COMPUTATION AND MENSURATION

THE MACMILLAN COMPANY
NEW YORK · BOSTON · CHICAGO
ATLANTA · SAN FRANCISCO

MACMILLAN & CO., Limited
LONDON · BOMBAY · CALCUTTA
MELBOURNE

THE MACMILLAN CO. OF CANADA, Ltd.
TORONTO

COMPUTATION AND MENSURATION

BY

P. A. LAMBERT, M.A.

PROFESSOR OF MATHEMATICS, LEHIGH UNIVERSITY

New York
THE MACMILLAN COMPANY
1907

Set up and electrotyped. Published September, 1907.

Norwood Press
J. S. Cushing Co. — Berwick & Smith Co.
Norwood, Mass., U.S.A.

PREFACE

THE transition from secondary school to college is disastrous for many students. This is due largely to the fact that the student has not been taught to make independent use of what he has learned. The transition should be accompanied by training in the application of the knowledge gained in the secondary school. Such training serves a threefold purpose — it makes the student realize that he has acquired increased power, it reviews the work of the secondary school in an interesting manner, and it gives an outlook into the work of the college.

Such a transition course in mathematics the author believes may be based on this text on Computation and Mensuration. This transition course would naturally come at the end of the secondary school course or at the beginning of the college course.

The student is expected to refer constantly to his texts on Algebra, Geometry, and Trigonometry. Formulas derived in all elementary text-books are neither proved nor tabulated in this text. The aim here is to build on the foundation already laid.

CONTENTS

CHAPTER I

Approximate Computation

CHAPTER II

Graphic Computation

CHAPTER III

The Method of Coördinates

CHAPTER IV

Volumes of Solids bounded by Planes

CHAPTER V

Computation and Use of Trigonometric Functions

CHAPTER VI

Computation and Use of Logarithms

CHAPTER VII

On Limits

CHAPTER VIII

Graphic Algebra

CHAPTER IX

Areas bounded by Curves

CHAPTER X

Volumes of Solids

COMPUTATION AND MENSURATION

CHAPTER I

APPROXIMATE COMPUTATION

Art. 1. — Approximate Numbers

Frequently numbers occurring in computation are known only approximately.

If the length of a line is to be measured, it may be absurd, either from the nature of the problem to be solved or the means of measurement employed, to attempt to determine the length of the line beyond hundredths of a foot.

The square root of a number not a perfect square cannot be expressed exactly in numbers, but its approximate value to any desired number of decimal places may be found.

For example, the approximate values of $\sqrt{5}$ are as follows:

to one decimal place 2.2^+
to two decimal places 2.24^-
to three decimal places 2.236^+
to four decimal places 2.2361^-
to five decimal places 2.23607^-
to six decimal places 2.236068^-
to seven decimal places 2.2360680^- *

* In these approximate values of $\sqrt{5}$ the figure in the last decimal place is always the nearest integer.

In each approximate value the sign + or − written after the last figure indicates that the error must be respectively added or subtracted. The error in every case is not greater than 5 units of the decimal place next following the last decimal place of the approximation.

The approximate value of $\sqrt{5}$ to eight decimal places cannot be determined until the figure in the ninth decimal place has been computed.

The ratio of the circumference of a circle to its diameter cannot be expressed exactly in numbers, but its value has been computed to 707 decimal places. This ratio is denoted by π, and its value to 15 decimal places is

$$\pi = 3.141592653589793.$$

The value of π most commonly used is 3.1416.

Problem 1. Find the limit of error when $\frac{22}{7}$ is used for π.

Problem 2. Find the limit of error when $\frac{355}{113}$ is used for π.

Problem 3. Find the limits of error of the sums of $\sqrt{26}$, $\sqrt{27}$, $\sqrt{28}$, $\sqrt{29}$, each computed approximately to four decimal places.

Problem 4. The yard is defined by law as $\frac{36.00}{39.37}$ meter.* Find the number of meters in a yard correct to 4 decimals.

Problem 5. The pound is defined by law as $\frac{1}{2.204622}$ kilogram.* Find the number of kilograms in a pound correct to 4 decimals.

* By decision of United States Superintendent of Weights and Measures, on April 5, 1903, with the approval of the Secretary of the Treasury.

Art. 2.—Direct Measurement

The direct measurement of the length of a line is effected by actually applying to the line the standard unit of length. However carefully several direct measurements are made, the results are generally not the same.

By universal agreement, the arithmetic mean of the results of several direct measurements, made with the same care, is accepted as the best result to be obtained from these measurements.

Problem 6. Eight measurements of a line give the values 186.4, 186.3, 186.2, 186.3, 186.2, 185.9, and 186.4 inches. Find the best value of the length of the line based on these measurements.

Problem 7. Four measurements of an angle give 62° 25′ 10″.0 ; 62° 25′ 7″.5 ; 62° 25′ 23″.3, and 62° 25′ 30″.0. Find the best result determined by these measurements.

Problem 8. Four measurements of a base line give 1472.34, 1471.99, 1472.25, and 1472.14 feet. Find the best approximate length of the line to be obtained from these measurements.

Art. 3.—Indirect Measurement

Example 1. Suppose A, B, C to be three stations forming the vertices of a right triangle, and let it be required to determine the length of the hypotenuse AB when an obstruction between A and B makes the direct measurement of this length impracticable.

If by direct measurement $AC = 1275.43$ feet and

$BC = 1526.56$ feet, AB may be computed by means of the proposition in Geometry —

The square on the hypotenuse equals the sum of the squares on the other two sides, or expressed as a formula,

$$\overline{AB}^2 = \overline{AC}^2 + \overline{BC}^2.$$

The number expressing the length of AB is found by extracting a square root and may be carried out to any desired number of decimal places; but since the measured lines AC and BC are determined only to two decimals, it would be absurd to attempt to determine by computation the length of AB beyond two decimals.

Example 2. Let it be required to determine the circumference of a circle whose diameter is found by direct measurement to be 425.63 feet.

The circumference of this circle is $\pi \times 425.63$ feet. Since the diameter of the circle is given to two decimals, it is impossible to determine by computation the circumference beyond two decimal places. It is therefore absurd to use more decimal places in the value of π than are necessary to determine the product $\pi \times 425.63$ to two decimal places. What is needed is a method of approximate computation which will determine the circumference correct to two decimal places with the least amount of computation.

Mensuration is chiefly concerned with problems of indirect measurement.

Problem 9. In a right triangle the sides about the right angle measure 875.27 feet and 565.45 feet. Compute the length of the hypotenuse.

PROBLEM 10. The diameter of a circle measures 365.18 feet. Compute the length of the circumference.

PROBLEM 11. The circumference of a circle measures 1000 feet. Compute the diameter.

ART. 4. — APPROXIMATE MULTIPLICATION

EXAMPLE. Let it be required to compute the product $\pi \times 425.63$ correct to two decimal places.

The computation is carried out to three decimal places in order that the second decimal place may be corrected.

The first step is to determine how many decimal places in the value of π must be used to give three decimal places in the product.

The computation consists in multiplying the approximate value of π by each figure of 425.63 and adding the partial products. These partial products are to contain three decimal places each.

In order that the partial product of the approximate value of π by the units' figure of 425.63 shall contain three decimal places, the value of π must be written to three decimal places.

For each integral place to the left of units' place in 425.63 the value of π must be extended one decimal place to the right in order that the corresponding partial product shall contain three decimal places.

In this example, therefore, the value of π must be written to five decimal places, and in order that the partial product by the figure in the hundredths' place in 425.63 may be corrected in the third decimal place the value of π is written to six decimal places.

The dot placed over the figure in the fifth decimal place of π indicates that this figure is the last figure to the right in the value of π to be used to form the partial product corresponding to the first figure to the left of 425.63.

As the figure of the multiplier for which a partial product is to be computed moves one place to the right, the last figure of the multiplicand to be used moves one place to the left.

There is considerable advantage in forming the partial products for the figures of 425.63 from right to left. The last figures of the partial products are corrected for the rejected part of the value of π. The signs + and − indicate the direction of the errors committed by these corrections. The sign + indicates that the error is to be added, the sign − that the error is to be subtracted.

The numerical value of the error cannot exceed $\frac{1}{2}$ of a unit of the third decimal place.

The computation now appears as follows:

$$\begin{array}{r} 3.141\dot{5}93 \\ 425.63 \\ \hline 1256.637^{+} \\ 62.832^{-} \\ 15.708^{-} \\ 1.885^{-} \\ .094^{+} \\ \hline 1337.156\phantom{^{+}} \end{array}$$

The error of this result must lie between $-1\frac{1}{2}$ and $+1$ units of the third decimal place. It is therefore uncertain if the figure in the second decimal place is 5 or 6.

It is necessary in this particular problem to carry out the approximate multiplication to 4 decimals as follows:

$$\begin{array}{r}
3.1415\dot{9}27 \\
425.63 \\
\hline
1256.6371^{-} \\
62.8319^{-} \\
15.7080^{-} \\
1.8850^{-} \\
.0942^{+} \\
\hline
1337.1562\phantom{^{-}}
\end{array}$$

The error of this result must lie between -2 and $+\frac{1}{2}$ units of the fourth decimal place. Hence the product correct to two places of decimals is 1337.16.

The following arrangement of this computation is perhaps to be preferred.

Place the units' figure of 425.63 under the third decimal place of the value of π and then write in inverse order the figures of 425.53 under the figures of the approximate value of π. This brings each figure of the multiplier in line with that figure of the multiplicand which is used to begin the corresponding partial product. The above approximate multiplications appear as follows:

$$\begin{array}{r}
3.141593 \\
36.524 \\
\hline
1256\ 637^{+} \\
62\ 832^{-} \\
15\ 708^{-} \\
1\ 885^{-} \\
94^{+} \\
\hline
1337.156\phantom{^{+}}
\end{array}
\qquad\qquad
\begin{array}{r}
3.1415927 \\
36.524 \\
\hline
12566\ 371^{-} \\
628\ 319^{-} \\
157\ 080^{-} \\
18\ 850^{-} \\
942^{+} \\
\hline
1337.1562\phantom{^{+}}
\end{array}$$

PROBLEM 12. Find the circumference of a shaft whose radius is 4.32 inches.

PROBLEM 13. If a yard is 1.904 meters, find the number of meters in 23.463 yards.

PROBLEM 14. Find the interest on $1525.75 for one year at $6\frac{1}{2}\%$, correct to $0.01.

PROBLEM 15. The distance of the moon from the earth is 59.97 times the earth's radius. If the earth's radius is 3962.824 miles, find the distance to the moon correct within 1 mile.

PROBLEM 16. Find the area of the circle whose radius is 16.27 feet.

PROBLEM 17. Find the volume of the sphere whose radius is 3.53 feet.

PROBLEM 18. Find the volume of the cone whose base is a circle, radius 2.35 feet, altitude 5.75 feet.

PROBLEM 19. Find the volume of the rectangular parallelopiped whose dimensions are 8.53 feet, 6.27 feet, and 4.65 feet.

PROBLEM 20. Find the length of the diagonal of the parallelopiped in Problem 19.

PROBLEM 21. Compute $168\sqrt{3}$ to two decimal places.

PROBLEM 22. Compute $\pi\sqrt{2}$ to three decimal places.

PROBLEM 23. Compute π^2 to four decimal places.

Art. 5.—Approximate Division

Example. Let it be required to compute correct to two places of decimals the diameter of the circle whose circumference measures 587.35 feet.

In order that the figure in the second decimal place of the quotient may be controlled, the quotient of 587.35 divided by π is computed to three places of decimals. By inspection the quotient has three integral places. Hence six figures of the quotient are to be computed. The nature of the problem shows that this computation requires six figures of the divisor and six figures in the dividend.

The successive partial products are formed as in the process of approximate multiplication in the preceding article. The computation is arranged as follows:

```
587.350   |   3.14159+
314 159+  |186.959
---------
273 191
251 327+
--------
 21 864
 18 850-
 -------
  3 014
  2 827-
  -----
    187
    157+
    ---
     30
     28+
     --
      2
```

The signs + and − placed after the partial products indicate the direction of the error of the partial products

The error in the sum of the partial products must be between + 2 and − 1 units of the last decimal place of the dividend. Hence the last remainder lies between 0 and + 3 of these units, and the error of the quotient lies between 0 and + 1 units of the third decimal place. Hence the quotient correct to two decimal places is 186.96.

There is some advantage in arranging the figures of the quotient in such a manner that the partial products begin with the product of the figure of the quotient and the figure of the divisor directly over it. The computation then takes this form,

$$\begin{array}{r|l}
587.350 & 3.14159^{+} \\
314\ 159^{+} & 959.681 \\
\cline{1-1}
273\ 191 & \\
251\ 327^{+} & \\
\cline{1-1}
21\ 864 & \\
18\ 850^{-} & \\
\cline{1-1}
3\ 014 & \\
2\ 827^{-} & \\
\cline{1-1}
187 & \\
157^{+} & \\
\cline{1-1}
30 & \\
28^{+} & \\
\cline{1-1}
2 &
\end{array}$$

Problem 24. Find the radius of the circle whose circumference is 425.76 feet.

Problem 25. Find the area of the circle whose circumference is 628.32 feet.

Problem 26. Find the circumference of the circle whose area is 3848.45 square feet.

PROBLEM 27. Find the diameter of a wheel which makes 373 revolutions in a mile.

PROBLEM 28. How many revolutions per mile are made by a locomotive drive wheel 4.5 feet in diameter?

PROBLEM 29. Compute $\frac{\pi}{\sqrt{2}}$ to four decimal places.

PROBLEM 30. Find correct to four decimal places the number of degrees in an angle of 1 radian.

PROBLEM 31. The moon revolves about the earth in 28 days, 7 hours, 43 minutes, 11.5 seconds. What is the average angle passed over in a day?

CHAPTER II

GRAPHIC COMPUTATION

Art. 6.—Graphic Representation of Numbers

On a straight line mark a series of equidistant points. Select one of these points and call it zero. Starting from zero point call the successive points in one direction $+1$, $+2$, $+3$, $+4$, $+5$, $\cdots$, the successive points in the opposite direction -1, -2, -3, -4, -5, $\cdots$. There is thus attached to each integral number, positive or negative, one point of the line.

Insert between the points attached to two consecutive integral numbers, 3 and 4, for example, nine equidistant points and call these points 3.1, 3.2, 3.3, 3.4, 3.5, 3.6, 3.7, 3.8, 3.9. There is thus attached to each number containing not more than one decimal place one point of the line.

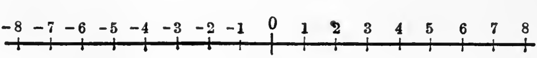


Fig. 1.

Theoretically, this operation of inserting between the points attached to two consecutive numbers nine equidistant points may be repeated indefinitely, but practically little is gained by attempting to attach points to numbers containing more than one decimal place.

Conversely, if any point is taken at random on the straight line, the number attached to the point can be determined to one decimal place with considerable certainty, but the attempt to determine the number beyond one decimal place would be accompanied by great uncertainty.

If the line segment joining the points attached to two consecutive whole numbers is taken as the unit of length, the number attached to any point of the line measures the length of the line segment from the zero point to this point. The algebraic sign of the number indicates the direction in which the line segment extends from the origin.

The straight line with numbers attached to successive equidistant points now becomes a scale for measuring the lengths of line segments.

Art. 7.—Addition and Subtraction of Segments

A line segment is a definite part of a straight line and extending in a definite direction along the line.

A line segment has an initial point and a terminal point. The segment extends from the initial point to the terminal point.

A line segment is transferred from one straight line to another by means of dividers.

Two line segments are equal when one can be transferred into the other.

When line segments are to be added or subtracted they are transferred to the same straight line.

To add two line segments having the same sign, lay off the two segments on a straight line both in the same direction, making the terminal point of the first segment the initial point of the second. The line segment which now extends from the initial point of the first segment to the terminal point of the second is the sum of the two.

To add two line segments having opposite signs, lay off the two segments on a straight line in opposite directions, making the terminal point of the first segment the initial point of the second. The line segment which now extends from the initial point of the first segment to the terminal point of the second is their sum.

To subtract two line segments, change the sign of the segment to be subtracted and add to the other segment.

To add or subtract numbers graphically, find from the scale of segments the line segments corresponding to the given numbers, then find the sum or difference of these line segments and from the scale determine the corresponding number.

Art. 8.—Multiplication and Division of Segments

To find the product of two line segments, define the product by the proportion

$$\text{product} : \text{multiplicand} = \text{multiplier} : \text{unity}.$$

The product of two line segments is therefore the line segment which is the fourth proportional to the two given line segments and the unit line segment.

This fourth proportional is constructed by drawing two straight lines through a common point making any convenient angle, the positive direction on each line being indicated by the arrowhead, and laying off from the common point the unit segment and multiplier on the first straight line, the multiplicand on the second straight line. Now join the terminal points of the unit segment and the multiplicand by a straight line and draw through the terminal point of the multiplier a parallel to this straight

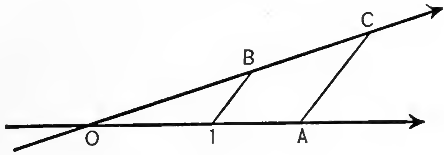


FIG. 2.

line. The point of intersection of this parallel with the second straight line is the terminal point, the common point the initial point of the product segment. In the figure

$$\frac{OC}{OA}=\frac{OB}{O1}, \text{ hence } OC = OA \cdot OB.$$

To find the quotient of two line segments, define the quotient by the proportion

$$\text{quotient} : 1 = \text{dividend} : \text{divisor}.$$

The quotient of two line segments is, therefore, the line segment which is the fourth proportional to the unit line segment and the two segments which are respectively the dividend and the divisor.

This fourth proportional is constructed by drawing two straight lines through a common point, making any convenient angle, laying off from the common point the dividend and divisor on the first straight line, the unit segment on the second straight line. Now join the terminal points of the unit segment and the divisor by a straight line and draw through the terminal point of the dividend a parallel to this straight line. The point where this parallel intersects the second straight line is the terminal point, the common point the initial point of the quotient segment.

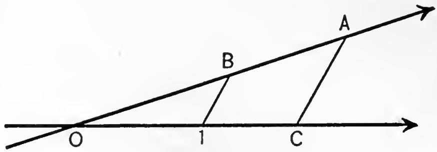


FIG. 3.

In the figure

$$\frac{OC}{O1}=\frac{OA}{OB}, \text{ hence } OC=\frac{OA}{OB}.$$

To find graphically the product or the quotient of two numbers, find by means of the scale of segments the line segments corresponding to the given numbers.

Then find the product or quotient of these line segments and from the scale the number corresponding to the product or quotient segment.

PROBLEM 32. Find graphically the product of 8.6 and 7.3.

PROBLEM 33. Find graphically the quotient of 47.5 divided by 11.7.

ART. 9.—SQUARE ROOT OF A SEGMENT

If the product of two equal line segments equals the given line segment, one of the equal segments is called the square root of the given line segment.

To construct the line segment which is the square root of the given line segment, find in any manner two line segments whose product equals the given line segment. On the sum of these two line segments as a diameter, construct a semicircumference. The perpendicular to the diameter at the point of meeting of the two line segments

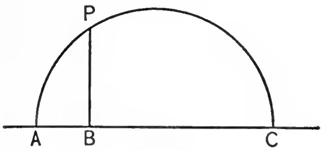


FIG. 4.

and terminating in the circumference is the line segment required.

Repeated applications of this operation make it possible to extract a root which is any power of 2 by the use of the straight line and circle.

PROBLEM 34. Find graphically $\sqrt{24}$.

PROBLEM 35. Add graphically $\sqrt{19}$, $\sqrt{21}$, and $\sqrt{28}$.

ART. 10. — GRAPHIC COMPUTATION OF AREAS

Construct as accurately as practicable the figure whose area is to be found on cross-section paper, taking the distance between consecutive lines to represent the unit of length in the measurement of the figure. The square of the cross-section paper now represents the unit of area, and the area of the figure is determined by counting the squares wholly within the boundary line of the figure and estimating the squares partially within the boundary line. The accuracy of the result depends upon the skill in estimating the sum of the partial squares.

PROBLEM 36. Find graphically the area of the triangle whose sides are 8, 12, 15 feet.

PROBLEM 37. Find graphically the area of the circle whose diameter is 9 inches.

PROBLEM 38. Find graphically the area of the equilateral triangle whose side is 7 feet.

ART. 11. — GRAPHIC SOLUTION OF TRIANGLES

The data necessary for the solution of a triangle is sufficient for the construction of the triangle.

If the triangle is constructed with the given parts as accurately as practicable by the aid of straight edge, protractor, and dividers, by means of these same instruments the values of the unknown parts of the triangle can be found with considerable accuracy.

PROBLEM 39. Two sides of a triangle are 7.8 feet and 12.5 feet, their included angle is 28° 30′. Find the remaining three parts graphically.

PROBLEM 40. Two sides of a triangle are 9.5 feet and 16.3 feet. The angle opposite the first side is 16° 15′. Find the other parts graphically.

PROBLEM 41. One side of a triangle is 10.6 feet, the adjacent angles are 65° 15′ and 80° 10′. Find graphically the other two sides.

PROBLEM 42. The sides of a triangle are 18 feet, 25 feet, and 29 feet. Find the angles graphically.

PROBLEM 43. Two sides of a triangle are 4.6 feet and 12.8 feet. The angle opposite the former sides is 60° 30′. Find the remaining parts graphically.

CHAPTER III

THE METHOD OF COÖRDINATES

Art. 12.—The Plane Coördinates of a Point

In a given plane draw two straight lines perpendicular to each other. On each of these lines, with their intersection at zero point, construct a linear scale, denoting the positive direction on each line by an arrowhead. Call these two lines of reference the X-axis and the Y-axis.

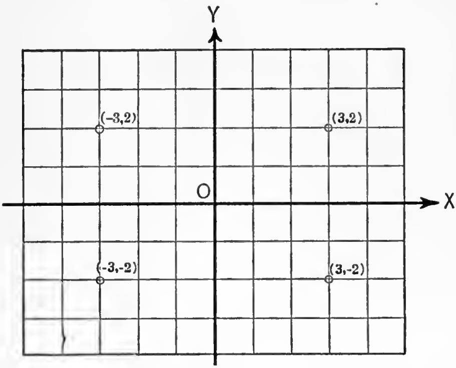


Fig. 5.

Select any point P in the plane, and through this point draw a line parallel to the Y-axis to meet the X-axis. The number which expresses the distance and direction of this meeting point from the zero point is called the abscissa of the point P and is denoted by x.

Through the point P draw a parallel to the X-axis to meet the Y-axis. The number which expresses the distance and direction of this meeting point from the zero point is called the ordinate of the point and is denoted by y.

The abscissa of a point may be defined as the number which expresses the distance and direction of the point from the Y-axis measured on a parallel to the X-axis; and the ordinate may be defined as the number which expresses the distance and direction of the point from the X-axis measured on a parallel to the Y-axis.

The abscissa of a point determines a straight line parallel to the Y-axis in which the point must be situated.

The ordinate of a point determines a straight line parallel to the X-axis in which the point must be situated.

The abscissa and ordinate of a point together determine the point in the plane and are called the coördinates of the point.

The point whose coördinates are $x = 3$, $y = -2$ is called the point $(3, -2)$.

PROBLEM 44. Where is the point situated whose abscissa is 4?

PROBLEM 45. Where is the point situated whose ordinate is -5?

PROBLEM 46. Locate the points $(3, 5)$, $(-4, 6)$, $(0, 8)$.

PROBLEM 47. Construct the triangle whose vertices are the points $(10, 4)$, $(6, -8)$, $(15, 12)$.

PROBLEM 48. Locate the points $(5, 3)$, $(10, 7)$ and find the distance between them.

PROBLEM 49. Find the area of the triangle whose vertices are $(3, 7)$, $(8, 5)$, $(13, 15)$.

ART. 13.—AREA OF TRIANGLE IN COÖRDINATES OF VERTICES

Suppose the vertices of the triangle to be the points $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, $P_3(x_3, y_3)$.

Two cases will be considered, Case I, when all the coördinates of the vertices are positive, Case II, when some of these coördinates are negative.

It must be remembered that line segments parallel to the X-axis are positive when they extend in the positive direction of the X-axis, and line segments parallel to the Y-axis are positive when they extend in the positive direction of the Y-axis.

In the propcsition

The area of a trapezoid is half the sum of the parallel bases by the altitude,

the bases and altitude are positive line segments.

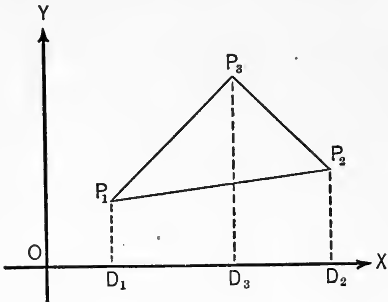


Fig. 6.

Case I. The area of the triangle $P_1P_2P_3$ equals the sum of the areas of the trapezoids $P_1D_1D_3P_3$, and $P_3D_3D_2P_2$ minus the area of the trapezoid $P_1D_1D_2P_2$.

Hence

$$2 \text{ Area triangle } P_1P_2P_3 = D_1D_3 \times (D_1P_1 + D_3P_3) + D_3D_2 \times (D_3P_3 + D_2P_2) - D_1D_2 \times (D_1P_1 + D_2P_2).$$

Now $x_1 = OD_1$, $x_2 = OD_2$, $x_3 = OD_3$,

$$y_1 = D_1P_1,\ y_2 = D_2P_2,\ y_3 = D_3P_3.$$

Hence $D_1D_3 = x_3 - x_1$, $D_3D_2 = x_2 - x_3$, $D_1D_2 = x_2 - x_1$, and

$$2 \text{ Area } P_1P_2P_3 = (x_3 - x_1)(y_1 + y_3) + (x_2 - x_3)(y_3 + y_2) - (x_2 - x_1)(y_1 + y_2).$$

Multiplying out and simplifying,

$$2 \text{ Area } P_1P_2P_3 = x_1y_2 + x_2y_3 + x_3y_1 - x_1y_3 - x_2y_1 - x_3y_2.$$

This expression for double the area of a triangle in terms of the coördinates of its vertices is readily reproduced by arranging the coördinates of the vertices in order in two vertical columns and repeating at the end of the column the coördinates of the first point. The three positive terms of the area are the products of each of the three abscissas by the next lower ordinate, and the three negative terms of the area are the products of each of the three ordinates by the next lower abscissa.

$$\begin{matrix} x_1 y_1 \\ x_2 y_2 \\ x_3 y_3 \\ x_1 y_1 \end{matrix}$$

The area of the triangle is positive when the perimeter is supposed to be described in such a manner that the area lies to the left of the boundary line.

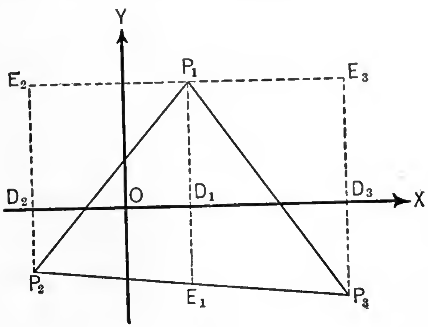


FIG. 7.

Case II. The area of the triangle $P_1P_2P_3$ equals the area of the trapezoid $P_2E_2E_3P_3$ minus the sum of the areas of the triangles $P_2E_2P_1$ and $P_3E_3P_1$.

Hence

$$2 \text{ Area triangle } P_1P_2P_3 = (P_2E_2 + P_3E_3) \times D_2D_3 - P_2E_2 \times E_2P_1 - P_3E_3 \times P_1E_3.$$

Now $x_1 = OD_1$, $x_2 = -D_2O$, $x_3 = OD_3$,

$$y_1 = D_1P_1,\ y_2 = -P_2D_2,\ y_3 = -P_3D_3.$$

Hence $P_2E_2 = -y_2 + y_1$, $P_3E_3 = -y_3 + y_1$,

$D_2D_3 = -x_2 + x_3$, $E_2P_1 = -x_2 + x_1$, $P_1E_3 = x_3 - x_1$, and

$$2 \text{ Area } P_1P_2P_3 = (2y_1 - y_2 - y_3)(-x_1 + x_3) - (-y_2 + y_1)(-x_2 + x_1) - (-y_3 + y_1)(x_3 - x_1).$$

This reduces to

$$2 \text{ Area } P_1P_2P_3 = x_1y_2 + x_2y_3 + x_3y_1 - y_1x_2 - y_2x_3 - y_3x_1,$$

the same result as was found for Case I.

This expression for double the area of a triangle in terms of the coördinates of its vertices is entirely general.

Problem 50. Find the area of the triangle whose vertices are the points (12, – 5), (– 8, 7), (10, 15), the coördinates being measured in inches.

Problem 51. Find the area of the triangle the coördinates of whose vertices measured in chains are (15.75, 4.26), (18.25, 20.63), (21.43, 16.52).

Problem 52. Show that the points (1, 4), (3, 2), (– 3, 8) lie in a straight line.

ART. 14. — AREAS OF ANY RECTILINEAR FIGURES

Let $P_1(x_1y_1)$, $P_2(x_2y_2)$, $P_3(x_3y_3)$, $P_4(x_4y_4)$ be the vertices of a quadrilateral. The straight line P_1P_3 divides this quadrilateral into the triangles $P_1P_2P_3$ and $P_1P_3P_4$.

$$2 \text{ Area } P_1P_2P_3 = x_1y_2 + x_2y_3 + x_3y_1 - y_1x_2 - y_2x_3 - y_3x_1.$$

$$2 \text{ Area } P_1P_3P_4 = x_1y_3 + x_3y_4 + x_4y_1 - y_1x_3 - y_3x_4 - y_4x_1.$$

Hence

$$2 \text{ Area } P_1P_2P_3P_4 = x_1y_2 + x_2y_3 + x_3y_4 + x_4y_1 - y_1x_2 - y_2x_3 - y_3x_4 - y_4x_1.$$

x_1y_1
x_2y_2
x_3y_3
x_4y_4
x_1y_1

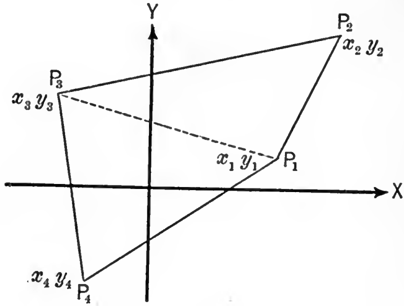


FIG. 8.

This expression for double the area of a quadrilateral is readily reproduced by writing the coördinates of the vertices in regular order in two vertical columns, the coördinates of the first vertex being repeated at the end of the columns. The positive terms of the expression for the double area are the products of each abscissa by the next lower ordinate; the negative terms are the products of each ordinate by the next lower abscissa.

This same method gives the area in terms of the coördinates of its vertices of any rectilinear figure.

Problem 53. The coördinates measured in chains of the angular points of a quadrilateral field are (10.25, 0), (3.21, 7.35), (– 8.75, 0), (2.37, – 9.13). Construct the figure and find the area of the field in acres.

Problem 54. Compute the area and construct the figure of the polygon the coördinates of whose angular points measured in feet are

(– 27, 0), (20, 3.75), (16.5, 4), (15, 7.5), (6.6).

Problems 55, 56, and 57. The coördinates measured in chains of the successive corners of three tracts of land bounded by straight lines are as follows. Compute the areas and construct the figures.

Problem 55

x	y
0	– 8.12
9.31	– 16.21
25.11	– 9.61
30.	0
23.	0
22.1	15.4
7.23	16.48
0	11.

Problem 56

x	y
13.31	6.30
24.09	7.21
14.11	12.40
26.	13.10
26.23	15.
20.	20.37
10.	16.20
1.01	16.
0	14.30

Problem 57

x	y
– 30	90
0	140
80	130
50	90
54	80
100	40
60	20
10	0
5	70

CHAPTER IV

VOLUMES OF SOLIDS BOUNDED BY PLANES

Art. 15. — The Prismatoid

The polyhedron two of whose faces are any two polygons in parallel planes and whose other faces are composed of triangles formed by so joining the vertices of these parallel polygons that each line in order forms a triangle with the preceding line and one side of either parallel polygon is called a prismatoid.

The parallel polygons are called the bases of the prismatoid. The section of the prismatoid formed by a plane midway between the bases is called the midsection. The perpendicular distance between the bases is called the altitude of the prismatoid.

Call the altitude of the prismatoid h, the areas of the bases b_1 and b_2, the area of the midsection m. Let O be any point in the midsection. Draw lines from O to the vertices of the midsection, and to the vertices of the prismatoid. The lines drawn from O to the vertices of the midsection divide the midsection into triangles. The planes determined by O and the edges of the prismatoid divide the prismatoid into two pyramids O-ABC and

O-DEFG, and a series of tetrahedrons *O-BDE*, *O-BEF*, *O-BFG*, *O-BCG*, *O-CDG*, *O-ACD*, and *O-ABD*.

The volume of the pyramid *O-ABC* is $\frac{1}{6}\, h \cdot b_1$; the volume of the pyramid *O-DEFG* is $\frac{1}{6}\, h \cdot b_2$.

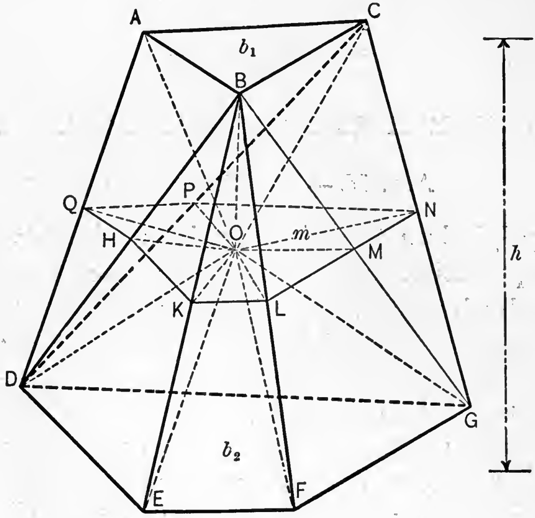


FIG. 9.

The triangle *BHK* is similar to the triangle *BDE* and *HK* is $\frac{1}{2}$ *DE*. Hence area of *BDE* is 4 times area of *BHK*.

The tetrahedrons *O-BDE* and *O-BHK* have the same altitude and therefore are proportional to their bases. Hence volume *O-BDE* = 4 times volume *O-BHK*.

Tetrahedron *O-BHK* = tetrahedron *B-HOK*. The volume of *B-OHK* = $\frac{1}{6}\, h \cdot$ *OHK*. Hence volume *O-BDE* = $\frac{4}{6}\, h \cdot$ *OHK*.

In like manner it is proved that the

$$\text{volume of } O\text{-}BEF = \tfrac{4}{6}\, h \cdot OKL.$$
$$\text{volume of } O\text{-}BFG = \tfrac{4}{6}\, h \cdot OLM.$$
$$\text{volume of } O\text{-}BGC = \tfrac{4}{6}\, h \cdot OMN.$$
$$\text{volume of } O\text{-}CDG = \tfrac{4}{6}\, h \cdot ONP.$$
$$\text{volume of } O\text{-}ACD = \tfrac{4}{6}\, h \cdot OPQ.$$
$$\text{volume of } O\text{-}ABD = \tfrac{4}{6}\, h \cdot OQH.$$

The sum of the volumes of all these tetrahedra is $\frac{4}{6}\, h \cdot m$.

It follows that the volume of the prismatoid is $\frac{1}{6}\, h \cdot (b_1 + b_2 + 4\, m)$, that is, the volume of the prismatoid is $\frac{1}{6}$ of the altitude times the sum of the areas of the two bases and four times the area of the midsection.

This result is called the prismoidal formula.

Art. 16. — Special Cases of Prismatoids

When the bases are equal polygons, with their corresponding sides parallel, the prismatoid is a prism.

When the bases are similar polygons, with their corresponding sides parallel, the prismatoid is a frustum of a pyramid.

When one base becomes a point, the prismatoid is a pyramid.

When one base becomes a line, the prismatoid is a wedge.

When one face is a rectangle and the bases are trapezoids perpendicular to the rectangular face, the prismatoid is called a prismoid. This is the general shape of a railway cutting or embankment.

In case of a cutting or embankment, when one face is not a plane, the error in computing the volume is diminished by measuring the areas of $2n+1$ equidistant cross sections $A_1, A_2, A_3, A_4, \cdots, A_{2n}, A_{2n+1}$ and finding the sum of the volumes of the prismoids whose bases are A_1 and A_3, A_3 and A_5, A_5 and A_7, $\cdots$, A_{2n} and A_{2n+1}. The approximation is made closer by increasing n.

PROBLEM 58. From the prismoidal formula derive the formulas for the volumes of a pyramid and a frustum of a pyramid.

PROBLEM 59. The base of the great pyramid of Egypt is a square 764 feet on a side and the altitude is 477.6 feet. Find the volume in cubic yards.

PROBLEM 60. The section of a canal is 32 feet wide at the top, 14 feet wide at the bottom, and 8 feet deep. How many cubic yards were excavated in a mile of the canal?

PROBLEM 61. Find the volume of a rectangular wedge whose base is 70 meters by 20 meters, length of edge 110 meters, and altitude 24.8 meters.

PROBLEM 62. Find the number of cubic yards of earth excavated from a railway cutting made through ground the original surface of which was an inclined plane running in the same direction as the rails. The length of the cutting is 4 chains 15 links, the breadth at bottom 30 feet, the breadth at top at one end 75 feet, and at the other end 135 feet, and the depths of these ends 20 feet and 46 feet respectively.

Problem **63.** A railway embankment is to be made on ground which slopes 20 feet in a mile in the direction of the rails and the rails themselves slope 1 in 700. The embankment is straight for $2\frac{1}{4}$ miles. The breadth at the top is 29 feet, the slopes of the sides 1 in 1, and the height at the upper end is 3 feet. Find the number of cubic yards of earth required for the embankment.

Problem **64.** A railway cutting is to be made 30 feet wide at the bottom, the slopes of the sides being $1\frac{1}{2}$ to 1. The depths in feet on one side of the cutting, taken at intervals of 1 chain, are 12, 11, 10, 9, 10, 12, 14, 15, 17, 20, 21, 23, 25, 27, 30, 33, 37, 41, 45, 50, 53; the corresponding depths on the opposite side, 15, 14, 13, 12, 14, 15, 17, 19, 22, 26, 28, 30, 33, 36, 39, 41, 44, 51, 57, 60, 58. Find the number of cubic yards of earth to be excavated.

CHAPTER V

COMPUTATION AND USE OF TRIGONOMETRIC FUNCTIONS

Art. 17. — On Angles

The figure formed by two straight lines proceeding from a point is called an angle.

The two straight lines are called the sides of the angle, and their direction is indicated by arrowheads. The point is called the vertex of the angle.

One side of the angle is called the initial side, the other the terminal side of the angle.

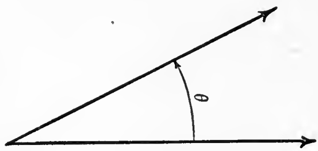


Fig. 10.

A circular arrow about the vertex between the two sides with the arrowhead at the terminal side, indicates which side is to be considered the terminal side of the angle.

The angle is called positive when this circular arrow is anticlockwise; negative when the arrow is clockwise.

When the terminal side of the angle coincides with the initial side the angle is called a perigon. An angle of one degree is one three hundred and sixtieth of a perigon.

The ratio of the circular arc, described with the vertex of the angle as a center and bounded by the sides of the angle, to the radius of the arc is called the circular measure of the angle.

Denoting the circular measure of the angle by θ and the radius of the arc by r, it follows that $\frac{\text{arc}}{r} = \theta$ and $\text{arc} = r\theta$.

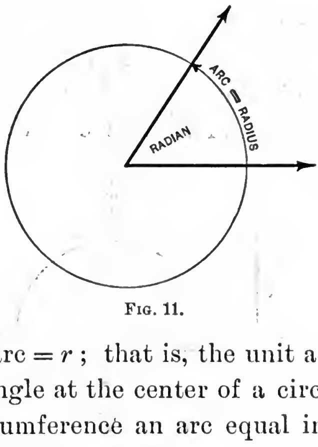


FIG. 11.

When $\theta = 1$, $\text{arc} = r$; that is, the unit angle in circular measure is the angle at the center of a circle which intercepts on the circumference an arc equal in length to the radius. The unit angle in circular measure is called the radian.

If the angle is four right angles, the arc is the circumference of the circle; that is, $\text{arc} = 2\pi r$. Hence, the circular measure of an angle of four right angles is 2π radians.

It follows that 2π radians are equivalent to 360°. Hence,

$$1 \text{ radian} = \frac{180°}{\pi} \text{ and } 1° = \frac{\pi}{180} \text{ radians.}$$

PROBLEM 65. Find the circular measure of an angle of 5° correct to four decimal places.

Problem 66. Find the circular measure of an angle of $14° 27' 25''$ correct to four decimal places.

Problem 67. Find in degrees, minutes, and seconds the angle whose circular measure is .357.

Problem 68. Find in degrees correct to four decimal places the angle 1.258.

Art. 18. — Trigonometric Functions

Call the vertex of the angle O and the indefinite straight line in which the initial side of the angle lies XX'. Call distances measured on XX' from O in the direction of the

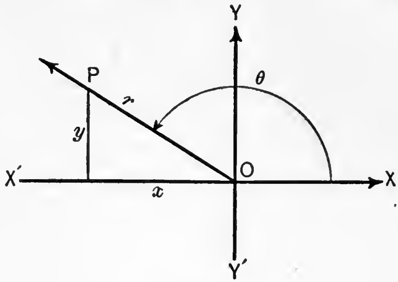


Fig. 12.

arrowhead on the initial side positive, distances measured in the opposite direction negative. Let OX be the positive side of XX'. Draw a straight line YY' through O perpendicular to XX'. If OY makes an angle of $+90°$ with OX, call distances measured from O towards Y positive and denote this by the arrowhead at Y.

Now take any point P in the terminal side of the angle. The distance from O to P is called the radial distance of the point P and is denoted by r. The radial distance is always positive.

The projection of the radial distance r of the point P on the line XX' is the abscissa x of the point P.

The projection of the radial distance r of the point P on the line YY' is the ordinate y of the point P.

The trigonometric functions of an angle are the following six ratios:

The sine of an angle is the ratio of the ordinate of any point in the terminal side to the radial distance of the same point; that is,

$$\sin \theta = \frac{y}{r}.$$

The cosine of an angle is the ratio of the abscissa of any point in the terminal side to the radial distance of the same point; that is,

$$\cos \theta = \frac{x}{r}.$$

The tangent of an angle is the ratio of the ordinate of any point in the terminal side to the abscissa of the same point; that is,

$$\tan \theta = \frac{y}{r}.$$

The cotangent θ is the reciprocal of the tangent θ.

The secant θ is the reciprocal of the cosine θ.

The cosecant θ is the reciprocal of the sine θ.

Two auxiliary trigonometric functions are the versed-sine and coversed-sine, defined as follows: versed-sine $\theta = 1 -$ cosine θ, coversed-sine $= 1 -$ sine θ.

PROBLEM 69. The tangent of an angle is $\frac{12}{13}$. Construct the angle and find the values of all the other functions of the angle.

PROBLEM 70 The sine of an angle is .375. Compute the cosine and the tangent of this angle.

PROBLEM 71. The cosine of an angle is $-.45$. Compute the sine and tangent of this angle.

ART. 19. — COMPUTATION OF TRIGONOMETRIC FUNCTIONS

If an angle is given in degrees or in radians, the angle may be constructed by means of a protractor and the lengths of the radial distance, abscissa, and ordinate of any point in the terminal side measured, and the trigonometric functions of the angle computed.

This method does not in general determine the value of the trigonometric functions beyond the first decimal place.

If θ is the circular measure of an angle, sine θ may be computed to any required degree of approximation by means of the following infinite series, derived in the Differential Calculus,

$$\sin\theta = \theta - \frac{\theta^3}{1\cdot 2\cdot 3} + \frac{\theta^5}{1\cdot 2\cdot 3\cdot 4\cdot 5} - \frac{\theta^7}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7} + \cdots.$$

Let it be required to compute sine 20° to four decimal places.

The circular measure of 20° is $\theta = \frac{\pi}{9} = .34907$.

Hence

$$\theta = .34907,\ \frac{\theta^3}{1\cdot 2\cdot 3} = .00707,\ \frac{\theta^5}{1\cdot 2\cdot 3\cdot 4\cdot 5} = .00004.$$

It follows that sine 20° = .3420.

The error committed by omitting the terms of the infinite series after the third may be written

$$(a)\quad \frac{\theta^7}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}\left[-1 + \left(\frac{\theta^2}{8\cdot 9} - \frac{\theta^4}{8\cdot 9\cdot 10\cdot 11}\right) + \cdots\right];$$

$$\text{or, } (b) \quad \frac{\theta^7}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}\left[-\left(1-\frac{\theta^2}{8\cdot 9}\right)-\left(\frac{\theta^4}{8\cdot 9\cdot 10\cdot 11} - \frac{\theta^6}{8\cdot 9\cdot 10\cdot 11\cdot 12\cdot 13}\right)-\cdots\right].$$

According to (b) the error is negative, and according to (a) the numerical value of the error is less than $\frac{\theta^7}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}$. Hence the error committed by omitting the terms of the infinite series after the third in computing sine 20° is negative and numerically less than .0000002.

When sine θ has been computed, the remaining functions of θ may be computed by means of the fundamental relations,

$$\sin^2\theta + \cos^2\theta = 1,\ \tan\theta = \frac{\sin\theta}{\cos\theta},\ \cot\theta = \frac{1}{\tan\theta},$$

$$\sec\theta = \frac{1}{\cos\theta},\ \operatorname{cosec}\theta = \frac{1}{\sin\theta}.$$

The trigonometric functions of any angle whatever may be found in terms of the trigonometric functions of an angle not greater than 45°. Therefore if the functions of angles from 0° to 45° are computed, the functions of all angles become known.

If θ denotes the circular measure of a small angle and θ'' the number of seconds in this angle,

approximately $\sin\theta = \tan\theta = \theta = \theta'' \times$ circular measure $1''$

$$= \frac{\theta''}{206264.8}.$$

The approximation is correct to six places of decimals from 0° to 38′; to five places of decimals to 1° 20′; to four places of decimals to 2° 20′′.

PROBLEM 72. Construct the angle 50° and find the trigonometric functions of the angle by measurement.

PROBLEM 73. Construct the angle 110° and find the trigonometric functions of the angle by measurement.

PROBLEM 74. Compute the sine of 18° to four decimal places.

PROBLEM 75. Compute the sine of 115° to four decimal places.

PROBLEM 76. Compute the sine of 214° to four decimal places.

PROBLEM 77. Show that the circular measure, sine, and tangent of any angle from 0° to 7° are the same to three decimal places.

PROBLEM 78. From the general definitions of the trigonometric functions of an angle, show that in a right triangle, if the hypotenuse is c, the oblique angles A and B and the sides opposite a and b,

$$\sin A = \frac{a}{c}, \ \cos A = \frac{b}{c}, \ \tan A = \frac{a}{b}.$$

If the side a is so small compared with the sides b and c that the angle A does not exceed two or three degrees,

$$\frac{a}{c} = \sin A = \frac{A''}{206264.8} \text{ approximately,}$$

and consequently

$$c = \frac{a}{A''} 206264.8 \text{ approximately.}$$

PROBLEM 79. Find the area of the equilateral triangle inscribed in a circle whose radius is 15.5 inches.

PROBLEM 80. The angle at the vertex of a cone is 25°, the altitude is 12 feet. Find the volume of the cone.

PROBLEM 81. A street railway track is 10 feet from the curbstone where the track is straight. In passing a corner where the street is deflected through an angle of 60°, the rail must be 4 feet from the corner. Find the radius of the circular curve.

PROBLEM 82. The greatest angle the radius of the earth, 3963.3 miles, subtends by lines drawn from the center of the sun is $8''.8$. Find the distance of the sun from the earth.

PROBLEM 83. Two sides of a triangle are 76 feet and 125 feet, their included angle is $35° 15'$. Find the third side of the triangle.

PROBLEM 84. The three sides of a triangle are 17.5 feet, 26.7 feet, and 35.25 feet. Find the angle opposite the last side.

PROBLEM 85. Two sides of a parallelogram are 125.75 feet and 153.25 feet, their included angle is $67° 45'$. Find the diagonals of the parallelogram.

ART. 20. — ON VECTORS

A vector is a quantity which is completely determined by assigning its magnitude and its direction.

A vector may therefore be represented by a line segment and the angle which this line segment makes with a fixed line of reference, the X-axis, for example.

The direction of a vector is indicated by the arrowhead placed at the terminal point of the line segment representing the vector.

Two vectors are equal when they have the same magnitude and the same direction.

The geometric representative of the vector will be used as the vector itself.

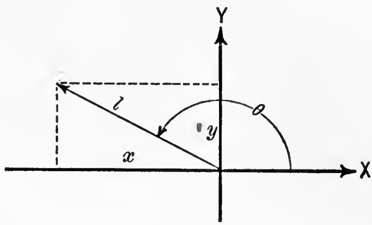


FIG. 13.

Let l denote the length of the vector, θ the angle the vector makes with the X-axis, x the projection of the vector on the X-axis, y the projection of the vector on the Y-axis. From the figure

$$(1)\ l\cos\theta = x, \quad (2)\ l\sin\theta = y.$$

Squaring equations (1) and (2) and adding,

$$l^2 = x^2 + y^2.$$

Dividing (2) by (1),

$$\tan\theta = \frac{y}{x}.$$

Hence from the projections of a vector on the axes of reference the length and direction of the vector may be computed.

If several vectors are drawn in such a manner that the initial point of each successive vector is the terminal point of the preceding vector, the vector drawn from the initial point of the first vector to the terminal point of the last vector is called the vector sum of the several vectors.

The sum of the projections of the several vectors on any straight line is the projection of the vector sum on the same straight line.

Let l_1, l_2, l_3, l_4 denote the lengths of the several vectors, θ_1, θ_2, θ_3, θ_4 the angles these vectors make with the X-axis.

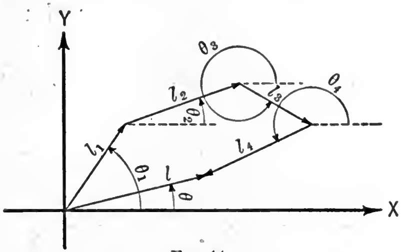


FIG. 14.

Let l denote the length of the vector sum, θ the angle this vector makes with the X-axis, x its projection on the X-axis, y its projection on the Y-axis.

From the figure

$$l\cos\theta = x = l_1\cos\theta_1 + l_2\cos\theta_2 + l_3\cos\theta_3 + l_4\cos\theta_4,$$

$$l\sin\theta = y = l_1\sin\theta_1 + l_2\sin\theta_2 + l_3\sin\theta_3 + l_4\sin\theta_4.$$

From these equations the length and direction of the vector sum may be computed.

The displacement of a body is determined by its magnitude and direction. A displacement may therefore be represented by a vector.

If a body has several successive displacements, each displacement may be represented by a vector, and the resulting displacement is represented by the vector sum.

The force acting at a point is determined by its magnitude and direction. A force may therefore be represented by a vector.

By experiment it is shown that if several forces acting at a point are represented by vectors, the vector sum represents the resultant of the several forces.

PROBLEM 86. Three sides of a quadrilateral taken in order are 11 feet, 7 feet, and 8 feet, and make angles of 18° 18′, 74° 50′, and 130° 20′ respectively with a fixed line. Find the length and direction of the fourth side of the quadrilateral.

PROBLEM 87. A man walks 8, 12, 15, and 20 miles along successive straight lines, making angles respectively of 30°, 70°, 120° 15′, and 155° with the E. W. line. Find the distance and bearing of his final position from the starting point.

PROBLEM 88. The following are the field notes of a survey. Bearing means the direction from one station to the next succeeding station.

STATIONS	BEARINGS	DISTANCES
A	N. $31\frac{1}{2}$ W.	10 chains
B	N. $62\frac{3}{4}$ E.	9.25 chains
C		
D	S. $45\frac{1}{2}$ W.	10.40 chains

Plot the survey, find the distance and bearing from *C* to *D*, and find the inclosed area.

PROBLEM **89**. Find the magnitude and direction of the resultant of three forces acting at a point of 11, 7, and 8 pounds and making angles respectively of 18° 18′, 74° 50′, and 130° 20′ with a fixed line.

PROBLEM **90**. Find the magnitude and direction of the resultant of four forces acting at a point of 8, 12, 15, 20 pounds, making angles respectively of 30°, 180°, 225°, 330° with a fixed line.

CHAPTER VI

COMPUTATION AND USE OF LOGARITHMS

Art. 21. — Nature of Logarithms

Examine the tables,

I. $10^0 = 1$	II. $10^0 = 1$
$10^1 = 10$	$10^{-1} = .1$
$10^2 = 100$	$10^{-2} = .01$
$10^3 = 1000$	$10^{-3} = .001$
$10^4 = 10000$	$10^{-4} = .0001$
$10^5 = 100000$	$10^{-5} = .00001$
. . . .	. . . .
. . . .	. . . .

Numbers with one integral place lie between 1 and 10, and it is evident that the exponent by which 10 must be affected to give numbers between 1 and 10 must lie between 0 and 1.

In like manner it becomes evident that the exponent by which 10 must be affected to give a number with two integral places must lie between 1 and 2; to give a number with three integral places the exponent must lie between 2 and 3; and in general to give a number with n integral places the exponent must lie between $n-1$ and n.

Decimal fractions with the first significant figure in the first decimal place lie between 1 and .1, and it is evident from table II that the exponent by which 10 must be affected to give numbers between 1 and .1 must lie between 0 and -1.

In like manner it becomes evident that the exponent by which 10 must be affected to give a decimal fraction with the first significant figure in the second decimal place must lie between -1 and -2; to give a decimal fraction with the first significant figure in the third decimal place, the exponent must lie between -2 and -3; and in general to give a decimal fraction with the first significant figure in the nth decimal place, the exponent must lie between $-(n-1)$ and $-n$.

The exponent by which 10 must be affected to give any number is called the logarithm of that number to base 10.

The logarithm to base 10 of a number is called the common logarithm of that number and is denoted by writing $\log_{10}$ before the number. For example, $\log_{10} 100 = 2$; $\log_{10} .01 = -2$.

PROBLEM 91. If the base of a system of logarithms is 3, what are the logarithms of 27, 243, $\frac{1}{27}$, $\frac{1}{243}$?

PROBLEM 92. Can 1 be used as the base of a system of logarithms?

PROBLEM 93. Can -5 be used as the base of a system of logarithms?

ART. 22. — COMPUTATION OF COMMON LOGARITHMS

(*a*) Let n_1, n_2, n_3, n_4 represent the logarithms of the numbers N_1, N_2, N_3, N_4 to base 10.

By definition $N_1 = 10^{n_1}$, $N_2 = 10^{n_2}$, $N_3 = 10^{n_3}$, $N_4 = 10^{n_4}$.

The product of the numbers N_1, N_2, N_3, N_4, by the law of indices in Algebra, is

$$N_1N_2N_3N_4 = 10^{n_1+n_2+n_3+n_4}.$$

Hence by the definition of the logarithm,

$$\log (N_1N_2N_3N_4) = n_1 + n_2 + n_3 + n_4$$
$$= \log N_1 + \log N_2 + \log N_3 + \log N_4,$$

that is, the logarithm of the product equals the sum of the logarithms of the factors.

(*b*) Any number in the decimal notation may be written as a common fraction whose numerator is integral and whose denominator is some power of 10. For example,

$$375.485 = \frac{375485}{1000} = 375485 \times 10^{-3}.$$

Applying the rule just proved to this product,

$$\log 375.485 = \log 375485 + \log 10^{-3}.$$

As a consequence of (*b*) the direct computation of the logarithms of whole numbers only is necessary.

As a consequence of (*a*) the direct computation of the logarithms of prime numbers only is necessary.

The logarithms to base 10 of the prime numbers may be computed to any required degree of approximation by using the infinite series

$$(1)\quad \log_{10}(1+x) = \log_{10} x + 2(0.43429448)\Big[\frac{1}{2x+1} + \frac{1}{3(2x+1)^3} + \frac{1}{5(2x+1)^5} + \frac{1}{7(2x+1)^7} + \cdots + \frac{1}{(2n-1)(2x+1)^{2n-1}} + \frac{1}{(2n+1)(2x+1)^{2n+1}} + \cdots\Big],$$

which is derived in the Differential Calculus.

The application of this formula requires the $\log_{10} x$, that is, the logarithm of the number one less than the given prime, to be known.

Let it be required to compute the common logarithms of 2, 4, 5 to five decimal places.

1. Place $x = 1$ in (1). There results

$$\log_{10} 2 = \log_{10} 1 + 2(0.43429448)\left[\frac{1}{3} + \frac{1}{3 \cdot 3^3} + \frac{1}{5 \cdot 3^5} + \frac{1}{7 \cdot 3^7} + \cdots\right].$$

The computations are carried out to six decimal places in order to obtain the correction in the fifth decimal place. The following is a convenient arrangement of the computation. Begin by computing the sum of the series in brackets.

$\frac{1}{3} = 0.333333$	$\frac{1}{3} = 0.333333$
$\frac{1}{3^3} = 00.37037$	$\frac{1}{3} \cdot \frac{1}{3^3} = 0.012246$
$\frac{1}{3^5} = 0.004115$	$\frac{1}{5} \cdot \frac{1}{3^5} = 0.000823$
$\frac{1}{3^7} = 0.000457$	$\frac{1}{7} \cdot \frac{1}{3^7} = 0.000065$
$\frac{1}{3^9} = 0.000051$	$\frac{1}{9} \cdot \frac{1}{3^9} = 0.000006$
$\frac{1}{3^{11}} = 0.000006$	$\frac{1}{11} \cdot \frac{1}{1^{11}} = 0.000001$
$\frac{1}{3^{13}} = 0.000007$	0.346574

The sum of all the omitted terms of the bracketed infinite series is

$$\frac{1}{13}\cdot\frac{1}{3^{13}}+\frac{1}{15}\cdot\frac{1}{3^{15}}+\frac{1}{17}\cdot\frac{1}{3^{17}}+\frac{1}{19}\cdot\frac{1}{3^{19}}+\cdots,$$

which is less than

$$\frac{1}{13}\cdot\frac{1}{3^{13}}\left(1+\frac{1}{3^2}+\frac{1}{3^4}+\frac{1}{3^6}+\cdots\right).$$

The series in parenthesis is an infinite decreasing geometric progression, whose sum is found by the formula $S=\frac{a}{1-r}$ to be $\frac{9}{8}$.

Hence the sum of all the omitted terms of the original infinite series is less than

$$\frac{1}{13}\cdot\frac{1}{3^{13}}\cdot\frac{9}{8}=0.00000006,$$

and does not influence the sixth decimal place.

It follows that

$$\begin{aligned}\log_{10}2 &= \log_{10}1+2\,(0.43429448)(0.346574)\\ &= 0.30103.\end{aligned}$$

2. $\log_{10}4=2\log_{10}2=0.60206.$

3. $\log_{10}5=\log_{10}(1+4)=\log_{10}4+2\,(0.43429448)$

$$\left[\frac{1}{9}+\frac{1}{3}\cdot\frac{1}{9^3}+\frac{1}{5}\cdot\frac{1}{9^5}+\frac{1}{7}\cdot\frac{1}{9^7}+\frac{1}{9}\cdot\frac{1}{9^9}+\cdots\right].$$

Computation of sum of infinite series to six decimal places:

$$\frac{1}{9} = 0.111111 \qquad\qquad \frac{1}{9} = 0.111111$$

$$\frac{1}{9^3} = 0.001372 \qquad\qquad \frac{1}{3}\cdot\frac{1}{9^3} = 0.000457$$

$$\frac{1}{9^5} = 0.000017 \qquad\qquad \frac{1}{5}\cdot\frac{1}{9^5} = 0.000003$$

$$\overline{0.111571}$$

$$\frac{1}{9^7} = 0.0000002$$

The error committed by omitting all the terms after the third is

$$\frac{1}{7}\cdot\frac{1}{9^7}+\frac{1}{9}\cdot\frac{1}{9^9}+\frac{1}{11}\cdot\frac{1}{9^{11}}+\frac{1}{13}\cdot\frac{1}{9^{13}}+\cdots,$$

which is less than

$$\frac{1}{7}\cdot\frac{1}{9^7}\left(1+\frac{1}{9^2}+\frac{1}{9^4}+\frac{1}{9^6}+\cdots\right)=0.00000003,$$

and does not affect the sixth decimal place.

It follows that

$$\begin{aligned}\log_{10}5 &= \log_{10}4 + 2(0.43429448)(0.111571)\\ &= 0.60206 + 0.9691\\ &= 0.69897.\end{aligned}$$

PROBLEM **94.** Compute $\log_{10} 3$.

PROBLEM **95.** Compute $\log_{10} 7$.

PROBLEM **96.** Compute $\log_{10} 11$.

PROBLEM **97.** Compute $\log_{10} 13$.

PROBLEM **98.** Compute $\log_{10} 17$.

Art. 23. — Arrangement of Tables of Logarithms

The common logarithm of a number consists of two parts, an integer and a decimal fraction.

From a table of five-place logarithms it is found that $\log_{10} 97756 = 4.99014$, hence $10^{4.99014} = 97756$.

Dividing the last equation by 10 several times in succession,

$10^{4.99014} = 97756$	hence $\log_{10} 97756 = 4.99014$
$10^{3.99014} = 9775.6$	hence $\log_{10} 9775.6 = 3.99014$
$10^{2.99014} = 977.56$	hence $\log_{10} 977.56 = 2.99014$
$10^{1.99014} = 97.756$	hence $\log_{10} 97.756 = 1.99014$
$10^{0.99014} = 9.7756$	hence $\log_{10} 9.7756 = 0.99014$
$10^{\bar{1}.99014} = 0.97756$	hence $\log_{10} 0.97756 = \bar{1}.99014$
$10^{\bar{2}.99014} = 0.097756$	hence $\log_{10} 0.097756 = \bar{2}.99014$
$10^{\bar{3}.99014} = 0.0097756$	hence $\log_{10} 0.0097756 = \bar{3}.99014$
$10^{\bar{4}.99014} = 0.00097756$	hence $\log_{10} 0.00097756 = \bar{4}.99014$

The minus sign placed over the 1, 2, 3, 4 in the last four logarithms denotes that only the integral part of the logarithm is negative. The decimal part of the logarithm is always positive.

This manner of writing the logarithm of a number is adopted in order that the decimal part of the logarithm shall depend only on the significant figures of the number and be independent of the position of the decimal point in the number.

When the logarithm is written in this manner, the integral part is called the characteristic; the decimal part, the mantissa.

The characteristic is determined by the rule:

If the number is greater than 1, the characteristic is positive and less by unity than the number of integral places in the number.

If the number is less than 1, the characteristic is negative and equal to the number of the decimal place occupied by the first significant figure of the number.

The mantissa of the logarithm of a number is taken from the table of logarithms.

If five-place logarithms are used, it is absurd to attempt to compute beyond six significant figures, and even the sixth figure and frequently the fifth figure is unreliable. The truth of this statement will become evident from the computations themselves.

Example 1. Let it be required to find the common logarithm of 375.658 from the five-place table.

By rule the characteristic is 2. The mantissa corresponding to the first four figures of the number is taken directly from the table. The logarithms of the two consecutive numbers of four significant figures between which the given number lies are

$$\log 375.7 = 2.57484$$
$$\log 375.6 = 2.57473$$

The difference in the mantissa corresponding to 1 unit of the fourth significant figure of the number is 11 units of the fifth decimal place of the mantissa.

Assuming that the mantissa increases uniformly while the number increases by one unit of the fourth significant figure, it follows that the increase of the mantissa due to the fifth figure of the number is $.5 \times 11 = 5.5$ units of the fifth decimal place, and the increase due to the sixth figure of the number is $.08 \times 11 = .88$ units of the fifth decimal place of the mantissa.

Collecting the results,

$$\begin{aligned} \log 375.6 &= 2.57473 \\ \text{increase for } .05 &= .000055 \\ \text{increase for } .008 &= \underline{.0000088} \\ \text{hence} \quad \log 375.658 &= 2.57479 \end{aligned}$$

Observe that if any figure other than zero stood in the sixth significant place of the number, the logarithm of the number to five decimal places would remain the same.

The method of finding the corrections of the logarithm for the fifth and sixth figures of the number is called interpolation.

EXAMPLE 2. Let it be required to find the number whose common logarithm is 3.72564.

The characteristic shows that there are four integral places in the number required. The figures of the number corresponding to the mantissa are found from the table of logarithms.

Take from the table the logarithm next lower than the given logarithm and subtract it from the given logarithm and from the next higher logarithm.

$$\begin{array}{ll} 3.72564 & \\ \underline{3.72558} = \log 5306 & \\ \quad\;\; 6 & \end{array} \qquad\qquad \begin{array}{ll} 3.72567 = \log 5307 \\ \underline{3.72558} = \log 5306 \\ \quad\;\; 9 \end{array}$$

A difference of 9 units in the fifth decimal place of the mantissa gives a difference of 1 unit in the fourth significant figure of the number. Assuming that the rate of change of the mantissa from log 5306 to log 5307 is uniform, the increase in the number due to the increase of the mantissa by 6 units in the fifth decimal place is $6 \div 9 = .67$ units of the fourth significant figure of the number. Hence $3.72564 = \log_{10} 5306.67$.

The sixth figure of this number is uncertain. It may have any value from 2 to 9. This makes the uncertainty in the number extend to the fifth significant figure, which may be either 6 or 7 if the number is to be expressed correct to five significant figures.

It appears from these examples that the use of five-place logarithmic tables limits the computation to four or five significant figures.

PROBLEM 99. From a five-place table find the logarithm of 1476.38. What other numbers of six figures have the same logarithm?

PROBLEM 100. From a five-place table find the logarithm of 8754.88. What other numbers of six figures have the same logarithm?

PROBLEM 101. From a five-place table find the number whose logarithm is 2.14896 and determine the limits of uncertainty of the number.

PROBLEM 102. From a five-place table find the number whose logarithm is 4.79658 and determine the limits of uncertainty of the number.

PROBLEM 103. How does the change in the mantissa corresponding to a change of 1 unit in the fourth significant figure in the number vary with the number?

Art. 24.—Computation by Means of Logarithms

Logarithms are used to shorten the operations of multiplication, division, involution, and evolution.

These operations are performed by applying the following rules:

Let $\log_{10} M = m$, $\log_{10} N = n$. By definition

$$M = 10^m,\ N = 10^n.$$

The product of M and N is

$$MN = 10^{m+n}.$$

Hence by definition

$$\log_{10}(MN) = m + n = \log_{10} M + \log_{10} N;\ \text{that is,}$$

I. The logarithm of the product equals the sum of the logarithms of the factors.

The quotient of M and N is

$$\frac{M}{N} = 10^{m-n}.$$

Hence by definition

$$\log_{10}\frac{M}{N} = m - n = \log_{10} M - \log_{10} N;\ \text{that is,}$$

II. The logarithm of the quotient equals the logarithm of the dividend minus the logarithm of the divisor.

Raising both sides of the equation $M = 10^m$ to the power p,

$$M^p = 10^{pm}.$$

Hence by definition

$$\log_{10} M^p = pm = p \log_{10} M;\ \text{that is,}$$

III. The logarithm of the power of a number equals the logarithm of the number multiplied by the exponent of the power.

Extracting the r root of both sides of the equation

$$M = 10^m,$$

$$M^{\frac{1}{r}} = 10^{\frac{m}{r}}.$$

Hence by definition

$$\log_{10} M^{\frac{1}{r}} = \frac{m}{r} = \frac{1}{r} \log_{10} M; \text{ that is,}$$

IV. The logarithm of the root of a number equals the logarithm of the number divided by the index of the root.

In working a numerical problem, first, decide on a method of computation; second, arrange a scheme of computation; third, perform the computation.

EXAMPLE. Let it be required to compute the lateral area and the volume of a cone of revolution, altitude 29.75 feet, radius of base 12.15 feet.

Denoting the lateral area by A, the volume by V, the radius of the base by R, and the altitude by H,

$$(a)\quad A = \pi R \sqrt{H^2 + R^2},$$

$$(b)\quad V = \tfrac{1}{3} \pi R^2 H.$$

Applying logarithms to (a),

$$\log A = \log \pi + \log R + \tfrac{1}{2} \log (H^2 + R^2).$$

Scheme of computation:

$$\begin{array}{ll} \log H = 1.47349 & H^2 = 885.08 \\ \log H^2 = 2.94698 & R^2 = 147.62 \\ \log R^2 = 2.16916 & R^2 + H^2 = 1032.70 \\ & \log (H^2 + R^2) = 3.01396 \end{array}$$

$$\begin{aligned} \log \pi &= 0.49715 \\ + \log R &= 1.08458 \\ + \tfrac{1}{2}\log (H^2 + R^2) &= \underline{1.50698} \\ \log A &= 3.08871 \\ A &= 1226.63 \text{ square feet.} \end{aligned}$$

Applying logarithms to (*b*),

$$\log V = \log \pi + 2 \log R + \log H - \log 3.$$

Scheme of computation:

$$\begin{aligned} \log \pi &= 0.49715 \\ 2 \log R &= 2.16916 \\ \log H &= \underline{1.47349} \\ & 4.13980 \\ - \log 3 &= \underline{0.47712} \\ \log V &= 3.66268 \\ V &= 4599.2 \text{ cubic feet.} \end{aligned}$$

In these values of A and V the fifth and sixth figures are uncertain.

PROBLEM 104. The chord of a circular segment of one base is 3.25 inches, the altitude of the segment is 1.15 inches. Find the diameter of the circle.

PROBLEM 105. Find the area of the surface of the sphere whose radius is 6.7 inches.

PROBLEM 106. Find the volume of the sphere whose diameter is 15.36 feet.

PROBLEM 107. Find the area of the equilateral triangle whose side is 27.16 feet.

PROBLEM 108. Find the capacity in gallons of a rectangular cistern whose dimensions are 10.5, 6.35, and 4.65 feet.

PROBLEM 109. Find the capacity in gallons of a tank in the shape of a circular cylinder, radius of base 8.35 feet, altitude 12.46 feet.

PROBLEM 110. Find the volume of a regular pyramid whose base is a square 4.18 feet on a side and whose lateral edge is 11.27 feet.

PROBLEM 111. The radius of a circle is 8.57 feet. Find the least chord that can be drawn through a point 3.25 feet from the center.

PROBLEM 112. The diameter of a circle is 15.28 inches. Find the length of the tangent to the circle from a point 20.15 inches from the center.

PROBLEM 113. In a right triangle the sides about the right angle are 5.75 feet and 8.17 feet. Find the perpendicular from the vertex of the right angle to the hypotenuse.

PROBLEM 114. An excavation 1.5 yards deep, rectangular at top and bottom, and in the form of a frustum of a pyramid has its upper base 10 yards wide and 16 yards long, and the lower base 7.5 yards wide. How many cubic yards were removed to make the excavation?

PROBLEM 115. If the atmosphere extends to a height of 45 miles above the earth's surface, what is the ratio of its volume to the volume of the earth, assuming the earth to be a sphere with a diameter of 7912 miles.

PROBLEM 116. The circumference of the base of a conic frustum is 49.3 feet, the diameter of the top is 12.5 feet, the altitude is 15.7 feet. Find the surface and volume.

PROBLEM 117. On a sphere whose radius is 11.75 inches, what is the area of a zone whose altitude is 3.25 inches?

PROBLEM 118. On a sphere whose radius is 28.5 feet find the area of a spherical triangle whose angles are 120°, 130°, and 140°.

PROBLEM 119. On a sphere whose diameter is 25 inches find the area of the spherical polygon whose angles are 110°, 120°, 130°, and 140°.

PROBLEM 120. The sides of a triangle are each 12.49 inches. What is the volume of the solid generated by revolving the triangle about one side?

PROBLEM 121. A dome is in the form of a spherical zone of one base. Its height is 30 feet and the diameter of the base 60 feet. Find its area.

PROBLEM 122. Find the volume of the frustum of the cone of revolution, the radii of whose bases are 12.5 feet and 7.25 feet and whose slant height is 9.75 feet.

PROBLEM 123. The base of a triangle is 6.5 feet, the altitude is 15 feet. Find at what distances from the base lines parallel to the base must be drawn to divide the triangle into three equal areas.

PROBLEM 124. The radius of the base of a cone is 7.2 feet, the altitude is 25 feet. Find at what distances from the base planes parallel to the base must be drawn to divide the cone into three equal volumes.

PROBLEM 125. Two parallel planes divide the diameter of a sphere into three equal parts. The radius of the sphere is 12 feet. Find the ratio of the volumes of the solids into which the sphere is divided.

ART. 25. — THE COMPOUND INTEREST FORMULA

If a sum of money P is invested at compound interest at R per cent, the amount at the end of the first year is $P(1+R)$.

This amount is the principal at the beginning of the second year, hence the amount at the end of the second year is $P(1+R)^2$.

In like manner the amount at the end of the third year is found to be $P(1+R)^3$.

In general the amount at the end of t years is

$$A = P(1+R)^t.$$

From this it follows that the present worth of a sum of money A due t years hence, if money is worth R per cent compound interest is

$$P = A(1+R)^{-t}.$$

EXAMPLE 1. What is the present worth of an annual pension of $500 to run for 7 years, the first payment to be made one year from date, and money being worth 5 % compound interest?

The present worth of the first payment is 500×1.05^{-1}
The present worth of the second payment is 500×1.05^{-2}
The present worth of the third payment is 500×1.05^{-3}
The present worth of the fourth payment is 500×1.05^{-4}
The present worth of the fifth payment is 500×1.05^{-5}
The present worth of the sixth payment is 500×1.05^{-6}
The present worth of the seventh payment is 500×1.05^{-7}
The present worth of the seven payments is

$$P = 500[1.05^{-1} + 1.05^{-2} + 1.05^{-3} + 1.05^{-4} + 1.05^{-5} + 1.05^{-6} + 1.05^{-7}]$$

$$= 500\frac{1.05^{-8} - 1.05^{-1}}{1.05^{-1} - 1} = 500\frac{1.05^{-7} - 1}{1 - 1.05}$$

$$= 500\frac{1 - 1.05^{-7}}{.05}$$

$$= 10000(1 - 1.05^{-7}).$$

To compute $x = 1.05^{-7}$, pass to logarithms,

$$\begin{aligned} \log x &= -7 \log 1.05 \\ &= -0.14833 \\ &= \bar{1}.85167. \\ \therefore x &= 0.710667. \end{aligned}$$

It follows that

$$\begin{aligned} P &= 10000(1 - 0.710667) \\ &= \$\,2893.33. \end{aligned}$$

EXAMPLE 2. In how many years will an annual payment of \$300 meet principal and interest of a debt of \$ 2000, money being worth 5 % compound interest?

Let t represent the required number of years. At the end of t years the debt amounts to 2000×1.05^t.

If the first annual payment is to be made at the end of the first year, its amount at the end of t years is $300 \times 1.05^{t-1}$. The amount at the end of t years of the second annual payment is $300 \times 1.05^{t-2}$, and so on for the successive payments. The last annual payment is to be made at the end of the t years and its value then is \$300.

Hence the sum of the amounts of the t annual payments is

$$A = 300[1 + 1.05 + 1.05^2 + 1.05^3 + \cdots + 1.05^{t-1}]$$

$$= 300\frac{1.05^t - 1}{1.05 - 1} = 6000(1.05^t - 1).$$

At the end of t years the amount of the debt is to equal the amount of the annual payments. Hence

$$2000 \times 1.05^t = 6000(1.05^t - 1).$$

Solving this equation for 1.05^t,

$$1.05^t = 1.5.$$

Passing to logarithms,

$$t \log 1.05 = \log 1.5.$$

Solving for t,

$$t = \frac{\log 1.5}{\log 1.05} = 8.31 \text{ years}.$$

PROBLEM 126. What is the present worth of \$2000 due 10 years hence without interest, if money is worth 6 % compound interest?

PROBLEM 127. A man pays a premium of \$104 per annum on a life policy of \$3500 for twenty years before death. Money being worth 5% compound interest, does the insurance company gain or lose, and how much?

PROBLEM 128. A town, whose property has the assessed valuation of \$7,325,000 and whose annual tax rate is 19 mills on \$1, issues bonds to the amount of \$225,000 to build a sewerage system, these bonds being at $4\frac{1}{2}$ per cent interest and maturing in 15 years. An extra tax is to be laid to meet this interest and to provide a sinking fund to redeem the bonds, the rate of interest in the sinking fund being $3\frac{1}{2}$ per cent compounded annually. How many mills must be added to the tax rate for this purpose?

PROBLEM 129. A town finds it necessary to build a bridge. The cost of a stone bridge is \$20,000, the cost of maintenance \$50 per year, the life of the bridge 50 years. The cost of a steel bridge is \$10,000, the cost of maintenance \$150 per year, the life of the bridge 15 years. The town can borrow money at 4 per cent compound interest and realize 4 per cent compound interest on its sinking fund. What is the cost per year to the town of each of the two bridges?

ART. 26. — THE SLIDE RULE

If line segments are laid off from one point of a straight line proportional to the mantissas of the logarithms of numbers from 100 to 1000, and the number whose logarithm is represented by the line segment is written over the terminal point of the line segment, there will be formed a logarithmic scale.

From a logarithmic scale nine inches long constructed in this manner the line segment which represents the logarithm of a number of three figures, and conversely the number to three figures whose logarithm is represented by a given line segment, can be read off with considerable accuracy.

To find the product of two numbers, apply to the logarithmic scale the sum of the line segments representing the logarithms of the numbers. Over the terminal point of the sum of these line segments stands the number which is the product of the two given numbers.

Fig. 15.

To find the quotient of two numbers, apply to the logarithmic scale the difference of the line segments representing respectively the dividend and divisor. Over the terminal point of the difference of these line segments stands the number which is the quotient of the two given numbers.

The slide rule consists of two logarithmic scales, one of which slides along the other.

To find the product of two numbers by means of the slide rule, locate one number on the fixed logarithmic scale and bring the initial point of the sliding scale over this number. Now locate the second number on the sliding scale and in line with this number on the fixed scale the product is found.

To find the quotient of two numbers by means of the slide rule, locate the dividend on the fixed scale and the divisor on the sliding scale and bring these two numbers over each other. The quotient is the number on the fixed scale in line with the initial point of the sliding scale.

To find the square root of a number by means of the slide rule, bisect the line segment which represents the logarithm of the number and find the number whose logarithm is represented by the corresponding line segment.

In like manner any root of a number may be found by means of the slide rule.

CHAPTER VII

ON LIMITS

Art. 27. — The Infinite Decreasing Geometric Progression

The formula for the sum of the first n terms of a geometric progression whose first term is a and ratio r is

$$S_n = \frac{ar^n - a}{r - 1} = \frac{a}{1 - r} - \frac{ar^n}{1 - r},$$

whence

$$S_n - \frac{a}{1 - r} = -\frac{ar^n}{1 - r}.$$

In these formulas a and r for a particular progression have fixed values; that is, they are constants; while n may be any positive integer and the value of S_n depends on n; that is, n and S_n are variables.

If r is less than unity and n increases without limit, the progression becomes an infinite decreasing geometric progression.

The difference between the variable S_n and the constant $\frac{a}{1 - r}$ is numerically equal to $\frac{ar^n}{1 - r}$. When r is less than one, if any numerical value, however small, be assigned in advance, a value of n can be computed such that for this value of n and all larger values of n the difference between the variable S_n and the constant $\frac{a}{1-r}$ shall be less than the value assigned.

This is expressed by saying that the limit of the variable S_n when n is infinitely increased is the constant $\frac{a}{1-r}$, and it is written

$$\lim_{n=\infty} S_n = \frac{a}{1-r}.$$

In general, if a variable approaches a constant in such a manner that the numerical value of the difference between the variable and the constant becomes and remains less than any value, however small, that can be assigned in advance, the variable is said to approach the constant as its limit.

PROBLEM 130. Find the limit of the sum of the first n terms of $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \cdots$ when n is infinitely increased.

PROBLEM 131. For what values of n does the sum of the first n terms of $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \cdots$ differ from the limit of the sum by less than .00001?

ART. 28. — THE LENGTH OF A CURVED LINE

The approximate length of a curved line is found by inscribing or circumscribing a broken straight line the parts of which fit the corresponding parts of the curved line as closely as may be practicable. The approximation is made closer by diminishing the lengths of the parts of the broken line. The limits of accuracy of direct measurement are soon reached.

The length of a curved line is defined as the limit of the inscribed or circumscribed broken line when the number of parts of the broken line is infinitely increased in such a manner that the length of each part of the broken line is infinitely diminished.

By using this definition the length of a curved line may be computed either exactly or to any required degree of approximation.

Art. 29. — The Computation of π

By definition

$$\pi = \frac{\text{circumference of circle}}{\text{diameter}} = \frac{\text{semicircumference}}{\text{radius}}.$$

Hence if the radius is made unity, π equals the length of the semicircumference.

If c_n represents the length of the side of the regular polygon of n sides inscribed in the circle, and t_n represents the side of the regular polygon of n sides circumscribed about the circle, the semicircumference always lies between $\frac{1}{2}nc_n$ and $\frac{1}{2}nt_n$ and the semicircumference is the common limit of $\frac{1}{2}nc_n$ and $\frac{1}{2}nt_n$ when n is infinitely increased.

It is readily proved that

(1) $$t_n = \frac{2\,c_n}{\sqrt{4 - c_n^2}}$$

(2) $$c_{2n} = \sqrt{2 - \sqrt{4 - c_n^2}}.$$

The side of the inscribed regular hexagon is the radius of the circle. Hence $c_6 = 1$. By formula (1) the side of the circumscribed regular hexagon, t_6, is computed. By formula (2) the side of the inscribed regular polygon of 12 sides, c_{12}, is computed.

By repeated successive application of formulas (1) and (2) the sides of inscribed and circumscribed regular polygons, the number of whose sides is six multiplied by any power of two, may be computed.

The results of twelve successive applications of formulas (1) and (2) give the following table of values for the semiperimeters of the polygons:

n	$\frac{1}{2} nc_n$	$\frac{1}{2} nt_n$
6	3.0000000	3.4641016
12	3.1058285	3.2153903
24	3.1326286	3.1596599
48	3.1393502	3.1460862
96	3.1410319	3.1427146
192	3.1414524	3.1418730
384	3.1415576	3.1416627
768	3.1415836	3.1416101
1536	3.1415904	3.1415970
3072	3.1415921	3.1415937
6144	3.1415925	3.1415929
12288	3.1415926	3.1415927

The semicircumference of the circle, and therefore π, always lies between the semiperimeters of the inscribed and circumscribed regular polygons of the same number of sides.

Hence the value of π correct to six decimal places is 3.141593.

By direct measurement the value of π could not be determined beyond the second or third decimal place. There is no limit to the approximation that may be reached by computation.

Art. 30. — An Important Limit

The derivation of many of the formulas for the determination, exact or approximate, of lengths, areas, and volumes is based on the limit, when n is infinitely increased, of the ratio of the sum of the p powers of the first n natural numbers to the $p+1$ power of n, or expressed as a formula,

$$\lim_{n=\infty} \frac{1^p + 2^p + 3^p + 4^p + 5^p + \cdots + n^p}{n^{p+1}}.$$

If p is any positive integer, and if $a > b$,

$$\frac{a^{p+1} - b^{p+1}}{a - b} = a^p + a^{p-1}b + a^{p-2}b^2 + \cdots + ab^{p-1} + b^p$$
$$> (p+1)b^p,$$

which may be written

(1) $$(p+1)b^p < \frac{a^{p+1} - b^{p+1}}{a - b}.$$

From (1), by placing b successively equal to

$$1, 2, 3, 4, 5, \cdots, n \text{ and } a = b + 1,$$
$$(p+1)\, 1^p < 2^{p+1} - 1^{p+1},$$
$$(p+1)\, 2^p < 3^{p+1} - 2^{p+1},$$
$$(p+1)\, 3^p < 4^{p+1} - 3^{p+1},$$
$$(p+1)\, 4^p < 5^{p+1} - 4^{p+1},$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$(p+1)\, n^p < (n+1)^{p+1} - n^{p+1}.$$

Adding these n inequalities,

$$(p+1)(1^p+2^p+3^p+4^p+\cdots+n^p)<(n+1)^{p+1}-1^{p+1}.$$

Dividing the last inequality by $(p+1)n^{p+1}$,

$$\frac{1^p+2^p+3^p+4^p+\cdots+n^p}{n^{p+1}}<\frac{1}{p+1}\left\{\left(1+\frac{1}{n}\right)^{p+1}-\frac{1}{n^{p+1}}\right\}$$

$$<\frac{1}{p+1}\left\{1+\frac{p+1}{n}+\frac{p(p+1)}{2!n^2}+\cdots+\frac{p+1}{n^{p-1}}\right\},$$

and finally

$$\frac{1^p+2^p+3^p+4^p+\cdots+n^p}{n^{p+1}}-\frac{1}{p+1}$$

$$<\frac{1}{p+1}\left\{\frac{p+1}{n}+\frac{p(p+1)}{2!n^2}+\cdots+\frac{p+1}{n^{p-1}}\right\}.$$

If now n is infinitely increased, since $\frac{1}{n}$ becomes infinitely small and the number of terms in the bracket is $p-1$, a finite number,

$$\lim_{n=\infty}\frac{1^p+2^p+3^p+4^p+\cdots+n^p}{n^{p+1}}=\frac{1}{p+1}.$$

Art. 31. — Length and Area of Involute of Circle

If a fine inextensible string, coiled around a circle, be uncoiled from the circle and the string be always kept taut, the free extremity of the string will trace the curve called the involute of the circle.

Denote the radius of the circle by r and suppose the length of the string uncoiled to be $r\theta$. Divide the arc $r\theta$ into n equal parts and inscribe in the arc a regular broken line. Denote by a the length of each part of the broken line. Then $a = 2r \sin \frac{\theta}{2n}$. If the string were wound around the broken line and then uncoiled, if S_n represents

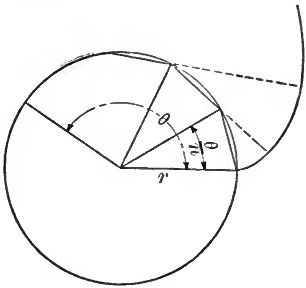


FIG. 16.

the length of the curve traced by the extremity of the string and A_n the area swept over by the string,

$$S_n = 2r\sin\frac{\theta}{2n}\left[\frac{\theta}{n} + \frac{2\theta}{n} + \frac{3\theta}{n} + \frac{4\theta}{n} + \cdots + \frac{n\theta}{n}\right],$$

$$A_n = 2r^2\sin^2\frac{\theta}{2n}\left[\frac{\theta}{n} + \frac{2^2\theta}{n} + \frac{3^2\theta}{n} + \cdots + \frac{n^2\theta}{n}\right].$$

When n is infinitely increased, the broken line approaches the circular arc $r\theta$ as its limit, and the involute of the broken line approaches the involute of the circular arc as its limit.

Since $\lim\limits_{n=\infty} \sin\frac{\theta}{2n} = \lim\limits_{n=\infty} \frac{\theta}{2n}$, it follows that

$$\lim_{n=\infty} S_n = r\theta^2 \lim_{n=\infty}\left[\frac{1+2+3+4+\cdots+n}{n^2}\right] = \tfrac{1}{2}r\theta^2,$$

$$\lim_{n=\infty} A_n = \tfrac{1}{2}r^2\theta^3 \lim_{n=\infty}\left[\frac{1^2+2^2+3^2+4^2+\cdots+n^2}{n^3}\right] = \tfrac{1}{6}r^2\theta^3.$$

Problem 132. Find the length and area of the involute of the circle radius r extending from $\theta = 0$ to $\theta = 2\pi$.

Problem 133. Find the length and area of the involute of the circle radius r extending from $\theta = 2\pi$ to $\theta = 4\pi$.

CHAPTER VIII

GRAPHIC ALGEBRA

Art. 32.—The Graph of an Equation

If an equation $f(x, y) = 0$ is satisfied by the coördinates of every point (x, y) of a line, and if every point (x, y) whose coördinates satisfy the equation is a point of the line, the line is called the graph of the equation.

To construct the graph of an equation $f(x, y) = 0$, compute the values of y for different values of x, and locate the points whose coördinates are the pairs of corresponding real values of x and y. The graph of the equation is the smooth curve drawn through the points located.

Example. Construct the graph of $x^2 + y^2 = 9$.

The corresponding pairs of real values of x and y are

$x =$	-3	-2	-1	0	$+1$	$+2$	$+3$
$y =$	0	± 2.24	± 2.83	± 3	± 2.83	± 2.24	$0.$

For $x > +3$ and for $x < -3$, y is imaginary. Hence the graph lies between the lines $x = 3$ and $x = -3$. The graph also lies between the lines $y = 3$ and $y = -3$.

Locating the points whose coördinates are the corresponding pairs of values of x and y and drawing a smooth line through these points, it becomes evident that the graph resembles a circle.

In fact the form of the equation shows at once that the graph is a circle whose radius is 3 and center the origin.

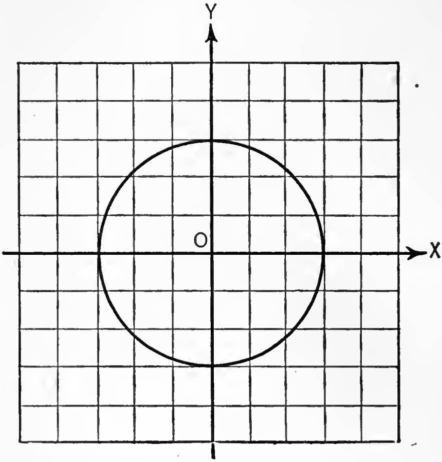


FIG. 17.

Notice that for all points (x, y) on the circumference of the circle $x^2 + y^2 - 9 = 0$; that for all points (x, y) within the circle $x^2 + y^2 - 9 < 0$; and that for all points (x, y) without the circle $x^2 + y^2 - 9 > 0$.

PROBLEM 134. Construct the graph of $3x - 4y = 12$.

PROBLEM 135. Construct the graph of $x^2 + y^2 = 16$.

PROBLEM 136. Construct the graph of $\frac{x^2}{4} + \frac{y^2}{9} = 1$.

PROBLEM 137. Construct the graph of $\frac{x^2}{4} - \frac{y^2}{9} = 1$.

PROBLEM 138. Construct the graph of $y^2 = 4x$.

PROBLEM 139. Construct the graph of $x^2 = 4y$.

PROBLEM 140. Construct the graph of $y = 3x - 5x^2$.

PROBLEM 141. Construct the graph of $y = 5 - x + 6x^2$.

ART. 33. — EQUATIONS OF LINES

Suppose the relation between the coördinates of any point $P(x, y)$ to be expressed by the first degree equation

$$Ax + By + C = 0. \tag{1}$$

Let $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, $P_3(x_3, y_3)$ be any three points whose coördinates satisfy this equation, that is,

$$Ax_1 + By_1 + C = 0,$$
$$Ax_2 + By_2 + C = 0,$$
$$Ax_3 + By_3 + C = 0.$$

The elimination of A, B, C gives

$$x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) = 0.$$

The left-hand member of this equation is double the area of the triangle whose vertices are P_1, P_2, P_3. It follows that the area of the triangle whose vertices are any three points whose coördinates satisfy equation (1) is zero; that is, these three points lie in a straight line. Hence the graph of every first degree equation is a straight line.

The equation

$$x^2 + y^2 = a^2 \tag{2}$$

expresses the fact that the distance from the origin to the point (x, y) is a. Therefore all points (x, y) whose coordinates satisfy equation (2) are located on the circle whose radius is a and center at origin.

From equation (2)

$$y = \sqrt{a^2 - x^2}.$$

Multiplying y by $\frac{b}{a}$, that is, multiplying each ordinate of the circle by the same factor $\frac{b}{a}$,

$$y = \frac{b}{a}\sqrt{a^2 - x^2},$$

which reduces to

(3) $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

The curve formed by the points (x, y) whose coördinates satisfy equation (3) is called the ellipse.

The curve formed by the points (x, y) whose coördinates satisfy the equation

(4) $$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

is called the hyperbola.

The curve formed by the points (x, y) whose coördinates satisfy the equation

(5) $$y^2 = 2px$$

is called the parabola.

PROBLEM 142. Show that the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is πab.

ART. 34. — GRAPHIC SOLUTION OF EQUATIONS

EXAMPLE 1. Solve the equation $x^3 - 7x + 7 = 0$ graphically.

Construct the graph of $y = x^3 - 7x + 7$.

The abscissas of the points where the graph intersects the X-axis are the roots of the equation $x^3 - 7x + 7 = 0$. These roots are found by measurement to be 1.3, 1.7, −3.1.

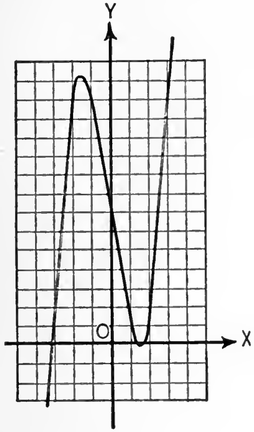


FIG. 18.

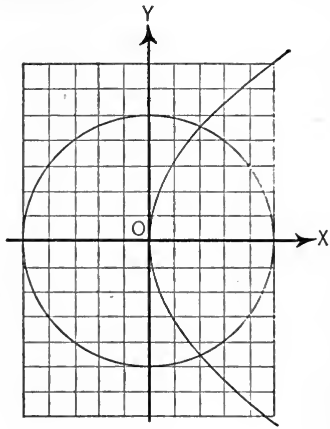


FIG. 19.

EXAMPLE 2. Solve graphically the simultaneous equations $y^2 = 10x$ and $x^2 + y^2 = 25$.

Construct the graphs of these two equations on the same axes of reference. The coördinates of the points of intersection of the graphs are the common solutions of the equations. The points of intersection are (2.1, 4.4) and (2.1, − 4.4).

PROBLEM **143.** Solve graphically $x^2 - 4x - 15 = 0$.

PROBLEM **144.** Solve graphically $x^3 + 7x - 7 = 0$.

PROBLEM **145.** Solve graphically $x^2 + y^2 = 25$,

$$y^2 = 10x - x^2.$$

PROBLEM **146.** Solve graphically $x^2 + y^2 = 25$,

$$z^2 - y^2 = 4.$$

ART. 35. — INEQUALITIES TREATED GRAPHICALLY

EXAMPLE. Show graphically for what pairs of values of x and y the inequalities

$$x - y + 1 > 0$$
$$2x + y - 6 > 0$$
$$-5x + 2y + 10 > 0$$

are true?

Construct the graphs of the equations

$$x - y + 1 = 0,\ 2x + y - 6 = 0,\ -5x + 2y + 10 = 0.$$

The graph of the equation $x - y + 1 = 0$ divides the XY-plane into two parts, such that for the coördinates of every point (x, y) in one part $x - y + 1 > 0$ and for the

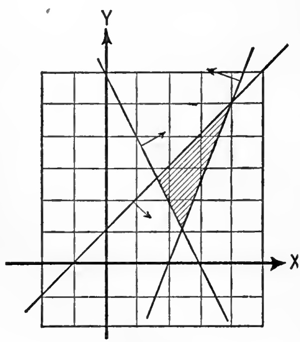


FIG. 20.

coördinates of every point (x, y) in the other part $x - y + 1 < 0$. Determine by trial for which part $x - y + 1 > 0$ and indicate this part of the plane by the arrowhead placed on the graph.

Repeat the same operation with the other two graphs. The shaded triangle in the figure contains all the points (x, y) whose coördinates satisfy simultaneously the three given inequalities.

PROBLEM **147**. Solve graphically the simultaneous inequalities $z - y + 4 > 0$, $x - 2y + 5 < 0$, $x + 5y - 4 < 0$.

PROBLEM **148**. Solve graphically the simultaneous inequalities $x^2 + y^2 - 25 < 0$, $2x + y - 10 > 0$.

CHAPTER IX

AREAS BOUNDED BY CURVES

Art. 36.—Exact Areas

If the equation of a curve is known, the ordinate corresponding to any abscissa may be computed.

If a curve is given whose equation is not known, the ordinate corresponding to any abscissa may be measured.

Denote by a and b the abscissas of the end points of a curve and assume $b > a$. Divide $b - a$ into m equal parts and call each part h, so that $mh = b - a$.

Denote by $y_0, y_1, y_2, y_3, \cdots, y_m$ the ordinates of the points of the curve whose distances from a are the multiples of h.

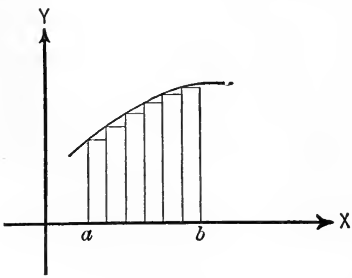


Fig. 21.

Construct rectangles on each successive ordinate and the adjacent part of $b - a$ located to the right of the respective ordinate.

The limit of the sum of the areas of these rectangles when m is infinitely increased is the area bounded by the curve, the ordinates of the end points of the curve and the X-axis.

If the relation between ordinate and abscissa is expressed by an equation of the form

(1) $$y = A + Bx + Cx^2 + Dx^3 + \cdots + Nx^n,$$

this limit can be determined exactly.

Consider the rectangle constructed on the ordinate y_l corresponding to the point whose distance from a is lh. The area of this rectangle is

$$A_l = Ah + Bh^2l + Ch^3l^2 + Dh^4l^3 + \cdots + Nh^{n+1}l^n.$$

Substituting for h its value $\frac{b-a}{m}$, the expression for the area bounded by the curve (1), the X-axis, and the ordinates corresponding to $x = a$, $x = b$ becomes

$$\text{Area} = \underset{m=\infty}{\text{limit}} \sum_{l=0}^{l=m-1} \Big[A(b-a)\frac{1}{m} + B(b-a)^2\frac{l}{m^2} + C(b-a)^3\frac{l^2}{m^3} + \cdots + N(b-a)^{n+1}\frac{l^n}{m^{n+1}} \Big]$$

and finally

$$\text{Area} = A(b-a) + \frac{B}{2}(b-a)^2 + \frac{C}{3}(b-a)^3 + \cdots + \frac{N}{n+1}(b-a)^{n+1}.$$

When $n = 1$, equation (1) represents a straight line and the area is given by the formula

$$\text{Area} = A(b-a) + \frac{B}{2}(b-a)^2.$$

When $n = 2$, equation (1) represents a parabola and the area is given by

$$\text{Area} = A(b-a) + \frac{B}{2}(b+a)^2 + \frac{C}{3}(b+a)^3.$$

PROBLEM 149. Find the area bounded by $y = 3 + 5x$, $x = 2$, $x = 10$, and $y = 0$.

PROBLEM 150. Find the area bounded by $y = 2x + 6x^2$, $x = 0$, $x = 5$, and $y = 0$.

PROBLEM 151. Find the area bounded by $x^2 = 2py$, $x = 0$, $x = a$, and $y = 0$.

ART. 37.—APPROXIMATE EQUATION OF A CURVE

If the equation of a curve is not of the form (1), it may be possible to express approximately the relation between ordinate and abscissa by means of an equation of the form (1) and thus obtain an approximate value of the area.

For example, the equation of the circle is

$$x^2 + y^2 = r^2,$$

from which

$$y = \sqrt{r^2 - x^2} = r\left[1 - \frac{1}{2}\frac{x^2}{r^2} - \frac{1}{2}\cdot\frac{3}{4}\frac{x^4}{r^4} - \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\frac{x^6}{r^6} - \cdots\right].$$

This infinite series is convergent for all values of x numerically less than r, hence for all such values y can be expressed in the form $A + Bx + Cx^2 + Dx^3 + \cdots + Nx^n$ to any required degree of approximation, and consequently the area of the circular segment may be computed to any desired degree of approximation by the methods of the previous article.

If a curve is given whose equation is not known, and if the abscissas of the end points are $x = a$ and $x = b$, divide

$b-a$ into n equal parts and call each part k, so that $b-a=nk$. The ordinates corresponding to the points of division may be measured. Denote by $y_0, y_1, y_2, y_3, \cdots, y_n$ the ordinates corresponding to the abscissas a, $a+k$, $a+2k$, $a+3k$, $\cdots$, $a+nk$.

It is possible to determine an equation of the form

$$y = A + Bx + Cx^2 + Dx^3 + \cdots + Nx^n, \tag{1}$$

which is satisfied by the coördinates of these $n+1$ points of the curve.

For the successive substitution of the coördinates of the $n+1$ points

$$(a, y_0), (a+k, y_1), (a+2k, y_3), \cdots, (a+nk, y_n)$$

in equation (1) furnishes $n+1$ equations of the first degree between the $n+1$ coefficients of equation (1). These equations determine the coefficients.

In general the higher the degree of equation (1), and consequently the greater the number of points the line represented by equation (1) has in common with the given curve, the more closely will the line represented by equation (1) fit the given curve and the closer will be the approximation of the area bounded by equation (1) to the area bounded by the given curve.

Art. 38. — Approximate Areas

First approximation. When $n=1$, equation (1) becomes

$$y = A + Bx,$$

and

$$\text{Area} = A(b-a) + \frac{B}{2}(b-a)^2.$$

Denoting the ordinates of the end points of the given curve by y_1 and y_2,

$$y_1 = A,$$
$$y_2 = A + B(b - a).$$

From these equations $A = y_1$, $B = \frac{-y_1 + y_2}{b}$.

Substituting these values,

$$\text{Area} = \frac{b-a}{2}(y_1 + y_2).$$

This is the approximation found by replacing the given curve by a straight line joining the end points of the curve.

Second approximation. When $n = 2$, equation (1) becomes

$$y = A + Bx + Cx^2,$$

and
$$\text{Area} = A(b-a) + \frac{B}{2}(b-a)^2 + \frac{C}{3}(b-a)^3.$$

Denoting the three equidistant ordinates of the given curve by y_1, y_2, y_3,

$$y_1 = A,$$
$$y_2 = A + \frac{B}{2}(b-a) + \frac{C}{4}(b-a)^2,$$
$$y_3 = A + B(b-a) + C(b-a)^2.$$

From these equations

$$A = y_1,$$
$$B = \frac{-3y_1 + 4y_2 - y_3}{b-a},$$
$$C = \frac{2y_1 - 4y_2 + 2y_3}{(b-a)^2}.$$

Substituting these values,

$$\text{Area} = \frac{b-a}{6}(y_1 + 4y_2 + y_3).$$

This is the approximation found by replacing the given curve by a parabola through three points of the curve with equidistant ordinates. It is known as Simpson's approximation for the area bounded by a curve.

Third approximation. When $n = 3$, equation (1) becomes

$$y = A + Bx + Cx^2 + Dx^3,$$

and

$$\text{Area} = A(b-a) + \frac{B}{2}(b-a)^2 + \frac{C}{3}(b-a)^3 + \frac{D}{4}(b-a)^4.$$

Denoting the four equidistant ordinates of the given curve by y_1, y_2, y_3, and y_4,

$$y_1 = A,$$
$$y_2 = A + \tfrac{1}{3}B(b-a) + \tfrac{1}{9}C(b-a)^2 + \tfrac{1}{27}D(b-a)^3,$$
$$y_3 = A + \tfrac{2}{3}B(b-a) + \tfrac{4}{9}C(b-a)^2 + \tfrac{8}{27}D(b-a)^3,$$
$$y_4 = A + B(b-a) + C(b-a)^2 + D(b-a)^3.$$

Solving these equations for A, B, C, D and substituting,

$$\text{Area} = \tfrac{1}{8}(b-a)\{y_1 + 3(y_2 + y_3) + y_4\}.$$

Other approximations. When $n = 4$ or 5, the resulting expression for the approximate area does not take a simple form.

When $n = 6$, if the equidistant ordinates are denoted by y_1, y_2, y_3, y_4, y_5, y_6, y_7, the expression for the area is

$$\text{Area} = \tfrac{1}{20}(b+a)\{y_1 + y_3 + y_5 + y_7 + 5(y_2 + y_6) + 6y_4\}.$$

This is Weddle's approximation for the area bounded by a curve.

The application of these several formulas for areas will not give close approximations if the equidistant ordinates have a wide range of values. It is therefore important to break up the given curve into parts such that for each part the equidistant ordinates have a small range of values and apply the formulas to each part separately.

PROBLEM 152. By using the value of π compute correct to two decimal places the area of the segment of the circle whose radius is 14 feet bounded by a diameter and a chord parallel to the diameter and distant 7 feet from the diameter.

PROBLEM 153. Compute six equidistant ordinates of the circular segment of Problem 149 and find the error of the first approximation.

PROBLEM 154. Using the six equidistant ordinates of Problem 150, find the error of Simpson's approximation.

PROBLEM 155. Compute seven equidistant ordinates of the circular segment of Problem 149 and find the error of Weddle's approximation.

PROBLEM 156. Find, approximately, the area of a plot of ground bounded on one side by a straight line and on the other by an irregular curve. The length of the straight line is 8.97 chains, and the lengths in chains of 31 equidistant ordinates taken in order, measured from this line to the opposite boundary, are

1.37	1.83	2.31	2.52	2.58	2.47	2.32
2.46	2.73	2.91	3.21	2.75	2.63	2.01
1.98	1.74	1.50	1.59	1.98	2.37	2.76
2.93	3.41	3.92	3.42	2.76	2.42	2.01
1.75	1.50	1.65				

CHAPTER X

VOLUMES OF SOLIDS

Art. 39. — Exact Volumes

Suppose two parallel planes to be drawn such that the solid extends from the one plane to the other and is contained wholly between the two planes.

Divide the distance between these two planes into n equal parts, and through the points of division pass planes parallel to the planes drawn at the start.

Starting from one of the original parallel planes, con-

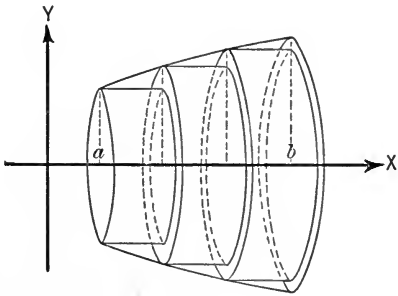


Fig. 22.

struct cylinders on the successive plane sections of the solid as bases with one of the n equal parts of the distance between the original parallel planes as altitude.

Denoting by X_0, X_1, X_2, X_3, X_4, $\cdots$, X_n the areas of the successive equidistant sections of the solid, and the distance between the parallel planes including the solid by $b-a$, the sum of the volumes of the cylinders is

$$(X_0+X_1+X_2+X_3+X_4+\cdots+X_{n-1})\frac{b-a}{n},$$

and the volume of the given solid is the limit of this sum when n is infinitely increased.

If the areas of the successive plane sections of the solid can be expressed in terms of the distance x from one of the end planes by an equation of the form

$$X=A+Bx+Cx^2+Dx^3+Ex^4+\cdots+Nx^n,$$

the limit of the sum can be determined exactly. In fact the determination of this limit is identical with the determination of the limit for areas of Article 36.

Art. 40. — Approximate Volumes

The volume of a solid is given by the expression

$$\text{Volume}=\lim_{n=\infty}(X_0+X_1+X_2+X_3+X_4+\cdots+X_{n-1})\frac{b-a}{n}.$$

It follows that the approximation to the volume is made closer by increasing the value of n.

First approximation. When $n=1$, denoting by X_1 and X_2 the areas of the end sections,

$$\text{Volume}=\frac{b-a}{2}(X_1+X_2).$$

Second approximation. When $n=2$, denoting by X_1, X_2, and X_3 the areas of the three equidistant sections,

$$\text{Volume}=\frac{b-a}{6}(X_1+4X_2+X_3).$$

This is the prismoidal formula.

Third approximation. When $n=3$, denoting the areas of the four equidistant sections by X_1, X_2, X_3, and X_4,

$$\text{Volume} = \tfrac{1}{8}(b-a)\,[X_1 + 3(X_2 + X_3) + X_4].$$

Fourth approximation. When $n=6$, denoting the areas of the seven equidistant sections by X_1, X_2, X_3, X_4, X_5, X_6, X_7,

$$\text{Volume} = \tfrac{1}{20}(b-a)\,[X_1 + X_3 + X_5 + X_7 + 5(X_2 + X_6) + 6X_4].$$

PROBLEM 157. The solid generated by revolving a circular segment of one base about its chord is called a circular spindle. Find the first, second, third, and fourth approximations of the volume of the circular spindle if the radius of the circle is 14 feet and the height of the circular segment is 7 feet.

PROBLEM 158. A vessel laden with a cargo floats at rest in still water and the line of flotation is marked. Upon the removal of the cargo every part of the vessel rises 3 feet, when the line of flotation is again marked. From the known lines of the vessel the areas of the two planes of flotation and of five equidistant sections are calculated and found to be as follows, expressed in square feet:

3918	3794	3661	3517	3361	3191	3004

Find the weight of the cargo removed, supposing 355 cubic feet of water to weigh 10 tons.

Art. 41. — Applicability of Prismoidal Formula

The prismoidal formula

$$\text{Volume} = \tfrac{1}{6}(b - a)(X_1 + 4X_2 + X_3)$$

gives the exact volume of any solid bounded by two parallel planes provided

$$X = A + Bx + Cx^2,$$

where X denotes the area of a section of the solid made by a plane parallel to the end planes, x denotes the distance from this plane section to one of the end planes, and A, B, C are constants.

a. For prisms and cylinders

$$X = A$$

and the prismoidal formula applies.

b. For pyramids and cones

$$X = Cx^2$$

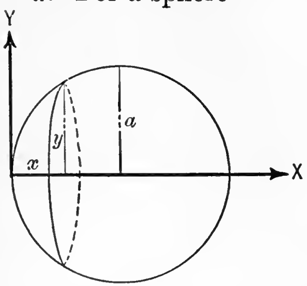
and the prismoidal formula applies.

c. For frustums of pyramids and cones

$$X = A + Bx + Cx^2$$

and the prismoidal formula applies.

d. For a sphere

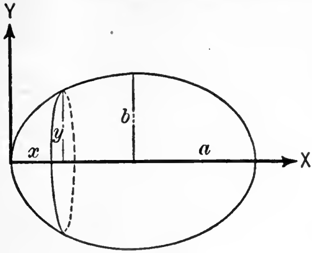


$$X = \pi(2ax - x^2)$$

Fig. 23.

and the prismoidal formula applies.

e. For an ellipsoid of revolution or spheroid

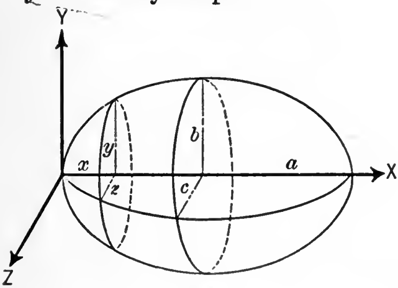


$$X = \pi \frac{b^2}{a^2}(2\,ax - x^2)$$

and the prismoidal formula applies.

Fig. 24.

f. For any ellipsoid

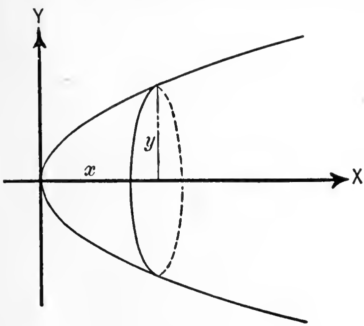


$$X = \pi \frac{bc}{a^2}(2\,ax - x^2)$$

and the prismoidal formula applies.

Fig. 25.

g. For the paraboloid of revolution



$$X = 2\,p\pi x$$

and the prismoidal formula applies.

Fig. 26.

PROBLEM 159. The radius of a sphere is 10 feet. Find the volume of the spherical segment whose bases are distant 2 feet and 4 feet from the center of the sphere.

PROBLEM 160. Find the volume of the solid generated by revolving the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ about the x-axis.

PROBLEM 161. Find the volume of the ellipsoid whose axes are 10, 14, 18 inches.

PROBLEM 162. Find the volume of the solid generated by revolving about the x-axis that part of the parabola $y^2 = 2px$ bounded by $x = 0$ and $x = a$.

PROBLEM 163. Find the volume of the sphere whose radius is R.

PROBLEM 164. Find the volume of the frustum of a cone of revolution whose altitude is h, the radii of the bases R and r.

PROBLEM 165. A cask is constructed in the shape of the middle frustum of a spheroid. The inside dimensions of the cask are length $1\frac{1}{3}$ meters, end diameters $\frac{2}{3}$ meter, bung diameter $\frac{3}{4}$ meter. Find the capacity of the cask in liters and in gallons.