3}{5}$ into $\frac{75}{5^3}$, we may multiply both numerator and denominator of $\frac{75}{5^3}$ by 8^3; this gives $\frac{38400}{5^3\,.\,8^3}$, the approximate root of which is $\frac{34}{5\,.\,8}-=\frac{34}{40}-$.

Art. 102. The root of a whole number, which is not a third power, may be approximated in a similar way, by first converting it into a fraction whose denominator is an exact third power. For example, $3 = \frac{3 . 5^3}{5^3} = \frac{375}{5^3}$, the approximate root of which is $\frac{7}{5}+$.

But the best mode of approximating the third root either of a whole number or of a fraction, is, to convert it into a fraction, whose denominator is the third power of 10, 100, or 1000, &c.; that is, convert it into 1000ths, 1000000ths, 1000000000ths, &c., and find the nearest root of the result. The root will then be found in decimals.

For example, $5 = \frac{5 . 10^3}{10^3} = \frac{5000}{1000}$, the root of which is $\frac{17}{10} + = 1{\cdot}7 +$. If a more accurate root is wanted, we may change 5 to 1000000ths; thus, $5 = \frac{5000000}{1000000}$, the root of which is $\frac{171}{100} - = 1{\cdot}71 -$.

Hence, it is evident that the denominator may be omitted, and that it is sufficient to annex three zeros to the number for every additional figure in the root. Nor is it necessary to write all the zeros at once, but we may annex three to the remainder, when an additional figure of the root is required, in the same manner as we bring down successive periods.

In like manner, to find the third root of a vulgar fraction, we change it to a decimal with thrice as many decimal figures as we want decimals in the root.

When the number whose root is sought contains whole numbers and decimals, and the number of decimal figures is not a multiple of three, make it so by annexing zeros; or, commencing at the decimal point, separate the whole numbers into periods by proceeding towards the left, and the decimals by proceeding towards the right, and then complete the right-hand period, if necessary, by annexing zeros.

These preparations being made, the root of a number containing decimals is found in the same way as that of an integral number, care being taken to point off one third as many decimals in the root as there are in the power, including the zeros annexed.

1 Extract the third root of 2, accurate to three decimal figures.

Operation.

```
2·  ( 1·259 +.
1
10′00 ( 3
          6
           4
        364
          2
  728
  2720′00 ( 432
               180
                 25
              45025
                  5
  225125
   468750′00 ( 46875
                   3375
                     81
                4721331
                      9
   42491979
    4383021.
```

Extract the third roots of the following numbers, accurate to two decimals.

2. 4.	6. 3·7.	10. $\frac{13}{15}$.
3 7.	7. 6·55.	11. $2\frac{3}{7}$.
4. 9.	8. 7·75.	12. $5\frac{3}{8}$.
5. 15.	9. $\frac{3}{4}$.	13. $9\frac{1}{2}$.

SECTION XXXIII.

QUESTIONS PRODUCING PURE EQUATIONS OF THE THIRD DEGREE.

Art. **103.** *An equation of the third degree* is such as, when reduced to its simplest form, contains at least one term in which there are three, but no term in which there are more than three, unknown factors.

When an equation with one unknown quantity contains the third power only of that quantity, it is called a *pure* equation of the third degree. Thus, $x^3 = 729$ is an equation of this kind.

1. In a package of cloth there are as many pieces as there are yards in each piece, and it is worth $\frac{1}{3}$ as many cents per yard as there are yards in a piece. Required the number of pieces and the price per yard, the whole being worth $90.

Let $x =$ the number of pieces, also the number of yards in a piece; then $\frac{x}{3} =$ the price per yard in cents. Hence,

$$x^2 = \text{the whole number of yards; and}$$

$$x^2 \cdot \frac{x}{3} = \frac{x^3}{3} = \text{the price of the whole in cents}$$

We have, therefore,

$$\frac{x^3}{3} = 9000, \therefore$$

$$x^3 = 27000,$$

$x = 30$ pieces, and there were 30 yards in a piece.

$\frac{x}{3} = 10$ cents, price per yard.

2. The length of a rectangular box is twice the breadth, and the depth is $\frac{4}{5}$ of the breadth. The box holds 200 cubic feet. Required the three dimensions.

Remark. The cubical contents of any rectangular space, or rectangular solid, are found by taking the product of the length, breadth, and depth.

3. A pile of wood is 27 feet long, 25 feet wide, and 5 feet high. If the same quantity of wood were in a cubical form, what would be the length of one side of the pile?

4. Two numbers are to each other as 4 to 5, and the sum of their third powers is 5103. Required the numbers.

5. What two numbers are such, that the second power of the greater multiplied by the less makes 75, and the second power of the less multiplied by the greater makes 45?

Let $x =$ the greater, and $y =$ the less. Then,

$$\left.\begin{aligned}(1)\quad x^2 y &= 75;\\ (2)\quad x y^2 &= 45.\end{aligned}\right\}$$

From the 1st

$$y = \frac{75}{x^2} \therefore y^2 = \frac{5625}{x^4}.$$

Substituting this value of y^2 in the 2d,

$$\frac{5625\,x}{x^4} = 45, \text{ or}$$

$$\frac{5625}{x^3} = 45. \quad \text{Hence,}$$

$$45\,x^3 = 5625;\ x^3 = 125, \therefore$$

$$x = 5, \text{ the greater, and}$$

$$y = \frac{75}{x^2} = \frac{75}{25} = 3, \text{ the less.}$$

6. The product of two numbers is 28, and 8 times the second power of the greater, divided by the less, gives 98 for a quotient. What are these numbers? 7 – 4

7. The sum of the third powers of two numbers is 2728, and the difference of those powers is 728. Required the numbers. 10 – 12.

8. The breadth of a piece of land is $\frac{1}{3}$ of its length, and it is worth $\frac{1}{5}$ as many dollars per square rod, as there are rods in the breadth. The whole piece being worth $9375, what are the dimensions? 75 – 25

9. A gallon being 231 cubic inches, what is the length of one side of a cubical box holding 5 gallons? 10.4+

10. A bushel being $2150\frac{2}{5}$ cubic inches, required the side of a cubical vessel containing 7 bushels. 24.6+

SECTION XXXIV.

POWERS OF MONOMIALS.

ART. **104.** Any power of a quantity is found by multiplying that quantity by itself as many times, *less one*, as there are units in the exponent of the power. The second power of a or a^1 is $a \cdot a = a^{1+1} = a^2$, (Art. 30); this is the same as $a^{1 \times 2}$.

The third power of a^2 is $a^2 \cdot a^2 \cdot a^2 = a^{2+2+2} = a^6$; this is the same as $a^{2 \times 3}$.

The second power of $a^2 b^3$ is $a^2 b^3 \cdot a^2 b^3 = a^{2+2} b^{3+3} = a^4 b^6$; this is the same as $a^{2 \times 2} b^{3 \times 2}$.

The third power of $3 m^2 n^3$ is $27 m^6 n^9$; this is the same as $27 m^{2 \times 3} n^{3 \times 3}$.

Thus we perceive that, in all these examples, we have multiplied the exponent of each letter by the exponent of the power to which the quantity was to be raised, and in the last example we actually raised the numerical coeffi cient to that power by multiplication. Hence we hav the following

RULE FOR RAISING A MONOMIAL TO ANY POWER.

Raise the numerical coefficient to the required power, and multiply the exponent of each letter by the number which marks the degree of that power.

It is manifest, moreover, that

Any power of a product is the product of that power of each of its factors. For example, the third power of $3\,a\,b\,c$ is $27\,a^3\,b^3\,c^3$, which is the product of the third powers of 3, a, b, and c.

From the rule for the signs in multiplication, it follows, that when the index of the power to which a quantity is to be raised is an even number, the power will always have the sign $+$; but when the index is an odd number, the power will have the same sign as the root. Thus the second, fourth, sixth, &c. powers of any quantity, whether positive or negative, will have the sign $+$; but the third, fifth, &c. powers of a negative quantity will have the sign $-$, while the same powers of a positive quantity have the sign $+$. For example, the second power of $+a$ is $+a^2$, and the second power of $-a$ is also $+a^2$; but the third power of a is $+a^3$, while the third power of $-a$ is $-a^3$.

1. Find the 2d power of $3\,x\,y^2$.
2. Find the 3d power of $2\,a^2\,b^3$.
3. Find the 2d power of $7\,a^2\,x\,y^2$.
4. Find the 5th power of $2\,a\,b^2\,c^3$.
5. Find the 10th power of $a\,x^2\,y^2$.

6. Find the mth power of $a^2 x$. Ans. $a^{2m} x^m$
7. Find the 2d power of $-7x^2y^3$.
8. Find the 3d power of $-3a^3y^4$.
9. Find the 5th power of $-2a^2m^3$.
10. Find the 3d power of $\frac{a^2x^4}{2b^3}$.
11. Find the 5th power of $-\frac{amx^2}{2bc^2y^3}$.
12. Find the 4th power of $-\frac{2x^2y}{3mn^3}$.
13. Find the 6th power of $-\frac{a^2m^3x}{b^3y^2}$.
14. Find the 3d power of a^m. Ans. a^{3m}.
15. Find the mth power of a^3b^2.

SECTION XXXV.

POWERS OF POLYNOMIALS.

ART. **105.** Any power of a polynomial is *indicated* by enclosing it in a parenthesis, or putting it under a vinculum, and, in both cases, putting the index of the power at the right. For example, $(a^2+2m)^3$, or $\overline{a^2+2m}^3$, represents the third power of a^2+2m.

Operations may be performed upon powers of polynomials thus represented, in the same manner as upon the powers of simple quantities. Thus, to raise $(x+2y)^2$ to the fourth power, we multiply the exponent by 4, and obtain $(x+2y)^8$. Also, when several quantities are represented as multiplied together, the whole is raised to any power, by raising each factor, whether monomial or polynomial, to the power required. For example, the third power of $(m-2n)(c^2+3d)^2$ is $(m-2n)^3(c^2+3d)^6$; and the fourth power of $3x(m^2-n^2)^3(x+y)^4$ is $81x^4(m^2-n^2)^{12}(x+y)^{16}$.

Note. The learner must be careful to distinguish factors from terms. In the quantity $3m(x-y)^2(a^2+b^2)^3$, the different factors are 3, m, $x-y$, and a^2+b^2.

Indicate the specified powers of the following quantities.

1. The 2d power of $x+2y$.
2. The 3d power of $(3m+n^2)^2$.
3. The 5th power of $(x+2b)^3$.
4. The 4th power of $a(b+c)^5$.
5. The 2d power of $5x^2(m-n)^7$.
6. The 2d power of $(a+b)^m$.
7. The mth power of $(x-y)^3$.
8. The 6th power of $(a-b)^2(m^2+n^2)$.
9. The 2d power of $3a^2(x-2y)^3(m+n)^4$.
10. The 3d power of $\frac{a+b}{m-n}$. Ans. $\left(\frac{a+b}{m-n}\right)^3$, or $\frac{(a+b)^3}{(m-n)^3}$.
11. The 4th power of $\frac{a^2+x}{3a+4b}$.
12. The 3d power of $\frac{(a-b)^2}{(x+y)^3}$.
13. The 2d power of $\frac{5(x+y)^4(m-n)}{(3x+y)^3}$.

ART **106.** If the powers of polynomials are required in a developed form, they may be found by multiplication. Thus, $(3x+y)^3=(3x+y)(3x+y)(3x+y)=27x^3+27x^2y+9xy^2+y^3$.

But the development of powers of polynomials by multiplication, when the powers are of a high degree, becomes very tedious. A mode has however been discovered, by which any power of a binomial may be developed with great facility. The principle of this method is called the Binomial Theorem, and was first discovered by Sir Isaac Newton. It is particularly adapted to binomials, but may be extended to quantities consisting of more than two terms.

Let us form some of the powers of the binomial $x+a$ by actual multiplication.

$$\begin{aligned}
(x+a)^1 &= x+a.\\
& x+a\\
& x^2+xa\\
& \quad +xa+a^2\\
(x+a)^2 &= x^2+2xa+a^2.\\
& x+a\\
& x^3+2x^2a+xa^2\\
& \quad +x^2a+2xa^2+a^3\\
(x+a)^3 &= x^3+3x^2a+3xa^2+a^3.\\
& x+a\\
& x^4+3x^3a+3x^2a^2+xa^3\\
& \quad +x^3a+3x^2a^2+3xa^3+a^4\\
(x+a)^4 &= x^4+4x^3a+6x^2a^2+4xa^3+a^4.\\
& x+a\\
& x^5+4x^4a+6x^3a^2+4x^2a^3+xa^4\\
& \quad +x^4a+4x^3a^2+6x^2a^3+4xa^4+a^5\\
(x+a)^5 &= x^5+5x^4a+10x^3a^2+10x^2a^3+5xa^4+a^5.
\end{aligned}$$

The developments of the powers of $x-a$ will be the same as those of $x+a$, except that the 2d, 4th, 6th, &c terms will have the sign $-$; that is, all the terms in which an odd power of the negative term enters, have the sign $-$, all the others having the sign $+$. This necessarily follows from the rule for the signs in multiplication; for, when the number of negative factors is even, the product has the sign $+$, but when the number of negative factors is odd, the product has the sign $-$.

We shall, therefore, obtain by multiplication the first five powers of $x - a$, as follows, viz.

$(x - a)^1 = x - a.$

$(x - a)^2 = x^2 - 2\,x\,a + a^2.$

$(x - a)^3 = x^3 - 3\,x^2\,a + 3\,x\,a^2 - a^3.$

$(x - a)^4 = x^4 - 4\,x^3\,a + 6\,x^2\,a^2 - 4\,x\,a^3 + a^4.$

$(x - a)^5 = x^5 - 5\,x^4\,a + 10\,x^3\,a^2 - 10\,x^2\,a^3 + 5\,x\,a^4 - a^5.$

ART. **107.** From an examination of the preceding developments, we shall be able to deduce the law, with regard to the letters, exponents, and coefficients.

1. We perceive that x, the first or *leading* term of the binomial, is found in every term of the development except the last; and that a, the second term of the binomial, is found in every term of the development except the first.

2. The exponent of x, or of the leading term, is, in the first term of the development, equal to the index of the power to which the binomial is raised, and goes on decreasing by unity in the succeeding terms.

The exponent of a is 1 in the second term of the development, and goes on increasing by unity in the succeeding terms, until in the last term it has the same exponent as x in the first. Thus the terms of the 5th power of $x + a$, without their coefficients, are x^5, $x^4\,a$, $x^3\,a^2$, $x^2\,a^3$, $x\,a^4$, a^5.

3. The coefficient of the first, as well as that of the last term of the development, is always 1.

The coefficient of the second term is always the same as the index of the power to which the binomial is raised. Thus the coefficient of the second term of the development of $(x + a)^3$ is 3; that of the second term of the development of $(x + a)^4$ is 4, &c.

The coefficient of the third term is found by multiplying the coefficient of the second term by the exponent of

x in the same term, and dividing the product by 2. The coefficient of the fourth term is found by multiplying that of the third term by the exponent of x in the same term, and dividing the product by 3. For example, the second term of the development of $(x+a)^4$ being $4x^3a$, the coefficient of the third term is $\frac{4.3}{2}=6$; annexing the letters with their proper exponents, we have $6x^2a^2$ for the third term. In like manner, $6x^2a^2$ being the third term, the coefficient of the fourth term is $\frac{6.2}{3}=4$; and we have for the fourth term $4x^3a$.

Thus, if we multiply the coefficient of any term by the exponent of x in the same term, and divide the product by the number marking the place of that term from the first inclusive, the result will be the coefficient of the succeeding term.

Art. **108.** Hence, having one term of any power of a binomial, the succeeding term may be found by the following

RULE.

Multiply the given term by the exponent of x in that term, that is, by the exponent of the leading quantity of the binomial, and divide the product by the number which marks the place of the given term from the first inclusive; diminish the exponent of x by 1, *and increase that of a by* 1.

This rule, which admits of a rigorous demonstration, enables us to develop any power of a binomial.

Let it be required to develop $(x+a)^9$.

We know, from what has been said, that the first term is x^9, and that the second is $9x^8a$. The succeeding terms may be found by the rule, and the process of finding them is exhibited below.

The 1st term is x^9,

2d " " $9\,x^8\,a$,

3d " " $\frac{8.9}{2}x^7\,a^2 = 36\,x^7\,a^2$,

4th " " $\frac{7.36}{3}x^6\,a^3 = 84\,x^6\,a^3$,

5th " " $\frac{6.84}{4}x^5\,a^4 = 126\,x^5\,a^4$,

6th " " $\frac{5.126}{5}x^4\,a^5 = 126\,x^4\,a^5$,

7th " " $\frac{4.126}{6}x^3\,a^6 = 84\,x^3\,a^6$,

8th " " $\frac{3.84}{7}x^2\,a^7 = 36\,x^2\,a^7$,

9th " " $\frac{2.36}{8}x\,a^8 = 9\,x\,a^8$,

10th " " $\frac{1.9}{9}a^9 = a^9$.

Hence,

$$(x+a)^9 = x^9 + 9\,x^8\,a + 36\,x^7\,a^2 + 84\,x^6\,a^3 + 126\,x^5\,a^4 + 126\,x^4\,a^5 + 84\,x^3\,a^6 + 36\,x^2\,a^7 + 9\,x\,a^8 + a^9.$$

Remembering that odd powers of negative quantities have the sign —, we shall find, by means of the rule, that the seventh power of $x - a$ is as follows.

$$(x-a)^7 = x^7 - 7\,x^6\,a + 21\,x^5\,a^2 - 35\,x^4\,a^3 + 35\,x^3\,a^4 - 21\,x^2\,a^5 + 7\,x\,a^6 - a^7.$$

ART. **109.** The labor of developing any power of a binomial may be facilitated by attention to the following principles.

From the preceding examples and the table of powers given in Article 106, we see,

1. That the number of terms in the development of

any power of a binomial, exceeds by 1 the index of that power. Thus the development of $(x+a)^6$ has 7 terms; that of $(x-a)^9$ has 10 terms.

2. If the number of terms in the development is odd, there is one coefficient, in the middle of the series, greater than any of the others; but if the number of terms is even, there are two coefficients, in the middle, of equal value and greater than any of the others. Moreover, those coefficients which precede and those which follow the greatest or greatest two, are the same, but are arranged in an inverse order.

Hence, after half or one more than half of the successive coefficients have been found, the rest may be written down without the trouble of calculation.

Develop the following quantities.

1. $(a+b)^6$.
2. $(x+y)^{10}$.
3. $(m-n)^9$.
4. $(m+n)^{11}$.
5. $(x-y)^{12}$.
6. $(x+2y)^5$.

In the 6th example, we must raise the numerical coefficient of y to the requisite powers. We first write the development, merely indicating the powers of $2y$. Thus,

$$(x+2y)^5 = x^5 + 5x^4 . 2y + 10x^3(2y)^2 + 10x^2(2y)^3 + 5x(2y)^4 + (2y)^5.$$

Now, raising $2y$ to the powers indicated, and putting the results instead of $2y$, $(2y)^2$, &c., we have

$$x^5 + 10x^4y + 40x^3y^2 + 80x^2y^3 + 80xy^4 + 32y^5.$$

7. $(5x+a)^4$.
8. $(a^2+x^2)^5$.

To develop the 8th, we first indicate the powers of a^2 and x^2. Thus,

$$(a^2+x^2)^5 = (a^2)^5 + 5(a^2)^4x^2 + 10(a^2)^3(x^2)^2 + 10(a^2)^2(x^2)^3 + 5(a^2)(x^2)^4 + (x^2)^5.$$

Note In this example, a^2 is the leading term, and the exponents of a^2 outside of the parentheses are to be used in finding the coefficients.

Now, raising a^2 and x^2 to the powers indicated, and substituting the results, we have

$$(a^2+x^2)^5 = a^{10} + 5\,a^8\,x^2 + 10\,a^6\,x^4 + 10\,a^4\,x^6 + 5\,a^2\,x^8 + x^{10}.$$

As an example in which the binomial theorem may be used to find any power of a quantity consisting of more than two terms, we shall develop $(a+b+c)^4$.

Substitute any single letter, x, for instance, instead of $b+c$; then $a+b+c$ becomes $a+x$.

Now, $(a+x)^4 = a^4 + 4\,a^3\,x + 6\,a^2\,x^2 + 4\,a\,x^3 + x^4$.

But $x = b+c$, $\therefore$

$$x^2 = (b+c)^2 = b^2 + 2\,b\,c + c^2;$$
$$x^3 = (b+c)^3 = b^3 + 3\,b^2\,c + 3\,b\,c^2 + c^3; \text{ and}$$
$$x^4 = (b+c)^4 = b^4 + 4\,b^3\,c + 6\,b^2\,c^2 + 4\,b\,c^3 + c^4.$$

Putting these values instead of x, x^2, &c., we have

$$a^4 + 4\,a^3\,(b+c) + 6\,a^2\,(b^2 + 2\,b\,c + c^2) + 4\,a\,(b^3 + 3\,b^2\,c + 3\,b\,c^2 + c^3) + b^4 + 4\,b^3\,c + 6\,b^2\,c^2 + 4\,b\,c^3 + c^4.$$

Lastly, performing the multiplication indicated, we have

$$(a+b+c)^4 = a^4 + 4\,a^3\,b + 4\,a^3\,c + 6\,a^2\,b^2 + 12\,a^2\,b\,c + 6\,a^2\,c^2 + 4\,a\,b^3 + 12\,a\,b^2\,c + 12\,a\,b\,c^2 + 4\,a\,c^3 + b^4 + 4\,b^3\,c + 6\,b^2\,c^2 + 4\,b\,c^3 + c^4.$$

SECTION XXXVI.

ROOTS OF MONOMIALS.

ART. **110.** From the manner in which a monomial is raised to any power, shown in Article 104, results the following

RULE FOR EXTRACTING THE ROOT OF ANY MONOMIAL.

Extract the root of the numerical coefficient, and divide the exponent of each literal factor by the number which marks the degree of the root.

The reason of this rule is manifest, as may be shown by an example. Thus, the second power of $3xy^2$ is $9x^{1\times 2}y^{2\times 2} = 9x^2y^4$; consequently, the second root of $9x^2y^4 = 3x^{\frac{2}{2}}y^{\frac{4}{2}} = 3xy^2$. In like manner, the third root of $27x^6y^9$ is $3x^{\frac{6}{3}}y^{\frac{9}{3}} = 3x^2y^3$.

It is to be remarked,

That every root of an even degree may have either the sign $+$ or $-$; but a root of an odd degree has the same sign as the power.

Thus, the 4th root of $+a^4$ may be either $+a$ or $-a$; because $(+a)^4 = +a^4$, and $(-a)^4 = +a^4$. Whereas the third root of $+a^3$ is $+a$, but the third root of $-a^3$ is $-a$; because $(+a)^3 = +a^3$, but $(-a)^3 = -a^3$.

We have already said, (Art. 86,) that the second root of a negative quantity is imaginary. The same may be said of any *even* root of a negative quantity. Thus, the fourth root of -81, and the sixth root of $-a$ are *imaginary quantities.*

1. Find the second root of a^2b^4.
2. Find the second root of $4x^2y^6$.
3. Find the second root of $9a^2x^4y^8$.
4. Find the third root of $a^3b^6c^9$.
5. Find the third root of $27x^3y^9z^{12}$.
6. Find the third root of $-125a^3x^6$.
7. Find the fourth root of $a^4b^8x^{12}$.
8. Find the fifth root of x^5y^{10}.
9. Find the second root of $\frac{9x^2}{16y^4}$.
10. Find the third root of $\frac{64x^3y^6}{125b^9}$.

Art. **111.** From the preceding examples, also from what was shown in Article 104, relative to any power of a product, we infer that

Any root of a product is the product of the roots, to the same degree, of each of the factors of that product.

For example, the third root of $27\,a^3\,b^6$ is $3\,a\,b^2$, which is the product of the third roots of 27, a^3, and b^6, the factors of $27\,a^3\,b^6$.

In a similar manner, if any numerical quantity is separated into factors that are exact powers of the required degree, which may always be done when the number itself is an exact power of that degree, we may extract separately the root of each factor, and afterwards multiply these roots together. Thus, $1296 = 9 \,.\, 144$, the second root of which is $3 \,.\, 12 = 36$.

Art. **112.** Since, in extracting the root of a monomial, we divide the exponent of each letter by the number expressing the degree of the root, it follows, that if any exponent is not divisible by that number, the division must be expressed, and this gives rise to fractional exponents For example, the third root of a is $a^{\frac{1}{3}}$, that of a^2 is $a^{\frac{2}{3}}$.

The expression $a^{\frac{2}{3}}$ represents either the third root of a^2, or the second power of $a^{\frac{1}{3}}$; for $(a^{\frac{1}{3}})^2 = a^{\frac{1}{3} \times 2} = a^{\frac{2}{3}}$. Also, $a^{\frac{3}{5}}$ denotes either the fifth root of a^3, or the third power of $a^{\frac{1}{5}}$.

The radical sign, as well as fractional exponents, may be used to indicate a root of any degree, provided we place over this sign a number expressing the degree of the root. Thus $\sqrt{\ }$, which is the same as $\sqrt[2]{\ }$, indicates the second root; $\sqrt[3]{\ }$, the third root; $\sqrt[4]{\ }$, the fourth root. Hence, we have the following equivalent expressions, viz.

$$\sqrt{m}=m^{\frac{1}{2}};\ \sqrt[3]{m^2}=m^{\frac{2}{3}};\ \sqrt[4]{m^3}=m^{\frac{3}{4}};\ \sqrt[6]{m^7}=m^{\frac{7}{6}},\ \&c.$$

We may, therefore, use indifferently either the radical sign or a fractional exponent, remembering that the number over the radical sign is the denominator, and that the exponent of the quantity under the sign is the numerator, of the fractional exponent.

SECTION XXXVII.

SECOND ROOTS OF POLYNOMIALS.

ART. **113.** It is required to extract the second root of $25x^2+60xy+36y^2$.

Operation.

$$\begin{array}{l} 25x^2+60xy+36y^2\ (\underline{5x+6y}.\quad \text{Root.} \\ \underline{25x^2\qquad\qquad\qquad\quad} \\ \qquad\quad 60xy+36y^2\ (\underline{10x+6y} \\ \qquad\quad \underline{60xy+36y^2} \\ \qquad\qquad\quad 0. \end{array}$$

By comparing this quantity with $a^2+2ab+b^2$, the second power of $a+b$, and recollecting the process of extracting the second roots of numbers, we shall readily see the mode of proceeding. The first term $25x^2$ corresponds to a^2; we therefore extract the root of $25x^2$, which is $5x$, (Art. 110,) place it at the right, and subtract its second power from the given quantity.

The remainder $60xy+36y^2$, which we regard as a dividend, corresponds to $2ab+b^2$, or $(2a+b)b$. Dividing the first term $60\,x\,y$ corresponding to $2\,a\,b$, by $10\,x$ corresponding to $2\,a$, we have $6\,y$ answering to b.

We now place $6\,y$, with its proper sign, in the root also at the right of our divisor, and have $10\,x + 6\,y$ answering to $2\,a + b$. Multiplying $10\,x + 6\,y$ by $6\,y$, we obtain $60\,x\,y + 36\,y^2$ corresponding to $(2\,a + b)\,b$. Subtracting this product from the dividend, we have no remainder. Consequently, $5\,x + 6\,y$ is the required root.

When a quantity consists of more than three dissimilar terms, its second root will consist of more than two terms. But the process of finding the second root of a polynomial is, in all cases, so similar to that of extracting the root of a number, that it hardly needs a separate explanation The following example will serve as an illustration.

What is the second root of $9\,a^4 - 24\,a^3\,b + 22\,a^2\,b^2 - 8\,a\,b^3 + b^4$?

Operation.

$$
\begin{array}{l}
9\,a^4 - 24\,a^3\,b + 22\,a^2\,b^2 - 8\,a\,b^3 + b^4\ (\,\underline{3\,a^2 - 4\,a\,b + b^2.}\\
\underline{9\,a^4 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}\\
\quad - 24\,a^3\,b + 22\,a^2\,b^2 - 8\,a\,b^3 + b^4\ (\,\underline{6\,a^2 - 4\,a\,b}\\
\quad \underline{- 24\,a^3\,b + 16\,a^2\,b^2 \qquad\qquad\qquad}\\
\qquad\qquad\qquad 6\,a^2\,b^2 - 8\,a\,b^3 + b^4\ (\,\underline{6\,a^2 - 8\,a\,b + b^2}\\
\qquad\qquad\qquad \underline{6\,a^2\,b^2 - 8\,a\,b^3 + b^4}\\
\qquad\qquad\qquad\qquad\quad 0.
\end{array}
$$

The process of finding the first two terms of the root is precisely the same as in the first example of this article.

Having obtained the second dividend, $6\,a^2\,b^2 - 8\,a\,b^3 + b^4$, we double the first two terms of the root, and have for a second divisor $6\,a^2 - 8\,a\,b$.

Performing the division, we obtain b^2 for the third term of the root, which we annex, with its proper sign, both to the preceding part of the root and to the divisor. Our divisor then becomes $6\,a^2 - 8\,a\,b + b^2$, which we multiply by b^2, and subtract the product from the second dividend. As there is no remainder, the root required is $3a^2 - 4ab + b^2$, or $4ab - 3a^2 - b^2$, for the second power of eithei will produce the given quantity.

ART. **114.** From the foregoing examples and explanations, we derive the following

RULE FOR EXTRACTING THE SECOND ROOT OF A POLYNOMIAL.

1. *Arrange the quantity according to the powers of some letter.*

2. *Find the root of the first term, and place it as the first term of the root sought; subtract the second power of this term from the given polynomial, and call the remainder the first dividend.*

3. *Double the term of the root already found for a divisor, by which divide the first term of the dividend, and place the quotient, with its proper sign, as the second term of the root, also at the right of the divisor. Multiply the divisor, with the term annexed, by the second term of the root, and subtract the product from the dividend.*

4. *The remainder will form a second dividend, which is to be divided by twice the whole root found, and the quotient is to be placed, as the next term of the root, also at the right of the divisor. Multiply the divisor, with the term last annexed, by the last term of the root, and subtract the product from the last dividend.*

5. *The remainder will form a new dividend, with which proceed as before; and thus continue, until all the terms of the root are found.*

Remark 1. As we at first arrange the given polynomial according to the powers of some letter, so the same arrangement must be preserved in each dividend.

Remark 2. In dividing, we merely divide the first term of the dividend by the first term of the divisor; and it is manifest, from the manner in which the divisors are obtained, as well as from inspection, that the successive divisors will have their first terms alike.

Extract the second roots of the following quantities.

1. $9\,m^2 + 24\,m\,c + 16\,c^2$.
2. $25\,x^2 + 70\,x\,y + 49\,y^2$.
3. $36\,b^2 - 48\,b\,x + 16\,x^2$.
4. $x^4 - 4\,x\,y^3 + y^4 - 4\,x^3\,y + 6\,x^2\,y^2$.
5. $9\,x^4 + 30\,x^3\,y + 25\,x^2\,y^2 - 42\,x^2 - 70\,x\,y + 49$.
6. $12\,x^5 + 5\,x^4 + 4\,x^6 + 7\,x^2 - 2\,x^3 - 2\,x + 1$.
7. $25\,x^4\,y^4 - 70\,x^3\,y^3 + 49\,x^2\,y^2$.
8. $9\,m^6 - 12\,m^5 + 34\,m^4 - 20\,m^3 + 25\,m^2$.

ART. **115.** The following additional remarks may be found useful.

1. *No binomial can be an exact second power;* for the second power of a monomial is a monomial, and the second power of a binomial necessarily contains three terms. Thus, $x^2 + y^2$ cannot be an exact second power. It wants $+2\,x\,y$ to make it the second power of $x + y$, and it wants $-2\,x\,y$ to make it the second power of $x - y$.

2. In order that a *trinomial* may be a perfect second power, it must be such, that, when it is arranged according to the powers of a particular letter, the extreme terms shall both be positive, and shall both be exact second powers, and the mean term shall be twice the product of the second roots of those powers.

When these conditions are fulfilled, the second root may be found in the following manner.

Extract the second roots of the extreme terms, writing these roots after each other, and giving them both the sign

+, *when the second term of the trinomial is positive, but giving one of them the sign* —, *when that second term is negative. The result will be the second root of the trinomial.*

For example, $49x^2 - 112xy + 64y^2$ is an exact second power. The root of the first term is $7x$, that of the third term is $8y$, and twice the product of these roots is $112xy$. But since $112xy$ in the given trinomial has the sign —, the required root is $7x - 8y$, or $8y - 7x$.

SECTION XXXVIII.

TRANSFORMATION AND SIMPLIFICATION OF IRRATIONAL QUANTITIES.

ART. **116.** We have already seen, in Article 86, that, when a quantity is not an exact power of the same degree as the root required, this root is expressed, either by the radical sign or by fractional exponents. Such expressions are in general called irrational quantities. Thus, $\sqrt{2}$, $\sqrt{a}$, $\sqrt[3]{a^2}$, $\sqrt[3]{ab^2}$, or the equivalent expressions $2^{\frac{1}{2}}$, $a^{\frac{1}{2}}$, $a^{\frac{2}{3}}$, $a^{\frac{1}{3}} b^{\frac{2}{3}}$, are irrational.

We have also seen, (Art. 110,) that to extract the root of a monomial, we divide the exponent of each factor by the number expressing the degree of the root; so that, when this division can only be represented, it gives rise to fractional exponents. But any root of a quantity is also expressed by writing the quantity under the radical sign, and putting the number denoting the degree of the root over the sign. Thus, $\sqrt[3]{a^2 b c^4} = a^{\frac{2}{3}} b^{\frac{1}{3}} c^{\frac{4}{3}}$, and $\sqrt[4]{a^3 x y^2} = a^{\frac{3}{4}} x^{\frac{1}{4}} y^{\frac{2}{4}}$. Hence, we may readily convert a quantity having a radical sign into an equivalent expression with fractional exponents, in the following manner.

Remove the radical sign, and divide the exponents of the different factors under the sign, by the number placed or supposed to be placed over it; for that number denotes the degree of the root required.

Transform the following expressions into equivalent ones having fractional exponents.

1. $\sqrt{a b}$. Ans. $(a b)^{\frac{1}{2}} = a^{\frac{1}{2}} b^{\frac{1}{2}}$
2. $\sqrt{a^2 b^3}$.
3. $\sqrt{x^3 y}$.
4. $\sqrt[3]{a^2 b^2 c}$.
5. $\sqrt[3]{m x^2 y}$.
6. $\sqrt[4]{a^3 b^2}$.
7. $\sqrt{a+b}$. Ans. $(a+b)^{\frac{1}{2}}$.
8. $\sqrt[3]{a^2 (x+y)}$. Ans. $a^{\frac{2}{3}} (x+y)^{\frac{1}{3}}$.
9. $\sqrt{(a+b) x y}$.
10. $\sqrt[4]{m x^3 y^5}$.
11. $\sqrt[5]{m^2 (x-y)}$.
12. $\sqrt[7]{m^5 y^4 (a+b)}$.

ART. **117.** On the other hand, it is manifest that expressions with fractional exponents may be converted into equivalent ones with the radical sign, by a reverse operation, as follows.

Take away the denominators to the fractional exponents of the different factors, supposing them to have a common denominator, placing the result under the radical sign, and putting the denominator over it. Thus, $a^{\frac{3}{4}} b^{\frac{7}{4}} = \sqrt[4]{a^3 b^7}$.

If the fractional exponents have not all the same denominator, they must be reduced to a common denominator; and integral exponents, if there are any, must be converted into fractions with the same denominator; after which proceed as before.

Thus, $a^{\frac{3}{4}} b^{\frac{5}{6}} = a^{\frac{9}{12}} b^{\frac{10}{12}} = \sqrt[12]{a^9 b^{10}}$; and $x^2 y^{\frac{1}{3}} z^{\frac{1}{2}} = x^{\frac{12}{6}} y^{\frac{2}{6}} z^{\frac{3}{6}} = \sqrt[6]{x^{12} y^2 z^3}$.

Remark. Taking away the denominators of the fractional exponents of all the factors, after all the exponents have been reduced to a common denominator, is equivalent to raising the quantity to the power denoted by that denominator.

Transform the following quantities into equivalent expressions with the radical sign.

1. $x^{\frac{1}{2}} y^{\frac{1}{2}}$.	6. $a^{\frac{3}{4}} b^{\frac{1}{3}} c$.
2. $x^{\frac{2}{3}} y^{\frac{4}{3}}$.	7. $m^{\frac{2}{3}} n^{\frac{3}{5}}$.
3. $x^{\frac{2}{3}} y^{\frac{1}{2}}$.	8. $x y^{\frac{2}{3}} z^{\frac{1}{4}}$.
4. $x^{\frac{1}{2}} y^{\frac{3}{4}}$.	9. $a^{\frac{1}{3}} (x+y)^{\frac{1}{3}}$.
5. $a^{\frac{1}{2}} b^{\frac{1}{3}} c^2$.	10. $(a+b)^{\frac{1}{2}} (x-y)^{\frac{1}{3}}$.

ART. **118.** We have shown, in Article 111, that the root of a product is formed by multiplying together the roots of all the factors of that product. Hence, we may extract the roots of all such factors as are exact powers of the requisite degree, and indicate the roots of the other factors.

Let it be required to find the second root of $32 a^3 b^5$.

This root is indicated thus, $\sqrt{32\,a^3\,b^5}$, or thus, $(32\,a^3\,b^5)^{\frac{1}{2}}$. But $32\,a^3\,b^5 = 16\,.\,2\,a^2\,b^4\,a\,b = 16\,a^2\,b^4\,.\,2\,a\,b$. Now, 16, a^2, and b^4 are exact second powers. We may, therefore, take the roots of these factors, and place their product as a coefficient to the expression for the second root of $2\,a\,b$. We then have $\sqrt{32\,a^3\,b^5} = \sqrt{16\,a^2\,b^4\,.\,2\,a\,b} = 4\,a\,b^2\sqrt{2\,a\,b}$ $= 4\,a\,b^2\,(2\,a\,b)^{\frac{1}{2}}$.

In a similar manner we have $(81\,a^4\,b^5)^{\frac{1}{3}} = (27\,a^3\,b^3)^{\frac{1}{3}} \times (3\,a\,b^2)^{\frac{1}{3}} = 3\,a\,b\,(a\,b^2)^{\frac{1}{3}} = 3\,a\,b\sqrt[3]{a\,b^2}$. In this case, we find all the factors which are exact third powers.

In order to separate an irrational quantity into factors, for the purpose of simplifying, we seek the greatest numerical factor that is an exact power of the degree required, and the greatest exponent of each literal factor, not exceeding its given exponent, that is divisible by the number which denotes the degree of the root.

For example, in the expression $(128\,a^7\,b^5)^{\frac{1}{2}}$, although 4 and 16 are factors of 128, and are exact second powers, yet the greatest factor of 128, which is also an exact second power, is 64. The exponent of a being 7, the greatest number less than 7, and divisible by 2, is 6; also the greatest number divisible by 2, and not exceeding 6, the exponent of b, is 6 itself. Hence $128\,a^7\,b^6$, when resolved into the requisite factors, becomes $64\,a^6\,b^6\,.\,2\,a$, the second root of which is $8\,a^3\,b^3\sqrt{2\,a}$, or $8\,a^3\,b^3\,(2\,a)^{\frac{1}{2}}$.

Simplify the following expressions.

1. $(a^2\,b)^{\frac{1}{2}}$.
2. $\sqrt{a^2\,b^3}$.
3. $\sqrt{8\,a^3\,x^3}$.
4. $(27\,a^4\,x^3)^{\frac{1}{2}}$.
5. $\sqrt[3]{108}$.
6. $(56\,a^3\,x^5)^{\frac{1}{2}}$.
7. $(72\,a^5)^{\frac{1}{3}}$.
8. $2\sqrt{a^2\,x^3}$.

9. $a\sqrt{8\,a^3\,b^5\,c}$.

10. $3\sqrt[3]{686\,a^3\,b}$.

11. $(3\,x^3\,m^4)^{\frac{1}{3}}$.

12. $\sqrt[3]{x^3\,(2\,a+b)}$.

13. $\sqrt{2\,x^2+3\,x^2\,y}$.

14. $(3\,a^3+4\,a^4)^{\frac{1}{3}}$.

ART. **119.** When the quantity which is under the radical sign, or which is enclosed in a parenthesis with a fractional exponent, is a fraction, the expression may be simplified in the following manner, viz.

Multiply the numerator and denominator of the fraction by such a quantity as will render the denominator an exact power of the requisite degree; then take the roots of the denominator and of such factors of the numerator as are exact powers.

Thus, to simplify $\left(\frac{a\,m^2}{3\,b}\right)^{\frac{1}{2}}$, we multiply both numerator and denominator of the fraction by $3\,b$; the expression then becomes $\left(\frac{3\,a\,b\,m^2}{9\,b^2}\right)^{\frac{1}{2}} = \left(\frac{m^2}{9\,b^2}\,.\,3\,a\,b\right)^{\frac{1}{2}}$. Taking the second root of the fraction $\frac{m^2}{9\,b^2}$ and placing it as a factor before the parenthesis, we have $\frac{m}{3\,b}(3\,a\,b)^{\frac{1}{2}}$.

In like manner $\left(\frac{7\,a^2}{25\,b^3}\right)^{\frac{1}{3}} = \left(\frac{35\,a^2}{125\,b^3}\right)^{\frac{1}{3}} = \left(\frac{1}{125\,b^3}\,.\,35\,a^2\right)^{\frac{1}{3}} = \frac{1}{5\,b}(35\,a^2)^{\frac{1}{3}}$.

Simplify in a similar manner the following expressions.

1. $\left(\frac{3\,m}{8\,x^2}\right)^{\frac{1}{2}}$

2. $\left(\frac{4\,x}{5\,y}\right)^{\frac{1}{2}}$.

3. $\sqrt{\frac{2}{7}}$.

4. $\sqrt[3]{\frac{5}{9}}$.

5. $2\sqrt{\frac{7}{8}}$.

6. $3\left(\frac{3\,a^4}{4\,b^2}\right)^{\frac{1}{3}}$.

7. $2\,a\left(\frac{m}{6\,x^2}\right)^{\frac{1}{2}}$.

8. $\frac{1}{3}\left(\frac{7\,a\,x}{27\,m^2}\right)^{\frac{1}{3}}$.

ART. **120.** As we can extract the root of any factor, and place it as a factor before the radical sign or the parenthesis with a fractional exponent, so we may put under the sign, or within the parenthesis, any factor standing before it, if we first raise that factor to a power of the same degree as the root.

Thus, $3a\sqrt{b} = \sqrt{9a^2b}$; to obtain this, we raise $3a$ to the second power, and place the result as a factor under the radical sign. In like manner, $\frac{2}{3}m(xy)^{\frac{1}{3}} = \left(\frac{8m^3xy}{27}\right)^{\frac{1}{3}}$; to obtain this, we raise $\frac{2m}{3}$ to the third power, and multiply xy within the parenthesis by the result.

We may, in a similar manner, reduce any rational quantity to the form of an irrational quantity. Thus, $2 = \sqrt{4}$ or $4^{\frac{1}{2}} = \sqrt[3]{8}$ or $8^{\frac{1}{3}}$, &c.; $ab = \sqrt{a^2b^2}$ or $(a^2b^2)^{\frac{1}{2}} = \sqrt[3]{a^3b^3}$ or $(a^3b^3)^{\frac{1}{3}}$, &c.

Reduce the following quantities entirely to an irrational form.

1. $ab\sqrt{x}$.
2. $2m\sqrt{bc}$.
3. $\frac{3}{4}\sqrt{a}$.
4. $\frac{2a}{3b}\sqrt{xy}$.
5. $3\sqrt[3]{m^2}$.
6. $4m(xy)^{\frac{1}{3}}$.
7. $\frac{2m}{5n}(a^2)^{\frac{1}{3}}$.
8. $\frac{1}{4}(a+b)^{\frac{1}{3}}$.

9. Reduce 2 to the form of a second root.

Ans. $\sqrt{4}$, or $4^{\frac{1}{2}}$.

10. Reduce ab to the form of a second root.

11. Reduce $2a$ to the form of a third root.

12 Reduce $\frac{2ab}{c}$ to the form of a third root.

SECTION XXXIX.

OPERATIONS ON IRRATIONAL QUANTITIES WITH FRACTIONAL EXPONENTS.

ART. **121.** Quantities having fractional exponents are, in general, to be treated in the same manner as if the exponents were whole numbers.

1. Add $3\,a^{\frac{1}{2}}$ and $7\,a^{\frac{1}{2}}$.

The sum is $3\,a^{\frac{1}{2}}+7\,a^{\frac{1}{2}}=10\,a^{\frac{1}{2}}$. (Art. 33.)

2. Add $a\,x^{\frac{1}{3}}$ and $3\,b\,x^{\frac{1}{3}}$.

The sum is $a\,x^{\frac{1}{3}}+3\,b\,x^{\frac{1}{3}}=(a+3\,b)\,x^{\frac{1}{3}}$. (Art. 59.)

3. From $10\,a^{\frac{1}{2}}$ subtract $3\,a^{\frac{1}{2}}$.

The difference is $10\,a^{\frac{1}{2}}-3\,a^{\frac{1}{2}}=7\,a^{\frac{1}{2}}$.

4. From $5\,a\,x^{\frac{1}{2}}\,y^{\frac{2}{3}}$ subtract $3\,m\,x^{\frac{1}{2}}\,y^{\frac{2}{3}}$.

The result is $5\,a\,x^{\frac{1}{2}}\,y^{\frac{2}{3}}-3\,m\,x^{\frac{1}{2}}\,y^{\frac{2}{3}}=(5\,a-3\,m)\,x^{\frac{1}{2}}\,y^{\frac{2}{3}}$.

5. Add $3\,(12)^{\frac{1}{2}}$ and $4\,(27)^{\frac{1}{2}}$.

The sum expressed is $3\,(12)^{\frac{1}{2}}+4\,(27)^{\frac{1}{2}}$.

But by simplifying these terms, the result may be obtained in a reduced form. For, $3\,(12)^{\frac{1}{2}}=3\,.\,4^{\frac{1}{2}}\,.\,3^{\frac{1}{2}}=3\,.\,2\,.\,3^{\frac{1}{2}}=6\,.\,3^{\frac{1}{2}}$, (Art. 118;) and $4\,(27)^{\frac{1}{2}}=4\,.\,9^{\frac{1}{2}}\,.\,3^{\frac{1}{2}}=4\,.\,3\,.\,3^{\frac{1}{2}}=12\,.\,3^{\frac{1}{2}}$. Therefore, $3\,(12)^{\frac{1}{2}}+4\,(27)^{\frac{1}{2}}=6\,.\,3^{\frac{1}{2}}+12\,.\,3^{\frac{1}{2}}=18\,.\,3^{\frac{1}{2}}$, the result in its most simple form

6. From $(250\,x^3)^{\frac{1}{3}}$ subtract $(54\,y^3)^{\frac{1}{3}}$.

The difference expressed is $(250\,x^3)^{\frac{1}{3}}-(54\,y^3)^{\frac{1}{3}}$

But $(250\,x^3)^{\frac{1}{3}} = (125\,x^3 \,.\, 2)^{\frac{1}{3}} = 5\,x \,.\, 2^{\frac{1}{3}}$; and $(54\,y^3)^{\frac{1}{3}} = (27\,y^3 \,.\, 2)^{\frac{1}{3}} = 3\,y \,.\, 2^{\frac{1}{3}}$. Hence, $(250\,x^3)^{\frac{1}{3}} - (54\,y^3)^{\frac{1}{3}} = 5\,x \,.\, 2^{\frac{1}{3}} - 3\,y \,.\, 2^{\frac{1}{3}} = (5\,x - 3\,y)\,2^{\frac{1}{3}}$, the result in its simplest form.

From the preceding examples we derive the following

RULE FOR ADDING AND SUBTRACTING IRRATIONAL QUANTITIES.

Express the addition or subtraction as usual by signs, simplify the terms if possible, and reduce similar terms.

Remark. Irrational quantities, expressed by means of fractional exponents, are called similar, when the factors having fractional exponents are alike in all, and have respectively the same exponents. Thus, $3\,a\,x^{\frac{1}{2}}$ and $m\,x^{\frac{1}{2}}$ are similar; but $3\,a^{\frac{1}{2}}\,x$ and $m\,x^{\frac{1}{2}}$ are not similar.

ART. **122.** 1. Multiply $a^{\frac{2}{7}}$ by $a^{\frac{3}{7}}$.

This is done by adding the exponents. Thus, $a^{\frac{2}{7}} \,.\, a^{\frac{3}{7}} = a^{\frac{2}{7} + \frac{3}{7}} = a^{\frac{5}{7}}$. (Art. 30.)

2. Multiply $2\,m^{\frac{1}{3}}\,x^{\frac{1}{5}}$ by $3\,m^{\frac{1}{3}}\,x^{\frac{2}{5}}$.

The product is $6\,m^{\frac{1}{3} + \frac{1}{3}}\,x^{\frac{1}{5} + \frac{2}{5}} = 6\,m^{\frac{2}{3}}\,x^{\frac{3}{5}}$.

3. Multiply $5\,a^{\frac{1}{2}}$ by $4\,a^{\frac{1}{3}}$.

In this example, in order to add the exponents, we must reduce them to a common denominator. We then have $5\,a^{\frac{1}{2}} = 5\,a^{\frac{3}{6}}$, and $4\,a^{\frac{1}{3}} = 4\,a^{\frac{2}{6}}$; hence, $5\,a^{\frac{1}{2}} \,.\, 4\,a^{\frac{1}{3}} = 5\,a^{\frac{3}{6}} \,.\, 4\,a^{\frac{2}{6}} = 20\,a^{\frac{5}{6}}$.

4. Multiply $2\,a^{\frac{1}{2}}\,b^{\frac{3}{4}}$ by $7\,a^{\frac{2}{3}}\,b^{\frac{4}{5}}$.

In this case, the exponents of a and those of b must be separately reduced to a common denominator. Then $2\,a^{\frac{1}{2}}\,b^{\frac{3}{4}}$ becomes $2\,a^{\frac{3}{6}}\,b^{\frac{15}{20}}$, and $7\,a^{\frac{2}{3}}\,b^{\frac{4}{5}}$ becomes $7\,a^{\frac{4}{6}}\,b^{\frac{16}{20}}$. Hence, $2\,a^{\frac{1}{2}}\,b^{\frac{3}{4}}\,.\,7\,a^{\frac{2}{3}}\,b^{\frac{4}{5}} = 2\,a^{\frac{3}{6}}\,b^{\frac{15}{20}}\,.\,7\,a^{\frac{4}{6}}\,b^{\frac{16}{20}} = 14\,a^{\frac{7}{6}}\,b^{\frac{31}{20}}$.

5. Divide $m^{\frac{7}{8}}$ by $m^{\frac{3}{8}}$.

This is done by subtracting the exponent of the divisor from that of the dividend. Thus, $\dfrac{m^{\frac{7}{8}}}{m^{\frac{3}{8}}} = m^{\frac{7}{8}-\frac{3}{8}} = m^{\frac{4}{8}} = m^{\frac{1}{2}}$.

6. Divide $6\,m^{\frac{1}{2}}\,x^{\frac{3}{4}}$ by $2\,m^{\frac{1}{3}}\,x^{\frac{2}{3}}$.

In this case we reduce the exponents of the same letter in each quantity to a common denominator. We then have $\dfrac{6\,m^{\frac{1}{2}}\,x^{\frac{3}{4}}}{2\,m^{\frac{1}{3}}\,x^{\frac{2}{3}}} = \dfrac{6\,m^{\frac{3}{6}}\,x^{\frac{9}{12}}}{2\,m^{\frac{2}{6}}\,x^{\frac{8}{12}}} = 3\,m^{\frac{1}{6}}\,x^{\frac{1}{12}}$. (Art. 48.)

7. Divide $3\,b^{\frac{1}{2}}\,c$ by $4\,b^{\frac{1}{7}}\,c^2$.

The quotient is $\dfrac{3\,b^{\frac{1}{2}}\,c}{4\,b^{\frac{1}{7}}\,c^2} = \dfrac{3\,b^{\frac{7}{14}}\,c}{4\,b^{\frac{2}{14}}\,c^2} = \dfrac{3\,b^{\frac{5}{14}}}{4\,c}$. (Art. 63)

8. Required the second power of $2\,m^{\frac{2}{5}}$.

This is performed by raising the coefficient to the second power and multiplying the exponent of m by 2. (Art. 104.) Thus, $(2\,m^{\frac{2}{5}})^2 = 4\,m^{\frac{4}{5}}$.

9. Required the third power of $3\,a^{\frac{1}{5}}\,b^{\frac{2}{7}}$.

Ans. $27\,a^{\frac{3}{5}}\,b^{\frac{6}{7}}$.

Conversely, to find the root of an irrational quantity, we either extract or express the root of the numerical coefficient, and divide the exponents of the other factors by the number which marks the degree of the root.

10. What is the second root of $4\,a^{\frac{2}{3}}$? Ans. $2\,a^{\frac{1}{3}}$.

11. What is the third root of $5\,a^{\frac{3}{4}}\,b^{\frac{1}{2}}$?

Ans. $5^{\frac{1}{3}}\,a^{\frac{1}{4}}\,b^{\frac{1}{6}}$.

From what precedes we infer that the following operations, viz., *multiplication, division, finding powers, and extracting roots, are performed upon quantities with fractional exponents, in the same manner as if the exponents were integral.* This is manifest; for there is no reason why exponents in a fractional form should not be subject to the same law as those in an integral form.

Art. **123.** We have assumed, in what precedes, that no change is made in the value of irrational quantities by reducing the fractional exponents to a common denominator. This is manifestly the case, since reducing to a common denominator does not change the value of fractions, but only their form.

Hence, we may reduce the exponents of all the factors in an irrational quantity to a common denominator, without changing the value of the quantity. Thus, $2\,x^{\frac{1}{2}}\,y^{\frac{1}{3}} = 2^{\frac{6}{6}}\,x^{\frac{3}{6}}\,y^{\frac{2}{6}} = (64\,x^3\,y^2)^{\frac{1}{6}}$.

It is evident also that fractional exponents may be converted into the decimal form, and used in that form as well as any other. Thus, $a^{\frac{1}{2}} = a^{0.5}$; $a^{\frac{3}{4}} = a^{0.75}$; and the product $a^{\frac{1}{2}}\,.\,a^{\frac{3}{4}} = a^{0.5}\,.\,a^{0.75} = a^{1.25}$.

1. Add $(8)^{\frac{1}{2}}$ and $(32)^{\frac{1}{2}}$. *Ans.* $6\,(2)^{\frac{1}{2}}$

2. Add $(27)^{\frac{1}{2}}$ and $(75)^{\frac{1}{2}}$.

3. Add $(135\,a)^{\frac{1}{3}}$ and $(40\,a)^{\frac{1}{3}}$.

4. Add $(250\,a^2)^{\frac{1}{3}}$ and $(128\,a^2)^{\frac{1}{3}}$

5. Add $(192\,a^3\,x)^{\frac{1}{3}}$ and $(24\,a^3\,x)^{\frac{1}{3}}$

6. From $(75)^{\frac{1}{2}}$ subtract $(48)^{\frac{1}{2}}$.

7. From $3(50)^{\frac{1}{2}}$ subtract $(18)^{\frac{1}{2}}$.

8. From $(320\,a^2)^{\frac{1}{3}}$ subtract $(40\,a^2)^{\frac{1}{3}}$.

9. From $(8)^{\frac{1}{2}}$ subtract $2\,(\frac{1}{2})^{\frac{1}{2}}$.

10. From $(54\,b^3)^{\frac{1}{2}}$ subtract $(24\,a^3)^{\frac{1}{2}}$.

11. Multiply $3\,x^{\frac{1}{2}}\,y^{\frac{2}{3}}$ by $5\,x^{\frac{1}{2}}\,y^{\frac{1}{3}}$.

12. Multiply $2\,x\,y^{\frac{1}{2}}$ by $3\,x^{\frac{1}{2}}\,y^{\frac{1}{3}}$.

13. Multiply $3\,(8)^{\frac{1}{2}}$ by $2\,(6)^{\frac{1}{2}}$.

The product is $6 \,.\, 8^{\frac{1}{2}} \,.\, 6^{\frac{1}{2}} = 6\,(8 \,.\, 6)^{\frac{1}{2}} = 6\,(16 \,.\, 3)^{\frac{1}{2}} = 6 \,.\, 4\,(3)^{\frac{1}{2}} = 24\,(3)^{\frac{1}{2}}$.

14. Multiply $4 \,.\, 5^{\frac{1}{2}}$ by $3 \,.\, 8^{\frac{1}{2}}$, and simplify.

15. Multiply $8\,(108)^{\frac{1}{3}}$ by $5\,(4)^{\frac{1}{3}}$, and simplify.

16. Divide $m^{\frac{3}{4}}$ by $m^{\frac{1}{4}}$.

17. Divide $a^{\frac{5}{8}}\,x^{\frac{3}{4}}$ by $a^{\frac{3}{8}}\,x^{\frac{1}{4}}$.

18. Divide $a^{\frac{1}{2}}$ by $a^{\frac{1}{3}}$.

19. Divide $4\,x\,y^{\frac{1}{4}}$ by $2\,x^{\frac{1}{2}}\,y^{\frac{1}{8}}$.

20. Divide $8\,(27)^{\frac{1}{2}}$ by $4\,(3)^{\frac{1}{2}}$.

The quotient is $\dfrac{8\,(27)^{\frac{1}{2}}}{4\,(3)^{\frac{1}{2}}} = \dfrac{2\,(27)^{\frac{1}{2}}}{(3)^{\frac{1}{2}}} = \dfrac{2\,(9 \,.\, 3)^{\frac{1}{2}}}{(3)^{\frac{1}{2}}} = \dfrac{6 \,.\, 3^{\frac{1}{2}}}{3^{\frac{1}{2}}} = 6$.

In another way; $\dfrac{8\,(27)^{\frac{1}{2}}}{4\,(3^{\frac{1}{2}})} = 2\,(\frac{27}{3})^{\frac{1}{2}} = 2 \,.\, 9^{\frac{1}{2}} = 2 \,.\, 3 = 6$.

21. Divide $4\,(512)^{\frac{1}{3}}$ by $2\,(3)^{\frac{1}{3}}$.

22. Find the 2d power of $a^{\frac{3}{4}}\,b^{\frac{1}{2}}$.

23. Find the 2d power of $2\,x^{\frac{1}{5}}\,y^{\frac{2}{7}}$.

24. Find the 3d power of $3\,m^{\frac{1}{7}}\,x\,y^{\frac{3}{5}}$.

25. Find the 2d power of $(a-x)^{\frac{1}{5}}$.

26. Find the 3d power of $5\,(m+n)^{\frac{5}{9}}$

27. Extract the 2d root of $a^{\frac{4}{5}}$.

28. Extract the 2d root of $4\,a^{\frac{6}{7}}\,b^{\frac{4}{3}}$.

29. Extract the 2d root of $9\,a^{\frac{1}{2}}\,b^{\frac{1}{3}}$.

30. Extract the 3d root of $2\,a^{\frac{3}{4}}\,b^{\frac{5}{7}}$.

Remark. Represent the 3d root of 2 in this example.

31. Extract the 2d root of $3\,(a+b)^{\frac{4}{7}}$.

32. Extract the 3d root of $40\,a^{\frac{2}{5}}\,(a-x)^{\frac{1}{2}}$.

SECTION XL.

OPERATIONS UPON IRRATIONAL QUANTITIES WITH THE RADICAL SIGN.

ART. **124.** Since irrational quantities with the radical sign may always be converted into equivalent expressions with fractional exponents, (Art. 116,) all operations might be performed upon them in this latter form.

But as the radical sign is used in many mathematical works, we shall show how to treat irrational quantities expressed by means of this sign.

Irrational quantities with the radical sign are commonly called *radical quantities.* The mode of simplifying irrational quantities in both forms has already been shown in Article 118.

The addition and subtraction of irrational quantities with the radical sign, are manifestly performed in the same manner as if fractional exponents were used. (Art. 121.)

We observe, however, that

Irrational quantities with the radical sign are said to be similar, when the indices over the sign are alike, and the quantities under the sign are in all respects the same.

Thus, $\sqrt{ab}$ and $3\sqrt{ab}$ are similar; also, $\sqrt[3]{a^2bc}$ and $m\sqrt[3]{a^2bc}$ are similar. But $\sqrt{ab}$ and $\sqrt[3]{ab}$ are not similar; neither are $\sqrt[3]{a^2bm}$ and $\sqrt[3]{a^3b^2m}$.

1. Add $\sqrt{288}$ and $3\sqrt{8}$.

The sum expressed is $\sqrt{288}+3\sqrt{8}$. But $\sqrt{288}=12\sqrt{2}$, and $3\sqrt{8}=6\sqrt{2}$. Hence, $\sqrt{288}+3\sqrt{8}=12\sqrt{2}+6\sqrt{2}=18\sqrt{2}$.

2. Add $m\sqrt[3]{27ax}$ and $3n\sqrt[3]{125ax}$.

The sum is $m\sqrt[3]{27ax}+3n\sqrt[3]{125ax}$. But when simplified, these terms become, respectively, $3m\sqrt[3]{ax}$ and $15n\sqrt[3]{ax}$. Hence, the sum in its simplest form is $3m\sqrt[3]{ax}+15n\sqrt[3]{ax}=(3m+15n)\sqrt[3]{ax}$.

3. From $\sqrt{48}$ subtract $\sqrt{27}$.

Expressing the subtraction, and simplifying, we have $\sqrt{48}-\sqrt{27}=4\sqrt{3}-3\sqrt{3}=\sqrt{3}$, the result in its simplest form.

4. From $5m\sqrt[3]{81x}$ subtract $2n\sqrt[3]{24x}$.

In this case we have $5m\sqrt[3]{81x} - 2n\sqrt[3]{24x} = 15m\sqrt[3]{3x} - 4n\sqrt[3]{3x} = (15m - 4n)\sqrt[3]{3x}$.

ART. **125.** We shall deduce rules for other operations on radicals, from the modes given in the preceding section, for corresponding operations on irrational quantities with fractional exponents.

The following principle will be of frequent use, viz.

The exponents of all the factors under the radical sign and the index over the sign, may both be multiplied or divided by the same number without affecting the value of the expression.

Remark. A numerical factor under the radical sign may either be considered as having an exponent, or it may be actually raised by multiplication to the power denoted by the number by which the exponents of the literal factors are multiplied, or the root may be extracted when the exponents of the letters are divided.

Thus, in the expression $\sqrt{m^3}$ or $\sqrt[2]{m^3}$, we may multiply the 2 and 3 both by 4, for example, which gives $\sqrt[8]{m^{12}}$; for, $\sqrt{m^3} = m^{\frac{3}{2}} = m^{\frac{12}{8}} = \sqrt[8]{m^{12}}$. (Arts. 116, 117.)

In like manner $\sqrt[3]{2a^2} = \sqrt[6]{2^2 a^4} = \sqrt[6]{4a^4}$.

On the other hand, $\sqrt[8]{m^{12}} = m^{\frac{12}{8}} = m^{\frac{3}{2}} = \sqrt[2]{m^3}$, or $\sqrt{m^3}$ which might have been obtained from $\sqrt[8]{m^{12}}$, simply by dividing the 8 and 12 both by 4.

In like manner, $\sqrt[6]{27a^3b^9} = \sqrt[2]{(27)^{\frac{1}{3}}ab^3} = \sqrt[2]{3ab^3}$, or $\sqrt{3ab^3}$.

Art. 126. Upon the principle explained in the foregoing Article, two or more radical expressions may be made to have the same index over the sign without affecting the value of these expressions.

Thus, $\sqrt{m}$ and $\sqrt[3]{x^2}$ are, respectively, the same as $\sqrt[6]{m^3}$ and $\sqrt[6]{x^4}$. The first of these is obtained by multiplying 2 supposed to be over the sign, and the exponent of m under it by 3, and the second is obtained by multiplying the 3 and 2 both by 2.

In like manner, $\sqrt[4]{a^2\,b^3}$ and $\sqrt[6]{a\,m^2}$ are, respectively, the same as $\sqrt[12]{a^6\,b^9}$ and $\sqrt[12]{a^2\,m^4}$.

This process is equivalent to reducing the corresponding fractional exponents to a common denominator, the indices over the sign being considered as denominators, and the exponents under the sign, as numerators. The common index will therefore be either the product of all the indices over the sign, or their least common multiple.

Art. 127. 1. Multiply $\sqrt{a}$ by $\sqrt{m}$.

The product is $\sqrt{a\,m}$; for $\sqrt{a} = a^{\frac{1}{2}}$, and $\sqrt{m} = m^{\frac{1}{2}}$; hence, $\sqrt{a} \cdot \sqrt{m} = a^{\frac{1}{2}}\,m^{\frac{1}{2}} = (a\,m)^{\frac{1}{2}} = \sqrt{a\,m}$.

2. Multiply $7\sqrt[3]{a^2}$ by $3\sqrt[4]{m^3}$.

We first render the indices over the radical sign alike. We then have $7\sqrt[3]{a^2} = 7\sqrt[12]{a^8}$, and $3\sqrt[4]{m^3} = 3\sqrt[12]{m^9}$; consequently, $7\sqrt[3]{a^2} \cdot 3\sqrt[4]{m^3} = 7\sqrt[12]{a^8} \cdot 3\sqrt[12]{m^9} = 7\,a^{\frac{8}{12}} \cdot 3\,m^{\frac{9}{12}} = 21\,a^{\frac{8}{12}}\,m^{\frac{9}{12}} = 21\,(a^8\,m^9)^{\frac{1}{12}} = 21\sqrt[12]{a^8\,m^9}$.

3. Divide $9\sqrt{a\,m}$ by $3\sqrt{a}$.

Representing the division, we have $\dfrac{9\sqrt{a\,m}}{3\sqrt{a}}$.

But, $9\sqrt{am} = 9a^{\frac{1}{2}}m^{\frac{1}{2}}$, and $3\sqrt{a} = 3a^{\frac{1}{2}}$; hence,

$$\frac{9\sqrt{am}}{3\sqrt{a}} = \frac{9a^{\frac{1}{2}}m^{\frac{1}{2}}}{3a^{\frac{1}{2}}} = 3m^{\frac{1}{2}} = 3\sqrt{m}.$$

4. Divide $7\sqrt{a}$ by $4\sqrt[3]{b^2}$.

Rendering the indices over the radical sign alike, and representing the division, we have

$$\frac{7\sqrt{a}}{4\sqrt[3]{b^2}} = \frac{7\sqrt[6]{a^3}}{4\sqrt[6]{b^4}} = \frac{7}{4} \cdot \frac{a^{\frac{3}{6}}}{b^{\frac{4}{6}}} = \frac{7}{4}\left(\frac{a^3}{b^4}\right)^{\frac{1}{6}} = \frac{7}{4}\sqrt[6]{\frac{a^3}{b^4}}.$$

From an examination of the results in the four preceding questions, we deduce the following

RULE FOR THE MULTIPLICATION AND DIVISION OF RADICALS.

Make the indices over the radical sign alike, if they are not so; then multiply or divide one coefficient by the other; also take the product or quotient of the quantities under the radical sign; place the latter result under the common sign, before which write the product or quotient of the coefficients previously found.

ART. **128.** 1. Find the 3d power of $3\sqrt[4]{a^3x^2}$.

Since $3\sqrt[4]{a^3x^2} = 3a^{\frac{3}{4}}x^{\frac{2}{4}}$, we have $(3\sqrt[4]{a^3x^2})^3 = (3a^{\frac{3}{4}}x^{\frac{2}{4}})^3 = 3^3 \cdot a^{\frac{3}{4}\times 3}x^{\frac{2}{4}\times 3}$, (Art. 104,) $= 27a^{\frac{9}{4}}x^{\frac{6}{4}} = 27\sqrt[4]{a^9x^6}$.

This result might have been obtained from $3\sqrt[4]{a^3x^2}$, by raising the coefficient 3 to the third power, and multiplying the exponent of each factor under the radical sign by 3, the number which marks the degree of the power required.

2. Find the 2d power of $7\sqrt[6]{a^5 m}$.

Since $7\sqrt[6]{a^5 m} = 7\,a^{\frac{5}{6}} m^{\frac{1}{6}}$, we have $(7\sqrt[6]{a^5 m})^2 = (7\,a^{\frac{5}{6}} m^{\frac{1}{6}})^2 = 7^2 \,.\, a^{\frac{5}{6}\times 2}\, m^{\frac{1}{6}\times 2} = 49\,a^{\frac{5}{3}} m^{\frac{1}{3}} = 49\sqrt[3]{a^5 m}$.

This result is the same as would have been produced, if we had merely raised the coefficient 7, of the given quantity, to the second power, and divided the index over the sign by 2.

From these results we derive the following

RULE FOR RAISING A RADICAL TO ANY POWER.

Raise the coefficient of the radical to the power required, and either raise the quantity under the radical sign to the same power, or divide the index over it by the number expressing the degree of the power.

ART. **129.** The process of extracting a root is manifestly just the reverse of that by which a power is found; hence, we have the following

RULE FOR EXTRACTING ANY ROOT OF A RADICAL.

Extract or express the root of the coefficient of the radical, and either extract the root of the quantity under the radical sign, or multiply the index over it by the number expressing the degree of the root.

For example, the third root of $64\sqrt{a^6 x^9}$ is $4\sqrt{a^2 x^3}$; also, the fourth root of $2\sqrt[3]{m^5 x}$ is $\sqrt[4]{2} \,.\, \sqrt[12]{m^5 x} = \sqrt[12]{2^3} \times \sqrt[12]{m^5 x} = \sqrt[12]{2^3 \,.\, m^5 x} = \sqrt[12]{8\,m^5 x}$. This is the result in its simplest form, although the answer given directly by the rule would be $\sqrt[4]{2} \,.\, \sqrt[12]{m^5 x}$.

ART. **130.** Let the learner perform the following questions, simplifying the results when possible.

1. Add $\sqrt{32}$ and $\sqrt{18}$.
2. Add $\sqrt{25x}$ and $\sqrt{16x}$.
3. Add $2\sqrt{49ax}$ and $3\sqrt{36ax}$
4. Add $\sqrt[3]{27a}$ and $\sqrt[3]{64a}$.
5. Add $3\sqrt[3]{250x}$ and $7\sqrt[3]{54x}$.
6. From $\sqrt{500}$ subtract $\sqrt{125}$.
7. From $5\sqrt{12x}$ subtract $\sqrt{18x}$.
8. From $3\sqrt{\frac{2}{5}}$ subtract $2\sqrt{\frac{1}{10}}$.
9. From $5\sqrt[3]{16}$ subtract $2\sqrt[3]{54}$.
10. From $2\sqrt[3]{192a}$ subtract $\sqrt[3]{24a}$.
11. Multiply $\sqrt{2}$ by $\sqrt{2}$.
12. Multiply $5\sqrt{3}$ by $4\sqrt{3}$.
13. Multiply $2\sqrt{a}$ by $3\sqrt{x}$.
14. Multiply $3\sqrt{a}$ by $5\sqrt[3]{a}$.
15. Multiply $5\sqrt[3]{3}$ by $7\sqrt{8}$.
16. Multiply $4+\sqrt{2}$ by $4-\sqrt{2}$.
17. Divide $\sqrt{mx}$ by $\sqrt{m}$.
18. Divide $6\sqrt{abc}$ by $3\sqrt{ab}$.
19. Divide $10\sqrt{108}$ by $5\sqrt{12}$
20. Divide $\sqrt{5}$ by $\sqrt[3]{5}$.
21. Divide $4\sqrt[3]{ax}$ by $2\sqrt[5]{ax}$.
22. Divide $5\sqrt[3]{a^2m}$ by $3\sqrt[4]{a^2m}$.
23. Find the 2d power of $\sqrt[3]{am}$.
24. Find the 3d power of $2\sqrt{3a}$.

25. Find the 2d power of $\sqrt[4]{a^3 b^3}$.

26. Find the 3d power of $3\sqrt[6]{m y^2}$.

27. Find the 5th power of $2 x y \sqrt[10]{a^3 b^3}$

28. Extract the 2d root of $25\sqrt[3]{a^2 m^2}$.

29. Extract the 3d root of $27\sqrt[5]{a^6 b^6}$.

30. Extract the 2d root of $16\sqrt[3]{a x}$.

31. Extract the 3d root of $\frac{1}{8}\sqrt[2]{m y}$.

32. Extract the 2d root of $9\sqrt{4 m^4 y^4}$.

33. Extract the 3d root of $125\sqrt{(a+b)^6}$.

34. Extract the 2d root of $144\sqrt[3]{a+b}$.

SECTION XLI.

RATIO AND PROPORTION.

Art. 131. The *ratio* of two quantities is the quotient arising from the division of one by the other, whether that division can be exactly performed, or whether it can only be expressed. It is sometimes called *ratio by division*, or *geometrical ratio*, to distinguish it from the difference of two quantities, which is called *ratio by subtraction*, or *arithmetical ratio*. But when the word *ratio* simply is used, it signifies ratio by division.

The most proper way of expressing a ratio is in the form of a fraction. Thus, $\frac{5}{7}$ is the ratio of 5 to 7, and $\frac{m}{n}$ is the ratio of m to n.

A *proportion* is an expression of equality between two equal ratios. Sometimes the term *geometrical proportion* is used to express the same thing. For example, $\frac{2}{5}=\frac{14}{35}$, and $\frac{a}{b}=\frac{c}{d}$ are proportions.

For the sake of convenience, two dots, thus :, placed between the quantities, are used to express division, and four dots, thus : :, are used instead of the sign $=$. Thus, $a : b : : c : d$ is read "*a* is to *b* as *c* is to *d*," and has the same meaning as $\frac{a}{b}=\frac{c}{d}$. The signification in both cases is, that *a* divided by *b* gives the same quotient as *c* divided by *d*. In this work we shall sometimes use the points to denote division, but shall always prefer the sign $=$ to express equality.

In any proportion $a : b = c : d$, the quantities *a*, *b*, *c*, and *d* are called the *terms* of the proportion. The two quantities *a* and *b* are the terms of the first ratio; *c* and *d* are the terms of the second ratio.

In the proportion $a : b = c : d$, the two quantities *a* and *c* are called the *antecedents*, and the two quantities *b* and *d* are called the *consequents* of the proportion; *a* is the antecedent of the first ratio, and *c* that of the second; *b* is the consequent of the first ratio, and *d* that of the second. Moreover, *a* and *d* are called the *extremes*, *b* and *c* the *means* of the proportion.

These names are expressive of the position in which the quantities stand with respect to each other, when the division is indicated by dots. The word *antecedent* signifies going before, and *consequent* means following after. Thus, in the ratio $a : b$, *a* goes before or stands first, and *b* follows after it. Also, *a* and *d* are called *extremes*, because they occupy the ends or extremities of the propor tion; b and c are called the *means*, because they occupy the middle place in the proportion.

ART. **132.** We shall now proceed to demonstrate those properties of proportions, which are most important and of most frequent use.

(I). Take any proportion $a:b=c:d$. This is the same as $\frac{a}{b}=\frac{c}{d}$, and if we multiply by the denominators b and d, we have $ad=bc$. But a and d are the extremes, and b and c are the means. Hence,

In any proportion, the product of the means is equal to the product of the extremes.

(II). Suppose we have the equation $ad=bc$. If we divide both members by b and d, we have $\frac{a}{b}=\frac{c}{d}$, or $a:b=c:d$. Therefore,

If the product of two quantities is equal to the product of two other quantities, the two factors of one product may be made the means, and the two factors of the other product, the extremes of a proportion.

(III). If any three terms of a proportion are known quantities, we can always find the value of the remaining term.

For take any proportion, $a:b=c:d$. This gives, by (I), $ad=bc$; hence, by division, $a=\frac{bc}{d}$, $d=\frac{bc}{a}$, $b=\frac{ad}{c}$, $c=\frac{ad}{b}$. Hence,

In any proportion, either mean is equal to the product of the extremes, divided by the other mean; and either extreme is equal to the product of the means, divided by the other extreme.

From this we infer that,

If three terms of one proportion are respectively equal to the three corresponding terms of another proportion, the remaining term of one must be equal to the remaining term of the other.

(IV). The proportion, $a : b = b : c$, in which the two mean terms are alike, is called a *continued proportion.* The term b, in this case, is called a *mean proportional* between a and c, and c is called a *third proportional* to a and b. From this proportion we have $b^2 = a\,c$, $\therefore b = \sqrt{a\,c}$. Hence,

The mean proportional between two quantities is equal to the second root of their product.

From this it follows that,

If the second power of any quantity is equal to the product of two other quantities, the first quantity is a mean proportional between the last two.

For, by (II), the equation $b^2 = a\,c$ gives $a : b = b : c$.

(V). Suppose we have the proportion $a : b = c : d$. (1).
This gives, by (I), $a\,d = b\,c$.

Now, by (II), the equation $a\,d = b\,c$ may, besides the given proportion, be converted into the four following, viz.

$$a : c = b : d, \ (2);$$
$$d : b = c : a, \ (3);$$
$$c : d = a : b, \ (4);$$
$$b : a = d : c, \ (5).$$

By comparing proportions (2), (3), (4), and (5) with the given proportion (1), we infer that,

In any proportion, the means may exchange places; the extremes may exchange places; the extremes may be made the means, and the means the extremes; both ratios may *at the same time, be inverted, that is, the antecedent and consequent of each ratio may exchange places.*

(VI). Since a ratio is a fraction, and since the value of a fraction is not changed, when both numerator and denominator are either multiplied or divided by the same quantity, it follows that,

In any proportion, we may multiply or divide both terms of either ratio by the same quantity, and we may multiply or divide all the terms of a proportion by the same quantity, without disturbing the proportion.

We may also multiply or divide both terms of the first ratio by one quantity, and both terms of the second ratio by another quantity, or we may multiply both terms of one ratio by any quantity, and divide both terms of the other ratio by the same or a different quantity, without disturbing the proportion.

(VII). *Both of the antecedents, or both of the consequents, of a proportion, may either be multiplied or divided by the same quantity, without disturbing the proportion.*

The reason is obvious; for, by multiplying the antecedents or dividing the consequents, we multiply the ratios or fractions; and by dividing the antecedents or multiplying the consequents, we divide the ratios or fractions (Arts. 56, 58.) But if equal quantities are both multiplied or both divided by the same quantity, the results must be equal.

(VIII). Suppose we have the two proportions,

$$a : b = c : d, \text{ and } a : b = m : n,$$

the ratio $a : b$ being found in both proportions. By Ax. 7, we have

$$c : d = m : n. \quad \text{Hence,}$$

If two proportions have a common ratio, or a ratio in *one proportion equal to a ratio in the other, the two remaining ratios are equal, and may form a proportion.*

(IX). Let there be given the two proportions

$$a:b=c:d, \text{ and } a:m=c:n,$$

in which the corresponding antecedents are alike. By changing the means in each, according to (V), the proportions become

$$a:c=b:d, \text{ and } a:c=m:n; \text{ hence, by (VIII),}$$
$$b:d=m:n, \text{ or } b:m=d:n.$$

But b, d, m, and n are the consequents of the given proportions. Hence,

If in two proportions the antecedents are alike or equal, the consequents will form a proportion.

Suppose now that we have the two proportions

$$a:b=c:d, \text{ and } m:b=n:d,$$

in which the corresponding consequents are alike.

By (V), these proportions become

$$a:c=b:d, \text{ and } m:n=b:d.$$

Consequently, by (VIII),

$$a:c=m:n, \text{ or } a:m=c:n.$$

But a, c, m, and n are the antecedents of the given proportions. Hence,

If in two proportions the consequents are alike or equal the antecedents will form a proportio

(X). Suppose we have the proportion $a:b=c:d$ which is the same as $\frac{a}{b}=\frac{c}{d}$.

Adding ± 1 to each member, we have

$$\frac{a}{b}\pm 1=\frac{c}{d}\pm 1.$$

Reducing each member wholly to a fraction,

$$\frac{a\pm b}{b}=\frac{c\pm d}{d}, \text{ or } a\pm b:b=c\pm d:d,$$

which by (v) becomes

$$a \pm b : c \pm d = b : d = a : c, \ (1),$$

since, from the given proportion, these last two ratios are equal.

If we take the given proportion, invert the ratios, so that it becomes $\frac{b}{a} = \frac{d}{c}$, and then proceed as above, we shall obtain

$$b \pm a : d \pm c = a : c = b : d. \quad (2).$$

Comparing proportions (1) and (2), which are essentially alike, with the given proportion, we infer that

In any proportion, the sum or difference of the first two terms is to the sum or difference of the last two, as the first term is to the third, or as the second is to the fourth.

(XI). From proportion (1) given above, by taking the sign $+$, we have

$a + b : c + d = b : d$. By taking the sign $-$ in (1),

$a - b : c - d = b : d$. Hence, by (VIII),

$a + b : c + d = a - b : c - d$, or by (V),

$a + b : a - b = c + d : c - d$.

Comparing the last two proportions with the original proportion $a : b = c : d$, we infer that,

In any proportion, the sum of the first two terms is to the sum of the last two, as the difference of the first two terms is to the difference of the last two; also, the sum of the first two terms is to their difference, as the sum of the last two terms is to their difference.

Remark. If we had taken proportion (2) in (x), we might have obtained from it

$$b + a : d + c = b - a : d - c, \text{ and}$$

$$b + a : b - a = d + c : d - c,$$

so that the principle stated above is entirely general.

(XII). *If, in any proportion, the antecedents are alike or equal, the consequents must be equal; also, if the consequents are alike or equal, the antecedents must be equal.*

The reason is plain; for equal fractions having equal numerators, must have equal denominators; and equal fractions having equal denominators, must have equal numerators.

Moreover, it is evident that,

If, in any proportion, the second term is greater than the first, the fourth must be greater than the third, and conversely; and if the first two terms are equal, the last two must also be equal.

(XIII). Suppose we have a series of equal ratios, as

$$a:b=c:d=e:f=g:h, \text{ or}$$

$$\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\frac{g}{h}.$$

Let q represent the value of each of these fractions Then,

$$\frac{a}{b}=q, \frac{c}{d}=q, \frac{e}{f}=q, \frac{g}{h}=q.$$

Removing the denominators,

$$a=bq, c=dq, e=fq, g=hq.$$

Adding these equations,

$$a+c+e+g=bq+dq+fq+hq, \text{ or}$$
$$a+c+e+g=(b+d+f+h)q.$$

Dividing by $b+d+f+h$,

$$\frac{a+c+e+g}{b+d+f+h}=q=\frac{a}{b}=\frac{c}{d}, \text{ \&c.; or}$$

$$a+c+e+g:b+d+f+h=a:b=c:d=e:f=g:h.$$

Now, the first term of this proportion is the sum of the antecedents, and the second is the sum of the consequents, of the given ratios. Hence,

In any series of equal ratios, the sum of the antecedents *is to the sum of the consequents, as any one of the antecedents is to its consequent.*

(XIV). Suppose we have the two proportions

$$a : b = c : d, \text{ and}$$
$$e : f = g : h.$$

These are the same as

$$\frac{a}{b} = \frac{c}{d}, \text{ and}$$
$$\frac{e}{f} = \frac{g}{h}.$$

By multiplying together the corresponding members of these two equations, we obtain

$$\frac{ae}{bf} = \frac{cg}{dh}, \text{ that is,}$$
$$ae : bf = cg : dh.$$

This proportion is the same as we should have obtained from multiplying together the corresponding terms of the two given proportions in their first form. This is called multiplying the proportions *in order;* and it is evident that any number of proportions might be combined in the same way. Hence,

If two or more proportions are multiplied in order, the result will form a proportion.

Since division is the reverse of multiplication, it follows that,

If proportions are divided in order, the result will form a proportion.

(XV). Given $a : b = c : d$.

Putting this proportion in the form of $\frac{a}{b} = \frac{c}{d}$, and raising both members to any power, the degree of which is denoted by m, we have

$$\frac{a^m}{b^m} = \frac{c^m}{d^m}, \text{ or } a^m : b^m = c^m : d^m. \quad \text{Hence,}$$

Similar powers of proportional quantities form a proportion.

Since extracting roots is the reverse of finding powers, it follows that

Similar roots of proportional quantities will form a proportion.

Art. **133.** The following exercises are designed to exemplify the foregoing principles of proportions. The correctness of any proportion may be verified by ascertaining that the product of the means is equal to that of the extremes.

1. Illustrate (I) by the proportion $7:10=21:30$.

2. Illustrate (II) by putting $12 \cdot 8 = 32 \cdot 3$ into a proportion; also by forming a proportion from $mn=xy$.

3. Illustrate (III) by finding the value of x in each of the following proportions.

$$x:7=9:21;$$
$$10:x=5:15;$$
$$7:4=x:20;$$
$$3:5=7:x.$$

4. According to (III), what is to be inferred respecting x and y in the proportions

$$3:7=12:x \text{ and}$$
$$3:7=12:y?$$

and what are the values of x and y?

5. According to (IV), what is the mean proportional between 5 and 20? Also, what is to be inferred from the equation

$$x^2=m(a+b)?$$

6. In the proportion

$$5:7=15:21$$

make all the changes authorized by (V).

7. Illustrate (VI) by the proportion

$$10:15=30:45.$$

8 Illustrate (VII) by the proportion

$$30:49=60:98.$$

9 Illustrate (VIII) by the two proportions

$$7:9=21:27,$$
$$7:9=14:18.$$

10. Illustrate (IX) by the two sets of proportions

$$\begin{cases}10:7=30:21,\\10:5=30:15;\end{cases}$$
$$\begin{cases}8:5=16:10,\\12:5=24:10.\end{cases}$$

11. Illustrate (X) by the proportion

$$3:7=9:21.$$

12. Illustrate (XI) by the proportion

$$12:8=60:40.$$

13. According to (XII), what is to be inferred from the proportion

$$9:x=9:3?$$

also, from the proportion

$$y:7=5:7?$$

14. Also, according to (XII), what is to be inferred with regard to x in each of the proportions

$$4:10=12:x,$$
$$6:\ 6=20:x?$$

15. Illustrate (XIII) by the equal ratios

$$1:2=3:6=4:8=9:18=12:24$$

16. Illustrate (XIV) by the two proportions

$$3:\ 5=21:35,$$
$$12:20=42:70.$$

17. Illustrate (XV) by the proportion

$$4:9=36:81.$$

SECTION XLII.

PROGRESSION BY DIFFERENCE.

Art. **134.** *A progression by difference,* or *an arithmetical progression,* is a series of quantities constantly increasing or constantly diminishing by a common difference; and these successive quantities are called the *terms* of the progression.

Thus, 1, 2, 3, 4, 5, &c., is a progression by difference. the common difference being 1; also, 3, 5, 7, 9, 11, &c., the common difference being 2.

A progression is called *increasing*, when the terms increase from left to right; and it is called *decreasing*, when the terms decrease in the same direction. Thus, 8, 11, 14, 17, &c., is an increasing, but 25, 20, 15, 10, &c., is a decreasing progression.

Art. **135.** To exhibit a progression by difference in its most general form, let a be the first term, and d the common difference.

Then, if the progression is increasing,

$$\overset{\text{1st}}{a},\ \overset{\text{2d}}{(a+d)},\ \overset{\text{3d}}{(a+2d)},\ \overset{\text{4th}}{(a+3d)},\ \overset{\text{5th}}{(a+4d)},\ \&c.,$$

will be the successive terms at the commencement of the series.

But if the progression is decreasing,

$$\overset{\text{1st}}{a},\ \overset{\text{2d}}{(a-d)},\ \overset{\text{3d}}{(a-2d)},\ \overset{\text{4th}}{(a-3d)},\ \overset{\text{5th}}{(a-4d,)}\ \&c.,$$

will be the initial terms.

If we examine either of these series, we shall perceive that the coefficient of d in the second term is 1, in the third term it is 2, in the fourth, 3, in the fifth, 4, &c.;

that is, the coefficient of d in any term is always less by 1 than the number which marks the place of the term. In other words, to find any term, we multiply the common difference by a number less by 1 than that which marks the place of the term, and add the product to the first term when the progression is increasing, but subtract the product from the first term when the progression is decreasing.

Hence if, in addition to our previous notation, we denote the number of terms by n, and the last term by l, we have the formula

$l = a + (n - 1)\,d$, in an increasing progression; and

$l = a - (n - 1)\,d$, in a decreasing progression.

If the double sign $\pm$ be used, the general formula for the last term is

$l = a \pm (n - 1)\,d$. Hence,

To find the last term, multiply the common difference by the number of terms minus one, and add the product to the first term if the progression is increasing, but subtract the product from the first term if the progression is diminishing.

1. Required the 12th term of the progression, 7, 10, 13, 16, &c.

In this example, $a = 7$, $d = 3$, and $n = 12$; and by substituting these numbers in the formula, $l = a + (n - 1)\,d$, we have

$$l = 7 + (12 - 1)\,3 = 7 + 11 \,.\, 3 = 7 + 33 = 40.$$

Therefore, the 12th or last term is 40.

2. Required the 9th term of 60, 55, 50, &c.

In this example, $a = 60$, $d = 5$, and $n = 9$, and the progression is decreasing. Hence, $l = a - (n - 1)\,d$ becomes, by substitution,

$$l = 60 - (9 - 1)\,5 = 60 - 8 \,.\, 5 = 60 - 40 = 20.$$

ART. **136.** Let us now proceed to find a formula for the sum of any number of terms. For this purpose, let S represent the sum of n terms of the progression, a, $a+d$, $a+2d$, &c. Then,

$$S=\overset{\text{1st}}{a}+(\overset{\text{2d}}{a+d})+(\overset{\text{3d}}{a+2d})+(\overset{\text{4th}}{a+3d})+\ldots\ldots+\overset{n\text{th}}{l}. \quad (1).$$

If we write the progression in the reverse order, beginning with the last term, it is plain that the successive terms of the same progression will be l, $l-d$, $l-2d$, &c. Hence,

$$S=\overset{n\text{th}}{l}+(\overset{(n-1)\text{st}}{l-d})+(\overset{(n-2)\text{d}}{l-2d})+(\overset{(n-3)\text{d}}{l-3d})+\ldots\ldots+\overset{\text{1st}}{a}. \quad (2).$$

Remark. It is manifest that the terms cannot all be written, unless some determinate value is given to n. We therefore use points to supply the place of the indefinite number of terms.

By adding equations (1) and (2), and observing that d, $2d$, $3d$, &c. in (1), are cancelled by $-d$, $-2d$, $-3d$, &c. in (2), we have

$$2S=(a+l)+(a+l)+(a+l)+(a+l)+\ldots\ldots+(a+l).$$

But since, in this last equation, the quantities included between the several parentheses are the same, and since this same quantity $a+l$ is repeated as many times as there are terms in the progression, that is, n times, the second member is the same as $n(a+l)$. Hence,

$$2S=n(a+l), \therefore$$

$$S=\frac{n(a+l)}{2}.$$ This is the same as

$$S=\frac{n}{2}(a+l), \text{ or } n\cdot\frac{(a+l)}{2}.$$ Hence,

To find the sum of any number of terms in progression by difference, multiply the sum of the first and last terms by half the number of terms, or multiply half the sum of the first and last terms by the number of terms.

Required the sum of 8 terms of the series 6, 10, 14, &c.

In this example, $a = 6$, $d = 4$, and $n = 8$. We are first to find the value of l, which by the preceding Article is $l = 6 + (8 - 1)\,4 = 34$. Then, substituting the values of a, l, and n in the formula for S, we have

$$S = \tfrac{8}{2}\,(6 + 34) = 4 \,.\, 40 = 160.$$

SECTION XLIII.

EXAMPLES IN PROGRESSION BY DIFFERENCE.

ART. **137.** 1. Required the 12th term of the series 10, 16, 22, &c.

2. Required the 20th term of the series 100, 98, 96, &c.

3. What is the sum of 100 terms of 1, 2, 3, 4, &c.?

4. Find the 8th term and the sum of the first 8 terms of 7, 10, 13, &c.

5. Required the sum of 10 terms of the series, in which the first term is 2, and the common difference $\frac{1}{2}$.

6. Required the 25th term, and the sum of the first 25 terms of the series 60, $59\frac{3}{4}$, $59\frac{1}{2}$, &c.

7. A man buys 10 sheep, giving 2 s. for the first, 4 s. for the second, 6 s. for the third, and so on. How much do they all cost him?

8. Twenty stones and a basket are in the same straight line, and 5 yards asunder; how far would a boy travel, if, starting from the basket, he were to pick up the stones, and carry them one by one to the basket?

9. Separate 39 into three parts which shall be in arithmetical progression, the common difference being 7.

Let x = the least part, or first term of the progression

10. Find three numbers in arithmetical progression such that their sum shall be 30, and their continued product 750.

Let $y =$ the common difference, and $x =$ the middle term. Then $x - y$, x, and $x + y$ will represent the numbers.

11. Two men, 189 miles asunder, set out at the same time to travel towards each other till they meet. One of them goes 10 miles each day; the other goes 3 miles the first day, 5 the second, 7 the third, and so on. In how many days will they meet?

Let $x =$ the number of days; then x will represent the number of terms, and will correspond to n in the formula.

12. Two travellers, 135 miles asunder, set out at the same time to travel towards each other. One travels 5 miles the first day, 8 the second, 11 the third, and so on; the other travels 20 miles the first day, 18 the second, 16 the third, and so on. In how many days will they meet?

SECTION XLIV.

PROGRESSION BY QUOTIENT.

ART. **138.** *A progression by quotient*, or *geometrical progression*, is a series of quantities such, that if any one of them be divided by that which immediately precedes it, the quotient will be the same, in whatever part of the series the two quantities are taken. The successive quantities are called *terms* of the progression.

The quotient arising from the division of any term by that which immediately precedes it, is called the *common ratio.*

For example, 3, 6, 12, 24, &c. is a progression by quotient, the common ratio being 2; also, 100, 20, 4, $\frac{4}{5}$, $\frac{4}{25}$, &c. is a similar progression, the common ratio being $\frac{1}{5}$.

A progression by quotient is called *increasing* or *decreasing*, according as the terms increase or diminish from left to right. The former of the preceding progressions is increasing, the latter decreasing.

ART. **139.** In order to exhibit a progression by quotient in its most general form, let a, b, c, d, &c. represent the successive terms at the commencement of the series, and let q be the common ratio.

Now, since from the definition of a progression by quotient, each term is equal to q times the preceding term we have

$$b = a\,q,\ c = a\,q^2,\ d = a\,q^3,\ e = a\,q^4,\ \&c.$$

Representing the last term by l, and supplying the place of the indefinite number of intermediate terms by dots, the terms of the progression will be

$$\overset{\text{1st}}{a},\ \overset{\text{2d}}{a\,q},\ \overset{\text{3d}}{a\,q^2},\ \overset{\text{4th}}{a\,q^3},\ \overset{\text{5th}}{a\,q^4},\ \overset{\text{6th}}{a\,q^5},\ \ldots\ldots,\ l.$$

We see that the exponent of q, in any term of this series, is less by 1 than the number which marks the place of the term. Thus, the 5th term is $a\,q^4$, the 6th is $a\,q^5$, &c. Hence, if n represent the number of terms, the nth or last term will be $a\,q^{n-1}$. But l also represents the last term. Therefore,

$$l = a\,q^{n-1}.$$

This is the formula for the last term. Hence,

To find any term of a progression by quotient, multiply the first term by that power of the common ratio, denoted *by a number less by* 1 *than that which marks the place of the term.*

1. What is the fifth term of the progression 5, 15, 45, &c.?

In this example, $a=5$, $q=3$, and $n=5$; hence,

$$l=a\,q^{n-1} \text{ becomes } l=5\,.\,3^4=5\,.\,81=405.$$

2. Required the seventh term of the series 12288, 3072, 768, &c.

In this case, $a=12288$, $q=\frac{1}{4}$, and $n=7$. Therefore, $l=12288\,.\,(\frac{1}{4})^6=12288\,.\,\frac{1}{4096}=3$.

Art. **140.** We now wish to find a formula for the sum of any number of terms of a progression.

Let S denote the sum of any number n of terms of the series a, $a\,q$, $a\,q^2$, &c. Then,

$$S=a+a\,q+a\,q^2+a\,q^3+a\,q^4+\ldots.+a\,q^{n-2}+a\,q^{n-1}.$$

Multiplying this equation by q, we have

$$q\,S=a\,q+a\,q^2+a\,q^3+a\,q^4+a\,q^5+\ldots.+a\,q^{n-1}+a\,q^n.$$

By observation and a little reflection we shall perceive, that, if the indefinite number of omitted terms were supplied, the terms in the second members of the two equations would all be alike, with the exception of a and $a\,q^n$. Hence, by subtracting the first equation from the second, all the terms in the second members will cancel each other except these two, and the subtraction gives

$q\,S-S=a\,q^n-a$; or $(q-1)\,S=a\,q^n-a$. (Art. 59.)

Hence, dividing by $q-1$, we have

$$S=\frac{a\,q^n-a}{q-1}, \text{ or}$$

$$S=\frac{a\,(q^n-1)}{q-1}.$$

Such is the formula for the sum of any number of terms; but we may also obtain another. In the preceding Article we had

$$l = a q^{n-1};\ \text{multiplying by } q, \text{ we have}$$
$$l q = a q^n.$$

Substituting $l q$ instead of $a q^n$ in

$$S = \frac{a q^n - a}{q - 1}, \text{ we have}$$

$$S = \frac{l q - a}{q - 1}.$$

We have then, for the sum of a geometrical progression, the two following formulæ, viz.,

$$S = \frac{a(q^n - 1)}{q - 1}, \text{ and}$$

$$S = \frac{l q - a}{q - 1}.$$

Hence,

To find the sum of a progression by quotient, raise the common ratio to the power denoted by the number of terms, subtract 1 *from this power, multiply the remainder by the first term, and divide the product by the ratio minus* 1; *or, multiply the last term by the ratio, subtract the first term from the product, and divide the remainder by the ratio minus* 1.

1. Required the sum of six terms of the series, 4, 8, 16, &c.

In this example, $a = 4$, $q = 2$, and $n = 6$. Hence,

$$S = \frac{a(q^n - 1)}{q - 1} \text{ becomes, by substitution,}$$

$$S = \frac{4(2^6 - 1)}{2 - 1} = \frac{4(64 - 1)}{1} = 4 . 63 = 252$$

Or we may find the last term, and then use the formula $S = \frac{l q - a}{q - 1}$. By the last Article, $l = 4 . 2^5 = 4 . 32 = 28$

Then,

$$S=\frac{128.2-4}{2-1}=\frac{256-4}{1}=252.$$

2. Required the sum of seven terms of the series 2, 1, $\frac{1}{2}$, $\frac{1}{4}$, &c.

In this case, $a=2$, $q=\frac{1}{2}$, and $n=7$. Hence, using the first formula for S, we have

$$S=\frac{2\left[\left(\frac{1}{2}\right)^7-1\right]}{\frac{1}{2}-1}=\frac{2\left(\frac{1}{128}-1\right)}{\frac{1}{2}-1}=\frac{2\left(-\frac{127}{128}\right)}{-\frac{1}{2}}=\frac{-\frac{127}{64}}{-\frac{1}{2}}=\frac{127}{32}=3\tfrac{31}{32}.$$

Art. **141.** Whenever, in the formula for S, the ratio q is a proper fraction, that is, a fraction less than 1, $q-1$ will be negative. Also, q^n-1 will be negative, because any power of a proper fraction, the index of the power being greater than 1, is always less than the fraction itself Thus, $\left(\frac{1}{2}\right)^4=\frac{1}{16}$ is less than $\frac{1}{2}$.

Changing the signs of numerator and denominator in the formula for S, which does not alter the value of the fraction, we have

$$S=\frac{a(1-q^n)}{1-q}, \text{ or}$$

$$S=\frac{a-aq^n)}{1-q}.$$

Now, since the powers of a proper fraction constantly diminish in value, as the exponent of the power is increased, it follows that if n, the number of terms, is infinitely great, q^n must be infinitely small, and may be considered zero. In this case, $S=\frac{a-aq^n}{1-q}$ becomes $S=\frac{a-a.0}{1-q}=\frac{a}{1-q}$.

Since q is supposed to be a fraction, let it be represented by $\frac{m}{n}$, so that $q=\frac{m}{n}$. Substituting $\frac{m}{n}$ instead of q in the last formula, we have

$S = \frac{a}{1 - \frac{m}{n}}$. Multiplying numerator and denominator by n,

$$S = \frac{na}{n - m}.$$

This is the formula for the sum of a decreasing progression by quotient, continued to infinity.

Hence,

To find the sum of an infinite decreasing series in progression by quotient, multiply the first term by the denominator, and divide the product by the difference between the denominator and numerator of the ratio.

1. Required the sum of the series 7, $\frac{21}{5}$, $\frac{63}{25}$, &c., continued to infinity.

In this example, $a = 7$, $q = \frac{m}{n} = \frac{3}{5}$. Hence,

$$S = \frac{7 . 5}{5 - 3} = \frac{35}{2} = 17\tfrac{1}{2}.$$

SECTION XLV.

EXAMPLES IN PROGRESSION BY QUOTIENT.

ART. **142.** 1. What is the sum of 10 terms of the series, 1, 2, 4, 8, &c.?

2. Required the 6th term and the sum of the first 6 terms of the series, 5, 20, 80, &c.

3. Required the sum of 10 terms of the progression, 8, 4, 2, 1, $\frac{1}{2}$, $\frac{1}{4}$, &c.

4. Required the sum of the preceding series, continued to infinity.

5. What is the sum of the series, 2, $\frac{6}{8}$, $\frac{18}{64}$, &c., continued to infinity?

6. What three numbers form a geometrical progression, in which the mean term is 8, and the sum of the extremes 34?

Let $x =$ the ratio. Then $\frac{8}{x}$, 8, and $8x$ will represent the terms.

7. Three numbers in progression by quotient are such, that the sum of the first two is 90, and the sum of the last two 180. Required these numbers.

Let $x =$ the least number, and $y =$ the ratio. Then, x, xy, and xy^2 will represent the numbers. Hence,

$$\begin{cases} x + xy = 90; \\ xy + xy^2 = 180. \end{cases}$$

Remark. One of these equations can be divided by the other.

8. Four numbers are in geometrical progression. The sum of the first three is 62, and the sum of the last three is 310. What are these numbers?

9. Separate 105 into three parts which shall form a geometrical progression, such that the third term shall exceed the first by 75.

10. The sum of three numbers in progression by quotient is 91, and the mean term is to the sum of the extremes as 3 to 10. Required these numbers.

END.

5 : 7 = 15 : 21 = 5 : 21 = 7 : 15

5 : 15 = 7 : 21
21 : 7 = 15 : 5
15 : 21 = 5 : 7
7 : 5 = 21 : 15

5 × 10 : 15 = 30 : 45

50 : 75 = 150 : 225 — (1) ×

2 : 3 = 6 : 9 (2) ÷

(6) 2 : 3 = 10 : 15 — (3) ×÷

20 : 30 = 6 : 9 (4) × ÷

$\frac{x}{y} : \frac{x+y}{y} = 0$, $= 1$

$\frac{x+y}{y} = 0$, $x+4 = y - 10$, $x = y$, $x + 4 = y$

$a+y)\,a^5+2a^4y+a^3y^2\,(a^4+a^3y$

a^5+a^4y

$+a^4y+a^3y^2$

$+a^4y+a^3y^2$

$a+e$

$y-b$

$ay+ey-a$

$a+y)\,a^3y+2a^2y^2+2ay^3+y^4\,(a^2y+ay^2+$

$a^3y+a^2y^2$

$+a^2y^2+2ay^3$

$+a^2y^2+ay^3$

$+ay^3+y^4$

$+ay^3+y^4$

$b-e$

$y+a$

$by-ey+ab$

$a+y)\,a^2y+ay^2+y^3\,(ay$

$a^2y\;\;ay^2$

$a+y)\,a^4+a^3y\,(a^3$

a^4+a^3y

[illegible]

$ax + cx - ab - cb -$

$bx - cx + ab - ac$

$ax + 2cx + 2ab - cb - bx + ac$

$5x + 3y = 14.50$ $\quad x = 2009$

$7x + 8 = 7y$ $\quad x + 2x + 3x + 4x = 5.$

$10x = 5.00$

$x = .50$

$3x + 3y = 14.50$

$-7x + 4y = -8$

$35x + 21y = 101.50$

$35x$

$20x + 12y = 58.00$

$-21x + 12y = -24.$

$41x = 82$

$x = 2$

$10 + 3y = 14.50$

$3y = 14.50 - 10$

$3y = 4.50$

$y = 1.50$

50
150

50
100
150
200
5.00

a

2
14

14
6.00
8.00

$x^2 + 11xy + y^2$

$25m^4 + 10m^2n + n^2$

$4x^2 - y^2$

$9a^2 - 4b^2$

$1 - 81x^2$

$a^6 - a^4c^2$

$a^8 - c^8$

$x^8 - y^4$

$175) 13.65$

$\frac{-4+4}{14} = \frac{0}{14} = 0 \quad \frac{-4}{14-10} = \frac{-4}{4} = 1$

$16 + 4 = 20$

-4