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COMPOSITE PERFORMANCE MEASURES AND RISK PROXIES: SAMPLE SIZE, INVESTMENT HORIZON AND MARKET CONDITION

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SAMPLE SIZE, INVESTMENT HORIZON AND MARKET CONDITION

by

Son-Nan Chen
Assistant Professor of Management
Jackson State University
Jackson, MS 39217

and

Cheng F. Lee
Associate Professor of Finance
University of Illinois
at Urbana-Champaign
Urbana, IL 61801

ABSTRACT

The statistical relationships between estimated composite performance measures and their risk proxies are derived in accordance with statistical distribution theory. It is found that the estimated composite performance measures are generally highly correlated with their risk proxies. In general, sample size, investment horizon and the market condition are three important factors in determining the degree of relationship above-mentioned. It is shown that a large historical observation and an appropriate investment horizon can generally be used to improve the usefulness of composite performance measures in both portfolio and mutual fund managements.

COMPOSITE PERFORMANCE MEASURES AND RISK PROXIES: SAMPLE SIZE, INVESTMENT HORIZON AND MARKET CONDITION

I. Introduction

Following the capital asset pricing theory developed by Sharpe (1964), Lintner (1965) and Mossin (1966) [SLM], Sharpe (1966), Treynor (1965) and Jensen (1968, 1969) have derived the composite performance measures--Sharpe, Treynor and Jensen performance measures for evaluating the performance of either portfolios or mutual funds. Friend and Blume (1970) [FB] have discussed the theoretical rationale of these one-parameter performance measures and their relationships by using the capital asset pricing theory. In addition, FB have also empirically shown that the risk-adjusted rates of return as measured by the composite performance measures are not successfully abstract from risk. In other words, they have found that the estimated composite performance measures are generally significantly correlated with the estimated risk proxies. Hence, FB have concluded that there exists strong biases associated with the estimated composite performance measures. Klemkosky (1973) has employed mutual funds instead of random portfolios to re-examine the biases of the estimated composite performance measures and found that there exists a relatively strong relationship between the estimated composite performance measures and estimated risk proxies. However, the possible sources of biases associated with estimated composite performance measures have not been carefully investigated.

The main purpose of this paper is to investigate the possible sources of bias associated with the empirical relationship between the estimated

composite performances and estimated risk proxies.[1] It is shown that sample size and the investment horizon are two important factors in determining the degree of empirical relationship between estimated composite measures and estimated risk proxies. In addition, it is also shown that the above-mentioned empirical relationships are generally not independent of the market condition associated with the sample period selected for the empirical studies. In the second section, the statistical relationships between estimated Sharpe measures and its estimated risk proxies are derived. It is found that the sample size and market condition are two important factors in determining the empirical relationship between Sharpe's measure and its risk proxy. In the third section, the impact of investment horizon on the bias associated with testing the theoretical relationship between the Sharpe measure and its risk proxy is developed. In the fourth section, it is shown that the conclusions associated with the relationship between estimated Sharpe's measure and its risk proxy can easily be extended to those of Treynor and Jensen performance measures. The implications of the statistical relationships between estimated composite performances and their risk proxies on portfolio managements are also explored. Finally, the results of the paper are summarized.

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[1] The term "bias" used in this study refers to the deviation of the empirical relationship from the theoretical relationship. Theoretically, one parameter performance measures are not expected to depend upon their risk proxies. However, it is empirically found that the estimated composite performance measures are generally highly correlated with their estimated risk proxies. To test bias associated with the capital asset pricing theory, Black, Jensen and Scholes (1972), Blume and Friend (1973), Fama and MacBeth (1973) and others have done numerous empirical studies. Most recently, Roll (1977) has carefully re-examined these empirical tests.

II. Statistical Relationship Between Estimated Sharpe Measure and Its Risk Proxy

Following Friend and Blume (1970), the theoretical relationship of the capital asset pricing model [CAPM] developed by SLM can be defined as:

$$E(R_i) - R_f = \alpha_i + \beta_i \,[E(R_m) - R_f] \tag{1}$$

where α_i is a measure of disequilibrium of portfolio or asset i, R_f is the risk-free rate for borrowing or lending, R_i is the rate of return of portfolio or asset i, and R_m is the market rate of return. If the risk-free rate R_f and the index of systematic risk β_i are constant overtime, then equation (1) can be rewritten in ex post or historical data as [see Jensen (1968)]:

$$R_{it} - R_f = \alpha_i + \beta_i \,[R_{mt} - R_f] + \varepsilon_{it} \tag{2}$$

where R_{it} is the rate of return for portfolio or asset i in period t, R_{mt} is the market rate of return in period t and ε_{it} is a random disturbance with mean zero and variance σ_ε^2 and is independent of R_{mt}. If n observations are used to estimated the parameters by ordinary least squares (OLS), equation (2) can be summed over n and averaged to obtain:

$$\bar{R}_i - R_f = \hat{\alpha}_i + \hat{\beta}_i \,[\bar{R}_m - R_f] \tag{3}$$

where the bar indicates an average and $\hat{\alpha}_i$ and $\hat{\beta}_i$ are least-squares estimates of α_i and β_i respectively. Assume that the standard deviation of the rates of return of portfolio i is constant over time. Then estimated Sharpe's measure can be derived from equation (3) as:

$$\frac{\bar{R}_i - R_f}{S_i} = \frac{\hat{\alpha}_i}{S_i} + \frac{[\bar{R}_m - R_f]}{S_i} \tag{4}$$

where S_i is the sample standard deviation of ex post data for portfolio

i and defined as:

$$S_i = \sqrt{\frac{\sum_{t=1}^{n} (R_{it} - \overline{R}_i)^2}{n}} \qquad (5)$$

If the rates of return of securities in the population (capital market) is normally distributed with mean μ and variance σ^2. Then a sample of n rates, R_{it}, of return of portfolio i can be considered as a random sample drawn from the normal population. Assume that the holding period coincides with the true investment horizon.[2] It follows that the sampling distribution of the average rate $\overline{R}_i$ of return, $\overline{R}_i = \sum_{t=1}^{n} R_{it}/n$, is normally distributed with mean μ and variance σ^2/n, where n is the sample size. The random variable nS_i^2/σ^2 has a χ^2--distribution with (n - 1) degrees of freedom [see Hogg and Craig(1970)]. For simplicity, $\overline{R}$ and nS^2/σ^2 denote $\overline{R}_i$ and nS_i^2/σ^2 respectively.

To investigate the degree of relationship between the estimated Sharpe's measure and the estimated risk measure, the probability density function (p.d.f.) of the estimated risk (S) should first be derived. Following Hogg and Craig (1970), the p.d.f. of the random variable $nS^2/\sigma^2(y_1)$, can be defined as

$$f_1\ (y_1) = \begin{cases} \dfrac{1}{\Gamma(\frac{n-1}{2})\ 2^{(n-1)/2}} \cdot y_1^{\frac{n-1}{2}-1} \cdot e^{-y_1/2} & \text{for } o < y_1 < \infty \\ 0 \text{ elsewhere} \end{cases} \qquad (6)$$

let $y_2 = \frac{\sigma\sqrt{y_1}}{\sqrt{n}}$. Then $y_2 = S$. The Jacobian of the transformation is

$$\frac{dy_1}{dy_2} = \frac{2n}{\sigma^2}\, y_2.$$

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[2] The "Investment Horizon" concept will be explored in Section III.

Thus, the p.d.f. of the estimated risk measure S is:

$$f_2\,(y_2) = \frac{1}{\Gamma(\frac{n-1}{2})\quad 2^{(n-1)/2}} \cdot \left[\frac{n}{\sigma^2}\,y_2\right]^{\frac{n-1}{2}-1} \cdot e^{\frac{ny_2^2}{2\sigma^2}} \cdot \frac{2n}{\sigma^2}\,y_2$$

which can be simplified as:

$$f_2\,(S) = \begin{cases} \dfrac{(\frac{n}{\sigma^2})^{\frac{1}{2}}}{\Gamma(\frac{n-1}{2})\quad 2^{(n-3)/2}} \cdot (\dfrac{nS^2}{\sigma^2})^{\frac{n}{2}-1} \cdot e^{-\frac{1}{2}(\frac{nS^2}{\sigma^2})}, & o < S < \infty \\ 0 \text{ elsewhere} & \end{cases} \tag{7}$$

Now the covariance and the correlation coefficient between the estimated Sharpe measure and the estimated risk proxy are derived as follows:

(i) The covariance is defined as

$$\mathrm{Cov}(\frac{\bar{R} - R_f}{S}, S)$$

$$\text{(a)} \quad = E(\bar{R} - R_f) - E(\frac{\bar{R} - R_f}{S})E(S)$$

$$\text{(b)} \quad = E(\bar{R} - R_f)\ .\ [1 - E(\frac{1}{S})E(S)]$$

$$\text{(c)} \quad = d(R_f - \mu), \tag{8}$$

where $\mu = E(\bar{R}) = \bar{R}_m$ = average market rates of return

$$d \quad = E(S)E(\frac{1}{S}) - 1 = \frac{\Gamma(n/2)\Gamma(\frac{n-2}{2})}{[\Gamma(\frac{n-1}{2})]^2} - 1 > o \qquad {}^{3} \tag{9}$$

$$E(\frac{1}{S}) = (\frac{n}{2\sigma^2})^{\frac{1}{2}}\ .\ \frac{\Gamma(\frac{n-2}{2})}{\Gamma(\frac{n-1}{2})}\ , \tag{10}$$

$$E(\frac{1}{S^2}) = (\frac{n}{2\sigma^2})\quad \frac{\Gamma(\frac{n-3}{2})}{\Gamma(\frac{n-1}{2})} \tag{11}$$

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[3] Following Hogg and Craig (1970), it can be shown that d converges to zero in probability. The result of d being positive for n > 2 is derived in the appendix (B).

and $$E(S) = (\frac{2\sigma^2}{n})^{\frac{1}{2}} \frac{\Gamma(n/2)}{\Gamma(\frac{n-1}{2})} \quad ^4 \tag{12}$$

The quality of (8b) is obtained by using the stochastical independence property between $(\overline{R} - R_f)$ and S. The independence property can be justified either intuitively or methodologically. Intuitively, the $\mathrm{Cov}(\overline{R} - R_f, S)$ will approach zero as n becomes large. Methodologically, the joint p.d.f. of $(\overline{R} - R_f)$ and S can be derived and it can be shown that the joint p.d.f. of $(\overline{R} - R_f)$ and S can be written as a product of marginal p.d.f. of $(\overline{R} - R_f)$ and conditional p.d.f. of $(\overline{R} - R_f)$ given S. Note that the stochastical independence assumption does not necessarily contradict to the capital market theory. The Capital Market line (CML) is a linear relationship between the expected return and total risk (σ) on which only efficient portfolios will lie. This is based upon some assumptions. Note that the population from which a random sample of n rates R_{it}, $1 \leq t \leq n$, of return is drawn consists of individual investments and all possible portfolios. This population is regarded theorectically as the opportunity set--the set of investment opportunities. The risk-return relationship exists for all efficient portfolios lying on the capital market line and the efficient frontier of the opportunity set. In general, no relationship between the expected return and risk can be described for all individual investments and portfolios in the opportunity set. This can be seen from the graphic representation of the opportunity set. The opportunity set in the risk return space, $[\sigma, E(R)]$, is escalloped, quarter-moon-shaped. Hence, no functional form can describe the risk-return relationship for all individual investments and portfolios in the opportunity set.

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[4] Equations (10), (11), and (12) are derived from equation (7). See appendix (A) for the details.

(ii) The correlation coefficient is defined as

$$\rho\left(\frac{\bar{R} - R_f}{S}, S\right) = \frac{d(R_f - \mu)}{\sigma_{sp}\,\sigma_s}^{5} \tag{13}$$

where $\sigma_{sp}^2 = \mathrm{Var}\left(\frac{\bar{R} - R_f}{S}\right)$

$$= \frac{\Gamma(\frac{n-3}{2})}{2\Gamma(\frac{n-1}{2})} + (\mu - R_f)^2 \left(\frac{n}{2\sigma^2}\right) \left\{ \frac{\Gamma(\frac{n-3}{2})}{\Gamma(\frac{n-1}{2})} - \left[\frac{\Gamma(\frac{n-2}{2})}{\Gamma(\frac{n-1}{2})}\right]^2 \right\} \tag{14}$$

$$\sigma_s^2 = \mathrm{Var}\,(S) = \frac{\sigma^2}{n} \left\{ (n - 1) - 2 \left[\frac{\Gamma(n/2)}{\Gamma(\frac{n-1}{2})}\right]^2 \right\} \tag{15}$$

Following from equations (8) and (13), one can see that the estimated Sharpe's measure is uncorrelated with the estimated risk measure only either when the risk-free rate is equal to the mean rate, μ, of the return on the market portfolio or when sample size n is infinite. But the risk-free rate is generally not equal to the mean rate of return on the market portfolio and the sample size associated empirical work is generally finite. These indicate that the estimated Sharpe's measure is in general correlated with the estimated risk measure (S) even when the holding period does coincide with the true investment horizon. Therefore, if the risk-free rate is less than the mean rate of return on the market portfolio, the estimated Sharpe's measure is negatively correlated with the estimated risk measure. Conversely, the estimated Sharpe's measure and the estimated risk measure are positively correlated if the risk-free rate is greater than the mean rate of return on the market portfolio.

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[5] Derivation of the correlation coefficient is given in the appendix (C).

[6] The mean μ is considered as the mean rate of return on the market portfolio which is unanimously wanted by all investors.

The quantity d shown in the covariance defined as in (9) is a decreasing function of the sample size n.[7] The value of d is .010 for the sample size of 50; .005 for 100. Thus the covariance defined as in (8) gets smaller when the sample size increases. This implies that the bias (indicated by nonzero covariance) associated with the estimated Sharpe's measure can be reduced by choosing a large sample of size n. Hence, to reduce the bias in the empirical research, a large sample of the rates of return is suggested to be drawn if the cost of sampling is permitted.

The analyses derived in this section have statistically demonstrated why both Friend and Blume (1970) and Klemkosky (1973) have found that the estimated Sharpe performance measure and its risk proxy is highly correlated. Now, the sign associated with the empirical relationship between the estimated Sharpe measure and its risk proxy is discussed. As the risk-free rate proxy--Treasury bill rates and the market rates of return proxy--rates of return associated with New York Stock, are measured with error, the sign associated with estimated $(R_f - \mu)$ is hardly determined.[8] Finally, it should be noted that the t statistic is generally used to test whether the estimated Sharpe's measure is significantly related with its risk proxy. The assumption of using t statistic to perform this kind of test is that the estimated Sharpe measure is normally distributed. Unfortunately, the estimated Sharpe measure does not belong to either a normal distribution or any other well-known distribution.[9]

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[7] d is shown as a decreasing function of n in appendix (B).

[8] The discussion of measurement errors associated with R_f can be found in Roll (1969), Friend and Blume (1970) and Lee and Jen (1977); the justification of measurement errors associated with μ can be found in Miller and Scholes (1970), Black, Jensen and Scholes (1972). Lee and Jen (1977) and Roll (1977).

[9] The distribution of estimated Sharpe's measure is derived in appendix (D).

It is expected that the distribution associated with estimated Sharpe's performance measure is highly skewed. As the t test is generally not robust to extreme skewed distribution. A distribution-free test may well be more appropriate to be used to test the relationship between the estimated Sharpe performance and its risk proxy.

In the following section the impact of investment horizon on testing the relationship between Sharpe's performance measure and its risk proxy is studied.

III. Impact of Investment Horizon on Testing the Bias of Estimated Sharpe's Measure

Observation horizon used in the empirical study of portfolio management is generally not necessarily identical to the true horizon.[10] Levy (1972) has found that the portfolio with the smallest variance (or mean) will have the highest performance index for an observation investment horizon being longer than the true investment horizon. The portfolio with the highest risk (and hence the highest mean) tends to have the highest performance index if an observation investment horizon is shorter than the true investment horizon. This implies that the covariance and correlation coefficient between the estimated performance index and its risk measure may well be affected if the observation investment horizon used in the empirical research does not coincide with the true investment horizon. The impact of the investment horizon on the degree of relationship between the estimated Sharpe's measure and its risk proxy is now investigated. Under the assumption of stationary returns over time and all investors having the same investment horizon, one has

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[10] Observation horizon refers either one day, one week, one month, one quarter or one year. The concept of "true" investment horizon implies that investors will all share the same horizon. The justification and the implication of this assumption can be found in either Lee (1976) or Levy (1972).

$$\mu_j = \mu \text{ and } \sigma_j^2 = \sigma^2 \quad (16)$$ [11]

for all $j = 1, 2, \ldots, m$ where μ_j and σ_j^2 are the expected rate of return and the variance associated with j-period observations. Following Tobin (1965), one assumes stationary over time and independence. It can be shown that

$$\tilde{E} = (1 + \mu)^m - 1 \quad (17)$$

and

$$\tilde{\sigma}^2 = [\sigma^2 + (1 + \mu)^2]^m - (1 + \mu)^{2m}, \; m > o \quad (18)$$

where, for the m-period case, $\tilde{E}$ and $\tilde{\sigma}^2$ are the expected rate of return and the variance, respectively, of the market portfolio. For simplicity, it is assumed that the true investment horizon is equal to one period of time (say, one month, one quarter, or one year. etc.). If the investment horizon assumed in empirical research does not coincide with the true investment horizon,[12] the covariance and the correlation coefficient between the estimated Sharpe's measure and the estimated risk measure are obtained as follows. Let a random sample of n rates, R_{jt}, $1 \leq t \leq n$, of return be drawn from the opportunity set. Then for the m-period case the average rate $\bar{R}_j^*$ of return is normally distributed with mean $\tilde{E}$ and variance $\tilde{\sigma}^2/n$. The random variable $nS^{*2}/\tilde{\sigma}^2$ has χ^2-distribution with (n - 1) degrees of freedom.[13] Following the similar analysis in the section II, the covariance and the correlation coefficient. for the m-period case, between the estimated Sharpe's measure and the estimated risk measure S^* can be written as:

$$\begin{aligned} \text{cov}\left(\frac{R^* - R_f^*}{S^*}, S^*\right) &= E(\bar{R}^* - R_f^*) \cdot [1 - E(\tfrac{1}{S^*}) \cdot E(S^*)] \\ &= d[E(\bar{R}^* - R_f^*)] \end{aligned}$$

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[11] μ and σ^2 have the same definitions as in the section II.

[12] In this case, m is not equal to 1.

[13] The sample variance S^{*2} in this section is the estimated total risk ($\tilde{\sigma}^2$) for the m-period case.

$$= d[1 + R_f)^m - (1 + \mu)^m]^{14} \qquad (19)$$

and
$$\rho(\frac{\bar{R}^* - R_f^*}{S^*}, S^*) = \frac{d[(1 + R_f)^m - (1 + \mu)^m]}{\sigma_{sp}^* \ \sigma_s^*} \qquad (20)$$

where d is defined as in (9), R_f^* is the risk-free rate for the m-period case, and

$$\sigma_{sp}^{*2} = \text{Var}\,(\frac{\bar{R}^* - R_f^*}{S^*})$$

$$= \frac{\Gamma(\frac{n-3}{2})}{2\Gamma(\frac{n-1}{2})} + [(1+\mu)^m - (1 + R_f)^m]^2 \cdot (\frac{n}{2\tilde{\sigma}^2}) \cdot \left\{\frac{\Gamma(\frac{n-3}{2})}{\Gamma(\frac{n-1}{2})} - \left[\frac{\Gamma(\frac{n-2}{2})}{\Gamma(\frac{n-1}{2})}\right]^2\right\}, \qquad (21)$$

$$\sigma_s^{*2} = \text{Var}\,(S^*)$$

$$= \frac{\tilde{\sigma}^2}{n}\left\{(n - 1) - 2\,[\frac{\Gamma(n/2)}{\Gamma(\frac{n-1}{2})}]^2\right\} \qquad (22)$$

It can be seen that equations (19) and (20) reduce to equations (8) and (13), respectively, if the observation horizon coincides with the true investment horizon (m = 1). Hence, an "improper" observation horizon will have impact on the degree of association between the estimated Sharpe's measure and the estimated risk measure.[15]

The impact of an improper observation horizon can be first investigated by finding the first derivative of the covariance defined as in (19) with respect to the holding period m. Thus,

$$\frac{d(\text{Cov})}{dm} = d[(1 + R_f)^m \ln(1 + R_f) - (1 + \mu)^m \ln(1 + \mu)]. \qquad (23)$$

---

[14] Similar to (17), $E(R_f^*)$ can be shown as $E(R_f^*) = (1 + R_f)^m - 1$ where R_f is the risk-free rate for the one-period case (the true investment horizon).

[15] An "improper" observation horizon refers to that the time horizon used in empirical research does not coincide with the true investment horizon.

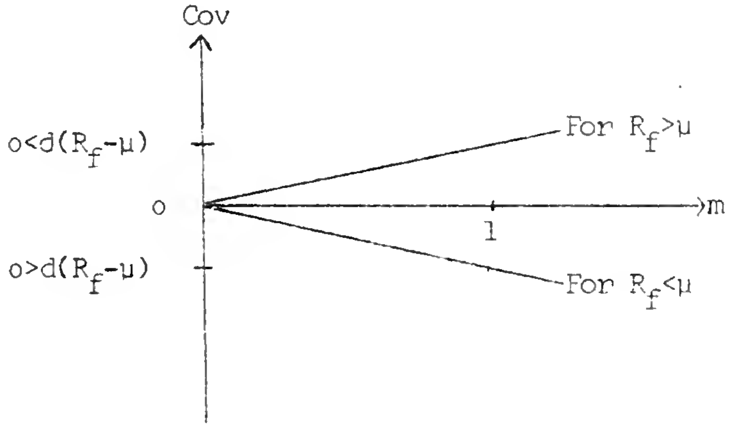
Then if R_f is greater than μ,

$$\begin{aligned}\frac{d(\mathrm{Cov})}{dm} &> d[(1 + R_f)^m \ln (1 + R_f) - (1 + \mu)^m \ln (1 + R_f)], \\ &\quad \text{since } \ln (1 + R_f) > \ln (1 + \mu) \text{ for } R_f > \mu, \\ &= d \cdot \ln (1 + R_f) \cdot [(1 + R_f)^m - (1 + \mu)^m] > o, \ (R_f > o), \\ &\quad \text{since the factors } d, \ \ln (1 + R_f), \text{ and} \\ &\quad [(1 + R_f)^m - (1 + \mu)^m] \text{ are all positive.}\end{aligned} \tag{24}$$

This implies that the covariance between the estimated Sharpe's measure and the estimated risk measure for the m-period case is a strictly increasing function of the number of oberation horizon, m if the risk-free rate R_f is greater than the mean rate μ of return on the market portfolio for the one-period case. Similarily, if R_f is less than μ,

$$\begin{aligned}\frac{d(\mathrm{Cov})}{dm} &< d[(1 + R_f)^m \ln (1 + \mu) - (1 + \mu)^m \ln (1 + \mu)] \\ &= d \ln (1 + \mu) [(1 + R_f)^m - (1 + \mu)^m] < o, \\ &\quad (\mu > R_f > o).\end{aligned} \tag{25}$$

Thus, the covariance for the m-period case is a strictly decreasing function of the number of observation horizon, m if R_f is less than μ. The graphic representation of the relationship between the m-period covariance and the number of observation horizon, m is shown as follows:



The impact of an improper observation horizon on the m-period covariance can be summarized as follows.

(i) A longer observation horizon than the true investment horizon

For the risk-free rate R_f being greater than the mean rate μ of return on the market portfolio, the m-period covariance becomes larger in the positive magnitude by moving the holding period m away from the true investment horizon-one period. Similarly, for R_f being less than μ, the m-period covariance increases its magnitude negatively when the observation horizon is longer than the true investment horizon. Hence, if a longer observation horizon than the true investment horizon is used, the m-period covariance is expected to increase in either positive or negative magnitude depending on the order of the magnitudes of the risk-free rate R_f and the mean rate μ of return on the market portfolio.

(ii) A shorter observation horizon than the true investment horizon

As the observation horizon is shorter than the true investment horizon, the m-period covariance becomes smaller if the risk-free rate R_f is greater than the mean rate of return on the market portfolio. And a weaker negative covariance is expected for the case where R_f is less than μ if the observation horizon approaches zero.

Therefore, in conducting empirical research, an improper observation horizon will have impact on the covariance between the estimated Sharpe's measure and the estimated risk measure. It is interesting to note that a shorter observation horizon than the true investment horizon will reduce the dependence of the estimated Sharpe's performance measure on its estimated risk measure. A longer observation horizon than the true investment horizon will magnify the dependence. These indicate that a shorter observation horizon should be used in the empirical research to reduce the bias associated with the estimated Sharpe's measure.

IV. Some Implications

In the sections II and III, it is shown that the estimated Sharpe's measure is generally highly correlated with the estimated risk proxy. It is also shown that the degree of relationship between the estimated Sharpe's measure is generally affected by the sample size, the market condition and the investment horizon.

Besides the Sharpe's performance measure, Treynor's measure and Jensen's measure are two other popular composite performance measures. Following equation (3), Treynor's performance associated with i^{th} portfolio can be defined as

$$\frac{\bar{R}_i - R_f}{\beta_i} = \frac{\hat{\alpha}_i}{\beta_i} + \hat{\beta}_i[\bar{R}_m - R_f] \qquad (26)$$

Similarly, the Jensen's performance measure associated with i^{th} portfolio can be defined as

$$\hat{\alpha}_i = (\bar{R}_i - R_f) - \hat{\beta}_i[\bar{R}_m - R_f] \qquad (27)$$

From equation (26), it is found that the formulation of Treynor's measure is identical to Sharpe's measure. In addition. Levhari and Levy (1976) have shown that the impact of investment horizon on Treynor's measure is identical to the impact of investment horizon on Sharpe's measure. Hence, it is not unacceptable to conclude that the results derived in the previous sections can be extended to examine the relationship between estimated Treynor's measure and its estimated risk proxy.

Now, the statistical relationship between estimated Jensen's measure and its estimated risk proxy is examined.

Following Heinen (1969) and the definitions defined in this paper, the covariance between estimated Jensen's measure and its estimated risk proxy can be defined as

$$\text{Cov}(\hat{\alpha}_i, \hat{\beta}_i) = \frac{-\bar{\chi}_{mt}\, S_\varepsilon^{\,2}}{\sum_{t=1}^{n} (\chi_{mt} - \bar{\chi})^2} \tag{28}$$

where $\chi_{mt} = R_{mt} - R_f$, $\bar{\chi}_m = \bar{R}_m - R_f$

$S_\varepsilon^{\,2}$ is the residual variance associated with equation (3). Equation (28) indicates that the estimated Jensen performance measure is generally highly correlated with its estimated risk proxy. This relationship has been found by both Friend and Blume (1970) and Klemkosky (1973). It is clear that the degree of relationship can also be affected by the sample size, market condition and the investment horizon.

The results of this study have shown that the estimated composite performance measures are generally highly correlated with its estimated risk proxies. In general, sample size, investment horizon and the market condition are three important factors in determining the degree of the above-mentioned relationships. This implies that the sampling errors associated with either sample size and investment horizon will affect the precision of estimated composite performance measures used in either portfolio or mutual fund managements.[16] Large sample and appropriate horizon can generally be used to improve the precision of the estimated composite performance measures. In addition, the possible impact of market condition on estimating composite performance measures should also be taken into consideration when these estimated performance measures are used in either portfolio or mutual fund managements. In conclusion, Friend and Blume (1970) have argued that the one parameter composite performance

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[16] Johnson and Burgess (1975) and Burgess and Johnson (1976) have investigated the effects of sample sizes and sampling fluctuation on the accuracy of both portfolio and security analyses. They have concluded that the number of historical observation is important to yield efficient portfolio performance characteristics. Their conclusion is similar to the result associated with the impact of sample size on the relationship between estimated composite measures and risk proxies derived in this study.

measure is more dangerous than the traditional two parameters--rates of return and risk to be used to measure the portfolio performance. The results derived in this paper can be used to explicitly justify this kind of argument.

V. Summary

In this paper, the statistical relationships between estimated one parameter composite performance measures and their risk proxies are derived in accordance with statistical distribution theory. It is found that the above-mentioned statistical relationships are generally affected by sample size, investment horizon and the market condition associated with the sample period selected for empirical studies. In addition, it is shown that large historical observations and an appropriate investment horizon can generally be used to improve the usefuleness of composite performance measures in both portfolio and mutual fund managements.

APPENDIX

(A) Derivation of equation (10), (11), and (12):

$$E(\frac{1}{S}) = \int_0^\infty \frac{(\frac{n}{\sigma^2})^{\frac{1}{2}}}{\Gamma(\frac{n-1}{2}) \quad 2^{(n-3)/2}} \quad (\frac{1}{S}) \quad (\frac{nS^2}{\sigma^2})^{\frac{n}{2}-1} \quad e^{-\frac{1}{2}(\frac{nS^2}{\sigma^2})} \quad ds$$

$$= C_1 \int_0^\infty (\frac{n}{\sigma^2})^{\frac{n}{2}-1} \quad S^{n-3} \quad e^{-\frac{1}{2}(\frac{nS^2}{\sigma^2})} \quad ds$$

where $C_1 = \frac{(\frac{n}{\sigma^2})^{\frac{1}{2}}}{\Gamma(\frac{n-1}{2}) \quad 2^{(n-3)/2}}$

$$= \frac{C_1}{2} \cdot \int_0^\infty (\frac{nS^2}{\sigma^2})^{\frac{n}{2}-2} \quad e^{-(\frac{nS^2}{2\sigma^2})} \quad d\,(\frac{nS^2}{\sigma^2})$$

$$= \frac{C_1}{2} \int_0^\infty Z^{\frac{n-2}{2}-1} \quad e^{-Z/2} \quad dZ \quad ,$$

where $Z = \frac{nS^2}{\sigma^2}$

$$= \frac{1}{2} \cdot \frac{(\frac{n}{\sigma^2})^{\frac{1}{2}}}{\Gamma(\frac{n-1}{2}) \quad 2^{(n-3)/2}} \cdot \Gamma(\frac{n-2}{2}) \quad 2^{\left(\frac{n-2}{2}\right)}$$

$$= (\frac{n}{2\sigma^2})^{\frac{1}{2}} \quad \frac{\Gamma(\frac{n-2}{2})}{\Gamma(\frac{n-1}{2})} \quad .$$

$$E\,(\frac{1}{S^2}) = \int_0^\infty C_1 \quad \frac{1}{S^2} \quad (\frac{nS^2}{\sigma^2})^{\frac{n}{2}-1} \quad e^{-\frac{1}{2}(\frac{nS^2}{\sigma^2})} \quad ds$$

$$= C_1 \int_0^\infty (S^2)^{-1} \quad (\frac{nS^2}{\sigma^2})^{\frac{n}{2}-1} \quad e^{-\frac{1}{2}(\frac{nS^2}{\sigma^2})} \cdot \frac{\sigma^2}{2nS} \quad d\,(\frac{nS^2}{\sigma^2})$$

$$= \frac{C_1}{2} \left(\frac{n}{\sigma^2}\right)^{\frac{1}{2}} \int_0^{\infty} \left(\frac{nS^2}{\sigma^2}\right)^{\frac{n-3}{2}-1} e^{-(nS^2/\sigma^2)/2} \, d\left(\frac{nS^2}{\sigma^2}\right)$$

$$= \frac{1}{2} \frac{\left(\frac{n}{\sigma^2}\right)^{\frac{1}{2}} \left(\frac{n}{\sigma^2}\right)^{\frac{1}{2}}}{\Gamma\left(\frac{n-1}{2}\right) \; 2^{(n-3)/2}} \cdot \Gamma\left(\frac{n-3}{2}\right) \; 2^{\frac{n-3}{2}}$$

$$= \frac{n}{2\sigma^2} \; \frac{\Gamma\left(\frac{n-3}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}$$

And

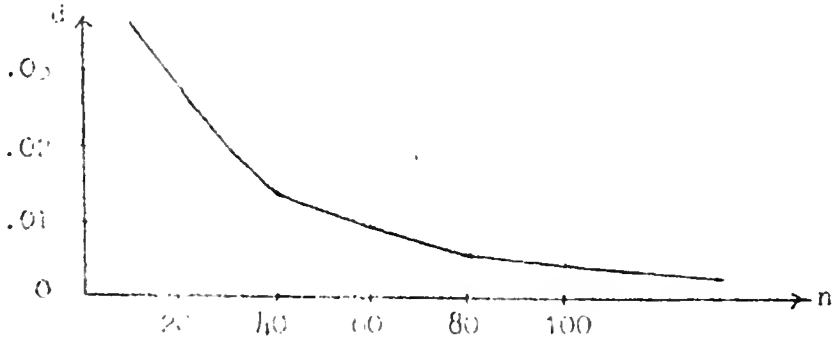
$$E(S) = \int_0^{\infty} C_1 \; S\left(\frac{nS^2}{\sigma^2}\right)^{\frac{n}{2}-1} e^{-\frac{1}{2}\left(\frac{nS^2}{\sigma^2}\right)} \, ds$$

$$= C_1 \int_0^{\infty} \frac{\sigma^2}{2n} \left(\frac{nS^2}{\sigma^2}\right)^{\frac{n}{2}-1} e^{-\frac{1}{2}\left(\frac{nS^2}{\sigma^2}\right)} \, d\left(\frac{nS^2}{\sigma^2}\right)$$

$$= \frac{\sigma^2 C_1}{2n} \cdot \Gamma\left(\frac{n}{2}\right) \; 2^{\frac{n}{2}} = \left(\frac{2\sigma^2}{n}\right)^{\frac{1}{2}} \cdot \frac{\Gamma(n/2)}{\Gamma\left(\frac{n-1}{2}\right)}$$

(B) $$d = \frac{\Gamma\left(\frac{n}{2}\right) \Gamma\left(\frac{n-2}{2}\right)}{\left[\Gamma\left(\frac{n-1}{2}\right)\right]^2} - 1 \quad :$$

The value of d is evaluated by digital computer for all values of n, $2 < n < 500$. The computed values of d indicate that d is a positive and decreasing function of n. The curve of d is depicted as follows:



Some values of d are tabulated as follows:

n	20	40	60	80	100	120	150	210	340	480
d	.028	.013	.009	.006	.005	.004	.003	.002	.0015	.0010

(C) Show that

$$\rho\left(\frac{\bar{R} - R_f}{S}, S\right) = \frac{d\ (R_f - \mu)}{\sigma_{sp}\ \sigma_s}$$

where d is defined as in (9), σ_{sp} and σ_s defined as in (14) and (15), respectively.

Proof:

$$\sigma_{sp}^2 = \mathrm{Var}\left(\frac{\bar{R} - R_f}{S}\right) = E\left[\left(\frac{\bar{R} - R_f}{S}\right)^2\right] - \left[E\left(\frac{\bar{R} - R_f}{S}\right)\right]^2$$

$$= E(\bar{R} - R_f)^2 \cdot E\left(\frac{1}{S^2}\right) - [E(\bar{R} - R_f)]^2 \cdot \left[E\left(\frac{1}{S}\right)\right]^2, \text{(a)}$$

using the property of independence between $\bar{R}$ and S.

$$= [\sigma^2/n + (\mu - R_f)^2] \cdot \left(\frac{n}{2\sigma^2}\right) \cdot \frac{\Gamma\left(\frac{n-3}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} - (\mu - R_f)^2 \cdot \left(\frac{n}{2\sigma^2}\right) \cdot \left(\frac{\Gamma\left(\frac{n-2}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}\right)^2,$$

by subtituting equations (10) and (11) in equation (a)

$$= \frac{\Gamma\left(\frac{n-3}{2}\right)}{2\Gamma\left(\frac{n-1}{2}\right)} + (\mu - R_f)^2 \cdot \left(\frac{n}{2\sigma^2}\right) \cdot \left[\frac{\Gamma\left(\frac{n-3}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} - \left(\frac{\Gamma\left(\frac{n-2}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}\right)^2\right],$$

which is equation (14)

And

$$\sigma_s^2 = \mathrm{Var}\ (S) = E(S^2) - [E(S)]^2$$

$$= \frac{n-1}{n} \cdot \sigma^2 - \frac{2\sigma^2}{n}\left[\frac{\Gamma(n/2)}{\Gamma(\frac{n-1}{2})}\right]^2$$

since $E\ (S^2) = \frac{n-1}{n}\ \sigma_s^2$ by Hogg and Craig [1969] and $E(S)$ is obtained as in equation (12).

$$= \frac{\sigma^2}{n}\left\{(n-1) - 2\cdot\left(\frac{\Gamma(n/2)}{\Gamma(\frac{n-1}{2})}\right)^2\right\}$$

which is equation (15).

Hence

$$\rho\left(\frac{\overline{R} - R_f}{S},\ S\right) = \frac{d\ (R_f - \mu_R)}{\sigma_{sp}\ \ \sigma_s}\ .$$

(D) Determine the distribution of the estimated Sharpe's performance measure: By the stochastical independence of the average excess rate of return and the estimated risk measure, the joint p.d.f. of $\overline{R} - R_f$ and S can be written as:

$$f(\overline{R}^*, S) = \begin{cases} C \cdot e^{-\frac{(\overline{R}^* - \mu^*)^2}{2\sigma^2/n}} \cdot \left(\frac{nS^2}{\sigma^2}\right)^{\frac{n}{2}-1} \cdot e^{-\frac{1}{2}\left(\frac{nS^2}{\sigma^2}\right)} \\ \text{for } -\infty < R < \infty \text{ and } o < S < \infty \\ 0 \text{ elsewhere} \end{cases}$$

where

$$C = \frac{1}{2\pi\,\sigma/\sqrt{n}} \cdot \frac{(n/\sigma^2)^{\frac{1}{2}}}{\Gamma(\frac{n-1}{2})\ 2^{(n-3)/2}}\ ,\quad \overline{R}^* = \overline{R} - R_f \text{ and } \mu^* = \mu - R_f$$

Next, let $Z_1 = \frac{\bar{R} - R_f}{S}$ and $Z_2 = S$. The Jacobian of the transformation is $J = Z_2 > o$. Thus, the joint p.d.f. of Z_1 and Z_2 is

$$h(Z_1 Z_2) = \begin{cases} C \cdot e^{\frac{-(Z_1 Z_2 - \mu^*)^2}{2\sigma^2/n}} \cdot \left(\frac{nZ_2^{\,2}}{\sigma^2}\right)^{\frac{n}{2}-1} \cdot e^{-\frac{1}{2}\left(\frac{nZ_2^{\,2}}{\sigma^2}\right)} \cdot Z_2 \\ 0 \text{ elsewhere} \end{cases}$$

Then, the marginal p.d.f. of the estimated Sharpe's measure is obtained by integrating the joint p.d.f. of Z_1 and Z_2 over the range of Z_2 (= S):

$$g(Z_1) = C \int_o^{\infty} Z_2 \exp\left[- \frac{(Z_1 Z_2 - \mu^*)^2}{2\sigma^2/n} - \frac{nZ_2^{\,2}}{2\sigma^2} \right] \left(\frac{nZ_2^{\,2}}{\sigma^2}\right)^{\frac{n}{2}-1} dZ_2,$$

$$\text{for } - \infty < Z_1 < \infty.$$

The integrand can not be integrated explicity. But it can be seen that the distribution of the estimated Sharpe's measure is not either a normal distribution or a well-known one.

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