Saml. A. Chambers

ECLECTIC EDUCATIONAL SERIES

A

COMPLETE ALGEBRA

TO ACCOMPANY

RAY'S SERIES OF MATHEMATICS

BY

GEORGE W. SMITH

Woodward High School, Cincinnati, Ohio

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PREFACE.

THIS work was commenced sixteen years ago at the earnest solicitation of numerous teachers, who were dissatisfied with the text-books then in use. That they were not alone in their opinion is evidenced by the number of new treatises, or revisions of old ones, printed since that time, and now used in the schools of this country. The crudeness of even the best Algebras of a quarter-century ago was mainly owing to the fact that, as a rule, mathematicians neglected the elementary branches for the more attractive fields of Higher and Applied Mathematics; hence blunders and inconsistencies were allowed which otherwise would not have been tolerated. The wonderful progress made in the Natural Sciences, and the extended use of Algebra in the treatment of Geometrical Magnitudes, have finally called the attention of educators to the necessity of improving the elementary treatises, and more rigidly limiting the meaning of the signs. That this agitation comes none too soon is evident to every thoughtful teacher, and can be readily seen by any one who compares the various text-books used in our schools. Note the following inconsistencies: In some text-books now before me, $6 : 7$ equals $\frac{6}{7}$; in others, $6 : 7$ equals $\frac{7}{6}$. In some, $6 + 4 \times 2 = 20$; in others, $6 + 4 \times 2 = 14$. Of course, the meaning and use of a sign depend upon *agreement*, but it is of extreme importance that WE DO AGREE in such matters. In the same work, too, statements incompatible with each other are made; thus, $a \div bc$ and $a \div b \times c$ are said to have *different* values, and yet bc and $b \times c$ are, *in all works*, said to have *one* and *the same* meaning. Since $a \div bc$ and $a \div b \times c$ differ *only* in the use of $b \times c$ for bc, it is plainly necessary that one or the other of these two statements be changed. One of the objects in writing this book is to urge the adoption of the following law for Numerical Values; viz., (1) *Find the value of each term separately;* thus, $6 + 4 \times 2 = 6 + 8 = 14$. (2) *In finding the value of a term, begin at the* RIGHT *and use the signs in their order;* thus, $6 \div 4 \times 2 = 6 \div 8 = \frac{3}{4}$. In other words, *the portion of the term to the* LEFT *of the division sign is the* DIVIDEND, *and the part to the* RIGHT *is the* DIVISOR. This law is at least as easy of application as the one in vogue in many American works, and has the merit of allowing no inconsistencies, besides permitting the use of the very important law that, *under all circumstances,* ab and $a \times b$ have the *same meaning.*

Owing to a complication of causes, the preparation of this work has been so long delayed that many of the improvements contemplated when it was planned have been made since by others. The author hopes, however, that his work is of sufficient value to warrant its being received favorably by the American public.

The task of preparing an elementary treatise is far from being an easy one, and this difficulty is further enhanced by the immaturity of many beginners. For this reason most authors have prepared their works in *two* or even *three* divisions: Part I. for beginners who are immature or badly prepared; Part II. for more advanced students. The opinion is rapidly gaining ground, however, that this method of studying Algebra consumes too much time and does not tend to produce the best results. In conformity with this opinion the present work is intended to combine in one volume all the principles a pupil needs from the time he begins the study of Algebra until he enters college, together with additional chapters for the benefit of such as do not have the advantages of a university education. Special care has been taken to adapt this treatise to the needs of those students who are without the aid of a teacher; hence every principle of importance is carefully explained and profusely illustrated by examples.

As in other text-books, the first chapter is devoted to *definitions.* This is partly done for convenience of reference, and partly because many teachers prefer that their pupils study the meanings of the various terms employed before proceeding to subsequent chapters. Some of the best teachers, however, believe that the *definition* should accompany the *word* when actually used; that the pupil should not be required to study the meaning of a term until he has some idea of the *purpose* of such study. For the benefit of this class, whenever a word defined in Chapter I. is employed thereafter, reference is made to the article in which the definition is found. This plan is pursued until it is assumed that the student has learned the meaning of each term used.

The propriety of introducing at such an early stage the study of *negative* and *fractional exponents* may be questioned by many able teachers; and with weak pupils it is perhaps best to postpone the consideration of such topics to a later period.

The sequence of subjects as herein contained has been carefully tested, with results which are eminently satisfactory; but some teachers may prefer to change the order of the topics. Thus, as mentioned above, Fractional and Negative Exponents may be treated many chapters later; Powers and Roots may also be considered later, whilst many of the principles of Factoring may be introduced immediately after Division, etc. The arrangement of the topics being to a considerable extent arbitrary, an effort has been made to render each chapter as nearly independent as possible, so that the order in which certain subjects are taken may be varied at the option of the instructor. Throughout this work it is assumed that the teacher would rather *omit* than *supply*, and would rather *postpone* topics for future consideration than be compelled to *introduce* them or bring them forward.

For reasons specified previously, the subject of Numerical Values has received unusual care and attention, and the same may be said of Factoring and Equations. The introduction of Equations so early and so often is due to the growing desire among teachers to cultivate the reason of the pupil at the same time that he is studying the mere mechanical parts of the Algebra. The subject of Detached Coefficients, also, is treated more fully than in other elementary works, and the use of this method of work is shown in the treatment of subjects involving division (the Greatest Common Measure, for instance). The time seems to be rapidly approaching when this plan will supplant the longer and more cumbrous ways now in vogue.

In the preparation of this work it was thought best not to dispense with rules, but the student is recommended not to commit them to memory. They are intended merely as guides to the pupil, and to aid him in *stating* the processes employed in solving the various problems. In writing these rules, no attempt has been made to secure conciseness or elegance at the expense of clearness, the object being to state the *successive steps* in the solution.

Many teachers prefer that the examples submitted for ordinary class work shall be accompanied by the answers; others prefer problems unaccompanied by answers; a third class adopts a middle course, by furnishing the results until the principle considered is understood, and then leaving the pupil unaided, to apply the principle. In this work the more difficult problems are worked out, then similar problems with their answers are given, and finally at the end of each chapter an "exercise" is appended, including examples unaccompanied by answers. It is hoped that a sufficient number of examples is submitted to satisfy each and every requirement. Only the strongest and best pupils are expected to solve *all* the problems, and it is well for the student to remember that quality, not quantity, is the test of excellence, and that one solution thoroughly understood is better than a dozen learned by rote. An examination of this work will show that all complicated problems have been carefully excluded, and that none are admitted which are too difficult for studious pupils of medium ability. As far as practicable, the examples are classified, the simplest of each class, as well as the simplest classes, preceding the more difficult.

The matter contained in this work is to some extent original, but the best English and American works have been freely consulted, the chief sources of supply being Colenso's Algebra of 1849 and Todhunter's Algebra of 1862. Care has been taken not to trespass upon the works of recent American authors; but whenever a proposition occurs in three or more books, it has been considered common property, and treated accordingly: certain propositions not coming under this rule are also included, because they were in manuscript, and in the hands of the pupils of Woodward, previous to their publication elsewhere.

If the student has a fair knowledge of Arithmetic before beginning the study of Algebra, he can easily accomplish the *whole work* of this text-book in three hundred recitations of one hour each. If but one school year be devoted to the subject, it is advisable to omit portions of these subjects: Numerical Values, Negative Numbers, Indeterminate Equations, Maxima and Minima, Factoring, Variation, Probabilities, Binomial Theorem, and Annuities. It might be well, also, to learn but *one* method of Multiplication, Division, and solving Quadratics, and to exclude Equations above the second degree.

Any amendments or corrections will be thankfully received.

G. W. SMITH.

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ALGEBRA.

CHAPTER I.

DEFINITIONS.

1. A **Definition** is such an explanation of the meaning of any word, sign, or phrase, as will perfectly distinguish the thing defined from every thing else. Thus, $+$ may be defined or explained by saying that it denotes *addition.*

QUANTITY AND NUMBER.

2. Quantity is whatever may be regarded as made up of parts, each like the whole. A quantity may be increased, diminished, and measured.

3. To Measure a Quantity is to find out how many times it contains some other known quantity of the same kind.

4. A **Unit** is the name given to the quantity with which other quantities of the same kind are compared. That is, the unit is the standard of comparison.

Thus, the yard, the square yard, the cubic yard, the day, the dollar, the ounce, are units for measuring lines, surfaces, space, time, money, and weight respectively.

5. Two or more quantities of the same kind are compared with each other, by finding out how many times each contains the same unit.

6. Number is the thought involving *how many times* the unit is contained in the quantity measured.

7. A **Quantity** is the answer to the question, "How *much?*" This answer must state the name of the unit.

8. A **Number** is the answer to the question, "How *many?*" Here the name of the unit is given in the question.

For example: A has a quarter of a dollar and a dime; how much money has he? The answer is the *quantity*—thirty-five cents. As the question does not specify the name of the unit, the answer must contain that name. If the question be, "How many cents has he?" the answer is the *number*—thirty-five.

Numbers may be either whole or fractional. The word *integer* is often used instead of *whole number*. *Integral* is the corresponding adjective.

9. Mathematics is the science which treats of the properties, relations, and measurements of quantities.

Arithmetic and *Algebra* are the branches of Mathematics which treat of numbers.

10. In Mathematics the subjects submitted for consideration are of two kinds; namely, *Theorems* and *Problems*.

11. A **Theorem** is a statement in words of a mathematical truth to be proved. The course of reasoning by which the truth of a theorem is shown, is its *demonstration*. A theorem whose truth is self-evident, is called an *Axiom*.

12. A **Problem** is something to be done. The doing of a problem is called its *solution*. To solve a problem may mean: (1) To perform some operation; as, for example, to add together two or more numbers: or, (2) To find the value of an unknown number from its given relations to known numbers; as, for example, to find the fourth term of a proportion, when the other three are known.

SYMBOLS OF NUMBERS.

13. The **Symbols** employed in Arithmetic to represent numbers are the ten digits or figures, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. Each of these figures, or any combination of them, has one value, and only one. Thus, 8 represents the number *eight,* 85 represents *eighty-five,* 581 represents the number *five hundred and eighty-one,* etc. Every figure has a simple value, depending upon its signification, and a local value, depending upon its place; but the same figures, in like positions, always express the same number. That is, *figures represent particular numbers.*

14. In studying the general properties of numbers, and in solving problems, it frequently becomes convenient, and even necessary, to use symbols which do *not* represent particular numbers. For this purpose letters are used. Each letter may represent *any* numerical value consistent with the given conditions.

15. **Algebra** is a method of solving problems, and of investigating the general relations and properties of numbers, by means of letters and other symbols.

16. **Particular Numbers** are represented in Algebra, as in Arithmetic, by figures; numbers in general are represented by letters. It is usual to represent known numbers by the first letters of the alphabet, a, b, c, d, etc., and unknown numbers by the last letters, x, y, z. This distinction is not necessary, and need not be strictly observed.

PRINCIPAL SIGNS.

17. **Signs** are the symbols employed in Algebra to express the relations of numbers, and the operations to be performed upon them.

18. The **Symbols of Relation** are *the sign of equality*, $=$, read *equals* or *is equal to*, and the *signs of inequality*, $>$ or $<$. Thus, $7 > 4$ is read, *seven is greater than four;* and $4 < 7$ is read, *four is less than seven.*

19. The **Symbols of Operation** are the sign of addition, $+$; the sign of subtraction, $-$; the sign of multiplication, $\times$; the sign of division, $\div$; the signs of involution, ^{2}, ^{3}, ^{4}, etc.; the signs of evolution, $\sqrt{\ }$, $\sqrt[2]{\ }$, $\sqrt[3]{\ }$, $\sqrt[4]{\ }$, etc. These signs have the same meanings in Algebra as in Arithmetic, but $+$ and $-$ are also used as signs of opposition. See Chapter on *Negative Numbers*, Art. 78.

20. The **Sign of Addition**, $+$ (read *plus*, meaning *more*), signifies that the number to which it is prefixed is to be added.

21. The **Sign of Subtraction**, $-$ (read *minus*, meaning *less*), signifies that the number to which it is prefixed is to be subtracted.

22. The **Double Sign**, $\pm$ or $\mp$, signifies that the number to which it is prefixed is to be both added and subtracted. Thus, 5 ± 3 is equivalent to the two statements $5 + 3 = 8$ and $5 - 3 = 2$; also, 6 ∓ 5 is equivalent to $6 - 5 = 1$ and $6 + 5 = 11$. The symbol $\pm$ is read *plus or minus*, and the symbol $\mp$ is read *minus or plus*. The upper sign shows which operation is to be performed first.

23. The **Sign of Multiplication**, $\times$ (read *times*, *into*, or *multiplied by*), signifies that the numbers between which it is placed are to be multiplied together. These numbers are called *factors*, and the result is called the *product*.

Multiplication may also be expressed by placing a period (.) between the factors. Thus, $2 . 3 . 4 = 2 \times 3 \times 4$; $6 . 8 . a . b = 6 \times 8 \times a \times b$, etc. The period is not used as a sign of multiplication whenever it can be mistaken for a decimal point. For example, 6.4 means $6\frac{4}{10}$, and not 6×4.

When not more than one of the factors is expressed arithmetically, multiplication is commonly denoted by writing the factors in close succession. Thus, $a \times b \times c = abc$; $45abc = 45 \times a \times b \times c$; $8 \times 6 \times a \times b \times c = 8 \times 6abc$; etc.

24. The **Sign of Division,** $\div$ (read *divided by,* or simply *by*), signifies that the number which precedes it, called the *dividend,* is to be divided by the number which follows it, called the *divisor.* The result is called the *quotient.*

Division may be expressed by writing the dividend above a horizontal line, and the divisor below the line; thus, $24 \div 6 = \frac{24}{6}$. The sign of a ratio, : (read *is to*), is also used as a sign of division; thus, $18 \div 6$, $\frac{18}{6}$, $18 : 6$, have the same meaning, namely, that 18 is to be divided by 6.

FACTORS, POWERS, AND ROOTS.

25. The **Factors** of a number are those numbers whose product is that number. Thus, 24 may be separated into the following sets of factors: 24×1; 12×2; 8×3; 6×4; $6 \times 2 \times 2$; $4 \times 3 \times 2$; $2 \times 2 \times 2 \times 3$.

Similarly, $6abc = 2 . 3 . a . b . c = 2 . 3abc = 2a . 3bc$, etc.

26. Factors expressed by letters are called *literal* factors. Factors expressed by figures are called *numerical* factors.

27. The **Co-efficient.**—When two factors form a product, either of the two is the *co-efficient* or *co-factor* of the other. In the product $6abc$, $2a$ is the co-efficient of $3bc$; $3b$ is the co-efficient of $2ac$; 6 is the co-efficient of abc; 1 is the co-efficient of $6abc$; $6c$ is the co-efficient of ab; etc.

28. The word co-efficient is frequently used in the sense of *numerical co-efficient.* Thus, in $6abc$ the numerical co-efficient is 6. When no numerical co-efficient is expressed, 1 is understood. Thus, abc has the same meaning as $1abc$.

29. When the co-efficient of a factor is an integer, that factor is called a *divisor* or *measure* of the product, and the product is called a *multiple* of the factor. For instance, $6 \times \frac{1}{2} = 3$. Here $\frac{1}{2}$ is a measure of 3, and 3 is a multiple of $\frac{1}{2}$, but 6 is *not* a divisor of 3. When *both* factors are integers, each is a measure of the product, and the product is a multiple of either.

30. When the product of two factors is unity, each factor is called the *reciprocal* of the other. Thus, $\frac{2}{3}$ and $\frac{3}{2}$ are reciprocals. Hence, the reciprocal of a number is unity divided by that number. The reciprocal of a is $1 \div a$; of 3 is $\frac{1}{3}$; of 1 is 1; of $\frac{4}{5}$ is $\frac{5}{4}$; of $\frac{1}{2}$ is 2; of $2\frac{1}{2}$ is $\frac{2}{5}$; etc.

31. When all the factors of a product are equal, the product is called a *power* of one of those factors, and each factor is called a *root* of the product. Thus, 5×5 is the second power of 5, and 5 is the second root of 25; $5 \times 5 \times 5$ is the third power of 5, and 5 is the third root of $5 \times 5 \times 5$. Similarly, 2 is the fourth root of 16, and 16 is the fourth power of 2. In like manner, aa is the second power of a, aaa is the third power of a, $aaaa$ is the fourth power of a, etc.; a is the second root of aa, the third root of aaa, the fourth root of $aaaa$, etc. Every number is the first power of itself, also the first root of itself. It is evident that the third power of the third root of any number is that number; that the fourth power of the fourth root of any number is that number, etc.

32. The **Exponent** is a small figure or letter placed above a number and a little to its right, to show how many times the number is taken as a factor. Thus, $a^1 = a$; $a^2 = aa$; $a^3 = aaa$; $a^2b = aab$; $ab^2 = abb$; $a^3b^2 = aaabb$; etc. a^2 is read, *the second power of a*, or *the square of a*, or *a squared*; a^3 is read, *the third power of a*, or *the cube of a*, or *a cubed*; a^4 is read, *the fourth power of a*, or *a to the fourth power*, or *a to the fourth*; etc.

33. The meaning of co-efficient must be carefully distinguished from that of exponent. Thus, $4a = a + a + a + a$; here 4 is the *co-efficient.* $a^4 = a \times a \times a \times a$; here 4 is the *exponent.* If $a = 3$, $4a = 4 \times 3 = 12$; whilst $a^4 = 3 \times 3 \times 3 \times 3 = 81$.

34. The exponent affects only the factor above which it is placed. Thus, $ab^3 = a \,.\, b \,.\, b \,.\, b$. If $a = 3$ and $b = 2$, $ab^3 = 3 \times 2^3 = 3 \times 8 = 24$.

35. *To extract the square root* of a number means to find a factor whose square or second power is equal to that number. To extract the *cube* root of a number means to find a factor whose cube or third power is equal to that number. To extract the *fourth* root of a number means to find a factor whose fourth power is equal to that number. To extract the *n*th root of any number means to find a factor whose *n*th power is equal to that number.

36. The **Radical Sign,** $\sqrt{\ }$, when placed before a number, denotes that a root of that number is to be extracted. The *index* is a small figure or letter placed over the radical sign to denote what root is to be extracted. When no index is written, 2 is understood. Thus, the square root of a is denoted by $\sqrt{a}$ or by $\sqrt[2]{a}$; the cube root of a is denoted by $\sqrt[3]{a}$; the fourth root of a is denoted by $\sqrt[4]{a}$; etc. If $a = 64$, $\sqrt{a} = 8$, because $8^2 = 64$; $\sqrt[3]{a} = 4$, because $4^3 = 64$; $\sqrt[6]{a} = 2$, because $2^6 = 64$; etc.

37. The root of a number is also denoted by the *fractional exponent.* Thus, $a^{\frac{1}{2}}$ denotes the square root of a; $a^{\frac{1}{3}}$ denotes the cube root of a; $a^{\frac{2}{3}}$ denotes the cube root of a^2; $a^{\frac{3}{4}}$ denotes the fourth root of a^3; etc. Hence, *the numerator of the fractional exponent denotes the power, and the denominator denotes the root.* For example, $8^{\frac{2}{3}} = 4$, because the cube root of 8 is 2, and the square of 2 is 4.

Similarly, $64^{\frac{2}{3}} = 16$, because the cube root of 64 is 4, and $4^2 = 16$, etc.

SYMBOLS OF AGGREGATION—MINOR SYMBOLS.

38. The **Symbols of Aggregation**, the *brackets* [], the *parenthesis* (), the *braces* { }, the *vinculum* ——, the *bar* | , denote that all the numbers inclosed are to be treated as a single number. Thus, $[6 + 4 - 7] \times 5 - 8$, $(6 + 4 - 7) \times 5 - 8$, $\{6 + 4 - 7\} \times 5 - 8$, $\overline{6 + 4 - 7} \times 5 - 8$, $\left.\begin{array}{r} 6 \\ +4 \\ -7 \end{array}\right| \times 5 - 8$, have the same meaning; namely, $6 + 4 - 7$ is to be taken as a single number, this result is to be multiplied by 5, and from the product 8 is to be taken. Solution: (1) $6 + 4 - 7 = 3$; (2) $3 \times 5 = 15$; (3) $15 - 8 = 7$. *Ans.* 7.

Similarly, $6 \times 5 + 2 \times 3 = 30 + 6 = 36$.

$$(6 \times 5 + 2) \times 3 = (30 + 2) \times 3 = 32 \times 3 = 96.$$
$$6 \times (5 + 2) \times 3 = 6 \times 7 \times 3 = 126.$$
$$6 \times (5 + 2 \times 3) = 6 \times (5 + 6) = 6 \times 11 = 66.$$

In like manner, $\frac{4 + 2 \times 3}{2 + 3} = \frac{4 + 6}{5} = \frac{10}{5} = 2$. It will be observed that the line separating the numerator from the denominator is a vinculum, used in a special sense.

39. The **Symbols of Continuation** are dots, . . ., or dashes, - - - -, and are read, *and so on.*

40. The **Symbol of Deduction** is ∴ (read *hence, therefore,* or *consequently*).

41. The **Symbol of Reason**, ∵, is read *since* or *because.*

42. The **Symbol** ∼ is sometimes used to denote the difference between two numbers. Thus, $a \sim b$ is equal to $a - b$ or to $b - a$, according as a is greater than b or less than b.

EXAMPLES.

1. Give the simplest factors of $18a^2b^3c$.

Ans. 2, 3, 3, a, a, b, b, b, c.

2. In $18a^2b^3c$, what is the co-efficient of $9b$? *Ans.* $2a^2b^2c$.

3. Write the reciprocal of a; $a - b$; $1\frac{1}{4}$.

Ans. $\frac{1}{a}$; $\frac{1}{a-b}$; $\frac{4}{5}$.

4. Give the second, third, and fifth powers of 3.

Ans. 9; 27; 243.

5. Give the second, third, and sixth roots of 64.

Ans. 8; 4; 2.

6. Give 2 the exponent 3; and 3 the exponent 2.

Ans. $2^3 = 8$; $3^2 = 9$.

7. From 3 with an exponent 5 take 3 with a co-efficient 5.

Ans. $3^5 - 5 \times 3 = 243 - 15 = 228$.

8. $[2 \times 6 - 3] \times (5 - 2) = ?$ *Ans.* $[12 - 3] \times (3) = 27$.

9. $\{5 \times 4 - 2 \times 3\} \times \overline{2 \times 5 - 4} = ?$

Ans. $\{20 - 6\} \times \overline{10 - 4} = 14 \times 6 = 84$.

10. Find the value of $16^{\frac{3}{4}}$; $8^{\frac{4}{3}}$; $9^{\frac{1}{2}}$. *Ans.* 8; 16; 3.

11. Find the value of $5a^{\frac{1}{2}}$ when $a = 16$. *Ans.* 20.

12. Simplify $a \sim b$, when $a = 3^4$ and $b = 4^3$.

Ans. $81 - 64 = 17$.

ALGEBRAIC EXPRESSIONS.

43. An **Algebraic Expression** is a collection of letters, figures, and signs used to denote a number. Thus, $6a$ is the algebraic expression for six times the number denoted by a.

44. A **Term** is an algebraic expression not connected with any other by the sign + or −, or it is any one of the parts of an expression which are so connected. Thus, $3a \div bc$ is a term; a^2 and $+ bc$ are the terms of $a^2 + bc$.

45. A **Positive Term** is a term which has the sign + prefixed to it. Thus, $+ a^2bc$ is a positive term.

When the first term of an expression is positive, its sign need not be expressed. Thus, $a^2 + bx$ has the same meaning as $+ a^2 + bx$.

46. A **Negative Term** is a term which has the sign − prefixed to it. Thus, in $ab - 3c$, $- 3c$ is a negative term.

47. A **Simple Term** is a single expression not having parts separated by + or −. Example: $3ab \div c$.

48. A **Compound Term** is a collection of terms united by one or more of the signs of aggregation. For example, each of the following is a compound term: $a(b + c)$; $(ab + c)$; $\{a + [b - c]\}$; $\{a + [b + (c - d)]\}$.

49. **Similar** or **Like Terms** are terms which have the same letters and the corresponding letters affected by the same exponents. Thus, $3a^2bc^3$ and $- 5a^2bc^3$ are like terms; but $3a^2bc^3$ and $5ab^3c^2$ are unlike terms.

50. The **Dimensions** of a term are its literal factors.

51. The **Degree** of a term is equal to the number of its dimensions, and is determined by taking the sum of the exponents of its literal factors. Thus, $5a^2bc^3$ is of the *sixth* degree, because the exponent of a is 2, of b is 1, and of c is 3, and the sum of these exponents is 6.

52. A **Monomial** is an expression which contains a single term, simple or compound; as, $5a^2b \div c$ or $7\,[a + b]$.

53. A **Polynomial** or **Compound Expression** is an expression which contains two or more terms.

54. A **Binomial** is a polynomial of two terms.

55. A **Trinomial** is a polynomial of three terms.

56. The terms of a polynomial may be written in any order if the sign of each term be retained. Thus, $a + b - c = b + a - c = b - c + a = a - c + b$, etc.

57. A **Homogeneous Polynomial** is a polynomial whose terms are all of the same degree. Thus, $6x^4 - 3x^3y + 4x^2y^2$ is homogeneous, for each term is of the fourth degree.

58. A polynomial is arranged according to the *descending* powers of some letter when the value of the exponent of that letter in each term is less than that of the preceding term. A polynomial is arranged according to the *ascending* powers of some letter when the value of the exponent of that letter in each term is greater than that of the preceding term. Thus, $x^4 - ax^3 + bx^2 - c$ is arranged according to the descending powers of x, and $a - bx + cx^2 - x^3$ is arranged according to the ascending powers of x.

59. The **Numerical Value** of an algebraic expression is the number obtained by giving a particular value to each letter, and then performing the operations indicated. Thus, if $a = 5$, $b = 4$, and $c = 2$, $ac - bc + ab = 5 \times 2 - 4 \times 2 + 5 \times 4 = 10 - 8 + 20 = 22$.

60. A **Formula** is an expression of a mathematical truth by means of symbols.

61. A **Rule** is a concise statement of the method of finding one side of a formula when the other side is known.

EXAMPLES.

1. How many terms in $2a + 3b \times c - d$? *Ans.* 3.
2. What is the degree of $3^5a^2b^4c$? *Ans.* Seventh.
3. What is the value of ab^3 when $a = 3$, $b = 2$? *Ans.* 24.
4. $6^2 + 4 \times 2 = ?$ *Ans.* 44.

5. $9^{\frac{1}{2}} \times 8^{\frac{2}{3}} \times 16^{\frac{3}{4}} = ?$ *Ans.* $3 \times 4 \times 8 = 96$.

6. In how many ways may $2a - 3b + c$ be written? *Ans.* 6.

7. Write twice the square of the cube root of a. *Ans.* $2a^{\frac{2}{3}}$ or $2\sqrt[3]{a^2}$.

8. From the cube of b take five times the square of a. *Ans.* $b^3 - 5a^2$.

9. Indicate the $(a + b)$ power of the fifth root of $(a - b)$. *Ans.* $(a - b)^{\frac{a+b}{5}}$ or $[\sqrt[5]{a - b}]^{a+b}$.

10. Indicate the a root of the cube of $\frac{1}{2}$. *Ans.* $(\frac{1}{2})^{\frac{3}{a}}$ or $\sqrt[a]{(\frac{1}{2})^3}$.

62. The principal **Axioms** employed in algebra are the following:

I. Numbers which are equal to the same number are equal to each other.

II. If equal numbers be added to equal numbers, the sums will be equal.

III. If equal numbers be subtracted from equal numbers, the remainders will be equal.

IV. If equal numbers be multiplied by equal numbers, the products will be equal.

V. If equal numbers be divided by equal numbers, the quotients will be equal.

VI. If the same number be both added to and subtracted from another, the value of the latter will not be altered.

VII. If a number be both multiplied and divided by another, the value of the former will not be altered.

VIII. If equal numbers be added to unequal numbers, the sums will be unequal.

IX. If equal numbers be subtracted from unequal numbers, the remainders will be unequal.

X. If unequal numbers be multiplied by equal numbers, the products will be unequal.

XI. If unequal numbers be divided by equal numbers, the quotients will be unequal.

XII. If unequal numbers be added to unequal numbers, the greater to the greater and the less to the less, the sums will be unequal.

XIII. If a be greater than b, and b greater than c, then a will be greater than c.

XIV. Like powers of equal numbers are equal.

XV. Like roots of equal numbers are equal.

XVI. A number may be substituted for an equal number in any expression.

63. The following principles, derived directly from the arithmetical definitions of Addition, Subtraction, Multiplication, and Division, though not strictly self-evident, merely require illustration to be perfectly understood:

I. Adding two or more numbers is equivalent to adding their sum. $4 + 6 + 4 + 5 = 4 + (6 + 4 + 5) = 4 + 15 = 19$.

II. Subtracting two or more numbers is equivalent to subtracting their sum. $17 - 6 - 4 - 3 = 17 - (6 + 4 + 3) = 17 - 13 = 4$.

III. If any number be diminished by an equal number, the remainder is zero (0). $6 - 6 = 0$.

IV. Adding any number and then subtracting a less number is equivalent to adding their difference. $7 + 8 - 5 = 7 + 3 = 10$.

V. Adding any number and then subtracting a greater number is equivalent to subtracting their difference. $21 + 3 - 7 = 21 - 4 = 17$.

VI. Multiplying two or more numbers by the same number and adding the products, is equivalent to multiplying the sum of the numbers by that number. Thus, $5 \times 6 + 3 \times 6 + 10 \times 6 = 30 + 18 + 60 = 108$, and $(5 + 3 + 10) \times 6 = 18 \times 6 = 108$. Similarly, it may be shown that for any values of a, b, c, and d, $ad + bd + cd = (a + b + c)d$.

VII. Multiplying two numbers by a number and taking the difference of the products, is equivalent to multiplying the difference of the numbers by that number. Thus, $5 \times 6 - 3 \times 6 = 30 - 18 = 12$, and $(5 - 3) \times 6 = 2 \times 6 = 12$. In like manner it may be shown that for all values of a, b, and c, $ac \sim bc = (a \sim b)c$.

VIII. If any number be divided by an equal number, the quotient is unity (1). That is, $\frac{a}{a} = 1$.

IX. Dividing two or more numbers by the same number and adding the quotients, is equivalent to dividing the sum of the numbers by that number. Thus, $\frac{24}{4} + \frac{8}{4} + \frac{16}{4} = 6 + 2 + 4 = 12$, and $\frac{24 + 8 + 16}{4} = \frac{48}{4} = 12$. Similarly, for all values of a, b, c, and d, $\frac{a}{d} + \frac{b}{d} + \frac{c}{d} = \frac{a + b + c}{d}$.

X. Dividing two numbers by the same number and taking the difference of the quotients, is equivalent to dividing the difference of the numbers by that number. Thus, $\frac{36}{4} - \frac{24}{4} = 9 - 6 = 3$, and $\frac{36 - 24}{4} = \frac{12}{4} = 3$. In like manner, for all values of a, b, and c, it may be shown that $\frac{a}{c} \sim \frac{b}{c} = \frac{a \sim b}{c}$.

EXAMPLES.

What number must x represent in each of the following?

1. x is 5 more than a. *Ans.* $x = a + 5$.
2. $x - 5 = 11$. *Ans.* $x = 16$.
3. $x = y$ and $y = 7$. *Ans.* $x = 7$.
4. $5x = 60$. *Ans.* $x = 12$.
5. $x \div 3 = 8$. *Ans.* $x = 24$.
6. The sum of x and 12 is 20. *Ans.* $x = 8$.
7. x is a less than b. *Ans.* $x = b - a$.
8. $3x + 4 = 16$. *Ans.* $x = 4$.
9. $5x - 2 = 48$. *Ans.* $x = 10$.
10. $3x + 4x = 42$. *Ans.* $x = 6$.
11. $\frac{2}{3}x = 12$. *Ans.* $x = 18$.
12. x is 6 less than $b + c$. *Ans.* $x = b + c - 6$.
13. $\sqrt{x} = 4$. *Ans.* $x = 16$.
14. $x^2 = 36$. *Ans.* $x = 6$.
15. $\sqrt[3]{x} = 3$. *Ans.* $x = 27$.
16. $x^{\frac{2}{3}} = 4$. *Ans.* $x = 8$.
17. $\sqrt[4]{x} + 5 = 21 - 14$. *Ans.* $x = 16$.
18. $3^x = 81$. *Ans.* $x = 4$.
19. $2^{x+1} = 64$. *Ans.* $x = 5$.
20. $2x^3 = \sqrt{64}$. *Ans.* $x = 2$.
21. $11 - x = 2$. *Ans.* $x = 9$.
22. $18 - 5x = 3$. *Ans.* $x = 3$.
23. $16^{\frac{1}{x}} = 2$. *Ans.* $x = 4$.

24. $36^{\frac{x}{2}} = 6.$ *Ans.* $x = 1.$

25. $12 - x = x - 4.$ *Ans.* $x = 8.$

26. The sum of x and 10 is a. *Ans.* $x = a - 10.$

27. $x^3 = 8^2.$ *Ans.* $x = 4.$

28. $x^{\frac{2}{3}} = 16^{\frac{3}{2}}.$ *Ans.* $x = 512.$

64. The preceding examples show how algebraic symbols are used for asking questions about number. The pupil should state what question is asked in each case. Thus:

Ex. 2. What number is that from which if 5 be taken, the remainder will be 11?

Ex. 5. What is the dividend if the divisor be 3 and the quotient 8?

Ex. 8. If 4 be added to three times a certain number, the sum is 16. What is the number?

Ex. 13. What number is that whose square root is 4?

Ex. 14. The square of what number is equal to 36?

Ex. 16. What number is that of which the square of the cube root is 4?

Ex. 18. To what power must 3 be raised to produce 81?

Ex. 19. The 19th is a combination of two questions. First, What power of 2 is 64? *Ans.* 6th power. Second, What number added to 1 will produce 6?

Ex. 23. 2 is what root of 16?

Ex. 24. To what power must the square root of 36 be raised to produce 6?

Ex. 25. 12 exceeds a certain number as much as that number exceeds 4. What is the number?

Ex. 27. What number is that whose cube is equal to the square of 8?

65. After translating the algebraic expression into ordinary language, it is well to reverse the process and learn how to express problems in algebraic language.

EXAMPLES.

66. The following examples show how Algebra is employed in the solution of problems:

1. John bought an apple and an orange for 12 cents, paying three times as much for the orange as for the apple. What was the price of each?

Let x represent the number of cents the apple cost.

Then $3x =$ the number of cents the orange cost.

Both cost the sum of x cents and $3x$ cents $= 4x$ cents.

But both cost 12 cents. $\therefore\ 4x = 12$, (Ax. I.)

whence $x = 3$, (Ax. V.)

and $3x = 9$. (Ax. IV.)

Therefore an apple cost 3 cents and an orange cost 9 cents.

2. The sum of two numbers is 36, and one of them is five times the other. What are the numbers?

Let x represent the smaller number.

Then $5x =$ the larger number.

Their sum $= x + 5x = 6x$.

But their sum is 36. $\therefore\ 6x = 36$, (Ax. I.)

whence $x = 6$, (Ax. V.)

and $5x = 30$. (Ax. IV.)

Therefore the numbers are 30 and 6.

3. The sum of three numbers is 90. The second is twice the first, and the third equals the sum of the first and second. What are the numbers?

Let x denote the first number.

Then $2x =$ the second number.

And $x + 2x = 3x$ denotes the third number.

The sum of the three numbers is $x + 2x + 3x = 6x$.

But their sum is 90. $\therefore\ 6x = 90$, (Ax. I.)

whence $x = 15$, (Ax. V.)

and $2x = 30$, (Ax. IV.)

$x + 2x = 45$. (Ax. II.)

Therefore the numbers are 15, 30, 45.

4. The sum of three numbers is 1888. The second is four times the first, and the third is 18 more than the sum of the other two. What are the numbers?

Let x denote the first number.

Then $4x =$ the second number.

And $x + 4x = 5x =$ the sum of the first and second numbers.

$\therefore$ $5x + 18 =$ the third number.

The sum of the three numbers is $x + 4x + 5x + 18 = 10x + 18$.

But the sum of the three numbers is 1888.

$\therefore$ (Axiom I.) $10x + 18 = 1888$.

Subtract 18 from each side.

$$\therefore \; 10x = 1870, \quad \text{(Ax. III.)}$$
$$\text{whence} \quad x = 187, \quad \text{(Ax. V.)}$$
$$\text{and} \quad 4x = 748. \quad \text{(Ax. IV.)}$$
$$\text{Also, } 5x + 18 = 5 \times 187 + 18 = 953.$$

Therefore the numbers are 187, 748, 953.

5. The sum of two numbers is 50, and one of them is two thirds of the other. What are the numbers?

Let $3x$ represent the larger number.

Then $2x =$ the smaller number.

Their sum $= 3x + 2x = 5x$.

But their sum is 50. $\therefore$ $5x = 50$, (Ax. I.)

whence $x = 10$, (Ax. V.)

$$3x = 3 \times 10 = 30, \text{ and } 2x = 2 \times 10 = 20.$$

Hence the numbers are 20 and 30.

6. The sum of two numbers is 60, and their difference is 10. What are the numbers?

Let x denote the smaller number; then the larger must be 10 more than x, that is, $x + 10$.

The sum of the numbers is $x + x + 10 = 2x + 10$.

But their sum is 60. $\therefore$ $2x + 10 = 60$, (Ax. I.)

$$\begin{aligned} \text{Subtract 10 from each,} \quad 2x &= 50, \text{ (Ax. III.)} \\ \text{whence} \quad x &= 25, \text{ (Ax. V.)} \\ \text{and} \quad x + 10 &= 35. \text{ (Ax. II.)} \end{aligned}$$

Therefore the numbers are 35 and 25.

7. A has as many quarters as B has dimes: together they have $2.10. How much money has each?

Let x represent the number of quarters A has.

Then x also denotes the number of dimes B has.

Since one quarter is worth 25 cents, ∴ x quarters are worth $25x$ cents; therefore the value of A's money is $25x$ cents.

Since one dime is worth 10 cents, ∴ x dimes are worth $10x$ cents; therefore the value of B's money is $10x$ cents.

A and B together have $25x + 10x$ cents.

But they together have 210 cents.

$$\begin{aligned} \therefore \quad 25x + 10x &= 210, \text{ (Ax. I.)} \\ \text{whence} \quad 35x &= 210, \\ \text{and} \quad x &= 6. \quad \text{(Ax. V.)} \end{aligned}$$

Therefore A has 6 quarters, worth $1.50, and B has 6 dimes, worth 60 cents.

It must be carefully noted that x denotes the number of units; thus, x represents the number of dollars, feet, persons, days, etc., as the case may be. Therefore, x must never be put for money, length, weight, etc., but only for the number of units of length, weight, etc.

8. Divide a line 42 inches long into two parts, so that one may be three fourths of the other. *Ans.* 18 in.; 24 in.

9. The property of two persons amounts to $4800, and one of them is seven times as rich as the other: what is the property of each? *Ans.* $4200; $600.

10. Divide $42 between A and B, so that for each dime A receives, B may receive a quarter. *Ans.* A, $12; B, $30.

11. The sum of two consecutive numbers is 21. What are they? *Ans.* 10 and 11.

Exercise I.

1. Divide \$760 among A, B, and C, so that B shall have \$50 more than A, and C \$135 more than B.

2. The difference of two numbers is 12, and one of them is 4 times the other. What are the numbers?

3. The difference of two numbers is 84, and one of them is three fourths of the other. What are the numbers?

4. In a company of 133 persons, there are four times as many women as children, and twice as many men as children. How many men are there?

5. Find three consecutive numbers whose sum is 45.

6. A has \$40 more than B, and three times the number of dollars A has equals five times the number B has. How many dollars has A?

7. A bought x horses at \$100 each, $2x$ cows at \$30 each, and $5x$ sheep at \$12 each. He paid \$1100 for all. Find the value of x.

8. Seven times a certain number exceeds four times the number by 84. Find the number.

9. A is twice as old as B, and three times as old as C. The sum of their ages is 66 years. How old is B?

10. If 6 times a certain number be diminished by 10, the remainder will be 10 more than twice the number. Find the number.

11. Find the number which exceeds the sum of its half, fourth, and sixth by 5.

12. A gave an equal number of nickels, dimes, and dollars, in payment of a bill of \$23. How many of each did he give?

CHAPTER II.

NEGATIVE NUMBERS.

67. THERE are certain quantities which are so opposed to each other in character that any number of units of the one taken together with the same number of units of the other, would neutralize each other. Thus, if a person's income were $600 and his outlay were $500, his capital would be increased by $100, because $500 of the income would be neutralized by the $500 outlay. If his income were $600 and his outlay $800, his capital would be diminished by $200, because his income would be neutralized by $600 of his outlay; hence *the rest of his outlay must be taken from his capital.*

If A gains $10 and loses $6, his net gain is $4, because $6 of his gains are neutralized by the $6 lost. If he gains $10 and loses $12, his net loss is $2. If a person go 100 yards in any direction and then retrace his steps for 60 yards, he will be 40 yards from his starting-point, because the 60 yards forwards are neutralized by the 60 yards backwards. If he go forwards 100 yards and then backwards 120 yards, he will be 20 yards from his starting-point, but in a contrary direction from his original motion.

If the mercury in a thermometer stand at 10° above zero, a fall of 15° will cause it to stand at 5° below zero. If a vessel in 5° north latitude sail south through 7°, she will then be in 2° south latitude. Similar relations exist between east longitude and west longitude; motion to the right and to the left; time before and after a fixed date, etc.

68. The discussion of the following problem will tend to explain the principles considered in this chapter: "How much will A's capital be increased if his income be $\$a$ and his outlay $\$c$?" Since A's income tends to increase his capital, $\$a$ must be added to his capital. Since his outlay tends to diminish his capital, $\$c$ must be subtracted: hence A's capital will be increased by as many dollars as a exceeds c. We say, therefore, that A's capital is increased by $(a - c)$ dollars.

If $a > c$, no difficulty will arise in considering this answer, for, whatever values be given to a and c, to find $a - c$ merely consists in subtracting a less number from a greater. If $a = c$, $a - c = 0$, and the answer is still true in an arithmetical sense. If $a < c$, in order to find $a - c$, we are required to subtract a greater number from a less, which, in an arithmetical sense, can not be done.

In arithmetic we deal only with the natural series of numbers; viz., 0, 1, 2, 3, 4, 5, etc.: these numbers increase to the right, each succeeding integer being obtained by adding one to the preceding integer. Since addition consists in counting to the right or forwards, subtraction, being the contrary of addition, must consist in counting to the left or backwards. To subtract 10 from 7, we count backwards from 7 to 0, but can go no farther, for there the natural series ends. Thus, we can subtract only 7, and there remain 3 units to be subtracted.

69. It is important that the answer to the problem in the preceding article, and to similar problems, shall in all cases be $(a - c)$, and that for all values of a and c, $a - c$ shall denote that a is to be added and c is to be subtracted. To enable us to subtract a greater number from a less, it is necessary to assume a new series of numbers, beginning at zero and extending to the left. To each of these numbers the sign $-$ is prefixed, and to each of the natural series the sign $+$ is prefixed. Numbers preceded by $-$ are called negative numbers, and numbers preceded by + are called positive numbers. The algebraic series of numbers is written thus:

$$\ldots\ldots -5, -4, -3, -2, -1, \pm 0, +1, +2, +3, +4, \ldots\ldots$$

or thus:

$$\ldots\ldots -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\ldots$$

The sign − is never omitted. 0 is either + or −, since it is the starting-point of both series.

70. A familiar example of the use of the algebraic series of numbers is furnished by the scale of an ordinary thermometer. A certain point is fixed upon as zero. Degrees above zero are $+1, +2, +3, +4, \ldots\ldots$; degrees below zero are $-1, -2, -3, -4, \ldots\ldots$

If the mercury descend 20° from a point 12° above zero, the result is found by subtracting 20° from 12°; that is, by counting downwards 20° from 12°: by so doing we arrive at 8° below zero, and the temperature is recorded as − 8°. If the mercury descend 5° more, we subtract 5° from − 8°; that is, we begin at − 8° and count 5° downwards, and arrive at − 13°. If now the mercury rise 10°, we add 10° to − 13°; that is, we begin at 13° below zero and count 10° upwards, and arrive at − 3°. A rise of 7° additional will give + 4°, etc.

71. (1) To add 6 to 5, we begin at + 5 and count 6 units in the positive direction; that is, forwards or to the right, and arrive at + 11; that is, $5 + 6 = 11$.

(2) To add 5 to − 3, we begin at − 3 in the negative series and count 5 units forwards and arrive at + 2; that is, $-3 + 5 = +2$.

(3) To add 4 to − 10, we begin at − 10 and count 4 units forwards and arrive at − 6; that is, $-10 + 4 = -6$.

(4) To subtract 10 from 7, we begin at 7 and count 10

units in the negative direction; that is, backwards or to the left in the algebraic scale of numbers, and arrive at -3; that is, $7-10=-3$.

(5) To subtract 4 from -3, we begin at -3 and count 4 units backwards and arrive at -7; that is, $-3-4=-7$.

In other words, (1) To add a positive number means to count so many units forwards or to the right, in the algebraic series of numbers. (2) To subtract a positive number means to count so many units backwards or to the left in the algebraic series of numbers.

72. The **Absolute Value** of a number is its arithmetical value, taken independently of the signs $+$ and $-$. Thus, $+4$ and -4 have the same absolute value; that is, four units.

73. The **Algebraic Value** of a number is its absolute or arithmetical value taken in connection with a sign $+$ or $-$. Thus, $+4$ and -4 have different algebraic values.

74. Two numbers which are both positive or both negative are said to have *like signs*. If one be positive and the other be negative, they are said to have *unlike signs*.

75. (1) To add -12 to 7. In (71), it was shown that in adding a positive number we follow the arithmetical meaning of addition, and count so many units to the right; we shall therefore agree that *to add a negative number means to count so many units to the left in the algebraic series of numbers.* Beginning at 7 and counting 12 units to the left, we arrive at -5; that is, $7+(-12)=-5$.
$\therefore 7+(-12)$ is equivalent to $7-12$.

(2) To add -5 to -4, we begin at -4 and count 5 units to the left and arrive at -9. That is, $-4+(-5)$ $=-4-5=-9$.

76. Since the subtraction of a positive number means to count so many units to the left (71), therefore we shall agree that *the subtraction of a negative number is performed by counting so many units to the right.*

(1) To subtract -6 from 4, we begin at $+4$, count 6 units to the right, and arrive at $+10$. That is, $4-(-6) = 4+6 = 10$.

(2) To subtract -7 from -3, we begin at -3, count 7 units to the right, and arrive at $+4$. That is, $-3-(-7) = -3+7 = +4$.

77. From the illustrations given in Articles 71, 75, and 76, we see:

(1) Adding a positive number and subtracting a negative number both mean counting to the right. Hence, *the addition of a positive number produces the same result as the subtraction of a negative number having the same absolute value.* Example: $3+(+4) = 7$. $3-(-4) = 3+4 = 7$.

(2) Subtracting a positive number and adding a negative number both mean counting to the left. Hence, *the subtraction of a positive number produces the same result as the addition of a negative number having the same absolute value.* Example: $6-(+2) = 6-2 = 4$. $6+(-2) = 6-2 = 4$.

78. It should be carefully noted that the signs $+$ and $-$ are used for two distinct purposes: (1) Arithmetically, as signs of operation, to connect numbers with each other by addition or subtraction. (2) Algebraically, as signs of opposition, to indicate in which series—the positive or the negative—a given number belongs.

It should be further noted that, (3) The sign $+$ placed before a term does not change its algebraic sign. Thus, $+(+a) = +a$, and $+(-a) = -a$. (4) The sign $-$ placed before a term changes the algebraic sign of that term. Thus, $-(+a) = -a$, and $-(-a) = +a$.

79. The principles considered in the preceding article may be still further extended. (1) Since prefixing a sign $+$ has no effect upon a term, therefore any number of $+$ signs may be prefixed without affecting the term. Thus, $+\{+[+(+a)]\}=+a$, and $+\{+[+(-a)]\}=-a$. (2) Since prefixing a sign $-$ changes the sign of a term, then prefixing a second minus changes the sign again, and thus gives the original sign; prefixing a third $-$ changes the sign, etc. Examples: $-[-(+a)]=+a$; $-[-(-a)] = -a$; $-\{-[-(-a)]\}=+a$; etc. Hence: *If a term be affected by an odd number of minus signs, its essential sign will be minus. If a term be affected by an even number of minus signs, its essential sign will be plus.*

80. By **Essential Sign** is meant the sign properly belonging to a term when affected by but one sign, $+$ or $-$.

81. The use of $+$ and $-$ as signs of opposition is carried into the treatment of exponents (32). Thus:

$$a^2=\frac{aa}{1}=\frac{aaa}{a}=\frac{aaaa}{aa}=\frac{aaaaa}{aaa}, \text{ etc.} \qquad \text{(Art. 62, Ax. VII.)}$$

$$a^3=\frac{aaa}{1}=\frac{aaaa}{a}=\frac{aaaaa}{aa}=\frac{aaaaaa}{aaa}, \text{ etc.}$$

$$a^4=\frac{aaaa}{1}=\frac{aaaaa}{a}=\frac{aaaaaa}{aa}, \text{ etc.}$$

$$a^5=\frac{aaaaa}{1}=\frac{aaaaaaaa}{aaa}, \text{ etc.}$$

In the various forms of a^2, there are two a's in the dividend more than in the divisor; of a^3, there are three more; of a^4, four more; of a^5, five more, etc. That is, *a positive exponent denotes that the number next to which it is placed occurs so many more times as a factor in the dividend than in the divisor.*

82. If any number occur as a factor an equal number of times in both dividend and divisor, the exponent of that number is zero (0). Thus:

$$\frac{a^n}{a^n} = a^0; \quad \frac{a^4}{a^4} = a^0; \quad \frac{b^5}{b^5} = b^0; \text{ etc.}$$

Since $\frac{a^n}{a^n} = 1$ (Art. 63, VIII.),

and since $\frac{a^n}{a^n} = a^0, \therefore a^0 = 1$ (Art. 62, Ax. I.)

Therefore, *any number with an exponent of zero is equal to unity.*

83. Since we use the positive exponent to denote that the number affected by the exponent occurs as a factor so many more times in the dividend than in the divisor (81), and since we use the zero exponent to denote that the number affected by it occurs an equal number of times in both dividend and divisor (82), therefore, *the negative exponent denotes that the number affected by it occurs as a factor in the dividend a fewer number of times than in the divisor.* Thus: $\frac{1}{2^3} = 2^{-3}$; $\frac{3}{a^4} = 3a^{-4}$; $\frac{5}{27} = \frac{5}{3^3} = 5 \times 3^{-3}$; $\frac{a}{b^2c^3} = ab^{-2}c^{-3}$; etc.

Exercise II.

Simplify the first five problems:

1. $6 + (-5)$; $-6 - 5$; $4 - (-7)$; $5 + (-3)$.
2. $-[-(-5)]$; $-\{-[-(-4)]\}$; $-\{-[-(-\overline{-8})]\}$.
3. $-2-[-4]$; $-5-\{-[-(-9)]\}$; $-(-6)-[-(-2)]$.
4. 5^0; $-(4^0)$; 4×6^0; $7^0 \times 3^0$; -2×9^0.
5. 3^{-1}; 4^{-2}; 3×5^{-1}; $4^{-1} \times 2^{-2}$; $3^0 \times 6^{-1}$.

6. A was born 10 B.C. and lived 67 years. When did he die?

7. D died 17 A.D. at the age of 49. When was he born?

8. E was born 205 B.C. and lived 80 years. When did he die?

9. F was born 43 B.C. and died 12 A.D. How many years did he live?

10. The zero of the Mohammedan calendar is the date of the Hegira, 622 A.D. Time since 622 A.D. being +, and before 622 A.D. being −, what are the following dates according to this calendar: 1888 A.D.? 300 A.D.? 1 B.C.? 753 B.C.? 218 B.C.?

11. A has $4750 and owes $927. How much is he worth?

12. B has $1800 and owes $3000. How much is he worth?

13. C went 125 steps forwards, then 49 steps backwards, then 20 steps forwards, then 74 steps backwards. How many steps did he take? How far is he from his starting-point?

14. A goes 75 steps in a minute, and B 60 steps. How far apart will they be at the end of five minutes, if they walk in the same direction from the same starting-point? In opposite directions?

CHAPTER III.

NUMERICAL VALUES.

84. IN Algebra, a letter may stand for any number which we wish it to represent. Thus, a may represent 2, 5, 10, 125, $\frac{1}{2}$, $\frac{3}{4}$, 0, -2, -7, $-\frac{2}{3}$, $-2\frac{1}{2}$, or any other number, positive or negative, fractional or integral. It must not be understood from this, however, that the letter has no determinate value. Its value is fixed for the time being; it can not represent two different numbers in the same problem, but on a different occasion the same letter may be put for any other number.

85. The numerical value of a simple term (44, 47), containing no sign of division, may be found from the preceding definitions and principles. To illustrate the various cases, let $a = 8$ in each of the following examples:

1. $a^{\frac{2}{3}} = ?$ This denotes that the cube root of 8 is to be extracted, and the result raised to the second power. The cube root of 8 is 2, and the second power of 2 is 4. $\therefore$ when $a = 8$, $a^{\frac{2}{3}} = 4$. (Arts. 31, 35, 37.)

2. $a^{-\frac{2}{3}} = ?$ Since $a^{\frac{2}{3}} = 4$, then $a^{-\frac{2}{3}} = \frac{1}{4}$. (Art. 83.)

3. $a^{-1} = \dfrac{1}{a} = \dfrac{1}{8}$. (Art. 83.)

4. $\sqrt[3]{a^4} = a^{\frac{4}{3}} = 8^{\frac{4}{3}} = (2)^4 = 16$.

In the following examples, let $a = 9$, and $b = 16$:

5. $a^{\frac{1}{2}} b^{-\frac{3}{4}} = 9^{\frac{1}{2}} \times 16^{-\frac{3}{4}} = 3 \times \frac{1}{8} = \frac{3}{8}$.

6. $\dfrac{a}{b^{\frac{1}{2}}} = \dfrac{9}{4}$.

7. $4a^{\frac{1}{2}} b^{-\frac{1}{4}} = 4 \times 3 \times \frac{1}{2} = 6$.

86. To find the numerical value of a simple term containing but one sign of division:

Find the value of the expression (43) *preceding the division sign* (85), *then of that following the division sign, and divide the former by the latter.*

EXAMPLES.—**1.** $ab^{-1} \div c^{\frac{1}{2}} d^{\frac{2}{3}} = ?$ when $a = 8$, $b = 2$, $c = 16$, $d = 8$.

$$ab^{-1} = 8 \times \tfrac{1}{2} = 4. \quad c^{\frac{1}{2}} d^{\frac{2}{3}} = 4 \times 4 = 16.$$

$$4 \div 16 = \tfrac{1}{4} \ \textit{Ans.}$$

2. $4a \div b^{\frac{1}{2}} c^{\frac{1}{3}} d^0 = ?$ when $a = 12$, $b = 9$, $c = 8$, $d = 5$.

$$4a = 4 \times 12 = 48. \quad b^{\frac{1}{2}} c^{\frac{1}{3}} d^0 = 3 \times 2 \times 1 = 6.$$

$$48 \div 6 = 8 \ \textit{Ans.}$$

After indicating the work, the dividend may be placed above a horizontal line, and the divisor below it, and cancellation may be employed.

87. To find the value of a simple term containing two or more signs of division:

(1) *Find the values of the expressions separated by the division signs.* (43.)

(2) *Commence at the right of the term, and use the signs in their order.*

EXAMPLES.—**1.** $16 \div 8 \div 2 = 16 \div 4 = 4$.

2. $18 \times 8 \div 12 \times 9 \div 8 = 144 \div 108 \div 8 = 10\frac{2}{3}$.

88. To find the numerical value of a polynomial (53) consisting of simple terms:

(1) *Find the simplest value of each term* (85, 86, 87).

(2) *Add together all the positive terms* (45).

(3) *Add together all the negative terms* (46).

(4) *Find the difference of the absolute values of these sums* (72).

(5) *Prefix to the remainder the sign of the sum whose absolute value is the greater.*

EXAMPLE.—Find the value of $a^{\frac{2}{3}} + b^{\frac{1}{4}}c^2 - b^{\frac{1}{2}} \div a^{\frac{1}{3}} - a^{-\frac{1}{3}}b^{\frac{1}{2}}c^{-1}$, when $a = 8$, $b = 16$, $c = 2$.

1st term $= a^{\frac{2}{3}} = 8^{\frac{2}{3}} = 2^2 = +4$.

2d term $= +b^{\frac{1}{4}}c^2 = +2 \times 4 = +8$.

3d term $= -b^{\frac{1}{2}} \div a^{\frac{1}{3}} = -4 \div 2 = -2$.

4th term $= -a^{-\frac{1}{3}}b^{\frac{1}{2}}c^{-1} = -\frac{1}{2} \times 4 \times \frac{1}{2} = -1$.

$$4 + 8 - 2 - 1 = 12 - 3 = 9.$$

89. To find the numerical value of a compound term (48):

(1) *Find the value of the expression inclosed by the innermost sign of aggregation* (43, 38).

(2) *Remove this sign of aggregation* (78, 79).

(3) *Proceed in like manner with the remaining signs of aggregation (removing one at a time); until all are removed.*

EXAMPLE.—If $a = 4$, $b = 3$, $c = -5$, $d = 0$, find the value of $-\{a - [b + 2c - (a + d)]\}$.

$a + d = 4 + 0 = 4$. $\therefore -(a + d) = -(+4) = -4$.

$b + 2c - 4 = 3 + 2(-5) - 4 = 3 - 10 - 4 = 3 - 14 = -11$. $\therefore -[b + 2c - 4] = -[-11] = +11$. (Art. 79.)

$a + 11 = 4 + 11 = 15$. $\therefore -\{a + 11\} = -15$.

90. From (88) and (89) the numerical value of any polynomial may be found.

EXAMPLES.

When $a = 4$, $b = 8$, $c = 1$, $d = 0$, find the value of:

1. $a^{\frac{1}{2}} \times b^{\frac{2}{3}} \times c^2$. *Ans.* 8.

2. $a^2b^{-\frac{2}{3}}c$. *Ans.* 4.

3. $a^0b^{\frac{1}{3}}c^3$. *Ans.* 2.

4. $ab^{\frac{1}{3}} \div c$. *Ans.* 8.

5. $a^{\frac{1}{2}}b^{\frac{1}{3}} \div c$. *Ans.* 4.

6. $c \div ab^{-\frac{2}{3}}$. *Ans.* 1.

7. $a^{\frac{1}{2}} - b + \frac{d}{c}$. *Ans.* -6.

8. $c \div a^{-1}b + d$. *Ans.* $\frac{1}{2}$.

9. $-abc^5 - d^2$. *Ans.* -32.

10. $(a^{\frac{3}{2}} + b - cd)^{\frac{1}{2}}$. *Ans.* 4.

11. $dac - b^{\frac{1}{3}}$. *Ans.* -2.

12. $ab - ac - bd$. *Ans.* 28.

EXERCISE III.

Given $a = 2$, $b = 3$, $c = 4$, $d = 5$, $m = 1$, $r = 0$, find the value of each of the following expressions:

1. $\sqrt[3]{\frac{acdm}{5}} + \sqrt{2acd^2} - \sqrt[5]{abcr}$.
2. $2\sqrt[3]{2c} + 5\sqrt[3]{9b} - \sqrt[4]{8am}$.
3. $bc\sqrt{5a + d + m} - (d - a)\sqrt[4]{4c + r}$.
4. $\{(a + c^{\frac{1}{2}})^2 + (a - c^{\frac{1}{2}})^2\}^2$.
5. $\sqrt[4]{(a + b + c)(2d - m)}$.
6. $[(a + b)^2(c + 1)]^{\frac{2}{3}} + (c + d)^{\frac{1}{2}}$.
7. $(c - a)(c + a) + am^2 - 2(a + c + m) - acr$.
8. $(c + 2d)^0 + (c + d)^{\frac{3}{2}} - [bc(2d + a)]^{\frac{1}{2}}$.
9. $(a + bc + 2m + r)^{\frac{3}{2}} - abmr$.
10. $a^2 + br + c^{\frac{1}{2}}m - (d + m + r)$.
11. $\sqrt[3]{a + [2b + c(d + mr)] - m^3}$.
12. $\sqrt[5]{3a + 2[3b - c(d - 2m) + 2a] + 2bc}$.

Show that the numerical values of the following are equal:

13. $(d + b)(d - b) = d^2 - b^2$.
14. $(d^3 - a^3) \div (d - a) = d^2 + ad + a^2$.
15. $(d^3 + a^3) \div (d + a) = d^2 - ad + a^2$.
16. $(d + a)^2 + (d - a)^2 = 2(d^2 + a^2)$.
17. $(d + a)^2 - (d - a)^2 = 4ad$.
18. $(d + a)^3 + (d - a)^3 = 2d(d^2 + 3a^2)$.
19. $(b + a)^3 - (b - a)^3 = 2a(3b^2 + a^2)$.
20. $(a^2 + b^2)(c^2 + d^2) = (ac + bd)^2 + (bc - ad)^2$.
21. $a^4 + a^2b^2 + b^4 = (a^2 + ab + b^2)(a^2 - ab + b^2)$.

SPECIMEN PAPER.

1. Find the value of $a^2c - 3ab^2 - 5(a - b + c)$, when $a = -2$, $b = 3$, $c = -4$.

	+	−
1st term $= a^2c = (-2)^2(-4) = (+4)(-4) = -16.$		16
2d term $= -3ab^2 = -3(-2)(3)^2 = -3(-2)(9) = +54.$	54	
3d term $= -5(a-b+c) = -5(-2-3-4) = -5(-9) = +45.$	45	
$+99 - 16 = +83.$	99	16

Ans. 83.

2. Find the value of $a - \{c - b[2a - b - c(a^2 + b)]\}$, when $a = -3$, $b = 4$, $c = -2$.

$(a^2 + b) = (+9 + 4) = (+13).$

$[2a - b - c(13)] = [-6 - 4 - (-2)(13)] = [-6 - 4 + 26] = [+16].$

$\{c - b[+16]\} = \{-2 - 4[16]\} = [-2 - 64] = [-66].$

$a - \{-66\} = -3 + 66 = +63.$

Ans. 63.

3. Find the value of $a^{\frac{4}{5}} - 3bc^{\frac{1}{2}} + 2b^{\frac{2}{3}}c - 5$, when $a = 32$, $b = -8$, $c = 9$.

	+	−
1st term $= a^{\frac{4}{5}} = 32^{\frac{4}{5}} = 2^4 = 16.$	16	
2d term $= -3bc^{\frac{1}{2}} = -3(-8)(9)^{\frac{1}{2}} = -3(-8)(3) = +72.$	72	
3d term $= +2b^{\frac{2}{3}}c = +2(-8)^{\frac{2}{3}}(9) = +2(-2)^2(9) = +2(+4)(9) = 72.$	72	
4th term $= -5.$		5
$+160 - 5 = +155.$	160	5

Ans. 155.

4. $(a - b)^2 - 2(a + b)^3 + 5(ab)^2 + (b - a)^3 = ?$, when $a = 2$, $b = -3$.

	+	−
1st term $= (a - b)^2 = (2 + 3)^2 = (5)^2 = +25.$	25	
2d term $= -2(a+b)^3 = -2(2-3)^3 = -2(-1)^3 = -2(-1) = +2.$	2	
3d term $= +5(ab)^2 = +5(-6)^2 = +5(36) = 180.$	180	
4th term $= +(b - a)^3 = +(-3 - 2)^3 = (-5)^3 = -125.$		125
$+207 - 125 = +82.$	207	125

Ans. 82.

CHAPTER IV.

REDUCTION OF TERMS.

91. Reduction is the process of changing the form of an expression without altering its value.

92. Reduction of Terms is the process of uniting terms and finding the simplest expression for the result.

93. (1) To simplify $3a + 5a$. If a stand for 12, it is required to find the sum of 3 dozen and 5 dozen, which is evidently $(3 + 5)$ dozen $= 8$ dozen. If $a = 20$, the sum of 3 score and 5 score is evidently $(3 + 5)$ score $= 8$ score. In like manner, for all values of a, $3a + 5a = (3 + 5)\,a = 8a$.

Similarly, $4ab + 7ab = (4 + 7)\,ab = 11ab$,

and $6ab^2 + 8ab^2 = (6 + 8)\,ab^2 = 14ab^2$.

Since the method of work is plainly the same for all other numbers, therefore, for every value of a, b, and c, $ac + bc = (a + b)c$. (Art. 63, VI.)

(2) To simplify $-3a - 7a$. Subtracting two numbers is equivalent to subtracting their sum. (63, II.)

$$\therefore -3a - 7a = -(3 + 7)\,a = -10a.$$

Therefore $-ca - ba = -(c + b)\,a$.

(3) To simplify $7a - 4a$. Reasoning as in the first problem, since 7 dozen $-$ 4 dozen $= (7 - 4)$ dozen, and since 7 score $-$ 4 score $= (7 - 4)$ score, and similarly for all values of a, therefore $7a - 4a = (7 - 4)\,a = 3a$,

If, instead of 7 and 4, any other numbers be substituted, the method of work is the same. $\therefore ba - ca = (b - c)\,a$.

94. (1) To simplify $4(b-c)+8(b-c)$. If $b=12$ and $c=3$, then $b-c=9$, and the sum of 4 nines and 8 nines is $(4+8)$ nines $=12$ nines. If $b=5$ and $c=2$, then $b-c=3$, and the sum of 4 threes and 8 threes is $(4+8)$ threes $=12\times 3$. In like manner, for all values of b and c, $4(b-c)+8(b-c)=(4+8)(b-c)=12(b-c)$.

Since the solution is evidently similar if any other particular numbers be substituted for 4 and 8, $\therefore m(b-c)+a(b-c)=(m+a)(b-c)$.

(2) To simplify $9(x-y)-4(x-y)$. Reasoning as before, we find the result to be $(9-4)(x-y)=5(x-y)$. Therefore $c(x-y)-b(x-y)=(c-b)(x-y)$.

(3) To simplify $a(x-y)+bx-by$. Since $bx-by=b(x-y)$, therefore the given expression is equal to $a(x-y)+b(x-y)=(a+b)(x-y)$.

95. (1) To simplify a^4-a. $a^4-a=aaaa-a=(aaa-1)a=(a^3-1)a$.

Similarly, $a^5-a^3=aaaaa-aaa=(aa-1)aaa=(a^2-1)a^3$.

(2) To simplify $6a^3-8a$. Since each term is divisible by $2a$, therefore $6a^3-8a=2a(3a^2-4)$.

96. (1) $a^3-a+a^4-a^2=a(a^2-1)+a^2(a^2-1)=$
$(a+a^2)(a^2-1)=a(1+a)(a^2-1)$.

(2) $3a(x+y)-2b(x+y)+(x-2y)(3a-2b)=$
$(3a-2b)(x+y)+(x-2y)(3a-2b)=$
$(3a-2b)(x+y+x-2y)=(3a-2b)(2x-y)$.

(3) $a(x-y)-b(x-y)-c(x-y)=(a-b-c)(x-y)$.

(4) $(a-b)(3x-y)-2b(3x-y)+(x-y)(a-3b)=$
$(3x-y)(a-b-2b)+(x-y)(a-3b)=$
$(3x-y)(a-3b)+(x-y)(a-3b)=$
$(a-3b)(3x-y+x-y)=(a-3b)(4x-2y)=$
$2(a-3b)(2x-y)$.

97. From the illustrations given in (93–96), inclusive, we derive the following Rule for the Reduction of Terms having a Common Factor:

(1) *Remove from each term the common factor.*

(2) *Inclose the resulting terms in parenthesis.*

(3) *Write the common factor, either before or after the parenthesis, with the sign* $\times$, *expressed or understood, between them.*

(4) *Write the sign* $+$ *before the whole.*

(5) *In like manner combine the resulting terms, and so continue until the expression is in its simplest form.*

Exercise IV.

Simplify each of the following expressions:

1. $5a + 9a - 2a - 6a$.

2. $3ab - ac$.

3. $a^2 - 3a$.

4. $5(b - c) - 2(b - c)$.

5. $-(a - b) + 7(a - b)$.

6. $ab + 3b$.

7. $a^3 - a$.

8. $ab + a + bc + c$.

9. $a^3 + a^2 + a + 1$.

10. $x^3 - x^2 + x - 1$.

11. $a^5 - a^4 + a^3 + a^2 - a + 1$.

12. $(a + b)x + (a - b)x$.

13. $(a - b)y + (a - b)z$.

14. $(a - c)bd - (a - c)ab$.

15. $(a + b)\sqrt{x} + (a - b)\sqrt{x}$.

16. $a^3 - a^2b + ab^2 - b^3$.

17. $(a - b + c)x^{\frac{1}{2}} + (2a + 3b - c)x^{\frac{1}{2}}$.

18. $(3a - 2b)(x - y) + (a - 2b)(x - y) + 4(a - b)(x + y)$.

19. $6a - 6b + ca - cb - 2a + 2b$.

20. $(2a - b)(c - 2d) + (2a - b)(2c + 5d)$.

CHAPTER V.

ADDITION; SUBTRACTION; BRACKETS.

98. Addition in Algebra is the process of uniting expressions with their proper signs, and reducing the result to its simplest form.

This definition agrees with the principle advanced in (78), that addition does not change the algebraic sign of a term. Thus: $+(+a) = +a$, and $+(-a) = -a$.

99. The arithmetical notion of addition applies only to the addition of a positive number to a positive number; but the use of negative numbers requires us to extend this definition. The addition of $+b$ to $+a$ gives $a+b$; of $-b$ to $+a$ gives $a-b$; of $+b$ to $-a$ gives $-a+b$; of $-b$ to $-a$ gives $-a-b$.

100. The result obtained by addition is called the *algebraic sum*. When the numbers to be added have like signs (74), the algebraic sum has the same absolute value as the arithmetical sum. When the numbers have unlike signs, the algebraic sum has the same absolute value as the arithmetical difference.

101. Subtraction in Algebra is the process of finding a number which, added to a given number, will produce a given sum. With reference to this operation, the given sum is called the *minuend*, the given number is called the *subtrahend*, and the required number is called the *remainder* or *difference*.

102. The algebraic definition of subtraction agrees with the arithmetical definition; but, as the algebraic sum has a more extended meaning than the arithmetical sum, so the operation of subtraction has a correspondingly extended meaning. In both Arithmetic and Algebra, subtraction is understood to be the contrary of addition; if the addition of a term has a certain effect, then the subtraction of that term has a contrary effect.

Since $+(+a) = +a$, then $-(+a) = -a$; also, since $+(-a) = -a$, then $-(-a) = +a$. These results agree with the principles advanced in Arts. 77 and 79.

103. Rule for Subtraction:

(1) *Write for the remainder all the terms of both minuend and subtrahend, giving to each term of the minuend its own sign, and to each term of the subtrahend a sign contrary to its own.*

(2) *Reduce the result to its simplest form* (Chapter IV.).

Verification: *Add the remainder to the subtrahend; the sum must be the minuend.*

Notes. In subtraction, and also in addition, it is convenient to write like terms under each other.

After the pupil has become expert in the use of the rule given above, it is recommended that the operations of changing the signs of the subtrahend, and of reducing like terms, be performed mentally.

104. The truth of the principle that the subtraction of $b - c$ is equivalent to the addition of $-b + c$, may also be shown as follows:

First Method. To subtract $b - c$ from a means that c is to be taken from b, and this remainder is to be taken from a. If b be taken from a, the remainder is $a - b$; but this remainder is c units too small, for c units too many have been taken from a. $\therefore$ the true remainder is found by adding c to $a - b$. That is,

$$a - (b - c) = a - b + c.$$

105. Second Method. $a = a + b - b + c - c$ (Art. 62, Ax. VI.).

Since adding $(b - c)$ to any number means to unite $b - c$ with that number (98), therefore subtracting $b - c$ means to remove $b - c$ from the sum. If $b - c$ be removed from $a + b - b + c - c$, the remaining terms will be $a - b + c$.

$$\therefore a - (b - c) = a - b + c.$$

106. Brackets.—The following laws for the use of brackets (38) depend upon the preceding principles:

(1) *When an expression within brackets is preceded by* $+$, *the brackets may be removed* (79).

(2) *Any number of terms in an expression may be inclosed by brackets, and the sign* $+$ *placed before the whole.*

(3) *When an expression within brackets is preceded by the sign* $-$, *the brackets may be omitted if the sign of every term within the brackets be changed* (79).

(4) *Any number of terms in an expression may be inclosed by brackets and the sign* $-$ *placed before the whole, provided the sign of every term within the brackets be changed.*

107. Expressions may occur with more than one pair of brackets; these may be removed in succession. Thus,

$$a-\{b-c-[-d+e-\overline{f-g}]\}=a-\{b-c-[-d+e-f+g]\}= a-\{b-c+d-e+f-g\}=a-b+c-d+e-f+g;$$

or, proceeding in a different order,

$$a-\{b-c-[-d+e-\overline{f-g}]\}=a-b+c+[-d+e-\overline{f-g}]= a-b+c-d+e-\overline{f-g}=a-b+c-d+e-f+g.$$

Reversing the above, we may introduce more than one pair of brackets.

108. The rules for removing more than one pair of brackets may be stated as follows:

First Method. *Remove the innermost pair of brackets; next, the innermost of all that remain, and so on.*

SECOND METHOD. *Remove the outermost pair of brackets; next, the outermost of all that remain, and so on.*

Make no change in any term until the brackets inclosing that term are removed.

109. Brackets may all be removed at once by observing the rules mentioned in (79). Every term inclosed in one pair of brackets is affected by two signs: (1) the sign prefixed to the term; (2) the sign prefixed to the brackets. If this pair of brackets be inclosed in another pair, each term is affected by three signs, and so on. If an odd number of these signs be negative, the essential sign (80) of the term will be *minus*. If an even number be negative, the essential sign of the term will be *plus*. Thus, in $a - \{b - c - [-d + e + (-d - e - \overline{f - g})]\}$, a is $+$, being affected by no minus; $b - c$, being affected by the sign prefixed to the brace, becomes $-b + c$. The terms inclosed by the brackets keep their own signs, because the $-$ before the brace and the $-$ before the brackets are together equivalent to $+$. $\therefore$ those terms become $-d + e - d - e - f + g$.

110. Since $+(a - b - c) = -(-a + b + c)$, therefore: *If the exponent of an expression inclosed in brackets be unity, the sign of each inclosed term may be changed, provided the sign prefixed to the brackets be changed.*

EXAMPLES.

1. Add $a - x$; $a - 3x$; $2a - 2$; $3 - 2x$; $x - 4$.
 Ans. $4a - 5x - 3$.
2. Add $(a + x)$; $2(a + x)$; $7(a + x)$. *Ans.* $10(a + x)$.
3. Add $3(a - b)$; $-5(a - b)$; $c(a - b)$.
 Ans. $(c - 2)(a - b)$.
4. Add $a(x - y)$; $-b(x - y)$; $c(x - y)$.
 Ans. $(a - b + c)(x - y)$.

5. From $2x - y - 3z$ take $x - 4y - z$.

Ans. $x + 3y - 2z$.

6. From ax take bx. *Ans.* $(a - b)\,x$.

7. From $2a^3 - b^3$ take $a^3 - a^2b + ab^2$.

Ans. $(a + b)(a^2 - b^2)$.

8. From $a(x - y)$ take $-b(x - y)$.

Ans. $(a + b)(x - y)$.

9. From $(a - b + c)\,d$ take $(2a - b - c)\,d$.

Ans. $(-a + 2c)\,d$.

10. From $ax - by + cx$ take $(1 + a)\,x + by$.

Ans. $(c - 1)\,x - 2by$.

EXERCISE V.

1. Add $(a + b)\,x;\ (b - a)\,x;\ cx$.
2. Add $(3b - 2c + 1)\,x;\ (-b + 3c)\,x$.
3. From $7(a - b)$ take $3(b - a)$.
4. From $(4a + 2b)(2x - y)$ take $(2x - y)(3a - b)$.

Simplify each of the following expressions:

5. $a - [3b - \{a - (2c - \overline{c - b}) - 3b\} - a + c]$.
6. $1 - \{1 - a - [(-a^2 + a) - 1] - [-a + 1 - \overline{a^3 - a^2}]\}$.
7. $1 - \{1 - a - [a - (1 - a^2) + (1 - a - \overline{a^2 - a})]\}$.
8. $2a - \{2a - (4a - 8 - \overline{4a - 4b})\}$.

Collect the co-efficients of the like powers of x.

9. $x^3 + ax^2 - bx^2 + ax - bx + a - 1$.
10. $x^2 + y^2 + z^2 + xy + xz + yz$.
11. $x^3 + x^2y - x^2z + x^2 + xy$.
12. $ax^3 - ax^2 - ax + a - bx^3 + bx^2 - bx - b$.

CHAPTER VI.

MULTIPLICATION.

111. Multiplication, as defined in Arithmetic, is a short method of adding equal numbers together. With reference to this operation, one of the equal numbers is called the *multiplicand,* the number of those equal numbers is called the *multiplier,* and the sum is called the *product.* The multiplicand and multiplier are the *factors* of the product. These words are used similarly in Algebra, but their meanings must be extended to include negative numbers.

112. Multiplication in Algebra is a process of adding as many numbers, each equal to the multiplicand, as there are units in the positive multiplier; and it is a process of subtracting as many numbers, each equal to the multiplicand, as there are units in the negative multiplier.

113. As adding a term does not change its sign, and as the sum of any number of positive terms is positive, and of any number of negative terms is negative; therefore, if the multiplier be positive, each term in the product and the corresponding term in the multiplicand will have like signs.

Thus, $+5 \times (+4) = +20$, and $+5 \times (-4) = -20$;
also, $+a \times (+b) = +ab$, and $+a \times (-b) = -ab$.

Since subtracting any number of terms is performed by changing their signs, therefore, if the multiplier be negative, the corresponding terms in the multiplicand and product will have unlike signs.

Thus, $-a \times (+b) = -ab$, and $-a(-b) = +ab$.

114. Since $+a(+b) = +ab$, and $-a(-b) = +ab$; therefore, *If two factors have like signs, the sign of their product is plus.*

Since $-a \times (+b) = -ab$, and $+a \times (-b) = -ab$; therefore, *If two factors have unlike signs, the sign of their product is minus.*

115. The product of any number of factors will be negative if the number of negative factors be odd (1, 3, 5, 7, 9, 11, etc.). (Art. 79.)

The product of any number of factors will be positive if the number of negative factors be even (0, 2, 4, 6, 8, 10, etc.). (Art. 79.) Thus,

$$2(-3)(-4)(-5) = -120.$$
$$-2(-3)(-4)(-5) = +120.$$

116. From (115) it follows that the signs of an even number of factors may be changed without affecting the sign of the product; but if the signs of an odd number of factors be changed, the sign of the product will be changed.

117. (1) $a^3 \times a^4 = aaa \times aaaa = a^7 = a^{3+4}$.

(2) $a^2 \times a^3 \times a^4 = aa \times aaa \times aaaa = a^9 = a^{2+3+4}$.

(3) $a^{-2} \times a^3 = \frac{1}{aa} \times aaa = \frac{aaa}{aa} = a^1 = a^{-2+3}$.

(4) $a^{-4} \times a = \frac{1}{aaaa} \times a = \frac{a}{aaaa} = \frac{1}{aaa} = \frac{1}{a^3} = a^{-3} = a^{-4+1}$.

(5) $a^{-2} \times a^{-3} = \frac{1}{aa} \times \frac{1}{aaa} = \frac{1}{aaaaa} = \frac{1}{a^5} = a^{-5} = a^{-2+(-3)}$.

In like manner, for all values of m and n, $a^m \times a^n = a^{m+n}$. That is, *The powers of a number are multiplied by adding the exponents.*

118. The product of the same powers of different letters may be indicated by writing the letters within brackets, and placing the exponent over the whole. Thus, $a^2 \times b^2 =$

$aabb = ab \times ab = (ab)^2$. Similarly, $a^3 \times b^3 \times c^3 = (abc)^3$. That is, $a^n \times b^n = (ab)^n$; $a^n \times b^n \times c^n = (abc)^n$; $a^n \times b^n \times c^n \times d^n = (abcd)^n$; and so on for any number of factors.

119. The principles explained in the preceding articles, being true for all numbers, must necessarily be true for particular numbers, expressions in brackets, etc. Thus, $2^3 \times 2^4 = 2^7$; $3^{\frac{1}{2}} \times 3^{\frac{2}{3}} = 3^{\frac{7}{6}}$; $(a-b)^5 \times (a-b)^3 = (a-b)^8$; $5^{-\frac{1}{2}} \times 5^{\frac{2}{3}} = 5^{\frac{1}{6}}$; $(ab)^3 \times (ab)^4 = (ab)^7$; $(a+b)^3 \times (a+b)^{-3} = (a+b)^{3-3} = (a+b)^0 = 1$; etc.

120. From the principles explained in Arts. 114 to 119 inclusive, we derive the following Rule for the Multiplication of two or more Monomials:

(1) SIGN. *If the number of negative factors be odd, the sign of the product is minus. Otherwise, the sign of the product is plus.*

(2) NUMERICAL CO-EFFICIENT. *Find the product of the numerical co-efficients as in Arithmetic.*

(3) LITERAL FACTORS. *Write in succession all the letters occurring in the factors, using each letter but once.*

(4) EXPONENTS. *To find the exponent of any letter in the product, add the exponents of that letter in the factors.*

121. To multiply a polynomial by a monomial: *Multiply each term of the multiplicand by the multiplier, and add the products* (63, VI. and VII.).

EXAMPLES.

1. $a^2bc \times a^4b^3c^5 = ?$ *Ans.* $a^6b^4c^6$.
2. $x^{\frac{1}{2}}y^{\frac{2}{3}}z \times xyz^{-3} = ?$ *Ans.* $x^{\frac{3}{2}}y^{\frac{5}{3}}z^{-2}$.
3. $-abc^2 \times a^4bc = ?$ *Ans.* $-a^5b^2c^3$.
4. $-a^2b^3c \times (-abc^{-3}) = ?$ *Ans.* $+a^3b^4c^{-2}$.
5. $-2a^2b^4c \times (-5ab^2c) = ?$ *Ans.* $+10a^3b^6c^2$.
6. $a^{\frac{1}{2}}b^{\frac{2}{3}}c^{-\frac{3}{4}} \times (-4ab^{-1}c) = ?$ *Ans.* $-4a^{\frac{3}{2}}b^{-\frac{1}{3}}c^{\frac{1}{4}}$.

7. $6^2 \times 6^{-3} = ?$ *Ans.* $6^{-1} = \frac{1}{6}$.

8. $(x + y - z)\, x = ?$ *Ans.* $x^2 + xy - xz$.

9. $(x - x^{\frac{1}{2}} - 1)x^{\frac{3}{2}} = ?$ *Ans.* $x^{\frac{5}{2}} - x^2 - x^{\frac{3}{2}}$.

10. $(a - b - c)\,(-5c) = ?$ *Ans.* $-5ac + 5bc + 5c^2$.

11. $(\frac{1}{2}x^2 - \frac{2}{3}x - 3)\,(-6x) = ?$ *Ans.* $-3x^3 + 4x^2 + 18x$.

12. $[(a-b)^3 - (a-b)](a-b) = ?$ *Ans.* $(a-b)^4 - (a-b)^2$.

13. $(6^x - 6^y - 6^z) \times 6^{x-y} = ?$ *Ans.* $6^{2x-y} - 6^x - 6^{x-y+z}$.

14. $[a - (b - c)]\,(-ab) = ?$ *Ans.* $-a^2b + ab^2 - abc$.

122. To multiply one polynomial by another: *Multiply each term of one factor by each term of the other, and add the partial products.*

This rule is a result of the preceding principles. Thus, multiplying by $a + b$ is equivalent to multiplying by a and by b separately, and adding the products; multiplying by $a - b$ is equivalent to multiplying by a and by b separately, and subtracting the second product from the first; but subtracting b times a number is equivalent to multiplying it by $-b$ and adding the result.

123. For convenience in collecting like terms, the following additional steps are recommended :

(1) *Arrange both expressions according to the descending powers of a letter common to both, or according to the ascending powers of that letter* (58).

(2) *Supply ciphers in place of the missing powers of the letter of arrangement, in order that the exponents of that letter may have a common difference in the successive terms.*

(3) *Write the multiplier under the multiplicand, the first term under the first, the second under the second, etc.*

(4) *Multiply each term of the multiplicand by each term of the multiplier, placing the first term of each partial product under the term of the multiplier which produces it.*

(5) *Add the columns.*

1. Multiply $a^2 - 2ab + 3b^2$ by $a^2 + 3ab - 2b^2$.

$$\begin{array}{lllll}
a^2 - 2ab & + 3b^2 & & & \\
a^2 + 3ab & - 2b^2 & & & \\
\hline
a^4 - 2a^3b & + 3a^2b^2 & & & \\
\quad + 3a^3b & - 6a^2b^2 & + 9ab^3 & & \\
 & - 2a^2b^2 & + 4ab^3 & - 6b^4 & \\
\hline
a^4 + a^3b & - 5a^2b^2 & + 13ab^3 & - 6b^4 &
\end{array}$$

2. Multiply $1 + 2a + a^4 - 3a^2$ by $a^3 - 1 - 2a$.

$$\begin{array}{lllllllll}
a^4 & + 0 & - 3a^2 & + 2a & + 1 & & & & \\
a^3 & + 0 & - 2a & - 1 & & & & & \\
\hline
a^7 & + 0 & - 3a^5 & + 2a^4 & + a^3 & & & & \\
 & & - 2a^5 & - 0 & + 6a^3 & - 4a^2 & - 2a & & \\
 & & & - a^4 & - 0 & + 3a^2 & - 2a & - 1 & \\
\hline
a^7 & + 0 & - 5a^5 & + a^4 & + 7a^3 & - a^2 & - 4a & - 1 &
\end{array}$$

3. Find the product of $(x + a)(x + b)(x + c)$.

$$\begin{array}{l}
x + a \\
x + b \\
\hline
x^2 + a \big| x \\
\quad\; + b \big| \; + ab
\end{array}
\qquad\qquad
\begin{array}{l}
x^2 + (a + b)x + ab \\
x + c \\
\hline
x^3 + a \big| x^2 + ab \big| x \\
\quad\; + b \big| \quad + ac \big| \\
\quad\; + c \big| \quad + bc \big| \; + abc
\end{array}$$

4. Multiply $x + x^{\frac{1}{2}} + 1$ by $x - x^{\frac{1}{2}} + 1$.

$$\begin{array}{llllll}
x & + x^{\frac{1}{2}} & + 1 & & & \\
x & - x^{\frac{1}{2}} & + 1 & & & \\
\hline
x^2 & + x^{\frac{3}{2}} & + x & & & \\
 & - x^{\frac{3}{2}} & - x & - x^{\frac{1}{2}} & & \\
 & & + x & + x^{\frac{1}{2}} & + 1 & \\
\hline
x^2 & + 0 & + x & + 0 & + 1 &
\end{array}$$

5. Multiply $x - 1 + x^{-1}$ by $x + 1 + x^{-1}$.

$$\begin{array}{l} x - 1 + x^{-1} \\ x + 1 + x^{-1} \\ \hline x^2 - x + 1 \\ \quad + x - 1 + x^{-1} \\ \qquad + 1 - x^{-1} + x^{-2} \\ \hline x^2 + 0 + 1 + 0 \quad + x^{-2} \end{array}$$

6. Multiply $9a^2 + b^2 + 3ab - 6a + 4 + 2b$ by $3a + 2 - b$.

$$\begin{array}{l} 9a^2 + 3a\,(b-2) + (b^2 + 2b + 4) \\ 3a - \quad (b - 2) \\ \hline 27a^3 + 9a^2(b-2) + 3a\,(b^2 + 2b + 4) \\ \quad - 9a^2(b-2) - 3a\,(b^2 - 4b + 4) - (b^3 - 8) \\ \hline 27a^3 + 0 \qquad + 3a\,(6b) \qquad - (b^3 - 8) = \\ \qquad 27a^3 + 18ab - (b^3 - 8). \end{array}$$

It will be observed that $(b - 2) \times (b - 2) = b^2 - 4b + 4$, and $(b^2 + 2b + 4)(b - 2) = b^3 - 8$.

7. Multiply $x^2 + y^2 - xy + x + y$ by $x + y - 1$.

$$\begin{array}{l} x^2 - x\,(y - 1) + (y^2 + y) \\ x + \quad (y - 1) \\ \hline x^3 - x^2(y-1) + x\,(y^2 + y) \\ \quad + x^2(y-1) - x\,(y^2 - 2y + 1) + (y^3 - y) \\ \hline x^3 + 0 \qquad + x\,(3y - 1) \qquad + (y^3 - y) \end{array}$$

8. Multiply $a^2 - x\,(a - b) + b^2$ by $a + b + x$.

$$\begin{array}{l} - x\,(a - b) + (a^2 + b^2) \\ \quad x \qquad + (a + b) \\ \hline - x^2(a-b) + x\,(a^2 + b^2) \\ \qquad - x\,(a^2 - b^2) + (a^3 + a^2b + ab^2 + b^3) \\ \hline - x^2(a-b) + \quad 2b^2x \quad + (a^3 + a^2b + ab^2 + b^3) \end{array}$$

EXAMPLES.

By multiplication show that:

1. $(a+b)^2 = a^2 + 2ab + b^2$.
2. $(a-b)^2 = a^2 - 2ab + b^2$.
3. $(a+b)(a-b) = a^2 - b^2$.
4. $(a^2 - ab + b^2)(a+b) = a^3 + b^3$.
5. $(a^2 + ab + b^2)(a-b) = a^3 - b^3$.
6. $(a^2 - 2a + 1)(a^2 + 2a + 1) = a^4 - 2a^2 + 1$.
7. $(a^3 + 2a^2 + 4a + 8)(a-2) = a^4 - 16$.
8. $(a^2 - 2ab + 2b^2)(a^2 + 2ab + 2b^2) = a^4 + 4b^4$.
9. $(a^4 - a^2 + 1)(a^2 + 1) = a^6 + 1$.
10. $(a^4 + a^2 + 1)(a^4 - a^2 + 1) = a^8 + a^4 + 1$.
11. $(a+b)^2 (a-b)^2 = a^4 - 2a^2b^2 + b^4$.
12. $(x-3)^2 (x+3)^2 = x^4 - 18x^2 + 81$.
13. $(a+x)(a-x)(a^2+x^2) = a^4 - x^4$.
14. $(a^x - 2)(a^x - 3) = a^{2x} - 5a^x + 6$.
15. $(3^x - 1)^2 = 9^x - 2(3^x) + 1$.
16. $(a^x - b^y)^2 = a^{2x} - 2a^xb^y + b^{2y}$.

MULTIPLICATION BY DETACHED CO-EFFICIENTS.

124. An examination of the solutions in (123) will show that if the terms of the two factors be arranged and graded (by supplying ciphers in place of the missing powers of the letter of arrangement, and collecting like powers in brackets):

(1) The like powers of the letter of arrangement will be in the same column.

(2) The powers of that letter will descend or ascend in the product as in the two factors.

(3) If both factors be arranged according to the powers of two or more letters, the product will also be so arranged.

By observing these laws, the letter or letters of arrangement may be omitted in the solution and placed in the result. Compare the following solutions with those in the preceding article:

1. $(a^2 - 2ab + 3b^2) \times (a^2 + 3ab - 2b^2) = ?$

$$\begin{array}{l}
1 - 2 + 3 \\
1 + 3 - 2 \\
\hline
1 - 2 + 3 \\
\quad + 3 - 6 + \ \ 9 \\
\qquad\quad - 2 + \ \ 4 - 6 \\
\hline
1 + 1 - 5 + 13 - 6
\end{array}$$

Ans. $a^4 + a^3b - 5a^2b^2 + 13ab^3 - 6b^4$.

2. $(a^4 - 3a^2 + 2a + 1) \times (a^3 - 2a - 1) = ?$

$$\begin{array}{l}
1 + 0 - 3 + 2 + 1 \\
1 + 0 - 2 - 1 \\
\hline
1 + 0 - 3 + 2 + 1 \\
\qquad\quad - 2 - 0 + 6 - 4 - 2 \\
\qquad\qquad\qquad - 1 - 0 + 3 - 2 - 1 \\
\hline
1 + 0 - 5 + 1 + 7 - 1 - 4 - 1
\end{array}$$

Ans. $a^7 + 0 - 5a^5 + a^4 + 7a^3 - a^2 - 4a - 1$.

3. $(x + a)(x + b)(x + c) = ?$

$$\begin{array}{l}
1 + a \\
1 + b \\
\hline
1 + a \\
\quad + b + ab \\
\hline
1 + (a + b) + ab \\
1 + c \\
\hline
1 + (a + b) + ab \\
\qquad\qquad + c + (ac + bc) + abc \\
\hline
1 + (a + b + c) + (ab + ac + bc) + abc
\end{array}$$

Ans. $x^3 + (a + b + c)x^2 + (ab + ac + bc)x + abc$.

4. $(9a^2 + b^2 + 3ab - 6a + 4 + 2b) \times (3a + 2 - b) = ?$
Arrange according to the descending powers of a, and omit a.

$$\begin{array}{llll}
9 + 3\,(b-2) + & (b^2 + 2b + 4) & & \\
3 - (b-2) & & & \\
\hline
27 + 9\,(b-2) + & 3\,(b^2 + 2b + 4) & & \\
 - 9\,(b-2) - & 3\,(b^2 - 4b + 4) & - (b^3 - 8) & \\
\hline
27 + 0 & + 3\,(6b) & - (b^3 - 8) &
\end{array}$$

Ans. $27a^3 + 18ab - (b^3 - 8)$.

125. The principles of the preceding article may be employed for the purpose of verifying results obtained by multiplication. Thus: *Omit the letter or letters of arrangement, and simplify the results. The sum of the co-efficients in one factor, multiplied by the sum of the co-efficients in the other, must be equal to the sum of the co-efficients in the product.* For instance, in Ex. 1 (124), the multiplicand is $1 - 2 + 3 = 2$; the multiplier is $1 + 3 - 2 = 2$; and the product is $1 + 1 - 5 + 13 - 6 = 4$.

EXAMPLES.

By detached co-efficients show that:

1. $(x^4 - 2x^3 + 3x^2 - 2x + 1)\,(x^4 + 2x^3 + 3x^2 + 2x + 1) = x^8 + 2x^6 + 3x^4 + 2x^2 + 1$.
2. $(x^5 - 5x^4 + 13x^3 - x^2 - x + 2)\,(x^2 - 2x - 2) = x^7 - 7x^6 + 21x^5 - 17x^4 - 25x^3 + 6x^2 - 2x - 4$.
3. $(x - 1)(x - 3)(x + 3)(x + 1) = x^4 - 10x^2 + 9$.
4. $(x^2 - x + 1)(x^2 + x + 1)(x^4 - x^2 + 1) = x^8 + x^4 + 1$.
5. $(x + a)^5 = x^5 + 5x^4a + 10x^3a^2 + 10x^2a^3 + 5xa^4 + a^5$.
6. $(x - 1)^6 = x^6 - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 6x + 1$.
7. $(x - 2)\,(x - 3)\,(x + 3)\,(x + 2) = x^4 - 13x^2 + 36$.
8. $(x - a)\,(x + b)\,(x - c) = x^3 - (a - b + c)\,x^2 + (ac - ab - bc)\,x + abc$.
9. $(- y - 1)^5 = - y^5 - 5y^4 - 10y^3 - 10y^2 - 5y - 1$.

EXERCISE VI.

Perform the multiplication indicated:

1. $(2x^3 + 4x^2 + 8x + 16)(3x - 6)$.
2. $(x^3 + 4x^2 + 5x - 24)(x^2 - 4x + 5)$.
3. $(x^3 - 4x^2 + 11x - 24)(x^2 + 4x + 5)$.
4. $(x^3 - 2x^2 + 3x - 4)(4x^3 + 3x^2 + 2x + 1)$.
5. $(x^2 - xy + y^2 + x + y + 1)(x + y - 1)$.
6. $(x^2 - 3x + 2)^2$.
7. $(x - a)(x + b)(x + c)$.
8. $(x^2 + 2x + 1)^3$.
9. $(x^3 - 3x^2 + 3x - 1)(x^2 - 2x + 1)(x - 1)$.
10. $(x^2 - x - 1)(2x^2 + 3)(x^2 + x - 1)(x - 4)$.
11. $(a^3 - 2 + a^{-3})(a^3 + 2 + a^{-3})$.
12. $(a^{\frac{3}{2}} + a^{\frac{1}{2}} - a^{-\frac{1}{2}})(a^{\frac{3}{2}} - a^{\frac{1}{2}} + a^{-\frac{1}{2}})$.
13. $(a^{\frac{1}{2}} + b^{\frac{1}{2}} + a^{-\frac{1}{2}}b)(ab^{-\frac{1}{2}} - a^{\frac{1}{2}} + b^{\frac{1}{2}})$.
14. $(a^{\frac{3}{4}} - a^{\frac{1}{2}}b^{\frac{1}{4}} + a^{\frac{1}{4}}b^{\frac{1}{2}} - b^{\frac{3}{4}})(a^{\frac{3}{4}} + a^{\frac{1}{2}}b^{\frac{1}{4}} + a^{\frac{1}{4}}b^{\frac{1}{2}} + b^{\frac{3}{4}})$.

Simplify the following expressions:

15. $(x + y)^3 - (x - y)^3$.
16. $(x^2 + 2x + 1)^2 - (x^2 - 2x + 1)^2$.
17. $(x + 2)^4 - (x - 2)^4$.
18. $(a + b)(b + c) - (c + d)(d + a) - (a + c)(b - d)$.
19. $(a + b + c)^3 - 3(a + b)(a + c)(b + c)$.
20. $(a + b + c)^2 - a(b + c - a) - b(a + c - b) - c(a + b - c)$.
21. $(x^2 + xy + y^2)^2 - (x^2 - xy + y^2)^2$.
22. $(x^2 + xy - y^2)^2 - (x^2 - xy - y^2)^2$.
23. $(x + y)^4 - 4xy(x + y)^2 + 2x^2y^2$.
24. $[(x + y)^3 + (x - y)^3]^2$.
25. Substitute $y + 2$ for x in $x^4 - 2x^2 - x^3 + 4$.

CHAPTER VII.

INVOLUTION.

126. Involution is the operation of raising an expression to any required power. Every case of involution is merely an example of multiplication in which the factors are equal.

Thus: $(-2a^2)^3 = (-2a^2)(-2a^2)(-2a^2) = -8a^6$.

MONOMIALS.

127. To raise a monomial to any power: *Multiply the exponent of each factor of the monomial by the exponent of the power. If the given expression be negative, and the exponent of the power be an odd number, the product will be negative; otherwise the product will be positive* (115).

(1) $(-2ab^2)^5 = -2^5a^5b^{10} = -32a^5b^{10}$.

(2) $(-3a^2bc^3)^4 = +3^4a^8b^4c^{12} = +81a^8b^4c^{12}$.

(3) $(2a^3b^2c)^6 = 2^6a^{18}b^{12}c^6 = 64a^{18}b^{12}c^6$.

(4) $(3^7a^2b)^4 = 3^{28}a^8b^4$.

(5) $(-3ab^{\frac{1}{2}}c^{-1})^4 = +3^4a^4b^2c^{-4} = 81a^4b^2c^{-4}$.

(6) $-(-2a^{\frac{1}{2}}b^{\frac{2}{3}}c^{-\frac{1}{3}})^6 = -(+2^6a^3b^4c^{-2}) = -64a^3b^4c^{-2}$.

(7) $(-2ab^2)^{-5} = -2^{-5}a^{-5}b^{-10} = -\frac{1}{32}a^{-5}b^{-10}$.

(8) $(-2ab^2)^n = +2^na^nb^{2n}$ if n be any even integer, but $(-2ab^2)^n = -2^na^nb^{2n}$ if n be any odd integer.

128. Since every even power is positive, therefore: *If any expression in brackets be affected by an even exponent, the sign of each inclosed term may be changed without altering the value.* Thus:

$$(a-b)^2=(b-a)^2;\ (a-b)^4=(b-a)^4;$$
$$(a-b-c)^6=(-a+b+c)^6;\text{ etc.}$$

If any expression in brackets be affected by an odd exponent, the sign of each inclosed term may be changed, provided the sign preceding the brackets be changed. Thus:

$(a-b)^3=-(b-a)^3$; $(a-b+c)^5=-(-a+b-c)^5$; etc.

BINOMIALS.

129. By multiplying $(a+b)$ by $(a+b)$, and that product by $(a+b)$, etc. (123), the following formulas are obtained:

$(a+b)^2=a^2+2ab+b^2.$
$(a+b)^3=a^3+3a^2b+3ab^2+b^3.$
$(a+b)^4=a^4+4a^3b+6a^2b^2+4ab^3+b^4.$
$(a+b)^5=a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5.$
$(a+b)^6=a^6+6a^5b+15a^4b^2+20a^3b^3+15a^2b^4+6ab^5+b^6.$

By multiplying $(a-b)$ by $(a-b)$, and that product by $(a-b)$, etc., the following formulas are obtained:

$(a-b)^2=a^2-2ab+b^2.$
$(a-b)^3=a^3-3a^2b+3ab^2-b^3.$
$(a-b)^4=a^4-4a^3b+6a^2b^2-4ab^3+b^4.$
$(a-b)^5=a^5-5a^4b+10a^3b^2-10a^2b^3+5ab^4-b^5.$
$(a-b)^6=a^6-6a^5b+15a^4b^2-20a^3b^3+15a^2b^4-6ab^5+b^6.$

By means of the *Binomial Theorem*, or *Newton's Theorem*, as it is sometimes called, any power of a binomial may be found without performing the successive multiplications. This theorem may be briefly stated thus:

In the expansion of $(a+b)^n$, n being any positive integer:

(1) The number of terms is greater by one than the exponent of the power to which the binomial is raised. Thus, if $n=8$, the number of terms is $n+1=8+1=9$.

(2) The exponent of a in the first term is the same as the exponent of the power to which the binomial is raised, and it decreases by one in each succeeding term.

(3) b appears in the second term with an exponent of 1, and its exponent increases by one in each succeeding term.

(4) The co-efficient of the first term is 1.

(5) The co-efficient of the second term is the same as the exponent of the power to which the binomial is raised.

(6) The co-efficient of each succeeding term is found by multiplying the co-efficient of the preceding term by the exponent of a in that term, and dividing the product by a number greater by one than the exponent of b in that term.

(7) Terms which are equidistant from the middle of the series, have equal co-efficients. If the number of terms be odd, there is one middle term, and its co-efficient is greater than that of any other term. If the number of terms be even, there are two middle terms whose co-efficients are equal, and greater than those of the other terms.

(8) If a be positive and b be negative, the terms in which odd powers of b occur, are negative.

130. Expanding an Expression is performing the multiplication indicated by multiplication-signs or exponents.

131. Given powers of any binomial may be expanded by means of Newton's Theorem. For example: Expand $(2x - 3y)^5$. For $2x$ put a, and for $3y$ put b. Then $(2x - 3y)^5 = (a - b)^5$.

By Newton's Theorem, $(a - b)^5 = a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5$.

$\therefore\ (2x - 3y)^5 = (2x)^5 - 5\,(2x)^4\,(3y) + 10\,(2x)^3\,(3y)^2 - 10\,(2x)^2\,(3y)^3 + 5\,(2x)\,(3y)^4 - (3y)^5 = 32x^5 - 240x^4y + 720x^3y^2 - 1080x^2y^3 + 810xy^4 - 243y^5$.

In like manner, powers of other polynomials may be expanded. Thus $(a + x - y)^3$ may be expanded by inclosing $x - y$ in brackets, and applying Newton's Theorem to the resulting binomial, as follows:

$[a + (x - y)]^3 = a^3 + 3a^2\,(x - y) + 3a\,(x - y)^2 + (x - y)^3 = a^3 + 3a^2x - 3a^2y + 3ax^2 - 6axy + 3ay^2 + x^3 - 3x^2y + 3xy^2 - y^3$.

132. It may be shown by actual multiplication that:

$$(a+b+c)^2 = a^2 + b^2 + c^2 + 2a\,(b+c) + 2bc.$$

$$(a-b+c)^2 = a^2 + b^2 + c^2 + 2a\,(-b+c) - 2bc.$$

The following rule may be observed to hold good in the above and similar examples: *The square of any polynomial consists of the square of each term, together with twice the product of each term by the sum of all the terms which follow it.*

133. Since $(a+b+c)^3 = a^3 + b^3 + c^3 + 3a^2\,(b+c) + 3b^2\,(a+c) + 3c^2\,(a+b) + 6abc,$

and $(a-b+c)^3 = a^3 - b^3 + c^3 + 3a^2\,(-b+c) + 3b^2\,(a+c) + 3c^2(a-b) - 6abc,$

therefore: *The cube of any polynomial consists of the cube of each term, plus three times the square of each term into the sum of the other terms, plus six times the product of the terms taken three at a time.*

134. By detaching the co-efficients, the results obtained in accordance with the preceding formulas may be verified, as may also the following:

(1) In the expansion of $(a+b)^n$ the sum of the co-efficients is $(1+1)^n = (2)^n$.

(2) In the expansion of $(a-b)^n$ the sum of the co-efficients is $(1-1)^n = (0)^n = 0$. Since the sum of all the co-efficients is zero, the sum of the co-efficients of the positive terms must be numerically equal to that of the negative terms. But all the odd terms are positive, and all the even terms are negative; therefore: *In the expansion of $(a \pm b)^n$ the sum of the co-efficients of the odd terms equals the sum of the co-efficients of the even terms.*

(3) In the expansion of $(a+b+c)^2$ the sum of the co-efficients is $(1+1+1)^2 = (3)^2 = 9$. In $(a+b+c)^3$ the sum of the co-efficients is $(1+1+1)^3 = (3)^3 = 27$.

(4) In the expansion of $(a+2b-3c)^n$ the sum of the co-efficients is $(1+2-3)^n = (0)^n = 0$; etc.

EXAMPLES.

Expand the following expressions:

1. $(x-y)^7$. *Ans.* $x^7 - 7x^6y + 21x^5y^2 - 35x^4y^3 + 35x^3y^4 - 21x^2y^5 + 7xy^6 - y^7$

2. $(x+y)^8$. *Ans.* $x^8 + 8x^7y + 28x^6y^2 + 56x^5y^3 + 70x^4y^4 + 56x^3y^5 + 28x^2y^6 + 8xy^7 + y^8$

3. $(x-y)^{-4}$. *Ans.* $(x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4)^{-1}$.

4. $(2a-3b)^3$. *Ans.* $8a^3 - 36a^2b + 54ab^2 - 27b^3$.

5. $(1-2x+x^2)^2$. *Ans.* $1 - 4x + 6x^2 - 4x^3 + x^4$.

6. $(1-2x+3x^2)^2$. *Ans.* $1 - 4x + 10x^2 - 12x^3 + 9x^4$.

7. $(a+b-c)^3$. *Ans.* $a^3 + b^3 - c^3 + 3a^2b - 3a^2c + 3ab^2 - 3b^2c + 3ac^2 + 3bc^2 - 6abc$.

8. $(1-3x+3x^2-x^3)^2$. *Ans.* $1 - 6x + 15x^2 - 20x^3 + 15x^4 - 6x^5 + x^6$

EXERCISE VII.

Expand and simplify the following expressions:

1. $(x-y)^9$.
2. $(x-y)^{-5}$.
3. $(a-2b)^4$.
4. $(3x-2y)^{-4}$.
5. $(x+y)^4 - (x-y)^4$.
6. $(x-1+x^{-1})^2$.
7. $(x^{\frac{1}{2}} - 2 + x^{-\frac{1}{2}})^2$.
8. $(a^2-2a+1)^3$.
9. $(x+y-1)^3$.
10. $(a-b-c+d)^2$.
11. $[(a+b)^2 + (a-b)^2]^2$.
12. $[(a+1)^3 + (a-1)^3]^{-2}$.
13. $(a^{\frac{1}{2}} - 3)^4$.
14. $(-2a^2 + a - 3a^{-1})^2$.
15. $(-x-y)^8$.
16. $(x - 2x^{\frac{1}{4}} + x^{-\frac{1}{2}})^3$.
17. $(bx \pm ay)^4$.
18. $(1 \pm 2x)^5$.
19. $(a+bx+cx^2)^2$.
20. $(a-2b+c)^3$.
21. $(ax+x^2)^{-4}$.
22. $(1+x+x^2)^2 - (1-x+2x^2)^2$.
23. $(3-2x+x^2)^2 - (2-x)^4$.
24. $(1+2x-3x^2+4x^3)^2$.

CHAPTER VIII.

DIVISION.

135. Division is the process of finding a factor which, being multiplied by a given factor, will produce a given product.

136. With reference to this operation, the product is called the *dividend,* the given factor is called the *divisor,* and the required factor is called the *quotient.*

137. Since the dividend is the product of the divisor and the quotient, it follows from the law of the signs in multiplication that if the dividend be positive, the divisor and quotient have like signs; if the dividend be negative, the divisor and quotient have unlike signs. That is:

(1) $+ab \div (+a) = +b$; (3) $-ab \div (+a) = -b$;

(2) $+ab \div (-a) = -b$; (4) $-ab \div (-a) = +b$.

From the first and fourth of these formulas we derive the rule: *If the dividend and the divisor have like signs, the sign of the quotient is plus.*

From the second and third of the above formulas we derive the rule: *If the dividend and the divisor have unlike signs, the sign of the quotient is minus.*

138. Since the numerical co-efficient of the dividend is equal to the product of the co-efficients of its two factors, therefore: *The numerical co-efficient of the quotient is found by dividing the numerical co-efficient of the dividend by that of the divisor.* Thus, since $5a \times 4b = 20ab$, therefore $20ab \div 4b = 5a$.

139. Since the exponent of any letter in the dividend is equal to the sum of the exponents of that letter in its two factors (117), therefore: *The exponent of any letter in the quotient is found by subtracting the exponent of that letter in the divisor from its exponent in the dividend.*

A letter which has the same exponent in both dividend and divisor, is not written in the quotient. (82.)

EXAMPLES.—1. Since $a^2 \times a^3 = a^5$, $\therefore a^5 \div a^2 = a^3$, and $a^5 \div a^3 = a^2$.

2. Since $a^5 \times a^{-2} = a^3$, $\therefore a^3 \div a^5 = a^{-2}$, and $a^3 \div a^{-2} = a^5$.

3. Since $a^{-2}b \times a^2b^3 = b^4$, $\therefore b^4 \div a^{-2}b = a^2b^3$, and $b^4 \div a^2b^3 = a^{-2}b$.

A MONOMIAL BY A MONOMIAL.

140. By means of (137, 138, 139) we may divide one monomial by another.

EXAMPLES.

1. $a^3b \div ab = a^2$.
2. $a^4b^3 \div a^2b^2 = a^2b$.
3. $a^m \div a^{m-2} = a^2$.
4. $a^{b+c} \div a^{b-c} = a^{2c}$.
5. $a^2b^3c \div a^2bc^2 = b^2c^{-1}$.
6. $(a-b)^5 \div (a-b)^3 = (a-b)^2$.
7. $a^{b+2}c^{b-2} \div ac^2 = a^{b+1}c^{b-4}$.
8. $(a-c)^{x+2} \div (a-c)^{x-1} = (a-c)^3$.
9. $a^2b^{-3}c^x \div a^4bc^2 = a^{-2}b^{-4}c^{x-2}$.
10. $2^{x+y} \div 2^{x+2} = 2^{y-2}$.

A POLYNOMIAL BY A MONOMIAL.

141. Since $(a-b)c = ac - bc$, $\therefore (ac - bc) \div c = a - b$.

Since $(a-b)(-c) = -ac + bc$, $\therefore (-ac + bc) \div (-c) = a - b$.

Hence, to divide a polynomial by a monomial, *Divide each term of the dividend by the divisor.*

EXAMPLES.

1. $(a^2b - a^3b^2 - ab^3) \div ab = a - a^2b - b^2.$
2. $(ac - ac^2 + ac^3) \div ac = 1 - c + c^2.$
3. $(ab^2 + a^2b^3 - a^3b^4) \div a^2b^2 = a^{-1} + b - ab^2.$
4. $(- a^m + a^{m+2} - a^{m+4}) \div a^{m-2} = - a^2 + a^4 - a^6.$
5. $(a^2b^3c - abc^3 + abc) \div (- ab^2c^3) = - abc^{-2} + b^{-1} - b^{-1}c^{-2}.$

A POLYNOMIAL BY A POLYNOMIAL.

142. If the divisor (given factor) $= a + b + c$,
and the quotient (required factor) $= x + y + z$,
then the dividend (product) $= \begin{cases} ax + bx + cx \\ + ay + by + cy \\ + az + bz + cz. \end{cases}$

The first term of the dividend, ax, is the product of the first term of the divisor, a, by the first term of the quotient, x; hence, to find the first term of the quotient, *Divide the first term of the dividend by the first term of the divisor.*

If the partial product formed by multiplying the entire divisor by x be subtracted from the dividend, the first term of the remainder, ay, is the product of a, the first term of the divisor, by y, the second term of the quotient. Hence, to find the second term of the quotient, *Divide the first term of the first remainder by the first term of the divisor.*

In like manner, if the partial product formed by multiplying the entire divisor by y be subtracted from the first remainder, the first term of the second remainder, az, is the product of a, the first term of the divisor, by z, the third term of the quotient. Hence, the third term of the quotient is found *by dividing the first term of the remainder by the first term of the divisor,* and so on.

143. To divide one polynomial by another:

(1) *Arrange both dividend and divisor according to the powers of some letter common to both.*

(2) *Divide the first term of the dividend by the first term of the divisor. The result is the first term of the quotient.*

(3) *Multiply all the terms of the divisor by the first term of the quotient.*

(4) *Subtract the product from the dividend.*

(5) *If there be a remainder, consider it as a new dividend, and proceed as before.*

It is convenient to supply ciphers in place of missing powers of the letter of arrangement.

Verification: *Multiply together the divisor and the quotient; the product must be equal to the dividend.*

EXAMPLES.

1. Divide $a^4 - 4a^3 - 19a^2 + 106a - 120$ by $a^2 - 5a + 6$.

$$\begin{array}{l|l}
a^4 - 4a^3 - 19a^2 + 106a - 120 & a^2 - 5a + 6 \\
\underline{a^4 - 5a^3 + 6a^2} & a^2 + a - 20 \quad \textit{Ans.} \\
\quad a^3 - 25a^2 + 106a - 120 & \\
\quad \underline{a^3 - 5a^2 + 6a} & \\
\qquad - 20a^2 + 100a - 120 & \\
\qquad \underline{- 20a^2 + 100a - 120} &
\end{array}$$

2. $[3x^4 + 2x^3 + x^6 - 4x^5 + 4x^2 - 15] \div [x^3 - x^2 - 3] = ?$

$$\begin{array}{l|l}
x^6 - 4x^5 + 3x^4 + 2x^3 + 4x^2 + 0 - 15 & x^3 - x^2 + 0 - 3 \\
\underline{x^6 - x^5 + 0 - 3x^3} & x^3 - 3x^2 + 0 + 5 \quad \textit{Ans.} \\
\quad - 3x^5 + 3x^4 + 5x^3 + 4x^2 + 0 - 15 & \\
\quad \underline{- 3x^5 + 3x^4 - 0 + 9x^2} & \\
\qquad + 0 + 5x^3 - 5x^2 + 0 - 15 & \\
\qquad \underline{+ 5x^3 - 5x^2 + 0 - 15} &
\end{array}$$

3. Divide $4x^3 - 32x^2 + 3x^4 + 49$ by $2x - 3x^2 + 7$.

$$\begin{array}{l|l}
3x^4 + 4x^3 - 32x^2 + 0 + 49 & -3x^2 + 2x + 7 \\
3x^4 - 2x^3 - \ \ 7x^2 & -\ x^2 - 2x + 7 \quad \textit{Ans.} \\
\hline
\quad + 6x^3 - 25x^2 + 0 \quad + 49 & \\
\quad + 6x^3 - \ \ 4x^2 - 14x & \\
\hline
\qquad\quad - 21x^2 + 14x + 49 & \\
\qquad\quad - 21x^2 + 14x + 49 & \\
\hline
\end{array}$$

4. Divide $x^3 + x^2y + x^2z - xyz - y^2z - yz^2$ by $x^2 - yz$.

$$\begin{array}{l|l}
x^3 + x^2(y+z) - xyz - (y^2z + yz^2) & x^2 + \ 0 - yz \\
x^3 + 0 \qquad\qquad - xyz & x + (y+z) \quad \textit{Ans.} \\
\hline
\quad + x^2(y+z) + 0 - (y^2z + yz^2) & \\
\quad + x^2(y+z) + 0 - (y^2z + yz^2) & \\
\hline
\end{array}$$

5. Divide $a^6 - y^6 - 3a^2y^2 - 1$ by $a^2 - y^2 - 1$.

$$\begin{array}{l|l}
a^6 + 0 - 3a^2y^2 - (y^6 + 1) & a^2 - (y^2 + 1) \\
a^6 - a^4(y^2 + 1) & a^4 + a^2(y^2+1) + (y^4 - y^2 + 1) \\
\hline
\quad + a^4(y^2+1) - 3a^2y^2 - (y^6 + 1) & \\
\quad + a^4(y^2+1) - a^2(y^4 + 2y^2 + 1) & \\
\hline
\qquad\qquad + a^2(y^4 - y^2 + 1) - (y^6 + 1) & \\
\qquad\qquad + a^2(y^4 - y^2 + 1) - (y^6 + 1) & \\
\hline
\end{array}$$

Observe that $(y^2 + 1)^2 = y^4 + 2y^2 + 1$, and that $(y^2 + 1)(y^4 - y^2 + 1) = y^6 + 1$. *Ans.* $a^4 + a^2y^2 + a^2 + y^4 - y^2 + 1$.

6. Divide $x^3 + a^3$ by $x - a$.

$$\begin{array}{l|l}
x^3 + 0 + 0 + a^3 & x - a \\
x^3 - ax^2 & x^2 + ax + a^2 + \dfrac{2a^3}{x-a} \quad \textit{Ans.} \\
\hline
\quad + ax^2 + 0 + a^3 & \\
\quad + ax^2 - a^2x & \\
\hline
\qquad\quad + a^2x + a^3 & \\
\qquad\quad + a^2x - a^3 & \\
\hline
\qquad\qquad\quad + 2a^3 & \\
\end{array}$$

Divide the first expression by the second :

7. $4x^5 + 4x - x^3$; $3x + 2x^2 + 2$. *Ans.* $2x^3 - 3x^2 + 2x$.

8. $1 - 6x^5 + 5x^6$; $1 - 2x + x^2$.
Ans. $1 + 2x + 3x^2 + 4x^3 + 5x^4$.

9. $x^2 + x^3 - 7x^4 + 5x^5$; $x - x^2$. *Ans.* $x + 2x^2 - 5x^3$.

10. $a^2 + (a - 1)\, x^2 + (a - 1)\, x^3 + (a - 1)\, x^4 - x^5$; $a - x$.
Ans. $a + x + x^2 + x^3 + x^4$.

11. $1 - 9x^8 - 8x^9$; $1 + 2x + x^2$.
Ans. $1 - 2x + 3x^2 - 4x^3 + 5x^4 - 6x^5 + 7x^6 - 8x^7$.

12. $a^2 + 2ab - c^2 - 2cd + b^2 - d^2$; $a + b + c + d$.
Ans. $a + b - c - d$.

13. $x^3 - (a + b + c)\, x^2 + (ab + ac + bc)\, x - abc$; $x - b$.
Ans. $x^2 - (a + c)\, x + ac$.

14. $a^2 - 2ab + b^2 - c^2$; $a - b + c$. *Ans.* $a - b - c$.

15. $a^2x^4 - (b^2 - 2ac)\, x^2 + c^2$; $ax^2 - bx + c$.
Ans. $ax^2 + bx + c$.

16. $a^3 + 8b^3 + c^3 - 6abc$; $a + 2b + c$.
Ans. $a^2 + 4b^2 + c^2 - ac - 2ab - 2bc$.

17. $a^3 - 3a^2b + 3ab^2 - b^3 + c^3$; $a - b + c$.
Ans. $a^2 - 2ab + b^2 - ac + bc + c^2$.

18. $a^2\,(b + c) + b^2\,(a - c) + c^2\,(a - b) + abc$; $a + b + c$.
Ans. $a\,(b + c) - bc$.

19. $(a - b)^2 - 2\,(a - b)\, c + c^2$; $a - b - c$. *Ans.* $a - b - c$.

20. $(a-b)^3 + 3(a-b)^2c + 3(a-b)c^2 + c^3$; $(a-b)^2 + 2(a-b)c + c^2$.
Ans. $a - b + c$.

21. $7x^3 - 24x + 58x^{-1} - 21x^{-3}$; $7x - 3x^{-1}$.
Ans. $x^2 - 3 + 7x^{-2}$.

22. $(x^3 + y^3)\,(x^2 - y^2)$; $(x^2 - xy + y^2)\,(x - y)$.
Ans. $x^2 + 2xy + y^2$.

SYNTHETIC DIVISION.

144. An examination of the solutions in (143) will show that they may be abbreviated in several respects:

(1) The products of the first term of the divisor by the various terms of the quotient are unnecessary, as they are the same as the first terms of the corresponding dividends.

(2) Since only the first term of each remainder is used for purposes of division, the other terms need not be found.

(3) By changing the signs of the terms of the divisor in the first instance, the subtraction of the various partial products is avoided.

(4) Since the like powers of the letter of arrangement occur in the same column and follow a uniform law, such letter may be omitted in the solution and supplied in the answer.

These suggestions are embodied in the following solutions of four problems considered in the preceding article. For the sake of conciseness, the divisor is written at the left in a vertical column. The first term of the divisor is separated from the others, because it is used as a divisor and not as a multiplicand. The signs of all the other terms of the divisor are changed. The quotient is written below the dividend and partial products.

1. Without detaching co-efficients.

$$\begin{array}{r|r|l} a^2 & a^4-4a^3-19a^2 & +106a-120 \\ \cline{1-1} +5a & +5a^3-\ 6a^2 & \\ -6 & +\ 5a^2 & -6a \\ & & -100a+120 \\ \cline{2-3} & a^2+\ a-20 & +\ 0\ \ +\ 0 \end{array}$$

Quo. a^2+a-20.

By detached co-efficients.

$$\begin{array}{r|r|l} 1 & 1-4-19 & +106-120 \\ \cline{1-1} +5 & +5-\ 6 & \\ -6 & +\ 5 & -6 \\ & & -100+120 \\ \cline{2-3} & 1+1-20 & +\ 0\ +\ 0 \end{array}$$

Quotient, a^2+a-20.

EXPLANATION.—After the terms are arranged and graded, divide the first term of the dividend, a^4, by the first term of the divisor, a^2, and write the first term of the quotient, a^2, under the dividend, as shown above. Multiply $5a - 6$ (the terms of the divisor, except the first, with contrary signs) by a^2, and write the product under the dividend in the second, third, etc., columns. Add the second column and divide the sum, a^3, by a^2, the first term of the divisor. Write the quotient, $+ a$, under the column added. Multiply $5a - 6$ by a, and write the product under the dividend, beginning in the third column. Add the third column, and divide the sum, $- 20a^2$, by a^2. Write the quotient, $- 20$, under the third column. Multiply $5a - 6$ by $- 20$, and write the product as before, beginning in the fourth column. The sum of the other columns being 0, the division is exact, and the quotient is $a^2 + a - 20$. It is well to draw a second vertical line, to denote where the division ends. The number of columns to the right of this line is one less than the number of terms in the divisor.

If the letter of arrangement be omitted, the work is still shorter, as is shown in the second solutions.

2.

$$
\begin{array}{r|rrrr|rrr}
x^3 & x^6 & -4x^5 & +3x^4 & +2x^3 & +4x^2 & +0 & -15 \\
\hline
+x^2 & & +x^5 & +0 & +3x^3 & & & \\
+0 & & & -3x^4 & -0 & -9x^2 & & \\
+3 & & & & & +5x^2 & +0 & +15 \\
\hline
& x^3 & -3x^2 & +0 & +5 & +0 & +0 & +0
\end{array}
$$

$$
\begin{array}{r|rrrr|rrr}
1 & 1 & -4 & +3 & +2 & +4 & +0 & -15 \\
\hline
+1 & & +1 & +0 & +3 & & & \\
+0 & & & -3 & -0 & -9 & & \\
+3 & & & & & +5 & +0 & +15 \\
\hline
& 1 & -3 & +0 & +5 & +0 & +0 & +0
\end{array}
$$

By detached co-efficients.

Quotient, $1x^3 - 3x^2 + 0x + 5 = x^3 - 3x^2 + 5$.

3. Divide $x^3 + x^2y + x^2z - xyz - y^2z - yz^2$ by $x^2 - yz$.

$$
\begin{array}{r|ll|l}
x^2 & x^3 + x^2 & (y+z) & -\,xyz - (y^2z + yz^2) \\
\hline
+0 & \quad + & 0 & +\,xyz \\
+yz & & & +\;0 \;+ (y^2z + yz^2) \\
\cline{2-4}
 & x \;+ & (y+z) & +\;0 \;+\quad 0
\end{array}
$$

$$
\begin{array}{r|l|l}
1 & 1 + (y+z) & -\,yz - (y^2z + yz^2) \\
\hline
+0 & \quad + \quad 0 & +\,yz \\
+yz & & +\;0 \;+ (y^2z + yz^2) \\
\cline{2-3}
 & 1 + (y+z). &
\end{array}
\qquad \textit{Quotient, } x + (y+z).
$$

4. Divide $x^3 + a^3$ by $x - a$.

$$
\begin{array}{r|l|l}
x & x^3 + \;0 \;+\; 0 & +\,a^3 \\
\hline
+a & \quad +\,ax^2 + a^2x & +\,a^3 \\
\hline
 & x^2 + ax \;+\; a^2 & +\,2a^3
\end{array}
$$

$$
\begin{array}{r|l|l}
1 & 1 + 0 + 0 & +1 \\
\hline
+1 & \quad +1 + 1 & +1 \\
\hline
 & 1 + 1 + 1 & +2
\end{array}
\qquad \textit{Quo., } x^2 + ax + a^2 + \frac{2a^3}{x-a}.
$$

EXAMPLES.

Divide the first expression by the second:

1. $-10x^4 + 15x^2 - 5x^3;\ -5x^2.$ *Ans.* $+2x^2 + x - 3.$

2. $x^3 - 40x - 63;\ x - 7.$ *Ans.* $x^2 + 7x + 9.$

3. $x^4 + x^3 - 9x^2 - 16x - 4;\ x^2 + 4x + 4.$ *Ans.* $x^2 - 3x - 1.$

4. $x^4 - ax^3 + (b-1)x^2 + ax - b;\ x^2 - 1.$
   Ans. $x^2 - ax + b.$

5. $1 + x^3y^{-\frac{1}{2}} - x^{-3}y - y^{\frac{1}{2}};\ x^{\frac{3}{2}} - y^{\frac{1}{2}}.$
   Ans. $x^{\frac{3}{2}}y^{-\frac{1}{2}} + 1 + x^{-\frac{3}{2}} + x^{-3}y^{\frac{1}{2}}.$

6. $x^4 - y^4;\ x^{-1} + y^{-1}.$ *Ans.* $x^4y - x^3y^2 + x^2y^3 - xy^4.$

7. $x^{\frac{4}{5}} - y^{\frac{4}{3}};\ x^{-\frac{1}{5}} - y^{-\frac{1}{3}}.$ *Ans.* $x^{\frac{4}{5}}y^{\frac{1}{3}} + x^{\frac{3}{5}}y^{\frac{2}{3}} + x^{\frac{2}{5}}y + x^{\frac{1}{5}}y^{\frac{4}{3}}.$

EXERCISE VIII.

Divide the first expression by the second:

1. $-8a^2bc^3;\ 4abc.$
2. $6a^2b - 12ab^3 + 18a^3b^2;\ -3ab.$
3. $(m-n)^3(m+n)^4;\ -(m-n)(m+n)^3.$
4. $a^6 - 6a + 5;\ a^2 - 2a + 1.$
5. $x^4 + 4x + 3;\ x^2 + 2x + 1.$
6. $a^5 - 4a^3b^2 - 8a^2b^3 - 17ab^4 - 12b^5;\ a^2 - 2ab - 3b^2.$
7. $x^3 - (a+b)x^2 + (c+ab)x - ac;\ x - a.$
8. $x^5 - ax^4 + bx^3 - bx^2 + ax - 1;\ x - 1.$
9. $a^3 - b^3 - c^3 - 3abc;\ a - b - c.$
10. $x^3 - 8y^3 + 6xy + 1;\ 1 - 2y + x.$
11. $a^6 - b^6;\ (a-b)(a+b).$
12. $(a+b)^3 + 8c^3;\ a + b + 2c.$
13. $2a^{-3} - 3a^{-2} + a^{-4} - a^{-5};\ -a^{-3}.$
14. $(a+1)^3 - 2(a+1)^2 + a + 1;\ a + 1.$
15. $(a-b)^{\frac{1}{2}} - (a-b)cd + a - b;\ (a-b)^{\frac{1}{2}}.$
16. $3^2(a-b) - 3^3(a-b)^2 - 3^4(a-b)^4;\ 3^{-1}(a-b)^{-2}.$
17. $(a+b-c)^x - (a+b-c)^{x-y} - (a+b-c)^y;\ (a+b-c)^{x+y}.$
18. $x^4 + x^2 + 1;\ x - x^{\frac{1}{2}} + 1.$
19. $(a^2 - 2bc)^3 - 8b^3c^3;\ a^2 - 4bc.$
20. $(a^3 - b^3)(a^2 - b^2);\ (a-b)^2.$
21. $a^{\frac{3}{2}} - 2a^{\frac{1}{2}} + 2a^{-1} - a^{-\frac{3}{2}};\ a^{\frac{1}{2}} + 1 - a^{-\frac{1}{2}}.$
22. $2a^3 + 2b^3 + 2c^3 - 6abc;\ (a-b)^2 + (b-c)^2 + (c-a)^2.$
23. $(a^3 - b^3)^2;\ (a^2 + ab + b^2)^2.$
24. $a^{-3} - (b-c)^{-3};\ a^{-1} - (b-c)^{-1}.$

CHAPTER IX.

EVOLUTION.

145. Evolution is the process of finding any required root of an expression; that is, evolution is the process of finding a number which, being raised to a proposed power, will produce a given number.

MONOMIALS.

146. Since evolution is the opposite of involution, therefore the rules for the extraction of roots are formed by reversing those given in Chapter VII.

Since $(a^2)^5 = a^{10}$ (Art. 127), therefore $\sqrt[5]{a^{10}} = a^2$.

Since $(-2a^2)^3 = -2^3a^6 = -8a^6$, $\therefore$ $\sqrt[3]{-8a^6} = -2a^2$.

Since $(\pm a^3)^4 = +a^{12}$, $\therefore$ $\sqrt[4]{a^{12}} = \pm a^3$.

Since $(\pm a^{-2})^4 = +a^{-8}$, $\therefore$ $\sqrt[4]{a^{-8}} = \pm a^{-2}$.

Since $\left(\frac{a}{b}\right)^3 = \frac{a^3}{b^3}$, therefore the cube root of $\frac{a^3}{b^3}$ is $\frac{a}{b}$.

Since no even power is negative, therefore an even root of a negative number is impossible.

Rule for Extracting any required Root of a Monomial:

(1) SIGN. *If the given monomial be positive, any odd root will be positive, and any even root will have the double sign* (±). *If the given monomial be negative, any odd root will be negative, and any even root will be impossible.*

(2) NUMERICAL CO-EFFICIENT. *Separate the given numerical co-efficient into as many equal factors as there are units in the index of the root. One of these factors will be the numerical co-efficient of the required root.*

(3) EXPONENT. *Divide the exponent of each factor by the index of the root; the quotient will be the exponent of that factor in the required root.*

147. If the root of a number expressed in figures is not readily detected, it may be found by resolving the number into its prime factors. For this purpose the following principles may be advantageously employed:

(1) Every number is divisible by 2 when its right-hand digit is 0, 2, 4, 6, or 8.

(2) Every number is divisible by $2^2 = 4$ when the number denoted by its two right-hand digits is divisible by 4.

(3) Every number is divisible by $2^3 = 8$ when the number denoted by its three right-hand digits is divisible by 8.

(4) Every number is divisible by 3 or 9 when the sum of its digits is divisible by 3 or 9.

(5) Every number is divisible by 5 when its right-hand digit is 0 or 5.

(6) Every number is divisible by 11 when the sum of the digits in the odd places and the sum of the digits in the even places differ by zero, or by a multiple of 11.

(7) Every number is divisible by a composite number when it is divisible by all the factors of that number.

148. An integer is *prime* when it is divisible by no integer except itself and unity. Examples: 2, 5, 7, etc. An integer is *composite* when it is divisible by some other integer besides itself and unity. Examples: 4, 6, 9, etc.

SQUARE ROOTS OF POLYNOMIALS.

149. Since the square of $a + b$ is $a^2 + 2ab + b^2$, $\therefore$ the square root of $a^2 + 2ab + b^2$ is $a + b$. Since the square of $a + b + c$ is $(a + b)^2 + 2(a + b)c + c^2$, $\therefore$ the square root of $(a + b)^2 + 2(a + b)c + c^2$ is $(a + b) + c$. Similarly, the square root of $(a + b + c)^2 + 2(a + b + c)d + d^2$ is $(a + b + c) + d$; etc. All of these squares are of the general form, $a^2 + 2ab + b^2$, in which a stands in succession for one term, two terms, three terms, etc. The rule for the extraction of the square roots of polynomials is formed, therefore, by finding in what manner $a + b$ is derived from $a^2 + 2ab + b^2$, or its equal, $a^2 + (2a + b)\,b$.

$$\begin{array}{r|l|l}
 & a^2 + 2ab + b^2 & a + b \\
 & a^2 & \\ \hline
2a + b & 2ab + b^2 & \\
 & 2ab + b^2 & \\ \hline
\end{array}$$

The first term, a, of the root is evidently the square root of the first term, a^2, of the given expression. If a^2 be subtracted from $a^2 + 2ab + b^2$, the remainder is $2ab + b^2$. Therefore b, the second term of the root, is obtained by dividing $2ab$, the first term of this remainder, by $2a$; that is, by twice the part of the root already found. Multiply $2a + b$ by b, and subtract the product, $2ab + b^2$, from the remainder. If the root contain more terms, we proceed with $a + b$ as we formerly did with a; its square, $a^2 + 2ab + b^2$, has already been subtracted from the given expression, and the first term of the remainder is $2\,(a + b)\,c$; so we divide this term by $2\,(a + b)$, and the quotient is c, the third term of the root. Multiply $2\,(a + b) + c$ by c, and subtract the product, $2\,(a + b)\,c + c^2$, from the second remainder. This process must be continued until the required root is found; the trial divisor in each case being double the part of the root already found, and the complete divisor being the sum of the trial divisor and the new term of the root.

EXAMPLES.—1. $\sqrt{36x^2 - 60xy + 25y^2} =$ what?

$$\begin{array}{r|l|l}
 & 36x^2 - 60xy + 25y^2 & 6x - 5y \\
 & 36x^2 & \\ \hline
12x - 5y & -60xy + 25y^2 & \\
 & -60xy + 25y^2 & \\ \hline
\end{array}$$

Ans. $\pm\,(6x - 5y)$.

The square root of the first term is $6x$, and $6x$ is placed at the right of the given expression for the first term of the root. The trial divisor is $12x$, and the second term of the root, $-5y$, is obtained by dividing $-60xy$ by $12x$. The complete divisor is $12x-5y$.

2. Find the square root of $x^4-2x^3-x^2+2x+1$.

$$
\begin{array}{l|rrrr|l}
 & x^4-2x^3 & -\ x^2 & +\ 2x & +\ 1 & \underline{x^2-x-1} \\
 & x^4 & & & & \\
\hline
2x^2-x & -2x^3 & -\ x^2 & +\ 2x & +\ 1 & \\
 & -2x^3 & +\ x^2 & & & \\
\cline{2-5}
 & & -2x^2 & +\ 2x & +\ 1 & \\
2x^2-2x-1 & & -2x^2 & +\ 2x & +\ 1 & \\
\end{array}
$$

Ans. $\pm(x^2-x-1)$.

The square root of x^4 is x^2, $\therefore x^2$ is the first term of the root. The first term of the remainder is $-2x^3$, and the first trial divisor is $2x^2$; $\therefore -2x^3 \div 2x^2 = -x$ is the second term of the root. The complete divisor is $2x^2-x$, and the product of $2x^2-x$ into the second term of the root is $-2x^3+x^2$. The second trial divisor is $2x^2-2x$, and the first term of the second remainder is $-2x^2$. Since $-2x^2 \div 2x^2 = -1$, $\therefore -1$ is the third term of the root. The second complete divisor is $2x^2-2x-1$; etc.

150. The fourth root of an expression may be found by extracting the square root of the square root; the eighth root, by extracting the square root of the fourth root; the sixteenth root, by extracting the square root of the eighth root; etc.

151. Since $(a \pm b)^2 = a^2 \pm 2ab + b^2$, therefore:

(1) *The square root of a trinomial, which is a perfect square, may be found by extracting the square roots of the terms which are squares, and placing the sign of the remaining term between them.* Thus, the square root of $9x^2-6x+1$ is $\pm(3x-1)$, because the square root of $9x^2$ is $3x$, of 1 is 1, and the sign of the remaining term is $-$.

(2) *If the first and third terms of a square be given, the second term is found by taking twice the product of the square roots of the given terms.* Example: The first and third terms of a square being $9x^4$ and $4y^6$, the second term is $\pm 2\sqrt{9x^4} \times \sqrt{4y^6} = \pm 2 \times 3x^2 \times 2y^3 = \pm 12x^2y^3$.

(3) *If the first and second terms of a square be given, the third term is found by extracting the square root of the first term, dividing the given second term by twice this root, and squaring the result.* Example: The first and second terms of a square are $16x^2$ and $\pm 24xy$. To complete the square proceed as follows: (1) $\sqrt{16x^2} = 4x$. (2) $2(4x) = 8x$. (3) $24xy \div 8x = 3y$. (4) $(3y)^2 = 9y^2$. $\therefore$ the third term is $+9y^2$, and the complete square is $16x^2 \pm 24xy + 9y^2$.

(4) A binomial can not be a perfect square, for the square of a monomial is a monomial, and the square of a binomial is a trinomial.

EXAMPLES.

Extract the square root of each of the following:

1. $4a^4 - 12a^3 + 25a^2 - 24a + 16$. *Ans.* $2a^2 - 3a + 4$.

2. $16x^6 - 24x^5 + 25x^4 - 20x^3 + 10x^2 - 4x + 1$.
Ans. $4x^3 - 3x^2 + 2x - 1$.

3. $9a^2 + 12ab + 4b^2 + 6ac + 4bc + c^2$. *Ans.* $3a + 2b + c$.

4. $1 - 6x + 15x^2 - 20x^3 + 15x^4 - 6x^5 + x^6$.
Ans. $1 - 3x + 3x^2 - x^3$.

5. $4a^6 + 5a^2 - 11a^4 - 4a^5 + 14a^3 - 12a + 4$.
Ans. $2a^3 - a^2 - 3a + 2$.

6. $25x^4 - 30ax^3 + 49a^2x^2 - 24a^3x + 16a^4$.
Ans. $5x^2 - 3ax + 4a^2$.

Extract the fourth root of:

7. $a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4$. *Ans.* $a - b$.

8. $a^4 - 8a^3 + 24a^2 - 32a + 16$. *Ans.* $a - 2$.

9. $16a^4 - 96a^3b + 216a^2b^2 - 216ab^3 + 81b^4$. *Ans.* $2a - 3b$.

Extract the eighth root of:

10. $x^8 - 8x^7 + 28x^6 - 56x^5 + 70x^4 - 56x^3 + 28x^2 - 8x + 1.$

Ans. $x - 1.$

11. $[(x^2 + x^{-2})^2 - 4(x + x^{-1})^2 + 12]^2.$ *Ans.* $x - x^{-1}.$

Complete the square in the following expressions:

12. $a^4 - 6a^2 + (\quad).$ *Ans.* $a^4 - 6a^2 + 9.$

13. $16a^8 - 24a^4b^2 + (\quad).$ *Ans.* $4^2a^8 - 24a^4b^2 + 9b^4.$

14. $9a^6 - (\quad) + 4b^4.$ *Ans.* $9a^6 - 12a^3b^2 + 4b^4.$

15. $(a + b)^4 - 4(a + b)^2(a - b) + (\quad).$

Ans. $(a + b)^4 - 4(a + b)^2(a - b) + 4(a - b)^2.$

16. $(a - b)^2 + (\quad) + 81c^2.$

Ans. $(a - b)^2 + 18(a - b)c + 81c^2.$

SQUARE ROOTS OF ARITHMETICAL NUMBERS.

152. The method of finding the square roots of arithmetical expressions is similar to the preceding, but in the first place it is convenient to mark off the figures in periods.

Since $1 = 1^2$, $100 = 10^2$, $10000 = 100^2$, $1000000 = 1000^2$, etc.; therefore the square root of any number between 1 and 100 lies between 1 and 10; the square root of any number between 100 and 10000 lies between 10 and 100; the square root of any number between 10000 and 1000000 lies between 100 and 1000; etc. In other words, the square root of any number expressed by one or two figures is a number of one figure; the square root of any number expressed by three or four figures is a number expressed by two figures; the square root of any number expressed by five or six figures is a number expressed by three figures; etc. If, therefore, a dot be placed over the units' figure of a number, and also over every alternate figure, the number of dots will be equal to the number of figures in its square root.

To extract the square root of 5329.

$$\begin{array}{r|l} & 5\dot{3}2\dot{9} \;|\; 70 + 3 \\ & 4900 \\ \hline 140 + 3 & 429 \\ & 429 \\ \hline \end{array}$$

Since the square of 70 is 4900, and the square of 80 is 6400, therefore the square root of 5329 must be greater than 70 and less than 80. Taking 70 as a in the general formula $a^2 + 2ab + b^2 = 5329$, and subtracting $a^2 = 4900$ from $a^2 + 2ab + b^2 = 5329$, we find $2ab + b^2 = 429$. Since $2a = 140$, then b can not be greater than 3. Taking $b = 3$, $2a + b = 143$, and $b(2a + b) = 429$. There being no remainder, 73 is the required square root.

In practice the ciphers are usually omitted; thus,

$$\begin{array}{r|l} & 5\dot{3}2\dot{9} \;|\; 73 \\ & 49 \\ \hline 143 & 429 \\ & 429 \\ \hline \end{array}$$

153. The same method will apply to numbers of more than two periods by considering a in the formula $a^2 + 2ab + b^2 = (a + b)^2$ at each step to represent the part of the root found; that is, a represents so many tens with respect to the next figure of the root.

1. Find the square root of 11607649.

$$\begin{array}{lr|r} & & 1\dot{1}6\dot{0}7\dot{6}4\dot{9} \;|\; 3407 \\ & & 9 \\ \hline 2a = 60 & 64 & 260 \\ & & 256 \\ \hline 2a_1 = 680 & & 47649 \\ 2a_2 = 6800 & 6807 & 47649 \\ \hline \end{array}$$

154. From the arithmetical rule for the multiplication of decimals, it is plain that if a number have any number of decimal places, its square will have twice as many; therefore the number of decimal places in every square decimal will necessarily be even, and the number of decimal places in the root will be half that number. Hence, if the given square number be a decimal, and therefore one of an even number of places, place a dot over the units' figure, and then over every alternate figure on both sides of it. The number of dots to the left of the decimal point will be the number of figures in the integral part of the root, and the number of dots to the right the number of decimal places. The square root of 11607649 being 3407, the square root of 116076.49 is 340.7; of 1160.7649 is 34.07; of 11.607649 is 3.407; etc.

155. If a number have an odd number of decimal places, its exact square root can not be determined. Thus, .144 is not a perfect square. In every case, if there be a remainder after finding as many figures in the root as there are periods in the proposed number, the exact root can not be determined. The approximate root may be found to any required degree of accuracy by annexing ciphers and continuing the operation.

2 and **3**. To find the square roots of 12 and of .144 to five places:

	$1\dot{2}.\dot{0}0\dot{0}0\dot{0}0\dot{0}0\dot{0}0$	3.46410
	9	
64	300	
	256	
686	4400	
	4116	
6924	28400	
	27696	
69281	70400	
	69281	
692820	111900	

$\sqrt{12}$ is between 3.46410 and 3.46411.

	$.1\dot{4}4\dot{0}0\dot{0}0\dot{0}0\dot{0}$	.37947
	9	
67	540	
	469	
749	7100	
	6741	
7584	35900	
	30336	
75887	556400	
	531209	
	25191	

$\sqrt{.144}$ is between .37947 and .37948.

EXAMPLES.

Extract the square root of :

4. 33124.	*Ans.* 182.	**8.** 6.5536.	*Ans.* 2.56.
5. 47961.	*Ans.* 219.	**9.** .194481.	*Ans.* .441.
6. 4643.0596.	*Ans.* 68.14.	**10.** 3.515625.	*Ans.* 1.875.
7. 285.61.	*Ans.* 16.9.	**11.** 3.	*Ans.* 1.7320 +.

Extract the fourth root of :

12. 6.5536.	*Ans.* 1.6.	**14.** .00028561.	*Ans.* .13.
13. 19.4481.	*Ans.* 2.1.	**15.** 4304.6721.	*Ans.* 8.1.

CUBE ROOTS OF POLYNOMIALS.

156. Since the cube of $(a+b)$ is $a^3+3a^2b+3ab^2+b^3$, $\therefore$ the cube root of $a^3+3a^2b+3ab^2+b^3$ is $a+b$. The cube being given, a general rule for the extraction of the cube root may be deduced by observing in what manner a and b may be derived from $a^3+3a^2b+3ab^2+b^3$ or its equal, $a^3+(3a^2+3ab+b^2)\,b$.

$$\begin{array}{lll} & a^3+3a^2b+3ab^2+b^3 & \underline{|\,a+b} \\ & a^3 & \\ 3a^2 & \quad 3a^2b+3ab^2+b^3 & \\ \quad +3ab+b^2 & \quad \underline{3a^2b+3ab^2+b^3} & \end{array}$$

The first term, a, of the root is evidently the cube root of the first term, a^3, of the given expression.

If the cube of a be subtracted, the remainder is $3a^2b+3ab^2+b^3$; therefore the second term, b, of the root is found by dividing $3a^2b$ by $3a^2$; that is, by three times the square of the first term of the root. To this trial divisor add $3ab+b^2$ for the complete divisor. Multiply the complete divisor by b, and subtract the product, $3a^2b+3ab^2+b^3$, from the remainder. If there be but two terms in the root, this finishes the operation. If the root contain more terms, the same method may be employed, by considering a to represent the part of the root already found, and b to represent the new term of the root. In every case, *The trial divisor is three times the square of the part of the root already found,* and *The complete divisor is formed by adding to the trial divisor three times the product of the part of the root already found times the new term of the root, and the square of the new term of the root.*

If the root be $a + b + c + d$,

(1) To find b, the trial divisor is $3a^2$, and the complete divisor is $3a^2 + 3ab + b^2$.

(2) To find c, the trial divisor is $3(a + b)^2$, and the complete divisor is $3(a + b)^2 + 3(a + b)c + c^2$.

(3) To find d, the trial divisor is $3(a + b + c)^2$, and the complete divisor is $3(a + b + c)^2 + 3(a + b + c)d + d^2$.

Ex. 1. Find the cube root of $27x^3 + 54x^2y + 36xy^2 + 8y^3$.

	$27x^3 + 54x^2y + 36xy^2 + 8y^3 \mid 3x + 2y$
	$27x^3$
$3a^2 = 3(3x)^2 = 27x^2$	$54x^2y + 36xy^2 + 8y^3$
$3ab + b^2 = \quad + 18xy + 4y^2$	$54x^2y + 36xy^2 + 8y^3$
$3a^2 + 3ab + b^2 = 27x^2 + 18xy + 4y^2$	

Cube root, $3x + 2y$.

The expression being arranged according to the descending powers of x, the first term, $27x^3$, corresponds to a^3 in the general formula; therefore the cube root of $27x^3$, which is $3x$, is the first term of the root. The trial divisor is three times the square of $3x$, that is, $27x^2$. Since $54x^2y \div 27x^2 = 2y$, therefore $2y$ is the second term of the root. To $27x^2$ add three times the product of the two terms of the root, $18xy$, also the square of the second term, $4y^2$, and the sum is the complete divisor, $27x^2 + 18xy + 4y^2$. Multiply this complete divisor by $2y$, and subtract the product from the remainder.

Ex. 2. Find the cube root of $x^6 - 6x^5 + 9x^4 + 4x^3 - 9x^2 - 6x - 1$. *Ans.* $x^2 - 2x - 1$.

$$
\begin{array}{lll|l}
 & & & x^6 - 6x^5 + 9x^4 + 4x^3 - 9x^2 - 6x - 1 \\
a^3 & = & & x^6 \\
\hline
3a^2 & = & 3x^4 & -6x^5 + 9x^4 + 4x^3 - 9x^2 - 6x - 1 \\
3ab + b^2 & = & \quad - 6x^3 + 4x^2 & -6x^5 + 12x^4 - 8x^3 \\
\hline
 & & & -3x^4 + 12x^3 - 9x^2 - 6x - 1 \\
3(a+b)^2 & = & 3x^4 - 12x^3 + 12x^2 & \\
3(a+b)c & = & \qquad\quad - 3x^2 + 6x & \\
+c^2 & = & \qquad\qquad\qquad\quad +1 & \\
\hline
 & & 3x^4 - 12x^3 + 9x^2 + 6x + 1 & -3x^4 + 12x^3 - 9x^2 - 6x - 1 \\
\hline
\end{array}
$$

The expression being arranged according to the descending powers of x, extract the cube root of x^6, and the result, x^2, is the first term of the root. Subtract x^6. The trial divisor being $3x^4$, and the first term of the remainder being $-6x^5$, $\therefore -6x^5 \div 3x^4 = -2x$ is the second term of the root. To $3x^4$ add $3(x^2)(-2x) = -6x^3$; also add $(-2x)^2 = 4x^2$, and multiply the sum, $3x^4 - 6x^3 + 4x^2$, by $-2x$. Subtract the product, $-6x^5 + 12x^4 - 8x^3$, from the first remainder. The second trial divisor is $3(x^2 - 2x)^2 = 3x^4 - 12x^3 + 12x^2$. The quotient of the first term of the remainder, $-3x^4$, by the first term of the trial divisor is -1; $\therefore -1$ is the third term of the root. To the trial divisor add $3(x^2 - 2x)(-1) + (-1)^2 = -3x^2 + 6x + 1$, and multiply the complete divisor by -1. Subtract the product from the second remainder.

CUBE ROOTS OF ARITHMETICAL NUMBERS.

157. Since $1 = 1^3$, $1000 = 10^3$, $1000000 = 100^3$, etc., it follows that the cube root of any number between 1 and 1000 is between 1 and 10; the cube root of any number between 1000 and 1000000 is between 10 and 100; etc. In other words, the cube root of any number which has one, two, or three figures, is expressed by one figure; the cube root of any number which has four, five, or six figures, is a number of two figures; the cube root of any number which has seven, eight, or nine figures, is a number of three figures; etc.

Hence, if a dot be placed over every third figure of a cube, beginning with units' figure, the number of dots will be equal to the number of figures in its cube root.

158. If a number contain any decimal figures, its cube will contain three times as many. Hence, there must be one decimal place in the cube root for every three decimal places in the cube. If a given cube have decimal places, and a dot be placed over the units' figure, and over every third figure on both sides of it, the number of dots to the left of the decimal point will be the number of figures in the integral part of the root, and the number of dots to the right will be the number of figures in the decimal part of the root.

159. From the preceding article it follows that if the number of decimal places in a given number be not exactly divisible by three, the number is not an exact cube. If there be a remainder left after finding as many figures in the root as there are periods, then the given number is not a perfect cube. The approximate cube root of an imperfect cube may be found to any required degree of accuracy by annexing ciphers and continuing the work.

160. After pointing off the number into periods of three figures each as explained in the preceding articles, the work is continued, as in the extraction of the cube root of algebraic expressions, by applying the formula $(a+b)^3 = a^3 + (3a^2 + 3ab + b^2)b$; a representing the first figure of the root, then the first two, next the first three, and so on. In every case a represents so many tens, and b represents so many units.

1. Find the cube root of 820025.856.

			$820\dot{0}25.\dot{8}5\dot{6}$ | 93.6
$a^3 = 9^3$	$=$		729
$3a^2 = 3\,(90)^2$	$=$	24300	91025
$3ab = 3\,(90)(3)$	$=$	810	
$b^2 = 3^2$	$=$	9	
$3a^2 + 3ab + b^2$	$=$	25119	75357
$3a'^2 = 3\,(930)^2$	$=$	2594700	15668856
$3a'b' = 3\,(930)(6)$	$=$	16740	
$b'^2 = 6^2$	$=$	36	
$3a'^2 + 3a'b' + b'^2$	$=$	2611476	15668856

161. If the successive terms of the root be represented by a, b, c, etc., it will be observed that the first complete divisor $= 3a^2 + 3ab + b^2$, and the second trial divisor $= 3\,(a + b)^2 = 3a^2 + 6ab + 3b^2$.

Hence: *To the first complete divisor add its second term and twice its third term, and the sum will be the second trial divisor.*

In like manner, each trial divisor may be found from the preceding complete divisor.

EXAMPLES.

Extract the cube root of:

1. $27x^3 - 54x^2y + 36xy^2 - 8y^3$. *Ans.* $3x - 2y$.
2. $8x^6 - 36x^5 + 66x^4 - 63x^3 + 33x^2 - 9x + 1$. *Ans.* $2x^2 - 3x + 1$.
3. $x^6y^{-3} - 6x^4 + 12x^2y^3 - 8y^6$. *Ans.* $x^2y^{-1} - 2y^2$.
4. $a^9 - 9a^8 + 36a^7 - 84a^6 + 126a^5 - 126a^4 + 84a^3 - 36a^2 + 9a - 1$. *Ans.* $a^3 - 3a^2 + 3a - 1$.
5. $x^6 + 6x^4y^2 + 12x^2y^4 + 8y^6 - 3\,(x^2 + 2y^2)^2 + 3x^2 + 6y^2 - 1$. *Ans.* $x^2 + 2y^2 - 1$.
6. $1 + 12x + 60x^2 + 160x^3 + 240x^4 + 192x^5 + 64x^6$. *Ans.* $1 + 4x + 4x^2$.

7. 34012.224.	*Ans.* 32.4.	**10.** 6321363049.	*Ans.* 1849.
8. 148.035889.	*Ans.* 5.29.	**11.** 1.073741824.	*Ans.* 1.024.
9. .244140625.	*Ans.* .625.	**12.** 3 (3 places).	*Ans.* 1.442.

HIGHER ROOTS.

162. Since the sixth power of a number is the square of the cube, or the cube of the square, therefore the sixth root of a number is the square root of the cube root, or the cube root of the square root. Since the ninth power of a number is the cube of the cube, therefore the ninth root of a number is the cube root of the cube root. In like manner, the tenth root is the fifth root of the square root, the twelfth root is the cube root of the fourth root, etc. That is, when the index of the root is a composite number, the root may be found by extracting the root indicated by one factor of the given index, then the root of the result as indicated by one of the remaining factors of the given index, and so on until all the factors have been used. The final result is the root required. Thus:

$$\sqrt[6]{64} = \sqrt[3]{\sqrt{64}} = \sqrt[3]{\pm 8} = \pm 2.$$

$$\sqrt[8]{256} = \sqrt{\sqrt{\sqrt{256}}} = \sqrt{\sqrt{16}} = \sqrt{4} = \pm 2.$$

163. The root of any expression may be found by applying the formula for the corresponding power. For example, since $(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 = a^4 + b\,(4a^3 + 6a^2b + 4ab^2 + b^3)$, therefore the first term of the root is the fourth root of the first term of the given power. The trial divisor is $4a^3$, and the second term of the root, b, is found by dividing $4a^3b$ (the first term of the remainder left after subtracting a^4 from the dividend) by the trial divisor. The complete divisor is $(4a^3 + 6a^2b + 4ab^2 + b^3)$. Similarly, the fifth root is found by extracting the fifth root of the first term of the power, dividing the first term of the remainder by $5a^4$, etc.; the sixth root is found by extracting the sixth root of the first term of the power, dividing the first term of the remainder by $6a^5$, etc. The trial divisor for finding the nth root is na^{n-1}. That is: *In the extraction of any root, the trial divisor is formed by raising the root found to a power whose exponent is less by unity than the index of the root, and multiplying this power by a number equal to that index.*

EXERCISE IX.

Extract the square root of:

1. $9x^8y^6$; $25a^6b^{12}$; $81a^4(b-c)^2$; $36a^{14}(x-y)^6$.
2. $(a^2-2ab+b^2)^3$; $(4x^2-4x+1)^5$; $(a^2-6a+9)^7$.
3. $(a-b+c)^{2n}$; $a^{4n-2}(b-c)^{6n-8}$; $x^{4n^2}(a-2b)^{8n}$.
4. $(x^2-2xy+y^2)^n$; $(9a^2-6a+1)^{5n}$; $\{(a-b)^2-2(a-b)+1\}^3$.
5. $9x^4-6x^3-5x^2+2x+1$.
6. $16x^6-40x^5+x^4+46x^3-11x^2-12x+4$.
7. 177241; 543169; 14356521; 17.338896.
8. To four decimal places: 2.5; 14.4; .169; .3; .03.

Extract the cube root of:

9. $8x^6+48x^5+60x^4-80x^3-90x^2+108x-27$.
10. $x^9-3x^8+6x^7-10x^6+12x^5-12x^4+10x^3-6x^2+3x-1$.
11. $a^3+b^3+c^3+3a^2b+3a^2c+3ab^2+3b^2c+3ac^2+3bc^2+6abc$.
12. 74088; 110592; 103.823; 884.736.
13. 12.812904; 33076.161; 102.503232.
14. To two decimal places: 1.25; .08; 16; 12.8.

Extract the sixth root of:

15. $a^{-6}-6a^{-5}+15a^{-4}-20a^{-3}+15a^{-2}-6a^{-1}+1$.
16. 729; .004096; $15625a^{12}b^{-18}$.

CHAPTER X.

SIMPLE EQUATIONS INVOLVING ONE UNKNOWN QUANTITY.

164. An **Equation** is a statement that two expressions are equal. Thus, $3x - 8 = 10$.

The expression to the left of the sign of equality is called the *first member* or *first side*, and the expression to the right, the *second member* or *second side*.

165. An **Identical Equation** is one in which the two sides are equal for all values of every letter employed. Thus, $(a + b)(a - b) = a^2 - b^2$ is an identical equation. An identical equation is also called an *identity*.

166. An **Equation of Condition** is one which is true only for particular values of one or more of the letters employed. Thus, $x + 4 = 7$ is true only when $x = 3$.

When the word *equation* is used in Algebra, it is understood that *equation of condition* is meant.

167. A letter to which a particular value must be given in order that the statement contained in an equation may be true, is called an *unknown quantity*. Such particular value of the unknown quantity is said to satisfy the equation, and is called a *root of the equation*.

168. To **Solve an Equation** is to find the value of the unknown quantity.

169. A **Simple Equation,** also called an *equation of the first degree,* is one which contains only the first power of the unknown quantity.

170. To solve an equation, we separate the known from the unknown quantities by using the contrary operation to that indicated by the connecting sign. That is, if the known and the unknown quantities be connected by $+$, they may be separated by subtraction; if connected by $\times$, they may be separated by division; etc. In performing these operations, the two sides remain equal throughout; if the value of one side be changed in any way, the same change must be made on the other side (Ax. II, III, IV, V, etc., Art. 62). To illustrate this principle, let it be required to find the value of x in the following examples:

1. $3x - 1 = 14$. Here 1 is connected with the unknown term, $3x$, by the sign $-$; therefore we add 1 to each side. Whence $3x = 15$. Now, since 3 and x are connected by the sign $\times$, we divide each side by 3. $\therefore x = 5$.

2. $5x + 6 = 21$. 6 being joined to $5x$ by $+$, we subtract 6 from each side. Whence $5x = 21 - 6 = 15$. x and 5 being connected by $\times$, we divide each side by 5. $\therefore x = 3$.

3. $4x + 3 = 2x + 11$. By subtracting $2x$ from each member, the unknown term disappears from the second side; that is, $4x - 2x + 3 = 11$. Now subtract 3 from each side, and $2x = 8$. $\therefore x = 4$.

4. $3x + 13 = 68 - 2x$. Here we add $2x$ to each side to cause x to disappear from the second member, and we subtract 13 from each side to cause the known term to disappear from the first member. Whence $3x + 2x = 68 - 13$; that is, $5x = 55$. $\therefore x = 11$.

171. If terms containing the unknown quantity occur on both sides of an equation, it is customary, by means of Ax. II and III, to place them all on the first side, and, in like manner, to put the known terms on the second side. This operation is called *transposition.* Thus, if we have $x + a = b$, by subtracting a from each side we find $x = b - a$; also, if $x - a = b$, by adding a to each side we find $x = b + a$. In each case a is transposed from one side to the other, but its sign is changed; hence, *Any term may be transposed from one side of an equation to the other, provided its sign be changed.*

172. The signs of all the terms on both sides of an equation may be changed, for this is equivalent to multiplying every term by (-1). (Ax. IV.)

In like manner, an equation may be *cleared of fractions* by multiplying every term of each side by a common multiple of the denominators (29). Thus:

5. $\frac{x}{5} + 2 = 4$; $\therefore \frac{x}{5} = 4 - 2 = 2$. Since x and 5 are connected by division, we multiply each term by 5. $\therefore x = 10$.

6. $\frac{x}{3} + \frac{x}{4} - \frac{x}{6} = 5$. Since x is connected by division with 3, 4, and 6, we multiply each term by 12. $\therefore 4x + 3x - 2x = 60$. Whence $5x = 60$, and $x = 12$.

7. $\frac{24}{x} + \frac{12}{x} + \frac{6}{x} = 7$. Here multiply each term by x. $\therefore 24 + 12 + 6 = 7x$. Whence $42 = 7x$, and $x = 6$.

173. To solve an equation of the first degree, containing one unknown quantity.

(1) *If necessary, clear the equation of fractions.* (172.)

(2) *Transpose all the terms containing the unknown quantity to one side, and all other terms to the other side, remembering to change the sign of each transposed term.* (171.)

(3) *Reduce each side to its simplest form.*

(4) *Divide both sides by the co-efficient of the unknown quantity.*

EXAMPLES.

Find the value of x in each of the following equations:

1. $3x + 7 = 9x - 5$. *Ans.* 2.
2. $3 + 2x = 17 - 5x$. *Ans.* 2.
3. $3(x - 2) = 4(2 - x)$. *Ans.* 2.
4. $5 - 3(4 - x) = 4(2x - 3)$. *Ans.* 1.
5. $13x - 21(x - 3) = 10 - 21(3 - x)$. *Ans.* 4.
6. $5(2x + 6) = 4(3x + 6)$. *Ans.* 3.
7. $4x - 40 = 40 - x$. *Ans.* 16.
8. $4(x + 16) = 10(x + 1)$. *Ans.* 9.
9. $2x - 22 = 3(x - 22)$. *Ans.* 44.
10. $ax + b^2 = bx + a^2$. *Ans.* $a + b$.
11. $5(x + 1) - 2(6 - x) - 2 = 3(x + 5)$. *Ans.* 6.
12. $3(x - 2) - 4(3 - x) = -4 + 2(x + 1) - 3x$. *Ans.* 2.
13. $\frac{x}{2} + \frac{x}{3} + \frac{x}{4} = 26 - 13(13 - x)$. *Ans.* 12.
14. $2x - 1 - 2(3x - 2) + 3(4x - 3) - 4(5x - 4) = 0$. *Ans.* $\frac{5}{6}$.
15. $\frac{1}{2}(3x - 1) - \frac{2}{3}(x - 1) = \frac{1}{4}(x - 3) - \frac{1}{6}(x - 5) + \frac{16}{3}$. *Ans.* 7.
16. $\frac{1}{6}x - 1\frac{2}{3} = 8\frac{3}{5} + 2(\frac{3}{5}x - 1) - \frac{1}{3}(x + 8)$. *Ans.* -8.
17. $\frac{x}{2} - \frac{2x}{3} + \frac{3x}{4} = 15 + \frac{x}{8} - \frac{x}{6}$. *Ans.* 24.
18. $\frac{x + 1}{2} + \frac{1}{5}(3x - 4) + \frac{1}{8} = \frac{6x + 7}{8}$. *Ans.* 3.
19. $\frac{5x - 11}{4} - \frac{1}{10}(x - 1) = \frac{11}{12}x - \frac{1}{12}$. *Ans.* 11.
20. $\frac{2x - a}{3} - \frac{2x + 3b}{5} = \frac{x - a}{2}$. *Ans.* $\frac{5a - 18b}{7}$.
21. $(a + x)(b + x) = (c + x)(d + x)$. *Ans.* $\frac{cd - ab}{a + b - c - d}$.

PROBLEMS.

174. One of the main uses of equations is to find numbers which shall satisfy certain given conditions. In order to find such a number, two steps are necessary:

(1) *To express the conditions of the problem in algebraic language; that is, to form the equation.*

(2) *To solve the equation thus formed.*

It is impossible to give a precise rule by means of which every question may be readily stated in the form of an equation. The first step is to understand fully the nature of the question; secondly, to denote the required quantity by one of the final letters of the alphabet; thirdly, to indicate, by means of signs, the same operations that it would be necessary to perform with the answer, to verify it.

EXAMPLES.

1. Five times a certain number exceeds twice the number by 30; find the number.

Let x represent the required number; then the problem becomes: Five times x exceeds twice x by 30; find x: or, $5x$ exceeds $2x$ by 30; find x. The corresponding equation is $5x - 2x = 30$. $\therefore x = 10$.

2. Divide 140 into two parts such that one part shall be $2\frac{1}{2}$ times the other.

Here it is convenient to represent the smaller part by $2x$. Since the larger part is $2\frac{1}{2}$ times the smaller, then $5x$ is the larger. But the sum of the two parts must be 140, and their sum is $2x + 5x = 7x$. We now have two expressions for the same thing. $\therefore$ (Ax. I) $7x = 140$. Whence $x = 20$, and the two numbers are 40 and 100.

3. Divide 27 cents between A and B so that $\frac{1}{5}$ of A's share shall equal $\frac{1}{4}$ of B's.

Let x represent the number of cents A shall receive. Since A and B together receive 27 cents, and since A alone receives x cents, therefore B must receive $27 - x$ cents.

But $\frac{1}{5}$ of A's share must equal $\frac{1}{4}$ of B's.

$\therefore\ \frac{1}{5}x = \frac{1}{4}(27 - x)$. Now multiply both sides by 20.

$4x = 5(27 - x)$; that is, $4x = 135 - 5x$.

$\therefore\ x = 15$, A's share, and $27 - x = 12$, B's share.

4. A and B together have \$7; $\frac{1}{2}$ of A's equals $\frac{2}{3}$ of B's. How many dollars has each?

Let $3x$ represent the number of dollars B has. Since $\frac{1}{2}$ of A's $= \frac{2}{3}$ of B's, $\therefore\ \frac{1}{2}$ of A's $= 2x$, and A's $= 4x$.

As A has $4x$ dollars and B $3x$, both have $7x$ dollars.

But together they have 7 dollars. $\therefore\ 7x = 7$. Therefore $x = 1$, $3x = 3$, B's share, and $4x = 4$, A's share.

5. Find two consecutive integers, the difference of whose squares is 25.

Let x represent the smaller; then, since the integers are consecutive, $x + 1$ is the greater.

$\therefore\ (x + 1)^2 - x^2 = 25$. *Ans.* 12 and 13.

6. A is three times as old as B, but in ten years he will be only twice as old; how old is A?

Let x represent B's age. Since A's age is three times B's, $\therefore\ 3x$ will represent A's age. In ten years B's age will be $x + 10$, and A's $3x + 10$. But then A's age will be twice B's.

$\therefore\ 3x + 10 = 2(x + 10)$. Whence $x = 10$, and $3x = 30$.

7. A has \$25 in half-dollars and dimes, the number of both being 170. How many half-dollars has he?

Let x represent the number of half-dollars; then $170 - x$ must be the number of dimes.

The x half-dollars must be worth $50x$ cents.

The $170 - x$ dimes must be worth $10(170 - x)$ cents.

$\therefore\ 50x + 10(170 - x) = 2500$. Whence $x = 20$. *Ans.*

8. Divide \$33 between A and B, in such a manner that 4 times what A receives shall equal 7 times what B receives.

Ans. A receives \$21, and B \$12.

9. How many dollars has A, if \$5 more than $\frac{1}{2}$ of his money is \$20 less than $\frac{2}{3}$ of it? *Ans.* 150.

10. A and B together have \$156; $\frac{3}{2}$ of A's equals $\frac{2}{3}$ of B's. How much has each? *Ans.* A, \$48; B, \$108.

11. A and B together have \$130; A and C together have \$60. B has \$20 more than twice as much as C. How many dollars has each? *Ans.* A has 10; B, 120; C, 50.

12. A and B had equal sums of money; A gave B \$30, and then had only half as much money as B. How much had each at first? *Ans.* \$90.

13. A is five times as old as B; in 15 years he will be only twice as old. How old is A? *Ans.* 25 years.

14. A is 50 years old, and B is 5. In how many years will B be $\frac{1}{4}$ as old as A? *Ans.* 10.

15. What o'clock is it, if the time past noon is $\frac{1}{2}$ the time to midnight? *Ans.* 4 P.M.

16. A can do a piece of work in 10 days; A and B do the same work in 8 days. How long would it take B alone?

Let $x =$ the number of days B would take to do the work.

Since A can do all the work in 10 days, he can do $\frac{1}{10}$ of it in one day, and $\frac{8}{10}$ of it in 8 days.

Since B can do all the work in x days, he can do $1 \div x$ of it in one day, and $8 \div x$ in 8 days.

In 8 days both do $(8 \div 10) + (8 \div x)$ (the work being represented by unity). But in 8 days both do once the work.

$\therefore$ (Ax. I) $\frac{8}{10} + \frac{8}{x} = 1.$ *Ans.* 40 days.

Verification: A can do $\frac{1}{10}$ in a day, and B $\frac{1}{40}$; together they can do $\frac{1}{10} + \frac{1}{40} = \frac{1}{8}$. $\therefore$ they together can do the work in 8 days, which agrees with the given conditions.

17. A cistern can be filled by a pipe, A, in 8 hours, and emptied by a pipe, B, in 12 hours. How long would it take to fill $\frac{3}{4}$ of the cistern if both pipes be left open?

Let x represent the required number of hours.

Since A can fill the cistern in 8 hours, it can fill $\frac{1}{8}$ of the cistern in 1 hour, and $x \div 8$ in x hours.

Since B can empty the cistern in 12 hours, it can empty $\frac{1}{12}$ of it in 1 hour, and $x \div 12$ in x hours.

$(x \div 8) - (x \div 12)$ will remain in x hours.

But $\frac{3}{4}$ remain in x hours. $\therefore$ (Ax. I) $\frac{x}{8} - \frac{x}{12} = \frac{3}{4}$.

Whence $x = 18$. *Ans.* 18 hours.

18. A can do a piece of work in 60 days. After working 20 days, B joins him, and together they complete the job in 16 days more. How long would it take B alone to do all the work? *Ans.* 40 days.

19. A can do a piece of work in $\frac{1}{2}$ the time that B can; B in $\frac{2}{3}$ the time that C can. The three can do the work in 18 days. How long would it take each of them alone?

Ans. A, 33 days; B, 66 days; C, 99 days.

20. A works half as fast as B; together they can finish a piece of work in 12 days. How long would it take each alone? *Ans.* A, 36 days; B, 18 days.

Exercise X.

Find the value of x in the following equations:

1. $5x - 3 = 8(x - 3)$.

2. $(4x - 2)(1 - x) = \frac{1}{2} - (2x - 1)^2$.

3. $\frac{2}{3}x + \frac{1}{5}(x + 8) = -x^0$.

4. $\frac{a + b}{x} = c$.

5. $\frac{1}{x} + \frac{1}{2x} + \frac{1}{3x} = -11$.

6. $ax - bx = cx - d$.

7. $\frac{x}{a} + \frac{x}{b} = \frac{1}{c}$.

8. $\frac{ax}{b} + \frac{cx}{d} = 1$.

9. $\frac{a - 2b}{2x} = 3 + \frac{2a - b}{x}$.

10. $4x - a = cx - b$.

11. $\frac{x + 1}{2} + \frac{1}{5}(4x + 1) = \frac{3x + 2}{2}$.

12. $(x - a)(x + b) = (x - b)(x - c)$.

13. $\frac{x}{8} - \frac{2(x - 1)}{5} = \frac{3x - 4}{15} + \frac{x}{12}$.

14. $\frac{x}{6} - 8\frac{3}{5} = 2\left(\frac{3}{5}x - 1\right) - \frac{x+8}{3} + \frac{5}{3} - 26\frac{3}{5}.$

15. $\frac{9x-1}{10} - \frac{x-9}{5} + \frac{2x-3}{15} - \frac{x-4}{6} = \frac{1}{2}.$

16. $5x + 16 - 2x + 3a = 5a + x + 8.$

Solve the following problems:

17. Find two numbers whose difference is 2, and whose product exceeds the square of the less by 8.

18. The half of what number exceeds the third by 6?

19. A cistern is supplied by two pipes; the smaller alone can fill it in half an hour, and the larger in 20 minutes. In what time will they both fill it when running together?

20. It takes an inlet-pipe, A, one third as long to fill a cistern as it takes an outlet-pipe, B, to empty it. When both run, it requires 6 hours to fill the cistern. How long would it take B to empty it?

21. A can perform a piece of work in 4 days, B in 6 days, and C in 8 days. How long would it take all to do the work? A and B? A and C? B and C?

22. Equal numbers of dollars, half-dollars, and quarters amount to \$700. How many of each are there?

23. Divide \$440 among three persons, so that A may have $\frac{3}{5}$ as much as B, and B $\frac{3}{4}$ as much as C.

24. A and B engage in trade with the same capital. A gains \$1600, and B loses \$1900, and A's capital is now 8 times B's. With how much money does each begin?

25. A gentleman divided a dollar among 12 children, giving to some 9 cents, and to the remainder 7 cents. How many children received 9 cents apiece?

26. Divide 12 into two such parts that the difference of their squares shall be 48.

27. A is 54 years old, and B 9 years. In how many years will A's age be just four times B's?

28. A is 3 times as old as B, and in 3 years the sum of their ages will be 30 years. How old is each?

29. Ten years ago A was twice as old as B; 20 years hence he will be $\frac{4}{3}$ as old. How old is each?

30. A is 15 years older than B, and in 10 years he will be twice as old as B. How old is each?

31. Two shepherds owning a flock of sheep agree to divide its value equally. A takes 20 sheep; B takes 30 sheep, and pays A $150. What is the value of a sheep?

32. One square field is 2 rods longer than another, and contains 1 acre more. How long is each?

33. A lent $500, part at 4 per cent and the rest at 5 per cent: the whole annual interest received was $22. How much was lent at 5 per cent?

34. A lent $900, part at 4 per cent and the rest at 5 per cent, and he received equal sums as interest from the two parts. How much did he lend at 5 per cent?

35. In a mixture of wine and water, the wine composed 25 gallons more than half, and the water 5 gallons less than a third. How many gallons were there of each?

36. Divide 144 into two parts such that one part may be five sevenths of the other.

37. The time past midnight is one third of the time to noon. What o'clock is it?

38. The time past noon is one fifth of the time past midnight. What o'clock is it?

39. If 20 men, 40 women, and 50 children receive $500 for a week's work, and 2 men receive as much as 3 women or 5 children, how much does each woman receive?

40. A bought a number of apples, half at 2 for a dime, and half at 3 for a dime. By selling all at 4 cents each, he lost half a dollar. What was the number of apples?

CHAPTER XI.

SIMULTANEOUS EQUATIONS OF THE FIRST DEGREE.

175. If one equation contains two unknown quantities, an unlimited number of pairs of values may be found that will satisfy the equation. Thus, in $x + y = 16$, if any value be given to y, a corresponding value for x may be found which will satisfy the equation. If $y = 15$, $x = 1$; if $y = 12$, $x = 4$; if $y = 2\frac{1}{2}$, $x = 13\frac{1}{2}$; etc.

But if a second equation be given, independent of the first (that is, expressing different relations between the unknown quantities), only one pair of values of x and y can be found that will satisfy both equations. Thus, whilst $x + y = 16$ and $2x - y = 2$, considered separately, may be satisfied by an indefinite number of pairs of values, there will be but one pair common to both; namely, $x = 6$, $y = 10$. These are therefore the roots of the pair of equations $x + y = 16$, and $2x - y = 2$.

176. Equations which are to be satisfied by the same values of the unknown quantities, are called *simultaneous equations*. To determine the values of the unknown quantities: (1) *The number of independent equations must be equal to the number of unknown quantities.* (2) *The given equations must be consistent;* thus, $2x + y = 11$ and $2x + y = 15$, can not both be satisfied by any pair of values.

177. Simultaneous equations are solved by combining the equations so as to obtain one equation containing one unknown quantity. This process is called *elimination*.

Three methods of elimination are generally given:

(1) By Addition or Subtraction.

(2) By Comparison.

(3) By Substitution.

ELIMINATION BY ADDITION OR SUBTRACTION.

178.—1. Solve: I. $3x - 2y = 11$.

II. $5x - 4y = 17$.

Multiply I by 2. III. $6x - 4y = 22$.

Subtract II from III. IV. $x \quad = 5$.

Substitute 5 for x in I. V. $15 - 2y = 11$. $\therefore y = 2$.

Verification: I. $3 \times 5 - 2 \times 2 = 15 - 4 = 11$.

II. $5 \times 5 - 4 \times 2 = 25 - 8 = 17$.

2. Solve: I. $5x + 2y = 30$.

II. $4x - 3y = 1$.

Multiply I by 3. III. $15x + 6y = 90$.

Multiply II by 2. IV. $8x - 6y = 2$.

Add III and IV. V. $23x \quad = 92$. $\therefore x = 4$.

Substitute 4 for x in I. $20 + 2y = 10$. $\therefore y = 5$.

Hence, to eliminate an unknown quantity by addition or subtraction:

Multiply the equations by such numbers as will make the co-efficients of this unknown quantity equal in the resulting equations.

If these equal quantities have unlike signs, add the resulting equations; if they have like signs, subtract one equation from the other.

ELIMINATION BY COMPARISON.

179.—1. Solve: I. $3x - 2y = 11$.

II. $5x - 4y = 17$.

In I transpose $3x$. III. $-2y = 11 - 3x$.

Multiply III by 2. IV. $-4y = 22 - 6x$.

In II transpose $5x$. V. $-4y = 17 - 5x$.

Expressions which are equal to $-4y$ must be equal to each other. $\therefore\ 22 - 6x = 17 - 5x$. Whence $x = 5$. Substituting 5 for x in III, $-2y = 11 - 15$. $\therefore\ y = 2$.

2. Solve:
I. $5x + 2y = 30$.
II. $4x - 3y = 1$.

In I transpose $5x$. III. $2y = 30 - 5x$.
Multiply III by 3. IV. $6y = 90 - 15x$.
In II transpose $4x$. V. $-3y = 1 - 4x$.
Multiply V by -2. VI. $6y = -2 + 8x$.

Since the first sides of IV and VI are equal, the second sides must be equal (Ax. I.). $\therefore\ 90 - 15x = -2 + 8x$.
Whence $x = 4$. In III substitute 4 for x.
Whence $2y = 30 - 20$. $\therefore\ y = 5$.

Hence, to eliminate an unknown quantity by comparison,
In each equation find the value of one of the unknown quantities (or of the same multiple of that quantity) in terms of the other, and place these values equal to each other.

ELIMINATION BY SUBSTITUTION.

180.—1. Solve:
I. $3x - 2y = 11$.
II. $5x - 4y = 17$.

In I transpose $3x$. III. $-2y = 11 - 3x$.
In II substitute for $-2y$ its value $= 11 - 3x$.
Whence, IV. $5x + 2(11 - 3x) = 17$. $\therefore\ x = 5$.
Substituting 5 for x in III. $-2y = 11 - 15$. $\therefore\ y = 2$.

2. Solve:
I. $5x + 2y = 30$.
II. $4x - 3y = 1$.

In I transpose $5x$. III. $2y = 30 - 5x$.

$\therefore$ IV. $y = 15 - \frac{5x}{2}$.

In II substitute for y its value.

$$\therefore\ \text{V.}\quad 4x - 3\left(15 - \frac{5x}{2}\right) = 1.$$

Multiply V by 2. $8x - 90 + 15x = 2$. $\therefore\ x = 4$.
In IV substitute 4 for x. $\therefore\ y = 15 - 10 = 5$.

Hence, to eliminate an unknown quantity by substitution:

From one of the equations find the value of one of the unknown quantities in terms of the other.

Substitute this value instead of that unknown quantity in the other equation.

NOTE. Each equation must be simplified, if necessary, before the elimination is performed.

EXAMPLES.

Find the values of x and y in the following pairs of equations:

1. $7x - 5y = 24;\ 4x - 3y = 11.$ *Ans.* $x = 17;\ y = 19.$
2. $3x + 2y = 32;\ 20x - 3y = 1.$ *Ans.* $x = 2;\ y = 13.$
3. $3x - 4y = 18;\ 3x + 2y = 0.$ *Ans.* $x = 2;\ y = -3.$
4. $2x - \frac{1}{2}y = 11;\ 2x - 6y = 0.$ *Ans.* $x = 6;\ y = 2.$
5. $3x = 2y;\ 6x - 5y = -3\frac{1}{2}.$ *Ans.* $x = 2\frac{1}{3};\ y = 3\frac{1}{2}.$
6. $\frac{x}{2} + \frac{y}{3} = 6;\ \frac{x}{3} - \frac{y}{9} = 1.$ *Ans.* $x = 6;\ y = 9.$
7. $\frac{1}{x} + \frac{1}{y} = \frac{5}{12};\ \frac{1}{x} - \frac{1}{y} = -\frac{1}{12}.$ *Ans.* $x = 6;\ y = 4.$
8. $2x^{-1} + 3y^{-1} = \frac{9}{20};\ 3x^{-1} + 2y^{-1} = \frac{7}{15}.$ *Ans.* $x = 10;\ y = 12.$
9. $\frac{x}{5} + 5y = 51;\ 5x + \frac{1}{5}y = 27.$ *Ans.* $x = 5;\ y = 10.$
10. $ax = by;\ x + y = c.$ *Ans.* $x = \frac{bc}{a+b};\ y = \frac{ac}{a+b}.$
11. $\frac{a}{x} + \frac{b}{y} = c;\ \frac{b}{x} + \frac{a}{y} = d.$ *Ans.* $x = \frac{a^2 - b^2}{ac - bd};\ y = \frac{a^2 - b^2}{ad - bc}.$
12. $5x + 7y = 43;\ 11x + 9y = 69.$ *Ans.* $x = 3;\ y = 4.$
13. $3y - 7x = 4;\ 5x + 2y = 22.$ *Ans.* $x = 2;\ y = 6.$
14. $11x - 10y = 14;\ 5x + 7y = 41.$ *Ans.* $x = 4;\ y = 3.$
15. $13x + 11y = 4a;\ 12x - 6y = a.$ *Ans.* $x = \frac{1}{6}a;\ y = \frac{1}{6}a.$

SIMULTANEOUS EQUATIONS CONTAINING THREE OR MORE UNKNOWN QUANTITIES.

181. If three simultaneous equations involving three unknown quantities be given, one of the three unknown quantities must be eliminated between two pairs of the equations; then a second between the resulting pair.

Solve:	I.	$3x - 2y + z = 5.$
	II.	$5x + 3y - 5z = 2.$
	III.	$2x - 5y + 4z = 7.$
IV =	I × 5.	$15x - 10y + 5z = 25.$
V =	II + IV.	$20x - 7y = 27.$ Contains only x and y.
VI =	I × 4.	$12x - 8y + 4z = 20.$
VII =	VI − III.	$10x - 3y = 13.$ Contains only x and y.
VIII =	VII × 2.	$20x - 6y = 26.$
IX =	VIII − V.	$y = -1.$

In VII put -1 for y. $10x + 3 = 13$. $\therefore x = 1$.

In I put -1 for y and 1 for x. $\therefore z = 0$.

182. If four simultaneous equations involving four unknown quantities be given, one of the unknown quantities must be eliminated between three pairs of the equations; there will result three equations containing three unknown quantities. Another unknown quantity is then eliminated between two pairs of these equations, the result being two equations containing two unknown quantities. A third unknown quantity is eliminated between this pair of equations. Proceed in a similar manner if there be more than four unknown quantities.

EXAMPLES.

Solve the following simultaneous equations:

1. $3x + 2y - 4z = 15$; $5x - 3y + 2z = 28$; $-x + 3y + 4z = 24$.
Ans. $x = 7$; $y = 5$; $z = 4$.

2. $4x - 3y + 2z = 9$; $2x + 5y - 3z = 4$; $5x + 6y - 2z = 18$.
Ans. $x = 2$; $y = 3$; $z = 5$.

3. $x + y - z = 8x + 3y - 6z = -4x - y + 3z = 1.$

Ans. $x = 2;\ y = 3;\ z = 4.$

4. $\frac{1}{x} + \frac{1}{y} = \frac{7}{12};\ \frac{1}{x} + \frac{1}{z} = \frac{1}{2};\ \frac{1}{y} + \frac{1}{z} = \frac{5}{12}.$

Ans. $x = 3;\ y = 4;\ z = 6.$

5. $\frac{2}{x} + \frac{1}{y} = \frac{3}{z};\ \frac{3}{z} - \frac{2}{y} = 2;\ \frac{1}{x} + \frac{1}{z} = \frac{4}{3}.$

Ans. $x = \frac{7}{6};\ y = -\frac{7}{2};\ z = \frac{21}{10}.$

PROBLEMS INVOLVING SIMULTANEOUS EQUATIONS.

183. A pound of tea and 3 pounds of sugar cost 96 cents, but if sugar rise 50 per cent and tea 25 per cent, they would cost $1.26. Find the price of tea and sugar.

Let $4x$ represent the number of cents 1 lb. of tea costs.

Let $2y$ represent the number of cents 1 lb. of sugar costs.

From the first condition, I. $4x + 6y = 96.$

If sugar rise 50 per cent, then the original cost, $6y$, would be increased by $3y$. If tea rise 25 per cent, then the original cost, $4x$, would be increased by $1x$. But the total increase in price is 30 cents. $\therefore$ II. $x + 3y = 30.$

Solving these equations, we find that a pound of tea costs 72 cents, and a pound of sugar 8 cents.

184. What fraction becomes 1 when 3 is added to its numerator, and $\frac{1}{2}$ when 2 is added to its denominator?

Let x represent the numerator, and y the denominator.

From the first condition, $\frac{x+3}{y} = 1.\quad \therefore$ I. $x + 3 = y.$

From the second condition, $\frac{x}{y+2} = \frac{1}{2}.\quad \therefore$ II. $2x = y + 2.$

$\therefore\ x = 5$, and $y = 8.$ *Ans.* $\frac{5}{8}$ is the fraction.

185. A purse holds 6 dimes and 19 nickels; $\frac{17}{21}$ of it holds 15 dimes and 12 nickels; how many of each will it hold?

Let $x =$ the number of dimes that will fill the purse.

Let $y =$ the number of nickels that will fill the purse.

The size of the purse may be represented by unity (1).

If x dimes or y nickels fill the purse, 1 dime will occupy $\frac{1}{x}$ of it, and 1 nickel will occupy $\frac{1}{y}$ of it.

Therefore 6 dimes occupy $\frac{6}{x}$, 19 nickels occupy $\frac{19}{y}$, and both occupy $\frac{6}{x}+\frac{19}{y}$.

But both just fill the purse. $\therefore$ I. $\frac{6}{x}+\frac{19}{y}=1.$

In a similar manner, II. $\frac{15}{x}+\frac{12}{y}=\frac{17}{21}.$

Multiply I by 5. III. $\frac{30}{x}+\frac{95}{y}=5.$

Multiply II by 2. IV. $\frac{30}{x}+\frac{24}{y}=\frac{34}{21}.$

Subtract IV from III. $\frac{71}{y}=\frac{71}{21}.$ $\therefore y=21.$

Substitute 21 for y in I. $\therefore x=63.$

Exercise XI.

Solve:

1. $6x-5y=1;\ 14x-8y=17.$
2. $2x+y=147;\ 2x-y=77.$
3. $7x+3y-2z=16;\ 2x+5y+3z=39;\ 5x-y+5z=31.$
4. $4x-5y+z=6;\ 7x-11y+2z=9;\ 6x-12y-z=-3.$
5. $x+y+z+v=14;\ x-z=2y-2v-2;\ 2y+2z+3v=19+x;$ $5x+4y+10z+\frac{20}{3}v=80.$

6. A, B, and C can together perform a piece of work in 15 days; A and B can together perform it in 16 days; and B and C can together perform it in 60 days. Find the time in which each alone can perform the work.

7. If B give A \$50, they will have equal sums of money; if A give B \$44, B's money will be double that of A. Find the money which each actually has.

8. Seven years ago A was three times as old as B, and seven years hence A will be twice as old as B will be. Find their present ages.

9. What fraction becomes equal to $\frac{1}{3}$ when the numerator is increased by 1, and equal to $\frac{1}{4}$ when the denominator is increased by 1?

10. A spends \$3 in apples and pears, buying the apples at 4 for a dime and the pears at 5 for a dime; he sells half his apples and one third of his pears at prime cost for \$1.30. How many apples and pears does he buy?

11. A owes \$240, and B \$510. Said A to B, "Give me $\frac{1}{8}$ of your money and I shall have enough to pay my debts." B answered, "I can pay my debts if you give me $\frac{1}{6}$ of your money." How much has each?

12. Divide \$640 between A and B, so that 7 times A's share shall equal 9 times B's share.

13. What numbers are those whose difference is 20, and quotient 3?

14. Find four numbers, such that the sum of the first, second, and third shall be 13; the sum of the first, second, and fourth, 15; the sum of the first, third, and fourth, 18; and the sum of the second, third, and fourth, 20.

15. Find three numbers which, being taken in pairs, shall produce sums of a, b, and c, respectively.

CHAPTER XII.

ELEMENTARY QUADRATIC EQUATIONS.

186. An equation which contains the second power of the unknown quantity, but no higher power, is called a *quadratic equation.* Ex. $x^2 = 25$; $x^2 + 4x = 60$.

187. If the equation contain the second power only, it is called a *pure quadratic;* but if it contain both second and first powers, it is called an *affected quadratic.*

Thus, $3x^2 = 36 - x^2$ is a pure quadratic,
and $5x^2 - 2x = 3$ is an affected quadratic.

188. To solve a pure quadratic:

(1) *Collect the unknown quantities on one side, and the known quantities on the other.* (2) *Divide both sides by the co-efficient of the unknown quantity.* (3) *Extract the square root of each side of the resulting equation.*

1. Solve $3x^2 = 36 - x^2$.

Transpose $-x^2$. $4x^2 = 36$.
Divide by 4. $x^2 = 9$.
Extract square root. $x = \pm 3$.

2. If $3x^2 - 48 = 0$, $\therefore 3x^2 = 48$, $x^2 = 16$, and $x = \pm 4$.

3. If $4x^2 - 9 = 19$, then $4x^2 = 28$, $x^2 = 7$, and $x = \pm \sqrt{7}$.

Here x represents a number whose value can not be exactly ascertained, but an approximate value of it to any assigned degree of accuracy may be found.

189. It will be observed that there are two roots of equal value but of opposite signs. The square root of $x^2 = 9$

really admits of four forms: $\pm x = \pm 3$; that is, $+x = +3$, $+x = -3$, $-x = +3$, $-x = -3$. But the equation $-x = +3$ gives $x = -3$, and the equation $-x = -3$ gives $x = +3$; hence the equation $x = \pm 3$ expresses all the values, and there are only two roots.

EXAMPLES.

Solve the following equations:

Answers.

4. $11x^2 - 30 = 5x^2 + 24$. $x = \pm 3$.

5. $\frac{1}{3}x^2 - 4 = \frac{1}{4}x^2 - 1$. $x = \pm 6$.

6. $(x + 3)^2 = 6x + 25$. $x = \pm 4$.

7. $(2x - 3)^2 = 18 - 12x$. $x = \pm \frac{3}{2}$.

8. $16x^2 = 4(3 - x)^2$. Extract sq. root. $x = 1$ or -3.

9. $\dfrac{x^2 - 24}{5} + \dfrac{x^2 - 37}{4} = 8$. $x = \pm 7$.

10. $(x - 4)(x + 4) = 9$. $x = \pm 5$.

11. $\dfrac{3(x^2 - 11)}{5} - \dfrac{2(x^2 - 60)}{7} = 36$. $x = \pm 9$.

12. $\dfrac{x^2}{120} = \frac{5}{6}(x - 9)^2$. $x = 10$ or $8\frac{2}{11}$.

Multiply by 120, and extract sq. root.

13. $\dfrac{x}{4} + \dfrac{4}{x} = \dfrac{x}{9} + \dfrac{9}{x}$. $x = \pm 6$.

14. $\dfrac{x}{4} + \dfrac{4}{x} = \dfrac{x}{2}$. $x = \pm 4$.

15. $\dfrac{x(2x + 9)}{15} = \dfrac{3x + 6}{5}$. $x = \pm 3$.

16. $(3x + \frac{1}{3})^2 = 2x + \frac{5}{9}$. $x = \pm \frac{2}{9}$.

17. $(x - 3)(x - 2) - (x - 4)(x + 5) = 3(x - 1)^2 + 11$.
Ans. $x = \pm 2$.

18. $(x + 1)(x - 1)(x^2 + 2) = (x^2 - 1)^2 + 9$. *Ans.* $x = \pm 2$.

19. $\frac{1}{8}(3x^2 + 5) - \frac{1}{4}(x - 1)(x + 1) = \frac{1}{3}(4x^2 + 9) - 13$.
Ans. $x = \pm 3$.

AFFECTED QUADRATIC EQUATIONS.

190. Every affected quadratic may be reduced to the form of $a^2x^2 + 2abx = c^2$. The first step in the solution of such an equation is to *complete the square* (151 (3)); that is, *Add to each side the square of the quotient found from dividing the second term by twice the square root of the first.* The first side will then be a perfect square. The second step is to *extract the square root of each side of the resulting equation.* By placing $\pm$ before the second side, two simple equations will result. The third step is to *find the value of x in each of these simple equations.*

1. Solve $x^2 - 4x = 5$.

(1) The square root of x^2 is x. (2) Twice this root is $2x$. (3) $4x \div 2x = 2$. (4) The square of this quotient is 4.

$\therefore$ Add 4 to each side of the given equation. Whence $x^2 - 4x + 4 = 9$.

Extract the square root, and use the double sign on the second side only, for the reasons explained in Art. 189. Whence $x - 2 = \pm 3$.

If $x - 2 = +3$, then $x = 5$.

If $x - 2 = -3$, then $x = -1$. $\therefore x = 5$ or -1.

2. Solve $x^2 + 13x = -12$.

Twice the square root of the first term $= 2x$, and $13x \div 2x = \frac{13}{2}$; therefore we add $(\frac{13}{2})^2$ to each side.

$\therefore x^2 + 13x + (\frac{13}{2})^2 = -12 + \frac{169}{4} = \frac{121}{4}$.

Extract the square root. $x + \frac{13}{2} = \pm \frac{11}{2}$.

$\therefore x = -\frac{13}{2} \pm \frac{11}{2} = -1$ or -12.

3. Solve $2x^2 + x = 3$.

Since $2x^2$ is not a perfect square, it is necessary to multiply or divide each term by such a number that the first term shall be a square.

Multiply each side by 2. $\therefore 4x^2 + 2x = 6$.

Now proceed as before. $\therefore 4x^2 + 2x + \frac{1}{4} = 6\frac{1}{4}$.

$\therefore x = 1$ or $-\frac{3}{2}$.

4. Solve $3x^2 - 3 + x = 3x - 2x^2$.

Transpose and combine so that the first term shall contain all the x^2's, the second term all the x's, and the second side all the known terms. $\therefore 5x^2 - 2x = 3$.

The first term not being a square, multiply by 5.

$\therefore 25x^2 - 10x = 15$, whence $25x^2 - 10x + 1 = 16$.

$\therefore 5x - 1 = \pm 4$, and $x = 1$ or $-\frac{3}{5}$.

EXAMPLES.

Solve:

	Answers.
5. $x^2 - 8x = -12$.	$x = 6$ or 2.
6. $x^2 + 5x - 14 = 0$.	$x = 2$ or -7.
7. $x^2 - 6x = 4x - 16$.	$x = 8$ or 2.
8. $x^2 + 10x + 21 = 0$.	$x = -3$ or -7.
9. $x^2 + x = 6$.	$x = 3$ or -3.
10. $(x - 2)(x + 6) = 33$.	$x = 5$ or -9.
11. $(3x - 5)^2 = 20x$.	$x = 5$ or $\frac{5}{9}$.
12. $12x^2 - 7x + 1 = 0$.	$x = \frac{1}{3}$ or $\frac{1}{4}$.
13. $2x^2 - 3 = \frac{1}{2}(x + 5) + 11$.	$x = 3$ or $-\frac{11}{4}$.
14. $3x - \frac{1}{4}x^2 = 5$.	$x = 10$ or 2.
15. $\frac{1}{2}x^2 - \frac{1}{4}x = 7$.	$x = 4$ or $-3\frac{1}{2}$.
16. $x = 1 + 110 \div x$.	$x = 11$ or -10.
17. $(x - 1)(x - 2) = 12$.	$x = 5$ or -2.
18. $\frac{2}{3}x^2 + 3\frac{1}{2} = \frac{1}{2}x + 8$.	$x = 3$ or $-\frac{9}{4}$.

Exercise XII.

Solve:

1. $x^2 - 5x = -4$.

2. $x^2 - 4x + 3 = 0$.

3. $2x^2 - 7x = -3$.

4. $x^2 + 10x + 9 = 0$.

5. $14x - x^2 = 45$.

6. $(x - 2)(x + 1) = 10$.

7. $(x - 4)(x - 5) = 0$.

8. $(x - 14)(x + 3) = 0$.

9. $(2x + 1)(x + 2) = 3x^2 - 4$.

10. $(x - 1)(x - 2) + (x - 2)(x - 4) = 15$.

11. $(2x-3)^2 = 6x+1.$

12. $(3x-2)(x-1) = 14.$

13. $(5x-3)^2 = 4(11x+3).$

14. $x^2 - 3 = \frac{1}{6}(x-3).$

15. $x^2 - 2ax + a^2 = b^2.$

16. $(x+a)^2 - b^2 = 0.$

SPECIMEN PAPER.

1. Solve $x^2 - 16x = -63.$

No.	Operation.	Axiom.	
I			$x^2 - 16x = -63.$
II	$\text{I} + (\frac{16}{2})^2$	2	$x^2 - 16x + 8^2 = 64 - 63 = 1.$
III	$\sqrt{\text{II}}$	15	$x - 8 = \pm 1.$
IV	III + 8	2	$x = 8 \pm 1 = 9$ or 7.

Ans. $x = 9$ or 7.

Verification: 1st, $x = 9$. 1st side: $(9)^2 - 16(9) = 81 - 144 = -63.$
2d side: $= -63.$
2d, $x = 7$. 1st side: $(7)^2 - 16(7) = 49 - 112 = -63.$
2d side: $= -63.$

2. Solve $4x^2 - 11x - 2 = 7x^2 - 20x - 32.$

No.	Operation.	Axiom.	
I			$4x^2 - 11x - 2 = 7x^2 - 20x - 32.$
II	I transposed	2 and 3	$4x^2 - 11x - 7x^2 + 20x = +2 - 32.$
III	II simplified		$-3x^2 + 9x = -30.$
IV	$\text{III} \div (-3)$	5	$x^2 - 3x = +10.$
V	$\text{IV} + (\frac{3}{2})^2$	2	$x^2 - 3x + \frac{9}{4} = \frac{49}{4}.$
VI	$\sqrt{\text{V}}$	15	$x - \frac{3}{2} = \pm \frac{7}{2}.$
VII	$\text{VI} + \frac{3}{2}$	2	$x = +\frac{3}{2} \pm \frac{7}{2} = 5$ or $-2.$

Ans. $x = 5$ or $-2.$

Verification:
1st, $x = 5$. 1st side: $4(5)^2 - 11(5) - 2 = 100 - 55 - 2 = 43.$
2d side: $7(5)^2 - 20(5) - 32 = 175 - 100 - 32 = 43.$
2d, $x = -2$. 1st side: $4(-2)^2 - 11(-2) - 2 = 16 + 22 - 2 = 36.$
2d side: $7(-2)^2 - 20(-2) - 32 = 28 + 40 - 32 = 36.$

CHAPTER XIII.

FACTORING.

191. Factoring is the process of finding the simplest numbers whose product is a given number.

192.—CASE I. Monomials.

The factors of a monomial consist of the factors of the numerical co-efficient, together with the various letters taken as many times as there are units in their exponents.

Thus: $6a^2bc^3 = 2 \cdot 3 \cdot a \cdot a \cdot b \cdot c \cdot c \cdot c$; $12a^2(b-c)^3 = 2 \cdot 2 \cdot 3 \cdot a \cdot a \cdot (b-c) \cdot (b-c) \cdot (b-c)$.

193.—CASE II. *A polynomial whose terms have a common factor is factored as in Chapter IV.* Thus:

1. $a^2 + 2ab = a(a+2b)$. **2.** $2a^2 - 6a^3 = 2a^2(1-3a)$.

3. $(x-y)^2 - (x-y)^3 = (x-y)^2\{1-(x-y)\}$.

4. $(a-b-c)^2 - (a-b-c) = (a-b-c)[(a-b-c)-1]$.

5. $2^7 - 2^4 = 2^4(2^3-1) = 2^4(7)$.

194.—CASE III. *A polynomial whose terms may be so combined that the resulting polynomial has a factor common to all its terms, is factored as in Chapter IV.* Thus:

1. $x^3 + x^2 + x + 1 = x^2(x+1) + (x+1) = (x+1)(x^2+1)$.

2. $ab - ac - b^2 + bc = a(b-c) - b(b-c) = (b-c)(a-b)$.

3. $a^2 - 2a - 1 - a^2b + 2ab + b = (a^2-2a-1) - b(a^2-2a-1) =$
$(a^2-2a-1)(1-b)$.

4. $(a-b)^4 - (b-a)^3 = (a-b)^4 + (a-b)^3 = (a-b)^3[(a-b+1)]$.

5. $(a-b-c)^3-(b+c-a)^2=(a-b-c)^3-(a-b-c)^2=$
$(a-b-c)^2[(a-b-c)-1]$.

195. After separating an expression into two factors, each of these factors must be separated, if possible, into factors, and so on until the various factors are prime.

196. Before applying any of the following cases to the factoring of a polynomial, the factors which can be found by the preceding rules are removed from the expression.

197.—CASE IV. *A trinomial which is a perfect square is factored according to* (149) *or* (151).

Notice that a trinomial is a perfect square when two of its terms are squares and positive, and the remaining term is twice the product of their square roots (151).

Formula: $a^2 \pm 2ab + b^2 = (a \pm b)(a \pm b)$.

EXAMPLES.

1. $a^2-4a+4=(a-2)(a-2)$.
2. $a^2b^4-6ab^2c+9c^2=(ab^2-3c)(ab^2-3c)$.
3. $4a^3-12a^{\frac{3}{2}}b+9b^2=(2a^{\frac{3}{2}}-3b)^2$.
4. $2a^3-8a^2+8a=2a(a-2)^2$.
5. $18a^2-12a+2=2(3a-1)(3a-1)$.
6. $(a-b)^2-2(a-b)+1=[(a-b)-1]^2$.
7. $x^{-2}-8x^{-1}+16=(x^{-1}-4)^2$.
8. $2a^3+12a^2+18a=2a(a+3)^2$.
9. $a^{2c}-10a^c+25=(a^c-5)^2$.
10. $3^{2x}-2(3^x)+1=(3^x-1)^2$.

198.—CASE V. *A polynomial which is the square of three or more terms is factored by* (149), *or thus:*

Transform the given expression into a trinomial such that two terms shall be squares and positive, and the other term shall be twice the product of their square roots.

1. To factor $a^2 + b^2 + c^2 - 2ab - 2ac + 2bc$.

Here there are six terms, three being squares and three being products with even co-efficients. Arranged according to a, the expression becomes $a^2 - 2a(b+c) + (b^2 + 2bc + c^2)$.

Now the first term is the square of a, and the third term is the square of $(b + c)$ (Art. 197); also, the second term is twice the product of these square roots; ∴ the given expression is the square of $a - (b + c) = a - b - c$.

2. $a^2 + b^2 - 4a + 4b - 2ab + 4 = a^2 - 2a(b + 2) + (b^2 + 4b + 4) = a^2 - 2a(b + 2) + (b + 2)^2 = (a - b - c)^2$.

3. To factor $a^4 + 2a^3 + 3a^2 + 2a + 1$. Here there are but five terms; ∴ one of the terms must be separated into two. In such cases it is convenient to use Art. 149, but the example may also be factored thus: $a^4 + 2a^3 + a^2 + 2a^2 + 2a + 1 = (a^2 + a)^2 + 2(a^2 + a) + 1 = (a^2 + a + 1)^2$.

4. $a^2 + b^2 + c^2 + d^2 - 2ab - 2ac + 2ad + 2bc - 2bd - 2cd = a^2 - 2a(b + c - d) + (b^2 + c^2 + d^2 + 2bc - 2bd - 2cd) = [a - (b + c - d)]^2$. See Ex. 1.

199.—Case VI. *A binomial which is the difference of two squares.* Formula: $a^2 - b^2 = (a + b)(a - b)$.

Rule. (1) *Extract the square root of each square.*

(2) *One factor will be the sum of these roots, and the other will be their difference.*

EXAMPLES.

1. $a^2 - 4 = (a + 2)(a - 2)$.
2. $a^3 - a = a(a^2 - 1) = a(a + 1)(a - 1)$.
3. $a^4 - b^4 = (a^2 + b^2)(a^2 - b^2) = (a^2 + b^2)(a + b)(a - b)$.
4. $(a - b)^2 - (c - d)^2 = [(a - b) + (c - d)][(a - b) - (c - d)]$.
5. $4a^3 - 9a = a(4a^2 - 9) = a(2a + 3)(2a - 3)$.
6. $(a - b)^2 - (c^2 - 2cd + d^2) = (a - b)^2 - (c - d)^2 = [(a - b) + (c - d)][(a - b) - (c - d)]$.
7. $9^x - 4^y = (3^x + 2^y)(3^x - 2^y)$.

8. $a^{-2} - 16 = (a^{-1} + 4)(a^{-1} - 4)$.

9. $a^{\frac{2}{3}} - 25 = (a^{\frac{1}{3}} + 5)(a^{\frac{1}{3}} - 5)$.

200.—CASE VII. *A polynomial which is the difference of two squares is factored by collecting the terms of one square in one parenthesis, and the terms of the other square in a second parenthesis, and then using Case VI.*

1. $a^2 - 2ab + b^2 - c^2 = (a^2 - 2ab + b^2) - c^2 = (a - b)^2 - c^2 =$
$[(a - b) + c][(a - b) - c]$.

2. $a^2 - 4a - b^2 + 4 = (a^2 - 4a + 4) - b^2 = (a - 2)^2 - b^2 =$
$(a - 2 + b)(a - 2 - b)$.

3. $a^2 + b^2 - c^2 - d^2 - 2ab - 2cd =$
$(a^2 - 2ab + b^2) - (c^2 + 2cd + d^2) =$
$(a-b)^2 - (c+d)^2 = [(a-b)+(c+d)][(a-b)-(c+d)] =$
$(a - b + c + d)(a - b - c - d)$.

4. $a^2 - b^2 - 4a - 6b - 5 = (a^2 - 4a + 4) - (b^2 + 6b + 9) =$
$(a-2)^2 - (b+3)^2 = [(a-2)+(b+3)][(a-2)-(b+3)] =$
$(a + b + 1)(a - b - 5)$.

201.—CASE VIII. *Since* $(x+c)(x+d) = x^2 + (c+d)x + cd$, *therefore a trinomial of the form* $x^2 + ax + b$ *can be resolved into two factors when* a *is the sum of two numbers whose product is* b. Thus:

1. $x^2 + 6x + 5 = (x + 1)(x + 5)$, since $5 + 1 = 6$,
and $5 \times 1 = 5$.

2. $x^2 - 3x + 2 = (x - 1)(x - 2)$, since $-1 + (-2) = -3$,
and $-1(-2) = +2$.

3. $x^2 - 5x - 14 = (x - 7)(x + 2)$, since $-7 + 2 = -5$,
and $-7(2) = -14$.

4. $x^2 + 4x - 5 = (x + 5)(x - 1)$, since $5 + (-1) = +4$,
and $5(-1) = -5$.

202. A composite expression of the form $x^2 + ax + b$ is really the difference of two squares, and may be factored by Case VII.

5. $x^2 - 4x + 3 = (x^2 - 4x + \ 4) - 1 = (x-2)^2 - \ 1 =$
$(x - 2 + 1)(x - 2 - 1) = (x-1)(x-3).$

6. $x^2 - 6x - 7 = (x^2 - 6x + \ 9) - 16 = (x-3)^2 - 16 =$
$(x - 3 + 4)(x - 3 - 4) = (x+1)(x-7).$

7. $x^2 + 5x - 6 = (x^2 + 5x + \frac{25}{4}) - \frac{49}{4} = (x + \frac{5}{2})^2 - \frac{49}{4} =$
$(x + \frac{5}{2} + \frac{7}{2})(x + \frac{5}{2} - \frac{7}{2}) = (x+6)(x-1).$

8. $x^4 - 5x^2 + 4 = (x^4 - 5x^2 + \frac{25}{4}) - \ \frac{9}{4} = (x^2 - \frac{5}{2})^2 - \ \frac{9}{4} =$
$(x^2 - \frac{5}{2} + \frac{3}{2})(x^2 - \frac{5}{2} - \frac{3}{2}) = (x^2-1)(x^2-4) =$
$(x+1)(x-1)(x+2)(x-2).$

203.—Rule. (1) *Arrange the given trinomial.*

(2) *Complete the square of the first two terms* (151, (3)).

(3) *Subtract the resulting third term from the given third term, and write the remainder as the fourth term.* (*If this fourth term is not minus a square, the given trinomial is prime.*)

(4) *Combine the first three terms in one, and continue as in Case VII.*

EXAMPLES.

9. $x^2 - 5x + \ 6 = (x-2)(x-3).$

10. $x^2 - 2x - \ 3 = (x+1)(x-3).$

11. $x^2 - 9x + 20 = (x-4)(x-5).$

12. $x^2 + 2x - \ 3 = (x+3)(x-1).$

13. $x^2 + 4x + \ 3 = (x+3)(x+1).$

14. $3x^2 - 3x - 6 = 3(x+1)(x-2).$

15. $x^3 - x^2 - 20x = x(x+4)(x-5).$

16. $x^4 + 5x^2 + 6 = (x^2+3)(x^2+2).$

17. $x^4 - 10x^2 + 9 = (x+1)(x-1)(x+3)(x-3).$

18. $2^{2x} - 2^{x+1} - 15 = (2^x - 5)(2^x + 3).$

204. In like manner, expressions of the form $(cx)^2 + a(cx) + b$ may be factored. It is convenient in this case to let y represent the square root of the first term.

EXAMPLES.

1. $4x^2 - 12x + 5 = (2x)^2 - 6(2x) + 5 = y^2 - 6y + 5 = (y-1)(y-5)$. $\therefore 4x^2 - 12x + 5 = (2x-1)(2x-5)$.

2 $9x^2 - 15x - 14 = (3x)^2 - 5(3x) - 14 = y^2 - 5y - 14 = (y+2)(y-7)$. *Ans.* $(3x+2)(3x-7)$.

3. $16x^4 + 8x^2 - 3 = (4x^2)^2 + 2(4x^2) - 3 = y^2 + 2y - 3 = (y+3)(y-1) = (4x^2+3)(4x^2-1) = (4x^2+3)(2x+1)(2x-1)$.

4. $9x^2 - 12x - 5 = (3x+1)(3x-5)$.

5. $4x^2 - 14x + 6 = 2(2x-1)(x-3)$.

6. $4x^2 - \ \ 7x + 3 = (x-1)(4x-3)$.

7. $16x^4 + \ 4x^2 - 2 = 2(2x^2+1)(2x+1)(2x-1)$.

8. $4x^4 - 17x^2 + 4 = (x+2)(x-2)(2x+1)(2x-1)$.

205.—CASE IX. *To factor a trinomial of the form* $ax^2 + bx + c$, *multiply the expression by* a, *factor the result according to Case VIII, and finally divide by* a. Thus:

1. $3x^2 - 2x - 16 = \dfrac{(3x)^2 - 2(3x) - 48}{3} = \dfrac{y^2 - 2y - 48}{3} =$

$\frac{1}{3}(y+6)(y-8) = \frac{1}{3}(3x+6)(3x-8) = (x+2)(3x-8)$.

2. $3x^2 - 7x - 20 = \dfrac{(3x)^2 - 7(3x) - 60}{3} = \dfrac{y^2 - 7y - 60}{3} =$

$\frac{1}{3}(y+5)(y-12) = \frac{1}{3}(3x+5)(3x-12) = (3x+5)(x-4)$.

3. $8x^2 - 2x - 3 = \frac{1}{2}(16x^2 - 4x - 6) = \frac{1}{2}(y^2 - y - 6) = \frac{1}{2}(y-3)(y+2) = \frac{1}{2}(4x-3)(4x+2) = (4x-3)(2x+1)$.

206. Example 3 shows that it is not always necessary to multiply the expression by the co-efficient of x^2. It is usually best to multiply by the least integer that will make the first term a perfect square.

207. If the multiplier be composite, it is sometimes necessary to divide one factor of the product by one factor of the multiplier, and to divide the other factor of the product by the other factor of the multiplier. Thus:

4. $6x^2-13x+6=\frac{1}{6}[(6x)^2-13(6x)+36]=\frac{1}{6}[y^2-13y+36]=$
$\frac{1}{6}[y-4][y-9]=\frac{1}{6}(6x-4)(6x-9)=(3x-2)(2x-3)$.

EXAMPLES.

5. $2x^2-7x+3=(2x-1)(x-3)$.
6. $6x^2+13x+6=(2x+3)(3x+2)$.
7. $3x^2+10x+3=(3x+1)(x+3)$.
8. $3x^2-20x+12=(3x-2)(x-6)$.
9. $3x^2+10x-57=(x-3)(3x+19)$.
10. $3x^2-2x-65=(x-5)(3x+13)$.
11. $3x^4-13x^2+4=(3x^2-1)(x+2)(x-2)$.
12. $\frac{1}{3}x^2-\frac{1}{10}x-\frac{1}{30}=\frac{1}{3}(x-\frac{1}{2})(x+\frac{1}{5})$.

208.—CASE X. *A trinomial, which would be a square if a square were added to its middle term, is factored like Case VII.* Thus:

1. $x^4+2x^2+9=(x^4+6x^2+9)-4x^2=(x^2+3)^2-4x^2=$
$(x^2+3+2x)(x^2+3-2x)$.

2. $9x^4-13x^2+4=(9x^4-12x^2+4)-x^2=(3x^2-2)^2-x^2=$
$(3x^2-2+x)(3x^2-2-x)=(x+1)(3x-2)(x-1)(3x+2)$.

3. $x^4-10x^2+9=(x^4-6x^2+9)-4x^2=(x^2-3)^2-4x^2=$
$(x^2-3+2x)(x^2-3-2x)=(x-1)(x+3)(x+1)(x-3)$.

Examples 2 and 3 may be factored by Case VIII. Thus:

$x^4-10x^2+9=(x^4-10x^2+25)-16=(x^2-5)^2-16=$
$(x^2-1)(x^2-9)=(x+1)(x-1)(x+3)(x-3)$.

209.—RULE. (1) *Arrange the expression.*

(2) *Write for the second term twice the product of the square roots of the extreme terms* (151, (2)).

(3) *Subtract the second term from the given second term, and write the remainder as the fourth term.*

(4) *Combine the first three terms, and continue as in Case VII.*

210. When the terms of a binomial are squares and positive, and twice the product of their square roots is a square, the binomial may be factored by the rule in (209).

4. $x^4 + 4 = (x^4 + 4x^2 + 4) - 4x^2 = (x^2 + 2)^2 - 4x^2 =$
$(x^2 + 2 + 2x)(x^2 + 2 - 2x).$

EXAMPLES.

5. $4x^4 + 1 = (2x^2 + 2x + 1)(2x^2 - 2x + 1).$
6. $64x^4 + y^4 = (8x^2 + 4xy + y^2)(8x^2 - 4xy + y^2).$
7. $x^4 + x^2 + 1 = (x^2 + x + 1)(x^2 - x + 1).$
8. $x^4 - 7x^2 + 1 = (x^2 + 3x + 1)(x^2 - 3x + 1).$
9. $x^4 - 23x^2 + 1 = (x^2 + 5x + 1)(x^2 - 5x + 1).$
10. $x^4 + 5x^2y^2 + 9y^4 = (x^2 + xy + 3y^2)(x^2 - xy + 3y^2).$
11. $x^4 - x^2 + 16 = (x^2 + 3x + 4)(x^2 - 3x + 4).$

211.—CASE XI. *A binomial of the form* $a^n - b^n$, n *being any integer* > 1, *can always be factored.*

(1) When n is an even number, $a^n - b^n$ is the difference of two squares, and can be factored as explained in Case VI.

1. $a^4 - 4b^2 = (a^2 + 2b)(a^2 - 2b).$
2. $4a^4 - 9b^4 = (2a^2 + 3b^2)(2a^2 - 3b^2).$
3. $a^4 - b^4 = (a^2 + b^2)(a^2 - b^2) = (a^2 + b^2)(a + b)(a - b).$

(2) To factor $a^n - b^n$ when n is an odd integer.

(1) *Extract that root of each term whose index is the least integer* > 1 *by which* n *is divisible.*

(2) *Divide the expression by the difference of these roots.*

4. $a^3 - b^3 = (a - b)(a^2 + ab + b^2).$
5. $a^5 - b^5 = (a - b)(a^4 + a^3b + a^2b^2 + ab^3 + b^4).$

6. $a^6 - 8 = (a^2 - 2)(a^4 + 2a^2 + 4)$.
7. $8a^3 - 27 = (2a - 3)(4a^2 + 6a + 9)$.
8. $a^9 - b^9 = (a^3 - b^3)(a^6 + a^3b^3 + b^6) =$
$(a - b)(a^2 + ab + b^2)(a^6 + a^3b^3 + b^6)$.
9. $x^6 - xy^5 = x(x - y)(x^4 + x^3y + x^2y^2 + xy^3 + y^4)$.
10. $8a^7 - a = a(8a^6 - 1) = a(2a^2 - 1)(4a^4 + 2a^2 + 1)$.
11. $x^{-3} - a^{-3} = (x^{-1} - a^{-1})(x^{-2} + x^{-1}a^{-1} + a^{-2})$.
12. $3^{3x} - 2^{3x} = (3^x - 2^x)(9^x + 6^x + 4^x)$.

212.—CASE XII. *A binomial of the form* $a^n + b^n$ *is composite when* n *is divisible by an odd integer* > 1. Thus:

1. $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$.
2. $a^5 + b^5 = (a + b)(a^4 - a^3b + a^2b^2 - ab^3 + b^4)$.
3. $a^6 + 8 = (a^2 + 2)(a^4 - 2a^2 + 4)$.
4. $a^6 + b^6 = (a^2 + b^2)(a^4 - a^2b^2 + b^4)$.
5. $a^4 + b^4$ is prime, because 4 has no odd factor > 1.
6. $a^8 + b^8$ is prime, because 8 has no odd factor > 1.
7. $a^{18} + 1 = (a^2 + 1)(a^4 - a^2 + 1)(a^{12} - a^6 + 1)$.

213.—RULE. (1) *Extract that root of each term whose index is the least odd number* > 1 *by which* n *is divisible.*

(2) *Divide the given expression by the sum of these roots.* That is, if the terms of the proposed binomial be cubes, the divisor will be the sum of their cube roots; if fifth powers, the divisor will be the sum of their fifth roots, etc.

EXAMPLES.

8. $27a^3 + 8 = (3a + 2)(9a^2 - 6a + 4)$.
9. $a^6 + 64 = (a^2 + 4)(a^4 - 4a^2 + 16)$.
10. $a^5 + 27a^2 = a^2(a + 3)(a^2 - 3a + 9)$.
11. $3^{3x} + 2^{3x} = (3^x + 2^x)(9^x - 6^x + 4^x)$.
12. $a^9 + 1 = (a + 1)(a^2 - a + 1)(a^6 - a^3 + 1)$.
13. $a^{12} + 1 = (a^4 + 1)(a^8 - a^4 + 1)$.

214. The following examples involve Cases XI and XII.

14. $a^6 - 64 = (a-2)(a^2+2a+4)(a+2)(a^2-2a+4)$.

15. $a^8 - a^2 = a^2(a-1)(a^2+a+1)(a+1)(a^2-a+1)$.

16. $a^{12} - b^6 = (a^2-b)(a^4+a^2b+b^2)(a^2+b)(a^4-a^2b+b^2)$.

17. $2x^{-6} - 128 = 2(x^{-6} - 64) =$

$2(x^{-1}-2)(x^{-2}+2x^{-1}+4)(x^{-1}+2)(x^{-2}-2x^{-1}+4)$.

18. $a^{-6} - 1 =$

$(a^{-1}-1)(a^{-2}+a^{-1}+1)(a^{-1}+1)(a^{-2}-a^{-1}+1)$.

19. $3^{6x} - 2^{6x} = (3^x - 2^x)(9^x + 6^x + 4^x)(3^x + 2^x)(9^x - 6^x + 4^x)$.

[NOTE.—Beginners may omit 215–220 inclusive.]

215. In Case XI we have assumed that $a^n - b^n$ is divisible by $a - b$, if n be any integer; and in Case XII that $a^n + b^n$ is divisible by $a + b$, if n be an odd integer. That these principles are true in any selected case may easily be shown by performing the division; but the general proof is somewhat difficult for beginners, as it depends upon *induction*, a kind of reasoning not heretofore employed.

216. Mathematical Induction may be thus described: We prove that if a theorem is true in one case, whatever that case may be, it is true in another case, which may be called the next case. Now, by trial we prove that the theorem is true in a certain case, hence it is true in the next case, and hence in the next to that, and so on without limit; therefore, the theorem must be true in every case after that with which we began. For example:

217. To prove that $a^n - b^n$ is divisible by $a - b$.

By division, $$\frac{a^n - b^n}{a - b} = a^{n-1} + \frac{b(a^{n-1} - b^{n-1})}{a - b}. \quad (143.)$$

The second term of the quotient is fractional in form, but, if $a^{n-1} - b^{n-1}$ be divisible by $a - b$, it will be integral in value; therefore, $a^n - b^n$ will be divisible by $a - b$ if $a^{n-1} - b^{n-1}$ be divisible by $a - b$. That is, *If the difference of the same powers of two quantities be divisible by the difference of the quantities themselves, then will the difference of the next higher powers of the same quantities be divisible by the difference of the quantities.* Hence, since $a^1 - b^1$ is divisible by $a - b$, $\therefore$ $a^2 - b^2$ is divisible by $a - b$; since $a^2 - b^2$ is divisible by $a - b$, $\therefore$ $a^3 - b^3$ is divisible by $a - b$; and so on without limit. Therefore, for every integral value of n, $a^n - b^n$ is divisible by $a - b$.

218. $\dfrac{a^n - b^n}{a - b} = a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \ldots + a^{n-r}b^{r-1} + \ldots + b^{n-1}$.

By carrying out the division of $a^n - b^n$ by $a - b$, it will be observed that the number of the term and the exponents of a and b have a constant relation to each other. This relation is expressed by the *general term*, $+ a^{n-r}b^{r-1}$. By substituting in this term 1, 2, 3, 4, etc., in succession, in place of r, all the terms are found.

Thus, the 15th term of $\dfrac{a^{20} - b^{20}}{a - b}$ is found by substituting 20 for n, and 15 for r. That is, $a^{n-r}b^{r-1}$ becomes $a^{20-15}b^{15-1} = a^5b^{14}$, which is the required 15th term.

219. To prove that $a^n + b^n$ is divisible by $a + b$ when n is an odd integer, and is not so divisible when n is even.

By division, $\dfrac{a^n + b^n}{a + b} = a^{n-1} - a^{n-2}b + \dfrac{b^2(a^{n-2} + b^{n-2})}{a + b}$.

Reasoning as in (217), we see that, *If the sum of the same powers of two quantities be divisible by the sum of the quantities themselves, then will the sum of the powers two degrees higher be divisible by the sum of the quantities.* Hence, since $a^1 + b^1$ is divisible by $a + b$, $\therefore$ $a^3 + b^3$ is divisible by $a + b$; since $a^3 + b^3$ is divisible by $a + b$, $\therefore$ $a^5 + b^5$ is divisible by $a + b$; and so on without limit. $\therefore$ if n be any odd integer, $a^n + b^n$ is divisible by $a + b$.

We see further that $a^n + b^n$ will not be divisible by $a + b$ unless $a^{n-2} + b^{n-2}$ is divisible by $a + b$.

Now since $a^2 + b^2$ is not divisible by $a + b$, therefore $a^4 + b^4$ is not divisible by $a + b$, therefore $a^6 + b^6$ is not divisible by $a + b$, and so on to infinity.

The quotient of $(a^n + b^n) \div (a + b)$ is like that in (218), excepting that the sign of each even term is $-$.

220. In like manner it can be shown: (1) $a^n - b^n$ is divisible by $a + b$ when n is even, and is not so divisible when n is odd; (2) $a^n + b^n$ is never divisible by $a - b$.

221.—CASE XIII. *A polynomial which is a power, the difference of like powers, or the sum of like powers, may be factored by using the Binomial Formula* (129) *in connection with the preceding cases.* Thus:

1. $a^3 - 3a^2b + 3ab^2 - b^3 = (a - b)^3.$

2. $a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4 = (a - b)^4.$

3. $2x^3 - 3x^2 + 3x - 1 = x^3 + (x^3 - 3x^2 + 3x - 1) =$
$x^3 + (x - 1)^3 = [x + (x - 1)][x^2 - x(x - 1) + (x-1)^2] =$
$(2x - 1)(x^2 - x + 1).$

4. $x^4 - 4x^3 + 5x^2 - 4x + 1 = (x^4 - 4x^3 + 6x^2 - 4x + 1) - x^2 =$
$(x - 1)^4 - x^2 = [(x - 1)^2 + x][(x - 1)^2 - x] =$
$(x^2 - x + 1)(x^2 - 3x + 1).$

5. $x^4 - 4x^3 + 5x^2 - 8x - 3 =$
$(x^4 - 4x^3 + 6x^2 - 4x + 1) - (x^2 + 4x + 4) = (x-1)^4 - (x+2)^2 =$
$[(x - 1)^2 + (x + 2)][(x - 1)^2 - (x + 2)] =$
$(x^2 - x + 3)(x^2 - 3x - 1).$

6. $x^3 + 3x^2 + 3x - 7 = (x^3 + 3x^2 + 3x + 1) - 8 =$
$(x + 1)^3 - 8 = [(x + 1) - 2][(x + 1)^2 + 2(x + 1) + 4] =$
$(x - 1)(x^2 + 4x + 7).$

7. $a^4 + b^4 + c^4 - 2a^2b^2 - 2a^2c^2 - 2b^2c^2 =$
$(a^4 + b^4 + c^4 - 2a^2b^2 - 2a^2c^2 + 2b^2c^2) - 4b^2c^2 =$
$(a^2 - b^2 - c^2)^2 - 4b^2c^2 = (a^2 - b^2 - c^2 + 2bc)(a^2 - b^2 - c^2 - 2bc) =$
$(a + b - c)(a - b + c)(a + b + c)(a - b - c).$

222.—Case XIV. *A polynomial which has a binomial divisor is readily factored by trial.*

Two methods are in common use:

(1) Synthetic Division (144); (2) Separation of Terms.

223.—(1) 1. To factor $x^3 - 9x^2 + 26x - 24$ by synthetic division. The first term of the binomial divisor must be x, and the second term must be one of the divisors of -24. It is usually best to try one of the smaller divisors, commencing with the one next to the square root. In this case try -4.

$$\begin{array}{r|l} & 1 - 9 + 26 - 24 \\ +4 & \quad + 4 - 20 + 24 \\ \hline & 1 - 5 + 6 \end{array}$$

The division being exact, one of the factors is $x - 4$, and the other is $x^2 - 5x + 6 = (x-2)(x-3)$. *Ans.* $(x-4)(x-2)(x-3)$.

224.—(2) To factor $x^3 - 9x^2 + 26x - 24$ by separation of terms, proceed as follows:

(1) *Arrange and grade the terms.* (2) *Write the extreme terms unchanged.* (3) *Write as the co-efficient of the second term a divisor of the product of the extremes.* In selecting this co-efficient, try the lesser divisors, commencing with the one next to the square root. Hence in this case try $-4x^2$ as the second term. (4) To find the third term: *Subtract this second term from the second term of the proposed expression.* In this example the third term is $-5x^2$ (because the sum of the second and third terms must be equal to $-9x^2$, the second term of the proposed expression). (5) To find the fourth term: *Divide the product of the second and third terms by the first term.* Hence the fourth term in this case is $-4x^2(-5x^2) \div x^3 = +20x$. (6) To find the fifth term: *Subtract this fourth term from the third term of the proposed expression.* Hence in this case the fifth term is $+6x$. (7) *So continue, each odd term being*

found from subtracting the preceding even term from the corresponding term of the proposed expression, and each even term being found from multiplying the preceding odd term by the quotient of the second term by the first.

If the last even term found as above be equal to the last term of the proposed expression, the proper second term has been selected, and the work is continued by combining the odd terms in one parenthesis and the even terms in another. If no mistake be made, the expressions in parentheses will be alike, and the work is finished as in Case III. Thus: $x^3 - 9x^2 + 26x - 24 = x^3 - 4x^2 - 5x^2 + 20x + 6x - 24 = x(x^2 - 5x + 6) - 4(x^2 - 5x + 6) = (x - 4)(x^2 - 5x + 6) = (x - 4)(x - 2)(x - 3)$.

If the last even term found from the rule be not equal to the last term of the proposed expression, the wrong second term has been selected, and another trial must be made; and so on.

EXAMPLES.

2. $2x^3 + 3x^2 - 8x + 3 = 2x^3 - 2x^2 + 5x^2 - 5x - 3x + 3 = x(2x^2 + 5x - 3) - (2x^2 + 5x - 3) = (x - 1)(2x^2 + 5x - 3) = (x - 1)(2x - 1)(x + 3)$.
3. $x^3 - 6x^2 + 11x - 6 = (x - 2)(x - 1)(x - 3)$.
4. $x^4 - 7x^2 - 6x = x(x + 2)(x + 1)(x - 3)$.
5. $x^6 - 21x^2 - 20 = (x^2 + 4)(x^2 + 1)(x^2 - 5)$.
6. $2x^3 - x^2 - 12 = (x - 2)(2x^2 + 3x + 6)$.
7. $x^3 - 3x - 18 = (x - 3)(x^2 + 3x + 6)$.
8. $x^3 - 6x + 9 = (x + 3)(x^2 - 3x + 3)$.
9. $x^3 - 7x + 6 = (x - 2)(x - 1)(x + 3)$.
10. $x^4 - 3x - 4 = (x + 1)(x^3 - x^2 + x - 4)$.

225.—CASE XV. *Certain polynomials of the general form* $x^4 + ax^3 + bx^2 + cx + d$ *can be decomposed into two trinomial factors.*

1. To factor $x^4 + x^3 - 4x^2 + 11x - 3$.

It is evident that the first term of each trinomial is x^2, that the second terms contain x with co-efficients at present unknown, and that the product of the third terms is equal to the last term of the proposed expression. The required factors may, therefore, be represented by $(x^2 + ax + b)(x^2 + cx + d)$, in which a, b, c, and d are to be found.

Multiplying these factors together, we find that the proposed expression must be equal to

$$x^4 + \left.\begin{matrix} a \\ +c \\ \end{matrix}\right| x^3 + \left.\begin{matrix} b \\ +ac \\ +d \end{matrix}\right| x^2 + \left.\begin{matrix} bc \\ +ad \\ \end{matrix}\right| x + bd.$$

$$\left.\begin{array}{lrcr} \text{Hence, I.} & a + c & = & 1. \\ \text{II.} & b + ac + d & = & -4. \\ \text{III.} & bc + ad & = & 11. \\ \text{IV.} & bd & = & -3. \end{array}\right\}$$

Since $bd = -3$, one of these numbers is ± 1 and the other is ∓ 3. Assume $b = -1$, then $d = +3$. We now find a and c by substituting in III, and combining the resulting equation with I. Thus:

$$\left.\begin{array}{ll} \text{I.} & a + c = 1. \\ \text{III.} & -c + 3a = 11. \end{array}\right\}$$

Add. $\therefore 4a = 12$ and $a = 3$, whence $c = -2$.

By trial, we find that the values $a = 3$, $b = -1$, $c = -2$, and $d = 3$ satisfy equation II; therefore the proper values of the letters have been found, and the required factors are $(x^2 + 3x - 1)(x^2 - 2x + 3)$.

If the values found do not satisfy equation II, other values must be assumed for b and d, and the corresponding values of a and c must be found as before; and so on.

2. To factor $x^4 - 13x^2 + 2x^3 + 34x - 15$.

Here I. $a + c = +2$; II. $b + ac + d = -13$; III. $bc + ad = 34$; IV. $bd = -15$.

These equations are satisfied by $a = 5$, $b = -3$, $c = -3$, $d = 5$; $\therefore$ the required factors are $(x^2 + 5x - 3)(x^2 - 3x + 5)$.

3. To factor

$2x^4 - 6x^3 + 9x^2 - 6x + 2 = 2(x^4 - 3x^3 + \frac{9}{2}x^2 - 3x + 1)$.

Here $a+c=-3$; $b+ac+d=\frac{9}{2}$; $bc+ad=-3$; $bd=1$; which are satisfied by $a=-1$; $b=\frac{1}{2}$; $c=-2$; $d=2$.

Ans. $2(x^2-x+\frac{1}{2})(x^2-2x+2)=(2x^2-2x+1)(x^2-2x+2)$.

EXAMPLES.

4. $x^4+x^3-5x^2-27x-30=(x^2-2x-5)(x^2+3x+6)$.
5. $x^4+5x^3+17x^2+27x+30=(x^2+2x+5)(x^2+3x+6)$.
6. $x^4+3x^3+9x^2+10x+12=(x^2+2x+4)(x^2+x+3)$.
7. $5x^4+17x^3+12x^2+5x+1=(5x^2+2x+1)(x^2+3x+1)$.
8. $6x^4+x^3+3x^2-3x-2=(2x^2+x+2)(3x^2-x-1)$.

[NOTE.—Beginners may omit Arts. 226, 227.]

226.—CASE XVI. *A binomial of the form* $a^n \pm b^n$, *where* n *is divisible by two or more odd numbers which are prime to each other, may be decomposed into at least four factors.* Ex. $a^{15} \pm b^{15}$; $a^{21} \pm b^{21}$; $a^{15} \pm b^{30}$; $a^{30}+1$; etc.

The solution of one of these examples will illustrate the method for all; hence we shall explain the method of decomposing $a^{15}+1$ into four factors.

By Case XII, $a^{15}+1=(a^5+1)(a^{10}-a^5+1)=$
$(a+1)(a^4-a^3+a^2-a+1)(a^{10}-a^5+1)$.

But $a^{15}+1=(a^3+1)(a^{12}-a^9+a^6-a^3+1)=$
$(a+1)(a^2-a+1)(a^{12}-a^9+a^6-a^3+1)$.

It is evident that a number can not be made up of two different sets of prime factors; hence one factor at least in each of the above sets is composite. In the first set $a^{10}-a^5+1$ is the only factor which can be composite; therefore $a^{10}-a^5+1$ is divisible by a^2-a+1, the prime factor of the second set not found in the first set.

$(a^{10}-a^5+1)\div(a^2-a+1)=a^8+a^7-a^5-a^4-a^3+a+1$.

$\therefore\ a^{15}+1=(a+1)(a^4-a^3+a^2-a+1)(a^2-a+1)$
$(a^8+a^7-a^5-a^4-a^3+a+1)$.

It is clear that the same result would be obtained by dividing the composite factor of the second set, $a^{12}-a^9+$

$a^6 - a^3 + 1$, by the prime factor of the first set not found in the second; viz., $a^4 - a^3 + a^2 - a + 1$.

227. Certain other expressions are factored in like manner. Thus: $a^{12} + 64 = (a^4 + 4)(a^8 - 4a^4 + 16)$. XII.

But by Case X, $a^4 + 4 = (a^2 + 2a + 2)(a^2 - 2a + 2)$.

$\therefore\ a^{12} + 64 = (a^2 + 2a + 2)(a^2 - 2a + 2)(a^8 - 4a^4 + 16)$.

But $a^{12} + 64 = (a^6 + 4a^3 + 8)(a^6 - 4a^3 + 8)$. X.

Hence the factors of the second set must be divisible by the prime factors of the first set. By trial we find

$$a^6 + 4a^3 + 8 = (a^2 - 2a + 2)(a^4 + 2a^3 + 2a^2 + 4a + 4),$$

and $a^6 - 4a^3 + 8 = (a^2 + 2a + 2)(a^4 - 2a^3 + 2a^2 - 4a + 4)$.

228. CASE XVII. *When the parts of a polynomial are factored by previous cases, and these parts have a common factor, the expression is factored by using Cases II and III in connection with the cases used in factoring the parts.*

1. $a^2 - b^2 - a + b = (a^2 - b^2) - (a - b) = (a - b)[(a + b) - 1]$. VI.

2. $a^3 + b^3 - a^2 + ab - b^2 = (a^3 + b^3) - (a^2 - ab + b^2) = (a^2 - ab + b^2)(a + b - 1)$. XII.

3. $a^2 - 2ab + b^2 - 3a + 3b = (a - b)^2 - 3(a - b) = (a - b)(a - b - 3)$. IV.

4. $a^2 - 5ab + 6b^2 - 2a + 4b = (a^2 - 5ab + 6b^2) - 2(a - 2b) = (a - 2b)(a - 3b) - 2(a - 2b) = (a - 2b)(a - 3b - 2)$. VIII.

5. $a^4 - 2a^2 + 4ab - 4b^2 + 4b^4 = (a^4 + 4b^4) - 2(a^2 - 2ab + 2b^2) = (a^2 + 2ab + 2b^2)(a^2 - 2ab + 2b^2) - 2(a^2 - 2ab + 2b^2) = (a^2 - 2ab + 2b^2)(a^2 + 2ab + 2b^2 - 2)$. X.

6. $a^2 - ab - a - 2b^2 - 7b - 6 = a^2 - a(b + 1) - (2b^2 + 7b + 6)$.

The two numbers whose sum is $-(b + 1)$, and whose product is $-(2b^2 + 7b + 6)$ are $-(2b + 3)$ and $(b + 2)$. (VIII and IX.) *Ans.* $(a - 2b - 3)(a + b + 2)$.

7. $5a^2 - 8ab + 7a + 3b^2 - 5b + 2 =$
$5a^2 - a(8b - 7) + (3b^2 - 5b + 2)$.

Multiply by 5, and substitute y for $5a$. (Art. 204.) $\therefore\ y^2 - (8b - 7)y + 5(3b^2 - 5b + 2)$ results. Now factor $5(3b^2 - 5b + 2)$ (Art. 205) into $5(3b - 2)(b - 1)$.

Since $(3b - 2) + 5(b - 1) = (8b - 7)$, $\therefore\ y^2 - (8b - 7)y + 5(3b - 2)(b - 1) = (y - 3b + 2)(y - 5b + 5) = (5a - 3b + 2)(5a - 5b + 5)$. Now divide by 5.

Ans. $(5a - 3b + 2)(a - b + 1)$.

8. $a^{12} + a^4 + 10 = (a^{12} + 8) + (a^4 + 2) =$
$(a^4 + 2)[(a^8 - 2a^4 + 5)]$. XII.

9. $x^4 + 2x^2y^2 + 9y^4 - 2x^2 + 4xy - 6y^2 =$
$(x^4 + 2x^2y^2 + 9y^4) - 2(x^2 - 2xy + 3y^2) =$
$(x^2 - 2xy + 3y^2)(x^2 + 2xy + 3y^2) - 2(x^2 - 2xy + 3y^2) =$
$(x^2 - 2xy + 3y^2)(x^2 + 2xy + 3y^2 - 2)$. X.

10. $a^3 - 3a^2b + 3ab^2 - b^3 - 3a^2 + 6ab - 3b^2 =$
$(a-b)^3 - 3(a-b)^2 = (a-b)^2(a-b-3)$. XIII and IV.

11. $a^9 + a^3 + 2 = (a^9 + 1) + (a^3 + 1) = (a^3 + 1)(a^6 - a^3 + 2) =$
$(a + 1)(a^2 - a + 1)(a^6 - a^3 + 2)$. XII.

12. $a^9 + a^3 - 10 = (a^9 - 8) + (a^3 - 2) = (a^3 - 2)(a^6 + a^3 + 5)$. XI.

GENERAL REMARKS ON FACTORING.

229. In order to become proficient in factoring, it is necessary, first, to learn the formulas; and secondly, to be able to apply the proper formula to the proposed expression.

This power can be acquired only by practice and study, but the following general rules may help the student to become expert in this difficult branch.

Before applying any of the following rules, it is necessary to remove from the proposed expression all monomial factors common to the terms (193, 196), and to arrange the result.

230.—RULE A. *Binomials.*

First notice whether the proposed binomial be the difference of two squares; if so, employ Case VI.

If the proposed binomial be not the difference of two squares, see if its terms be both third powers, or both fifth powers, or both seventh powers, etc.; if so, Case XI or Case XII will apply. If VI, XI, and XII fail, try Art. 210.

231.—Rule B. *Trinomials.*

First notice if the proposed trinomial be a square (197). If it be so, factor according to Case IV.

If the trinomial be not a square, then, if the first term be a square, try VIII; and if the first term be not a square, try IX. If two of the terms be squares, and twice the product of their square roots exceed the middle term by a square, then X will apply. Instead of IV, VIII, and IX, XIV may be employed, and instead of X we may use XV; but XIV and XV may frequently be used where the others fail. Finally, in some examples, III and XVII will apply.

232.—Rule C. *Expressions having four or more terms.*

Try the various cases in the following order: III; V; VII; XIII; XIV; XVII; XV.

233.—Rule D. After resolving an expression into factors, each factor must be examined for the purpose of determining if it be prime (195).

234. It is important to notice that the signs of an even number of factors may be changed (116). Thus:

$$a^2 - b^2 = (a - b)(a + b) = (-a + b)(-a - b);$$
$$x^2 - x - 6 = (x - 3)(x + 2) = (-x + 3)(-x - 2); \text{ etc.}$$

It is usual to give the factors whose first terms are +.

235. Formulas to be memorized:

1. $a^2 - b^2 = (a + b)(a - b)$.
2. $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$.
3. $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$.
4. $a^2 + 2ab + b^2 = (a + b)(a + b)$.
5. $a^2 - 2ab + b^2 = (a - b)(a - b)$.

6. $x^2 + (a + b)x + ab = (x + a)(x + b)$.
7. $x^2 - (a + b)x + ab = (x - a)(x - b)$.
8. $x^2 + (a - b)x - ab = (x + a)(x - b)$.
9. $x^4 + x^2y^2 + y^4 = (x^2 + xy + y^2)(x^2 - xy + y^2)$.
10. $x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - xz - yz)$.

Exercise XIII.

Resolve the following expressions into prime factors:

1. $a^5 - 8a^4 + 3a^3$.
2. $(a - b)^5 - 3(a - b)^2$.
3. $x^4 - x^3 + x^2 - x$.
4. $a^6 + a^5 + a^4 + a^3 + a^2 + a$.
5. $(a - b)^4 - (b - a)^3 + (a - b)^2$.
6. $a^4 - a^2 + \frac{1}{4}$.
7. $a^5 - 7a^3 + \frac{49}{4}a$.
8. $a^8 - 9$.
9. $16a^4 - 9$.
10. $(a - b - c)^2 - d^2$.
11. $a^2 - 2ab + b^2 - 6a + 6b + 9$.
12. $a^2 - 4ab + 4b^2 + 2a - 4b - 8$.
13. $a^2 - 4a + 4 - b^2 - 2bc - c^2$.
14. $a^2 + 4a - b^2 + 6b - 5$.
15. $a^2 - 14a + 13$.
16. $x^2 - 10xy + 16y^2$.
17. $x^5 - 2x^3 + x$.
18. $x^4 - 9x^2 + 8$.
19. $9x^2 + 3x - 2$.
20. $4x^2 - 8x + 3$.
21. $6x^2 - 5x + 1$.
22. $6x^2 - x - 1$.
23. $x^4 - 3x^2 + 1$.
24. $4x^4 - 5x^2 + 1$.
25. $9x^4 - 16x^2 + 4$.
26. $a^{12} - 8$.
27. $a^{12} - 1$.
28. $a^{24} + 1$.
29. $a^{12} + 27$.
30. $a^3 - 6a^2 + 12a - 9$.
31. $a^4 - 4a^3 - 3a^2 - 4a + 1$.
32. $2x^4 - x^3 - 12x$.
33. $x^3 - x - 6$.
34. $x^5 + x + 2$.

CHAPTER XIV.

COMMON FACTORS AND MULTIPLES.

236. A **Common Factor** of two or more expressions is an expression which is contained in each of them without a remainder. Thus: $4a$ is a common factor of $12a$ and $20a$; $x + y$ is a common factor of $x^2 - y^2$ and $(x + y)^2$.

237. Two expressions which have no common integral factor except 1 are said to be *prime to each other*.

238. The **Highest Common Factor** (H. C. F.) of two or more expressions consists of all the prime factors common to the expressions.

Thus: $5a^2$ is the H. C. F. of $10a^2$, $15a^3$, and $25a^4$.

The Highest Common Factor is also called the Greatest Common Divisor (G. C. D.); the Greatest Common Measure (G. C. M.); the Highest Common Divisor (H. C. D.); and the Highest Common Measure (H. C. M.).

239. If two or more expressions be divided by their Highest Common Factor, the quotients will have no common integral factor except 1.

240. To find the Highest Common Factor by means of factoring:

(1) *Resolve each expression into its prime factors.*

(2) *Select from these the lowest power of each common factor, and find the product of these powers.*

1. Find the H. C. F. of $24a^4b^5c^2$ and $36a^2bc^3$.

$$24a^4b^5c^2 = 2^3.3^1.a^4.b^5.c^2;\quad 36a^2b^1c^3 = 2^2.3^2.a^2.b^1.c^3.$$

$\therefore$ the H. C. F. $= 2^2 \times 3^1 \times a^2 \times b^1 \times c^2 = 12a^2bc^2$.

2. Find the H. C. F. of $4x^3 - 4xy^2$ and $6x^4 + 6xy^3$.

$$4x^3 - 4xy^2 = 2^2x(x^2 - y^2) = 2^2.x.(x+y)(x-y);$$
$$6x^4 + 6xy^3 = 2.3.x.(x^3 + y^3) = 2.3.x.(x+y)(x^2 - xy + y^2).$$

$\therefore$ the H. C. F. $= 2^1.x^1.(x+y)^1 = 2x(x+y) = 2x^2 + 2xy$.

3. Find the H. C. F. of $a^3 - a$; $a^2 + 2a + 1$; and $4a^3 + 4$.

$$a^3 - a = a.(a^2 - 1) = a.(a+1)(a-1).$$
$$4(a^3 + 1) = 2^2(a+1)(a^2 - a + 1).$$

$a^2 + 2a + 1 = (a+1)^2$; $\therefore$ the H. C. F. $= a + 1$.

4. Find the H.C.F. of $x^2-(a+b)x+ab$; $x^2-(a+c)x+ac$; and $x^2 - (a-b)x - ab$.

$$x^2-(a+b)x+ab=(x-a)(x-b).$$
$$x^2-(a+c)x+ac=(x-a)(x-c).$$

$x^2-(a-b)x-ab=(x-a)(x+b)$. $\therefore$ the H.C.F. $= x - a$.

EXAMPLES.

Find the H. C. F. of the following expressions:

5. $8a^6b$; $6a^2b^3$. *Ans.* $2a^2b$.

6. $a^2 - b^2$; $a^2 - 3ab + 2b^2$. *Ans.* $a - b$.

7. $a^3 - 8$; $a^2 - 4a + 4$. *Ans.* $a - 2$.

8. $a^3 + 27$; $a^2 + 2a - 3$. *Ans.* $a + 3$.

9. $a^5 - a^2$; $a^3 - 5a^2 + 4a$. *Ans.* $a^2 - a$.

10. $x^3 - 1$; $x^4 + x^2 + 1$. *Ans.* $x^2 + x + 1$.

11. $x^3 + 8$; $x^4 + 4x^2 + 16$. *Ans.* $x^2 - 2x + 4$.

12. $a^9 + 1$; $a^9 - 1$. *Ans.* 1.

241. The method of finding the H. C. F. of two or more expressions by division is similar to the corresponding case in Arithmetic, and depends upon the following principles:

242. *A factor of an expression is a factor of any multiple of that expression.* Thus: a divides ab, and a also divides cab.

243. *A common factor of two expressions is a factor of the sum or difference of any multiples of the expressions.* Thus: a divides ab and ac, and a also divides $mab \pm rac$.

244. *A common factor of two expressions is a factor of their remainder after division.*

This is merely a special case of the preceding principle, because the remainder after division is found by subtracting from the dividend the product of the divisor and quotient. Let the dividend be ab and the divisor be ac, and let the quotient be q; then $ab - acq$ is the remainder, which evidently contains a, the common divisor of ab and ac.

245. *The H. C. F. of two or more expressions will not be altered if they be multiplied by expressions which are prime to each other.*

This follows from (238), since introducing factors which are prime to each other can not change the common factors.

246. *Factors which are not part of the H. C. F. may be removed from the proposed expressions without affecting the H. C. F.* (238). The H. C. F. of 12, 36, and 27, is 3; if 4 be removed from 12 and 36, the resulting numbers, 3, 9, and 27, have also 3 for their H. C. F.

247. *If a common factor of two or more expressions be removed, their H. C. F. is found from multiplying the factor removed by the H. C. F. of the quotients.*

To find the H. C. F. of $5a^2(a^3 - b^3)$ and $3a(a^2 - b^2)$. The factor $5a^2$ may be removed from the first expression, and $3a$ from the second, and their common factor, a, is part of the H. C. F.; but the other factors may be discarded. The H. C. F. of the quotients, $a^3 - b^3$ and $a^2 - b^2$, is $a - b$; therefore, the H. C. F. of the proposed expressions is $a(a - b) = a^2 - ab$.

248. The solutions of the following examples illustrate the method of finding the H. C. F. of two expressions by division (Common and Synthetic). Before beginning the operation of division, remove all monomial factors from the polynomials (246, 247). Then arrange the proposed expressions according to the descending powers of some letter, and use the expression containing the highest power, as the dividend.

1. Find the H. C. F. of $4x^4 + 6x^3 + x - 2x^2 - 1$ and $4x^3 + 2x^2 - 1$.

(1)
$$\begin{array}{l|l}
4x^4 + 6x^3 - 2x^2 + x - 1 & 4x^3 + 2x^2 + 0 - 1 \\
4x^4 + 2x^3 + 0 \quad - x & x + 1 \\
\hline
+ 4x^3 - 2x^2 + 2x - 1 & \\
\quad 4x^3 + 2x^2 + 0 - 1 & \\
\hline
- 4x^2 + 2x = - 2x\,(2x - 1). &
\end{array}$$

$$\begin{array}{l|l}
4x^3 + 2x^2 + 0 - 1 & 2x - 1 \\
4x^3 - 2x^2 & 2x^2 + 2x + 1 \\
\hline
+ 4x^2 + 0 - 1 & \\
+ 4x^2 - 2x & \\
\hline
+ 2x - 1 & \\
+ 2x - 1 \quad \therefore\ 2x - 1 = \text{H. C. F.} &
\end{array}$$

(2)
$$\begin{array}{r|r|l}
4 & 4 + 6 & - 2 + 1 - 1 \\
\hline
- 2 & - 2 & - 0 + 1 \\
- 0 & & - 2 - 0 + 1 \\
+ 1 & & \\
\hline
 & 1 + 1 & - 4 + 2 = - 2\,(2 - 1)
\end{array}
\qquad
\begin{array}{r|l}
2 & 4 + 2 + 0 - 1 \\
\hline
+ 1 & \quad + 2 + 2 + 1 \\
\hline
 & 2 + 2 + 1
\end{array}$$

2. Find the H. C. F. of $30x^4 - 64x^3 - 17x^2 + 49x + 2$ and $10x^3 - 23x^2 - x + 14$.

(1)
$$\begin{array}{l|l}
30x^4 - 64x^3 - 17x^2 + 49x + 2 & 10x^3 - 23x^2 - x + 14 \\
30x^4 - 69x^3 - \quad 3x^2 + 42x & 3x + \frac{1}{2} \\
\hline
+ \quad 5x^3 - 14x^2 + \quad 7x + 2 & \\
\quad 5x^3 - 11\frac{1}{2}x^2 - \quad \frac{1}{2}x + 7 & \\
\hline
- \quad 2\frac{1}{2}x^2 + 7\frac{1}{2}x - 5 = - 2\frac{1}{2}\,(x^2 - 3x + 2). &
\end{array}$$

$$\begin{array}{l|l} 10x^3 - 23x^2 - \quad x + 14 & x^2 - 3x + 2 \\ \cline{2-2} 10x^3 - 30x^2 + 20x & 10x + 7 \\ \cline{1-1} \qquad 7x^2 - 21x + 14 & \\ \qquad 7x^2 - 21x + 14 & \end{array} \qquad \text{H. C. F.} = x^2 - 3x + 2.$$

(2)

$$\begin{array}{r|r|l} 10 & 30 - 64 & -17 \ +49 + 2 \\ \hline +23 & +69 & +\ 3 \ -42 \\ +\ 1 & & +11\tfrac{1}{2} + \tfrac{1}{2} - 7 \\ -14 & & \\ \hline & 3 + \tfrac{1}{2} & -\ 2\tfrac{1}{2} + 7\tfrac{1}{2} - 5 = \\ & & -\ 2\tfrac{1}{2}(1 - 3 + 2) \end{array} \qquad \begin{array}{r|l} 1 & 10 - 23 - \ 1 + 14 \\ \hline +3 & \quad +30 - 20 \\ -2 & \qquad\quad +21 - 14 \\ \hline & 10 + \ 7 \end{array}$$

3. Find the H. C. F. of $2x^3 + (2a - 9)x^2 - (9a + 6)x + 27$ and $2x^2 - 13x + 18$.

(1)

$$\begin{array}{l|l} 2x^3 + (2a - 9)x^2 - \ (9a + \ 6)x + 27 & 2x^2 - 13x + 18 \\ \cline{2-2} 2x^3 - 13x^2 \qquad + \qquad 18x & x + (a + 2) \\ \cline{1-1} \quad + (2a + 4)x^2 - \ (9a + 24)x + 27 & \\ \quad + (2a + 4)x^2 - (13a + 26)x + (18a + 36) & \\ \cline{1-1} \qquad\qquad + \ (4a + \ 2)x - (18a + \ 9) = & \\ \quad 2(2a + 1)x - 9(2a + \ 1) = (2a + 1)(2x - 9) & \end{array}$$

$$\frac{2x^2 - 13x + 18}{2x - 9} = x - 2.$$

Ans. $2x - 9$.

(2)

$$\begin{array}{r|l|l} 2 & 2 + (2a - 9) & - \ (9a + 6) \ + 27 \\ \hline +13 & + \quad 13 & - \qquad 18 \\ -18 & & + (13a + 26) - (18a + 36) \\ \hline & 1 + \ (a + 2) & + \ (4a + 2) \ - (18a + 9) = \\ & & \quad 2(2a + 1) \ - 9(2a + 1) = \\ & & \quad (2a + 1) \ (2 - 9) \end{array}$$

$$\begin{array}{r|l} 2 & 2 - 13 + 18 \\ \hline +\ 9 & \ + \ 9 - 18 \\ \hline & 1 - \ 2 \end{array}$$

4. Find the H. C. F. of $x^4 - ax^3 - a^2x^2 - a^3x - 2a^4$ and $3x^3 - 7ax^2 + 3a^2x - 2a^3$.

Since the first term of the dividend is not divisible by the first term of the divisor, in order to avoid fractions, multiply the dividend by such a number as will render the first term divisible. The least multiplier is 3; but, since the division will be carried out to two terms in the quotient, it is preferable to multiply by $3^2 = 9$. If there were three terms in the quotient, multiply by $3^3 = 27$, etc.

(1)
$$\begin{array}{l|l}
9x^4 - 9ax^3 - 9a^2x^2 - 9a^3x - 18a^4 & 3x^3 - 7ax^2 + 3a^2x - 2a^3 \\
\underline{9x^4 - 21ax^3 + 9a^2x^2 - 6a^3x} & 3x + 4a \\
\quad +12ax^3 - 18a^2x^2 - 3a^3x - 18a^4 & \\
\quad \underline{+12ax^3 - 28a^2x^2 + 12a^3x - 8a^4} & \\
\qquad\qquad +10a^2x^2 - 15a^3x - 10a^4 = 5a^2(2x^2 - 3ax - 2a^2). &
\end{array}$$

For the same reason as before, we multiply the new dividend by $4 = 2^2$.

$$\begin{array}{l|l}
12x^3 - 28ax^2 + 12a^2x - 8a^3 & 2x^2 - 3ax - 2a^2 \\
\underline{12x^3 - 18ax^2 - 12a^2x} & 6x - 5a \\
\quad -10ax^2 + 24a^2x - 8a^3 & \\
\quad \underline{-10ax^2 + 15a^2x + 10a^3} & \\
\qquad\qquad + 9a^2x - 18a^3 = 9a^2(x - 2a) &
\end{array}$$

$(2x^2 - 3ax - 2a^2) \div (x - 2a) = 2x + a$. *Ans.* $x - 2a$.

(2)
$$\begin{array}{r|r|l}
3 & 9 - 9 & -9 - 9 - 18 \\
+7 & +21 & -9 + 6 \\
-3 & & +28 - 12 + 8 \\
+2 & & \\
\hline
 & 3 + 4 & +10 - 15 - 10 = \\
 & & 5(2 - 3 - 2).
\end{array}$$

Multiply previous divisor by 4 for new dividend.

$$\begin{array}{r|r|l}
2 & 12 - 28 & +12 - 8 \\
+3 & +18 & +12 \\
+2 & & -15 - 10 \\
\hline
 & 6 - 5 & 9 - 18 = \\
 & & 9(1 - 2).
\end{array}$$

$$\begin{array}{r|l}
1 & 2 - 3 - 2 \\
+2 & \quad + 4 + 2 \\
\hline
 & 2 + 1
\end{array}$$

5. Find the H. C. F. of $a^4 - ba^3 + (c-1)a^2 + ba - c$ and $a^4 - ca^3 + (b-1)a^2 + ca - b$.

In this case it is not necessary to go through the form of dividing one expression by the other, because the quotient will be unity. Therefore subtract one expression from the other, use the remainder as the divisor, and one of the proposed expressions as the dividend. From the remainder $(c-b)a^3 + (c-b)a^2 - (c-b)a - (c-b)$, remove the factor $(c-b)$, which can not be part of the H. C. F.

$$\begin{array}{llll}
(1) & a^4 - ca^3 + (b-1)a^2 + ca - b & | & a^3 + a^2 - a - 1 \\
 & a^4 + a^3 - a^2 - a & & a - (c+1) \\
 & -(c+1)a^3 + ba^2 + (c+1)a - b & & \\
 & -(c+1)a^3 - (c+1)a^2 + (c+1)a + (c+1) & & \\
 & +(b+c+1)a^2 + 0 - (b+c+1) = & & \\
 & (b+c+1)(a^2 + 0 - 1). & &
\end{array}$$

$$\begin{array}{lll}
a^3 + a^2 - a - 1 & | & a^2 + 0 - 1 \\
a^3 + 0 - a & & a + 1 \\
+a^2 + 0 - 1 & & \\
+a^2 + 0 - 1 & &
\end{array}$$

$\therefore \quad a^2 - 1 = \text{H. C. F.}$

$$\begin{array}{r|l|l}
(2) \quad 1 & 1 - c & +(b-1) + c \\
-1 & \quad -1 & + \quad 1 \quad +1 \qquad -b \\
+1 & & +(c+1) - (c+1) - (c+1) \\
+1 & & \\
\hline
 & 1 - (c+1) & +(b+c+1) + 0 - (b+c+1) = \\
 & & \quad (b+c+1)(1 + 0 - 1)
\end{array}$$

$$\begin{array}{r|l}
1 & 1 + 1 - 1 - 1 \\
-0 & \quad -0 + 1 \\
+1 & \qquad -0 + 1 \\
\hline
 & 1 + 1
\end{array}$$

249. Rule for finding the H. C. F. of two polynomials.

(1) *Remove all monomial factors from the proposed expressions, and set aside the H. C. F. of these monomials as a factor of the H. C. F. sought.*

(2) *Arrange the resulting polynomials according to the descending powers of a common letter: take for the dividend that expression which has the highest power of the letter of arrangement; or, if both have the same power, take either for the dividend.*

(3) *Divide as in division of polynomials, until a remainder is obtained which is of lower degree than the divisor: if the first term of any dividend be not divisible by the first term of the divisor, multiply that dividend by such an expression as will make it thus divisible.*

(4) *Remove from the remainder every factor common to all the terms. Take the resulting polynomial as a divisor, and the divisor as a dividend, and proceed as before.*

(5) *So continue, always dividing the last divisor by the last remainder, until nothing remains.*

(6) *The product of the final divisor, and the H. C. F. of the monomials set aside in the first place, is the H. C. F. sought.*

Verification: *Divide each of the proposed expressions by the answer found. If the division be exact, and the quotients have no factor in common, the correct answer has been found.*

250. To find the H. C. F. of three expressions: *First find the H. C. F. of two of them, and then of that result and the third expression.* For the H. C. F. of two of the expressions contains all the factors common to those two, and the H. C. F. of this result and the third expression contains such of those factors as are found in the third.

Of four expressions: *First find the H. C. F. of three of them, and then of that result and the fourth. Proceed in like manner if there be more than four expressions.*

EXAMPLES.

Find the H. C. F. of the following expressions:

6. $5x^2 - 15x - 50;\ x^2 - 6x + 5.$ *Ans.* $x - 5.$

7. $2x^2 - 4x - 16;\ 2x^2 - 10x + 8.$ *Ans.* $2x - 8.$

8. $2x^3 + 9x^2 + 7x - 3;\ 3x^3 + 5x^2 - 15x + 4.$ *Ans.* $x^2 + 3x - 1.$

9. $x^4 + 3x^3 - x^2 - 8x + 5;\ x^4 + 2x^3 - 5x^2 - 10x + 12.$ *Ans.* $x - 1.$

10. $3x^4 + 10x^3 - x^2 + 4x + 9;\ x^4 + 3x^3 - x^2 + 2x + 2.$ *Ans.* $x^2 - x + 1.$

11. $4x^4 - 8x^3 - 20x^2 + 24x + 20;\ 3x^3 + 6x^2 - 24x - 45.$ *Ans.* $x^2 - x - 5.$

12. $x^5 + 2x^4 - x^3 + x^2 - 3;\ x^4 - x^3 + 2x^2 + x - 3.$ *Ans.* $x^2 - 1.$

13. $x^4 - 3x^3 + 7x^2 - 7x - 6;\ x^4 + x^3 - 8x^2 + 7x - 6.$ *Ans.* $x - 2.$

14. $x^4 - 5x^3 + 8x^2 - 7x + 3;\ 2x^3 - 9x^2 + 10x - 3.$ *Ans.* $x^2 - 4x + 3.$

15. $2x^4 - 3x^3 + 3x^2 - 3x + 1;\ 6x^3 - 7x^2 + 4x - 1.$ *Ans.* $2x - 1.$

16. $2x^2 + 3x - 2;\ 2x^3 + 3x^2 + 4x - 3;\ 4x^3 + 2x^2 - 1.$ *Ans.* $2x - 1.$

17. $x^3 + x^2 - 3x - 2;\ x^3 - 3x^2 + x + 2;\ x^4 - x^3 - 5x^2 + 4x + 4.$ *Ans.* $x^2 - x - 1.$

251. A Common Multiple of two or more expressions is an expression which is exactly divisible by each of them.

252. The Lowest Common Multiple, or the **Least Common Multiple** (L. C. M.), of two or more expressions, is an expression which is exactly divisible by each of them, and the quotients have no integral factor in common except unity.

253. Every common multiple of two or more expressions must contain all the factors of the proposed expressions, and may contain other factors also; but the L. C. M.

consists of the factors of the proposed expressions only, each factor being written with its highest exponent. Thus:

1. Find the L. C. M. of $8a^3bc$; $12a^2b^2c$; $36ab^3c^2$.

$8a^3bc = 2^3a^3bc$; $12a^2b^2c = 2^2 . 3a^2b^2c$; $36ab^3c^2 = 2^2 . 3^2ab^3c^2$.

$\therefore$ the L. C. M. $= 2^3 \times 3^2 . a^3b^3c^2 = 72a^3b^3c^2$.

2. Find the L. C. M. of $x^2 - 2x + 1$; $4x^2 - 4$; x^2+2x+1.

$$x^2 - 2x + 1 = (x-1)^2;\ x^2 + 2x + 1 = (x+1)^2;$$
$$4x^2 - 4 = 2^2(x-1)(x+1).$$

$\therefore$ *Ans.* $4(x-1)^2(x+1)^2$.

254. The L. C. M. of two expressions may also be found by means of their H. C. F. For example, the L. C. M. of ab and ac is abc. This is evidently obtained from multiplying ab by c, or ac by b.

Hence: *Divide one of the two proposed expressions by their H. C. F., and multiply the other expression by the quotient. The product is their L. C. M.*

3. Find the L. C. M. of $x^2 - (a-b)x - ab$ and $x^2 - (a+c)x + ac$.

(1) The H. C. F. is $x - a$.

(2) The quotient of $x^2 - (a-b)x - ab$ by $x - a$ is $x+b$.

$\therefore$(3) The L. C. M. $= [x^2 - (a+c)x + ac][x+b]$.

4. Find the L. C. M. of x^3+x^2-4x-4 and $x^3+6x^2+11x+6$.

(1) The H. C. F. is $x^2 + 3x + 2$.

(2) The quotient of x^3+x^2-4x-4 by x^2+3x+2 is $x-2$.

$\therefore$(3) The L. C. M. $= (x^3 + 6x^2 + 11x + 6)(x-2)$.

255. To find the L. C. M. of three expressions: *First find the L. C. M. of two of the proposed expressions; then find the L. C. M. of this result and the third expression.*

5. Find the L. C. M. of $x^2 - x - 2$; $x^2 + x - 6$; $x^2 - 4$.

(1) The H. C. F. of $x^2 - x - 2$ and $x^2 + x - 6$ is $x - 2$.

(2) The quotient of $x^2 + x - 6$ by $x - 2$ is $x + 3$.

$\therefore$ (3) The L. C. M. of the first and second expressions is $(x^2 - x - 2)(x + 3)$.

(4) The H. C. F. of this L. C. M. and $x^2 - 4$ is $x - 2$.

(5) The quotient of $x^2 - 4$ by $x - 2$ is $x + 2$.

$\therefore$ (6) The required L. C. M. is $(x^2 - x - 2)(x + 3)(x + 2)$.

256. To find the L. C. M. of four or more expressions: *Find the L. C. M. of three of them, and then the L. C. M. of this result and the fourth. Proceed in like manner if there be more than four expressions.*

257. The operation of finding the L. C. M. may be abbreviated when certain of the proposed expressions are divisors of other given expressions, by omitting such divisors. For: *A multiple of an expression is a multiple of every divisor of that expression.*

6. Find the L. C. M. of $x^4 - y^4$; $x^2 + xy + y^2$; $x^2 + y^2$; $x + y$; $x - y$; $x^3 - y^3$.

Since $x^2 + xy + y^2$ is a divisor of $x^3 - y^3$, and since $x^2 + y^2$, $x + y$, and $x - y$ are divisors of $x^4 - y^4$, therefore the L. C. M. of $x^3 - y^3$ and $x^4 - y^4$ contains all the proposed expressions. *Ans.* $(x^4 - y^4)(x^2 + xy + y^2)$.

EXAMPLES.

Find the L. C. M. of the following expressions:

7. $x^2 - x$; $x^2 - 4x + 3$; $x^2 - 3x + 2$; $x^2 - 5x + 6$.
Ans. $x^4 - 6x^3 + 11x^2 - 6x$.

8. $6x^2 + 7x - 3$; $3x^2 + 5x - 2$; $2x^2 + 7x + 6$.
Ans. $6x^3 + 19x^2 + 11x - 6$.

9. $6x^2 - 5x + 1$; $2x^2 - 5x + 2$; $3x^2 - 7x + 2$.
Ans. $6x^3 - 17x^2 + 11x - 2$.

10. $1 - x^3$; $1 + x + x^2$; $1 - x + x^2 - x^3$.
Ans. $1 + x^2 - x^3 - x^5$.

11. $x^4 - 5x^3 + 8x^2 - 7x + 3$; $2x^3 - 9x^2 + 10x - 3$.
Ans. $2x^5 - 11x^4 + 21x^3 - 22x^2 + 13x - 3$.

12. $x^4 - 4x^3 + 3x^2 + 4x - 12$; $2x^3 - x^2 - 18x + 9$.
Ans. $2x^6 - 3x^5 - 17x^4 + 35x^3 - 13x^2 - 72x + 36$.

13. $x^2 - 4$; $x^2 - 5x + 6$; $x^2 - 2x - 8$; $x^2 - 7x + 12$.
Ans. $x^4 - 7x^3 + 8x^2 + 28x - 48$.

Exercise XIV.

Find the H. C. F. of the following expressions:

1. $x^4 + 2x^2y^2 + y^4$; $x^4 + 3x^2y^2 + 2y^4$.
2. $x^4 + x^2y^2 + y^4$; $x^4 + x^3y - xy^3 - y^4$.
3. $x^3 - 5x^2y + 13xy^2 - 9y^3$; $x^3 - 2x^2y + 4xy^2 - 3y^3$.
4. $x^4 + 2x^2y^2 + 9y^4$; $7x^3 - 11x^2y + 15xy^2 + 9y^3$.
5. $9x^4 - 3xy^3 + 3x^2y^2 - 9x^3y$; $2y^3 - 10xy^2 + 8x^2y$.
6. $3x^3 - 22x - 15$; $5x^4 + x^3 - 54x^2 + 18x$.
7. $x^3 - 8x + 3$; $x^6 + 3x^5 + x + 3$.
8. $-5x^3 - 2x^2 + 15x + 6$; $-7x^3 + 4x^2 + 21x - 12$.
9. $20x^4 + x^2 - 1$; $25x^4 + 5x^3 - x - 1$.
10. $-x^4 + 2x^3 - x^2 + 8x - 8$; $4x^3 - 12x^2 + 9x - 1$.

Find the L. C. M. of the following expressions:

11. $x^3 + 1$; $x^2 - x - 2$; $x^4 + x^2 + 1$.
12. $x^3 - x$; $x^3 - 1$; $x^3 + 1$; $x^4 + x^2 + 1$.
13. $x^2 - 4$; $x^3 + 2x^2 + 4x + 8$; $x^3 - 2x^2 + 4x - 8$.
14. $x^4 - 3x^3 + 2x^2 + x - 1$; $x^3 - x^2 - 2x + 2$.
15. $4x^4 + 9x^3 + 2x^2 - 2x - 4$; $3x^3 + 5x^2 - x + 2$.
16. $x^3 - 1$; $3x^2 - 5x + 2$; $4x^3 - 4x^2 - x + 1$.
17. $6x^4 + x^3 - x$; $4x^3 - 6x^2 - 4x + 3$.
18. $12x^2 - 15xy + 3y^2$; $6x^3 - 6x^2y + 2xy^2 - 2y^3$.

CHAPTER XV.

FRACTIONS.

258. The expression $\frac{a}{b}$ is employed to indicate that one unit is divided into b equal parts, and that a of these parts are taken, or that a units are divided into b equal parts, and that one of these parts is taken.

259. The expression $\frac{a}{b}$ is called a *fraction*. a is called the *numerator*, because it *numbers* the parts. b is called the *denominator*, because it *names* the parts. The numerator and denominator are called the *terms of the fraction*.

260. Every integer may be considered as a fraction whose denominator is unity; thus, $a = \frac{a}{1}$.

261. To multiply a fraction by an integer, we may either multiply the numerator or divide the denominator by it; and, conversely, to divide a fraction by an integer, we may either divide the numerator or multiply the denominator by it.

For, to multiply the numerator multiplies the number of parts taken without affecting their size; hence the fraction is multiplied.

To divide the denominator increases the size of the parts, without affecting the number of parts taken; hence the fraction is multiplied.

Again, to divide the numerator divides the number of parts taken without affecting their size; hence the fraction is divided.

To multiply the denominator decreases the size of the parts without affecting the number taken; hence the fraction is divided.

262. As a result of the preceding article: *If both terms of a fraction be multiplied by the same number, or if both be divided by the same number, the value of the fraction will remain unaltered.* Thus: $\frac{a}{b}=\frac{ac}{bc}=\frac{a^2c}{abc}$, etc.

263. Since $a=\frac{a}{1}$ (Art. 260), $\therefore\ a\div b=\frac{a}{1}\div b=\frac{a}{b}$ (Art. 261). That is, a fraction represents the quotient of the numerator by the denominator (24).

REDUCTION OF FRACTIONS.

264. To reduce a fraction, multiply or divide both terms by the same number (262).

265. CASE I. To reduce a fraction to *lower* terms:
Divide both terms by any common factor.

Thus: $\frac{a^4bc^3}{a^7b^3c}=\frac{a^2bc^2}{a^5b^3}=\frac{ac^2}{a^4b^2}=\frac{c^2}{a^3b^2}$.

To reduce a fraction to its *lowest* terms:
Divide the numerator and denominator by their H. C. F.

EXAMPLES.

1. $\frac{12x^2-3}{20x-10}=\frac{3(2x-1)(2x+1)}{10(2x-1)}=\frac{3(2x+1)}{10}$.

2. $\frac{x^2-8x+12}{x^2-4}=\frac{(x-2)(x-6)}{(x-2)(x+2)}=\frac{x-6}{x+2}$.

3. $\frac{x^4+x^3-x-1}{x^4-x^3-x+1}=\frac{(x+1)(x^3-1)}{(x-1)(x^3-1)}=\frac{x+1}{x-1}$.

4. $\frac{x^4+a^2x^2+a^4}{x^4+ax^3-a^3x-a^4}=\frac{x^2-ax+a^2}{x^2-a^2}$.

5. $\frac{6x^3-5x^2+4}{2x^3-x^2-x+2}=\frac{3x+2}{x+1}$.

6. $\dfrac{2x^3+9x^2+7x-3}{3x^3+5x^2-15x+4}=\dfrac{2x+3}{3x-4}.$

7. $\dfrac{x^4-4x^3+4x^2-9}{x^4+2x^2+9}=\dfrac{x^2-2x-3}{x^2+2x+3}.$

8. $\dfrac{x^4-x^3-x+1}{x^4-2x^3-x^2-2x+1}=\dfrac{x^2-2x+1}{x^2-3x+1}.$

9. $\dfrac{x^2+(a+b)\,x+ab}{x^2+(a+c)\,x+ac}=\dfrac{x+b}{x+c}.$

10. $\dfrac{ax^m+bx^{m+1}}{a^2bx-b^3x^3}=\dfrac{x^{m-1}}{b\,(a-bx)}.$

11. $\dfrac{a^3+(a+b)\,ax+a^0bx^2}{a^4-b^2x^2}=\dfrac{a+x}{a^2-bx}.$

12. $\dfrac{bx+2}{2b+b^2x-4x-2bx^2}=\dfrac{1}{b-2x}.$

13. $\dfrac{(a-b)^2+(b-c)^2+(c-a)^2}{a^3+b^3+c^3-3abc}=\dfrac{2}{a+b+c}.$

266. Case II. To reduce an integer to a fraction with a given denominator:

(1) *Express the integer in the form of a fraction* (260).

(2) *Multiply both terms by the given denominator.*

Thus: $a=\dfrac{a}{1}=\dfrac{ax}{x}=\dfrac{a\,(b+c)}{b+c}$, etc.

EXAMPLES.

1. Reduce x^2-1 to a fraction whose denominator is x^2-3.

Ans. $\dfrac{x^4-4x^2+3}{x^2-3}.$

2. Reduce $(a-b)^2$ to a fraction having a denominator $(a+b)^2$.

Ans. $\dfrac{(a^2-b^2)^2}{(a+b)^2}.$

3. Reduce x^4-x^2+1 to a fraction having a denominator x^4+x^2+1.

Ans. $\dfrac{x^8+x^4+1}{x^4+x^2+1}.$

267. Case III. To reduce a fraction to an integral or mixed expression:

Arrange both terms according to the descending powers of a letter common to both.

Divide the numerator by the denominator. If the division be exact, the value of the fraction is integral; but if not exact, continue the division until the degree of the remainder is less than that of the divisor: write the remainder as the numerator and the divisor as the denominator of the fractional expression, reduce this fraction to its lowest terms and annex it to the quotient by means of the sign +.

Thus: (1) $\frac{x^3-1}{x+1}=x^2-x+1+\frac{2}{x+1}$.

(2) $\frac{x^2+2x-3}{x-1}=x+3$.

(3) $\frac{x^2-4x+3}{x+2}=x-6+\frac{15}{x+2}$.

(4) $\frac{x^2-3x+2}{x^2-x-1}=1+\frac{-2x+3}{x^2-x-1}$.

268. In order that a fraction may be reduced to an integral or mixed expression, the degree of the numerator must not be less than that of the denominator; and if the degrees be equal, the co-efficient of the term containing the highest power in the numerator must not be less than that of the corresponding term in the denominator. Such fractions may be called *improper* fractions.

269. The signs of both terms of a fraction may be changed without altering the value of the fraction, because this is equivalent to multiplying both terms by -1 (Art. 262). If the sign of but one term of a fraction be changed, the sign of the fraction is changed.

Thus, $\frac{a}{b}=\frac{-a}{-b}=-\frac{a}{-b}=-\frac{-a}{b}$;

also, $$-\frac{a}{b} = -\frac{-a}{-b} = +\frac{a}{-b} = +\frac{-a}{b}.$$

Similarly, $$\frac{a-b}{a-c} = \frac{b-a}{c-a} = -\frac{a-b}{c-a} = -\frac{b-a}{-c+a}.$$

270. CASE IV. To reduce a mixed expression to the form of a fraction:

Multiply the denominator of the fraction by the integer; to the product annex the numerator by means of the sign of the fraction. Under the result write the denominator, and reduce the expression to its simplest form.

EXAMPLES.

1. $a + \frac{b}{c} = \frac{ac+b}{c}.$

2. $a + \frac{-b}{c} = \frac{ac+(-b)}{c} = \frac{ac-b}{c}.$

3. $a + \frac{b}{-c} = \frac{-ac+b}{-c} = \frac{ac-b}{c}.$

4. $a - \frac{b}{c} \quad = \frac{ac-b}{c}.$

5. $a - \frac{-b}{c} = \frac{ac-(-b)}{c} = \frac{ac+b}{c}.$

6. $a - \frac{b}{-c} = \frac{-ac-(-b)}{-c} = \frac{-ac+b}{-c} = \frac{ac-b}{c}.$

7. $a - \frac{-b}{-c} = \frac{-ac-(-b)}{-c} = \frac{-ac+b}{-c} = \frac{ac-b}{c}.$

8. $x^4 + x^2y^2 + y^4 - \frac{x^8+y^8}{x^4-x^2y^2+y^4} = \frac{x^4y^4}{x^4-x^2y^2+y^4}.$

9. $(x-y)^2 - \frac{6x^2y+2y^3}{y-x} = \frac{(x+y)^3}{x-y}.$

10. $x^3 + xy^2 + \dfrac{y^4(x-y)}{x^2-y^2} = \dfrac{x^4 + x^3y + x^2y^2 + xy^3 + y^4}{x+y}.$

11. $x^2 + x + 1 - \dfrac{1+x^3}{1-x} = \dfrac{2x^3}{x-1}.$

12. $x^2 - 4x + 8 - \dfrac{(x^2-8)^2}{x^2+4x+8} = \dfrac{16x^2}{x^2+4x+8}.$

271. CASE V. To reduce a complex fraction to a simple one.

A *complex fraction* is one which has a fraction in the numerator or in the denominator, or in both.

A complex fraction does not come under the definition of *fraction* as given in (258). It is merely the quotient of the numerator by the denominator (263).

RULE: *Multiply both terms by the L. C. M. of the denominators of the fractions contained in the numerator and denominator.*

Thus: (1) $\dfrac{4\frac{1}{2}}{5\frac{1}{3}} = \dfrac{4\frac{1}{2}}{5\frac{1}{3}} \times \dfrac{6}{6} = \dfrac{27}{32}.$

(2) $$\frac{a+\frac{b}{c}}{a-\frac{c}{b}} = \frac{a+\frac{b}{c}}{a-\frac{c}{b}} \times \frac{bc}{bc} = \frac{abc+b^2}{abc-c^2}.$$

(3) $$\frac{x-\frac{1}{x}}{y-\frac{1}{y}} = \frac{x-\frac{1}{x}}{y-\frac{1}{y}} \times \frac{xy}{xy} = \frac{x^2y-y}{xy^2-x}.$$

(4) $$\frac{a+\frac{b^2}{a-b}}{b-\frac{a^2}{a+b}} = \frac{a+\frac{b^2}{a-b}}{b-\frac{a^2}{a+b}} \times \frac{(a-b)(a+b)}{(a-b)(a+b)} =$$

$$\frac{a^3-ab^2+b^2(a+b)}{a^2b-b^3-a^2(a-b)} = \frac{a^3-ab^2+ab^2+b^3}{a^2b-b^3-a^3+a^2b} = \frac{a^3+b^3}{-a^3+2a^2b-b^3}.$$

EXAMPLES.

Reduce the following complex fractions to simple ones:

		Answers.			Answers.
5.	$\dfrac{\frac{a}{a+b}}{\frac{b}{a-b}}$.	$\dfrac{a(a-b)}{b(a+b)}$.	8.	$\dfrac{\frac{2y^2}{x^3+y^3}}{\frac{y}{x+y}}$.	$\dfrac{2y}{x^2-xy+y^2}$.
6.	$\dfrac{\frac{a}{x}+\frac{x}{a}-2}{x-a}$.	$\dfrac{x-a}{ax}$.	9.	$\dfrac{\frac{4ab}{a+b}+2a}{\frac{4ab}{a+b}-2a}$.	$\dfrac{a+3b}{b-a}$.
7.	$\dfrac{x+\frac{1}{x}}{x^2+\frac{1}{x^2}+2}$.	$\dfrac{x}{x^2+1}$.	10.	$\dfrac{\frac{a}{a+b}+\frac{b}{a-b}}{\frac{a}{a-b}-\frac{b}{a+b}}$	1.

272. CASE VI. To reduce fractions to equivalent fractions having a common denominator:

Multiply both terms of each fraction by the product of the denominators of the other fractions.

Thus: 1. Reduce $\frac{a}{b}, \frac{b}{c}, \frac{1}{a}$ to a common denominator.

$$\frac{a}{b}=\frac{a}{b}\times\frac{ac}{ac}=\frac{a^2c}{abc};\ \frac{b}{c}=\frac{b}{c}\times\frac{ab}{ab}=\frac{ab^2}{abc};\ \frac{1}{a}=\frac{1}{a}\times\frac{bc}{bc}=\frac{bc}{abc}.$$

2. Reduce to a common denominator $\frac{4}{a+b}, \frac{3}{a-b}, \frac{5}{a}$.

$$\frac{4}{a+b}=\frac{4}{a+b}\times\frac{a(a-b)}{a(a-b)}=\frac{4a^2-4ab}{a^3-ab^2};$$

$$\frac{3}{a-b}=\frac{3}{a-b}\times\frac{a(a+b)}{a(a+b)}=\frac{3a^2+3ab}{a^3-ab^2};$$

$$\frac{5}{a}=\frac{5}{a}\times\frac{(a+b)(a-b)}{(a+b)(a-b)}=\frac{5a^2-5b^2}{a^3-ab^2}.$$

273. CASE VII. To reduce fractions to equivalent fractions having the lowest common denominator (L. C. D.):

(1) *Find the L. C. M. of the denominators.*

(2) *Divide this L. C. M. by the denominator of each fraction.*

(3) *Multiply both terms of the first fraction by the first quotient; both terms of the second fraction by the second quotient; both terms of the third fraction by the third quotient; and so on.*

NOTE. Before reducing fractions to the L. C. D. it is usually best to reduce them to their lowest terms, as in examples 3, 4, 5, 7, below.

EXAMPLES.

Reduce the following fractions to the L. C. D.:

1. $\dfrac{3a}{a-b}, \dfrac{2b}{a+b}, \dfrac{1}{a^2-b^2}.$

The L. C. M. of the denominators is (a^2-b^2), and the quotients are $a+b$, $a-b$, 1.

$$\frac{3a}{a-b}\times\frac{a+b}{a+b}=\frac{3a^2+3ab}{a^2-b^2};\quad \frac{2b}{a+b}\times\frac{a-b}{a-b}=\frac{2ab-2b^2}{a^2-b^2};$$

Ans. $\dfrac{3a^2+3ab}{a^2-b^2}, \dfrac{2ab-2b^2}{a^2-b^2}, \dfrac{1}{a^2-b^2}.$

2. $\dfrac{bc}{(a-b)(a-c)}, \dfrac{ac}{(a-b)(b-c)}, \dfrac{ab}{(a-c)(b-c)}.$

The L. C. D. is $(a-b)(a-c)(b-c)$, and the quotients are $(b-c)$, $(a-c)$, $(a-b)$. *Ans.* $\dfrac{bc(b-c)}{(a-b)(a-c)(b-c)},$

$$\frac{ac(a-c)}{(a-b)(a-c)(b-c)}, \frac{ab(a-b)}{(a-b)(a-c)(b-c)}.$$

3. $\dfrac{x^2-9}{x^2+x-12}, \dfrac{x^2-x-20}{x^2-25}.$

Ans. $\dfrac{x^2+8x+15}{x^2+9x+20}, \dfrac{x^2+8x+16}{x^2+9x+20}.$

4. $\dfrac{x^2+4x+3}{x^2+x-6}$, $\dfrac{x^2+8x+15}{x^2-25}$.

Ans. $\dfrac{x^2-4x-5}{x^2-7x+10}$, $\dfrac{x^2+x-6}{x^2-7x+10}$.

5. $\dfrac{x-1}{x^2-5x+4}$, $\dfrac{x+1}{x^2-3x-4}$. *Ans.* $\dfrac{1}{x-4}$, $\dfrac{1}{x-4}$.

6. $\dfrac{4}{3(x-2)}$, $\dfrac{1}{2(x-1)}$, $\dfrac{1}{6(x+1)}$.

Ans. $\dfrac{8(x^2-1)}{6(x-2)(x^2-1)}$, $\dfrac{3(x^2-x-2)}{6(x-2)(x^2-1)}$, $\dfrac{x^2-3x+2}{6(x-2)(x^2-1)}$.

7. $\dfrac{1}{1-x}$, $\dfrac{1}{(1-x)^2}$, $\dfrac{x-1}{(1-x)^3}$.

Ans. $\dfrac{1-x}{(1-x)^2}$, $\dfrac{1}{(1-x)^2}$, $\dfrac{-1}{(1-x)^2}$.

8. $\dfrac{1}{(a-b)(a-c)}$, $\dfrac{1}{(b-a)(b-c)}$.

Ans. $\dfrac{b-c}{(a-b)(a-c)(b-c)}$, $\dfrac{c-a}{(a-b)(a-c)(b-c)}$.

9. $\dfrac{a+c}{(a-b)(x-a)}$, $\dfrac{b+c}{(a-b)(x-b)}$.

Ans. $\dfrac{(a+c)(x-b)}{(a-b)(x-a)(x-b)}$, $\dfrac{(b+c)(x-a)}{(a-b)(x-a)(x-b)}$.

ADDITION AND SUBTRACTION OF FRACTIONS.

274. To add or subtract fractions:

(1) *Reduce the fractions to equivalent fractions having the lowest common denominator* (273).

(2) *Combine the numerators of the resulting fractions according to the rule for addition* (98) *or subtraction* (103).

(3) *Write the result over the L. C. D.*

(4) *Reduce this fraction to its simplest form.*

EXAMPLES.

1. $\frac{a}{a+b}+\frac{b}{a-b}=\frac{a^2-ab}{(a+b)(a-b)}+\frac{ab+b^2}{(a+b)(a-b)}=\frac{a^2+b^2}{a^2-b^2}.$

2. $\frac{a}{2a-2b}+\frac{b}{2b-2a}=\frac{a}{2(a-b)}+\frac{-b}{2(a-b)}=\frac{a-b}{2(a-b)}=\frac{1}{2}.$

3. $\frac{2}{x}-\frac{3}{2x-1}=\frac{4x-2}{x(2x-1)}-\frac{3x}{x(2x-1)}=\frac{x-2}{x(2x-1)}.$

4. $\frac{x^2+x-5}{2x^2-11x+12}-\frac{x^2+x-1}{2x^2+5x-12}=\frac{x^2+x-5}{(2x-3)(x-4)}-$

$$\frac{x^2+x-1}{(2x-3)(x+4)}=\frac{x^3+5x^2-x-20}{(2x-3)(x-4)(x+4)}-$$

$$\frac{x^3-3x^2-5x+4}{(2x-3)(x-4)(x+4)}=\frac{8x^2+4x-24}{(2x-3)(x-4)(x+4)}=$$

$$\frac{4(2x-3)(x+2)}{(2x-3)(x-4)(x+4)}=\frac{4(x+2)}{x^2-16}.$$

5. $\frac{b}{(a-b)(a-c)}+\frac{a}{(b-a)(b-c)}=\frac{b}{(a-b)(a-c)}+$

$$\frac{-a}{(a-b)(b-c)}=\frac{b(b-c)}{(a-b)(a-c)(b-c)}+\frac{-a(a-c)}{(a-b)(a-c)(b-c)}=$$

$$\frac{b^2-bc-a^2+ac}{(a-b)(a-c)(b-c)}=\frac{c-a-b}{(a-c)(b-c)}.$$

6. $\frac{1}{x^2-4}-\frac{1}{x^3-8}+\frac{1}{x^3+8}=\frac{x^4+4x^2}{x^6-64}.$

7. $\frac{x^2+x-2}{x^2+4x+4}+\frac{x^2-4}{x^2+5x+6}=\frac{2x^2+2x-7}{x^2+5x+6}.$

8. $\frac{1}{x^2-y^2}+\frac{1}{(x+y)^2}-\frac{1}{(x-y)^2}+\frac{4xy}{(x^2-y^2)^2}=\frac{1}{x^2-y^2}.$

9. $\frac{1}{x-3a}-\frac{1}{x+3a}+\frac{3}{x+a}-\frac{3}{x-a}-\frac{48ax^2}{x^4-10a^2x^2+9a^4}.$

Ans. $\frac{48a}{9a^2-x^2}.$

275. In many examples it is preferable not to combine all the fractions at once.

10. To simplify $\frac{x^{3n}}{x^n-1} - \frac{x^{2n}}{x^n+1} - \frac{1}{x^n-1} + \frac{1}{x^n+1}$.

The sum of the first and third is $\frac{x^{3n}-1}{x^n-1} = x^{2n} + x^n + 1$.

The sum of the second and fourth is $\frac{-x^{2n}+1}{x^n+1} = 1 - x^n$.

Ans. $x^{2n} + 2$.

MULTIPLICATION OF FRACTIONS.

276. To multiply by a fraction means to take such a part of the multiplicand as the multiplier is part of unity. Thus, to multiply by $\frac{3}{4}$ means to divide the multiplicand into four equal parts and to take three of those parts. To multiply $\frac{2}{3}$ by $\frac{4}{5}$ means, therefore, to divide $\frac{2}{3}$ into five equal parts, and to take four of those parts. Now $\frac{2}{3} = \frac{10}{15}$ (Art. 262). If $\frac{10}{15}$ be divided into five equal parts, each part will be $\frac{2}{15}$. If $\frac{2}{15}$ be taken four times, the sum will be $\frac{2}{15} + \frac{2}{15} + \frac{2}{15} + \frac{2}{15} = \frac{8}{15}$. $\therefore$ $\frac{4}{5}$ of $\frac{2}{3}$ or $\frac{2}{3} \times \frac{4}{5} = \frac{8}{15}$.

Similarly, $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$.

277. If there be more than two fractions to be multiplied together, the same method will apply. Thus:

$$\frac{2}{3} \times \frac{4}{5} \times \frac{7}{9} = \frac{8}{15} \times \frac{7}{9} = \frac{56}{135}, \text{ and } \frac{a}{b} \times \frac{c}{d} \times \frac{e}{f} = \frac{ace}{bdf}.$$

Hence, to find the product of two or more fractions:

278. *Multiply the numerators together for the numerator of the product, and the denominators together for the denominator of the product.*

EXAMPLES.

1. $\frac{(a-b)^2}{a+b} \times \frac{b}{x(a-b)} = \frac{b(a-b)^2}{x(a+b)(a-b)} = \frac{b(a-b)}{x(a+b)}$.

2. $\left(\frac{1-x^2}{1+y}\right) \times \left(\frac{1-y^2}{x+x^2}\right) \times \left(1+\frac{x}{1-x}\right) = \frac{1-y}{x}.$

3. $\frac{x(a-x)}{a^2+2ax+x^2} \times \frac{a(a+x)}{a^2-2ax+x^2} = \frac{ax}{(a+x)(a-x)}.$

4. $\frac{a^3-b^3}{a^3+b^3} \cdot \frac{a+b}{a-b} \cdot \left(\frac{a^2-ab+b^2}{a^2+ab+b^2}\right)^2 = \frac{a^2-ab+b^2}{a^2+ab+b^2}.$

5. $(x^2-x+1)\left(\frac{1}{x^2}+\frac{1}{x}+1\right) = \frac{x^4+x^2+1}{x^2} = x^2+1+\frac{1}{x^2}.$

6. $\frac{a^2-a-6}{a^2+a-30} \times \frac{a^2-a-42}{a^2-7a+12} \times \frac{a^2-9a+20}{a^2-5a-14} = 1.$

7. $\frac{x^4-y^4}{x^2-2xy+y^2} \times \left(x-\frac{2xy}{x+y}\right) = x(x^2+y^2).$

8. $\frac{a^2-b^2-c^2+2bc}{a^2+c^2-b^2+2ac} \times \frac{a+b+c}{a+b-c} = 1.$

9. $\frac{(a+b)^m}{x^{m+1}} \times \frac{(ax-bx)^m}{(a^2-b^2)^m} = \frac{1}{x}.$

DIVISION OF FRACTIONS.

279. Since $\frac{2}{3} \times \frac{4}{5} = \frac{8}{15}$, $\therefore \frac{8}{15} \div \frac{2}{3} = \frac{4}{5}$. Similarly, since $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$, $\therefore \frac{ac}{bd} \div \frac{c}{d} = \frac{a}{b}$.

Hence, to divide one fraction by another:

Divide the numerator of the dividend by the numerator of the divisor for the numerator of the quotient, and divide the denominator of the dividend by the denominator of the divisor for the denominator of the quotient.

NOTE 1. If the terms of the dividend be not divisible by the corresponding terms of the divisor, multiply both terms of the dividend by an expression which will render them divisible. See Ex. 2.

NOTE 2. If the denominators be the same, divide the numerator of the dividend by that of the divisor. See Ex. 3.

EXAMPLES.

1. $\frac{2y^3}{x^3+y^3} \div \frac{y}{x+y} = \frac{2y^2}{x^2-xy+y^2}.$

2. $\frac{x}{a-x} \div \frac{x}{a+x} = \frac{x(a+x)}{(a-x)(a+x)} \div \frac{x}{a+x} = \frac{a+x}{a-x}.$

3. $\left(x+\frac{2x}{x-3}\right) \div \left(x-\frac{2x}{x-3}\right) = \frac{x^2-x}{x^2-5x} = \frac{x-1}{x-5}.$

4. $\frac{x^6-3x^4y^2+3x^2y^4-y^6}{(x^2-y^2)^4} \div \frac{(x+y)^3}{(x-y)^4} = \frac{(x-y)^3}{(x+y)^4}.$

280. The rule in Art. 279 is the most convenient when the terms of the dividend are multiples of the corresponding terms of the divisor, as in Ex. 1 and 4 above. In other cases one of the following rules is preferable:

281. Second Rule. *Multiply the dividend by the reciprocal of the divisor.*

Thus: To divide $\frac{8}{15}$ by $\frac{2}{3}$. Multiply $\frac{8}{15}$ by $\frac{3}{2}$ (Art. 30).
Ans. $\frac{4}{5}$.

Similarly, $\frac{ac}{bd} \div \frac{a}{b} = \frac{ac}{bd} \times \frac{b}{a} = \frac{abc}{abd} = \frac{c}{d}.$

This rule depends upon the following reasoning: Let it be required to divide $\frac{a}{b}$ by $\frac{c}{d}$. If $\frac{a}{b}$ be divided by c, the quotient is $\frac{a}{bc}$ (Art. 261). But the divisor c being d times as large as the true divisor, $\frac{c}{d}$, the quotient thus obtained is d times too small; hence $\frac{a}{bc}$ must be multiplied by d for the true quotient. $\therefore \frac{a}{b} \times \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}.$

282. Third Rule. (1) *Reduce the fractions to a common denominator.*

(2) *Divide the numerator of the dividend by the numerator of the divisor.*

Thus: $\frac{8}{15} \div \frac{4}{5} = \frac{8}{15} \div \frac{12}{15} = \frac{8}{12} = \frac{2}{3}.$

Similarly, $\frac{a}{b} \div \frac{c}{d} = \frac{ad}{bd} \div \frac{bc}{bd} = \frac{ad}{bc}.$

283. It is evident that a complex fraction may be reduced to a simple one by considering the numerator as the dividend and the denominator as the divisor, and finding the quotient according to one of the preceding rules. Thus:

1. $$\frac{\frac{x}{a} + \frac{a}{x} - 2}{x - a} = \frac{x^2 - 2ax + a^2}{ax} \div (x - a) = \frac{x - a}{ax}.$$

2. $$\frac{\dfrac{x^2 + (a + c)x + ac}{x^2 + (b + c)x + bc}}{\dfrac{x^2 - a^2}{x^2 - b^2}} = \frac{(x+a)(x+c)}{(x+b)(x+c)} \div \frac{(x-a)(x+a)}{(x-b)(x+b)} = \frac{x-b}{x-a}.$$

EXAMPLES.

By each of the three methods show that:

1. $$\frac{ax - x^2}{a^2 + 2ax + x^2} \div \frac{a^2 - 2ax + x^2}{a^2 + ax} = \frac{ax}{a^2 - x^2}.$$

2. $$\left(\frac{x}{a} - \frac{a}{x}\right) \div \frac{x + a}{ax} = x - a.$$

3. $$\left(x + \frac{4x}{x - 3}\right) \div \left(x - \frac{2x}{x - 3}\right) = \frac{x + 1}{x - 5}.$$

4. $$\frac{x^6 - 3x^4y^2 + 3x^2y^4 - y^6}{(x^2 - y^2)^4} \div \frac{(x - y)^3}{(x + y)^4} = \frac{(x + y)^3}{(x - y)^4}.$$

5. $$\left(\frac{x + 2y}{y} + \frac{2y}{x}\right) \div \left(\frac{x + 2y}{y} - \frac{x}{x + y}\right) = \frac{x + y}{x}.$$

MISCELLANEOUS PROPOSITIONS.

284. In each of the following propositions let $\frac{a}{b}=\frac{c}{d}$:

I. $1 \div \frac{a}{b} = 1 \div \frac{c}{d}$, Ax. V; $\therefore \frac{b}{a}=\frac{d}{c}$.

II. $\frac{a}{b}\times\frac{b}{c}=\frac{c}{d}\times\frac{b}{c}$, Ax. IV; $\therefore \frac{a}{c}=\frac{b}{d}$.

III. $\frac{a}{b}+1=\frac{c}{d}+1$, Ax. II; $\therefore \frac{a+b}{b}=\frac{c+d}{d}$.

IV. $\frac{a}{b}-1=\frac{c}{d}-1$, Ax. III; $\therefore \frac{a-b}{b}=\frac{c-d}{d}$.

V. From I $\frac{b}{a}=\frac{d}{c}$, whence from III $\frac{a+b}{a}=\frac{c+d}{d}$.

VI. From I $\frac{b}{a}=\frac{d}{c}$, whence from IV $\frac{b-a}{a}=\frac{d-c}{c}$.

VII. If III be divided by IV, $\therefore \frac{a+b}{a-b}=\frac{c+d}{c-d}$.

Other results may be obtained from propositions III to VII, inclusive, by using *inversion* (I) or *alternation* (II). Thus: from III, $\frac{a+b}{b}=\frac{c+d}{d}$, whence, from I, $\frac{b}{a+b}=\frac{d}{c+d}$; also, from II, $\frac{a+b}{c+d}=\frac{b}{d}$; etc. Hence:

If two fractions are equal, we may combine by addition or subtraction, in any way, the numerator and denominator of the one, provided that we do the same with the other.

285. If $\frac{a}{b}=\frac{c}{d}$, then $\frac{a^n}{b^n}=\frac{c^n}{d^n}$ (Ax. XIV). Hence the preceding results hold with a^n, b^n, c^n, d^n, instead of a, b, c, d. For example, VII would read $\frac{a^n+b^n}{a^n-b^n}=\frac{c^n+d^n}{c^n-d^n}$.

286. If a, b, and c represent positive numbers, then if $a > b$, $\frac{a+c}{b+c} < \frac{a}{b}$; if $a = b$, $\frac{a+c}{b+c} = \frac{a}{b}$; if $a < b$, $\frac{a+c}{b+c} > \frac{a}{b}$.

For, in order to compare fractions, we reduce them to a common denominator and compare the numerators of the resulting fractions.

Now $\frac{a+c}{b+c} = \frac{ab+bc}{b(b+c)}$, and $\frac{a}{b} = \frac{ab+ac}{b(b+c)}$. When $a > b$, $ab + bc < ab + ac$; when $a = b$, $ab + bc = ab + ac$; when $a < b$, $ab + bc > ab + ac$: which proves the proposition.

287. If a, b, and c represent positive numbers, then if $a > b$, $\frac{a-c}{b-c} > \frac{a}{b}$; if $a = b$, $\frac{a-c}{b-c} = \frac{a}{b}$; if $a < b$, $\frac{a-c}{b-c} < \frac{a}{b}$.

For $\frac{a-c}{b-c} = \frac{ab-bc}{b(b-c)}$, and $\frac{a}{b} = \frac{ab-ac}{b(b-c)}$. When $a > b$, $ab - bc > ab - ac$; when $a = b$, $ab - bc = ab - ac$; when $a < b$, $ab - bc < ab - ac$: which proves the proposition.

Exercise XV.

Simplify the following fractions:

1. $\frac{x^2 - 3x + 2}{x^3 - 6x^2 + 11x - 6}$.

2. $\frac{x^2 + 2xy + y^2}{x^3 + 3x^2y + 3xy^2 + y^3}$.

3. $\frac{3x^3 - 16x^2 + 23x - 6}{2x^3 - 11x^2 + 17x - 6}$.

4. $\frac{x^3 + 9x^2 + 26x + 24}{x^4 + 10x^3 + 35x^2 + 50x + 24}$.

5. $\frac{x^4 + 2x^3 + 2x^2}{x^5 + 4x}$.

6. $\frac{x^2 + x^{-2} - 2}{x - x^{-1}}$.

7. $\frac{x^2 + x^{-2} + 1}{x^{-1} + 1 + x}$.

8. $\frac{x + x^{-1}}{x^3 + x^{-3}}$.

In the following examples, perform the operations indicated:

9. $\frac{5}{2x+2} + \frac{1}{10(1-x)} - \frac{24}{5(2x+3)}$.

10. $\frac{3}{(x+2)^2}+\frac{1}{x+2}-\frac{1}{x-1}$.

11. $\frac{3+2x}{2-x}+\frac{3x-2}{x+2}-\frac{x^2-16x}{x^2-4}$.

12. $\frac{4-20x}{4x^2-1}+\frac{7}{2x+1}-\frac{3}{1-2x}$.

13. $\frac{1}{x+y}+\frac{y}{x^2-y^2}-\frac{x}{x^2+y^2}$.

14. $\frac{a-b}{b-x}+\frac{2b-a}{b+x}+\frac{3x(a-b)}{x^2-b^2}$.

15. $\frac{1}{a(a-b)(a-c)}+\frac{1}{b(b-a)(b-c)}+\frac{1}{c(c-a)(c-b)}$.

16. $\frac{a^2+a+1}{(a-b)(a-c)}+\frac{b^2+b+1}{(b-a)(b-c)}+\frac{c^2+c+1}{(c-a)(c-b)}$.

17. $\frac{a^2+x^2}{(a-b)(a-c)}+\frac{b^2+x^2}{(b-a)(b-c)}+\frac{c^2+x^2}{(c-a)(c-b)}$.

18. $\frac{(a^2+y^2)(b+c)}{(a-b)(a-c)}+\frac{(b^2+y^2)(a+c)}{(b-a)(b-c)}+\frac{(c^2+y^2)(a+b)}{(c-a)(c-b)}$.

19. $\frac{bc(a^2+m^2)}{(a-b)(a-c)}+\frac{ac(b^2+m^2)}{(b-a)(b-c)}+\frac{ab(c^2+m^2)}{(c-a)(c-b)}$.

20. $\frac{3ax}{4by}\times\frac{a^2-x^2}{c^2-x^2}\times\frac{bc+bx}{a^2+ax}\times\frac{c-x}{a-x}$.

21. $\left(\frac{b}{c}+\frac{c}{b}\right)^2+\left(\frac{c}{a}+\frac{a}{c}\right)^2+\left(\frac{a}{b}+\frac{b}{a}\right)^2-6$.

22. $\left(\frac{a}{b}+\frac{b}{a}\right)\left(\frac{c}{d}+\frac{d}{c}\right)+\left(\frac{a}{b}-\frac{b}{a}\right)\left(\frac{c}{d}-\frac{d}{c}\right)$.

23. $\left(\frac{x+y}{x-y}-\frac{x-y}{x+y}-\frac{4y^2}{x^2-y^2}\right)\frac{x+y}{2y}$.

24. $\frac{4(a^2-ab)}{b(a+b)^2}\div\frac{6ab}{a^2-b^2}$.

25. $\dfrac{a^2-b^2}{x+y} \times \dfrac{x^2-y^2}{a-b} \div \dfrac{(x-y)^2}{a^2}$.

26. $\left(\dfrac{1}{x-1} - \dfrac{1}{2(x+1)} - \dfrac{x+3}{2(x^2-1)}\right)(x^2-1)$.

27. $\left(\dfrac{1}{1+x} + \dfrac{x}{1-x}\right) \div \left(\dfrac{1}{1-x} - \dfrac{x}{1+x}\right)$.

28. $\left(\dfrac{1}{1+x} + \dfrac{x}{1-x}\right) \div \dfrac{1}{1-x} - \dfrac{x}{1+x}$.

29. $\dfrac{1}{1+x} + \dfrac{x}{1-x} \div \dfrac{1}{1-x} - \dfrac{x}{1+x}$.

30. $\left(\dfrac{a-b}{a+b} - \dfrac{a^3-b^3}{a^3+b^3}\right) \div \left(\dfrac{a+b}{a-b} + \dfrac{a^2+b^2}{a^2-b^2}\right)$.

31. $\left(\dfrac{x^2+y^2}{x^2-y^2} - \dfrac{x^2-y^2}{x^2+y^2}\right) \div \left(\dfrac{x+y}{x-y} - \dfrac{x-y}{x+y}\right)$.

32. $\left(\dfrac{a+b}{a-b} + \dfrac{a-b}{a+b}\right) \div \left(\dfrac{a^2+b^2}{a^2-b^2} - \dfrac{a^2-b^2}{a^2+b^2}\right)$.

33. $\dfrac{\dfrac{a^2+b^2}{b} - a}{b^{-1}-a^{-1}} \times \dfrac{a^2-b^2}{a^3+b^3}$.

34. $\dfrac{x}{x-a} - \dfrac{x}{x+a} - \dfrac{\dfrac{x+a}{x-a} - \dfrac{x-a}{x+a}}{\dfrac{x+a}{x-a} + \dfrac{x-a}{x+a}}$.

35. $\dfrac{1}{x + \dfrac{1}{1 + \dfrac{x+1}{3-x}}}$.

36. $\dfrac{a}{b + \dfrac{c}{d + \dfrac{e}{f}}}$.

CHAPTER XVI.

SIMPLE EQUATIONS.

CONTINUED FROM CHAPTER X.

288. 1. SOLVE $a - \frac{bx}{a} = \frac{ax - b^2}{c}$.

Multiply by ac, the L. C. M. of the denominators.

Then $\quad a^2c - bcx = a^2x - ab^2$.

Transpose. $\quad -bcx - a^2x = -a^2c - ab^2$.

Divide both sides by $-bc-a^2$. $\therefore\ x = \frac{a(ac + b^2)}{a^2 + bc}$.

2. Solve $\frac{ax + b}{c} - \frac{a}{b} = \frac{cx + d}{c}$.

Multiply by bc, the L. C. M. of the denominators.

Then $\quad abx + b^2 - ac = bcx + bd$.

Transpose. $\quad bx(a - c) = ac - b^2 + bd$.

Divide both sides by $b(a - c)$. $\therefore\ x = \frac{ac - b^2 + bd}{b(a - c)}$.

289. 3. Solve $\frac{8x + 5}{14} + \frac{7x - 3}{6x + 2} = \frac{4x + 6}{7}$.

When an equation contains simple and compound denominators, it is usually best to get rid of the simple denominators first, and then of each compound denominator in turn, being careful to simplify as much as possible after each step. In this example, therefore, transpose $\frac{8x + 5}{14}$ and simplify.

Then $\quad \frac{(7x - 3)}{6x + 2} = \frac{8x + 12 - 8x - 5}{14} = \frac{1}{2}$.

Multiply by $(6x + 2)$. $\therefore\ 7x - 3 = 3x + 1$, whence $x = 1$.

4. Solve $\frac{6x+13}{15} - \frac{2x}{5} = \frac{3x+5}{5x-25}$.

Multiply both sides by 15.

Then $6x + 13 - 6x = \frac{9x+15}{x-5}$. $\therefore 13 = \frac{9x+15}{x-5}$.

Multiply both sides by $(x-5)$.

Then $13x - 65 = 9x + 15$. $\therefore x = 20$.

5. Solve $\frac{x}{a+x} = \frac{a+x}{x} - \frac{2a-b}{2x}$.

Multiply both sides by $2x$.

Then $\frac{2x^2}{a+x} = 2a + 2x - 2a + b = 2x + b$.

Now multiply both sides by $a + x$.

$\therefore 2x^2 = 2x^2 + (2a+b)x + ab$, whence $x = \frac{-ab}{2a+b}$.

290. 6. Solve $\frac{17}{6x-17} - \frac{10}{3x-10} = \frac{1}{1-2x}$.

Multiply both sides by $(6x-17)(3x-10)$.

$\therefore 51x - 170 - 60x + 170 = \frac{(6x-17)(3x-10)}{1-2x}$.

$\therefore -9x = \frac{18x^2 - 111x + 170}{1-2x}$.

$\therefore -9x + 18x^2 = 18x^2 - 111x + 170$; whence $x = 1\frac{2}{3}$.

291. 7. Solve $\frac{x+7}{11} - \frac{2x-16}{3} + \frac{2x+5}{4} = 5\frac{1}{3} + \frac{3x+7}{12}$.

Here it is convenient to clear of fractions partially, and to reduce the resulting expression. Hence, multiply by 12.

$\therefore 12\left(\frac{x+7}{11}\right) - 8x + 64 + 6x + 15 = 64 + 3x + 7$.

$\therefore 12x + 84 = 55x - 88$; whence $x = 4$.

292. 8. $(x-a)(x-b) - (x-b)(x-c) = 2(x-a)(a-c)$.

Simplify the first side: $(x-b)[(x-a)-(x-c)] = (x-b)(-a+c) = (b-x)(a-c)$.

$\therefore (b-x)(a-c) = 2(x-a)(a-c)$.

Divide by $(a - c)$. $\therefore\ b - x = 2(x - a)$; whence $x = \frac{2a + b}{3}$.

293. **9.** Solve $\frac{x-1}{x-2} - \frac{x-2}{x-3} = \frac{x-5}{x-6} - \frac{x-6}{x-7}$.

Simplify each side.

$$\therefore \frac{(x^2-4x+3)-(x^2-4x+4)}{(x-2)(x-3)} = \frac{(x^2-12x+35)-(x^2-12x+36)}{(x-6)(x-7)};$$

that is, $$\frac{-1}{(x-2)(x-3)} = \frac{-1}{(x-6)(x-7)}.$$

$\therefore\ -x^2 + 13x - 42 = -x^2 + 5x - 6$ and $x = 4\frac{1}{2}$.

10. Solve $\frac{1}{2}\left(\frac{2x}{3} + 4\right) - \frac{7\frac{1}{2} - x}{3} = \frac{x}{2}\left(\frac{6}{x} - 1\right)$.

Expand. $\frac{1}{3}x + 2 - \frac{5}{2} + \frac{1}{3}x = 3 - \frac{1}{2}x$.

Transpose and simplify. $\frac{7}{6}x = \frac{7}{2}$. $\therefore\ x = 3$.

294. **11.** Solve $\frac{66x + 1}{1.5x + 1} + \frac{4x + 5}{.5x - 1} = 52$.

Multiply each term by $\frac{1}{2}$.

$$\therefore \frac{66x + 1}{3x + 2} + \frac{4x + 5}{x - 2} = 26.$$

Clear of fractions.

$66x^2 - 131x - 2 + 12x^2 + 23x + 10 = 78x^2 - 104x - 104$.

Transpose and simplify. $-4x = -112$. $\therefore\ x = 28$.

295. **12.** Solve

$$\frac{x}{2} - \frac{\frac{1}{3}(2x-3) - \frac{1}{4}(3x-1)}{\frac{1}{2}(x-1)} = \frac{3}{2}\left(\frac{x^2 - \frac{1}{3}x + 2}{3x - 2}\right).$$

Transpose $\frac{x}{2}$ and simplify each side.

$$\therefore \frac{x+9}{6(x-1)} = \frac{3x^2 - x + 6}{2(3x-2)} - \frac{x}{2} = \frac{x+6}{2(3x-2)}.$$

Now take $\frac{x+9}{6(x-1)} = \frac{x+6}{2(3x+2)}$, and clear of fractions.

$\therefore\ 3x^2 + 25x - 18 = 3x^2 + 15x - 18$; whence $x = 0$.

296. **13.** Solve $\frac{x-\frac{3}{2}}{\frac{3}{2}x-\frac{3}{2}}+\frac{x-\frac{5}{2}}{\frac{5}{2}(x+1)}=1+\frac{1}{15(1-x^{-2})}$.

Multiply both terms of the first fraction by 2, of the second by 2, and of the last fraction by x^2.

$$\therefore \frac{2x-3}{3(x-1)}+\frac{2x-5}{5(x+1)}=1+\frac{x^2}{15(x^2-1)}.$$

Multiply every term by the L. C. D. $15(x-1)(x+1)$.

$\therefore 5(2x-3)(x+1)+3(2x-5)(x-1)=15(x-1)(x+1)+x^2$.

Expand. $10x^2-5x-15+6x^2-21x+15=15x^2-15+x^2$.

Transpose and simplify. $\therefore -26x=-15$; whence $x=\frac{15}{26}$.

Exercise XVI.

Solve the following equations. (For the solutions of the simultaneous equations [13–24 inclusive] consult Chapter XI.)

1. $\frac{a-b}{x-c}=\frac{a+b}{x+2c}$.

2. $\frac{3x+1}{9}+\frac{3x-6}{2x-5}=\frac{3x+28}{9}$.

3. $\frac{x}{a}-1-\frac{dx}{c}=3ab$.

4. $\frac{x}{a}+\frac{x}{b-a}=\frac{a}{a+b}$.

5. $\frac{3x-1}{x+2}+\frac{6+x}{4}-2\frac{1}{4}=\frac{3x-9}{12}+\frac{3x+9}{x+7}$.

6. $\frac{1}{x-2}-\frac{1}{x-4}=\frac{1}{x-6}-\frac{1}{x-8}$.

7. $\frac{2}{2x-5}+\frac{1}{x-3}-\frac{6}{3x-1}=0$.

8. $\frac{a+b}{x-c}=\frac{a}{x+a}+\frac{b}{x-b}$.

9. $\frac{25-\frac{1}{3}x}{x+1}+\frac{80x+21}{5(3x+2)}=\frac{5x+28}{x+1}$.

10. $\frac{x-7}{x+7}=\frac{2x-15}{2x-6}-\frac{1}{2(x+7)}$.

11. $\frac{9(7x+1)}{x-1}=\frac{35(x+4)}{x+2}+28$.

12. $\frac{11}{12x+11}+\frac{5}{6x+5}=\frac{7}{4x+7}.$

13. $\frac{4}{5+y}=\frac{5}{12+x};\quad 2x+5y=35.$

14. $\frac{2x+6}{3y+2}=\frac{8}{7};\quad 8x=9y+4.$

15. $\frac{a}{x}+\frac{b}{y}=5;\quad \frac{b}{x}+\frac{a}{y}=12.$

16. $(a^2-b^2)(5x+3y)=8a^2b-2ab^2;$

$$b(a+b+c)x+(a^2-b^2)y=\frac{ab^2c}{a+b}+ab(a+2b).$$

17. $2x-y-z=12;$
$3y-x-z=16;$
$5z-x-y=24.$

18. $x+y+z=4;$
$3x-y+2z=1;$
$4x+3y-z=18.$

19. $x+y+t=0;$
$x+z+u=-1;$
$3y+u+t=5;$
$3z+t=7;$
$x+y-z-u+2t=17.$

20. $x+2y+3z+5u=2;$
$3x-4y+z+4u=-9;$
$4x-y-2z+u=15;$
$4y-2z+5u=-1.$

21. $2x+3y-4z=20;$
$x+2y-3z=8;$
$3x+y-2z=26.$

22. $x^{-1}+y^{-1}=a;$
$x^{-1}+z^{-1}=b;$
$y^{-1}+z^{-1}=c.$

23. $x+y+z+t+v=0;$
$3x+4y+z+5t+2v=0;$
$4x-2y+4t+v=0;$
$2x+y-z-2t=23;$
$-6y+3z-5t=-1.$

24. $x+y+z-v=a;$
$x+y-z+v=b;$
$x-y+z+v=c;$
$-x+y+z+v=d.$

CHAPTER XVII.

PROBLEMS INVOLVING SIMPLE EQUATIONS.

297.—1. AFTER losing $\frac{1}{5}$ of my money and $\frac{1}{4}$ of what was left, I gained $\frac{1}{3}$ of the remainder, and then had \$280. How much had I at first?

Let $5x$ represent the number of dollars I had at first.

I lost $\frac{1}{5}$ of my money, hence I had left $\frac{4}{5}$ of $5x = 4x$.

I lost $\frac{1}{4}$ of what was left, so I had then left $\frac{3}{4}$ of $4x = 3x$.

Finally, I gained $\frac{1}{3}$ of $3x = x$, and then had $3x + x = 4x$.

But I then had \$280. $\therefore$ (Ax. I.) $4x = 280$, whence $x = 70$ and $5x = 350$. $\therefore$ I had \$350 at first.

298.—2. A's money was $\frac{6}{7}$ of B's; A gained \$2, and B lost \$3, when A had $\frac{10}{9}$ as many dollars as B. How much had each at first?

Let $7x$ represent the number of dollars B had at first. Since A's money was $\frac{6}{7}$ of B's, $\therefore$ A had $\frac{6}{7}$ of $7x = 6x$.

A gained \$2; $\therefore$ A then had $6x + 2$ dollars.

B lost \$3; $\therefore$ B then had $7x - 3$ dollars.

$\therefore 6x + 2 = \frac{10}{9}(7x - 3)$, whence $x = 3$.

Ans. B had 21 dollars, and A had 18 dollars.

299.—3. A lease was given for 60 years. Three fourths of the time it has already run equals $\frac{9}{8}$ of the time it has yet to run. How many years has it yet to run?

Let $8x$ represent the number of years the lease has yet to run; then $60 - 8x$ is the number of years it has already run.

$\therefore \frac{3}{4}(60 - 8x) = 9x$, whence $x = 3$, and $8x = 24$, *Ans.*

300.—4. E was hired for a days, at b cents for each day he worked, and at a forfeit of c cents for each day he was idle. At the end of the time he received d cents. How many days did he work?

Let x represent the number of days E worked. Since he was hired for a days, and worked x days, he must have been idle $a - x$ days. Since he worked x days at b cents a day, his total wages amounted to bx cents. As he forfeited c cents a day for $\overline{a - x}$ days, he forfeited $c(a - x)$ cents.

The amount due him for wages, diminished by the amount forfeited, must be the amount of money actually received.

$$\therefore\ bx - c(a - x) = d, \text{ whence } x = \frac{d + ac}{b + c}.$$

301. In solving problems involving *time, rate,* and *distance,* observe that: (1) *The time* $\times$ *the rate* = *the distance;* Formula $t \times r = d$. (2) $\frac{\textit{The distance}}{\textit{the time}} =$ *the rate;* Formula $\frac{d}{t} = r$. and (3) $\frac{\textit{The distance}}{\textit{the rate}} =$ *the time;* Formula $\frac{d}{r} = t$. Thus, if A go 5 miles per hour for 8 hours, his distance will be 40 miles. If A go 40 miles at the rate of 5 miles per hour, his time will be 8 hours. If A go 40 miles in 8 hours, his rate will be 5 miles per hour.

5. How many miles can A walk at the rate of 3 miles an hour, so as to return to the starting-point in 11 hours, riding back at the rate of 8 miles an hour?

Let x represent the required number of miles. Since he walks x miles at the rate of 3 miles per hour, $\therefore$ his time equals $\frac{1}{3}x$ hours. Since he rides back x miles at the rate of 8 miles per hour, $\therefore$ his time returning is $\frac{1}{8}x$ hours.

Since he takes $\frac{1}{3}x$ hours to go, and $\frac{1}{8}x$ hours to return, $\therefore$ his entire time $= (\frac{1}{3}x + \frac{1}{8}x)$ hours. But according to the given conditions, his entire time is 11 hours. $\therefore$ (Ax. I.) $\frac{1}{3}x + \frac{1}{8}x = 11$, whence $x = 24$, *Ans.*

302.—6. A, who traveled at the rate of $31\frac{1}{2}$ miles in 5 hours, left a certain city 8 hours before B, who traveled at the rate of $22\frac{1}{2}$ miles in 3 hours. In what time did B overtake A, and at what distance from the place of departure?

Let x represent the number of hours B traveled.

Then $x+8$ represents the number of hours A traveled.

B's rate is $22\frac{1}{2} \div 3 = 7\frac{1}{2}$ miles per hour.

A's rate is $31\frac{1}{2} \div 5 = 6\frac{3}{10}$ miles per hour.

Since $d = rt$, $\therefore$ B's distance is $\frac{15}{2}x$, and A's is $\frac{63}{10}(x+8)$.

But they traveled equal distances.

$\therefore \frac{15}{2}x = \frac{63}{10}(x+8)$, whence $x = 42$.

Ans. B's time is 42 hours, and his distance is 315 miles.

303.—7. A takes 5 steps while B takes 4, but 3 of B's steps are equivalent to 4 of A's. How many steps must B take to overtake A, who has a start of 80 steps?

In problems of this class it is convenient to consider *a step of the pursued* as the unit of length. That is, in this case, one of A's steps is the unit. Let $12x$ represent the number of steps B must take in order to overtake A. Since A takes $\frac{5}{4}$ as many steps as B, then A takes $15x$ steps; $\therefore$ A goes $15x$ units. As three of B's steps are equivalent to 4 units, then $12x$ steps of B are equivalent to $16x$ units.

But A's distance is 80 units less than B's distance;

$\therefore 16x - 15x = 80$, whence $x = 80$, and $12x = 960$, *Ans.*

304.—8. Find the time between 4 and 5 o'clock when the hour and minute hands of a watch are:

I. Together; II. At right angles;
III. Opposite; IV. Equidistant from VII.

I. Let CM and CH denote the positions of the minute and the hour hands at 4 o'clock, and let CA denote the position of both hands when together.

Since the hour-hand goes from IV to V while the minute-hand makes a complete revolution, therefore the hour-hand moves only $\frac{1}{12}$ as fast as the minute-hand. If, therefore, the distance the hour-hand goes in any time be x, then the distance the minute-hand goes in the same time will be $12x$.

FIG. 1.

$\therefore$ Let the arc HA be x minute-spaces in length.

Then the arc MA will be $12x$ minute-spaces in length.

But $MA = MH + HA = 20 + x$.

$\therefore$ $12x = 20 + x$, whence $x = 1\frac{9}{11}$, and $20 + x = 21\frac{9}{11}$.

Ans. The hands are together at $21\frac{9}{11}$ minutes past IV.

II. Let CA be the position of the minute-hand, and CB be the position of the hour-hand when the hands are at right angles.

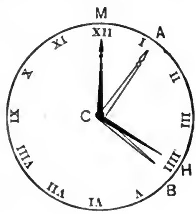

FIG. 2.

Let x represent the number of minute-spaces in the arc HB.

Then $12x$ represents the number of minute-spaces in the arc MA.

But MA also equals $5 + x$. $\therefore$ (Ax. 1.)

$12x = 5 + x$, whence $x = \frac{5}{11}$, and $12x = 5\frac{5}{11}$.

Ans. The hands form a right angle at $5\frac{5}{11}$ minutes past IV.

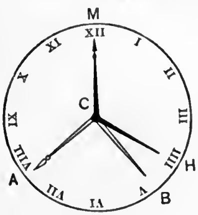

FIG. 3.

But the hands will be again at right angles between IV and V, when the minute-hand is 15 minute-spaces ahead of the hour-hand. If, as before, the number of minute-spaces in the arc $HB = x$, then the number of minute-spaces in the arc $MHBA$ is $12x = 35 + x$.

$\therefore$ $x = 3\frac{2}{11}$ and $12x = 38\frac{2}{11}$.

Ans. The hands are at right angles a second time between IV and V at $38\frac{2}{11}$ minutes past IV.

III. Let CB be the position of the hour-hand, and CA be the position of the minute-hand when the hands are opposite.

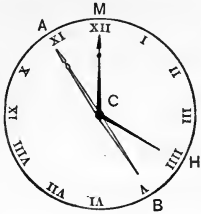

FIG. 4.

If $HB = x$, then $MHBA = 12x$. But $MHBA$ also equals $50 + x$. $\therefore 12x = 50 + x$. $\therefore x = 4\frac{6}{11}$, and $12x = 54\frac{6}{11}$.

Ans. The hands are opposite at $54\frac{6}{11}$ minutes past IV.

IV. Let CB and CA be the positions of the hands when VII is midway between them; also, let $HB = x$.

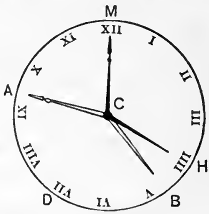

FIG. 5.

$\therefore DB = 15 - x$, because $HD = 15$, and $DB = DH - BH$.

Since $DA = DB$, $\therefore DA = 15 - x$, whence $MDA = 35 + DA = 50 - x$.

But $MDA = 12x$. $\therefore 12x = 50 - x$, whence $x = 3\frac{11}{13}$, and $12x = 46\frac{2}{13}$.

Ans. The hands are equidistant from VII at $46\frac{2}{13}$ minutes past IV.

PROBLEMS INVOLVING TWO OR MORE UNKNOWN QUANTITIES.

305.—9. A and B work together for 2 hours, after which B leaves, and A completes the task in 6 hours more. Had A left, and B continued to work, he could have completed the work in 3 hours more. How long would it take each alone to do the entire work?

Let x represent the number of hours A would require, and let y represent the number of hours B would require.

Then A does $\frac{1}{x}$ of the work in an hour, and B does $\frac{1}{y}$.

From the first condition the work is completed when A works for 8 hours, and B for 2 hours. $\therefore$ (1) $\frac{8}{x} + \frac{2}{y} = 1$.

From the second condition the work is completed when A works for 2 hours and B for 5 hours. $\therefore$ (2) $\frac{2}{x} + \frac{5}{y} = 1.$

Multiply (2) by 4, and subtract (1) from the product.

$\therefore \frac{18}{y} = 3$, whence $y = 6$. $\therefore x = 12$.

Therefore A would take 12 hours, and B 6 hours.

306.—**10.** The sum of three digits expressing a certain number, is 12. The middle digit equals the difference of the other two, and the number would be reversed by adding 396. Find the number.

Let x represent the hundreds' digit, y the tens', and z the units'.

From the first condition, I. $x + y + z = 12$.

From the second condition, II. $y = z - x$, because we know from the third condition that z is larger than x.

In order to express the third condition, we must recollect that the required number consists of x hundreds + y tens + z units; thus, $574 = 500 + 70 + 4$, etc. Similarly, the reverse of the required number consists of z hundreds + y tens + x units. Hence, from the third condition,

III. $100x + 10y + z + 396 = 100z + 10y + x$.

From equation III. $x - z = -4$.

From combining I and II we get $z = 6$. Therefore, from III, $x = 2$, and, from I, $y = 4$. *Ans.* 246.

11. What number of two digits contains the sum of its digits four times and their product twice?

From the first condition, I. $\frac{10x + y}{x + y} = 4$.

$\therefore 10x + y = 4x + 4y$, and $2x = y$.

From the second condition, II. $\frac{10x + y}{xy} = 2$.

In II substitute $2x$ for y, whence $12x = 4x^2$, and $x = 3$.

Ans. 36.

Exercise XVII.

Solve 1 like Ex. 1, Art. 297.

1. A lost $\frac{2}{7}$ of his money and then gained $10; he then lost $50 more than $\frac{3}{5}$ of this amount; he next gained $\frac{1}{2}$ of this remainder, but found that his total loss amounted to $233. How many dollars had A at first?

Solve 2 and 3 like Ex. 2, Art. 298.

2. A's age is $\frac{3}{4}$ of B's; in 15 years it will be $\frac{6}{7}$ of B's. How old is each?

3. A's money equals $\frac{4}{5}$ of B's. If A give B $12, he will have $\frac{1}{2}$ as much as B. How much has each?

Solve 4, 5, and 6 like Ex. 3, Art. 299.

4. Divide 45 into two parts such that one part shall be $\frac{1}{4}$ of the other.

5. A debt of $100 was paid in silver dollars and paper dollars. The number of silver dollars was $\frac{9}{16}$ of the number of paper dollars. How many of each kind were used?

6. It took A 40 hours to go from C to D and return. His rate going was $\frac{2}{3}$ of his rate returning. How many hours did it take him to go from C to D?

Solve 7 and 8 like Ex. 16, Art. 174.

7. A's time to do a certain work is $\frac{2}{3}$ of B's. Both together can do the work in 5 hours. How long will it take each alone?

8. A and B together do a piece of work in c days, which A alone can do in d days. How long would it take B?

Solve 9 and 10 like Ex. 4, Art. 300.

9. A was hired for 40 days, at 50 cents a day and his board. For each day he was idle he forfeited 25 cents for his board. At the end of the time he received $11. How many days was he idle?

10. A agreed to work for a year for $200 and a suit of clothes. At the end of 5 months he left, receiving for his wages $60 and the clothes. What was the suit worth?

Solve 11–16 inclusive like Ex. 5, Art. 301, and 6, Art. 302.

11. A starts from C at the same instant that B starts from D, 60 miles from C. A's rate is 3 miles, and B's 2 miles, an hour. In how many hours will they be together: (1) If they travel toward each other? (2) If A pursue B?

12. A walks at the rate of 9 miles in 4 hours, and starts from C 2 hours before B does; but B overtakes him 45 miles from C. Find B's rate and time.

13. A and B start at the same instant from C to go to F. A, by traveling half a mile an hour faster than B, arrives at F 40 minutes before him. A's time being 16 hours, find B's rate and the distance from C to F.

14. A boatman rows with the tide 36 miles in 4 hours, and returns against a tide half as strong in 8 hours. Find the rate of the tide in each case. (See note to the 15th.)

15. A boatman rows 40 miles down stream and back again in 15 hours, and his rate down stream is twice his rate up stream. Find his time down, the rate of the current, and the rate of rowing.

NOTE.—The rate down equals the rate of rowing plus the rate of the current; the rate up equals the rate of rowing minus the rate of the current. Formulas: (1) $r + c = d$; (2) $r - c = u$.

16. A and B set out at the same instant and travel toward each other, A at the rate of 3, and B at the rate of 4 miles, an hour. At the same time C sets out with B, at the rate of 5 miles an hour, travels till he meets A, then turns about, and in 10 hours after setting out, meets B. How far apart were A and B at first?

Solve 17, 18, and 19 like Ex. 7, Art. 303.

17. A takes 5 steps while B takes 7, but 3 of A's steps are equivalent to 5 of B's. B has a start of 80 steps: how many steps must A take to overtake him?

18. C takes 5 steps while B takes 6, but 4 of C's steps are equivalent to 5 of B's. B has a start of 30 steps: how many steps can he take before C overtakes him?

19. A takes 2 steps while B takes 3, but A's steps are twice as long as B's. How many steps must A take to overtake B, if A take 120 steps to reach B's starting-point?

Solve 20, 21, 22, and 23 like Ex. 8, Art. 304.

20. At what time between 7 and 8 will the hands of a clock be: I. Together? II. At right angles? III. Opposite? IV. Equidistant from v? V. Ten minute-spaces apart? VI. Twice as far apart as they were ten minutes before?

21. When between 3 and 4 o'clock will the minute-hand be as far past VIII as the hour-hand is past II?

22. When between 5 and 6 o'clock will the hour-hand be as far past IV as the minute-hand lacks of being at XI?

23. At what time between a and $a + 1$ o'clock will the hands of a clock be together? Opposite?

Solve 24, 25, and 26 like Ex. 9, Art. 305.

24. A and B work together for 3 hours, after which A alone completes the task in 5 hours more. B can do as much work in 2 hours as A can in 3. How long would it take B alone to do all the work?

25. After A and B have worked together for 4 hours, B completes the task in 6 hours more, while A would take 8 hours to complete it. How long would it take B alone to do the entire work?

26. An empty cistern has an inlet and an outlet pipe. Both are left open for 16 hours, when the outlet pipe is stopped, and the cistern is filled in 6 hours more. Had the inlet pipe been stopped instead of the outlet pipe, the cistern would have been emptied in 4 hours. How long would it take the inlet pipe to fill the cistern if the outlet pipe were closed? If both pipes be left open, how long would it take to fill the cistern?

Solve 27–31 inclusive like Ex. 10, Art. 306.

27. A number of two digits is reversed by subtracting 18, and the sum of both numbers is 110. Find the number.

28. A number is $\frac{5}{6}$ of its reverse, and the sum of its two digits is 9. Find the number.

29. Find a number of two places which equals three times the sum of its digits, and the difference of whose digits is 5.

30. A certain number equals 12 times the difference of the two digits expressing it, and the digits will be reversed by adding 27. Find the number.

31. Find a number of three places which will be reversed by adding 297. The right-hand digit equals the sum of the middle digit and twice the left-hand one, and the sum of the three digits is 7.

32. If a rectangular field were 10 rods longer and 4 rods narrower, it would contain one acre less; if it were 20 rods wider and 36 rods shorter, it would contain one acre more. Find the length, the breadth, and the area of the field.

33. A bought a certain number of apples; had he bought 20 fewer for the same total price, each would have cost 1 cent more; had he bought 30 more for the same total price, each would have cost 1 cent less. How many apples did he buy, and at what price?

CHAPTER XVIII.

INDETERMINATE SIMPLE EQUATIONS.

307. WE have seen (175) that if a single equation contain two unknown quantities, the number of solutions is unlimited. Such an equation is said to be *indeterminate.*

308. If one solution be given of the equation $ax \pm by = c$, all the others may be readily found.

For, let $x = r$ and $y = n$ be a pair of values of the equation $ax + by = c$; then $ar + bn = c$.

$$\therefore \text{(Ax. I.) } ax + by = ar + bn.$$
$$\therefore a(x - r) + b(y - n) = 0.$$

This equation is satisfied by $(x - r) = -bt$, $(y - n) = at$, where t may be any quantity whatever, positive or negative. Since $x - r = -bt, \quad \therefore x = r - bt.$

Since $y - n = at, \quad \therefore y = n + at.$

If the equation be of the form $ax - by = c$, we obtain in the same way, $x = r + bt$, $y = n + at$.

309. If only integral values of x and y be required, the number of solutions will sometimes be limited. The above general values of x and y will still apply, only r, n, and t must all be integral.

There can be no integral values of x and y in $ax \pm by = c$, if a and b have any common factor not common also to c. For, let $a = md$, $b = nd$, while c does not contain d.

Then $mdx \pm ndy = c. \quad \therefore mx \pm ny = \frac{c}{d}.$

But m and n are integers; if, therefore, x and y be also integers, mx will be integral, so also will ny, and we shall have the sum or difference of two integers equal to a fraction, which is evidently impossible.

Therefore no integral values of x and y can be found.

310. To solve $ax \pm by = c$ in integers.

1. Find the integral solutions of $3x + 5y = 73$.

Transpose the term having the larger co-efficient.

$$\therefore\ 3x = 73 - 5y, \text{ whence } x = 24 - y + \frac{1-2y}{3}.$$

Since the value of x is integral, then $\frac{1-2y}{3}$, although fractional in form, must be integral; and so also is any multiple of it. Now, multiply the numerator by such a number that the co-efficient of y shall be 1 more than a multiple of the denominator. In this case multiply by 2.

$\therefore\ \frac{2-4y}{3}$ or $\frac{2-y}{3} - y$ is integral. $\therefore\ \frac{2-y}{3}$ is integral.

Let $\frac{2-y}{3} = t$, an integer. Then $2 - y = 3t$, and $y = 2 - 3t$. $\therefore\ x = 73 - 5y = 21 + 5t$.

If $t = 0$, then $x = 21$, $y = 2$; if $t = 1$, then $x = 26$, $y = -1$; if $t = -1$, then $x = 16$, $y = 5$; etc.

311. If the solution of $3x + 5y = 73$ be required in positive integers, there will be but five pairs of values that will satisfy the equation. For, in order that $y = 2 - 3t$ shall be a positive integer, t can not be greater than 0. In order that $x = 21 + 5t$ shall be a positive integer, t can not be less than -4. $\therefore\ t$ may be $0, -1, -2, -3, -4$, giving the following pairs of values:

$$x = 21;\ 16;\ 11;\ \ 6;\ \ 1.$$
$$y = \ \ 2;\ \ 5;\ \ 8;\ 11;\ 14.$$

312.—**2.** Solve in positive integers $39x - 56y = 11$.

Here $x = y + \frac{17y + 11}{39}$. $\therefore$ $\frac{17y + 11}{39}$ is integral.

To find a multiple of 17 which is 1 more than a multiple of 39, takes too much time; hence proceed as follows:

$$\frac{17y + 11}{39} = \frac{39y - 22y + 11}{39} = y - \frac{11(2y - 1)}{39}.$$

Since $\frac{11(2y - 1)}{39}$ is integral, then $\frac{2y - 1}{39}$ must be integral, and $\frac{40y - 20}{39} = y + \frac{y - 20}{39}$ must be integral, $= y + t$.

$\therefore$ $y = 39t + 20$, whence $x = 56t + 29$.

Here t can not be a negative integer, but it may be 0, or any positive integer. $\therefore$ the number of values is unlimited. When $t = 0$, $x = 29$, and $y = 20$, which are their least positive integral values.

313. From the preceding examples it appears that when only positive integral values of x and y are required, the number of them will be limited when the equation is of the form $ax + by = c$, but the number of solutions is unlimited if the equation is of the form $ax - by = c$.

314. The necessity for a multiplier may often be avoided by changing the form of the given equation.

3. Solve in positive integers $3x + 5y = 73$.

$$\text{Add } y. \quad \therefore \; 3x + 6y = 73 + y.$$

$$\therefore \quad x + 2y = 24 + \frac{1 + y}{3}$$

$\therefore$ $\frac{1 + y}{3}$ is an integer $= t$. $\therefore$ $y = 3t - 1$, $x = 26 - 5t$.

Here t may be any integer from 1 to 5, giving the same pairs of values as by the method in (310).

315.—4. To find the least number which, when divided by 14 and 5, will leave remainders 1 and 3 respectively.

Let x represent the quotient of the number divided by 14.

$$\therefore N = 14x + 1.$$

Let y represent the quotient of the number divided by 5.

$$\therefore N = 5y + 3.$$

$$\therefore 5y + 3 = 14x + 1;$$

$$5y = 14x - 2 = 15x - x - 2;$$

$$\therefore y = 3x - \frac{x+2}{5}. \quad \therefore \frac{x+2}{5} = \text{an integer} = t.$$

$$x = 5t - 2, \text{ and } y = 14t - 6.$$

The least value of t is 1. $\therefore x = 3$, and $N = 43$.

316.—5. Find the least integer which is divisible by 2, 3, 4, with remainders 1, 2, 3.

Let the quotients be respectively x, y, and z.

Then $N = 2x + 1 = 3y + 2 = 4z + 3$.

Since $2x+1=3y+2$, $\therefore 2x = 3y+1$; $\therefore x = y + \frac{y+1}{2}$;

$\therefore \frac{y+1}{2} = t$; $\therefore y = 2t - 1$; whence $x = 3t - 1$.

In $2x + 1 = 4z + 3$, for x substitute its value $(3t - 1)$.

$$\therefore 6t - 1 = 4z + 3; \therefore z = t + \frac{t}{2} - 1;$$

$$\therefore \frac{t}{2} = \text{an integer} = m, \text{ and } t = 2m.$$

Whence $x = 6m - 1$; $y = 4m - 1$; $z = 3m - 1$.

The least value of m is 1.

Then $x = 5$, and $N = 2x + 1 = 2 \times 5 + 1 = 11$.

Exercise XVIII.

Solve the following equations in positive integers:

1. $8x + 65y = 81$.
2. $7x + 10y = 297$.
3. $3x + 7y = 250$.
4. $5x + \frac{1}{2}y = 64$.
5. $\frac{x}{7} + \frac{y}{5} = \frac{53}{35}$.
6. $3x + 7y = 100$.

Find the least integral values of x and y which satisfy the following equations:

7. $7x - 9y = 29.$

8. $9x - 11y = 8.$

9. $19x - 5y = 119.$

10. $17x - 49y + 8 = 0.$

11. In how many ways can $100 be paid in dollar bills and five-dollar bills?

12. Divide 200 into two parts, such that if one of them be divided by 6 and the other by 11, the respective remainders may be 5 and 4.

13. What is the least number which divided by 28 leaves a remainder 21, and divided by 19 leaves a remainder 17?

14. What is the least number which, divided by 28, 19, and 15, leaves remainders 13, 2, and 7?

15. Find a number of two places such that if it be divided by 19, the quotient will equal the units' digit, and the remainder will equal the tens' digit.

16. The difference between a certain multiple of ten and the sum of its digits is 99. Find it.

17. A number is expressed by three digits whose sum is 20; if 16 be taken from the required number, and the remainder be divided by 2, the digits will be reversed. Find the number.

18. A buys for $100 sheep, turkeys, and chickens, 100 in all. Each sheep costs $10, each turkey $3, and each chicken 50 cents. How many of each does he buy?

CHAPTER XIX.

RADICAL EXPRESSIONS.

317. AN indicated root that can not be exactly obtained is called a *radical, surd,* or *irrational expression.* Thus, $\sqrt[3]{a^2}$ or $a^{\frac{2}{3}}$ is called a *surd.* An indicated root that can be exactly ascertained is said to have the *form* of a *surd.* Thus: $\sqrt[3]{a^6}$ has the *form* of a surd; but, since the indicated root can be exactly obtained, it is a *rational expression,* and not a surd.

318. It was stated in Art. 146 that there can not be an even root of a negative number. Such roots, however, may be expressed in the form of surds, and are then called *impossible* or *imaginary* expressions. Examples: $\sqrt{-5}$; $\sqrt[4]{-16}$; $\sqrt[6]{-a}$; $(-3)^{\frac{1}{2}}$; etc.

319. Surds are named *quadratic, cubic, biquadratic,* according as the second, third, or fourth roots are required.

320. A *mixed surd* is the indicated product of a rational factor and a surd factor; as, $2\sqrt{5}$; $a\sqrt{b}$.

In this case the rational factor is called the *co-efficient* of the *surd.*

321. An *entire surd* is one whose co-efficient is unity.

322. A surd is in its *simplest form* when the surd factor is integral, and as small as possible.

323. *Similar surds* are those which, reduced to their simplest form, have the same surd factor; as $3\sqrt{a}$ and $-5\sqrt{a}$. Otherwise the surds are *dissimilar*.

324. Surds are of the *same order* when they have a common index; as, $\sqrt[3]{a^2}$ and $\sqrt[3]{5}$.

325. *Any rational number may be expressed in the form of a given surd by raising it to a power whose exponent is equal to the given index, and indicating the required root of the product;* as, $2 = 4^{\frac{1}{2}} = 8^{\frac{1}{3}}$, etc.; $a = \sqrt{a^2} = \sqrt[3]{a^3} = a^{\frac{4}{4}}$, etc.; $a + x = (a^2 + 2ax + x^2)^{\frac{1}{2}} = (a + x)^{\frac{3}{3}} = \sqrt[n]{(a + x)^n}$, etc. The rational numbers are here raised to certain powers, and the roots of those powers are then taken, so that the values of the expressions are unchanged.

326. *The product of two or more surds of the same order is found by taking the product of the numbers under the radical signs, and retaining the common index.* Thus: Since $(a^{\frac{1}{n}} b^{\frac{1}{n}})^n = ab$, and $(\sqrt[n]{ab})^n = ab$, $\therefore\ a^{\frac{1}{n}} b^{\frac{1}{n}} = \sqrt[n]{ab} = (ab)^{\frac{1}{n}}$. Similarly: $\sqrt{2} \,.\, \sqrt{5} = \sqrt{10}$; $\sqrt{a+b} \,.\, \sqrt{a-b} = \sqrt{a^2-b^2}$; $\sqrt[n]{a} \,.\, \sqrt[n]{b^2} \,.\, \sqrt[n]{a^3b} = \sqrt[n]{a^4b^3}$; etc.

The product of mixed surds of the same order is found by prefixing the product of the co-efficients to the product of the surd factors. Thus: $a\sqrt{b} \times c\sqrt{d} = ac\sqrt{bd}$; $5\sqrt{2} \times 3\sqrt{8} = 15\sqrt{16} = 15 \times 4 = 60$.

327. *A mixed surd may be expressed as an entire surd by raising the rational factor to a power whose exponent is equal to the surd-index, and placing beneath the radical sign the product of this power and the surd factor.* Thus: $2\sqrt{3} = \sqrt{4} \,.\, \sqrt{3} = \sqrt{12}$; $3 \,.\, 2^{\frac{2}{3}} = \sqrt[3]{27} \,.\, \sqrt[3]{4} = \sqrt[3]{108}$; $2a\sqrt{3b} = \sqrt{4a^2} \,.\, \sqrt{3b} = \sqrt{12a^2b}$; etc.

328. Conversely, a factor may be removed from under the radical sign by extracting the indicated root. Thus: $\sqrt{12} = \sqrt{4 \times 3} = 2\sqrt{3}$; $\sqrt[3]{24} = \sqrt[3]{8 \times 3} = 2\sqrt[3]{3}$; $\sqrt[4]{32} = \sqrt[4]{16 \times 2} = 2\sqrt[4]{2}$; $\sqrt{ab - b^2} = \sqrt{b(a-b)} = b^{\frac{1}{2}}\sqrt{a-b}$; etc.

Hence: *An integral surd is reduced to its simplest form by removing from under the radical sign all factors of which the indicated root can be exactly obtained.*

Thus: $\sqrt{72} = \sqrt{36 \times 2} = 6\sqrt{2}$;

$$\sqrt{75a^3 - 50a^2b} = \sqrt{25a^2(3a - 2b)} = 5a\sqrt{3a - 2b}.$$

If the surd factor be a fraction, its numerator and denominator should both be multiplied by such a number that the indicated root of the denominator can be exactly ascertained. For example:

$$\sqrt{\frac{2}{3}} = \sqrt{\frac{6}{9}} = \frac{\sqrt{6}}{3}; \quad \sqrt{\frac{8}{5}} = \sqrt{\frac{40}{25}} = \frac{2\sqrt{10}}{5};$$

$$\sqrt{\frac{a^3b}{c}} = \sqrt{\frac{a^3bc}{c^2}} = \frac{a\sqrt{abc}}{c}; \quad \sqrt[5]{\frac{3}{8}} = \sqrt[5]{\frac{12}{32}} = \frac{\sqrt[5]{12}}{2}; \text{ etc.}$$

If the index be a composite number, and if the root indicated by a factor of the index can be extracted, the surd is simplified by extracting that root. Thus: $\sqrt[4]{144} = \sqrt{12} = 2\sqrt{3}$; $\sqrt[9]{8} = \sqrt[3]{2}$; $\sqrt[6]{27} = \sqrt{3}$; etc.

329. *The order of a surd may be changed by multiplying both exponent and index by the same number.* Thus:

$$\sqrt{a} = \sqrt[4]{a^2} = \sqrt[6]{a^3} = \sqrt[10]{a^5}, \text{ etc.}; \quad \sqrt[3]{a^2} = \sqrt[6]{a^4} = \sqrt[3n]{a^{2n}};$$

$$\sqrt[5]{a(a-b)} = \sqrt[10]{a^2(a-b)^2} = \sqrt[15]{a^3(a-b)^3}; \quad a^{\frac{4}{3}} = a^{\frac{8}{6}} = a^{\frac{4n}{3n}}.$$

330. To reduce surds of different orders to the same order:

(1) *Find the L. C. M. of the indices.*

(2) *Divide this L. C. M. by each index in succession.*

(3) *Multiply both exponent and index of the first radical*

by the first quotient, of the second radical by the second quotient, etc.

Thus: $\sqrt{2}$, $\sqrt[3]{4}$, $\sqrt[5]{3}$, and $\sqrt[10]{5}$ may be reduced to equivalent surds having an index of 30.

$$\sqrt{2} = \sqrt[30]{2^{15}};\ \sqrt[3]{4} = \sqrt[30]{4^{10}};\ \sqrt[5]{3} = \sqrt[30]{3^6};\ \sqrt[10]{5} = \sqrt[30]{5^3}.$$

331. To compare surds of the same order:

Express them as entire surds, and compare the resulting surd-factors.

Example. Compare $6\sqrt{3}$ and $4\sqrt{7}$.

$$6\sqrt{3} = \sqrt{108};\ 4\sqrt{7} = \sqrt{102}.\quad \therefore 6\sqrt{3} > 4\sqrt{7}.$$

To compare surds of different orders:

Reduce them to entire surds of the same order, and compare the resulting surd-factors.

Example. Compare $\sqrt{5}$ and $\sqrt[3]{11}$.

$$\sqrt{5} = \sqrt[6]{5^3} = \sqrt[6]{125};\ \sqrt[3]{11} = \sqrt[6]{11^2} = \sqrt[6]{121}.\quad \therefore \sqrt{5} > \sqrt[3]{11}.$$

332. To add or subtract surds:

(1) *Reduce each surd to its simplest form* (328).

(2) *If the resulting surds be similar, prefix the sum or difference of the co-efficients to the common surd-factor.*

Dissimilar surds can only be connected by their signs.

Example. $\sqrt{12} + \sqrt{18} + \sqrt{27} + \sqrt[6]{27} + 2\sqrt[10]{32} =$

$$2\sqrt{3} + 3\sqrt{2} + 3\sqrt{3} + \sqrt{3} + 2\sqrt{2} = 6\sqrt{3} + 5\sqrt{2}.$$

Similarly:

$$\sqrt[3]{40} - \tfrac{1}{2}\sqrt[3]{320} + \sqrt[3]{135} = 2\sqrt[3]{5} - \tfrac{1}{2}(4\sqrt[3]{5}) + 3\sqrt[3]{5} =$$
$$2\sqrt[3]{5} - 2\sqrt[3]{5} + 3\sqrt[3]{5} = 3\sqrt[3]{5}.$$

333. To multiply surds:

(1) *Reduce the surds to the same order* (330).

(2) *Prefix the product of the co-efficients to the product of the surd-factors, retaining the common surd-index.*

Thus: $\sqrt{8} \times 5\sqrt{2} = 5\sqrt{16} = 5 \times 4 = 20;$

$$2\sqrt{3} \times 3\sqrt{2} \times 4\sqrt{5} = 24\sqrt{30};$$

$$2\sqrt{3} \times 3\sqrt[3]{2} = 2\sqrt[6]{27} \times 3\sqrt[6]{4} = 6\sqrt[6]{108};$$

$$2^{\frac{3}{2}} \times 3^{\frac{1}{2}} = 8^{\frac{1}{2}} \times 3^{\frac{1}{2}} = (24)^{\frac{1}{2}};$$

$$2^{\frac{1}{2}} \times 3^{\frac{1}{3}} = 2^{\frac{3}{6}} \times 3^{\frac{2}{6}} = 8^{\frac{1}{6}} \times 9^{\frac{1}{6}} = (72)^{\frac{1}{6}}.$$

334. *If two or more surds have the same rational expression under the radical sign, their product is found by making the sum of the fractional exponents the exponent of that expression.*

Thus: $\sqrt{a} \times \sqrt[3]{a} = a^{\frac{1}{2}} \times a^{\frac{1}{3}} = a^{\frac{5}{6}};$

$$\sqrt[2]{2} \times \sqrt[3]{2} \times \sqrt[4]{2} = 2^{\frac{1}{2}} \times 2^{\frac{1}{3}} \times 2^{\frac{1}{4}} = 2^{\frac{1}{2}+\frac{1}{3}+\frac{1}{4}} = 2^{\frac{13}{12}}.$$

335. *Division of surds is performed, when the divisor is a monomial, by a process similar to that for multiplication* (333, 334). Thus:

$$\sqrt{8} \div \sqrt{2} = \sqrt{4} = 2; \ (8\sqrt{2} - 12\sqrt{3} - 4\sqrt{6} + 6) \div 2\sqrt{3} = 4\sqrt{\tfrac{2}{3}} - 6 - 2\sqrt{2} + \sqrt{3} = \tfrac{4}{3}\sqrt{6} - 6 - 2\sqrt{2} + \sqrt{3};$$

$$\sqrt{2} \div \sqrt[3]{2} = 2^{\frac{1}{2}} \div 2^{\frac{1}{3}} = 2^{\frac{1}{2}-\frac{1}{3}} = 2^{\frac{1}{6}};$$

$$3\sqrt{2} \div 2\sqrt[3]{3} = 3\sqrt[6]{8} \div 2\sqrt[6]{9} = \frac{3}{2}\sqrt[6]{\frac{8}{9}} = \frac{3}{2}\sqrt[6]{\frac{8 \times 3^4}{3^6}} = \tfrac{1}{2}\sqrt[6]{8 \times 3^4} = \tfrac{1}{2}\sqrt[6]{648};$$

$$3 \div \sqrt{6} = \frac{3}{\sqrt{6}} = \frac{3\sqrt{6}}{6} = \tfrac{1}{2}\sqrt{6}.*$$

336. In case the dividend is not exactly divisible by the divisor, express the quotient in the form of a fraction, and

* $\frac{1}{2}\sqrt{6}$ is considered simpler than $3 \div \sqrt{6}$; because if we wish to find the approximate value, it is easier to take $\frac{1}{2}\sqrt{6}$ than to divide 3 by $\sqrt{6}$, as can be ascertained by trial.

multiply both numerator and denominator by a factor that will render the denominator rational. The last example of the preceding article is solved in this manner. Similarly: $\frac{1-\sqrt{5}}{\sqrt{6}}=\frac{\sqrt{6}-\sqrt{30}}{6}$. Here both terms are multiplied by $\sqrt{6}$.

337. If the divisor be compound, express the quotient in the form of a fraction, then multiply both numerator and denominator by a factor that will render the denominator rational. To do this, proceed as follows:

338.—First Case. If the denominator be of the form $\sqrt{a} \pm \sqrt{b}$, multiply both terms by $(\sqrt{a} \mp \sqrt{b})$. The denominator will then become $a-b$, a rational expression.

Thus:
$$\frac{1}{2\sqrt{2}-\sqrt{5}}=\frac{1}{2\sqrt{2}-\sqrt{5}}\times\frac{2\sqrt{2}+\sqrt{5}}{2\sqrt{2}+\sqrt{5}}=$$
$$\frac{2\sqrt{2}+\sqrt{5}}{8-5}=\frac{2\sqrt{2}+\sqrt{5}}{3}.$$

Similarly:
$$\frac{4\sqrt{7}+3\sqrt{2}}{5\sqrt{2}+2\sqrt{7}}=\frac{4\sqrt{7}+3\sqrt{2}}{5\sqrt{2}+2\sqrt{7}}\times\frac{5\sqrt{2}-2\sqrt{7}}{5\sqrt{2}-2\sqrt{7}}=$$
$$\frac{20\sqrt{14}+15(2)-8(7)-6\sqrt{14}}{25\times 2-4\times 7}=\frac{14\sqrt{14}-26}{22}=\frac{7\sqrt{14}-13}{11}.$$

Therefore: *When the denominator is a binomial involving only quadratic surds, the required multiplier will consist of the same terms as the given denominator, but with a different sign between them.*

339.—Second Case. If the denominator be of the form $\sqrt[3]{a} \pm \sqrt[3]{b}$, multiply both terms by $a^{\frac{2}{3}} \mp a^{\frac{1}{3}} b^{\frac{1}{3}} + b^{\frac{2}{3}}$, for the product of these factors is $a \pm b$, a rational expression.

Thus: $\frac{\sqrt[3]{6}}{\sqrt[3]{3}-\sqrt[3]{2}} = \frac{\sqrt[3]{6}}{\sqrt[3]{3}-\sqrt[3]{2}} \times \frac{\sqrt[3]{9}+\sqrt[3]{6}+\sqrt[3]{4}}{\sqrt[3]{9}+\sqrt[3]{6}+\sqrt[3]{4}} =$

$$\frac{\sqrt[3]{54}+\sqrt[3]{36}+\sqrt[3]{24}}{3-2} = 3\sqrt[3]{2}+\sqrt[3]{36}+2\sqrt[3]{3}.$$

Therefore: *When the denominator is a binomial involving only cubic surds, if of the form* $(x-y)$, *the multiplier will be* x^2+xy+y^2; *if of the form* $(x+y)$, *the multiplier will be* x^2-xy+y^2.

As a result of this rule: If the denominator be of the form x^2+xy+y^2, and involve only cubic surds, both terms should be multiplied by $x-y$. If it be of the form x^2-xy+y^2, and involve only cubic surds, the multiplier will be $x+y$.

Thus: $\frac{1}{\sqrt[3]{9}+\sqrt[3]{3}+1} = \frac{\sqrt[3]{3}-1}{3-1} = \frac{\sqrt[3]{3}-1}{2}$;

also, $\frac{1}{\sqrt[3]{16}-2+\sqrt[3]{4}} = \frac{\sqrt[3]{4}+\sqrt[3]{2}}{4+2} = \frac{\sqrt[3]{4}+\sqrt[3]{2}}{6}$.

340. The following general law covers the two preceding cases, and all others involving binomial denominators:

(1) Suppose the denominator $a^{\frac{1}{p}}+b^{\frac{1}{q}}$. Put $x=a^{\frac{1}{p}}$ and $y=b^{\frac{1}{q}}$; let n be the least common multiple of p and q, then x^n and y^n are both rational. If n be even, the multiplier is found from dividing x^n-y^n by $x+y$. If n be odd, the required factor is found from dividing x^n+y^n by $x+y$. (Consult Articles 219 and 220.)

(2) Suppose the denominator $a^{\frac{1}{p}}-b^{\frac{1}{q}}$. Take x, y, and n as before. The required factor is found from dividing x^n-y^n by $x-y$. (Consult Articles 217 and 218.)

341. By two operations we may rationalize the denominator of a fraction when that denominator consists of three quadratic surds. For, suppose the denominator to be $\sqrt{a}+\sqrt{b}+\sqrt{c}$: first multiply both numerator and denominator by $\sqrt{a}+\sqrt{b}-\sqrt{c}$, thus the denominator becomes $a+2\sqrt{ab}+b-c$; then multiply both numerator and denominator by $(a+b-c)-2\sqrt{ab}$, and we obtain the rational denominator $(a+b-c)^2-4ab$. Thus:

$$\frac{\sqrt{10}}{\sqrt{5}+\sqrt{2}-\sqrt{6}}=\frac{\sqrt{10}}{(\sqrt{5}+\sqrt{2})-\sqrt{6}}\times\frac{(\sqrt{5}+\sqrt{2})+\sqrt{6}}{(\sqrt{5}+\sqrt{2})+\sqrt{6}}=$$

$$\frac{5\sqrt{2}+2\sqrt{5}+2\sqrt{15}}{2\sqrt{10}+1}=\frac{(5\sqrt{2}+2\sqrt{5}+2\sqrt{15})(2\sqrt{10}-1)}{(2\sqrt{10}+1)(2\sqrt{10}-1)}=$$

$$\frac{18\sqrt{5}+15\sqrt{2}+20\sqrt{6}-2\sqrt{15}}{39}.$$

EXAMPLES.

Simplify:

1. $\sqrt{63}$; $\sqrt{28}$; $\sqrt{275}$; $\sqrt{52}$; $\sqrt{245}$.

Ans. $3\sqrt{7}$; $2\sqrt{7}$; $5\sqrt{11}$; $2\sqrt{13}$; $7\sqrt{5}$.

2. $\sqrt{5a^4}$; $\sqrt{3a^7}$; $\sqrt{7a^5}$; $\sqrt{24a^6}$; $\sqrt{48a^9}$.

Ans. $a^2\sqrt{5}$; $a^3\sqrt{3a}$; $a^2\sqrt{7a}$; $2a^3\sqrt{6}$; $4a^4\sqrt{3a}$.

3. $\sqrt{\frac{1}{32}}$; $\sqrt{\frac{9}{48}}$; $\sqrt{\frac{7}{18}}$; $\sqrt{\frac{54}{32}}$; $\sqrt{2\frac{1}{25}}$.

Ans. $\frac{1}{8}\sqrt{2}$; $\frac{1}{4}\sqrt{3}$; $\frac{1}{6}\sqrt{14}$; $\frac{3}{4}\sqrt{3}$; $\frac{1}{5}\sqrt{51}$.

4. $\sqrt[3]{56}$; $\sqrt[3]{8a^4}$; $\sqrt[3]{27a^2}$; $\sqrt[3]{a^8}$; $\sqrt[3]{16a^7}$.

Ans. $2\sqrt[3]{7}$; $2a\sqrt[3]{a}$; $3\sqrt[3]{a^2}$; $a^2\sqrt[3]{a^2}$; $2a^2\sqrt[3]{2a}$.

5. $2\sqrt{80}$; $\frac{1}{3}\sqrt[3]{48}$; $2\sqrt{\frac{1}{3}}$; $2a^2\sqrt{a^7}$; $\frac{1}{3}a\sqrt[3]{8a^4}$.

Ans. $8\sqrt{5}$; $\frac{2}{3}\sqrt[3]{6}$; $\frac{2}{3}\sqrt{3}$; $2a^5\sqrt{a}$; $\frac{2}{3}a^2\sqrt[3]{a}$.

6. $\sqrt{a^3 + a^2x}$; $\sqrt{a^3 - 2a^2x + ax^2}$; $\sqrt{(x-y)(y-x)}$.

Ans. $a\sqrt{a+x}$; $(a-x)\sqrt{a}$; $(x-y)\sqrt{-1}$.

7. $\sqrt[4]{\frac{1}{25}}$; $\sqrt[6]{\frac{27}{8}}$; $\sqrt[3x]{a^{3x}b^x}$; $\sqrt[6]{-64a^{14}}$; $\sqrt[10]{-32a^5}$.

Ans. $\frac{1}{5}\sqrt{5}$; $\frac{1}{2}\sqrt{6}$; $a\sqrt[3]{b}$; $-2a^2\sqrt[3]{a}$; $\sqrt{-2a}$.

8. $\sqrt{24} + \sqrt{\frac{3}{2}} - \frac{2}{3}\sqrt{\frac{2}{3}} + 3\sqrt{\frac{1}{6}}$. *Ans.* $\frac{25}{9}\sqrt{6}$.

9. $3\sqrt[3]{-81} - 5\sqrt[3]{-24} - 3\sqrt[3]{-\frac{1}{9}} + \sqrt[3]{-375}$.

Ans. $-3\sqrt[3]{3}$.

10. $\sqrt{3a(2a-b)^2} - \sqrt{3a(a-2b)^2} + \sqrt{12a^3}$.

Ans. $(3a+b)\sqrt{3a}$.

11. $3\sqrt{\frac{1}{6}} \times 2\sqrt{\frac{2}{3}} \times 5\sqrt{7} \div \sqrt{3\frac{1}{2}}$. *Ans.* $10\sqrt{2}$.

12. $\sqrt{63} \div \sqrt{14} + \frac{1}{2}\sqrt{18}$. *Ans.* $3\sqrt{2}$.

13. $\sqrt{a - \sqrt{b}} \times \sqrt{a + \sqrt{b}}$. *Ans.* $\sqrt{a^2 - b}$.

14. $\frac{5}{3\sqrt{3}}$; $\frac{6}{\sqrt{18}}$; $\frac{3}{\sqrt[4]{9}}$; $\frac{5}{\sqrt[5]{27}}$; $\frac{2}{\sqrt[3]{-4}}$.

Ans. $\frac{5}{9}\sqrt{3}$; $\sqrt{2}$; $\sqrt{3}$; $\frac{5}{3}\sqrt[5]{9}$; $-\sqrt[3]{2}$.

15. $\frac{1}{\sqrt{5}-\sqrt{2}}$; $-\frac{2}{\sqrt{7}+1}$; $\frac{3}{\sqrt{11}-\sqrt{7}}$.

Ans. $\frac{1}{3}(\sqrt{5}+\sqrt{2})$; $-\frac{1}{3}(\sqrt{7}-1)$; $\frac{3}{4}(\sqrt{11}+\sqrt{7})$.

16. $\frac{2}{\sqrt{2}+\sqrt{3}-1}$. *Ans.* $\frac{1}{2}(\sqrt{6}-2+\sqrt{2})$.

17. $\frac{1}{\sqrt[3]{5}-\sqrt[3]{2}}$. *Ans.* $\frac{1}{3}(\sqrt[3]{25}+\sqrt[3]{10}+\sqrt[3]{4})$.

18. $\frac{1}{\sqrt[3]{5}+1}$. *Ans.* $\frac{1}{6}(\sqrt[3]{25}-\sqrt[3]{5}+1)$.

19. $\frac{1}{\sqrt[3]{25}-\sqrt[3]{20}+2\sqrt[3]{2}}$. *Ans.* $\frac{1}{9}(\sqrt[3]{5}+\sqrt[3]{4})$.

IMAGINARY EXPRESSIONS.

342. All imaginary square roots may be reduced to one of the two forms, $a\sqrt{-1}$ or $b^{\frac{1}{2}}\sqrt{-1}$. Thus: $\sqrt{-9} = \sqrt{9(-1)} = 3\sqrt{-1}$; $\sqrt{-6} = \sqrt{6(-1)} = \sqrt{6}\sqrt{-1}$.

It is frequently convenient to use $a\sqrt{-1}$ instead of $\sqrt{-a^2}$, also $b^{\frac{1}{2}}\sqrt{-1}$ instead of $\sqrt{-b}$.

343. Since the square of the square root of any number will produce that number, $\therefore (\sqrt{-1})^2 = -1$. Since the third power of any number is equal to the product of the square and the first power of that number, $\therefore (\sqrt{-1})^3 = (\sqrt{-1})^2\sqrt{-1} = (-1)\sqrt{-1} = -\sqrt{-1}$. Since the fourth power is equal to the square of the square, $\therefore$ $(\sqrt{-1})^4 = (\sqrt{-1})^2(\sqrt{-1})^2 = (-1)(-1) = +1$. In like manner, we find the fifth power of $\sqrt{-1} = \sqrt{-1}$; the sixth power of $\sqrt{-1} = -1$; the seventh power of $\sqrt{-1} = -\sqrt{-1}$; etc.

Thus all the powers of $\sqrt{-1}$ may be expressed by four formulas: I. $(\sqrt{-1})^{4n+1} = \sqrt{-1}$; II. $(\sqrt{-1})^{4n+2} = -1$; III. $(\sqrt{-1})^{4n+3} = -\sqrt{-1}$; IV. $(\sqrt{-1})^{4n} = +1$.

For example: $(\sqrt{-1})^{49} = \sqrt{-1}$; $(\sqrt{-1})^{50} = -1$; $(\sqrt{-1})^{51} = -\sqrt{-1}$; $(\sqrt{-1})^{52} = +1$.

344. The rules for addition and subtraction of imaginary expressions are the same as those for ordinary surds; but before applying the rules for multiplication and division (333, 335), the factor $\sqrt{-1}$ must be removed as explained in (343).

EXAMPLES.

1. $2\sqrt{-4}+3\sqrt{-9}+5\sqrt{-16}-8\sqrt{-25}=$
$4\sqrt{-1}+9\sqrt{-1}+20\sqrt{-1}-40\sqrt{-1}=-7\sqrt{-1}.$

2. $2\sqrt{-12}+\frac{1}{\sqrt{-3}}-\sqrt{-18}=$
$4\sqrt{-3}-\frac{1}{3}\sqrt{-3}-3\sqrt{-2}=\frac{11}{3}\sqrt{-3}-3\sqrt{-2}.$

3. $\sqrt{-6}\times\sqrt{-3}\times\sqrt{-2}=$
$(\sqrt{6}\sqrt{-1})(\sqrt{3}\sqrt{-1})(\sqrt{2}\sqrt{-1})=6(-1)^{\frac{3}{2}}=-6\sqrt{-1}.$

4. $\sqrt{-ab}\div\sqrt{-b}=\sqrt{a}.$

5. $\sqrt{ab}\div\sqrt{-a}=-\sqrt{-b}.$

6. $\frac{3-3\sqrt{-1}}{2-2\sqrt{-1}}=\frac{3(1-\sqrt{-1})}{2(1-\sqrt{-1})}=\frac{3}{2}.$

7. $\frac{3-\sqrt{-1}}{2+\sqrt{-1}}=\frac{(3-\sqrt{-1})(2-\sqrt{-1})}{(2+\sqrt{-1})(2-\sqrt{-1})}=1-\sqrt{-1}.$

8. $(1-\sqrt{-1})^5=[(1-\sqrt{-1})^2]^2(1-\sqrt{-1})=$
$(-2\sqrt{-1})^2(1-\sqrt{-1})=-4(1-\sqrt{-1}).$

9. $2\sqrt{-24}-4\sqrt{-\frac{3}{2}}-3\sqrt{-\frac{1}{6}}=\frac{3}{2}\sqrt{-6}.$

10. $3\sqrt{-\frac{1}{6}}\times4\sqrt{-\frac{2}{3}}\times5\sqrt{-7}\div2\sqrt{-\frac{7}{2}}=-10\sqrt{2}.$

SQUARE ROOT OF A BINOMIAL SURD.

345. *The product or quotient of two dissimilar quadratic surds will be a quadratic surd.* Thus: (1) $\sqrt{ab}\times\sqrt{abc}=ab\sqrt{c}$; (2) $\sqrt{abc}\div\sqrt{ab}=\sqrt{c}$; (3) $\sqrt{ab}\times\sqrt{ac}=a\sqrt{bc}$.

For, since the surds are dissimilar, one or more of the factors under the two radical signs must be unlike (323);

hence in the product or quotient these unlike factors remain under the radical sign (326, 333, 335).

346. *The sum or difference of two dissimilar quadratic surds can be neither a rational number nor a single surd.*

For, if possible, let $\sqrt{a} \pm \sqrt{b} = \sqrt{c}$, in which $\sqrt{a}$ and $\sqrt{b}$ are dissimilar surds, and $\sqrt{c}$ either rational or a single surd. Then, by squaring, $a \pm 2\sqrt{ab} + b = c$.

That is, $\pm 2\sqrt{ab} = c - a - b$.

But $\sqrt{ab}$ is not rational (345); $\therefore \pm 2\sqrt{ab}$ can not be equal to $c - a - b$, a rational expression. Hence the supposition that $\sqrt{a} \pm \sqrt{b} = \sqrt{c}$ is false. Therefore $\sqrt{a} \pm \sqrt{b}$ can not be equal to a rational expression nor to a single surd.

347. *A quadratic surd can not equal the sum of a rational expression and a surd.*

If possible, let $\sqrt{a} = b \pm \sqrt{c}$; then $a = b^2 \pm 2b\sqrt{c} + c$.

$\therefore a - b^2 - c = \pm 2b\sqrt{c}$; that is, a rational number is equal to a surd, which is impossible. Therefore $\sqrt{a}$ can not equal $b \pm \sqrt{c}$.

348. If $a + \sqrt{b} = x + \sqrt{y}$, in which a and x are rational and $\sqrt{b}$ and $\sqrt{y}$ are quadratic surds, then will $a = x$ and $\sqrt{b} = \sqrt{y}$.

For, by transposing, $\sqrt{b} - \sqrt{y} = x - a$. Now if b and y were unequal, we would have the difference of two unequal surds equal to a rational expression, which is impossible (346). $\therefore b = y$, and therefore $a = x$.

349. If $\sqrt{a + \sqrt{b}} = x + \sqrt{y}$, then $\sqrt{a - \sqrt{b}} = x - \sqrt{y}$.

For, by squaring, $a + \sqrt{b} = x^2 + 2x\sqrt{y} + y$.

$\therefore a = x^2 + y$ and $\sqrt{b} = 2x\sqrt{y}$. (Art. 348.)

By subtracting, $a - \sqrt{b} = x^2 - 2x\sqrt{y} + y$. Extract square root, $\therefore \sqrt{a - \sqrt{b}} = x - \sqrt{y}$.

Similarly it may be shown that if $\sqrt{a + \sqrt{b}} = \sqrt{x} + \sqrt{y}$,

then $\sqrt{a - \sqrt{b}} = \sqrt{x} - \sqrt{y}$.

350. To extract the square root of a binomial surd, $a + \sqrt{b}$.

Let I. $\sqrt{a + \sqrt{b}} = \sqrt{x} + \sqrt{y}$;

then (349) II. $\sqrt{a - \sqrt{b}} = \sqrt{x} - \sqrt{y}$.

By multiplication, III. $\sqrt{a^2 - b} = x - y$.

By squaring I., IV. $a + \sqrt{b} = x + 2\sqrt{xy} + y$.

$\therefore$ V. $a = x + y$.

VI = V + III. $2x = a + \sqrt{a^2 - b}$. $\therefore x = \frac{1}{2}(a + \sqrt{a^2 - b})$.

VII = V − III. $2y = a - \sqrt{a^2 - b}$. $\therefore y = \frac{1}{2}(a - \sqrt{a^2 - b})$.

Therefore x and y are known; $\therefore \sqrt{x} + \sqrt{y}$ is known, and $\sqrt{x} - \sqrt{y}$ is known. $\therefore \sqrt{a + \sqrt{b}}$ and $\sqrt{a - \sqrt{b}}$ are known.

For example, extract the square root of $11 + 6\sqrt{2}$.

Here $x + y = 11$ and $x - y = \sqrt{121 - 72} = \sqrt{49} = 7$.

$\therefore x = 9$ and $\sqrt{x} = 3$; also, $y = 2$ and $\sqrt{y} = \sqrt{2}$.

$$\therefore \sqrt{11 + 6\sqrt{2}} = 3 + \sqrt{2}.$$

Or thus: $11 + 6\sqrt{2} = 9 + 6\sqrt{2} + 2 = (3 + \sqrt{2})^2$.

$$\therefore \sqrt{11 + 6\sqrt{2}} = 3 + \sqrt{2}.$$

351. From (350) it appears that $\sqrt{x} = \sqrt{\frac{1}{2}(a + \sqrt{a^2 - b})}$ and $\sqrt{y} = \sqrt{\frac{1}{2}(a - \sqrt{a^2 - b})}$; hence unless $a^2 - b$ be a perfect square, the values of $\sqrt{x}$ and $\sqrt{y}$ will be complex surds, and the expression $\sqrt{x} + \sqrt{y}$ will not be as simple as $\sqrt{a + \sqrt{b}}$ itself.

352. A binomial surd of the form $\sqrt{a^2c}+\sqrt{bc}$ may be written thus: $\sqrt{c}(a+\sqrt{b})$. If then a^2-b be a perfect square, the square root of $a+\sqrt{b}$ may be expressed in the form $\sqrt{x}+\sqrt{y}$. $\therefore$ the square root of $\sqrt{a^2c}+\sqrt{bc}$ is of the form $(\sqrt{x}+\sqrt{y})\sqrt[4]{c}$.

For example, find the square root of $4\sqrt{2}+2\sqrt{6}$.

$$4\sqrt{2}+2\sqrt{6}=\sqrt{2}(4+2\sqrt{3})=\sqrt{2}(\sqrt{3}+1)^2.$$

$$\therefore \sqrt{4\sqrt{2}+2\sqrt{6}}=\sqrt[4]{2}(4+2\sqrt{3})^{\frac{1}{2}}=\sqrt[4]{2}(\sqrt{3}+1).$$

Also, $\sqrt{11\sqrt{2}+12}=\sqrt{\sqrt{2}(11+6\sqrt{2})}=$

$$\sqrt[4]{2}\times\sqrt{11+6\sqrt{2}}=\sqrt[4]{2}(3+\sqrt{2}).$$

353. The square root of a binomial surd may frequently be found thus: *Find two numbers whose sum is the rational term, and whose product is the square of half the radical term. The square roots of these numbers, connected by the sign of the radical term, is the required root.*

For example, to determine by inspection the square root of $4-2\sqrt{3}$. The two numbers whose sum is 4 and whose product is 3, are 3 and 1. $\therefore \sqrt{4-2\sqrt{3}}=\sqrt{3}-1$.

Also, $\sqrt{18+8\sqrt{5}}=\sqrt{10}+\sqrt{8}=\sqrt{10}+2\sqrt{2}$, because 10 and 8 are the numbers whose sum is 18, and whose product is $(4\sqrt{5})^2=80$.

354. The square root of a binomial surd may also be found by solving the equation x^2- (rational term) $\times x=$ $-\frac{1}{4}$ (square of radical term), and connecting the square roots of the values of x by the sign of the radical term. Thus: Extract the square root of $7+2\sqrt{10}$.

Solve the equation $x^2-7x=-10$;

$$\therefore x^2-7x+\tfrac{49}{4}=\tfrac{9}{4};\quad \therefore x-\tfrac{7}{2}=\pm\tfrac{3}{2};\ x=5 \text{ or } 2.$$

$$\therefore \sqrt{7+2\sqrt{10}}=\sqrt{5}+\sqrt{2}.$$

In like manner the square root of $16 - 5\sqrt{7}$ is found by solving the equation $x^2 - 16x = -\frac{175}{4}$. Here $x = \frac{25}{2}$ or $\frac{7}{2}$.

$$\therefore \sqrt{16 - 5\sqrt{7}} = \sqrt{\tfrac{25}{2}} - \sqrt{\tfrac{7}{2}} = \tfrac{1}{2}(5\sqrt{2} - \sqrt{14}).$$

EXAMPLES.

Extract the square root of each of the following expressions:

1. $13 - 4\sqrt{10}$.	*Ans.* $2\sqrt{2} - \sqrt{5}$.
2. $10 - 2\sqrt{21}$.	*Ans.* $\sqrt{7} - \sqrt{3}$.
3. $9 + 4\sqrt{5}$.	*Ans.* $\sqrt{5} + 2$.
4. $21 - 4\sqrt{5}$.	*Ans.* $2\sqrt{5} - 1$.
5. $11 - 4\sqrt{7}$.	*Ans.* $\sqrt{7} - 2$.
6. $2\frac{1}{4} - \frac{3}{2}\sqrt{2}$.	*Ans.* $\frac{1}{2}(\sqrt{6} - \sqrt{3})$.
7. $8\sqrt{15} - 30$.	*Ans.* $\sqrt[4]{15}(\sqrt{5} - \sqrt{3})$.
8. $16\sqrt{7} - 42$.	*Ans.* $\sqrt[4]{7}(3 - \sqrt{7})$.
9. $11\sqrt{30} - 60$.	*Ans.* $\sqrt[4]{30}(\sqrt{6} - \sqrt{5})$.
10. $8\sqrt{2} + 2\sqrt{30}$.	*Ans.* $\sqrt[4]{2}(\sqrt{5} + \sqrt{3})$.
11. $9\sqrt{5} + 20$.	*Ans.* $\sqrt[4]{5}(\sqrt{5} + 2)$.
12. $15\sqrt{2} + 12\sqrt{3}$.	*Ans.* $\sqrt[4]{2}(3 + \sqrt{6})$.

EQUATIONS INVOLVING RADICALS.

355. To solve an equation containing a single radical: *Arrange the terms so as to have the radical alone on one side, and then raise both sides to a power corresponding to the order of the radical.* Thus:

1. Solve I. $\sqrt{x^2 - 6x + 24} = x + 2$.

II. Square I. $x^2 - 6x + 24 = x^2 + 4x + 4$. $\therefore x = 2$.

2. Solve $2x - \sqrt{x^2 - 1} = 3$.

II.	Transpose I.	$2x - 3 = \sqrt{x^2 - 1}$.
III.	Square II.	$4x^2 - 12x + 9 = x^2 - 1$.
IV.	Transpose III.	$3x^2 - 12x = -10$.
V.	Multiply IV by 12.	$36x^2 - 144x = -120$.
VI.	Complete the square.	$36x^2 - 144x + 144 = 24$.
VII.	Extract square root.	$6x - 12 = \pm 2\sqrt{6}$. $\therefore x = 2 \pm \frac{1}{3}\sqrt{6}$.

3. Solve $2x - \sqrt[3]{8x^3 + 26} + 2 = 0$.

II.	Transpose I.	$2x + 2 = \sqrt[3]{8x^3 + 26}$.
III.	Cube II.	$8x^3 + 24x^2 + 24x + 8 = 8x^3 + 26$.
IV.	Transpose III.	$24x^2 + 24x = 18$.
V.	Divide IV by 6.	$4x^2 + 4x = 3$.
VI.	Add 1.	$4x^2 + 4x + 1 = 4$.
VII.	Extract root.	$2x + 1 = \pm 2$. $\therefore x = \frac{1}{2}$ or $-\frac{3}{2}$.

356. If an equation contain two radicals, two steps may be necessary in order to clear the equation of radicals. It is usually best to have the larger radical term alone on one side; but if the radicals are reciprocals, both are placed on one side, and the remaining terms on the other. As before, the equation is raised to a power corresponding to the order of the radicals.

4. Solve $\sqrt{x + 4} + \sqrt{2x - 1} = 6$.

Transpose I.	∴ II.	$\sqrt{2x - 1} = 6 - \sqrt{x + 4}$.
Square II.	∴ III.	$2x - 1 = 36 - 12\sqrt{x + 4} + x + 4$.
Transpose III.	∴ IV.	$x - 41 = -12\sqrt{x + 4}$.
Square IV.	∴ V.	$x^2 - 82x + 1681 = 144x + 576$.
Transpose V.	∴ VI.	$x^2 - 226x = -1105$. $\therefore x = 221$ or 5.

357. If we attempt to verify the above values, we find that 5 satisfies the given equation, but that 221 does not.

We find further that the four equations,

I. $\sqrt{x + 4} + \sqrt{2x - 1} = 6$; II. $-\sqrt{x + 4} + \sqrt{2x - 1} = 6$;
III. $\sqrt{x + 4} - \sqrt{2x - 1} = 6$; IV. $-\sqrt{x + 4} - \sqrt{2x - 1} = 6$,

all reduce to the quadratic $x^2 - 226x = -1105$, whose roots are 221 and 5. Now 5 satisfies the first of these forms and 221 the second, whilst neither 5 nor 21 satisfies the third or fourth forms.

These examples show that when an equation has been reduced to a rational form by squaring, the roots found may or may not satisfy the equation in the form originally given. Both 221 and 5 would satisfy all the forms, however, if we agree that every indicated square root may be either + or −, and that we take whichever sign produces the proper result.

358.—5. Solve $\dfrac{ax - b^2}{\sqrt{ax} + b} = c + \dfrac{\sqrt{ax} - b}{c}$.

II. Simplify each side of I. $\sqrt{ax} - b = c + \dfrac{\sqrt{ax}}{c} - \dfrac{b}{c}$.

III. Transpose II and clear. $\sqrt{ax}(c - 1) = c^2 + bc - b$.

IV. Square III. $ax(c - 1)^2 = (c^2 + bc - b)^2$.

V. Divide IV by $a(c - 1)^2$. $x = \dfrac{(c^2 + bc - b)^2}{a(c - 1)^2}$.

359.—6. Solve $\sqrt{2x - 1} + \sqrt{3x - 2} = \sqrt{4x - 3} + \sqrt{5x - 4}$.

II. Transpose I. $\sqrt{2x - 1} - \sqrt{5x - 4} = \sqrt{4x - 3} - \sqrt{3x - 2}$.

III. Square II. $2x - 1 - 2\sqrt{(2x - 1)(5x - 4)} + 5x - 4 =$
$4x - 3 - 2\sqrt{(4x - 3)(3x - 2)} + 3x - 2$.

$\therefore$ IV. $-2\sqrt{(2x - 1)(5x - 4)} = -2\sqrt{(4x - 3)(3x - 2)}$.

V. Divide IV by -2 and square the result.
$10x^2 - 13x + 4 = 12x^2 - 17x + 6$. $\therefore x = 1$.

EXAMPLES.

Find the value of x in the following:

7. $\sqrt{x + 2} - \sqrt{x - 3} = 1$. *Ans.* $x = 7$.

8. $\sqrt{3x + 1} - \sqrt{2x - 1} = 1$. *Ans.* $x = 5$ or 1.

9. $\sqrt[3]{5x - 3} + 2 = 5$. *Ans.* $x = 6$.

10. $\dfrac{5x-1}{\sqrt{5x}-1}=4+\dfrac{\sqrt{5x}+1}{3}$. *Ans.* $x=5$.

11. $\sqrt[3]{8x^3+61}-1=2x$. *Ans.* $x=2$ or $-2\frac{1}{2}$.

12. $\dfrac{1}{x+\sqrt{2-x^2}}+\dfrac{1}{x-\sqrt{2-x^2}}=\dfrac{1}{2}$. *Ans.* $x=1\pm\sqrt{2}$.

13. $\dfrac{a-\sqrt{a^2-x^2}}{a+\sqrt{a^2-x^2}}=c^2$. *Ans.* $x=\pm\dfrac{2ac}{c^2+1}$.

14. $x+\sqrt{9+x^2}=\dfrac{18}{\sqrt{9+x^2}}$. *Ans.* $x=\pm\sqrt{3}$.

EXERCISE XIX.

Simplify (328):

1. $\sqrt{8}$; $\sqrt{48}$; $\sqrt{125}$; $\sqrt{18a^2}$; $\sqrt{72x^2y}$; $\sqrt{45(a-b)^3}$; $\sqrt{96a^2b^{-2}}$.

2. $\sqrt[3]{54x^3y^6}$; $\sqrt[3]{40a^4}$; $\sqrt[3]{686}$; $\sqrt[3]{16(x-y)^3}$; $\sqrt[3]{81(x-y)^{-6}}$.

3. $4\sqrt{18}$; $3\sqrt{12}$; $\frac{2}{3}\sqrt{27}$; $\frac{1}{2}\sqrt{72}$; $\dfrac{1}{a}\sqrt{a^3}$; $(a-b)\sqrt{(a+b)^3}$.

4. $\frac{1}{3}\sqrt[3]{27a^{-4}}$; $(a-b)^{-1}\sqrt[3]{(a-b)^4}$; $\sqrt[3]{(b^2-a^2)(a-b)^2}$.

5. $\sqrt{\frac{4}{3}}$; $\sqrt{\frac{27}{8}}$; $4\sqrt{\frac{40}{3}}$; $(a-b)\sqrt{\dfrac{a+b}{a-b}}$; $(a+b)(a-b)^{-\frac{1}{2}}$.

6. $3\sqrt[3]{\frac{1}{9}}$; $\sqrt[3]{3\frac{3}{4}}$; $(a+b)\sqrt[3]{\dfrac{1}{(a+b)^{\frac{1}{2}}}}$; $\sqrt[3]{(a+b)(a-b)^{-2}}$.

7. $\sqrt[4]{32}$; $\sqrt[5]{64}$; $a\sqrt[4]{32a^5}$; $\sqrt[4]{a^6-a^5b}$; $\sqrt[n]{4a^nb^{2n}}$; $\sqrt[x]{a^{2x}b^{x+1}}$.

8. $\sqrt[4]{9}$; $\sqrt[6]{8}$; $\sqrt[8]{16}$; $\sqrt[4]{a^2-2ab+b^2}$; $\sqrt[10]{(a-b)^5}$; $\sqrt[6]{-8(a+b)^3}$.

9. $\sqrt[9]{a^3-3a^2b+3ab^2-b^3}$; $\sqrt[8]{81a^4b^8(a-b)^4}$; $\sqrt[6]{\frac{4}{25}}$: $3\sqrt[8]{\frac{1}{16}}$.

Reduce to the same order (330):

10. $\sqrt[3]{4}$ and $\sqrt{3}$; $\sqrt[3]{16}$ and $\sqrt{2}$; $\sqrt[3]{2}$ and $\sqrt{\frac{1}{3}}$; $\sqrt[4]{6}$ and $\sqrt{3}$.

Compare the following surds (331):

11. $3\sqrt{3}$ and $2\sqrt{7}$; $3\sqrt[3]{3}$ and $2\sqrt[3]{10}$; $4\sqrt[3]{2}$ and $3\sqrt[3]{5}$;

12. $\frac{1}{2}\sqrt{2}$ and $\frac{1}{3}\sqrt[4]{27}$; $\sqrt{5}$ and $2\sqrt[3]{\frac{3}{2}}$; $2\sqrt{5}$ and $\sqrt[3]{88}$.

Simplify (332):

13. $4\sqrt{24} + \sqrt{54} - \sqrt{96} + 3\sqrt{50} - 2\sqrt{98} - 5\sqrt{72}$.

14. $\sqrt{\frac{1}{3}} + \sqrt{\frac{75}{9}} - \sqrt{3} - \frac{1}{2}\sqrt{\frac{1}{2}} + \frac{1}{4}\sqrt{18} + \sqrt[4]{4}$.

15. $5\sqrt{a^3 - a^2b} + \sqrt{9a^3 - 9a^2b} - 8\sqrt{ab^2 - b^3}$.

16. $\sqrt{a^2(a - 3b) + b^2(3a - b)} - \sqrt{(a + b)(a^2 - b^2)}$.

17. $\sqrt[6]{25} - \frac{5}{3}\sqrt[3]{\frac{27}{25}} - \sqrt[3]{\frac{5}{8}} + \sqrt[3]{40} - \sqrt[3]{-625}$.

Simplify (333):

18. $5\sqrt{12} \times 2\sqrt{2}$; $3\sqrt{\frac{1}{8}} \times \frac{1}{3}\sqrt{16}$; $2\sqrt{12} \times \frac{1}{2}\sqrt{20}$.

19. $\frac{1}{3}\sqrt[3]{2} \times 6\sqrt{\frac{1}{2}}$; $(\sqrt{8} + \sqrt{18}) \times \sqrt{6}$; $(\sqrt{3} - \sqrt{2})^2$.

20. $\sqrt{3} \times \sqrt[3]{3}$; $\sqrt[3]{4} \times \sqrt{8} \times \sqrt[4]{9}$; $3\sqrt[4]{2} \times 2\sqrt[4]{4} \times \frac{1}{6}\sqrt[8]{9}$.

21. $\sqrt{(a - b)} \times \sqrt{(a - b)^2}$; $(a - b)\sqrt{(a - b)} \times \sqrt{(a - b)^4}$.

22. $(\sqrt{3} - \sqrt{2})(\sqrt{3} + \sqrt{2})$; $\sqrt[a]{x + y} \times \sqrt[b]{x + y}$.

Simplify (335):

23. $(2\sqrt{3} - 3\sqrt{2} - \sqrt{30}) \div \sqrt{6}$; $(\frac{5}{7}\sqrt[3]{\frac{4}{9}} \div \frac{10}{21}\sqrt[3]{\frac{2}{3}}) \times 2\sqrt[3]{3}$.

24. $(\sqrt{6a^2 + 6a^2b^2} + \sqrt{24a^2 + 24a^2b^2}) \div 6\sqrt{1 + b^2}$.

25. $(\sqrt[6]{54} \div \sqrt[3]{3}) \times \sqrt[6]{6} + \sqrt[3]{2} \times \sqrt[3]{3} + \frac{1}{2}\sqrt[6]{72} \div \frac{1}{4}\sqrt[6]{2}$.

Rationalize the denominators (336–341, inclusive):

26. $\frac{2}{\sqrt{2}}$; $\frac{3}{2\sqrt{2}}$; $\frac{2}{\sqrt[3]{4}}$; $\frac{4}{\sqrt[4]{8}}$; $\frac{3}{\sqrt[5]{9}}$; $\frac{\sqrt{a} + \sqrt{c}}{\sqrt{ac}}$.

27. $\frac{\sqrt{6}}{\sqrt{3} + \sqrt{2}}$; $\frac{\sqrt{5} + \sqrt{3}}{\sqrt{5} - \sqrt{3}}$; $\frac{2 + \sqrt{3}}{2 - \sqrt{3}}$; $\frac{1}{\sqrt{3} + 2\sqrt{2}}$.

28. $\dfrac{4}{\sqrt{5}-1}$; $\dfrac{4\sqrt{7}+2}{2\sqrt{7}-1}$; $\dfrac{\sqrt{12}+\sqrt{8}}{\sqrt{3}-\sqrt{2}}$; $\dfrac{\sqrt{27}-\sqrt{12}}{\sqrt{6}-\sqrt{3}}$.

29. $\dfrac{1}{\sqrt[3]{4}-1}$; $\dfrac{\sqrt[3]{2}}{\sqrt[3]{4}-\sqrt[3]{2}}$; $\dfrac{1}{\sqrt[3]{16}+2+\sqrt[3]{4}}$.

30. $\dfrac{3}{\sqrt{3}-\sqrt{2}+\sqrt{5}}$; $\dfrac{\sqrt{5}}{\sqrt{5}+\sqrt{2}+\sqrt{7}}$; $\dfrac{\sqrt{10}+2}{\sqrt{5}-\sqrt{3}+\sqrt{2}}$.

Simplify (342, 343, 344):

31. $\sqrt{-18}-\sqrt{-8}+\frac{1}{2}\sqrt{-\frac{1}{2}}-\frac{3}{2}\sqrt{-2}-2\sqrt[6]{-8}$.

32. $2\sqrt{-3}\times 3\sqrt{-2}\times\sqrt[6]{-8}\times\sqrt{-\frac{3}{4}}$.

33. $(3\sqrt{-6}\times\frac{1}{3}\sqrt{-2})^3$; $(-1-\sqrt{-3})^2$; $(\sqrt{-28}-\sqrt{-7})^2$.

34. $\sqrt{-4}\div\sqrt{\frac{1}{2}}$; $\frac{1}{2}\sqrt{-3}\div\frac{1}{3}\sqrt{-2}$; $6\sqrt{-10}\div\frac{1}{2}\sqrt{-5}$.

35. $\sqrt[4]{-16}+\sqrt[4]{-81}+3\sqrt[4]{2\sqrt{-3}+2}\times\sqrt[4]{2\sqrt{-3}-2}$.

36. $\sqrt[4]{6+\sqrt{-13}}\times\sqrt[4]{6-\sqrt{-13}}-35\sqrt{\frac{1}{7}}$.

Simplify (350, 352, 353, 354):

37. $\sqrt{5-2\sqrt{6}}+\sqrt{14-4\sqrt{6}}-\sqrt{30-12\sqrt{6}}$.

38. $\sqrt[4]{17-12\sqrt{2}}+\sqrt[4]{49+20\sqrt{6}}-\sqrt[4]{28-16\sqrt{3}}$.

39. $\dfrac{1}{(4-2\sqrt{3})^{\frac{1}{2}}}+\dfrac{1}{\sqrt{10+4\sqrt{6}}}-\dfrac{1}{\sqrt{7-4\sqrt{3}}}$.

40. $\sqrt{4+3\sqrt{2}}\div\sqrt{2\sqrt{2}+\sqrt{6}}-\sqrt{4\sqrt{2}-2\sqrt{6}}$.

41. $\sqrt{3-2\sqrt{2}}$; $\sqrt{6-4\sqrt{2}}$; $\sqrt{7-2\sqrt{10}}$; $\sqrt{9-4\sqrt{2}}$.

42. $(8+2\sqrt{15})^{\frac{1}{2}}$; $\sqrt{7-2\sqrt{6}}$; $\sqrt{11-4\sqrt{7}}$; $(2-\sqrt{3})^{\frac{1}{2}}$.

Solve (355–359, inclusive):

43. $\sqrt{x-7}=-3$.

44. $\sqrt{x^2-6x+7}=x+2$.

45. $\sqrt{2x^2-x+1}=x+1$.

46. $\sqrt{x+12}-\sqrt{x-12}=2$.

47. $\sqrt{x+5} + \sqrt{x-3} = 4.$

48. $\sqrt[6]{x + 2\sqrt{1+x}} = \sqrt[3]{1 - \sqrt{1+x}}.$

49. $\sqrt{5x - 7a} + \sqrt{5x} = 7a^{\frac{1}{2}}.$

50. $\sqrt{x} + \sqrt{1+x} = \dfrac{2}{\sqrt{1+x}}.$

51. $a\sqrt{x} - b\sqrt{x} = (a+b)(a-b).$

52. $\frac{1}{2}\sqrt{2+x} + \dfrac{1}{x}\sqrt{2+x} = \dfrac{1}{b^{\frac{3}{2}}}\sqrt{2x}.$

53. $\dfrac{1}{x+\sqrt{x^2-1}} + \dfrac{1}{x-\sqrt{x^2-1}} = 6.$

54. $\sqrt{\dfrac{x-2}{x+2}} + \sqrt{\dfrac{x+2}{x-2}} = 3.$

55. $\sqrt{a+x} = \sqrt{b} - \sqrt{a-x}.$

56. $x + \sqrt{x+5} = 7.$

57. $3x - 2\sqrt{3x+7} = 8.$

58. $2x + 3\sqrt{5x} = 25.$

59. $-\sqrt{3x+7a} + 7\sqrt{a} = \sqrt{3x}.$

60. $\sqrt{2x+7} + \sqrt{3x-18} = \sqrt{7x+1}.$

61. $\sqrt{\dfrac{x^2-16}{x-3}} + \sqrt{x+3} = \dfrac{7}{\sqrt{x-3}}.$

62. $x\sqrt{6x^{-1}+x} = \dfrac{1+x^2}{x^{\frac{1}{2}}}.$

63. $\sqrt{x} + \sqrt{\{x - \sqrt{[1-x]}\}} = 1.$

64. $\dfrac{x^3+1}{x^2-1} = x + \sqrt{6x^{-1}}.$

CHAPTER XX.

QUADRATIC EQUATIONS.

CONTINUED FROM CHAPTER XII.

360. THE **General Method** of solving affected quadratics (187, 190) is as follows:

(1) *Reduce the equation (by clearing of fractions, transposition, and combining like terms) to the form* $ax^2 + bx = c$.

(2) *Multiply or divide both sides by such a number that the co-efficient of* x^2 *shall be a square.* Thus: $a^2x^2 + abx = ac$.

(3) *Complete the square of the first side by adding to each member the square of the quotient obtained from dividing the second term by twice the square root of the first term.* Thus: $a^2x^2 + abx + \left(\frac{b}{2}\right)^2 = ac + \frac{b^2}{4}$.

(4) *Extract the square root of each side.* Thus:

$$ax + \frac{b}{2} = \pm\sqrt{ac + \frac{b^2}{4}}.$$

(*Place the double sign before one side, as explained in Art.* 189.)

(5) *Transpose the known term to the second side, and divide both sides by the co-efficient of* x. Thus:

$$ax = -\frac{b}{2} \pm \sqrt{ac + \frac{b^2}{4}} = -\frac{b}{2} \pm \frac{1}{2}\sqrt{4ac + b^2}.$$

$$\therefore x = \frac{1}{a}\left(-\frac{b}{2} \pm \frac{1}{2}\sqrt{4ac + b^2}\right) = \frac{1}{2a}(-b \pm \sqrt{4ac + b^2}).$$

361. Two modifications of the general method are often employed,—the *Common Method* and the *Hindoo Method.*

COMMON METHOD OF SOLVING QUADRATICS.

362. RULE. (1) *Reduce the equation to the form* $ax^2+bx=c$. (2) *Divide both sides by the co-efficient of* x^2. Thus: $x^2+\frac{b}{a}x=\frac{c}{a}$.

(3) *Add to each side the square of half the co-efficient of* x. Thus: $x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2=\frac{c}{a}+\frac{b^2}{4a^2}=\frac{4ac+b^2}{4a^2}$.

(4) *Extract the square root of each side.* Thus:

$$x+\frac{b}{2a}=\pm\frac{\sqrt{4ac+b^2}}{2a}.$$

(5) *Transpose the known term to the second side.* Thus:

$$x=\frac{-b\pm\sqrt{4ac+b^2}}{2a}.$$

363.—1. Solve $2x^2-7x+3=0$.

Transpose $+3$. $\therefore\ 2x^2-7x=-3$.

Divide by 2. $\therefore\ x^2-\frac{7}{2}x=-\frac{3}{2}$.

Add $(\frac{7}{4})^2$ to each side. $\therefore\ x^2-\frac{7}{2}x+(\frac{7}{4})^2=\frac{49}{16}-\frac{3}{2}=\frac{25}{16}$.

Extract square root. $\therefore\ x-\frac{7}{4}=\pm\frac{5}{4}$. $\therefore\ x=3$ or $\frac{1}{2}$.

Verification: (1) $2(3)^2-7(3)+3=18-21+3=0$.

(2) $2(\frac{1}{2})^2-7(\frac{1}{2})+3=\frac{1}{2}-\frac{7}{2}+3=0$.

364.—2. Solve $\frac{1}{x-1}-\frac{1}{x+3}=\frac{1}{35}$.

Clear of fractions. $\therefore\ 35x+105-35x+35=x^2+2x-3$.

Transpose. $\therefore\ -x^2-2x=-143$.

Divide by -1. $\therefore\ x^2+2x=143$.

Add $1^2=1$ to each side. $\therefore\ x^2+2x+1=144$.

Extract square root. $\therefore\ x+1=\pm 12$. $\therefore\ x=11$ or -13.

Verification: (1) $\frac{1}{11-1}-\frac{1}{11+3}=\frac{1}{10}-\frac{1}{14}=\frac{1}{35}$.

(2) $\frac{1}{-13-1}-\frac{1}{-13+3}=\frac{1}{-14}-\frac{1}{-10}=\frac{1}{10}-\frac{1}{14}=\frac{1}{35}$.

365.—3. Solve $\frac{2x+9}{9}+\frac{4x-3}{4x+3}=3+\frac{3x-16}{18}$.

Transpose $\frac{2x+9}{9}$ and multiply each term by 18.

$$\therefore \frac{72x-54}{4x+3}=54+3x-16-4x-18=20-x.$$

Clear of fractions. $\therefore 72x-54=-4x^2+77x+60.$

Transpose and simplify. $\therefore 4x^2-5x=114.$

Divide by 4. $\therefore x^2-\frac{5}{4}x=\frac{114}{4}.$

Add $(\frac{5}{8})^2$. $\therefore x^2-\frac{5}{4}x+(\frac{5}{8})^2=\frac{25}{64}+\frac{114}{4}=\frac{1849}{64}.$

Extract root. $\therefore x-\frac{5}{8}=\pm\frac{43}{8}. \quad \therefore x=6$ or $-\frac{19}{4}.$

HINDOO METHOD OF SOLVING QUADRATICS.

366. RULE. (1) *Reduce the equation to the form* $ax^2+bx=c$.

(2) *Multiply both sides by four times the co-efficient of* x^2. Thus: $4a^2x^2+4abx=4ac$.

(3) *Add to each side the square of the co-efficient of* x *in the equation* $ax^2+bx=c$; *that is, add* b^2. Thus: $4a^2x^2+4abx+b^2=4ac+b^2$.

(4) *Extract the square root of each side.* Thus: $2ax+b=\pm\sqrt{4ac+b^2}$.

(5) *Transpose the known term to the second side, and divide both sides by the co-efficient of* x. Thus:

$$2ax=-b\pm\sqrt{4ac+b^2}.$$

$$\therefore x=\frac{-b\pm\sqrt{4ac+b^2}}{2a}.$$

367.—4. Solve $(2x-3)^2=8x$.

Expand. $\therefore 4x^2-12x+9=8x.$

Transpose. $\therefore 4x^2-20x=-9.$

Multiply by 16. $\therefore (8x)^2-320x=-144.$

Add $(20)^2$. $\therefore (8x)^2-320x+20^2=400-144=256.$

Extract root. $\therefore 8x-20=\pm 16. \quad \therefore x=4\frac{1}{2}$ or $\frac{1}{2}.$

368.—5. Solve $\frac{6}{x}+\frac{x}{6}=\frac{5}{4}(x-1)$.

Multiply by $12x$.	$\therefore\ 72+2x^2=15x^2-15x$.
Transpose and simplify.	$\therefore\ -13x^2+15x=-72$.
Multiply by $4(-13)$.	$\therefore\ (26x)^2-(\quad)=3744$.
Add $(15)^2$ to each side.	$\therefore\ (26x)^2-(\quad)+15^2=3969$.
Extract the square root.	$\therefore\ 26x-15=\pm 63$.
	$\therefore\ x=3$ or $-\frac{24}{13}$.

369. It will be observed that, after multiplying by four times the co-efficient of x^2, (1) The co-efficient of the first term is the square of twice the co-efficient of x^2 in the original equation, since $4a^2$ is $(2a)^2$; (2) The second term of the product is not used in the work. Advantage is taken of these truths to save multiplication, as in Ex. 5 above.

SPECIAL METHOD OF SOLVING QUADRATICS.

370.—6. I. Solve $x^2+4=5x$.

II. Divide by x.	$x+\frac{4}{x}=5$.
III. Square each side.	$x^2+8+\frac{16}{x^2}=25$.
IV. Subtract 2×8 from each side.	$x^2-8+\frac{16}{x^2}=9$.
V. Extract the square root.	$x-\frac{4}{x}=\pm 3$.
VI. Add V and II.	$2x=5\pm 3=8$ or 2.
	$\therefore\ x=4$ or 1.

371.—7. I. Solve $x^2+24=10x$.

II. Divide by x.	$x+24x^{-1}=10$.
III. Square II.	$x^2+48+(\quad)=100$.
IV. Subtract 2×48.	$x^2-48+(\quad)=100-96=4$.
V. Extract square root.	$x-24x^{-1}=\pm 2$.
VI. Add V and II.	$2x=10\pm 2=12$ or 8.
	$\therefore\ x=6$ or 4.

372.—8. I. Solve $\frac{x-6}{x-12} - \frac{x-12}{x-6} = \frac{5}{6}$.

II. Square. $\left(\frac{x-6}{x-12}\right)^2 - 2 + \left(\frac{x-12}{x-6}\right)^2 = \frac{25}{36}$.

III. Subtract -2×2. $\left(\frac{x-6}{x-12}\right)^2 + 2 + \left(\frac{x-12}{x-6}\right)^2 = 4\frac{25}{36}$.

IV. Extract square root. $\frac{x-6}{x-12} + \frac{x-12}{x-6} = \pm\frac{13}{6}$.

V. Add IV and I. $2\left(\frac{x-6}{x-12}\right) = \frac{5}{6} \pm \frac{13}{6} = 3$ or $-\frac{4}{3}$.

$\therefore 2x - 12 = 3x - 36$, whence $x = 24$;
or $6x - 36 = -4x + 48$, whence $x = 8.4$.

373.—9. I. Solve $x^2 + (a^2 - b^2) = 2ax$.

II. Divide by x. $x + \left(\frac{a^2 - b^2}{x}\right) = 2a$.

III. Square. $x^2 + 2(a^2 - b^2) + (\quad) = 4a^2$.

IV. Subtract $2 \times 2(a^2 - b^2)$. $x^2 - 2(a^2 - b^2) + (\quad) = 4b^2$.

V. Extract square root. $x - (\quad) = \pm 2b$.

VI. Add V and II. $2x = 2a \pm 2b$. $\therefore x = a \pm b$.

374. Rule for solving quadratics by the special method.

(1) *Reduce the equation to the form* $ax^2 - c = -bx$.

(2) *Divide both sides by* x. Thus: $ax - \frac{c}{x} = -b$.

(3) *Square the resulting equation.* Thus:

$$(ax)^2 - 2ac + \left(\frac{c}{x}\right)^2 = b^2.$$

(4) *Subtract twice the second term from each side; that is, add* $4ac$. Thus: $(ax)^2 + 2ac + \left(\frac{c}{x}\right)^2 = b^2 + 4ac$.

(5) *Extract the square root.* $ax + \frac{c}{x} = \pm\sqrt{b^2 + 4ac}$.

(6) *Add the equations found by using the second and fifth steps.* Thus: $2ax = -b \pm \sqrt{b^2 + 4ac}$.

(7) *Solve the resulting simple equation.* Thus:

$$x = \frac{-b \pm \sqrt{b^2 + 4ac}}{2a}.$$

The last term on the first side being the same in each step, need not be expressed, as in Examples 7 and 9 above.

In Example 8, above, an expression takes the place of x.

375. The general equation $ax^2 + bx = c$ has been solved by each of the preceding methods, and it is found that $x = \frac{-b \pm \sqrt{b^2 + 4ac}}{2a}$. We may make use of this formula, therefore, instead of going through the process of finding x.

10. Find x in the equation, $-x^2 + 14x = 33$.

Here $a = -1$, $b = +14$, $c = +33$. $b^2 + 4ac = 196 - 132 = 64$.

$$\therefore x = \frac{-14 \pm \sqrt{64}}{-2} = \frac{-14 \pm 8}{-2} = +3 \text{ or } +11.$$

In Example 1, Art. 363 (solve $2x^2 - 7x + 3 = 0$),

$a = 2$, $b = -7$, $c = -3$. $4ac + b^2 = -24 + 49 = 25$.

$$\therefore \frac{-b \pm \sqrt{4ac + b^2}}{2a} = \frac{+7 \pm 5}{4} = 3 \text{ or } \tfrac{1}{2}.$$

376. In the examples heretofore considered we have found two different roots of a quadratic equation; in some cases, however, we shall find really only one root. For example:

11. If $x^2 - 6x + 9 = 0$, by extracting the square root $x - 3 = 0$, and therefore $x = 3$. It is, however, convenient in such cases to say that the equation has two equal roots.

377. If the quadratic equation be represented by $ax^2 + bx = c$, we have found that the roots are $\frac{-b + \sqrt{4ac + b^2}}{2a}$ and $\frac{-b - \sqrt{4ac + b^2}}{2a}$.

(1) These roots are different, unless $4ac + b^2 = 0$, and then each of them is $-\frac{b}{2a}$.

(2) Since the two roots have the same expression, $\sqrt{4ac + b^2}$, both will be real or both will be imaginary: real when $4ac + b^2$ is positive; imaginary when $4ac + b^2$ is negative.

(3) Both roots will be rational if $4ac + b^2$ be a perfect square. Both roots will be surds if $4ac + b^2$ be not a perfect square.

Exercise XX.

Solve by the General Method (360):

1. $4x^2 - 3(x - 1) = 10x.$
2. $2x^2 + x = 3.$
3. $15x^2 - 14x = -3.$
4. $(3x - 2)(x - 1) = 14.$
5. $(x - 7)(x - 4) + (2x - 3)(x - 5) = 103.$
6. $8x + 11 + \frac{7}{x} = \frac{21 + 65x}{7}.$
7. $\frac{x + 2}{x - 1} - \frac{4 - x}{2x} = \frac{7}{3}.$
8. $\frac{x + 4}{x - 4} + \frac{2x - 12}{x + 4} = \frac{10}{3}.$

Solve by the Common Method (362):

9. $x^2 + 7x = 8.$
10. $x^2 - \frac{1}{3}x = 34.$
11. $2x^2 - 3x = 54.$
12. $2x^2 + 1 = 11(x + 2).$
13. $\frac{1}{3} + \frac{1}{3 + x} + \frac{1}{3 + 2x} = 0.$
14. $\frac{3}{5 - x} + \frac{1}{4 - x} = \frac{8}{x + 2}.$
15. $x - \frac{x^3 - 8}{x^2 + 5} = 2.$
16. $\frac{x}{x - 1} = \frac{3}{2} + \frac{x - 1}{x}.$

Solve by the Hindoo Method (366):

17. $(2x+3)^2 = -8x$.

18. $\frac{5}{x+2} + \frac{3}{x} = \frac{14}{x+4}$.

19. $(5x-3)^2 - 7 = 44x + 5$.

20. $\frac{3}{4}(x^2-3) = \frac{1}{8}(x-3)$.

21. $\frac{x+22}{3} - \frac{4}{x} = \frac{9x-6}{2}$.

22. $\frac{x+4}{3} - \frac{7-x}{x-3} = \frac{4x+7}{9} - 1$.

23. $\frac{3x-7}{x} + \frac{4x-10}{x+5} = \frac{7}{2}$.

24. $\frac{4x+7}{19} + \frac{5-x}{3+x} = \frac{4x}{9}$.

Solve by the Special Method (374):

25. $2x^{\frac{1}{2}} + \frac{3}{x^{\frac{1}{2}}} = 7$.

26. $x^{\frac{1}{4}} - \frac{56}{x^{\frac{1}{4}}} = -1$.

27. $\sqrt{x} - 3x^{-\frac{1}{2}} = -2$.

28. $6x^{-\frac{1}{2}} - x^{\frac{1}{2}} = -1$.

29. $x^{\frac{1}{2}} - \frac{20}{x^{\frac{1}{2}}} = -3\sqrt{5}$.

30. $x + c(a-b)x^{-1} = a-b+c$.

31. $2x + \frac{a^2-b^2}{x} = 3a - b$.

32. $3x(b+c) - \frac{2a^2}{x(b+c)} = a$.

Solve by Formula (375):

33. $\frac{5}{7}x^2 + \frac{5}{7}x = -\frac{9}{140}$.

34. $780x^2 - 73x = -1$.

35. $a^2(x^2+a^2) = 2a^3x + 1$.

36. $x^2 - x(2a-b) = 2ab$.

37. $2x^2 = \frac{x+5}{2} + 14$.

38. $\frac{3}{x+2} + \frac{5}{x+4} = \frac{14}{x+6}$.

39. $4x + \frac{a^2}{x} - \frac{b^2}{x} = 4a$.

40. $x^2 - x\left(\frac{2a^2+2b^2}{a^2-b^2}\right) = -1$.

CHAPTER XXI.

PROBLEMS INVOLVING QUADRATICS.

378.—1. A SELLS a box for $24, and by so doing he gains as much per cent as the box cost. What was the cost of the box?

Let x denote the number of dollars the box cost.

Then $\frac{x}{100}$ denotes one per cent of the price, and x times $\frac{x}{100} = \frac{x^2}{100}$ denotes x per cent of the price.

Since the cost is x, and the gain is $\frac{x^2}{100}$, therefore $x + \frac{x^2}{100}$ is the selling price.

But 24 equals the selling price.

$\therefore$ (Ax. 1) $x + \frac{x^2}{100} = 24$, whence $x = 20$ or -120.

Only the positive value of x is admissible; therefore the box cost $20.

379.—2. A rows 7 miles down stream and back again in 3 hours 20 minutes. Supposing the river to have a current of 2 miles per hour, find the rate of rowing.

Let x denote the rate of rowing; that is, the number of miles A could row per hour in still water.

Then $x + 2$ is the rate down stream, and $x - 2$ is the rate up stream. Since the distance down is 7, and the rate down is $x + 2$, $\therefore \frac{7}{x+2}$ is the number of hours required to go 7 miles down stream. Similarly, $\frac{7}{x-2}$ is the time required to go 7 miles up stream.

The time down being $\frac{7}{x+2}$, and the time up being $\frac{7}{x-2}$, therefore the entire time is $\frac{7}{x+2}+\frac{7}{x-2}$. But the entire time is $3\frac{1}{3}$. $\therefore$ (Ax. 1) $\frac{7}{x+2}+\frac{7}{x-2}=\frac{10}{3}$, whence $x=5$ or $-\frac{4}{5}$.

The negative answer to the quadratic, $-\frac{4}{5}$, does not apply to the question proposed, hence it is rejected.

Ans. The rate of rowing is 5 miles an hour.

380.—**3.** A left C for D at the same instant that B left D for C. A reached D 9 hours, and B reached C 16 hours, after they met on the road. Find the time A required for the journey.

Let x denote the number of hours A required to perform the journey. Since B evidently required 7 hours longer, $\therefore$ B's time was $x+7$ hours.

Let d denote the distance from C to D; then A's rate was $\frac{d}{x}$, and B's rate was $\frac{d}{x+7}$. From the time they started until they met was $x-9$ hours, hence A traveled in that time $\frac{d(x-9)}{x}$ miles. But B performed this distance in 16 hours; $\therefore$ $\frac{d(x-9)}{16x}$ was B's rate. We now have two expressions for B's rate; $\therefore$ $\frac{d(x-9)}{16x}=\frac{d}{x+7}$, whence $x^2-2x-63=16x$ and $x=25$ or -3. As before, reject the negative root; therefore A's time was 25 hours.

381.—**4.** A worked x days, and received as wages \$96; B worked $x-6$ days for \$54. Had A worked $x-6$ days and B x days, they would have received equal sums. Find x; also find the daily wages of each.

Since A received \$96 for x days' work, he received $\frac{96}{x}$ dollars per day, and for $x - 6$ days he would have received $\frac{96(x-6)}{x}$ dollars. Since B received \$54 for $x - 6$ days' work, he received $\frac{54}{x-6}$ dollars per day, and for x days he would have received $\frac{54x}{x-6}$ dollars. $\therefore \frac{96(x-6)}{x} = \frac{54x}{x-6}$, whence $16(x-6)^2 = 9x^2$ and $4(x-6) = \pm 3x$, from which $x = 24$ or $\frac{24}{7}$. But $\frac{24}{7}$ is not admissible, because B's time would then be $(\frac{24}{7} - 6)$ days $= -\frac{18}{7}$ days.

Ans. $x = 24$; A received \$4 a day and B \$3 a day.

382. In solving problems, as in the preceding examples of this chapter, results will sometimes be obtained which do not apply to the question actually proposed. This is owing to the fact that the algebraic language is more general than ordinary language; hence the equation not only expresses the conditions of the given problem, but it will sometimes apply to other conditions. When this is the case, the conditions of the second problem are usually the contrary of those given in the proposed problem. Frequently, however, it will be found that only one root applies to the problem proposed, and that no obvious interpretation occurs for the other. To illustrate the method of interpreting negative answers, take the following problem.

383.—5. A bought a certain number of apples for 80 cents; had he bought 4 more for the same sum, each apple would have cost 1 cent less. How many apples did he buy?

Let x denote the number he bought, then $\frac{80}{x}$ is the price of each; if he had bought $x + 4$, the price of each would have been $\frac{80}{x+4}$. $\therefore \frac{80}{x+4} = \frac{80}{x} - 1$, whence $x = 16$ or -20.

If A bought -20 apples, then he must have sold 20 apples, hence the answer, 20, applies to the following problem:

6. A sold a certain number of apples for 80 cents; had he sold 4 fewer for the same sum, each apple would have sold for 1 cent more. How many apples did he sell?

Let x denote the number he sold: then, reasoning as in Problem 5, we have $\frac{80}{x+4} = \frac{80}{x} + 1$, whence $x = 20$ and -16; thus the number 20, which appeared with a negative sign as a result in Example 5 and was in that case rejected, is here the admissible result.

Exercise XXI.

1. Find two numbers whose sum is 10 and product 21.

2. What number added to its square gives 132?

3. Find two numbers whose sum is 16, and the sum of whose squares is 146.

4. A sold a horse for $24, and by so doing he lost as much per cent as the horse cost. Required the price of the horse.

5. A bought a certain number of oxen for $240, and after losing 3 sold the remainder for $8 apiece more than they cost him, thus gaining $59. How many did he buy?

6. Divide a into two parts such that their product shall be equal to the difference of their squares.

7. A field which is 6 rods longer and 2 rods narrower than a certain square field, contains 128 square rods. How long is each field?

8. A banker had two kinds of money with which he paid a bill of $40, giving 60 of the more valuable coins and 100 of the less valuable. It required 8 more of the latter than of the former to make one dollar. What was the value of each coin?

9. A receives from a banker 60 francs and 140 marks for $47, and it requires one more of the former than of the latter to make one dollar. What is the value of each coin?

10. A sold 16 pounds of mace and 20 pounds of cloves for $65. He sold 12 pounds more of cloves for $20 than he did of mace for $10. What was the price of a pound of cloves?

11. A and B had jointly $1000 in business. A's money was in trade 9 months, and B's 6 months. When they shared stock and gain, A received $1140 and B $640. What was each man's investment?

NOTE. $\frac{\text{A's gain for } n \text{ months}}{\text{B's gain for } n \text{ months}} = \frac{\text{A's capital}}{\text{B's capital}}.$

12. A cistern has two pipes, one of which will fill it in 4 hours less than the other, and both together can fill it in $6\frac{6}{7}$ hours. How long will it take each separately to fill it?

13. A pasture was rented at a fixed weekly price by A and B, who agreed to share the rent in proportion to the number of animals put to pasture. The first week A put in 4 animals and B paid $20. The next week B put in 2 additional animals and paid $4 more. At what price was the pasture hired?

14. Find a number of two places, such that the units' digit, increased by 2, shall equal three times the tens' digit, and the square of the units' digit is 12 more than the required number.

15. Find the number of two places whose units' digit is 3 more than the tens', and 125 times the units' digit equals the square of the required number.

16. The square of the time past noon, increased by the time to midnight, equals the time past midnight. What o'clock is it?

17. The hands of a clock are together between two consecutive numbers, the difference of whose squares is 1 more than the square of the less. What o'clock is it?

18. A traveled 48 miles in a certain time. Had he traveled 2 miles an hour faster, his time would have been 4 hours less. Find his rate.

19. A set out from C toward D at the same instant that B left D for C. They met 18 miles nearer to D than to C. A arrived in D in 8 hours, and B in D in 18 hours, after they met. Find the distance from C to D.

20. A set out from C, and B from D, distant 60 miles, at the same instant. A arrived in D 2 hours, and B in C 8 hours, after they met. How far from D did they meet?

21. A rowed 20 miles and back in 14 hours, going out with a tide of 2 miles an hour, and back against a tide of 1 mile per hour. How long would it have taken him to row the same distance in still water?

22. A steamer went 40 miles down a river and back in 16 hours, the current being 3 miles per hour. Owing to an accident to her machinery, she could steam back with only half the power she had when going down. How long did she take to return?

23. The height of a room is two thirds of the breadth, and the breadth is two yards less than the length. The plastering at 50 cents per square yard cost $80. Find the dimensions of the room.

24. A loaned two sums of money, which differed by $200. His rate on each was one per cent of its principal, and the interest on both was $52. Find each principal.

CHAPTER XXII.

PROPERTIES OF QUADRATICS. MAXIMA AND MINIMA.

384. FROM (362) it follows that every quadratic equation can be reduced to the form $x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{c}{a} + \frac{b^2}{4a^2}$.

For convenience, let $\frac{b}{a} = 2p$, and $\frac{c}{a} = q$; also, let $\frac{c}{a} + \frac{b^2}{4a^2} = m^2$. Then the general quadratic reduces to the form

$$x^2 + 2px + p^2 = q + p^2.$$

$\therefore (x+p)^2 = q + p^2 = m^2$; $\therefore (x+p)^2 - m^2 = 0$, and, from (199) and (200), $(x + p + m)(x + p - m) = 0$.

385. *Every quadratic equation has two roots, and only two.*

For, $(x + p + m)(x + p - m) = 0$ can be satisfied by putting each factor equal to zero, and it can be satisfied in no other way. From $x + p + m = 0$, $x = -p - m$; from $x + p - m = 0$, $x = -p + m$. Hence every quadratic has two roots, and only two.

386. *In every quadratic equation, reduced to the form $x^2 + 2px = q$, the sum of the roots is equal to the co-efficient of the second term with its sign changed, and the product of the roots is equal to the known term with its sign changed.*

For the roots are $-p-m$ and $-p+m$, whose sum is $-2p$, and product $p^2 - m^2 = -q$.

387. If the two roots be represented by r and s, then it follows from (386) that $x^2 - (r + s)x = -rs$.

$\therefore x^2 - (r + s)x + rs = 0$; $\therefore (x - r)(x - s) = 0$, a result which agrees with that obtained in (384). Whence:

Every quadratic reduced to the form $x^2 + \frac{b}{a}x - \frac{c}{a} = 0$ *can be decomposed into two binomial factors, the first term of each being x, and the second terms the two roots with their signs changed.*

388. An expression of the form $ax^2 + bx + c$ is called a *quadratic expression.* A method of factoring such expressions is given in (205), but it is evident from (387) that the factors are readily found from placing $ax^2 + bx + c = 0$, and finding the roots of the resulting quadratic. If these roots be represented by r and s, the required factors will be $a(x - r)(x - s)$.

If r and s be not rational, the proposed expression is prime.

389. *If both sides of an equation be divisible by a common factor involving x, the equation is satisfied by putting that factor equal to zero.*

For, if $(x-a)k = (x-a)m$, then $(x-a)k-(x - a)m = 0$.
$\therefore (x - a)(k - m) = 0$. Whence $x - a = 0$, and $x = a$.
In like manner, if $k - m$ involve x, then $k - m = 0$.

390. If r, s, and t represent three values of x, then:
I. $x - r = 0$; II. $x - s = 0$; III. $x - t = 0$.
$\therefore (x - r)(x - s)(x - t) = 0$.
This is a cubic equation, as may be seen by performing the indicated multiplication. Hence we may infer that a cubic equation has three roots, and only three. In like manner, *the number of roots of any equation is equal to the degree of the equation.* Similarly, it may be inferred that if r is a root of an equation whose second side is zero, then $x - r$ is a factor of the first side.

391. From (389) and (390) it follows that quadratics and many equations of higher degree may be solved by factoring.

EXAMPLES.

1. Solve $x^4 + 2x^3 - x = 2$.

By transposing, $x^4 + 2x^3 - x - 2 = 0$.

$\therefore$ $x^3(x+2)-(x+2)=0$, and $(x+2)(x-1)(x^2+x+1)=0$.

By putting each factor in succession equal to 0, we find $x = -2, 1, \frac{1}{2}(-1 \pm \sqrt{-3})$.

The equation is of the fourth degree; it has four roots.

2. Solve $3x^3 + 5x^2 - x + 2 = 0$.

$$3x^3 + 6x^2 - x^2 - 2x + x + 2 = 0;$$
$$\therefore (x+2)(3x^2 - x + 1) = 0.$$

Since $x + 2 = 0$, $\therefore x = -2$. Since $3x^2 - x + 1 = 0$, $\therefore x = \frac{1}{6}(1 \pm \sqrt{-11})$.

3. If $x^3 - 8x^2 + 19x = 12$, then $(x-3)(x-1)(x-4)=0$.

$$\therefore x = 3, 1, \text{ or } 4.$$

4. If $6x^3 - x^2 - x = 0$, then $x(2x-1)(3x+1) = 0$.

$$\therefore x = 0, \tfrac{1}{2}, -\tfrac{1}{3}.$$

392. From the preceding articles it is clear that any equation may be formed if its roots be known. Thus:

If the roots be s and r, the equation is

$$(x-s)(x-r) = 0; \text{ that is, } x^2 - (s+r)x = -sr.$$

If the roots be s, r, and t, the equation is

$$(x-s)(x-r)(x-t) = 0; \text{ and so on.}$$

EXAMPLES.

1. The equation whose roots are 5 and -2 is $x^2 - (5-2)x = -(5)(-2)$; that is, $x^2 - 3x = 10$.

2. If the roots are -1 and $\frac{2}{3}$, the equation is

$$x^2 - (-1 + \tfrac{2}{3})x = -(-1)\tfrac{2}{3}, \text{ or } x^2 + \tfrac{1}{3}x = \tfrac{2}{3}.$$

3. If the roots are $3 \pm \sqrt{5}$, the equation is $x^2 - 6x = -4$, because the sum of $3 + \sqrt{5}$ and $3 - \sqrt{5}$ is 6, and their product is $(9 - 5) = 4$.

4. Form the equation having two roots, whose sum is s, and their product p. *Ans.* $x^2 - sx = -p$.

5. If the roots are 2, 1, and -3, the equation is $(x-2)(x-1)(x+3)=0$; that is, $x^3-7x+6=0$.

6. The equation whose roots are 0, 2, -3, -4, is $(x-0)(x-2)(x+3)(x+4)=0$; that is, $x^4+5x^3-2x^2-24x=0$.

393. From (384) it follows that every quadratic equation may be written in the form $x^2+2px=q$, in which p and q may represent any numbers, positive or negative, integral or fractional. Let r represent the first value of x, and s the second; then we have:

$$(1)\quad x^2+2px=+q,\text{ whence }\begin{cases} r=-p+\sqrt{p^2+q},\\ s=-p-\sqrt{p^2+q}.\end{cases}$$

$$(2)\quad x^2-2px=+q,\text{ whence }\begin{cases} r=+p+\sqrt{p^2+q},\\ s=+p-\sqrt{p^2+q}.\end{cases}$$

$$(3)\quad x^2+2px=-q,\text{ whence }\begin{cases} r=-p+\sqrt{p^2-q},\\ s=-p-\sqrt{p^2-q}.\end{cases}$$

$$(4)\quad x^2-2px=-q,\text{ whence }\begin{cases} r=+p+\sqrt{p^2-q},\\ s=+p-\sqrt{p^2-q}.\end{cases}$$

Since $p^2+q>p^2$, then $\sqrt{p^2+q}>p$.

Since $p^2-q<p^2$, then $\sqrt{p^2-q}<p$.

In the first and second forms, the first term of each root is numerically less than the second term, while in the third and fourth forms the first term of each root is numerically greater than the second.

In the first, second, and fourth forms r is positive; and in the fourth form s is also positive.

In the first, second, and third forms s is negative, and in the third form r is also negative.

Hence in the first and second forms r is positive, and s is negative; in the third form both are negative, and in the fourth form both are positive.

In each case the root whose terms have like signs is numerically greater than the other, unless $2p=0$, in which case the two roots are numerically equal.

If $q = 0$, one root becomes zero.

If $q > p^2$, then in the third and fourth forms r and s both become impossible quantities.

MAXIMA AND MINIMA.

394. It is often useful to determine the greatest or least values which a quadratic expression can have.

A *quadratic expression* is of the general form ax^2+bx+c, in which a, b, and c are fixed or constant quantities, and x may have any value we please to assign to it (388).

395. To determine the maximum or minimum value of a quadratic expression, put it equal to m, and solve this equation; by this means x will be expressed in terms of m, and it will be easy to see what will be the greatest or least values allowable for m, so that x shall be a possible quantity.

EXAMPLES.

Find the maximum or minimum value of the following:

1. Of $3x^2 - 3x + 2$.

$$3x^2 - 3x + 2 = m; \quad \therefore x = \tfrac{1}{6}(3 \pm \sqrt{12m - 15}).$$

In order for x to represent a possible number, the expression $12m - 15$ can not be negative; hence the least value of $12m - 15$ is zero, in which case $m = \frac{5}{4}$. Hence the minimum value is $\frac{5}{4}$.

2. Of $x^2 - 6x + 16$.

$$x^2 - 6x + 16 = m; \quad \therefore x = 3 \pm \sqrt{m - 7}.$$

Reasoning as in Ex. 1, m can not be less than 7. Hence the minimum value of the expression is 7, in which case $x = 3$.

3. Of $x^2 + 3x + 4$.

$$x^2 + 3x + 4 = m; \quad \therefore x = \tfrac{1}{2}(-3 \pm \sqrt{4m - 7}).$$

Here the least value of m is $\frac{7}{4}$, which is therefore the minimum value of the expression, and for this value $x = -\frac{3}{2}$.

4. Of $7x - x^2 - 12$.

$$-x^2 + 7x - 12 = m; \quad \therefore x = \tfrac{1}{2}(7 \pm \sqrt{1 - 4m}).$$

Here m can not be greater than $\frac{1}{4}$. Hence the maximum value of the expression is $\frac{1}{4}$, in which case $x = \frac{7}{2}$.

Exercise XXII.

Form the equation whose roots are :

1. 3 and 4.

2. 3 and -2.

3. $\frac{1}{2}$ and $\frac{1}{3}$.

4. $\frac{2}{3}$ and $-\frac{1}{2}$.

5. ± 5.

6. $2 \pm \sqrt{3}$.

7. $\frac{1}{2}(3 \pm \sqrt{6})$.

8. $1 \pm \sqrt{-3}$.

9. $a \pm b$.

10. $\sqrt{a} \pm \sqrt{b}$.

11. -2, 3, and -4.

12. 3 and $1 \pm \sqrt{5}$.

13. ± 2 and $1 \pm \sqrt{3}$.

14. $a + 2$ and $a - 1$.

15. $a \pm 1$ and a.

16. $a \pm 2$, $a \pm 1$.

Solve the following equations by factoring :

17. $x^2 - 6x + 8 = 0$.

18. $2x^2 - 4x - 16 = 0$.

19. $2x^2 - 2x - 24 = 0$.

20. $x^3 + 1 = 0$.

21. $3x^2 - 5x = -2$.

22. $6x^3 - 3x^2 - 2x + 1 = 0$.

23. $5x^2 - 5a^2 - 3x + 3a = 0$.

24. $x^4 + 2x^2 + 9 = 0$.

Find the maximum or minimum value of the following :

25. $x^2 - 3x + 4$.

26. $x^2 + 3x + 2$.

27. $\dfrac{x - 4}{(x - 1)^2}$.

28. $\dfrac{4 + x^2}{4 - x^2}$.

CHAPTER XXIII.

EQUATIONS WHICH MAY BE SOLVED LIKE QUADRATICS.

396. ANY equation which can be reduced to the form $ax^{2n} + bx^n = c$ may be solved like quadratics. Several examples have already been given (Exercise XX, examples 25–29 inclusive).

In this chapter we shall consider special forms of $ax^{2n} + bx^n = c$, together with certain methods of reducing equations of various degrees to the form of quadratics.

397.—1. I. Solve $ax^{2n} + bx^n = c$.

II. $= \text{I} \times 4a$. $\quad 4a^2x^{2n} + 4abx^n = 4ac$.

III. Add b^2. $\quad 4a^2x^{2n} + 4abx^n + b^2 = b^2 + 4ac$.

IV. Extract square root. $\quad 2ax^n + b = \pm\sqrt{b^2 + 4ac}$.

V. Transpose and divide. $$x^n = \frac{-b \pm \sqrt{b^2 + 4ac}}{2a}.$$

VI. Extract nth root. $$x = \sqrt[n]{\frac{-b \pm \sqrt{b^2 + 4ac}}{2a}}.$$

2. Solve $x^4 - 5x^2 = -4$.

Add $(\frac{5}{2})^2$. $\quad x^4 - 5x^2 + (\frac{5}{2})^2 = \frac{25}{4} - 4 = \frac{9}{4}$.

Extract square root. $\quad x^2 - \frac{5}{2} = \pm\frac{3}{2}$. $\quad \therefore x^2 = 4$ or 1;

whence $\quad x = \pm 2$ or ± 1.

398.—3. Solve $x + 6\sqrt{x} = 16$.

II. Add 9. $x + 6\sqrt{x} + 9 = 25$.

III. Extract square root. $x^{\frac{1}{2}} + 3 = \pm 5$.

$\therefore x^{\frac{1}{2}} = 2$ or -8, and $x = 4$ or 64.

399.—4. Solve $x^{-1} - 4x^{-\frac{1}{2}} = 21$.

II. Add 4. $x^{-1} - 4x^{-\frac{1}{2}} + 4 = 25$.

III. Extract square root. $x^{-\frac{1}{2}} - 2 = \pm 5$. $\therefore x^{-\frac{1}{2}} = 7$ or 3; whence $x^{-1} = 49$ or 9, and $x = \frac{1}{49}$ or $\frac{1}{9}$.

400.—5. Solve $x^6 - 9x^3 = -8$.

II. Add $(\frac{9}{2})^2$. $x^6 - 9x^3 + (\frac{9}{2})^2 = \frac{81}{4} - 8 = \frac{49}{4}$.

III. Extract square root. $x^3 - \frac{9}{2} = \pm \frac{7}{2}$.

IV. Transpose. $x^3 = \frac{9}{2} \pm \frac{7}{2} = 8$ or 1.

Since $x^3 = 8$, $\therefore x^3 - 8 = 0$. $\therefore (x-2)(x^2 + 2x + 4) = 0$. Therefore (Art. 389) $x - 2 = 0$, whence $x = 2$;

and $x^2 + 2x + 4 = 0$, whence $x = -1 \pm \sqrt{-3}$.

In like manner, since $x^3 - 1 = 0$, $\therefore (x-1)(x^2 + x + 1) = 0$. If $x - 1 = 0$, $x = 1$; if $x^2 + x + 1 = 0$, $x = \frac{1}{2}(-1 \pm \sqrt{-3})$.

401.—6. Find six sixth roots of 64.

Let x represent a sixth root of 64.

$$\therefore x^6 = 64, \text{ whence } x^6 - 64 = 0.$$

$$\therefore (x-2)(x^2 + 2x + 4)(x+2)(x^2 - 2x + 4) = 0.$$

Ans. $x = 2, \quad -1 \pm \sqrt{-3}, \quad -2, \quad 1 \pm \sqrt{-3}$.

402.—7. Solve $x + \sqrt{x+2} = 10$.

II. Add 2. $(x+2) + \sqrt{x+2} = 12$. This is plainly in the form of a quadratic, because if $\sqrt{x+2} = y$, then $x + 2 = y^2$.

III. Add $\frac{1}{4}$. $(x+2) + \sqrt{x+2} + \frac{1}{4} = 12\frac{1}{4}$.

IV. Extract square root. $\sqrt{x+2} + \frac{1}{2} = \pm 3\frac{1}{2}$.

V. Transpose. $\sqrt{x+2} = 3$ or -4.

VI. Square. $x + 2 = 9$ or 16. $\therefore x = 7$ or 14.

8. Solve $2x + \sqrt{4x + 8} = \frac{7}{2}$.

II. Multiply by 2. $4x + 2\sqrt{4x + 8} = 7$.

III. Add 8. $(4x + 8) + 2\sqrt{4x + 8} = 15$.

Add 1 and continue as in Ex. 7. *Ans.* $x = \frac{1}{4}$ or $4\frac{1}{4}$.

9. Solve $3x^2 + 15x - 2\sqrt{x^2 + 5x + 1} = 2$.

II. Divide by 3. $x^2 + 5x - \frac{2}{3}\sqrt{x^2 + 5x + 1} = \frac{2}{3}$.

III. Add 1. $(x^2 + 5x + 1) - \frac{2}{3}\sqrt{x^2 + 5x + 1} = \frac{5}{3}$.

IV. Add $(\frac{1}{3})^2$. $(x^2 + 5x + 1) - \frac{2}{3}\sqrt{x^2 + 5x + 1} + \frac{1}{9} = \frac{16}{9}$.

V. Extract square root. $\sqrt{x^2 + 5x + 1} - \frac{1}{3} = \pm\frac{4}{3}$.

Ans. $x = \frac{1}{3}, -\frac{16}{3}, 0, -5$.

403. An equation which will remain unaltered when $\frac{1}{x}$ is substituted for x, is called a *reciprocal equation.* Example: $x^4 - x^3 - x^2 - x + 1 = 0$.

Every reciprocal equation of odd degree is divisible by $x - 1$ or $x + 1$, according as the last term is negative or positive.

Every reciprocal equation of even degree with its last term negative, is divisible by $x^2 - 1$.

In every case the reduced equation after the division will be reciprocal, of an even degree, and with its last term positive.

By this means a reciprocal cubic may be reduced to a quadratic, and one of the fifth or sixth degree to a biquadratic, which latter may be solved as follows:

404.—10. Solve $x^4 + x^3 - 4x^2 + x + 1 = 0$.

II. Divide by x^2. $x^2 + x - 4 + \frac{1}{x} + \frac{1}{x^2} = 0$.

III. Add 6. $x^2 + 2 + \frac{1}{x^2} + x + \frac{1}{x} = 6$.

IV. Combine terms. $\left(x + \frac{1}{x}\right)^2 + \left(x + \frac{1}{x}\right) = 6$.

V. Complete the square. $\left(x+\frac{1}{x}\right)^2+\left(x+\frac{1}{x}\right)+\frac{1}{4}=6\frac{1}{4}.$

VI. Extract square root. $x+\frac{1}{x}+\frac{1}{2}=\pm 2\frac{1}{2}.$

$\therefore x+\frac{1}{x}=2$ or -3, which is best solved by the Special Method (374). *Ans.* $x=1,\ 1,$ or $\frac{1}{2}(-3\pm\sqrt{5}).$

11. Solve $x^4+x^3-2x^2-x+1=0.$

Ans. $x=1,\ -1,\ \frac{1}{2}(-1\pm\sqrt{5}).$

405. Cubic equations with the second power of x missing may sometimes be solved as follow:

12. Solve $8x^3+16x=9.$

II. Multiply by $2x$. $16x^4+32x^2=18x.$

III. Add $(2x)^2$. $16x^4+36x^2=4x^2+18x.$

IV. Add $(\frac{9}{2})^2$. $16x^4+36x^2+(\frac{9}{2})^2=4x^2+18x+(\frac{9}{2})^2.$

V. Extract root. $4x^2+\frac{9}{2}=\pm(2x+\frac{9}{2}).$

From $4x^2+\frac{9}{2}=+(2x+\frac{9}{2})$, $x=0$, or $\frac{1}{2}$.

From $4x^2+\frac{9}{2}=-(2x+\frac{9}{2})$, $x=\frac{1}{4}(-1\pm\sqrt{-35}).$

The value of $x=0$ satisfies the equation after being multiplied by $2x$, but is not a root of the given equation.

406. Rule for solving cubics of the form $ax^3+bx=c$.

(1) *Multiply each side by such a number that the first term shall be a perfect square.* Thus: $a^2x^4+bx^2=cx.$

(2) *Add to each side x^2 with a co-efficient which is the square of a divisor of ac, and such that the number which completes the square on one side also completes the square on the other side.*

(3) *Extract the square root of each side (prefix $\pm$ to the square root of the second side), and solve the resulting quadratics.*

13. Solve $x^3-3x=2.$

II. Multiply by x. $x^4-3x^2=2x.$

III. Add $(1x)^2$. $x^4-2x^2=x^2+2x.$

IV. Complete the square. $x^4 - 2x^2 + 1 = x^2 + 2x + 1.$
V. Extract square root. $x^2 - 1 = \pm (x + 1).$
Ans. $x = 2, -1, -1.$

No. III may also be formed by adding $(2x)^2 = 4x^2$.

14. Solve $x^3 - 6x = 9$.
II. Multiply by x. $x^4 - 6x^2 = 9x$.
III. Add $(3x)^2$. $x^4 + 3x^2 = 9x^2 + 9x$.
IV. Add $(\frac{3}{2})^2$. $x^4 + 3x^2 + (\frac{3}{2})^2 = 9x^2 + 9x + (\frac{3}{2})^2$.
Ans. $x = 3, \frac{1}{2}(-3 \pm \sqrt{-3})$.

15. Solve $x^3 - 13x = 12$.
II. Multiply by x. $x^4 - 13x^2 = 12x$.
III. Add $(1x)^2$. $x^4 - 12x^2 = x^2 + 12x$.
IV. Add 36. $x^4 - 12x^2 + 36 = x^2 + 12x + 36$.
Whence $x = 4, -3, -1$.
III may be formed by adding either $(3x)^2$ or $(4x)^2$.

407. Certain biquadratics, with x^3 missing, may be solved in a similar manner. Thus:

16. Solve $x^4 - 6x^2 - 8x - 3 = 0$.
II. Transpose. $x^4 - 6x^2 = 8x + 3$.
III. Add $(2x)^2$. $x^4 - 2x^2 = 4x^2 + 8x + 3$.
IV. Add 1. $x^4 - 2x^2 + 1 = 4x^2 + 8x + 4$.
Whence $x = 3, -1, -1, -1$.

408.—17. Solve $x^4 - 10x^3 + 35x^2 - 50x = -24$.

II. Complete the square of the first two terms by writing $25x^2$ as the third term, and the equation becomes

$$x^4 - 10x^3 + 25x^2 + 10x^2 - 50x = -24.$$

III. $\therefore (x^4 - 10x^3 + 25x^2) + 10(x^2 - 5x) = -24.$
IV. Add 25. $(x^2 - 5x)^2 + 10(x^2 - 5x) + 25 = 1.$
V. Extract square root. $(x^2 - 5x) + 5 = \pm 1.$
Solve $x^2 - 5x + 5 = +1$, whence $x = 4$ or 1.
Solve $x^2 - 5x + 5 = -1$, whence $x = 3$ or 2.

18. Solve $x^4 - 2x^3 + x = 132$.
II. $x^4 - 2x^3 + x^2 - x^2 + x = 132.$
III. Combine. $(x^2 - x)^2 - (x^2 - x) = 132.$

IV. Add $\frac{1}{4}$. $(x^2 - x)^2 - (x^2 - x) + \frac{1}{4} = 132\frac{1}{4}$.

V. Extract root. $(x^2 - x) - \frac{1}{2} = \pm 11\frac{1}{2}$.

Ans. $x = 4$ or -3, or $\frac{1}{2}(1 \pm \sqrt{-43})$.

19. Solve $4x^4 - 4x^3 + 5x^2 - 2x = 8$ in a similar manner.

Ans. $x = \frac{1}{4}(1 \pm \sqrt{17})$ or $\frac{1}{4}(1 \pm \sqrt{-31})$.

The solutions of Examples 17, 18, 19, illustrate the method of solving biquadratics, such that if the square of the first two terms be completed, the fourth and fifth terms will contain the square root of the first three terms.

409.—20. Solve $x^4 - 10x^3 + 120x = 144$.

Transpose $120x$ and complete the square of the first two terms. $\therefore x^4 - 10x^3 + 25x^2 = 25x^2 - 120x + 144$.

Both sides being perfect squares, extract the square roots. $x^2 - 5x = \pm (5x - 12)$.

If $x^2 - 5x = + (5x - 12)$, $x = 5 \pm \sqrt{13}$.

If $x^2 - 5x = - (5x - 12)$, $x = \pm 2\sqrt{3}$.

21. Solve $x^4 - 10x^3 + 9x^2 + 40x = 25$ in a similar manner.

Ans. $x = \frac{1}{2}(9 \pm \sqrt{61})$ or $\frac{1}{2}(1 \pm \sqrt{21})$.

Equations (20) and (21) are examples of biquadratics such that if the first and second terms be on one side and the remaining terms on the other, the number which completes the square on one side will complete the square on the other side also.

410.—22. Solve $\sqrt{x^2 + 8} + \sqrt{x^2 - 8} = \sqrt{17} + 1$.

II. Take the identical equation $(x^2 + 8) - (x^2 - 8) = 16 = 17 - 1$, and divide this equation by the given equation. The given equation is of the form $a + b = m + n$, and the assumed identity is of the form $a^2 - b^2 = m^2 - n^2$.

$\therefore$ III. $\sqrt{x^2 + 8} - \sqrt{x^2 - 8} = \sqrt{17} - 1$.

IV. Subtract III from I. $2\sqrt{x^2 - 8} = 2$. $\therefore x = \pm 3$.

This equation is introduced for the sake of illustrating the artifice employed in the solution. Similar methods may often be employed with advantage.

411. The principles explained in Art. 284 may frequently be used to abbreviate the solutions of equations. Thus:

23. Solve $\dfrac{x^4+6x^2+1}{4x(x^2+1)}=\dfrac{81x^4+16}{81x^4-16}$.

II. From Art. 284, Case VII,

$$\frac{x^4+4x^3+6x^2+4x+1}{x^4-4x^3+6x^2-4x+1}=\frac{162x^4}{32}.$$

III. $\therefore\ \dfrac{(x+1)^4}{(x-1)^4}=\dfrac{81x^4}{16}$.

IV. Extract the fourth root. $\dfrac{x+1}{x-1}=\pm\dfrac{3x}{2}$.

When $\dfrac{x+1}{x-1}=+\dfrac{3x}{2}$, $x=2$ or $-\frac{1}{3}$.

When $\dfrac{x+1}{x-1}=-\dfrac{3x}{2}$, $x=\frac{1}{6}(1\pm\sqrt{-23})$.

412. When none of the methods heretofore explained will apply, if the equation have rational roots they may be found by factoring, as follows: Transpose all the terms to one side; resolve this expression into its prime factors, and place each factor equal to zero. The values found from solving these equations are the required roots (391).

413. Conversely: In order to find the factors of an expression, place it equal to zero, and solve the resulting equation. Thus: The factors of $x^4-2x^3+x-132$ may be found as in Ex. 18. One root being 4, $\therefore$ $x-4$ is one factor; another root being -3, $\therefore$ $x+3$ is a factor; the other two roots being irrational, the quadratic, x^2-x+11, of which they are the roots, is prime. Hence the required factors are $(x-4)(x+3)(x^2-x+11)$.

In like manner, $4x^4-4x^3+5x^2-2x-8$ may be factored as in Ex. 19. The four roots being irrational, the two quadratics, $2x^2-x-2$ and $2x^2-x+4$, are the required prime factors.

EXERCISE XXIII.

Solve like Example 2, Art. 397 :

1. $x^4 - 6x^2 = -8$.

2. $x^4 - (a+b)x^2 = -ab$.

3. $x^{\frac{4}{3}} - 5x^{\frac{2}{3}} = -4$.

4. $x^5 - 33x^{\frac{5}{2}} = -32$.

Solve like Example 3, Art. 398 :

5. $x - 10x^{\frac{1}{2}} = -16$.

6. $x^{\frac{1}{2}} - 3x^{\frac{1}{4}} = -2$.

7. $x^{\frac{1}{3}} - 2x^{\frac{1}{6}} = 3$.

8. $3x + 2\sqrt{x} = 1$.

Solve like Example 4, Art. 399:

9. $x^{-\frac{1}{4}} + 5x^{-\frac{1}{8}} = 22$.

10. $3x^{-\frac{3}{2}} - 4x^{-\frac{3}{4}} = 7$.

11. $x^{-2} - 35x^{-1} = -216$.

12. $x^{-\frac{1}{n}} - x^{-\frac{2}{n}} = -2$.

Solve like Example 5, Art. 400:

13. $x^6 - 7x^3 = 8$.

14. $x^3 + 14x^{\frac{3}{2}} = 1107$.

15. $x^{\frac{3}{2}} - 7x^{\frac{3}{4}} = -10$.

16. $x^8 + x^4 = 2$.

Solve like Examples 7, 8, and 9, Art. 402:

17. $x - \sqrt{x+5} = 1$.

18. $x + \sqrt{5x+10} = 8$.

19. $x + 3\sqrt{5x} = 20$.

20. $2x + \sqrt{2x+3} = 9$.

21. $x^2 + 3 = 2\sqrt{x^2 - 2x + 2} + 2x$.

22. $x^2 + 5x + 4 = 5\sqrt{x^2 + 5x + 28}$.

23. $\sqrt{x^2 - 2x + 9} - \dfrac{x^2}{2} = 3 - x$.

24. $x^2 - x + 3\sqrt{2x^2 - 3x + 2} = \dfrac{x}{2} + 7$.

Solve like Example 10, Art. 404 :

25. $x^2 + x + x^{-1} + x^{-2} = 10$.

26. $x^2 + 3x - 3x^{-1} + x^{-2} = 6$.

27. $x + x^{\frac{1}{2}} + \dfrac{1}{2x^{\frac{1}{2}}} + \dfrac{1}{4x} = 11$.

28. $x^4 - x^2 - x^{-2} + x^{-4} = 18$.

Solve like Examples 13, 14, and 15, Art. 406:

29. $4x^3 - 39x = -35.$

30. $x^6 - 3x^2 = 18.$

31. $9x^2 - 4x^{-1} = 13.$

32. $x - x^{-\frac{1}{2}} = 2.$

Solve like Example 16, Art. 407:

33. $x^4 + 5x^2 - 2x + 8 = 0.$

34. $x^4 + 3x^2 + 6x = 5.$

35. $4x^4 - 5x^2 + 8x = 15.$

36. $x^2 - 5x + 6x^{\frac{1}{2}} + 3 = 0.$

Solve like Examples 17 and 18, Art. 408:

37. $x^4 - 4x^3 + 8x^2 - 8x = 5.$

38. $x^4 + 8x^3 + 13x^2 - 12x = 10.$

39. $x^8 - 10x^6 + 35x^4 - 50x^2 = 24.$

40. $4x^4 - 4x^3 - 5x^2 + 3x = 4.$

Solve like Example 20, Art. 409:

41. $x^4 - 6x^3 + 8x^2 - 8x = 16.$

42. $x^4 - 4x^3 - 5x^2 + 6x = 1.$

43. $4x^4 - 8x^3 + 3x^2 - 8x = 16.$

44. $3x^4 - 13x^3 - 117x = 243.$

Solve like Example 22, Art. 410:

45. $\sqrt{x^2 + 21} - \sqrt{x^2 + 5} = 2.$

46. $\sqrt{v - \frac{1}{x}} + \sqrt{1 - \frac{1}{x}} = x.$

47. $\sqrt{x^3 + 9} - \sqrt{x^3 - 11} = 2.$

48. $\sqrt{3a - 4x} + 2\sqrt{3ax - x} = 3a\sqrt{1 - 4x}.$

Solve like Example 23, Art. 411:

49. $\frac{a + x + \sqrt{a^2 - x^2}}{a + x - \sqrt{a^2 - x^2}} = \frac{c}{x}.$

50. $\frac{ax + 1 + \sqrt{a^2x^2 - 1}}{ax + 1 - \sqrt{a^2x^2 - 1}} = \frac{x}{2}.$

51. $\frac{3x + \sqrt{4x - x^2}}{3x - \sqrt{4x - x^2}} = 2.$

52. $\frac{\sqrt{7x^2 + 4} + \sqrt{12x - 4}}{\sqrt{7x^2 + 4} - 2\sqrt{3x - 1}} = 7.$

CHAPTER XXIV.

SIMULTANEOUS EQUATIONS INVOLVING QUADRATICS.

414. QUADRATIC equations involving two unknown quantities require different methods for their solutions, according to the form of the proposed equations; hence no general rule can be given. There are, however, three cases of frequent occurrence, for which the following observations will be useful:

415.—CASE I. When one of the unknown quantities can be eliminated, and the resulting equation solved according to preceding methods.

This can usually be done when one of the equations is of the first degree, or is capable of being changed into an equation of the first degree, and it can frequently be done in other classes of problems.

Elimination by substitution is the method commonly employed (180).

416.—1. Solve: I. $x^2 + y^2 = 25$. II. $3x + 4y = 24$.

III. From II, $y = \frac{1}{4}(24 - 3x)$.

IV. $\therefore$ I becomes $x^2 + \frac{1}{16}(24 - 3x)^2 = 25$, whence $x = 4$ or $\frac{44}{25}$.

From III, when $x = 4$, $y = 3$, and when $x = \frac{44}{25}$, $y = 4\frac{17}{25}$.

417.—2. Solve: I. $x^2 + xy + y^2 = 7$.

II. $2x + 3y = 8$.

III. From II, $x = \frac{1}{2}(8 - 3y)$.

IV. $\therefore$ I becomes $\left(\frac{8 - 3y}{2}\right)^2 + \left(\frac{8 - 3y}{2}\right) y + y^2 = 7$.

V. Expand IV, and simplify. $\frac{7}{4}y^2 - 8y = -9$.

Whence $y = 2\frac{4}{7}$ or 2; and from III, $x = \frac{1}{7}$ or 1.

418.—**3.** Solve: I. $x + y = x^2$. II. $3y - x = y^2$.
III. From I, $y = x^2 - x$.
IV. By substituting in II, $3(x^2 - x) - x = (x^2 - x)^2$.
V. Transpose and simplify IV. $x^4 - 2x^3 - 2x^2 + 4x = 0$.
VI. By factoring V, $x(x-2)(x^2-2) = 0$.

$$\therefore\ x = 0,\ 2,\ \pm\sqrt{2}.$$

From III, when $x = 0$, $y = 0$; when $x = 2$, $y = 2$; when $x = \pm\sqrt{2}$, $y = 2 \mp \sqrt{2}$.

419.—**4.** Solve: I. $x^2 + y^2 - x - y = 6 - 2xy$.
II. $x^2 + 3y^2 = 19$.

The first equation is of the form $(x+y)^2 - (x+y) = 6$.
$\therefore (x+y)^2 - (x+y) + \frac{1}{4} = 6\frac{1}{4}$, whence $x + y = 3$ or -2, and $x = 3 - y$ or $-2 - y$.

In II for x put $3 - y$. $\therefore$ III. $(3-y)^2 + 3y^2 = 19$.
$\therefore 4y^2 - 6y = 10$, whence $y = 2\frac{1}{2}$ or -1.
Since $x = 3 - y$, $\therefore x = 3 - 2\frac{1}{2} = \frac{1}{2}$, or $3 - (-1) = 4$.
Similarly, in II substitute the value $-2 - y$ for x, and solve.

Ans. $x = \frac{1}{2},\quad 4,\quad -3\frac{1}{2},\quad \frac{1}{2}$;
$y = 2\frac{1}{2},\quad -1,\quad 1\frac{1}{2},\quad -2\frac{1}{2}$.

420.—**5.** Given I. $x^3y^3 + xy = 10$. To find x and y.
II. $2x + 3y = 7$.

By solving I according to Art. 406,
III. $xy = 2$ or $-1 \pm \sqrt{-4}$.
From II, $x = \frac{1}{2}(7 - 3y)$.

Substitute in III. $y\left(\dfrac{7-3y}{2}\right) = 2$ or $-1 \pm \sqrt{-4}$.

$$\therefore\ 3y^2 - 7y = -4 \text{ or } 2 \mp 4\sqrt{-1},$$

whence $y = \frac{4}{3}$, 1, or $\frac{1}{6}(7 \pm \sqrt{73 \mp 48\sqrt{-1}})$,

and $x = \frac{3}{2}$, 2, or $= \frac{1}{4}(7 \mp \sqrt{73 \mp 48\sqrt{-1}})$.

The impossible values of x and y (318) are usually rejected.

421.—CASE II. When the two equations are symmetrical with respect to x and y, put $a + b$ for x, and $a - b$ for y.

An expression is said to be *symmetrical* with respect to x and y when these quantities are similarly involved in it. Thus: $x^3 - x^2y - xy^2 + y^3$; $2xy + 3x + 3y - 5$; $x^2 + y^2 - x - y$.

In like manner, $x^3 + x^2y - xy^2 - y^3$, $-2xy + 3x - 3y - 1$, $x^2 + y^2 - x + y$, are symmetrical with respect to x and $-y$.

Also, $4x^2 + 9y^2 - 2x - 3y$ is symmetrical with respect to $2x$ and $3y$.

422.—**6.** Solve: I. $x^3 + y^{-3} = 27\frac{1}{8}$. Let $x = a + b$.
II. $x + y^{-1} = 3\frac{1}{2}$. Let $y^{-1} = a - b$.

III. Substitute in I. $(a + b)^3 + (a - b)^3 = 27\frac{1}{8}$.
$\therefore 2a^3 + 6ab^2 = 27\frac{1}{8}$.

IV. Substitute in II. $(a + b) + (a - b) = 3\frac{1}{2}$. $\therefore a = \frac{7}{4}$.

V. Substitute $\frac{7}{4}$ for a in III. $\therefore 2(\frac{7}{4})^3 + 6(\frac{7}{4})b^2 = 27\frac{1}{8}$.

VI. Simplify V. $\therefore \frac{343}{32} + \frac{42}{4}b^2 = 27\frac{1}{8}$, whence $b = \pm\frac{5}{4}$.

$x = a + b = \frac{7}{4} \pm \frac{5}{4} = 3$ or $\frac{1}{2}$.

$y^{-1} = a - b = \frac{7}{4} \mp \frac{5}{4} = \frac{1}{2}$ or 3. $\therefore y = 2$ or $\frac{1}{3}$.

423.—**7.** Solve: I. $x^2 + y^2 = 10$. Substitute for x, $a + b$,
II. $x^3 + y^3 = 28$, and for y, $a - b$.

III. $(a + b)^2 + (a - b)^2 = 10$, $\therefore 2a^2 + 2b^2 = 10$.

IV. $(a + b)^3 + (a - b)^3 = 28$, $\therefore 2a^3 + 6ab^2 = 28$.

V. Multiply III by $3a$. $\therefore 6a^3 + 6ab^2 = 30a$.

VI. From V take IV. $\therefore 4a^3 = 30a - 28$.

VII. Multiply VI by a. $4a^4 - 30a^2 = -28a$.

VIII. Add $(4a)^2$. $4a^4 - 14a^2 = 16a^2 - 28a$.

IX. Add $(\frac{7}{2})^2$. $4a^4 - 14a^2 + (\frac{7}{2})^2 = 16a^2 - 28a + (\frac{7}{2})^2$.

X. Extract square root. $2a^2 - \frac{7}{2} = \pm(4a - \frac{7}{2})$, whence

$$a = 2 \text{ or } -1 \pm \tfrac{3}{2}\sqrt{2}.$$

From III, when $a = 2$, $b = \pm 1$. $\therefore x = 3$ or 1.

When $a = -1 \pm \frac{3}{2}\sqrt{2}$, $b = \pm\sqrt{-\frac{1}{2} \pm 3\sqrt{2}}$.

Ans. $x = -1 \pm \frac{3}{2}\sqrt{2} \pm \sqrt{-\frac{1}{2} \pm 3\sqrt{2}}$, 3, 1;
$y = -1 \pm \frac{3}{2}\sqrt{2} \mp \sqrt{-\frac{1}{2} \pm 3\sqrt{2}}$, 1, 3.

424.—8. Solve: I. $(x + y)(x - y)^2 = 160.$
II. $(x + y)(x^2 + y^2) = 580.$

Let $x = a + b$, and $y = a - b$.

Then $x + y = 2a$, $x - y = 2b$, $x^2 + y^2 = 2a^2 + 2b^2$.

III. Substitute in I. $2a(2b)^2 = 160$. $\therefore ab^2 = 20$.

IV. Substitute in II. $2a(2a^2 + 2b^2) = 580$. $\therefore a^3 + ab^2 = 145$.

Subtract III from IV. $\therefore a^3 = 125$, and $a = 5$.

From III, $b = \pm 2$; whence $x = 7$ or 3, $y = 3$ or 7.

Having $a^3 = 125$, three values of a may be found (401).

425. Many equations which are not symmetrical with respect to x and y, may be solved in like manner.

9. Solve: I. $x - y = 1$.
II. $x^3 + y^3 = 35$.

Substitute $a + b$ for x, and $a - b$ for y.

III. I becomes $2b = 1$, whence $b = \frac{1}{2}$.

IV. II becomes $2a^3 + 6ab^2 = 35$.

V. In IV put $\frac{1}{2}$ for b. $\therefore 2a^3 + \frac{3}{2}a = 35$.

Solve V as explained in Art. 406.

$$\therefore a = 2\tfrac{1}{2} \text{ or } \tfrac{1}{4}(-5 \pm \sqrt{-87}).$$
$$\therefore x = 3 \text{ or } \tfrac{1}{4}(-3 \pm \sqrt{-87}).$$
$$y = 2 \text{ or } \tfrac{1}{4}(-7 \pm \sqrt{-87}).$$

426.—Case III. When the equations are homogeneous with respect to x and y, and of the second degree, substitute tx for y, and divide one equation by the other.

10. Solve: I. $x^2 + xy = 15$. $\therefore$ III. $x^2 + tx^2 = 15$.
II. $xy - y^2 = 2$. $\therefore$ IV. $tx^2 - t^2x^2 = 2$.

V. Divide III by IV. $\dfrac{1 + t}{t - t^2} = \dfrac{15}{2}$.

From Chapter XX. $t = \frac{2}{3}$ or $\frac{1}{5}$.

In III put $\frac{2}{3}$ for t. $x^2 + \frac{2}{3}x^2 = 15$. $\therefore x = \pm 3$.

$y = \frac{2}{3}x$. $\therefore y = \pm 2$.

In III put $\frac{1}{5}$ for t. $\therefore \frac{6}{5}x^2 = 15$.

$$\therefore x = \pm \tfrac{5}{2}\sqrt{2}. \quad y = \tfrac{1}{5}x = \pm \tfrac{1}{2}\sqrt{2}.$$

427. Equations of this class may also be solved by eliminating the known quantity, then finding the value of one of the unknown quantities in terms of the other, and substituting this value in one of the given equations. Thus, to solve Ex. 10, multiply I by 2, and II by 15.

$$\text{III} = \text{I} \times 2. \quad 2x^2 + 2xy = 30.$$

$$\text{IV} = \text{II} \times 15. \quad 15xy - 15y^2 = 30.$$

Subtract IV from III. $2x^2 - 13xy + 15y^2 = 0.$

$$16x^2 - (\quad) + 169y^2 = -120y^2 + 169y^2 = 49y^2.$$

$$4x - 13y = \pm 7y. \quad \therefore x = 5y \text{ or } \tfrac{3}{2}y.$$

Substitute in II, and solve the resulting quadratics.

428. It frequently happens that a new equation, simpler in form than the two given equations, can be formed by combining them. Especially is this the case when one equation is divisible by the other. Three examples are appended illustrating this artifice, but skill in the use of such methods must be acquired by experience.

429.—11. Solve: I. $x^2 - xy + y^2 = 7.$

II. $x^4 + x^2y^2 + y^4 = 133.$

III = II ÷ I. $x^2 + xy + y^2 = 19.$

IV = III − I. $2xy = 12. \quad \therefore xy = 6.$

V = III + $(xy = 6)$. $(x + y)^2 = 25. \quad \therefore x + y = \pm 5.$

VI = I − $(xy = 6)$. $(x - y)^2 = 1. \quad \therefore x - y = \pm 1.$

$$\therefore x = \pm 3, \pm 2. \quad y = \pm 2, \pm 3.$$

430.—12. Solve: I. $x^2 + y^2 + x + y = 18.$

II. $xy = 6.$

III. Add twice the second equation to the first.

$\therefore (x + y)^2 + (x + y) = 30,$ whence $x + y = 5$ or $-6.$

Now take $x + y = 5$ or -6, and $xy = 6$.

Solving this pair of equations (421), we find

$$x = 3,\ 2,\ -3 \pm \sqrt{3}; \quad y = 2,\ 3,\ -3 \mp \sqrt{3}.$$

431.—13. Solve: I. $x^2 + xy - 6y^2 = 21$.

II. $xy - 2y^2 = 4$.

III = I ÷ II. $\dfrac{x + 3y}{y} = \dfrac{21}{4}$.

IV. From III, $x = \frac{9}{4}y$.

In II substitute for x its value, $\frac{9}{4}y$. $\therefore$ $\frac{1}{4}y^2 = 4$, whence $y = \pm 4$, and, from IV, $x = \pm 9$.

432. Simultaneous equations involving three or more unknown quantities usually require special methods for their solutions. In addition to the artifices heretofore explained, two others are submitted for the consideration of advanced students.

433.—14. Solve: I. $x + y + z = 2$.

II. $xy + xz + yz = -5$.

III. $xyz = -6$.

Assume a new unknown quantity, v, whose values shall be x, y, and z.

$\therefore$ IV. $v - x = 0$. V. $v - y = 0$. VI. $v - z = 0$.

VII. Multiplying IV, V, and VI together,

$$v^3 - (x + y + z)v^2 + (xy + xz + yz)v - xyz = 0.$$

But $x + y + z = 2$; $xy + xz + yz = -5$; $xyz = -6$.

Substituting in VII, $\therefore$ VIII. $v^3 - 2v^2 - 5v + 6 = 0$.

$\therefore$ IX. $(v - 1)(v - 3)(v + 2) = 0$.

$\therefore$ $v = 1,\ 3,\ -2$.

The given equations being symmetrical with respect to x, y, and z, it is evident that each unknown quantity may have any of the values of v.

When $x = 1$, $y = 3$ or -2, and $z = -2$ or 3, etc.

434.—15. Solve: I. $\dfrac{xyz}{x + y} = 2$. $\therefore$ IV. $\dfrac{x + y}{xyz} = \dfrac{1}{2}$.

II. $\dfrac{xyz}{x + z} = \dfrac{18}{7}$. $\therefore$ V. $\dfrac{x + z}{xyz} = \dfrac{7}{18}$.

III. $\dfrac{xyz}{y + z} = \dfrac{9}{2}$. $\therefore$ VI. $\dfrac{y + z}{xyz} = \dfrac{2}{9}$.

$$\text{VII} = (\text{IV}+\text{V}+\text{VI}) \div 2. \quad \frac{x+y+z}{xyz} = \frac{5}{9}.$$

From VII take IV, V, and VI, respectively.

$$\therefore \text{ VIII. } \frac{1}{xy} = \frac{1}{18}. \quad \text{IX. } \frac{1}{xz} = \frac{1}{6}. \quad \text{X. } \frac{1}{yz} = \frac{1}{3}.$$

Divide each of these equations by the product of the other two.

$$\therefore x = \pm 6, \quad y = \pm 3, \quad z = \pm 1.$$

Exercise XXIV.

Solve:

1. $3x^2 + 5y^2 = 47,\ 2x^2 - 3y^2 = 6.$
2. $x - y = 3,\ 2x^2 - 5y^2 = 27.$
3. $x^2 + xy = 15,\ 3xy + 4y^2 = 34.$
4. $2x + 3y = 14,\ x^{-1} + y^{-1} = \frac{3}{4}.$
5. $x^2 - 4y^2 = 9,\ xy + 2y^2 = 18.$
6. $\dfrac{x+y}{x-y} + \dfrac{x-y}{x+y} = \dfrac{10}{3},\ x^2 + y^2 = 20.$
7. $x + y = 9,\ x^3 + y^3 = 189.$
8. $x - y = 4,\ x^3 - y^3 = 124.$
9. $x - y = 19,\ x^{\frac{1}{3}} - y^{\frac{1}{3}} = 1.$
10. $x + y = 126,\ x^{\frac{1}{3}} + y^{\frac{1}{3}} = 6.$
11. $x + y = 10,\ \sqrt{xy^{-1}} + \sqrt{x^{-1}y} = \frac{5}{2}.$
12. $x^2 + y^2 - x + y = 2xy + 2,\ x^2 + y^2 = 26.$
13. $x + \sqrt{xy} + y = 14,\ x^2 + y^2 + xy = 84.$
14. $(x + y)^3 - x - y = 60,\ xy = 3.$
15. $x + y = 1,\ x^3 - y^3 = 9.$
16. $x - y = 3,\ x^3 + y^3 = 27.$
17. $x^2 - y^2 = 3,\ x^3 + y^3 = 9$; (find one pair of values.)
18. $x^2 + y^2 = 17,\ x^3 + y^3 = 63$; (find two pairs of values.)
19. $x^2 + y^2 = 20,\ x^3 - y^3 = 56$; (find two pairs of values.)

20. $x - y + xy(x - y) + x^2y^2 = 17,$
$xy + (x - y)^2 + xy(x - y) = 13.$

21. $x^3 + y^2 = 33,\ 5x + y^2 = 35.$

22. $x(x+y+z) = 50,\ y(x+y+z) = 30,\ z(x+y+z) = 20.$

23. $x^3 + y^3 + z^3 = x^2 + y^2 + z^2 = x + y + z = 1.$

24. $x + y + z = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{13}{3},\ xyz = 1.$

25. $x + y + z = 7,\ x^2 + y^2 + z^2 = 21,\ xyz = 8.$

26. Find two numbers whose difference is 4 and whose product is 45.

27. The sum of two numbers multiplied by their product is 70, and their difference multiplied by their product is 30. Find the numbers.

28. The product of two required numbers is 12 times their difference, and the sum of their squares is 52.

29. If A had worked 5 days less and had received $4 less per day, he would have earned $9; if he had worked 4 days less and had daily earned $5 less, he would have earned $8. How much did he earn?

30. A number is equal to 3 times the product of its two digits, and is 18 less than its reverse. Find the number.

31. A went from C to H, a distance of 24 miles, in 5 hours, riding half the way and walking the other half. Returning, he rode half the way at a rate 2 miles faster than when he went out, and walked the other half at a rate 2 miles slower than before. He reached C in $7\frac{1}{2}$ hours. Find his rates of riding and walking.

32. A and B gained by trading, $100. Half of A's investment was less than B's by $100, and A's gain was $\frac{3}{20}$ of B's investment. Find the investment and gain of each.

33. Find three numbers whose sum is 10, whose product is 18, and the sum of whose squares is 46.

CHAPTER XXV.

INEQUALITIES.

435. Two expressions containing the same letter will have their values changed when different values are assigned to that letter, and the expressions may be so related that one will be larger than, equal to, or less than the other, according to the values of the given letter.

For example, when $x = \pm 1$, $x^3 + 1 = x^2 + x$; when $x < -1$, $x^3 + 1 < x^2 + x$. For all other values of x, $x^3 + 1 > x^2 + x$.

In other cases, however, the relations may be such that one of the two can not be greater than the other. Thus: $2x$ can not be greater than $x^2 + 1$, whatever value be given to x.

436. The axioms employed in the consideration of inequalities have been already given in Art. 62 (numbered from 8 to 13 inclusive).

In using the axioms for multiplication and division (Axioms X and XI), it must be observed that if the multiplier or divisor be negative, the inequality will be reversed. Thus: 6 being greater than 2, if both be multiplied by -3, we shall have -18 less than -6. Similarly, if both be divided by -2, we shall have -3 less than -1.

437. From Axioms VIII and IX it is evident that *a term may be transposed from one side of an inequality to the other, provided its sign be changed.* Thus:

$$16 - 4 > 11. \quad \therefore 16 > 11 + 4.$$

$$\text{Also } 16 + 4 > 11. \quad \therefore 16 > 11 - 4.$$

438. From (436) it follows that *the signs of all the terms of an inequality may be changed, provided the sign of inequality be reversed;* that is, $>$ *changed to* $<$, *or* $<$ *changed to* $>$. Thus: $6 > 4$, but $-6 < -4$.

439. Two inequalities are said to subsist in the *same sense* when both are read *is greater than,* or both *is less than.* Thus: $6 > 4$ and $5 > 2$ subsist in the same sense.

440. Two inequalities are said to subsist in a *contrary sense* when one is read *is greater than,* and the other *is less than.* Thus: $6 > 4$ and $2 < 5$ subsist in a contrary sense.

441. *Both members of an inequality may be raised to the same odd power without changing the sense.* Thus:
$4 > 2. \therefore 64 > 8.$ Also, $-2 > -3. \therefore -8 > -27.$

442. *If both members of an inequality be raised to the same even power, the sense will be unchanged if both be positive, but the sense will be changed if both be negative.*
Thus: $5 > 3. \therefore 25 > 9.$
Also, $-5 < -3$, but $(-5)^2 > (-3)^2$.

If one member be positive and the other negative, both members may be raised to the same even power without changing the sense, if the positive member be numerically the greater. If the negative member be numerically the greater, the sense will be changed. If the two members be numerically equal, an equation will result. Thus:

$$5 > -3, \text{ then } 25 > 9;\; 3 > -5, \text{ but } 9 < (-5)^2;$$
$$3 > -3, \text{ but } 9 = (-3)^2.$$

443. For finding which of two given expressions is the greater, the following is a fundamental theorem:

I. *The sum of the squares of two unequal numbers is greater than twice their product.*

Formula: If $a - b$ is not zero, $a^2 + b^2 > 2ab$.

For, $(a - b)^2$ must be positive, whatever the values of a and b (114).

Since every positive number is greater than zero, $\therefore (a-b)^2 > 0$. Expanding, $a^2 - 2ab + b^2 > 0$.

Add $2ab$ to each member (Ax. VIII). $\therefore a^2 + b^2 > 2ab$.

If $a = b$, then $a^2 + b^2 = 2ab$.

444. The following are evidently special forms of the above general case, and result from substituting particular values for a and b in the formula $a^2 + b^2 > 2ab$:

$$\text{II.}\quad a^n + b^n > 2a^{\frac{n}{2}}b^{\frac{n}{2}}.$$

$$\text{III.}\quad a + b > 2a^{\frac{1}{2}}b^{\frac{1}{2}}.\quad \therefore \frac{a+b}{2} > a^{\frac{1}{2}}b^{\frac{1}{2}}.$$

$$\text{IV.}\quad a^{\frac{1}{2}} + b^{\frac{1}{2}} > 2a^{\frac{1}{4}}b^{\frac{1}{4}}.$$

445. If ab be positive, both members of I, $a^2 + b^2 > 2ab$, may be divided by ab without changing the sense.

$$\therefore \text{V.}\quad \frac{a}{b} + \frac{b}{a} > 2.$$

Whence: *The sum of any fraction and its reciprocal is greater than two, unless the value of the fraction is unity.*

446. By adding $2ab$ to both sides of the inequality $a^2 + b^2 > 2ab$, we have:

$$\text{VI.}\quad (a+b)^2 > 4ab.$$

Whence: *The square of the sum of two unequal numbers is greater than four times their product.*

447. In order to prove any proposed inequality, the following method is recommended:

In the first place, analyze the proposed statement as follows:

(1) Assume the statement to be true.

(2) Simplify both members as far as possible by adding equals to both, subtracting equals from both, etc.; the object being to reduce the proposed inequality to one whose truth is evident.

To prove the truth of the proposed inequality, *begin* *where the analysis ended, and proceed by steps the reverse of those taken in the analysis.* For example:

448.—VII. To show that $a^3 + b^3 > a^2b + ab^2$.

Analysis: If $a^3 + b^3 > a^2b + ab^2$, then by factoring each member we find $(a + b)(a^2 - ab + b^2) > ab(a + b)$. Divide both members by the common factor $a + b$.

$$\therefore\ a^2 - ab + b^2 > ab.$$

Subtract ab from each member.

$$\therefore\ a^2 - 2ab + b^2 > 0.$$

That is, $(a - b)^2 > 0$, a known truth (443).

By reversing this analysis, we show that the proposed inequality is true, as follows:

Since $(a - b)^2 > 0$, $\therefore\ a^2 - 2ab + b^2 > 0$.

Add ab to each member. $\therefore$ (Ax. VIII) $a^2 - ab + b^2 > ab$.

Multiply each member by $(a + b)$.

$\therefore$ (Ax. X) $a^3 + b^3 > a^2b + ab^2$, which proves the proposition.

449. *No inequality involving letters is true under all circumstances.* Thus: $6a > 4a$ if a be any positive number, but $6a < 4a$ if a be any negative number. In order to determine in what cases a proposed inequality is untrue, it is necessary to examine carefully every step of the proof. Take, for instance, the proof of formula VII (448).

(1) $(a - b)^2 > 0$ for *all unequal values of a and b.*

(2) $a^2 - ab + b^2 > ab$ for *all unequal values of a and b,* because adding equals to unequals never changes the sense.

(3) To derive $a^3 + b^3$ from $a^2 - ab + b^2$, it is necessary to multiply by $a + b$. If, therefore, $a + b$ be any positive number, the sense is unchanged (436); if $a + b$ be equal to 0, an equation is formed; if $a + b$ be negative, the sense is changed (436).

Whence: I. $a^3 + b^3 = a^2b + ab^2$, if $a = \pm b$.

II. $a^3 + b^3 < a^2b + ab^2$, if $a + b$ be a negative number.

In all other cases $a^3 + b^3 > a^2b + ab^2$.

450. If three or more letters occur symmetrically (421), the proof of the proposed inequality usually depends upon a corresponding inequality with two letters. Thus:

VIII. To show that $a^2 + b^2 + c^2 > ab + ac + bc$.

From I (443), I. $a^2 + b^2 > 2ab$.

Similarly, II. $a^2 + c^2 > 2ac$,

and III. $b^2 + c^2 > 2bc$.

Adding I, II, and III (Ax. XII),

$$\text{IV. } 2a^2 + 2b^2 + 2c^2 > 2ab + 2ac + 2bc.$$

Divide IV by 2 (Ax. XI).

$$\therefore a^2 + b^2 + c^2 > ab + ac + bc.$$

451.—IX. To show that

$$a^3 + b^3 + c^3 > \tfrac{1}{2}(a^2b + ab^2 + a^2c + ac^2 + b^2c + bc^2).$$

From VII (448), I. $a^3 + b^3 > a^2b + ab^2$.

Similarly, II. $a^3 + c^3 > a^2c + ac^2$,

and III. $b^3 + c^3 > b^2c + bc^2$.

Adding I, II, and III (Ax. XII),

$$\text{IV. } 2a^3 + 2b^3 + 2c^3 > a^2b + ab^2 + a^2c + ac^2 + b^2c + bc^2.$$

Divide IV by 2 (Ax. XI).

$$\therefore a^3 + b^3 + c^3 > \tfrac{1}{2}(a^2b + ab^2 + a^2c + ac^2 + b^2c + bc^2).$$

Formulas VIII and IX are also true when one of the three fundamental inequalities becomes an equation, and the same remark may be made in regard to all formulas similarly derived.

452. If both members of two or more inequalities be positive, and if the inequalities subsist in the same sense, the product of the corresponding members will form an inequality in the same sense. If $6 > 5$, and $9 > 8$, then $6 \times 9 > 5 \times 8$.

453.—X. To show that $(a + b)(a + c)(b + c) > 8abc$.

From formula III (444), I. $a + b > 2a^{\frac{1}{2}}b^{\frac{1}{2}}$.

Similarly, II. $a + c > 2a^{\frac{1}{2}}c^{\frac{1}{2}}$,

and III. $b + c > 2b^{\frac{1}{2}}c^{\frac{1}{2}}$.

Multiply together I, II, and III.

$$\therefore (a+b)(a+c)(b+c) > 8abc.$$

EXERCISE XXV.

Show that each of the following propositions is true for all unequal positive values of the various letters employed.

Advanced students should also state the conditions under which the propositions are untrue.

1. $a^2c + b^2c > 2abc;\ a^2b + c^2b > 2abc;\ b^2a + c^2a > 2abc.$
2. $a^2 > 3(2a - 3)$, unless $a = 3$.
3. $\dfrac{a+b}{2} > \dfrac{2ab}{a+b}.$
4. $a^3b + ab^3 > 2a^2b^2.$
5. $a^4 + b^4 > 2a^2b^2.$
6. $a^2c + b^2c + a^2b + c^2b + b^2a + c^2a > 6abc.$
7. $a^3 + b^3 + c^3 > 3abc.$
8. $a^3 > a^2 + a - 1$, unless $a = 1$.
9. $a^3 > 3(a^2 + 3a - 9)$, unless $a = 3$.
10. $a + b + c > 3a^{\frac{1}{3}}b^{\frac{1}{3}}c^{\frac{1}{3}}.$
11. $(a + b + c)^3 > 27abc.$
12. $a^4 + b^4 + c^4 + d^4 > 2a^2b^2 + 2c^2d^2.$
13. $a + b + c + d > 4a^{\frac{1}{4}}b^{\frac{1}{4}}c^{\frac{1}{4}}d^{\frac{1}{4}}.$
14. $(a + b + c + d)^4 > 4^4abcd.$
15. $2(ab+ac+bc) > a^2+b^2+c^2$, if $a+b>c$, $a+c>b$, $b+c>a$.
16. $a^4 + b^4 + c^4 > abc(a + b + c).$
17. $\dfrac{a+c+e}{b+d+f} > \dfrac{c}{d}$, if $\dfrac{a}{b} > \dfrac{e}{f} > \dfrac{c}{d}.$
18. $\dfrac{a+c+e}{b+d+f} < \dfrac{a}{b}$, if $\dfrac{a}{b} > \dfrac{e}{f} > \dfrac{c}{d}.$
19. $(a + b + c)(a^2 + b^2 + c^2) > 9abc.$
20. $\left(\dfrac{a+b+c}{3}\right)^3 < \frac{1}{3}(a^3 + b^3 + c^3).$

CHAPTER XXVI.

LOGARITHMS.

454. THE **Logarithm** of a number is the exponent of the power to which a fixed number, called the *base*, must be raised to be equal to the number: that is, if $b^l = n$, then l is called *the logarithm of* n *to the base* b.

455. The logarithm of n to the base b is written $\log_b n$; thus, $\log_b n = l$ expresses the same relation as $b^l = n$. For example, $4^3 = 64$ and $\log_4 64 = 3$ may both be translated, 3 *is the logarithm of* 64 *to the base* 4.

COMMON LOGARITHMS.

456. In practical calculations the only base that is used is 10; logarithms to the base 10 are called *common* logarithms. Thus, since $10^2 = 100$, therefore the common logarithm of 100 is 2, written thus, $\log_{10} 100 = 2$, or more briefly, $\log 100 = 2$.

457. Since $10^0 = 1$, $\therefore$ the log of 1 is 0.
Since $10^1 = 10$, $\therefore$ the log of 10 is 1.
Since $10^3 = 1000$, $\therefore$ the log of 1000 is 3.
Since $10^{-1} = \frac{1}{10} = .1$, $\therefore$ the log of .1 is -1.
Since $10^{-2} = \frac{1}{100} = .01$, $\therefore$ the log of .01 is -2.
Since $10^{-3} = \frac{1}{1000} = .001$, $\therefore$ the log of .001 is -3,
and so on.

It is evident that the logarithm of any numbers between
1 and 10 is 0 + a fraction;
10 and 100 is 1 + a fraction;
100 and 1000 is 2 + a fraction;

1 and .1 is $-1+$ a fraction;
.1 and .01 is $-2+$ a fraction;
.01 and .001 is $-3+$ a fraction; etc.

458. The fractional part of a logarithm can not be expressed exactly, either by common or decimal fractions, but the approximate value may be found, true to as many decimal places as may be desired. For instance:

Let it be required to find the logarithm of 2; that is, to find x in the equation $10^x = 2$. Since $10^0 = 1$ and $10^1 = 10$, $\therefore x > 0$ and < 1. Let $x = 0 + y^{-1}$; then $10^{\frac{1}{y}} = 2$, whence, by raising each side to the y power, $10 = 2^y$. Since $2^3 = 8$ and $2^4 = 16$, $\therefore y > 3$ and < 4. Let $y = 3 + z^{-1}$; then $2^{3+\frac{1}{z}} = 10$, or $8 \times 2^{\frac{1}{z}} = 10$, or $2^{\frac{1}{z}} = \frac{5}{4}$. $\therefore 2 = (\frac{5}{4})^z$. Since $(\frac{5}{4})^3 = \frac{125}{64} < 2$, and since $(\frac{5}{4})^4 = \frac{625}{256} > 2$, therefore $z > 3$ and < 4. Let $z = 3 + v^{-1}$. $\therefore 2 = (\frac{5}{4})^{3+\frac{1}{v}}$. $\therefore 2 = \frac{125}{64} \times (\frac{5}{4})^{\frac{1}{v}}$. $\therefore \frac{128}{125} = (\frac{5}{4})^{\frac{1}{v}}$. $\therefore (\frac{128}{125})^v = \frac{5}{4}$.

By trial v is found to be greater than 9 and less than 10.

If we take $v = 9$, then $z = 3 + v^{-1} = 3 + \frac{1}{9} = \frac{28}{9}$; $y = 3 + z^{-1} = 3 + \frac{9}{28} = \frac{93}{28}$; $x = y^{-1} = \frac{28}{93} = .30107 +$.

If we take $v = 10$, then $z = 3 + v^{-1} = 3 + \frac{1}{10} = \frac{31}{10}$; $y = 3 + z^{-1} = 3 + \frac{10}{31} = \frac{103}{31}$; $x = y^{-1} = \frac{31}{103} = .30097 +$.

Now the true value of x must be between .30107 and .30097. By shorter methods of Higher Mathematics, the logarithm of 2 is found to be .3010300, true to the seventh place.

459. The logarithm of a number consists of an integral part, called *the characteristic*, and a fractional part, called *the mantissa*. Thus, $\log 2 = 0.30103$; here the characteristic is 0, and the mantissa is .30103: $\log 200 = 2.30103$; here the characteristic is 2, and the mantissa is .30103.

460. *The mantissa is always positive;* hence a negative logarithm consists of a negative characteristic and a positive mantissa. Thus, the log of $\frac{2}{10} = -1 + .3010300$; the log of $.03 = -2 + .477121$ (Art. 457).

461. When the characteristic is negative, it is usual to write the minus sign over the characteristic, and to omit the plus sign before the mantissa. Thus:

$$\log .2 = \bar{1}.301030\ ;\ \log .03 = \bar{2}.477121.$$

A negative logarithm may also be expressed by adding 10 to the characteristic, and indicating the subtraction of 10 from the resulting logarithm. Thus :

$$\log .2 = 9.301030 - 10;\ \log .02 = 8.301030 - 10.$$

462. *The characteristic of a logarithm of an integer, or of that of a mixed number, is one less than the number of integral digits.* Thus, the characteristic of the logarithm of 50 is 1, because $50 > 10$ and < 100; hence the logarithm of 50 is between 1 and 2 (Art. 457). Similarly, the characteristic of the logarithm of 320.624 is 2, and of $5476\frac{1}{3}$ is 3.

463. *The characteristic of a logarithm of a decimal fraction is negative, and is numerically one more than the number of ciphers between the decimal point and the left-hand significant figure of the decimal.*

Thus (457): The characteristic of the logarithm of any number between .1 and 1 is -1; of any number between .01 and .1 is -2; of any number between .001 and .01 is -3; etc.

464. Rule for determining the characteristic of a logarithm of an integer, of a mixed number, or of a decimal.

If the left-hand significant figure be n places to the left of units' place, the characteristic of the logarithm will be $+n$; if the left-hand significant figure be n places to the right of units' place, the characteristic of the logarithm will be $-n$.

The characteristic of the logarithm of 6435 is $+3$, because the left-hand significant figure, 6, is three places to the left of units' place. The characteristic of the logarithm of

.0065 is -3, because the left-hand significant figure, 6, is the third place to the right of units. If the left-hand figure be in units' place, the characteristic will be 0.

465. *Let n represent any number, and let w represent any whole number, either positive or negative; then will the logarithms of n and $10^w \times n$ have equal mantissas.*

For, let c be the characteristic and m the mantissa of the logarithm of n; then, from the definition of a common logarithm (456), $10^{c+m} = n$. Multiply both sides of this equation by 10^w; $\therefore$ (Art. 117) $10^{c+w+m} = 10^w \times n$. Whence $c + w + m$ is the logarithm of $10^w \times n$, in which $(c+w)$ is the characteristic and m is the mantissa.

466. From (465) it follows that changing the position of the decimal point in any number does not affect the mantissa of the logarithm. For, moving the decimal point to the right is equivalent to multiplying by 10^1, 10^2, 10^3, etc., according as the point is moved one place, two places, three places, etc. In like manner, moving the decimal point to the left is equivalent to multiplying by $10^{-1} = \frac{1}{10}$, $10^{-2} = \frac{1}{100}$, $10^{-3} = \frac{1}{1000}$, etc., according as the point is moved one place, two places, three places, etc. Thus, the logarithm of 3570 being 3.552668, then $\log 35.7 = 1.552668$, $\log .357 = \bar{1}.552668$, $\log .00357 = \bar{3}.552668$, etc.

467. $\log (ab) = \log a + \log b$; i.e.: *The logarithm of any product equals the sum of the logarithms of the factors.*

For, if $10^x = a$ and $10^y = b$, then $10^{x+y} = ab$ (Art. 117). But $x = \log a$, $y = \log b$, and $x + y = \log (ab)$ (Art. 456). $\therefore \log (ab) = \log a + \log b$.

Similarly, $\log (abc) = \log a + \log b + \log c$; etc.

EXAMPLES.

Given $\log 2 = 0.301$, $\log 3 = 0.477$, $\log 7 = 0.845$, and $\log 10^x = x$ (Art. 457); find the logarithms of the following numbers by adding the logarithms of the factors:

1. 6.	*Ans.* .778.		8. 24.	*Ans.* 1.380.
2. 8.	*Ans.* .903.		9. 27.	*Ans.* 1.431.
3. 9.	*Ans.* .954.		10. 3.2.	*Ans.* 0.505.
4. 12.	*Ans.* 1.079.		11. .36.	*Ans.* $\bar{1}.556$.
5. 14.	*Ans.* 1.146.		12. .042.	*Ans.* $\bar{2}.623$.
6. 16.	*Ans.* 1.204.		13. 490.	*Ans.* 2.690.
7. 18.	*Ans.* 1.255.		14. .0056.	*Ans.* $\bar{3}.748$.

468. Log $a^p = p \log a$; i.e.: *The logarithm of any power of a number is obtained from multiplying the logarithm of the number by the exponent of the power.*

For, if $10^x = a$, then $(10^x)^p = a^p$. $\therefore 10^{xp} = a^p$ (Art. 127); $\therefore \log a^p = px = p \log a$.

EXAMPLES.

Given $\log 2 = 0.301$, $\log 3 = 0.477$, $\log 7 = 0.845$, and $\log 10^x = x$; find the logarithms of the following numbers:

1. 64.	*Ans.* 1.806.	5. 196.	*Ans.* 2.292.
2. 81.	*Ans.* 1.908.	6. .036.	*Ans.* $\bar{2}.556$.
3. 343.	*Ans.* 2.535.	7. .000128.	*Ans.* $\bar{4}.107$.
4. 72.	*Ans.* 1.857.	8. 25600.	*Ans.* 4.408.

469. Log $a \div b = \log a - \log b$; i.e.: *The logarithm of a quotient is obtained from subtracting the logarithm of the divisor from the logarithm of the dividend.*

For, if $10^x = a$ and $10^y = b$, then $10^{x-y} = a \div b$ (Art. 139). $\therefore (x - y)$, which is the logarithm of $a \div b$, equals $\log a - \log b$.

Similarly, $\log b^{-1} = \log 1 - \log b = 0 - \log b = -\log b$.
$\text{Log } \frac{3}{7} = \log 3 - \log 7 = (-1 + 1.477) - .845 = \bar{1}.632$.

EXAMPLES.

Given logarithms of 2, 3, and 7 as before, also log 10 = 1, find the logarithms of the following numbers:

1. 5. *Ans.* .699.
2. $\frac{1}{2}$. *Ans.* $\bar{1}.699$.
3. $\frac{2}{3}$. *Ans.* $\bar{1}.824$.
4. $3\frac{1}{2}$. *Ans.* .544.
5. $\frac{2}{7}$. *Ans.* $\bar{1}.456$.
6. $\frac{5}{7}$. *Ans.* $\bar{1}.854$.
7. 1.25. *Ans.* 0.097.
8. $\frac{4}{49}$. *Ans.* $\bar{2}.912$.

470. Log $\sqrt[r]{a} = \frac{\log a}{r}$; i. e.: *The logarithm of any root of a number is equal to the logarithm of the number divided by the index of the root.*

For, if $10^x = a$, then $10^{\frac{x}{r}} = \sqrt[r]{a}$. $\therefore \log \sqrt[r]{a} = \frac{x}{r} = \frac{\log a}{r}$.

Thus, if $\log 5^7 = 4.893$, then $\log \sqrt[7]{5^7} = \frac{1}{7}(4.893) = .699$.

Similarly, if $\log 2 = .301$, then $\log \sqrt[5]{2} = \frac{1}{5}(.301) = .060$.

Also, $\log \sqrt[5]{.003} = \frac{1}{5}(-5 + 2.477) = \frac{1}{5}(\bar{3}.477) = \bar{1}.495$.

471. If a negative characteristic be not exactly divisible by the index of the root, add to the characteristic the least negative number which will render it so divisible, and prefix an equal positive number to the mantissa. Thus:

$$\tfrac{1}{4}(\bar{3}.624) = \tfrac{1}{4}(-\ 4 + 1.624) = \bar{1}.406;$$

$$\tfrac{1}{5}(-7 + .835) = \tfrac{1}{5}(-10 + 3.835) = \bar{2}.767.$$

EXAMPLES.

Given log 2 = 0.3010, log 3 = 0.4771, log 7 = 0.8451, log 10 = 1; find the logarithms of the following:

1. $\sqrt{2}$. *Ans.* 0.1505.
2. $\sqrt[3]{7}$. *Ans.* 0.2817.
3. $3^{\frac{1}{4}}$. *Ans.* 0.1193.
4. 3^{-1}. *Ans.* $\bar{1}.5229$.

5.	$6^{\frac{2}{3}}$.	*Ans.* 0.5187.	9.	$\sqrt[4]{.007}$.	*Ans.* $\bar{2}.9226$.
6.	$5^{\frac{1}{5}}$.	*Ans.* 0.1398.	10.	$\sqrt[4]{.021}$.	*Ans.* $\bar{1}.7603$.
7.	$15^{-\frac{1}{3}}$.	*Ans.* $\bar{1}.6080$.	11.	$(.0084)^{\frac{4}{5}}$.	*Ans.* $\bar{2}.3394$.
8.	$.7^{\frac{3}{4}}$.	*Ans.* 0.8838.	12.	$7^{\frac{1}{3}} \times 3^{\frac{1}{4}}$.	*Ans.* 0.4010.

472. The remainder obtained by subtracting the logarithm of a number from 10 is called the *cologarithm* of the number, or the *arithmetical complement* of the logarithm of the number (abbreviated into *colog* or *a. c. log*). Thus:

$$\begin{aligned}
\log 3 &= 0.4771 \text{ and } \operatorname{colog} 3 = 9.5229;\\
\log 7 &= 0.8451 \text{ and a. c. } \log 7 = 9.1549;\\
\log 200 &= 2.3010 \text{ and } \operatorname{colog} 200 = 7.6990;\\
\log .005 &= \bar{3}.6990 \text{ and } \operatorname{colog} .005 = 12.3010;\\
\log .0009 &= \bar{4}.9542 \text{ and a. c. } \log .0009 = 13.0458.
\end{aligned}$$

To find the cologarithm: *Begin at the left, subtract each figure from* 9 *down to the last significant figure, and subtract that figure from* 10.

473. Let a and b represent any two logarithms, then it is evident that $a - b = a + (10 - b) - 10 = a + \operatorname{colog} b - 10$.

Therefore: *To subtract one logarithm from another, add to the minuend the cologarithm of the subtrahend, and subtract* 10 *from the result.*

In like manner, $a - b - c = a + (10 - b) + (10 - c) - 20 = a + \operatorname{colog} b + \operatorname{colog} c - 20$; and so on for any number of subtrahends.

EXAMPLE. Find the logarithm of $\sqrt[7]{a \div bc}$, having given $\log a = \bar{1}.7321$, $\log b = \bar{2}.4792$, $\log c = 5.4771$.

$$(\log a + \operatorname{colog} b + \operatorname{colog} c) - 20 = \bar{5}.7758.$$

$$\tfrac{1}{7}(\bar{5}.7758) = \tfrac{1}{7}(-7 + 2.7758) = \bar{1}.3965. \quad \textit{Ans.}$$

TABLE OF LOGARITHMS.

474. The following table of four-place logarithms contains logarithms of all numbers up to 1000, the decimal point and characteristic being omitted. The logarithms of single digits, 2, 3, 4, etc., will be found at 20, 30, 40, etc.

N	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396

N	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996

USE OF THE TABLE WHEN A NUMBER IS GIVEN.

475.—1. To find the logarithm of 74.6.

In the column headed *N* look for the first two figures, and at the top of the table for the third figure. In the line with 74 and in the column headed 6 is seen 8727. To this number prefix the decimal point, and before the decimal point place the characteristic (464). *Ans.* 1.8727.

2. To find the logarithm of 9.

The mantissa of the logarithm of 9 is the same as that of 900 (Art. 465). In the column headed *N* look for 90, and in the column headed 0, on the line with 90, is seen 9542. $\therefore \log 9 = 0.9542$.

3. To find the logarithm of 47.236.

The position of the decimal point does not affect the mantissa (466). Therefore, suppose the point to be placed after the third figure, thus: 472.36. In the line with 47 and in the column headed 2 is seen 6739. Hence the mantissa of the logarithm of 472 is .6739. In like manner, the mantissa of the logarithm of 473 is .6749.

The number 472.36 being between 472 and 473, the mantissa of its logarithm must be between .6739 and .6749. These mantissas differ by .0010, which therefore represents the difference for one unit. Since 472.36 exceeds 472 by .36, it is assumed that the logarithm of 472.36 exceeds that of 472 by .36 of .0010 = .00036. As this table is prepared for four places only, when the fifth figure is less than 5, it is rejected ; when it is 5 or more the preceding figure is increased by 1, $\therefore$ .00036 is taken as .0004. Add .0004 to .6739, and the required mantissa is .6743. *Ans.* 1.6743.

In practice it is customary to omit the ciphers before the 36, and to arrange the work as in the following example:

4. To find the logarithm of .008248.

$$\begin{aligned} &\text{The mantissa for } 824 && = .9159 \\ &\text{Difference for one unit} = 6. \quad .8 \times 6 = 4.8 = && 5 \\ &\therefore \log \text{ of } .008248 = && \bar{3}.9164 \end{aligned}$$

EXAMPLES.

Find from the table the logarithms of the following:

1. 605. *Ans.* 2.7818.
2. 43.2. *Ans.* 1.6355.
3. .0129. *Ans.* $\bar{2}.1106$.
4. .0021. *Ans.* $\bar{3}.3222$.
5. 4815. *Ans.* 3.6826.
6. 49.875. *Ans.* 1.6979.
7. .00007. *Ans.* $\bar{5}.8451$.
8. .09263. *Ans.* $\bar{2}.9668$.
9. 50008. *Ans.* 4.6991.
10. $\sqrt{624}$. *Ans.* 1.3976.
11. $\sqrt[4]{6.02}$. *Ans.* .1949.
12. $(578)^{\frac{2}{3}}$. *Ans.* 1.8413.
13. $\frac{476}{541}$. *Ans.* $\bar{1}.9444$.
14. 47×51. *Ans.* 3.3797.
15. $\sqrt[3]{.018 \times 63}$. *Ans.* 0.0182.
16. $(3.1416)^3$. *Ans.* 1.4913.

USE OF THE TABLE WHEN A LOGARITHM IS GIVEN.

476.—1. To find the number whose logarithm is $\bar{2}.8543$.

Look for 8543 in the table. It is found on the line with 71, and in the column headed 5. Hence .8543 corresponds to 715. The characteristic being -2, it is necessary to prefix one cipher (463). *Ans.* .0715.

2. To find the number of which the logarithm is 1.7384. 7384 is not found in the table, but 7380 is found on the line with 54 and in the column headed 7. ∴ 7380 corresponds to 547. But 7388 corresponds to 548. The difference for one unit being 8, the required number exceeds 547 by $\frac{4}{8} = .5$.

Annex 5 to the number 547, and place the decimal point by the rule (462). *Ans.* 54.75.

3. To find the number whose logarithm is $\bar{3}.4563$.

The next less mantissa is 4548, corresponding to 285. The difference for one unit is 16, and the given mantissa exceeds the next less by 15. ∴ $\frac{15}{16} = .9 +$ must be annexed to the number 285; thus 4563 corresponds to the sequence of figures 2859. Now place the decimal point according to the rule (463). *Ans.* .002859.

In working with four-place logarithms, it is useless to carry the work for finding the *antilogarithms* (numbers corresponding to given logarithms) beyond the fourth place.

477. The following is the rule for finding antilogarithms:

(1) *If the given mantissa be found in the table, the first two figures of the required number will be seen on the same line with the mantissa and in the column headed N; the third figure is seen at the head of the column in which the mantissa is found.*

(2) *If the given mantissa be not found in the table, find the next less mantissa, and set aside the corresponding number as the first three figures of the required number.*

Subtract the next less mantissa from the given mantissa, and annex one cipher to the remainder.

Divide this result by the difference between the next lower and the next higher mantissas; the quotient will be the fourth figure of the required number.

(3) *Insert the decimal point according to the rule* (464).

EXAMPLES.

Find the antilogarithms to the following logarithms :

1. 2.4698. *Ans.* 295.

2. 1.5977. *Ans.* 39.6.

3. $\bar{1}.8573$. *Ans.* .72.

4. 0.8648. *Ans.* 7.325.

5. $\bar{2}.9413$. *Ans.* .08736.

6. $\bar{3}.6888$. *Ans.* .004884.

7. 0.9949. *Ans.* 9.883.

8. $\bar{4}.7310$. *Ans.* .0005383.

9. $\bar{2}.5979$. *Ans.* .03962.

10. $\bar{1}.8994$. *Ans.* .7932.

11. .4821. *Ans.* 3.035.

12. .4672. *Ans.* 2.932.

LOGARITHMS IN GENERAL.

478. Any positive number excepting 1 may be the base of a system of logarithms. In the general equation $b^l = n$, if b be greater than unity, l will be positive when $n > 1$, and l will be negative when $n < 1$. If b be less than unity, l

will be negative when $n > 1$, and l will be positive when $n < 1$. For every base the logarithm of 1 is 0.

479. In every system of logarithms the rules for I, *multiplication* (467); II, *powers* (468); III, *division* (469); and IV, *roots* (470), are the same as those given for common logarithms.

Thus: I. If $b^x = m$ and $b^y = n$, $\therefore$ $b^{x+y} = mn$.

II. If $b^x = n$, then $b^{px} = n^p$.

III. If $b^x = m$ and $b^y = n$, $\therefore$ $b^{x-y} = \frac{m}{n}$.

IV. If $b^x = n$, then $b^{\frac{x}{r}} = \sqrt[r]{n}$.

480. *The difference between the logarithms of two consecutive numbers becomes less as the numbers themselves become greater.*

For, $\log (x + 1) - \log x = \log \frac{x+1}{x}$. As x increases, the value of the fraction $\frac{x+1}{x}$ decreases, hence the logarithm of $\frac{x+1}{x}$ decreases; therefore its equal, $\log (x+1) - \log x$, decreases.

EXPONENTIAL EQUATIONS.

481. An **Exponential Equation** is one in which the unknown quantity is an exponent or index. Thus, $a^x = b$; $\sqrt[x]{a} = b$. Such equations usually require logarithms for their solution.

EXAMPLES.

1. If $2^x = 32$, find x. Since $2^x = 32 = 2^5$, $\therefore x = 5$.

2. If $2^x = 12$, find x.

Since $2^x = 12$, then $\log 2^x = \log 12$; that is,

$$x \log 2 = \log 12; \ \therefore x = \frac{\log 12}{\log 2} = \frac{1.0792}{.301} = 3.58538 + .$$

3. Given $20^x = 100$, find x.

$$\log 20^x = \log 100; \therefore x \log 20 = \log 100;$$

$$\therefore x = \frac{\log 100}{\log 20} = \frac{2}{1.301} = 1.537 + .$$

4. Find log 144 in the system whose base is $2\sqrt{3}$.

Let x represent the required logarithm.

Then, from Art. 454, $(2\sqrt{3})^x = 144 = 12^2$; hence by squaring each side $12^x = 144^2 = 12^4$; $\therefore x = 4$.

5. Find $\log_5 2$. Given common logarithm of $2 = .3010$.

Let x represent the required logarithm.

Then $5^x = 2$; $\therefore x \log 5 = \log 2$; $\therefore x = \dfrac{\log 2}{\log 5}$.

Since $\log 10 = 1$ and $\log 2 = .3010$, $\therefore \log (5 = \frac{10}{2}) = 1 - .3010 = .6990$. $\therefore x = \dfrac{.3010}{.6990} = \dfrac{301}{699} = .4306 + .$

6. Find $\log_{\frac{2}{3}} \frac{3}{4}$; that is, find x in the equation $(\frac{2}{3})^x = \frac{3}{4}$.

$$x = \frac{\log \frac{3}{4}}{\log \frac{2}{3}} = \frac{\bar{1}.8751}{\bar{1}.8239} = \frac{-.1249}{-.1761} = \frac{1249}{1761} = .7092 + .$$

7. Given $4^x + 2^x = 72$, find x.

If $2^x = y$, then $4^x = y^2$.

$\therefore y^2 + y = 72$, $\therefore y = 8$ or -9.

Discarding the negative value of y, we have $2^x = 8$, $\therefore x = 3$.

8. Given $4^x + 2^x = 12$, find x.

As before, $y^2 + y = 12$ and $y = 3$ or -4.

$\therefore 2^x = 3$, whence $x = \dfrac{\log 3}{\log 2} = \dfrac{.4771}{.3010} = 1.585$, nearly.

9. What is the logarithm of 3125 to the base 5? *Ans.* 5.

10. What is the logarithm of 8 to the base $\frac{1}{2}$? *Ans.* -3.

11. Given $\log 2 = .301$, find the number of digits in the sixty-fourth power of 2. *Ans.* 20.

12. Find $\log_3 243$. *Ans.* 5.

13. Given $(\frac{4}{5})^x = .5$, find x. *Ans.* 3.103.

14. $25^x + 5^x = 650$, find x. *Ans.* 2.

15. $3^{2x} + 3^x = 42$, find x. *Ans.* 1.631.

Exercise XXVI.

1. $3.1416 \times 672 = ?$
2. $3.1416 \times (2.7)^4 = ?$
3. $70642 \div 82.3 = ?$
4. $21.06 \div \overline{6.9 \times .541} = ?$
5. If $10^{.301} = 2$, find $\log .0625$.
6. $(1\frac{2}{5})^{21} = ?$
7. $(2\frac{2}{3})^{\frac{1}{3}} = ?$
8. $(6.7)^{2.4} = ?$
9. Find $\log_4 125$.
10. $21^{\frac{2}{3}} = ?$

Find the value of x in each of the following:

11. $6^x + 6^{\frac{x}{2}} = 30.$
12. $a^{bx+c} = d.$
13. $a^{bx}c^{dx} = n.$
14. I. $2^{5x}.3^{3z} = 2000$; II. $5x = 3z$.
15. $a^{2x} - a^x = b^2 - \frac{1}{4}.$
16. $(a^2 - b^2)^{4(x-1)} = (a-b)^{4x} \div (a+b)^4.$
17. $3^{x^2-4x+4} = 1200.$
18. $(\frac{4}{9})^x - (\frac{2}{3})^x = 6.$
19. $(\frac{16}{9})^x - (\frac{4}{3})^x = 2.$
20. $8^x - 3(2^x) = 2.$

CHAPTER XXVII.

RATIO, PROPORTION, AND VARIATION.

482. The **Ratio** of two numbers is their relative magnitude expressed by the fraction of which the first is the numerator and the second is the denominator.

Thus the ratio of 6 to 2 is indicated by the fraction $\frac{6}{2}$; the ratio of 2 to 6 is indicated by the fraction $\frac{2}{6}$; the ratio of $\frac{2}{3}$ to $\frac{3}{4}$ is indicated by the fraction $\frac{8}{9} = \frac{2}{3} \div \frac{3}{4}$.

483. The ratio of a to b is usually written $a : b$ (read a is to b). The first term of the ratio is called the *antecedent*, and the second term is called the *consequent*.

When the antecedent is greater than the consequent, the ratio is called a *ratio of greater inequality;* when the antecedent is equal to the consequent, the ratio is called a *ratio of equality;* when the antecedent is less than the consequent, the ratio is called a *ratio of less inequality.*

484. When two quantities are *commensurable* (that is, can be expressed in integers in terms of a common unit), their ratio is equal to the ratio of the two numbers by which they are expressed. Thus, the ratio of \$9 to \$12 is $\frac{9}{12}$; of $2\frac{1}{2}$ ft. to $3\frac{1}{4}$ ft. is $2\frac{1}{2} \div 3\frac{1}{4} = \frac{10}{13}$.

485. *Two quantities of different kinds can have no ratio.* Thus, 2 hours and 3 inches have no relative magnitude.

486. When two quantities of the same kind are *incommensurable,* it is not possible to express their ratio by a fraction both of whose terms are integers; yet a fraction of this kind may be found which shall express the ratio of the two quantities to any required degree of accuracy.

For, let a and b represent two incommensurable numbers; also, let $b = nx$, where n is an integer, and let a be greater than mx and less than $(m+1)x$. Then $\frac{a}{b} > \frac{mx}{nx}$; that is, $\frac{a}{b} > \frac{m}{n}$; also, $\frac{a}{b} < \frac{m+1}{n}$. Thus the difference between $\frac{a}{b}$ and $\frac{m}{n}$ is less than $\frac{1}{n} = n^{-1}$. Since $nx = b$, when x is diminished n is increased and n^{-1} is diminished. Hence, by taking x small enough, n^{-1} can be made less than any assigned magnitude, and therefore the difference between $\frac{a}{b}$ and $\frac{m}{n}$ can be made less than any assigned magnitude.

487. *A ratio will not be altered if both its terms be multiplied or divided by the same number.*

For, $a : b = \frac{a}{b}$, and $ma : mb = \frac{ma}{mb}$. Since $\frac{a}{b} = \frac{ma}{mb}$, $\therefore a : b = ma : mb$.

488. *A ratio of greater inequality will be diminished, and a ratio of less inequality will be increased, by adding the same number to both terms of the ratio.*

For, let $\frac{a}{b}$ represent any ratio, and let a new ratio be formed by adding any positive number, x, to both terms of the original ratio; then $\frac{a+x}{b+x} >$ or $< \frac{a}{b}$, according as $\frac{ab+bx}{b(b+x)} >$ or $< \frac{ab+ax}{b(b+x)}$; that is, according as $ab + bx >$ or $< ab + ax$; that is, according as $bx >$ or $< ax$; that is, according as $b >$ or $< a$.

489. In like manner it may be shown that *a ratio of greater inequality will be increased, and a ratio of less inequality will be diminished, by taking from both terms of the ratio any number which is less than each of these terms.*

490. Ratios are *compounded* by multiplying together the fractions that represent them. Thus, $ac : bd$ is said to be compounded of the two ratios $a : b$ and $c : d$.

491. When the ratio $a : b$ is compounded with itself, the resulting ratio, $a^2 : b^2$, is called the *duplicate* ratio of $a : b$. Similarly, $a^3 : b^3$, the ratio compounded of $a : b$, $a : b$, $a : b$, is called the *triplicate* ratio of $a : b$.

The ratio $\sqrt{a} : \sqrt{b}$ is called the *subduplicate* ratio of $a : b$; and $\sqrt[3]{a} : \sqrt[3]{b}$ is called the *subtriplicate* ratio of $a : b$.

492. *A ratio of greater inequality compounded with another increases it, and a ratio of less inequality compounded with another diminishes it.*

Let the ratio $x : y$ be compounded with the ratio $a : b$; the compound ratio $ax : by$ is greater or less than $a : b$, according as $\frac{ax}{by}$ is greater or less than $\frac{a}{b}$; that is, according as x is greater or less than y.

493. Ratios are compared by reducing the fractions which represent them to a common denominator, and comparing the numerators of the resulting fractions. Thus, $4 : 5$ is compared with $7 : 9$ by comparing $\frac{4}{5}$ with $\frac{7}{9}$. Now $\frac{4}{5} = \frac{36}{45}$, and $\frac{7}{9} = \frac{35}{45}$; $\therefore$ $4 : 5$ is greater than $7 : 9$.

EXAMPLES.

1. Compare the ratios $5 : 7$ and $2 : 3$. *Ans.* $5 : 7 > 2 : 3$.
2. Find the value of $\frac{5}{7} : \frac{2}{3}$. *Ans.* $15 : 14 = \frac{15}{14}$.
3. Compound $2 : 3$, $3 : 4$, $6 : 7$. *Ans.* $\frac{3}{7} = 3 : 7$.

4. Compound the ratios $a+b : a-b$, $a^2+b^2 : (a+b)^2$, $(a^2-b^2)^2 : a^4-b^4$. *Ans.* 1.

5. Write the duplicate ratio of 3 : 2, and the subduplicate ratio of 100 : 169. *Ans.* 9 : 4, 10 : 13.

6. Two numbers are in the ratio of 6 : 7, and if 4 be subtracted from each, the remainders will be in the ratio of 4 : 5. Find the numbers. *Ans.* 12 and 14.

7. Compound the duplicate ratio of 3 : 4, the triplicate ratio of 2 : 3, the subduplicate ratio of 16 : 9, and the subtriplicate ratio of 27 : 64. *Ans.* 1 : 6.

8. Show that $\sqrt{2} : \sqrt{3} > \sqrt{3} : \sqrt{5}$.

PROPORTION.

494. **Proportion** is an equality of ratios. Thus, if a, b, c, d are four numbers such that $\frac{a}{b} = \frac{c}{d}$, these four numbers are called *proportionals*, or are said to be *in proportion*. This proportion may be written $a : b = c : d$ (read, *the ratio of a to b is equal to the ratio of c to d*), or $a : b :: c : d$ (read, *a is to b as c is to d*).

495. When $a : b :: c : d$, a and d are called the *extremes*; b and c are called the *means*; a and c are called the *antecedents*; b and d are called the *consequents*; a and b are together called the *first couplet*; c and d are together called the *second couplet.*

496. *When four numbers are in proportion, the product of the extremes is equal to the product of the means.*

For, if $a : b :: c : d$, then $\frac{a}{b} = \frac{c}{d}$. $\therefore$ (Ax. IV) $ad = bc$.

497. Hence, if $a : b :: b : d$, then $ad = b^2$.

In this case b is said to be a *mean proportional* between a and d, and d is called a *third proportional* to a and b.

498. If any three terms in a proportion are given, the fourth may be determined from the equation $ad = bc$.

499. *If the product of two numbers be equal to the product of two others, the four are in proportion; the two factors of either product being taken for the means, and the two factors of the other product for the extremes.*

For, if $ad = bc$, by dividing by bd, $\frac{ad}{bd} = \frac{bc}{bd}$;

that is, $\frac{a}{b} = \frac{c}{d}$. $\therefore a : b :: c : d$.

500. If $ad = bc$, then it follows from (499) that:

I. $a : b :: c : d$, since $\frac{ad}{bd} = \frac{bc}{bd}$.

II. $a : c :: b : d$, since $\frac{ad}{dc} = \frac{bc}{dc}$.

III. $d : b :: c : a$, since $\frac{ad}{ba} = \frac{bc}{ba}$.

IV. $d : c :: b : a$, since $\frac{ad}{ac} = \frac{bc}{ac}$.

V. $b : a :: d : c$, since $\frac{bc}{ac} = \frac{ad}{ac}$.

VI. $b : d :: a : c$, since $\frac{bc}{cd} = \frac{ad}{cd}$.

VII. $c : a :: d : b$, since $\frac{bc}{ab} = \frac{ad}{ab}$.

VIII. $c : d :: a : b$, since $\frac{bc}{bd} = \frac{ad}{bd}$.

It is evident, therefore, that if four numbers be proportionals, the relations between them may be expressed in eight ways.

To change one of the above forms into another: *Take the product of the extremes equal to the product of the means, and divide both sides of the resulting equation by the product of the consequents of the required proportion.*

501. *If four numbers are proportionals, they are in proportion when taken by* ALTERNATION; that is, *the first is to the third as the second is to the fourth.*

If $a : b :: c : d$, then $a : c :: b : d$.

For, $\frac{a}{b} = \frac{c}{d}$; multiply by $\frac{b}{c}$; $\therefore \frac{a}{c} = \frac{b}{d}$; or, $a : c :: b : d$.

502. *If four numbers are proportionals, they are in proportion when taken by* INVERSION; that is, *the second is to the first as the fourth is to the third.*

If $a : b :: c : d$, then $b : a :: d : c$.

For, $\frac{a}{b} = \frac{c}{d}$; $\therefore 1 \div \frac{a}{b} = 1 \div \frac{c}{d}$; that is, $\frac{b}{a} = \frac{d}{c}$, or $b : a :: d : c$.

503. *If four numbers are in proportion, they are in proportion by* COMPOSITION; that is, *the sum of the first and second is to the second as the sum of the third and fourth is to the fourth.*

If $a : b :: c : d$, then $a + b : b :: c + d : d$.

For, $\frac{a}{b} = \frac{c}{d}$; $\therefore \frac{a}{b} + 1 = \frac{c}{d} + 1$; that is, $\frac{a+b}{b} = \frac{c+d}{d}$, or $a + b : b :: c + d : d$.

If the given proportion be taken by *inversion*, it can be shown similarly that $a + b : a :: c + d : c$. In like manner, by *alternation* and *composition*, $a + c : c :: b + d : d$.

504. *If four numbers are in proportion, they are in proportion by* DIVISION; that is, *the difference between the first and second is to the second as the difference between the third and fourth is to the fourth.*

If $a : b :: c : d$, then $a - b : b :: c - d : d$.

For, $\frac{a}{b} = \frac{c}{d}$; $\therefore \frac{a}{b} - 1 = \frac{c}{d} - 1$, whence $\frac{a-b}{b} = \frac{c-d}{d}$, or $a - b : b :: c - d : d$.

Similarly, by *inversion* and *division*, $b - a : a :: d - c : c$; also, by *alternation* and *division*, $a - c : c :: b - d : d$.

505. *If four numbers are in proportion, they are in proportion by composition and division.*

If $a : b :: c : d$, then $a + b : a - b :: c + d : c - d$.

From (503), $\dfrac{a+b}{b} = \dfrac{c+d}{d}$; and from (504), $\dfrac{a-b}{b} = \dfrac{c-d}{d}$.

$\therefore$ (Ax. V) $\dfrac{a+b}{a-b} = \dfrac{c+d}{c-d}$; or, $a+b : a-b :: c+d : c-d$.

506. *In a series of equal ratios, the sum of the antecedents is to the sum of the consequents as any antecedent is to its consequent.*

For example, if $\dfrac{a}{b} = \dfrac{c}{d} = \dfrac{e}{f} = \dfrac{g}{h}$, then

$$a + c + e + g : b + d + f + h :: c : d.$$

$$\text{Since } \frac{a}{b} = \frac{c}{d},\ \therefore ad = bc.$$

$$\text{Since } \frac{c}{d} = \frac{c}{d},\ \therefore cd = dc.$$

$$\text{Since } \frac{e}{f} = \frac{c}{d},\ \therefore ed = fc.$$

$$\text{Since } \frac{g}{h} = \frac{c}{d},\ \therefore gd = hc.$$

Whence, by addition, $d(a + c + e + g) = c(b + d + f + h)$.
$\therefore$ (499), $a + c + e + g : b + d + f + h :: c : d$.

507. If a, b, c, d be in *continued proportion*, that is, if $a : b :: b : c :: c : d$, then $a : c :: a^2 : b^2$, and $a : d :: a^3 : b^3$.

For, $\dfrac{a}{b} = \dfrac{b}{c}$; $\therefore \dfrac{b}{c} \times \dfrac{a}{b} = \dfrac{a}{b} \times \dfrac{a}{b}$; $\therefore \dfrac{a}{c} = \dfrac{a^2}{b^2}$;

that is, $$a : c :: a^2 : b^2.$$

Similarly, $\dfrac{a}{b} \times \dfrac{b}{c} \times \dfrac{c}{d} = \dfrac{a}{b} \times \dfrac{a}{b} \times \dfrac{a}{b}$; $\therefore \dfrac{a}{d} = \dfrac{a^3}{b^3}$;

that is, $$a : d :: a^3 : b^3.$$

508. *If four numbers are proportionals, the first couplet may be multiplied or divided by any number, as also the second couplet, and the resulting numbers will be proportionals.*

If $a : b :: c : d$, then $ma : mb :: nc : nd$.

For, $\frac{a}{b} = \frac{c}{d}$; $\therefore \frac{ma}{mb} = \frac{nc}{nd}$, or $ma : mb :: nc : nd$.

509. *If four numbers are proportionals, the antecedents may be multiplied or divided by any number, as also the consequents, and the resulting numbers will be proportionals.*

If $a : b :: c : d$, then $ma : nb :: mc : nd$.

For, $\frac{a}{b} = \frac{c}{d}$; therefore $\frac{ma}{b} = \frac{mc}{d}$; $\therefore \frac{ma}{nb} = \frac{mc}{nd}$;

or, $$ma : nb :: mc : nd.$$

510. *If two proportions have a common couplet, the other couplets form a proportion.*

If $a : b :: c : d$, and if $a : b :: e : f$, then $c : d :: e : f$.

For, $\frac{a}{b} = \frac{c}{d}$, and $\frac{a}{b} = \frac{e}{f}$. $\therefore \frac{c}{d} = \frac{e}{f}$; or, $c : d :: e : f$.

511. *If the antecedents in two proportions be the same, the consequents will be in proportion.*

If $a : b :: c : d$, and if $a : e :: c : f$, then $b : d :: e : f$.

If $a : b :: c : d$, then (Art. 501) $a : c :: b : d$.

If $a : e :: c : f$, then (Art. 501) $a : c :: e : f$.

Whence (Art. 510) $b : d :: e : f$.

Similarly: *If the consequents in two proportions be the same, the antecedents will be in proportion.*

512. *The products of the corresponding terms of two or more proportions are in proportion.*

If $a : b :: c : d$, $e : f :: g : h$, $k : l :: m : n$,

then $$\frac{a}{b} = \frac{c}{d};\ \frac{e}{f} = \frac{g}{h};\ \frac{k}{l} = \frac{m}{n}.$$

The product of the first sides is equal to the product of the second sides; $\therefore \frac{aek}{bfl} = \frac{cgm}{dhn}$; $\therefore aek : bfl :: cgm : dhn$.

513. *If four numbers be in proportion, their like powers, or like roots, will be in proportion.*

If $a : b :: c : d$, then $a^n : b^n :: c^n : d^n$.

For, $\frac{a}{b} = \frac{c}{d}$; $\therefore \frac{a^n}{b^n} = \frac{c^n}{d^n}$, where n may be integral or fractional; $\therefore a^n : b^n :: c^n : d^n$.

514. *If two quantities be increased or diminished by like parts of each, the results will be in the same ratio as the quantities themselves.*

For, $$\frac{a}{b} = \frac{ma}{mb} = \frac{\left(1 \pm \frac{c}{d}\right)a}{\left(1 \pm \frac{c}{d}\right)b} = \frac{a \pm \frac{c}{d}a}{b \pm \frac{c}{d}b}.$$

$$\therefore a : b :: a \pm \frac{c}{d}a : b \pm \frac{c}{d}b.$$

515. If a and b are incommensurable, and if c and d are also incommensurable; also, if when $\frac{a}{b}$ lies between $\frac{m}{n}$ and $\frac{m+1}{n}$, then $\frac{c}{d}$ also lies between $\frac{m}{n}$ and $\frac{m+1}{n}$, however the numbers m and n are increased, then $\frac{a}{b} = \frac{c}{d}$.

For, if $\frac{a}{b}$ and $\frac{c}{d}$ are not equal, they must have some assignable difference, and since each of them lies between $\frac{m}{n}$ and $\frac{m+1}{n}$, this difference must be less than $\frac{1}{n} = n^{-1}$. But since n may, by supposition, be increased without limit, n^{-1} may be diminished without limit; that is, n^{-1} may be made less than any assignable magnitude. Therefore, $\frac{a}{b}$ and $\frac{c}{d}$ have no assignable difference, so that we may say $\frac{a}{b} = \frac{c}{d}$. Hence all the propositions respecting proportionals are true of the four numbers a, b, c, d.

EXAMPLES.

1. If $4 : 6 :: x : 36$, find x. *Ans.* $x = 24$.

2. If 3, x, and 867 are in continued proportion, find x. *Ans.* $x = 51$.

3. Find a third proportional to 25 and 40. *Ans.* 64.

4. Find a mean proportional to 16 and 9. *Ans.* 12.

5. If $5y - 8x : 7x - 5y :: 6 : 1$, find the ratio of x to y. *Ans.* $7 : 10$.

6. If $a^2 - b^2 : a + b :: (a - b)^2 : x$, find x. *Ans.* $x = a - b$.

7. What number added to each of the numbers 4, 8, 10, 16 will make the results proportional? *Ans.* 8.

8. $\sqrt[3]{x + 5} : \sqrt[3]{x - 5} :: 5 : 3$. Find x. *Ans.* $x = \frac{380}{49}$.

9. Given $x + y : x :: 5 : 3$, and $x : 2 :: 3 : y$, to find x and y. *Ans.* $x = \pm 3$, $y = \pm 2$.

10. Given $x + y : x - y :: 3 : 1$, and $x^3 - y^3 = 56$, to find x and y. *Ans.* $x = 4$, $y = 2$.

VARIATION.

516. Variation is an abridged form of proportion. It is employed when two quantities are so related that if the value of one be changed, the value of the other is necessarily changed.

Thus, if A work for \$4 a day, his total wages will depend upon the number of days he works. If he work twenty days, he will receive \$80; if he work twice as many days, he will receive twice as many dollars; etc. That is, as *one* day is to n days, so is A's *daily* salary to A's *total* salary for n days. Hence we say that if A's salary for one day be fixed, *his total salary varies as the number of days he works.*

517. Variable Quantities are such as admit of various values in the same computation. In this chapter variables are represented by capital letters.

518. Constant or **Invariable Quantities** are such as have only one fixed value. In this chapter constants are represented by small letters.

519. Two variable quantities are said to *vary directly* when their *ratio* is constant. Thus, the length of a shadow varies directly as the height of the object which casts it. If $\frac{A}{B} = c$, a constant quantity, we say that $A \propto B$, the symbol $\propto$ being used instead of *varies* or *varies as*.

Since $\frac{A}{B} = c, \therefore A = cB$.

520. Two variable quantities are said to *vary inversely* when their *product* is constant. Thus, the distance being fixed, the rate and time vary inversely; for, as the one is increased, the other is diminished in the same ratio.

If $AB = c$, then $A = \frac{c}{B} = c \times \frac{1}{B}$, and $A \propto \frac{1}{B}$.

521. If $A = cBD$, where c is constant, then A is said to *vary as B and D jointly*, written thus: $A \propto BD$.

For example, A's total wages will vary jointly as the number of days he works, and his wages per day.

522. If $A = \frac{cB}{D}$, where c is constant, then A is said to vary directly as B, and inversely as D, written thus: $A \propto \frac{B}{D}$.

For example, the time occupied in a journey varies directly as the distance, and inversely as the rate of travel.

523. Besides the four kinds of variation heretofore mentioned, many other forms are to be noted; thus, A may vary as the square or the cube of B; inversely as the square or the cube; directly as the square and inversely as the cube; etc. For instance, the intensity of the light shed by any luminous body upon an object varies directly as the mass of the luminous body, and inversely as the square of its distance from the object. Thus, $A \propto \frac{M}{D^2}$.

524. If $A \propto B$ and $B \propto C$, then $A \propto C$.

For, let $A = mB$ and $B = nC$, where m and n are constants; then $A = mnC$, and, as mn is constant, $A \propto C$.

525. If $A \propto C$ and $B \propto C$, then $A \pm B \propto C$, and $\sqrt{AB} \propto C$.

For, let $A = mC$, and $B = nC$; then $A \pm B = (m \pm n)C$; hence, since $m \pm n$ is constant, $A \pm B \propto C$.

Also, $AB = mnC^2$; $\therefore\ \sqrt{AB} = C\sqrt{mn}$; $\therefore\ \sqrt{AB} \propto C$.

526. If $A \propto BC$, then $B \propto \frac{A}{C}$ and $C \propto \frac{A}{B}$.

For, let $A = mBC$, where m is constant; therefore $B = \frac{A}{mC} = \frac{1}{m} \cdot \frac{A}{C}$; $\therefore\ B \propto \frac{A}{C}$. Similarly, $C \propto \frac{A}{B}$.

527. If $A \propto B$ and $C \propto D$, then $AC \propto BD$.

For, let $A = mB$ and $C = nD$, where m and n are constants; then $AC = mnBD$; $\therefore\ AC \propto BD$.

528. If $A \propto B$, then $A^n \propto B^n$.

For, let $A = mB$; then $A^n = m^nB^n$; $\therefore\ A^n \propto B^n$.

529. If $A \propto B$, then $AP \propto BP$, where P is any quantity, variable or invariable.

For, let $A = mB$; then $AP = mBP$; $\therefore\ AP \propto BP$.

530. If, when C is constant $A \propto B$, and when B is constant $A \propto C$; then $A \propto BC$.

For, since $A \propto B$ when C is constant, A must be of the form mB, where m is some constant which may, therefore, contain the constant C, but not B. Hence A must contain B as a factor, but not B^2, B^3, B^4, etc., and may contain C.

Since $A \propto C$ when B is constant, A must also be of the form nC, where n is some constant which may, therefore, contain the constant B, but not C. Hence A must contain C as a factor, but not C^2, C^3, etc., and may contain B.

Therefore A contains B and C as factors, but no other powers of B or C; hence A must be of the form pBC, where p is constant, containing neither B nor C; $\therefore A \propto BC$.

Similarly it may be shown that *if any quantity vary separately as each of several others when the rest are constant, it varies as their product when they may all vary together.*

EXAMPLES.

1. If $A \propto B$, and when $A = 4$, $B = 8$, what is the equation between A and B? *Ans.* $B = 2A$.

2. If $A \propto B$, and when $A = 4$, $B = 1$, what is the value of A when $B = 5$? *Ans.* $A = 20$.

3. If A varies inversely as B, and when $A = \frac{1}{2}$, $B = 8$, find the equation between A and B. *Ans.* $AB = 4$.

4. If $A^2 \propto B^3$, and $A = 2$ when $B = 3$, find the equation between A and B. *Ans.* $27A^2 = 4B^3$.

5. If $A \propto B$, and $A = \frac{3}{4}$ when $B = \frac{4}{3}$, what will be the value of A when $B = 12$? *Ans.* $A = 6\frac{3}{4}$.

6. If $B^2 \propto \dfrac{A}{C}$, and when $A = 1$, $B = 2$, and $C = 3$, express A in terms of B and C. *Ans.* $A = \frac{1}{12}B^2C$.

7. If $A = B + C$, and if $B \propto D$ and $C \propto D^2$, also when $D = 1$, $A = 6$, and when $D = 2$, $A = 20$, express A in terms of D. *Ans.* $A = 2D + 4D^2$.

8. If $AB \propto A^2 + B^2$, and when $A = 3$, $B = 4$, express AB in terms of $A^2 + B^2$. *Ans.* $AB = \frac{12}{25}(A^2 + B^2)$.

Exercise XXVII.

1. Compare $4 : 5$ and $11 : 13$, $13 : 14$ and $25 : 27$.

2. Compare the ratios $a - b : a + b$ and $a^2 - b^2 : a^2 + b^2$.

3. Compound $x^2 - 9x + 20 : x^2 - 6x$,
and $x^2 - 13x + 42 : x^2 - 5x$.

4. Compound the duplicate ratio of $3 : 2$, the subduplicate ratio of $27 : 48$, and the triplicate ratio of $\frac{2}{3} : \frac{3}{4}$.

5 What number must be added to each term of the ratio $8 : 11$ that it may become equal to $4 : 5$?

6. Find two numbers in the ratio of $3 : 2$, such that if 9 be added to each the sums will be in the ratio of $4 : 3$.

7. Find a fourth proportional to 6, 10, 12; to 12, 5, 10.

8. $\frac{3}{4} : \frac{5}{6} :: \frac{2}{7} : x$. Find x.

9. Find a third proportional to 8, 12; to 6, 9.

10. If $a : b :: b : c$, show that $a^2+b^2 : a+c :: a^2-b^2 : a-c$.

11. If $a : b :: c : d$, and $m : n :: p : q$, show that $ma + nb : ma - nb :: pc + qd : pc - qd$.

12. If $a : b :: b : c$, show that $a^2 - b^2 : a :: b^2 - c^2 : c$.

13. If $a : b :: b : c$, show that $a + b + c : a - b + c :: (a + b + c)^2 : a^2 + b^2 + c^2$.

14. What number added to each of the numbers a, b, c, d, will make the results proportionals?

15. If $a : b :: c : d$, show that $a^2+b^2 : \frac{a^3}{a+b} :: c^2+d^2 : \frac{c^3}{c+d}$.

16. If four numbers are proportionals, and if the sum of the extremes be not equal to the sum of the means, show that there is no number which being added to each will leave the resulting four numbers proportionals.

17. If $a : b = c : d$, $e : f = g : h$, show that $\frac{a}{e} : \frac{b}{f} :: \frac{c}{g} : \frac{d}{h}$.

18. If $a : b :: c : d$, $a^3 + b^3 : a^3 - b^3 :: c^3 + d^3 : c^3 - d^3$.

19. If $(a + b + c + d)(a - b - c + d) = (a - b + c - d)(a + b - c - d)$, prove that a, b, c, and d are proportionals.

20. If four numbers are proportionals, show that the sum of the greatest and the least of them is greater than the sum of the other two.

21. If $a : b = c : d$, show that $7a + b : 3a + 5b = 7c + d : 3c + 5d$.

22. If $a : b = c : d$, show that $a - c - (b - d) = \frac{(a - b)(a - c)}{a}$.

23. If $A \propto B$, and $B = 21$ when $A = 3$, find the equation between A and B.

24. If $AB \propto A^2 + B^2$, and $A = 3$ when $B = 6$, find the equation between AB and $A^2 + B^2$.

25. If $C^2 \propto D^2 - a^2$, and $C = \frac{b^2}{a}$ when $D = \sqrt{a^2 + b^2}$, find the equation between C and D.

26. If $D = a + B$, in which $B \propto \frac{1}{C}$, and when $C = 2$, $D = 0$, also when $C = 3$, $D = 1$, find the value of D when $C = 6$.

27. If $C = a + B$, in which $B \propto CD$, and when $C = 2$, $D = -2\frac{2}{3}$; when $C = -2$, $D = 1$; express D in terms of C.

28. If $AB^2 \propto C$, and $AC^2 \propto B$, show that $A \propto \frac{1}{B} \propto \frac{1}{C}$.

29. If $D = A + B$, in which $A \propto C$, and $B \propto \frac{1}{C}$, and if when $D = 4$, $C = 1$; when $D = 5$, $C = 2$; find the equation between C and D.

30. If $S \propto t^2$ when f is constant, and $S \propto f$ when t is constant, and when $t = 1$, $f = 2S$, find the equation between S, f, and t.

CHAPTER XXVIII.

SERIES.

531. A **Series** is a succession of numbers, called the *terms,* each of which is derived from one or more of the preceding ones by a fixed law, called the *law of the series.*

Thus, 1, 3, 5, 7, 9, etc., is a series, and each term is found by adding 2 to the term immediately before it.

532. An **Infinite Series** is one that consists of an *unlimited* number of terms. A **Finite Series** contains a *limited* number of terms.

533. A **Converging Series** is an infinite series in which the sum of the first n terms can not numerically exceed some finite number, however great n may be.

Thus, 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, $\frac{1}{32}$, etc., is a converging series, because the sum of any number of its terms is less than 2.

534. The **Sum** of a converging series is the *limit* to which we approach by adding together more terms, but which can not be exceeded by adding together any number of terms whatever. By taking n large enough, the sum of the first n terms of the series can be made to differ from the limit by less than any assigned magnitude.

535. A **Diverging Series** is an infinite series in which, by taking n large enough, the sum of the first n terms can be made larger than any finite number.

Thus, 1, 2, 3, 4, 5, 6, etc., is a diverging series.

536. If we divide 1 by $1 - x$, the quotient is the infinite series $1 + x + x^2 + x^3 + x^4 + \ldots.$ If x be any positive number less than unity, this series is convergent. Thus, if $x = \frac{1}{2}$, then $1 \div (1 - x) = 1 \div \frac{1}{2} = 2$, and the series is $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$, whose sum can be made to differ from 2 by as little as we please, by increasing indefinitely the number of terms taken.

By taking x small enough, we can make $1 \div (1 - x)$ approach as near unity as we please. Thus, if $x = .01$, $1 \div (1 - x) = \dfrac{100}{99}$; if $x = .001$, $1 \div (1 - x) = \dfrac{1000}{999}$; etc.

By taking x nearly equal to 1, the sum of the series can be made very great. Thus, if $x = .99$, $1 \div (1 - x) = 100$; if $x = .999$, $1 \div (1 - x) = 1000$; etc.

If $x = 1$, $\dfrac{1}{1 - x} = 1 + x + x^2 + x^3 + \ldots$ becomes $\dfrac{1}{1 - 1} = 1 + 1 + 1 + 1 + \ldots$ That is, $\frac{1}{0}$ = an unlimited number, expressed thus: $\frac{1}{0} = \infty$.

The only meaning to be attached to this expression is, that, *by taking a denominator small enough, we can make the value of a fraction as large as we please, without limit.*

If $x > 1$, the series is divergent. Thus, if $x = 4$, $\dfrac{1}{1 - x} = -\dfrac{1}{3}$; but $1 + x + x^2 + x^3 + x^4 + \ldots = 1 + 4 + 16 + 64 + 256 + \ldots$, and the more terms of the series we take, the more does their sum diverge from $-\frac{1}{3}$.

537. There are certain fractions which, on a certain supposition, assume the form $\frac{0}{0}$. Such fractions are called *vanishing fractions*. Thus, if $x = 1$, $\dfrac{1 - x^3}{1 - x^2} = \dfrac{0}{0}$.

The symbol $\frac{0}{0}$ denotes *indetermination,* and may be interpreted in one of two ways:

(1) The expression which equals $\frac{0}{0}$ is really indeterminate, and may be satisfied by any value.

(2) The fraction may have a factor common to both terms, and, by the supposition made, this factor becomes 0, whence each term becomes 0. Thus, $\frac{1-x^3}{1-x^2}$ becomes $\frac{0}{0}$ when $x=1$. If $x=1$, then $x-1=0$. Now see if $x-1$ or $1-x$ is a factor of each term. If so, divide each term by the common factor, and then the real value of the fraction is readily found. Thus:

$$\frac{1-x^3}{1-x^2}=\frac{1+x+x^2}{1+x}=\frac{3}{2}\ (\text{if } x=1).$$

538. The number of different series is unlimited, but three kinds are of primary importance, called *Arithmetical, Geometrical,* and *Harmonical Progressions.*

ARITHMETICAL PROGRESSION.

539. An **Arithmetical Progression** is a series each term of which is derived from the preceding, by adding a constant number. Thus, 1, 3, 5, 7, 9, . . . is an arithmetical progression.

540. Let a represent the first term, and d the common difference; then the terms of the arithmetical progression are $a, a+d, a+2d, a+3d, a+4d, \ldots$, in which the series will be *increasing* or *decreasing* according as d is *positive* or *negative.*

Since the co-efficient of d in each term is less by unity than the number of the term, $\therefore$ the nth term is $a+(n-1)d$.

If l represent the nth term, then $l=a+(n-1)d$. (I)

541. Let s represent the sum of the series.

Then $s=a+(a+d)+(a+2d)+\ldots+l$.

By writing the terms in reverse order, we have also

$$s=l+(l-d)+(l-2d)+\ldots+a.$$

Therefore, by addition,

$$2s=(a+l)+(a+l)+(a+l)+\ldots \text{ to } n \text{ terms.}$$

$$\therefore 2s=n(a+l), \text{ whence } s=\frac{n}{2}(a+l). \qquad \text{(II)}$$

542. *In an arithmetical progression, the sum of any two terms equidistant from the beginning and the end, is equal to the sum of the first and last terms.*

For, the rth term from the beginning is $a + (r - 1)d$,
and the rth term from the end is $l - (r - 1)d$,
and the sum of these two terms is $(a + l)$.

543. *If the number of terms in an arithmetical progression be odd, the middle term is equal to half the sum of the extreme terms.*

Every odd number is of the form $2r + 1$; therefore, let $n = 2r + 1$. Thus the middle term, m, has r terms before it and r terms after it; hence m is the $(r + 1)$ term, counting from each end.

Therefore, $m = a + rd$,
and $m = l - rd$, (Art. 540, I);
whence, by addition, $2m = a + l$;
$\therefore m = \frac{1}{2}(a + l)$.

544. The **Arithmetical Mean** between two numbers is the number which, when placed between them, will form with the given numbers an Arithmetical Progression. Thus, 7 is the arithmetical mean between 5 and 9, and $a + d$ is the arithmetical mean between a and $a + 2d$.

From Art. 543 it follows that *the arithmetical mean between two numbers is half their sum.*

545. *To insert a given number of arithmetical means between two numbers.*

Let a and l be the two given numbers, and let m represent the number of terms to be inserted. Then the meaning of the problem is, that we are to find $m + 2$ terms in arithmetical progression, a being the first term and l the last. Since $l = a + (m + 1)d$, (540, I,) $\therefore d = \frac{l - a}{m + 1}$. This finds d, and the m required means are $a + d$, $a + 2d$, $a + 3d, \ldots, a + md$.

546. From the two equations,

$$\text{(I) } l = a + (n-1)d; \quad \text{(II) } s = \frac{n}{2}(a + l);$$

any two of the quantities a, d, l, n, s may be found when the other three are given. For example,—

To find l when a, d, s, are given.

Since n is neither given nor required, eliminate n.

From I, $n = \frac{l-a}{d} + 1$. From II, $n = \frac{2s}{a+l}$.

$$\therefore \frac{l-a}{d} + 1 = \frac{2s}{a+l}, \text{ in which } l \text{ is required.}$$

Clear of fractions. $l^2 - a^2 + ad + dl = 2ds$.

Transpose and add $\frac{d^2}{4}$. $l^2 + dl + \frac{d^2}{4} = 2ds + a^2 - ad + \frac{d^2}{4}$.

Extract square root. $l + \frac{1}{2}d = \pm\sqrt{2ds + (a - \frac{1}{2}d)^2}$.

$$\therefore l = -\tfrac{1}{2}d \pm \sqrt{2ds + (a - \tfrac{1}{2}d)^2}.$$

EXAMPLES.

1. Given $a = 8$, $d = 2$, $s = 44$, find n. *Ans.* $n = 4$.

2. Sum 2, 5, 8, 11, . . . to 20 terms. *Ans.* 610.

3. Find d when $a = 2$, $l = 50$, $s = 520$. *Ans.* $d = 2\frac{10}{19}$.

4. Find the sum of the first n odd numbers. *Ans.* n^2.

5. The first term of an arithmetical progression being 2, and the fifth term being 10, how many terms must be taken that the sum may be 90? *Ans.* 9.

6. If $s = 36$, $a = 12$, $d = -2$, find n. *Ans.* $n = 9$ or 4.

7. Insert six arithmetical means between -1 and 20.
Ans. 2, 5, 8, 11, 14, 17.

8. The sum of the first two terms of an A. P. is 4, and the fifth term is 9; find the series. *Ans.* 1, 3, 5, 7, 9, etc.

9. In the series 1, 3, 5, 7, . . . the sum of r terms : the sum of $2r$ terms :: 1 : x; determine the value of x. *Ans.* 4.

10. Divide 3 into 5 parts whose common difference is $\frac{1}{5}$.
Ans. $\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1$.

GEOMETRICAL PROGRESSION.

547. A **Geometrical Progression** is a series of terms each of which is derived from the preceding, by multiplying it by a constant factor, called the *ratio*.

Thus, 1, 2, 4, 8, 16, . . . is an *increasing* geometrical progression whose ratio is 2; also, 18, 6, 2, $\frac{2}{3}$, . . . is a *decreasing* geometrical progression whose ratio is $\frac{1}{3}$.

If a be the first term and r the ratio, then $a, ar, ar^2, ar^3, \ldots$ is a geometrical progression which is an *increasing* series when $r > 1$, but a *decreasing* series when $r < 1$.

548. Since the exponent of r is less by unity than the number of the term, therefore the nth term $= ar^{n-1}$.

If l denote the nth term, then $\qquad l = ar^{n-1}$. (I)

If s denote the sum of the series,

$$\therefore \; s = a + ar + ar^2 + ar^3 + \ldots + ar^{n-1};$$

$$\therefore \; sr = ar + ar^2 + ar^3 \quad + \ldots + ar^{n-1} + ar^n.$$

Hence, by subtraction, $sr - s = ar^n - a$.

Whence $$s = \frac{a(r^n - 1)}{r - 1}. \qquad \text{(II)}$$

Since $ar^n = rl$, $\therefore \; s = \dfrac{rl - a}{r - 1}$. (III)

When $r < 1$, formulas II and III will be more convenient if written: II. $s = \dfrac{a(1 - r^n)}{1 - r}$. III. $s = \dfrac{a - rl}{1 - r}$.

549. The **Geometrical Mean** between two numbers is that number which when placed between them will form with the given numbers a geometrical progression.

Let the two numbers be a and b, and let G represent their geometrical mean; then, a, G, and b being in geometrical progression, $\dfrac{G}{a} = \dfrac{b}{G}$. $\therefore \; G^2 = ab$, whence $G = \sqrt{ab}$.

That is: *The geometrical mean between two numbers is equal to the square root of their product.*

550. *To insert a given number of geometrical means between two given terms.*

Let a and l be the two given terms, r the ratio, and m the number of terms to be inserted.

Then the meaning of the problem is, to find $m + 2$ terms in geometrical progression, a being the first term and l the last. From (548, I), $l = ar^{m+1}$; $\therefore\ r = \left(\frac{l}{a}\right)^{\frac{1}{m+1}}$. This finds r, and the required terms are $ar, ar^2, ar^3, \ldots, ar^m$.

551. If $r < 1$, then the larger n is the smaller will r^n be; and by taking n large enough r^n can be made as small as we please. If n be taken so large that r^n may be neglected in comparison with unity, the value of $s = \frac{a(1 - r^n)}{1 - r}$ reduces to $s = \frac{a(1 - 0)}{1 - r} = \frac{a}{1 - r}$. Hence:

By taking n large enough, the sum of n terms of a decreasing geometrical series can be made to differ as little as we please from $a \div (1 - r)$.

This statement is sometimes abbreviated into the following: *The sum of an infinite number of terms of a decreasing geometrical progression is $a \div (1 - r)$.*

552. In (548) we have five quantities occurring, namely, a, r, l, n, s; and these are connected by the equations I and II, or II and III, there given. We might, therefore, propose to find any two of these quantities when the other three are given; it will, however, be found that some of the cases of this problem are too difficult to be solved. The following four cases present no difficulty:

(1) Given a, r, n. (3) Given r, n, l.
(2) Given a, n, l. (4) Given r, n, s.

In the four cases in which n is required, the unknown quantity is an exponent, and may be found by the use of logarithms.

EXAMPLES.

1. Sum to n terms $3 + 2 + \frac{4}{3} + \ldots$ *Ans.* $9\{1 - (\frac{2}{3})^n\}$.

2. Sum to infinity $3 + 2 + \frac{4}{3} + \ldots$ *Ans.* 9.

3. Sum to infinity $\frac{3}{2} - \frac{2}{3} + \frac{8}{27} - \ldots$ *Ans.* $\frac{27}{26}$.

4. Given a, r, s, to find l. *Ans.* $l = \frac{a + (r - 1)s}{r}$.

5. Given a, r, l, to find n. *Ans.* $n = \frac{\log l - \log a}{\log r} + 1$.

6. Find two numbers whose sum is 13, and geometrical mean 6. *Ans.* 9 and 4.

7. Find three numbers in geometrical progression whose sum is 26, and the sum of their squares 364. *Ans.* 2, 6, 18.

8. The sum of two numbers is 10, and their geometrical mean is $\frac{3}{8}$ of their difference; find the numbers.
Ans. 9 and 1.

9. Find three numbers in geometrical progression whose sum is 21, and the sum of the first and second : the sum of the second and third :: 1 : 4. *Ans.* 1, 4, 16.

10. The sum of three numbers in G. P. is 7, and the sum of their reciprocals is $\frac{7}{4}$; find the numbers. *Ans.* 1, 2, 4.

11. There are four numbers in G. P., the sum of the extremes being 18 and of the means 12; find them.
Ans. 2, 4, 8, 16.

12. There are four numbers in A. P., and if 2 be added to the third and 8 to the fourth, the first, second, and the two sums will be in geometrical progression; find the numbers. *Ans.* 2, 4, 6, 8.

HARMONICAL PROGRESSION.

553. A **Harmonical Progression** is a series of numbers whose reciprocals are in arithmetical progression. Hence the general representative of such a series will be

$$\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}, \frac{1}{a+3d}, \ldots, \frac{1}{a+(n-1)d}.$$

554. Questions relating to harmonical progression are readily solved by writing the reciprocals of the terms so as to form an arithmetical progression.

555. If a and b denote two numbers, and H their harmonic mean, then $\frac{1}{H} - \frac{1}{a} = \frac{1}{b} - \frac{1}{H}$;

$$\therefore \frac{2}{H} = \frac{1}{a} + \frac{1}{b} = \frac{a+b}{ab};$$

$$\therefore H = \frac{2ab}{a+b}.$$

556. The following solution illustrates the method of inserting two or more harmonical means between two numbers:

EXAMPLE. To insert five harmonical means between 3 and 15.

Find the five arithmetical means between $\frac{1}{3}$ and $\frac{1}{15}$.

$$d = \frac{l-a}{m+1} = \frac{\frac{1}{15} - \frac{1}{3}}{6} = -\frac{2}{45};$$

Hence the arithmetical means are $\frac{13}{45}, \frac{11}{45}, \frac{9}{45}, \frac{7}{45}, \frac{5}{45}$; therefore the harmonical means are $\frac{45}{13}, \frac{45}{11}, 5, \frac{45}{7}, 9$.

557. If A, B, and C are in harmonical progression, then $A : C :: A - B : B - C$.

For, $\frac{1}{A} - \frac{1}{B} = \frac{1}{B} - \frac{1}{C}$; (553)

$$\therefore C(B-A) = A(C-B), \text{ or } C(A-B) = A(B-C);$$

$$\therefore A : C :: A - B : B - C.$$

558. Let a and b be any two unequal numbers; let A be their arithmetical mean, G their geometrical mean, H their harmonical mean. Then : I. $A = \frac{1}{2}(a+b)$;

II. $G = \sqrt{ab}$, $\therefore G^2 = ab$; III. $H = \frac{2ab}{a+b}$;

IV. = I × III. $\therefore A \times H = ab$; but II. $G^2 = ab$;

$$\therefore A \times H = G^2, \therefore A : G :: G : H.$$

Hence: *The geometrical mean is a mean proportional between the arithmetical and harmonical means.*

Therefore the absolute value (72) of G is between A and H. Now A is numerically greater than H.

For, $(a-b)^2>0$; $\therefore a^2+2ab+b^2>4ab$; $\therefore \frac{a+b}{2}>\frac{2ab}{a+b}$;

$$\therefore A>H.$$

559. The three numbers, a, b, and c are in arithmetical, geometrical, or harmonical progression according as $\frac{a-b}{b-c}=\frac{a}{a}$ or $=\frac{a}{b}$ or $=\frac{a}{c}$, respectively.

For, (1) If $\frac{a-b}{b-c}=\frac{a}{a}=1$, $\therefore a-b=b-c$.

(2) If $\frac{a-b}{b-c}=\frac{a}{b}$, then $ab-b^2=ab-ac$, $\therefore b^2=ac$.

(3) If $\frac{a-b}{b-c}=\frac{a}{c}$, $\therefore \frac{1}{b}-\frac{1}{a}=\frac{1}{c}-\frac{1}{b}$.

EXAMPLES.

1. Insert five harmonical means between $\frac{1}{3}$ and $\frac{1}{15}$.

Ans. $\frac{1}{5}$, $\frac{1}{7}$, $\frac{1}{9}$, $\frac{1}{11}$, $\frac{1}{13}$.

2. Find two numbers whose arithmetical mean is 3, and whose harmonical mean is $\frac{8}{3}$. *Ans.* 2 and 4.

3. Given the first two terms, a and b, of a harmonical progression, to find the nth term.

Ans. $l=\frac{ab}{(n-1)a-(n-2)b}$.

4. The first term of a harmonical progression is $\frac{1}{2}$, and the sixth term is $\frac{1}{12}$; find the intermediate terms.

Ans. $\frac{1}{4}$, $\frac{1}{6}$, $\frac{1}{8}$, $\frac{1}{10}$.

5. What number added to each of the numbers a, b, c, will give results in harmonical progression?

Ans. $\frac{ab-2ac+bc}{a-2b+c}$.

Exercise XXVIII.

When $x = a$, find the value of :

1. $\dfrac{x^4 - a^4}{x - a}$. 2. $\dfrac{x^4 - a^4}{x^3 - a^3}$. 3. $\dfrac{x^n - a^n}{x - a}$.

4. Find the sum of n terms of the series 1, 2, 3, 4, 5, . . .

5. The sum of five numbers in A. P. is 35, and the sum of their squares is 335; find the numbers.

Note. Let the numbers be represented by $x - 2y, x - y, x, x + y, x + 2y$. In general, if the number of terms in an A. P. be odd, it is convenient to represent the middle term by x. If the number of terms be even, it is convenient to represent the two middle terms by $x - y$ and $x + y$.

6. Find four numbers in A. P. such that the sum of the squares of the extremes is 50, and of the means is 34.

7. Find four numbers in A. P. such that the product of the extremes is 22, and of the means 40.

8. The sum of the fourth powers of three consecutive natural numbers is 962; find them.

9. The sum of a certain number of terms of the series $21 + 19 + 17 + \ldots$ is 120; find the number of terms.

10. Sum to n terms the series 5, 9, 13, . . .

11. Sum to n terms the series whose rth term is $2r - 1$.

12. Sum to n terms the A. P. whose first term is $n^2 - n + 1$, and the common difference 2.

13. Find the sum of five terms of a geometrical progression whose second term is 3 and fourth term 27.

14. The sum of four terms in geometrical progression is 700, and the difference of the extremes is $\frac{37}{12}$ of the difference of the means; find the numbers.

15. In a G. P. show that the product of any two terms equidistant from a given term is constant.

16. In a G. P. show that if each term be subtracted from the succeeding term, the successive differences are in G. P.

17. The sum of the first three terms of a G. P. is 28, and of the first four terms is 60; find the seventh term.

18. Three numbers whose sum is 21 are in A. P.; if 2, 3, and 9 be added to them respectively, the sums are in G. P.; find the numbers.

19. The fifth term of a G. P. is 8 times the second, and the third term is 12; find the series.

20. The sum of an infinite G. P. is 3, and the sum of the first two terms is $2\frac{2}{3}$; find the series.

21. Show that b^2 is greater than, equal to, or less than ac, according as a, b, c are in A. P., G. P., or H. P.

22. If x is the harmonic mean between a and b, show that $\dfrac{1}{x-a}+\dfrac{1}{x-b}=\dfrac{1}{a}+\dfrac{1}{b}$.

23. If a, b, c are in H. P., show that $\dfrac{a}{b+c}$, $\dfrac{b}{a+c}$, $\dfrac{c}{a+b}$, are in H. P.

24. Find two numbers whose difference is 8, and whose harmonic mean is $1\frac{4}{5}$.

25. If a, b, c are in arithmetical progression, and b, c, d in harmonical progression, prove that $a : b :: c : d$.

26. A sets out from a certain place and travels one mile the first day, two miles the second day, three miles the third day, and so on. B sets out five days after A, and travels the same road at the rate of 12 miles a day. How far will A travel before he is overtaken by B?

27. From 256 gallons of wine a certain number are drawn and replaced with water; this is done a second, a third, and a fourth time, and 81 gallons of wine are then left. How many gallons are drawn out each time?

CHAPTER XXIX.

PERMUTATIONS, COMBINATIONS, AND CHANCE.

560. THE **Permutations** of quantities are the different orders in which they can be arranged.

Quantities may be arranged singly, in pairs, in groups of three, and so on. Thus the permutations of the letters a, b, c, taken two at a time, are ab, ba, ac, ca, bc, cb.

Permutations are also called *variations*, or *arrangements*.

561. *The number of permutations of n letters taken r at a time is* $n(n-1)(n-2)(n-3)\ldots(n-\{r-1\})$.

Let $a, b, c, d, \ldots,$ be the n letters; and let P_1, P_2, P_3, P_4, $\ldots$, P_r, respectively denote the whole number of permutations where the letters are taken *singly*, in *pairs*, *three* together, *four* together, $\ldots$, r together.

The number of permutations of n things taken singly is evidently equal to the number of things; $\therefore P_1 = n$.

The number of permutations of n letters, taken in pairs, is $n(n-1)$. For, from the n letters, $a, b, c, \ldots,$ if we remove a, there will remain $(n-1)$ letters. Writing a before each of these we shall have $ab, ac, ad, \ldots$; that is, there will be $(n-1)$ permutations in which a stands first.

In like manner, if a be replaced and b be removed, it is readily shown that there are $(n-1)$ permutations in which b stands first; similarly, $(n-1)$ permutations in which c stands first; and so on for the n letters. That is:

$P_2 = (n-1) + (n-1) + (n-1) + \ldots$ to n terms;
$\therefore P_2 = n(n-1)$.

The number of permutations of n letters, taken three at a time, is $n(n-1)(n-2)$. For, if we remove a, there will remain $(n-1)$ letters, $b, c, d, \ldots$ If these $(n-1)$ letters be taken two at a time, according to the preceding paragraph $(n-1)(n-2)$ permutations can be formed. Put a before each of these, and we have $(n-1)(n-2)$ permutations, each of three letters, in which a stands first. Similarly, there are $(n-1)(n-2)$ permutations, each of three letters, in which b stands first; similarly, there are $(n-1)(n-2)$ permutations, in which c stands first; and so on. $\therefore P_3 = (n-1)(n-2)+(n-1)(n-2)+(n-1)(n-2)+\ldots$ to n terms.

Whence, $\qquad P_3 = n(n-1)(n-2)$.

In like manner, $P_4 = n(n-1)(n-2)(n-3)$,

$$P_5 = n(n-1)(n-2)(n-3)(n-4),$$

and so on.

Hence we may conclude that

$$P_r = n(n-1)(n-2)(n-3)\ldots(n-r+1).$$

562. The general proof of the formula in (561) depends upon *mathematical induction* (216). We have shown in (561) that this formula holds if $r = 1, 2,$ or 3, and similarly, by taking successive values for r, we can show that it is true in any selected case. We shall now show that if this formula be true for any selected value of r, it will be true for the next higher value; that is,

if $\qquad P_s = n(n-1)(n-2)\ldots(n-[s-1])$,

then $P_{s+1} = n(n-1)(n-2)\ldots(n-[s-1])(n-s)$.

For, in the value of P_s, substitute $n-1$ for n, then out of the $(n-1)$ letters, $b, c, d, \ldots,$ we can form $(n-1)(n-2)(n-3)\ldots(n-1-[s-1])$ permutations, each of s letters. Put a before each of these, and we obtain as many permutations, each of $(s+1)$ letters, in which a stands first. Similarly, we have as many in which b stands first, as many in which c stands first, and so on for the n letters. P_{s+1} is therefore equal to the sum

of n terms, each equal to $(n-1)(n-2)\ldots(n-s)$.
$\therefore P_{s+1} = n(n-1)(n-2)\ldots(n-s)$.

Since this formula holds when the letters are taken three at a time, therefore it holds when they are taken four at a time, therefore it holds when they are taken five at a time, and so on; thus it holds for every value of r.

563. If $r = n$, then $P_n = n(n-1)(n-2)\ldots(n-n+1)$; that is, $P_n = n(n-1)(n-2)\ldots 1$.
Whence: *The number of permutations of n things, taken all together, is equal to the product of the natural numbers from 1 up to n.*

For the sake of brevity, $1.2.3\ldots(n-1)n$ is denoted by $\lfloor n$, which is read, *factorial n.* Thus, $1.2.3.4. = \lfloor 4$.

564. It is evident that $n(n-1)(n-2)\ldots(n-r+1)=$

$$\frac{[n(n-1)\ldots(n-r+1)][(n-r)(n-r-1)\ldots 1]}{(n-r)(n-r-1)\ldots 1} = \frac{\lfloor n}{\lfloor n-r};$$

$$\therefore P_r = \frac{\lfloor n}{\lfloor n-r}.$$

565. *The number of permutations of n letters, taken all together, of which p are a's, q are b's, r are c's, ..., is*

$$\lfloor n \div (\lfloor p \lfloor q \lfloor r \ldots)$$

For, suppose N to represent the entire number of permutations. If in any one of these permutations the p a's were changed into p new letters different from any of the rest, then these p letters could be arranged in $\lfloor p$ different ways; hence, without altering the situation of any of the remaining letters, we could from the single permutation produce $\lfloor p$ different permutations. The same would be true for each of the N permutations; hence, if the p a's were different letters, the whole number of permutations would be $N \times \lfloor p$, of which still q are b's, r are c's, etc. Similarly, if the q b's were also changed into different letters, the whole number of permutations would be $N \times \lfloor p \times \lfloor q$; and if their c's were also changed, the whole number would be $N \times \underline{|p} \times \underline{|q} \times \underline{|r}$; and so on until all the n letters are different. But when this is case we know that their whole number of permutations equals $\underline{|n}$. (563.)

$$\therefore N \times \underline{|p} \times \underline{|q} \times \underline{|r} \ldots, = \underline{|n};$$

$$\therefore N = \underline{|n} \div (\underline{|p}\,\underline{|q}\,\underline{|r} \ldots)$$

566. *The number of permutations of n different letters, when each may occur once, twice, thrice, . . . , r times, is n^r.*

Let the n letters be $a, b, c, \ldots$ First take them singly: this gives n permutations. Next take them two at a time: here a may stand before itself, or before any one of the remaining letters, as $aa, ab, ac, \ldots$; that is, there are n permutations of this kind in which a stands first. Similarly, there are n permutations of the form $bb, ba, bc, bd, \ldots$, and so on for the n letters: thus there are $n \times n = n^2$ permutations of the letters taken two together. Similarly, by putting successively $a, b, c, \ldots$ before each of the permutations of the letters taken two at a time, we obtain $n \times n^2 = n^3$ permutations of the letters taken three at a time, and so on for all values of r.

567. The sum of such permutations of n letters, taken $1, 2, 3, \ldots, r$ together, equals $n + n^2 + n^3 + \ldots n^r = n(1 + n + n^2 + \ldots n^{r-1}) = n\left(\frac{n^r - 1}{n - 1}\right)$.

EXAMPLES.

1. How many permutations can be formed with the letters a, b, c, d, e?

$P_1 = 5,\ P_2 = 20,\ P_3 = 60,\ P_4 = 120,\ P_5 = 120.$

Ans. 325.

2. How many permutations can be formed out of *h o m e*?

Ans. $4 + 12 + 24 + 24 = 64$.

3. In how many ways, taken all together, can the seven prismatic colors be arranged? *Ans.* $\lfloor 7 = 5040$.

4. How many numbers of two figures each can be expressed by the ten digits? *Ans.* 90.

5. How many words, each containing two consonants and one vowel, can be formed from the alphabet?

The 20 consonants can be arranged in $20^2 = 400$ ways.

Since a may be placed before, between, or after the consonants, therefore 3×400 words can be formed containing a. Similarly, 1200 words can be formed containing each of the other vowels. Therefore the required number of words is 7200.

If repetitions be not allowed, the number of words will be $7200 - 360 = 6840$.

6. Out of 10 consonants and 4 vowels, how many words can be formed, each containing 3 different consonants and 2 different vowels?

The 10 consonants can be arranged in $10 \times 9 \times 8$ ways.

The 4 vowels can be arranged in 4×3 ways.

Each pair of vowels can form 10 different words with each group of consonants.

Ans. $(10 \times 9 \times 8) \times (10) \times (4 \times 3) = 86,400$.

7. How many different words, each containing eight letters, can be formed from the letters of *mammalia?*

Ans. $\lfloor 8 \div (\lfloor 3 \lfloor 3) = 1120$.

8. The number of permutations of n things taken 3 together : the number of permutations of $(n + 2)$ things taken 3 together :: 5 : 12; find n. *Ans.* $n = 7$.

9. The number of permutations of n letters taken 3 together is $20n$; find n. *Ans.* $n = 6$.

10. How many different words can be formed out of the letters of the word *Mississippi*, taken all together?

Ans. $\lfloor 11 \div (\lfloor 4 \lfloor 4 \lfloor 2) = 34,650$.

11. How many words of six letters might be made out of the first ten letters of the alphabet, no letter being used more than once in any word, and each word containing two vowels? *Ans.* $(7 . 6 . 5 . 4) \times (15) \times (3 . 2) = 75,600$.

COMBINATIONS.

568. The **Combinations** of things are the different collections that can be formed out of them, without regarding the order in which the things are placed.

Thus, the combinations of the three letters *a, b, c,* taken two at a time, are *ab, ac, bc; ab* and *ba,* though different permutations, forming the same combination.

569. *The number of combinations of n things, taken r at a time, is* $\dfrac{n(n-1)(n-2)\ .\ .\ .\ (n-r+1)}{\lfloor r}$.

For, each combination of r letters produces $\lfloor r$ permutations (563). Hence, to find C_r (the number of combinations of n things taken r together), we divide the number of permutations (Art. 561) by factorial r.

$$\therefore C_r = \frac{n(n-1)(n-2)\ .\ .\ .\ (n-r+1)}{\lfloor r}.$$

570. Since $P_r = \dfrac{\lfloor n}{\lfloor n-r}$ (564), and since $C_r = \dfrac{P_r}{\lfloor r}$ (569),

$$\therefore C_r = \frac{\lfloor n}{\lfloor n-r \lfloor r}.$$

571. *The number of combinations of* **n** *things taken r at a time is equal to the number of combinations of n things taken* $(n-r)$ *at a time.*

For, from (570), $C_r = \dfrac{\lfloor n}{\lfloor r \lfloor n-r}$, $C_{n-r} = \dfrac{\lfloor n}{\lfloor n-r \lfloor r}$.

$\therefore C_r = C_{n-r}$.

Such combinations are called *complementary*.

572. *To find for what value of r the number of combinations of n things, taken r together, is the greatest.*

From (569), $C_r = \frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4} \cdots \frac{n-r+1}{r}$.

Similarly, $C_{r+1} = \frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdots \frac{n-r+1}{r} \cdot \frac{n-r}{r+1}$.

As r changes its value from 1 to n, each successive value of C_{r+1} is obtained from multiplying C_r by $\frac{n-r}{r+1}$. As r increases, the value of this multiplier decreases; but so long as the multiplier > 1, each product will be larger than the multiplicand, and $C_{r+1} > C_r$. For some value of r, however, the multiplier changes from being greater than 1, to being either equal to 1, in which case $C_{r+1} = C_r$, or less than 1, when $C_{r+1} < C_r$.

(1) When $\frac{n-r}{r+1}=1$, then $r=\frac{1}{2}(n-1)$ and $r+1=\frac{1}{2}(n+1)$.

Since r is integral, its value, $\frac{1}{2}(n-1)$, must be integral; $\therefore$ n must be an *odd* number. Hence: *When n is an odd integer, the greatest number of combinations is obtained when the n things are taken* $\frac{1}{2}(n-1)$ *at a time or* $\frac{1}{2}(n+1)$ *at a time, the results being the same in these two cases.*

Thus, if $n=5$, then $C_1=5$; $C_2=10$; $C_3=10$; $C_4=5$; $C_5=1$. If $n=7$, then $C_1=7$; $C_2=21$; $C_3=35$; $C_4=35$; $C_5=21$; etc.

(2) If $\frac{n-r}{r+1} < 1$, then $r > \frac{1}{2}(n-1)$. C_r will be the greatest when r is the *first* integer greater than $\frac{1}{2}(n-1)$. When n is even, this integer is $\frac{n}{2}$. Hence: *When n is even, the greatest number of combinations is obtained when the n things are taken* $\frac{n}{2}$ *at a time.* Thus:

If $n=6$, then $C_1=6$; $C_2=15$; $C_3=20$; $C_4=15$; $C_5=6$; $C_6=1$.
If $n=8$, then $C_1=8$; $C_2=28$; $C_3=56$; $C_4=70$; $C_5=56$; etc.

EXAMPLES.

1. How many combinations can be formed of nine things taken five at a time? three at a time? *Ans.* 126; 84.

2. Four persons are chosen by lot out of ten; in how many ways can this be done? *Ans.* 210.

3. Find the whole number of combinations of six things.
Ans. $6 + 15 + 20 + 15 + 6 + 1 = 63$.

4. The number of combinations of $\frac{1}{3}n$ things taken two at a time is 28; find n. *Ans.* $n = 24$.

5. The number of combinations of n things taken three at a time is $\frac{5}{18}$ of the number taken five at a time. Find n.
Ans. $n = 12$.

6. In how many ways can 12 men be divided into two classes of 8 and 4? *Ans.* $\lfloor 12 \div (\lfloor 8 \lfloor 4) = 495$.

7. How many words of six letters might be formed out of the first ten letters of the alphabet, if each word differ from every other in at least one letter? *Ans.* 210.

8. How many parties of 8 men each can be formed from a company of 40 men? *Ans.* $\lfloor 40 \div (\lfloor 8 \lfloor 32)$.

9. The number of combinations of $(n + 1)$ things, 4 together, is 9 times the number of combinations of n things, 2 together; find n. *Ans.* $n = 11$.

10. A person wishes to make up as many different dinner-parties as he can out of 20 friends. How many should he invite at a time? *Ans.* 10.

11. How many combinations can be formed from the letters in the word *notation*, taken three together?

Ten combinations can be formed, in each of which the three letters are different. Four combinations contain two n's, four contain two o's, and four contain two t's.
Ans. 22.

CHANCE.

573. If an event may happen in a ways, and fail in b ways, and all these ways are equally likely to occur, then the probability or chance of the event's happening is $\frac{a}{a+b}$, and the probability or chance of its failing is $\frac{b}{a+b}$.

This may be regarded as the definition of the word *chance* in mathematical works. This definition may be further illustrated as follows: If an event may happen in a ways and fail in b ways, it is evident that the chance of its happening is to the chance of its failing as a is to b; therefore, the chance of its happening is to the sum of the chances of its happening and failing as a is to $(a+b)$. But the event must either happen or fail; hence the sum of the chances of its happening and failing is *certainty*. Therefore, the chance of its happening is to certainty as a is to $(a+b)$. If, therefore, we represent certainty by *unity*, the probability of the happening of the event is represented by $\frac{a}{a+b}$, and of its failing by $\frac{b}{a+b}$.

574. Hence, if p be the chance of the happening of an event, $(1-p)$ is the chance of its failing.

575. When the probability of the happening of an event is to the probability of its failing as a is to b, the fact is expressed in ordinary language thus: the *odds* are a to b *for* the event, or b to a *against* the event. When $a=b$, then the *odds are even;* that is, the event is as likely to happen as not.

576. If there be any number of events A, B, C, etc., such that *one must happen* and *only one can happen;* and if a, b, c, etc., be the number of ways in which these events can respectively happen, and each way be equally likely to occur, then the chances of the events are proportional to a, b, c, etc., respectively. Consider, for example, three events: then A can happen in a ways out of $(a + b + c)$ ways, and fail in $(b + c)$ ways; therefore, by Art. 573, the chance of A's happening is $\frac{a}{a + b + c}$. Similarly, the chance of B's happening is $\frac{b}{a + b + c}$, and the chance of C's happening is $\frac{c}{a + b + c}$.

577. If four white balls, five black balls, and six red balls are thrown promiscuously into a bag, and a person draws out one of them, the chance that this will be a white ball is $\frac{4}{15}$, that it will be a black ball is $\frac{5}{15} = \frac{1}{3}$, and that it will be a red ball is $\frac{6}{15} = \frac{2}{5}$. The chance that the ball drawn will be either white or black is $\frac{9}{15} = \frac{3}{5}$, that it will be either white or red is $\frac{2}{3}$, and that it will be either black or red is $\frac{11}{15}$.

Now let us consider the chances of the different cases if two balls be drawn. The number of pairs that can be formed out of 15 things is $\frac{15 \times 14}{2} = 105$. The number of pairs that can be formed out of the four white balls is $\frac{4 \times 3}{2} = 6$; out of the five black balls is $\frac{5 \times 4}{2} = 10$; and out of the six red balls is $\frac{6 \times 5}{2} = 15$. Therefore, the chance that the two balls drawn will be both white is $\frac{6}{105} = \frac{2}{35}$; that they will be both black is $\frac{10}{105} = \frac{2}{21}$, and that they will be both red is $\frac{15}{105} = \frac{1}{7}$. Also, since each white ball might be associated with each black ball, the number of pairs consisting of one white ball and one black ball is $4 \times 5 = 20$; hence the chance of drawing such a pair is $\frac{20}{105} = \frac{4}{15}$. Similarly, the chance of drawing a white ball and a red one is $\frac{8}{35}$, and the chance of drawing a black ball and a red one is $\frac{2}{7}$. The sum of these six chances is, of course, equal to unity.

The chance that at least one of the two balls drawn will be white is $\frac{50}{105} = \frac{10}{21}$. For this may occur in any one of three ways: (1) Both may be white, the chance being $\frac{6}{105}$; (2) One may be white and one red, the chance being $\frac{24}{105}$; (3) One may be white and one black, the chance being $\frac{20}{105}$. Therefore, out of the 105 ways, there are $6 + 24 + 20 = 50$ ways in which a white ball may be drawn. The chance that neither ball is white is $(1 - \frac{10}{21}) = \frac{11}{21}$. In like manner, the chance that at least one of the two balls drawn is black is $\frac{60}{105} = \frac{4}{7}$, and that at least one is red is $\frac{69}{105} = \frac{23}{35}$.

If three balls be drawn, the chances of the various cases may be calculated similarly. The chance that the three balls will all be white is $\frac{4}{455}$; that all will be black is $\frac{10}{455}$; that all will be red is $\frac{15}{455}$; and so on for the other cases.

578. In a bag there are four white and six black balls. Find the chance that, out of five drawn, two only shall be white.

The number of combinations of ten things taken five at a time is 252. The four white balls may be taken two at a time in 6 ways, and the six black balls may be taken three at a time in 20 ways. Since each pair of white balls may be drawn with any three of the black balls, therefore, out of the 252 possible ways in which five balls may be drawn, there are $6 \times 20 = 120$ ways in which two are white and three are black. Hence, the required chance is $\frac{120}{252} = \frac{10}{21}$.

Suppose that we are required to find the chance that at least two shall be white. The chance that two will be white and three black is $\frac{120}{252}$; that three will be white and two black is $\frac{60}{252}$; and that four will be white and one black is $\frac{6}{252}$. The sum of these chances is the required chance, i.e., $\frac{186}{252} = \frac{31}{42}$. The chance of not drawing as many as two white balls is $1 - \frac{31}{42} = \frac{11}{42}$.

579. A's chance of winning a prize is $\frac{1}{4}$, and B's $\frac{1}{6}$; what is the chance that neither will obtain a prize?

The chance that one will obtain a prize is $(\frac{1}{4} + \frac{1}{6}) = \frac{5}{12}$; hence the chance that neither will win is $1 - \frac{5}{12} = \frac{7}{12}$.

580. A series of events such that only one of them can happen, may be called a series of *exclusive* or *dependent* events.

Two or more events such that both or all may happen, are called *non-exclusive* or *independent* events.

For example, if a copper be tossed twice, it may fall head up both times; if it be tossed six times, it is possible for it to fall head up six times.

581. If there are two or more independent events, the occurrence of all of them simultaneously or in succession may be regarded as a single *compound* event. Thus, in tossing a copper twice, the event of its falling head up at both trials may be regarded as an event *compounded of two simple events;* namely, with head up at the first trial, and with head up at the second trial.

The chance that the coin will fall head up at each trial is $\frac{1}{2}$. If these separate chances were added the sum would be 1, that is, *certainty;* a result obviously false. A little consideration will show that the chances should be multiplied instead of added. For, in the double fall there are four possibilities equally likely to occur:

1. Both times head up.
2. First time head up, second time head down.
3. First time head down, second time head up.
4. Both times head down.

Only one of these four ways gives "heads" both times; hence the chance of heads both times is $\frac{1}{4} = \frac{1}{2} \times \frac{1}{2}$; that is, *The chance that two independent events both happen is the product of their separate chances of happening.*

In general, suppose that there are two independent events, C and D, and let C happen in a ways and fail in b ways; also, let D happen in a' ways and fail in b' ways, all of these ways being equally likely to occur. Each of the $(a+b)$ ways may be associated with each of the $(a'+b')$ ways; thus there are $(a+b)(a'+b')$ compound cases which are equally likely to occur. In aa' of these compound cases *both events happen;* in ab' of them the *first happens* and the *second fails;* in $a'b$ of them the *first fails* and the *second happens;* and in bb' of them *both fail.* Thus:

$\frac{a}{a+b} \times \frac{a'}{a'+b'}$ is the chance that both events happen.

$\frac{a}{a+b} \times \frac{b'}{a'+b'}$ is the chance that C happens and D fails.

$\frac{b}{a+b} \times \frac{a'}{a'+b'}$ is the chance that C fails and D happens.

$\frac{b}{a+b} \times \frac{b'}{a'+b'}$ is the chance that both fail.

582. In like manner, *if there be any number of independent events, the chance that they will all happen is the product of their respective chances of happening, and the chance that all fail is the product of their respective chances of failing.* For example:

The chance that A can solve a certain problem is $\frac{1}{4}$; the chance that B can solve it is $\frac{1}{2}$; and the chance that C can solve it is $\frac{2}{3}$. What is the chance that the problem will be solved if they all try?

The problem will be solved unless they all fail.

A's chance of failing is $\frac{3}{4}$; B's is $\frac{1}{2}$; and C's is $\frac{1}{3}$.

The chance of all failing is $\frac{3}{4} \times \frac{1}{2} \times \frac{1}{3} = \frac{1}{8}$. Therefore, the chance that the problem will be solved is $1 - \frac{1}{8} = \frac{7}{8}$.

The chance that all will succeed is $\frac{1}{4} \times \frac{1}{2} \times \frac{2}{3} = \frac{1}{12}$.

583. Suppose that four white balls, five red balls, and six black balls be thrown promiscuously into a bag; required the chance that in two successive trials two red balls will be drawn, *the ball first drawn being replaced before the second trial.* Here the chance of drawing a red ball at the first trial is $\frac{1}{3}$, and the chance at the second trial is again $\frac{1}{3}$; hence the chance of drawing two red balls is $(\frac{1}{3})^2 = \frac{1}{9}$.

Let us now consider the case when the ball first drawn is not replaced before the second trial. Here the chance of drawing a red ball at the first trial is $\frac{1}{3}$; if a red ball be drawn at first, out of the 14 balls remaining 4 are red; hence the chance that a red ball will be drawn at the second trial is $\frac{2}{7}$; therefore, the chance of drawing two red balls is $\frac{1}{3} \times \frac{2}{7} = \frac{2}{21}$. This result corresponds with the one obtained in (577), when two balls are drawn simultaneously, and a little consideration will show that the two processes are really identical.

584. A bag contains five white and six black balls. If five balls be drawn in succession, and no one of them be replaced, what is the chance that the first three will be white and the last two black?

The chances for the successive trials are $\frac{5}{11}, \frac{4}{10}, \frac{3}{9}, \frac{6}{8}, \frac{5}{7}$. Hence the chance for the compound event is $\frac{5}{11} \times \frac{4}{10} \times \frac{3}{9} \times \frac{6}{8} \times \frac{5}{7} = \frac{5}{154}$.

If the three white and two black balls were drawn in *any order*, instead of being drawn in an *assigned order*, the chance would be $10 \times \frac{5}{154} = \frac{25}{77}$. For, the five things of which three are alike and the other two are alike, may be arranged in $\frac{\lfloor 5}{\lfloor 3 \lfloor 2} = 10$ ways (Art. 565). As in (583), this result corresponds with the one obtained when the five balls are drawn simultaneously.

585. One bag, A, contains three white balls and four black balls; another bag, B, contains two white balls and three black balls; required the probability of obtaining a white ball by a single drawing from one of the bags taken at random.

Since each bag is equally likely to be taken, the chance of taking A is $\frac{1}{2}$, and the chance then of drawing a white ball from it is $\frac{3}{7}$; hence the chance of obtaining a white ball so far as it depends upon A is $\frac{1}{2} \times \frac{3}{7} = \frac{3}{14}$. Similarly, the chance of obtaining a white ball so far as it depends on B is $\frac{1}{5}$. Therefore, the chance of obtaining a white ball is $(\frac{3}{14} + \frac{1}{5}) = \frac{29}{70}$.

586. If a person is to receive a sum of money m in case a particular event happen, and if p represent the chance that the event will happen, then pm is called his *expectation*.

Thus, suppose that there is a lottery with 20 tickets, and one prize worth $100. If a person own three tickets, his *chance* of drawing the prize is $\frac{3}{20}$, and his *expectation* is worth $\frac{3}{20}$ of $100 = $15.

587. A and B draw from a bag containing three white balls and three black balls. A is to draw a ball, then B, and so on alternately; and whichever draws a white ball first is to receive $60. Find A's expectation: (I) if the ball drawn be replaced; (II) if it be not replaced.

I. A's chance of drawing a white ball on the first trial is $\frac{1}{2}$. B's chance of having a trial is the same as A's chance of drawing a black ball, that is, $\frac{1}{2}$; and since the ball drawn by A is replaced, B's chance of drawing a white ball is $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$. If B draw a black ball, of which the chance is $\frac{1}{4}$, then A will have a second trial, and his chance of drawing a white ball on this trial is $\frac{1}{4} \times \frac{1}{2} = \frac{1}{8}$. B's chance of having a second trial is $\frac{1}{8}$, and his chance of drawing a white ball on the second trial is $\frac{1}{16}$, etc. Evidently A's chance of drawing a white ball is the sum of the infinite series $\frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \ldots = \frac{2}{3}$ (Art. 551). Hence A's expectation is $\frac{2}{3}$ of $60 = $40.

II. A's chance on the first trial is $\frac{1}{2}$. B's chance of having a trial is $\frac{1}{2}$; and if A drew a black ball, B's chance of drawing a white ball is $\frac{3}{5}$; therefore his chance of having a trial *and* drawing a white ball is $\frac{1}{2} \times \frac{3}{5} = \frac{3}{10}$; also, his chance of drawing a black ball is $\frac{2}{10}$. A's chance of having a second trial is the same as B's chance of drawing a black ball, i.e., $\frac{2}{10}$; and if two black balls have been drawn, A's chance of drawing a white ball is $\frac{3}{4}$; hence A's chance of having a second trial *and* drawing a white ball in that trial is $\frac{2}{10} \times \frac{3}{4} = \frac{3}{20}$; also, A's chance of drawing a black ball is $\frac{2}{10} \times \frac{1}{4} = \frac{1}{20}$. B's chance of having a second trial is equal to A's chance of drawing a black ball, i.e., $\frac{1}{20}$, and then there are only white balls left; hence B's chance of drawing a white ball, if he have a second trial, is certainty $= 1$, and his chance of having a second trial *and* drawing a white ball is $\frac{1}{20} \times 1 = \frac{1}{20}$. A's chance, therefore, is the sum of $\frac{1}{2}$ and $\frac{3}{20} = \frac{13}{20}$, and his expectation is $\frac{13}{20}$ of \$60 = \$39.

EXAMPLES.

1. In a bag are three white and five red balls: find the chance that, *one* being drawn, it shall be (I) white, (II) red; *two* being drawn, they shall be (III) both white, (IV) both red. *Ans.* I. $\frac{3}{8}$; II. $\frac{5}{8}$; III. $\frac{3}{28}$; IV. $\frac{5}{14}$.

2. In a bag are three white, four red, and five black balls: if *one* be drawn, find the chance of its being (I) white, (II) red, (III) black; if *two* be drawn, of their being (IV) white and red, (V) both black, (VI) one at least red; if *three* be drawn, of their being (VII) one of each color, (VIII) two of them black, (IX) one of them white.

Ans. I. $\frac{1}{4}$; II. $\frac{1}{3}$; III. $\frac{5}{12}$; IV. $\frac{2}{11}$; V. $\frac{5}{33}$; VI. $\frac{19}{33}$; VII. $\frac{3}{11}$; VIII. $\frac{7}{22}$; IX. $\frac{27}{55}$.

3. The odds against a certain event are 3 to 2, and the odds in favor of another event independent of the former are 4 to 3. What are the chances (I) that both happen; (II) that the first happens and the second fails; (III) that the first fails and the second happens; (IV) that neither happens? I. $\frac{8}{35}$; II. $\frac{6}{35}$; III. $\frac{12}{35}$; IV. $\frac{9}{35}$.

4. If from a lottery of 30 tickets marked 1, 2, 3, 4, . . . , four tickets be drawn, what is the chance that 3 and 4 will be among them? *Ans.* $\frac{2}{145}$.

5. A has three shares in a lottery where there are three prizes and six blanks; B has one share in another lottery where there is but one prize and two blanks. What are their comparative chances of drawing a prize?

A's chance of drawing three blanks is $\frac{5}{21}$; ∴ A's chance of drawing at least one prize is $\frac{16}{21}$. B's chance of drawing one prize is $\frac{1}{3} = \frac{7}{21}$. ∴ A's chance : B's :: 16 : 7.

6. When n coins are tossed up, what is the chance that one, and only one, will turn up head? *Ans.* $n \div 2^n$.

7. The skill of A is double that of B; what is the chance that A will win four games before B wins two?

A's chance of winning any one game is $\frac{2}{3}$. His chance of winning four games in succession is $(\frac{2}{3})^4 = \frac{16}{81}$. His chance of winning four out of the first five games is $\frac{64}{243}$. In order to win four before B wins two, it is evident that A must either win four straight or four out of the first five. His chance of doing one or the other is $\frac{64}{243} + \frac{16}{81} = \frac{112}{243}$. *Ans.* $\frac{112}{243}$.

8. A draws five times from a bag containing three white and seven black balls, replacing the ball drawn after each trial. Every time he draws a white ball he is to receive \$2, and every time he draws a black ball he is to pay \$1. What is his expectation?

A's chance of drawing a white ball the first trial is $\frac{3}{10}$; hence his chance of *receiving* \$2 is $\frac{3}{10}$, and his expectation is $\frac{3}{10}$ of \$2 = .60. A's chance of drawing a black ball on the first trial is $\frac{7}{10}$; hence his chance of *paying* \$1 is $\frac{7}{10}$, and his expectation is $-\frac{7}{10}$ of \$1 = − .70. Hence his expectation is to *lose* 10 cents each trial. *Ans.* − 50 cents.

9. It is 3 to 1 that A speaks the truth, 4 to 1 that B does, and 6 to 1 that C does; what is the probability of the happening of an event, if both A and B assert that it happened, and C denies it?

If the event happened, A and B tell the truth and C is mistaken, the separate chances being respectively $\frac{3}{4}$, $\frac{4}{5}$, and $\frac{1}{7}$; hence the compound chance is $\frac{3}{4} \times \frac{4}{5} \times \frac{1}{7} = \frac{6}{70}$.

If the event did *not* happen, both A and B are mistaken and C tells the truth, of which the separate chances are $\frac{1}{4}$, $\frac{1}{5}$, $\frac{6}{7}$; hence the compound chance is $\frac{1}{4} \times \frac{1}{5} \times \frac{6}{7} = \frac{3}{70}$.

Now, the event either *did* or *did not* happen, and there are six chances in favor of its happening to three chances against it; hence the chance of the happening of the event is $\frac{6}{9} = \frac{2}{3}$.

Exercise XXIX.

1. How many permutations can be formed out of the letters in the word *Milton?*

2. How many different words can be formed out of the letters in the word *Cincinnati,* taken all together?

3. The number of permutations of $2n$ things taken three at a time is twenty times as great as the number of permutations of n things taken two at a time; find n.

4. How many different words can be formed from the letters in the expression $ab^2c^3d^4$?

5. The number of permutations of n things taken three together is $\frac{1}{6}$ the number taken four together; find n.

6. If the number of permutations of n things taken three together be 12 times the number of permutations of $\frac{1}{2}n$ things taken three together, what is the number of permutations of n things taken all together?

7. The number of permutations of n things taken all together is 720; find n.

8. If 210 different words can be formed from seven letters, of which a certain number are a's and the others are all different, how many a's are there?

9. How many combinations can be made of 11 things taken four at a time? five at a time? ten at a time?

10. How many combinations can be made out of the letters in the word *Homer?*

11. The number of combinations of n things taken two at a time is 15; find n.

12. The number of combinations of $3n$ things taken four at a time is $\frac{15}{4}$ of the number of combinations of $2n$ things taken three at a time; find n.

13. How many different sums of money can be formed from a *three-cent piece*, a *half-dime*, a *dime*, and a *dollar?*

14. From a company of soldiers numbering 96, a picket of 10 is to be selected; determine in how many ways it can be done, (I) so as always to *include* a particular man; (II) so as to *exclude* the same man.

15. If the number of combinations of r things taken $(a-b)$ together be equal to the number of combinations of r things taken $(a+b)$ together, find r.

16. A boat's crew consists of eight men, three of whom can row only on one side and two only on the other; find the number of ways in which the crew can be arranged.

17. There are ten tickets, five of which are blanks, and the others are marked 1, 2, 3, 4, 5. What is the probability of drawing 10 in three trials, the tickets being replaced?

18. If the tickets be not replaced?

19. Two players of equal skill, A and B, are playing a set of games for a prize of \$80. A needs two games to win the set, and B needs three games. Find the expectation of each.

20. A bag contains eight tickets numbered 1, 2, 3, . . . , 8. A draws two tickets: what is his expectation, if each *odd* number be worth \$1, and each *even* number be worth 5 cents?

CHAPTER XXX.

CONTINUED FRACTIONS. INDETERMINATE CO-EFFICIENTS. BINOMIAL THEOREM.

588. A **Continued Fraction** is one of the general form $\cfrac{1}{a+\cfrac{1}{b+\cfrac{1}{c}}}$ + etc., in which *a, b, c,* . . . represent positive integers. When the number of *terms a, b, c,* . . . is finite, the fraction is said to be *terminating.*

For the sake of abbreviation the continued fraction is sometimes written, $\frac{1}{a+}, \frac{1}{b+}, \frac{1}{c+}$, etc.

In this chapter we shall consider mainly *proper* continued fractions, as indicated above; but an integer plus a continued fraction may also be regarded as a continued fraction and treated accordingly.

589. To convert any proper fraction into a continued fraction: *Proceed as in finding the H. C. F. of the terms of the fraction.* Thus: to convert $\frac{9}{25}$ into a continued fraction, divide both terms by 9; whence $\frac{9}{25} = \frac{1}{2+\frac{7}{9}}$.

Similarly, divide both terms of $\frac{7}{9}$ by 7: $\therefore \frac{7}{9} = \frac{1}{1+\frac{2}{7}}$.

Now divide both terms of $\frac{2}{7}$ by 2; $\therefore \frac{2}{7} = \frac{1}{3+\frac{1}{2}}$.

Ans. $\frac{9}{25} = \frac{1}{2+}, \frac{1}{1+}, \frac{1}{3+}, \frac{1}{2}$.

It is evident that every proper fraction may be converted into a terminating continued fraction; for, by continuing the process indicated above, we must finally arrive at a point where the remainder is zero, and the operation terminates. In like manner, an improper fraction may be converted into an integer plus a terminating continued fraction.

590. To find the value of a terminating continued fraction: *Perform the operations indicated, beginning at the right.*

Thus: $$\cfrac{1}{3+\cfrac{1}{2+\cfrac{1}{4+\frac{1}{5}}}} = \cfrac{1}{3+\cfrac{1}{2+\frac{5}{21}}} = \frac{1}{3+\frac{21}{47}} = \frac{47}{162}.$$

591. To find the approximate value of an infinite continued fraction:

Omit all its terms beyond any assumed term, and obtain the value of the resulting fraction as in the preceding article.

Let the fraction be $\frac{1}{a+}, \frac{1}{b+}, \frac{1}{c+} \ldots$

If we omit all the terms after the first, the approximate result thus obtained, $\frac{1}{a}$, is greater than the continued fraction, because the denominator a is less than the true denominator $a + \frac{1}{b + \text{etc.}}$

If we omit all the terms after the second, the approximate result thus obtained, $\cfrac{1}{a+\cfrac{1}{b}} = \frac{b}{ab+1}$, is too small; for

$b < b + \frac{1}{c + \text{etc.}}$; hence $\frac{1}{b}$ is too large, and therefore $a + \frac{1}{b}$ is too large, whence $\cfrac{1}{a+\cfrac{1}{b}}$ is too small. Similarly,

$$\frac{1}{a+\cfrac{1}{b+\cfrac{1}{c}}} = \frac{1}{a+\cfrac{c}{bc+1}} = \frac{bc+1}{abc+a+c} \text{ is too large, etc.}$$

592. The fractions $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$, etc., are called *integral fractions.* The fractions formed by taking one, two, three, etc., of the integral fractions are called *converging fractions* or *convergents.* Thus, the first convergent is $\frac{1}{a}$; the second is $\frac{1}{a+\frac{1}{b}} = \frac{b}{ab+1}$; the third is $\frac{1}{a+\cfrac{1}{b+\cfrac{1}{c}}} = \frac{bc+1}{abc+a+c}$; and so on.

From the preceding article it is evident that *the convergents taken in order are alternately greater and less than the continued fraction.*

593. *To convert a quadratic surd into a continued fraction.*

Let it be required to convert $\sqrt{19}$ into a continued fraction.

$$\sqrt{19} = \sqrt{16+3} = 4 + \frac{1}{a}.$$

Since $\sqrt{19} = 4 + \frac{1}{a}$, then $\frac{1}{a} = \sqrt{19} - 4$;

$$\therefore a = \frac{1}{\sqrt{19}-4} = \frac{\sqrt{19}+4}{3} = 2 + \frac{1}{b}.$$

Since $\frac{\sqrt{19}+4}{3} = 2 + \frac{1}{b}$, then $\frac{1}{b} = \frac{\sqrt{19}-2}{3}$;

$$\therefore b = \frac{3}{\sqrt{19}-2} = \frac{3(\sqrt{19}+2)}{15} = 1 + \frac{1}{c}.$$

Since $\frac{1}{c} = \frac{\sqrt{19}-3}{5}$, $\therefore c = \frac{\sqrt{19}+3}{2} = 3 + \frac{1}{d}$.

Proceeding in like manner, we find that $d = 1 + \frac{1}{e}$; $e = 2 + \frac{1}{f}$; $f = 8 + \frac{1}{g}$; and that $g = a$; hence the succeeding values will occur in the same order as before.

Therefore, $\sqrt{19} = 4 + \frac{1}{2+}, \frac{1}{1+}, \frac{1}{3+}, \frac{1}{1+}, \frac{1}{2+}, \frac{1}{8+}$, etc.

The converging values of $\sqrt{19}$ are 4, $4\frac{1}{2}$, $4\frac{1}{3}$, $4\frac{4}{11}$, etc.

EXAMPLES.

Convert into continued fractions:

1. $\frac{7}{11}$. *Ans.* $\frac{1}{1+}, \frac{1}{1+}, \frac{1}{1+}, \frac{1}{3}$.

2. $\frac{130}{421}$. *Ans.* $\frac{1}{3+}, \frac{1}{4+}, \frac{1}{5+}, \frac{1}{6}$.

3. Find a series of fractions converging to $\frac{1}{3.1416}$.

Ans. $\frac{1}{3}$; $\frac{7}{22}$; $\frac{113}{355}$.

4. Express approximately the ratio of 5 hrs., 48 min., 51 sec., to one day. *Ans.* $\frac{1}{4}$; $\frac{7}{29}$; $\frac{8}{33}$; $\frac{39}{161}$.

5. Show that $\sqrt{5} > \frac{682}{305}$; also that $\sqrt{5} < \frac{2889}{1292}$.

6. If $8^x = 32$, find x. *Ans.* $\frac{5}{3}$.

7. If $3^x = 15$, find x. *Ans.* 2.465.

8. Show that $\sqrt{a^2+1} = a + \frac{1}{2a+}, \frac{1}{2a+}, \frac{1}{2a+}$, etc.

9. Show that $\sqrt{\frac{5}{3}} = 1 + \frac{1}{3+}, \frac{1}{2+}, \frac{1}{3+}, \frac{1}{2+}$, etc.

10. Find the value of $\frac{1}{3+}, \frac{1}{22+}, \frac{1}{1+}, \frac{1}{4}$. *Ans.* $\frac{114}{347}$.

INDETERMINATE CO-EFFICIENTS.

594. The method of developing algebraic expressions, by assuming a series with unknown co-efficients, and finding the values of the assumed co-efficients, is termed the method of **Indeterminate** or **Undetermined Co-efficients.**

595. If $A + Bx + Cx^2 + Dx^3 + \ldots = A' + B'x + C'x^2 + D'x^3 + \ldots$ for all possible values of x (the co-efficients of the variable, x, being finite quantities and independent of x), then $A = A'$, $B = B'$, $C = C'$, $D = D'$, etc.

Since the equation is true for all possible values of x, it must be true when $x = 0$; then $A = A'$.

If $A = A'$ be subtracted from both sides, then

$$Bx + Cx^2 + Dx^3 + \ldots = B'x + C'x^2 + D'x^3 + \ldots$$

Divide by x; $\therefore B + Cx + Dx^2 + \ldots = B' + C'x + D'x^2 + \ldots$

This equation is true for all values of x, because the various co-efficients are independent of x; whence, as before, put $x = 0$, and $B = B'$.

In a similar manner, $C = C'$, $D = D'$, etc.

596. If $A + Bx + Cx^2 + Dx^3 + \ldots = 0$, for all possible values of x (the co-efficients A, B, C, D, etc., being finite quantities and independent of x), then $A = 0$, $B = 0$, $C = 0$, $D = 0$, etc.

Since the equation is true for all values of x, it must be true when $x = 0$; $\therefore A = 0$.

As before, divide by x, and then put $x = 0$ in the resulting equation; $\therefore B = 0$. In like manner, $C = 0$, $D = 0$, etc.

597. To develop $\frac{a}{a + bx}$ into a series by means of indeterminate co-efficients.

The series will consist of the powers of x, with certain co-efficients depending upon a or b, and it is evident that

x will not occur in the first term. Therefore, assume

$$\frac{a}{a+bx} = A + Bx + Cx^2 + Dx^3 + \ldots$$

Clear of fractions by multiplying both sides by $a + bx$.

$$\therefore a = aA + \left.\begin{matrix} aB \\ +Ab \end{matrix}\right| x + \left.\begin{matrix} aC \\ +bB \end{matrix}\right| x^2 + \left.\begin{matrix} aD \\ +bC \end{matrix}\right| x^3 + \begin{matrix} \ldots \\ +\ldots \end{matrix}$$

I. $\therefore aA = a$, from which $A = 1$.

II. $aB + Ab = 0$; that is, $aB + b = 0$; $\therefore B = -\frac{b}{a}$.

III. $aC + bB = 0$; that is, $aC + b\left(-\frac{b}{a}\right) = 0$; $\therefore C = +\frac{b^2}{a^2}$.

IV. $aD + bC = 0$; that is, $aD + b\left(\frac{b^2}{a^2}\right) = 0$; $\therefore D = -\frac{b^3}{a^3}$.

Whence $\frac{a}{a+bx} = 1 - \frac{b}{a}x + \frac{b^2}{a^2}x^2 - \frac{b^3}{a^3}x^3 + \ldots$

If additional terms be required, they are readily supplied by observing the law of the series; viz., Each successive term is found from multiplying the next preceding term by $-\frac{b}{a}x$, *the general term* being $\left(-\frac{b}{a}x\right)^{n-1}$.

598. To develop $\frac{1}{3x - x^2}$ into a series.

The first term will evidently contain x^{-1}.

Hence put $\frac{1}{3x - x^2} = Ax^{-1} + Bx^0 + Cx + Dx^2 + \ldots$

Clearing of fractions, $1 = 3A + \left.\begin{matrix} 3B \\ -A \end{matrix}\right| x + \left.\begin{matrix} 3C \\ -B \end{matrix}\right| x^2 + \left.\begin{matrix} 3D \\ -C \end{matrix}\right| x^3 + \ldots$

I. $1 = 3A$; $\therefore A = \frac{1}{3}$.

II. $0 = 3B - A$; that is, $0 = 3B - \frac{1}{3}$; $\therefore B = \frac{1}{9}$.

III. $0 = 3C - B$; that is, $0 = 3C - \frac{1}{9}$; $\therefore C = \frac{1}{27}$, etc.

That is, $\frac{1}{3x - x^2} = \frac{1}{3}x^{-1} + \frac{1}{9}x^0 + \frac{1}{27}x^1 + \frac{1}{81}x^2 + \ldots$, the general term being $(\frac{1}{3})^n x^{n-2}$.

PARTIAL FRACTIONS.

599. An algebraic fraction may be sometimes decomposed into two or more simpler fractions.

To decompose $\frac{2x-3}{x^2-3x+2}$ into two fractions whose denominators shall be the factors of x^2-3x+2.

The factors of x^2-3x+2 are $x-1$ and $x-2$.

Assume $\frac{2x-3}{x^2-3x+2}=\frac{A}{x-1}+\frac{B}{x-2}$.

Clear of fractions. $\therefore\ 2x-3=A(x-2)+B(x-1)$.

Transpose. $(2A+B-3)+x(2-A-B)=0$.

From (596), I. $2A+B-3=0$. II. $2-A-B=0$.

III = I + II. $A-1=0$; $\therefore A=1$. $\therefore B=1$.

$$\therefore \frac{2x-3}{x^2-3x+2}=\frac{1}{x-1}+\frac{1}{x-2}.$$

EXAMPLES.

Prove the following identities:

1. $\frac{1+2x}{1-3x}=1+5x+15x^2+45x^3+\ldots+(5x)(3x)^{n-2}$.

2. $\frac{1}{3-2x}=\frac{1}{3}+\frac{2}{9}x+\frac{4}{27}x^2+\frac{8}{81}x^3+\ldots+\frac{1}{3}(\frac{2}{3}x)^{n-1}$.

3. $\frac{1}{1-2x+x^2}=1+2x+3x^2+4x^3+\ldots+nx^{n-1}$.

4. $\frac{x^2}{(x+1)(x^2-3x+2)}=\frac{4}{3(x-2)}+\frac{1}{6(x+1)}-\frac{1}{2(x-1)}$.

5. $\sqrt{a^2+x^2}=a+\frac{x^2}{2a}-\frac{x^4}{8a^3}+\frac{x^6}{16a^5}\ldots\ldots$

6. $\sqrt{1+x+x^2}=1+\frac{x}{2}+\frac{3x^2}{8}-\frac{3x^3}{16}+\ldots\ldots$

7. Decompose $\frac{5x-14}{x^2-6x+8}$. *Ans.* $\frac{2}{x-2}+\frac{3}{x-4}$.

THE BINOMIAL THEOREM.

PECK'S PROOF.

600. LEMMA.

$$\left(\frac{x^n - y^n}{x - y}\right)_{y=x} = nx^{n-1}, \text{ for all values of } n.$$

In other words, if $y = x$, then, for all values of n, $\frac{x^n - y^n}{x - y} = nx^{n-1}$.

I. When n is a positive integer.

By division, $\frac{x^n - y^n}{x - y} = x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \ldots + y^{n-1}$.

We have shown that the division is exact (Art. 217). The number of terms in the quotient is n, because there are $n - 1$ terms containing y, and one term containing x only.

If $x = y$, each of these terms becomes x^{n-1}, and their sum is nx^{n-1}. $\therefore \left(\frac{x^n - y^n}{x - y}\right)_{y=x} = nx^{n-1}$.

II. When $n = \frac{a}{b}$, a positive fraction.

Let $x = r^b$; $\therefore x^{\frac{a}{b}} = r^a$, and $x^{\frac{a}{b} - 1} = x^{\frac{a-b}{b}} = r^{a-b}$.

Let $y = s^b$; $\therefore y^{\frac{a}{b}} = s^a$, and if $y = x$, $s = r$.

$$\therefore \frac{x^{\frac{a}{b}} - y^{\frac{a}{b}}}{x - y} = \frac{r^a - s^a}{r^b - s^b} = \left\{\frac{\dfrac{r^a - s^a}{r - s}}{\dfrac{r^b - s^b}{r - s}}\right\}_{s=r} = \frac{ar^{a-1}}{br^{b-1}} =$$

$$\frac{a}{b} r^{a-b} = \frac{a}{b} x^{\frac{a}{b} - 1} = nx^{n-1}.$$

III. When n is a negative integer $= -m$.

$$\frac{x^{-m} - y^{-m}}{x - y} = -x^{-m}y^{-m}\left(\frac{-y^m + x^m}{x - y}\right)_{y=x} =$$

$$-x^{-m}x^{-m}(mx^{m-1}) = -mx^{-m-1} = nx^{n-1}.$$

IV. When $n = -\frac{a}{b}$, a negative fraction.

$$\frac{x^{-\frac{a}{b}} - y^{-\frac{a}{b}}}{x - y} = -x^{-\frac{a}{b}}y^{-\frac{a}{b}}\left(\frac{-y^{\frac{a}{b}} + x^{\frac{a}{b}}}{x - y}\right)_{y=x} =$$

$$-x^{-\frac{a}{b}}x^{-\frac{a}{b}}\left(\frac{a}{b}x^{\frac{a}{b}-1}\right) = -\frac{a}{b}x^{-\frac{a}{b}-1} = nx^{n-1}.$$

601. To expand $(a + x)^n$.

I. Assume $(a + x)^n = A + Bx + Cx^2 + Dx^3 + \ldots$

Making $x = 0$, we find $a^n = A$; hence

II. $(a + x)^n = a^n + Bx + Cx^2 + Dx^3 + \ldots$

Substitute y for x, then

III. $(a + y)^n = a^n + By + Cy^2 + Dy^3 + \ldots$

IV = II − III. $(a + x)^n - (a + y)^n = B(x - y) + C(x^2 - y^2) + D(x^3 - y^3) + \ldots$

Divide both members by the identity

$$(a + x) - (a + y) = x - y.$$

$\therefore$ V. $$\frac{(a + x)^n - (a + y)^n}{(a + x) - (a + y)} = B + C(x + y) + D(x^2 + xy + y^2) + \ldots$$

But if $x = y$, then, for all values of n, V becomes

VI. $n(a + x)^{n-1} = B + 2Cx + 3Dx^2 + 4Ex^3 + \ldots$

Multiply both members of VI by $(a + x)$; hence

VII. $$n(a+x)^n = aB + \begin{array}{c|c|c|l} 2aC & x + 3aD & x^2 + 4aE & x^3 + \ldots \\ + B & + 2C & + 3D & \end{array}$$

Multiply both members of II by n; hence

VIII. $n(a+x)^n = na^n + nBx + nCx^2 + nDx^3 + nEx^4 + \ldots$

Equating the second members of VII and VIII,

IX. $$aB + \begin{array}{c|c|c|l} 2aC & x + 3aD & x^2 + 4aE & x^3 + \ldots \\ + B & + 2C & + 3D & \end{array}$$
$$= na^n + nBx + nCx^2 + nDx^3 + \ldots$$

Hence, by the principle of indeterminate co-efficients,

X. $aB = na^n$; $\qquad \therefore B = na^{n-1}$.

XI. $2aC + B = nB$; $\therefore C = B\dfrac{(n-1)}{2a} = \dfrac{n(n-1)}{2}a^{n-2}$.

XII. $3aD + 2C = nC$;

$$\therefore D = C\frac{(n-2)}{3a} = \frac{n(n-1)}{2}\frac{(n-2)}{3}a^{n-3};$$

and so on.

Substituting these values in II, we have, for any exponent,

$$(a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{2}a^{n-2}x^2 + \frac{n(n-1)}{2}\frac{(n-2)}{3}a^{n-3}x^2 + \dots$$

$$\dots\left[\frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \dots \frac{n-r+2}{r-1}a^{n-r+1}x^{r-1}\right] + \dots + x^n.$$

602. If x be negative, then the terms containing odd powers of x are negative;

$$\therefore (a-x)^n = a^n - na^{n-1}x + \frac{n(n-1)}{2}a^{n-2}x^2 - \dots$$

603. *To find the numerically greatest term in the expansion of* $(a+b)^n$.

I. If n be a positive integer.

An examination of the series will show that the $(r+1)$th term is derived from the rth term by multiplying the latter by $\dfrac{n-r+1}{r}\cdot\dfrac{b}{a} = \left(\dfrac{n+1}{r} - 1\right)\dfrac{b}{a}$. Evidently this multiplier diminishes as r increases, but as long as it is greater than unity, the $(r+1)$th term is greater than the rth term. The rth term will be the greatest when the multiplier first becomes less than unity. When $\left(\dfrac{n+1}{r} - 1\right)\dfrac{b}{a} < 1$, then $r > \dfrac{(n+1)b}{a+b}$; hence the rth term is the greatest when r is the first integer greater than $\dfrac{(n+1)b}{b+a}$.

If $\frac{(n+1)b}{a+b}$ be an integer, then the $(r+1)$th term is equal to the rth, and each of these is greater than any other term.

II. If n be a positive fraction.

If $\frac{b}{a} > 1$, there is no greatest term, for the series will evidently diverge. If $\frac{b}{a} < 1$, the greatest term or terms may be ascertained as in I.

III. If n be negative, either integral or a fraction greater than 1.

The multiplier that changes the rth term into the $(r+1)$th is $\frac{-n-r+1}{r} \cdot \frac{b}{a} = -\frac{n+r-1}{r} \cdot \frac{b}{a}$. As the numerically greatest term is sought, reject the negative sign before the multiplier; then, as in I, the rth term will be the greatest when $\frac{n+r-1}{r} \cdot \frac{b}{a}$ is first less than 1; that is, when r is first greater than $\frac{b(n-1)}{a-b}$.

As in I, if $\frac{b(n-1)}{a-b}$ be an integer, there are two equal terms each greater than any other; and if, as in II, $\frac{b}{a} > 1$, there is no greatest term.

IV. If n be negative and < 1, and $\frac{b}{a} < 1$, the first term is the greatest; for in this case $\frac{n+r-1}{r} \cdot \frac{b}{a} < 1$ for all values of r; that is, each term is less than the preceding.

The same rules apply to the expansion of $(a-b)^n$, because the terms differ from those of $(a+b)^n$ only in the signs, and the sign does not affect the arithmetical value.

EXAMPLES.

Expand to four terms each of the following expressions:

1. $(1+x)^{\frac{1}{2}}$. *Ans.* $1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3-\dots$
2. $(1+x)^{-\frac{1}{2}}$. *Ans.* $1-\frac{1}{2}x+\frac{3}{8}x^2-\frac{5}{16}x^3+\dots$
3. $(a^3-x)^{\frac{1}{3}}$. *Ans.* $a-\dfrac{x}{3a^2}-\dfrac{x^2}{9a^5}-\dfrac{5x^3}{81a^8}-\dots$
4. $\sqrt{a^2-x^2}$. *Ans.* $a-\dfrac{x^2}{2a}-\dfrac{x^4}{8a^3}-\dfrac{x^6}{16a^5}-\dots$
5. $\dfrac{a^3}{(a^3-x^3)^{\frac{2}{3}}}$. *Ans.* $a+\frac{2}{3}a^{-2}x^3+\frac{5}{9}a^{-5}x^6+\frac{40}{81}a^{-8}x^9+\dots$

Find the greatest term in each of the following:

6. $(3+\frac{2}{3})^6$. *Ans.* Second term $=810$.
7. $(2-\frac{1}{4})^{\frac{10}{3}}$. *Ans.* First.
8. $(\frac{2}{3}+\frac{5}{3})^6$. *Ans.* Fifth and sixth.
9. $(1+\frac{1}{5})^{-12}$. *Ans.* Third.
10. $(1-\frac{7}{12})^{-\frac{8}{3}}$. *Ans.* Third.

Exercise XXX.

Reduce each of the following to a continued fraction, and find the successive convergents:

1. $\frac{3900}{10963}$. 2. $\frac{4900}{11283}$. 3. $\frac{157}{972}$. 4. $\frac{445}{612}$.

Find the first four convergents in the following:

5. $\sqrt{8}$. 6. $\sqrt{17}$. 7. $\sqrt{53}$. 8. $\sqrt{101}$. 9. $\sqrt{46}$.

10. Simplify $\dfrac{1}{2+}, \dfrac{1}{3+}, \dfrac{1}{5+}, \dfrac{1}{6+}, \dfrac{1}{7}$.

11. Expand $(1+x)^{-1}$.

12. Expand $(1-x)^{-1}$.

13. Expand $\dfrac{2}{3+4x}$.

14. Expand $\dfrac{1+x}{1+2x+3x^2}$.

15. Decompose $\frac{3x}{x^2 - 8x + 15}$.

16. Decompose $\frac{2x^2 + x + 1}{x^3 - 2x - 1}$.

17. Decompose $\frac{2x}{x^2 - 12x + 35}$.

18. Decompose $\frac{2}{1 - x^4}$.

19. Expand $(x + y)^{-2}$.

20. Expand $(1 - x)^{\frac{1}{3}}$.

21. Expand $\left(\frac{a}{a^2 + b^2}\right)^{\frac{2}{3}}$.

22. Write the 49th term of $(a - x)^{50}$.

23. Find the greatest term of $(1 + x)^n$, when $x = \frac{2}{3}$ and $n = 4$.

24. Extract the square root of 24 approximately.

25. Find the number of terms in the expansion of $(a + b + c + d)^{10}$.

26. What is the sum of the co-efficients in the expansion of $(2a - b + 3c)^{10}$?

27. In the expansion of $(3a - b - 2c)^{16}$, what is the sum of the co-efficients of the terms containing odd powers of a?

28. Simplify $\frac{1}{(a - b)^{-7}} + \frac{1}{(a + b)^{-7}}$.

CHAPTER XXXI.

BUSINESS FORMULAS.

604. **Interest** is money paid for the use of money. The *principal* is the sum lent. The *amount* is the sum of the principal and interest at the end of any time. The *rate* is the interest of one dollar for one year.

605. Interest is of two kinds, *simple* and *compound.*

Simple Interest is interest charged upon the principal only.

Compound Interest is interest charged upon the sum of the principal and the interest due.

SIMPLE INTEREST.

606. Let the principal be represented by P;
the interest on $1 for one year by r;
the amount of $1 for one year by R;
the number of years by n;
the amount of P dollars for n years by A.

Then: I. $R = 1 + r$.

II. Interest on P for one year $= Pr$.

III. Amount of P for one year $= P(1+r) = PR$.

IV. Interest on P for n years $= Pnr$.

V. Amount of P for n years $= P(1+nr)$.

From (V), $A = P(1+nr)$.

607. If any three of the quantities A, P, n, r, be given, the fourth may be found by substituting the given quantities in the equation (V), $A = P(1+nr)$. Thus:

Ex. 1. At what rate will \$500 amount to \$650 in 3 years 4 months?

Substituting in Formula V, $650 = 500\left(1 + \frac{10}{3}r\right)$.

$$\therefore \tfrac{13}{10} = 1 + \tfrac{10}{3}r. \quad \therefore r = \tfrac{9}{100}.$$

Ans. The rate is 9%.

Ex. 2. At 6% per annum, what principal will amount to \$650 in 5 years?

As before, $650 = P\left(1 + 5 \times \dfrac{6}{100}\right)$. $\therefore P = \dfrac{650}{1.3} = 500.$

Ans. \$500.

608. Since P will, in n years, amount to A, it is evident that P at the present time may be considered equivalent in value to A due in n years; thus P may be regarded as the *present worth* of a sum A due in n years. Thus, from Ex. 2 in the preceding article, \$500 is the present worth of \$650 due in 5 years, if money be worth 6% per annum.

609. Discount is an allowance made for the payment of a sum of money before it is due.

From the definition of present worth, it follows that a debt due at some future time is equitably discharged by paying the present value at once; hence: *The* TRUE DISCOUNT *is equal to the amount due diminished by its present worth.*

Thus: VI. $D = A - P = A - \dfrac{A}{1 + nr} = \dfrac{Anr}{1 + nr}.$

For examples in simple interest consult any arithmetic.

COMPOUND INTEREST.

610. When the interest is compounded *annually.*

The amount of P in one year is PR;

in two years is $PR(1 + r) = PR^2$;

in three years is $PR^2(1+r) = PR^3$; etc.

$\therefore$ the amount of P in n years is PR^n.

That is: VII. $A = PR^n = P(1 + r)^n$.

611. Having given any three of the quantities A, P, r, n, the fourth may be found by substituting the given quantities in VII. $A = P(1+r)^n$.

Thus: VIII. $P = \frac{A}{(1+r)^n}$; IX. $r = \sqrt[n]{\frac{A}{P}} - 1$.

Since $R^n = \frac{A}{P}$, $\therefore$ X. $n = \frac{\log A - \log P}{\log R}$.

612. The interest is equal to $A - P = PR^n - P = P(R^n - 1)$.

$\therefore D = A - P = P(R^n - 1)$; or, since $P = \frac{A}{R^n}$, therefore,

XI. $D = A - P = A - \frac{A}{R^n} = \frac{A(R^n - 1)}{R^n}$.

613. When the interest is due more frequently than once a year.

(1) When due *semi-annually*.

The amount of P in $\frac{1}{2}$ year is $P\left(1 + \frac{r}{2}\right)$;

in 1 year is $P\left(1 + \frac{r}{2}\right)^2$;

in n years is $P\left(1 + \frac{r}{2}\right)^{2n}$.

$$\therefore \text{(XII)}\ A = P\left(1 + \frac{r}{2}\right)^{2n}.$$

(2) When due *quarterly* (XIII), $A = P\left(1 + \frac{r}{4}\right)^{4n}$.

(3) When due *monthly* (XIV), $A = P\left(1 + \frac{r}{12}\right)^{12n}$.

(4) When due *q times a year* (XV), $A = P\left(1 + \frac{r}{q}\right)^{qn}$.

XVI. As before, $D = A - P = P\left[\left(1 + \frac{r}{q}\right)^{qn} - 1\right]$.

XVII. The present worth, $P = \frac{A}{\left(1 + \frac{r}{q}\right)^{qn}}$.

ANNUITIES.

614. An **Annuity** is a sum of money payable annually, or at fixed periods in the year.

615. *To find the amount of an annuity left unpaid for any number of years, allowing compound interest.*

The sum due at the end of first year $= S$;

of second year $= S + SR$;

of third year $= S + SR + SR^2$;

and so on for n years.

$$\therefore A = S + SR + SR^2 + SR^3 + \ldots + SR^{n-1}.$$

$$\therefore A = S(1 + R + R^2 + R^3 + \ldots + R^{n-1}).$$

$$\therefore \text{XVIII.}\quad A = \frac{S(R^n - 1)}{R - 1} = \frac{S(R^n - 1)}{r}.$$

616. By means of Formula XVIII, problems relating to *sinking funds* may be solved. Thus:

A city owes \$300,000, payable at the end of 10 years, without interest. What sum must be set apart annually, as a sinking fund, to pay this debt, money being worth 6%, compounded annually?

$$\text{From XVIII, } S = \frac{Ar}{R^n - 1} = \frac{\$300{,}000 \times .06}{(1.06)^{10} - 1} = \$22{,}670.$$

617. *To find the present worth of an annuity to continue a certain number of years, allowing compound interest.*

Let P denote the present worth; then the amount of P in n years is equal to the amount of the annuity in the same time; $\therefore PR^n = \dfrac{S(R^n - 1)}{R - 1}$.

$$\therefore \text{XIX.}\quad P = \frac{S(1 - R^{-n})}{R - 1} = \frac{S\{1 - (1 + r)^{-n}\}}{r}.$$

618. If the annuity is perpetual, then, in Formula XIX, $(1 + r)^{-n}$ differs from zero by less than any assignable quantity. $\therefore$ XX. $P = \dfrac{S\{1 - 0\}}{r} = \dfrac{S}{r}$.

619. To find the present worth of an annuity to commence at the end of p years, and then to continue q years.

Subtract the present worth of the annuity for p years from the present worth of the annuity for $p + q$ years.

Thus:

$$P = \frac{S(1 - R^{-p-q})}{R-1} - \frac{S(1 - R^{-p})}{R-1} = \frac{S}{R-1}(R^{-p} - R^{-p-q});$$

or, $$\text{XXI. } P = \frac{S}{r}(R^{-p} - R^{-p-q}) = \frac{S(R^q - 1)}{R^{p+q}(R-1)}.$$

620. If the annuity is to commence at the end of p years and then to continue forever, we must suppose q to be infinite; hence, in XXI, R^{-p-q} differs from zero by less than any assignable limits; hence,

$$\text{XXII. } P = \frac{S}{r}(R^{-p} - 0) = \frac{SR^{-p}}{r} = \frac{S}{R^p(R-1)}.$$

621. *To find the annuity when the present worth, the time, and the rate per cent are given.*

From XIX, $P = \dfrac{S(1 - R^{-n})}{R-1}$;

$$\therefore \text{XXIII. } S = \frac{P(R-1)}{1 - R^{-n}} = \frac{PR^n(R-1)}{R^n - 1} = \frac{PrR^n}{R^n - 1}.$$

EXAMPLES.

In all the following examples days of grace are not considered, and one month is counted equivalent to $\frac{1}{12}$ of a year:

1. The sum of \$300 amounts in 25 years to \$900. What is the rate: (1) at simple interest? (2) If the interest be compounded annually? *Ans.* (1) 8%; (2) $4\frac{1}{2}$%.

2. The interest on a certain sum is \$18, and the discount on the same sum for the same time and at the same rate is \$15; find the sum. *Ans.* \$90.

3. In how many years will \$10 amount to \$105 at 5% compounded annually? *Ans.* 48.17.

4. In how many years will any sum double itself at compound interest, at 5%? *Ans.* 14.2066.

5. In how many years will a sum of money treble itself, if compounded annually at $3\frac{1}{2}$%? *Ans.* Nearly 32.

6. What is the amount of an annuity of $120 for 20 years at 6% per annum? *Ans.* $4412.80.

7. At 5% per annum, what is the present worth of an annuity of $500 for 30 years? *Ans.* $7687.86.

8. What is the present worth of a perpetual annuity of $600 at 6% per annum? *Ans.* $10,000.

9. At 4%, what is the present worth of an annuity of $450, to commence at the end of 10 years, and to continue 20 years? *Ans.* $4130.10.

10. At 6% per annum, what is the present worth of a perpetual annuity of $5000, commencing at the end of 15 years? *Ans.* $34,780.

11. At 8% per annum, compounded quarterly, what will be the amount of $1000 in 2 years 7 months?

Ans. $1225.11.

Exercise XXXI.

1. What is the amount of $1 for 200 years at 5% per annum, compound interest?

2. Find the logarithm corresponding to the amount of $1 for 1000 years, compounded at 5% per annum.

3. In what time will any sum double itself at 8% per annum, if the interest be compounded quarterly?

4. In how many years, at 5% compounded annually, will $1 amount to $10?

5. What is the compound discount on $1000, due in 20 years, at 5%?

6. What is the present worth of an annuity of \$1000, at 5%, for 30 years?

7. What is the present worth of a perpetual annuity of \$1000, at 5% per annum?

8. A debt of \$40,000, at 6% compound interest, is discharged by eight equal annual payments; required the annual payment.

9. Find the present worth of an annual pension of \$90, at 5% compound interest, to continue forever.

10. What is the present worth of a perpetual annuity of \$1000, at 4%, payments to begin at the end of 10 years?

11. A debt of \$3700 is discharged by two payments of \$2000 each, at the end of one and two years; find the rate of interest paid.

12. At 4%, what annual sum should be paid for 30 years, at the beginning of each year, to secure the payment of \$10,000 at the end of that time?

13. An annual premium of \$300 is paid to a life insurance company for insuring \$10,000. For how many years must the premium be paid in order that the company shall sustain no loss, money being worth 4 per cent compounded annually?

14. A debt of \$$d$, compounded at r% per annum, is discharged in n years by equal annual payments of \$$b$; find n.

CHAPTER XXXII.

TEST EXAMPLES.

Time, 2 hours for each set.

I.

1. When $a = 5$, $b = 4$, $c = 2$, find the value of $3a - 2[b - 2c - 3(a - b + 2c) + (-a - 2b + c)] - \frac{ab}{c}$.

2. Simplify $4a - 3\{a - b - [2(2a - 3b) - (-a - 2b)] - 3c\}$, and verify the result when $a = 4$, $b = -2$, $c = 3$.

3. Simplify $(a - 2b)(2a - 3b) - (2b - a)(b - 4a)$.

4. Expand $(x + 1)(2x - 1)(3x - 2)$.

5. Expand $(-c - 2d)^4$.

6. Divide $(-a - b)^5$ by $-(a + b)^2$.

7. Divide $-a^3 - b^3 - c^3 + 3abc$ by $a + b + c$.

8. Simplify $5\sqrt[5]{a^{10}b^5c^{15}} - 2\sqrt[3]{a^6b^3c^9} - \sqrt{a^4b^2c^6}$.

9. A and B are each entitled to half a box of oranges; A, however, takes three more than B, and pays B six cents to equalize accounts. What is the value of an orange?

10. If $3x - \frac{2x - 3}{4} - \frac{x - 1}{3} = 26$, find x.

II.

1. If $a = \frac{4}{5}$, $b = \frac{2}{3}$, $c = \frac{1}{2}$, find the value of
$$35a - 2[6b - 4c - 3\{5a - 3b + 2(4c - 6b)\}].$$

2. Simplify
$$(2a - 3b)(a + b) + (-a - b)(a - 2b) - (b - a)(3a - 2b).$$

3. Verify the answer to Ex. 2 when $a = 6$, $b = 3$.

4. Expand $[a^2 - (b - c)a - bc][a^2 + (b - c)a + bc]$.

5. Expand $(-2a - 3b - c)^3$.

6. Divide $(-a - b)^n$ by $(a + b)^n$.

7. Simplify $(a - b)^4 - (b - a)^3 + (b - a)^2$.

8. Simplify $\sqrt{a^2 + b^2 + c^2 - 2ab - 2ac + 2bc}$.

9. Divide the cube of $(a^2 - b^2)$ by the cube of $(a + b)$.

10. A can do in 7 days what would take B 11 days; how long would it take both, working together, to do a piece of work which A alone can do in 21 days?

III.

1. Interpret $6^x = 36$, and find the value of x.

2. Simplify $3(a - b)a^2 - a^3 + a^2b$.

3. Expand $(a - b)(a-c)-(a-b)(a + c)-(a-c)(a + c)$.

4. Expand $(a^2 - ab + b^2)^2 + (a^2 + ab + b^2)^2$.

5. Divide
$(a - b - c)^{2m}-(a - b - c)^m-(a - b-c)^n$ by $(a-b-c)^{m-n}$.

6. Divide $(-a- b -c)(ab + ac + bc) + abc$ by $(-a-b)$.

7. Given $4ax - 2b = 3ab - 6a^2x$; find x.

8. The sum of a certain number of quarter-dollars and an equal number of dimes is $21. How many dimes are there?

9. A is three times as old as B; 10 years ago he was four times as old. How old is A?

10. A bought a certain number of oranges at two for five cents, and an equal number at five cents each: by selling the lot at three for a dime he lost 15 cents. How many did he buy?

IV.

1. Interpret $x^{\frac{4}{5}} = \frac{1}{16}$, and find the value of x.

2. Simplify $2a - 2\{a - 2 - 2[a - 2 - 2(a - 2)]\}$.

3. Expand $(x - 2)(x - b)(x + c)$.

4. Show that $(a^4 - a^2 + 1)(a^4 - 1) + a^2(a^2 - 1)(a^2 + 1) = (a^2 - 1)(a^6 + a^4 + a^2 + 1)$.

5. Divide $-x^3 - y^3 + 3xy + 2x + 2y + 1$ by $-1 - x - y$.

6. Divide $(x^3 + y^3)(x^3 - y^3)$ by $(x^2 + xy + y^2)(x^2 - xy + y^2)$.

7. Simplify $(a + b - c)^3 + (-a - b + c)^3$.

8. Given $\frac{1}{2}(7x + 9) - 2(x - \frac{1}{9}[2x - 1]) = 14$; find x.

9. A has as much money as B and C together; B has twice as much as C; and A has \$40 more than C. How much has A?

10. A can walk from C to D in 20 hours, and he can ride the same distance in 6 hours. He goes from C to D, walking half the time and riding half the time. How long does it take him to go from C to D?

V.

1. Expand $(1 + 3x + 3x^2 + x^3)(1 + 2x + x^2)$.

2. Expand and simplify $(x^2 + x + 1)^3 - (x^2 - x + 1)^3$.

3. Divide $-a^3 - b^3 - 1 + 3ab$ by $-a - b - 1$.

4. Show that $(a^2 + b^2)(c^2 + d^2) = (ac \pm bd)^2 + (ad \mp bc)^2$.

5. If $\frac{5x - 7}{2} - \frac{2x + 7}{3} = 3x - m$, what must be the value of m in order that $x = 7$?

6. A can do twice as much as B, and B twice as much as C; working together they complete a piece of work in 7 days; how long would it take A alone to do the work?

7. The time past noon equals one third the time past midnight. What o'clock is it?

8. If each side of a square field were four rods longer, the area would be one acre more; find the area of the field.

9. Given $\left.\begin{array}{l}\frac{x}{2} + \frac{y}{3} = 7, \\ \frac{x}{4} - \frac{y}{9} = 1;\end{array}\right\}$ find x and y.

10. Given $\left.\begin{array}{r} x + 2y + 3z = \ \ 8, \\ 2x - \ \ y - \ \ z = 15, \\ 3x - 4y - 2z = 16; \end{array}\right\}$ find x, y, and z.

VI.

1. Reduce $[(x^2 - 2ab)^3 + a^3b^3] \div (x^2 - ab)$ to its simplest form.

2. From the cube of $(2a - b)$ take $(a - 2b)^3$.

3. $\sqrt[3]{x^6 - 6x^5y + 15x^4y^2 - 20x^3y^3 + 15x^2y^4 - 6xy^5 + y^6} = ?$

4. $(a^{\frac{1}{3}} - 1)^5 = ?$

5. A's age is three times the united ages of his three sons, whose ages differ by two years respectively. In four years A's age will be $\frac{5}{3}$ their united ages. How old is A?

6. Given $\left.\begin{array}{r} \frac{x}{4} - \frac{3y}{8} = 1, \\ \frac{x}{2} - \frac{5y}{8} = 3; \end{array}\right\}$ find x and y.

7. If $x^2 - 3x = 0$, find x.

8. If $x^2 - 8x = -15$, find x.

9. The square of A's age is 75 more than 10 times his age. How old is A?

10. A's age is now equal to the square of B's, and in four years it will be four times B's. How old is A?

VII.

1. Expand $(-2a - 3b)^4$.

2. Extract the fourth root of 5 to 2 decimal places.

3. A travels three times as fast as B, and requires 5 hours longer to go 60 miles than B does to go 10 miles; find A's rate of travel.

4. A number expressed by two digits equals 7 times the sum of its digits, and if 18 be subtracted from it, the digits will change places; find the number.

5. If a rectangular field were 2 rods longer and 4 rods wider, it would be a square containing 64 sq. rods more; find the area.

6. Solve $2x^2 - 3x = 20$.

7. $2x^2 + x = 21$; find x.

8. Factor $x^6 - x^2$.

9. Factor $-x^3 - 3x + x^2 + 3$.

10. Factor $x^2 - 11x + 18$.

VIII.

1. Find the G. C. M. of $x^4 - 3x^3 - 5x^2 + 3x + 4$ and $2x^4 - 6x^3 - 8x^2 + 6x + 6$.

2. Find the L. C. M. of $(a-b)^3$, $a^3 - b^3$, $a^2 - 2ab + b^2$.

3. Solve $25x^2 - 15x = -2$.

4. Form the quadratic whose roots are $5 \pm \sqrt{3}$.

5. Form the cubic whose roots are $1, -2, 2$.

6. Find three numbers whose sum is 11: the sum of the first and third is one less than twice the second, and the smallest number is the positive root of the equation $2x^2 + x = 3$.

7. Expand $(1 + \sqrt{3})(2 - \sqrt{3})(\sqrt{3} - 1)$.

8. Solve $x^3 + 2x^2 - x = 2$.

9. Simplify $\dfrac{\sqrt{2}}{\sqrt{3} - \sqrt{2}}$.

10. Find four fourth roots of 4.

IX.

1. Factor $a^3 + 3a^2 - 4a - 12$.

2. Find the H. C. F. of $a^{12} + 1$ and $a^8 - a^6 - a^2 - 1$.

3. Find two numbers whose sum is 4, and the sum of whose squares is 10.

4. The square of $\frac{2}{3}$ of a certain number is one more than the number itself; find the number.

5. A's rate in going 80 miles exceeds his time by 2; find his rate.

6. Solve $x^3 - 5x = 12$.

7. Form the equation whose roots are the factors of $x^2 - x - 2$.

8. The time past noon is the square root of the time past midnight; what time is it?

9. $2\sqrt{5} - \sqrt{80} + \sqrt{45} - \sqrt{125} = ?$

10. If $x^2 + y^2 = 10$ and $xy = 3$, find x and y.

X.

1. The log of 5 to the base x is $\frac{1}{2}$; find x.

2. The log of 2 is .3010; what is the log of $2\frac{1}{2}$?

3. If $x^2 - 4 : x - 3 :: 2x + 1 : 2$, find x.

4. $\sqrt{x^2 + 11} - \sqrt{x^2 - 9} = 2$; find x.

5. Show that $a^2 + 3b^2 > 2b(a + b)$, unless $a = b$.

6. Simplify $5\sqrt[3]{4} \times 2\sqrt[3]{32} \div \sqrt[3]{108}$.

7. Simplify

$$(5 + \sqrt{-3})(5 - \sqrt{-3})(\sqrt{3} - 2\sqrt{-2})(\sqrt{3} + 2\sqrt{-2}).$$

8. If $a : b :: c : d$, show that

$$a^2 + ab : ab - b^2 :: c^2 + cd : cd - d^2.$$

9. Given $x + x^{-1} = 2.9$, find the value of x.

10. A sold a horse for \$144, and gained as much per cent as the horse cost him; find the cost.

XI.

1. Show that $a^2 + b^2 + 1 > ab + a + b$ unless $a = b = 1$.

2. Extract the square root of $20\sqrt{6} - 48$.

3. Given $\log 2 = .3010$; how many figures express 2^{100}?

4. Given $25^x + 5^x = 650$; find x.

5. If $x + 5 : 2x - 3 :: 5x + 1 : 3(x - 1)$, find x.

6. If $a : b :: b : c$, prove $a : c :: (a + b)^2 : (b + c)^2$.

7. The eighth term of an A. P. whose common difference is 3, is 14; find the third term.

8. Find two numbers whose sum is 10 and whose geometric mean is 3.

9. Sum the infinite series $\frac{1}{2} - \frac{1}{3} + \frac{2}{9} - \frac{4}{27} + \ldots$

10. Insert three harmonical means between 2 and 10.

XII.

1. Simplify $\dfrac{\sqrt[3]{2}}{\sqrt[3]{10} - \sqrt[3]{2}}$.

2. Find x when

$$x^2 - 4x + 2 : x^2 - 4x :: x^2 - 2x - 1 : x^2 - 2x - 2.$$

3. In an A. P., given d, l, s; find a.

4. Find three numbers in G. P. whose sum is 13, and the sum of whose squares is 91.

5. If $2^x 3^y = 20^3$ and $5x = 3y$, find x and y; having given $\log 2 = .3010$ and $\log 3 = .4771$.

6. How many different sums of money can be formed with 2 cents, a three-cent piece, a half-dime, a dime, and a quarter-dollar?

7. If $\log xy = 1.4982$ and $\log (x \div y) = \bar{1}.6543$, find $\log x$ and $\log y$.

8. How many figures will express the amount of $1 for 1000 years at 6% per annum, compound interest? (Use table of logarithms.)

9. What is the present worth of a perpetual annuity of $1200 at 4% per annum?

10. Of the same annuity, beginning after 10 years?

XIII.

1. Divide $a + b + c - 3\sqrt[3]{abc}$ by $a^{\frac{1}{3}} + b^{\frac{1}{3}} + c^{\frac{1}{3}}$.

2. Multiply $x^{\frac{1}{3}} + 1 + x^{-\frac{1}{3}}$ by $x^{\frac{1}{3}} - 1 + x^{-\frac{1}{3}}$.

3. If $a = -3$, $b = -5$, $c = \frac{1}{2}$, find the value of

$$(-a)^2 - 2\{b - 4c - [a + b + 12c - (a - b + 8c^2)]\}.$$

4. Resolve into prime factors $x^6 - 7x^3 - 8$.

5. $\sqrt{16x^{-4} - 32x^{-3} + 24x^{-2} - 8x^{-1} + 1} = ?$

6. Find two consecutive numbers, the difference of whose cubes is 61.

7. Expand $(a^{\frac{1}{2}} - 3b)^4$.

8. Expand $(2a^{-\frac{1}{2}} - 3b^{-\frac{2}{3}})^4$.

9. Factor $a^8 - 256^{-1}$.

10. Simplify $\sqrt{5} \times \sqrt[3]{2} \times \sqrt[4]{9}$.

XIV.

1. Divide $x^4 - y^4$ by $x^{-1} + y^{-1}$.

2. Find x, if $\dfrac{x + \sqrt{2 - x^2}}{x - \sqrt{2 - x^2}} = \dfrac{4}{3}$.

3. $\sqrt{28 - 5\sqrt{12}} = ?$

4. Expand to four terms $\sqrt{1 \pm 3x}$.

5. Divide $21 - 5\sqrt{3}$ by $\sqrt{3} + 1$.

6. Sum 3, $3\frac{1}{3}$, $3\frac{2}{3}$, . . . , to fifteen terms.

7. Form the equation whose roots are double those of $x^2 - 2x = 3$.

8. Insert six arithmetical means between 2 and 23.

9. If $x + y = a$ and $x^2 + xy + y^2 = b$, find x and y.

10. Find the fifth term of $(x + a)^{-n}$.

XV.

1. Solve $\sqrt[3]{x + 22} - \sqrt[3]{x + 3} = 1$.

2. Find three cube roots of 27.

3. If $2^x = \sqrt[3]{16}$, find x.

4. If the base is 25, of what number is $-\frac{1}{2}$ the logarithm?

5. In the common system, what is the log of 0?

6. If $(\frac{2}{3})^x = \frac{5}{6}$, find x.

7. Simplify $\dfrac{1}{2 +}, \dfrac{1}{1 +}, \dfrac{1}{4 +}, \dfrac{1}{5}$.

8. By indeterminate co-efficients, find four terms of the quotient of $\frac{3 + 2x}{5 + 7x}$.

9. Expand $(a - b)^{11}$.

10. If $\frac{1}{a + b + x} = a^{-1} + b^{-1} + x^{-1}$, find x.

XVI.

1. Given $\log 2 = .3010$; find $\log 2^{-\frac{1}{3}}$.

2. Expand to four terms $(a^n - b^x)^{-\frac{1}{2}}$.

3. If $6^{\frac{2}{x}} = 36$, find x.

4. Form the equation whose roots are the reciprocals of the roots of $x^2 - 7x = -10$.

5. Divide $a^3 + 3a^2b + 3ab^2 + b^3 - c^3$ by $a + b - c$.

6. If $7(x - y) = 3(x + y)$, what is the ratio of x to y?

7. Simplify $\frac{1}{x - 1} - \frac{2}{x} + \frac{1}{x + 1}$.

8. If $a+b : a-b :: c + d : c - d$, show that $a : b :: c : d$.

9. Solve $x^2 + (a^{-1} - b^{-1})x = (ab)^{-1}$.

10. If d is the least term in the proportion $a : b :: c : d$, prove that $a^3 + d^3 > b^3 + c^3$.

XVII.

1. Extract the fourth root of $7 - 4\sqrt{3}$.

2. Solve $\sqrt[3]{14 - x} + \sqrt[3]{14 + x} = 4$.

3. Simplify $(a^3 - b^3) \times (a^2 - b^2) \div (a - b)^2$.

4. Simplify $(x^3-3x^2-15x+25)^{-1} - (x^3+7x^2+5x-25)^{-1}$.

5. What must be added to $9x^4 + 12x^3 + 20x + 25$ to make it a perfect square?

6. Expand $(2x - a)^5$.

7. Given $6\left(x^2+\frac{1}{x^2}\right)+5\left(x+\frac{1}{x}\right)=38$, find x.

8. The sum of five numbers in A. P. is 15, and the sum of their squares is 65; find them.

9. If $x^2(x^2-y^2)=25$ and $y^2(x^2+y^2)=19\frac{1}{8}$, find x and y.

10. Divide $a+a^{\frac{1}{2}}b^{\frac{1}{2}}+b$ by $a^{\frac{1}{2}}+\sqrt[4]{ab}+b^{\frac{1}{2}}$.

XVIII.

1. Extract the cube root of $16.\dot{3}131\ldots$ to two places.

2. Find the maximum or minimum value of $\frac{2x^2-2x+5}{x^2-2x+3}$.

3. Solve: I. $(x+y)(x^2+y^2)=203$;
II. $(x^2-y^2)(x-y)=63$.

4. Solve: I. $x+y+z=10$;
II. $x^2+y^2+z^2=38$;
III. $xy-xz-yz=-1$.

5. Simplify $(1-a)^5\times(a-1)^{-3}$.

6. Divide $x^{-1}+x^{-\frac{1}{2}}y^{-\frac{1}{2}}+y^{-1}$ by $x^{-\frac{1}{2}}+x^{-\frac{1}{4}}y^{-\frac{1}{4}}+y^{-\frac{1}{2}}$.

7. Find a fourth proportional to a^{-1}, b^{-1}, c^{-1}.

8. Sum to infinity $9,\ -3,\ +1,\ -\frac{1}{3},\ +\ldots$

9. In a bag 9 white and 3 red balls are placed; if 5 balls be drawn, find the chance that all the red balls will be among those removed.

10. Solve $15x^2-9x-4=2x\sqrt{9x+4}$.

XIX.

1. Solve $x^3+10x=57$. **2.** Given $(2\ 5)^x=\frac{4}{25}$, find x.

3. Simplify $(4x^{-\frac{2}{3}})^{-\frac{3}{2}}$.

4. $(x^{-1}-y^{-1})\div(x^{\frac{1}{3}}-y^{\frac{1}{3}})=?$

5. $\dfrac{1}{x-1+\dfrac{1}{1+\dfrac{x}{x-2}}}=\dfrac{3}{10}$; find x.

6. A's rate exceeds B's by $\frac{1}{3}$ of a mile per hour, and he goes 39 miles in 45 minutes less than B does; find A's rate.

7. Find the fourteenth term in the expansion of $(2^{-1}-b)^{16}$.

8. Solve $\sqrt{x^2+1} - \sqrt{-x^2+1} = \sqrt{1-x^4}$.

9. Resolve into partial fractions $\dfrac{1-2x}{x^2-x-20}$.

10. Find an A. P. such that the sum of n terms shall be equal to n^2.

XX.

1. Sum the A. P. whose first and twelfth terms are 6 and 61, and the entire number of terms is sixteen.

2. Find the A. P. such that the first, third, and thirteenth terms are in G. P.

3. Insert five harmonic means between $-\frac{1}{3}$ and $-\frac{1}{15}$.

4. Solve $\dfrac{x^2}{y} + \dfrac{y^2}{x} = 9$, and $x + y = 6$.

5. Expand $(1+\sqrt{-2})^6$. 6. Solve $20^x = 100$.

7. If $\dfrac{1}{x} + \dfrac{1}{y} = 5$ and $\dfrac{1}{x^2} + \dfrac{1}{y^2} = 13$, find x and y.

8. How many words, each containing one vowel and three consonants, can be formed out of the first thirteen letters of the alphabet?

9. If $ax^2 + bx + c$ becomes 24, 37, 60, respectively, when x becomes 2, 3, 4, what will it become when $x = -3$?

10. Prove that the difference of the squares of any two odd numbers is exactly divisible by 8.

ANSWERS.

Exercise I.

1. A, \$175; B, \$225; C, \$360. **2.** 16; 4. **3.** 252; 336. **4.** 38. **5.** 14, 15, 16. **6.** 100. **7.** 5. **8.** 28. **9.** 18 years. **10.** 5. **11.** 60. **12.** 20.

Exercise II.

1. 1; − 11; 11; 2. **2.** − 5; 4; − 8. **3.** 2; 4; 4. **4.** 1; − 1; 4; 1; − 2. **5.** $-\frac{1}{3}$; $\frac{1}{16}$; $\frac{3}{5}$; $-\frac{1}{16}$; $\frac{1}{6}$. **6.** 57 A.D. **7.** 32 B.C. **8.** 125 B.C. **9.** 55. **10.** + 1266; − 322; − 623; − 1375; − 840. **11.** \$3823. **12.** − \$1200. **13.** 268; 22 steps. **14.** 75 steps; 675 steps.

Exercise III.

1. 22. **2.** 17. **3.** 42. **4.** 256. **5.** 3. **6.** 28. **7.** 0. **8.** 16. **9.** 64. **10.** 0. **11.** 3. **12.** 2.

Exercise IV.

1. $6a$. **2.** $a(3b-c)$. **3.** $a(a-3)$. **4.** $3(b-c)$. **5.** $6(a-b)$. **6.** $b(a+3)$. **7.** $a(a^2-1)$. **8.** $a(b+1)+c(b+1) = (a+c)(b+1)$. **9.** $a^2(a+1)+(a+1) = (a^2+1)(a+1)$. **10.** $(x^2+1)(x-1)$. **11.** $(a^3+1)(a^2-a+1)$. **12.** $x[(a+b)+(a-b)] = 2ax$. **13.** $(a-b)(y+z)$. **14.** $b(a-c)(d-a)$. **15.** $[(a+b)+(a-b)]\sqrt{x} = 2a\sqrt{x}$. **16.** $(a^2+b^2)(a-b)$. **17.** $(3a+2b)x^{\frac{1}{2}}$. **18.** $8x(a-b)$. **19.** $(4+c)(a-b)$. **20.** $3(2a-b)(c+d)$.

EXERCISE V.

1. $(2b+c)x$. 2. $(2b+c+1)x$. 3. $10(a-b)$. 4. $(2x-y)(a+3b)$. 5. $3a-7b-2c$. 6. $a-a^3=a(1-a^2)$. 7. $2a$. 8. $4b-8=4(b-2)$. 9. $x^3+(a-b)x^2+(a-b)x+(a-1)$. 10. $x^2+x(y+z)+(y^2+yz+z^2)$. 11. $x^3+x^2(y-z+1)+xy$. 12. $(a-b)x^3-(a-b)x^2-(a+b)x+(a-b)$.

EXERCISE VI.

1. $6x^4-96$. 2. $x^5-6x^3-24x^2+121x-120$. 3. $x^5-41x-120$. 4. $4x^6-5x^5+8x^4-10x^3-8x^2-5x-4$. 5. $x^3+3xy+y^3-1$. 6. $x^4-6x^3+13x^2-12x+4$. 7. $x^3+(b-a+c)x^2+(-ac-ab+bc)x-abc$. 8. $x^6+6x^5+15x^4+20x^3+15x^2+6x+1$. 9. $x^6-6x^5+15x^4-20x^3+15x^2-6x+1$. 10. $2x^7-8x^6-3x^5+12x^4-7x^3+28x^2+3x-12$. 11. a^6-2+a^{-6}. 12. $a^3-a+2-a^{-1}$. 13. $a^{\frac{3}{2}}b^{-\frac{1}{2}}+a^{\frac{1}{2}}b^{\frac{1}{2}}+a^{-\frac{1}{2}}b^{\frac{3}{2}}$. 14. $a^{\frac{3}{2}}+ab^{\frac{1}{2}}-a^{\frac{1}{2}}b-b^{\frac{3}{2}}$. 15. $6x^2y+2y^3=2y(3x^2+y^2)$. 16. $8x(x^2+1)$. 17. $16x(x^2+4)$. 18. b^2-d^2. 19. $a^3+b^3+c^3$. 20. $2a^2+2b^2+2c^2=2(a^2+b^2+c^2)$. 21. $4xy(x^2+y^2)$. 22. $4xy(x^2-y^2)$. 23. x^4+y^4. 24. $[2x(x^2+3y^2)]^2=4x^6+24x^4y^2+36x^2y^4$. 25. $y^4+7y^3+16y^2+12y+4$.

EXERCISE VII.

1. $x^9-9x^8y+36x^7y^2-84x^6y^3+126x^5y^4-126x^4y^5+84x^3y^6-36x^2y^7+9xy^8-y^9$. 2. $(x^5-5x^4y+10x^3y^2-10x^2y^3+5xy^4-y^5)^{-1}$. 3. $a^4-8a^3b+24a^2b^2-32ab^3+16b^4$. 4. $(81x^4-216x^3y+216x^2y^2-96xy^3+16y^4)^{-1}$. 5. $8x^3y+8xy^3=8xy(x^2+y^2)$. 6. $x^2-2x+3-2x^{-1}+x^{-2}$. 7. $x-4x^{\frac{1}{2}}+6-4x^{-\frac{1}{2}}+x^{-1}$. 8. $a^6-6a^5+15a^4-20a^3+15a^2-6a+1$. 9. $x^3+y^3-1+3x^2y-3x^2+3xy^2-3y^2+3x+3y-6xy$. 10. $a^2+b^2+c^2+d^2-2ab-2ac+2ad+2bc-2bd-2cd$. 11. $[2(a^2+b^2)]^2=4a^4+8a^2b^2+4b^4$. 12. $(4a^6+24a^4+36a^2)^{-1}$. 13. $a^2-12a^{\frac{3}{2}}+54a-108a^{\frac{1}{2}}+81$.

14. $4a^4-4a^3+a^2+12a-6+9a^{-2}$. **15.** $x^8+8x^7y+28x^6y^2+56x^5y^3+70x^4y^4+56x^3y^5+28x^2y^6+8xy^7+y^8$. **16.** $x^3-6x^{\frac{9}{4}}+15x^{\frac{3}{2}}-20x^{\frac{3}{4}}+15-6x^{-\frac{3}{4}}+x^{-\frac{3}{2}}$. **17.** $b^4x^4 \pm 4b^3x^3ay+6b^2x^2a^2y^2 \pm 4bxa^3y^3+a^4y^4$. **18.** $1 \pm 10x+40x^2 \pm 80x^3+80x^4 \pm 32x^5$. **19.** $a^2+b^2x^2+c^2x^4+2abx+2acx^2+2bcx^3$. **20.** $a^3-8b^3+c^3-6a^2b+3a^2c+12ab^2+12b^2c+3ac^2-6bc^2-12abc$. **21.** $(a^4x^4+4a^3x^5+6a^2x^6+4ax^7+x^8)^{-1}$. **22.** $x(2-x)(2+3x^2)$ or $4x-2x^2+6x^3-3x^4$. **23.** $-7+20x-14x^2+4x^3$. **24.** $1+4x-2x^2-4x^3+25x^4-24x^5+16x^6$.

Exercise VIII.

1. $-2ac^2$. **2.** $-2a+4b^2-6a^2b$. **3.** $-(m-n)^2(m+n)$. **4.** $a^4+2a^3+3a^2+4a+5$. **5.** x^2-2x+3. **6.** $a^3+2a^2b+3ab^2+4b^3$. **7.** x^2-bx+c. **8.** $x^4-(a-1)x^3-(a-b-1)x^2-(a-1)x+1$. **9.** $a^2+b^2+c^2+ab+ac-bc$. **10.** $x^2+x(2y-1)+(4y^2+2y+1)$. **11.** $a^4+a^2b^2+b^4$. **12.** $(a+b)^2-2c(a+b)+4c^2$. **13.** $-2+3a-a^{-1}+a^{-2}$. **14.** $(a+1)^2-2(a+1)+1=a^2$. **15.** $1-cd(a-b)^{\frac{1}{2}}+(a-b)^{\frac{1}{2}}$. **16.** $3^3(a-b)^3-3^4(a-b)^4-3^5(a-b)^6$. **17.** $(a+b-c)^{-y}-(a+b-c)^{-2y}-(a+b-c)^{-x}$. **18.** $x^3+x^{\frac{5}{2}}-x^{\frac{3}{2}}+x^{\frac{1}{2}}+1$. **19.** $a^4-2a^2bc+4b^2c^2$. **20.** $a^3+2a^2b+2ab^2+b^3$. **21.** $a-a^{\frac{1}{2}}-a^{-\frac{1}{2}}+a^{-1}$. **22.** $a+b+c$. **23.** $a^2-2ab+b^2$. **24.** $a^{-2}+a^{-1}(b-c)^{-1}+(b-c)^{-2}$.

Exercise IX.

1. $3x^4y^3$; $5a^3b^6$; $9a^2(b-c)$; $6a^7(x-y)^3$. **2.** $(a-b)^3$; $(2x-1)^5$; $(a-3)^7$. **3.** $(a-b+c)^n$; $a^{2n-1}(b-c)^{3n-4}$; $x^{2n^2}(a-2b)^{4n}$. **4.** $(x-y)^n$; $(3a-1)^{5n}$; $(a-b-1)^3$. **5.** $3x^2-x-1$. **6.** $4x^3-5x^2-3x+2$. **7.** 421; 737; 3789; 4.164. **8.** 1.5811 +; 3.7947 +; .4110 + ; .5477 +; .1732 +. **9.** $2x^2+4x-3$. **10.** x^3-x^2+x-1. **11.** $a+b+c$. **12.** 42 ; 48 ; 4.7 ; 9.6. **13.** 2.34 ; 32.1 ; 4.68.

14. 1.07 +; .43 +; 2.51 +; 2.33 +. **15.** $a^{-1} - 1$. **16.** 3; .4; $5a^2b^{-3}$.

EXERCISE X.

1. 7. **2.** $\frac{3}{4}$. **3.** -3. **4.** $\frac{a+b}{c}$. **5.** $-\frac{1}{6}$. **6.** $d \div (b+c-a)$.
7. $\frac{ab}{c(a+b)}$. **8.** $bd \div (ad+bc)$. **9.** $-\frac{a}{2}$. **10.** $(a-b) \div (4-c)$.
11. $-\frac{3}{2}$. **12.** $b(a+c) \div (2b+c-a)$. **13.** $\frac{80}{67}$. **14.** 30.
15. $-2\frac{1}{2}$. **16.** $a - 4$. **17.** 6, 4. **18.** 36. **19.** 12 minutes.
20. 12 hours. **21.** $1\frac{11}{13}$ days; $2\frac{2}{5}$ days; $2\frac{2}{3}$ days; $3\frac{3}{7}$ days.
22. 400. **23.** A, \$90; B, \$150; C, \$200. **24.** \$2400.
25. 8. **26.** 8, 4. **27.** 6. **28.** A, 18 years; B, 6 years.
29. A, 40 years; B, 25 years. **30.** A, 20 years; B, 5 years.
31. \$30. **32.** 39 rods; 41 rods. **33.** \$200. **34.** \$400.
35. Wine, 85; water, 35. **36.** 60, 84. **37.** 3 A.M. **38.** 3 P.M.
39. \$5. **40.** 300.

EXERCISE XI.

1. $x = 3\frac{1}{2}, y = 4$. **2.** $x = 56, y = 35$. **3.** $x = 2, y = 4, z = 5$. **4.** $x = 2, y = 1, z = 3$. **5.** $x = 4, y = 5, z = 2, v = 3$. **6.** A in 20 days, B in 80 days, C in 240 days. **7.** A has \$232, B has \$332. **8.** A, 49 years; B, 21 years. **9.** $\frac{4}{15}$. **10.** 72 apples and 60 pears. **11.** A has \$180; B, \$480. **12.** A, \$360; B, \$280. **13.** 30, 10. **14.** 2, 4, 7, 9. **15.** $\frac{a+b-c}{2}$, $\frac{a-b+c}{2}$, $\frac{-a+b+c}{2}$.

EXERCISE XII.

1. 4, 1. **2.** 3, 1. **3.** $3\frac{1}{2}$. **4.** $-1, -9$. **5.** 9, 5. **6.** 4, -3. **7.** 4, 5. **8.** 14, -3. **9.** 6, -1. **10.** 5, $-\frac{1}{2}$. **11.** 4, $\frac{1}{2}$. **12.** 3, $-\frac{4}{3}$. **13.** 3, $-\frac{1}{25}$. **14.** $\frac{5}{3}, -\frac{3}{2}$. **15.** $a \pm b$. **16.** $-a \pm b$.

EXERCISE XIII.

1. $a \cdot a \cdot a \cdot (a^2 - 8a + 3)$. **2.** $(a-b)(a-b)[(a-b)^3-3]$. **3.** $x(x^2+1)(x-1)$. **4.** $a(a^2+a+1)(a+1)(a^2-a+1)$. **5.** $(a-b)^2[(a-b)^2+(a-b)+1]$. **6.** $(a^2-\frac{1}{2})(a^2-\frac{1}{2})$. **7.** $a(a^2-\frac{7}{2})(a^2-\frac{7}{2})$. **8.** $(a^4-3)(a^4+3)$. **9.** $(4a^2-3)(4a^2+3)$. **10.** $(a-b-c-d)(a-b-c+d)$. **11.** $(a-b-3)(a-b-3)$. **12.** $(a-2b+4)(a-2b-2)$. **13.** $(a-2-b-c)(a-2+b+c)$. **14.** $(a+2)^2-(b-3)^2=(a+2-b+3)(a+2+b-3)=(a-b+5)(a+b-1)$. **15.** $(a-13)(a-1)$. **16.** $(x-8y)(x-2y)$. **17.** $x(x-1)(x+1)(x-1)(x+1)$. **18.** $(x^2-8)(x-1)(x+1)$. **19.** $(3x+2)(3x-1)$. **20.** $(2x-3)(2x-1)$. **21.** $(3x-1)(2x-1)$ **22.** $(3x+1)(2x-1)$. **23.** $(x^2-x-1)(x^2+x-1)$. **24.** $(2x^2-x-1)(2x^2+x-1)=(2x+1)(x-1)(2x-1)(x+1)$. **25.** $(3x^2-2x-2)(3x^2+2x-2)$. **26.** $(a^4-2)(a^8+2a^4+4)$. **27.** $(a-1)(a^2+a+1)(a+1)(a^2-a+1)(a^2+1)(a^4-a^2+1)$. **28.** $(a^8+1)(a^{16}-a^8+1)$. **29.** $(a^4+3)(a^8-3a^4+9)=(a^4+3)(a^4-3a^2+3)(a^4+3a^2+3)$. **30.** $(a-2)^3-1=(a-3)(a^2-3a+3)$. **31.** $(a-1)^4-9a^2=(a^2-5a+1)(a^2+a+1)$. **32.** $x(x-2)(2x^2+3x+6)$. **33.** $(x-2)(x^2+2x+3)$. **34.** $(x+1)(x^4-x^3+x^2-x+2)$.

EXERCISE XIV.

1. x^2+y^2. **2.** x^2+xy+y^2. **3.** $x-y$. **4.** $x^2-2xy+3y^2$. **5.** $x-y$. **6.** $x-3$. **7.** $x+3$. **8.** x^2-3. **9.** $5x^2-1$. **10.** $x-1$. **11.** $x^6-x^5-x^4-x^3-x^2-x-2$. **12.** x^7-x. **13.** x^4-16. **14.** $x^6-3x^5+7x^3-5x^2-2x+2$. **15.** $12x^6+23x^5+x^4+x^3-8x^2+2x-4$. **16.** $12x^6-8x^5-3x^4-10x^3+8x^2+3x-2$. **17.** $12x^6-10x^5-20x^4-5x^3+2x^2+3x$. **18.** $72x^4-90x^3y+42x^2y^2-30xy^3+6y^4$.

EXERCISE XV.

1. $\dfrac{1}{x-3}$. **2.** $\dfrac{1}{x+y}$. **3.** $\dfrac{3x-1}{2x-1}$. **4.** $\dfrac{1}{x+1}$. **5.** $\dfrac{x}{x^2-2x+2}$.

6. $x - \frac{1}{x}$. 7. $x - 1 + x^{-1}$. 8. $\frac{1}{x^2 - 1 + x^{-2}} = \frac{x^2}{x^4 - x^2 + 1}$.

9. $\frac{2x-3}{(x^2-1)(2x+3)}$. 10. $\frac{9}{(1-x)(x+2)^2}$. 11. $\frac{1}{x+2}$.

12. 0. 13. $\frac{2xy^2}{x^4 - y^4}$. 14. $\frac{b^2 - ax}{b^2 - x^2}$. 15. $\frac{1}{abc}$. 16. 1. 17. 1.

18. 0. 19. m^2. 20. $\frac{3x}{4y}$. 21. $\frac{a^2+b^2}{c^2} + \frac{a^2+c^2}{b^2} + \frac{b^2+c^2}{a^2}$.

22. $2\left(\frac{ac}{bd} + \frac{bd}{ac}\right)$. 23. 2. 24. $\frac{2(a-b)^2}{3b^2(a+b)}$. 25. $\frac{a^2(a+b)}{x-y}$.

26. 0. 27. 1. 28. $\frac{1-x+x^2}{1+x}$. 29. $\frac{1+x^2}{1+x}$. 30. $\frac{-ab(a-b)^2}{a^4+a^2b^2+b^4}$.

31. $\frac{xy}{x^2+y^2}$. 32. $\frac{(a^2+b^2)^2}{2a^2b^2}$. 33. a. 34. $\frac{4a^3x}{x^4-a^4}$.

35. $\frac{4}{3(x+1)}$. 36. $\frac{a(df+e)}{bdf+be+cf}$.

Exercise XVI.

1. $\frac{c(3a-b)}{2b}$. 2. 3. 3. $\frac{ac(3ab+1)}{c-ad}$. 4. $\frac{a^2(b-a)}{b(b+a)}$.

5. 5. 6. 5. 7. $2\frac{21}{29}$. 8. $ab \div (a+b-c)$. 9. $3\frac{3}{8}$. 10. 8.
11. -107. 12. $-\frac{7}{8}$. 13. $x = 2$, $y = \frac{31}{5}$. 14. $x = 5$, $y = 4$.

15. $x = \frac{a^2-b^2}{5a-12b}$, $y = \frac{a^2-b^2}{12a-5b}$. 16. $x = \frac{ab}{a+b}$, $y = \frac{ab}{a-b}$.

17. $x = 20$, $y = 16$, $z = 12$. 18. $x = 2$, $y = 3$, $z = -1$.
19. $x = -15$, $y = -16$, $z = -8$, $t = 31$, $u = 22$.
20. $x = 6$, $y = 4$, $z = 1$, $u = -3$. 21. $x = 10$, $y = 8$, $z = 6$. 22. $x = \frac{2}{a+b-c}$, $y = \frac{2}{a-b+c}$, $z = \frac{2}{-a+b+c}$.

23. $x = 5$, $y = 2$, $z = -3$, $t = -4$, $v = 0$. 24. $x = \frac{1}{4}(a+b+c-d)$, $y = \frac{1}{4}(a+b-c+d)$, $z = \frac{1}{4}(a-b+c+d)$, $v = \frac{1}{4}(-a+b+c+d)$. If $\frac{1}{2}(a+b+c+d)$ be represented by s, then $x = \frac{1}{2}(s-d)$; $y = \frac{1}{2}(s-c)$; $z = \frac{1}{2}(s-b)$; $v = \frac{1}{2}(s-a)$.

EXERCISE XVII.

1. 287. **2.** A, 15 years; B, 20 years. **3.** A, \$48; B, \$60. **4.** 9, 36. **5.** 36, 64. **6.** 24. **7.** A, $8\frac{1}{3}$ hours; B, $12\frac{1}{2}$ hours. **8.** $\frac{cd}{d-c}$ days. **9.** 12. **10.** \$40. **11.** 12, 60. **12.** *rate*, $2\frac{1}{2}$ miles an hour; *time*, 18 hours. **13.** B's *rate* is 12 miles an hour, and CF is 200 miles. **14.** 3 miles an hour and $1\frac{1}{2}$ miles an hour. **15.** *Time down*, 5 hours; *rate of current*, 2 miles an hour; *rate of rowing*, 6 miles an hour. **16.** 72 miles. **17.** 300. **18.** 720. **19.** 480. **20.** I. $38\frac{2}{11}$ minutes past 7. II. $21\frac{9}{11}$ minutes past 7 and at $54\frac{6}{11}$ minutes past 7. III. $5\frac{5}{11}$ minutes past 7. IV. $13\frac{11}{13}$ minutes past 7. V. $27\frac{3}{11}$ minutes past 7 and $49\frac{1}{11}$ minutes past 7. VI. $44\frac{28}{33}$ minutes past 7 and $58\frac{2}{11}$ minutes past 7. **21.** $49\frac{1}{11}$ minutes past 3 o'clock. **22.** $46\frac{2}{13}$ minutes past 5. **23.** *Together* at $\frac{60}{11}a$ minutes past a o'clock; *opposite* at $\frac{60}{11}(a \pm 6)$ minutes past a o'clock. **24.** $8\frac{1}{3}$ hours. **25.** 13 hours. **26.** $9\frac{1}{5}$ hours; 46 hours. **27.** 64. **28.** 45. **29.** 27. **30.** 36. **31.** 124. **32.** *Length*, 80 rods; *width*, 20 rods; *area*, 10 acres. **33.** 120 @ 5 cents each.

EXERCISE XVIII.

1. $x = 2$, $y = 1$. **2.** Formula: $x = 41 + 10t$, $y = 1 - 7t$, in which $t = 0, -1, -2, -3, -4$. $\therefore x = 41; 31; 21; 11; 1.$
$y = 1; 8; 15; 22; 29.$

3. Formula: $x = 81 + 7t$, $y = 1 - 3t$, in which t may be $0, -1, -2, -3, -4, -5, -6, -7, -8, -9, -10, -11$.
$\therefore x = 81; 74; 67; 60; 53; 46; 39; 32; 25; 18; 11; 4.$
$y = 1; 4; 7; 10; 13; 16; 19; 22; 25; 28; 31; 34.$

4. Formula: $x = 12 + t$, $y = 8 - 10t$, in which t may be $0, -1, -2, -3, -4, -5, -6, -7, -8, -9, -10, -11$.
$\therefore x = 12; 11; 10; 9; 8; 7; 6; 5; 4; 3; 2; 1.$
$y = 8; 18; 28; 38; 48; 58; 68; 78; 88; 98; 108; 118.$

5. $x = 12 - 7t = 5$. $y = 5t - 1 = 4$. **6.** $x = 31 + 7t = 31; 24; 17; 10; 3.$ $y = 1 - 3t = 1; 4; 7; 10; 13.$

7. $x = 8,\ y = 3$. **8.** $x = 7,\ y = 5$. **9.** $x = 11,\ y = 18$. **10.** $x = 37,\ y = 13$. **11.** 19. **12.** 185 and 15; 119 and 81; 53 and 147. **13.** 245. **14.** 97. **15.** 21; 42; 63; 84. **16.** 100. **17.** 974. **18.** 5 sheep, 1 turkey, 94 chickens.

Exercise XIX.

1. $2\sqrt{2}$; $4\sqrt{3}$; $5\sqrt{5}$; $3a\sqrt{2}$; $6x\sqrt{2y}$; $3(a-b)\sqrt{5(a-b)}$; $4ab^{-1}\sqrt{6}$.

2. $3xy^2\sqrt[3]{2}$; $2a\sqrt[3]{5a}$; $7\sqrt[3]{2}$; $2(x-y)\sqrt[3]{2}$; $3(x-y)^{-2}\sqrt[3]{3}$.

3. $12\sqrt{2}$; $6\sqrt{3}$; $2\sqrt{3}$; $3\sqrt{2}$; $\sqrt{a}$; $(a^2-b^2)\sqrt{a+b}$.

4. $a^{-1}\sqrt[3]{a^{-1}} = a^{-2}\sqrt[3]{a^2}$; $\sqrt[3]{a-b}$; $(b-a)\sqrt[3]{b+a}$.

5. $\frac{2}{3}\sqrt{3}$; $\frac{3}{4}\sqrt{6}$; $\frac{8}{3}\sqrt{30}$; $\sqrt{a^2-b^2}$; $\dfrac{a+b}{a-b}\sqrt{a-b}$.

6. $\sqrt[3]{3}$; $\frac{1}{2}\sqrt[3]{30}$; $\sqrt[3]{a+b}$; $\dfrac{1}{a-b}\sqrt[3]{a^2-b^2}$.

7. $2\sqrt[4]{2}$; $2\sqrt[5]{2}$; $2a^2\sqrt[4]{2a}$; $a\sqrt[4]{a^2-ab}$; $ab^2\sqrt[n]{4}$; $a^2b\sqrt[x]{b}$.

8. $\sqrt{3}$; $\sqrt{2}$; $\sqrt{2}$; $\sqrt{a-b}$; $\sqrt{a-b}$; $\sqrt{-2(a+b)}$.

9. $\sqrt[3]{a-b}$; $b\sqrt{3a(a-b)}$; $\frac{1}{5}\sqrt[3]{50}$; $\frac{3}{2}\sqrt{2}$.

10. $\sqrt[6]{16}$ and $\sqrt[6]{27}$; $\sqrt[6]{256}$ and $\sqrt[6]{8}$; $\sqrt[6]{4}$ and $\sqrt[6]{\frac{1}{27}}$; $\sqrt[4]{6}$ and $\sqrt[4]{9}$.

11. $2\sqrt{7} > 3\sqrt{3}$; $3\sqrt[3]{3} > 2\sqrt[3]{10}$; $3\sqrt[3]{5} > 4\sqrt[3]{2}$.

12. $\frac{1}{3}\sqrt[4]{27} > \frac{1}{2}\sqrt{2}$; $2\sqrt[3]{\frac{3}{2}} > \sqrt{5}$; $2\sqrt{5} > \sqrt[3]{88}$.

13. $7\sqrt{6} - 29\sqrt{2}$. **14.** $\sqrt{3} + \frac{3}{2}\sqrt{2}$. **15.** $8(a-b)\sqrt{a-b}$.

16. $-2b\sqrt{a-b}$. **17.** $\frac{13}{2}\sqrt[3]{5}$. **18.** $20\sqrt{6}$; $\sqrt{2}$; $4\sqrt{15}$.

19. $\sqrt[6]{32}$; $10\sqrt{3}$; $5 - 2\sqrt{6}$. **20.** $3^{\frac{5}{6}}$; $4\sqrt[6]{54}$; $\sqrt[4]{24}$.

21. $(a-b)\sqrt{a-b} = (a-b)^{\frac{3}{2}}$; $(a-b)^3\sqrt{a-b}$.

22. 1; $(x+y)^{\frac{a+b}{ab}}$. **23.** $\sqrt{2} - \sqrt{3} - \sqrt{5}$; $3\sqrt[3]{2}$. **24.** $\dfrac{a}{2}\sqrt{6}$.

25. $4\sqrt[3]{6}$. **26.** $\sqrt{2}$; $\frac{3}{4}\sqrt{2}$; $\sqrt[3]{2}$; $2\sqrt[4]{2}$; $\sqrt[5]{27}$; $\dfrac{a\sqrt{c} + c\sqrt{a}}{ac}$.

27. $3\sqrt{2}-2\sqrt{3}$; $4+\sqrt{15}$; $7+4\sqrt{3}$; $\frac{1}{5}(2\sqrt{2}-\sqrt{3})$.
28. $\sqrt{5}+1$; $\frac{2}{27}(29+4\sqrt{7})$; $10+4\sqrt{6}$; $\sqrt{2}+1$.
29. $\frac{1}{3}(2\sqrt[3]{2}+\sqrt[3]{4}+1)$; $\sqrt[3]{4}+\sqrt[3]{2}+1$; $\frac{1}{2}(\sqrt[3]{4}-\sqrt[3]{2})$.
30. $\frac{1}{4}(-3\sqrt{2}+2\sqrt{3}+\sqrt{30})$; $\frac{1}{4}(\sqrt{10}+2-\sqrt{14})$; $\frac{1}{2}(\sqrt{5}+\sqrt{3}+\sqrt{2})$. **31.** $-\frac{9}{4}\sqrt{-2}$. **32.** 18.
33. $-24\sqrt{3}$; $2\sqrt{-3}-2$; -7. **34.** $2\sqrt{-2}$; $\frac{3}{4}\sqrt{6}$; $12\sqrt{2}$.
35. $11\sqrt[4]{-1}$. **36.** $-4\sqrt{7}$. **37.** $5(\sqrt{3}-\sqrt{2})$. **38.** $2\sqrt{2}$.
39. $\frac{1}{2}(\sqrt{6}-\sqrt{3}-5)$. **40.** $\frac{1}{2}[3\sqrt{2}+4+\sqrt{6}-2\sqrt{3}]\sqrt[4]{2}$.
41. $\sqrt{2}-1$; $2-\sqrt{2}=(\sqrt{2}-1)\sqrt{2}$; $\sqrt{5}-\sqrt{2}$; $2\sqrt{2}-1$.
42. $\sqrt{5}+\sqrt{3}$; $\sqrt{6}-1$; $\sqrt{7}-2$; $\frac{1}{2}(\sqrt{6}-\sqrt{2})$. **43.** 16.
44. $\frac{3}{10}$. **45.** 3 or 0. **46.** 37. **47.** 4. **48.** $-\frac{3}{4}$. **49.** $\frac{16a}{5}$.
50. $\frac{1}{3}$. **51.** $(a+b)^2$. **52.** $\frac{2b}{2-b}$. **53.** 3. **54.** $\pm\frac{6}{5}\sqrt{5}$.
55. $\pm\frac{1}{2}\sqrt{4ab-b^2}$. **56.** 4. **57.** 6. **58.** 5. **59.** $3a$. **60.** 9.
61. 5. **62.** $\pm\frac{1}{2}$. **63.** $\frac{16}{25}$. **64.** $\frac{3}{2}$.

Exercise XX.

1. $3, \frac{1}{4}$. **2.** $1, -\frac{3}{2}$. **3.** $\frac{3}{5}, \frac{1}{3}$. **4.** $3, -\frac{4}{3}$. **5.** $10, -2$. **6.** $7, -\frac{7}{5}$. **7.** $3, -\frac{4}{5}$. **8.** $8, -44$. **9.** $1, -8$. **10.** $6, -\frac{17}{3}$. **11.** $6, -\frac{9}{2}$. **12.** $7, -1\frac{1}{2}$. **13.** $\frac{3}{2}(-3\pm\sqrt{3})$. **14.** $\frac{1}{8}(27\pm\sqrt{57})$. **15.** $2, \frac{1}{2}$. **16.** $2, \frac{1}{3}$. **17.** $-\frac{1}{2}$ or $-4\frac{1}{2}$. **18.** $3, -\frac{4}{3}$. **19.** $3, -\frac{1}{25}$. **20.** $\frac{5}{3}, -\frac{3}{2}$. **21.** $2, \frac{12}{25}$. **22.** 21, 5. **23.** $7, -\frac{10}{7}$. **24.** $3, -\frac{87}{10}$. **25.** $9, \frac{1}{4}$. **26.** $64^2, 49^2$. **27.** 1, 9. **28.** 9, 4. **29.** 5, 80. **30.** $a-b, c$. **31.** $a-b, \frac{a+b}{2}$. **32.** $\frac{a}{b+c}, -\frac{2a}{3(b+c)}$. **33.** $-\frac{2}{5}, -3\frac{3}{5}$. **34.** $\frac{1}{13}, \frac{1}{60}$. **35.** $a\pm\frac{1}{a}$. **36.** $2a, -b$. **37.** $3, -\frac{11}{4}$. **38.** $1, -\frac{10}{3}$.
39. $\frac{a\pm b}{2}$. **40.** $\frac{a+b}{a-b}, \frac{a-b}{a+b}$.

EXERCISE XXI.

1. 7, 3. **2.** 11, -12. **3.** 11, 5. **4.** \$60 or \$40. **5.** 16. **6.** $\frac{a}{2}(3 \pm \sqrt{5})$ and $-\frac{a}{2}(1 \pm \sqrt{5})$. **7.** The square, 10 rods; the rectangle, 16 rods. **8.** 50 cents, 10 cents. **9.** A franc, 20 cents; a mark, 25 cents. **10.** \$$1\frac{1}{4}$. **11.** A's, \$600; B's, \$400. **12.** 12 hours, 16 hours. **13.** \$40 per week. **14.** 37. **15.** 25. **16.** 2 P.M., or 12 M. **17.** $10\frac{10}{11}$ minutes past 2. **18.** 4 miles an hour. **19.** 90 miles. **20.** 20 miles. **21.** $13\frac{1}{3}$ hours. **22.** $13\frac{1}{3}$ hours. **23.** Height 4 yards, breadth 6 yards, length 8 yards. **24.** \$600, \$400.

EXERCISE XXII.

1. $x^2 - 7x = -12$. **2.** $x^2 - x = 6$. **3.** $x^2 - \frac{5}{6}x = -\frac{1}{6}$. **4.** $x^2 - \frac{1}{6}x = \frac{1}{3}$. **5.** $x^2 = 25$. **6.** $x^2 - 4x = -1$. **7.** $x^2 - 3x = -\frac{3}{4}$. **8.** $x^2 - 2x = -4$. **9.** $x^2 - 2ax = b^2 - a^2$. **10.** $x^2 - 2x\sqrt{a} = b - a$. **11.** $x^3 + 3x^2 - 10x = 24$. **12.** $x^3 - 5x^2 + 2x = -12$. **13.** $x^4 - 2x^3 - 6x^2 + 8x = -8$. **14.** $x^2 - (2a+1)x = (1-a)(a+2)$. **15.** $x^3 - 3ax^2 + x(3a^2 - 1) = a^3 - a$. **16.** $x^4 - 4ax^3 + x^2(6a^2 - 5) - 2ax(2a^2 - 5) = -a^4 + 5a^2 - 4$. **17.** $(x-4)(x-2) = 0$, $\therefore x = 4$ or 2. **18.** 4, -2. **19.** 4, -3. **20.** -1, $\frac{1}{2}(1 \pm \sqrt{-3})$. **21.** 1, $\frac{2}{3}$. **22.** $\frac{1}{2}$, $\pm\frac{1}{3}\sqrt{3}$. **23.** a, $\frac{1}{5}(3-5a)$. **24.** $1 \pm \sqrt{-2}$, $-1 \pm \sqrt{-2}$. **25.** Minimum $\frac{7}{4}$, when $x = \frac{3}{2}$. **26.** Minimum $-\frac{1}{4}$, when $x = -\frac{3}{2}$. **27.** Maximum $\frac{1}{12}$, when $x=7$. **28.** Minimum positive value 1, when $x = 0$.

EXERCISE XXIII.

1. ± 2, $\pm\sqrt{2}$. **2.** $\pm\sqrt{a}$, $\pm\sqrt{b}$. **3.** ± 1, ± 8. **4.** 4, 1. **5.** 64, 4. **6.** 16, 1. **7.** 729, 1. **8.** $\frac{1}{9}$, $(-1)^2 = 1$. **9.** $\frac{1}{16}$, $(-\frac{5}{11})^4$. **10.** 1, $(\frac{3}{7})^{\frac{4}{3}}$. **11.** $\frac{1}{8}$, $\frac{1}{27}$. **12.** $\frac{1}{2^n}$, $(-1)^n$. **13.** 2,

-1. **14.** $9, (-41)^{\frac{2}{3}}$. **15.** $2\sqrt[3]{2}, 5^{\frac{4}{3}}$. **16.** $\pm\sqrt[4]{-2}, \pm 1$. **17.** $4, -1$. **18.** $18, 3$. **19.** $5, 80$. **20.** $3, \frac{13}{2}$. **21.** 1. **22.** $4, -9, \frac{1}{2}(-5 \pm \sqrt{-51})$. **23.** $2, 0, 1 \pm \sqrt{-7}$. **24.** $2, -\frac{1}{2}, \frac{1}{4}(3 \pm \sqrt{505})$. **25.** $\frac{1}{2}(3 \pm \sqrt{5}), -2 \pm \sqrt{3}$. **26.** $-2 \pm \sqrt{5}, \frac{1}{2}(1 \pm \sqrt{5})$. **27.** $\frac{1}{2}(8 \pm 3\sqrt{7}), \frac{1}{2}(15 \mp 4\sqrt{14})$. **28.** $\frac{1}{2}\sqrt{10 \pm 2\sqrt{21}} = \pm\frac{1}{2}(\sqrt{7} \pm \sqrt{3}), (-2 \pm \sqrt{3})^{\frac{1}{2}} = \pm\frac{1}{2}(\sqrt{-2} \mp \sqrt{-6}) = \frac{1}{2}(1 \mp \sqrt{3})\sqrt{-2}$. **29.** $1, 2\frac{1}{2}, -3\frac{1}{2}$. **30.** $\pm\sqrt{3}, \pm\sqrt{\frac{1}{2}(-3 \pm \sqrt{-15})}$. **31.** $\frac{4}{3}, -\frac{1}{3}, -1$. **32.** $\frac{1}{2}(3 \pm \sqrt{5}), 1$. **33.** $\frac{1}{2}(1 \pm \sqrt{-7}), \frac{1}{2}(-1 \pm \sqrt{-15})$. **34.** $\frac{1}{2}(1 \pm \sqrt{-19}), \frac{1}{2}(-1 \pm \sqrt{5})$. **35.** $\frac{1}{4}(1 \pm \sqrt{-23}), \frac{1}{4}(-1 \pm \sqrt{41})$. **36.** $\frac{3}{2}(1 \pm \sqrt{-3}), \frac{1}{2}(7 \mp 3\sqrt{5})$. **37.** $1 \pm \sqrt{2}, 1 \pm 2\sqrt{-1}$. **38.** $1, -5, -2 \pm \sqrt{2}$. **39.** $\pm\frac{1}{2}\sqrt{10 \pm 2\sqrt{33}}, \pm\frac{1}{2}\sqrt{10 \pm 2\sqrt{-23}}$. **40.** $\frac{1}{4}(1 \pm \sqrt{33}), \frac{1}{4}(1 \pm \sqrt{-7})$. **41.** $2(1 \pm \sqrt{2}), 1 \pm \sqrt{-3}$. **42.** $\frac{1}{2}(-1 \pm \sqrt{5}), \frac{1}{2}(5 \pm \sqrt{21})$. **43.** $\frac{1}{4}(3 \pm \sqrt{41}), \frac{1}{4}(1 \pm \sqrt{-31})$. **44.** $\frac{1}{6}(13 \pm \sqrt{493}), \pm 3\sqrt{-1}$. **45.** ± 2. **46.** $\frac{1}{2}(1 \pm \sqrt{5})$. **47.** $3, \frac{3}{2}(-1 \pm \sqrt{-3})$. **48.** $\frac{3a-1}{12(a+1)}$. **49.** $\pm\sqrt{2ac-c^2}, 0$. **50.** $\pm\frac{2}{\sqrt{4a-1}}$. **51.** 2. **52.** $\frac{50}{21}, \frac{2}{3}$.

Exercise XXIV.

1. $x = \pm 3;\ y = \pm 2$. **2.** $x = 6, 4;\ y = 3, 1$. **3.** $x = \pm 10, \pm 3;\ y = \mp\frac{17}{2}, \pm 2$. **4.** $x = 4, \frac{7}{3};\ y = 2, 3\frac{1}{9}$. **5.** $x = \pm 5;\ y = \pm 2$. **6.** $x = \pm 4;\ y = \pm 2$. **7.** $x = 5, 4;\ y = 4, 5$. **8.** $x = 5, -1;\ y = 1, -5$. **9.** $x = 27, -8;\ y = 8, -27$. **10.** $x = 125, 1;\ y = 1, 125$. **11.** $x = 8, 2;\ y = 2, 8$. **12.** $x = 1 \pm 2\sqrt{3}, \frac{1}{2}(-1 \pm \sqrt{51});\ y = -1 \pm 2\sqrt{3}, \frac{1}{2}(1 \pm \sqrt{51})$. **13.** $x = 8, 2;\ y = 2, 8$. **14.** $x = 3, 1;\ y = 1, 3$. **15.** $x = 2, \frac{1}{4}(-1 \pm \sqrt{-39});\ y = -1, \frac{1}{4}(5 \mp \sqrt{-39})$. **16.** $x = 3, \frac{3}{4}(1 \pm \sqrt{-15});\ y = 0, \frac{3}{4}(-3 \pm \sqrt{-15})$.

17. $x = 2$; $y = 1$. **18.** $x = 4, -1$; $y = -1, 4$. **19.** $x = 4, -2$; $y = 2, -4$. **20.** $x = 3, -1, \frac{1}{14}(-19 \pm \sqrt{-283})$; $y = 1, -3, \frac{1}{14}(19 \pm \sqrt{-283})$. **21.** $x = 2, -1 \pm \sqrt{2}$; $y = \pm 5, (40 \mp 5\sqrt{2})^{\frac{1}{2}}$. **22.** $x = \pm 5$; $y = \pm 3$; $z = \pm 2$. **23.** $x = 1$, $y = 0$, $z = 0$, and these values may be interchanged. **24.** $x = 3$, $y = 1$, $z = \frac{1}{3}$; and these values may be interchanged. **25.** $x = 4$, $y = 2$, $z = 1$, and interchanged. **26.** $\pm 9, \pm 5$. **27.** 5, 2. **28.** 6, 4. **29.** \$56. **30.** 24. **31.** Rates of riding, 6, 8; of walking, 4, 2. **32.** A's investment, \$600; B's, \$400; A's gain, \$60; B's, \$40. **33.** 1, 3, 6.

Exercise XXVI.

1. 2111. **2.** 167. **3.** 858.4. **4.** 5.6417. **5.** $\bar{2}.796$. **6.** 1169.72. **7.** 1.387. **8.** 96.08. **9.** 3.483. **10.** 7.6117. **11.** 1.7963. **12.** $\dfrac{\log d - c \log a}{b \log a}$. **13.** $\dfrac{\log n}{b \log a + d \log c}$. **14.** $x = .8483$. **15.** $\dfrac{\log(\frac{1}{2} \pm b)}{\log a}$. **16.** $\dfrac{\log(a-b)}{\log(a+b)}$. **17.** 4.54 or $-.54$. **18.** -1.7092. **19.** 2.408. **20.** $y^3 - 3y = 2$; $\therefore y = 2$ or -1; $\therefore 2^x = 2$ or -1, $x = 1$.

Exercise XXVII.

1. $\frac{11}{13} > \frac{4}{5}$, $\frac{13}{14} > \frac{25}{27}$. **2.** If $a > b$, $\dfrac{a-b}{a+b} < \dfrac{a^2-b^2}{a^2+b^2}$; if $a = b$, both ratios become zero; if $a < b$, $\dfrac{a-b}{a+b} > \dfrac{a^2-b^2}{a^2+b^2}$. **3.** $x^2 - 11x + 28 : x^2$. **4.** 32 : 27. **5.** 4. **6.** 27, 18. **7.** 20, $4\frac{1}{6}$. **8.** $\frac{20}{63}$. **9.** 18, $13\frac{1}{2}$. **14.** $\dfrac{bc - ad}{a - b - c + d}$.

23. $B = 7A$. **24.** $5AB = 2(A^2 + B^2)$. **25.** $C = \dfrac{b}{a}\sqrt{D^2 - a^2}$. **26.** 2. **27.** $D = -\dfrac{5C + 22}{6C}$. **29.** $D = 2C + \dfrac{2}{C}$. **30.** $S = \frac{1}{2}ft^2$.

EXERCISE XXVIII.

1. $4a^3$. **2.** $\frac{4}{3}a$. **3.** na^{n-1}. **4.** $\dfrac{n(n+1)}{2}$. **5.** 1, 4, 7, 10, 13. **6.** $\pm 1, \pm 3, \pm 5, \pm 7$. **7.** $\pm 2, \pm 5, \pm 8, \pm 11$. **8.** 3, 4, 5. **9.** 10 or 12. **10.** $n(2n+3)$. **11.** n^2. **12.** $\dfrac{n}{2}(n^2 + n - 1)$. **13.** 121. **14.** 108, 144, 192, 256. **17.** 256. **18.** 3, 7, 11, or 18, 7, -4. **19.** 3, 6, 12, 24, 48, . . . **20.** 2, $\frac{2}{3}$, $\frac{2}{9}$, . . . , or 4, $-\frac{4}{3}$, $+\frac{4}{9}$, $-\frac{4}{27}$, . . . **24.** 9, 1 or $\frac{8}{10}$, $-7\frac{2}{10}$. **26.** 36 miles. (B will be overtaken afterwards by A at a distance of 120 miles from the starting-point.) **27.** 64, 48, 36, 27, wine; 0, 16, 28, 37, water.

EXERCISE XXIX.

1. 1956. **2.** 50400. **3.** 3. **4.** 12600. **5.** 9. **6.** $\underline{|10}$. **7.** 6. **8.** 4. **9.** 330, 462, 11. **10.** 31. **11.** 6. **12.** 2. **13.** 15. **14.** $\dfrac{\underline{|95}}{\underline{|9}\,\underline{|86}}$, $\dfrac{\underline{|95}}{\underline{|10}\,\underline{|85}}$. **15.** $2a$. **16.** 1728. **17.** .033. **18.** $\frac{1}{60}$. **19.** A's expectation, \$55; B's, \$25. **20.** \$1.05.

EXERCISE XXX.

1. Integral fractions, $\dfrac{1}{2+}, \dfrac{1}{1+}, \dfrac{1}{4+}, \dfrac{1}{3+}, \dfrac{1}{2+}, \dfrac{1}{2+}, \dfrac{1}{1+}, \dfrac{1}{30}$.
Convergents, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{5}{14}$, $\frac{16}{45}$, $\frac{37}{104}$, $\frac{90}{253}$, $\frac{127}{357}$.

2. Integral fractions, $\dfrac{1}{2+}, \dfrac{1}{3+}, \dfrac{1}{3+}, \dfrac{1}{3+}, \dfrac{1}{2+}, \dfrac{1}{7+}, \dfrac{1}{1+}, \dfrac{1}{1+}, \dfrac{1}{1+}, \dfrac{1}{2}$.
Convergents, $\frac{1}{2}$, $\frac{3}{7}$, $\frac{10}{23}$, $\frac{33}{76}$, $\frac{76}{175}$, $\frac{565}{1301}$, $\frac{641}{1476}$, $\frac{1206}{2777}$, $\frac{1747}{4023}$.

3. Integral fractions, $\frac{1}{6+}, \frac{1}{5+}, \frac{1}{4+}, \frac{1}{3+}, \frac{1}{2}$.
Convergents, $\frac{1}{6}, \frac{5}{31}, \frac{21}{130}, \frac{68}{421}$.

4. Integral fractions, $\frac{1}{1+}, \frac{1}{2+}, \frac{1}{1+}, \frac{1}{1+}, \frac{1}{1+}, \frac{1}{55}$.
Convergents, $1, \frac{2}{3}, \frac{3}{4}, \frac{5}{7}, \frac{8}{11}$.

5. Convergents, $2, 3, \frac{14}{5}, \frac{17}{6}, \frac{82}{29}$, etc. 6. Convergents, $4, \frac{33}{8}, \frac{268}{65}, \frac{2177}{528}$, etc. 7. Convergents, $7, \frac{22}{3}, \frac{29}{4}, \frac{51}{7}$, etc. 8. Convergents, $10, \frac{201}{20}, \frac{4030}{401}, \frac{80801}{8040}$, etc. 9. Convergents, $6, 7, \frac{27}{4}, \frac{34}{5}$, etc. 10. $\frac{709}{1640}$. 11. $1 - x + x^2 - x^3 + x^4 - \ldots$ 12. $1 + x + x^2 + x^3 + \ldots$ 13. $\frac{2}{3} - \frac{8}{9}x + \frac{32}{27}x^2 - \frac{128}{81}x^3 + \ldots$ 14. $1 - x - x^2 + 5x^3 - 7x^4 - \ldots$ 15. $\frac{15}{2(x-5)} - \frac{9}{2(x-3)}$.

16. $\frac{2}{x+1} + \frac{1}{x^2-x-1}$. 17. $\frac{7}{x-7} - \frac{5}{x-5}$. 18. $\frac{1}{2(1-x)} + \frac{1}{2(1+x)} + \frac{1}{1+x^2}$. 19. $x^{-2} - 2x^{-3}y + 3x^{-4}y^2 - 4x^{-5}y^3 + \ldots$ 20. $1 - \frac{x}{3} - \frac{x^2}{9} - \frac{5x^3}{81} - \ldots$ 21. $a^{-\frac{2}{3}} - \frac{2}{3}a^{-\frac{8}{3}}b^2 + \frac{5}{9}a^{-\frac{14}{3}}b^4 - \frac{40}{81}a^{-\frac{20}{3}}b^6 + \ldots$ 22. $+ 1225a^2x^{48}$. 23. 2d and 3d terms, each $\frac{8}{3}$. 24. 4.89898 —. 25. 286. 26. 4^{10}. 27. $3(6)^{15}$. 28. $2a^7 + 42a^5b^2 + 70a^3b^4 + 14ab^6 = 2a(a^6 + 21a^4b^2 + 35a^2b^4 + 7b^6)$.

Exercise XXXI.

1. \$17380. 2. 21.2000. 3. $8\frac{3}{4}$ years. 4. 47 years. 5. \$623.35. 6. \$15375.73. 7. \$20000. 8. \$6440.40. 9. \$1800. 10. \$16903.31. 11. 5.451. 12. \$172. 13. 22.

14. $\frac{\log b - \log (b - rd)}{\log (1 + r)}$.

ANSWERS TO TEST EXAMPLES.

I.

1. 57. **2.** $16a - 9(b-c)$; 109. **3.** $-2(a+b)(a-2b)$. **4.** $6x^3 - x^2 - 5x + 2$. **5.** $c^4 + 8c^3d + 24c^2d^2 + 32cd^3 + 16d^4$. **6.** $(a+b)^3$. **7.** $-a^2 - b^2 - c^2 + ab + ac + bc$. **8.** $2a^2bc^3$. **9.** Four cents. **10.** $11\frac{1}{2}$.

II.

1. 12. **2.** $(a-b)(4a-b)$. **3.** 63. **4.** $a^4 - (b-c)^2a^2 - 2abc(b-c) - b^2c^2$. **5.** $-8a^3 - 27b^3 - c^3 - 36a^2b - 12a^2c - 54b^2a - 27b^2c - 6ac^2 - 9bc^2 - 36abc$. **6.** $(-1)^n$. **7.** $(a-b)^2[(a-b)^2 + (a-b) + 1]$. **8.** $\pm(a-b-c)$. **9.** $(a-b)^3$. **10.** $12\frac{5}{6}$ days.

III.

1. 2. **2.** $2a^2(a-b)$. **3.** $2bc - a^2 - 2ac + c^2$. **4.** $2a^4 + 6a^2b^2 + 2b^4$. **5.** $(a-b-c)^{m+n} - (a-b-c)^n - (a-b-c)^{2n-m}$. **6.** $ab + ac + bc + c^2$. **7.** $\frac{b}{2a}$. **8.** 60. **9.** 90 years. **10.** 36.

IV.

1. $\frac{1}{32}$. **2.** $-4(a-3)$. **3.** $x^3 + (c-b-2)x^2 + (2b-2c-bc)x + 2bc$. **4.** Each equals $a^8 - 1$. **5.** $x^2 - x(y+1) - (-y^2 + y + 1)$. **6.** $x^2 - y^2$. **7.** 0. **8.** 5. **9.** \$60. **10.** $9\frac{3}{13}$ hours.

V.

1. $1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5$. **2.** $2x(3x^4 + 7x^2 + 3)$. **3.** $a^2 + b^2 + 1 - ab - a - b$. **5.** 14. **6.** $12\frac{1}{4}$ days. **7.** 6 P.M. **8.** 324 square rods. **9.** $x = 8$, $y = 9$. **10.** $x = 8$, $y = 3$, $z = -2$.

VI.

1. $x^4 - 5abx^2 + 7a^2b^2$. 2. $7a^3 - 6a^2b - 6ab^2 + 7b^3$. 3. $x^2 - 2xy + y^2$. 4. $a^{\frac{5}{3}} - 5a^{\frac{4}{3}} + 10a - 10a^{\frac{2}{3}} + 5a^{\frac{1}{3}} - 1$. 5. 36 years. 6. $x = 16$, $y = 8$. 7. 3, 0. 8. 5, 3. 9. 15 years. 10. 36 years.

VII.

1. $16a^4 + 96a^3b + 216a^2b^2 + 216ab^3 + 81b^4$. 2. 1.49 +. 3. 6 miles an hour. 4. 42. 5. 80 square rods. 6. 4, $-2\frac{1}{2}$. 7. 3, $-3\frac{1}{2}$. 8. $x \,.\, x \,.\, (x - 1)(x + 1)(x^2 + 1)$. 9. $(x^2 + 3)(1 - x)$. 10. $(x - 2)(x - 9)$.

VIII.

1. $x^2 - 1$. 2. $(a^3 - b^3)(a - b)^2$. 3. $\frac{2}{5}$ or $\frac{1}{5}$. 4. $x^2 - 10x = -22$. 5. $x^3 - x^2 - 4x = -4$. 6. 1, 4, 6. 7. $4 - 2\sqrt{3}$. 8. -2, ± 1. 9. $2 + \sqrt{6}$. 10. $\pm\sqrt{2}$, $\pm\sqrt{-2}$.

IX.

1. $(a + 3)(a + 2)(a - 2)$. 2. $a^4 + 1$. 3. 3, 1. 4. 3, $-\frac{3}{4}$. 5. 10 miles. 6. 3, $\frac{1}{2}(-3 \pm \sqrt{-7})$. 7. $y^2 \mp (2x - 1)y = -x^2 + x + 2$. 8. 4 P.M. or 9 A.M. 9. $-4\sqrt{5}$. 10. $x = \pm 3$ or ± 1, $y = \pm 1$ or ± 3.

X.

1. 25. 2. .3980. 3. 1. 4. ± 5. 6. $\frac{20}{3}\sqrt[3]{4}$. 7. 308. 9. $2\frac{1}{2}$, $\frac{2}{5}$. 10. $80.

XI.

2. $2(\sqrt{3} - \sqrt{2})\sqrt[4]{6}$. 3. 31. 4. 2. 5. 3 or $\frac{4}{7}$. 7. -1. 8. 9, 1. 9. $\frac{3}{10}$. 10. $2\frac{1}{2}$, $3\frac{1}{3}$, 5.

XII.

1. $\frac{1}{4}(\sqrt[3]{25} + \sqrt[3]{5} + 1)$. 2. ± 2. 3. $\frac{1}{2}d \pm \sqrt{(l + \frac{1}{2}d)^2 - 2ds}$. 4. 1, 3, 9. 5. $x = 3.5606$, $y = 5.9343$. 6. 41. 7. $\log x = .57625$, $\log y = .92195$. 8. 26. 9. $30,000. 10. $20,284 —.

XIII.

1. $a^{\frac{2}{3}} + b^{\frac{2}{3}} + c^{\frac{2}{3}} - \sqrt[3]{ab} - \sqrt[3]{ac} - \sqrt[3]{bc}$. 2. $x^{\frac{2}{3}} + 1 + x^{-\frac{2}{3}}$. 3. 11. 4. $(x-2)(x^2+2x+4)(x+1)(x^2-x+1)$. 5. $4x^{-2} - 4x^{-1} + 1$. 6. ± 5, ± 4. 7. $a^2 - 12a^{\frac{3}{2}}b + 54ab^2 - 108a^{\frac{1}{2}}b^3 + 81b^4$. 8. $16a^{-2} - 96a^{-\frac{3}{2}}b^{-\frac{2}{3}} + 216a^{-1}b^{-\frac{4}{3}} - 216a^{-\frac{1}{2}}b^{-2} + 81b^{-\frac{8}{3}}$.

9. $(a^4 + \frac{1}{16})(a^2 + \frac{1}{4})(a + \frac{1}{2})(a - \frac{1}{2})$. 10. $\sqrt[6]{13500}$.

XIV.

1. $x^4y - x^3y^2 + x^2y^3 - xy^4$. 2. $\pm \frac{7}{5}$. 3. $\pm(5 - \sqrt{3})$.

4. $1 \pm \frac{3x}{2} - \frac{9}{8}x^2 \pm \frac{27}{16}x^3 - \ldots$ 5. $13\sqrt{3} - 18$. 6. 80.

7. $x^2 - 4x = 12$. 8. 5, 8, 11, 14, 17, 20.

9. $x = \frac{1}{2}[a \pm \sqrt{4b - 3a^2}]$, $y = \frac{1}{2}[a \mp \sqrt{4b - 3a^2}]$.

10. $\frac{n(n+1)}{2}\frac{(n+2)}{3}\frac{(n+3)}{4}x^{-n-4}a^4$.

XV.

1. 5, -30. 2. 3, $\frac{3}{2}(-1 \pm \sqrt{-3})$. 3. $\frac{4}{3}$. 4. $\frac{1}{5}$. 5. $-\infty$. 6. $.449+$. 7. $\frac{26}{73}$. 8. $\frac{3}{5} - \frac{11}{25}x + \frac{77}{125}x^2 - \frac{539}{625}x^3 \ldots$ 9. $a^{11} - 11a^{10}b + 55a^9b^2 - 165a^8b^3 + 330a^7b^4 - 462a^6b^5 + 462a^5b^6 - 330a^4b^7 + 165a^3b^8 - 55a^2b^9 + 11ab^{10} - b^{11}$.
10. $-a$, $-b$.

XVI.

1. $\bar{1}.8997$. 2. $a^{-\frac{1}{2}n} + \frac{1}{2}a^{-\frac{3}{2}n}b^x + \frac{3}{8}a^{-\frac{5}{2}n}b^{2x} + \frac{5}{16}a^{-\frac{7}{2}n}b^{3x} +$. 3. 1. 4. $10x^2 - 7x = -1$. 5. $(a+b)^2 + (a+b)c + c^2$.

6. $x : y :: 5 : 2$. 7. $\frac{2}{x^3 - x}$. 9. b^{-1}, $-a^{-1}$.

XVII.

1. $\pm \frac{1}{2}(\sqrt{6} - \sqrt{2})$. 2. ± 13. 3. $(a^2 + ab + b^2)(a + b)$.

4. $\frac{10}{(x^2 - 25)(x^2 + 2x - 5)}$. 5. $34x^2$. 6. $32x^5 - 80x^4a +$

$80x^3a^2 - 40x^2a^3 + 10xa^4 - a^5$. **7.** $x = 2, \frac{1}{2}, -\frac{1}{3}, -3$. **8.** $3 - 2\sqrt{2}, 3 - \sqrt{2}, 3, 3 + \sqrt{2}, 3 + 2\sqrt{2}$. **9.** $x = \pm\frac{5}{2}$, $y = \pm\frac{3}{2}$. **10.** $a^{\frac{1}{2}} - \sqrt[4]{ab} + b^{\frac{1}{2}}$.

XVIII.

1. 2.53. **2.** Minimum $\frac{3}{2}$. **3.** $x = 5$ or 2, $y = 2$ or 5. **4.** $x = 5$ or 3; $y = 3$ or 5; $z = 2$. **5.** $-(1 - a)^2$. **6.** $x^{-\frac{1}{2}} - x^{-\frac{1}{4}}y^{-\frac{1}{4}} + y^{-\frac{1}{2}}$. **7.** $\frac{a}{bc}$. **8.** $6\frac{3}{4}$. **9.** $\frac{1}{22}$. **10.** $\frac{4}{3}, -\frac{1}{3}, \frac{1}{50}(9 \pm \sqrt{481})$.

XIX.

1. $3, \frac{1}{2}(-3 \pm \sqrt{-67})$. **2.** -2. **3.** $\frac{1}{8}x$. **4.** $-x^{-1}y^{-\frac{1}{3}} - x^{-\frac{2}{3}}y^{-\frac{2}{3}} - x^{-\frac{1}{3}}y^{-1}$. **5.** $4, \frac{5}{6}$. **6.** $4\frac{1}{3}$ miles an hour. **7.** $-70b^{13}$. **8.** $\pm\sqrt[4]{-3 \pm 2\sqrt{3}}$. **9.** $\frac{1}{5 - x} - \frac{1}{4 + x}$. **10.** 1, 3, 5, 7, . . .

XX.

1. 696. **2.** $a, 3a, 5a, 7a$, . . . **3.** $-\frac{1}{5}, -\frac{1}{7}, -\frac{1}{9}, -\frac{1}{11}, -\frac{1}{13}$. **4.** $x = 4$ or 2, $y = 2$ or 4. **5.** $23 - 10\sqrt{-2}$. **6.** $1.5372 +$. **7.** $x = \frac{1}{2}$ or $\frac{1}{3}$, $y = \frac{1}{3}$ or $\frac{1}{2}$.

8. { 8640, if no repetitions of consonants be allowed;
12000, if repetitions of consonants be allowed.

9. 109.