CONCISE
MATHEMATICAL OPERATIONS;

BEING A

SEQUEL

TO THE AUTHOR'S CLASS BOOKS:

WITH MUCH ADDITIONAL MATTER.

A WORK ESSENTIALLY PRACTICAL, DESIGNED TO GIVE THE LEARNER A PROPER APPRECIATION OF THE UTILITY OF MATHEMATICS; EMBRACING THE GEMS OF SCIENCE FROM COMMON ARITHMETIC, THROUGH ALGEBRA, GEOMETRY, THE CALCULUS, AND ASTRONOMY.

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PREFACE.

This book is not designed to teach Mathematical Principles, but to apply and enforce them. It contains collections and groups of mathematical problems which show the utility of science, and place its fruits in the foreground.

Let no one expect to find any close connection between the different parts of this book, or even in any one part of it. System and connection is essential in every theoretical work, but it would be as absurd to look for it here as to look for a composition in a dictionary.

That there is need for such a work as this, all would be convinced who could see but a tenth part of the letters that every accommodating mathematician is constantly receiving, requesting the solution of problems or the exposition of principles.

Indeed, much important matter to be found in this volume, has been suggested and brought to the immediate notice of the author by letters received requiring his aid ; and to save the trouble of answering such letters in future was one inducement to publish this work.

There is a great deal of perfectly barren mathematical knowledge in this country; particularly among those who have studied, not for science, but for a diploma.

Not unfrequently do we meet persons who can demonstrate many, if not all the elementary problems in common Geometry, who, at the same time, cannot make the least application of them, and who seem to be unaware that they were ever intended for any practical use.

Knowledge, so confined and abstract, is of doubtful utility, even as a mental discipline. Unless we take in a broad expanse, and unite both theory and practice, we perceive nothing of the beauties of the Mathematics. Detached propositions and abstract mathematical principles, give us no better idea of true and living science, than detached words and abstract grammar would give us of poetry and rhetoric. Small acquirements in the Mathematics serve only to make us timid, cautious, and distrustful of our own powers—but a step or two further gives us life, confidence, and power.

The efforts of the great mass, who attempt the study of the Mathematics, are very inefficient and feeble, because the motive is not sufficiently pointed and pressing. *They study for the discipline of mind.*

Now, we venture to assert, that those who study for any object so indirect and indefinite, can never be decidedly successful. And those who teach with no other view than giving discipline to the minds of their pupils, never more than half teach. The object, and the *only object,* should be to understand the subject studied, and if that understanding is attained, the highest mental discipline that the subject can yield, will surely come with it.

Let a person undertake the study of any science, Trigonometry for example, with no other object than the discipline of the mind, and our word for it, the science will come to him with the utmost difficulty; and however long he may study, the spirit of the science will never find a lodgment with him. But let him be determined to understand it, for the purpose of being an architect, an engineer, or a navigator, and all is changed—beauties are now seen where none were discovered before, and the student is now sensible of possessing both knowledge and mental discipline.

Let a person commence Astronomy, simply with a view to mental discipline, and when will he obtain a sound knowledge of that science? We answer, never. But let him commence the study with a determination to understand it, and his efforts will be well directed, and science will come to him with ease, and with it will come a discipline of mind, the most pure and lasting that man can attain.

There is another erroneous impression which serves, as far it goes, to obstruct the progress of sound mathematical learning in this country. It is a vague, yet general idea, that Arithmetic, Algebra, Geometry, Trigonometry, and the Calculus, are distinct and separate sciences, and each is to be learned by itself and then carefully laid aside. The truth is, they are but different sections of the same science, and each one in turn may be used to illustrate the other; and studied as a whole, under the direction of a philosophic teacher, the labor of acquisition would be very much reduced.

Were we to say nothing in respect to our method of treating the square and cube roots in this volume, the mere arithmetician would undoubtedly depreciate it. He will perhaps still regard the method as unscientific, and call it a mere "cut and try" operation; but when he finds the same thing in Geometry, and there finds lines which may represent all the different factors in any case, and sees the geometrical reason why the exact square root is always a little less than the half sum of two unequal factors, he must then admit that the cut and try method is not very unscientific after all. The truth is, in the hands of those who can take the geometrical view of it, and who can use it with judgment, this method is as scientific as any, and in many cases far more practical than the common rules.

The first principles of Geometry are, to a certain degree, abstract; but the application of Geometry, as appears in this work, is far from being so; and he must be a very practical mathematician who cannot find something here to amuse, to interest, or to instruct him.

To the subject of finding sines and cosines, both natural and logarithmic, for every minute of the quadrant, we call special attention—as strict attention to that subject in all its bearings, will so readily impress upon the mind of a learner, the importance of theoretical Geometry.

To the practial application of Interpolation, we also call attention. Some problems in Mensuration and Plane Trigonometry will be found very interesting to those who possess a taste for the Mathematics, and we have extracted several different solutions from the works of others, to show how differently different persons present the same thing. There are few mathematical students who could not be greatly benefitted by a close perusal of Spherical Trigonometry and Astronomy as presented in this work. Any person who has the outlines of Astronomy and Elementary Mathematics can here have a view of all the details of a solar eclipse, in a comprehensible shape.

There has been a great deal of unnecessary controversy about the Differential and Integral Calculus, which we think can and ought to be wiped away. And we have here given a little foretaste of what we shall attempt if circumstances prompt us to write a work on that subject.

It is not for us to assume that we can make science clearer than others, but we have yet to see the works of an author who has made the least attempt to show the simple elementary nature of this science. They at once commence with the definition of constants and variables, and then direct what to do.

We have yet to see the first book that expends a word in giving an idea of what the Calculus is, or what is the utility and object of the science, and we charge more than half the obscurity to this fact alone: hence we could not forbear being a little elementary when we came to that subject, and we leave it to those readers, who have formerly studied other works on this science, to say whether we have or can dispel any of the obscurity that has so long hovered around it.

All sciences are obscure until they are applied. Even Arithmetic would be so in the abstract, and being alive to this fact we have extended the application of the Calculus to more subjects than we have hitherto observed in other works. For example, see the method of clearing lunar distances, and the use we made of the same principle in computing an eclipse.

But neither in the Differential nor the Integral Calculus do we pretend to be any thing like full or perfect, even for a work of this kind.

We have only thrown out a few practical remarks and problems, in our own unique manner, more to learn what is desired, and what can be appreciated, than for any thing else.

When we commenced, we did not intend to produce so large a volume; it grew on our hands; but we believe that this result will not be regretted by generous patrons.

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PART FIRST.

ARITHMETIC.

SECTION I.

We shall be very brief in this work on the subject of arithmetic, only touching on such points as are generally neglected in the class room.

Formerly all kinds of problems and puzzles were to be found in arithmetics; but pure science, good taste, and the rapid advancement of the pupils, require that the works on arithmetic should be concise and clear, and take no undue proportion of the student's time and attention.

Severe problems do not teach science—but science will subdue all severe problems, and we would use problems only as a means of elucidating science. Algebraic problems, and problems in geometry and mensuration, should never appear in arithmetic, but old custom will not yet tolerate their expulsion.

We shall pay particular attention to the metaphysique of the science.

Numbers only of the same kind can be added together or subtracted from each other.

Numbers are either abstract or concrete. Abstract numbers are unapplied and are mere numerals. Concrete numbers bring to the mind the particular number of things to which they refer.

Arithmetic proper, comprises the system of notation, and the operations to be performed with abstract numbers only—*without any reference to their application whatever.*

The application of arithmetic includes all kinds of numerical computations, and they are therefore endless in variety and character.

In the application of arithmetic, there are two distinct operations, the *logical one* and the *mechanical one;* the thinking and the doing.

The undisciplined direct their attention more to the doing than to the thinking, when it should be the reverse; and nearly all the efforts of a good teacher are directed to make his pupils reason correctly.

If a person fails in an arithmetical problem, the failure is always in *the logic,* for false logic directs to false operations, and true logic points out true operations.

Abstract arithmetic we shall not touch, except when necessary to illustrate a point before us.

With these introductory remarks we commence with the following principles:

1. *Multiplication is the repetition of one number as many times as there are units in another.*

This is general, whether the numbers be large or small, whole or fractional. The rules in whole numbers and in fractions, apply to the mechanical operations only, and not to the one fundamental principle.

2. When the multiplicand and multiplier are both abstract numbers, the product is abstract, or a mere numeral *without a name.*

3. *No two things can be multiplied together.*

A multiplicand may have a name, as dollars, yards, men, &c.; then the multiplier must be a mere numeral, and the product will have the same *name* as the multiplicand.

4. *Division is finding how many times one number can be subtracted from another of the same kind.*

There are other definitions to be found in books, which do very well in the main, but this is the only truly logical definition I can find. Division should never be considered in the light of separating a number into parts, for this is not true in all cases, and confusion often arises in fractions by this view of the subject.

5. *Division corresponds to multiplication conversely, when we take the product for a dividend, the multiplicand for a divisor, and the quotient for a multiplier.*

6. *In multiplication it is indifferent which of the two factors is called the multiplicand, the other must be an abstract multiplier.* The name of the product (when known) is an infallible index to show which of the two factors is really the multiplicand.

To illustrate principles *three* and *six*, we give the following

EXAMPLES.

1. *What will* 763 *pounds of pork come to at* 8 *cents per pound?*

At first view, this example seems to conflict with principle three, for, says the pupil, we multiply the pounds by the price per pound; but it is not so.

Two pounds would cost twice as many cents as one pound, and 763 pounds would cost 763 *times* as many cents as one pound; therefore 763 is the abstract multiplier in the operation, and 8 cents is the true multiplicand, and the product will be cents, as required.

In the act of multiplying, it is indifferent how the numbers are written.

2. *Reduce* 5£ 13*s*. 6*d*. *to pence.*

Here 20 and 5, as abstract numbers, must be multiplied together and 13 added, making 113 shillings,—but which of the two factors 20 or 5, is the multiplicand?

Here nine-tenths of those who teach arithmetic would call the 5£ the multiplicand and 20 the multiplier; but this is not so. A multiplicand suffers *no change of name* by being multiplied, and as the name of the product is unquestionably shillings, 20 shillings is the multiplicand, and 5, as an abstract number, is the multiplier, there being 5 *times* as many shillings in 5£ as in 1£.

By the same logic, to reduce 113 shillings to pence, 12, the number of pence in a shilling, is the true multiplicand, which must be repeated 113 times, and then the product must of course be pence as required.

Dollars can be divided by dollars, and by nothing else. Yards can be divided by yards, and by nothing else, and so on, for any other thing that might be mentioned.

This fact has not been sufficiently attended to; indeed, it has scarcely been recognized by many teachers.

It is true we can divide a number of dollars, yards, &c. into equal parts, but we do so *indirectly*, in point of logic, while the mechanical operation is direct.

That dollars can only be divided by dollars arises from the fact that division is but a short process of finding how many times one quantity can be subtracted from another, *and we can subtract only dollars from dollars*, therefore we can divide dollars only by dollars.

Example.—Divide $42 equally among 6 men.

Now we cannot divide $42 by 6 men nor by 6; but if we give each man a dollar, that will require $6, and $6 can be subtracted from $42 seven times. Hence we can give each man a dollar seven times, or we can give him $7 at one time. After the operation is performed, we may call the 7, seven dollars, then the 6 will be a mere number, and thus, indirectly, we may divide $42 by 6. Practically, however, all such operations are performed abstractly, as 42, 6, and 7, taken as mere numbers, and then mere logic decides upon the names. For another example,

Divide 11£ 7*s*. 8*d*. into 4 equal parts.

Lay out a £ in four different places—this will require 4£. Now we are to consider how many times 4£ can be subtracted from 11£, which is 2 times and 3£ over, which reduced to shillings and 7 added, makes 67 shillings; this divided by 4 shillings gives 16, and 3 shillings over, which reduced to pence and 8 added, makes 44 pence, which for the same reason, divided by 4 pence gives 11. The operation stands thus:

```
4)11£ 7s. 8d.(2
   8
  ---
   3
  20
  ---
4)67(16
  64
  ---
   3
  12
  ---
4)44(11
  44
```

The first divisor is 4£, the second 4*s.*, and the third 4*d.*; but as the divisor and the quotient may change names, we may say 2£ 16*s.* 11*d* is the original sum divided into four equal parts.

The following example will further illustrate this philosophy:

Divide 421£ 14*s.* 8*d. among* 3 *men,* 5 *women, and* 7 *boys, giving each woman three times as much as a boy, and each man double the sum to a woman.* Required the shares for each.

I will commence by giving one boy 1£.

One woman 3£.

One man 6£.

But there are 7 boys; to give each 1£ would require 7£

5 women each 3£ would require 15£

3 men each 6£ would require 18£

—— 40£.

Thus going once round giving each boy 1£ and each man and woman their due proportion, would require 40£. We are now to consider how many such rounds it would take to consume 421£. In other words, we must divide 421£ by 40£. When each boy can no longer take a pound, we must in like manner go the rounds with shillings, then with pence, &c. The operation is thus:

40£)421£ 14*s.* 8*d.*(10
400
——
21
20
——
40)434(10
400
——
34
12
——
40)416($10\frac{2}{5}$
416

Here then it is clear that each boy must have 1£ ten different times, or which is the same thing in effect, 10£ at one time. Hence, simply changing the *names* of the divisor and quotient,

Each boy is to have 10£ 10*s.* $10\frac{2}{5}$*d.*

Each woman 3 times as much, or 31£ 12*s.* $7\frac{1}{5}$*d.*

And each man 63£ 5*s.* $2\frac{2}{5}$*d.*

We will give but one more example to show that the divisor and dividend must be of the same name.

The moon describes an arc in the heavens of 197° 38′ 45″ *in* 15 *days; how great an arc will it describe in* 1 *day?*

The fifteenth part of 197° 38′ 45″ is obviously the sum required, but how will the number 15 measure 197 degrees? If we drop the name of degrees and say 197, then we can divide it by 15, and this is the usual way—*during the operation all names are practically destroyed*—and after the operation is over, the proper name is given according to the logic or philosophy involved in the question, and it is in this logic or philosophy where the unthinking fail, if they fail at all.

We may also solve this problem by conceiving the moon to move 15° in one day, and then dividing 197° 38′ 45″ by 15°, we shall obtain an abstract number, each unit of which corresponds to a day. Then changing the names between the divisor and the quotient, 15° will become an abstract number, and the quotient will be degrees, &c. as required.

```
15°)197° 38′ 45″(13
    15
    ──
     47
     45
     ──
      2
     60
     ──
  15)158(10
     150
     ───
       8
      60
     ───
  15)525(35
      45
     ───
      75
      75
```

Our first divisor was 15°, second 15′, third 15″, but by making these abstract numbers, the quotient will become 13° 10′ 35″, the answer.

CANCELING.

Within a few years the subject of canceling has been brought to the special notice of teachers and others, and like every other improvement, it has been opposed by some, and looked upon with distrust and indifference by others. But still, it being a real and substantial improvement, it is working its way; and even at this day intelligent pupils are astonished that teachers should oppose it as they sometimes do. Indeed, that teacher who would in any degree discountenance cancellation should be dismissed at once from the class room.

Some few educationists had private reasons of a pecuniary nature for opposing cancellation; but the chief opposition arose from the disinclination of persons to break into old habits.

Cancellation does not change the process of reasoning on a problem, but it requires a more general perception at a glance, and more rapidity of thought, than the old methods; hence the naturally dull did and do yet oppose it as a *matter of course.*

The architect makes the design of a proposed building on paper, represents it inside and out, estimates the cost, suggests changes and improvements, and has it all in his mind before a stick of timber is prepared, or any serious labor commenced. *It is economy to do so.* An arithmetician should do the same; he should be able to represent what he proposes to do, on paper, look at it and consider it fully before he commences real labor. *It is economy to do so,* for then he may see counter operations that will cancel or abridge each other. Desirable as all this is, it is rarely thought of;—no sooner is an operation decided upon than the operator hastens to perform it, without thinking further until that is done. He then decides upon another step and performs it,—then another, and so on through the problem.

Now as a general rule we would have each step of an operation distinctly indicated before it be performed, and then examined as a whole, the same as an engineer would examine a map, or an architect the plan of a building. We give the following examples to exercise this faculty:

1. *A merchant bought* 526 *barrels of flour at* \$4.50 *per barrel, and paid in cloth at* \$2.25 *per yard. How many yards did it require?*

Ans. 1052.

The following is the common method of thought and operation. We must find what the flour will amount to, and as soon as that thought is defined, the operator commences the multiplication. When that is done, then comes the thought about the cloth, and it is decided to divide the amount by \$2.25 for the required result, and the operation stands thus:

```
       526
        450
     ------
      26300
     2104
     ------
225)236700(1052
    225
    -----
     1170
     1125
     -----
       450
       450
```

Now we would not change the direction of the thought in the least, but we would have it continued to the end, and each operation indicated as we go along.*

The map of the whole operation stands thus:

$$\frac{526 \cdot 450}{225}$$

Here is a fraction, the numerator consisting of two factors, the denominator of one factor which is contained twice in 450. Hence twice 526 is the required result, and the mechanical operation is just nothing at all.

---

*When two numbers are to be multiplied together, we write them with a point between, thus 4.6 indicates 4 multiplied by 6. If this is to be divided by any other number, say 3, we would write $\frac{4.6}{3}$. When two numbers are to be added, we write (+) plus between them; when one is to be subtracted, we write (—) minus before that one.

2. *How much will* 540 *yards of cloth cost at* 3*s.* 4*d., in dollars at* 6 *shillings each?*

The map of the operation is thus: $\frac{540 \cdot 3\frac{1}{3}}{6}$.

This reduces to $90 \cdot 3\frac{1}{3}$. Multiply one factor by 3, and divide the other by 3, (which will not change the value of the product), then $30 \cdot 10 = 300$, the result.

N. B. In this work we do not pretend to explain prime and composite numbers, what numbers will cancel with each other, and what will not. These things must be learned elsewhere.

3. *At* $12\frac{1}{2}$ *cents per pound what must be paid for four boxes of sugar, each containing* 136 *pounds?*

Map of the operation, $\frac{136 \cdot 4}{8} = \frac{136}{2} = 63$ dollars.

4. *What will one hogshead, or* 63 *gallons of wine, cost at* $6\frac{1}{4}$ *cents a gill?* Ans. $126.

Map of the operation, $\frac{63 \cdot 4 \cdot 2 \cdot 4}{16} = 126$.

The multiplication indicated in the numerator reduces the 63 gallons to gills, and as $6\frac{1}{4}$ cents is one-sixteenth of a dollar, we divide by 16, which cancels the product of the fours in the numerator and leaves 63 to be doubled for the result.

5. *At* $1\frac{1}{2}$ *cents a gill, how many gallons of cider can be bought for* $24? *Ans.* 50.

Map of part of the operation $\frac{24 \cdot 100}{1\frac{1}{2}}$ = the number of gills, or $\frac{24 \cdot 200}{3}$ = the gills.

Map of the whole operation, $\frac{24 \cdot 200}{3 \cdot 4 \cdot 2 \cdot 4} = \frac{200}{4} = 50$ Ans.

6. *If a man travel* 39 *miles* 20 *rods in a day, how many days will be required to traverse* 25000 *miles?* *Ans.* 640.

As 320 rods make a mile, the following is the m

$$\frac{320 \cdot 25000}{320 \cdot 39 + 20}$$

Here numerator and denominator can be divided by 20, which reduces the operation to $\frac{16 \cdot 25000}{16 \cdot 39 + 1}$.

Because no further reduction can be made, this last indicated operation must be performed in full.

In all cases, whether reduction can be made or not, we would insist on having the operations first indicated; and in practice, nine-tenths of the operations can be reduced. There is now and then one that cannot be reduced. Even when the plan of an arithmetical operation is laid down, judgment should be used in drawing out the final result, as the following example will illustrate:

Required the value of the following expression:

$$\left(\frac{43}{80}\right)^2 \cdot \left(\frac{144}{95}\right)^2 \cdot \left(\frac{4900}{24}\right)^2.$$

This occurs on page 156 of Robinson's University Algebra, and we have seen it literally carried out as indicated, in several of the best schools in the country; no reductions being made until after the numbers were squared; thus making a long and tedious process.

The proper way is to take the square root of the expression; then we shall have $\frac{43}{80} \cdot \frac{144}{95} \cdot \frac{4900}{24}$.

Reducing does not change the value of the expression; the first obvious reduction, is to divide the numerator and denominator by 10 and 24; then the expression will stand thus, $\frac{43}{8} \cdot \frac{6}{95} \cdot \frac{490}{1}$.

A still further reduction gives $\frac{43}{2} \cdot \frac{3}{19} \cdot \frac{49}{1} = 166$, nearly.

Now the square of 166.3, is the value of the required expression.

We square, because the square root was taken in the first step. We may do this, because we have no where changed the value of the expession, except in taking the root.

PROPORTION.

This manner of expressing an operation is most efficacious and practical in proportion.

We shall make no attempt to elucidate the principles of proportion, our attention for the present being entirely on numerical operations.

EXAMPLES.

1. *If 2cwt. 3qr. 21lb. of sugar, cost* 6£ 1*s.* 8*d.*, *what will* 35*cwt.* 1*qr.** *cost?*

	cwt.	*qr.*	*lb.*		*cwt.*	*qr.*		£	*s.*	*d.*
Statement.	2	3	21	:	35	1	: :	6	1	8

This example is taken from an old but popular book, in which the solution covers about two pages. The sugar is reduced to pounds, and the money, to pence. The result of the proportion is then obtained in pence, which being reduced, gives 73£.

We do it thus: Reduce the sugar to qrs. Then the proportion is

$$11\tfrac{3}{4} : 141 :: 6£\ 1s.\ 8d.$$

Multiplying the two first terms of this proportion by 4, which does not change the proportion, then we have

$$47 : 141\cdot4 :: 6£\ 1s.\ 8d.$$
or
$$1 : 3\cdot4 :: 6£\ 1s.\ 8d.$$

Therefore 12 times the third term is the result, 73£.

2. *If* 3*cwt. of sugar cost* 9£ 2*s.*, *what will* 4*cwt.* 3*qr.* 26*lb. cost at the same rate?* *Ans.* 15£ 2*s.* 3*d.*

We give the following solution just as it appears in a very popular book:

	cwt.		*cwt.*	*qr.*	*lb.*		£	*s.*	*d.*
	3	:	4	3	26	: :	9	2	0
	4		4				20		
4·7	12		19				182		
	7		7				12		
	84		133				2184		
	4		4						
	336*lb.*	:	558*lb.*			: :	2184		

*We take the old scale of 28 pounds to the quarter.

```
336lb. : 558lb. : : 2184
                     558
                   -----
                   17472
                  10920
                 10920
                 -------
           336)1218672(3627
                1008
                ----
                 2106            12)3627
                 2016            -------
                 ----             20)302  3d.
                  907             -------
                  672                 15£ 2s. 3d. Ans.
                 ----
                 2352
                 2352
```

If the question had called for the cost of 2 pounds more of sugar, it would have called for the price of 5*cwt.*

Then the proportion would have been

$$3 : 5 : : 9.1£.$$

		£	s.	d.
Whence the cost of 5*cwt.* would be		15	3	4
For the cost of 2 pounds we have	2*lb.* cost		1	1
3·112 : 2 : : 182 shillings.		15	2	3 *Ans.*
or 168 : 1 : : 182				
or 84 : 1 : : 91 : $1\frac{1}{12}$ shillings.				

3. *If* $15\frac{5}{8}$ *bushels of clover cost* $\$156\frac{1}{4}$, *how many bushels can be bought for* $\$95\frac{3}{4}$? *Ans.* $9\frac{575}{1000}$.

Statement, \$156.25 : \$95.75 : : 15.625.

When a statement is properly made, drop all names and operate as abstract numbers; then the proper name can be given to the result by the rules of logic, or rather, the true name comes as a matter of course. Those who operate by rule and without thought and close *observation*, would make very tedious work of this.

The map of the operation is thus: $\frac{95.75(15.625)}{(156.25)}$.

The factor in the denominator is 10 times one of those in the numerator, therefore the operation reduces to

$$\frac{95.75}{10}=9.575 \textit{ Ans.}$$

4. *If* 240 *bushels of wheat can be purchased at the rate of* $\$22\frac{1}{2}$ *for* 18 *bushels, and sold at the rate of* $\$33\frac{3}{4}$ *for* $22\frac{1}{2}$ *bushels, what would be the profit?* *Ans.* \$60.

$$18 : 240 :: 22\tfrac{1}{2} : \text{cost}=\frac{240\cdot 22\frac{1}{2}}{18}$$

$$22\tfrac{1}{2} : 240 :: 33\tfrac{3}{4} : \text{sale}=\frac{240\cdot 33\frac{3}{4}}{22\frac{1}{2}}$$

$$\text{Cost}=\frac{240\cdot 45}{18\cdot 2}=\frac{40\cdot 45}{3\cdot 2}=20\cdot 15=300 \text{ dollars.}$$

$$\text{Sale}=\frac{3\cdot 240\cdot 11\frac{1}{4}}{22\frac{1}{2}}=\frac{3\cdot 240}{2}=3\cdot 120=360 \text{ dollars.}$$

A complete proportion consists of four terms; and in problems, the unknown answer is generally one of them ; and were it not for old prejudices, it would be conducive to perspicuity to represent the unknown term by a symbol, say x. Then a problem stated, would no longer consist of *three* terms, but of *four*.

At first a young learner will not comprehend a symbol nor an equation, *and his confusion arises from the very simplicity of the thing.*

Notwithstanding the aversion of learners to the use of symbols, the aversion must be overcome before they can enter the first portals of science ; and a little firmness on the part of the teacher will remove every difficulty in a very short time.

When a proportion is complete, the *ratio* between the first couplet is the same as the *ratio* between the second couplet. Thus,

$$3 : 6 :: 8 : 16.$$

Here the proportion is true, because 6 *divided by* 3 *gives the same quotient as* 16 *divided by* 8.

Such a trial will test any proportion.

Suppose in this proportion that 16 is not known, and represented by x; then it becomes $3 : 6 :: 8 : x$.

Whence $\frac{6}{3}=\frac{x}{8}$. Or, $x=\frac{6.8}{3}$.

That is, when the three first terms of a proportion are given, the fourth is found by multiplying the second and third terms together, and dividing by the first.

In any proportion the product of the extremes is equal to the product of the means; and from this principle any one of the terms of a proportion can be found, provided the other three are given.

A term may consist of two or more factors, and one of those factors unknown: in such cases, the unknown factor may always be found from *an equation formed by the product of the extremes and means.*

Thus $3 : 6 :: 2x : 16$. Whence $6 \cdot 2 \cdot x = 3 \cdot 16$.

$$\text{Or } x = \frac{3 \cdot 16}{6 \cdot 2} = 4.$$

The foregoing is designed to prepare the way for such problems as are usually found under compound proportion, which we shall call

CAUSE AND EFFECT.

After several years reflection, we have come to the conclusion that the only clear and scientific method of presenting compound proportion is that of cause and effect.

It is an *axiom* in philosophy that equal causes produce equal effects; a double cause a double effect, &c. In short, *effects are proportional to their causes.*

Now causes and effects that admit of computation, that is, involve the idea of quantity, may be represented by numbers, which numbers have the same relation to each other as the things they represent.

EXAMPLES.

1. *If* 7 *men in* 12 *days dig a ditch* 60 *feet long,* 8 *feet wide, and* 6 *feet deep, in how many days can* 21 *men dig a ditch* 80 *feet long,* 3 *feet wide, and* 8 *feet deep?*

Here 7 men in 12 days perform 84 days work; the force or *cause* of removing $60 \cdot 8 \cdot 6$ cubic feet of earth, which is the *effect.* In how many days (we say x days) can 21 men remove $80 \cdot 3 \cdot 8$ cubic feet of earth? Hence we have this proportion:

$$\begin{array}{cccc} \textit{Cause.} & \textit{Effect.} & \textit{Cause.} & \textit{Effect.} \\ 7 \cdot 12 : & 60 \cdot 8 \cdot 6 :: & 21x : & 80 \cdot 3 \cdot 8. \end{array}$$

Here is a case where a factor in one of the terms is unknown, and that factor is the answer to the question.

A proportion is equally true when the same factors are rejected from corresponding terms. *This is but another form of canceling.* In this proportion, we observe *the factor* 7 in each cause, and the *factor* 8 in each effect. Expunging these, the proportion becomes

$$12 : 60 \cdot 6 :: 3x : 80 \cdot 3$$

Similarly $2 : 6 :: x : 8$ Whence $x = 2\frac{2}{3}$ days, *Ans.*

2. *If* 6 *men build a wall in* 12 *days, how long would it require* 20 *men to build it?* *Ans.* $3\frac{3}{5}$ *days.*

Questions of this kind are usually placed under the *rule of three inverse;* they do in fact belong to compound proportion, or rather, to cause and effect; but the effect being the same in the supposition, and in the demand, (that is the building of one wall,) it may be omitted and only three quantities used.

The following statement banishes all confusion:

Cause.		*Effect.*		*Cause.*		*Effect.*
$6 \cdot 12$	:	1	::	$20x$	:	1

As effect=effect, therefore cause=cause, that is, $20x = 6 \cdot 12$.

3. *If* 4 *men in* $2\frac{1}{2}$ *days, working* $8\frac{1}{4}$ *hours a day, mow* $6\frac{2}{3}$ *acres of grass, how many acres (ans.* x *acres,) will* 15 *men mow in* $3\frac{3}{4}$ *days by working* 9 *hours a day?* *Ans.* $40\frac{10}{11}$ *acres.*

Here the unit of cause is one hour's work for a man.

C.		*E.*		*C.*		*E.*
$4 \cdot 2\frac{1}{2} \cdot 8\frac{1}{4}$	:	$6\frac{2}{3}$	::	$15 \cdot 3\frac{3}{4} \cdot 9$	:	x

Multiply the 1st and 3d terms by 4, then

$$4 \cdot 2\tfrac{1}{2} \cdot 33 : 6\tfrac{2}{3} :: 15 \cdot 15 \cdot 9 : x$$

Because $2\frac{1}{2}$ is contained in 15 six times; and the 1st and 3d terms contain the factor 3: therefore

$$4 \cdot 11 : 6\tfrac{2}{3} :: 6 \cdot 15 \cdot 3 : x$$

Or $2 \cdot 11 : 3\frac{1}{3} :: 6 \cdot 15 \cdot 3 : x$

Or $6 \cdot 11 : 10 :: 6 \cdot 15 \cdot 3 : x$

Or $11 : 10 :: 15 \cdot 3 : x$ $x = \frac{450}{11} = 40\frac{10}{11}$ *Ans.*

The reader will observe that we give but specimen examples; one of a kind: the preceding one was given on account of the fractional factors.

4. *What is the interest of* \$240 *for* $3\frac{1}{2}$ *years, at* 6 *per cent.?*

This question simply demands the effect of loaning \$240 for $3\frac{1}{2}$ years, in case \$100 in one year yields \$6.

Cause.	*Effect.*	*Cause.*	*Effect.*

$$100\cdot1 : 6 :: 240\cdot3\tfrac{1}{2} : x.$$

$$\text{Whence } x=\frac{240\cdot6\cdot3\frac{1}{2}}{100}.$$

This equation *shows the common rule* for computing interest. That is:

Multiply the principal by the rate per cent.; that product by the time, and divide by 100.

Now let us take this same example and reduce the time to months, then the proportion will stand thus:

$$100\cdot12 : 6 :: 240\cdot42 : x.$$

Cast out the factor 6 from the first couplet, then

$$100\cdot2 : 1 :: 240\cdot42 : x.$$

Divide the 1st and 3d terms by 2, then we shall have

$$100 : 1 :: 240\cdot21 : x \qquad x=\frac{240\cdot21}{100}.$$

This equation *shows* a special rule to compute interest at 6 per cent. which is,

Multiply the principal by half the number of months and divide by 100.

5. *What is the interest of* \$1248, *for* 16 *days*, 30 *days taken for a month, and* 12 *months in a year?*

	Cause.	*Effect.*	*Cause.*	*Effect.*
	100. :	6 ::	1248 :	x
Days	360.		16.	

Divide the first couplet by 6, then

$$100\cdot60 : 1 :: 1248\cdot16 : x \qquad x=\frac{1248\cdot16}{100\cdot60}.$$

This equation shows a special rule to compute interest for days at six per cent. which is thus,

Multiply the principal by the number of days, divide by 60, *and that quotient by* 100.

6. *The interest on* \$98, *at* 8 *per cent., was* \$25.48 : *what was the time?* *Ans.* 3 *years* 3 *months.*

$$\begin{array}{ccccccc} \textit{Cause.} & & \textit{Effect.} & & \textit{Cause.} & & \textit{Effect.} \\ 100\cdot 1 & : & 8 & :: & 98\cdot x & : & 25.48 \end{array}$$

$$\text{Whence} \quad x = \frac{2548}{8\cdot 98} = 3 \text{ y. } 3 \text{ m.}$$

This equation shows the following rule to find the time in interest problems when the other elements are given:

RULE. *Multiply the interest by* 100 *and divide by the product of the principal and rate.*

These general rules *refer only to forms.* It is not intended that they should be literally followed. In the last equation 8 and 98, the principal and rate, are multiplied in *form* as they stand, and the fraction can be canceled down.

The great detriment to improvement has been, that both teacher and taught, have clung close to the letter of the rules.

SECTION II.

We shall touch on but few points in this section ; and only such as will bear on conciseness of operations.

We give but one example in Exchange and per centage—it is the following :

A merchant bought sugar in New York at 6 *pence a pound, New York currency ; and while on his hand the wastage was estimated at* 5 *per cent.; and interest on first cost at* 2 *per cent.; how many cents shall he ask per pound to gain* 25 *per cent.* *Ans.* $8\frac{17}{128}$.

To reduce pence, New York currency, to cents, we must multiply by $\frac{25}{24}$; to increase any quantity 5 per cent. we must multiply by $\frac{105}{100}$, and so on for any other per cent.; hence the index of the operation is as follows :

$$\frac{6}{1} \cdot \frac{25}{24} \cdot \frac{105}{100} \cdot \frac{102}{100} \cdot \frac{125}{100}$$

This will cancel to a considerable extent. This form is a general rule for all problems of the kind. A *loss* in any problem, of 3 per cent. for example, is brought in by the factor $\frac{97}{100}$, and so on for any other estimated loss, expressed as per centage.

COMPOUND FELLOWSHIP.

Under this head gains and losses must be proportioned by the products of capital and time. We give a few *peculiar* examples.

1. *Two men commenced partnership for a year; one put in* \$1,000 *at the commencement, and four months afterwards the other put in his capital: at the close of the year they divided their gains equally. What capital did the second put in?* *Ans.* \$1,500.

For a mere arithmetical student, who had never been taught the use of symbols, this would be a very puzzling problem. We are therefore opposed to taking up the time of students with difficult problems, except so far as may be necessary to show them the necessity and advantages of symbols and true science. This is a very good example to illustrate the utility of symbols:

Let $x =$ the required capital. It was in trade 8 months, and as their gains were equally divided, therefore the products of capital and time of each must be equal; that is,

$$8x = 12 \cdot 1000 \quad \text{or,} \quad x = \frac{3000}{2} = 1500.$$

2. A, B *and* C *had a capital stock of* \$5762. A's *money was in trade* 5 *months,* B's 7 *months, and* C's 9 *months. They gained* \$780, *which was divided in the proportion of* 4, 5 *and* 3. *Now* B *received* \$2087 *and absconded. What did each gain and put in, and did* A *and* C *gain or lose by* B's *misconduct, and how much?*

$$A\text{'s share} = \frac{780 \cdot 4}{12} = \frac{780}{3}. \quad B = \frac{780 \cdot 5}{12}. \quad C = \frac{780 \cdot 3}{12} = \frac{780}{4}.$$

As gains are divided in proportion to capital *multiplied* by the time it is in trade; conversely then, capital must be in proportion to the respective *gains, divided* by the respective times.

Their proportional gains are 4, 5, 3, which divided by the times 5, 7, and 9, give $\frac{4}{5}$, $\frac{5}{7}$, and $\frac{1}{3}$ for their proportional shares of the capital.

But these numbers being fractional are inconvenient. We will multiply each by $5 \cdot 7 \cdot 3$ or 105, which gives the proportional numbers 84, 75, 35, the sum of which (194) may be taken as the number of shares composing the capital, \$5762; and A's capital is 84 such shares, B's 75, and C's 35. That is,

$$A\text{'s capital} = \frac{5672 \cdot 84}{194 \cdot 1}. \quad B\text{'s} = \frac{5762 \cdot 75}{194}. \quad C\text{'s} = \frac{5762 \cdot 35}{194}.$$

A and C gain by B, \$465.57+.

3. *In a certain factory were employed, men, women, and boys. The boys received* 3 *cents per hour, the women* 4, *and the men* 6; *the boys worked* 8 *hours a day, the women* 9, *and the men* 12; *the boys received* \$5 *as often as the women* \$10, *and for every* \$10 *paid to the women* \$24 *were paid to the men: how many men, women and boys were there, the whole number being* 59?

Ans. 24 men, 20 women and 15 boys.

	Boys.	*Women.*	*Men.*
Sums per hour,	3 cts.	4 cts.	6 cts.
No. of hours,	8	9	12
Sums paid to one of each class,	24	36	72
Proportional sums paid to one of each,	2	3	6

The sums paid to *all* of each class divided by the sums paid one of each class will give the proportional number in each class. That is $\frac{5}{2} : \frac{10}{3} \cdot \frac{24}{6}$ are the proportional numbers of persons respectively. Multiply by 6 to clear of fractions, for fractional numbers cannot apply to persons. Then the proportional numbers will be 15, 20, 24, and as these numbers make 59 they are the numbers in fact.

4. A, B, *and* C, *are employed to do a piece of work for* \$26.45: A *and* B *together are supposed to do* $\frac{3}{4}$ *of it,* A *and* C $\frac{9}{10}$, *and* B *and* C $\frac{13}{20}$, *and are paid proportionally to that supposition: what is each man's share?* *Ans.* A \$11.50, B \$5.75, C \$9.20.

This problem is algebraic, and the operation is algebraic whether the symbols be used or not, and this is true of many other problems found in Arithmetics.

Here A works with B and with C, and we must discover what he is supposed to do, working alone. It is done thus:

$$A+B = \tfrac{3}{4}. \qquad (1)$$
$$A+C = \tfrac{9}{10} \qquad (2)$$
$$B+C = \tfrac{13}{20}. \qquad (3)$$

By addition, $2(A+B+C)=\frac{15}{20}+\frac{13}{20}+\frac{18}{20}=\frac{46}{20}$.

Dividing by 2, $A+B+C=\frac{23}{20}$. (4) This equation shows that the amount each one was supposed to do was overestimated.

Equation (3) taken from (4) gives $A=\frac{23}{20}-\frac{13}{20}=\frac{10}{20}$.

" (2) from (4) " $B=\frac{23}{20}-\frac{18}{20}=\frac{5}{20}$.

" (1) from (4) " $C=\frac{23}{20}-\frac{15}{20}=\frac{8}{20}$.

For *A's* portion of the money we have the following proportion:

$$\tfrac{23}{20} : \tfrac{10}{20} :: 26.45 : A\text{'s part.}$$

$$\text{Or,}\quad 23 : 10 :: 26.45 : \frac{264.5}{23}=\$11.50.$$

5. *A person after doing* $\frac{3}{5}$ *of a piece of work in* 30 *days, calls in an assistant, and together they complete it in* 6 *days: in what time could the assistant alone do the whole work?* *Ans.* $21\frac{3}{7}$ days.

If the person could do $\frac{3}{5}$ in 30 days, he could do $\frac{1}{5}$ in 10 days, and in one day he could do $\frac{1}{50}$ of the whole work. Therefore, it would require 50 days for him to do the whole work alone. Again, $\frac{3}{5}$ of the work being done $\frac{2}{5}$ remained to be done; on this the first person worked 6 days and did $\frac{6}{50}$ of it. Then $(\frac{2}{5}-\frac{6}{50})$ or $\frac{14}{50}$ remained for the assistant to do in 6 days; hence he must do $\frac{14}{300}$ in one day, or $\frac{1}{21\frac{3}{7}}$ in one day. Therefore, to do the whole he would require $21\frac{3}{7}$ days, the answer required.

PROBLEMS IN MENSURATION AND THE ROOTS.

Mensuration and the Roots belong to Geometry and Algebra, but custom requires that some practical problems under these principles, should appear in every Arithmetic. We select such examples as will illustrate numerical brevities.

1. *How many feet in a board* 22 *inches wide at one end,* 8 *inches wide at the other, and* 14 *feet long?* *Ans.* $17\frac{1}{2}$ feet.

Index to the operation, $\frac{14 \cdot 15}{12}$

2. *A man bought a farm* 198 *rods long,* 150 *rods wide, at* \$32 *per acre; what did the farm come to?* Ans. \$5940.

Index to the operation, $\frac{198\cdot150\cdot32}{160}=198\cdot30.$

3. *If the forward wheels of a coach are four feet in diameter, and the hind wheels* 5 *feet, how many more times will the former revolve than the latter in going a mile, estimating the diameter of a circle to the circumference, as* 7. *to* 22.?

Circumference of the fore wheels, $=\frac{22\cdot4}{7}$; hind wheels, $\frac{22\cdot5}{7}$.

There are 5280 feet in a mile; this, divided by the circumference of each wheel will give the number of revolutions of each wheel.

The fore wheels revolve $\frac{5280\cdot7}{22\cdot4}$ times.

The hind wheels revolve $\frac{5280\cdot7}{22\cdot5}$ times.

Difference $\frac{5280\cdot7}{22}\left(\frac{1}{4}-\frac{1}{5}\right)$

That is $\frac{5280\cdot7}{22\cdot20}=\frac{264\cdot7}{22}=12\cdot7=84$ *Ans.*

4. *The bin of a granary is* 10 *feet long,* 5 *feet wide, and* 4 *feet high; allowing the cubical contents of a dry gallon to contain* $268\frac{1}{5}$ *cubic inches, how many bushels will it contain?* *Ans.* $161\frac{1}{149}$.

Index to the operation, $\frac{10\cdot12\cdot5\cdot12\cdot4\cdot12}{268\frac{1}{5}\cdot8}=\frac{50\cdot12\cdot5\cdot12\cdot6}{1341}$.

5. *A man wishes to make a cistern* 8 *feet in diameter to contain* 60 *barrels, at* 32 *gallons each and* 231 *cubic inches to a gallon: what must be the depth of the cistern?* *Ans.* $61\frac{1}{2}$ *inches.*

The diameter of the cistern is 96 inches;
Its area is $96\cdot96\cdot(0.7854.)$

The index to the operation is $\frac{60\cdot32\cdot231}{96\cdot96\cdot(0.7854)}$.

6. *What will it cost to build a wall* 240 *feet long,* 6 *feet high, and* 3 *feet thick, at* \$3.25 *per* 1000 *bricks, each brick being* 9 *inches long,* 4 *inches wide, and* 2 *inches thick?* *Ans.* \$336.96.

$$\text{Index } \frac{240\cdot12\cdot6\cdot12\cdot3\cdot12\cdot(3.25)}{1000\cdot9\cdot4\cdot2}.$$

7. *The bung diameter of a cask is* 38 *inches, the head diameters inside the staves* 28 *inches, and the length* 45 *inches: how many wine gallons will it contain?* *Ans.* 167.89+.

N. B. The cask is conceived to be two equal *frustums* of cones joined by their greater diameters. (*See Geometry.*)

$$\text{Index to solution } \frac{(\overline{38}^2+\overline{28}^2+28\cdot38),7854\cdot45}{3\cdot231}.$$

Observe that the decimal 0.7854 is divisible by 231: quotient .0034. *Therefore we may have the following rule to find the number of wine gallons in a cask:*

Rule. *To the square of the head diameter add the square of the bung diameter, and the product of the two diameters: multiply that sum by* $\frac{1}{3}$ *of the length of the cask and by the decimal* .0034.

8. *A man bought a grindstone which was* 48 *inches in diameter and* 5 *inches in thickness, for* \$10. *When he had ground down* 3 *inches of its radius, a neighbor proposed to purchase it from him at the same proportional price, in case he would deduct* 4 *inches each way from the center, allowed to be the limit to which it could be used. What should the purchaser pay?* *Ans.* \$7.58+.

Statement $(\overline{48}^2-8^2).7854 : (\overline{42}^2-8^2).7854 :: 10 : Ans.$

Or $(\overline{48}^2-8^2) : (\overline{42}^2-8^2) :: 10 : Ans.$

Or $56\cdot40 : 50\cdot34 :: 10 : Ans.$

Or $56\cdot2 : 5\cdot17 :: 10 : Ans.=\frac{850}{112}.$

9. *If a man* 6 *feet in height travel round the earth, how much further must his head travel than his feet?*

Ans. $37\frac{7}{10}$ *feet nearly.*

Let $D=$ the diameter of the earth in feet; then $\pi D=$ the circumference in feet. $(D+12)=$ the diameter, and $\pi D+12\pi=$ the circumference traveled by the man's head.

The difference $=.12(3.1416)=Ans.$

SECTION III.

POWERS AND ROOTS.

The common methods of operation, as taught under this head, are in general the best. One object in this work is to show some peculiarities which will in some instances abridge labor, awaken investigation, and inspire originality of thought.

We give the following definitions:

1. Any number multiplied into itself is called the *square* of that number. Or we may say *the product of two equal factors* produces a square. Either factor is called the root.

2. The product of three equal factors is *a cube* or third power,—of four equal factors, a fourth power, and so on. One of the equal factors is a root in all cases.

3. A square number multiplied by a square number, will produce a square number.

N. B. This is obvious in Algebra for a^2 multiplied by b^2 produces $a^2 b^2$, obviously a square, whatever numbers may be represented by a and b.

4. A square number divided by a square number will give a square number, either whole or fractional.

5. A cube number multiplied by a cube number will give a cube number.

6. If a root is a composite number, its power (square or cube as the case may be) can be separated in square or cube factors: but if the root is a prime number, the power cannot be so separated.

We will soon show the practical utility of these principles.

While operating in powers and roots we should have the following table before us:

Numbers,	1	2	3	4	5	6	7	8	9	10
Sq. or 2d power,	1	4	9	16	25	36	49	64	81	100
Cube or 3d power,	1	8	27	64	125	216	343	512	729	1000

Powers being obtained from roots by simple multiplication, there is no room for much artifice.

Sometimes the application of the following properties of numbers will be useful :

The square of the difference of two numbers is equal to the sum of the squares, less twice the product of the two numbers.

Algebraically, $(a-b)^2=a^2+b^2-2ab$.

The square of the sum of two numbers is equal to the sum of the squares added to twice the product of the number.

Algebraically $(a+b)^2=a^2+b^2+2ab$.

EXAMPLES.

1. What is the square of 79? *Ans.* 6241.
   $(79)^2=(80-1)^2=6401-160=6241$.
2. What is the square of 83? *Ans.* 6889.
   $(83)^2=(80+3)^2=6409+480=6889$.
3. What is the square of 97? *Ans.* 9409.
   $(97)^2=(100-3)^2=10009-600=9409$.
4. What is the square of 971? *Ans.* 942841.
   $(971)^2=(970+1)^2=940901+1940=942841$.
5. What is the square of 29? *Ans.* 841.
   $(29)^2=(30-1)^2=901-60=841$.

These formulas are useful when one or all of the integers are large.

We shall now turn our attention to the extraction of *square* root. We suppose the reader understands the common method, *which as a general operation is the best.*

To call out thought, however, we will require the square root of 9409, on the supposition that we know nothing of the common rule, *and only know that two equal factors of the numbers* 9409 are required.

The first thought is, *that if we divide any number by any factor, the quotient will be another factor.*

Take 100 for one factor. Divide 9409 by 100, and the other factor is 94, omitting the decimal ; but these factors *are not equal.* The factors sought then, or rather one of them, is more than 94, and less than 100. Hence it must be near the half sum of these two numbers ; that is, near 97.

By trial we find 97 correct.

N. B. The half sum of two unequal factors, is always a little greater than one of the equal factors, *because the sum of two unequal factors which form a product, is always greater than the sum of two equal factors.*

EXAMPLES.

1. *Find the square root of* 841 ; *that is, we demand two equal factors, which, multiplied together, will produce* 841.

Assume any factor : say 25.

25)841(33, (plus a fraction, which we omit) is the corresponding factor. But these factors are not equal, and the equal factors must be near their half sum ; that is, near 29. By trial, 29 is found to be the number exactly.

2. *Find the square root, or two equal factors of the number* 444889.

```
Divide by 6.      6)444889
                    74148
```

Here the two factors are *very unequal,* but we can bring them to a proximate equality, by conceiving one multiplied by 100, and the other divided by 100. The factors will then be 600 and 741, nearly. The half sum of these is 670, which must be near one of the equal factors sought.

```
Now divide.   670)444889(664
                  4020
                  ----
                   4288
                   4020
                   ----
                    2689
                    2680
                    ----
                       9
```

These factors being so nearly equal, and there being a slight remainder, the half sum of the two (667) may be relied upon as the true root.

3. *Find the square root of* 3. *Ans.* 1.7320508.

The only two factors in whole numbers are 1 and 3 ;* these are so unequal that their half sum, 2, will be entirely too large. Hence I will assume one factor to be 1.7.

1.7)3. (1.7647	
1.7	
130	1.7647
119	1.7
110	2)3.4647
102	1.7323 root nearly.
80	
68	
120	

Making another trial with the assumed factor, 1.732, we find the result as stated in the answer.

4. *Find the square root of* 181. *Ans.* 13.45362+.

If we allow ourselves to have some knowledge of square numbers, we can find a factor near in value to one of the equal factors sought. Thus the square of 12 is 144, and of 13, 169 ; therefore one of the equal factors of 181 is more than 13. Assume it 13.5, the other factor is, then, 13.4074 ; the mean of these is 13.4537.

Taking this as the assumed factor, we approximate still nearer to the root by a like operation ; and thus we can approximate to any degree of accuracy required.

By admitting that every figure in a root demands two places in its second power, we can come near the root at the first assumption. For example :

5. *Find the square root of* 617796. *Ans.* 786.

Separate the power into periods, as in the common operation ; the superior period is 61 ; the square root of this is near 8, and being three periods the root is near 800. Assume 780, then divide by it, thus,

* In our geometrical problems we shall give a scientific and satisfactory method of reducing unequal to the equivalent equal factors.

```
780)617796(792
     5460
     ----
      7179
      7020
      ----
       1596
       1560
       ----
         36
```

The half sum of 780 and 792, is 786 ; the answer.

By the last example we perceive that the square root of the product of two factors *which are nearly equal,* is very nearly equal to the half sum of the two factors. *It is a little less.* In the last example there was a small remainder, which was rejected; had there been no remainder, 786 would have been too great for the root.

The square root of the product of two square factors is equal to the product of the square root of those factors. That is, the square root of a^2b^2 is the square root of a^2 into the square root of b^2; in short it is $a\times b$.

To apply this principle I adduce the following examples :

1. *A section of government land is a square of* 640 *acres. What is the length, in rods, of one of its sides?* *Ans.* 320.

This problem requires the square root of the product of the two factors, 640 and 160.

The product of two factors is not affected by multiplying one and dividing the other by the same number.

Now multiply the factor 160 by 10, and divide the other by 10, then $1600\cdot64$ will be the equivalent factors; both square factors; their roots are 40 and 8. Hence, the value sought is $40\cdot8=320$.

Again. Take the original factors, 640, 160. Divide 640 by 2, and multiply 160 by 2, which gives 320, 320.

As the factors are *now equal,* one of them is the root sought.

2. *A man has* $50\frac{5}{8}$ *acres of land in a square form; what is the length of one of its sides?* *Ans.* 90 rods.

Index. $\sqrt{50\frac{5}{8}\cdot160}=\sqrt{\frac{405}{8}\cdot160}=\sqrt{405\cdot20}=\sqrt{8100}=90.$

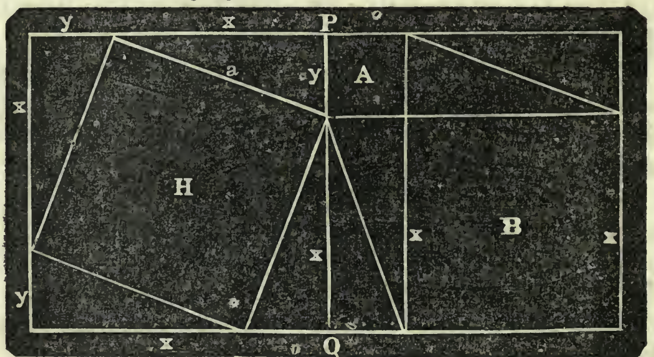
3. *Find the square root of the product of the two factors,* 18 *and* 32.

Equivalent factors, 9 and 64 roots $3 \cdot 8 = 24$.
" Or, 36 and 16 roots $6 \cdot 4 = 24$.

Again, $\frac{18+32}{2} = 25$, which is too great for one of the equal factors by 1, because the factors are so unequal.

In working square root, it is important that the teacher should be able to show to his intelligent pupils, that *the square on the hypotenuse of a right angled triangle is equal to the sum of the squares on the other two sides,* notwithstanding they have never been students in geometry.

To give an ocular demonstration of this importaut truth, we present the following figure;



The line PQ separates two equal squares. The triangle a is the right angled triangle in question, its right angle at P. x and y are its two sides, and the side opposite the right angle P is called thc hypotenuse. In each square are four equal right angled triangles. Let them be taken away from each square, and in one square the square H will be left, and in the other square the two squares A and B will be left.

Now, from each of the two equal squares on each side of PQ, we took equal sums—which must leave equal sums. That is, $A+B=H$.

When we operate on a right angled triangle, we may divide the two given sides by the same number, if we can do so without a remainder on either side, and then operate with the quotients as we would with the original numbers. But in conclusion we must multiply the result by the number which we divided by. This is working on a similar reduced triangle.

EXAMPLES.

1. *The two sides of a right angled triangle are* 312 *and* 416; *what is the hypotenuse?* *Ans.* 520.

Divide by 52)312. 416
Divide by 2) 6 8
3 4

Square 3 and 4, add those squares which make 25; the square root of 25 is 5.

Multiply $5\cdot 2\cdot 52=520$. *Ans.*

2. *A hawk, perched on a tree* 77 *feet high, was brought down by a sportsman* 14 *rods distant on a level with the base; what distance in yards did he shoot?* *Ans.* 81.15+yards.

14 rods reduced to feet is $14\cdot 16\frac{1}{2}=7\cdot 33$.

Now without reduction we shall be obliged to square 77 and 231. But we may operate thus,

7)7·33 77
11) 33 11
3 1

$3^2+1^2=10$. $\sqrt{10}=3.1622+$.

Ans. in feet, = 77 (3.1622). *Ans.* in yards, 77 (1.054).

CUBE ROOT.

The object of the Cube Root is to find *three* equal factors, exactly or approximately, whose product will give any required sum. The reason of its being called *cube* is because the three factors may be correctly represented by the length, breadth, and height of a geometrical cube. The product of three *unequal* factors may be represented by a *geometrical solid* of unequal length, breadth, and height, called a *parallelopipedon.*

While operating for cube root it is convenient to have the cube numbers before us.

Roots,	1	2	3	4	5	6	7	8	9	10
Cubes,	1	8	27	64	125	216	343	512	729	1000

We see by these cubes that one figure in a root may have three places in its corresponding power.

Therefore separate the power into periods of three figures each, beginning at the units; the number of periods will show the number of figures in the root.

Now as we are to have nothing to do with the common methods of extracting cube root, all we are permitted to know is the division of the power into periods, *and the fact that three equal factors* of the power are required.

EXAMPLES.

1. *Extract the cube root of* 84604519; *or in other words, find three equal factors whose product will produce this number.* *Ans.* 439.

Here are three periods 84′604′519, which show that there must be three figures in the root. The superior period is 84, and 84 referred to the line of cubes, its place would be between 64 and 125, whose roots are 4 and 5. Hence the root sought for is greater than 400 and less than 500; I should judge it not far from 440. Therefore I assume 440 as one of the factors of the number.

```
440)84604519(192283
    440
    ----
    4060
    3960
    ----
     1004
      880
     ----
     1245
      880
     ----
      3651
      3520
      ----
       1319
       1320
       ----
```

Now if one factor is 440, the product of the other two is 192283, *very* nearly; (not *exactly*, for the last figure 3 is too large by a very small fraction.)

We will now operate for two *equal* factors of the number 192283, and if our first factor is near an equal factor, that same factor is near an equal factor in this number; therefore try it thus,

```
440)192283(437
    1760
    ----
     1628
     1320
     ----
      3083
      3080
```

Here we have three factors, 440, 440, 437, whose product will give the number 84604519, within 3 units. These factors are not *all equal*, and of course are not the factors required; but they are so nearly equal that *one-third of their sum* will be one of the equal factors required. That is,

```
   440
   440
   437
3)1317
   439  Ans.
```

2. *Find the cube root, or three equal factors of the number* 32461759. *Ans.* 319.

By the aid of the periods we perceive that the factors must be greater than 300, but nearer 300 than 400.

Assume then 312 to be near one of the equal factors sought for. Divide by 312 twice, or once by the square of 312. That is,

```
97344)32461759(333.3
      292032
      ------
       325855
       292032
       ------
        338239
        292032
        ------
         46207
```

It is now obvious that the product of the three factors, 312, 312, and 333.3, will produce very nearly the given power; but these factors are not all equal, and equal factors are required;

but they are so nearly equal that $\frac{1}{3}$ of their sum, 319.1, can be relied upon as *extremely near* the root required. *A factor, or root, determined in this manner from unequal factors, will always be a little in excess of the true value required.* Hence, in this case we will omit the one-tenth and take 319 as nearer the root sought, and on trial find it to be the root exactly.

We will now give one of the most difficult examples.

3. *Find the approximate cube root of* 16. *Ans.* 2.519842.

The factors of 16 are $2 \cdot 2 \cdot 4$; the sum of these is 8, which divided by 3 gives 2.66 for the first approximation to equal factors, but as these factors are so unequal, 2.66 must be in excess. Therefore we assume 2.5 to be near one of the equal factors required. To find the other two factors, divide twice by 2.5, or once by 6.25.

Thus,
```
6.25)16.00(2.56
     12 50
     -----
      3500
      3125
      ----
       3750
       3750
       ----
```

Here we have three factors, 2.5, 2.5, 2.56, whose product will give 16 *exactly;* they are not all equal however, but being nearly so $\frac{1}{3}$ of their sum, 2.52, is *very nearly* equal to the root sought: (*it must be a very little in excess*).

Now if we repeat the operation with 2.52 as an assumed factor and find two other corresponding factors, $\frac{1}{3}$ of the sum of the three will be the root to a high degree of approximation.

4. *Find an approximate cube root of* 66. *Ans.* 4.041240.

By the cube numbers we find that 4 must be near one of the equal factors, therefore divide by the square of 4.

```
16)66(4.125
   64
   --
    20
    16
    --
     40
     32
     --
     80
```

Hence the root sought must be a very little less than $\frac{1}{3}$ of the sum of 4, 4, 4.125; that is, a very little less than 4.0416.

For a nearer approximation take 4.041 as one of the factors of 66, and find the other two, &c.

5. *Find an approximate cube root of* 21. *Ans.* 2.758923.

That is, find three factors, as near equal as possible, whose product will be 21, or very nearly 21.

We know that 27 has three factors, each equal to 3; therefore the equal factors of 21 must be each less than 3, and as we cannot expect to find the equal factors at the first trial, we will assume 2.7 and 2.8 to be two of the factors, their product is 7.56; hence, the third corresponding factor is found by the following division:

```
7.56)21.00(2.777
     15 12
     -----
      5880
      5292
      ----
       5880
       5292
       ----
        5880
        5292
        ----
         588
```

Here we have three factors nearly equal, whose product is *very near* 21; one-third of their sum is 2.759, which must be a little greater than the root required. We will therefore assume 2.75 as one of the equal factors sought, and find the other two corresponding factors, and one-third of their sum will be an approximate cube root of 21.

It is not necessary to give more examples.

When it is necessary to multiply several numbers together and extract the cube root of their product, we may often *evade* or *abridge* the operation by resolving the numbers into cube factors.

EXAMPLES.

1. *What is the side of a cubical mound, equal to one* 288 *feet long,* 216 *feet broad, and* 48 *feet high?* *Ans.* 144.

$$288=2\cdot12\cdot12$$
$$216=6\cdot\ 6\cdot\ 6$$
$$48=4\cdot12$$

Product, $288\cdot216\cdot48=12^3\cdot6^3\cdot8$

Whence, $\sqrt[3]{288\cdot216\cdot48}=12\cdot6\cdot2=144.$ *Ans.*

2. *Required the cube root of the product of* 448 *by* 392 *in a brief manner.*

N. B. Divide by the cube number 8; then it will appear that

$$448=8\cdot8\cdot7$$
and $$392=8\cdot7\cdot7$$

Product, $448\cdot392=8^3\cdot7^3$

Whence, $\sqrt[3]{448\cdot392}=8\cdot7=56$ *Ans.*

3. *Find the cube root of the product of the two factors* 192 *and* 1025 *in as brief a manner as possible.* *Ans.* 60.

The three last examples are rare cases; nevertheless they serve to awaken thought, and for this purpose they were introduced.

ALLIGATION ALTERNATE.

No arithmetical rule is more difficult to be comprehended by young pupils than this.

The operations are generally very trifling, but the *rationale* is rarely discovered.

For this reason we shall be a little *unique* in our exposition of the principle—we shall resort to an experiment in philosophy.

It is clear to the comprehension of almost every one, that two bodies balanced on a *fulcrum*, the heavier body must be nearer the fulcrum than the lighter body.

Thus two bodies balanced on the fulcrum *F*, 2 pounds at the distance of 6, will balance 6 pounds at the distance of 2.

Or when we have the distances, we can take those distances, or their proportion for corresponding weights if we *alternate* them. That is, the long distance must go on the opposite side of the fulcrum, and there become weight, and so of the other distance, and there will be a *balance.*

We shall apply this principle in the following example.

1. *A grocer has two kinds of sugar, one at* 9 *cts., the other at* 16 *cts. per pound; he wishes to make a mixture worth* 11 *cts. per pound: what proportion of the two kinds shall he take?*

Here two quantities are to be balanced on the *fulcrum* 11.

The difference between 9 and 11 is 2; place the 2 opposite 16. The difference between 11 and 16 is 5; place this opposite 9.

$$11\left\{\begin{array}{ll}9 & 5\\ 16 & 2\end{array}\right.$$

The result is, that 5 pounds at 9 cts. = 45 cts.
and 2 pounds at 16 cts. = 32 cts.
Makes 7 pounds worth 77 cts., which is 11 cts. per pound as required.

We may now expand the problem and add another kind of sugar, worth 10 cts. per pound.

Then make a mixture, worth 11 cts., with sugars worth 9, 10, and 16 cts. per pound.

Link each price below 11 to the one above. Make a balance between 9 and 16 as before, then between 10 and 16, and all will be balanced as required. The result is, 5 pounds at 9, 5 at 10, and 3 pounds at 16 cts. That is, 13 pounds of this mixture is worth 143 cts., which is 11 cts. per pound as required.

$$11\left\{\begin{array}{ll}9 & 5\\ 10 & 5\\ 16 & 2+1\end{array}\right.$$

On the same principle, any number of ingredients may be reduced to any *given mean* price or quality.

We give but one example.

Mix 6 *bushels of oats, worth* 20 *cts. per bushel, with* 8 *bushels of oats worth* 25 *cts. per bushel, with rye at* 70 *cts. per bushel, and wheat at* 80 *cts., and sell the mixture at* 75 *cts. per bushel; what proportion of rye and wheat will there be in the mixture?*

Ans. Rye 14 *bushels, wheat* 160 *bushels.*

6 bushels at 20 cts. will cost 120 cts. and 8 at 25 cts. will cost

200 cts.; whence the 14 bushels of oats will cost 320 cts., or $22\frac{6}{7}$ cents per bushel.

75	$22\frac{6}{7}$	5	5 bushels of oats.
	70	5	5 bushels of rye.
	80	$5+52\frac{1}{7}$	$57\frac{1}{7}$ bushels of wheat.

Here we have a true mixture worth 75 cts. per bushel, but the mixture contains only 5 bushels of oats: it must contain 14, therefore multiply each of these quantities by $\frac{14}{5}$. Then $5\cdot\frac{14}{5}=14$. $57\frac{1}{7}\cdot\frac{14}{5}=160$.

Alligation is of little or no practical utility, yet it serves as well as any other arithmetical operation to discipline the mind.

POSITION.

SINGLE POSITION—DOUBLE POSITION.

Before Algebra became a popular study many algebraic problems appeared in common Arithmetics, and were solved by *special rules,* which were drawn from the results of algebraic investigations. But at the present day all such problems in Arithmetic are improper; as much so as to travel 500 miles in a private carriage by the side of a railroad track.

Problems in Single Position produce equations reduceable to this form:

$$ax=m. \qquad (1)$$

Problems in Double Position produce equations in this form:

$$ax+b=m \qquad (2)$$

Not knowing the value of x in equation (1) we assume some known number, x', which may not be the true one, and if it is not, the result will not be the given number m; let it be m'. Then we shall have:

$$ax'=m' \qquad (3)$$

Divide equation (1) by (3), then

$$\frac{x}{x'}=\frac{m}{m'}.$$

Converting this into a proportion, we have

$$m' : m :: x' : x.$$

The result of this proportion, put into words, is the rule of Single Position given in all the old Arithmetics.

Rule. *Assume a number and find the result of the supposition; then say: As the result of the supposition is to the given result, so is the supposed number to the true number.*

We give but a single example.

A and *B* have the same income. *A* contracts an annual debt amounting to $\frac{1}{7}$ of it: *B* lives on $\frac{4}{5}$ of his income, and at the end of 10 years lends to *A* money enough to pay off his debts and has $160 to spare: what is the income of each? *Ans.* $280.

For the sake of convenience we will take some number divisible by 5 and 7; therefore take 35 for the supposed income of each.

Then *A*'s debt in one year is $5, in 10 years $50.

B saves $\frac{1}{5}$, or $7, in one year, in 10 years $70.

B lends *A* 50 and has $20 left as the result of the supposition.

Then, $20 : 160 :: 35 : 280$. *Ans.*

Now let us suppose the income to be 1, or unity. Then *A*'s debts in 10 years amount to $\frac{10}{7}$.

B saves in 10 years $\frac{10}{5}$, or 2.

B pays *A*'s debts; he then has $(2-\frac{10}{7})=\frac{4}{7}$.

Whence, $\frac{4}{7}=160$, or $\frac{1}{7}=40$, or $1=280$. *Ans.*

This manner of working by fractions some teachers call Arithmetic, but it is Algebra in disguise.

Let x be the income, in place of 1, and the identity will be obvious.

To show the arithmetical rule for Double Position we take the equation

$$ax+b=m. \qquad (1)$$

1st. Suppose x to be represented by the assumed number x', and $m+e'$ the result of this supposition, e' being the excess, or *error*. Then,

$$ax'+b=m+e'. \qquad (2)$$

Again, assume another number, say x'' and e'' the second error.

$$ax''+b=m+e''. \qquad (3)$$

Subtract (1) from (2) and

$$a\,(x'-x)=e'. \qquad (4)$$

(1) from (3) $$a\,(x''-x)=e''. \qquad (5)$$

Divide (4) by (5) and we have,

$$\frac{x'-x}{x''-x}=\frac{e'}{e''}.$$

Whence, $$e''x'-e''x=e'x''-e'x.$$

And $$x=\frac{e'x''-e''x'}{e'-e''}.$$

This last equation, put in words, is the rule given in all the Arithmetics of a former day. It is substantially this :

Make two distinct suppositions and note the results. Take the difference between the given result and the result of each supposition, which difference call *error*. Then,

Multiply the first error by the last supposition, and the last error by the first supposition. Divide the difference of these products by the difference of the errors, and the quotient will be the number required.*

EXAMPLE.

A *has* \$20; B *has as many as* A *and half as many as* C; *and* C *has as many as* A *and* B *both. How many dollars had each?*

Ans. A \$20; B \$60; C \$80.

Suppose *C* had 60.
Then *B* had 30+20=50.
A had 20. But 20+50 is not 60, the error therefore is 10.

Again, suppose *C* had 66.
Then *B* had 33+20=53.
A had 20.
But 20+53 is not 66. Error 7.

1st sup. 60.	Error 10.	660
2d sup. 66.	Error 7.	420
		3)240
	Ans.	80

By Algebra,

Let $2x$= *C's*. Then $x+20$=*B's*. 20=*A's*.

Then $2x=x+40$. $x=40$. $2x=80$. *Ans.*

By comparing the last operation with the operation of the arithmetical rule, and then applying it, we perceive the folly of retaining the old rules for mere arithmetical purposes.

The rule of Double Position, however, is of importance in solving exponential equations in Algebra.

* The difference is *Algebraic* $\pm$; hence some Arithmetics give two cases to the rule—one when the errors are alike, the other when unlike.

PART SECOND

ALGEBRA.

SECTION I.

In Algebra, as in Arithmetic, we shall only touch here and there on such points as might come up in the school-room, and present some difficulties. Hence this work will seem to want connection.

When we indicate the solution of a problem in Robinson's Algebra, University edition, we shall refer to it by Article and number of the problem, and not write the problem in full.

When the problem is to be found only in some other book, or is original here, it will be written out in full.

ROBINSON'S ALGEBRA.

SECTION II.

CHAPTER I.

EQUATIONS.

None of the questions in this chapter require the aid of a key, until we come to the 15th, page 65.

(15.) $\left(\frac{4x-4a}{3}-a\right)\frac{4}{3}=\frac{16x-16a}{9}-\frac{4a}{3}=$ his stock at the commencement of the third year, before his expenses are taken out. Hence,

$$\left(\frac{16x-16a}{9}-\frac{4a}{3}-a\right)\frac{4}{3}=2x.$$

Reduced gives $x=14800$, *Ans.*

(16.) Put $a=99$, $x=$time past. Then $a-x=$time to come, and per question,

$$\frac{2x}{3}=\frac{4a-4x}{5}\ldots\ldots\ldots x=54.$$

(17.) Let $x=$ the whole composition.

Then per question,

$$\frac{2x}{3}+10=\text{nitre.}$$

$$\frac{x}{6}-4\tfrac{1}{2}=\text{ sulphur.}$$

$$\frac{2x}{21}+\frac{10}{7}-2=\text{charcoal.}$$

By addition, $\frac{2x}{3}+\frac{x}{6}+\frac{2x}{21}+3\tfrac{1}{2}+\frac{10}{7}=x.$

Multiply by 6, and drop $5x$ from both sides, and we have

$$\frac{4x}{7}+21+\frac{60}{7}=x. \quad \text{Or, } 4x+21\cdot7+60=7x\ldots\ldots x=69.$$

(18.) Put $a=183$; x=what the first received; then $a-x=$ 2d received.

Per question, $\frac{4x}{7}=\frac{3a-3x}{10}\ldots\ldots\ldots\ldots x=63.$

(19.) Put $a=68$, $x=$ the greater part, and $a-x=$ the less.

$$84-x=3(40-a+x)\ldots\ldots\ldots\ldots x=42.$$

(20.) The distance from A to B put $=2x$.
The distance from C to D " $=3x$.
Then, 3 times the distance from B to C must be

$\frac{x}{2}+\frac{3x}{2}$ or the distance is, $\frac{x}{6}+\frac{x}{2}$.

Hence the whole distance is, $5x+\frac{x}{6}+\frac{x}{2}=34.$

(21.) Let x=the flock.

The first party left him $\frac{2x}{3}-6$. The second left $\frac{x}{3}-3-10=2$.

(23.) Observe that for every vessel he broke he lost 12 cents: 3 cents fee and 9 cents forfeiture.

$$300-12x=240\ldots\ldots\ldots\ldots x=5.$$

(24.) Had he not been idle he would have been entitled to ab cents. But he was idle x days at a loss of $(b+c)$ cents.

Hence, $ab-(b+c)x=d.$ $\qquad x=\frac{ab-d}{b+c}.$

(25.) Put $5x=$ less part. Then $a-5x=$ the greater part.

Per question, $a-7x=20x-\frac{3}{7}(a-5x)$

$7a-49x=140x-3a+15x$

or, $204x=10a=10\cdot204$

or, $x=10$

Therefore, $5x=50=$ the less part.

(26.) Let $8x=$the price of the horse.
Then $a-8x=$ the chaise. $a=341$.

Per question, $2a-16x-3x=24x-\frac{5}{7}(a-8x)$

or, $2a=43x-\frac{5}{7}(a-8x)$

$14a=301x-5a+40x$

$19a=341x$ or, $x=19$

$8x=152$. *Ans.*

(29.) Let $5x=$ his money.
After he first lost and won 4*s*., he had $4x+4$.
He again lost and won, and then had $3x+3+3$.
$\frac{5}{6}$ of this must equal 20, or, $3x+6=24$. $x=6$

$5x=30$. *Ans.*

(30.) Let $3x=$ the income.
Then $2x=$ the family support.

$x-\frac{2x}{3}=\frac{x}{3}=70$. Hence, ... $3x=70\cdot9$.

31, 32, and 33 require no explanation.

(35.) Last year the rent was x dollars.

This year it is $x+\frac{8x}{100}=1890$.

(36.) Is the (35) in general terms.

(37.) Let $7x=$ equal the income.

Then $x=$ A's annual debt. $\frac{7x}{5}=$what B saves.

$\frac{7x}{5}-x=16$ or $x=40$. $7x=280$. *Ans.*

In general terms,

$$\frac{7x}{5}-x=\frac{a}{2} \qquad x=\frac{5a}{4} \qquad 7x=\tfrac{1}{4}(35a).$$

(38.) $$\frac{x}{3}+\frac{x}{4}+\frac{2x}{7}=a.$$

(39.) Let $10x=$ the income.

Then $2x+100=$ the sum spent.

$5x+\ 35=$ " sum left.

$7x+135=10x$ the whole,

or, $45=x$ . . . . 450. *Ans.*

(40.) Let $21x=$ the income.

Then $3x+a=$ the sum spent.

$7x+b=$ the sum left.

$10x+a+b=21x=$ the whole. $x=\frac{(a+b)}{11}.$

(41.) $2x+4 \ : \ 3x+4 \ :: \ 5 \ : \ 7.$

(42.) Let $x^2-7=$ the number.

Then, per conditions, $x-1=\sqrt{x^2-7}$

$x^2-2x+1=x^2-7$

or, $x=4$ and $x^2-7=9.$

(43.) A's rate of travel is $\frac{7}{5}$ miles per hour.

B's rate of travel is $\frac{5}{3}$ miles per hour.

A is in advance when B sets out, $\frac{56}{5}$ miles.

Let $x=$ the hours after B starts.

Then, $\frac{5x}{3}=\frac{7x}{5}+\frac{56}{5}$ Reduced gives . . . . $x=42.$

CHAPTER II.

EQUATIONS IN TWO UNKNOWN QUANTITIES.

(6.) Add the two equations together, representing $(x+y)$ by s, and we have $\frac{1}{3}s+3s=50$ or . . . . $s=5\cdot3.$

But $x+9y=21\cdot3$

Subtract $x+\ y=\ 5\cdot3$

$8y=16\cdot3$ or, ... $y=6.$

(7.) By adding the two equations we have

$$5s=50$$

$$\begin{array}{ll} \text{or,} & x+y=10 \\ \text{but} & 4x+y=34 \\ \hline \text{Hence} & 3x \quad =24 \end{array} \quad \text{or,} \ldots\ldots\ldots x=8.$$

(9) and (10) are resolved same as (6) and (7).

(11.) From the first equation we have

$$y=2x-80.$$

Transpose —8 in the second equation and we have

$$\frac{x+y}{5}+\frac{x}{3}=\frac{2y-x}{4}+35.$$

Multiply by 60 and we have

$$12x+12y+20x=30y-15x+35\cdot 60$$

$$\text{or} \quad 47x=18y+35\cdot 60$$

Substituting the value of $18y$, we have

$$47x=36x-18\cdot 80+35\cdot 60$$

$$\text{or} \quad 11x=-240\cdot 6+350\cdot 6=110\cdot 6$$

Hence,..........................$x=60$.

(14.) Bringing unknown terms to the first members of the equations and we have

$$\frac{4}{x}-\frac{4}{y}=-1 \qquad \frac{4}{y}-\frac{2}{x}=\frac{3}{2}$$

By addition, $\frac{2}{x}=\frac{1}{2}$ or..................$x=4$.

(15.) Put $a=50$.

Then, $x+3a : y-a :: 3 : 2$

And $x-a : y+2a :: 5 : 9$

$$2x+6a=3y-3a \qquad (1)$$

$$9x-9a=5y+10a \qquad (2)$$

Multiply (1) by 5, and (2) by 3; then,

$$10x+30a=15y-15a \qquad (3)$$

$$27x-27a=15y+30a \qquad (4)$$

Subtract (3) from (4) and

$$17x-57a=45a$$

$$17x=102a$$

$$x=6a=6\cdot 50=300.$$

(16.) Divide the numerator of the second member of the first equation by its denominator, and we have

$$3x+6y+1=3x+6y+1+\frac{-11x-14y+127}{2x-4y+3}.$$

Hence, $11x+14y=127$ (1)

Multiply the second equation by $(3y-4)$ and we shall have

$$9xy-12x=\frac{(151-16x)\ (3y-4)}{4y-1}+9xy-110.$$

or, $$110-12x=\frac{(151-16x)(3y-4)}{4y-1}.$$

$$440y-48xy-110+12x=453y-48xy-604+64x.$$

$$0=13y+52x-494$$

or, $4x+y=38$ (2)

Add (1) and (2), and we have

$$15(x+y)=165$$

or, $x+y=11$ (3)

(3) from (2) gives $3x=27$............$x=9$.

(17.) Multiply the 1st equation by 14 and we have

$$42x-7y=49$$

Add $-x+7y=33$

$41x \quad =82$ or,.......$x=2$.

(18.) $$x+\frac{2}{3}y=16$$

$$x-\frac{3y}{5}=-3$$

Subtract $\frac{2}{3}y+\frac{3}{5}y=19$

$10y+9y=19\cdot15$........$y=15$.

(19.) Divide 2d by the 1st, and $x-y=2$.

But $x+y=8$.

(20.) Multiply the first equation by $(x+y)$, and the second by 9, and we have

$4(x+y)^2=9(x^2-y^2)$ $\quad 9(x^2-y^2)=9\cdot36$.

$4(x+y)^2=9\cdot36$.

Hence, $x+y=9$.

Divide the first equation by this last, and we have $x-y=4$.

(21.) $x=\frac{4y}{3}$, $x^3=\frac{64y^3}{27}$

$$\frac{64y^3}{27}-y^3=37$$

$$37y^3=37\cdot27 \ldots\ldots\ldots\ldots\ldots\ldots y=3.$$

(22) and (23) require no remarks.

(24.) The first equation gives

$$x+24y=91 \qquad (1)$$

Add $\quad 40x+y=763 \qquad (2)$

Multiply (1) by 40, and subtract equation (2), and

$$959y=2877 \quad \text{or} \ldots\ldots\ldots\ldots\ldots\ldots y=3.$$

(25.) From 1st equation take the 2d, and we have

$$2\tfrac{1}{2}x+5y=60.$$

Divide by $2\frac{1}{2}$ and we have $\quad x+2y=24$

But $\quad \frac{1}{2}x+2y=19$

$$\tfrac{1}{2}x \quad =5 \quad \text{or,} \ldots x=10.$$

(26.) Add the two equations, and

$$\tfrac{1}{2}(x+y)+\tfrac{1}{3}(x+y)+20=x+y$$

or, $\quad \frac{s}{2}+\frac{s}{3}+20=s \ldots\ldots\ldots\ldots\ldots\ldots s=120.$

By 2d equation, $\quad \frac{1}{3}(120)-5=y=35.$ *Ans.*

CHAPTER III.

EQUATIONS OF THREE OR MORE UNKNOWN QUANTITIES.

(7.) Given $\left\{\begin{array}{l} 2x=u+y+z \\ 3y=u+x+z \\ 4z=u+x+y \\ u=x-14 \end{array}\right\}$ to find u, x, y and z.

Subtract 2d from the 1st, and

$$2x-3y=y-x \quad \text{or} \quad 3x=4y \quad (1)$$

Subtract 2d from the 3d, and

$$4z-3y=y-z \quad \text{or} \quad 5z=4y \quad (2)$$

Add 3d and 4th and $\quad 4z=2x+y-14 \quad (3)$

Multiply (3) by 5 and (2) by 4 and

$$\begin{array}{r} 202=10x+5y-70 \\ 202=\quad\quad 16y \\ \hline 0=10x-11y-70 \end{array}$$

or $$30x-33y=210$$

From equation (1) $$\begin{array}{r} 30x-40y=0 \\ \hline 7y=210 \\ y=30 \end{array}$$

(8.) $$\left\{\begin{array}{l} \frac{x}{2}+\frac{y}{3}+\frac{z}{4}=62 \\ \frac{x}{3}+\frac{y}{4}+\frac{z}{5}=47 \\ \frac{x}{4}+\frac{y}{5}+\frac{z}{6}=38 \end{array}\right\} \begin{array}{l} =a+b \\ =a-\frac{b}{4} \\ =a-b \end{array}$$

To avoid numerical multiplication, and really to understand *algebra* as applied here, observe that $62+38=100$.

Put $a=50$; then $62=a+12=a+b$.

Clearing of fractions, we have

$$6x+4y+3z=12a+12b \quad (1)$$
$$20x+15y+12z=60a-15b \quad (2)$$
$$15x+12y+10z=60a-60b \quad (3)$$

Multiply (1) by 4, and subtract (2).

Then, $4x+y=63b-12a$ (4)

Subtract (3) from (2), and

$$\begin{array}{r} 5x+3y+2z=45b \\ 3 \\ \hline 15x+9y+6z=135b \end{array}$$

Subtract $$\begin{array}{r} 12x+8y+6z=24b+24a \\ \hline 3x+y \quad =111b-24a \end{array} \quad (5)$$

Subtract (5) from (4), and we have

$$x=(12a-48b)=12(a-4b)=12\cdot 2. \textit{ Ans.}$$

That is, $x=24$ or $2b$.

Now, equation (4) gives us

$$8b+y=63b-ba$$
$$y=(55-a)b=5\cdot 12=60.$$

(9.) By adding the three equations and reducing, we have

$$4x+3y+2z=3a \quad (1)$$

By adding the 2d and 3d, reducing and doubling, we have

$$10x+4y+2z=4a \qquad (2)$$

Subtracting (1) from (2), and we have

$$6x+y=a \qquad (3)$$

Adding the 1st and 3d, and reducing, we have

$$x+2y=a \qquad (4)$$

From (3) and (4) we readily find x and y.

(10.) $$\begin{cases} 2x+y-2z=40 & (1) \\ 4y-x+3z=35 & (2) \\ 3u+t=13 & (3) \\ y+u+t=15 & (4) \\ 3x-y+3t-u=49 & (5) \end{cases}$$

It is easier to eliminate t than any other letter.

Subtract (3) from (4), and we have

$$y-2u=2 \qquad (6)$$

Three times (3) taken from (5), gives

$$3x-y-10u=10 \qquad (7)$$

Add (6) and (7) and divide the sum by 3, and

$$x-4u=4 \qquad (8)$$

Double (6), and subtract it from (8), and we have

$$x=2y \qquad (9)$$

Eliminate z from equations (1) and (2), and we have

$$4x+11y=190$$

But $4x=8y$. Then $19y=190$, or.............. $y=10$.

PROBLEMS PRODUCING SIMPLE EQUATIONS, INVOLVING TWO OR MORE UNKNOWN QUANTITIES.

(1.) Let x, y, and z represent the numbers.

$$xy=600 \qquad (1)$$
$$xz=300 \qquad (2)$$
$$yz=200 \qquad (3)$$

Multiply (1) and (2), and divide the product by equation 3, and we have $x^2=900$.................... $x=30$.

(2.) Let x, y, and z represent the numbers. Then per question

$$x+\frac{y}{2}+\frac{z}{2}=120 \qquad (1)$$

$$y+\frac{x-z}{5}=70 \qquad (2)$$

$$x+y+z=190 \qquad (3)$$

Double (1) and subtract (3), gives............. $x=50$.

This problem calls the pupil's judgment into exercise. He does not know in the first place which is greater, x or z; hence he must try both suppositions, and the one that verifies equation (2) is right.

(3.) Let x, y, and z represent the shares, and put $a=120$

$$x-\tfrac{4}{7}(y+z)=a$$
$$y-\tfrac{3}{8}(x+z)=a$$
$$z-\tfrac{2}{9}(x+y)=a$$

Clearing of fractions, we have

$$7x-4y-4z=7a \qquad (1)$$
$$-3x+8y-3z=8a \qquad (2)$$
$$-2x-2y+9z=9a \qquad (3)$$

Double (1), and to the product add (2), and we have

$$11x-11z=22a$$
$$x-\ z=\ 2a \qquad (4)$$

Double (3), and to the product add (1), and we have

$$-11x+22z=11a$$
$$-x+\ 2z=\ a \qquad (5)$$

Add (4) and (5), and we have............. $z=3a=360$.

Another solution.

Assume, $x+y+z=s$.

By the question as before, $\begin{cases} x-\tfrac{4}{7}(y+z)=a \\ y-\tfrac{3}{8}(x+z)=a \\ z-\tfrac{2}{9}(x+y)=a \end{cases}$

Clearing of fractions, and we have

$$7x-4y-4z=7a \qquad (1)$$
$$8y-3x-3z=8a \qquad (2)$$
$$9z-2x-2y=9a \qquad (3)$$

To $4x+4y+4z=4s$

Add (1) $7x-4y-4z=7a$

And $11x=4s+7a$ (4)

To $3x+3y+3z=3s$

Add (2) $8y-3x-3z=8a$

And $11y=3s+8a$ (5)

To $2x+2y+2z=2s$

Add (3) $9z-2x-2y=9a$

And $11z=2s+9a$ (6)

Add (4), (5) and (6), observing that $(x+y+z)=s$, and

$$11s=9s+24a$$

Whence, $s=12a$

This value of s, put in (4), gives

$$11x=48a+7a=55a$$

or $x=5a=600$. *Ans.*

(4.) Resolved in the book.

Let x, y, u, and z represent their ages, and s their sum.

Then,

$$s-z=18$$
$$s-u=16$$
$$s-y=14$$
$$s-x=12$$

By addition, $3s=60$ .... $s=20$.

(5.) Let $x=A$'s shillings.
$y=B$'s "
$z=C$'s "

After the first game they will have as here represented:

$$x-y-z=A$$
$$2y=B$$
$$2z=C$$

After the second game,

$$2x-2y-2z=A$$
$$3y-x-z=B$$
$$4z=C$$

After the third game

$$4x-4y-4z=16 \qquad (1)$$
$$6y-2x-2z=16 \qquad (2)$$
$$7z-x-y=16 \qquad (3)$$

Sum, $x+y+z=3\cdot16$ (4)

Add (3) and (4) and we have

$8z=4\cdot16$ . . . . $z=8$.

(6.) This problem is resolved in the book, by equation 7, Art. 53.

(7.) Let x represent the better horse, and y the poorer

$$x+15=\tfrac{4}{3}(y+10)$$
$$x+10=\tfrac{15}{13}(y+15)$$

Therefore, $\tfrac{4}{3}(y+10)=\tfrac{15}{13}(y+15)+5$

Reduced gives . . . . $y=50$.

(8.) Let $x=$ the price of the sherry.

$y=$ . . . . brandy.

Put $a=78$.

$$2x+y=3a$$
$$7x+2y=9a+9$$
$$3x=3a+9$$
$$x=a+3= \ldots 81s.\ \textit{Ans.}$$

(9.) Let $x=A$'s time. $y=B$'s time.

Then, $\frac{1}{y}=$the part that B can do in one day.

$$\frac{4}{x}+\frac{4}{y}=\frac{4}{16}=\frac{1}{4}$$

$\frac{36}{y}=\frac{3}{4}$ Hence . . . . $y=48$.

(10.) $\frac{2x}{y+7}=\frac{2}{3}$ $\qquad$ $\frac{x+2}{2y}=\frac{3}{5}$

$3x=y+7$ $\qquad$ $5x+10=6y$.

(11.) $x+\frac{2y}{3}=a$ $\qquad$ $y+\frac{3x}{4}=a$

(12.) Let $x=$ the greater, and $(24-x)=$ the less.

$$\frac{x}{24-x} : \frac{24-x}{x} :: 4 : 1$$

$$x^2 : (24-x)^2 :: 4 : 1$$

By evolution, $x : 24-x :: 2 : 1$

(13.) Let $x=$ the number of persons.

$y=$ what each had to pay.

Then, $xy=$ the amount of the bill.

Put $(x+4)(y-1)=$ ....the bill.

Also, $(x-3)(y+1)=$ ....the bill.

$$xy+4y-x-4=xy$$
$$xy-3y+x-3=xy$$

$$4y-x-4=0$$
$$-3y+x-3=0$$

By addition, $y \quad -7=0$

(14.) $10x+y=4x+4y$ or, ... $y=2x$.

$$10x+y+27=10y+x$$

or, $9x \quad +27= \ 9y$

$x \quad + \ 3= \ y=2x$, hence, $x=3$.

(15.) Let $x=$ the digits in the place of 100's.

$y=$ " in the place of 10's.

$z=$ " the units.

$x+y+z=11$ $z=2x$

$$100x+10y+z+297=100z+10y+x$$

$$99x+297= \ 99z$$

$x+3=z=2x$ Hence .... $x=3$.

(16.) Let $\frac{x-40}{2}$, $\frac{x-20}{3}$, and $\frac{x-10}{4}$ represent the parts.

Then, $\frac{x-40}{2}+\frac{x-20}{3}+\frac{x-10}{4}=90$ .... $x=100$.

(17.) Let x represent the part at 5 per cent, and $(a-x)$ the part at 4 per cent. Then

$$\frac{5x}{100}+\frac{4a-4x}{100}=b$$

Hence .... $x=100b-4a$.

(18.) To avoid high numerals, and of course a tedious operation, Put $a=5000$; then $2a=10000$, $3a=15000$,

$$\frac{3a}{10}=1500, \text{ and } \frac{16a}{100}=800.$$

Put $x=$ A's capital, and $r-1=A$'s rate.

$x+2a=$ B's " $r=B$'s rate.

$x+3a=$ C's " $r+1=C$'s rate.

By conditions,
$$\begin{cases} \dfrac{rx-x}{100}+\dfrac{16a}{100}=\dfrac{rx+2ar}{100} & (1) \\ \dfrac{rx-x}{100}+\dfrac{3a}{10}=\dfrac{rx+3ar+x+3a}{100} & (2) \end{cases}$$

Reducing (1), gives $x=(16-2r)a$

" (2), " $x=\left(\dfrac{27-3r}{2}\right)a$

Hence, $32-4r=27-3r$, or........$r=5$.

(19.) Put $a=1000$, x and y to represent the two parts, and r and t the rates expressed in *decimals*.

Then by conditions,
$$\begin{cases} x+y=13a & (1) \\ rx=\ ty & (2) \\ tx=360 & (3) \\ ry=490 & (4) \end{cases}$$

Divide (3) by (4), and we have

$$\left(\frac{t}{r}\right)\left(\frac{x}{y}\right)=\frac{36}{49} \qquad (5)$$

From (2) we have $\dfrac{x}{y}=\dfrac{t}{r}$

Substitute the value of $\dfrac{t}{r}$ in equation (5), and

$$\frac{x^2}{y^2}=\frac{36}{49} \qquad \text{or} \qquad x=\frac{6}{7}$$

By returning to equation (1) we have $\dfrac{6y}{7}+y=13a$.

$13y=13a\cdot 7$ or.....................$y=7a$.

(20.) Let x, y, and z, represent their respective ages.

Then by conditions given,
$$\begin{aligned} x-y&=\ z \\ 5y+2z-x&=147 \\ x+y+z&=\ 96 \end{aligned}$$

(21.) Let x, y, and z, represent the respective property of each, and put $s=$ their sum.

Conditions, $\begin{cases} x+3y+3z=47a \\ y+4x+4z=58a \\ z+5x+5y=63a \end{cases}$ $a=100.$

Add $2x$ to the 1st equation, $3y$ to the 2d, and $4z$ to the 3d, observing that $x+y+z=s$; then we shall have

$$3s=47a+2x \qquad (1)$$

$$4s=58a+3y \qquad (2)$$

$$5s=63a+4z \qquad (3)$$

or,

$$x=\frac{3s-47a}{2}$$

$$y=\frac{4s-58a}{3}$$

$$z=\frac{5s-63a}{4}$$

By addition, $s=\frac{3s-47a}{2}+\frac{4s-58a}{3}+\frac{5s-63a}{4}$

Hence, ...$s=19a.$

This value of s, put in equation (1), gives $x=5a=500.$

(23.) Let x, y, and z represent the respective sums.

$$x+\frac{y}{2}=a \qquad (1)$$

$$y+\frac{z}{3}=a \qquad (2)$$

$$z+\frac{x}{4}=a \qquad (3)$$

$2x+y=2a$

$3y+z=3a$ or, $4z+12y=12a$

$4z+x=4a$ or, $4z+x=4a$

$-x+12y=8a$

From the 1st $24x+12y=24a$

$25x=16a$

(24.) This problem is resolved in the work, by the 13th example, page 80, (Art. 51.)

(25.) Let x=the greater, and y the less.

$\frac{1}{2}x+\frac{1}{3}y=13$

$\frac{1}{3}x-\frac{1}{2}y=0$ or,................$2x=3y$.

(26.)

$$x+\tfrac{1}{2}(y+z)=a=51$$
$$y+\tfrac{1}{3}(x+z)=a$$
$$z+\tfrac{1}{4}(x+y)=a$$
$$x+(x+y+z)=2a$$
$$2y+(x+y+z)=3a$$
$$3z+(x+y+z)=4a$$

$$x=2a-s \quad (1)$$
$$y=\tfrac{1}{2}(3a-s) \quad (2)$$
$$z=\tfrac{1}{3}(4a-s) \quad (3)$$
$$s=2a-s+\tfrac{1}{2}(3a-s)+\tfrac{1}{3}(a-s)$$
$$6s=12a-6s+9a-3s+2a-2s$$

$17s=29a$ or................$s=29\cdot3=87$.

Now equations (1), (2), and (3), will readily give x, y and z.

(27.) Let x=A's, y B's, and z=C's sheep.

Then by the conditions,

$$x+8-4=y+z-8$$
$$\tfrac{1}{2}(y+8)=x+z-8$$
$$\tfrac{1}{3}(z+8)=x+y-8$$
$$x+12=y+z \quad (1)$$
$$y+24=2x+2z \quad (2)$$
$$z+32=3x+3y \quad (3)$$

Add (1) and (3), and we have $x+44=3x+4y$.

Double (1), and subtract (2), and we have

$2x-y=2y-2x$ or, $4x=3y$

But $2x+4y=44$

$4x+8y=44\cdot2$

$11y=44\cdot2$ or........$y=8$.

(28.) $\frac{x+1}{y}=\frac{1}{3}$ $\frac{x}{y+1}=\frac{1}{4}$

(29.) $\frac{x+2}{y}=\frac{5}{7}$ $\frac{x}{y+2}=\frac{1}{3}$

(30.) This is a repetition of the 10th example, page 89, inserted here by oversight.

(31) Let $x=A$'s money, and $y=B$'s.

$$x-5=\tfrac{1}{2}(y+5) \qquad (1)$$
$$x+5=3y-15 \qquad (2)$$

Subtract (1) from (2), and we have

$$10=3y-15-\frac{y}{2}-\frac{5}{2} \qquad \text{or} \ldots\ldots\ldots\ldots\ldots y=11.$$

(32.) Let $x=$ the number of bushels of wheat flour.
And $y=$ " " barley "
Then the cost of the whole will be expressed by

$$10x+4y$$

The sale at 11 shillings will be $11x+11y$
Now by the conditions,

$$10x+4y \;:\; 11x+11y \;::\; 100 \;:\; 143\tfrac{3}{4}$$

Multiply the last two terms by 4, and

$$10x+4y \;:\; 11x+11y \;::\; 400 \;:\; 575$$

Divide the two last terms by 25, and

$$10x+4y \;:\; 11x+11y \;::\; 16 \;:\; 23$$
$$5x+2y \;:\; 11x+11y \;::\; 8 \;:\; 23$$
$$115x+46y=88x+88y$$
$$27x=42y$$
$$9x=14y$$

These co-efficients, 9 and 14, give the lowest proportion in whole numbers. The proportion was only required.

(33.) Let $10x+y$ represent the number.

$$\tfrac{1}{5}(10x+y)=Q+\tfrac{1}{5}$$
$$\tfrac{1}{8}(10x+y)=Q'+\tfrac{1}{8}$$

Now the question gives us $Q=2x$
And $Q'=5y$

$$\tfrac{1}{5}(10x+y)=2x+\tfrac{1}{5} \qquad \text{or} \ldots\ldots\ldots\ldots\ldots y=1.$$

INTERPRETATION OF NEGATIVE VALUES.

(Art. 55.)

(4.) Let x represent the years to elapse.

Then $30+x=3(15+x)\ldots\ldots\ldots\ldots\ldots x=-7\tfrac{1}{2}$.

To make this equation true, the years required must be taken *subtractively*.

(5.) Let $x=$ the man's daily wages, and $y=$ the son's.

$$7x+3y=22 \quad (1)$$
$$5x+\ y=18 \quad (2)$$
$$2x+4y=40$$
$$3x+\ y=18 \quad x=4, \quad y=-2.$$

(6.) Let $x=$ man's wages, $y=$ wife's, and $z=$ the son's.

$$10x+\ 8y+\ 6z=1030 \text{ cts.} \quad (1)$$
$$12x+10y+\ 4z=1320 \text{ cts.} \quad (2)$$
$$15x+10y+12z=1385 \text{ cts.} \quad (3)$$

Subtract (2) from (3), and we have

$$3x+8z=65 \quad (4$$

Multiply (1) by 5, and (2) by 4, and take their difference, and we have $2x+14z=\ -130$

$$x+\ 7z=\ -65 \quad (5)$$
$$3x+21z=-3\cdot 65$$
$$(4)\ldots\ldots\ldots\ldots 3x+\ 8z=\ 65$$

By subtraction, $13z=-4\cdot 65$

$$z=-4\cdot 5=-20.$$

As z comes out with a *minus* value, it shows that the son had no wages, but the reverse of it, he was on expense.

(7.)

$$10x+4y+3z=1150 \quad (1)$$
$$9x+8y+6z=1200 \quad (2)$$
$$7x+6y+4z=\ 900 \quad (3)$$

Double (1), and subtract (2), and $11x=1100\ldots x=100$ cts.

(8.) $\dfrac{x+1}{y}=\dfrac{3}{5}$ $\qquad$ $\dfrac{x}{y+1}=\dfrac{5}{7}$

$$5x+5=3y \quad (1)$$
$$7x\ \ =5y+5 \quad (2)$$

Add $12x\ \ =8y$

$$3x\ \ =2y \quad (3)$$

Subtract (2) from (1) and we have

$$-2x+5=-2y-5 \quad (4)$$
$$x=-10 \text{ by adding (3) and (4).}$$
$$y=-15$$

The result coming out *minus*, shows that there is no such arithmetical fraction. Algebraically, however, $\frac{-10}{-15}$ will answer the conditions.

FINDING AND CORRECTING ERRORS.

This subject has been suggested to us by circumstances. Those who have not been as severely disciplined in this matter as ourselves, are too much inclined to assume that the error must be in the answer, when it is more likely to be in some portion of the data. In short, we believe the following exposition will be of use to many. We shall illustrate by examples.

1. *A young man, who had just received a fortune, spent* $\frac{5}{8}$ *of it the first year, and* $\frac{4}{5}$ *of the remainder the next year; when he had* \$1420 *left. What was his fortune?* *Ans.* \$11360.

Let $x=$ the income; then if he spent $\frac{5}{8}$ of it, $\frac{3}{8}$ would be left, and the next year he spent $\frac{3}{8}\times\frac{4}{5}=\frac{3}{10}$

Then, $$\frac{3x}{8}-\frac{3x}{10}=1420.$$

The value of x in this equation is \$38933$\frac{1}{3}$.

This result shows a great error, somewhere. If an error existed in the answer, it is probable it would be in one or two figures, at most. But every figure of our result differs from the given answer; and besides, it comes out with a fraction, which is against probability.

We will therefore assume, that the stated answer is correct, and that the error is in 1420.

To test this, write 11360 for x, and a for 1420 in the equation.

Then, $$a=\left(\frac{3}{8}-\frac{3}{10}\right)11360=\frac{3}{4}\cdot 1136=852.$$

By this supposition, 1420 should be 852; but we can conceive of no mistake, either in the writer or the printer, that would be likely to change 1420 to 852, they are so entirely unlike in all respects. We therefore assume, that 1420, as well as the answer, is correct.

Now, it only remains to find the error in one of the fractions, $\frac{5}{8}$ or $\frac{4}{5}$. To try $\frac{5}{8}$ let it be represented by m.

Then, $(1-m)=$ the portion he saved the first year; $\frac{4}{5}$ of this, or $\frac{4-4m}{5}=$ the portion he spent the second year.

Hence, $$(1-m)x-\frac{4(1-m)x}{5}=1420$$

$$5(1-m)x-4(1-m)x=1420\cdot5$$

Or, $$(1-m)x=1420\cdot5$$

Write the value of x in this equation and we have

$$(1-m)11360=1420\cdot5$$

$$\therefore\ 1-m=\frac{142\cdot5}{1136.}=\tfrac{5}{8}$$

Whence, $$m=\tfrac{3}{8}$$

Here is the error; $\frac{5}{8}$ was written or printed, by mistake, in the example, when it should have been $\frac{3}{8}$. The error was in one figure, only, and this is generally the case.

2. *A company at a tavern, when they came to pay their bill, found that had there been* 4 *more in company, they would have had a shilling a piece less to pay: but had there been* 8 *less in company, they must have paid a shilling a piece less. How many were in company, and what did each have to pay?*

Ans. 24 *persons. Each paid* 7 *shillings.*

Let $x=$ the number of persons.

$y=$ the number of shillings each had to pay.

Then, $xy=$ the amount of the bill.

By the 1st condition $(x+4)(y-1)=xy$ (1)

By the 2d " $(x-8)(y+1)=xy$ (2)

Expanding, and omitting xy on both sides, we have

$$4y-x-4=0$$

$$-8y+x-8=0$$

By addition, $-4y-12=0$ or, $y=-3$.

This value of y, substituted in one of the preceding equations, gives $x=-16$.

That is, there were 16 less than *no* persons in company, and each paid 3 less than no shillings; in short we have a complete absurdity in all respects. No change in numbers expressed in the answer will remedy the matter, and indeed with the present data, no other values can be assigned to x and y.

Now to find where the error is in the data, let m represent 4, and n stand in the place of 8, in equations (1) and (2), and in place of x write 24, and of y write 7.

Then (1) becomes $(24+m)6=7\cdot 24$ (3)

And (2) becomes $(24-n)8=7\cdot 24$ (4)

Then $m=4.$ and $n=3.$

Here then we find that 8 in the example was printed in the place of 3. This correction being made every thing corresponds.

In an Algebra recently published, I find the following example given for solution. It contains an error—find that error.

3. Given

$$\left\{\begin{array}{l}\dfrac{y}{(x+y)^{\frac{2}{3}}}+\dfrac{\sqrt{x+y}}{y}=\dfrac{17}{4\sqrt{x+y}} \\ x=y^2+2.\end{array}\right\}\text{to find } x \text{ and } y \left\{\begin{array}{l}Ans.\ x=6, \text{ or } 3. \\ \\ y=2, \text{ or } 1.\end{array}\right.$$

This example is given under equations of the second degree; but when we attempt a solution, the resulting equation will produce an equation of a higher degree.

As the answers are given in small commensurate numbers, it is probably *highly probable* that they are correct, or rather should not be changed.

Because the second equation verifies with both answers, we must regard that as correct.

It is also very improbable that an error should exist in regard to the radical sign of square root, but in regard to the exponent $\frac{2}{3}$, an error there is very possible. On this supposition we will substitute the values of x and y, in the first equation, and write m for the exponent; then we shall have,

$$\frac{2}{(6+2)^m}+\frac{\sqrt{6+2}}{2}=\frac{17}{4\sqrt{6+2}}$$

Multiplying by $\sqrt{8}$ and we shall have

$$\frac{2\sqrt{8}}{8^m}+\frac{8}{2}=\frac{17}{4} \qquad \text{or,} \quad \frac{8\sqrt{8}}{8^m}+16=17$$

Whence, $\quad 8\sqrt{8}=8^{\frac{3}{2}}=8^m \qquad$ That is, $\quad m=\frac{3}{2}$

From this we learn that the printer inverted the terms of the fractional exponent. This being corrected, all is harmonious.

We shall give other examples in finding errors as we naturally come in contact with them.

PURE EQUATIONS.

We omit all the examples in Robinson's Algebra to the 17th, page 136. From thence we touch all that can require any notice in a work like this.

(17.) Divide the numerator by its denominator, in each member, and we have

$$1-\frac{4}{\sqrt{6x}+2}=1-\frac{15}{4\sqrt{6x}+6}$$

Drop 1 and change signs, and clear of fractions, and

$16\sqrt{6x}+24=15\sqrt{6x}+30$. Hence........$x=6$.

(18.) Cube both members, and

$$64+x^2-8x=\frac{(4+x)^3}{(4+x)}=(4+x)^2$$

Hence, $64+x^2-8x=16+8x+x^2$ or........$x=3$.

(19.) By clearing of fractions, $5+x+\sqrt{x^2+5x}=15$

By reduction, $\sqrt{x^2+5x}=10-x$ Square, &c.

(20.) $$\sqrt{x+\sqrt{x}}-\sqrt{x-\sqrt{x}}=\frac{3}{2}\frac{\sqrt{x}}{\sqrt{x+\sqrt{x}}}$$

Multiply by $\sqrt{x+\sqrt{x}}$, and we have

$$x+\sqrt{x}-\sqrt{x^2-x}=\tfrac{3}{2}\sqrt{x}$$

$$2x+2\sqrt{x}-2\sqrt{x^2-x}=3\sqrt{x}$$

$$2x-\sqrt{x}=2\sqrt{x^2-x}$$

$$4x^2-4x\sqrt{x}+x=4x^2-4x \qquad x=\tfrac{25}{16}.$$

(21.) Resolved in the work.

(22.) Resolved the same as 17.

$$1-\frac{2b}{\sqrt{ax}+b}=1-\frac{7b}{3\sqrt{ax}+5b}$$

Hence, $6\sqrt{ax}+10b=7\sqrt{ax}+7b$ or........$x=\dfrac{9b^2}{a}$

(*U.* 136.)

(23.) Square both members and we have

$$1+x\sqrt{x^2+12}=1+2x+x^2$$

$\sqrt{x^2+12}=2+x$ Square, &c. $x=2$.

(24.) Multiply numerator and denominator by the numerator, and we have

$$\frac{(\sqrt{4x+1}+\sqrt{4x})^2}{1}=9.$$

Take square root and transpose $\sqrt{4x}$, and $\sqrt{4x+1}=3-\sqrt{4x}$.

Square, $4x+1=9-6\sqrt{4x}+4x$. Hence, ... $x=\frac{4}{9}$.

(25.) Square root gives $a-x=\sqrt{b}$, or $x=a-\sqrt{b}$.

(26.) Clearing of fractions and we have

$$1+\sqrt{1-x^2}-1+\sqrt{1-x^2}=\sqrt{3}.$$

$$2\sqrt{1-x^2}=\sqrt{3}$$

$$4-4x^2=3 \quad \ldots\ldots x^2=\tfrac{1}{4}. \quad x=\pm\tfrac{1}{2}.$$

(27.) Take the square root of both members, and

$$\frac{2}{x-1}=\frac{1}{2} \text{ or } \ldots\ldots\ldots\ldots\ldots\ldots x=5.$$

(28.) Resolved the same as the 21st.

(29.) Clearing of fractions we have

$$\sqrt{x^2-9x}+x-9=36$$

$$\sqrt{x^2-9x}=45-x$$

$$x^2-9x=45^2-90x+x^2$$

$$81x=45\cdot45$$

$$9\cdot9x=5\cdot9\cdot5\cdot9 \ldots\ldots\ldots\ldots x=5\cdot5=25.$$

(30.) Resolved the same as (17) and (22.)

Dividing numerators by denominators, we have

$$3-\frac{10}{\sqrt{x}+2}=3-\frac{105}{\sqrt{x}+40}$$

Drop 3 from both sides, change signs, and divide by 5. and clear of fractions, and then

$$2\sqrt{x}+80=21\sqrt{x}+42. \text{ Hence} \ldots\ldots\ldots\ldots x=4.$$

(31.) Multiply numerator and denominator of the first member by $(\sqrt{x}+\sqrt{x-a})$

Then, $\dfrac{(\sqrt{x}+\sqrt{x-a})^2}{a}=\dfrac{an^2}{x-a}$

Multiply by a and take the square root, and

$$\sqrt{x}+\sqrt{x-a}=\frac{an}{\sqrt{x-a}}$$

$$\sqrt{x^2-ax}+x-a=an$$

$$\sqrt{x^2-ax}=(n+1)a-x$$

$$x^2-ax=(n+1)^2a^2-2a(n+1)x+x^2.$$

Drop x^2, and divide by a, and

$$-x=(n+1)^2a-2nx-2x$$

$$(1+2n)x=(n+1)^2a$$

$$x=\frac{(n+1)^2a}{1+2n}$$

(32.) Resolved in the work.

(Art. 90.)

(4.) Observe that $180=9\cdot20$ $189=9\cdot21$. Put $a=9$.

$$x^2y+xy^2=20a \qquad x^3+y^3=21a$$

Multiply the first equation by 3, and add it to the second,

and $x^3+3x^2y+3xy^2+y^3=81a=a^3$

cube root, $x+y=a=9$

The rest of the operation is obvious.

(5.) Divide the first equation by $(x+y)$ and

$$x^2-xy+y^2=xy$$

$$x^2-2xy+y^2=0 \qquad \text{or} \qquad x-y=0.$$

Hence............................$x=2$ $y=2$.

(6.) $x+y : x :: 7 : 5$ $xy+y^2=126$.

$$5x+5y=7x$$

$$5y=2x \qquad \text{or} \qquad x=\tfrac{5}{2}y.$$

Put this value of x in the second equation, and

$$\tfrac{5}{2}y^2+y^2=126$$

$$7y^2=126\cdot2$$

$$y^2=18.2=36\ldots\ldots y=\pm6.$$

(7.) From the first equation we have

$$5x-5y=4y \qquad 5x=9y$$

$$x=\tfrac{9}{5}y$$

$$x^2+4y^2=181$$
$$\tfrac{81}{25}y^2+4y^2=181$$
$$81y^2+100y^2=181\cdot25$$
$$181y^2=181\cdot25 \quad \text{or} \ldots\ldots\ldots\ldots\ldots\ldots y^2=25.$$

(8.) From the proportion we have

$$\sqrt{x}+\sqrt{y}=4\sqrt{x}-4\sqrt{y}$$
$$5\sqrt{y}=3\sqrt{x} \quad \text{or} \ldots\ldots\ldots\ldots\ldots\ldots 25y=9x.$$

The rest of the operation is obvious.

(9.) Extract square root and

$$\tfrac{1}{3}x+\tfrac{1}{2}=3 \quad \text{or} \ldots\ldots\ldots\ldots\ldots\ldots x=7\tfrac{1}{2}.$$

(10.) From the first proportion

$$x+y=3x-3y \qquad \text{or} \quad 4y=2x$$
$$\text{Hence} \ldots\ldots\ldots\ldots\ldots\ldots 8y^3=x^3.$$
$$8y^3-y^3=56 \qquad y^3=8 \qquad y=2.$$

(11), (12) and (13) resolved in the work.

(14.) $$\frac{x^2-y^2}{x-y}=6 \quad \text{or} \ldots\ldots\ldots\ldots\ldots\ldots x+y=6$$

From the second, $\ldots\ldots\ldots\ldots\ldots\ldots xy=5.$

(15.) Divide the first equation by $(x+y)$, and

$$x^2-xy+y^2=2xy$$
$$x^2-2xy+y^2=xy=16 \qquad (1)$$
$$\text{Add} \quad 4xy \qquad =64$$
$$x^2+2xy+y^2= \qquad 80=16\cdot5$$

Square root, $\quad x+y=4\sqrt{5}$

Square root of (1) $\quad x-y=4$

$$2x \quad =4\sqrt{5}+4.$$

(19.) Double the 2d equation, and add and subtract it from the 1st, then

$$x^2+2xy+y^2=a+2b$$
$$x^2-2xy+y^2=a-2b$$
$$x+y=\sqrt{a+2b}$$

(Art. 92.)

(5.) Add the two equations and extract square root, and we have $x^{\frac{1}{2}}+y^{\frac{1}{2}}=\pm 4$ (1)

Separate the first member of the first equation into factors, and we have $x^{\frac{1}{2}}(x^{\frac{1}{2}}+y^{\frac{1}{2}})=12$ (2)

Divide (2) by (1) and $x^{\frac{1}{2}}=\pm 3$ $x=9$.

(6.) Is of the same form and resolved the same as (5.)

(7.) Add the two equations, and extract square root, and we have $x^{\frac{3}{4}}+y^{\frac{3}{4}}=\sqrt{a+b}$

But $x^{\frac{3}{4}}(x^{\frac{3}{4}}+y^{\frac{3}{4}})=a$

$$x^{\frac{3}{4}}=\frac{a}{\sqrt{a+b}}$$

$$x^3=\frac{a^4}{(a+b)^2} \quad \text{or} \ldots\ldots\ldots\ldots x=\left(\frac{a^4}{(a+b^2}\right)^{\frac{1}{3}}$$

(8.) Resolved in (Art. 90,) of this Key.

(9.) Square the first equation, and

$$x+2x^{\frac{1}{2}}y^{\frac{1}{2}}+y=25$$

$$x \qquad +y=13 \qquad (1)$$

Difference, $2x^{\frac{1}{2}}y^{\frac{1}{2}} \quad =12$ (2)

Subtract (2) from (1) and

$$x-2x^{\frac{1}{2}}y^{\frac{1}{2}}+y=1$$

By evolution, $x^{\frac{1}{2}}-y^{\frac{1}{2}} \quad =\pm 1$

But, $x^{\frac{1}{2}}+y^{\frac{1}{2}} \quad = \quad 5$

$2x^{\frac{1}{2}} \qquad =6$ or 4.

The following are not in Robinson's Algebra. They are mostly from Bland's Problems. We shall number them in order.

(1.) Given $x^2+3x-7=x+2+\frac{18}{x}$ to find the values of x.

Ans. $+3$, -3, -2.

Reducing $x^2+2x=9+\frac{18}{x}$

Factoring $x^2(x+2)=9(x+2)$

This equation will be verified by putting $x+2=0$

Then will $x=-2$

Again dividing both members by $(x+2)$ and $x^2=9$

Whence $x=\pm 3.$

As a general thing we shall not give all the roots to equations. Imaginary roots we shall not pretend to give, except in rare cases, or unless we have an ulterior object in view.

(2.) Given $\frac{\sqrt{a+x}}{a}+\frac{\sqrt{a+x}}{x}=\frac{\sqrt{x}}{c}$ to find x.

Let $P=\sqrt{a+x}$ then $P^2=a+x$ and the equation becomes

$$\frac{P}{a}+\frac{P}{x}=\frac{\sqrt{x}}{c}$$

Or $\frac{Px+Pa}{ax}=\frac{\sqrt{x}}{c}$

That is $P(x+a)=\frac{a}{c}\cdot x^{\frac{3}{2}}$

Whence $P^3=\frac{a}{c}\cdot x^{\frac{3}{2}}$

By taking the cube root we shall have

$$P=\left(\frac{a}{c}\right)^{\frac{1}{3}}x^{\frac{1}{2}}$$

By squaring and replacing the value of P^2 we shall have

$$a+x=\left(\frac{a}{c}\right)^{\frac{2}{3}}x$$

Let the known co-efficient $\left(\frac{a}{c}\right)^{\frac{2}{3}}$ be represented by m.

Then $a+x=mx$ or, $x=\frac{a}{m-1}.$

(3.) Given $\frac{(a+x)^{\frac{1}{n}}}{a}+\frac{(a+x)^{\frac{1}{n}}}{x}=\frac{x^{\frac{1}{n}}}{c}$ to find x.

This is the same as example 2, in case $n=2$, therefore we may jump to the conclusion at once.

Thus, $$a+x=\left(\frac{a}{c}\right)^{\frac{n}{n+1}}x.$$

(4.) Given $\begin{cases} x^3+y^3=(x+y)xy \\ x^2y+xy^2=4xy \end{cases}$ to find the values of x and y.

Dividing the first equation by $(x+y)$ and the second by xy we obtain

$$x^2-xy+y^2=xy \qquad (1)$$
$$x+y=4 \qquad (2)$$

Transposing xy in (1) from the second member to the first gives $$x^2-2xy+y^2=0$$

Whence $x-y=0$ or, $x=y$

These values substituted in (2) give $x=2$ $y=2$

In this example we divided one of the equations by $(x+y)$, therefore $(x+y)$ must contain a root of that equation. (See theory of equations.) That is, $x+y=0$, or, $x=-y$, and $-y$ substituted for x in either equation will verify it.

(5.) Given $\begin{cases} x^2+y^2+xy(x+y)=68 \\ x^3+y^3-3x^2-3y^2=12 \end{cases}$ to find the values of x and y.

Ans. $x=4$ or 2.
$y=2$ or 4.

Multiply the first equation by 3 and to the product add the second; then

$$x^3+3xy(x+y)+y^3=216 \qquad (1)$$

Cube root $$x+y=6 \qquad (2)$$

By squaring and transposing (2) becomes

$$x^2+y^2=36-2xy \qquad (3)$$

By the aid of (2) and (3) we perceive that the first equation is equivalent to

$$36-2xy+6xy=68 \quad \text{or} \quad xy=8 \qquad (4)$$

From (2) and (4) we find x and y.

(6.) Given $\left\{\begin{array}{l}\frac{9}{8}\frac{(x+y)^{\frac{1}{3}}}{y}+\frac{9}{8}\frac{(x+y)^{\frac{1}{3}}}{x}=\frac{8}{7}\\ \frac{7}{4}\frac{(x-y)^{\frac{1}{3}}}{y}-\frac{7}{4}\frac{(x-y)^{\frac{1}{3}}}{x}=\frac{1}{9}\end{array}\right\}$ To find the values of x and y.

Ans. $x=\frac{9}{2}$ $y=\frac{7}{2}$.

Put $P=(x+y)^{\frac{1}{3}}$ $P^3=(x+y)$

And $Q=(x-y)^{\frac{1}{3}}$ $Q^3=(x-y)$

Divide the first equation by $\frac{9}{8}$, and the second by $\frac{7}{4}$, then we shall have

$$\frac{P}{y}+\frac{P}{x}=\frac{64}{63} \quad (1)$$

And $$\frac{Q}{y}-\frac{Q}{x}=\frac{4}{63} \quad (2)$$

Equation (1) reduced, becomes

$$\frac{(x+y)P}{xy}=\frac{64}{63}$$

That is $$\frac{P^4}{xy}=\frac{64}{63} \quad (3)$$

Also (2) becomes $$\frac{Q^4}{xy}=\frac{4}{63} \quad (4)$$

Divide (3) by (4) and we shall have

$$\frac{P^4}{Q^4}=16$$

Whence $P=2Q$

And $P^3=8Q^3$

That is $x+y=8x-8y$ or $9y=7x$

From this last, $xy=\frac{7x^2}{9}$, which being substituted in both (3) and (4) gives $\frac{9P^4}{7x^2}=\frac{64}{63}$ or, $\frac{9P^4}{x^2}=\frac{64}{9}$

Whence $P^2=\frac{8x}{9}$, That is $(x+y)^{\frac{2}{3}}=\frac{8x}{9}$ (5)

Substituting $\frac{7x}{9}$ for y in equation (5) and we have

$$\left(x+\frac{7x}{9}\right)^{\frac{2}{3}}=\frac{8x}{9}$$

Cubing $\left(x+\frac{7x}{9}\right)^2=\frac{8^3}{9^3}x^3$

That is $x^2\left(1+\frac{7}{9}\right)^2=\frac{8^3}{9^3}x^3$

Or, $\left(\frac{16}{9}\right)^2=\frac{8^3}{9^3}x$

$$\frac{2^2 8^2}{9^2}=\frac{8^3}{9^3}x$$

$$4=\frac{8}{9}x \quad \text{or} \quad x=\frac{9}{2}. \quad \textit{Ans}$$

The following solutions refer to problems in Robinson's Algebra, Chapter V., Art. 93. They number from (5) to (13).

QUESTIONS PRODUCING PURE EQUATIONS.

(5.) Let $x+y=$ the greater number.
And $x-y=$ the less.

Difference $2y=4$ Sum $=2x$

$2x(4xy)=1600$ Hence....$x=10$.

(6.) Let $x+y=$ the greater number.
$x-y=$ the less.

$2y : x-y :: 4 : 3$ or $x=\frac{5}{2}y$.

$(x^2-y^2)(x-y)=504$

$(\frac{25}{4}y^2-y^2)(\frac{5}{2}y-y)=504$

$(\frac{21}{4}y^2)(\frac{3}{2}y) =504$ Hence $y=4$.

(7.) Let $8x=$ the length of the field, and $5x=$ its breadth.

Then $\frac{40x^2}{160}=\frac{x^2}{4}=$ the acres.

$\frac{1}{4}x^2\times 8x=$ the whole cost.

$26x=$ the rods around the field.

$13\times 26x=$ the whole cost.

Hence $2x^3=13\cdot 26x$ or $x=13$. $8x=104$. *Ans.*

(8.) Let $5x=$ the length of the stack.
$4x=$ the breadth.

Then, $\frac{7x}{2}=$ the height.

$$5x\cdot 4x\cdot \frac{7x}{2}\cdot 4x = \text{ the cost in cents.}$$

Also, $5x\cdot 4x\cdot 224 =$ the cost in cents.

Hence, $5x\cdot 4x\cdot \frac{7x}{2}\cdot 4x = 5x\cdot 4x\cdot 224$

$7x\cdot 2x = 224$ or .... $x=4$.

(9.) Put $x^2-7=$ the number.

Then, $x+\sqrt{x^2+9}=9$

$\sqrt{x^2+9}=9-x$ or .... $x=4$.

(10.) Let x represent A's eggs; then $100-x=$ B's eggs.

$\frac{18}{100-x}=A$'s price. $\frac{8}{x}=B$'s price.

Hence, $\frac{18x}{100-x}=\frac{8}{x}(100-x)$

$9x^2=4(100-x)^2$

$3x=2(100-x)$ .... $x=40$.

(11.) Let $x+y=$ the greater number,

$x-y=$ the less.

$2x=6$ or $x=3$

$$(x+y)^3=x^3+3x^2y+3xy^2+y^3$$
$$(x-y)^3=x^3-3x^2y+3xy^2-y^3$$
$$2x^3 \qquad +6xy^2=72$$

Divide by $2x$ and $x^2+3y^2=12$. Hence, ... $y=1$.

(12.) Let $x=$ one number.

Then, $a^2x=$ the other.

$a^2x^2=b^2$ $\qquad ax=b$ $\qquad x=\frac{b}{a}$.

(13.) Let x^2 and y^2 represent the numbers.

Then $x^2+y^2=100$

$x+y=14$.

The following are additional problems, and for the sake of distinction we shall mark them (*a*), (*b*), &c.

(a) *Two men,* A *and* B, *lay out some money on speculation.* A *disposes of his bargain for* \$11, *and gains as much* PER CENT. *as* B *lays out;* B'*s gain is* \$36, *and it appears that* A *gains four times as much* PER CENT. *as* B. *Required the capital of each?*

Ans. A'*s capital* \$5, B'*s* \$120.

Let $x=B$'s gain per cent.

Then $4x=A$'s gain per cent; also what B lays out.

Per question, $100 : x :: 4x : 36$

$$4x^2=36\cdot100 \quad \text{or} \quad x=30.$$

Whence, $4x$ or 120 is A's gain per cent.

Therefore, $220 : 100 :: 11 : A\text{'s capital} =\frac{10\cdot11}{22}=5.$

(b) *A vintner draws a certain quantity of wine out of a full cask which holds* 256 *gallons; and then filling the vessel with water, draws off the same quantity of liquid as before, and so on four times, when* 81 *gallons of pure wine was left. How much wine did he draw each time?* *Ans.* 64, 48, 36, *and* 27 *gallons.*

Let $a=256$. $x=$ the number of gallons of wine drawn the first time; then $a-x=$the wine left.

It is obvious that the wine drawn out the second time will be found by the following proportion:

$$a : a-x :: x : \frac{(a-x)x}{a}= \text{wine in 2d drawing.}$$

Then, $(a-x)-\frac{(a-x)x}{a}=\frac{(a-x)^2}{a}=$ wine left after the second drawing.

Again, $a : \frac{(a-x)^2}{a} :: x : \frac{(a-x)^2x}{a^2}=$ the third drawing.

Whence, $\frac{(a-x)^2}{a}-\frac{(a-x)^2x}{a^2}=\frac{(a-x)^3}{a^2}=$ the wine left after the third drawing. Whence we conclude that $\frac{(a-x)^n}{a^{n-1}}$ would be the wine left after n drawings.

After four drawings, $\frac{(a-x)^4}{a^3}=81$ by conditions.

$$(a-x)^4=81\cdot256\cdot a^2$$

Square root, $(a- \ )^2=9\cdot16a=9\cdot16\cdot256$

Square root again, $a-x=3\cdot4\cdot16$

That is, $16\cdot16-x=12\cdot16$

Or, $16\cdot16-12\cdot16=4\cdot16=64=x.$

(c) A *and* B *have two rectangular tracts of land, their lengths being as* 7 to 6, *and the difference between the areas is* 150 *acres; B's being the greater. Now had A's been as broad as B's, it would have been* 672 *rods long; but had B's been as broad as A's, it would have been* 900 *rods long. How many acres were there in each?*

Ans. A's 2100 *acres*, B's 2250 *acres.*

Let $7x=$ the length of A's lot in rods,

And $y=$ the breadth of the same,

Then, $7xy=$ the square rods in A's tract.

Again, let $6x=$ the length of B's tract in rods,

And, $v=$ the breadth of the same,

Then, $6vx=$ the square rods in B's tract.

By the given conditions,

$$6vx-7xy=150\cdot160 \qquad (1)$$

Now had A's been v in breadth, it would have been 672 rods long, therefore

$$672v=7xy \qquad (2)$$

Also, $900y=6vx$ (3) by the last given condition.

By multiplying (2) and (3), omitting common factors,

$$112\cdot900=7x^2$$

Or, $16\cdot900=x^2$

Whence, $x=4\cdot30=120$

Substituting $900y$ for $6vx$ in (1) and 120 for x, we shall have

$$900y-840y=150\cdot160$$

Or, $60y=150\cdot160$

$$y=400$$

Lastly, $\dfrac{7\cdot120\cdot400}{160}=2100$ A's acres.

(d) A *and* B *engaged to work for a certain number of days. At the end of the time*, A, *who had been absent* 4 *days, received* \$18.75, *while* B, *who had been absent* 7 *days, received only* \$12. *Now, had* B *been absent* 4 *and* A 7 *days, each would have been entitled to the same sum.*

How many days were they engaged, and at what rate?

Ans. They were engaged for 19 *days*, A *at* \$1.25, B *at* \$1 *per day.*

Let $t=$ the time or number of days.
$x=$ the daily compensation of A.
$y=$ the " " B.

Then by the given conditions

$$(t-4)x=18\tfrac{3}{4} \qquad (1)$$
$$(t-7)y=12 \qquad (2)$$
$$(t-7)x=(t-4)y \qquad (3)$$

From (3) $x=\frac{(t-4)}{t-7}y$. This value put in (1) gives

$$\frac{(t-4)^2y}{t-7}=18\tfrac{3}{4} \qquad (4)$$

Dividing (4) by (2) gives

$$\frac{(t-4)^2}{(t-7)^2}=\frac{18\frac{3}{4}}{12}=\frac{75}{12\cdot 4}=\frac{25}{4\cdot 4}$$

Square root $\frac{t-4}{t-7}=\frac{5}{4}$ or $t=19$

The value of t put in (1) and (2) gives.. $x=1.25$. $y=1$.

SECTION II.

QUADRATIC EQUATIONS.

The following are but hints to the solutions of Equations in Robinson's Algebra, University Edition, commencing at example 10, page 167.

(10.) Put $(2x-4)^2=y$.

Then, $\frac{8}{y}=1+\frac{16}{y^2}$

$8y=y^2+16$

$y^2-8y+16=0$ or .... $y-4=0$.

(11.) Multiply by 16. Rule 2.

Then, $64x^{\frac{1}{3}}+16x^{\frac{1}{6}}+1=39\cdot16+1=625$

$$8x^{\frac{1}{6}}+1=25 \qquad x^{\frac{1}{6}}=3 \qquad x=729.$$

(12.) Add 5 to each member.

Then $(x^2-2x+5)+(x^2-2x+5)^{\frac{1}{2}}=16$

By substitution, $y^2+6y+9=25$..........$y=2$ or -8.

Hence $x^2-2x+5=4$ $\qquad x=1$.

(13.) By (Art. 99) we have

$$\frac{x^2}{361}-\frac{12x}{19}+t^2=t^2-32$$

$$\frac{xt}{19}=-\frac{6x}{19} \qquad t=-6 \quad t^2=36$$

$$\frac{x^2}{361}-\frac{12x}{19}+36=4$$

$$\frac{x}{19}-6=\pm2\text{..............}x=152 \text{ or } 76.$$

(14) Observe that $81x^2$ and $\frac{1}{x^2}$ are both squares, and if these are taken for the first and last terms of a binomial square, the middle term must be $9x\frac{1}{x}.2=18$.

This indicates to add one to each member. Then extract the square root $9x+\frac{1}{x}=\pm10$. Hence, $x=1$ or -1

(15.) The first member of (15) is the same as (14.) Hence, add unity to each member and extract square root; we then have $9x+\frac{1}{x}=\frac{29}{x}+4$

$$9x^2-4x=28 \qquad \text{Put } x=\frac{u}{9}.$$

$$u^2-4u=28\cdot9=252$$

$$u-2=\pm16\text{..............................}x=2.$$

(Art. 105)

(4.) Multiply every term by x, and

$$x^4+8x^3+19x^2-12x=0.$$

Operate for square root thus ;

$$x^4-8x^3+19x^2-12x\ (x^2-4x$$
$$x^4$$
$$2x^2-4x)\ \ -8x^3+19x^2$$
$$-8x^3+16x^2$$
$$3x^2-12x$$
$$3(x^2-4x)$$
$$(x^2-4x)^2+3(x^2-4x)=0.$$

Divide by (x^2-4x), and

$x^2-4x+3=0$ .... $x=1$ or 3.

But the factor x^2-4x gives $x=0$ or 4.

(5.) $x^4-10x^3+35x^2-50x+24=0\ (x^2-5x$

$$x^4$$
$$2x^2-5x)\ \ -10x^3+35x^2$$
$$-10x^3+35x^2$$
$$10x^2-50x+24$$
$$(x^2-5x)^2+10(x^2-5x)+24=0$$

If we add unity to each member, we shall have complete squares. Extract the square root, and

$$(x^2-5x)+5=\pm 1$$
$$x^2-5x=-4 \text{ or } -6.$$

From these two equations we find $x=1$, 2, 3, or 4.

(6.) By mere inspection we perceive that this equation can take the form $(x^2-x)^2-(x^2-x)=132$.

$$y^2-y=132 \qquad y=12 \text{ or} -11.$$
$$x^2-x=12 \text{ or } -11.$$

If we take -11, the value of x will become imaginary. 12 gives $x=4$ or -3.

(7.) This equation may be put into this form :

$$(y^2-cy^2)-(2y^2-cy)=c^2$$

from which the reduction is easy.

(Art. 107.)

(3.) Taken from the work we have

$$(a+1)x^2-a^2x=a^2$$

Or, $$(a+1)x^2=(x+1)a^2.$$

Both members are of exactly the same form, and of course the equation could not be verified unless $x=a$.

EXAMPLES.

(1.) $x^2+11x=80$. Multiply by 4, &c.

$4x^2+A+11^2=329+121=441$

$2x+11=\pm 21$ .... $x=5$ or -16.

(2.) Drop $2x$ from each member, and divide by 3; then

$$x-\frac{x-1}{x-3}=\frac{x-2}{2}$$

Clearing of fractions and

$$2x^2-6x-2x+2=x^2-3x-2x+6$$

$x^2-3x=4.$ Put $2a=3$.

Hence, (Art. 106) .... $x=4$ or -1.

(3.) Multiply the equation by $6x$; then

$$\frac{6x^2}{x+1}+6x+6=13x$$

$$\frac{6x^2}{x+1}+6=7x$$

$$6x^2+6x+6=7x^2+7x$$

Hence $x^2+x=6$ .... $x=2$ or -3.

(4.) Clearing of fractions

$$70x-21x^2+72x=500-150x$$

$$21x^2-292x=-500.$$

Or, $21x^2-42x=250x-500$. or $21x(x-2)=250(x-2)$.

(5.) Put $\left(\frac{6}{y}+y\right)=x$. Then

$x^2+x=30$. Or, $x=5$ or -6.

Now, $$\left(\frac{6}{y}+y\right)=5 \text{ or } -6$$

$y^2-5y=-6$, or $y^2+6y=-6$

$4y^2-A+25=25-24$

$2y-5=\pm 1$ .... $y=3$ or 2.

(6.) Put $x^{\frac{2}{3}}=y$; Then $y^2+7y=44$

$4y^2+A+49=225$

$2y+7=\pm 15$ $y=4$ or -11.

$x=(4)^{\frac{2}{3}}$ or $(-11)^{\frac{2}{3}}$

(7) $x^2+x=42$. Hence $x=6$ or -7.

That is $y^2+11=36$ or 49...................... $y=5$ or $\sqrt{38}$.

(8.) $11-\frac{x+7}{x-7}=\frac{x}{3}$

$33x-231-3x-21=x^2-7x$

$x^2-37x=-252$

$4x^2-A+37^2=1369-1008=361$

$2x-37=\pm 19$ .... $x=28$ or 9.

(9.) $3x^2-9x=84$

12

$36x^2-A+81=12\cdot 84+81=1089$

$6x-9=\pm 33$.

(10.) Clearing of fractions we have

$2x+2\sqrt{x}=16-x$

$3x+2\sqrt{x}=16$

Multiply by 12, &c.

$6\sqrt{x}+2=\pm 14$ .... $x=4$ or $7\frac{1}{9}$.

(11.) $\frac{6(2x-11)}{x-3}+4x=26$

$\frac{3(2x-11)}{x-3}+2x=13$

$6x-33+2x^2-6x=13x-39$

$2x^2-13x=-6$

$16x^2-A+13^2=169-48=121$

$4x-13=\pm 11$

(12.) Multiply by x^2 and we have

$$10x-14+2x=\frac{22x^2}{9}$$

$$6x-7=\frac{11x^2}{9}$$

$$11x^2-54x=-63$$

Put $x=\frac{u}{11}$; then $u^2-54u=-693$

$$u^2-54u+27^2=36$$

$$u-27=\pm 6 \ldots\ldots\ldots\ldots\ldots u=33 \text{ or } 21.$$

(13.) Clearing of fractions we have

$$x^3-10x^2+1=x^3-6x^2+9x-3x^2+18x-27$$

$$-x^2=27x-28$$

$$x^2+27x=28. \qquad \text{Put } 2a=27.$$

$$x^2+2ax=2a+1$$

$$x^2+2ax+a^2=a^2+2a+1$$

$$x+a=\pm(a+1) \ldots\ldots\ldots\ldots\ldots x=1 \text{ or } -28.$$

(14.) Given $mx^2-2mx\sqrt{n}=nx^2-mn$, to find x.

By transposition, $mx^2-2mx\sqrt{n}+mn=nx^2$

Square root, $\sqrt{m}x-\sqrt{mn}=\pm\sqrt{n}x$

By transposition, $(\sqrt{m}\pm\sqrt{n})x=\sqrt{mn}$

$$x=\frac{\sqrt{mn}}{\sqrt{m}\pm\sqrt{n}}.$$

The following are not in Robinson's Algebra. They are too severe for learners in general, and are, therefore, not proper in an elementary work.

We commence again with No. 1.

(1.) Given $x^{\frac{7}{3}}+\frac{41x^{\frac{1}{3}}}{x}=\frac{97}{x^{\frac{2}{3}}}+x^{\frac{5}{6}}$, to find the values of x.

We observe that the lowest root of x is the 6th; therefore, put $x^{\frac{1}{6}}=y$; then $x^{\frac{1}{3}}=y^2$. $x=y^6$. $x^{\frac{2}{3}}=y^4$. $x^{\frac{5}{6}}=y^5$. $x^{\frac{7}{3}}=y^{14}$.

Then $y^{14}+\frac{41}{y^4}=\frac{97}{y^4}+y^5$

Whence, $y^{18}=56+y^9$.

Therefore, $y^9-\frac{1}{2}=\pm\frac{15}{2}$. Or, $y^9=8$ or -7.

$y^3=2$ or $(-7)^{\frac{1}{3}}$.

$x=y^6=4$, or $(-7)^{\frac{2}{3}}$. *Ans.*

(2.) Given $\sqrt[6]{\frac{1}{x^4}}+\sqrt[3]{\frac{1}{x}}=\frac{3-x^{\frac{2}{3}}}{x}$, to find the values of x.

Ans. $x=1$ or $-\frac{27}{8}$.

Multiply by x, and then we shall have

$$x\left(\sqrt[6]{\frac{1}{x^4}}\right)+x\left(\sqrt[3]{\frac{1}{x}}\right)=3-x^{\frac{2}{3}}.$$

Placing the value of x under the radical signs, then

$$\sqrt[6]{x^2}+\sqrt[3]{x^2}=3-x^{\frac{2}{3}}.$$

That is $x^{\frac{1}{3}}+x^{\frac{2}{3}}=3-x^{\frac{2}{3}}$

$$2x^{\frac{2}{3}}+x^{\frac{1}{3}}=3.$$

Whence, $x^{\frac{1}{3}}=1$ or $-\frac{3}{2}$.

Whence, $x^{\frac{1}{3}}=1$ or $-\frac{3}{2}$. Or, $x=1$ or $-\frac{27}{8}$. *Ans.*

(4.) Given $\left(x-\frac{1}{x}\right)^{\frac{1}{2}}+\left(1-\frac{1}{x}\right)^{\frac{1}{2}}=\frac{x}{1}$ to find x.

Put $P=\left(x-\frac{1}{x}\right)^{\frac{1}{2}}$ and $Q=\left(1-\frac{1}{x}\right)^{\frac{1}{2}}$; then

$$P+Q=x \qquad (1)$$

Multiply this last equation by $(P-Q)$, then

$$P^2-Q^2=x(P-Q)$$

But $P^2-Q^2=(x-1)$; therefore, $x(P-Q)=(x-1)$

Or, $P-Q=1-\frac{1}{x}$ (2)

Add (1) and (2), then $2P=\left(x-\frac{1}{x}\right)+1$

That is, $2P=P^2+1$

Or, $P^2-2P+1=0$, or $P-1=0$

Whence, $x-\frac{1}{x}=1$, or $x=\frac{1}{2}(1\pm\sqrt{5})$.

(5.) Given $\left(x^2-\frac{a^4}{x^2}\right)^{\frac{1}{2}}+\left(a^2-\frac{a^4}{x^2}\right)^{\frac{1}{2}}=\frac{x^2}{a}$; to find the values of x.

We observe that this equation is in the same form as the preceding, and would be identical if we changed x^2 to x, a^4 to 1. Therefore the value of x^2 in this equation will be of the same form as the value of x in the last example, except it will contain the factor a^2, because the square root has been once extracted:

That is $$x^2=\frac{a^2}{2}(1\pm\sqrt{5}),$$

$$x=\mp a\left(\frac{1\pm\sqrt{5}}{2}\right)^{\frac{1}{2}}.$$

But this conclusion is too summary to satisfy the young algebraist; therefore it is proper to take some of the intermediate steps.

Put $$\left(x^2-\frac{a^4}{x^2}\right)^{\frac{1}{2}}=P, \qquad \left(a^2-\frac{a^4}{x^2}\right)^{\frac{1}{2}}=Q. \qquad (1)$$

then the equation becomes

$$P+Q=\frac{x^2}{a}. \qquad (2)$$

Multiply (2) by $(P-Q)$, then we have

$$P^2-Q^2=\frac{x^2}{a}(P-Q).$$

But the value of (P^2-Q^2) drawn from (1), is (x^2-a^2);

therefore $$\frac{x^2}{a}(P-Q)=x^2-a^2$$

or $$P-Q=a-\frac{a^3}{x^2} \qquad (3)$$

By adding equations (2) and (3), we find

$$2P=a-\frac{a^3}{x^2}+\frac{x^2}{a} \qquad (4)$$

Multiply this equation by a, then

$$2aP=a^2-\frac{a^4}{x^2}+x^2$$

that is, $$2aP=a^2+P^2$$

or $$0=a^2-2aP+P^2.$$

Square root, $0=a-P$, or $P=a$

$$P^2=a^2.$$

From the first of equations (1), we find

$$x^2-\frac{a^4}{x^2}=a^2.$$

From this equation, we find $x=\mp a\left(\frac{1\pm\sqrt{5}}{2}\right)^{\frac{1}{2}}$.

(6.) Given $x^4(1+\frac{1}{3x})^2-(3x^2+x)=70$, to find the values of x.

Observe that $(3x^2+x)=3x^2(1+\frac{1}{3x})$. Put $(1+\frac{1}{3x})=y$; then the given equation becomes $x^4y^2-3x^2y=70$.

Completing the square, $x^4y^2-3x^2y+\frac{9}{4}=\frac{289}{4}$

$$x^2y-\frac{3}{2}=\pm\frac{17}{2}$$

$$x^2y=10, \text{ or } -7$$

That is $x^2+\frac{x}{3}=10$, or -7.

Whence $x=3$, or $-\frac{10}{3}$, or $\frac{1}{6}(1\pm\sqrt{-251})$.

(7.) Given $\frac{54-9\sqrt{x}}{x+2\sqrt{x}}=\frac{23(x-2\sqrt{x})}{6+\sqrt{x}}+\frac{7x^2-3x+4}{(x+2\sqrt{x})(6+\sqrt{x})}$ to find the values of x.

Multiply by $(6+\sqrt{x})$, then

$$\frac{9(36-x)}{x+2\sqrt{x}}=23(x-2\sqrt{x})+\frac{7x^2-3x+4}{x+2\sqrt{x}}$$

Multiply by $(x+2\sqrt{x})$, and we shall have

$$9(36-x)=23(x^2-4x)+7x^2-3x+4$$

Reducing, $15x^2-43x=160$. Whence $x=5$, or $-\frac{32}{15}$.

(8.) Given $x^2-\frac{5x}{2}+15=\frac{25x^2}{16}-\frac{64}{x^2}$, to find the values of x.

By transposition, $x^2+15+\frac{64}{x^2}=\frac{25x^2}{16}+\frac{5x}{2}$

Add 1 to each member, (see Robinson's Algebra, Art. 99,)

then $$x^2+16+\frac{64}{x^2}=\frac{25x^2}{16}+\frac{5x}{2}+1$$

By evolution, $$x+\frac{8}{x}=\pm\left(\frac{5x}{4}+1\right)$$

Taking the plus sign, $\frac{8}{x}=\frac{x}{4}+1.$ (1)

Taking the minus sign, $\frac{8}{x}=-\frac{9x}{4}-1.$ (2)

From (1) $x^2+4x=32.$ Whence $x=4$ or $-8.$

From (2) $9x^2+4x=-32$, and $x=\frac{-2\pm\sqrt{-71}}{9}$

(9.) Given $(x^2+5)^2-4x^2=160$, to find the values of x.

Subtract 20 from both sides, then

$$(x^2+5)^2-4(x^2+5)=140.$$

Whence, $x^2+5-2=\pm12.$

Therefore, $x=\pm3$ or $\pm\sqrt{-15}.$

(10.) Given $\frac{x^2}{(x^2-4)^2}+\frac{6}{(x^2-4)}=\frac{351}{25x^2}$, to find the values of x.

Multiply by x^2, and put $\frac{x^2}{(x^2-4)}=y$: then

$$y^2+6y=\frac{351}{25}$$

Whence $y+3=\pm\frac{24}{5}.$ $y=\frac{9}{5}$ or $-\frac{39}{5}.$

That is $\frac{x^2}{x^2-4}=\frac{9}{5}$, or $\frac{x^2}{x^2-4}=-\frac{39}{5}$

$$x=\pm3, \text{ or } x=\pm\sqrt{\tfrac{39}{11}}.$$

(11.) Given $(x-2)^2-6\sqrt{x}\,(x-2)=24-14x+15\sqrt{x}$, to find the values of x.

Expanding and reducing, gives

$$x^2-6x\sqrt{x}=20-10x+3\sqrt{x}.$$

Add $9x$ to both sides, and the first member will be a square; that is,

$$x^2-6x\sqrt{x}+9x=20-x+3\sqrt{x}.$$

Or, $$(3\sqrt{x}-x)^2=20-x+3\sqrt{x}$$

Now put $3\sqrt{x}-x=y$; then

$$y^2=20+y.$$

Whence, $$y=5 \text{ or } -4.$$

Then $$x-3\sqrt{x}=4, \text{ or } -5.$$

Whence, $x=16$, or 1, or $\dfrac{\pm 3\sqrt{-11}-1}{2}$.

(12.) Given $(4x+1)^2+4x^{\frac{1}{2}}(4x+1)=1912-(10x+3x^{\frac{1}{2}})$ to find the values of x.

Add $4x$ to both sides; then

$$(4x+1)^2+4x^{\frac{1}{2}}(4x+1)+4x=1912-6x-3x^{\frac{1}{2}}$$

That is $$[(4x+1)+2\sqrt{x}]^2+6x+3\sqrt{x}=1912$$

Or, $$[2(2x+\sqrt{x})+1]^2+3(2x+\sqrt{x})=1912.$$

Now put $y=2x+\sqrt{x}$; then

$$(2y+1)^2+3y=1912$$

Or, $$4y^2+7y=1911$$

$$64y^2+A+49=1911\cdot 16+49=30625$$

$$8y+7=\pm 175$$

$$y=21, \text{ or } -\tfrac{91}{4}.$$

That is, $$2x+\sqrt{x}=21, \text{ or } -\tfrac{91}{4}.$$

From these last, we find $x=9$, or $\frac{49}{4}$, or $\dfrac{-90\mp\sqrt{-181}}{8}$.

(13.) Given $8x^2-13=\dfrac{3x}{2}+\sqrt{6x^3+52x^2}$, to find the values of x.

Double the equation and remove the factor x^2 from under the radical sign; then

$$16x^2-26=3x+2x\sqrt{6x+52},$$

That is, $$16x^2=(3x+26)+2x\sqrt{2(3x+26)}.$$

Now put $y=\sqrt{(3x+26)}$; then the equation becomes

$$16x^2=y^2+(2\sqrt{2})xy$$

Add $2x^2$ to both members, and

$$18x^2=y^2+2\sqrt{2}\cdot xy+2x^2.$$

By evolution, $3x\sqrt{2}=y+x\sqrt{2}$ (*a*)

Or. $2x\sqrt{2}=y=\sqrt{3x+26}$

By squaring, $8x^2=3x+26.$

This equation gives $x=2$, or $-\frac{13}{8}$.

By taking the minus sign to the second member of (*a*), we would find $x=\dfrac{3\pm\sqrt{3337}}{64}$.

(14.) Given $4x^2+21x+8x^{\frac{1}{2}}\sqrt{7x^2-5x}=207-\dfrac{4x^2}{3}$, to find the values of x

Transposing $-\dfrac{4x^2}{3}$ and removing the factor x from under the radical sign, will give

$$\frac{16x^2}{3}+21x+8x\sqrt{7x-5}=207$$

Subtract 15 from each member, then

$$\frac{16x^2}{3}+(21x-15)+8x\sqrt{7x-5}=192$$

That is $\dfrac{16x^2}{3}+3(7x-5)+8x\sqrt{7x-5}=64\cdot 3$

Put $y=\sqrt{(7x-5)}$.

Then $\dfrac{16x^2}{3}+3y^2+8xy=64\cdot 3.$

Clearing of fractions, and changing terms,

$$16x^2+24xy+9y^2=64\cdot 9$$

By evolution, $4x+3y=8\cdot 3=\pm 24$

That is $4x+3\sqrt{(7x-5)}=\pm 24$

Taking the plus sign and transposing $4x$, we have

$$3\sqrt{(7x-5)}=(6-x)4$$

By squaring $9(7x-5)=(36-12x+x^2)16$

Reduced, $16x^2-255x=-621.$

This equation gives $x=3$, or $-\frac{207}{16}$. By taking -24, we obtain $x=\frac{123\pm3\sqrt{(-2567)}}{32}$.

(15.) Given $a^2b^2x^{\frac{1}{n}}-4(ab)^{\frac{3}{2}}x^{\frac{m+n}{2mn}}=(a-b)^2x^{\frac{1}{n}}$, to find the values of x.

Put $\quad P^2=x^{\frac{1}{n}} \quad (1)$

And $\quad Q^2=x^{\frac{1}{m}} \quad (2)$

Then $\quad P^2Q^2=x^{\frac{m+n}{mn}} \quad (3)$

And $\quad PQ=x^{\frac{m+n}{2mn}} \quad (4)$

Substitute these quantities in the given equation, and

$$a^2b^2P^2-4(ab)^{\frac{3}{2}}PQ=(a-b)^2Q^2 \quad (5)$$

Now let $\quad P=tQ$

Then $\quad a^2b^2t^2Q^2-4(ab)^{\frac{3}{2}}tQ^2=(a-b)^2Q^2$

Dividing by Q^2 gives

$$a^2b^2t^2-4(ab)^{\frac{3}{2}}t=(a-b)^2=a^2-2ab+b^2$$

Add $(4ab)$ to the first member to complete the square, (see Art. 99, Robinson's Algebra.)

Then $\quad a^2b^2t^2-4(ab)^{\frac{3}{2}}t+4ab=a^2+2ab+b^2$

By evolution, $\quad abt-2\sqrt{(ab)}=a+b$, or $-a-b$

Whence $\quad abt=(a+2\sqrt{ab}+b)=(\sqrt{a}+\sqrt{b})^2$

Or, $\quad t=\frac{(\sqrt{a}+\sqrt{b})^2}{ab}$ or, $\frac{-(\sqrt{a}-\sqrt{b})^2}{ab}$

But $\quad t=\frac{P}{Q}=\frac{x^{\frac{1}{2n}}}{x^{\frac{1}{2m}}}=x^{\frac{m-n}{2mn}}$

Therefore, $\quad x^{\frac{m-n}{2mn}}=\frac{(\sqrt{a}+\sqrt{b})^2}{ab}$ or, $\frac{-(\sqrt{a}-\sqrt{b})^2}{ab}$

Whence, $\quad x=\left\{\frac{(\sqrt{a}+\sqrt{b})^2}{ab}\right\}^{\frac{2mn}{m-n}}$ or, $\left\{\frac{-(\sqrt{a}-\sqrt{b})^2}{ab}\right\}^{\frac{2mn}{m-n}}$

SECTION III.

QUADRATIC EQUATIONS CONTAINING MORE THAN ONE UNKNOWN QUANTITY.

We commence by showing the outlines of the solution of the (3), (4), (5), (6), (7), and (8) equations in Robinson's Algebra, Art. (111), page 182.

(3.) Put $x^{\frac{1}{3}}=P$, and $y^{\frac{1}{3}}=Q$. Then the equations become

$$P+Q=8 \qquad (1)$$

$$P^2+Q^2+P^2Q^2=259. \qquad (2)$$

Square (1) and we have $P^2+2PQ+Q^2=64$ (3)

Subtract (3) from (2), and we have $P^2Q^2-2PQ=195$.

Hence, $PQ=15$ or -13.

Now we have $P+Q=8$, and $PQ=15$, whence $P=5$ or 3, and $Q=3$ or 5. That is, $x^{\frac{1}{3}}=5$ or 3, &c.

(4.) Put $x^{\frac{2}{3}}=P$, and $y^{\frac{1}{5}}=Q$; then the equations become

$$P^2+Q^2+P+Q=26, \text{ and } PQ=8$$

$$2PQ \qquad =16$$

$$(P+Q)^2+(P+Q)=42. \quad \text{Hence, } P+Q=6.$$

(5.) Put $\frac{x^{\frac{1}{2}}}{y^{\frac{1}{2}}}=u$; then $u^2+4u=\frac{33}{4}$ $u=\frac{3}{2}$ or $-\frac{11}{2}$.

The remaining operation is obvious.

(6.) Given $y^2-8x^{\frac{1}{2}}y=64$, and $y-2x^{\frac{1}{2}}y^{\frac{1}{2}}=4$, to find x and y.

To both members of the first equation add $16x$, and to the second add x, to complete the squares; then extract square root, and we have

$$y-4x^{\frac{1}{2}}=4(x+4)^{\frac{1}{2}} \text{ and } y^{\frac{1}{2}}-x^{\frac{1}{2}}=(x+4)^{\frac{1}{2}}$$

Four times the last equation subtracted from the preceding, gives $y-4y^{\frac{1}{2}}=0$. Or, ... $y=16$.

(7.) Multiply in the first equation as indicated, and subtract the second equation; we then have

$$x+y+2x^{\frac{1}{2}}y^{\frac{1}{2}}=25 \quad \text{or} \quad x^{\frac{1}{2}}+y^{\frac{1}{2}}=5$$

But from the second equation we have

$$(x^{\frac{1}{2}}+y^{\frac{1}{2}})x^{\frac{1}{2}}y^{\frac{1}{2}}=30. \quad \text{Hence,} \ldots\ldots\ldots\ldots\ldots x^{\frac{1}{2}}y^{\frac{1}{2}}=6$$

(8.) Divide the first equation by $y^{\frac{3}{2}}$, and $x^{\frac{2}{3}}=2y^{\frac{1}{2}}$, or $y^{\frac{1}{2}}=\frac{1}{2}x^{\frac{2}{3}}$
This put in the second equation gives

$$8x^{\frac{1}{3}}-\tfrac{1}{2}x^{\frac{2}{3}}=14.$$

$$x^{\frac{2}{3}}-16x^{\frac{1}{3}}+64=64-28=36.$$

We continue this section by adding other and more severe equations, commencing with number one.

(1.) Given $\left\{\begin{array}{l} y-\sqrt{y}=16-x \\ 28-y = x+4\sqrt{x} \end{array}\right\}$ to find the values of x and y.

By adding the two equations, omitting 16 on both sides, gives

$$12-\sqrt{y}=4\sqrt{x}$$

Squaring, $\quad 144-24\sqrt{y}+y=16x \qquad (1)$

Multiply the first equation by 16, and substitute the value of $-16x$ from (1), then we shall have

$$16y-16\sqrt{y}=256-144+24\sqrt{y}-y$$

Whence, $\quad 17y-40\sqrt{y}=112$

$$y-\tfrac{40}{17}\sqrt{y}=\tfrac{112}{17}$$

$$\sqrt{y}=\tfrac{20}{17}\pm\sqrt{(\tfrac{112}{289}\cdot\tfrac{17}{}+\tfrac{400}{289})}=\tfrac{20}{17}\pm\tfrac{48}{17}=4 \text{ or } -\tfrac{28}{17}.$$

Therefore, $\quad y=16 \text{ or } \tfrac{784}{289}.$

These values of y put in the first equation, give

$$x=\sqrt{y}=4, \text{ or } (\tfrac{58}{17})^2.$$

(2.) Given $\left\{\begin{array}{l} \dfrac{y}{(x+y)^{\frac{3}{2}}}+\dfrac{\sqrt{(x+y)}}{y}=\dfrac{17}{4\sqrt{(x+y)}} \\ x=y^2+2 \end{array}\right\}$ to find the values of x and y.

Multiply the first equation by $\sqrt{(x+y)}$, then

$$\frac{y}{x+y}+\frac{x+y}{y}=\frac{17}{4}$$

Clearing of fractions, and

$$4y^2+4x^2+8xy+4y^2=17xy+17y^2$$

Reducing, $4x^2=9xy+9y^2$.

Adding, $\frac{9x^2}{4}$ to both sides, (Robinson's Algebra, Art. 99) and

$$\frac{25x^2}{4}=\frac{9x^2}{4}+9xy+9y^2$$

By evolution, $\frac{5x}{2}=\pm\left(\frac{3x}{2}+3y\right)$

Whence $x=3y$ or $-\frac{3}{4}y$.

These values of x put in the second equation, readily give

$$x=6, \text{ or } 3, \text{ or } \frac{9\mp3\sqrt{(-119)}}{32}$$

$$y=3, \text{ or } 1, \text{ or } \frac{-3\pm\sqrt{(-119)}}{8}$$

(3.) Given and $\left\{\begin{array}{l} x+4\sqrt{x}+4y=21+8\sqrt{y}+4\sqrt{(xy)} \\ \sqrt{x}+\sqrt{y}=6 \end{array}\right\}$ to find the values of x and y.

From the first, $x-4\sqrt{(xy)}+4y=21+8\sqrt{y}-4\sqrt{x}$

That is $(2\sqrt{y}-\sqrt{x})^2=21+4(2\sqrt{y}-\sqrt{x})$

Let $P=2\sqrt{y}-\sqrt{x}$; then

$$P^2-4P=21$$

$$P=2\pm\sqrt{25}=7 \text{ or } -3.$$

That is $2\sqrt{y}-\sqrt{x}=7$ or -3. But $\sqrt{x}=6-\sqrt{y}$.

Therefore $3\sqrt{y}-6=7$ or -3.

$$\sqrt{y}=\tfrac{13}{3} \text{ or } 1.$$

$$y=\tfrac{169}{9} \text{ or } 1.$$

(4.) Given and $\left\{\begin{array}{l} 3x+\frac{2}{3}\sqrt{xy^2+9x^2y}=(x-\frac{1}{3})y \\ 6x+y : y : : x+5 : 3 \end{array}\right\}$ to find the values of x and y.

From the first $9x+2\sqrt{xy^2+9x^2y}=3xy-y$

That is $(y+9x)+2\sqrt{xy}\,(y+9x)^{\frac{1}{2}}=3xy$

Add xy to both members and extract square root, then

$$\sqrt{y+9x}+\sqrt{xy}=2\sqrt{xy} \qquad (1)$$

Whence $y+9x=xy$ (2)

From the second $-2y+18x=xy$ (3)

By subtraction, $3y-9x=0$

Or, $y=3x$

This value of y put in (2) gives $12x=3x^2$.

Or, $x=4$. Whence $y=12$.

By taking the *minus* sign to the second member of (1), other values of x and y can be found.

(5.) Given and $\left\{\begin{array}{l} x+y-\sqrt{\dfrac{x+y}{x-y}}=\dfrac{6}{x-y} \\ x^2+y^2=41 \end{array}\right\}$ to find the values of x and y.

The first cleared of fractions is

$$x^2-y^2-\sqrt{x^2-y^2}=6$$

Whence, $\sqrt{x^2-y^2}=3$, or -2

$$x=\pm 5, \text{ or } \pm 3\sqrt{\tfrac{5}{2}}$$

(6.) Given and $\left\{\begin{array}{l} \sqrt{(1+x)^2+y^2}+\sqrt{(1-x)^2+y^2}=4 \\ (4-x^2)^2=18-4y^2 \end{array}\right\}$ to find the values of x and y.

From the first

$$\sqrt{(1+x)^2+y^2}=4-\sqrt{(1-x)^2+y^2}$$

Squaring,

$$1+2x+x^2+y^2=16-8\sqrt{(1-x)^2+y^2}+1-2x+x^2+y^2$$

Reducing, $x=4-2\sqrt{(1-x)^2+y^2}$

Transposing 4 and squaring, gives

$$x^2-8x+16=4(1-2x+x^2+y^2)$$

Reducing, $12=3x^2+4y^2$ (1)

That is $4-x^2=\dfrac{4y^2}{3}$, $(4-x^2)^2=\dfrac{16y^4}{9}$

Comparing this last result with the second equation, we perceive that

$$\frac{16y^4}{9}+4y^2=18 \qquad (2)$$

Add $\frac{9}{4}$ to both members, (Art. 99, Algebra,) then

$$\frac{16y^4}{9}+4y^2+\frac{9}{4}=\frac{81}{4}$$

By evolution, $\frac{4y^2}{3}+\frac{3}{2}=\pm\frac{9}{2}$

Whence $4y^2=9$, or -18

$y=\pm\frac{3}{2}$, or $\pm\frac{3}{2}\sqrt{-2}$

The value of $4y^2$, that is 9, put in (1), gives $x=1$.

(7.) Given and $\left\{\begin{array}{l}\frac{x+\sqrt{x^2-y^2}}{x-\sqrt{x^2-y^2}}=\frac{17}{4}-\frac{x-\sqrt{x^2-y^2}}{x+\sqrt{x^2-y^2}} \\ x^2+xy=52-\sqrt{x^2+xy+4}\end{array}\right\}$ to find the values of x and y.

Add 4 to both members of the last equation, and transpose the radical, then

$$(x^2+xy+4)+(x^2+xy+4)^{\frac{1}{2}}=56$$

This is a quadratic, and

$$\sqrt{x^2+xy+4}+\tfrac{1}{2}=\pm\sqrt{\tfrac{225}{4}}=\pm\tfrac{15}{2}$$

Whence, $\sqrt{x^2+xy+4}=7$, or -8

$x^2+xy=45$, or 60. (1)

Now take the first equation, and multiply numerator and denominator of each of the literal fractions by its numerator, then

$$\frac{(x+\sqrt{x^2-y^2})^2}{y^2}=\frac{17}{4}-\frac{(x-\sqrt{x^2-y^2})^2}{y^2}$$

$$(x+\sqrt{x^2-y^2})^2+(x-\sqrt{x^2-y^2})^2=\frac{17y^2}{4}$$

Expanding and uniting, and we have

$$4x^2-2y^2=\frac{17y^2}{4}$$

$$16x^2=25y^2$$

$$4x=\pm 5y, \text{ or } y=\pm\frac{4x}{5}$$

This value of y put in (1), gives

$$x^2+\frac{4x^2}{5}=45, \text{ or } 60. \qquad (2)$$

Also,
$$x^2-\frac{4x^2}{5}=45, \text{ or } 60, \qquad (3)$$

From (2), $9x^2=9\cdot5\cdot5.$ Or, $x=\pm5.$

Or, $9x^2=25\cdot12=25\cdot4\cdot3$

$3x=5\cdot2\sqrt{3}.$ Or, $x=\pm10\sqrt{\tfrac{1}{3}}$

From (3), $x^2=9\cdot5\cdot5.$ Or, $x=\pm15$

Or, $x^2=300.$ Or, $x=\pm10\sqrt{3}.$

Here we have 8 different values of x, each of which being substituted in $y=\pm\frac{4x}{5}$, will give 8 different values to y.

(8.) Given
$$\left\{\begin{array}{l}\sqrt{\dfrac{x+y^2}{4x}}+\dfrac{y}{\sqrt{y^2+x}}=\dfrac{y^2}{4}\sqrt{\dfrac{4x}{y^2+x}} \\ \text{and} \quad \dfrac{\sqrt{x}+\sqrt{x-y-1}}{\sqrt{x}-\sqrt{x-y-1}}=y+1\end{array}\right\}$$
to find the values of x and y.

Clearing the first of fractions, gives

$$x+y^2+2y\sqrt{x}=xy^2 \qquad (1)$$

In the second equation, multiply the numerator and denominator of the fraction by the numerator; then

$$\frac{(\sqrt{x}+\sqrt{x-y-1})^2}{y+1}=y+1$$

Multiply by $(y+1)$, then extract square root, and we shall have

$$\sqrt{x}+\sqrt{x-y-1}=y+1 \qquad (2)$$

Or, $\sqrt{x-y-1}=(y+1)-\sqrt{x}$

By squaring, $x-y-1=y^2+2y+1-2\sqrt{x}(y+1)+x$

Reduced, $0=(y^2+y)+2y+2-2\sqrt{x}(y+1)$

Dividing by $(y+1)$, $0=y+2-2\sqrt{x}$ (3)

As we can divide by the binomial $(y+1)$ without a remainder, it follows, by the theory of equations that $(y+1)$ contains a root, that is $y+1=0.$ $y=-1.$

Corresponding with this value of y, equation (3) or (2) will give the value of x. $2\sqrt{x}=1,$ $x=\tfrac{1}{4}.$

To find other values, we must continue the solutions.

Return to equation (1) and extract the square root of both members, and we shall have

$$\sqrt{x}+y=\pm y\sqrt{x} \qquad (4)$$

From (3), $2\sqrt{x}=y+2$ (5)

Double (4), and $2\sqrt{x}+2y=\pm 2\sqrt{x}(y)$ (6)

That is, $y+2+2y=y^2+2y$

Or, $y^2-y=2$. Whence, ... $y=2$, or -1.

The value -1 we found before; which shows two roots equal to -1. The other value 2, put in (5), gives $x=4$. If we take the minus sign in (6), we shall have

$$y+2+2y=-y^2-2y$$

Or, $y^2+5y=-2$

Whence, $y=-\frac{5}{2}\pm\frac{1}{2}\sqrt{17}$

(9.) Given $\left\{\begin{array}{l}\frac{y}{x}\left(\frac{x}{y}\right)^{\frac{1}{2}}+\frac{1}{2}\left(\frac{x}{y}\right)^{\frac{1}{2}}\left(\frac{y}{x}\right)^{\frac{3}{4}}=5 \\ \frac{2x^2}{y}-\frac{x}{3\sqrt{y}}=\frac{1}{3}\end{array}\right\}$ to find the values of x and y.

The first equation can be put in this form

$$\left(\frac{y^2}{x^2}\cdot\frac{x}{y}\right)^{\frac{1}{2}}+\frac{1}{2}\left(\frac{x^2}{y^2}\right)^{\frac{1}{4}}\left(\frac{y^3}{x^3}\right)^{\frac{1}{4}}=5$$

That is $\left(\frac{y}{x}\right)^{\frac{1}{2}}+\frac{1}{2}\left(\frac{y}{x}\right)^{\frac{1}{4}}=5.$

The solution of this quadratic gives

$$\left(\frac{y}{x}\right)^{\frac{1}{4}}+\frac{1}{4}=\pm\frac{9}{4}, \quad \text{Or} \left(\frac{y}{x}\right)^{\frac{1}{4}}=2, \text{ or } -\frac{5}{2}$$

Whence, $\frac{y}{x}=16$, or $\frac{625}{16}$. $y=16x$. $\sqrt{y}=4\sqrt{x}$.

Substituting the values of y and $\sqrt{y}$, in the second equation, we find

$$\frac{x}{8}-\frac{x}{12\sqrt{x}}=\frac{1}{3}$$

The double is $$\frac{x}{4}-\frac{1}{6}\sqrt{x}=\frac{2}{3}$$

Add $\frac{1}{36}$ to both members to complete the square, (Art. 99, Algebra,) then

$$\tfrac{1}{4}x-\tfrac{1}{6}\sqrt{x}+\tfrac{1}{36}=\tfrac{25}{36}$$

By evolution, $$\tfrac{1}{2}\sqrt{x}-\tfrac{1}{6}=\pm\tfrac{5}{6}$$

Whence, $x=4$, or $\frac{16}{9}$; but $y=16x=64$, or $\frac{256}{9}$.

If in the second equation we write $\frac{625}{16}x$ for the value of y, and $\frac{25}{4}\sqrt{x}$ for the value of $\sqrt{y}$, we shall find

$$x=\tfrac{625}{64},\text{ or }\tfrac{625}{144}.$$

(10.) Given $\left\{\begin{array}{l} x^4+y^4=1+2xy+3x^2y^2 \\ x^3+y^3=2y^2x+2y^2+x+1 \end{array}\right\}$, to find the values of x and y.

Transposing $2x^2y^2$ in the the first equation, and we have

$$x^4-2x^2y^2+y^4=1+2xy+x^2y^2$$

By evolution, $$x^2-y^2=\pm(1+xy) \qquad (1)$$

The second equation can be put in this form,

$$(x^2-xy+y^2)(x+y)=2y^2(x+1)+(x+1)=$$
$$(2y^2+1)(x+1) \qquad (2)$$

Taking the plus sign in (1), we can put it into this form

$$x^2-xy+y^2=2y^2+1 \qquad (3)$$

By the help of (3) we perceive the equal factors in (2). Suppress them, and (2) becomes

$$x+y=x+1. \text{ Or } y=1.$$

This value of y put in (1), gives $x=2$, or -1.

(11.) Given $\left\{\begin{array}{l} \dfrac{x^2y^2}{2}-40y^2=136-y^2\sqrt{x^2-\dfrac{272}{y^2}} \\ \text{and } x^2-\dfrac{2}{y}\left(\dfrac{3}{y}+15x\right)=\dfrac{30}{y^2}+\dfrac{5x}{y} \end{array}\right\}$ to find the values of x and y.

It is obvious that the first equation can be put into this form

$$x^2y^2-80y^2=272-2y\sqrt{x^2y^2-272}$$

By transposition

$$(x^2y^2-272)+2y\sqrt{x^2y^2-272}=80y^2$$

By adding y^2 to both members, and extracting square root we have

$$(x^2y^2-272)^{\frac{1}{2}}+y=\pm 9y$$

Whence, $x^2y^2-272=64y^2$, or $100y^2$ (1)

Clearing the second of the given equations of fractions, and reducing, we have

$$x^2y^2-35xy=36 \qquad (2)$$

Put $2a=35$, (see Art. 106, Robinson's Algebra.)

Then $x^2y^2-2axy=2a+1$

Adding a^2, and taking square root, gives

$$xy-a=\pm(a+1)$$

Whence, $xy=(2a+1)=36$, or -1 (3)

These values of xy put in (1), give

$$64y^2=36\cdot 36-272, \text{ or } 64y^2=-271$$

$$64y^2=1024, \text{ or } 8y=\pm 32. \quad y=4, \text{ or } -4.$$

These values of y put in (3), give $x=9$, or -9.

Again, by observing (1), we perceive that we may put

$$100y^2=1024, \text{ or } 10y=\pm 32. \quad y=3.2, \text{ or } -3.2.$$

(12.) Given $\left\{\begin{array}{l}\dfrac{2y^2-8\sqrt{x}}{\sqrt{x}}+2\sqrt{y^2-4\sqrt{x}}=\dfrac{3\sqrt{x}}{2} \\ \text{and } \sqrt{x}+\sqrt{8(y-\sqrt{x})-4}=y+1\end{array}\right\}$ to find the values of x and y.

Put $\sqrt{y^2-4\sqrt{x}}=P$, in the first equation.

Then $\dfrac{2P^2}{\sqrt{x}}+2P=\dfrac{3\sqrt{x}}{2}$

$$4P^2+4\sqrt{x}\cdot P=3x$$

By adding x to both members to complete the square, we have

$$4P^2+4\sqrt{x}\cdot P+x=4x$$

$$2P+\sqrt{x}=\pm 2\sqrt{x}$$

Or, $2P=\sqrt{x}$, or $-3\sqrt{x}$

Restoring the value of P, we find

$$2\sqrt{y^2-4\sqrt{x}}=\sqrt{x}, \text{ or } -3\sqrt{x}$$

Whence, $4y^2-16\sqrt{x}=x, \text{ or } 9x$ (1)

From the second of the given equations, we have

$$\sqrt{8(y-\sqrt{x})-4}=(y-\sqrt{x})+1$$

Squaring, $8(y-\sqrt{x})-4=(y-\sqrt{x})^2+2(y-\sqrt{x})+1$

Whence, $(y-\sqrt{x})^2-6(y-\sqrt{x})=-5$

And $y-\sqrt{x}-3=\pm2$ (2)

Taking the plus sign $y-5=\sqrt{x}$ (3)

Taking the minus sign $y-1=\sqrt{x}$ (4)

Substituting the values of $\sqrt{x}$ and x taken from (3) in (1), we have

$$4y^2-16y+80=y^2-10y+25\text{; or, } 9y^2-90y+225$$

Whence, $y=\dfrac{\sqrt{3}\pm2\sqrt{-13}}{\sqrt{3}}$, or $y=\dfrac{37\pm\sqrt{644}}{5}$

Taking the values of the same from (4), and substituting, as before, we have

$$4y^2-16y+16=y^2-2y+1, \text{ and } 9y^2-18y+9$$

Whence, $y=3$, or $\dfrac{5}{3}$, and $y=\dfrac{1\pm\sqrt{34}}{5}$

Substituting the values of y in (3) and (4), we have the values of x.

(13.) Given $\left\{\begin{array}{l} 5y+\dfrac{\sqrt{x^2-15y-14}}{5}=\dfrac{x^2}{3}-36 \\ \text{and } \dfrac{x^2}{8y}+\dfrac{2x}{3}=\sqrt{\dfrac{x^3}{3y}+\dfrac{x^2}{4}}-\dfrac{y}{2} \end{array}\right\}$ to find the values of x and y.

Multiply the first equation by 3, transpose, &c., and we have

$$\frac{3\sqrt{x^2-15y-14}}{5}=(x^2-15y-14)-94$$

Put $P=\sqrt{x^2-15y-14}$; then we shall have

$$P^2-\tfrac{3}{5}P=94$$

Whence, $P=\frac{3}{10}\pm\frac{97}{10}=10, \text{ or } -9\frac{4}{10}$

That is, $x^2-15y-14=100$, or $\frac{8836}{100}$.

Or, $x^2=15y+114$; and $x^2=15y+\frac{10236}{100}$. (1)

The second equation may be written thus,

$$\frac{x^2}{8y}+\left(\frac{2x}{3}+\frac{y}{2}\right)=x\sqrt{\frac{x}{3y}+\frac{1}{4}}$$

Uniting the fractions, and

$$\frac{x^2}{8y}+\left(\frac{4x+3y}{6}\right)=x\sqrt{\frac{4x+3y}{12y}}$$

Dividing every term by $2y$, and we have

$$\frac{x^2}{16y^2}+\left(\frac{4x+3y}{12y}\right)=\frac{x}{2y}\left(\frac{4x+3y}{12y}\right)^{\frac{1}{2}}$$

For the sake of perspicuity, put $P=\left(\frac{4x+3y}{12y}\right)^{\frac{1}{2}}$, then

$$\frac{x^2}{16y^2}-\frac{x}{2y}P+P^2=0$$

By evolution, $\frac{x}{4y}-P=0$

Whence, $\frac{x^2}{16y^2}=P^2=\frac{4x+3y}{12y}$

Clearing of fractions, $3x^2=16xy+12y^2$

Whence, $9x^2-48xy=36y^2$

Add $64y^2$ to both members, to complete the squares, then

$$9x^2-48xy+64y^2=100y^2$$

By evolution, $3x-8y=\pm10y$

Whence, $x=6y$, and $x=-\frac{2}{3}y$ (2)

Substituting the first of these values of x in equation (1), we have

$$36y^2-15y=114$$

By adding $\frac{25}{16}$ to both members, (Art. 99, Algebra,) we shall have

$$36y^2-15y+\tfrac{25}{16}=114+\tfrac{25}{16}=\tfrac{1849}{16}$$

By evolution, $6y-\frac{5}{4}=\pm\frac{43}{4}$

Whence, $y=2$, or $-\frac{19}{12}$.

These values put in the first of equations (2), give

$$x=12, \text{ or } -\tfrac{19}{2}.$$

By taking the second set of equations in (1) and (2), we shall find other values of x and y.

(14.) Given $\left\{\begin{array}{l} x^2y-4=4x^{\frac{1}{2}}y-\frac{1}{4}y^3 \\ x^{\frac{3}{2}}-3=x^{\frac{1}{2}}y^{\frac{1}{2}}(x^{\frac{1}{2}}-y^{\frac{1}{2}}) \end{array}\right\}$ to find the values of x and y.

Ans. $x=1$. $y=4$.

Put $x^{\frac{1}{2}}=P$, and $y^{\frac{1}{2}}=Q$, and we have

$$P^4Q^2-4=4PQ^2-\tfrac{1}{4}Q^6 \qquad (1)$$

$$P^3-3 \;\;=PQ(P-Q) \qquad (2)$$

Now put $P=nQ$, and equation (1) becomes

$$(4n^4+1)Q^6-16nQ^3=16.$$

Conceive n to be a known quantity, then the last equation is quadratic, and a solution gives

$$Q^3=\frac{4(2n^2+2n+1)}{4n^4+1}=\frac{4}{2n^2-2n+1}=\frac{4}{2n^2-2n+2-1}.$$

But from (2), $Q^3=\dfrac{3}{n^3-n^2+n}=\dfrac{3}{n(n^2-n+1)}$

Put the two values of Q^3 equal, and put $n^2-n+1=R$, (3)

Then $\dfrac{4}{2R-1}=\dfrac{3}{nR}$. Whence, $2n=\dfrac{6R-3}{2R}$. (4)

But from (3) resolved as a quadratic,

$$2n=1\pm\sqrt{(4R-3)} \qquad (5)$$

From (4) and (5), $2R\pm2R\sqrt{(4R-3)}=6R-3$

Or, $\pm2R\sqrt{(4R-3)}=4R-3$

Put $\sqrt{(4R-3)}=S.$

Then $S^2\pm2RS=0$. Or, $S(S\pm2R)=0$.

This last equation may be verified by taking either factor equal to zero; and as the first factor only gives a rational quantity, we take that which gives $R=\frac{3}{4}$.

By retracing, we easily find x and y.

We now add a few unwrought examples for the benefit of those who may wish to test their own unaided powers in these difficult operations.

None of these that follow are as severe as many of the preceding.

(15.) Given and $\left\{\begin{array}{l}\sqrt{\dfrac{3x-2y}{2x}}+\sqrt{\dfrac{2x}{3x-2y}}=2 \\ x^2-18=x(4y-9)\end{array}\right\}$ to find the values of x and y.

Ans. $x=6$, or 3.
$y=3$, or $\frac{3}{2}$.

(16.) Given and $\left\{\begin{array}{l}\dfrac{(x+y)+\sqrt{(x^2-y^2)}}{(x+y)-\sqrt{(x^2-y^2)}}=\dfrac{9(x+y)}{8y} \\ (x^2+y)^2+(x-y)=2x(x^2+y)+506\end{array}\right\}$ to find the values of x and y.

Ans. $x=5$, or $-\frac{23}{5}$.
$y=3$, or $-\frac{69}{25}$.

(17.) Given and $\left\{\begin{array}{l}\dfrac{x+y}{x-y}-\dfrac{x-y}{x+y}=\dfrac{24}{5} \\ \sqrt{x-y}+x=\dfrac{4x^2}{9\sqrt{(x-y)}}\end{array}\right\}$ to find the values of x and y.

Ans. $x=3$, or $\frac{45}{2}$; or $\frac{3}{16}$, or $\frac{45}{32}$.
$y=2$, or $-\frac{135}{4}$; or $\frac{1}{8}$, or $-\frac{135}{64}$.

(18.) Given and $\left\{\begin{array}{l}\sqrt{6\sqrt{x}+6\sqrt{y}}+\frac{1}{2}\sqrt{x}=9-\frac{1}{2}\sqrt{y} \\ x-y=12\end{array}\right\}$ to find the values of x and y.

Ans. $x=16$, or $\frac{59536}{81}$.
$y=4$, or $\frac{58564}{81}$.

(19.) Given and $\left\{\begin{array}{l}x+\sqrt{3y^2-11+2x}=7+2y-y^2 \\ \sqrt{3y-x+7}=\dfrac{x+y}{x-y}\end{array}\right\}$ to find the values of x and y.

Ans. $x=4$. $y=2$.

(20.) Given and $\left\{ \begin{array}{l} x^2 - y^2 = 3 \\ (x^4 + y^4)^2 + x^2y^2(x^2 - y^2)^2 + x^2 - y^2 = 328 \end{array} \right\}$ to find the values of x and y.

Ans. $x = \pm 2$, or $\pm\sqrt{(-1)}$.
$y = \pm 1$, or $\pm 2\sqrt{(-1)}$.

(21.) Given and $\left\{ \begin{array}{l} \dfrac{x+\sqrt{x+y}}{x-\sqrt{x+y}} - \dfrac{\sqrt{x}-x-y}{\sqrt{x}+x+y} = \dfrac{89}{40} \\ y^2 - \sqrt{xy^2} = \dfrac{4x}{9} \end{array} \right\}$ to find the values of x and y.

Ans. $x = 9$, or $\frac{196}{9}$, or $\frac{289}{9}$, or 16.
$y = 4$, or $-\frac{14}{9}$, or $-\frac{68}{9}$, or $\frac{4}{3}$.

SECTION IV.

PROBLEMS PRODUCING QUADRATIC EQUATIONS CONTAINING MORE THAN ONE UNKNOWN QUANTITY.

The following outlines of operations, refer to problems in Robinson's Algebra, Chapter III, page 183. We pass on to the sixth problem, page 186, and only include those which serve to illustrate brevity and elegance in operation.

The figures in parenthesis refer to the number of the problem in the book.

(6.) Let $t =$ the time (hours) he traveled, and $r =$ his rate per hour; then $rt = 36$ (1)

But if r becomes $(r+1)$, t must become $(t-3)$, and then

$$(r+1)(t-3) = 36 \qquad (2)$$

Or,
$$rt + t - 3r - 3 = 36$$
$$rt \qquad\qquad = 36$$
$$\overline{t = 3(r+1)}$$

Hence, $r^2 + r = 12$, and $r = 3$.

(7.) Let $x =$ the number of children,
and $y =$ the original share of each.

Then $xy = 46800$ (1)

$$(x-2)(y+1950)=46800 \qquad (2)$$

$$xy+1950x-2y-2\cdot 1950=46800$$

$$1950(x-2)=2y$$

Or, $975(x-2)x=xy=46809$

By division, $x^2-2x=48$ .... $x=8$.

(8.) Let $x=$ the number of pieces,

Then $\frac{675}{x}=$ the cost of each piece.

$$48x-\frac{675}{x}=675$$

$$48x^2-675x=675$$

$$16x^2-225x=225.$$

(9.) Let $x=$ the purchase money.

Then $\frac{104x}{100}=$ the cost, and $390-\frac{104x}{100}=$ his whole gain.

Then $\frac{104x}{100} : 390-\frac{104x}{100} :: 100 : \frac{x}{12}$

Product of extremes and means,

$$\frac{26x^2}{300}=39000-104x$$

$$\frac{2x^2}{300}=3000-8x$$

Put $a=300$ and divide by 2; then

$$\frac{x^2}{a}=5a-4x$$

$$x^2+4ax=5a^2$$

$$x^2+4ax+4a^2=9a^2$$

$x+2a=3a$ .... $x=a=300$.

(10.) Put $x+y=$ the greater part,
and $x-y=$ the less part.

Then $2x=60$, $x=30$, and $x^2-y^2=704$.

(11.) Let $x=$ the cost; then $39-x=$ the whole gain.

$x : 39-x :: 100 : x$. *Ans.* $x=10$.

(12.) Let $(x-20)=$ the number of persons relieved by A.
Then $x+20=$ the number of persons relieved by B.

$$\frac{1200}{x+20}+5=\frac{1200}{x-20}$$

Divide by 5, and put $a=240$; then

$$\frac{a}{x+20}+1=\frac{a}{x-20}$$

$$ax-20a+x^2-400=ax+20a$$

$$x^2=40a+400=40(a+10)=40\cdot 250$$

Or, $x^2=400\cdot 25$ .... $x=20\cdot 5=100$.

Hence 80 is B's number, and 120 A's.

(13.) Let $x=$ the price of a dozen sherry
and $y=$ the price of a dozen claret.

$$7x+12y=50 \qquad (1)$$

$\frac{10}{x}=$ the number of dozen of sherry for 10£.

$\frac{6}{y}=$ the number of dozen of claret for 6£.

Then $$\frac{10}{x}=3+\frac{6}{y} \qquad (2)$$

Or, $$x=\frac{10}{3+\frac{6}{y}}=\frac{10y}{3y+6}$$

By substitution, $$\frac{70y}{3y+6}+12y=50$$

$$70y+36y^2+72y=150y+50\cdot 6$$

$$36y^2-8y=300$$

$9y^2-2y=75$. Hence, ... $y=3$.

(14.) Let $19x=$ the whole journey.
Then $x=$ B's days, also his rate per day.
Or $x^2=$ B's distance.
Also, $7x+32=$ A's distance.

$$x^2+7x+32=19x$$

$$x^2-12x=-32.$$

Hence, ... $x=8$ or 4.

And .... $19x=152$ or 76.

If we put x for the whole journey, we shall obtain the 13th equation, (Art. 104.)

(15.) Let $x=$ the bushels of wheat,
and $x+16=$ the bushels of barley.

$$\frac{24}{x}=\frac{24}{x+16}+\frac{1}{4}$$

$$24x+16\cdot24=24x+\frac{x^2+16x}{4}$$

$$x^2+16x=16\cdot96=16\cdot16\cdot6$$

Put $2a=16$. Then $2a\cdot2a\cdot6=24a^2$

$$x^2+2ax=24a^2$$

$$x+a=\pm5a.\ldots\ldots\ldots\ldots x=4a=32.$$

(16.) A put in 4 horses, and B put in x horses.

Then $\frac{18}{x}=$ the rate per head.

$$\frac{4\cdot18}{x}+18=\text{ the price of the pasture.}$$

$$\frac{4\cdot20}{x+2}+20=\text{ the price of the pasture.}$$

Hence, $\frac{4\cdot18}{x}=\frac{4\cdot20}{x+2}+2$

$$\frac{36}{x}=\frac{40}{x+2}+1.\ldots\ldots\ldots\ldots x=6.$$

(17.) Let $4x=$ the price per yard,
and $9x=$ the number of yards.

$$36x^2=324.\ldots\ldots\ldots\ldots x=3.$$

(18.) Let $10x+y=$ the number.

Then $\frac{10x+y}{xy}=2.$ (1)

And $10x+y+27=10y+x$ (2)

From (1), $10x=(2x-1)y$

From (2), $x+3=y$

By division, $\frac{10x}{x+3}=2x-1$

$$10x=2x^2+6x-x-3$$
$$2x^2-5x=3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots x=3.$$

(19.) Let $(x-y)$, x, and $(x+y-6)$, represent the numbers. Then $3x-6=33$, or $x=13$.

$$(x-y)^2=x^2-2xy+y^2$$
$$x^2=x^2$$
$$(x+y-6)^2=x^2+2xy+y^2-12x-12y+36$$
$$3x^2+2y^2-12x-12y+36=441$$

By subtracting the value of $3x^2-12x+36$, we have

$$2y^2-12y=54. \quad \text{Hence,} \ldots\ldots\ldots\ldots\ldots y=9.$$

(25.) Let $x+y=$ the greater, and $x-y=$ the less.

Then $(x^2-y^2)(2x^2+2y^2)=1248$ (1)

Or, $x^4-y^4=624$

Also, $4xy=20$ (2)

Whence, $y=\frac{5}{x}$. $y^4=\frac{625}{x^4}$

$$x^4-\frac{625}{x^4}=624$$

$x^8-624x^4=625$. Put $2a=624$.

Then $x^8-2ax^4+a^2=a^2+2a+1$

$x^4-a=\pm(a+1)$

Whence, $x^4=2a+1=625$. $x^2=\pm 25$.

$x=5$, or -5.

From (2), $y=1$.

(27.) Let $x=$ A's stock. $a=1000$. Then $a-x=$ B's stock. Observe that $780=$ the whole gain.

Then $9x+(6a-6x)=6a+3x : 9x :: 780 : 1140-x$.

Or, $2a+x : 3x :: 780 : 1140-x$.

This proportion will produce a laborious equation to work through. Therefore we will try $2x$ to represent A's stock; then $9\cdot 2x=18x$. $(a-2x)6=6a-12x$.

$$18x+(6a-12x)=6a+6x : 18x :: 780 : 1140-2x.$$

Reducing, gives us

$$a+x : 3x :: 390 : 570-x.$$
$$570a-ax+570x-x^2=1170x.$$

Whence, $x^2+1600x=570000.$

$$x^2+1600x+(800)^2=1210000.$$

$$x+800=1100.$$

$$x=300. \quad 2x=600, \ A\text{'s stock}.$$

When the Algebra was first published, the 6 months in the problem was printed 8 months, by mistake. How could we discover that mistake?

We look at the answer and see that the numbers 600 and 400, make the stated sum 1000; therefore we will assume that these three numbers are correct. We will now take m to represent 9, and n to represent B's time. Then the preceding proportion becomes

$$2mx-2nx+na : 2mx :: 780 : 1140-2x.$$

Also, $2mx-2nx+na : na-2nx :: 780 : 640+2x-a.$

Now give to x its value 300, and to a its value 1000, and these proportions will give $m=9$, and $n=6$.

(28.) Let $x^2=$ half the number in the first

Then $2x^2=$ the number in the first.

And $4x+4=$ the number in the second.

$3(2x^2+4x+4)=$ the number in the third.

$3(x^2+2x+2)+10=$ the number in the fourth.

Sum, $11(x^2+2x+2)+10=1121$, the given sum.

Whence, $x^2+2x+2=101.$

Or, $x^2+2x+1=100.$

By evolution, $x+1=\pm10$, or $x=9$, for the minus sign will not apply. Then $2x^2=162$, the number in the first.

(31.) Let $x=$ the greater of the two numbers,

and $y=$ the less.

Then per conditions, $xy=x^2-y^2$ (1)

And $x^2+y^2=x^3-y^3$ (2)

From (1), $x^2-xy=y^2.$

Conceive y a known quantity and complete the square thus;

$$4x^2-4y\cdot x+y^2=5y^2$$

$$2x-y=\pm\sqrt{5}\cdot y$$

Or, $2x=(1\pm\sqrt{5})y$. Let $(1\pm\sqrt{5})=a$.

Then $x=\frac{ay}{2}$ (3)

Let this value of x be substituted in (2), and we have

$$\frac{a^2y^2}{4}+y^2=\frac{a^3y^3}{8}-y^3$$

Dividing by y^2 and clearing of fractions, and

$$2a^2+8=(a^3-8)y$$

Whence, $y=\frac{2a^2+8}{a^3-8}$

But $a^2=6\pm2\sqrt{5}$. $a^3=16\pm8\sqrt{5}$. Therefore,

$$y=\frac{20\pm4\sqrt{5}}{8\pm8\sqrt{5}}=\frac{1}{2}\left(\frac{5\pm\sqrt{5}}{1\pm\sqrt{5}}\right)=\pm\tfrac{1}{2}\sqrt{5}.$$

This last operation may not be obvious to some ; it will be seen by multiplying $(1\pm\sqrt{5})$, by $\sqrt{5}$; that is, the numerator in parenthesis is $\sqrt{5}$ times the denominator.

To find x we must simply multiply y by $\frac{1}{2}a$, see (3) ; that is, $x=\frac{1}{4}(1\pm\sqrt{5})\sqrt{5}=\frac{1}{4}(\sqrt{5}\pm5)$.

The following are not in Robinson's Algebra, but selected from every source,—mostly from Bland's Problems.

(1.) *The sum of two numbers is 2, and the sum of their fifth powers is 32. What are the numbers?*

Let $x=$ one number, and $y=$ the other.

Then $x+y=2$ (1)

And $x^5+y^5=32$ (2)

As the 5th power of 2 is 32, therefore

$$(x+y)^5=x^5+y^5$$

That is, $x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5=x^5+y^5$

Or, $5x^4y+10x^3y^2+10x^2y^3+5xy^4=0$

By division, $x^3+2x^2y+2xy^2+y^3=0$

That is, $x^3+y^3+2xy(x+y)=0$

Dividing by $(x+y)$, and we have

$$x^2-xy+y^2+2xy=0$$

Or, $x^2+xy+y^2=0$

From (1), $x^2+2xy+y^2=4$

By subtraction, $xy \quad =4$ (3)

Multiplying (1) by x, gives $x^2+xy=2x$

That is, $x^2-2x=-4$

Whence, $x=1\pm\sqrt{-3}$

Then $y=1\mp\sqrt{-3}$

Here we have obtained two *expressions*, the sum of whose 5th powers is 32, but not two numbers.

We have not so clear an idea of $(1\pm\sqrt{-3})$, as we have of x itself. If we compare equations (1) and (3), we shall perceive an *impossibility;* for two numbers whose sum is only 2, can never make a product of 4. In the same manner when a sum is but 2, the sum of the 5th powers of any two of its parts, can never make 32.

To test our quantities, we will verify (2) with them. To save trouble, we will put $a=\sqrt{-3}$, then $a^2=-3$, $a^4=9$.

$$x^5=(1+a)^5=1+5a+10a^2+10a^3+5a^4+a^5$$
$$y^5=(1-a)^5=1-5a+10a^2-10a^3+5a^4-a^5$$
$$x^5+y^5=2+20a^2+10a^4=2-60+90=32.$$

(2.) *The fore wheels of a carriage make* 6 *revolutions more than the hind wheels in going* 120 *yards; but if the periphery of each wheel be increased by one yard, then the fore-wheels will make only* 4 *revolutions more than the hind wheels, in running over the same distance. Required the circumference of each wheel?*

Ans. Fore wheels, 4, hind wheels 5 yards.

Let $x=$ the yards in the circumference of the larger wheels, and $y=$ the yards in the circumference of the smaller.

Put $a=120$.

Then per question, $\frac{a}{x}=\frac{a}{y}-6.$ (1)

And $\frac{a}{x+1}=\frac{a}{y+1}-4$ (2)

Clearing of fractions, $ay=ax-6xy.$ (3)

$ay+a=ax+a-4xy-4x-4y-4.$ (4)

Suppressing a in both members of (4), and then subtracting it from (3), we have

$$0=-2xy+4x+4y+4. \qquad (5)$$

From (3), $x=\frac{ay}{a-6y}=\frac{120y}{120-6y}=\frac{20y}{20-y}$

From (5), $x=\frac{2y+2}{y-2}$

Therefore, $\frac{y+1}{y-2}=\frac{10y}{20-y}$

$$20y-y^2+20-y=10y^2-20y$$

Whence, $11y^2-39y=20.$

If we work out this quadratic, we shall find $y=4$; but the operation would be a little troublesome, because the numbers are prime to each other.

In cases like these, when a practical operator is only in pursuit of results, he looks at the absolute term, (in this example, 20), and observes its factors, 2, 10, 4, 5, and conceives y to represent one of them; and if it verifies the equation, then y is really that factor.

I will now conceive y to be 4, and divide the first member by y, the second by 4; then

$$11y-39=5$$

Or, $11y=44$, or $y=4.$

Therefore, as this supposition verifies the equation, the supposition itself is truth.

Now let us suppose y to be 5; then operate as before, and

$$11y-39=4$$
$$11y=43$$

Now as y does not come out equal to 5, the supposition was not true.

For x, we have $x=\frac{20y}{20-y}=\frac{20\cdot 4}{16}=5.$

(3.) A *and* B *engaged to reap a field for* \$24; *and as* A *could reap it alone in* 9 *days, they promised to complete it in* 5 *days. Finding, however, that they were unable to finish it, they called in* C *to assist them the last two days, in consequence of which,* B *received*

$1 *less than he otherwise would have done. In what time could* B *or* C *alone have reaped the field?*

Let $x=$ the number of days in which B could reap the field, and $y=$ the number of days in which C could reap it.

As A could do it in 9 days, for one day's work he should have $\frac{1}{9}$ of the money; and as B could do it in x days, for one day's work he should have $\frac{1}{x}$ of the money. A and B then working together one day would do $\frac{1}{9}+\frac{1}{x}$ of the work, and in 5 days they would do $5\left(\frac{1}{9}+\frac{1}{x}\right):\frac{5}{x}::24:\frac{9\cdot 24}{x+9}=$ the number of dollars B would have received had C not been called in.

But as B can reap the field in x days, for one day's work he should have $\frac{24}{x}$ dollars, and for five day's work, $\frac{5\cdot 24}{x}$ dollars, the sum he did receive.

Therefore, $\frac{9\cdot 24}{x+9}-\frac{5\cdot 24}{x}=1$

Whence, $x^2-87x=-1080$.

Here, as we are only in pursuit of results, we try inspection. We perceive that 10 for the value of x would not be large enough, and 20, too large; and as 1080 terminates in a cipher, we will try dividing by 15; then

$x-87=-72$. Whence, $x=15$, *Ans.*

Also, $x=72$; but this will not apply to the problem.

Again, as A could do the work in 9 days, for one day's work he should have $\frac{24}{9}$ dollars, and for 5 day's work, $\frac{5\cdot 24}{9}$ dollars. B should have $\frac{5\cdot 24}{15}$ dollars, and C, $\frac{2\cdot 24}{y}$ dollars, and the sum to the three is 24; therefore,

$$\frac{5\cdot 24}{9}+\frac{5\cdot 24}{15}+\frac{2\cdot 24}{y}=24$$

Or, $\frac{5}{9}+\frac{1}{3}+\frac{2}{y}=1$. Whence, $y=18$, *Ans.*

(4.) *Bacchus caught Silenus asleep by the side of a full cask, and seized the opportunity of drinking, which he continued for two-thirds of the time Silenus would have taken to empty the whole cask. After that, Silenus awoke and drank what Bacchus left. Had they both drank together, it would have been emptied two hours sooner, and Bacchus would have drank only half what he left Silenus. Required the time in which each would have emptied the cask separately.*

Ans. Bacchus in 6 hours, and Silenus in 3 hours.

Let $a=$ the volume of the cask.

$x=$ the time Bacchus would require to drink it alone.

$y=$ the time Silenus would require to drink it alone.

Then $\frac{a}{x}=$ the volume Bacchus drank per hour.

And $\frac{a}{y}=$ the volume Silenus drank per hour.

$\frac{a}{x}\cdot\frac{2y}{3}=$ the volume Bacchus drank; then

$\left(a-\frac{2ay}{3x}\right)=$ the quantity left to Silenus; and this quantity divided by the volume Silenus drank per hour, will give the hours he employed in drinking.

That is $\left(a-\frac{2ay}{3x}\right)\frac{y}{a}$, or $\left(y-\frac{2y^2}{3x}\right)=$ the time Silenus drank.

Had they both drank together, $\left(\frac{xy}{x+y}\right)$ would express the time.

Now by the given conditions,

$$\frac{2y}{3}+y-\frac{2y^2}{3x}=\frac{xy}{x+y}+2 \qquad (1)$$

And
$$\left(\frac{a}{2}-\frac{ay}{3x}\right)\frac{x}{a}=\frac{xy}{x+y} \qquad (2)$$

Reducing (2), and $\frac{1}{2}-\frac{y}{3x}=\frac{y}{x+y}$

Or,
$$3x^2-2y^2=5xy$$
$$9x^2-15xy=6y^2$$

$$9x^2-15xy+\frac{25y^2}{4}=\frac{49y^2}{4}$$

$$3x-\frac{5y}{2}=\pm\frac{7y}{2}$$

Whence, $x=2y$.

This value put in (1), gives

$$\frac{2y}{3}+y-\frac{2y^2}{6y}=\frac{2y^2}{3y}+2$$

$$\frac{2y}{3}+y-\frac{y}{3}=\frac{2y}{3}+2. \qquad y=3.$$

(5.) *A Banker has two kinds of money; it takes* a *pieces of the first to make a crown, and* b *pieces of the second to make the same sum. Some one offers him a crown for* c *pieces: how many of each kind shall he take?*

Ans. Of the first kind $\frac{(b-c)a}{(b-a)}$, of the second, $\frac{(a-c)b}{(a-b)}$.

This problem is more of a puzzle than most others, yet it is a fair scientific question.

Let $x=$ the number of pieces of the kind a,

and $y=$ the number of pieces of the kind b.

Then $x+y=c$. (1)

As a pieces are worth 1 crown, one piece is worth $\frac{1}{a}$, and x pieces are worth $\frac{x}{a}$.

By a parity of reasoning, y pieces of the second are worth $\frac{y}{b}$ and the worth of both together is just 1 crown; therefore,

$$\frac{x}{a}+\frac{y}{b}=1 \qquad (2)$$

Whence, $bx+ay=ab$

From (1), $bx+by=bc$

By subtraction, $(a-b)y=(a-c)b$. $y=\frac{(a-c)b}{a-b}$.

In like manner we find $x=\frac{(b-c)a}{(b-a)}$.

(6.) A *and* B *traveled on the same road, and at the same time, from Huntington to London. At the* 50*th mile stone from London,* A *overtook a drove of geese which were proceeding at the rate of* 3 *miles in* 2 *hours; and two hours afterwards, met a stage wagon, which was moving at the rate of* 9 *miles in* 4 *hours.* B *overtook the same drove of geese at the* 45*th mile stone, and met the same stage wagon exactly forty minutes before he came to the* 31*st mile stone. Where was* B *when* A *reached London?*

Ans. 25 miles from London.

Let $x=$ the rate which A and B traveled per hour.

Then $50-2x=$ the distance from London where A met the stage.

$$\overset{m}{3} : \overset{h}{2} : : \overset{m}{5} : \tfrac{10}{3} = \text{ the hours required for the geese to travel 5 miles.}$$

Then when B was 45 miles from London, A must have been $\left(50-\frac{10x}{3}\right)$ miles from the same place, and the distance between the two travelers must have been $\left(\frac{10x}{3}-5\right)$ miles, and the value of this expressson is the answer demanded.

Now let t be the hours elapsed between the times that A and B met the stage.

The motion per hour for the stage was $\frac{9}{4}$ miles.

B met the stage $\left(31+\frac{2x}{3}\right)$ miles from London; but A met it before, nearer to London by $\frac{9t}{4}$ miles. That is, A met the stage $\left(31+\frac{2x}{3}-\frac{9t}{4}\right)$ miles from London. We have before determined that A met the stage $(50-2x)$ miles from London; therefore,

$$31+\frac{2x}{3}-\frac{9t}{4}=50-2x \qquad (1)$$

Now after A met the stage he traveled in one direction, and the stage in another for t hours, before the stage met B. Then their distance asunder must have been $\left(\frac{9t}{4}+tx\right)$. But the distance the two travelers are asunder, has been expressed by $\left(\frac{10x}{3}-5\right)$.

Therefore, $$\frac{9t}{4}+tx=\frac{10x}{3}-5 \qquad (2)$$

From (1), $t=\frac{32x-228}{27}$. From (2), $t=\frac{40x-60}{27+12x}$.

Therefore, $$\frac{32x-228}{27}=\frac{40x-60}{27+12x}$$

Or, $$\frac{8x-57}{9}=\frac{10x-15}{9+4x}$$

Clearing of fractions,

$$32x^2-228x+72x-513=90x-135$$

Or, $$16x^2-123x=189$$

Here the obvious whole number factors of 189 are 3, 9, and 21; and as we are only in pursuit of results, we will try one or two of them. 21 we perceive at once is too large, therefore, try 9; then

$$16x-123=21$$

$$16x=144, \text{ or } x=9, \text{ a true result.}$$

Now because $x=9$, $\frac{10x}{3}-5=25$, the answer to the question.

SECTION V.

PROBLEMS IN PROPORTION, AND IN ARITHMETICAL, GEOMETRICAL AND HARMONICAL PROGRESSION.

The problems contained in Robinson's Algebra are not written out; they are only referred to by article, and number of the problem, and a mere outline of the solution indicated.

We commence with Art. 117, example 3.

(3.) Let $x-3y$, $x-y$, $x+y$, and $x+3y$ represent the numbers; then $2y=4$.

The product of the 1st and 4th, is

x^2-9y^2 ; of the 2d and 3d, is (x^2-y^2).

$$\begin{array}{l} x^2-y^2 \\ \hline x^4-9x^2y^2 \\ \quad -x^2y^2+9y^4 \\ \hline x^4-10x^2y^2+9y^4=176985 \\ \qquad\qquad\qquad 9y^4=\quad 144 \\ x^4-40x^2 \qquad\qquad =176841 \end{array}$$

(4.) The same notation as in the last example.

$$2x=8. \quad x=4. \quad x^2-y^2=15.$$

(5.) Let $n=$ the number of days.

Then $L=1+(n-1)1=n.$

$S=(1+n)\frac{1}{2}n=$ the whole distance.

Also, $(n-6)15 \quad =$ the whole distance.

$$n^2+n=30n-180.$$

$$n^2-29n=-180.........n=9 \text{ or } 20.$$

$$9-6=3. \quad 20-6=14.$$

(6.) The first day he must pay $1+i$; i representing the interest of one dollar for one day.

First day,	$1+\ i.$
2d day,	$1+\ 2i.$
3d day,	$1+\ 3i.$
Last day,	$1+60i.$

$(2+61i)30=$ the whole sum to be paid; but as this sum is to be paid in 60 equal payments, each payment must be

$$1+\frac{61i}{2}= \textit{Ans. } \$1 \text{ and } \tfrac{5}{6} \text{ of a cent, nearly.}$$

(7.) Let $x-3y$, $x-y$, $x+y$, and $x+3y$ represent the numbers; then

$$\begin{array}{r} 2x^2+18y^2=50 \\ 2x^2+\ 2y^2=34 \\ \hline 16y^2=16 \end{array} \;........................y=1.$$

GEOMETRICAL PROGRESSION AND HARMONICAL PROPORTION.

(Art. 124.)

(1.) Let x represent the mean sought.

$$x=\frac{2\cdot6\cdot12}{18}=8$$

(2.) Let $x=$ the number sought. Then, by harmonical proportion

$$234 : x :: 90 : 144-x$$

$$90x=234\cdot144-234x$$

$$324x=234\cdot144. \text{ Hence,} \ldots\ldots\ldots\ldots x=104.$$

(3.) Let $x=$ the number sought.

Then $24 : x :: 8 : 4-x$

Or, $3 : x :: 1 : 4-x \ldots\ldots\ldots\ldots\ldots x=3.$

(4.) Let $x=$ the second.

Then $16 : 2 :: 16-x : 1 \ldots\ldots\ldots\ldots\ldots x=8.$

(5.) Let $x=$ the first number, and $y=$ the ratio.

Then $x+xy+xy^2=210 \quad (1)$

$xy^2-x=90 \quad (2)$

By subtraction, $2x+xy=120$, or $x=\frac{120}{2+y}$

From (2), we have $x=\frac{90}{y^2-1}$

$$\frac{4}{2+y}=\frac{3}{y^2-1}, \text{ or } 4y^2-3y=10 \ldots\ldots\ldots\ldots\ldots y=2.$$

(5.) Let x, xy, xy^2, and xy^3 represent the numbers.

Then $$\frac{xy^3}{xy+xy^2}=\frac{y^2}{1+y}=\frac{4}{3}$$

From this equation we perceive at once that $y=2$; then

$$x+2x+4x+8x=15x=30 \ldots\ldots\ldots\ldots\ldots x=2.$$

(6.) Let x, xy, xy^2, and xy^3 represent the numbers.

$$x+xy^2=148 \quad (1)$$

$$xy+xy^3=888 \quad (2)$$

Or, $$x(1+y^2)=4\cdot37 \quad (3)$$

$$xy(1+y^2)=4\cdot222 \quad (4)$$

Divide (4) by (3), and, ... $y=6$.

(7.) Let x, $\surd(xy)$, and y represent the numbers; then

$$x+\surd(xy)+y=14 \quad (1)$$

And $$x^2+xy+y^2 \quad =84 \quad (2)$$

Put $x+y=s$, and $\surd(xy)=p$;

Then $x^2+xy+y^2=s^2-p^2$, and equations (1) and (2) become

$$s+p=14 \quad (3)$$

$$s^2-p^2=84 \quad (4)$$

Divide (4) by (3), and we have $s-p=6$ (5)

Add (3) to (5), and divide by 2, and $s=10$.

Hence, ... $p=4$.

(8.) Let x, xy, xy^2, and xy^3 represent the numbers; then

$$xy^3-xy=24$$

$$xy^3+x : xy^2+xy :: 7 : 3$$

Or, $$y^3+1 : y^2+y :: 7 : 3$$

Divide the first couplet by $(y+1)$, and we have

$$y^2-y+1 : y :: 7 : 3$$

$$3y^2-3y+3=7y, \text{ or } 3y^2-10y=-3.$$

From this equation we have $y=3$, the ratio.

(9.) Let x, xy, xy^2, and xy^3 represent the numbers;

Then $$x(1+y+y^2+y^3)=y+1$$

And $x=\frac{1}{10}$. Put $(y+1)=A$.

Then $$\tfrac{1}{10}(A+Ay^2)=A$$

$A+Ay^2=10A$. $Ay^2=9A$, or .... $y=3$.

Hence, $\frac{1}{10}$, $\frac{3}{10}$, &c. are the numbers.

(10.) Let x, $\frac{2xy}{x+y}$, and y represent the numbers; then

$$x+\frac{2xy}{x+y}+y=26 \qquad (1)$$

And $$xy=72$$

Put $x+y=s$; then equation (1) becomes

$$s+\frac{144}{s}=26, \text{ or } s^2-26s=-144 \ldots\ldots\ldots\ldots s=18.$$

(11.) Let x, xy, and xy^2 represent the numbers;

Then $$x^3y^3=216 \qquad (1)$$

$$x^2+x^2y^4=328 \qquad (2)$$

From (1) $$xy=6, \text{ or } x^2=\frac{36}{y^2}$$

From (2) $$x^2=\frac{328}{1+y^4}$$

$$\frac{36}{y^2}=\frac{328}{1+y^4}, \text{ or } \frac{9}{y^2}=\frac{82}{1+y^4}$$

$$9y^4-82y^2=-9. \text{ Hence, } \ldots\ldots\ldots\ldots y=3.$$

(12.) Let x, $\sqrt{xy}$, and y represent the numbers: then

$$x+\sqrt{xy}+y=13 \qquad (1)$$

$$(x+y)\sqrt{xy}=30 \qquad (2)$$

$$x+y=13-\sqrt{xy} \qquad (3)$$

$$x+y=\frac{30}{\sqrt{xy}} \qquad (4)$$

$$13-\sqrt{xy}=\frac{30}{\sqrt{xy}}. \text{ Hence } \sqrt{xy}=3.$$

(13.) Let x, $\frac{2xy}{x+y}$, and y represent the numbers; then

$$x+y=18 \qquad (1)$$

$$\frac{2x^2y^2}{18}=576 \qquad (2)$$

$$\frac{xy}{3}=24. \quad xy=72 \qquad (3)$$

From (1) and (3), we find x and y.

(14.) Let x, xy, and xy^2 represent the numbers; then $(xy^2-xy)\,(xy-x)$ are the first differences, and

$$xy^2-2xy+x=6$$
$$xy^2+xy+x=42$$

Difference, $3xy=36$ .... $xy=12$

(15.) Let x, $\frac{2xy}{x+y}$, and y represent the numbers. If y is supposed greater than x, then $\left(y-\frac{2xy}{x+y}\right)\left(\frac{2xy}{x+y}-x\right)$ are the 1st differences, and $y-\frac{4xy}{x+y}+x=2$, the 2d differences.

$xy=72$. Put $(x+y)=s$;

Then $$s-\frac{4\cdot 72}{s}=2$$

$$s^2-2s+1=289$$

$$s-1=17. \qquad s=x+y=18.$$

(17.) Let x^2, xy, and y^2 represent the numbers; then $x^2+xy+y^2=31$, and $x^2+y^2=26$.

(18.) Let x, xy, xy^2, xy^3, xy^4, and xy^5, represent the numbers. Then, by the conditions, we have

$$x+xy+xy^2+xy^3+xy^4+xy^5=189=a \qquad (1)$$

And $$xy+xy^4=54=b \qquad (2)$$

But equation (1) may be put into this form

$$(1+y+y^2)x+(1+y+y^2)xy^3=a$$

Or, $$x+xy^3=\frac{a}{1+y+y^2}$$

Multiply this last equation by y, and its first member will be the same as the first member of equation (2). Therefore, $\frac{ay}{1+y+y^2}=b$; a quadratic from which we obtain y, the ratio.

(19.) Take the same notation as for (18); then we have

$$(x+xy)+(x+xy)\,y^4=189-36=153=a. \qquad (1)$$

And $(x+xy)y^2=36=b.$ (2)

Divide (1) by (2), and we have

$$\frac{1+y^4}{y^2}=\frac{153}{36}=\frac{51}{12}. \quad \text{Hence} \ldots\ldots\ldots\ldots\ldots y=2.$$

CHAPTER III.—PROPORTION.

(5.) Let x and y represent the numbers; then

$$x-y : x+y :: 2 : 9$$

$$x+y : xy :: 18 : 77$$

From the first, $2x : 2y :: 11 : 7$, or $x=\frac{11}{7}y$.

$$\frac{18y}{7} : \frac{11y^2}{7} :: 18 : 77$$

$$y : 11y^2 :: 1 : 77. \quad y=7.$$

(6.) Let x and y represent the numbers.

$$x+4 : y+4 :: 3 : 4 \quad (1)$$

$$x-4 : y-4 :: 1 : 4 \quad (2)$$

From (2) we have $4x-16=y-4$, or $y=4x-12$. This value of y put in (1), gives

$$x+4 : 4x-8 :: 3 : 4$$

$$x+4 : x-2 :: 3 : 1$$

$$x+4=3x-6 \ldots\ldots\ldots\ldots\ldots\ldots x=5.$$

(7.) Let x and y represent the numbers;

Then $x+y=16$

And $xy : x^2+y^2 :: 15 : 34$

Double the first and third terms, then add and subtract, (Theorem 4), and $2xy : x^2+y^2 :: 30 : 34$

$$x^2+2xy+y^2 : x^2-2xy+y^2 :: 64 : 4$$

$$x+y : x-y :: 8 : 2$$

$$16 : x-y :: 4 : 1$$

Or, $x-y=4.$

9

(8.) Let $x=$ the gallons of rum.

And $y=$ the gallons of brandy,

$$x-y : y :: 100 : x$$
$$x-y : x :: 4 : y$$

Product, $(x-y)^2 : xy :: 400 : xy$

Dividing the second and fourth by xy, and

$$(x-y)^2 : 1 :: 400 : 1$$
$$x-y : 1 :: 20 : 1, \text{ or } x-y=20.$$

(9.) Let $x+y=$ the greater number,

And $x-y=$ the less.

Then $x^2-y^2=320.$ (1)

$$(x+y)^3=x^3+3x^2y+3xy^2+y^3$$
$$(x-y)^3=x^3-3x^2y+3xy^2-y^3$$

$6x^2y+2y^3=$ diff. of the cubes.

$2y=$ difference. Cube of $(2y)=8y^3$

$$6x^2y+2y^3 : 8y^3 :: 61 : 1$$
$$3x^2+y^2 : 4y^2 :: 61 : 1$$
$$3x^2+y^2=244y^2. \quad 3x^2=243y^2.$$
$$x^2=81y^2.$$

This value of x^2 put in equation (1), gives

$80y^2=320$, or .... $y=2$.

We now give additional problems.

(1.) *The sum of four whole numbers in arithmetical progression is* 20, *and the sum of their reciprocals is* $\frac{25}{24}$. *What are the numbers?*

Let $x-3y$, $x-y$, $x+y$, and $x+3y$ be the numbers ;

Then $4x=20$, and $x=5$.

Again $$\frac{1}{x-3y}+\frac{1}{x-y}+\frac{1}{x+y}+\frac{1}{x+3y}=\frac{25}{24}.$$

Uniting the 1st and 4th, and the 2d and 3d, we have

$$\frac{2x}{x^2-9y^2}+\frac{2x}{x^2-y^2}=\frac{25}{24}$$

Dividing by $x=5$, and then clearing of fractions, reduces the equation to $2x^2-2y^2+2x^2-18y^2=\frac{5}{24}(x^2-9y^2)(x^2-y^2)$.

Or, $4x^2-20y^2=\frac{5}{24}(x^4-10x^2y^2+9y^4)$

$$96x^2-480y^2=5x^4-50x^2y^2+45y^4$$

Putting the value of x^2 in the 2d member, we have

$$96x^2-480y^2=125x^2-1250y^2+45y^4$$

Whence, $0=29x^2-770y^2+45y^4$

Dividing by 5, $0=29x-154y^2+9y^4$

Or, $0=145-154y^2+9y^4$

Here *the sum of the coefficients is the same* in both members; therefore one of the values of the unknown quantity is 1; and as $y=1$ answers the conditions of the problem, we are not required to find the other roots.

Hence the numbers are 2, 4, 6, and 8.

(2.) *The sum of six numbers in arithmetical progression is* 33, *and the sum of their squares is* 199. *What are the numbers?*

Ans. 3, 4, 5, 6, 7, and 8.

Let $(x-y)$ represent the third term, and $(x+y)$ the fourth term; then $2y$ will be the common difference, and

$(x-5y)$, $(x-3y)$, $(x-y)$, $(x+y)$, $(x+3y)$, and $(x+5y)$

will represent the numbers.

Then $6x=33$, or $2x=11$. (1)

And $6x^2+70y^2=199$. (2)

From (1), $4x^2=121$, or $12x^2=363$.

Double (2), and write 363 for $12x^2$, then we have

$$363+140y^2=398$$

$$140y^2=35. \qquad y=\tfrac{1}{2}.$$

(3.) *Find four numbers in proportion such that their sum shall be* 20, *the sum of their squares* 130, *and the sum of their cubes* 980.

Ans. 6, 9, 2, and 3.

Let w, x, y, and z represent the numbers

Then because they are in proportion,

$$wz=xy \quad (1)$$

By conditions, $w+x+y+z=20=a.$ (2)

And $w^2+x^2+y^2+z^2=130.$ (3)

And $w^3+x^3+y^3+z^3=980.$ (4)

From (2) we have $w+z=a-(x+y)$ (5)

$$w^2+2wz+z^2=a^2-2a(x+y)+x^2+2xy+y^2$$

Suppressing $2wz$ in the first member, and its equal $2xy$ in the second member, and adding (x^2+y^2) to both members, we have

$$w^2+x^2+y^2+z^2=a^2-2a(x+y)+2x^2+2y^2$$

That is $130=400-2a(x+y)+2x^2+2y^2$

Whence, $a(x+y)=135+x^2+y^2$ (6)

By cubing (5), we have

$$w^3+3wz(w+z)+z^3=a^3-3a^2(x+y)+3a(x+y)^2$$
$$-(x^3+3xy(x+y)+y^3)$$

By transposing, and observing that $3wz$ equal $3xy$, we have

$$(w^3+x^3+y^3+z^3)+3xy(w+x+y+z)=a^3-3a^2(x+y)$$
$$+3a(x+y)^2$$

That is, $980+3axy=a^3-3a^2(x+y)+3a(x+y)^2$

Dividing by 20, or by a, which is the same thing, and we have

$$49+3xy=a^2-3a(x+y)+3(x+y)^2$$

Or, $3xy=351-3a(x+y)+3(x+y)^2$

Dividing by 3, and expanding the last term, gives

$$xy=117-a(x+y)+x^2+2xy+y^2$$

Or, $a(x+y)=117+x^2+xy+y^2$ (7)

By comparing (6) and (7), we perceive that

$$xy+117=135$$

Or, $xy=18$ (8)

By the aid of (8), (6) becomes

$$x^2+2xy+y^2+135=20(x+y)+36$$

Or, $(x+y)^2-20(x+y)=-99$

Whence, $(x+y)-10=\pm 1$

Therefore, $x+y=11$, or 9

From this last equation and equation (8), we find $x=9$, or 6, and $y=2$ or 3.

(4.) *The sum of five numbers in geometrical progression is* 31, *and the sum of their squares,* 341. *What are the numbers?*

Ans. 1, 2, 4, 8, and 16.

Let x, xy, xy^2, xy^3, xy^4 represent the numbers.

Put $a=31$ $11a=341$

Then $$x+xy+xy^2+xy^3+xy^4=a \qquad (1)$$

And $$x^2+x^2y^2+x^2y^4+x^2y^6+x^2y^8=11a \qquad (2)$$

By the formula for the sum of a geometrical series, we have

$$\frac{xy^5-x}{y-1}=a. \qquad (3)$$

And $$\frac{x^2y^{10}-x^2}{y^2-1}=11a. \qquad (4)$$

Dividing (4) by the square of (3), gives

$$\left(\frac{y^{10}-1}{y^2-1}\right)\left(\frac{y-1}{y^5-1}\right)^2=\frac{11}{a}$$

Factoring the first fraction,

$$\frac{(y^5+1)(y^5-1)(y-1)(y-1)}{(y+1)(y-1)(y^5-1)(y^5-1)}=\frac{11}{a}$$

Suppressing common factors,

$$\left(\frac{y^5+1}{y+1}\right)\left(\frac{y-1}{y^5-1}\right)=\frac{11}{a}$$

Divide the first fraction, numerator and denominator, by $(y+1)$ and the second fraction, numerator and denominator, by $(y-1)$, and we then have

$$\frac{y^4-y^3+y^2-y+1}{y^4+y^3+y^2+y+1}=\frac{11}{31}$$

Clearing of fractions and reducing,

$$10y^4-21y^3+10y^2-21y+10=0$$

Here we observe the same coefficients whether we begin to the right or the left of the expression. In such cases, divide by half the power of the unknown quantity. In this example, divide by y^2.

Then $10y^2-21y+10-\frac{21}{y}+\frac{10}{y^2}=0$ (5)

Now put $21y+\frac{21}{y}=P$ (6)

Then $y+\frac{1}{y}=\frac{P}{21}$

Squaring, $y^2+2+\frac{1}{y^2}=\frac{P^2}{441}$

$$10y^2+20+\frac{10}{y^2}=\frac{10P^2}{441} \qquad (7)$$

Comparing (5), (6), and (7), we perceive that

$$\frac{10P^2}{441}-P-10=0$$

$$10P^2-441P=4410$$

$$P^2-\frac{bP}{10}=b. \text{ By putting } b=441.$$

Then $P^2-\frac{b}{10}P+\frac{b^2}{400}=\frac{b^2}{400}+b=\frac{b(b+400)}{400}$

By evolution, $P-\frac{b}{20}=\sqrt{\frac{441\cdot 841}{400}}=\pm\frac{21\cdot 29}{20}$

$$P=\frac{21\cdot 21+21\cdot 29}{20}=\frac{21\cdot 50}{20}=\frac{105}{2}$$

Or, $P=\frac{-21\cdot 8}{20}=-\frac{42}{5}$

Now from (6), $y+\frac{1}{y}=\frac{5}{2}$, or $-\frac{2}{5}$

$$y^2-\frac{5y}{2}=-1. \quad y=2, \text{ or } \frac{1}{2}$$

To find x, we must return to equation (3), and in place of y, put in its value 2, and $x=1$.

Hence, the numbers are 1, 2, 4, 8, and 16.

(5.) *There are six numbers in geometrical progression; the sum of the extremes is* 99, *and the sum of the four means is* 90. *What are the numbers?* *Ans.* 3, 6, 12, 24, 48, 96.

Let x, xy, xy^2, &c. represent the numbers.

Then $xy^5+x=99$ (1)

And $xy^4+xy^3+xy^2+xy=90$ (2)

Dividing (1) by (2), gives

$$\frac{y^5+1}{y^4+y^3+y^2+y}=\frac{11}{10}$$

Dividing numerator and denominator by $(y+1)$, then

$$\frac{y^4-y^3+y^2-y+1}{y^3+y}=\frac{11}{10}$$

Clearing of fractions,

$$10y^4-10y^3+10y^2-10y+10=11y^3+11y$$

Or, $$10y^4-21y^3+10y^2-21y+10=0$$

This is the same equation as (5), in the preceding example; therefore, as in that example, $y=2$.

Now from equation (1), we have $33x=99$, or $x=3$, the first number.

(6.) *The number of deaths in a besieged garrison amounted to* 6 *daily; and allowing for this diminution, their stock of provisions was sufficient to last* 8 *days. But on the evening of the sixth day* 100 *men were killed in a sally, and afterwards, the mortality increased to* 10 *daily. Supposing their stock of provisions unconsumed at the end of the sixth day, to support* 6 *men for* 61 *days; it is required to find how long it would support the garrison, and the number of men alive when the provisions were exhausted.*

Ans. 6 days, and 26 men alive when the provisions were exhausted.

Let $x=$ the number of men at first,

and $p=$ the amount of provisions consumed by each per day.

By the question, we have a decreasing arithmetical series, whose common difference is 6, and number of terms 8.

For the amount of provisions at first in store, we have px for the first term, and $(px-42p)$ for the last term. Then the sum of the terms must be $(8px-168p)$.

Hence, $8px-168p=$ the amount of provisions at first.

$6px-\ 90p=$ the provisions consumed in 6 days.

$2px-\ 78p=6p\cdot 61$, the provisions left.

Whence, $x=222$.

During the 6 days, 36 men died, and 100 men were killed; therefore, at the end of the sixth day but 86 men were alive. Now the mortality increased to 10 daily, and at the end of n days their provisions were exhausted.

Here we have another decreasing arithmetical series, the first term 86, the common difference 10, and the number of terms n. The last term is, therefore,

$$86-10(n-1)$$

First term $\quad 86$

Sum of the terms $\quad (2\cdot 86-10n+10)\frac{n}{2}$

This number of men would require

$(86n-5n^2+5n)p$ amount of provisions.

Therefore, $\quad (91n-5n^2)p=6p\cdot 61$

Whence, $\quad 5n^2-91n=-366$

$$100n^2-A+(91)^2=(91)^2-366\cdot 20=8281-7320=961$$

By evolution, $\quad 10n-91=\pm 31$

Whence, $n=6$, or $12\frac{1}{5}$; but the last number cannot apply to the problem.

(7.) *Out of a vessel containing* 24 *gallons of pure wine, a vintner drew off at three successive times a certain number of gallons, which formed an increasing arithmetical progression, in which the difference between the squares of the extremes was equal to* 16 *times the mean, and filled up the vessel with water after each draught, till he found what he last drew off, reduced to one-sixth of its original strength. Required the number of gallons of pure wine drawn off each time.*

Ans. 12, 8, and $3\frac{1}{3}$.

Let $x-y=$ the number of gallons first drawn.

$x \quad =$ the number at second drawing.

and $x+y=$ the number at the third drawing.

By the first given condition

$$4xy=16x. \quad y=4.$$

$(x-y)$, or $(x-4)=$ the pure wine at the first drawing.

He then filled the cask with water, making 24 gallons of liquid, which contained $(24-x+4)$, or $(28-x)$ gallons of wine.

Liquid. *Wine.* *Liquid.*

Now by proportion, $24 : 28-x :: x : \frac{(28-x)x}{24}$ = the wine taken out at the second drawing.

Therefore, $(28-x)-\frac{(28-x)x}{24}$ = the wine left after the second drawing, which is $\frac{(24-x)(28-x)}{24}$, by reduction.

For the wine taken out at the third drawing, we have the following proportion;

$$24 : \frac{(24-x)(28-x)}{24} :: x+4 : \frac{(24-x)(28-x)(x+4)}{24 \cdot 24}$$

The number of gallons of liquid at the third drawing was $(x+4)$, but only one-sixth of it was wine; therefore,

$$\frac{(24-x)(28-x)(x+4)}{24 \cdot 24}=\frac{(x+4)}{6}$$

Whence, $$\frac{(24-x)(24-x+4)}{4 \cdot 24}=1.$$

Put $(24-x)=P$; then

$$P(P+4)=96$$
$$P^2+4P+4=100$$
$$P+2=\pm 10$$
$$P=8 \text{ or } -12.$$

That is, $24-x=8$, or $x=16$; the minus sign is not applicable. Hence the wine at the first drawing was $x-4$, or 12; at the 2d. $\frac{(28-x)x}{24}$, or 8; and at the 3d $\frac{x+4}{6}$, or $3\frac{1}{3}$, the answer.

(8.) *There are four numbers in geometrical progression such that the sum of the extremes is* 56, *and the sum of the means* 24. *What are the numbers?* *Ans.* 2, 6, 18, 54.

Let x, xy, xy^2, and xy^3 represent the numbers.

Then by the conditions,

$$xy+xy^2=24=3a \quad (1)$$
$$x+xy^3=56=7a \quad (2)$$

Dividing (1) by (2), and

$$\frac{y(1+y)}{1+y^3}=\frac{3}{7}$$

Dividing numerator and denominator by $(1+y)$, and we have

$$\frac{y}{1-y+y^2}=\frac{3}{7}$$

Or, $$3y^2-10y+3=0$$

If y is a whole number it is a factor of 3; that is 1 or 3. It is obviously not 1. Try 3; then

$$3y-10+1=0, \text{ or } y=3.$$

This value put in (1), gives $12x=24$. $x=2$.

Another Solution.

Let x, and y be the means, and $\frac{x^2}{y}$, and $\frac{y^2}{x}$ the extremes.

Then $$\frac{x^2}{y}+\frac{y^2}{x}=7a \quad (1)$$

And $$x+y=3a \quad (2)$$

From (1) $$x^3+y^3=7axy \quad (3)$$

(2) cubed, $$x^3+y^3+3xy(x+y)=27a^3 \quad (4)$$

That is, $$7axy+9axy=27a^3$$

$$16xy=27a^2 \quad (5)$$

Square of (2) gives $$x^2+2xy+y^2=9a^2$$

$$4xy=\frac{27a^2}{4}$$

By subtraction, $$x^2-2xy+y^2=\frac{9a^2}{4}$$

$$x-y=\tfrac{3}{2}a \quad (6)$$

From (2) and (6), we readily find x and y.

SECTION VI.

SOLUTIONS OF EQUATIONS OF THE HIGHER DEGREES.

We shall take equations and solve them. The most difficult equations in the common popular books will be selected, beginning with

NEWTON'S METHOD OF APPROXIMATION.

(1.) Given $x^3+2x^2-23x=70$, to find one value of x.

By trial we find that one value of x is between 5 and 6, nearer 5 than 6; therefore, let $a=5$ and $y=$ the remaining part of the root. Then $x=a+y$.

Expand, neglecting all the terms containing the powers of y after the first, and we shall have

$$\begin{aligned} x^3 &= a^3+3a^2y+\&c. \\ 2x^2 &= 2a^2+4ay+\&c. \\ -23x &= -23a-23y \end{aligned}$$

By addition,

$$x^3+2x^2-23x=a^3+2a^2-23a+(3a^2+4a-23)y=70$$

In this last equation we observe that a has the same powers and coefficients as x, and the coefficients to y may be found by the following

RULE. *Multiply each coefficient of* x *by its exponent, diminish each exponent by unity, and change* x *to* a.

Now $y=\dfrac{70+23a-2a^2-a^3}{3a^2+4a-23}$. Giving a its value, 5, we have $y=\frac{10}{72}=.1+$. Now make $a=5.1$, and substitute again in the preceding formula, we have a new value of y.

Thus $y=\dfrac{2\cdot629}{75\cdot43}=.03$. Now make $a=5.13$, and substitute again, and our new value of y will be $.004578+$. Hence, $a+y$ or $x=5.134578+$.

(2.) Given $x^4-3x^2+75x=10000$, to find one value of x.

By trial we find x must be near ten. Hence, put $a=10$ and $x=a+y$. Then by the preceding rule

$$y=\frac{10000-75a+3a^2-a^4}{4a^3-6a+75}=\frac{-450}{4015}=-.11$$

Now make $a=10-.11=9.89$. If we have the patience to substitute this value for a in the equation, we shall have a new value to y, true to 6 or 7 places of decimals, and of course a value to x to the same degree of exactness.

(3.) Given $3x^4-35x^3-11x^2-14x+30=0$, to find one value of x.

By trial we find that x must be near 12. Let $a=12$, and $x=a+y$. Then by the rule

$$y=\frac{-30+14a+11a^2+35a^3-3a^4}{12a^3-105a^2-22a-14}=\frac{-6}{5338}=-.00112$$

Hence, ... $x=12-.00112=11.99888.$

(4.) Given $5x^3-3x^2-2x=1560$, to find x.

We find by trial that one value of x is more than 7. Put $x=a+y$, and $a=7$. Then by the rule

$$y=\frac{1560+2a+3a^2-5a^3}{15a^2-6a-2}=\frac{6}{689}=.00867+$$

Hence, ... $x=7.00867+$

We give Newton's method on account of its simplicity in principle; it is easily understood, and can long be retained; but its numerical application is laborious and tedious.

A more modern, delicate, scientific, and practical method is Horner's, of Bath, England, first given to the world in 1819. The principle is that of transforming one equation into another whose roots shall be less in value by a given quantity, and again transforming that equation into another whose roots may be still less, &c. The theory is fully explained in Robinson's Algebra, last two chapters.

We give the following examples, commencing with quadratics. Some of the equations here solved are in the author's class book, and are numbered as in that work, pages 313—333.

(6.) Given $x^2+7x=1194$. to find the values of x by Horner's method.

We must first find an approximate value of x by trial; but the inexperienced might be at a loss how to make the trial. We suggest this method; separate the first member into factors thus:

$$x(x+7).$$

Here two factors of 1194 differ by 7. If the factors were equal, each one would be the square root of 1194.

Now one of the factors is a little less than the square root of 1194, and the other a little greater; but we want the less factor.

In short, the square root of 1194 *is a little below the arithmetical mean between x and $(x+7)$.*

This principle will do for a guide, when the coefficient of x is small in relation to the absolute term. The square root of 1194 is 34.5; from this we will subtract the half of 7, 3.5, and the approximate value of x will be 31; therefore, $r=31$. $a=7$.

		$r \quad st$
		1194 (31.231
$a+r$	38	1178
$r+s$	31.2	1600
$a+2r+s$	69.2	1384
$s+t$	23	21600
$a+2r+2s+t$	69.43	20829
	31	77100
	69461	

The operation may now be carried on as in simple division; thus

```
69461) 77100(111
       69461
        76390
        69461
         69290
```

and the figures thus obtained annexed to the portion of the root already obtained. Hence, $x=31.231111$.

As the sum of the two roots is equal to -7, the other root will be -38.231111.

(7.) Given $x^2-21x=214591760730$, to find the values of x.

Conceive $-21x$ not to exist; then the value of x will be the square root of the absolute term; but this term has six periods of two figures each, and the superior period is 21, the greatest square in this is 16, root 4; hence, x must be over 400000; take $r=400000$. $a=-21$.

$-a+r$	=399979	214591760730(400000	$=r$
$r+s$	460000	1599916	60000 $=s$
$-a+2r+s$	859979	5460016	3000 $=t$
$s+t$	63000	5159874	200 $=u$

$-a+2r+2s+t$	922979	3001420	$50=v$
	3200	2768937	$1=w$
	926179	2324837	
	250	1852358	
	926429	4724793	
	51	4632145	$x=463251.$
	926480	926480	
		926480	

As the algebraic sum of the two roots must make 21, the other root must be -463230.

We can find the negative root directly as well as indirectly, by taking r minus; then s, t, u, v, &c., will be minus. The divisors and quotients both being minus, their products will be plus.

The following example is in direct contrast to the preceding.

(*a*) Given $x^2-32141x=131$, to find the values of x.

Here the coefficient of the first power of the unknown quantity is large, and the absolute term comparatively very small. The factors x and $(x-32141)$ are so very unequal, that a resort to the square root of the absolute term for an approximate value of x, as in the preceding equation, would be useless. In this, and in similar cases, we can obtain an approximate value, by conceiving the absolute term to diminish to zero. Then

$$x^2-32141x=0.$$

This equation will be verified by putting $x=0$, and $x=32141$; and from this consideration we conclude that one value of x in our equation must be very small, and the other, over 30000. Hence, put $r=30000$, and the solution is as follows.

		131 (	$30000=r$
$-a+r$	-2141	-64230000	
$r+s$	32000	$+64230131$	$2000=s$
$-a+2r+s$	29859	59718000	
$s+t$	2100	4512131	$100=t$
$-a+2r+2s+t$	31959	3195900	$40=u$
$t+u$	140	1316231	$1=v$

	1316231
32099	1283960
41	32271
32140	32140
	131

Here we observe that the last divisor is numerically the same as the coefficient to x, and the last dividend is 131, the same as the absolute term.

Now if we divide 131 by 32140, we shall obtain decimal places in the root, and the positive value of x will be $32141\frac{131}{32141}$, very nearly.

The first four or five decimal places will be *exactly:* then the figures will be too large, because the divisor accurately corrected, will increase a little at every step. From this example, we learn that when we have an equation in the form

$$x^2-ax=\pm b,$$

and a numerically greater than b, the positive value of x will be $\left(a\pm\frac{b}{a}\right)$, very nearly, the value being a very little in excess of the true value, and if $\frac{b}{a}$ is a small fraction, this approximate value of x will be sufficiently near to call it the true value.

When the equation is in the form

$$x^2+ax=\pm b,$$

and a greater than b, then the negative value of x will be expressed by $-\left(a\pm\frac{b}{a}\right)$, very nearly, and if b is much greater than a, we may say accurately, *in a practical point of view.*

If we take the equation $x^2-ax=b$, and take $x=a+\frac{b}{a}$, and attempt to verify the equation with this value, we shall have

$$a^2+2b+\frac{b^2}{a^2}-a^2-b=b. \quad \text{Or,} \quad \frac{b^2}{a^2}=0$$

The error then, is $\frac{b^2}{a^2}$; and thus we perceive that if $\frac{b}{a}$ is a small fraction, the approximate value of x is really found; but if b is greater than a, it can hardly be called an approximation.

EXAMPLES.

(*b*) Given $x^2-3165x=632$, to find the approximate values of x. *Ans.* $x=3165\frac{632}{3165}$, or $-\frac{632}{3165}$.

(*c*) Given $x^2-2178x=-69$, to find the approximate values of x. *Ans.* $x=2178-\frac{69}{2178}$, or $\frac{69}{2178}$.

(*d*) Given $x^2+3116x=141$, to find the approximate values of x. *Ans.* $x=-(3116+\frac{141}{3116})$, or $\frac{141}{3116}$,

(*e*) Given $x^2+591x=-71$, to find the approximate values of x. *Ans.* $x=-591+\frac{71}{591}$, or $-\frac{71}{591}$.

The foregoing values are so near the true values, that they would be taken for true values, in any practical application.

(*f*) Given $x^2+\frac{17}{2}x=-4$, to find the values of x.
Ans. $x=-\frac{17}{2}+\frac{8}{17}=-8$ nearly.

-8 is the exact negative value of x, and $\frac{1}{2}$ is the positive value. We get $\frac{8}{17}$ for the approximate positive value.

(*g*) Given $x^2-\frac{5}{2}x=-1$, to find the approximate values of x.
Ans. $x=\frac{5}{2}-\frac{2}{5}=\frac{21}{10}$; the true value is 2.

(*h*) Given $x^2-\frac{26}{5}x=-1$, to find the approximate values of x.
Ans. $x=\frac{26}{5}-\frac{5}{26}=5$ nearly; 5 is the true value.
The other value is $\frac{1}{5}$.

When a and b are equal, or nearly equal, in the equation

$$x^2\pm ax=\pm b,$$

it is most difficult to find the value of r, or the approximate value of x.

Having now sufficiently explained the means of finding r in the different cases, we resume the application of Horner's method of operation.

(8.) Given $7x^2-3x=375$, to find one value of x.

Or $x^2-\frac{3}{7}x=\frac{375}{7}$. Put $x=\frac{1}{7}y$. (Art. 166.)

Then $\frac{y^2}{49}-\frac{3y}{49}=\frac{375}{7}$. Or, $y^2-3y=375\times 7=2625$.

In this equation we perceive that y must be more than the square root of 2625, that is, more than 50. Hence, put $r=50$.

		$r\ s\ \ t$
$-a+r$	47	2625 (52.756+
$r+s$	52	235
$-a+2r+s$	99	275
$s+t$	2.7	198
	1017	7700
	75	7119
	10245	58100
		51225
		6875

Hence $x=\frac{52.756+}{7}=7.+$

(10.) Given $x^2-\frac{3}{11}x=8$, to find x true to seven places of decimals.

Put $x=\frac{y}{11}$; then $x^2=\frac{y^2}{121}$, and the given equation is transformed into $\frac{y^2}{121}-\frac{3y}{121}=8$.

Or, $$y^2-3y=968.$$

It is obvious that y must be between 30 and 40; therefore, $r=30$.

$-a+r$	27	968 (30.
$r+s$	32	810
$-a+2r+s$	59	158 (2.64883625
$s+t$	2.6	118
	61.6	4000
	64	3696
	6224	30400
	48	24896

62288	550400
88	498304
62|2|9|6|8	5209600
	4983744
	225856
	186890
	38966
	37376
	1590
	1245
	345

After the 4th decimal, the operation was carried on by contracted division, giving $y=32.64883625$.

But $x=\frac{1}{11}$ of y; therefore, $x=2.96807602$.

(11.) Given $4x^2+\frac{7}{9}x=\frac{1}{5}$, to find one value of x.

This equation is the same as $x^2+\frac{7}{36}x=\frac{1}{20}$.

Put $x=\frac{y}{36}$; then $x^2=\frac{y^2}{(36)^2}$, and the equation becomes

$$\frac{y^2}{(36)^2}+\frac{7y}{(36)^2}=\frac{1}{20}$$

Or, $$y^2+7y=64.8$$

It is obvious that the value of y is between 5 and 6; therefore $r=5$. $a=7$.

$a+r$	12	64.8 (5.277812946
$r+s$	5.2	60
$a+2r+s$	17.2	480
$s+t$	2.7	344
	17.47	13600
	77	12229
	17547	137100
	78	122829
	175548	1427100
		1404384

```
                    227160
                    175548
                    ------
   17|5|5|4          51612
                     35108
                     -----
                     16504
                     15788
                     -----
                       816
                       702
                       ---
                       114
```

Whence, $y=5.277812946$, or -12.277812946.

And $x=0.146605915$, or -0.341050359.

(12.) Given $\frac{3}{4}x^2+\frac{3}{5}x=\frac{7}{11}$, to find one value of x.

$$x^2+\frac{4x}{5}=\frac{28}{33}. \quad \text{Put } x=\frac{y}{5}$$

Then $$\frac{y^2}{25}+\frac{4y}{25}=\frac{28}{33}$$

Or, $$y^2+4y=\frac{700}{33}=21.21212121.$$

$r=3$. $a=4$.

```
a+r          7.        21.21212121, &c. ( 3.021186235.
  r+s        3         21
----------   ----      ----
a+2r+s       10.0        2121
      s+t     02         2002
             ----        -----
             1002         11921
               21         10041
             -----        ------
             10041        188021
                11        100421
             ------       -------
             100421        8760021
                 18        8033824
             -------       -------
             10042|2|8      626197
                            602536
                            ------
                             23661
                             20084
                             -----
                              3577
```

$y=3.021186235$. $x=0.604237257$.

(13.) Given $115-3x^2-7x=0$, to find one value of x.

$$x^2+\frac{7x}{3}=\frac{115}{3}.\quad \text{Put } x=\frac{y}{3}.$$

Then,
$$\frac{y^2}{9}+\frac{7y}{9}=\frac{115}{3}$$

$$y^2+7y=345.\qquad r=15.\qquad a=7.$$

$a+r$	22	345 (15.40158.
$r+s$	15.4	330
$a+2r+s$	37.4	1500
$s+t$	40	1496
$a+2r+2s+t$	37|801	40000
		37801
		2199
		1890
		309
		302

$y=15.40158$. $x=5.13386$.

We will now apply Horner's method to cubic and the higher equations. For the theory, we must go to the class books.

CUBIC EQUATIONS.

The first example here, is the third on page 319 of the author's class book. Hence,

(3.) Given $x^3+2x^2-23x=70$, to find one value of x.

By trial we find x must be a little over 5; therefore,

$$r=5,\ A=2,\ B=-23,\ N=70$$

B	-23	
$r(r+A)$	35	$r\ s\ t$
1st Divisor	12	70 (5.134
r^2	25	60

B'	72	10000
$s(s+3r+A)$	171	7371
2d Divisor	7371	2629000
s^2	1	2276697
B''	7543	352303000
*$(3Q+t)t$	4599	305649104
3d Divisor	758899	46653896
t^2	9	
B'''	763507	
	61576	
4th Divisor	76412276	
	16	

Common division will give three or four more figures to perfect accuracy.

(4.) Given $x^3-17x^2+42x=185$, to find one value of x.

Here $A=-17$, $B=42$, $N=185$, and we find by trial that x must be between 15 and 16; therefore, $r=15$.

B	42	
		$r\ s\ t$
$r(r+A)$	-30	185 (15.02
1st Divisor	12	180
r^2	225	
B'	207	5000000
$s(s+3r+A)$	0	4154008
2d Divisor	207	2077) 845992 (407
s^2	0	8298
B''	207	16192
$t(3Q+t)$	7004	14539
3d Divisor	2077004	1653.0

Hence, ... $x=15.02407+$

(5.) Given $x^3+x^2=500$, to find one value of x.

Here $A=1$, $B=0$, $r=7$.

---

*Q represents the root as far as previously determined.

B	0	
$r(r+A)$	56	500 (7.61
1st Divisor	56	392
r^2	49	108
B'	161	104736
$(3r+s+A)s$	1356	3264
2d Divisor	17456	1887181
s^2	36	) 1376819
	18848	
	2381	
3d Divisor	1887181	Continue by common division.
	1	
	1889563	

(6.) Given $x^3+10x^2+5x=2600$, to find one value of x.

Here $A=10$, $B=5$, $r=11$.

B	5	2600 (11.006
$r(r+A)$	231	2596
1st Divisor	236	4
r^2	121	3529188216
B'	588	470811784
$(3R+u)u$	198036	Continue by common
4th Divisor	588198036	division.

(6.) Find one value of x from $5x^3-6x^2+3x=-85$.

As the result is negative, we will change the second and every alternate sign of the equation, (Art. 178), and find a value of x from the equation $5x^3+6x^2+3x=85$.

Use the formula of (Art. 194). $c=5$, $A=6$, $B=3$, and by trial we find $r=2$.

		r s
B	3	85 (2.1
$(cr+A)r$	32	70
1st Divisor	35	15
cr^2	20	9.065
	87	5.935
$(3cr+cs+A)s$	3.65	
2d Divisor	90.65	
cs^2	5	
	94.35	

Continuing this, we shall find the value of x to be $2.16399+$, and its sign changed will be the value of x in the original equa.

(7.) Find x from the equation $12x^3+x^2-5x=330$.

Here $c=12$, $A=1$, $B=-5$, $r=3$.

B	-5		$r\ s\ t$
$(cr+A)r$	111		330 (3.036
1st Divisor	106		318
cr^2	108		
B'	325		12
$(3cR+ct)t$	11208		9783624
3d Divisor	3261208		2216376
ct^2	108		Continue thus.
	3272524		

In the same manner perform 8 and 9.

(Art. 195.) Page 323.

(3.) Extract the cube root of 1·352·605·460·594·688

For the sake of brevity, take $r=11$, in place of 1.

1st Divisor	121		$r\ s\ t$
$B'=3r^2$	363		1·352·605·460·594·688 (110592.
$(3R+t)t$	16525		1 331
2d Divisor	3646525		21 605 460
	25		18 232 625
	3663075		3 372 835 594
$(3R+u)u$	298431		3 299 453 379
3d Divisor	366605931		73 382 215 688
	81		73 382 215 688
	366904443		
$(3R+v)v$	663544		
	36691107844		

(4.) By the table of cubes which run to 8000, we perceive at once that r in this example is 17.

1st Divisor	289		$r\ s\ t$
$3r^2=B'$	867		5382674 (175.2
$(3R+s)s$	2575		4913
2d Divisor	89275		469674
	25		446375

	91875	23299000
$(3R+t)t$..........	10504	18396008
3d Divisor...........	9198004	4902992
	4	Complete another divisor, then
	9208512	continue as in simple division.

(5.) Find x from the equation $x^3=15926.972504$.

For the sake of brevity, let r represent the value of the two superior digits. That is, let $r=25$.

	15926.972504 (25.16002549
1st Divisor....625	15625
$B'=3r^2$..... 1875	301 972
$(3r+s)s$..... 751	188 251
2d Divisor... 188251	113 721504
s^2......... 1	113 673096
B''..........189003	48408 000 000
$(3R+t)t$..... 45216	Common division.
3d Divisor.... 18945516	189.91) 48408 (2549
36	37982
18990768	10426
7548	9495
4th Divisor 1899084348	931
	759
	172

It is not important to show a solution to the remaining examples under this article.

From Robinson's Algebra, page 324.

(1.) Given $x^4+x^3+x^2-x=500$, to find one value of x.

By trial, we find $r=4$.

1	1	1	—1	=500 (4.
	4	20	84	332
	5	21	83	168
	4	36	228	

1	9	57	311
	4	52	
	13	109	
	4		
	17		

A new transformed equation is

$$s^4+17s^3+109s^2+311s=168. \qquad t=\tfrac{168}{311}=.4$$

1	17.	109.	311.	=168. (0.4
	.4	6.96	46.384	142.9536
	17.4	115.96	357.384	25.0464
	4	7.12	49.232	
	17.8	123.08	406.616	
	4	7.28		
	18.2	130.36		
	4			
	18.6			

The next transformed equation is

t^4	$+18.6t^3$	$+130.36t^2$	$+406.616t$	$=25.0464$ (0.06
	.06	1.12	7.888	24.8602
	18.66	131.48	414.504	.1862
	.06	1.12	7.956	
	18.72	132.60	422.460	
	.06	1.13		
	18.78	133.73		
	.06			
	18.84			

Several other decimal places of the root may be found by the following division,—the powers of u above the first being considered valueless.

```
423. ) 0.1862 ( 0.00044019
       1692
       ----
        1700
        1692
        ----
           800
```

Hence the root is 4.46044019.

(2.) Given $x^4-5x^3+0x^2+9x=2.8$, to find one value of x. $r=0.3$, found by trial.

				r
1	−5.	+0.	+9.	=2.8 (0.3
	0.3	−1.41	−.423	2.5731
	−4.7	−1.41	8.577	.2269
	.3	−1.32	−.819	
	−4.4	−2.73	7·758	
	.3	−1.23		
	−4.1	−3.96		
	.3			
	−3.8			

$$s^4-3.8s^3-3.96s^2+7.758s=0.2269.$$

1	−3.8	−3.96	+7.758	=0.2269 (0.02
	.02	−.0756	−0.081	0.15354
	−3.78	−4.0356	7.677	.07336
	.02	.075	.082	
	−3.76	−4.11	7.595	

The remaining figures may be found by division, thus :

```
7.5 ) .07336 ( 0.00978
       675
       ---
        586
        525
        ---
         610
```

Hence the approximate value of x is 0.32978.

(3.) Given $x^4-9x^3-11x^2-20x=-4$, to find one value of x.

				r
1	−9.	−11.	−20.	=−4 (.1
	.1	−.89	−1.189	−2.1189
	−8.9	−11.89	−21.189	−1.8811
	.1	−.88	− 1.277	
	−8.8	−12.77	−22.466	
	.1	−.87		
	−8.7	−13.64		
	.1			
1	−8.6	−13.64	−22.466	−1.8811 (.07 &c.

1	—8.6	—13.64	—22.466	=—1.8811 (.07
	.07	—.597	—.9966	—1.642382
	—8.53	—14.237	—23.4626	— .238718
	.07	—.592	—1.0380	
	—8.46	—14.829	—24.5006	
	.07	—.587		
	—8.39	—15.416		
	.07			
	—8.32			

1	—8.32	—15.416	—24.5	=—0.238718 (0.009
	.009	—.0748	—.139	— .2217547
	—8.311	—15.4908	—24.639	— .0169633
	9	—.0747	—.140	
	—8.302	—15.5655	—24.779	
	9	—.0746		
	—8.293	—15.6401		
	9			
	—8.284			

—24.78) —.0169633 (0.000684
14868
20953
19824
11290
9912
13780

Whence, x=0.179684, nearly.

(4.) Given $x^5=5000$, to find one value of x, or we may say find the fifth root of 5000.

Here all the coefficients are zero, except the first, and $r=5$.

					r
1	0	0	0	0	=5000 (5
	5	25	125	625	3125
	5	25	125	625	1875
	5	50	375	2500	
	10	75	500	3125	
	5	75	750		
	15	150	1250		
	5	100			
	20	250			
	5				
	25				

1	25.	250.	1250.	3125.	=1875. (.4
	.4	10.16	104.06	541.624	1466.6496
	25.4	260.16	1354.06	3666.624	408.3504
	.4	10.32	108.19	584.900	
	25.8	270.48	1462.25	4251.524	
	.4	10.48	112.38		
	26.2	280.96	1574.63		
	.4	10.64			
	26.6	291.60			
	.4				
	27.0				

1	27.	291.6	1574.6	4251.52	=408.3509(.09
	.09	22.38	28.26	144.26	395.6202
	27.09	313.98	1602.86	4395.78	12.7307
	.09	24.46	30.46	146.998	
	27.18	338.44	1633.32	4542.778	

Now by common division,

4542.7|78) 12.7307 (.0028, nearly.
9.0855
3.6452

Whence, $x = 5.4928$, nearly.

It is practically useless to solve such equations as the preceding, because solutions are so simple and direct by logarithms.

(5.) Given $x^3=\left(\frac{8x}{x^2+1}\right)^2$, or $x^3=\frac{64x^2}{x^4+2x^2+1}$, to find one value of x.

$$x^5+2x^3+x=64. \qquad r=2$$

1	0	2	0	1	=64 (2.
	2	4	12	24	50
	2	6	12	25	14
	2	8	28	80	
	4	14	40	105	
	2	12	52		
	6	26	92		
	2	16			
	8	42			
	2				
	10				

1	10.	42.	92.	105.	=14 (0.1
	.1	1.01	4.3	9.63	11.463
	10.1	43.01	96.3	114.63	2.537
	.1	1.02	4.403	10.07	
	10.2	44.03	100.703	124.70	
	.1	1.03	4.506		
	10.3	45.06	105.209		
	.1	1.04			
	10.4	46.1			
	.1				
	10.5				

1	10.5	46.1	105.21	124.7	=2.537(0.02
	.02	.21	.926	2.1227	2.536454
	10.52	46.31	106.136	126.8227	.000546
	.02	.21	.93	2.141	
	10.54	46.52	107.066	128.9637	

128.96) .00054600 (0.000004+
51584

Whence, $x=2.120004$, nearly.

By an exact solution, the last figure would be 3, in place of 4.

OBSERVATION. When we observe that the sum of the coefficients in any equation is zero, we may be sure that *unity* is one root of the equation.

Then we can depress the equation one degree by division. For example, we are sure that the equation

$$x^3-7x^2+7x-1=0$$

has a root $=1$, because $1-7+7-1=0$; and 1 put for x will neither increase nor decrease any of the terms.

The equation $x^4+2x^3-7x^2-8x+12=0$, also has one root $=1$, for the same reason.

In Bland's problems I find the following problem, (page 426). One root of the equation $x^4-5x^3-x+5=0$, is 5; determine all the roots. Here we perceive that another root is 1; therefore, the equation is divisible by $(x-5)(x-1)$, or by x^2-6x+5; thus

$$\begin{array}{l}
x^2-6x+5\;)\;x^4-5x^3-x+5\;(\;x^2+x+1 \\
\qquad\qquad\quad\; x^4-6x^3+5x^2 \\
\qquad\qquad\qquad\quad\;\; \overline{\qquad\; x^3-5x^2-x} \\
\qquad\qquad\qquad\qquad\quad\;\; x^3-6x^2+5x \\
\qquad\qquad\qquad\qquad\qquad\;\; \overline{\qquad x^2-6x+5} \\
\qquad\qquad\qquad\qquad\qquad\qquad x^2-6x+5
\end{array}$$

Whence $x^2+x+1=0$, and $x=\frac{1}{2}(\pm\sqrt{-3}-1)$.

To solve some of the following problems, it may be necessary to see how the roots combine to form the coefficients.

We shall consider all the roots as positive; and represent them by a, b, c, d, e, &c.

Then an equation of the second degree will be represented by

$$\begin{array}{r|l} x^2-a & x+ab=0. \\ -b & \end{array} \qquad (1)$$

An equation of the third degree, by

$$\begin{array}{r|r|l} x^3-a & x^2+ab & x-abc=0. \\ -b & +ac & \\ -c & +cb & \end{array} \qquad (2)$$

An equation of the fourth degree, by

$$\begin{array}{l|l|l|l}
x^4-a & x^3+ab & x^2-abc & x+abcd=0. \quad (3)\\
\quad\;-b & \quad+ac & \quad-abd & \\
\quad\;-c & \quad+ad & \quad-acd & \\
\quad\;-d & \quad+cb & \quad-cbd & \\
 & \quad+cd & & \\
 & \quad+bd & &
\end{array}$$

Now let $A=a+b+c+d$. $\quad B=a(b+c+d)+b(c+d)+cd.$

$C=a(bc+bd+cd)+cbd. \qquad D=abcd.$

Then the equation of the fourth degree will become

$$x^4-Ax^3+Bx^2-Cx+D=0. \qquad (4)$$

This equation multipled by $(x-E)$, gives the representative of an equation of the fifth degree, as follows:

$$\begin{array}{l|l|l|l|l}
x^5-A & x^4+\;B & x^3-\;C & x^2+\;D & x-ED=0 \qquad (5)\\
\quad\;-E & \quad+EA & \quad-EB & \quad+ED &
\end{array}$$

In these equations we observe that the coefficient of the highest power of x is 1; and that the coefficient of the next inferior power is the sum of all the roots, with their signs changed;—the absolute term is the product of all the roots.

EXAMPLES.

(1.) The roots of the equation

$$6x^4-43x^3+107x^2-108x+36=0,$$

are of the form a, b, $\frac{b}{a}$, $\frac{a}{b}$, find them.

Divide the equation by 6, so that it may compare with the fundamental equations (3) or (4), then

$$x^4-\tfrac{43}{6}x^3+\tfrac{107}{6}x^2-18x+6=0.$$

Now if the roots are of the form a, b, $\frac{b}{a}$, $\frac{a}{b}$, we may take these symbols to represent the roots.

Then $\qquad a+b+\frac{a^2+b^2}{ab}=\frac{43}{6}$, and $ab=6$.

That is, $\qquad 6(a+b)+a^2+b^2=43.$

By adding $2ab=12$ to the last equation, we have

$$(a+b)^2+6(a+b)=55.$$

Whence, $a+b=5$; but $ab=6$. $a=2$. $b=3$.

And the roots are 2, 3, $\frac{3}{2}$, $\frac{2}{3}$.

(2.) The roots of the equation

$$x^4-10x^3+35x^2-50x+24=0,$$

are of the form $(a+1)$, $(a-1)$, $(b+1)$, $(b-1)$, find them.

Here $2a+2b=10$. $(a^2-1)(b^2-1)=24$.

That is, $a+b=5$. $a^2b^2-a^2-b^2+1=24$.

But $2ab+a^2+b^2=25$.

By addition, $a^2b^2+2ab+1=49$.

$$ab+1=7, \text{ or } ab=6.$$

But $a+b=5$; hence, $a=2$, $b=3$, and the roots are 1, 2, 3, and 4.

The roots of the following equations are in arithmetical progression; find them.

1. $x^3-6x^2-4x+24=0$.
2. $x^3-9x^2+23x-16=0$.
3. $x^3-6x^2+11x-6=0$.
4. $x^4-8x^3+14x^2+8x-15=0$.
5. $x^4+x^3-11x^2+9x+18=0$.

We work out the fifth and last example; it being the only difficult one.

Let $(a-3b)$, $(a-b)$, $(a+b)$, and $(a+3b)$, represent the roots; then $4a=-1$, and $(a^2-b^2)(a^2-9b^2)=18$.

That is, $a^4-10a^2b^2+9b^4=18$. But $a^2=\frac{1}{16}$, $a^4=\frac{1}{256}$.

Therefore, $\frac{1}{256}-\frac{10b^2}{16}+9b^4=18$.

$$\tfrac{1}{16}-10b^2+144b^4=288.$$

Or, $144b^4-10b^2=288-\frac{1}{16}$.

Add $\frac{25}{144}$ to both members to complete the square,

Then $144b^4-10b^2+\frac{25}{144}=288\frac{16}{144}=\frac{41488}{144}$

By evolution, $12b^2-\frac{5}{12}=\frac{203.68603}{12}$

Whence, $b^2=1.449209$

And $b=1.20383$. But $a=-0.25$.

Therefore, $a-3b=-3.86149$, log. 0.586761.

$a-b=-1.45383$, log. 0.162515.

$a+b=+0.95383$, log. -1.979458.

$a+3b=+3.36149$, log. 0.526510.

Log. 18, 1.255244, nearly.

The sum of these numbers is -1, as it ought to be, and the product of the roots is 18.

Negative numbers have no logarithms, because there are in fact no such numbers. The product of several numbers is *numerically* the same, whether the numbers be positive or negative; therefore, we took the logarithm of each root as though it were positive. The product in every case will be positive or negative, according as the number of minus factors is even or odd.

The roots of the following equations are in geometrical progression: find them.

1. $x^3-7x^2+14x-8=0$.
2. $x^3-13x^2+39x-27=0$.
3. $x^3-14x^2+56x-64=0$.
4. $x^3-26x^2+156x-216=0$.

Let a, ar, and ar^2 represent the three roots.

Then $a+ar+ar^2=26$, and $a^3r^3=216$.

From these equations we find $a=2$ and $r=3$, and the roots sought are 2, 6, and 18.

Problems like the preceding are impractical, because there is no natural method of finding the *form of roots, a priori*, and *to give the form*, is nearly equivalent to giving the roots themselves.

RECURRING EQUATIONS.

A recurring equation is one in which there is a symmetry among the coefficients;—the terms which are equally distant from the extremes, have the same numerical coefficient. For example,

$$x^5-11x^4+17x^3+17x^2-11x+1=0$$

is a recurring equation, for the coefficients are 1, —11, +17, which recur in the inverse order 17, —11, 1.

Here it is obvious that $x=-1$ will satisfy the equation ; and if we change the second and every alternate sign, then $x=1$ will satisfy the equation ; that is, 1 is a root of the equation

$$x^5+11x^4+17x^3-17x^2-11x-1=0.$$

Here the sum of the coefficients is 0, and consequently $x=1$ must satisfy the equation.

Now we arrive at this general truth :

A recurring equation of an odd degree will have either —1 or +1 for one of its roots.

It will have —1, if the corresponding coefficients have like signs— and +1, if they have unlike signs.

Hence, every recurring equation of an odd degree will be divisible by $(x+1)$ or $(x-1)$, and can thus be reduced to an equation of an even degree, and one degree lower than the original equation.

Every binomial equation is also a recurring equation.

Every recurring equation of an even degree above the second, can be depressed one half by the following artifice :

Take the equation,

$$x^4+5x^3+2x^2+5x+1=0.$$

Divide every term by the square root of the highest power, in this example by x^2 ; then

$$x^2+5x+2+\frac{5}{x}+\frac{1}{x^2}=0$$

Then $$\left(x^2+\frac{1}{x^2}\right)+5\left(x+\frac{1}{x}\right)+2=0.$$

Put $x+\frac{1}{x}=y$; then $x^2+2+\frac{1}{x^2}=y^2$.

Whence, $y^2+5y=0$, an equation of only half the degree of the given equation.

This can be verified by $y=0$, or $y=-5$.

Therefore, $x+\frac{1}{x}=0$, or $x+\frac{1}{x}=-5$.

Whence, $x=\pm\sqrt{-1}$, or $x=\frac{1}{2}(-5\pm\sqrt{21})$

Find all the roots of the equation $x^5-1=0$.

One is obviously one root, therefore divide by $x-1=0$.

Then $x^4+x^3+x^2+x+1=0$. Divide by x^2.

$$x^2+x+1+\frac{1}{x}+\frac{1}{x^2}=0$$

$$\left(x^2+\frac{1}{x^2}\right)+\left(x+\frac{1}{x}\right)+1=0$$

Put $x+\frac{1}{x}=y$; then $x^2+2+\frac{1}{x^2}=y^2$

Whence, $y^2+y-1=0$.

$$y=-\tfrac{1}{2}+\tfrac{1}{2}\sqrt{5}, \text{ or } y=-\tfrac{1}{2}-\tfrac{1}{2}\sqrt{5}.$$

Put $2a=-\frac{1}{2}+\frac{1}{2}\sqrt{5}$; then $-(1+2a)=-\frac{1}{2}-\frac{1}{2}\sqrt{5}$.

Now $x+\frac{1}{x}=2a$, and $x+\frac{1}{x}=-(1+2a)$

From the first, $x=a+\sqrt{a^2-1}$, or $x=a-\sqrt{a^2-1}$

That is, $x=\frac{1}{4}(\sqrt{5}-1+\sqrt{-10-2\sqrt{5}})$

Or, $x=\frac{1}{4}(\sqrt{5}-1-\sqrt{-10-2\sqrt{5}})$

Taking the second,

$$x=-\tfrac{1}{4}(\sqrt{5}+1-\sqrt{-10+2\sqrt{5}})$$

Or, $x=-\frac{1}{4}(\sqrt{5}+1+\sqrt{-10+2\sqrt{5}})$

Given $(x^3+1)(x^2+1)(x+1)=30x^3$, to find the values of x.

Multiply as indicated, the product is

$$x^6+x^5+x^4+2x^3+x^2+x+1=30x^3$$

Dividing by x^3, and

$$x^3+x^2+x+2+\frac{1}{x}+\frac{1}{x^2}+\frac{1}{x^3}=30$$

Or, $\left(x^3+\frac{1}{x^3}\right)+\left(x+\frac{1}{x}\right)^2+\left(x+\frac{1}{x}\right)=30$

Put $\left(x+\frac{1}{x}\right)=y$. Then $\left(x^3+\frac{1}{x^3}\right)+3\left(x+\frac{1}{x}\right)=y^3$

Whence, $y^3-3y+y^2+y=30$

Or, $y^3+y^2-2y=30.$

The first attempt at solving this by Horner's method shows us that r is 3 exactly; that is, $y=3$.

Then $x+\frac{1}{x}=3$, and $x=\frac{1}{2}(3\pm\sqrt{5}.)$

The other values of y are imaginary.

Given $(x+y)(xy+1)=18xy$ (1)

and $(x^2+y^2)(x^2y^2+1)=208x^2y^2$ (2)

to find the values of x and y.

Multiply as indicated, then

$$x^2y+xy^2+x+y=18xy \quad (3)$$

$$x^4y^2+x^2y^4+x^2+y^2=208x^2y^2 \quad (4)$$

Divide (3) by xy, and (4) by x^2y^2, then

$$x+y+\frac{1}{y}+\frac{1}{x}=18 \quad (5)$$

And $$x^2+y^2+\frac{1}{y^2}+\frac{1}{x^2}=208 \quad (6)$$

Now put $P=\left(x+\frac{1}{x}\right)$, and $Q=\left(y+\frac{1}{y}\right)$.

Then $$P+Q=18 \quad (7)$$

And $$P^2+Q^2=212 \quad (8)$$

From (7), $$P^2+2PQ+Q^2=324 \quad (9)$$

$$2PQ=112 \quad (10)$$

(10) from (8) gives

$$P^2-2PQ+Q^2=100$$

$$P-Q=\pm10$$

Whence, $P=14$ or 4, and $Q=4$ or 14.

That is, $x+\frac{1}{x}=14$, or 4. $y+\frac{1}{y}=4$, or 14.

Whence, $x=(7\pm4\sqrt{3})$ or $(2\pm\sqrt{3})$. $y=(7\mp4\sqrt{3})$ or $(2\mp\sqrt{3})$.

We conclude this subject by the following equations:

Given $5x^5+11x^4-88x^3-88x^2+11x+5=0$, to find its roots.

This equation necessarily has one root equal to -1; therefore we divide by $(x+1)$, and we find

$$5x^4+6x^3-94x^2+6x+5=0.$$

Now $$5x^2+6x-94+\frac{6}{x}+\frac{5}{x^2}=0.$$

Or, $$5\left(x^2+\frac{1}{x^2}\right)+6\left(x+\frac{1}{x}\right)=94.$$

If we put $x+\frac{1}{x}=y$, then $x^2+\frac{1}{x^2}=y^2-2$.

And $$5(y^2-2)+6y=94.$$

$$y^2-2+\frac{6y}{5}=\frac{94}{5}.$$

$$y^2+\frac{6y}{5}=\frac{104}{5}.$$

$$y+\tfrac{3}{5}=\pm\tfrac{23}{5}.\quad y=4,\text{ or }-\tfrac{26}{5}.$$

$$x+\frac{1}{x}=4.\quad \text{Whence } x=2\pm\sqrt{3}.$$

Or, $x+\frac{1}{x}=-\frac{26}{5}$, whence $x=\frac{1}{5}$, or -5; and the five roots are $-1, -5, \frac{1}{5}, (2+\sqrt{3})$, and $(2-\sqrt{3}.)$

Given $\left\{\begin{array}{l}(x^6+1)y=(y^2+1)x^3\\(y^6+1)x=9(x^2+1)y^3\end{array}\right\}$ to find at least one of the values of x and y.

By division, the equations may be reduced to

$$x^3+\frac{1}{x^3}=y+\frac{1}{y} \qquad (1)$$

And $$y^3+\frac{1}{y^3}=9\left(x+\frac{1}{x}\right). \qquad (2)$$

Now put $x+\frac{1}{x}=P$, and $y+\frac{1}{y}=Q$.

Then $x^3+\frac{1}{x^3}=P^3-3P$, and $y^3+\frac{1}{y^3}=Q^3-3Q$.

Substituting these results, and (1) and (2) become

$$P^3-3P=Q \qquad (3)$$

$$Q^3-3Q=9P \qquad (4)$$

Assume $P=nQ$; then (3) and (4) become

$$n^3Q^3-3nQ=Q$$

And $$Q^3-3Q=9nQ$$

Dividing by Q,

$$n^3Q^2-3n=1 \qquad (5)$$

And $$Q^2-3=9n \qquad (6)$$

From (5), $$n^3Q^2=3n+1 \qquad (7)$$

From (6), $$n^3Q^2=(3n+1)3n^3 \qquad (8)$$

Dividing (8) by (7), gives $3n^3=1$.

Whence, $9n=3(9)^{\frac{1}{3}}$; and substituting this in (6), and we have $Q^2=3(9)^{\frac{1}{3}}+3$. Or, $Q=\sqrt{3(9)^{\frac{1}{3}}+3}$.

Having the value of n and Q, we have $P=nQ$. The values of Q and P will give us x and y.

$$Ans.\ x=\tfrac{1}{2}[(\sqrt[3]{3}+3)^{\frac{1}{2}}+(\sqrt[3]{3}-1)^{\frac{1}{2}}]$$

$$y=\tfrac{1}{2}[(\sqrt[3]{3}\cdot\sqrt{\sqrt[3]{3}+3}\pm\sqrt{\sqrt[3]{9}-1}]$$

SECTION VII.

INDETERMINATE ANALYSIS.

Preliminary to this subject it is proper to call to mind a few facts in the theory of numbers; for the Indeterminate and Diophantine analysis is but an application of that theory.

(1.) The sum of any number of even numbers is even.

(2.) The sum of any even number of odd numbers is even.

(3.) The sum of an even and an odd number is odd.

(4.) The product of any number of factors, one of which is even, will be an even number; but the product of any odd number is odd; hence,

(5.) Every power of an even number is even, and every power of an odd number is an odd number; hence,

(6.) The sum and difference of any power and its root is an even number. For the power and its root will be both even, or both odd, and the sum or difference of either, in either case, is an even number.

PROPOSITIONS.

1. *If an odd number divide an even number, it will divide the half of it.*

Every number is either odd or even—even numbers are in the form $2n$, and odd numbers are in the form $2n'+1$. Now by hypothesis, let

$$\frac{2n}{2n'+1}=q, \text{ and } q \text{ a whole number.}$$

Then $$2n=q(2n'+1)$$

It is obvious that one factor in the second member is odd, therefore the other factor q, must be even, otherwise the product $2n$ could not be even; hence, q may be expressed by $2q'$, and $2n=2q'(2n'+1)$. Then $\frac{n}{2n'+1}=q'$, and the odd number $(2n'+1)$ divides half the even number $2n$, which was to be demonstrated.

2. *If a number* p *divide each of two numbers* a *and* b, *it will divide their sum and difference.*

$$\frac{a}{p}=q. \quad \frac{b}{p}=q'.$$

That is q and q', the quotients, are whole numbers by hypothesis. Now by addition we have $\frac{a+b}{p}=q+q'$, and by subtraction, $\frac{a-b}{p}=q-q'$.

But the sum of two whole numbers is a whole number; therefore, $\left(\frac{a+b}{p}\right)$ is a whole number: and it is obvious that $\left(\frac{a-b}{p}\right)$ is also a whole number. Q. E. D.

3. *If two numbers are prime to each other, their sum is prime to each of them.*

Let a and b be the two numbers prime to each other, $(a+b)$ their sum, is prime to each of them.

For by the last proposition if $(a+b)$ and a have a common divisor, their difference b must have the same divisor; but b is not divisible by a by the supposition; therefore, if two numbers, &c. Q. E. D.

Corollary. In the same manner it may be demonstrated that if a and b be prime to each other, their difference $(a-b)$ will be prime to each of them.

4. *If two numbers are prime to each other, their sum and difference will have the common measure 2, but no other, or their sum and difference will be prime to each other.*

Let a and b be prime to each other, their sum is $(a+b)$, and difference, $(a-b)$; and if these numbers have a common divisor, their sum $2a$ and difference $2b$ will have the same; but the only common divisor to $2a$ and $2b$ is 2.

If one of these numbers, a or b, be even, and the other odd, then $(a+b)$ and $(a-b)$ are both odd and prime to each other. For if $(a+b)$ and $(a-b)$ are not prime, let them have the common measure n. Then by proposition 2, n will be the common measure of their sum and difference; that is, the common measure of $2a$ and $2b$; but the only common measure of these numbers is 2; therefore, $(a+b)$ and $(a-b)$ are prime to each other, or have the common measure 2. Q. E. D.

5. *If two numbers* a *and* b *be prime to each other,* b *being the greater, then* b *may always be represented by the formula* $b=aq+r$, *in which* r *is less than* a, *and prime to it.*

The formula arises from the actual division of b by a, the result is q, the integer quotient, and the remainder, r; that is,

$$\frac{b}{a}=q+\frac{r}{a}.$$

Multiplying this equation by a, gives the formula; r is necessarily less than a, if we suppose q to be the greatest quotient.

We are now to show that r and a are prime to each other. If they are not prime to each other, they have a common measure.

Let us suppose a common measure and reduce the fraction $\frac{r}{a}$ by it, giving $\frac{r'}{a'}$; then the equation becomes

$$\frac{b}{a}=q+\frac{r'}{a'}=\frac{qa'+r'}{a'}.$$

But by hypothesis the fraction $\frac{b}{a}$ is *irreducible.* Yet admitting a common measure to $\frac{r}{a}$, we have the reduced fraction $\frac{qa'+r'}{a'}$ which is absurd; therefore, a commom measure to $\frac{r}{a}$ is inadmissible, and r and a are prime to each other.

6. *If any number be prime to each of two others, it will be prime to their product.*

Let c be prime to both a and b; then we are to show that c is also prime to ab.

By the hypothesis $\frac{a}{c}$ is an irreducible fraction; multiply this fraction by b, and we shall have $\frac{ab}{c}$.

Now if this last fraction is reducible, some common measure must exist between b and c; but by hypothesis there is none; therefore, ab and c are prime to each other. Q. E. D.

Corollary 1. If $a=b$, then $\frac{ab}{c}=\frac{a^2}{c}$; and if c is prime to a, it is prime to a^2, a^3, and any power of a.

Corollary 2. If c is prime to any number of factors as a, b, d, e, &c. it will be prime to their product.

7. *If two numbers,* a *and* b, *be prime to each other, then* mb *divided by* a, *and* m′b *divided by* a, *will have different remainders for all values of* m *less than* a.

Let us admit that the two operations in division will produce the same remainder,—then by performing the division we shall have

$$\frac{mb}{a}=q+\frac{r}{a}$$

And $$\frac{m'b}{a}=q'+\frac{r}{a}$$

By subtraction, $$\frac{b}{a}\left(m-m'\right)=q-q'$$

Or, $$\frac{b}{a}=\frac{q-q'}{m-m'}$$

But by hypothesis, $\frac{b}{a}$ is irreducible, at the same time $(m-m')$ is less than a, which is absurd; therefore, the two divisions cannot have the same remainder.

8. *If two numbers,* a *and* b, *be prime to each other, the equation* ax—by=±1, *is always possible in integers. That is, positive integral values of* x *and* y *may be found which will satisfy it.*

By the preceding proposition $\frac{mb}{a}=q+\frac{r}{a}$, and as r may be of any value less than a, according to the magnitude of m, $r=a-1$ is possible. Whence,

$$mb=aq+a-1$$

Or, $$mb+1=(q+1)a$$

Now let $y=m$, and $x=(q+1)$; then

$$by+1=ax$$

Or, $$ax-by=1$$

But m is a whole number, therefore its equal y is a whole number, and $(q+1)$ is a whole number, and consequently its equal x is a whole number; that is, $ax-by=1$ is possible, x and y being whole numbers.

We now come more directly to the indeterminate analysis.

For a perfect and definite solution of a problem, there must be as many independent equations as unknown quantities to be determined; and when this is not the case, the problem is said to be indeterminate.

For instance, $x+y=20$. Here x may have any value whatever, and the equation will give the corresponding value to y, and the number of solutions may be infinite; but if we restrict the values of x and y to whole numbers, then only 19 different solutions can be found; for x may be equal to 1, or 2, or 3, &c., to 19, and y will equal the remaining part of 20.

In some cases, the number of solutions is unlimited or infinite. $ax-by=c$, represents a general equation of the kind, and a solution gives $x=\frac{c+by}{a}$ in which y may be any whole number whatever that will make $(c+by)=a$, or any multiple of a; but numberless such values of y may be found, and consequently numberless values of x.

N. B. Such equations are generally restricted to the least values of x and y.

Equations in the form $ax+by=c$, may be very limited in the number of their solutions,—may have only one solution, or a solution may be impossible, when x and y are restricted to whole numbers.

A solution gives $x=\frac{c-by}{a}$.

Now if c is very large, and b and a small, y may take a great number of integral values, before the numerator becomes so small as not to be divisible by a.

If we make $y=1$, and then find that $\left(\frac{c-b}{a}\right)$ is a proper fraction, a solution is impossible in integers.

The equation $ax+by=c$ is always possible in integers, when c *is greater* than $(ab-a-b)$, and a and b prime to each other.

The equation $ax+by=c$, is possible *sometimes*, or rather with *some numbers*, when c is less than $(ab-a-b)$. For example, $7x+13y=71$, is impossible in integers, because $(7\cdot 13-20)$ is not greater than 71. But $7x+13y=27$, is possible, giving $x=2$, and $y=1$. That is, $7x+13y$ put equal to any whole number greater than 71, will admit of a solution in integers; and put equal *some numbers* less than 71, will admit of a solution.

In all these equations a and b are supposed to be prime to each other. If they have a common divisor, that same divisor must divide c, or a solution is impossible in integer numbers.

For if $ax+by=\frac{c}{n}$, it is obvious that ax is a whole number, also by is a whole number, and the sum of two whole numbers can never be equal to an irreducible fraction, as the preceding indicates.

For a particular example, $6x+9y=32$, is impossible in whole numbers. Dividing by 3, and $2x+3y=\frac{32}{3}$. As x is to be a whole number, $2x$ must be a whole number, and $3y$ must also be a whole number; but *it is impossible* for any two whole numbers to be equal to $\frac{32}{3}$.

In cases where solutions are possible, our rules of operation rest entirely on the following facts:

1st. *A whole number added to a whole number, the sum is a whole number.*

2d. *A whole number taken from a whole number, the remainder is a whole number.*

3d. *Multiply a whole number by a whole number, and the product is a whole number.*

EXAMPLES.

(1.) Given $3x+5y=35$, to find the values of x and y in whole numbers.

$$x=\frac{35-5y}{3}.$$

Because x must be a whole number, the fractional form $\frac{35-5y}{3}$ must be a whole number, or $(11-y)+\frac{2-2y}{3}$ must be a whole number. But $(11-y)$ is obviously a whole number; therefore, the other part, $\left(\frac{2-2y}{3}\right)$ must be a whole number also.

Again, as y is a whole number, $\frac{3y}{3}$ is in fact a whole number, which added to $\left(\frac{2-2y}{3}\right)$, and the sum is $\frac{2+y}{3}$, a whole number.

In this last expression the coefficient of y is reduced to 1 understood, *and the operation* had that end in view.

Put this last expression equal n; that is *any whole number*, or rather *some whole number*.

$$\frac{2+y}{3}=n.$$

Or, $$y=3n-2.$$

In this last equation we can take $n=0$, 1, 2, 3, &c., as far as such substitutions will correspond to the values of x.

If we take $n=0$, then $y=-2$, an inadmissible result; for we demanded positive values. Therefore, we take $n=1$, then $y=1$, and $x=10$. If $n=2$, then $y=4$, and $x=3$. If $n=3$, $y=7$, and $x=0$. Hence the last is not a full solution, and the equation only admits of two solutions; namely, $x=10$ or 3, and $y=1$ or 4.

(2.) Given $35x-24y=68$, to find the *least* values of x and y in whole numbers.

We require the least values, because an unlimited number of solutions may be found.

From the equation, $x=\frac{68+24y}{35}=1+\frac{33+24y}{35}$.

Hence, $\frac{33+24y}{35}=$ some whole number: but $\frac{35y}{35}=$ some whole number; therefore, by taking the difference of these two whole numbers we have $\frac{11y-33}{35}=$ some whole number.

Three times a whole number is a whole number; therefore, $\frac{33y-99}{35}=\frac{33y-29}{35}-2=$ some whole number.*

$$\frac{33y-29}{35}=wh. \quad \text{Also, } \frac{35y}{35}=wh.$$

*Subsequently we shall put *wh* to represent the phrase, some whole number.

Whence, $\frac{35y}{35}-\frac{33y-29}{35}=\frac{2y+29}{35}=wh.$

$$\left(\frac{2y+29}{35}\right)18=y+14+\frac{y+32}{35}=wh.$$

Having thus deprived y of its numerical coefficient, we may put $\frac{y+32}{35}=n$. Whence, $y=35n-32$.

Taking $n=1$. $y=3$, and $x=\frac{68+24\cdot 3}{35}=4$, the least possible values of x and y in integers, as was required.

The next values are $x=28$, and $y=38$.

(3.) *A man proposed to lay out* \$500 *for cows and sheep; the cows at the rate of* \$17 *per head, and the sheep at* \$2. *How many of each could he purchase?*

Let $x=$ the number of cows, and $y=$ the number of sheep.

Then $17x+2y=500$, is the only equation that can be obtained, and x and y must be whole numbers by the nature of the problem—he could not purchase a part of a cow, or a part of a sheep.

To find the least number of cows, transpose $17x$.

Then $y=250-8x-\frac{x}{2}$.

Now as x and y must be whole numbers, $\frac{x}{2}$ must be a whole number, and the least number corresponding to x must be 2; corresponding to this, $y=233$.

Under all suppositions, the number of cows must be divisible by 2.

Now if the object was to purchase as many cows and as few sheep as possible, we would transpose the other term thus:

$$x=\frac{500-2y}{17}=29+\frac{7-2y}{17}$$

Whence, $\frac{7-2y}{17}=wh.$ Or, $\frac{56-16y}{17}=wh$; to this add $\frac{17y}{17}$, and $\frac{56+y}{17}=3+\frac{5+y}{17}$

Therefore $\frac{5+y}{17}=n$, or $y=17n-5$.

Put $n=1$, then $y=12$, the smallest number of sheep. The corresponding value of $x=28$.

The number of cows may be any one of the even numbers from 2 to 28.

(4.) *A man wished to spend* 100 *dollars in cows, sheep, and geese; cows at* 10 *dollars a piece, sheep at* 2 *dollars, and geese at* 25 *cents, and the aggregate number of animals to be* 100. *How many must he purchase of each?*

Let $x=$ the number of cows, y the sheep, and z the geese.

Then $$10x+2y+\frac{z}{4}=100. \quad (1)$$

And $$x+y+z=100. \quad (2)$$

Clear equation (1) of fractions, and

$$\begin{array}{r} 40x+8y+z=400. \\ x+\ y+z=100. \\ \hline 39x+7y\quad=300. \end{array}$$

$$x=\frac{300-7y}{39}=7+\frac{27-7y}{39}=\text{ a whole number.}$$

Or, $5\left(\frac{27-7y}{39}\right)=\frac{135-35y}{39}=$ a whole number; add $\frac{39y}{39}$ and $\frac{4y+135}{39}$, or $\frac{40y+1350}{39}=y+34+\frac{y+24}{39}=$ a whole number.

Therefore, $\frac{y+24}{39}=p$. Or, $y=39p-24=15$.

This value of y, gives $x=5$. Hence, $z=80$.

If we take $p=2$, we shall have $y=54$: then x will come a minus quantity, an inadmissible circumstance in any problem like this. Therefore, 5 cows, 15 sheep, and 80 geese, is the only solution.

(5.) *A person spent* 28 *shillings in ducks and geese; for the geese he paid* 4*s.* 4*d. a piece, and for the ducks,* 2*s.* 6*d. a piece. What number had he of each?*

Let $x=$ the number of geese, and y the number of ducks.

Then $52x+30y=28\cdot 12$. Or, $26x+15y=168$.

$$y=11-x+\frac{3-11x}{15}$$

Whence, $\frac{3-11x}{15}=wh.$ But $\frac{15x}{15}=wh.$

By addition, $\left(\frac{4x+3}{15}\right)4=wh.$ $\frac{16x+12}{15}=x+\frac{x+12}{15}$

$\frac{x+12}{15}=n.$ $x=15n-12.$ $x=3$, when $n=1$.

Then $y=8-2=6$. When $n=2$, $x=18$, and $y=-20$. But this is inadmissible; therefore, $x=3$ and $y=6$, is the only possible solution.

(6.) *Divide the number* 100 *into two such parts that one of them may be divisible by* 7, *the other by* 11.

Let $7x=$ one part, and $11y$ the other part.

Then $7x+11y=100.$

$$y=\frac{100-7x}{11}=9+\frac{1-7x}{11}$$

$\frac{1-7x}{11}+\frac{11x}{11}=\frac{4x+1}{11}=wh.$ $\frac{12x+3}{11}=x+\frac{x+3}{11}=wh.$

$$\frac{x+3}{11}=n. \quad x=11n-3.$$

Now put $n=1$, then $x=8$ and $y=4$. 56 and 44 are, therefore, the required parts.

(7.) *Find the least number which being divided by* 6 *shall leave the remainder* 3, *and being divided by* 13 *shall leave the remainder* 2.

Let $N=$ the number sought, and x and y the quotients arising from the divisions.

Then $\frac{N-3}{6}=x$, and $\frac{N-2}{13}=y$

Whence $N=6x+3$, and $N=13y+2$.

Consequently, $6x+1=13y$.

$$x=\frac{13y-1}{6}=2y+\frac{y-1}{6}$$

Then $$\frac{y-1}{6}=n, \text{ or } y=6n+1$$

For the least values of y we must take $n=0$, then $y=1$ and $x=2$. But $N=6x+3=15$.

We may determine N more directly without the aid of x and y, thus:

$$\frac{N-3}{6}=\text{ some whole number; also, } \frac{N-2}{13}=wh.$$

As N does not contain a coefficient to be *worked off*, we may put $\frac{N-3}{6}=p$, and $N=6p+3$ in which any integral value put for p will satisfy the first whole number, but the other must be satisfied also; then put the value of N in $\frac{N-2}{13}$; that is,

$$\frac{6p+3-2}{13}=\text{ some whole number.}$$

$$\frac{6p+1}{13}=wh. \quad \frac{12p+2}{13}=wh. \quad \frac{13p}{13}-\frac{12p+2}{13}=\frac{p-2}{13}=q.$$

Whence, $p=13q+2$. Assume $q=0$, then $p=2$ and $N=6p+3=15$, as before.

(8.) *What number is that which being divided by* 11, *leaves a remainder of* 3, *divided by* 19, *leaves a remainder of* 5, *and divided by* 29, *shall leave a remainder of* 10?

Let N be the required number, and x, y, and z the several quotients, and of course they must be whole numbers.

Then $11x+3=N$, and $19y+5=N$, and $29z+10=N$.

Hence, $x=\frac{29z+7}{11}$, and $x=\frac{19y+2}{11}$. $19y=29z+5$

$$y=\frac{29z+5}{19}=z+\frac{10z+5}{19}.$$

$$\frac{20z+10}{19}=wh, \text{ or } \frac{z+10}{19}=n. \quad z=19n-10.$$

Any integer written in place of n will give integer values to z and y, but x, or $\frac{29z+7}{11}$ must be a whole number also.

Hence such a value of n must be found as will make

$$\frac{29(19n-10)+7}{11}=wh.$$

Or, $\frac{551n-290+7}{11}=\frac{551n-283}{11}=50n-25+\frac{n-8}{11}=wh.$

Whence, $\frac{n-8}{11}=p.\quad n=11p+8.$

For the least value of n put $p=0$; then $n=8$.

But $z=19n-10=19\cdot8-10=152-10=142$.

And $N=29\cdot142+10=4128$, the number required.

(9.) *Required the least number that can be divided by each of the nine digits, without remainders.*

Let $x=$ the number.

Then $\frac{x}{2}, \frac{x}{3}, \frac{x}{4}, \frac{x}{5}, \frac{x}{6}, \frac{x}{7}, \frac{x}{8}, \frac{x}{9}$, must all be whole numbers.

Now if we make $\frac{x}{8}$ a whole number, $\frac{x}{4}$ and $\frac{x}{2}$, the double and quadruple, will be whole numbers of course. Also, if $\frac{x}{9}$ is a whole number, $\frac{x}{3}$ *will be* a whole number.

Therefore, we have only to find such a value of x as will make $\frac{x}{9}, \frac{x}{8}, \frac{x}{7}, \frac{x}{6}, \frac{x}{5}$ whole numbers. $\frac{x}{6}$, may also be cast out, on consideration that $6=2\cdot3$, and $2\cdot3$ are factors, one of 9, the other of 8, in the preceding expressions.

Hence we have only to find such a value of x as will make each of the fractions $\frac{x}{9}, \frac{x}{8}, \frac{x}{7}$, and $\frac{x}{5}$ whole numbers; and as these denominators contain no common factors, their product is the least number that will answer the condition.

Whence, $x=72\cdot35=2520$.

(10.) *A market woman has some eggs, which when counted out by twos, threes, fours, and fives, still left one; but when counted by sevens, there was none left. What was the least possible number of eggs she could have had?* *Ans.* 301.

This problem requires $\frac{x-1}{2}, \frac{x-1}{3}, \frac{x-1}{4}, \frac{x-1}{5}$, and $\frac{x}{7}$ to be whole numbers.

If $\left(\frac{x-1}{4}\right)$ is a whole number, $\left(\frac{x-1}{2}\right)$, its double, must be a whole number of course; hence, we have only to make

$$\frac{x-1}{3},\ \frac{x-1}{4},\ \frac{x-1}{5},\ \text{and}\ \frac{x}{7}\ \text{whole numbers.}$$

Put the first expression equal to the whole number n; then $x=3n+1$. This will satisfy the first expression; but the others must be satisfied also; therefore, substitute $(3n+1)$ for x in them.

Then $\frac{3n}{4}$, $\frac{3n}{5}$, and $\frac{3n+1}{7}$, must be whole numbers.

If $\frac{3n}{5}=wh.$ $\frac{6n}{5}=n+\frac{n}{5}=wh$; or $\frac{n}{5}=wh=p$.

That is, $n=5p$; and substituting this value of n in the other two expressions, we have

$$\frac{15p}{4},\ \text{and}\ \frac{15p+1}{7},\ \text{to be made whole numbers.}$$

$\frac{15p+1}{7}=2p+\frac{p+1}{7}=wh.$ Whence, put $\frac{p+1}{7}=q$. $p=7q-1$.

Finally, this value of p put in $\frac{15p}{4}$, gives

$$\frac{15\cdot 7q-15}{4},\ \text{which must be made a whole number}$$

before we can be sure of a result which will satisfy all the conditions.

$$\frac{105q-15}{4}=(26q-3)+\frac{q-3}{4}=wh.$$

Whence, $\frac{q-3}{4}=t$. $q=4t+3$.

For the least value of q, put $t=0$; then $q=3$.

But $p=7q-1=20$. $n=5p$. Then $n=100$. $x=3n+1=301$.

(11.) *Required the year of the Christian Era in which the solar cycle was, or will be* 15, *the lunar cycle* 12, *and the Roman Indiction* 12.

N. B. The operator must of course know the exact import of these terms, and the facts in the case, before he can be required to solve the problem.

The solar cycle is a period of 28 years, at the expiration of which, the days of the week return to the same days of the month, (provided a common centurial year has not intervened.) The first year of the Christian Era was the tenth of this cycle; therefore, we must add 9 to the year and divide the sum by 28, and the remainder will be the number of the cycle.

The lunar cycle, or Golden number, as it is sometimes called, is a period of 19 years, after which the eclipses return in the same order as in the previous 19 years. The first year of the Christian Era was the second of this period; therefore, we must add 1 to the year and divide by 19, and the remainder is the year of the lunar cycle.

The Roman Indiction is not astronomic. It is a period of 15 years, the first of our Era being the 4th of the Indiction; therefore, add 3 to the year and divide by 15, and the remainder is the Indiction.

The reader is now prepared to solve this or any other similar problem.

Let x represent the required year; then the problem requires that

$$\frac{x+9-15}{28}, \quad \frac{x+1-12}{19}, \quad \frac{x+3-12}{15}$$

should be whole numbers; that is, $\frac{x-6}{28}$, $\frac{x-11}{19}$, $\frac{x-9}{15}$, must be whole numbers.

The first expression will be satisfied by putting it equal to any whole number n; then $x=28n+6$.

But the other two expressions must be satisfied also.

Therefore $\frac{28n+6-11}{19}$ and $\frac{28n+6-9}{15}$

Or, $\frac{28n-5}{19}$ and $\frac{28n-3}{15}$ must be whole numbers.

Or, $\frac{9n-5}{19}$ and $\frac{13n-3}{15}$ must be whole numbers.

$$\frac{15n}{15}-\frac{13n-3}{15}=\frac{2n+3}{15}=wh. \quad \frac{7(2n+3)}{15}=wh.$$

$$\frac{15n}{15}-\frac{14n+21}{15}=wh. \quad \frac{n-21}{15}=p.$$

Whence, $n=15p+21$. Every expression is now satisfied, except $\frac{9n-5}{19}$; to satisfy this, write in the value of n; then

$$\frac{9(15p+21)-5}{19}=wh.$$

Or,
$$\frac{135p+184}{19}=wh.=7p+9+\frac{2p+13}{19}.$$

Whence, $\frac{2p+13}{19}=wh.$, and $\frac{20p+130}{19}=p+6+\frac{p+16}{19}$

$$\frac{p+16}{19}=q. \quad p=19q-16$$

We cannot take q less than 1. The least value of p will then be 3. But $n=15p+21=66$.

$$x=28n+6=28\cdot 66+6=1854.$$

If no interruption was to be made by the centurial years, the coincidence of these cycles would not occur again until the year 9834, which we find by making $q=2$.

SECTION VIII.

TO DETERMINE THE NUMBER OF SOLUTIONS AN EQUATION IN THE FORM $AX+BY=C$ WILL ADMIT OF.

An equation in the form $ax-by=c$, will admit of an unlimited number of solutions, because x and y can increase together; but not so with equations in the form $ax+by=c$; for in them an increase of x will cause a decrease of y, and an increase of y, a decrease of x; but neither x nor y are permitted to fall below 1. If c is very large in relation to a and b, the equation may have a great number of solutions, and we are now about to show a summary method of determining the solutions in any given case.

The equation $ax'-by'=1$ is always possible in whole numbers, (Prop. 8, sec. vii;) therefore, c times the equation is also possible; that is,

$$acx'-cby'=c, \text{ is possible.}$$

But $\qquad ax+by=c$, is a general equation.

Put these two values of c equal to each other, then

$$ax+by=acx'-cby' \qquad (1)$$

From (1), $$x=cx'-\left(\frac{cy'+y}{a}\right)b. \qquad (2)$$

For the sake of perspicuity, put $\frac{cy'+y}{a}=m$.

Then (2), becomes $$x=cx'-bm \qquad (3)$$

Multiply (3) by a, and substitute the value of ax in (1),

Then $$acx'-abm+by=acx'-cby'$$

Whence, $$y=am-cy' \qquad (4)$$

From (3), we find $$m=\frac{cx'}{b}-\frac{x}{b} \qquad (5)$$

From (4), we find $$m=\frac{cy'}{a}+\frac{y}{a} \qquad (6)$$

Equations (5) and (6) show that m must be *greater* than $\frac{cy'}{a}$ and at the same time *less* than $\frac{cx'}{b}$.

Therefore, *the limits to* m *are found.*

Now let us observe equations (3) and (4); x must be a whole number, and as c, x', and b are whole numbers, m must be taken in whole numbers, and it may be any whole number between $\frac{cy'}{a}$ and $\frac{cx'}{b}$.

The number of solutions will, therefore, correspond with the difference between the integral parts of the fractions $\frac{cx'}{b}$ and $\frac{cy'}{a}$ except when $\frac{cx'}{b}$ is a whole number, in that case x' becomes b, and $\frac{b}{b}$ must be considered a fraction, and rejected. If, however, we intend to include 0 among the integral values, *this precaution need not be observed.*

EXAMPLES.

(1.) Required the number of integral solutions of the equation,

$$7x+9y=100.$$

First find the least value of x' and y' in the equation

$$7x'-9y'=1.$$

The result will be $x'=4$. $y'=3$.

Then $\frac{cx'}{b}=\frac{100\cdot 4}{9}=44\frac{4}{9}$. $\frac{cy'}{a}=\frac{100\cdot 3}{7}=42\frac{6}{7}$.

Disregarding the fractions, the difference of the *integral parts is* 2, showing two integral solutions to the equation.

If we had taken the difference between $\frac{400}{9}$ and $\frac{300}{7}$, thus; $\frac{2800}{63}-\frac{2700}{63}=1\frac{37}{63}$, we might have come to the conclusion that the equation would admit of only one integral solution.

The *integral difference* in this case is not the *difference of the integrals.*

When the fractional part of $\frac{cx'}{b}$ is not less than the fractional part of $\frac{cy'}{a}$, but equal or greater than it, we can find the number of solutions by taking the difference, thus $\frac{cx'}{b}-\frac{cy'}{a}=\frac{c(ax'-by')}{ab}$

But $ax'-by'=1$; therefore, $\frac{cx'}{b}-\frac{y'}{a}=\frac{c}{ab}$.

Whence we conclude that $\frac{c}{ab}$ will in this case show the number of solutions.—In all cases it will be the number, *or one less.*

(2.) What number of integral solutions will the equation $9x+13y=2000$ admit of? *Ans.* 17.

$$9\cdot 13=117)2000(17.$$

That is, we are sure of 17 different solutions, and there may be 18.

The equation $5x+9y=40$ admits of no solution in whole numbers. c, 40, is not divisible by $5\cdot 9=45$, that is, no unit in the quotient. Yet the equation

$5x+9y=37$, admits of one solution.

The auxiliary equation $5x'-9y'=1$, gives $x'=2$, and $y'=1$.

Therefore, $\frac{cx'}{b}=\frac{74}{9}=8\frac{2}{9}$. $\frac{cy'}{a}=\frac{37}{5}=7\frac{2}{5}$.

Here the difference of the integrals is 1, indicating one solution. In fact $x=2$, $y=3$.

How many solutions will the equation $2x+5y=40$ admit of?

The auxiliary equation, $2x'-5y'=1$, gives $x'=3$, $y'=1$.

$$\frac{cx'}{b}=24. \quad \frac{cy'}{a}=20. \quad \text{Or, 4 solutions.}$$

But observe that $\frac{cx'}{b}$ in this case, is a complete integral, 24; according to previous considerations, we must deduct one, and the number of solutions are 3, as follows: $x=5.\ 10.\ 15.$ $y=6.\ 4.\ 2$, and no other solution can be found.

(3.) What number of solutions in whole numbers can be found for the equation $3x+5y+7z=100$.

As x and y each cannot be less than one, z cannot be greater than $\frac{100-3-5}{7}=13\frac{1}{7}$. That is, z cannot be greater than 13, in whole numbers. Now suppose $z=1$, and the equation becomes $3x+5y=93$.

The number of solutions for this equation, found as previously directed, is 6. That is $\begin{cases} x = 26.\ 21.\ 16.\ 11.\ 6.\ 1. \\ y = 3.\ 6.\ 9.\ 12.\ 15.\ 18. \end{cases}$

Now x and y can make these six changes, and z be constantly equal to 1, *and satisfy the primitive equation.*

We may observe here that x diminishes from one solution to another by the coefficient of y, and y increases by the coefficient of x, *but this is not a general principle.*

Take $z=2$, and the equation becomes $3x+5y=86$.

This equation has also 6 solutions, z being through all the changes of x and y equal to 2.

Now take $z=3$, then the original equation is $3x+5y=79$. This equation has five solutions.

Now take $z=4$, then $3x+5y=72$. This equation has four solutions.

Take $z=5$, then $3x+5y=65$. This equation has four solutions.

Take $z=6$, then $3x+5y=58$. This equation has four solutions.

In this manner, by taking z equal to all the integers up to 12 in succession, we find 41 solutions to the primitive equation.

(4.) Required some of the *integral solutions* to the equation

$$14x+19y+21z=252.$$

Here we observe that 14 and 21 and 252, are all divisible by 7, therefore y must be 7, or one of its multiples; suppose it 7, and divide the whole by 7.

Then $2x+19+3z=36$.

Or, $2x+3z=17$

Whence, x may equal 1, and $z=5$, or $x=7$ and $z=1$. Or, $x=4$ and $z=3$.

$$\text{Hence, } \begin{cases} x=7. & 4. & 1. \\ y=7. & 7. & 7. \\ z=1. & 3. & 5. \end{cases}$$

As these examples are of little practical utility, we give no more of them.

SECTION IX.

DIOPHANTINE ANALYSIS.

Diophantus was a Greek mathematician, who flourished in the early days of science: and the analysis that bears his name, mostly refers to squares and cubes.

The object of this analysis is to assign such values to the unknown quantities in any algebraic expression, as to render the whole a square or a cube, as may be required.

The first principles of this branch of science are very simple, but in their application, they expand into the region of impossibility.

To Euler and Lagrange, we are indebted for most that has appeared on this subject.

Case 1st. The most simple expression to be made a square, is of this form:

$$ax+b.$$

All we have to do, is to put this expression equal to *some* square, say n^2; then

$$x=\frac{n^2-b}{a}, \text{ and } n, a, \text{ and } b \text{ may be taken at pleasure.}$$

The result will give a value of x which will render $(ax+b)$, a square as required.

EXAMPLES.

(1.) *Three times a certain number increased by* 10, *is a square. What is the number?*

Here $a=3$, $b=10$, and $x=\frac{n^2-10}{3}$.

If we put $n=1$, then $x=-3$, and $ax+b=-9+10=1$, a square. If we put $n=2$, then $x=-2$, and $ax+b=4$, a square; and by taking different values for n, we can find as many squares as we please.

(2.) *Find such values of* x *as will render the following expressions squares:*

$$(9x+9),\ (7x+2),\ (3x-5),\ (2x-\tfrac{1}{2}).$$

All these expressions correspond to the general expression,

$$(ax+b.)$$

Case 2d. To advance another step, we require such values of x as will render any expression in the form

$$(ax^2+bx) \text{ a square.}$$

Because x is a factor in every term of the power, we will make it a factor in the root: that is, put the root equal nx; then

$$ax^2+bx=n^2x^2.$$
$$ax+b=n^2x.$$
$$x=\frac{b}{n^2-a}.$$

EXAMPLES.

(3.) *Six times the square of a certain number, added to five times the same number, is a square. What is the number?*

Ans. $x=\frac{5}{n^2-6}$, that is, the number is expressed by $\frac{5}{n^2-6}$ with the liberty to give any value to n that we please.

If we make $n=1$, then $x=-1$, and (ax^2+bx) will become 1, a square as required.

If we make $n=2$, then $x=\frac{5}{-2}=-\frac{5}{2}$, and $ax^2+bx=6.\frac{25}{4}-\frac{25}{2}=25$, a square as required, and many other results may be obtained.

(4.) *Find what values of* x *will render the following expressions squares:*

$$(5x^2-3x),\ (7x^2-15x),\ (12x^2-\tfrac{1}{2}x),\ (\tfrac{1}{2}x^2-3x.)$$

(5.) *Find such a number that if its square be divided by* 12, *and one-third of the number be taken from the quotient, the remainder will be a square number.*

Ans. 16 is one number, and there are many other numbers that will correspond to the conditions.

The practical utility of this analysis may be exemplified in forming examples in quadratic equations.

Thus $ax^2-bx=N$, is a quadratic, and we would assign such values to N, as will make the values of x commensurable quantities.

Completing the square, gives $x^2-\frac{b}{a}x+\frac{b^2}{4a^2}=N+\frac{b^2}{4a^2}$.

To make the values of x commensurable, it is necessary to make the expression $\left(N+\frac{b^2}{4a^2}\right)$ a perfect square, and this we can do by putting it equal to some definite square, (by case 1st a and b being known quantities,) and deducting the values of N.

Case 3d. Expressions in the form $(x^2\pm ax+b)$, can be made perfect squares, by putting them equal to the square of $(x-n.)$

This hypothesis assumes $(x-n)$ to be the square root of $(x^2\pm ax+b)$, and as this expression may be any number between zero and infinity, we now enquire whether it be possible that $(x-n)$ can always represent the root, whatever it may be. We reply, it can.

If x is large and n small, $(x-n)$ will be large. If $x=n$, then $(x-n)$ will be zero. If n is numerically greater than x, then $(x-n)$ will be negative; but the square of a negative quantity is positive; therefore, n can be so assumed that $(x-n)^2$ can be equal to any positive quantity whatever.

That is, $$x^2\pm ax+b=x^2-2nx+n^2.$$

Or, $$x=\frac{n^2-b}{2n\pm a}.$$

An expression in which n may be taken of any value whatever, and we shall have a corresponding value of x.

CASE 4TH. An expression in the form $(ax^2 \pm bx + c^2)$, can be made a complete square, by assuming its square root to be $(c - nx.)$

Because c^2 is in the power, c must be in the root, and it is obvious that x must be a factor in the other part of the root.

Whence, $$ax^2 \pm bx + c^2 = c^2 - 2cnx + n^2x^2.$$

$$ax \pm b = -2cn + n^2x.$$

$$x = \frac{2cn \pm b}{n^2 - a}.$$

EXAMPLE.

What value shall be given to x *to make* $8x^2+17x+4$ *a perfect square?*

Here $a=8,\ b=17,\ c=2.$

Whence, $$x = \frac{4n+17}{n^2-8}.$$

If $n=1$, then $x=-3$, and $8x^2+17x+4=25$, a square. If $n=3$, then $x=29$, and the value of the expression is 7225, the square of 85.

CASE 5TH. An expression in the form (ax^2+bx+c), in which *neither the first nor the last term* is a square, neither *branch* of the root can be taken, and the expression cannot be made a square, unless we can separate it into two rational factors, or unless we can find some square to subtract from it, which will leave a remainder susceptible of being separated into two rational factors.

By placing the expression (ax^2+bx+c) equal to zero, and solving the quadratic, we shall have two factors of the expression, but whether *rational* or *commensurable factors or not,* is the subject of enquiry.

To find the factors which make the product ax^2+bx+c, put this expression equal to 0, and work out the values of x thus, $ax^2+bx+c=0$. Or, $ax^2+bx=-c$. Complete the square, and

$$4a^2x^2+4abx+b^2=b^2-4ac.$$

Or, $$2ax+b=\pm\sqrt{(b^2-4ac.)}$$

Or $$x=\pm\frac{1}{2a}\sqrt{(b^2-4ac)}-\frac{b}{2a}.$$

We now perceive that the values of x must be rational, provided $\surd(b^2-4ac)$ is a complete square. If it be so, let

$$\frac{1}{2a}\surd(b^2-4ac)-\frac{b}{2a}=m, \text{ and } -\frac{1}{2a}\surd(b^2-4ac)-\frac{b}{2a}=n.$$

Then the two values of x are $x=m$ and $x=n$, and $(x-m)(x-n)$, are the factors which will give the expression ax^2+bx+c.

EXAMPLES.

(1.) *Find such a value of* x *as will make* $6x^2+13x+6$ *a square.*

Here $a=6$, $b=13$, $c=6$. $b^2=169$, $4ac=144$, $b^2-4ac=25$, and $\surd(b^2-4ac)=5$. Now $12x+13=\pm5$. $x=-\frac{2}{3}$. Or, $x=-\frac{3}{2}$. Or, $3x+2=0$, and $2x+3=0$.

That is, $(3x+2)(2x+3)$, will produce the expression

$$6x^2+13x+6.$$

Now to find such values of x, as will make the expression a square, put

$$(3x+2)(2x+3)=n^2(3x+2)^2.$$

That is, take *either factor of the expression* for a factor in its square root.

Then $$2x+3=n^2(3x+2.)$$

$$x=\frac{2n^2-3}{2-3n^2}.$$

Take $n=1$, then $x=1$, and the expression becomes

$$6+13+6=25, \text{ a square.}$$

Take $n=2$, then $x=-\frac{1}{2}$, and the expression becomes 1, a square.

Take $n=3$, then $x=-\frac{3}{5}$, and the expression becomes $\frac{9}{25}$, a square.

(2.) *Find such a value of* x *as shall render the expression* $(13x^2+15x+7)$, *a square.*

Here as neither the first nor the last term is square, nor (b^2-4ac) a square, we cannot find the required values of x, unless we can find a square, which subtracted from the expression, will leave a remainder divisible into rational factors. But in this case, $4ac$ is greater than b^2, we must therefore subtract such a square as to diminish a and c, and increase b.

To accomplish this object, we will subtract the square of $(x-1)$, and not the square of $(x+1.)$

That is, from $\quad 13x^2+15x+7,$
Subtract $\quad x^2-\ 2x+1.$

Difference, $\quad 12x^2+17x+6.$

In this last expression, $a=12$, $b=17$, and $c=6$.

Hence, $\quad b^2-4ac=289-288=1$, a square.

We are now sure the *difference* is divisible into rational factors, and to obtain the factors, we put

$$12x^2+17x+6=0.$$

A solution of the quadratic, gives $x=-\frac{2}{3}$, or $x=-\frac{3}{4}$.

Whence, $(3x+2)=0$, and $(4x+3)=0$, and our original expression becomes

$$(x-1)^2+(3x+2)(4x+3.)$$

It is obvious that $(x-1)$ must be in the root, and one of the factors may be in the other branch of the root; that is, put

$$(x-1)^2+(3x+2)(4x+3)=[(x-1)+n(3x+2)]^2.$$

By reduction, $\quad 4x+3=2n(x-1)+n^2(3x+2.)$

Or, $$x=\frac{2n+3-2n^2}{3n^2+2n-4}.$$

Take $n=1$, then $x=3$, and $13x^2+15x+7=169$, a square.

(3.) *Find such a value of* x *as will render* $14x^2+5x-39$, *a square.*

After a few trials this expression is found to be the same as $(2x-1)^2+(5x-8)(2x+5.)$ Assuming its root to be $2x-1+p(5x-8)$, then by squaring the root, making it equal to its power, and reducing, we find $x=\dfrac{8p^2+2p+5}{5p^2+4p-2}$.

Assuming $p=1$, $x=\frac{15}{7}$, and the expression equals 36, a square. Other values can be found, by assuming different values to p.

(4.) *Find such a value of* x *as will make* $2x^2+21x+28$, *a square.*

After a little inspection, we find this expression equal to $(x+4)^2+(x+1)(x+12.)$ Now if we make

$$(x+4)^2+(x+1)(x+12)=[(x+4)-p(x+1.)]^2$$

After reduction, we shall find $x=\frac{12+8p-p^2}{p^2-2p-1}$.

Assume $p=4$, then $x=4$, and the original expression is 144, a square.

If $(x+4)^2+(x+1)(x+12)=[(x+4)-p(x+12)]^2$, we shall find $x=\frac{12p^2-8p-1}{1+2p-p^2}$. If we take $p=1$, $x=\frac{3}{2}$. If we take $p=\frac{3}{2}$ then $x=8$, and we might find many other values of x that would answer the required condition.

Case 6th. Expressions in the form $a^2x^4+bx^3+cx^2+dx+e$, can be rendered square, provided we can extract three terms of their square roots.

Assume such terms as the whole root, making its square equal to the given expression, and the resulting value of x will make the whole expression a square.

EXAMPLE.

Find such a value of x *as will make* $4x^4+4x^3+4x^2+2x-6$, *a square.*

We commence by extracting the square root as far as *three* terms, and find them to be $(2x^2+x+\frac{3}{4}.)$

Therefore, $4x^4+4x^3+4x^2+2x-6=(2x^2+x+\frac{3}{4})^2$.

Expanding and reducing, $2x-6=\frac{3}{2}x+\frac{9}{16}$.

And $x=13\frac{1}{8}$.

Essentially the same method must be performed in other examples under this form.

Case 7th. Find such a value of x as will make ax^2+c, a square.

Expressions in this form, where $b=0$, and where neither a nor c are squares, nor (b^2-4ac) a square, *present impossible cases:* unless we can first find by inspection, some simple value of x that will answer the condition.

EXAMPLE.

Find such a value of x *as will make* $2x^2+2$, *a square.*

It is obvious that if $x=1$, the expression is a square. Now, having found that 1 will make the expression a square, we can find other values as follows :

Let $x=1+y$; then $x^2=1+2y+y^2$.

And $2x^2+2=4+4y+2y^2$.

Here the original expression is transformed into another expression, *having a square* for its first term.

Now we must find such a value of y as shall make $4+4y+2y^2$, a square.

Assume $4+4y+2y^2=(2-my)^2=4-4my+m^2y^2$. Or, $4+2y=-4m+m^2y$. Hence, $y=\frac{4(m+1)}{m^2-2}$, m may be any number greater than one. Put $m=2$. Then $y=6$, and $x=1+y=7$, and the original expression, $2x^2+2=98+2=100$, a square.

N. B. It often occurs incidentally in the solution of problems, that we must make a square of two other squares. This can be done thus: Let it be required to make x^2+y^2, a square.

Assume $x=p^2-q^2$, and $y=2pq$.

Then $x^2=p^4-2p^2q^2+q^4$.

And $y^2=\quad 4p^2q^2$.

Add, and $x^2+y^2=p^4+2p^2q^2+q^4$, which is evidently a square, whatever be the values of p and q. We can, therefore, assume p and q at pleasure, provided p be greater than q.

SECTION X.

DOUBLE AND TRIPLE EQUALITIES.

We have thus far confined our attention to finding a value of x that would render a single expression a square. Now we propose finding a value of x that will render several expressions squares at the same time.

CASE 1ST. As a general expression for double equality, let it be required to find such a value of x, that will make $(ax+b)$ and $(cx+d)$, squares at the same time.

$$\left.\begin{array}{ll}\text{Let} & ax+b=t^2 \\ \text{And} & cx+d=p^2\end{array}\right\} \text{then} \left\{\begin{array}{l} x=\dfrac{t^2-b}{a}. \\ x=\dfrac{p^2-d}{c}\end{array}\right.$$

Whence, $\dfrac{t^2-b}{a}=\dfrac{p^2-d}{c}$, or $ct^2-cb=ap^2-ad$.

Transposing cb, and multiplying by c, gives

$$c^2t^2=acp^2+c^2d-acd.$$

As the first member of this equation is square for all values of c and t, it is only requisite to find such a value of p, as to make the second member a square; which can be done by some of the artifices heretofore explained.

To illustrate, we give the following definite problem:

The double of a certain number increased by 4, makes a square; and five times the same number increased by 1, makes a square. What is that number?

Let x be the number; then

$$\left.\begin{array}{l} 2x+4=t^2 \\ 5x+1=p^2\end{array}\right\} \text{Whence,} \left\{\begin{array}{l} x=\dfrac{t^2-4}{2}. \\ x=\dfrac{p^2-1}{5}\end{array}\right.$$

Then $\quad 5t^2-20=2p^2-2.$

And $\quad 25t^2=10p^2+90.$

The first member of this equation is a square, whatever be the value of t; and all the conditions will be satisfied, provided we can find such a value of p as to make the second member a square.

This expression corresponds to case 7, and we *cannot proceed,* unless we find by trial, by *intuition* as it were, some simple value of p that will make $10p^2+90$, a square; and we do perceive that $p=1$ will make the whole expression 100, a square.

Now if $p=1$ will give a definite and positive value to x, which will answer the required conditions, the problem is solved. If not, we must find other values of p.

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Here $x=\frac{p^2-1}{5}$, and if $p=1$, $x=0$, and the expressions, $2x+4$ and $5x+1$, become 4 and 1. Squares, it is true, which answer the *technicalities*, but not the spirit of the problem.

To find another value of p, put $p=1+q$. Then

$$10p^2+90=100+20p+2q^2.$$

To make this a square, assume

$$100+20q+2q^2=(10-nq)^2=100-20nq+n^2q^2.$$

By reduction, $q=\frac{20(n+1)}{n^2-10}$. Now n must be so taken, that n^2 will be greater than 10; take $n=5$ and $q=8$, $p=9$, then $x=16$, and the original expressions, $2x+4=36$, a square, and $5x+1=81$, a square.

CASE 2D. A double equality in the form $ax^2+bx=\square$* and $cx^2+dx=\square$, may be resolved by making $x=\frac{1}{y}$, then the expressions will become $\frac{1}{y^2}\left(a+by\right)$ and $\frac{1}{y^2}\left(c+dy\right)$, which must be made squares.

But if we multiply a square by a square, or divide a square by a square, the product or quotient will be square.

Now as each of the preceding expressions are to be squares, and as they obviously have a square factor $\frac{1}{y^2}$, it is only necessary to make $a+by$, and $c+dy$, squares, as in the first case.

We may also take another course and assume $ax^2+bx=p^2x^2$, which gives $x=\frac{b}{p^2-a}$, which value put in the other expression, and we have $c\left(\frac{b}{p^2-a}\right)^2+d\left(\frac{b}{p^2-a}\right)=\square$.

Multiplying this by the square $(p^2-a)^2$, and the expression becomes $cb^2-dbd+abp^2=$ some square, from which the value of p can be found, and afterwards x.

EXAMPLE.

Find a number whose square increased by the number itself, and

*This symbol is read a square.

whose square diminished by the number itself, the sum and difference shall be squares.

Let $x=$ the number; then by the conditions,

$$x^2+x=\square, \text{ and } x^2-x= \text{ some other square.}$$

Assume $x=\frac{1}{y}$; then $\frac{1}{y^2}\left(1+y\right)=\square$.

And $\frac{1}{y^2}\left(1-y\right)=\square$.

The first members of these equations are obviously squares, provided the factors $(1+y)$ and $(1-y)$ are squares.

To make these factors squares, put

$$1+y=p^2, \text{ and } 1-y=q^2.$$

Whence, $y=p^2-1$, and $y=1-q^2$.

$$p^2=2-q^2.$$

All the conditions will be satisfied, when we discover what value must be given to q to make $(2-q^2)$, a square; and $q=1$ satisfies that condition.

This value of q makes $p=1$, and $y=0$.

But $x=\frac{1}{y}=\frac{1}{0}=$ infinity.

If x is infinite, x^2 can neither be increased or diminished by adding and subtracting x; therefore $x^2+x=\square$, and $x^2-x=\square$, because x^2 is obviously a square.

But practically, we say that this value of q will not answer the conditions; therefore, we will find other values as follows:

Put $q=1+t$.

Then $2-q^2=1-2t-t^2=(1-nt)^2=1-2nt+n^2t^2$.

Or, $t=\frac{2(n-1)}{n^2+1}$. Take $n=2$.

Then $t=\frac{2}{5}$. $q=1+\frac{2}{5}=\frac{7}{5}$. $y=1-\frac{49}{25}=-\frac{24}{25}$.

But $x=\frac{1}{y}=-\frac{25}{24}$, the number sought.

Those who desire a positive number, can take n negative.

CASE 3D. To resolve a triple equality.

Equations in the form $cx+by=t^2$, $ax+dy=u^2$, $ex+fy=s^2$, can be resolved thus:

By eliminating x, we find $y=\frac{au^2-ct^2}{ad-bc}$.

By eliminating y, we find $x=\frac{dt^2-bu^2}{ad-bc}$.

Substituting these values of x and y in the equation $ex+fy=s^2$, we shall have

$$\frac{(af-bc)u^2+(de-cf)t^2}{ad-bc}=s^2.$$

Assume $u=\pm tz$; then $u^2=t^2z^2$, and put this value in the above, and divide by t^2, then

$$\frac{(af-bc)z^2+(de-cf)}{ad-bc}=\frac{s^2}{t^2}.$$

As the second member of this equation is a perfect square, all the conditions will be satisfied when we find a value of z that will render the first member a square. This, *when possible*, can be done by case 7, of section viii.

After z is found, t can be assumed of any covenient value whatever. When u and t are known, x and y will be known.

We are now through with theory,—not that we have presented the whole, there are some cases in practice that no general rules will meet, and the operator must depend on his own judgment and penetration.

Much, *very much*, will depend on the skill and foresight displayed at the commencement of a problem, by assuming convenient expressions to satisfy one or two conditions at once, and the remaining conditions can be satisfied by some one of the preceding rules.

EXAMPLES.

(1.) *It is required to find three numbers in arithmetical progression, such that the sum of every two of them may be a square.*

Let x, $x+y$, and $x+2y$, represent the numbers.

Then by the general formula,

$$2x+y=t^2,\ 2x+2y=u^2,\ 2x+3y=s^2.$$

By exterminating x, we have $\frac{t^2-y}{2}=\frac{u^2-2y}{2}$.

Continuing thus according to the general equations, we must go through a long and troublesome process, and in conclusion we shall find the numbers to be 482, 3362, and 6242.

Another Solution.

Observing the remark immediately preceding the example, we put $\frac{x^2}{2}-y$, $\frac{x^2}{2}$, and $\frac{x^2}{2}+y$ to represent the numbers.

Then (x^2-y), (x^2+y), and x^2 must be the squares. But x^2 is a square for all values of x; therefore, we have only to make squares of (x^2-y) and $(x^2+y.)$

Let $y=2x+1$; then $x^2+y=x^2+2x+1$, a square for all values of x. Hence, all we have to do is to find a value of x that will make a square of the expression x^2-y, or x^2-2x-1. Assume the square root to be $(x-n)$; then

$$x^2-2x-1=x^2-2nx+n^2.$$

$$x=\frac{n^2+1}{2(n-1)}.$$

Take $n=11$, then $x=6.1$, and $\frac{1}{2}x^2=18.605$, $y=13.2$, and this numbers are 5.405, 18.605, and 31.805. Various other numbers may be found, by giving different values to n.

(2.) *Find two numbers such that if to each, as also to their sum, a given square* a² *be added, the three sums shall all be squares.*

Let x^2-a^2, and y^2-a^2 represent the numbers; then the first conditions are satisfied.

It now remains to make $x^2+y^2-2a^2+a^2$ a square, or, $x^2+y^2-a^2=\square$. Assume $y^2-a^2=2ax+a^2$. This assumption will make the expression a square, whatever be the values of either x or a. But the assumed equation gives $y^2=2ax+2a^2$, and as y^2 is a square, we must find *such values* of x and a, as shall make $2ax+2a^2$, a square. Put $x=na$. Then $2na^2+2a^2=\square$, or, $a^2(2n+2)=\square$. Hence it is sufficient that we put $2n+2=$ some square. Therefore, assume $2n+2=16$. Hence, $n=7$ and $x=7a$. Now take a equal to any number whatever. If $a=1$, $x=7$, $y=4$, and 48 and 15 are the numbers, add 1 to each, and we have 49 and 16, squares; sum, $63+1=64$, a square.

(3.) *Find three square numbers whose sum shall be a square.*

Let $x^2+y^2+z^2=\square$. Assume $y^2=2xz$. Then $x^2+2xz+z^2$ is a square. But $2xz=\square$. Let $x=uz$, then $2uz^2=\square$, or $2u=\square=16$, $u=8$, $x=8z$, $z=1$, $x=8$, $y=4$.

Therefore $64+16+1=81=9^2$.

(4.) *Find three square numbers in arithmetical progression.*

Let x^2-y, x^2, and x^2+y represent the numbers. Assume $x^2=y^2+\frac{1}{4}$, then the first and last will be squares, and it only remains to make $(y^2+\frac{1}{4})$, a square.

Therefore, put $y^2+\frac{1}{4}=(y-p)^2$. Whence, $y=\dfrac{p^2-\frac{1}{4}}{2p}$.

Take $p=1$, then $y=\frac{3}{8}$, and $y^2+\frac{1}{4}=\frac{25}{64}=x^2$.

Consequently, $\frac{1}{64}$, $\frac{25}{64}$, and $\frac{49}{64}$ are the numbers; but we can multiply them all by the same square number, 64, without changing their *arithmetical relation*, and their products will still be squares, 1, 25, and 49. Multiplying these numbers by any square number, will give other numbers that will answer the condition.

(5.) *Find two whole numbers, such that the sum and difference of their squares when diminished by unity, shall be a square.*

Let $x+1=$ one number, and $y=$ the other.

Then by the conditions, $x^2+y^2+2x=\square$ (1)

And $x^2-y^2+2x=\square$ (2)

Assume $2x=a^2$, and $y^2=2ax$; then (1) and (2) become

$$(x^2+2ax+a^2) \text{ and } (x^2-2ax+a^2),$$

obvious squares whatever may be the values of x and a.

But the equations $2x=a^2$, and $y^2=2ax$, must be satisfied. Take $a=4$, then $x=8$, $x+1=9$, and 9 and 8 are the numbers required.

(6.) *Find three whole numbers, such that if to the squares of each, the product of the other two be added, the three sums shall be squares.*

Let x, xy, and xv be the numbers.

Then by the conditions, $x^2+x^2vy=\square$.

$x^2y^2+x^2v=\square$.

And $x^2v^2+x^2y=\square$.

Omitting the common square factor x^2, it will be sufficient to make squares of the following expressions:

$$1+vy=\square.$$
$$y^2+v=\square.$$
$$v^2+y=\square.$$

Assuming $y=4v+4$ will make the first and third expressions square.

Substituting the value of y^2 in the second expression, we shall have $16v^2+33v+16$, which must be made a square.

Whence, $16v^2+33v+16=(4-pv)^2$.

Reduced, gives $v=\frac{33+8p}{p^2-16}$. Take $p=5$.

Then $v=\frac{73}{9}$. Now take $x=9$, and 9, 73, 328, will be the required numbers.

(7.) *Find two whole numbers whose sum shall be an integral cube, and the sum of their squares increased by thrice their sum, shall be an integral square.*

Let $x+y=n^3$, that is, some cube. Then $x^2+y^2+3n^3=\square$. Put $2xy=3n^3$, then $x^2+2xy+y^2$ is a square, whatever may be the values of x and y. But x and y must conform to the equations $x+y=n^3$, and $2xy=3n^3$. Work out the value of x from these equations, on the supposition that n is known, and we shall find $2x=n^3+\sqrt{(n^6-6n^3)}$.

Now x will be rational, provided we can find such a value of n as shall render n^6-6n^3 a square, but if we add 9 to this, we perceive it must be a square, and we have two squares, which differ by 9. Therefore one must be 16, the other 25, as these are the only two *integral squares* which differ by 9. Hence, $n^6-6n^3+9=25$. Or, $n^3-3=5$. $n^3=8$, $n=2$, and $x=6$, $y=2$.

(8.) *Find three numbers such that their sums, and also the sum of every two of them, may all be squares.*

Let $x^2-4x=$ the first, $4x=$ second, and $2x+1=$ third. By this notation, all the conditions will be satisfied, except the sum of the last two. That is $6x+1$ must be a square, but to have three *different whole numbers*, no square will answer under 121, the square of 11. Hence, put $6x+1=121$. Or, $x=20$. And the numbers will be 320, 80, and 41.

(9.) *Find two numbers such that their difference may be equal to the difference of their squares, and the sum of their squares shall be a square number.*

Let x and y be the numbers. Then $x-y=x^2-y^2$. Divide by $x-y$, and $1=x+y$. Hence $x=1-y$, and $x^2+y^2=1-2y+2y^2$. Which last expression, $1-2y+2y^2$, must be made a square. For this purpose put $1-2y+2y^2=(1-ny)^2$. Hence, $y=\frac{2(n-1)}{n^2-2}$.

Take n any value to render y less than one, in order to make x positive. Take $n=3$, then $y=\frac{4}{7}$, and $x=\frac{3}{7}$, the answer.

The following are not difficult, and we leave them as a pleasant exercise for learners.

(10.) *Find three numbers in geometrical progression, such that if the mean be added to each of the extremes, the sums in both cases shall be squares.* *Ans.* 5, 20, and 80.

(11.) *Find three numbers, such that their product increased by unity shall be a square, also the product of any two increased by unity, shall be a square.* *Ans.* 1, 3, and 8.

Assume 1 for the first number, and x and y for the other two.

(12.) *Find two numbers, such that if the square of each be added to their product, the sums shall be both squares.* *Ans.* 9 and 16.

(13.) *Find three integral square numbers in harmonical proportion.* *Ans.* 25, 49, and 1235.

(14.) *Find two numbers in the proportion of* 8 *to* 15, *and such that the sum of their squares shall be a square number.*

Ans. 136 and 255. Bonnycastle's answer is 476 and 1080.

(15.) *Find two numbers such that if each of them be added to their product, the sums shall be both square.* *Ans.* $\frac{1}{3}$ and $\frac{5}{3}$.

We have given as much on this topic as will be profitable, save the following remote and partial application.

EXAMPLES.

(1.) Given $\left\{\begin{array}{l} x^2+y=7 \\ y^2-x=7 \end{array}\right\}$ to find the values of x and y.

$x^2=(7-y.)$ $y^2=(7+x.)$ Here $(7-y)$ and $(7+x)$, must be squares. $x=2$, and $y=3$, will evidently answer the conditions; and as these values will verify the given equations, the solution is accomplished.

(2.) Given $\left\{\begin{array}{l} 2x^2-3xy+y^2=4 \\ x^2+3y^2-2xy=9 \end{array}\right\}$ to find values of x and y.

As 4 and 9 are squares, the first members are square in fact, though not in form. But we can make the first members square in form, by assuming

$$2x^2-3xy=0, \text{ and } 3y^2-2xy=0.$$

Then $y^2=4$ and $x^2=9$, or $y=2$ and $x=3$; values which verify all the equations.

(3.) Find such integral values of x, y, and z, as will verify the equations

$$x^2+y^2+xy=37,$$

And $$x^2+z^2+xz=49.$$

If we add xy to the first equation, and xz to the second, the first members will be square; and, of course, the second members will be square in fact, though not in form.

We have then to make $37+xy$, and $49+xz$, squares.

To accomplish this, put $37+xy=49$, or $xy=12$ (1)

And $49+xz=64$, or $xz=15$ (2)

From (1), $x=\frac{12}{y}$; from (2), $x=\frac{15}{z}$.

Hence, $12z=15y$, or $z=\frac{5y}{4}$.

Take $y=4$, then $z=5$, and $x=3$; values which will verify the given equations.

(4.) Find such integral values of y and z that will verify the equation $y^2+z^2+yz=61$.

Add yz to both members, then put $61+yz=n^2$.

Now if we assume $n=8$, $yz=3$.

But $yz=3$ will give $y+z=8$, and these two equations will not give integral values to y and z. Therefore, take $n=9$, then $n^2=81$, $yz=20$, $y+z=9$. Hence, $z=4$ or 5, and $y=5$ or 4.

(5.) Given $\begin{cases} 4x^2-2xy=12 \\ 2y^2+3xy=\ 8 \end{cases}$ to find the values of x and y.

Put $xy=p$, transpose, &c.

Then $4x^2=12+2p$. $4y^2=16-6p$.

Now if we find such a value of p as will make $(12+2p)$ and $(16-6p)$, squares at the same time, it is *highly probable* that such a value will verify the original equations. It is obvious that $p=2$, will make the expressions squares; then $4x^2=16$, $x=2$ and $y=1$, and these values will verify all the equations.

(6.) Given $\begin{cases} 6x^2+2y^2=5xy+12 \\ 3x^2+2xy=3y^2-\ 3 \end{cases}$ to find one value of x and y.

This problem is under (Art. 110, alg.)

Add the equations together, and reduce, and we have

$$9x^2=y^2+3xy+9.$$

The first member of this equation is a square; therefore the second member is a square, but to make it a square in *form*, as well as in *fact*, we perceive it is only necessary to make $x=2$. Then $9x^2=y^2+6y+9$, and $3x=y+3$; whence $y=3$, and these values verify the given equations.

This method of operation must be used with great caution, and taken for just what it is worth.

(7.) Given $x+y=35$, and $x^{\frac{2}{3}}-y^{\frac{2}{3}}=5$, to find the values of x and y.

Put $x^{\frac{1}{3}}=P$, and $y^{\frac{1}{3}}=Q$.

Then $P^3+Q^3=35$, and $P^2-Q^2=5$.

Or, $P^3=35-Q^3$, and $P^2=5+Q^2$.

The equations can all be verified, provided we find can such a value of Q that will make $(35-Q^3)$ a cube, and $(5+Q^2)$, a square.

We will try the next less integral cube below 35. That is, we will assume $35-Q^3=27$. Then $Q=2$, and $(5+Q^2)=9$, a square. Then $P=3$, and $x^{\frac{1}{3}}=3$, or $x=27$, and $y=8$.

This problem was given in the first editions of Robinson's Algebra, page 147, under the head of pure equations, but it was out of place and is now changed.

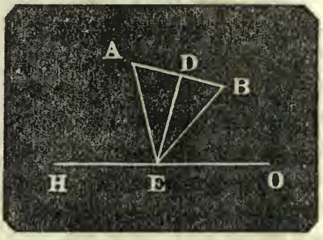
PART THIRD.

SECTION I.

GEOMETRY.

Thirty-one of the following problems will be found in Robinson's Geometry, commencing on page 100.

(1.) *From two given points, draw two equal straight lines which shall meet in the same point in a line given in position.*

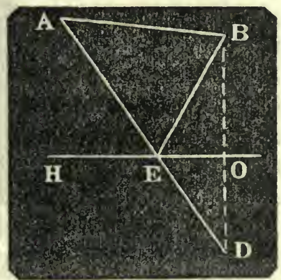


Let A and B be the two given points, *taken at pleasure,* and HO the line given in position.

Join AB and bisect it in D. Draw DE perpendicular to AB, to meet the line HO in E. Join AE and BE, the lines required. Because $AD=DB$, and DE common to the two △'s ADE, BDE, and the angles at D right angles, therefore $AE=BE$. Q. E. D.

N. B. For simple and obvious demonstrations, we shall not go through the steps in full, but refer to Robinson's Geometry for the proposition that applies.

(2.) *From two given points on the same side of a line given in position, to draw two lines which shall meet in that line and make equal angles with it.*

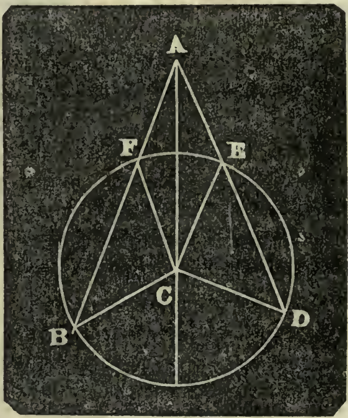


Let A and B be the two given points, and HO the line given in position.

From one of the given points as B, let fall the perpendicular BO, to the given line, and produce it to D, making $OD=BO$.

Then join AD: this line will necessarily cut the line HO in some point E. Join EB, and AE and EB are the required lines. $\angle BEO=\angle DEO$, (Book 1, Th. 13.) $\angle AEH=\angle DEO$, (Th. 3, Book 1.) Whence, $\angle BEO=\angle AEH$. Q.E.D.

(3.) *If from any point without a circle, two straight lines be drawn to the concave part of the circumference, making equal angles with the line joining the same point and the center; the parts of these lines which are intercepted within the circle, are equal.*

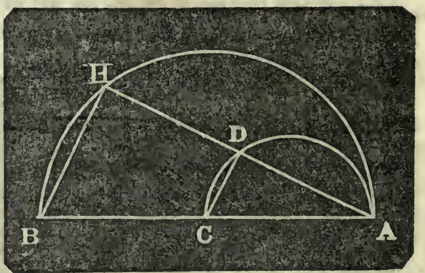


Let A be the point without the circle. Join AC and draw any other line to cut the circle as AD; then draw AB so that the angle $CAB=CAD$. Then *we are to show that* $FB=ED$.

The two △'s, ABC and ADC, having two sides AC, CB, of the one, equal to AC, CD, of the other, and their respective angles at A equal, the two △'s are equal. That is, $AB=AD$. For the same reason the two △'s ACF, ACE are equal, and $AF=AE$.

Whence, $AB-AF=AD-AE$, or $BF=DE$. Q. E. D.

(4.) *If a circle be described on the radius of another circle, any straight line drawn from the point where they meet, to the outer circumference, is bisected by the interior one.*

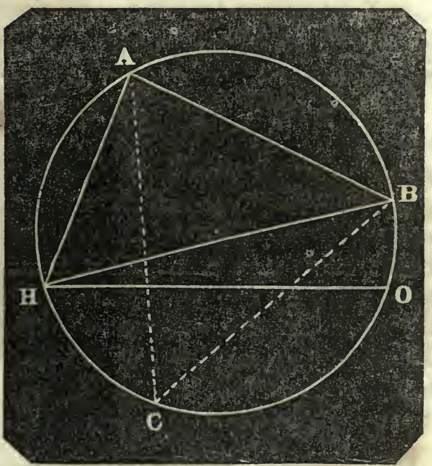


Let AC be the radius of one circle and the diameter of another, as represented in the figure. From the point of contact A, of the two circles, draw any line, as AH; *this line is bisected in D.* Join DC and HB. Then ADC being in a semicircle, is a right angle; also, AHB is a right angle, for the same reason: therefore, DC and HB are parallel. Whence,

$$AD : AH :: AC : AB$$

But as AB is double of AC, therefore AH is double of AD, or AH is bisected in D. Q. E. D.

(5.) *From two given points on the same side of a line given in position, to draw two straight lines which shall contain a given angle, and be terminated in that line.*



Let A and B be the two given points and HO the line given in position. For the sake of perspicuity, we will require two lines drawn from the two points, A and B, to meet in HO, and make an angle of 50°. Subtract 50 from 180, and divide the remainder by 2, this produces 65°.

At A make the angle $BAC=65°$, and at B make the angle $ABC=65°$; these two lines will meet in C, making an angle of 50°. About the $\triangle ABC$ describe a circle, cutting HO in H and O. Join AH, BH. AHB is equal ACB, (th. 9, b. iii, scholium,) the angle required.

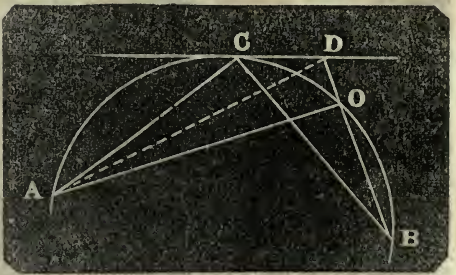
Lines drawn from A and B, to meet the line in O, would also answer the conditions.

N. B. When the given angle is not sufficiently small to cause the angle C to fall below the line HO, the problem is impossible.

(6.) *If from any point without a circle, lines be drawn touching it, the angle contained by the tangents is double of the angle contained by the line joining the points of contact, and the diameter drawn through one of them.*

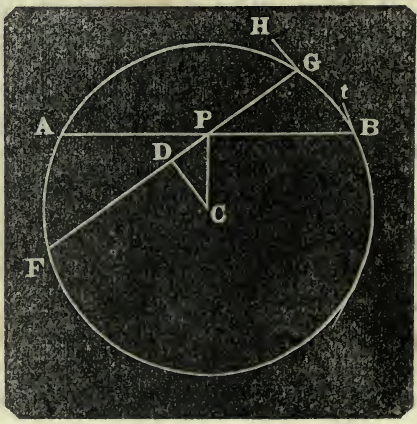
This problem requires no figure. Imagine a point without a circle, a line drawn from that point to the center of the circle, and lines drawn to touch the circle on each side. Join the points of contact and the center of the circle. Thus we have two equal right angled triangles, having the same hypotenuse, the line from the given point without the circle to the center of the circle. With the correct figure in the mind, the truth of the proposition is obvious.

(7.) *If from any two points in the circumference of a circle, there be drawn two straight lines to a point in a tangent to that circle, they will make the greatest angle when drawn to the point of contact.*



Let A and B be the two points in the circle, and CD a tangent line. The proposition requires us to demonstrate *that the angle ACB is greater than the angle ADB.* $ACB = AOB$, (th. 9, b. iii, sch.) But AOB is greater than ADB, (th. 11, b. i, cor. 1), therefore, ACB is also greater than ADB. Q. E. D

(8.) *From a given point within a given circle, to draw a straight line which shall make with the circumference an angle less than any angle made by any other line drawn from that point.*

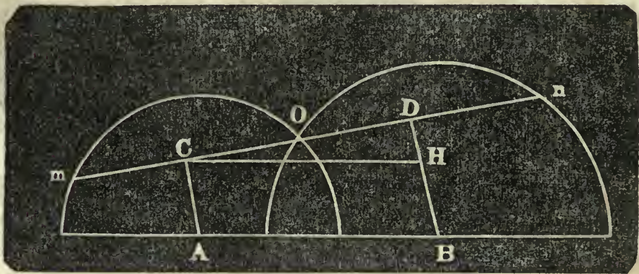


Let P be the given point within the circle, and C the center. Join PC. Through P draw APB at right angles to PC. Also, through P draw any other line as PG; *then we are to show* that PBt is *less* than PGH.

From C let fall the perpendicular CD on the chord FG. PC is the hypotenuse of the right angled triangle PDC; therefore, PC is greater than CD, consequently the chord FG is greater than the chord AB, (th. 3, b. iii.) and the arc GAF is greater than the arc BGA. The angle PGH is measured by half the arc GAF, and PBt is measured by half the arc BGA; therefore, the angle PGH is greater than the angle PBt, or PBt is less than PGH. Q. E. D.

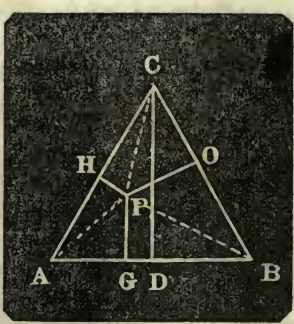
N. B. The angle which any chord makes with the circumference, is the same as between the chord and tangent,—because the circumference and tangent unite as they meet the chord.

(9.) *If two circles cut each other, the greatest line that can be drawn through the point of intersection, is that which is parallel to the line joining their centers.*



Let A and B be the center of two circles which intersect in O. Through O draw mn inclined to AB,—then *we are to prove that mn is less than it would be if it were parallel to AB.* Draw AC and BD perpendicular to mn, then $CD=\frac{1}{2}mn$. Draw CH parallel to AB, then $CH=AB$; and CH being the hypotenuse of the right angled $\triangle$ CDH, CH, or its equal AB, is greater than CD. Now conceive mn to revolve on the center O, until CD becomes parallel to AB; CD will then become equal to AB. But mn will be all the while double of CD: therefore, mn will be the greatest when parallel to AB. Q. E. D.

(10.) *If from any point within an equilateral triangle, perpendiculars be drawn to the sides, they are together, equal to a perpendicular drawn from any of the angles to the opposite side.*



Let ABC be the equilateral $\triangle$, CD a perpendicular from one of the angles on the oposite side; then the area of the $\triangle$ is expressed by $\frac{1}{2}AB \times CD$. Let P be any point within the triangle, and from it let drop the three perpendiculars PG, PH, PO.

The area of the triangle APB is expressed by $\frac{1}{2}AB \times PG$. The area of the $\triangle$ CPB is expressed by $\frac{1}{2}CB \times PO$: and the area of the $\triangle$ CPA is expressed by $\frac{1}{2}CA \times PH$. By adding these three expressions together, (observing that CB and CA are each equal to AB,) we have for the area of the whole $\triangle$ ACB, $\frac{1}{2}AB(PG+PH+PO.)$

Therefore, $\frac{1}{2}AB \times CD = \frac{1}{2}AB(PG+PH+PO.)$

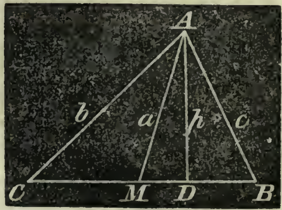
Dividing by $\frac{1}{2}AB$, gives $CD = PG+PH+PO$. Q. E. D.

(11.) *If the points, bisecting the sides of any triangle be joined, the triangle so formed will be one-fourth of the given triangle.*

If the points of bisection be joined, the triangle so formed will be similar to the given $\triangle$, (th. 19, b. ii.)

Then, the area of the given $\triangle$ will be to the area of the $\triangle$ formed by joining the bisecting points, as the square of a line is to the square of its half; that is, 2^2 to 1, or as 4 to 1. Hence the $\triangle$ cut off is $\frac{1}{4}$ of the given $\triangle$. Q. E. D.

(12.) *The difference of the angles at the base of any triangle, is double the angle contained by a line drawn from the vertex perpendicular to the base, and another bisecting the angle at the vertex.*



Let ABC be a $\triangle$. Draw AM bisecting the vertical angle, and draw AD perpendicular to the base.

The theorem requires us to prove that the difference between the angles B and C is double of the angle MAD.

By hypothesis, the angle $CAM = MAB$. That is, $CAM = MAD + DAB$. (1)

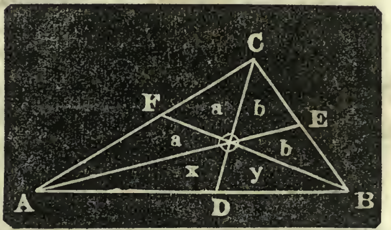
By (th. 11, b. i, cor. 4.) $\left\{\begin{array}{ll} C+CAM+MAD=90°. & (2) \\ B+DAB \qquad =90°. & (3) \end{array}\right.$

Therefore, $B+DAB=C+CAM+MAD.$ (4)

Taking the value of CAM from (1), and substituting it in (4), gives $B+DAB=C+MAD+DAB+MAD.$

Reducing, $(B-C)=2MAD.$ Q. E. D.

(13.) *If from the three angles of a triangle, lines be drawn to the middle of the opposite sides, these lines will intersect each other in the same point.*



Let ABC be a △, bisect BC in E, AC in F.

Join AE and BF, and through their point of intersection O, draw the line CD. *Now if we prove $AD=DB$, the theorem is true.*

Triangles whose bases are in the same line, and vertex in the same point, are to one another as their bases; and when the bases are equal, the triangles are equal. For this reason the △ AFO=△FCO, and the △ COE = △ EOB.

Put △ $AFO=a$; then △ $FCO=a$. Also, put △ $COE=b$, as represented in the figure.

Because CB is bisected in E, the △ ACE is half of the whole △ ABC. Because AC is bisected in F, the △ BFC is half the whole △ ABC.

That is, $2a+b=2b+a$.

Whence, $a=b$, and the four triangles above the point O are equal to each other.

Let the area of the △ ADO be represented by x, and the area of DOB by y.

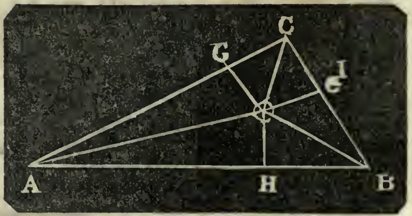
Now taking COD as the base of the triangles, we have

$$2a : x :: CO : OD$$

Also, $2b=$ $2a : y :: CO : OD$

Whence, $2a : x :: 2a : y$. Or, $x=y$.
Therefore, $AD=DB$. Q. E. D.

(14.) *The three straight lines which bisect the three angles of a triangle, meet in the same point.*

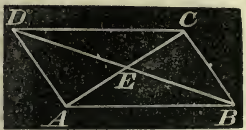


Let ABC be the $\triangle$, bisect two of the angles A and C—the bisecting lines will meet in the same point O. Join OB; *we are required to demonstrate that OB bisects the angle B.*

From O, let fall the perpendiculars on to the sides. The two right angled $\triangle$'s AOH and AOG, are equal in all respects, because they have the same hypotenuse AO, and equal angles by construction. In the same manner we prove that the $\triangle CGO = \triangle COI$. Whence, $GO=OI$. But $GO=OH$; therefore, $OH=OI$.

Now in the two right angled triangles OHB and OIB, we have $OH=OI$, and OB common, therefore, the triangles are equal, and $HBO=OBI$. Q. E. D.

(15.) *The two triangles formed by drawing straight lines from any point within a parallelogram to the extremities of the opposite sides, are together half the parallelogram.*



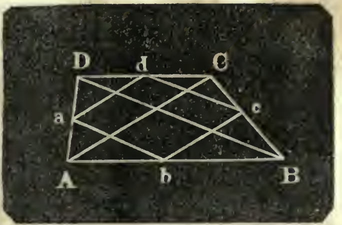
Let $ABDC$ be a parallelogram, E any point within.

We are to show that the triangles AEB, CED, are together half the parallelogram.

Through the point E draw a line parallel to AB or CD, thus forming two parallelograms.

The $\triangle AEB$ is half the lower parallelogram, and the $\triangle CED$ is half the upper parallelogram; therefore, the sum of the two $\triangle$'s is half the whole parallelogram. Q. E. D.

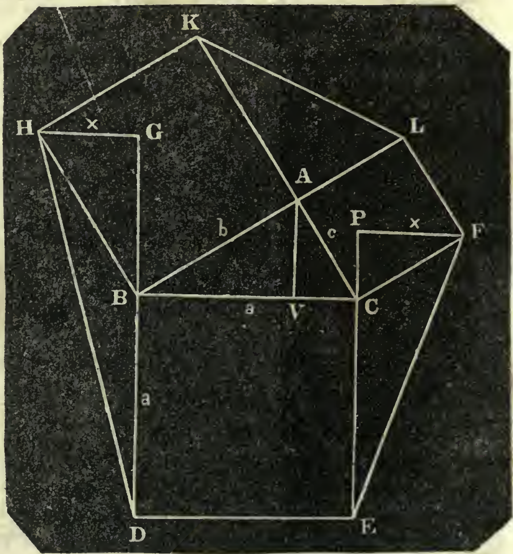
(16.) *The figure formed by joining the points of bisection of the sides of any trapezium, is a parallelogram.*



Let $ABCD$ be a trapezium. Draw the diagonals AC, BD. Bisect the sides in a, b, c, and d. Join $abcd$. We are to prove that this figure is a parallelogram.

ABD is a $\triangle$ whose sides are bisected in a and b; therefore, the $\triangle$ Aba is equiangular to the $\triangle$ ABD, (th. 19, b ii), and ab is parallel to BD, and by (th. 18, b. ii), $ab=\frac{1}{2}BD$. In the same manner we can prove that dc is parallel to BD and equal to half of it. Consequently ab and dc are parallel and equal. Therefore, by (th. 23, b. i), the figure $abcd$ is a parallelogram. Q. E. D.

(17.) *If squares be described on the three sides of a right angled triangle, and the extremities of the adjacent sides be joined, the triangles so formed are equal to the given triangle, and to each other.*



Let ABC be the given right angled triangle and construct the figure as here represented. It is obvious that the vertical right angled $\triangle$ AKL is equal to ABC.

Draw AV perpendicular to BC, and call it x. *We now propose to show that HG* $=x$. BD is produced to G, the angles VBG and ABH are right angles, and from these equals take away the common part ABG; thus showing that $ABV=HBG$.

The two right angled triangles ABV, HBG are equal, because they have equal angles, and the hypotenuse $AB=$ the hypotenuse HB, because they are sides of the same square. Therefore, $HG=AV$, and if one is in value x, the other has the same value.

Now we designate any side of the square on BC by a, then twice the area of the $\triangle ABC$ is ax, and the double area of the triangle HBD is obviously ax.

Therefore, HBD is equal in area to ABC.

In the same manner we can prove that $FCE=ABC$. Q.E.D.

(18.) *If squares be described on the hypotenuse and sides of a right angled triangle, and the extremities of the sides of the former, and the adjacent sides of the others be joined, the sum of the squares of the lines joining them will be equal to five times the square of the hypotenuse.*

(See figure to the last Theorem.)

In the right angled triangle HGD, we have

$$x^2+(BG+a)^2=(HD)^2 \qquad (1)$$

In the right angled triangle PFE, we have

$$x^2+(PC+a)^2=(FE)^2 \qquad (2)$$

Expanding (1) and (2), and observing that $GB=BV$, $PC=CV$, we shall have

$$x^2+(BV)^2+2a(BV)+a^2=(HD)^2 \qquad (3)$$

And $$x^2+(PC)^2+2a(PC)+a^2=(FE)^2 \qquad (4)$$

By adding (3) and (4), and observing that $x^2+(BV)^2=b^2$, and $x^2+(PC)^2=c^2$, then

$$(b^2+c^2)+2a(BV+VC)+2a^2=(HD)^2+(FC)^2$$

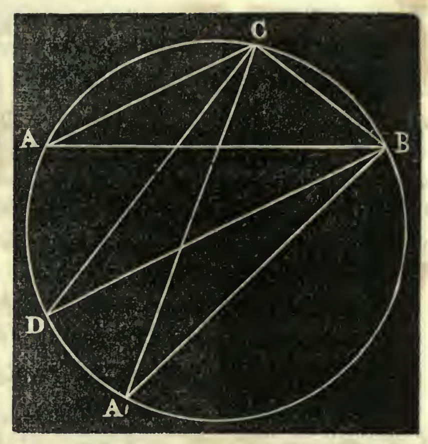
That is, $$a^2+2a(a)+2a^2=$$

Or, $$5a^2=(HD)^2+(FC)^2$$

SCHOLIUM. The sum of the squares of the sides of the last figure is $8a^2$.

(19.) *The vertical angle of an oblique-angled triangle, inscribed in a circle, is greater or less than a right angle, by the angle contained between the base and the diameter drawn from the extremity of the base.*

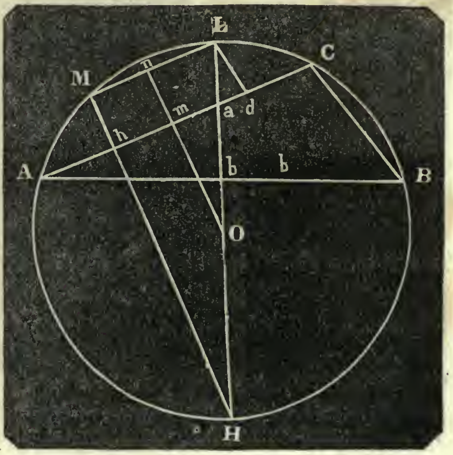
Let ABC be a △ having the angle ACB greater than a right angle, and describe a circle about it. From one extremity of the base as B draw the diameter BD.



The angle DBC is a right angle, because it is in a semicircle. The vertical angle ACB is greater than a right angle by ACD; but ACD is equal ABD, because each is measured by half the arc AD. Therefore ACB is greater than a right angle by ABD.

Next let $A'CB$ be the △; the angle $A'CB$ is less than a right angle by the angle $DCA'=DBA'$, because each is measured by half the arc DA'. Therefore, the vertical angle, &c.

(20.) *If the base of any triangle be bisected by the diameter of its circumscribing circle, and from the extremity of that diameter, a perpendicular be let fall upon the longer side, it will divide that side into segments, one of which will be equal half the sum, and the other half the difference of the sides.*



Let ABC be the △, bisect its base by the diameter of the circle drawn at right angles to AB.

From the center O let fall Om at right angles to AC, it will then bisect AC. From the extremity of the diameter H, draw Hh perpendicular to AC, and consequently parallel to Om. Produce

Hh to M and join ML. Complete and letter the figure as represented.

The two triangles Aab and Hha are equiangular. The angle a is common to them, and each has a right angle by construction, therefore the angle $H=$ the angle A. But equal angles at the circumference of the same circle subtend equal chords, (th. 2, b. iii;) therefore $CB=ML$. The angle HML is a right angle, because it is in a semicircle, therefore ML is parallel to AC, and ML is bisected in n.

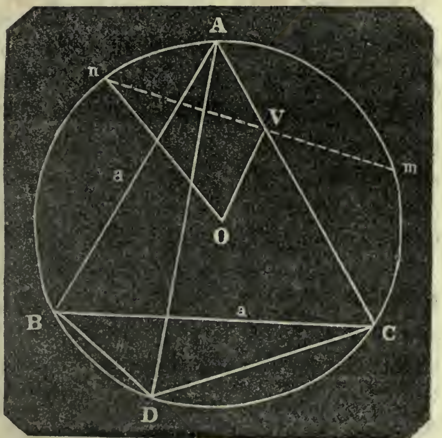
Now $Am=\frac{1}{2}AC$. $nL=md=\frac{1}{2}ML=\frac{1}{2}BC$.

Therefore by addition, $Am+md=\frac{1}{2}(AC+CB.)$

Or, $Ad=\frac{1}{2}(AC+CB.)$ Q. E. D.

Cor. If Ad is the half sum of the sides, dc or Ah must be the half difference; for the half sum and half difference make the greater of any two quantities.

(21.) *A straight line drawn from the vertex of an equilateral triangle, inscribed in a circle, to any point in the opposite circumference, is equal to the two lines together, which are drawn from the extremities of the base to the same point.*



Let ABC be the equilateral △ in a circle. Take D any point in the arc between B and C, and join AD, BD, and DC.

Designate each side of the given triangle by a.

Now $ABDC$ is a quadrilateral in a circle, AD is one diagonal and BC the other, and by (th. 21, b. iii) we have

$$a(AD)=a(BD)+a(DC.)$$

Dividing by a, and $AD=BD+DC$. Q. E. D.

(22.) *The straight line bisecting any angle of a triangle inscribed in a given circle, cuts the circumference in a point which is equidistant from the extremities of the sides opposite to the bisected angle, and from the center of a circle inscribed in the triangle.*

(See the figure to the last Theorem.)

The angle BAD is measured by half the arc BD, (th. 8, b. iii) and the angle DAC is measured by half the arc DC; therefore, if $BAD=DAC$, the arc BD must equal the arc DC.

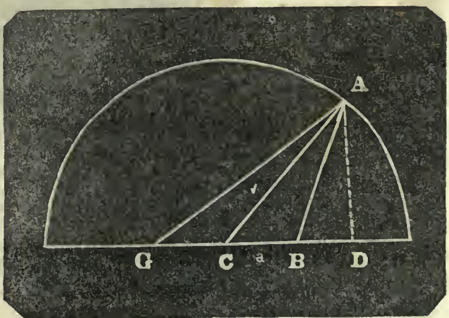
(23.) *If from the center of a circle a line be drawn to any point in the chord of an arc, the square of that line, together with the rectangle contained by the segments of the chord, will be equal to the square described on the radius.*

(See the figure to the 21st Theorem.)

From the center O draw OV to any point in AC, and through the point V draw nm at right angles to OV, and join Om; then OVm is a right angled triangle. Therefore, $(OV)^2+(Vm)^2=(Om)^2$. But $(Vm)^2=(nV)(Vm)=(AV)(VC)$, (th. 17, b. iii.) Therefore, by substitution,

$$(OV)^2+(AV)(VC)=(Om)^2. \quad \text{Q. E. D.}$$

(24.) *If two points be taken in the diameter of a circle, equidistant from the center, the sum of the squares of the two lines drawn from these points to any point in the circumference will be always the same.*



Let C be the center of a circle, and A any point in the circumference. $CA=r$, the radius.

Put $AD=y$, $DC=x$, and CB and CG each $=a$. Then $BD=(x-a)$, and $DG=(x+a)$.

Now in the triangle ADB we have $\quad y^2+(x-a)^2=(AB)^2$.

And in the triangle ADG, $\quad y^2+(x+a)^2=(AG)^2$.

By expanding and adding, we find

$$2y^2+2x^2+2a^2=(AB)^2+(AG)^2.$$

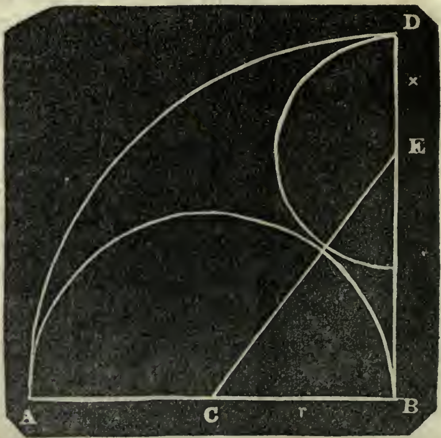
The triangle ADC gives $2y^2+2x^2=2r^2$; therefore,

$$2r^2+2a^2=(AB)^2+(AG)^2.$$

Because the first member of this equation is the same for all values of x and y—that is, because it is invariable; therefore the second member must also be invariable. Q. E. D.

(25.) *If on the diameter of a semicircle two equal circles be described, and in the space included by the three circumferences, a circle be inscribed, its diameter will be two-thirds the diameter of either of the equal circles.*

It is sufficient to represent a portion of the figure.



Let B be the center of the semicircle, and BA the diameter of one of the equal circles, and E the center of the circle sought—BD being at right angles to AB from the point B.

Put $CB=r$, and $DE=x$. Then $BD=2r$, $BE=2r-x$, and $CE=r+x$.

Now in the right angled triangle BEC, we have $(CB)^2+(BE)^2=(CE)^2$.

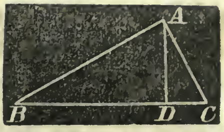
That is, $r^2+(2r-x)^2=(r+x)^2$.

By expanding, $r^2+4r^2-4rx+x^2=r^2+2rx+x^2$.

Reducing, $4r^2-4rx=2rx$.

Whence, $x=\frac{2}{3}r$. Q. E. D.

(26.) *If a perpendicular be drawn from the vertical angle of any triangle to the base, the difference of the squares of the sides is equal to the difference of the squares of the segments of the base.*



Let ABC be any triangle. Let fall AD perpendicular to the base. Now the two right angled triangles give us

$$(AD)^2+(BD)^2=(AB)^2.$$

And $$(AD)^2+(DC)^2=(AC)^2.$$

By subtraction, $(BD)^2-(DC)^2=(AB)^2-(AC)^2$. Q. E. D.

By factoring, $(BD+DC)(BD-DC)=(AB+AC)(AB-AC.)$

By observing that $(BD+DC)=BC$, and converting this equation into a proportion, we have

$$BC : (AB+AC) :: (AB-AC) : (BD-DC.)$$

(This is Prop. 6, Plane Trigonometry, page 149, Robinson's Geometry.)

Scho. This proportion is true whatever be the relation of AB to AC. It is true then when $AB=AC$. Making this supposition, then BD becomes equal to DC, and the proportion becomes

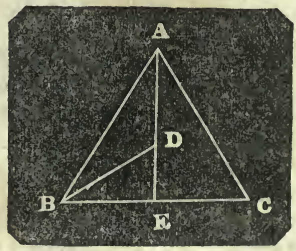
$$BC : AB+AC :: 0 : 0.$$

Now $(AB+AC)$ being two sides of a triangle are greater than the third side BC; *therefore the last zero is greater than the first*, an apparent absurdity.

But this is no more than saying that zero divided by zero can have a positive quotient—for we can subtract zero from zero as many times as we please, and still have zero left.

The proportion is obviously true, for 0 times BC is equal to 0 times $(AB+AC.)$ Indeed 0 may be to 0, as a to any quantity whatever.

(27.) *The square described on the side of an equilateral triangle is equal to three times the square of the radius of the circumscribing circle.*



Let ABC be the equilateral triangle. Let fall the perpendicular AE on the base; it will bisect the base. Draw BD bisecting the angle at B. D will be the center of the circumscribing circle, and AD or BD will be the radius.

We are to prove $AD=BD$, and find the value of BD in terms of AB.

Each angle of an equilateral triangle is 60°, ($\frac{1}{3}$ of 180°.)

If we bisect these, each division will be 30°.

Hence $BAD=30°$, and $ABD=30°$; therefore, $AD=BD$.

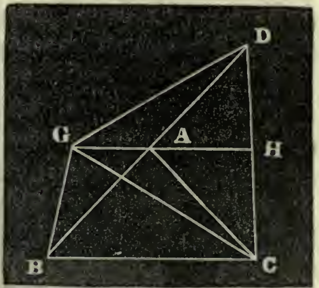
Put $AB=2a$, then $BE=a$. Also put $BD=x$, then $DE=\frac{1}{2}x$.* Now in the right angled triangle BDE, we have

$$a^2+\tfrac{1}{4}x^2=x^2.$$

Whence, $4a^2=3x^2$. But $4a^2=(AB)^2$.

Therefore, $(AB)^2=3(BD)^2$. Q. E. D.

(28.) *The sum of the sides of an isosceles triangle, is less than the sum of any other triangle on the same base, and between the same parallels.*



Let ABC be the isosceles triangle. $AB=AC$. Through the point A draw GAH parallel to BC.

Take G any other point on the line GH, and draw BG and GC.

We are to show that $AB+AC$ are less than $BG+GC$. Produce AB to D, making $AD=AB$, or AC.

Then by reason of the parallels GH and BC, the angle DAH is equal to the angle ABC, and $HAC=ABC$.

Because $AD=AC$, the angle $ADH=$ the angle ACH.

Whence the two triangles ADH and ACH, are equal in all respects, and GH is perpendicular to DC; whence any point in the line GH is equally distant from the two points D and C.

Now the straight line $BD=BA+AC$, and because $DG=GC$, $DG+GB=GB+GC$. But $DG+GB$, the two sides of a △ are greater than the third side DB; therefore, $GB+GC$ are greater than BD, that is, greater than $BA+AC$. Q. E. D.

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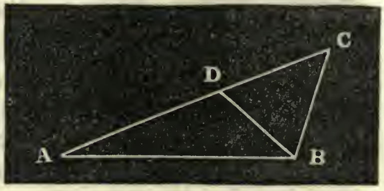
*This might not be admitted, at the same time the reader would readily admit that BE was one-half AB. ABE is a right angled triangle, one angle being 30 deg. the side opposite that angle is half the hypotenuse, ***and this is a general truth***. Now the angle **DBE** equals 30 deg., therefore **DE** is half **BD**.

GEOMETRICAL CONSTRUCTIONS.

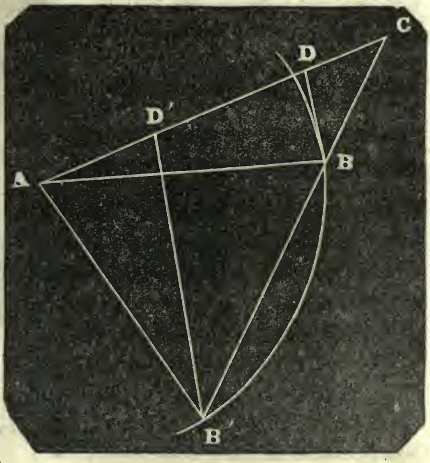
(29.) *In any triangle, given one angle, a side adjacent to the given angle, and the difference of the other two sides, to construct the triangle.*



Let AB represent the given side, and from one extremity as A, make the angle $BAC=$ to the given angle, (prob. 5, b. iv.)

Take $AD=$ to the given difference of the sides, and join DB. From the point B make the angle DBC equal to the angle BDC, then $CB=CD$, and AD is the given difference of the sides, and ABC is the triangle required.

(30.) *In any triangle, given the base, the sum of the other two sides, and the angle opposite the base, to construct the triangle.*



Draw AC equal to the sum of the sides. From the point A as a center, with a radius equal to the given base AB, describe an arc as represented in the figure.

From the point C in the line AC, make the angle ACB equal *to half* the given angle.

If the problem is possible, this line CB will cut the circular arc in two points, B and B'. From B and B' make the angles CBD and $CB'D'$, each equal to the angle at C. Join AB, AB', and either $\triangle$ ABD or $AB'D'$, fulfils the required conditions.

For $CD=DB$, and $CD'=B'D'$, (because they are sides of a $\triangle$ opposite equal angles,) therefore $AD+DB=AC$; also $AD'+$

$DB' = AC$. The angle ADB is double the angle C, (th. 11, b. i,) therefore it is the angle required.

(31.) *In any triangle, given the base, the angle opposite to the base, and the difference of the other two sides, to construct the triangle.*

Subtract the given angle from 180°, and divide the remainder by 2, designating the result by a.

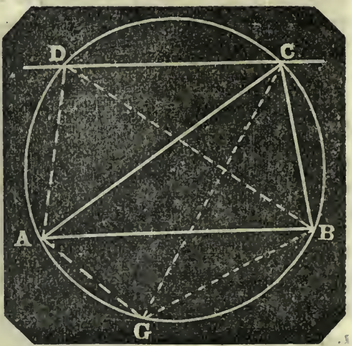
Draw an indefinite line as AC, (see figure to 29,) and take AD equal to the given difference of the sides.

From the point D, make the angle $CDB = a$.

From A as a center, with a radius equal to the given base AB, strike an arc, cutting DB in B.

At B make the angle $DBC = a$; then $DC = BC$, and ABC will be the triangle required.

(32.) *In any triangle, given the base, the perpendicular, and the angle opposite to the base, to construct the triangle.*



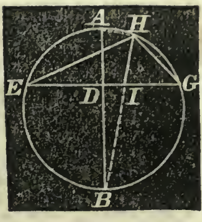
Draw AB equal to the given base, and DC parallel to it at the given perpendicular distance.

On the other side of the base AB, make the angle BAG equal to *part* of the given angle, and ABG equal to the *remaining part*, thus forming the $\triangle AGB$. About the $\triangle ABG$, describe a circle cutting DC in the points D and C. Join AC, CB, and ACB is the triangle required.

The angle $BCG = BAG$, (th. 9, b. iii, scho.), and the angle $ACG = ABG$. Therefore by addition, $ACB = BAG + ABG$; that is, $ACB =$ the given angle.

The triangle ADB will also answer the conditions; for $ACB = ADB$.

(33.) *In any triangle, given the base, the ratio of the two sides, and the line bisecting the vertical angle, to construct the triangle.*



Draw the base EG, and bisect it in D, Draw DB at right angles to EG.

Divide EG in the point I, so that EI shall be to IG in the ratio of EH to HG.

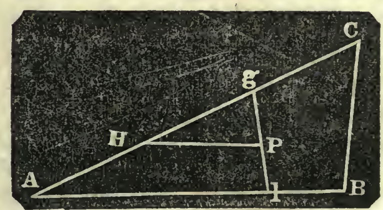
Find IB of such a value that

$$HI : GI :: EI : IB.$$

The three first terms are given; therefore the fourth is known. From I as a center, with the distance IB as radius, strike an arc, cutting DB in B. Join BI and produce it to H, making IH equal to the given distance. Join EH, HG, and EHG is the △ required.

Because $HI\times IB=EI\times IG$, a circle which passes through the points E, B, and G, will also pass through the point H, and the angle $EHI=IHG$, and for that reason $EH : HG :: EI : IG$, as required. (See th. 25, b. ii.)

(34.) *To draw a straight line through any given point within a triangle to meet the sides, or the sides produced, so that the given point shall bisect the line so drawn.*



Let ABC be the △, and P the given point within it.

Through P *it is required to draw the straight line gl, so that gP shall be equal Pl.*

From P draw PH parallel to AB. Take $gH=AH$. Join gP and produce it to l, and gl is the line required. Because PH is parallel to Al.

$$gH : HA :: gP : Pl.$$

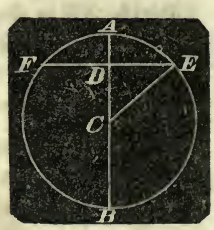
But $gH=HA$; therefore $gP=Pl$.

Scho. Had we taken Hg double of AH, then gP would have been double of Pl, and we might have required gP to be any number of times Pl.

(35.) *Find the square root of* 13 *or any other number, by a geometrical construction.*

Divide the number into any two factors, (say 2 and $6\frac{1}{2}$,) add the factors together, for the diameter of a circle.

Take the *half sum of* the two factors for the radius of a circle, and describe the circle as represented in the margin.



Let AB be a diameter, and take AD for one factor, and DB for the other; and through D, draw FE at right angles to AB. DE or DF *represents the square root required.*

In the present example, if $AD=2$ and $DB=6\frac{1}{2}$, then the length of DE applied to the same scale will show the square root of 13. Because $AD\times DB=(DE)^2$.

When the two factors are very nearly equal, D will be very near the center of the circle, and DE will be very nearly the radius of the circle,—always *a little* less, unless the factors are absolutely equal; in that case each one is a root. *On this principle we extracted square root in the first part of this volume.*

Observe the $\triangle$ DCE. CE is the half sum of the two factors, and DC is their half difference.

Also, DE is the sine of the arc AE, and DC is the cosine of the same arc; therefore, *we can if we desire it, bring in the aid of a table of natural sines and cosines.*

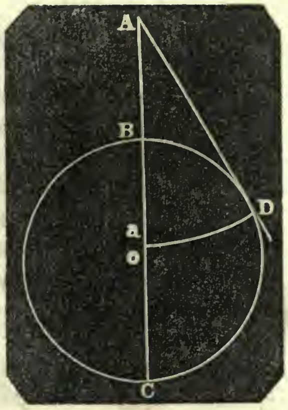
But the tables of natural sines are adapted to *radius unity*; here the radius is $4\frac{1}{4}$, therefore to have corresponding values of CD and DE, we have this proportion,

$$4\tfrac{1}{4} : 1 :: 2\tfrac{1}{4} : .52941.$$

The result of this proportion carried to the table of natural sines gives .848365 for the corresponding cosine, and this multiplied by $4\frac{1}{4}$, gives 3.605551 for the square root of 13.

Another Construction.

(36.) *Let it be required to find the square root of* 250, (*or any other number,*) *by a geometrical construction.*



Divide the number into two factors. Let one factor be represented by AB, the other by AC; BC being their difference. On the difference as a diameter, describe a circle.

From the extremity A, draw AD touching the circle. AD represents the square root required.

By (th. 18, b. iii, scho. 1),

$$AB \times AC = (AD)^2.$$

Therefore AD is the square root of the product of the two factors AC and AB.

Remark. When the two factors are nearly equal, the circle will be very small, and AD will be very nearly Ao. But Ao is the half sum of the factors AB and AC, hence we know that the square root of the product of two factors is always a little less than their half sum, unless the factors are absolutely equal.

In the proposed example, we divide 250 into the two factors, 25 and 10—their difference is 15. Hence $7\frac{1}{2}$ is the radius of the circle. Take $7\frac{1}{2}$ from any scale of equal parts in the dividers, and describe a circle.

Draw any diameter as BC, and produce it to A, making $AB=10$. From A draw AD to touch the circle; take that distance in the dividers and apply it to the scale, and the result will be the square root of 250.

The practical difficulty in this construction is to decide exactly where the point D is, therefore the first method of construction is the best.

Geometrical constructions are not to be relied upon for numerical accuracy, but they are invaluable to impress theory, and are sure guides to numerical operations.

Scho. If it were required to make a square equal to a given rectangle, either of the two preceding constructions may be applied. Let AC be one side of the rectangle, AB the other; then AD will be a side of the required square.

PROBLEMS.

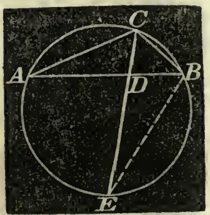
The following problems do not admit of geometrical constructions, in the sense of some of the preceding—they require *algebra applied to geometry.*

We take the problems from Robinson's Geometry, pages 105 to 109. For theory, the reader must look elsewhere.

We omit the first two problems, and number them as they are numbered in the geometry.

PROBLEM 3.

In a triangle, having given the sides about the vertical angle, and the line bisecting that angle and terminating in the base, to find the base.



Let ABC be the $\triangle$, and let a circle be circumscribed about it. Divide the arc AEB into two equal parts at the point E, and join EC. This line bisects the vertical angle, (th. 9, b. iii, scho.) Join BE.

Put $AD=x$, $DB=y$, $AC=a$, $CB=b$, $CD=c$, and $DE=w$. The two $\triangle$'s, ADC and EBC, are equiangular; from which we have,

$$w+c \ : \ b \ : \ : \ a \ : \ c. \quad \text{Or, } cw+c^2=ab. \quad (1)$$

But as EC and AB are two chords that intersect each other in a circle, we have, $cw=xy$ (th. 17, b. iii.)

Therefore, $xy+c^2=ab$ (2)

But as CD bisects the vertical angle, we have,

$$a \ : \ b \ : \ : \ x \ : \ y \quad \text{(th. 23, b. ii.)}$$

Or, $x=\frac{ay}{b}$ (3)

Hence, $\frac{a}{b}y^2+c^2=ab$: or, $y=\sqrt{b^2-\frac{c^2b}{a}}$

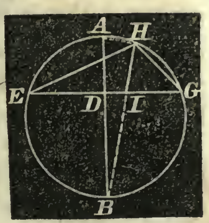
And, $x=\frac{a}{b}\sqrt{b^2-\frac{c^2b}{a}}$

Now as x and y are determined, the base is determined.

N. B. Observe that equation (2) is theorem 20, book iii.

PROBLEM 4.

To determine a triangle, from the base, the line bisecting the vertical angle, and the diameter of the circumscribing circle.



Describe the circle on the given diameter AB, and divide it in two parts, in the point D, so that $AD \times DB$ shall be equal to the square of one-half the given base.

Through D draw EDG at right angles to AB, and EG will be the given base of the △.

Put $AD=n$, $DB=m$, $AB=d$, $DG=b$.

Then, $n+m=d$, and $nm=b^2$; and these two equations will determine n and m; and therefore, n and m we shall consider as known.

Now suppose EHG to be the required △, and join HIB and HA. The two △'s AHB, DBI, are equiangular, and therefore, we have,

$$AB : HB :: IB : DB.$$

But HI is a given line, that we will represent by c; and if we put $IB=w$, we shall have $HB=c+w$; then the above proportion becomes, $d : c+w :: w : m$.

Now w can be determined by a quadratic equation; and therefore IB is a known line.

In the right angled △ DBI, the hypotenuse IB, and the base DB, are known; therefore, DI is known, (th. 36, b. i); and if DI is known, EI and IG are known.

Lastly, let $EH=x$, $HG=y$, and put $EI=p$, and $IG=q$.

Then by theorem 20, book iii, $pq+c^2=xy$ (1)

But, $x : y :: p : q$ (th. 25, b. ii.)

Or, $x=\frac{py}{q}$ (2)

And from equations (1) and (2) we can determine x and y, the sides of the △; and thus the determination has been attained, carefully and easily, step by step.

PROBLEM 5.

Three equal circles touch each other externally, and thus inclose one acre of ground; what is the diameter in rods of each of these circles?

Draw three equal circles to touch each other externally, and join the three centers, thus forming a triangle. The lines joining the centers will pass through the points of contact, (th. 7, b. iii.)

Let R represent the radius of these equal circles; then it is obvious that each side of this △ is equal to $2R$. The triangle is therefore equilateral, and it incloses the given area, and three equal sectors.

As each sector is a third of two right angles, the three sectors are, together, equal to a semicircle; but the area of a semicircle, whose radius is R, is expressed by $\frac{\pi R^2}{2}$ (th. 3, b, v, and th. 1, b. v); and the area of the whole triangle must be $\frac{\pi R^2}{2}+160$; but the area of the △ is also equal to R multiplied by the perpendicular altitude, which is $R\sqrt{3}$.

Therefore, $R^2\sqrt{3}=\frac{\pi R^2}{2}+160.$

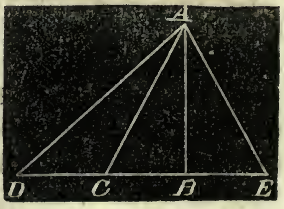
Or, $R^2(2\sqrt{3}-\pi)=320.$

$$R^2=\frac{320}{2\sqrt{3}-3.1415926}=\frac{320}{0.3225}=992.248.$$

Hence, $R=31.48+$ rods for the result.

PROBLEM 6.

In a right angled triangle, having given the base and the sum of the perpendicular and hypotenuse, to find these two sides.



Let ABC be the △. Put $CB=b$, $AB+AC=a$, $AB=x$; then $AC=a-x$.

By (th. 36, b. i),

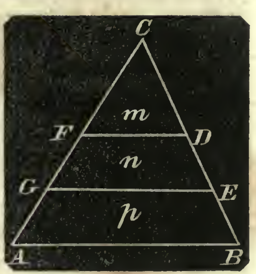
$$x^2+b^2=a^2-2ax+x^2.$$

Whence, $x=\frac{a^2-b^2}{2a}$.

Now the numerical value of x being known, the triangle can be constructed geometrically.

PROBLEM 7.

Given the base and altitude of a triangle, to divide it into three equal parts, by lines parallel to the base.



Let ABC represent the △. Conceive a perpendicular let drop from C to the base AB, and represent it by b. Put $2a=AB$. Then $ab=$ the area of the triangle.

Let x be the distance from C to FD; then by (th. 22, b. ii), we have,

$$x^2 : b^2 :: \tfrac{1}{3}ab : ab$$

Whence, $x : b :: 1 : \sqrt{3}$. Or, $x=\dfrac{b}{\sqrt{3}}$.

If x represents the distance from C to GE, then

$$x^2 : b^2 :: \tfrac{2}{3}ab : ab.$$

Or, $x : b :: \sqrt{2} : \sqrt{3}$. $x=\dfrac{\sqrt{2}\cdot b}{\sqrt{3}}$

We perceive by this that the divisions of the perpendicular are independent of the base, and that we may divide the triangle into any required number of parts, m, n, p, &c., equal or unequal.

PROBLEM 8.

In any equilateral triangle, given the length of the three perpendiculars drawn from any point within, to the three sides, to determine the sides.

Let ABC be the △. We have shown in this volume that $CD=PG+PH+PO=a$.

Put AD or $DB=x$; then $BC=2x$.

Then by the right angled △ CDB, we have $a^2+x^2=4x^2$, or

$$x=\frac{a}{\sqrt{3}}.$$

PROBLEM 9.

In a right angled triangle, having given the base (3), *and the difference between the hypotenuse and perpendicular* (1), *to find these sides.*

(See figure to Problem 6.)

Let $CB=3$. $AC-AB=1$. $AB=x$. Then $AC=x+1$, and

$$x^2+9=x^2+2x+1. \quad x=4.$$

PROBLEM 10.

In a right angled triangle, having given the hypotenuse (5), *and the difference between the base and perpendicular* (1), *to determine both of these two sides.*

(See figure to Problem 6.)

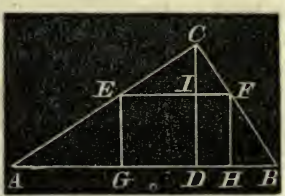
Let $CB=x$. $AB=x+1$. Then

$$x^2+(x+1)^2=25.$$

Or, $2x^2+2x=24$. Whence, $x=3$. $AB=4$.

PROBLEM 11.

Having given the area or measure of the space of a rectangle inscribed in a given triangle, to determine the sides of the rectangle.



When we say that a triangle is given, we mean that the base and perpendicular are given.

Let ABC be the triangle, $AB=b$, $CD=p$, $CI=x$; then $ID=p-x$.

By proportional triangles we have

$$CI : EF :: CD : AB$$

That is, $x : EF :: p : b$. $EF=\frac{bx}{p}$.

By the problem $\frac{bx}{p}\left(p-x\right)=a$. The symbol a being the given area.

Whence, $x^2-px=-\frac{ap}{b}$. $x=\frac{1}{2}p\pm\sqrt{\frac{1}{4}p^2-\frac{ap}{b}}$.

PROBLEM 12.

In a triangle having given the ratio of the two sides, together with both the segments of the base, made by a perpendicular from the vertical angle, to determine the sides of the triangle.

Let ACB be the $\triangle$, (see last figure.) $AD=a$, $BD=b$, and $CD=x$. Then $AC=\sqrt{a^2+x^2}$, and $CB=\sqrt{b^2+x^2}$.

The ratio of AC to CB is given, and let that ratio be as 1 to r; then

$$\sqrt{a^2+x^2} : \sqrt{b^2+x^2} :: 1 : r.$$

Whence, $$a^2+x^2 : b^2+x^2 :: 1 : r^2.$$

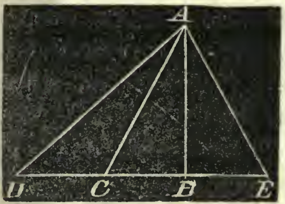
Or, $$b^2+x^2=a^2r^2+r^2x^2.$$

Or, $$x^2=\frac{a^2r^2-b^2}{1-r^2}.$$

But $AC=\sqrt{a^2+x^2}$, and as x^2 is now known, AC is known.

PROBLEM 13.

In any triangle having given the base, the sum of the other two sides and the length of a line drawn from the vertical angle to the middle of the base, to find the sides of the triangle.



Let ADE be the $\triangle$. Suppose C to be the middle of the base.

Put $AC=a$, DC or $CE=b$, $AE=x$, $DA+AE=c$; then $DA=c-x$.

Now by (th. 39, b. i), we have

$$(DA)^2+(AE)^2=2(AC)^2+2(DC)^2$$

That is, $c^2-2cx+2x^2=2a^2+2b^2$.

Or, $$4x^2-4cx+c^2=4a^2+4b^2-c^2.$$

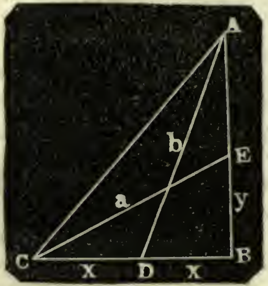
$$2x-c=\sqrt{4a^2+4b^2-c^2}.$$

Whence x becomes known, and consequently the sides become known.

PROBLEM 14.

To determine a right angled triangle, having given the length of two lines drawn from the acute angles to the middle of the opposite sides.

Let ABC be the triangle. Letter it as represented—$CE=a$, $AD=b$, &c.



Then $\left\{\begin{array}{l} 4x^2+y^2=a^2. \\ x^2+4y^2=b^2. \end{array}\right\}$

By add. $5x^2+5y^2=a^2+b^2=5m.$

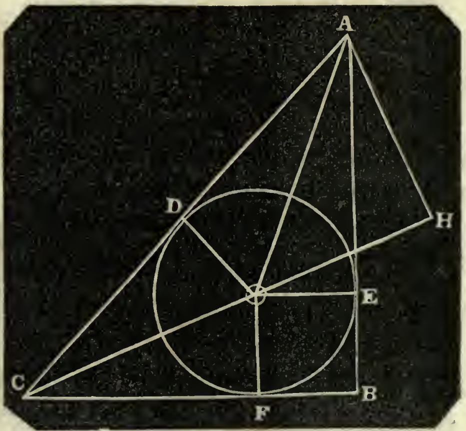
$$x^2+y^2=m.$$

$$3x^2=a^2-m.$$

$$x=\sqrt{\frac{a^2-m.}{3}} \quad y=\sqrt{\frac{b^2-m.}{3}}$$

PROBLEM 15.

To determine a right angled triangle, having given the perimeter and the radius of its inscribed circle.



Let ABC be the $\triangle$, OE the radius of the circle.

It is obvious that $AE=AD$, $CF=CD$. Put $AE=x$, $CF=y$, $FB=r$, $2p=$ the perimeter. Then by the conditions,

$x+y+r=p$ (1)

From the right angled $\triangle$ ABC, we have

$$(x+r)^2+(y+r)^2=(x+y)^2 \qquad (2)$$

By reduction,

$$rx+ry+r^2=xy \qquad (3)$$

That is, $(x+y+r)r=rp=xy$ (4)

Equation (4) expresses the area of the triangle.

From (1), $x^2+2xy+y^2=p^2-2pr+r^2.$

From (4), $4xy = 4pr$

By subtraction, $x^2-2xy+y^2=p^2-6pr+r^2.$

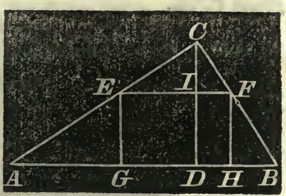
Whence, $x-y=\pm\sqrt{p^2-6pr+r^2}$.

But $x+y=p-r$.

Therefore, $x=\frac{1}{2}(p-r)\pm\frac{1}{2}\sqrt{p^2-6pr+r^2}$.

PROBLEM 16

To determine a triangle, having given the base, the perpendicular, and the ratio of the two sides.



Let ABC be the $\triangle$. $AB=b$, $CD=a$, $DB=x$. Then

$$AC=\sqrt{(b-x)^2+a^2}.$$

$$CB=\sqrt{a^2+x^2}.$$

Let the given ratio of the sides be as m to n; then

$$\sqrt{(b-x)^2+a^2} : \sqrt{a^2+x^2} :: m : n.$$

This proportion will give the value of x, then AC and CB will be known.

PROBLEM 17.

To determine a right angled triangle, having given the hypotenuse, and the side of the inscribed square.

Let ADC be the $\triangle$. (See last figure.) Put $CI=x$, $IE=a$, $AG=y$, and $AC=b$. Then by proportional triangles, we have

$$CI : IE :: EG : GA.$$

That is, $x : a :: a : y$. Whence, $xy=a^2$.

In the right angled $\triangle$'s AGE, ECI, we have

$$AE=\sqrt{y^2+a^2}. \quad EC=\sqrt{x^2+a^2}.$$

Observing that $AE+EC=AC=b$, and $a^2=xy$, we perceive that

$$\sqrt{x^2+xy}+\sqrt{y^2+xy}=b.$$

Whence, $\sqrt{x}+\sqrt{y}=\dfrac{b}{\sqrt{x+y}}$.

By squaring, $x+y+2\sqrt{xy}=\dfrac{b^2}{x+y}$.

Put $(x+y)=s$, and observe that $2\sqrt{xy}=2a$; then

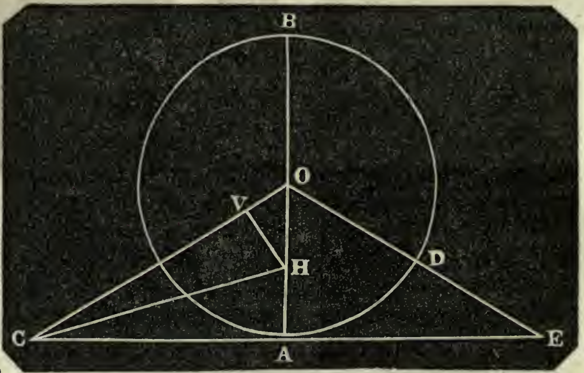
$$s^2+2as=b^2.$$

Whence, $s+a=\pm\sqrt{a^2+b^2}$.

Now having the value of $(x+y)$, and (xy) the separate values of x and y can be determined, which is a solution of the problem.

PROBLEM 18.

To determine the radii of three equal circles, inscribed in a given circle to touch each other, and also to touch the circumference of the given circle.



Let ADB be the given circle. Divide the circumference 360 deg. into 3 equal parts. BD is one of those parts 120° ; then the arc $AD=60°$. A circle inscribed in the △ COE, will be one of the equal circles required.

Let $AO=a$, $AH=x$; H being the center of the circle. From H, draw HV perpendicular to CO, then $AH=HV$.

Hence $HV=x$, $OH=a-x$, and $OV=\frac{1}{2}OH$, because the angle $VHO=30°$. (See prop. 1, plane trig., page 139.)

Now by the right angled △ OVH, we have

$$(OV)^2+(VH)^2=(OH)^2.$$

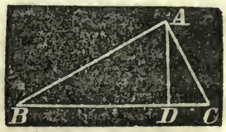
That is, $$\left(\frac{a-x}{2}\right)^2+x^2=(a-x)^2.$$

Whence, $$x=(2\sqrt{3}-3)a.$$

PROBLEM 19.

In a right angled triangle, having given the perimeter, or sum of all the sides, and the perpendicular let fall from the right angle on the hypotenuse, to determine the triangle, that is, its sides.

Let ABC be the $\triangle$, and represent its perimeter by p. Put $AD=a$, $AB=x$, $AC=y$. Then $BC=p-x-y$.



Because BAC is a right angle,

$$x^2+y^2=p^2-2p(x+y)+x^2+2xy+y^2 \qquad (1)$$

And, $$a(p-x-y)=xy \qquad (2)$$

Reducing (1), $$2p(x+y)=p^2+2xy \qquad (3)$$

Double (2), $$2ap-2a(x+y)=2xy \qquad (4)$$

By subtraction, $$(2a+2p)(x+y)-2ap=p^2 \qquad (5)$$

Whence, $$x+y=\frac{p^2+2ap}{2a+2p} \qquad (6)$$

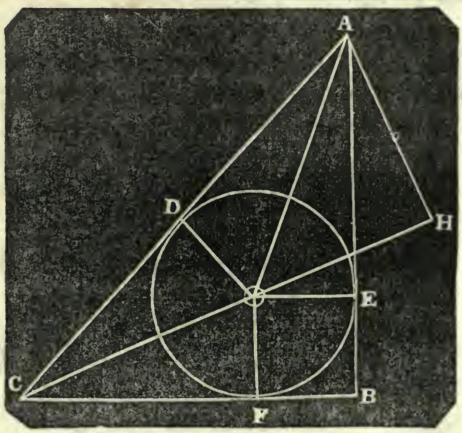
Because $BC=p-x-y$, $BC=p-\frac{p^2+2ap}{2a+2p}=\frac{p^2}{2a+2p}$

From (2) we observe that $xy=\frac{ap^2}{2a+2p}$ (7)

Equations (6) and (7), will readily give x and y.

PROBLEM 20.

To determine a right angled triangle, having given the hypotenuse and the difference of two lines, drawn from the two acute angles to the center of the inscribed circle.



Let ABC be the $\triangle$, O the center of the inscribed circle; then AO bisects the angle CAB, and CO bisects the angle C.

The angle AOH, being the exterior angle of the triangle AOC, it is equal to $CAO+ACO$, that is, AOH is equal to half the sum of the angles CAB, BCA, or to 45°. Produce CO to H; from A let fall AH

perpendicular on CA. Now in the $\triangle\ AOH$, because $H=90°$, and $AOH=45°$, $OAH=45°$, and consequently $AH=OH$.

Put $AC=a$, $AO=x$, $OC=x+d$, OH and AH, each equal to y. Then $CH=x+y+d$.

In the $\triangle\ AHO$, we have $2y^2=x^2$ (1)

In the $\triangle\ AHC$, we have

$$(x+y+d)^2+y^2=a^2 \qquad (2)$$

Expanding,

$$x^2+y^2+d^2+(2x+2d)y+2dx+y^2=a^2 \qquad (3)$$

Substituting the value of y^2 and y from (1), and

$$2x^2+(2x+2d)\frac{x}{\sqrt{2}}+2dx=a^2-d^2.$$

Or, $$2x^2+\sqrt{2}\cdot x^2+\sqrt{2}dx+2dx=a^2-d^2.$$

Dividing by $(2+\sqrt{2})$, and we have

$$x^2+dx=\frac{a^2-d^2}{2+\sqrt{2}}=m.$$

Whence, $$x=-\frac{d}{2}\pm\sqrt{m+\frac{d^2}{4}}.$$

PROBLEM 21

To determine a triangle, having given the base, the perpendicular, and the difference of the two sides.

(See figure to Problem 19.)

Let ABC be the $\triangle$. Put $BD=x$, $DC=y$, $AC=z$, $AB=z+d$, $AD=a$, $BC=b$.

By the conditions, $x+y=b$ (1)

$$x^2+a^2=z^2+2dz+d^2 \qquad (2)$$

$$y^2+a^2=z^2 \qquad (3)$$

By subtraction, $x^2-y^2=2dz+d^2$ (4)

Factoring, $(x+y)\ (x-y)=d(2z+d)$

That is, $b(x-y)=d(2z+d)$

From this we have the proportion,

$$b\ :\ (2z+d)\ ::\ d\ :\ (x-y)$$

This proportion is the following rule given in trigonometry, viz:

In any plane triangle, as the base is to the sum of the sides, so is the difference of the sides to the difference of the segments of the base.

We return to the solution. From (1) we have

$$x=a-y, \text{ whence } x^2-y^2=a^2-2ay.$$

From (3), $z=\sqrt{y^2+a^2}$. These values put in (4), give

$$a^2-2ay=2d\sqrt{y^2+a^2}+d^2$$

$$(a^2-d^2)-2ay=2d\sqrt{y^2+a^2}$$

Squaring, $(a^2-d^2)^2-4a(a^2-d^2)y+4a^2y^2=4d^2y^2+4a^2d^2$

Or, $(a^2-d^2)^2-4a(a^2-d^2)y+4(a^2-d^2)y^2=4a^2d^2$

$$a^2-d^2-4ay+4y^2=\frac{4a^2d^2}{a^2-d^2}$$

$$a^2-4ay+4y^2=d^2+\frac{4a^2d^2}{a^2-d^2}=\frac{5a^2d^2-d^4}{a^2-d^2}$$

$$a-2y=\pm d\sqrt{\frac{5a^2-d^2}{a^2-d^2}}$$

Whence, $y=\frac{a}{2}\mp\frac{d}{2}\left(\frac{5a^2-d^2}{a^2-d^2}\right)^{\frac{1}{2}}$

PROBLEM 22.

To determine a triangle, having given the base, the perpendicular, and the rectangle, or product of the two sides.

(See figure to Problem 19.)

Let ABC be the $\triangle$. Put $BD=x$, $DC=y$, $BC=b$, $AD=a$, and the recangle, $(AB)(AC)=c$.

Now in the right angled triangles, ADB, ADC, we have

$$AB=\sqrt{x^2+a^2}. \quad AC=\sqrt{y^2+a^2}.$$

Whence, $(\sqrt{x^2+a^2})(\sqrt{y^2+a^2})=c$ (1)

And, $x+y=b$ (2)

From (1), $x^2y^2+a^2(x^2+y^2)+a^4=c^2$ (3)

From (2), $x^2+y^2=b^2-2xy$ (4)

This value substituted in (3), gives

$$x^2y^2+a^2b^2-2a^2xy+a^4=c^2$$

$$x^2y^2-2a^2xy+a^4=c^2-a^2b^2$$

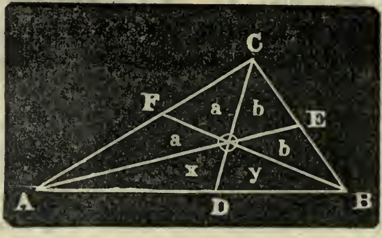
$$xy-a^2=\pm\sqrt{c^2-a^2b^2}$$

$$xy=a^2\pm\sqrt{c^2-a^2b^2} \quad (5)$$

From equations (2) and (5) the values of x and y can be determined.

PROBLEM 23.

To determine a triangle, having given the length of the three lines drawn from the three angles to the middle of the opposite sides.



Let ABC be the △. Bisect the sides AB in D, AC in F, CB in E.

Put $AE=a$, $BF=b$, $CD=c$, $AD=u$, $AF=x$, $BE=y$.

Now by (th. 39, b. i), we have,

$$u^2+c^2=4x^2+4y^2 \quad (1)$$

$$x^2+b^2=4u^2+4y^2 \quad (2)$$

$$y^2+a^2=4u^2+4x^2 \quad (3)$$

By addition, $a^2+b^2+c^2=7(x^2+y^2+u^2)$

Whence, $\frac{4}{7}(a^2+b^2+c^2)=4x^2+4y^2+4u^2 \quad (4)$

From (1), $u^2+c^2=4x^2+4y^2$

By subtraction, $\frac{4}{7}(a^2+b^2+c^2)-u^2-c^2=4u^2$

Or, $4a^2+4b^2+4c^2-7u^2-7c^2=28u^2$

$$4a^2+4b^2-3c^2=35u^2$$

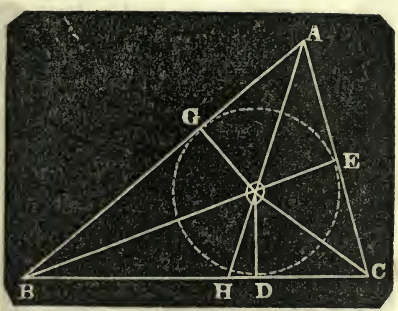
$$u=\pm\sqrt{\frac{4a^2+4b^2-3c^2}{35}}$$

By *inference* $x=\pm\sqrt{\frac{4a^2+4c^2-3b^2}{35}}$

And, $y=\pm\sqrt{\frac{4c^2+4b^2-3a^2}{35}}$

PROBLEM 24.

In a triangle, having given the three sides, to find the radius of the inscribed circle.



Let ABC be the $\triangle$. From the center of the circle O, let fall the perpendiculars OG, OE, OD, on the sides.

These perpendiculars are all equal, and each equal to the radius required.

Let the side opposite to the angle A, be represented by a, the side opposite B by b, and opposite C by c. Put OE, OD, &c. equal to r.

It is obvious that the double area of the $\triangle$ BOC is expressed by ar; the double area of AOB by cr; the double area of AOC by br; Therefore, the double area of ABC is $(a+b+c)r$.

From A let drop a perpendicular on BC, and call it x.

Then $ax=$ the double area of ABC. Consequently,

$$(a+b+c)r=ax \qquad (1)$$

The perpendicular from A will divide the base BC into two segments, one of which is $\sqrt{c^2-x^2}$, the other, $\sqrt{b^2-x^2}$, and the sum of these is a; therefore,

$$\sqrt{c^2-x^2}+\sqrt{b^2-x^2}=a \qquad (2)$$

$$\sqrt{c^2-x^2}=a-\sqrt{b^2-x^2}$$

$$c^2-x^2=a^2-2a\sqrt{b^2-x^2}+b^2-x^2$$

$$2a\sqrt{b^2-x^2}=a^2+b^2-c^2$$

$$\sqrt{b^2-x^2}=\frac{a^2+b^2-c^2}{2a}=m$$

Whence, $\quad b^2-x^2=m^2$

Or, $\quad x=\sqrt{b^2-m^2}$

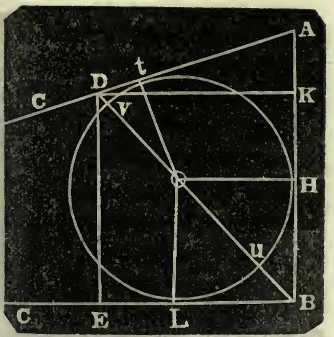
This value of x put in (1), gives

$$(a+b+c)r=a\sqrt{b^2-m^2}$$

Whence, $\quad r=\dfrac{a\sqrt{b^2-m^2}}{a+b+c}$, the required result.

PROBLEM 25.

To determine a right angled triangle, having given the side of the inscribed square, and the radius of the inscribed circle.



Let O be the center of a circle, OH or $OL=r$, the given radius, BE or $ED=a$, a side of the given square.

BO is the diagonal of r^2, and BD is the diagonal of a^2, and BOD is one continuous line.

The point D of the given square may be in the circle, in that case the hypotenuse touches the circle and the square in the same point, and that point is the middle of the hypotenuse. If the point D is not on the circumference, it must be without, as here represented.

Draw Dt to touch the circle in t, and that line produced both ways will define the hypotenuse.

AC and BC will meet if produced, and ABC will be the triangle required.

$OB=\sqrt{2}r$, $BD=\sqrt{2}a$, $DO=(a-r)\sqrt{2}$, $DV=(a-r)\sqrt{2}-r$, $DU=(a-r)\sqrt{2}+r$.

Now as D is a point without a circle, and Dt touching it, we have by (th. 18, b. iii), $(Dt)^2=DV\times DU$; that is,

$$(Dt)^2=[(a-r)\sqrt{2}-r]\,[(a-r)\sqrt{2}+r]=2a^2-4ar+r^2.$$

Whence, $$Dt=\sqrt{2a^2-4ar+r^2}=c.$$

Because A is a point without a circle, and AH, At, lines drawn touching the circle, $AH=At$, (th. 18, b. iii, scho. 2.)

Observe that $KH=a-r=d$. Put AH, At, each equal to x; then in the $\triangle\ AKD$ we have $AK=x-d$, $AD=x+c$, $KD=a$.

Whence, $(x+c)^2=a^2+(x-d)^2$.

$$x^2+2cx+c^2=a^2+x^2-2dx+d^2$$

$$x=\frac{a^2+d^2-c^2}{2c+2d}$$

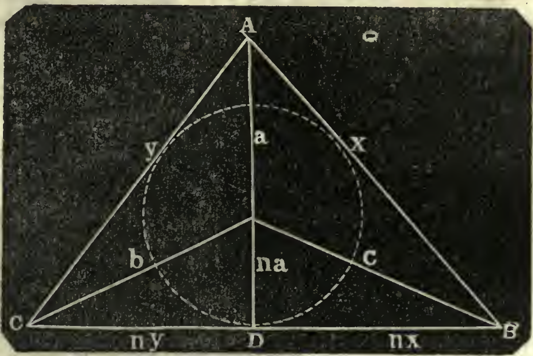
Now the value of x being known, AB is known, and all the sides of the $\triangle\ AKD$. But the $\triangle\ AKD$ is proportional to the triangle ABC, and gives us

$$AK\ :\ KD\ ::\ AB\ :\ BC$$

The first three terms of this proportion being known, the last is known, and the triangle is fully determined.

PROBLEM 26.

To determine a triangle and the radius of the inscribed circle, having given the lengths of three lines drawn from the three angles to the center of that circle.



Let ABC be the $\triangle$, O the center of the circle.

Put $AO=a$, $OB=c$, $OC=b$. AO bisects the angle A.

Produce AO to D. Then because the angle A is bisected, $CD : DB :: AC : AB$.

Put $AB=x$, $AC=y$, and let the ratio of AB to BD be n; then $nx=BD$ and $ny=CD$.

Now as the angle C is bisected by CO, we have

$$AC : CD :: AO : OD$$

That is, $\quad y : ny :: a : OD$

Whence, $\quad OD=na.$

Because AD bisects the angle A, we have, (th. 20, b. iii),

$$xy=a^2(1+n)^2+n^2xy \qquad (1)$$

Also, $\quad nx^2=c^2+na^2 \qquad (2)$

And, $\quad ny^2=b^2+na^2 \qquad (3)$

From (1), $\quad xy=\frac{a^2(1+n)^2}{1-n^2}=\frac{a^2(1+n)}{1-n} \qquad (4)$

The product of (2) and (3), gives

$$n^2x^2y^2=(c^2+na^2)(b^2+na^2) \qquad (5)$$

Squaring (4), and multiplying the result by n^2, also gives

$$n^2x^2y^2=\frac{a^4(1+n)^2n^2}{(1-n)^2} \qquad (6)$$

Equating (5) and (6), gives

$$(c^2+na^2)(b^2+na^2)(1-n)^2=a^4(1+n)^2n^2$$

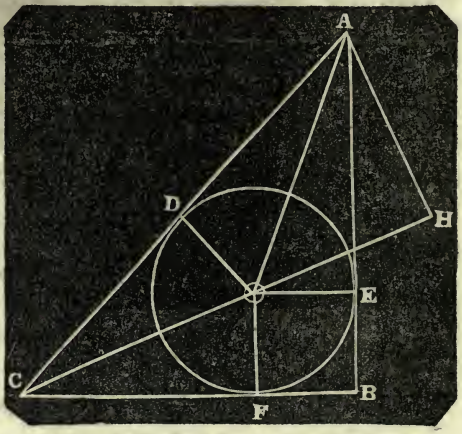
This equation contains only one unknown quantity n, but it rises to the fourth power—hence this problem is not susceptible of a solution from this notation short of an equation of the fourth degree.

In cases where a, b, and c are numerically given, the solution may be possible through an equation of the second or third degree.

We perceive by the figure, that if $b=c$, x must equal y.

PROBLEM 27.

To determine a right angled triangle, having given the hypotenuse and the radius of the inscribed circle



Let ABC be the $\triangle$, EO the radius of the circle. $AE=AD=x$, $CD=CF=y$. Then $AB=x+r$. $BC=y+r$.

By the right angled triangle,

$$(x+r)^2+(y+r)^2=(x+y)^2 \quad (1)$$

$$x+y=a \quad (2)$$

Reducing (1), gives $xy=rx+ry+r^2$.

That is, $$xy=ar+r^2 \quad (3)$$

From (2) and (3), x and y are easily found.

In numerical problems, great advantage can be taken of multiple numbers, the same as we have shown in common algebra. The following example will be sufficient.

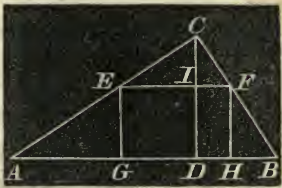
The sum of the two sides of a plane triangle is 1155, *the perpendicular drawn from the angle included by these sides to the base, is*

300 ; *the difference of the segments of the base is* 495. *What are the lengths of the three sides?* *Ans.* 945, 375, 780.

Write the given numbers in order, thus, 300, 495, 1155. Divide them by 15, and their relation is 20, 33, 77.

The two latter numbers have a common factor 11, which call a. Put $b=20$.

Then the three given lines will be b, $3a$, and $7a$.



Let $CB=x$, then $AC=7a-x$. $BD=y$, then $AD=y+3a$. $CD=b$.

In the right angled $\triangle$ CDB, we have $y^2+b^2=x^2$ (1)

ADC gives

$(y+3a)^2+b^2=(7a-x)^2$ (2)

Expanding (1) and subtracting (2) from it, gives

$$6ay+9a^2=49a^2-14ax$$

$$3ay=20a^2-7ax$$

Divide by a and write b in the place of 20, then

$$3y=ab-7x$$

Squaring, $9y^2=a^2b^2-14abx+49x^2$

From (1), $9y^2=-9b^2 \quad +9x^2$

By subtraction, $0=(a^2+9)b^2-14abx+40x^2$

Divide by b (or 20), then

$$0=(a^2+9)b-14ax+2x^2$$

$$4x^2-28ax=-2a^2b-18b$$

Add $(49a^2)$, $4x^2-28ax+49a^2=49a^2-2a^2b-18b$

$$=9a^2-360=729$$

By evolution, $2x-7a=\pm 27$

$$2x=77\pm 27=50, \text{ or } 104$$

$$x=25, \text{ or } 54$$

Here 25 is the number that corresponds with the problem; therefore, $BC=25\cdot 15=375$. We multiply by 15, because we reduced the numbers in the first place by dividing by 15.

16

SECTION II.

TRIGONOMETRY.

We shall here attempt to show the most *practical* method of finding the circumference of a circle to radius unity; and of finding the sines and cosines.

The trigonometrical equations that we may call into immediate use, are the following:

We number them as they are numbered in Robinson's Trigonometry.

$$\sin.(a+b)=\sin.a\cos.b+\cos.a\sin.b \qquad (7)$$

$$\sin.(a-b)=\sin.a\cos.b-\cos.a\sin.b \qquad (8)$$

$$\cos.(a+b)=\cos.a\cos.b-\sin.a\sin.b \qquad (9)$$

$$\cos.(a-b)=\cos.a\cos.b+\sin.a\sin.b \qquad (10)$$

$$\sin.^2a+\cos.^2a=1 \qquad (1)$$

$$\sin.2a=2\sin.a\cos.a \qquad (30)$$

Or, $$\sin.a=2\sin.\tfrac{1}{2}a\cos.\tfrac{1}{2}a$$

By problem 23, book iv. of Robinson's Geometry, we learn that if we divide the radius into extreme and mean ratio, and take the greater segment, that segment will be the chord of 36°.

Let 1 be the radius of a circle, and x the greater segment required; then

$$1 : x :: x : 1-x$$

Whence, $x=-\frac{1}{2}+\frac{1}{2}\sqrt{5}=0.6180340$, the chord of 36° in a circle whose radius is unity.

We learn by theorem 5, book v, Robinson's Geometry, that when c represents any chord of a circle, and x a chord of one-third of that arc, the following equation will exist:

$$x^3-3x=-c.$$

Put $c=0.618034000$, and a solution of the equation gives the chord of 12°. Again, put c equal to the chord of 12°, and another application of the equation will give the chord of 4°, and thus by the successive application of this equation, we have found the following values:

1. The chord of $36°=0.618034000$.
2. The chord of $12°=0.209056903$.
3. The chord of $4°=0.069798981$.
4. The chord of $80'=0.023270528$.

By theorem 4, book v, we learn that if c represent the chord of any arc, the chord of *half* that arc will be represented by

$$\sqrt{2-\sqrt{4-c^2}}.$$

Having the chord of 80′ the preceding expression gives us the chord of 40′, 20′, and 10′, as follows:

Chord of $40'=0.0116355131$.
Chord of $20'=0.0058177579$.
Chord of $10'=0.0029088819$.

The *chord* of 10′ so nearly coincides with the *arc* of 10′, that for all practical purposes, we may consider the chord and *arc* the same; then the semicircumference must be 1080 times 0.0029088819, or 3.141592452. A more exact determination gives 3.141592653+ for the length of 180°, when the radius is unity.

The chords of all arcs under 10′ *can be found from that chord, directly by division.*

As the sine of an arc is half the chord of double the arc, therefore, we can have the natural sine of 18° by dividing the chord of 36° by 2.

Having the sine of any arc, we can find its cosine by the following equation:

$$\cos. a=\sqrt{1-\sin.^2 a}$$

When $\sin.^2 a$ is a *very small* fraction, as it is for all *arcs* under 10′, then $\sqrt{1-\sin.^2 a}$ is very nearly equal to $(1-\frac{1}{2}\sin^2 a)$.

By the foregoing we find the following *sines* and *cosines*:

$\sin. 1'=.0002908881$	$\cos. 1'=.9999999576$
$\sin. 2'=.0005817762$	$\cos. 2'=.9999998802$
$\sin. 3'=.0008726643$	$\cos. 3'=.9999996692$
$\sin. 4'=.0011635524$	$\cos. 4'=.9999993231$

sin. 5′=.0014544405	cos. 5′=.9999989423
sin. 6′=.0017453286	cos. 6′=.9999984769
sin. 7′=.0020362167	cos. 7′=.9999979269
sin. 8′=.0023271036	cos. 8′=.9999972926
sin. 9′=.0026179916	cos. 9′=.9999965731
sin. 10′=.0029088789	cos. 10′=.9999957689
sin. 20′=.0058177378	cos. 20′=.9999830770
sin. 30′=.0087265343	cos. 30′=.9999618877
sin. 40′=.0116352640	cos. 40′=.9999323090

From the chords of 4°, 12°, and 36°, we readily find

sin. 2°=.0348995000	cos. 2°=.9993908139
sin. 6°=.1045284515	cos. 6°=.9945218389
sin. 18°=.3090170000	cos. 18°=.9510546619

Having the foregoing sines and cosines, we can find the sines and cosines of certain other arcs as follows :

Put $2a=$ to any arc whose sine is known, then we can obtain the sines and cosines of the half of $2a$, or a, by the following general equations:

$$\cos.^2 a+\sin.^2 a=1 \qquad (1)$$

$$2\cos. a \sin. a = \sin. 2a \qquad (2)$$

Now if we suppose $2a=18°$, we have $\sin. 2a=.3090170000$; and by substituting this value of $\sin. 2a$, and adding and subtracting the equations, we shall have

$$\cos.^2 a+2\cos. a \sin. a+\sin.^2 a=1.3090170000 \qquad (3)$$

and $$\cos.^2 a-2\cos. a \sin. a+\sin.^2 a=0.6909830000 \qquad (4)$$

By extracting the square root,

$$\cos. a+\sin. a=1.1441228508 \qquad (5)$$

$$\cos. a-\sin. a=0.8312532699 \qquad (6)$$

By adding (5) and (6), and dividing by 2, we find

$$\cos. a=\cos. 9°=.9876880603$$

Subtracting (6) from (5), and dividing by 2, gives

$$\sin.a=\sin.9°=.1564342904$$

If we put $2a=6°$, a like operation will give the cosine and sine of 3°. If we put $2a=2°$, a like operation will give the cosine and sine of 1°, and so on.

Again, we must not overlook the fact that the cosine of 2° is the same value as the sine of 88°; therefore, if we put $2a=88°$, an operation like the preceding will give us the cosine and sine of 44°. Another operation will give us the cosine and sine of 22°, and still another of 11°, and so on.

If we require the cosine and sine of any particular *arc* that we cannot arrive at, by any of these subdivisions, we may apply the following equations:

$$\sin.(a+b)=\sin.a\cos.b+\cos.a\sin.b \qquad (7)$$

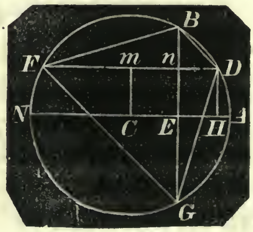
$$\sin.(a-b)=\sin.a\cos.b-\cos.a\sin.b \qquad (8)$$

For example, if $a=6°$ and $b=1°$, and we have the cosine and sine of 6° and 1°; then (7) will give us the sine of 7°, and equation (8) will give us the sine of 5°.

Whence by equations (1), (2), and (7), (8), the sine and cosine of every degree of the quadrant can be obtained without the trouble of fractional parts of degrees.

But there is a better method to continue the table after a beginning has been made, which we illustrate by the following example:

Suppose we have the sine and cosine of 15° and 16°, and also the sine and cosine of each of the small arcs from zero to 5° or 6°, and require the sine and cosine of 17° or of any other arc under 20°, we would operate as follows:



Let the arc $AD=15°$, $BD=2°$; then $AB=17°$, $AG=17°$, $DG=17°+15°=32°$.

Draw the chord BD. Now because an angle at the circumference is measured by half its subtended arc, therefore, the angle $nBD=16°$. The chord BD is double the sine of 1°; and it is obvious that we have BD and all the angles of the small right angled triangle nBD; and if we compute Bn, and add it to DH, the sine of 15°, we shall have BE, the sine of 17°; and nD subtracted from CH, will give the cosine of 17°.

The computation is as follows:

(We use the logarithmic sines and cosines, diminishing the indices by 10. to correspond with radius unity in the table of natural sines.)

Log. sine 1°....—2.241855
Log. 2........... .301030
Log. of BD....—2.542885...........—2.542885
sine 16°....—1.440338 cosine...—1.982842
nD....009621 —3.983223 nB .033554—2.525727
Nat. cos. 15°....96593 ....Nat. sin. 15° .258820
Nat. cos. 17°....95631 ....Nat. sin. 17° .292374

Thus we can go on and compute the sine and cosine of 19°.

Remark. When the triangle nBD is taken sufficienly small, the chord BD is confounded with the arc, and the triangle is then called the *differential* triangle, and figures largely in the *differential* calculus; and by it we can readily compute the sine and cosine of (15° 1′), (15° 2′), &c., having the sine and cosine of the degree, whatever it may be.

If we were making a table of sines and cosines for every minute of the quadrant, it would require too much labor to use the foregoing equations for every minute, we would use them for every degree, and then fill up the sines and cosines for the intermediate minutes by

INTERPOLATION.

In the appendix to Robinson's University Algebra, standard edition, is the following formula for inserting any intermediate term of a series:

$$a_n = a + nb + n\left(\frac{n-1}{2}\right)c + n\left(\frac{n-1}{2}\right)\left(\frac{n-2}{3}\right)d + \text{\&c.}$$

In this formula a is the first term of a series consisting of a, a_1, a_2, a_3, &c., terms, b is the first term of the first difference, c is the first term of the second difference, and so on. The interval between two given numbers in the series is always to be taken as unity, therefore, n is a fractional part of that unit.

The following example will clearly illustrate.

1. *Given the sines of* 1°, 2°, 3°, 4°, 5°, *and* 6°, *to find the sines of* 1° 12′, 2° 12′, 3° 12′, *and* 1° 24′, 2° 24′, *or to find the sine of any arc between* 1° *and* 3° *by interpolation.*

(a)	1st diff. (+b)	2d diff. (—c)	3d diff. (—d)
sin. 1°=.0174524035			
sin. 2°=.0348995000	.0174470965		
sin. 3°=.0523359508	.0174364508	106457	
sin. 4°=.0697564685	.0174205177	159331	52874
sin. 5°=.0871557450	.0173992765	212412	53081
sin. 6°=.1045284515	.0173727165	265600	53188

To interpolate 12′, we must put n of the formula $=\frac{12}{60}$.

Whence, $n=\frac{2}{10}$ $\quad \frac{n-1}{2}=-\frac{4}{10}$ $\quad \frac{n-2}{3}=-\frac{6}{10}$

And, $n=\frac{2}{10}$ $\quad n\cdot\frac{n-1}{2}=-\frac{8}{100}$ $\quad n\cdot\frac{n-1}{2}\cdot\frac{n-1}{3}=\frac{48}{1000}$

The products will be positive or negative according to the rules of multiplication. For the sine of any arc between 1° and 2°, we take the first line of the column under a for the first term of the series, and the first line of the column under b for the first difference, and so on.

To find the sine of any arc between 2° and 3°, we must take the second line of the column under a for the first term of the series, and the second line of the column under b for the first difference, and so on.

Whence, the following equations :

$$\text{sin. } 1°\ 12'=.0174524034+\tfrac{2}{10}(.0174470965)+\tfrac{8}{100}(\ 106457)-\tfrac{48}{1000}(\ 52874)=.0209424308$$

$$\text{sin. } 2°\ 12'=.0348995000+\tfrac{2}{10}(.0174364508)+\tfrac{8}{100}(\ 159331)-\tfrac{48}{1000}(\ 53081)=.0383878114$$

$$\sin. 3^\circ\ 12' = .0523359508 + \tfrac{2}{10}(.0174205177) + \tfrac{8}{100}(.212437) - \tfrac{48}{1000}(.53213) = .0558214985$$

If we put $n=\frac{24}{60}=\frac{4}{10}$, we can find the sine of 1° 24′, 2° 24′, and 3° 24′, in precisely the same manner.

In short, if we put n equal any number of times $\frac{1}{60}$, we can find the sine of the degree and that number of minutes, but it is best to be regular, and find the sines to 1° 12′, 1° 24′, 1° 36′, and so on, and then interpolate again between the numbers thus found.

Little attention has been paid to this subject of late, because the labor when once done, is done forever; and it has all been done in the preceding age; our object has been to present a systematic view of the whole matter, and show the student that the task of computing a trigonometrical table is not so great as is generally supposed.

We have thus far computed *natural* sines and cosines, but we generally use *logarithmic* sines and cosines.

To find the logarithmic sine, we simply take the logarithm of the natural sine from a table of the logarithms of numbers, increasing the index by 10.

After a few logarithmic sines have been found at equal intervals of *arc*, then the intermediate logarithms can be found directly by *interpolation.*

To make a table of logarithmic sines true to six places of decimals, we must compute with at least eight decimal places; and to make a table *true to nine places of decimals,* we must compute with twelve decimal places.

To show the advantage of working on a large scale, we will require the log. sines of 1°, 2°, 3°, 4°, 5°, and 6°, true to *nine* places of decimals.

The natural sines we already have, and the necessary tables of logarithms are in the latter part of this volume, the same as are to be found in our Surveying and Navigation.

Most operators would take out the logarithm of each natural sine separately, having no connection with each other, but this would require much unnecessary labor, and it is to explain the artifices, that we bring forward the example.

In the first place we will take the sine of 6°, that is, find the log. of the number .1045284515.

Log.	.104	....	—1.017033339299
Factor,	1.005....	Table *A*, ...	.002166071750
Log. prod.,	.104520	....	.019199411049
	1.00008...	Table *C*, ...	34742166
	.104528 36160	log.........	.019234153215

Dividing the given number by this number, we find another factor to be 1.0000008. The log. of this factor corresponds to 8(*c*) in table *C*; therefore,

—1.019234153215
347432

Log. .1045284515=.........—1.019234500647, nearly.
Add 10.

Tabular log. sine 6°= 9.019234500647

Having the logarithm of .1045284515, and requiring that of .0871557425, we first consider whether we cannot find some convenient divisor to the first that will produce the second for a quotient, or produce a number very near the second. To find definitely what this divisor is, represent it by *D*; then

$$\frac{.104528}{D}=0.087155. \quad \text{Whence, } D=1.2, \text{ nearly.}$$

	Log. .1045284515	....	—1.019234500647
Divide by 1.2)		log.	0.079181246048
	Gives .087107043	log.	—2.940053254599
Factors,	1.0005	....	217099966
	1.00005	Table *C*, ...	21714178
	1.000008	....	3474352
	1.0000008	....	347435
Prod. nearly=	.0871557425	log. nearly=	—2.940295890530

We found these factors by taking .0871557425 for a dividend, and .087107043 for a divisor; the quotient is 1.0005588, which we directly separate into the single factors, 1.0005, 1.00005, &c. (See Art. 14, Robinson's Surveying and Navigation.)

We can easily find the logarithmic sine of 1°, because the number .0174524035 happens to be peculiarly favorable ; $174 = 87 \cdot 2$, and 174 multiplied by 1003, gives 174522. Whence, $.87 \times .02 \times (1.003) = .0174522$.

Again, $\frac{.0174524035}{.017422} = 1.00001166.$

Factors,	Log. .02	....	—2.301 029 995 644	
	Log. .87	....	—1.939 519 252 619	
	Log. 1.003	....	.001 300 943 017	
	Log. 1.00001	Table *A*,	4 342 923	(*a*)
	Log. 1.000001	....	434 294	(*b*)
	Log. 1.0000006	....	260 574	(6*c*)
	Log. 1.00000006	....	26 058	(6*d*)
Product nearly	.0174524035	log.	—2.241 855 255 129	
			Add 10.	
Tabular log. of 1°, therefore, is			8.241 855 255 129	

To find the logarithmic sine of 2°, we proceed thus : To the log. of the sine of 1°, add the log. of the number 2, then we shall have the logarithm of a number a little above the one required, which can be reduced by division.

Log. .0174524035...........	—2.241855255129	
Add log. 2,........................	.301029995644	
Log. .034904807............	—2.542885250773	
Divide by 1.000152 Sub. log. 1.0001	43427277	
	—2.542841822496	
Also sub. log. 1.00005..............	21714178	
	—2.542820108318	
Also sub. log. 1.000002..............	868588	(2*b*)
Log. .0348995=..............	—2.542819239730	very
Add	10.	nearly.
Tabular log. 3° (true to 10 places),=...	8.5428192397	

We found the divisor 1.000152 by the following equation.

Calling D the divisor sought, then

$$\frac{.034904807}{D}=.0348995.\quad \text{Whence, } D=1.000152.$$

By a little examination, we shall find that if we multiply the sine of 1° by 3, and divide the product by 1.000406, the quotient will be the sine of 3° *very nearly*.

To the log..................	—2.241 855 255 129
Add log. 3,.................	477 121 254 720
	—2.718 976 509 849
Sub. log 1.0004..............	173 690 053
	—2.718 802 819 796
Also, sub. log. 1.000006...........	2 605 764 (2*b*)
Log. sine of 3°=.................	—2.718 800 214 032
Add 10.	
Tabular log. sine 3°=.............	8.718 800 214 032

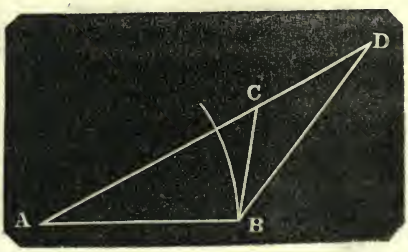
In a similar manner we can find the logarithmic sine of 4°.

If it were our object to compute a table of logarithmic sines and cosines for every degree and minute of the quadrant, we would first compute each degree and half degree in natural numbers—and take the logarithms of those numbers. Then we would *interpolate* for the intermediate logarithms.

We now proceed to solve problems in Trigonometry and Mensuration.

(Problems 1 and 2 are on page 167, Robinson's Geometry.)

1. *Given* AB 428, *the angle* C 49° 16′, *and* (AC+CB), 918, *to find the other parts.*



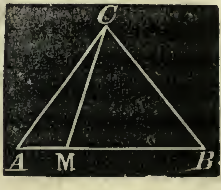
Let ABC represent the △. Draw AD=918. From D draw DB so that the angle ADB shall be half the angle ACB, that is, 24° 38′. From A as a center with AB as a radius, strike an arc cutting BD in B. From B make the angle DBC=24° 38′: then ACB will be the △ required.

In the △ADB we have AB : AD : : sin. D : sin. ABD.
That is, 428 : 918 : : sin. 24° 38′ : sin. ABD.

Sin. 24° 38′..............	9.619938
Log. 918................	2.962843
	12.582781
Log. 428................	2.631444
Sin. 63° 22′ 48″ or its supplement 116° 37′ 12″..	9.951337
From this take DBC...........	24° 38′
ABC=...............	91° 59′ 12″

Having now two angles of the △ ABC, we have the third angle A=38° 44′ 48″, and with all the angles, and the side AB, we find AC=564.49, and conseqently BC=354.51.

(2.) *Given a side and its opposite angle, and the difference of the other two sides, to construct the triangle and find the other parts.*



Let ABC be the △. AC=126, B= 29° 46′, and AM, the difference between AB and BC, =43.

From 180° take 29° 46′ and divide the remainder by 2. This gives the angle BMC or BCM. BMC taken from 180°, gives AMC.

Now in the △ AMC, we have the two sides AC, 126, AM, 43, and the angle AMC, to find the angle A. The computation is as follows: 180°—29° 46′=150° 14′; half, =75° 7′=BMC. 180°—75° 7′=104° 53′=AMC. Now in the △ AMC, we have

AC : AM : : sin. 104° 53′ : sin. ACM

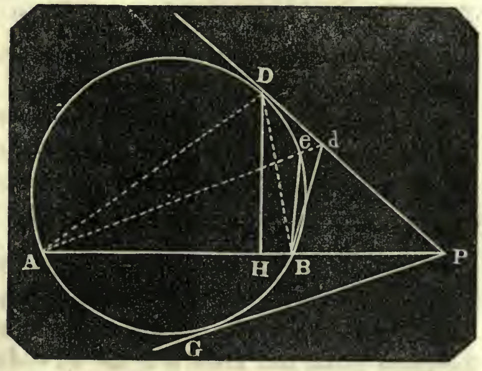
Sin. 104° 53′=cos. 14° 53′.

126 : 43 : : cos. 14° 53′ : sin. ACM

Cos. 14° 53′.......................	9.985180
Log. 43..........................	1.633468
	11.618648
Log. 126.........................	2.100371
Sin. ACM=sin. 19° 22′ 28″.................	9.518277

Whence, $ACB=75° 7'+19° 22' 28''=94° 29' 28''$. Consequently $A=55° 51' 32''$. Now we have all the angles, and AC, of the $\triangle ABC$.

(3.) *Two lines meet, making an angle of* 50°. *On one line are two objects, one* 200, *the other* 500 *yards from the angular point. Where abouts on the other line will these two objects appear under the greatest possible angle, and what will that angle be?*



Let PA and PD be the two lines, and A and B the two objects.

Let D be the required point on the other line. Then $PD=\sqrt{PA\times PB}$ $=\sqrt{500\times 200}=$ 316.226+ yards; but this requires demonstration.

If we make $PD=\sqrt{PA\times PB}$, and then pass a circle through the points A, B, and D, PD will touch the circle in the point D. (Th. 18, b. 3, scho.) And because PD is a tangent, the angle ADB at the point of contact, is greater than any other angle AdB, on either side of D, (see th. 7, page 101 of this volume.) Or we may prove it here. $AeB=ADB$, (th. 9, b. iii, scho.); but AeB is greater than AdB, therefore, ADB is greater than AdB; that is, greater than any angle drawn from any point between P and D. The same demonstration will apply on the other side of D.

The computation for the angle, is as follows:

From D let drop the perpendicular DH, then in the $\triangle PDH$, we have

As radius,	10.000000		10.000000
To PD.	2.500000		2.500000
So is sine 50°,	9.884252	cosine	9.808067
To DH, 242.24,	2.384252	PH, 203.26,	2.308067

From PH take PB, and we have $HB=3.26$. From PA take PH, and we have $AH=296.34$.

Now $\quad HB : HD :: R : \text{tang. } ABD.$

$\quad AH : HD :: R : \text{tang. } BAD.$

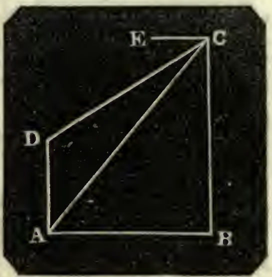
	12.384252		12.384252
HB	0,513218	AH	2.471800
Tan. ABD 89° 14′	11.871034	tan. BAD 39° 16′	9.912452

$180°-(89°\ 14'+39°\ 16')=51°\ 30'=ADB$, the greatest angle required.

At the point G on the line PG, the objects A and B would extend the greatest possible angle, and in that case also, $PG=\sqrt{PA\times PB}$. But the angle AGB must be of such a value that $ADB+AGs=180°$; therefore, $AGB=128°\ 30'$.

(The following are on page 174, Robinson's Geometry.)

(3.) *From an eminence of* 268 *feet in perpendicular height, the angle of depression of the top of a steeple which stood on the same horizontal plane, was found to be* 40° 3′, *and of the bottom* 56° 18′. *What was the height of the steeple?* *Ans.* 117.8 feet.



Let BC be the eminence 268 feet, and AD the steeple. Draw CE parallel to the horizontal AB. Then $ECD=40°\ 3'$, $ECA=CAB=56°\ 18'$.

$DCA=56°\ 18'-40°\ 3'=16°\ 15'$.

$DAC=90°-56°\ 18'=33°\ 42'$

In the $\triangle\ ABC$, we have

$\text{sin. } 56°\ 18' : 268 :: \text{sin. } 90° : AC.$

$$AC=\frac{268\times R}{\text{sin. } 56°\ 18'}$$

In the $\triangle\ ADC$, we have the supplement to the angle ADC equal to 16° 15′ added to 33° 42′, or 49° 57′; therefore,

As $\quad \text{sin. } ADC : AC :: \text{sin. } DCA : AD$

That is, $\text{sin. } 49°\ 57' : \dfrac{268\times R}{\text{sin. } 56°\ 18'} :: \text{sin. } 16°\ 15' : AD$

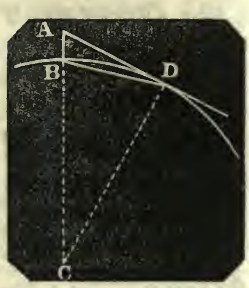
$$AD = \frac{268 \cdot R \cdot \sin. 16° 15'}{\sin. 49° 57' \cdot \sin. 56° 18'} = \frac{2.428135 + 10. + 9.446893}{9.883836 + 9.920099} =$$

$$21.875028 - 19.803935 = 2.071093$$

Log. $AD=2.071093$. Whence, $AD=117.78$ feet.

(4.) *From the top of a mountain three miles in height, the visible horizon appeared depressed 2° 13′ 27″. Required the diameter of the earth, and the distance of the boundary of the visible horizon.*

Ans. Diameter of the earth 7958 miles, distance of the horizon 154.54 miles.



Let AB represent the mountain, and AD the visible distance. AB produced will pass through the center of the earth at C. From D draw CD perpendicular to AD. Join BD. ADC is a right angled triangle.

$CAD=90°-2°\ 13'\ 27''=87°\ 46'\ 33''$. $ACD=2°\ 13'\ 27''$. $ADB=\frac{1}{2}ACD=1°\ 6'\ 44''$. $ABD=91°\ 6'\ 44''$.

Now in the △ ABD, we have

sin. 1° 6′ 44″ : 3 : : sin. 91° 6′ 44″ : AD.

Sin. 91° 6′ 44″=cos. 1° 6′ 44″................	9.999919
Log 3.........................	0.477121
	10.477030
Sin. 1° 6′ 44″............................	8.288029
Log. 154.54......................	2.189001

In the triangle ADC, we have

sin. ACD : AD : : cos. ACD : CD ·

Cos. ACD=cos. 2° 13′ 27″.................	9.999674
AD............................	2.189001
	12.188675
Sin. ACD=sin. 2° 13′ 27″.................	8.588932
	3.599743
Double	0.301030
Diameter....log. 7958 miles, nearly.........	3.900773

(Several of the following problems in Mensuration are taken from the Surveying and Navigation, page 60.)

(5.) *Find the length of an arc of* 30°, *the radius being* 9.

When the radius is 1, an arc of 180°=3.141592 ; therefore, an arc of 30° and radius 1 must be $\frac{3.141592}{6}$, and this multiplied by 9 must be the required result.

Hence, $\frac{3.141592\cdot3}{2}=4.712388$, *Ans.*

(6.) *Find the area of a circular sector whose arc is* 18°, *and radius* $1\frac{1}{2}$.

We must first find the length of the arc, as in the last problem, then multiply its half by the radius.

Whence, $\frac{3.141592}{180}\times18\times\frac{3}{4}=.3141592\times\frac{3}{4}=.235619=\frac{1}{2}$ arc.

Therefore the area *must be* $\frac{3}{2}\times0.235619=0.353427$, *Ans.*

The arc of 1° and radius unity is .0174533.

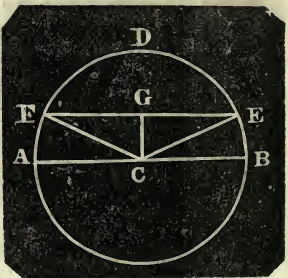
Therefore, that of 9° is .0174533×9, and this multiplied by the square of the radius will give the true result.

That is, $.0174533\times9\times\frac{9}{4}=0.353403$.

(7.) *Required the area of a sector whose radius is* 25, *and ar:* 147° 29′.

$$\frac{.0174533\times147.4833\times625}{2}=804.3986.$$

(8.) *What is the length of a chord which cuts off one-third of the area from a circle whose diameter is* 289 ? *Ans.* 278.6716.



Like many problems in relation to the circle, this can be solved *only by approximation.*

As circles are in all respects proportional to their radii, I will operate on radius unity, and in conclusion, multiply by $\frac{289}{2}$.

If the segment $EFD=\frac{1}{3}$ of the whole circle, $ABDE$ will equal $\frac{1}{6}$ of the whole. Because $\frac{1}{2}-\frac{1}{6}=\frac{1}{3}$.

The space $ABDE$ contains two equal sectors, DCB, ACE, and the triangle ECD. Put the arc $BD=x$, $CB=1$. Then $CG=\sin. x$, $GD=\cos. x$.

The area of the two sectors together is x.

The area of the triangle ECD is $\sin. x \cos. x$.

Therefore, $x+\sin. x \cos. x=\frac{1}{6}\pi$. $\pi=3.141592653+$

Double, $2x+2\sin. x \cos. x=\frac{1}{3}\pi$.

But $\sin. 2x=2\sin. x \cos. x$. (See eq. (30), page 143, Geom.)

Therefore, $2x+\sin. 2x=\frac{1}{3}\pi$ (1)

Here we have a correct and definite equation, but we cannot solve it, as it contains an *arc* and its *sine*, and they are not united by any definite numerical law; we must, therefore, resort to *approximation*.

We know that $\sin. 2x$ is not much less than $2x$.

Therefore, $4x=\frac{1}{3}\pi$ is *not far* from the truth.

Also, $2 \sin. 2x=\frac{1}{3}\pi$ is not far from the truth.—The one too small, the other too large.

That is, $x=\frac{1}{12}\pi$, *approximately*, and $\sin. 2x=\frac{1}{6}\pi$, *approximately*.

To find the arc BD approximately, we have this proportion:

$$\pi : \tfrac{1}{12}\pi :: 180^\circ : \text{Arc } BD. \quad \text{Whence, } BD=15^\circ.$$

By the table of natural sines we find $\sin. 2x = \sin. 31^\circ\ 34'$ nearly. Or, $x=15^\circ\ 47'$ nearly.

We now know that to make the area $ABDE=\frac{1}{6}\pi$, the arc BD must be *greater* than 15° and *less* than $15^\circ\ 47'$.

I will now suppose the arc $BD=15^\circ\ 20'$, and compute the area $ABDE$, corresponding to that supposition.

For the numerical value of the arc $15^\circ\ 20'$, we have the following proportion:

$$180^\circ : 15\tfrac{1}{3}^\circ :: 3.14159265 : \text{Arc } BD$$

Or, $540 : 46 :: 3.14159265 : \text{Arc } BD=0.2676175$.

The tables will give us $(\sin. 15^\circ\ 20')(\cos. 15^\circ\ 20')$ thus:

Sin. 15° 20′................9.422318
Cos. 15° 20′................9.984259

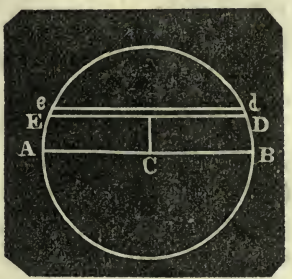
Sum less 20=log...........—1.406577=0.2550200 nearly.

$BD=x$=0.2676175 nearly.

Area $ABDE$=.................0.5226375 nearly.

But the required area of $ABDE$ is $\frac{1}{6}\pi$=....0.5235987 nearly.

Hence, 15° 20′ for BD, gives an area too small by 0.0009612



Now we wish to increase the area $ABDE$ by the *little narrow space* $EDde$, and this is so narrow that Dd and Ee are in respect to *practical* or *numerical* purposes, right lines, and *EDde is a trapezoid*, and its paralel sides may be taken as equal; it is then practically a parallelogram whose *area is given* and its longer side equal to 2(cos. 15° 20′).

Let y= the width of this parallelogram or trapezoid, (as we may call it either.) Then we shall have the following equation:

$$2\cos.(15° 20')y=0.0009612$$

Or, $$\cos.(15° 20')y=0.0004806$$

That is, $0.9644y=0.0004806$. Whence, $y=0.000498$

That is, we must increase the *natural sine* of BD 15° 20′, by 0.000498.

The natural sine of 15° 20′ is............0.264434
To which add........................0.000498

N. sin. of 15° 21′ 47″ cor. to sum..........0.264932

Thus we learn that the arc Bd corresponds to 15° 21′ 47″ as nearly as a table of natural sines computed to 6 decimal places will give it.

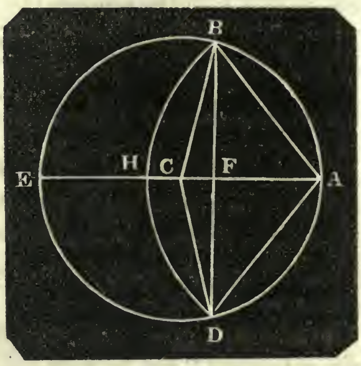
Twice the cosine of 15° 21′ 47″, to a radius of ½(289) is the chord sought, which we compute as follows:

Cos. 15° 21′ 47″..................9.984184
Log. 289.........................2.460898

Log. 278.67+....................2.445082

(9.) *What is the radius of a circle whose center being taken in the circumference of another containing an acre, shall cut off half of its contents?**

This problem is the same for circles of every magnitude; therefore, we will operate on a circle of radius *unity*.



Let x represent the number of degrees in the arc AB, and $\frac{\pi}{180}$ the length of each degree; then $\frac{\pi x}{180}$ represents the length of the *arc* AB.

$BF=\sin. x$. $CF=\cos. x$. $FA=1-\cos. x$. $(AB)^2=(1-\cos. x)^2+\sin.^2 x$, or $AB=\sqrt{2-2\cos. x}$, which equals the radius of the cutting circle.

The area of the sector $CBAD$, is measured by the arc $AB\cdot CA$; that is, $\frac{\pi x}{180}$. From this take the triangle CBD, or $\sin. x \cos. x$, and the segment ABD will be left. That is,

$$\text{Segment } ABFD=\frac{\pi x}{180}-\sin. x\cos. x. \quad (\pi=3.141592.)$$

---

*One reason for the appearance of this work is that it is required, because able mathematicians have written so obscurely. They seem to have written as I should, were I indifferent whether the reader, or rather the learner, understood me or not. Do not the following extracted solutions justify this observation? they are brief, to be sure, and no one sets a higher value on brevity than does the author of this work;—but nothing is meritorious which is wanting in perspicuity.

The following extracts are from the Mathematical Diary, published by James Ryan, 1825.

SOLUTION.—*By Robert Adrain, LL. D.*

In a circle to radius unity, let $2z$ be the arc of which the chord is the required radius, then π being the area of the given circle to radius unity, if we express analytically the area cut off by the radius sought and divide by 2, we obtain the transcendental equation

$$\left(\frac{\pi}{2}-z\right)\cos. 2z+\tfrac{1}{2}\sin. 2z=\frac{\pi}{4}.$$

Again, as $x=$ the degrees in AB, $(180-x)=$ the degrees in BE. Because the angle BAH is at the circumference, it is measured by $\frac{1}{2}$ of $(180-x)$, or $(90-\frac{1}{2}x)$.

Whence, the *arc* $BH=90-\frac{1}{2}x$, *measured in degrees.*

For the length of the *arc* BH, we observe that 180° of the circumference would be measured by $\pi\sqrt{2-2\cos.x}$.

for 1° then, we have $\frac{\pi\sqrt{2-2\cos.x}}{180}$, this multiplied by the number of degrees, $\left(90-\frac{x}{2}\right)$ will produce $\left(\frac{\pi\sqrt{2-2\cos.x}}{180}\right)\left(90-\frac{x}{2}\right)$ for the *linear* measure of the *arc* BH.

This multiplied by the radius AB, or $\sqrt{2-2\cos.x}$, will give the *area* of the sector $ABHD$; that is,

$$\text{Sector } ABHD=\left(\frac{\pi(2-2\cos.x)}{180}\right)\left(90-\frac{x}{2}\right)$$

From this subtract the triangle ABD, which is measured by $\sin.x(1-\cos.x)$, and we have the segment $BHDF$.

That is,

$$\text{Segment } BHDF=\frac{\pi(1-\cos.x)}{90}\left(\frac{180-x}{2}\right)-\sin.x+\sin.x\cos.x.$$

But segment $ABFD=\frac{\pi x}{180}-\sin.x\cos.x.$

The sum of these two segments is the double circular space, $ABHD$ required; that is,

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Hence, $z=35°\ 24'$, and therefore, if R=the radius of the given circle, the radius sought $=2\text{R}\sin.(35°\ 24')=1.158\text{R}$.

SOLUTION.—*By Dr. Henry J. Anderson.*

Let the radius of the given circle be represented by unity, and of the two portions of its circumference terminated at the intersections of the two circles let the greater be denoted by 2ϕ. Then, by the rules of mensuration, it will be found that the two parts into which the given circle is divided are equal, each to $2\phi\cos.^2\frac{1}{2}\phi+\pi-\phi-\sin.\phi$. Putting this equal $\frac{1}{2}\pi$, the area of the semicircle, and transposing, we have

$$\sin.\phi-(2\cos.^2\tfrac{1}{2}\phi-1)=\frac{\pi}{2},\quad \text{Or by trigo., } \sin.\phi-\phi\cos.\phi=\frac{\pi}{2},$$

whence, $\phi=109°\ 11'\ 17''$ and the required radius$=1.15874$.

If the contents of the given circle be one acre, then the required radius will be 206.7336 links, or about 45.4814 yards.

The *area* $ABHD=\frac{\pi x}{180}$—sin. $x+\frac{\pi}{180}\Big((1-\cos.x)(180-x)\Big)$

This reduces to $\left(1-\cos.x+\frac{x\cos.x}{180}\right)\pi-\sin.x$.

If we put this expression equal to the given quantity, $\frac{\pi}{2}$ we cannot resolve the equation, because it would contain the linear quantity x and the transcendental quantities sin. x and cos. x. Therefore if we solve the problem at all, we must do it indirectly, by approximation, as we are obliged to do with nearly all problems pertaining to the circle.

This expression is a general one, and if we assume x any number of degrees, we can readily obtain the corresponding value of the expression, and if any assumption corresponds to a given value, the problem is solved; and if it nearly corresponds, we shall have nearly the radius required, which can be increased or decreased, as we are about to explain.

For the *area* $ABHD$ to contain half of the circle, it is our judgment that the *arc* AB should contain about 75°; therefore, we assume $x=75°$ and the expression becomes

$$\left(1-\cos.75+\frac{10\cos.75}{24}\right)\pi-\sin.75.$$

By the table of natural sines we find

$$(0.74118+0.10787)\pi-0.96593$$

The final result of this supposition is that the area $ABHD=$ 1.701370. But the half of the circle is 1.570796; therefore, we have taken x too great to obtain the area of half the circle.

We will now take $x=70°$.

Then $\left(1-\cos.70+\frac{7\cos.70}{18}\right)\pi-\sin.70°$ will be an expression for a less area than before.

By log.	log. 0.79099................	—1.898175
	log. 3.1415926..............	0.497149
	2.48500	0.395324
Nat. sin. 70°..........	.93969	
Area ABHD..........	1.54531	
Given result..........	1.57079	
Error too small........	0.02548	

This error must be conceived to be a *winding parallelogram*, whose length is BHD. Dividing .02548 by BHD, will give the amount to be added to the radius AH. The radius AH or AB is $2\sin.35° = 1.14716$.

The angle $BAD = 180 - 70 = 110°$.

The *linear* value of $BHD = \frac{(1.14716)(3.141592)110}{180}$.

Now the amount that the radius must be increased is expressed by $\frac{(.02548)18}{(1.14716)(3.141592)11}$.

By logarithms. Log. .02548..............—2.406199

Log. 18............... 1.255273

—1.661472

Log. 1.14716........0.0596187

Log. 3.141592.......0.4971499

Log. 11.............1.0413927

1.5981613......... 1.598161

0.011568.........................—2.063311

Add...1.14716

1.158728= the required radius AB, which will cut the circle into two equal parts.

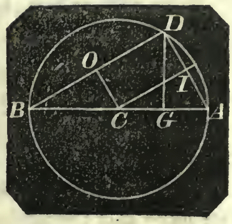
If the radius of the given circle is (a), in place of unity, then the radius of the cutting circle must be

$$(1.15828)a$$

To find the number of degrees and minutes in AB, divide 1.15828 by 2, which gives .57914 for the sine of half AB, or 35° 23′ 30″ or $AB = 70°\ 47'$.

The following theorems are extracted from pages 219 and 220 of Robinson's Geometry,

(1.) *Show geometrically, that* $R(R+\cos. A) = 2\cos.^2\frac{1}{2}A$; *and that* $R(R-\cos. A) = 2\sin.^2\frac{1}{2}A$.



Let CB or CA represent the radius of a circle and call it R. Let the arc $AD = A$, and draw the lines here represented.

Then $GD = \sin. A$, $CG = \cos. A$, $BG = R + \cos. A$, $GA = R - \cos. A$, $AI = \sin. \frac{1}{2}A$, $CI = \cos. \frac{1}{2}A$, $BD = 2\cos. \frac{1}{2}A$.

From C draw CO perpendicular to BD; then $BO = OD$, and the two $\triangle$'s BOC, BDG, are equiangular; therefore,

$$BG : BD :: BO : BC.$$

That is, $R + \cos. A : 2\cos. \frac{1}{2}A :: \cos. \frac{1}{2}A : R$.

Whence, $R(R + \cos. A) = 2\cos.^2 \frac{1}{2}A$. Q. E. D.

Again, by the similar $\triangle$'s AGD, ADB, we have

$$AG : AD :: AD : AB.$$

That is, $R - \cos. A : 2\sin. \frac{1}{2}A :: 2\sin. \frac{1}{2}A : 2R$.

Whence, $R(R - \cos. A) = 2\sin.^2 \frac{1}{2}A$. Q. E. D.

(2.) *Show that* $R \cdot \sin. A = 2\sin. \frac{1}{2}A \cos. \frac{1}{2}A$.

By similar triangles, we have

$$AC : CI :: AD : DG.$$

That is, $R : \cos. \frac{1}{2}A :: 2\sin. \frac{1}{2}A : \sin. A$.

Whence, $R \sin. A = 2\sin. \frac{1}{2}A \cos. \frac{1}{2}A$. Q. E. D.

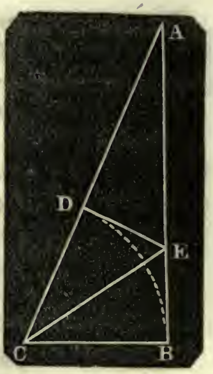
(3.) *Prove that* $\tan. A + \tan. B = \frac{\sin.(A+B)}{\cos. A \cos. B}$ *radius being unity.*

It is admitted that $\tan. A = \frac{\sin. A}{\cos. A}$, and $\tan. B = \frac{\sin. B}{\cos. B}$.

By addition, $\tan. A + \tan. B = \frac{\sin A}{\cos. A} + \frac{\sin. B}{\cos. B}$.

$$\frac{\sin. A \cos. B + \cos. A \sin. B}{\cos. A \cos. B} = \frac{\sin. (A+B)}{\cos. A \cos. B}. \quad \text{Q. E. D.}$$

(4.) *Demonstrate geometrically, that* $R \sec. 2A = \tan. A \tan. 2A + R^2$.



Take CB radius, let the arc $BD=2A$. Then $BA=\tan.2A$, $AC=\sec.2A$. Draw CE bisecting the angle ACB, then $BE=\tan.A$. Also, $DE=\tan.A$, because the two triangles CBE and CDE, are in all respects equal.

Now by the similar $\triangle$'s ADE, ABC, we have this proportion,

$$AD : DE :: AB : BC.$$

That is, $AD : \tan.A :: \tan.2A : R$

Whence, $AD \cdot R=\tan.A \cdot \tan.2A$

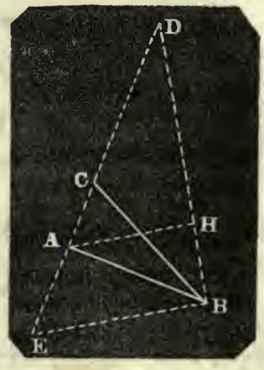
By adding R^2 to both members and factoring, we have

$$(AD+R)R=\tan.A \tan.2A+R^2.$$

But $(AD+R)=AC=\sec.2A$; therefore,

$$R \sec.2A=\tan.A \tan.2A+R^2. \qquad \text{Q. E. D.}$$

(5.) *Show that in any plane triangle, the base is to the sum of the other two sides, as the sine of half the vertical angle is to the cosine of half the difference of the angles at the base.*



Let ABC be the $\triangle$. Call AB the base, and produce AC the shorter side so that $CD=CB$ and $CE=CB$. Then if C be taken as the center of a circle and CB radius, that circle must pass through the points E, B, and D, and the angle EBD must, therefore, be a right angle.

Because ACB is the exterior angle of the $\triangle$ CDB, and that $\triangle$ isosceles, the angle ACB must equal $2D$, or the angle D is half the vertical angle.

Because BAC is the exterior angle of the $\triangle$ AEB, we have

$$BAC=AEB+ABE \qquad (1)$$

But $AEB=CBE=CAB+ABE$. This value of AEB, substituted in (1), gives $BAC=CBA+ABE+ABE$ (2)

Whence, $\qquad BAC-CBA=2ABE \qquad$ (3)

This last equation shows us that ABE is half the difference of the angles at the base.

Now in the $\triangle\ ABD$, we have

$$AB : AD :: \sin. D : \sin. ABD.$$

But the $\sin. ABD = \cos. ABE$; because the sum of these two angles make 90°. Hence the preceding proportion becomes

$AB : (AC + CB) :: \sin. \frac{1}{2} ACB : \cos. \frac{1}{2}(BAC - CBA)$. Q.E.D.

SCHO. 1. The $\triangle\ AEB$ gives us this proportion,

$$AB : AE :: \sin. E : \sin. ABE.$$

Because the angles E and D together make 90°, $\sin. E = \cos. D$.

Hence, $AB : AE :: \cos. D : \sin. ABE$.

That is, in relation to the triangle ABC, and generally,

*The base of any plane triangle, is to the difference of the other two sides, as the cosine of half the angle opposite to the base, is to the sine of half the difference of the other two angles.**

SCHO. 2. Draw AH parallel to EB, and of course perpendicular to DB; then we have

$$DA : AE :: DH : HB.$$

If AH be made radius, DH is tangent to the angle DAH, and HB is tangent to the angle BAH.

Because AH is parallel to EB, the angle BAH is equal to ABE; but ABE has been demonstrated to be equal to the half difference of the angles CAB, CBA; therefore, DAH is the half sum of the same angles, for the half sum and half difference of any two quantities make the greater of the two. Therefore, the preceding proportion becomes the following theorem:

As the sum of the sides is to their difference, so is the tangent of the half sum of the angles at the base, to the tangent of half their difference.

This theorem is demonstrated in some form in every treatise on plane trigonometry. It is the 7th prop., page 149, Robinson's Geometry

(6.) *The difference of two sides of a triangle, is to the difference of the segments of a third side, made by a perpendicular from the opposite angle, as the sine of half the vertical angle is to the cosine of half the difference of the angles at the base; required the proof.*

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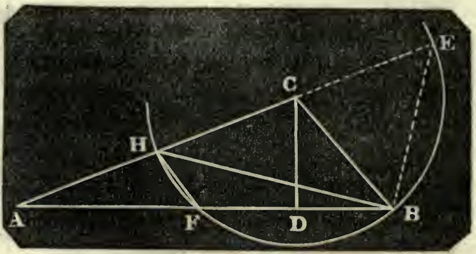
*This is theorem v, Robinson's Geometry, page 220.



Let ABC be the $\triangle$. On the shorter side CB as radius, describe a circle, cutting AB in F, AC in H, and produce AC to E. Draw CD perpendicular to the base, then DB is one segment of the base, AD is the other, and AF is their difference. AH is obviously the difference of the sides.

Now in the $\triangle$ AHF, we have

$$AH : AF :: \sin. AFH : \sin. AHF \quad (1)$$

This proportion demonstrates the theorem, as will appear when we show the values of these angles.

Because $CHFB$ is a quadrilateral in a circle, the angles

$$HFB+E=180°.$$

But $\qquad HFB+AFH=180°.$

By subtraction, $\quad E-AFH=0$, or $AFH=E$.

In the same manner we prove that $AHF=ABE$.

Substituting these equals in proportion (1), it becomes

$$AH : AF :: \sin. E : \sin. ABE \quad (2)$$

The angle E is half the angle ACB, because ACB is at the center of the circle, and E at the circumference intercepting the same arc.

Also, sin. ABE=cos. ABH, *because HBE is a right angle, and the sine of an arc over* 90° *is equal to the cosine of the excess over* 90°.

Again, BCE= the sum of the angles at the base. BHC, or its equal CBH, is half BCE; therefore, CBH = the half sum of the angles at the base of the triangle ACB, and HBA is their half difference. Whence proportion (2) becomes

$$AH : AF :: \sin.\tfrac{1}{2}(ACB) : \cos.\tfrac{1}{2}(ABC-A). \quad \text{Q.E.D.}$$

SCHO. Because A is a point without a circle, &c.

$$AB\times AF=AE\times AH.$$

Whence, $\quad AB : AE :: AH : AE.$

The two $\triangle$'s ABE, AHF, have the common angle A, and the sides about the equal angle proportional, therefore, (th. 20, b. ii.) the two $\triangle$'s are similar, and $AFH=E$. $AHF=ABE$.

(7.) *Given the base, the difference of the other two sides, and the difference of the angles at the base, to construct the triangle.*

(See figure to Theorem 5.)

Draw AB equal to the given base. From B on the opposite side of the base, make the angle $ABE=$ half the difference of the angles at the base.

Take AE, the given difference of the sides, in the dividers; put one foot on A, and strike an arc cutting BE in E. Join AE, and produce EA.

Make the angle $EBD=90°$. BD and EA produced will meet in D. Bisect ED in C, and join BC, and ACB will be the $\triangle$ required.

N. B. This problem was suggested by the investigation of theorem v.

(8.) *Prove that* $\sin.^{-1}\sqrt{\frac{x}{a+x}}=\tan.^{-1}\sqrt{\frac{x}{a}}$.

Remark. The notation $\sin.^{-1}u$, signifies an arc of a circle whose radius is unity, and sine u, &c., &c.

Hence the above proposition in plain English is this:

The radius of a circle is unity, the sine of an arc in that circle is $\sqrt{\frac{x}{a+x}}$. *Prove that the tangent of the same arc must be* $\sqrt{\frac{x}{a}}$.

Let $y=$ the cosine of the arc in question.

Then $y^2+\frac{x}{a+x}=1$. Whence, $y^2=\frac{a}{a+x}$.

But to every arc we have the following proportion:

$$\cos. : \sin. :: 1 : \tan.$$

That is, $\sqrt{\frac{a}{a+x}} : \sqrt{\frac{x}{a+x}} :: 1 : \tan.$

Or, $\sqrt{a} : \sqrt{x} :: 1 : \tan.=\sqrt{\frac{x}{a}}$. Q. E. D.

(9.) *If* $\frac{\tan.(a-b)}{\tan.a}=1-\frac{\sin.^2 c}{\sin.^2 a}$, *then* $\tan.a\tan.b=\tan.^2 c$.

To perform the reduction, multiply by tan. a, and in the last term take its equal $\frac{\sin.a}{\cos.a}$; then

$$\tan.(a-b)=\tan.a-\frac{\sin.^2 c}{\sin.a\cos.a}.$$

That is, $\frac{\tan.a-\tan.b}{1+\tan.a\tan.b}=\tan.a-\frac{\sin.^2 c}{\sin.a\cos.a}$.

$$\tan.a-\tan.b=\tan.a+\tan.^2 a\tan.b-\left(\frac{\sin.^2 c+\sin.^2 c\tan.a\tan.b}{\sin.a\cos.a}\right)$$

Whence, $(1+\tan.^2 a)\tan.b=\frac{\sin.^2 c+\sin.^2 c\tan.a\tan.b}{\sin.a\cos.a}$.

Multiplying by cos.a, and observing that $(1+\tan.^2 a)=\sec.^2 a$, and $\cos.a\sec.a=1$; then

$$\sec.a\tan.b=\frac{\sin.^2 c}{\sin.a}+\frac{\sin.^2 c\tan.a\tan.b}{\sin.a}.$$

Or, $\sec.a\tan.b\sin.a=\sin.^2 c+\sin.^2 c\tan.a\tan.b$.

Take $\frac{\sin.a}{\cos.a}$ for tan.a in the last term, then

$$\sec.a\tan.b\sin.a=\sin.^2 c+\frac{\sin.^2 c\sin.a\tan.b}{\cos.a}.$$

Or, $(\cos.a\sec.a)\tan.b\sin.a=\sin.^2 c\cos.a+\sin.^2 c\sin.a\tan.b$.

Observing again that $(\cos.a\sec.a)=1$, we have

$$\tan.b\sin.a=\sin.^2 c\cos.a+\sin.^2 c\sin.a\tan.b$$

Divide each term by cos.a, and taking tan.a for $\frac{\sin.a}{\cos.a}$, we find

$$\tan.a\tan.b=\sin.^2 c+\sin.^2 c\tan.a\tan.b.$$

$$(1-\sin.^2 c)\tan.a\tan.b=\sin.^2 c.$$

But $(1-\sin.^2 c)=\cos.^2 c$, because $\sin.^2 c+\cos.^2 c=1$; therefore,

$$\cos.^2 c\tan.a\tan.b=\sin.^2 c$$

$$\tan.a\tan.b=\frac{\sin.^2 c}{\cos.^2 c}=\tan.^2 c.$$ Q. E. D.

SECTION III.

PROBLEMS IN SPHERICAL TRIGONOMETRY AND ASTRONOMY

Let ABC be a right angled triangle, right angled at B. a the side opposite A, b the side opposite B, and c the side opposite C.

Taking the *complement* of the oblique angles A and C, calling them A', C', and the complement of b calling it b'.

Then Napier's Circular Parts give us the following equations. We retain the same numbers for the equations as in our Geometry, page 186.

(11)	$R\sin.c=\tan.a\tan.A'$	(16)	$R\sin.A'=\tan.b'\tan.c$
(12)	$R\sin.a=\tan.c\tan.C'$	(17)	$R\sin.A'=\cos.a\cos.C'$
(13)	$R\sin.a=\cos.b'\cos.A'$	(18)	$R\sin.b'=\cos.a\cos.c$
(14)	$R\sin.c=\cos.b'\cos.C'$	(19)	$R\sin.C'=\tan.b'\tan.a$
(15)	$R\sin.b'=\tan.A'\tan.C'$	(20)	$R\sin.C'=\cos.c\cos.A'$

These equations are written in the present form to assist the memory, the second members being the products of two cosines or two tangents; but in practice, we often modify an equation by taking sine for cosine, and cotangent for tangent, and the reverse.

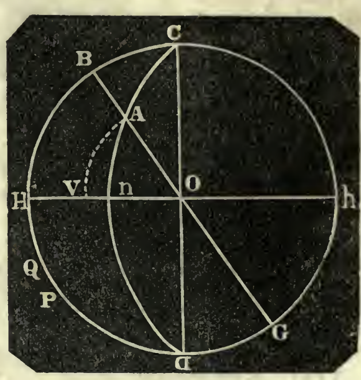
For instance, in equation (18), we invariably take $\cos.b$ for $\sin.b'$, it being the same, *which saves the trouble of finding the complement to the hypotenuse.* The same may be said of other complements.

In all spherical triangles, right angled or oblique angled, the sine of the sides are to each other as the sines of the angles opposite to them.

When two sides of a spherical triangle are given, there can be but one result, that is, there can be no ambiguity about the parts required; but when only one side is given, and one of the oblique angles in a spherical triangle, the conditions correspond equally to *two* triangles, and the answer is said to be ambiguous. For a learner fully to comprehend this, it is necessary to learn to construct his triangles as follows:

We shall illustrate by examples, beginning with the 10th example, page 199, Robinson's Geometry, which will sufficiently illustrate several others.

(1.) *In the right angled triangle* ABC, *right angle at* B, *given* AB 29° 12′ 50″ *and the angle* C 37° 26′ 21″, *to find the other parts.*



To construct a spherical △, the operator should have a scale of chords and semitangents; but he can do all with a ruler and dividers.

Take *OC* in the dividers equal to the chord of 60°, (or any distance if no scale is at hand), and from any point *O* as a center describe the circle *CHDh*. Draw *CD* and *Hh* at right angles through the center. Each of the lines *OC*, *OD*, *OH*, as well as the curve *HC*, *HD*, &c. represent 90° on a sphere. *OHC* is a right angle,—that is, any line from *O* to the circumference will make a right angle with the circumference.

Now from *C* we propose to make the angle *HCA*=37° 26′ 21″. Divide the quadrant *HD* into degrees, beginning at *H*. Take *HP* equal to 37° 26′ 21″; or if the scale is used, take the chord of 37° 26′ 21″ from the scale and apply it from *H* to *P*. Apply the ruler from *C* to *P*, and through the point *n*, where this line crosses *HO*, describe the curve *CnD*.

From *H*, set off *HQ*=29° 12′ 50″, apply the ruler between *Q* and *C*, and mark *V* where this line cuts *HO*. From the center *O*, with *OV* radius, strike the arc *VA*. Lastly, through *A* and *O* draw *BAG* through the center. The △ *ABC* or its supplement *DBA*, is the one required. The side *AC* is measured by the arc, but neither *AC* nor the angle *A* can be measured instrumentally. To measure sides, they must either be on the circumference or on the straight lines through the center.

Remark. If the angle *BAC* had been given, we should call the triangle *ADG* the supplemental triangle, for *AG* is the supplement to AB, AD to AC, and the angle ADG is supplemental to ADB or its equal BCA.

When we have all the parts of the triangle ABC, we in effect have all the parts of the triangle DAB, also all parts of the $\triangle$ ADG and all parts of the triangle GAC.

That is, when one spherical triangle is determined, we have three others, the whole four making up a hemisphere.

For the numerical computation of AC we take equation (14) modified thus:

$$\sin. AC = \sin. b = \frac{R \sin. c \ldots\ldots\ldots\ldots 19.688483}{\sin. C \ldots\ldots\ldots\ldots 9.783843}$$

$$\sin. 53^\circ\ 24'\ 13'' \ldots\ldots\ldots\ldots\ldots\ldots 9.904640$$

To find BC or a, we take a modification of (18).

$$\cos. a = \frac{R \cos. b \ldots\ldots\ldots\ldots 19.775374}{\cos. c \ldots\ldots\ldots\ldots 9.940917}$$

$$\cos. 46^\circ\ 55'\ 2'' \ldots\ldots\ldots\ldots\ldots\ldots 9.834457$$

To find the angle A, we take a modification of equation (13).

$$\sin. A = \frac{R \sin. a \ldots\ldots\ldots\ldots 19.863539}{\sin. b \ldots\ldots\ldots\ldots 9.904640}$$

$$\sin. 65^\circ\ 27'\ 50'' \ldots\ldots\ldots\ldots\ldots\ldots 9.958899$$

Whence,

$AC = 53^\circ\ 24'\ 13''$, $BC = 46^\circ\ 55'\ 2''$, and the $\angle\ A$, $65^\circ\ 27'\ 50''$.

$AD = 126^\circ\ 35'\ 47''$, $BD = 133^\circ\ 4'\ 58''$, and the $\angle\ BAD$, $114^\circ\ 32'\ 10''$

The same figure will sufficiently illustrate example 12, page 199, Robinson's Geometry.

(2.) *In the right angled triangle* ABC, *given* AB, 54° 21′ 35″, *and the angle* C, 61° 2′ 15″, *to find the other parts.*

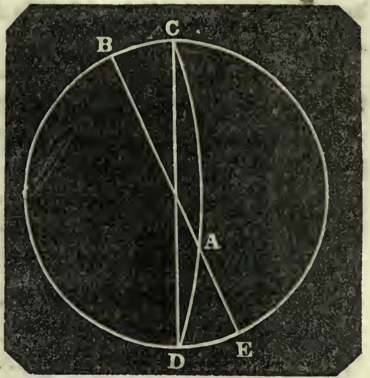
(14) $$\sin. AC = \sin. b = \frac{R \sin. c \ldots\ldots\ldots\ldots 19.909916}{\sin. C \ldots\ldots\ldots\ldots 9.941976}$$

$$\sin. 68^\circ\ 15'\ 16'' \ldots\ldots\ldots\ldots\ldots\ldots 9.967940$$

Whence, $AC = 68^\circ\ 15'\ 16''$, and $AD = 111^\circ\ 44'\ 44''$. The answers given in the book correspond to the triangle ADB,— and those answers were given to exercise the judgment of the learner.

The other parts are found as in the last example.

(3.) *In the right angled spherical triangle, given* AB, 100° 10′ 3″ *and the angle* BCA, 90° 14′ 20″, *to find the other parts.*



Because the sines, cosines, &c., of the tables correspond to arcs under 90°; therefore we will operate on the supplemental triangle, *ADE*. $BC=DE$. $180°-AB=79°\ 49'\ 57''=AD=c$.

The angle $ADE=90°-(14'\ 20'')=89°\ 45'\ 40''$.

$AC=b$, and $AD=b'$ in the equations. $AB=c$, $AED=90°$, $ADE=C'=89°\ 45'\ 40''$.

To solve this △, we use equation (20).

$\sin. A=\dfrac{R\cos. C}{\cos. c}$	17.620026
....	9.246810
sin. 1° 21′ 12″	8.373216

To compute *AD*, we take equation (15); *AD* the supplement of $AC=b$.

$R\cos. b=\cot. A\cot. C$.

cot. $A=1°\ 21'\ 12''$	11.626819
cot. $C=89°\ 45'\ 40''$	7.619860
AD cos. 79° 50′ 5″	9.246679
AC 100° 9′ 55″	

To find *DE*, or its equal *BC*, we take equation (13).

$R\sin. a=\sin. b\cdot\sin. A$.

sin. 79° 50′ 5″	9.993128
sin. 1° 21′ 12″	8.373216
BC, sin. 1° 19′ 52″	8.366344

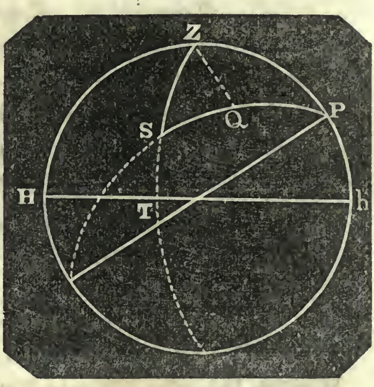
These examples give a sufficient key to the solution of all other examples in right angled spherical trigonometry.

We now turn to the application of spherical trigonometry—taking the examples from page 215, Geometry.

MISCELLANEOUS ASTRONOMICAL PROBLEMS.

(1.) *In latitude* 40° 48′ *north, the sun bore south* 78° 16′ *west, at* 3h. 38m. P. M., *apparent time. Required his altitude and declination, making no allowance for refraction.*

Ans. The altitude, 36° 46′, and declination, 15° 32′ north.



Let Hh be the horizon, Z the zenith of the observer, P the north pole, and PS a meridian through the sun.

PZ is the co-latitude, 49° 12′, and PS is the co-declination or polar distance, one of the arcs sought. ZS is the co-altitude or ST is the altitude of the sun at the time of observation.

The angle ZPS is found by reducing 3h. 38m. into degrees at the rate of 4m. to one degree ; hence, ZPS=54° 30′.

Because HZS=78° 16′, PZS=101° 44′. From Z let fall the perpendicular ZQ on PS. Then in the right angled spherical △ PZQ, equation (13) gives us*

$$R \sin. ZQ = \sin. PZ \sin. P.$$

sin. PZ=sin. 49° 12′	....	9.879093
sin. P =sin. 54° 30′	....	9.910686
sin. ZQ=sin. 38° 2′ 33″	....	9.789779

To obtain the angle PZQ, we apply equation (19), which gives

$$R \cos. PZQ = \cot. PZ \tan. ZQ.$$

That is, $R \cos. PZQ = \tan. 40° 48′ \tan. 38° 2′ 33″$.

* To apply the equations without confusion, letter each right angled spherical triangle ABC, right angled at B, then A must be written in place of P; and when operating on ZSQ, write A in place of S, and C for the angle SZQ

9.936100
9.893464

PZQ= cos. 47° 30′ 50″....................9.829564
PZS= 101° 44′
SZQ= 54° 14′ 10″

To obtain *ZS* or its complement, we again apply (19)

(19) R cos. SZQ=cot. ZS tan. ZQ.

That is, R cos. 54° 14′ 10″=tan. ST tan. 38° 2′ 33″.

R cos. 54° 14′ 10″=..............19.766744
tan. 38° 2′ 33″ =.............. 9.893464
tan. 36° 46′, nearly,........................ 9.873280

To find *PS*, we take the following proportion:

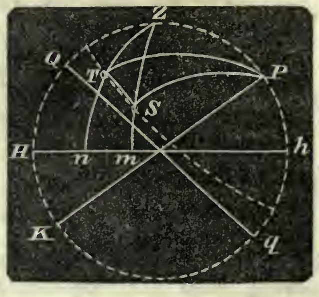
sin. P : sin. ZS : : sin. PZS : sin. PS

That is, sin. 54° 30′ : cos. 36° 46′ : : sin. 101° 44′ : sin. PS

cos. 11° 44′......................9.990829
cos. 36° 46′......................9.903676
19.894505
sin. 54° 30′......................9.910686
PS, 74° 28′ sin...........................9.983819

Whence, the sun's distance from the equator must have been 15° 32′ north.

(2.) *In north latitude, when the sun's declination was* 14° 20′ *north, his altitudes, at two different times on the same forenoon, were* 43° 7′+, *and* 67° 10′+ ; *and the change of his azimuth, in the interval,* 45° 2′. *Required the latitude.* *Ans.* 34° 20′ north.



Let *PK* be the earth's axis, *Qq* the equator, and *Hh* the horizon.

Also, let *Z* be the zenith of the observer, *Sm* the first altitude, *Tn* the second, and the angle *TZS*=45° 2′. Our first operation must be on the triangle *ZTS*. *ZT*=22° 50′, *ZS*=46° 53′, and we must find *TS*, and the ∠ *TSZ*.

From T, conceive TB let fall on ZS making two right angled $\triangle$'s; and to avoid confusion in the figure, we will keep the arc TB in mind, *and not actually draw it.*

Then the $\triangle\ ZTB$ furnishes this proportion:

$$R \ : \ \sin. 22° \ 50' \ : \ : \ \sin. 45° \ 2' \ : \ \sin. TB = \sin. 15° \ 56' \ 8''$$

To find ZB we have the following proportion, (see p. 185 Geo.)

$$R \ : \ \cos. ZB \ : \ : \ \cos. 15° \ 56' \ 8'' \ : \ \cos. 22° \ 50'$$

Whence, we find $ZB = 16° \ 34' \ 20''$. Now in the right angled spherical $\triangle\ TBS$, we have $TB = 15° \ 56' \ 8''$, $BS = 46° \ 53' - 16° \ 34' \ 20''$, or $BS = 30° \ 18' 40''$; and TS is found from the following proportion:

$$R \ : \ \cos. 15° \ 56' \ 8'' \ : \ : \ \cos. 30° \ 18' \ 40'' \ : \ \cos. TS$$

This gives $TS = 33° \ 53' \ 16''$. To find the angle TSZ, we have the proportion, $\sin. 33° 53' 16'' : R : : \sin. TB \ 15° 56' 8'' : \sin. TSZ$.

Whence, the angle $TSZ = 29° \ 30'$.

The next step is to operate on the *isosceles* spherical $\triangle\ PTS$. We require the angle TSP.

Conceive a meridian drawn bisecting the angle at P, it will also bisect the base TS, forming two equal right angled spherical triangles.

Observe that $PS = 75° \ 40'$ and $\frac{1}{2}\ TS = 16° \ 56' \ 38''$.

To find the angle TSP we apply equation (19), in which $a = 16° \ 56' \ 38''$, $b = 75° \ 40'$, and the equation becomes

$$R \cos. TSP = \cot. 75° \ 40' \tan. 16° \ 56' \ 38''$$

Whence, $TSP = 85° \ 31' \ 40''$, and $PSZ = 85° \ 31' \ 40'' - 29° \ 30' = 56° \ 1' \ 40''$.

The third step is to operate on the $\triangle\ ZSP$; we *now* have its two sides ZS and SP, and the included angle.

From Z *conceive* a perpendicular arc let fall on SP, calling it ZB; then the right angled spherical triangle SZB, gives

$$R \ : \ \sin. ZS \ : \ : \ \sin. ZSB \ : \ \sin. ZB$$

That is, $R : \sin. 46° \ 53' : : \sin. 56° \ 1' \ 40'' : \sin. ZB = \sin. 37° \ 15' 20''$

To find SB we have the following proportion, (see Geo. p. 185.)

$$R \ : \ \cos. SB \ : \ : \ \cos. ZB \ : \ \cos. ZS$$

That is, $R : \cos. SB : : \cos. 37° \ 15' \ 20'' : \cos. 46° \ 53'$

Whence SB=30° 49′ 40″. Now from PS, 75° 40′, take SB, 30° 49′ 40″, and the difference must be BP, 44° 50′ 20″.

Lastly, to obtain PZ, and consequently ZQ the latitude, we have

$$R \;:\; \cos. ZB \;::\; \cos. BP \;:\; \cos. ZP = \sin. ZQ$$

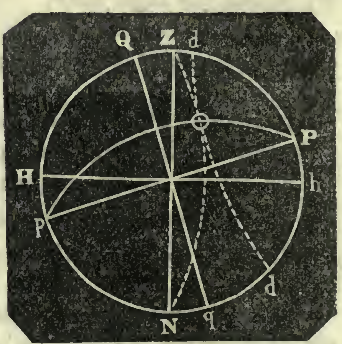
That is, R : cos. 37° 15′ 20″ : : cos. 44° 50′ 20″ : sin. ZQ= sin. 34° 21′ north.

This computation differs *one mile* from the given answer, but any two operators will differ about this much, unless each observe the utmost nicety.

This is a modification of latitude by double altitudes, but in real double altitudes the arc TS is measured from the elapsed time between the observations, and the angle TZS is not given.

(3.) *In latitude* 16° 4′ *north, when the sun's declination is* 23° 2′ *north. Required the time in the afternoon, and the sun's altitude and bearing when his azimuth neither increases nor decreases.*

Ans. Time, 3h. 9m. 26s. P. M., altitude, 45° 1′, and bearing north 73° 16′ west.



Let Pp be the earth's axis, Hh the horizon, Qq the equator, QZ and Pp, each equal to 16° 4′ north, and Qd, qd, each equal to 23° 2′; then the dotted curve dd represents the parallel of the sun's declination.

Through Z and N an infinite number of vertical circles can be drawn, one of these will touch the curve dd ; let it be ZON.

At the point O where this circle touches the curve dd will be the position of the sun at the time required, and POZ will be a right angled spherical △, right angled at O. The problem requires the complement of ZO, and the time corresponding to the angle ZPO.

In the spherical △ POZ, we have

$$R \;:\; \cos. PO \;::\; \cos. ZO \;:\; \cos. PZ$$

That is, R : sin. 23° 2′ : : sin. altitude : sin. 16° 4′

Whence, $\sin.\text{alt.} = \frac{R\sin. 16° 4'}{\sin. 23° 2'} = \sin. 45° 1'$ nearly. *Ans.*

To find the angle at P, we have the following proportion :

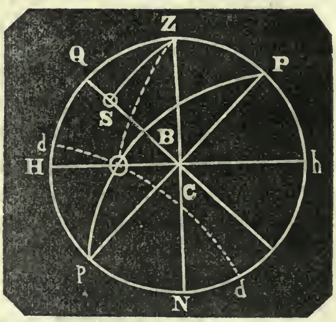
$$\cos. 16° 4' \ : \ R \ : : \ \cos. 45° 1' \ : \ \sin. P$$

Whence, $\sin. P = \sin. 47° 21' 30''$, and $ZPO = 47° 21' 30''$, which being changed into time, at the rate of 15° to one hour, gives 3h. 9m. 26s.

To find the angle PZO, we have this proportion :

$$\cos. 16° 4' \ : \ R \ : : \ \cos. 23° 2' \ : \ \sin. PZO = \sin. 73° 16'$$

(4.) *The sun set south-west ½ south, when his declination was* 16° 4′ *south. Required the latitude.* *Ans.* 69° 1′ north.



Draw a circle as before. Let Hh be the horizon, Z the zenith, P the pole. The great circle PZH is the meridian, and ZCN at right angles to it, and of course east and west. Let BC be a portion of the equator, and BO the arc of declination. The position on the horizon where the sun set is the arc $HO = 45° - 5° 37' 30'' = 39° 22' 30''$.

Consequently, the arc $OC = 50° 37' 30''$.

In the right angled spherical triangle BOC, we have BC, BO given to find the angle BCO, which is the complement of the latitude, or the complement of the angle BCZ.

To find the angle BCO, we apply equation (14).

$$R\sin. BO = \sin. OC \sin. BCO$$

That is, $R\sin. 16° 4' = \sin. 50° 37' 30'' \sin. BCO$

$R\sin. 16° 4'$	19.442096
$\sin. 50° 37' 30''$	9.888184
$\cos. 69° 1'$ nearly,	9.553912

Scho. The arc BC on the equator measures the angle BPC, corresponding to the time from 6 o'clock to sun rise or sun set.

This arc is called the arc of ascensional difference in astronomy. The time of sun set is before six if the latitude is north and the declination south, as in this example, but after six, if the latitude and declination are both north or both south.

To obtain this arc, the latitude and declination must be given ; that is, BO and the angle BCO, the complement of the latitude. Here we apply (12), that is,

$$R \sin. BC = \tan. D \tan. L$$

an equation in which D represents the declination, and L the latitude.

(5.) *The altitude of the sun, when on the equator, was* 14° 28′+, *bearing east* 22° 30′ *south. Required the latitude and time.*

Ans. Latitude 56° 1′, and time 7h. 46m. 12s. A. M.

Let S be the position of the sun on the equator. (See the last figure.) Draw the arc ZS, and the right angled spherical △ ZQS is the one we have to operate upon.

Then ZS is the complement of the given altitude, and the angle QZS, is the complement of 22° 30′. The portion of the equator between Q and S, changed into time, will be the required time from noon, and the arc QZ will be the required latitude.

First for the arc QS.

$$R \ : \ \sin. ZS \ : : \ \sin. QZS \ : \ \sin. QS$$

That is, R : cos. 14° 28′ : : cos. 22° 30′ : sin. QS = 73° 27′ 38″

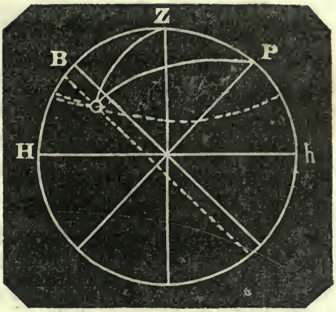
But 73° 27′ 38″ at the rate of 4m. to one degree, corresponds to 4h. 13m. 48s. from noon,—and as the altitude was marked +, rising, it was before noon, or at 7h. 46m. 12s. in the morning.

To find the arc QZ we have the following proportion :

$$R \ : \ \cos. 63° 27' 38'' \ : : \ \cos. QZ \ : \ \sin. 14° 28'$$

Whence, cos. QZ=cos. 56° 1′ nearly, and 56° 1′ is the latitude sought.

(6.) *The altitude of the sun was* 20° 41′ *at* 2h. 20m. P. M. *when his declination was* 10° 28′ *south. Required his azimuth and the latitude.* *Ans.* Azimuth south 37° 5′ west, latitude 51° 58′ north.



This problem furnishes the spherical △ PZO, in which the side ZO is the complement of 20° 41′ or 69° 19′. PO=90° +10° 28′ = 100° 28′, and the angle ZPO is 2h. 20m., changed into degrees at the rate of 15° to one hour, or ZPO=35°.

Now in the triangle ZPO, we have

sin. ZO : sin. ZPO : : sin. PO : sin. PZO That is,
cos. 20° 41′ : sin. 35° : : cos. 10° 28′ : sin. BZO = cos. 37° 5′.

In the right angled spherical △ BOZ, we apply equation (16).

(16). R cos.37° 5′=tan. 20° 41′ tan. BZ.

R cos. 37° 5′	19.901872
tan. 20° 41′	9.576958
tan. BZ=tan. 64° 40′ 40″	10.324914

To find PB in the right angled △ BPO, we apply the same equation, (16). R cos. 35°=tan. 10° 28′ tan. PB.

R cos. 35°	19.913365
tan. 10° 28′	9.266555
tan. 12° 42′ 40″	10.646810

But PB is obviously greater than 90°, therefore the point B is 12° 42′ 40″ below the equator, but from B to Z is 64° 40′ 40″; therefore from Z to the equator, or the latitude, is the difference between 64° 40′ 40″ and 12° 42′ 40″, or 51° 58′ north.

Ans. Lat. 51° 58′ north.

(7.) *If in August* 1840, *Spica was observed to set* 2h. 26m. 14s. *before Arcturus, what was the latitude of the observer? Taking no account of the height of the eye above the sea, nor of the effect of refraction.* *Ans.* 36° 48′ north.

By a catalogue of the stars to be found in the author's Astronomy, or in any copy of the English Nautical Almanac, we find the positions of these stars in 1840 to have been as follows:

Spica, right ascension, 13h. 16m. 46s. Dec. 10° 19′ 40″ south.
Arcturus, " " 14h. 8m. 25s. Dec. 20° 1′ 4″ north.

Let $L=$ the latitude sought. Put $d=10° 19' 40''$, and $D=20° 1' 4''$.

The difference in right ascensions is 51m. 39s., and this would be about the time that Arcturus would set after Spica, provided the observer was near the equator or a little south of it; but as the interval observed was 2h. 26m. 14s., the observer must have been a considerable distance in *north latitude.* In high southern latitudes Arcturus sets before Spica.

When an observer is north of the equator, and the sun or star south of it, the sun or star will set within six hours after it comes to the meridian.

When the observer and the object are both north of the equator, the interval from the meridian to the horizon is greater than six hours.

The difference between this interval and six hours, is called the ascensional difference, and it is measured in *arc* by BC in the figure to the 4th example.

Now let $x=$ the ascensional difference of Spica corresponding to the latitude L, and $y=$ the ascensional difference corresponding to the same latitude; then by the scholium to the 4th example, calling radius unity, we shall have

$$\sin. x=\tan. L \tan. d \qquad (1)$$

$$\sin. y=\tan. L \tan. D \qquad (2)$$

The star Spica came to the observer's meridian at a certain time that we may denote by M.

Then $$M+\left(6-\frac{x}{15}\right)=\text{ the time Spica set.}$$

And $$M+51\text{m. }39\text{s.}+\left(6+\frac{y}{15}\right)=\text{ the time Arcturus set.}$$

By subtracting the time Spica set from the time Arcturus set we shall obtain an expression equal to 2h. 26m. 14s. That is

$$51\text{m. }39\text{s. }+\frac{x}{15}+\frac{y}{15}=2\text{h. }26\text{m. }14\text{s.}$$

Or, $$\frac{x}{15}+\frac{y}{15}=1\text{h. }34\text{m. }35\text{s.} \qquad (3)$$

$$x+y=15(1\text{h. }34\text{m. }35\text{s.}) \qquad (4)$$

Equation (3) expresses time. Equation (4) expresses arc.

When we divide arc by 15 we obtain time, one degree being the *unit* for arc, and one hour the *unit* for time; therefore, when we multiply time by 15 we obtain arc; that is, 1h. multiplied by 15 gives 15°; hence (4) becomes

$$x+y=23° \, 39'=a$$
$$x=a-y \qquad (5)$$

That is, the *arc* x is equal to the difference of the *arcs* a and y; but to make use of these *arcs* and avail ourselves of equations (1) and (2), we must take the *sines* of the arcs, (see equation (8), plane trigonometry); then (5) becomes

$$\sin. x=\sin. a \cos. y-\cos. a \sin. y \qquad (6)$$

Substituting the values of sin. x and sin. y from (1) and (2), (6) becomes

$$\tan. L \tan. d=\sin. a \cos. y-\cos. a \tan. L \tan. D \qquad (7)$$

Squaring (2), $\sin.^2 y=\tan.^2 L \tan.^2 D.$

Subtracting each member from unity, and observing that $(1-\sin.^2 y)$ equals $\cos.^2 y$, then

$$\cos.^2 y=1-\tan.^2 L \tan.^2 D.$$

Or, $\cos. y=\sqrt{1-\tan.^2 L \tan.^2 D}.$

This value of cos. y put in (7), gives

$$\tan. L \tan. d=\sin. a\sqrt{1-\tan.^2 L \tan.^2 D}-\cos. a \tan. L \tan. D \quad (8)$$

By transposition and division,

$$\left(\frac{\tan. d+\cos. a \tan. D}{\sin. a}\right) \tan. L=\sqrt{1-\tan.^2 L \tan.^2 D}$$

Squaring, $\left(\frac{\tan. d+\cos. a \tan. D}{\sin. a}\right)^2 \tan.^2 L=1-\tan.^2 L \tan.^2 D$

Dividing by $\tan.^2 L$ and observing that $\frac{1}{\tan.^2 L}=\cot.^2 L$ we have

$$\left(\frac{\tan. d+\cos. a \tan. D}{\sin. a}\right)^2=\cot.^2 L-\tan.^2 D$$

Or, $$\cot.^2 L=\tan.^2 D+\left(\frac{\tan. d+\cos. a \tan. D}{\sin. a}\right)^2$$

$$=\tan.^2 D+\left(\frac{\tan. d}{\sin. a}+\frac{\tan. D}{\tan. a}\right)^2$$

We must now find the numerical value of the second member. Using logarithmic sines, cosines, tangents, &c., we must diminish the indices by 10, because the equation refers to radius unity.

log. tan. $D.=-1.561460.$ tan. $^2D=-1.122920=0.132712$ num.

log. tan. d....$\overline{1}$.260623	log. tan. D....$\overline{1}$.561460
sin. a....$\overline{1}$.603305	tan. a....$\overline{1}$.641404
0.45424........$\overline{1}$.657318	0.83188........$\overline{1}$.920056

$0.45424+0.83188=1.28612 \qquad (1.28612)^2=1.654105$

Whence, $\cot.^2 L=0.132712+1.654105=1.786817$

Square root, $\cot. L=1.33672$

Taking the log. of this number, increasing its index by 10 will give the log. cot. in our tables.

log. $1.33672=0.126076+10.=10.126076=\cot. 36° 48'$

(8.) *On the* 14th *of November*, 1829, *Menkar was observed to rise* 48m. 3s. *before Aldebaran : what was the latitude of the observer?*

Ans. 39° 34′ north.

The positions of these two stars in the heavens, Nov. 1829, were as follows:

Menkar, right ascension, 2h. 53m. 21s. Dec. 3° 24′ 52″ north.
Aldebaran, " 4h. 26m. 7s. Dec. 16° 19′ 31″ north.

Aldebaran passes the meridian 1h. 32m. 46s. after Menkar. Now let M represent the time Menkar was on the meridian, then $M+$1h. 32m. 46s. represents the time Aldebaran was on the meridian. Also, let $x=$ the arc of ascensional difference corresponding to the latitude and the star Menkar, and y that of the star Aldebaran.

Then $$M-\left(6+\frac{x}{15}\right)=\text{the time Menkar rose.}$$

And $$M+1\text{h. }32\text{m. }46\text{s}-\left(6+\frac{y}{15}\right)=\text{the time Aldebaran rose.}$$

Subtracting the upper from the lower, the difference must be 48m. 3s.; that is,

$$1\text{h.}+32\text{m. }46\text{s.}-\frac{y}{15}+\frac{x}{15}=48\text{m. }3\text{s.}$$

Whence, $$\frac{x}{15}-\frac{y}{15}=-44\text{m. }43\text{s.}=-0.74527.$$

That is, 1h. being the unit, 44m. 43s. = 0.74527 of an hour, and multiplying by 15, we shall have as many degrees of arc as we have units; therefore,

$$x-y=-(0.74527)15=-11° \; 10' \; 45''=-a.$$

$$x=y-a.$$

$$\sin. x=\sin. y \cos. a-\cos. y \sin. a \qquad (1)$$

Put $d=3°\ 24'\ 52''$, $D=16°\ 19'\ 31''$, and $L=$ the required latitude. Then by scholium to the 4th example,

$$\sin. x=\tan.d \tan.L. \qquad \sin.y=\tan.D \tan.L.$$

These values of $\sin.x$ and $\sin.y$, substituted in (1), give

$$\tan.d \tan.L=\cos.a \tan.D \tan.L-\cos.y \sin.a \qquad (2)$$

But $\sin.^2 y=\tan.^2 D \tan.^2 L$, and $1-\sin.^2 y=1-\tan.^2 D \tan.^2 L$.

Or, $$\cos.^2 y=1-\tan.^2 D \tan.^2 L.$$

Or, $$\cos.y=\sqrt{1-\tan.^2 D \tan.^2 L.}$$

By substituting this value of $\cos.y$ in (2) and transposing, we find

$$\sin.a\sqrt{1-\tan.^2 D \tan.^2 L}=(\cos.a \tan.D-\tan.d)\tan.L$$

Dividing by $\sin.a$, and observing that $\frac{\cos. a}{\sin. a}=\frac{1}{\tan.a}$, we have

$$\sqrt{1-\tan.^2 D \tan.^2 L}=\left(\frac{\tan. D}{\tan.a}-\frac{\tan.d}{\sin. a}\right) \tan.L.$$

Squaring and dividing by $\tan.^2 L$, and at the same time observing that $\frac{1}{\tan.L}=\cot.L$, and we shall have

$$\cot.^2 L-\tan.^2 D=\left(\frac{\tan. D}{\tan.a}-\frac{\tan.d}{\sin. a}\right)^2$$

We will now find the numerical values of the known quantities.

Log. $\tan.D$...	-1.466696	Log. $\tan.d$...	-2.775685
Log. $\tan.a$....	-1.296179	Log. $\sin.a$...	-1.287617
Log. 1.48089...	0.170517	Log. 0.3076...	-1.488068
$\tan.^2 D=0.085778$		$1.48089-0.3076=1.17329$	

Whence, $$\cot.^2 L-0.085778=(1.17329)^2.$$

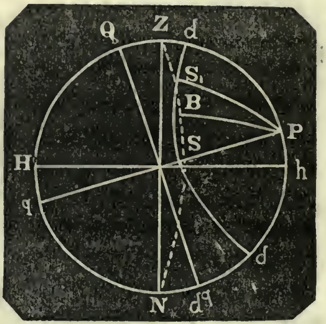
Or, $$\cot.^2 L= 1.462293.$$

$$\cot. L = 1.20925.$$

Log. $\cot.L+10=10.082785=\cot. 39°\ 34'$. *Ans.*

(9.) *In latitude* 16° 40′ *north, when the sun's declination was* 23° 18′ *north, I observed him twice, in the same forenoon, bearing north* 68° 30′ *east. Required the times of observation, and his altitude at each time.*

Ans. Times 6h. 15m. 40s. A. M., and 10h. 32m. 48s. A. M., altitudes 9° 59′ 36″, and 68° 29′ 42″.



Let Z be the zenith, P the north pole, and the curve dd be the parallel of the sun's declination along which it appears to revolve. Make the angle PZS' equal to 68° 30′; then the sun was at S at the time of the first observation, and at S' at the time of the second.

In the spherical △ PZS' there is given PZ, PS' and the angle PZS'; also, in the △ PZS' there is given PZ, PS, and the angle PZS. Observe that PSS' is an *isosceles* △.

Describe the meridian PB bisecting the angle $S'PS$, and then we have three right angled spherical triangles, BPS, BPS', and BPZ; taking the last, we have the following proportion:

$$R \ : \ \sin. PZ \ : : \ \sin. PZB \ : \ \sin. PB$$

That is, R : cos. 16° 40′ : : sin. 68° 30′ : sin. PB=sin.63° 2′ 30″.

To find ZB, we take the following proportion, (see page 185, observation 1, Robinson's Geometry):

$$R \ : \ \cos. ZB \ : : \ \cos. BP \ : \ \cos. PZ$$

That is, R : cos. ZB : : cos. 63° 2′ 30″ : sin. 16° 40′

R sin. 16° 40′	19.457584
cos. 63° 2′ 30″	9.656411
cos. 50° 45′ 48″	9.801173

To find $S'B$, we have

$$R \ : \ \cos. S'B \ : : \ \cos. 63° 2' 30'' \ : \ \sin. 23° 18'$$

R sin. 23° 18′	19.597196
cos. 63° 2′ 30″	9.656411
cos. 29° 14′ 38″	9.940785

Observe that $S'B=BS$; therefore, ZS= 50° 45′ 48″+29° 14′ 38″=80° 0′ 26″, and ZS'=50° 45′ 48″−29° 14′ 38″=21° 31′ 10″, the complement of the altitudes. Consequently the altitude at the first observation was 9° 59′ 34″, and at the second, 68° 28′ 50″.*

* Our results differ a little from the given answer, owing, perhaps, to our not being minute in taking out the logarithms, or finding the nearest second corresponding to a given logarithm.—Experienced men on these matters do not pretend to work to seconds.

To find the time from noon at the first observation, we have the following proportion:

sin. PS : sin. PZS : : sin. ZS : sin. ZPS. That is,
cos.23° 18′ : sin.68° 30′ : : sin.80° 0′ 26″ : sin.ZPS=sin.86° 5′ 30″

Had the angle been 90°, the time would have been just 6h. but the angle 3° 54′ 30″ less; this corresponds to 15m. 38s. in time. Therefore, the time was 6h. 15m. 38s. For the time at the second observation, we have

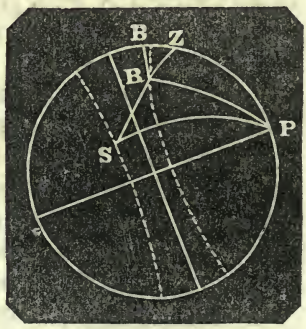
cos.23° 18′ : sin.68° 30′ : : sin.21° 31′ 10″ : sin.ZPS'=sin.21° 48′ 40″

21° 48′ 40″=1h. 31m. 14s. from noon, or 10h. 32m. 46s. apparent time in the morning.

(10.) *An observer in north latitude marked the time when the stars Regulus and Spica were eclipsed by a plumb line, that is, they were both in the same vertical plane passing through the zenith of the observer. One hour and ten minutes afterwards, Regulus was on the observer's meridian. What was the observer's latitude?*

The positions of the stars in the heavens were

Regulus,	right ascension	10h. 0m. 10s.	Dec. 12° 43′ north.
Spica,	" "	13h. 17m. 2s.	Dec. 10° 21′ 20″ south.



Let R be the position of Regulus, S the position of Spica, P the pole, and Z the zenith.

Then the side PS=100° 21′ 20″, PR=77° 17′, and the angle RPS =3h. 16m. 52s., converted into degrees; that is, RPS=49° 13′.

One hour and ten minutes reduced to arc, give 17° 30′; but the stars revolve according to *siderial*, not solar time, and to reduce solar to siderial arc we must increase it by about its $\frac{1}{365}$th part; this gives about 3′ to add to 17° 30′, making 17° 33′ for the angle ZPR. Our ultimate object is to find PZ, the complement of the latitude.

In the △ PRS, we have the two sides PR, PS, and the included angle P, from which we must find RS and the angle SRP, and we can let a perpendicular fall from R on to the side PS and solve it in the usual way; but to show that a wide field is open for a bold operator; we will put the unknown arc $RS=x$, the side opposite $R=r$, and opposite $S=s$, and apply one of the equations in formula (S), page 191, Robinson's Geometry.

That is, $\cos. P=\frac{\cos.x-\cos.r\cos.s}{\sin.r\sin.s}$

Whence, $\cos.P\sin.r\sin.s+\cos.r\cos.s=\cos.x$

We now apply this equation, recollecting that radius is unity, which will require us to diminish indices of the logarithms by 10.

$\cos.P=\cos. 49° 13'$	-1.815046		
$\sin. r=\sin.100° 21' 20''$ ...	-1.992868	$-\cos.$ ...	-1.254579*
$\sin. s=\sin.77° 17'$	-1.989214	$\cos.$ ...	-1.342679
0.6268	-1.797128	.03956 ...	-2.597258

$\cos.x=0.6268-0.03956=.58724.$

Whence, by the table of natural cosines, we find $x=54° 2' 20''$.

To find the angle SRP or ZRP, we have

$\sin. 54° 2' 20'' : \sin. 49° 13' :: \sin. 100° 21' 20'' : \sin. ZRP$

Whence, $ZRP=66° 57' 30''$.

Let fall the perpendicular RB on PZ produced, then the right angled spherical △ PBR gives this proportion:

$R : \sin. 77° 17' :: \sin. 17° 33' : \sin. RB=\sin. 17° 6' 22''$

To find PB we have

$R : \cos. PB :: \cos. 17° 6' 22'' : \cos. 77° 17'$

Whence, $PB=76° 41'$. Now to find the angle BRP, we have

$\sin. 77° 17' : R :: \sin. 76° 41' : \sin. BRP=\sin. 86° 1'$

From PRB take PRZ, and ZRB will remain; that is,

From 86° 1' take 66° 57' 30'', and $ZRB=19° 3' 30''$.

By the application of equation (12), we find that

$R \sin. 17° 6' 22''=\tan. BZ \cot. 19° 3' 30''$

Whence, $BZ=5° 48'$ And $PZ=76° 41'-5° 48'=70° 53'$.

The complement of 70° 53' is 19° 7', the latitude sought.

By this example we perceive that by the means of a meridian line, a good watch, and a plumb line, any person having a knowledge of spherical trigonometry, and having a catalogue of the stars at hand, can determine his latitude by observation.

*Observe that r is greater than 90°, its cosine is therefore, negative in value, rendering the product $\cos. r \cos. s$, or .03956, negative.

PART FOURTH.

PHYSICAL ASTRONOMY.

KEPLER'S LAWS.

1. *The orbits of the planets are ellipses, of which the sun occupies one of the foci.*

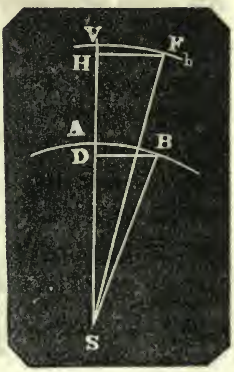
2. *The radius vector in each case describes areas about the focus which are proportional to the times.*

3. *The squares of the times of revolution are to each other as the cubes of the mean distances from the sun.*

The first of these is a mere fact drawn from observation. The second is also an observed fact—but susceptible of mathematical demonstration, under strict geometrical principles, and the law of inertia. The demonstration is to be found in Robinson's Astronomy, and in various philosophical works.

The third is also susceptible of demonstration by means of the calculus—and by simple geometrical proportion, if we suppose the orbits circular.

We now propose to investigate and determine the relative times of revolutions of two bodies about the sun, on the supposition that they revolve in circles, (which is not far from the truth,) and are attracted towards the center inversely proportional to the squares of their distances.



Let S be the center of the sun, AS the radius vector of one planet, and SV that of another.

Let m be the mass of the sun, $SA=r$, and $SV=R$. Then $\frac{m}{r^2}$ is the force which is exerted on the planet at A, and $\frac{m}{R^2}$ is the force exerted on the other planet at V.

If we take any small interval of time, say one minute, and let AD represent the distance the first planet falls from the tangent of its orbit in unity of time, and VH the distances the other falls in the same time,

Then $\frac{m}{r^2} : \frac{m}{R^2} :: AD : VH$ (1)

That the planets may maintain themselves in their orbits, the first must run over the arc AB in the unit of time, and the second must run over the arc VF. But this interval or unit of time can be taken ever so short; and when very short, as a minute or a second, AB and VF, may, yea *must be* considered straight lines, *chords* coinciding with the arc.

But if we take any *chord* of an arc, as AB, and from one extremity draw the diameter, and from the other let fall the perpendicular BD, we shall have

$$AD : AB :: AB : 2r$$

Whence, $AD=\frac{\overline{AB}^2}{2r}$, and in like manner $VH=\frac{\overline{VF}^2}{2R}$

Substituting these equals in proportion (1), and dividing the first couplet by m, and multiplying the last couplet by 2, we have

$$\frac{1}{r^2} : \frac{1}{R^2} :: \frac{\overline{AB}^2}{r} : \frac{\overline{VF}^2}{R}$$

Or, $\frac{1}{r} : \frac{1}{R} :: \overline{AB}^2 : \overline{VF}^2$ (2)

Because the first planet is supposed to run along the arc AB, in one minute, the number of minutes it will require to make its revolution will be found by dividing the whole circumference by AB. The circumference is expressed by $2r\pi$, and put t to represent the time of revolution; then $t=\frac{2r\pi}{AB}$, or $AB=\frac{2r\pi}{t}$. In the same manner if T represents the time of revolution of the second planet, we must have $VF=\frac{2R\pi}{T}$. By squaring these expressions and substituting the values of $\overline{AB}^2$ and $\overline{VF}^2$ in proportion (2), we have

$$\frac{1}{r} : \frac{1}{R} :: \frac{4r^2\pi^2}{t^2} : \frac{4R^2\pi^2}{T^2}$$

Or, $\frac{1}{r} : \frac{1}{R} :: \frac{r^2}{t^2} : \frac{R^2}{T^2}$.

Multiply the first couplet by rR, then

$$R : r :: \frac{r^2}{t^2} : \frac{R^2}{T^2}$$

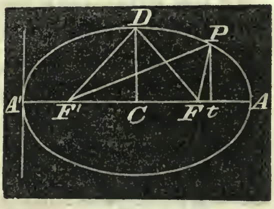
Or, $\frac{r^3}{t^2}=\frac{R^3}{t^2}$. Whence, $t^2 : T^2 :: r^3 : R^3$.

This last proportion corresponds with Kepler's third law.

The following propositions are to be found on page 146 of Robinson's Astronomy. The frequent requests we have received to demonstrate them, suggested the propriety of publishing the demonstrations in this connection.

The propositions are as follows:

(1.) *If two comets move in parabolic orbits, the areas described by them in the same time are proportional to the square roots of their perihelion distances.*



Conceive a comet to revolve in an ellipse, F' the position of the sun, and $A'F'$ the perihelion distance.

Let $F'D=r$, $F'C=x$, $DC=y$, and put t to represent the number of hours required by the comet to make a revolution.

Now $\pi ry=$ the area of the ellipse. This area divided by t, will express the area described by the comet about the sun in one hour. Let that area be represented by a.

Then $\frac{\pi ry}{t}=a$. Let R, x', y', T, and A, represent similar quantities pertaining to another orbit, and by parity of reasoning, $\frac{\pi Ry'}{T}=A$.

Whence, $\frac{ry}{t} : \frac{Ry'}{T} :: a : A$

By squaring, $\frac{r^2y^2}{t^2} : \frac{R^2y'^2}{T^2} :: a^2 : A^2$ (1)

By Kepler's 3d law, $t^2 : T^2 :: r^3 : R^3$, or $t^2=\frac{r^3T^2}{R^3}$

The value of t^2 substituted in (1), and reduced, will give

$$\frac{y^2}{r} : \frac{y'^2}{R} :: a^2 : A^2 \qquad (2)$$

By inspecting the right angled triangle $F'CD$, we readily perceive that $y^2=r^2-x^2=(r+x)(r-x)$. Similarly, $y'^2=(R+x')\times(R-x')$.

Now if we suppose the ellipse to be infinitely eccentric, (as we must when it becomes a parabola,) $(r+x)=2r$ nearly, and $(r-x)=A'F'=p$ exactly, (calling p the perihelion distance of one comet, and P the perihelion distance of the other.)

Similarly, $(R+x')=2R$, and $(R-x')=P$.

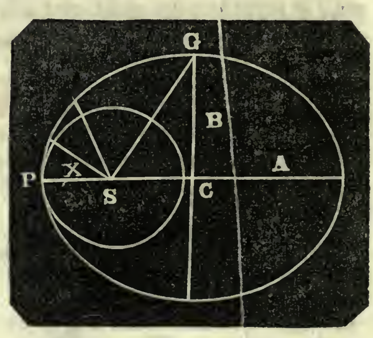
Substituting these values in (2), we have

$$\frac{2rp}{r} : \frac{2RP}{R} :: a^2 : A^2$$

Or, $$p : P :: a^2 : A^2$$

Or, $$\sqrt{p} : \sqrt{P} :: a : A \qquad \text{Q. E. D.}$$

(2.) *If we suppose a planet moving in a circular orbit, whose radius is equal to the perihelion distance of a comet moving in a parabola, the areas described by these two bodies, in the same time, will be to each other as 1 to the square root of 2. Thus are the motions of comets and planets connected.*



Let S be the position of the sun, P the perihelion point of a comet revolving in an *ellipse*.

Put $SP=x$, and let $t=$the time in which the planet would revolve in the circle, and $T=$ the time required for the comet to revolve in the ellipse.

By the first law of Kepler the same body describes equal areas in equal times; therefore if we divide the area of the circle by the number of units in the time of revolution, we shall have the area described in one unit of time.

The area of the circle is πx^2, and this divided by t, gives

$\frac{\pi x^2}{t}$ = the sector described by the planet in unity of time. Also,

$\frac{\pi AB}{T}$ = the sector described by the comet in the same time.

Conceive these two sectors to commence on the line SP, then

$$\text{(sector in circle)} : \text{(sector in ellipse)} :: \frac{x^2}{t} : \frac{AB}{T} \qquad (1)$$

A and B are the semi-conjugate axes of the ellipse.

By Kepler's third law,

$$t^2 : T^2 :: x^3 : A^3 \qquad (2)$$

Multiplying the last couplet of (1) by tT, gives

$$\text{(sector in circle)} : \text{(sector in ellipse)} :: Tx^2 : (tA)B$$

By squaring, we have

$$\text{(sec. in circle)}^2 : \text{(sec. in ellipse)}^2 :: (T^2x^3)x : (t^2A^2)B^2 \qquad (3)$$

From (2) we find $T^2x^3 = t^2A^3$, and substituting the value of T^2x^3 in (3), we have

$$\text{(sec. in circle)}^2 : \text{(sec. in ellipse)}^2 :: t^2A^3x : (t^2A^2)B^2$$

$$\text{"} \qquad \text{"} \qquad :: Ax : B^2 \qquad (5)$$

Observe the right angled triangle CSG. $SG=A$, $CS=A-x$, $CG=B$.

$$A^2-(A-x)^2=B^2$$

Or,

$$2Ax-x^2=B^2$$

Substituting this value of B^2 in (5), and dividing the last couplet, gives

$$\text{(sector in circle)}^2 : \text{(sector in ellipse)}^2 :: A : 2A-x$$

Dividing the last couplet by A, and extracting the square root, gives

$$\text{(sector in circle)} : \text{(sector in ellipse)} :: 1 : \sqrt{2-\frac{x}{A}} \qquad (6)$$

When the ellipse is very eccentric, A is very great in relation to x, and the fraction, $\frac{x}{A}$ is then very insignificant in value. As an ellipse becomes more and more eccentric, its curve approaches nearer and nearer to a *parabola*, and when it becomes a parabola, A is infinite in respect to x, and the fraction $\frac{x}{A}$ is then absolutely *zero*, and proportion (6) becomes

$$\text{(sector in circle)} : \text{(sector in parabola)} :: 1 : \sqrt{2} \qquad \text{Q. E. D.}$$

The following inquiry has frequently come to us. We now give it in the words of a correspondent.

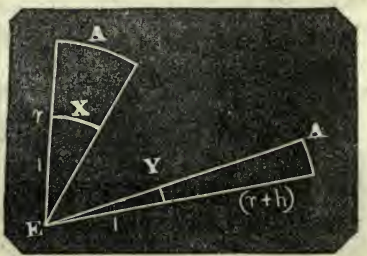
Mr. Robinson :

Dear Sir.—On page 192, Art. 180 of your Astronomy, it is stated that because the mean radial force causes the moon to circulate at $\frac{1}{358}$ part greater distance from the earth than it otherwise would, its periodical revolution is increased by its 179th part. The question is, where does the fraction $\frac{1}{179}$ come from ?

REPLY.

The mean radial force acting in the direction of the radius vector does not prevent the moon from describing equal areas in equal times. Therefore the moon describes the same area with, as it would without this action ; but the radius is increased, and consequently the angular velocity diminished.

We will now give the increase of radius, and require the corresponding decrease of angular velocity, and we shall find the *ratio* of one will be double that of the other, on the condition that the increase or decrease of either, is small in relation to the whole.



Let E be the angular point of two equal sectors, r the radius of one, and A its arc. x its angle on the radius of unity.

Let $(r+h)$ be the radius of the other sector, A_1 its arc and y its angle.

Then by reason of the two equal sectors, $rA=(r+h)A_1$ (1)

From one sector, $1 : x :: r : A$. Or, $A=rx$.

From the other, $1 : y :: (r+h) : A_1$ Or, $A_1=(r+h)y$.

Substituting the values of A and A_1 in (1), we have

$$r^2x=(r+h)^2y$$

Or, $x : y :: (r+h)^2 : r^2$

Or, $x : y :: r^2+2rh+h^2 : r^2$

Because h is a very small fraction in relation to r, h^2 can be omitted ; then

$$x : y :: r^2+2rh : r^2$$

Or, $x : y :: r+2h : r$

This last proportion shows that if the radius r is increased by h, the angular velocity and consequently the periodic time must be diminished by $2h$.

PROPOSITION.

Given the position of the earth as seen from the sun, the position of any other planet as seen from the sun, to find the position of that planet as seen from the earth.

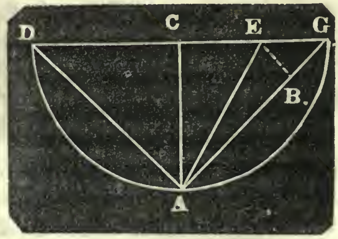
The motion of the earth and planets being known, and the elements of their orbits, the astronomical tables give the position of the earth and any planet for any given instant of time. The position of the planet from the earth must then be computed by plane trigonometry. But before we give a definite example, we adduce the following

LEMMA.

1. *In any plane triangle the greater of two sides is to the less, as radius to the tangent of a certain angle.*

2. *Radius is to the tangent of the difference between this angle and* 45°, *as the tangent of half the sum of the angles at the base of the triangle is to the tangent of half their difference.*

To obtain that certain angle, we must place the two sides at right angles to each other.



Let CA be the greater of two sides of a $\triangle$, and CE a less side placed at right angles; then CAE is the certain angle spoken of, less than 45°, and EAB is the difference between it and 45°.

From C as a center with the longer side as radius, describe the semicircle. Then $DE=$ the sum of the sides, and EG their difference. Join DA, AG, and from E draw EB parallel to DA. DAG is a right angle because it is in a semicircle; therefore, EB being parallel to DA, EBG is a right angle also. $DA=AG$, and $EB=BG$.

Let a be the greater side of a triangle represented in *magnitude* but not in position, by CA, and c the shorter side, represented in *magnitude by* CE; then it is obvious that

$$a \; : \; c \; :: \; R \; : \; \tan. CAE$$

This angle taken from the table and subtracted from 45° will give the angle EAB. By proportional triangles we have

$$DE : EG :: AB : BG = EB$$

That is, $a+c : a-c :: AB : EB$

But. $AB : EB :: R : \tan. EAB$

Whence, $a+c : a-c :: R : \tan. EAB$

By proportion 7, page 149, Robinson's Geometry, we find that

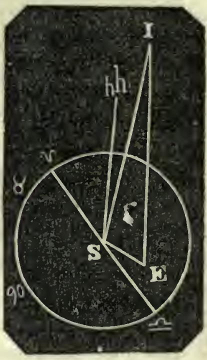
$$a+c : a-c :: \tan. \tfrac{1}{2} \text{sum ang. at base} : \tan. \tfrac{1}{2} \text{their diff.}$$

Therefore by comparison,

$R : \tan. EAB :: \tan. \frac{1}{2}$ sum ang. at base $: \tan. \frac{1}{2}$ their diff. Q. E. D.

The application of this proposition is very advantageous when the logarithms of the two sides of a triangle are given and not the sides themselves. It obviates the necessity of finding the numerical values of the sides. This proposition is almost solely used in Astronomy, and we give the following example as an illustration.

In the Nautical Almanac for 1854, *I find that on the first day of April at noon, mean time at Greenwich, the sun's longitude is* 11° 26′ 28″, *and the logarithm of the radius vector of the earth is* 0.0000224. *At the same time the heliocentric longitude of Jupiter is* 283° 46′ 7″, *south latitude* 6′ 41″, *and logarithm of its radius vector* 0.7145152. *Required the geocentric latitude and longitude of Jupiter, and the logarithm of its true distance from the earth.*



Let S be the sun, ♈♎ the line made by Aries and Libra in the plane of the ecliptic, ♈♉ &c., the direction of counting longitude.

Place the earth at E, so that the sun at S will appear to be in 11° 26′ 28″ of longitude. Then E, the earth, will appear from the sun to be in 191° 26′ 28″ of longitude. SE is a very little over a unit in distance, as we see by the log. 0.0000224.

The longitude of Jupiter as seen from the sun is 283° 46′ 7″; hence, draw SI so that the angle ♎SI will be 103° 46′ 7″, and its distance from S to I a little over S times SE. The angle ESI will be 92° 19′ 39″, and log. of SI is given at 0.7145152. Our object is to find the position of the line EI, or Sh which is supposed parallel to EI, and the logarithm of the distance EI.

Jupiter not being in the plane of the ecliptic, we must reduce it to that plane, by multiplying its distance by the cosine of its inclination.

Thus, to the log	0.7145152
Add cos. 6′ 41″	9.9999998
Log. of distance in the ecliptic,	0.7145150

Now by the first part of the Lemma,

As	0.7145150
To	0.0000224
So is radius	10.0000000
To tan. 10° 55′ 21″	9.2855074

This arc from 45° gives 34° 4′ 39″. The angle ESI from 180° gives 87° 40′ 21″ for the sum of the angles E and I. Their half sum is therefore, 43° 50′ 10″5. Let their half difference be denoted by x. Then by the last part of the Lemma,

R : tan. 34° 4′ 40″ : : tan. 43° 50′ 10″5 : tan. x

tan. 34° 4′ 39″	9.830254
tan. 43° 50′ 10″5	9.982352
tan.x 33° 0′ 19″	9.812606

The angle E=43° 50′ 10″+33° 0′ 19″=76° 50′ 29″.

The angle I=43° 50′ 10″—33° 0′ 19″=10° 49′ 51″.

The angle ISh=10° 49′ 51″; therefore, the geocentric longitude of Jupiter is 283° 46′ 7″+10° 49′ 51″ or 294° 35′ 58″.

For the log. of EI we have the following proportion:

sin. I : SE : : sin. ISE : EI

Or, sin. 10° 49′ 51″ : SE : : sin. 92° 19′ 39″ : EI

Log. sin. 92° 19′ 39″	9.999641
Log. SE	0.0000224
	9.9996634
Log. sin. 10° 49′ 51″	9.2739400
Log. EI	0.7257234

This result is the logarithm of the distance from E along in the plane of the ecliptic to *the point* where the perpendicular falls from Jupiter,—the hypotenusal or absolute distance is a little greater, but it is hardly perceptible in this case, as Jupiter is so near the ecliptic. Indeed it would increase the last decimal figure in the logarithm by 2, making it 6.

Jupiter appears from the sun at this time to be 6′ 41″ south of the ecliptic, but from the earth, the angle between it and the ecliptic would not be so great, because *EI* is greater than *SI*. But to compute the geocentric latitude of Jupiter or any other planet exactly, we have the following principle :

We refer in particular to this example, but the principle is general.

Conceive the perpendicular distance of the planet from the ecliptic to be represented by *D*, and let this distance be made radius; then *IS* will be the cotangent of the heliocentric latitude, and *IE* the cotangent of the geocentric latitude.

Denote the geocentric latitude by x; then

$$D : R :: IS : \text{cot. } 6' 41''$$

And $$D : R :: IE : \text{cot. } x$$

Whence, $$IS : \text{cot } 6' 41'' :: IE : \text{cot. } x = \frac{IE}{IS} \text{ cot. } 6' 41''$$

That is, *From the log. of the planet's distance from the earth, subtract the log. of its distance from the sun, and to the difference add the log. cotangent of the heliocentric latitude, and the sum is the log. cot. of the planet's geocentric latitude.*

To apply this equation with accuracy, requires some little tact in using logarithms. Observe that cot. 6′ 41″ is the tan. of 6′ 41″, subtracted from 20.0000. To find the tan. of 6′ 41″ or 401″, first find the tangent of 1″, then add the log. of 401.

tan. 1′=60″	6.463726
sub. log.60	1.778151
tan. 1″	4.685575
log. 401	2.603144
tan. 6′ 41″	7.288719

Log. *EI*—log. *IS*=0.7257234—0.7145150=0.0112084.

The log. cot. must be increased by this quantity, therefore the log. tan. must be diminished by the same ; hence

7.2887190
0.0112084

Log. tan. of geocentric latitude is............7.2775106
Subtract log. tan. of 1″..................4.685575

Log. of 390″8, or 6′ 31″ nearly,.............. 2.5919356

Thus we find the geocentric latitude of Jupiter to be 6′ 31″ south at this particular time.

Having the planet's latitude and longitude, we can compute its corresponding right ascension and declination, and the following results will be obtained :

Right ascension, 19h. 46m. 9s. South declination 21° 19′ 41″.

SOLAR ECLIPSES.

We will now show the computation to determine the times of beginning and end, and other circumstances attending a solar eclipse as seen from any assumed locality on the earth. No person can do this with any safety, depending on the rules of another, he must understand the nature and scope of the problem for himself. It requires a general knowledge of astronomy and philosophy, and a familiar knowledge of both plane and spherical trigonometry.

The mathematical philosophy of the subject is explained generally on page 214 of Robinson's Geometry, and here we will illustrate it by an example.

As near as we can determine by some rough projections, the eclipse of May 26, 1854, will* be nearly central and annular as seen from Burlington in Vermont. Curiosity has, therefore, led us to make minute calculations for that place.

We take the elements from the English Nautical Almanac.— Let the reader observe that the elements here correspond to the mean time of conjunction in *right ascension.* The elements in Robinson's Astronomy correspond to conjunction in *longitude,* the difference is 8m. 37s. in time.

1854, *May,* 26.

Greenwich mean time ☌ in R. A......... 8h. 55m. 43.8s.
Sun and moon's R. A...................... 4h. 13m. 7.41s.
Moon's Declination North,..................21° 33′ 31″8

* This was written in April, 1853, and therefore spoken of in the future tense.

Sun's Declination North,....................21° 11′ 16″8
Moon's Horary motion in R. A................ 31′ 18″9
Sun's Horary motion in R. A................. 2′ 31″8
Moon's Horary motion in Declination N........ 8′ 7″3
Sun's Horary motion in Declination N.......... 25″9
Moon's Equatorial Horizontal parallax......... 54′ 32″6
Sun's " " " .... 8″5
Moon's semidiameter 14′ 53″5. Sun's S. D. 15′ 48″9.
Lat. of Burlington 44° 28′ N. West Long. 73° 14′=4h. 52m. 56s

Greenwich mean time of ☌.....8h. 55m. 44s.
Long. in time..................4h. 52m. 56s.

Mean time of ☌ at Burlington..... 4h. 2m. 48s. P. M.
Equation of time, add .... 3m. 15s.

Conjunction at B., apparent time....4h. 6m. 3s.=61° 30′ 45″.

As the earth is not a perfect sphere, (the equatorial diameter being the largest,) the equatorial horizontal parallax requires reduction for other latitudes, and latitude itself requires a reduction at all points, except at the equator and the poles.

The horizontal semidiameter of the moon requires augmentation, as the moon rises in altitude, for the nearer the moon is to the zenith of the observer, the nearer it is in absolute distance.

The following tables correct the elements in these particulars.

Reduction of the Parallax and also of the Latitude.

Lat.	Red. of par.	Red. of Lat.	Lat.	Red. of par.	Red. of Lat.	Lat.	Red. of par.	Red. of Lat.
°	″	′ ″	°	″	′ ″	°	″	′ ″
0	0.0	0 0.0						
3	0.0	1 11.8	33	3.3	10 28.3	63	8.8	9 18.3
6	0.1	2 22.7	36	3.8	10 54.3	66	9.2	8 32.9
9	0.3	3 32.1	39	4.4	11 13.2	69	9.7	7 42.0
12	0.5	4 39.3	42	4.9	11 24.7	72	10.0	6 45.9
15	0.7	5 43.4	45	5.5	11 28.7	75	10.3	5 45.4
18	1.0	6 43.7	48	6.1	11 25.2	78	10.6	4 41.0
21	1.4	7 39.7	51	6.7	11 14.1	81	10.8	3 33.5
24	1.8	8 30.7	54	7.2	10 55.7	84	11.0	2 23.7
27	2.3	9 16.1	57	7.8	10 30.0	87	11.1	1 12.3
30	2.7	9 55.4	60	8.3	9 57.4	90	11.1	0 0.0

Augmentation of the Moon's Semi-diameter.

Alt.	Horizon. Semi-diameter.				Alt.	Horizon. Semi-diameter.			
	14′30″	15′	16′	17′		14′30″	15′	16′	17′
°	″	″	″	″	°	″	″	″	″
2	0.6	0.6	0.7	0.8	42	9.2	9.8	11.2	12.6
4	1.0	1.1	1.3	1.5	45	9.7	10.4	11.8	13.3
6	1.5	1.6	1.9	2.1	48	10.2	10.9	12.4	14.0
8	2.0	2.1	2.4	2.7	51	10.6	11.4	13.0	14.7
10	2.4	2.6	3.0	3.4	54	11.1	11.8	13.5	15.2
12	2.9	3.1	3.6	4.0	57	11.5	12.3	14.0	15.8
14	3.4	3.6	4.1	4.7	60	11.8	12.7	14.4	16.3
16	3.8	4.1	4.7	5.3	63	12.2	13.0	14.9	16.8
18	4.3	4.6	5.2	5.9	66	12.5	13.4	15.2	17.2
21	4.9	5.3	6.0	6.8	69	12.8	13.7	15.6	17.6
24	5 6	6.0	6.8	7.7	72	13.0	13.9	15.9	17.9
27	6.2	6.7	7.6	8.6	75	13.2	14.1	16.1	18.2
30	6.9	7.3	8.4	9.5	78	13.4	14.3	16.3	18.4
33	7.5	8.0	9.1	10.3	81	13.5	14.4	16.5	18.6
36	8.1	8.6	9.8	11.1	84	13.6	14.5	16.6	18.7
39	8.6	9.2	10.5	11.9	90	13.7	14.6	16.7	18.8

Equatorial horizontal parallax.............54′ 32″6
Reduction for latitude.................. 5″4

Reduced h. p...........................54′ 27″2
Subtract sun's h. p..................... 8″5

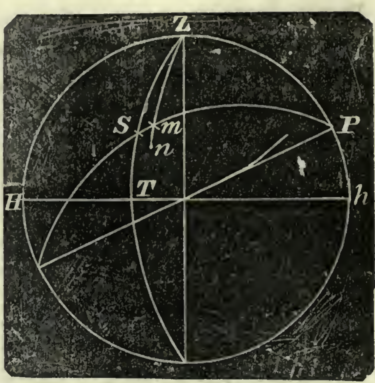
Relative or effective h. p................54′ 18″7=3258″7

Lat. of Burlington..................44° 28′
Reduction...................... 11′ 27

Reduced latitude....................44° 16′ 33″

As the sun and moon are west of the meridian, therefore the moon is apparently thrown back by the effects of parallax, and consequently the beginning of the eclipse will not take place until about, or after, the time of conjunction.

To decide this point, let m represent the place of the moon at 4h. 6m. 3s., and S the place of the sun at the same time. Sm=22′ 15″, their difference in declination, mn the parallax in altitude, and n will be the apparent place of the moon. Our



object is to find the distance between S and n, and we accomplish it by the aid of the spherical triangle PZm*. We have PZ, Pm, and the angle ZPm. We must find Zm and the angle ZmP.

We use the following equation taken from page 209, Geometry, in which A represents the moon's altitude, L the latitude of the place, and D the moon's polar distance.

$$\cos. P = \frac{\sin. A - \sin. L \cos. D}{\cos. L \sin. D}.$$

Whence, $\sin. A = \cos. Zm = \cos. P \cos. L \sin. D + \sin. L \cos. D$.

cos. P=cos. 61° 30′ 45″....9.678489		
cos. L=cos. 44° 16′ 33″.....9.854910	sin.....9.843917
sin. D=sin. 68° 26′ 28″.....9.968853	cos.....9.562944
0.31787.................—1.502252	.255192 —1.406861

Whence, the *natural sine* of the moon's true altitude, or cos. Zm=0.31787+0.25519=.57306.

---

* As the moon is at m, and the sun at S. on the same meridian PS, which is 61° 30′ 45″ west of Burlington, or 134° 44′ 45″ west of Greenwich; therefore, this is the meridian on which the sun will be centrally eclipsed at apparent noon; and the latitude will be such that mS must be the moon's parallax in altitude. To find that latitude is a very easy and interesting problem. Let the latitude be such that the apparent altitude of the moon shall be represented by x. Then 3257″ cos. x=mS=1335″. (Rad. 1.)

$$\cos. x = \frac{1335 \cdot R}{3258.7} \quad \begin{matrix} \ldots\ldots 13.125481 \\ \ldots\ldots 3.513044 \end{matrix}$$

cos. x=sin. moon's app. zenith distance 24° 11′ 5″	sin.....	9.612437
Moon's declination north.............	21° 33′ 32″	
Lat. (apparent,)......................	45° 44′ 37″	
Reduction of Lat......................	11′ 29″	
True Lat..............................	45° 33′	north.

Hence the sun will be centrally eclipsed at noon in longitude 134° 45′ west and latitude 45° 33′ north.

By taking out the corresponding arcs, we find

Moon's true altitude=34° 58′ 10″ Zm=55° 1′ 50″.

sin. Zm : sin. P : : sin. ZP : sin. ZmP= Smn.

Or, sin. 55° 1′ 50″ : sin. 61° 30′ 45″ : : cos. 44° 16′ 33″ : sin. Smn =sin. 50° 10′ 8″.

The next and most delicate step is to obtain mn, the parallax in altitude.

If we use the *true* altitude of the moon for the apparent altitude, we can find the approximate value of mn as follows: (see page 201, Surveying and Navigation.)

Horizontal parallax 3258″7	log.	3.513044
cos. 34° 58′ 10″	add.	9.913526
1st approx. value of mn 44′ 30″=2670″		3.426570

Moon's true alt. . . . . . .34° 58′ 10″
Sub. parallax in alt. . . . 44′ 30″
Moon's appa. alt. nearly . . 34° 13′ 40″

H. p. as before,	3.513044
cos. 34° 13′ 40″	9.917380
2d approx. value of mn, 44′ 54″=2694″	3.430424
H. p. as before,	3.513044
Moon's appa. alt. . . . . cos. 34° 13′ 16″	9.917422
True value of mn, 44′ 56″6=2694″6	3.430466

Now in the △ Smn, (which we may conceive to be a plane triangle,) we have Sm=1335″, mn=2694″6, and the angle Smn, 50° 10′ 8″, to find Sn.

If from n we conceive a perpendicular to be let fall on to the meridian Sm, and designate it by p, and the other side of the right angled triangle thus formed by q, then we shall have

R : 2694.6 : : sin. 50° 10′ 8″ : p

And R : 2694.6 : : cos. 50° 10′ 8″ : q

mn	3.430466		3.430466
sin. 50° 10′ 8″	9.885322	cos.	9.806537
p 2069.2	3.315788	q 1725.8	3.237003

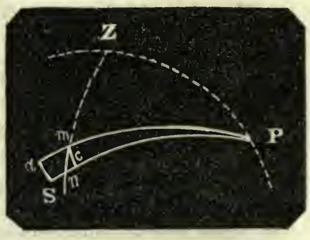
Observe that p is the effect of parallax perpendicular to the lunar meridian at that time; and q is the parallax in declination.

Moon's true declination north of the sun.......	1335″
Moon's parallax in declination south...........	1725″8
Moon's apparent dec. south of the sun.........	390″8

The apparent distance between sun and moon is, therefore,

$$\sqrt{(2069.2)^2+(390.8)^2}=2105''7$$

Moon's S. D., 14′ 53″5. Augmentation for altitude 8″. Sun's S. D. 15′ 48″9. Sum = 30′ 50″4=1850″4. But the distance (apparent) from center to center, we have just determined to be 2105″7; therefore the distance from *limb* to *limb* must be 255″3, and the eclipse has not yet commenced, and cannot commence, until the moon gains 255″ on the sun's motion, which will require more than ten minutes of time.



We now require the apparent distance between the centers of the sun and moon, ten or twelve minutes later, so as to get the apparent rate of approach. The rate is continually changing, but during any short interval of ten or twelve minutes, it may be considered uniform, without any sensible error.

If we vary the time, the angle ZPS will vary 1° to 4 minutes, but in that variable time the moon will move from c to m, and the angle ZPm will vary, but not quite so much as ZPS. The question now is, *If we make a small difference in the angle ZPm, what corresponding difference will it make in the arc Zm*; and this is a question in the *differential calculus*,* although we can work it out at large by spherical trigonometry.

We will take the interval of 12m., then the angle ZPS will increase 3°. But in one hour the moon's motion in right ascension exceeds that of the sun 28′ 47″; this in 12m. will be 5′ 45″4, therefore the angle ZPm varies in that interval of time, 2° 54′ 14″6=2.90405, taking one degree as the unit.

* The *differential calculus* is the science of minute variations, or of corresponding *small differences*—a science which owes its birth to the varying elements of astronomy.

The equation as before is

$$\sin. A = \cos. Zm = \cos. P \cos. L \sin. D + \sin. L \cos. D$$

But we have caused P to vary, while L and D remain constant. *What variation will this give to the altitude* A?

Taking the differential of the equation, we find

$$dA = -\frac{\sin. P \cos. L \sin. D \cdot dP}{\cos. A}$$

But we have assumed $dP=2.90405$, while P, L, D, and A, in this equation, have the same values as before. That is,

$P=61° 30' 45''$, $L=44° 16' 33''$, $D=68° 26' 28''$, and $A=34° 58' 10''$.

log. 2.90405	0.463000	
sin. P	—1.943954	(radius unity.)
cos. L	—1.854910	
sin. D	—1.968853	
cos. complement A	0.086445	
dA=2.0755	0.317162	

Thus we find that the moon changes its altitude at this time, in the interval of 12 minutes, 2° 4′ 32″, and because the second member of the last equation is *minus*, the altitude has diminished.

Moon's altitude was	34° 58′ 10″
Variation	2° 4′ 32″
Moon's altitude at this time	32° 53′ 38″

The angle $ZPm=61° 30' 45''+2° 54' 15''=64° 25'$.

cos. 32° 53′ 38″ : sin. 64° 25′ : : cos. 44° 16′ 33″ : sin. ZmP=sin. 50° 16′ 30″.

To find mn, or the parallax in altitude,

To log. of the horizontal parallax		3.513044
Add cos. 32° 53′ 38″		9.924100
Approximate value of mn 45′ 36″=2736″		3.437144
From the moon's true alt.	32° 53′ 38″	
Subtract apparent parallax	45′ 36″	3.513044
Moon's appa. alt. nearly	32° 8′ 8″ cos.	9.927877
True value of mn 2760″		3.440921

As before,

R : 2760 : : sin. 50° 16′ 30″ : p

R : 2760 : : cos. 50° 16′ 30″ : q

	3.440921		3.440921
sin. 50° 16′ 30″	9.885996	cos.	9.805420
p 2123″	3.326917	q 1763″5	3.246341

During the 12 minutes the moon moves over the oblique *small. arc cm,* (in relation to the sun, as conceived to be stationary,) which is 5′ 45″4 in right ascension, or the difference between the two meridians *PS* and *Pm on the equator,* is 5′ 45″4, or 345″4.

The perpendicular distance at the point *m* is therefore found by multiplying 345″4 by the cosine of the moon's declination to radius unity. Therefore,

Log. 345″4	2.538322
Moon's dec. 21° 35′ nearly, cos.	9.968429
Perpendicular dis. between *PS* and *Pm* 321″2	2.506751

During *one* hour the moon's relative motion in declinatian is 7′ 41″4. During 12 minutes it is therefore 92″2, which added to 22′ 15″ or 1335″ makes 1427″2 for the distance represented by *ma*. But q 1763″5 is the effect of parallax on the line or the effect in declination, and it being greater than 1427″2, their difference, 336″2 is the apparent distance in declination of n below S, or of the center of the moon below the center of the sun.

Again, p is the parallax in right ascension, projecting the moon 2123″ west of its true place, while it is 321″2 east of the sun; therefore the apparent right ascension of the moon is 1801″8 west of the sun. Consequently the apparent distance of the two centers is

$$\sqrt{(1801''8)^2+(336''2)^2}=1833''2.$$

But the semidiameter of the sun and the augmented semidiameter of the moon at this time amount to 1850″4, differing only 17″2. The distance between the centers being less than the sum of the semidiameters, shows that the eclipse has already commenced

Twelve minutes before this time, the distance between the centers was . . . . . 2105″7

Now it is . . . . . 1833″2

Moon's apparent motion in 12 minutes . . . . . 272″5

or 22″7 in one minute.

Then 22″7 : 17″2 : : 60s. : 45.4 seconds.

That is, the eclipse commences 11m. 14.6 sec. after the apparent time of conjunction at Burlington, or at 4h. 17m. 17.6 sec.

If 5 seconds be taken from the sum of the semidiameters for *irradiation* and *inflection*, as most astronomers recommend, the eclipse will commence at 4h. 17m. 30s.

THE POINT OF FIRST CONTACT.

The point n, the apparent place of the center of the moon is nearly west of S and the angle $ZmP=50°$. Therefore the point of first contact, from the sun's vertex, must be $(50°+90°)$, 140° towards the right, but if viewed through an inverting telescope, the appearance will be directly opposite.

GREATEST OBSCURATION.

The time of greatest obscuration will take place not far from 1h. 20m. after conjunction at Burlington, or not far from 5h. 26m. 3s.; we will therefore compute the apparent distance between the two centers for this time. We could compute it by proportion, provided the apparent motion of the moon was uniform, and in a straight line; but that motion being neither uniform nor in a straight line, we are compelled to compute it by *points* to obtain any thing like accuracy.

Using the last figure, the angle $ZPS=$5h. 26m. 3s.$=81°\ 30'\ 45''$; but during 1h. 20m. the moon will gain $38'\ 22''$ in right ascension; therefore the angle $ZPm=80°\ 52'\ 23''$.

In 1h. 20m. the moon will increase her declination $10'\ 49''$, making it $21°\ 44'\ 21''$, or $Pm=68°\ 15'\ 39''$, and am is now $32'\ 30''=1950''$.

As before,

$$\text{sin. } A=\text{cos. } Zm=\text{cos. } P \text{ cos. } L \text{ sin. } D+\text{sin. } L \text{ cos. } D.$$

cos. P=cos. 80° 52′ 23″.....	9.200404		
cos. L=cos. 44° 16′ 33″.....	9.854910	sin.....	9.843917
sin. D=sin. 68° 15′ 39″......	9.967959	cos.....	9.568656
0.10555.........	—1.023273	.25856	—1.412571

Nat. sin. $A=0.10555+0.25856=.36411$.

Whence, A, moon's true alt.$=21°\ 21'\ 12''$. $Zm=68°\ 38'\ 48''$.

sin. 68° 38′ 48″ : sin. 80° 52′ 23″ : : cos. 44° 16′ 33″ : sin. ZmP= sin. 49° 22′ 45″.

To find mn. Moon's horizontal par. log........3.513044
cos. 21° 21′ 12″..........9.969114

Approximate value of mn 50′ 35″=3035″.......3.482158

From moon's true alt...21° 21′ 12″
Take.............. 50′ 35″ 3.513044

Moon's appa. alt. nearly 20° 30′ 37″ cos..... 9.970630

True value of mn 3045″....................3.483674

R : 3045″ : : sin. 49° 22′ 45″ : p
R : 3045″ : : cos. 49° 22′ 45″ : q

	3.483674		3.483674
sin. 49° 22′ 45″......	9.880265	cos..........	9.813620
p=2311″8...........	3.363939	q=1982″6....	3.297294

In the 1h. and 20m. which elapses after conjunction, the moon gains 38′ 22″ or 2302″ in right ascension on the sun; but this is arc on the equator, it is not perpendicular distance, the two meridians PS and Pm, drawn from m; but that distance is required and it is found thus:

Log. 2302..............................3.362105
Add cos. of moon's declination 21° 44′ 21″......9.967959
mc........2138″5...............3.330064
p.........2311″8

Moon apparently west.. 173″3
Moon's declination north of sun am............1950″
Moon's parallax in declination q..............1982″6
Moon apparently south of the sun.............. 32″6

Distance between centers$=\sqrt{(173''3)^2+(32''6)^2}=176''3$.

We know by comparing this result with the last, that the greatest obscuration or nearest approach of the centers, must take place about 7 minutes after this time. We will, therefore, *differentiate* for 10 minutes.

In 10 minutes the sun's polar angle will increase from the meridian 2° 30′
For the ☽'s motion in R. A. sub. 4′ 47″
The angle ZPm will increase 2° 25′ 13″=2°.4202.

As before,

$$d.A = -\frac{\sin. P \cos. L \sin. D\ (2.4202)}{\cos. A}.$$

An equation in which $A=21°\ 21'\ 12''$, $P=80°\ 52'\ 23''$, $L=44°\ 16'\ 33''$, and $D=68°\ 15'\ 39''$.

sin. P	—1.994465 (radius 1)
cos. L	—1.854910
sin. D	—1.967959
log. 2.4202	0.383851
cos. complement A	0.030886
dA=1.7062	0.232071

The minus sign before the second member shows that this must be subtracted from A.

A	21° 21′ 12″
1.7062=	1° 42′ 22″
Moon's true altitude at this time, ...	19° 38′ 50″

Log. Horizontal parallax	....	3.513044
	cos. 19° 38′ 50″	9.973950
	51′ 9″	3.486994
Moon's app. alt. nearly	18° 47′ 41″	

		3.513044
	cos. 18° 47′ 41″	9.976154
True value of mn	3084″5	3.489198

cos. 19° 38′ 50″ : sin. 83° 17′ 36″ : : cos. 44° 16′ 33″ : sin. ZmP =sin. 49° 1′ 40″.

R : 3084″5 : : sin. 49° 1′ 40″ : p

R : 3084″5 : : cos. 49° 1′ 40″ : q

	3.489198		3.489198
sin.49° 1′ 40″	9.877978	cos.	9.816700
p 2329″....	3.367176	q 2022″7	3.305898

At the last point, am was 1950″ which has increased 77″ by the moon's motion; therefore it is now 2027″.

At the last point, Sa was 2302″ of *arc* which has increased 287″, making 2589″, which must be reduced to the arc of a great circle as before.

Log. 2589 . . . . 3.413132
Moon's declination 21° 45′ cos. 9.967927

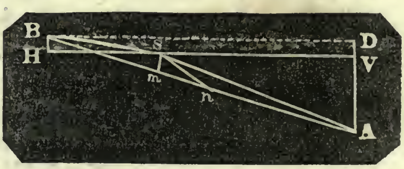
Moon east of sun . . . . . . 2404″5 . . . . 3.381059
Parallax in R. A. p . . . 2329″

Moon east of sun, apparently 75″5
Moon north of sun . . . . 2027″
Parallax in declination, q . . . . 2022″7

Moon north of sun, apparently . . . . 4″3

Distance between centers $=\sqrt{(75''5)^2+(4''3)^2}=75''6$, apparently.

Now to find the nearest approach of the centers, the time of forming the ring, its continuance, &c., we have a very delicate and simple problem in plane geometry.



Let S be the center of the sun. Take $SV=$ 173″3, VA=32″6. Then AS=176″3. Also take SH=75″5, HB = 4″3; then SB= 75″6, BD= 173″3+75″5=248″8. DA=37″ nearly; then AB the moon's apparent motion on the face of the sun during 10 minutes must be

$$\sqrt{(248''8)^2+(37'')^2}=251''5.$$

Therefore the apparent motion per minute is 25″15.

We must now find Sm, the distance between the two centers at the time of their nearest approach. In the triangle ABS, we have all the sides, therefore by (Prop. 6, page 149, Geom.), we have $AB : AS+SB :: AS-SB : Am-mB$

That is, 251″5 : 251″9 : : 100.7 : 100.86

$$Am+mB=251.5$$
$$Am-mB=100.86$$
$$2mB=150.64 \qquad mB=75.32.$$

Whence, Am=176″18. In the right angled triangle BmS, we have

$$Sm=\sqrt{(75''6)^2-(75''32)^2}=6''5$$

Moon's semidiameter from given elements . . . . 14′ 53″5
Augmentation for alt. 18° 4′ (see table) . . . . . . 4″7

Augmented semidiameter............	14′ 58″2
Sun's semidiameter from given elements........	15′ 48″9
Difference,................................	50″7 *

It is obvious that the ring will form when the distance between the two centers comes within the difference of the semidiameters. Suppose it to form when the moon's center passes n; then in the right angled triangle Smn, $Sn=50''7$. $Sm=6''5$.

And $mn=\sqrt{(50''7)^2-(6''5)^2}=50''28.$

$An=Am-mn=126''9$, and at the rate of 25″15 per minute, this will be passed over in 5m. 2.7 seconds, nm in 2 minutes very nearly, and an equal line on the other side of m in 2 minutes more.

The appa. time the ☽'s center arrives at A is	5h. 26m. 3s.
To which add........................	5m. 2.7s.
Ring forms at........................	5h. 31m. 5.7s.
Time of nearest approach..............	5h. 33m. 5.7s.
Rupture of the ring....................	5h. 35m. 5.3s.

At the time of nearest approach the breadth of the ring on the north limb of the sun will be 31″8, and on the south limb 18″8; but if the customary allowance be made for irradiation and inflection, these quantities reduce to 31″3 and 18″3, and the duration of the ring must be reduced from 3m. 59.6s. to 3m. 55.2s.

THE END OF THE ECLIPSE.

We know by the moon's apparent motion (25″15 per minute, which is continually increasing) that more than an hour will be required from the time of nearest approach, for the eclipse to pass off. We will therefore compute the apparent distance between the two centers, *one hour and ten minutes* after the moon passes A, of the last figure.

By referring back we shall find that the point A corresponds with 5h. 26m. 3s. or 81° 30′ 45″ for the angle ZPS. One hour and ten minutes later will be 6h. 36m. 3s. and will correspond to 99° 0′ 45″ for the angle ZPS. (See next figure.)

* Astronomers recommend a diminution of 3″ for the sun's semidiameter for *irradiation*, and a diminution of 2″ of the moon's semidiameter for *inflection*, this would make 49″7 for their difference instead of 50″7.

But the difference between the right ascensions of the sun and moon is now 1° 21′ 56″; therefore the angle ZPm=97° 38′ 49″.

At 5h. 26m. 3s. the value of *ma* was 1950″, in one hour and ten minutes it increased 540″, it is now 2490″.

Sa in arc=1° 21′ 56″=4916″ which we reduce to distance.

Log. 4916 .... 3.691612
Sun's dec. cos. 21° 12′ 9.969567
Sa .... 4583″ 3.661179

As before,

sin. A=cos. Zm=cos. P cos. L sin. D+sin. L cos. D

cos. P=cos. 97° 38′ 49″..—1.124088
cos. L=cos. 44° 16′ 33″..—1.854910 sin...—1.843917
sin. D=sin. 68° 7′ 13″..—1.967531 cos..—1.571327
—0.088416 * .... —2.946529 0.26015 –1.415244

sin. A or cos. Zm=0.26015—0.08842=.17173

Whence, A=9° 53′ 21″. Zm=80° 6′ 39″.

sin. 80° 6′ 39″ : sin. 97° 38′ 49″ : : cos. 44° 16′ 33″ : sin. ZmP= sin. 46° 5′.

Log. Horizontal parallax 3.513044
cos. 9° 53′ 21″ 9.993500
53′ 30″ 3.506544

Moon's app. alt. 8° 59′ 51″ cos. 9.994618
3.513044
mn 3.507662

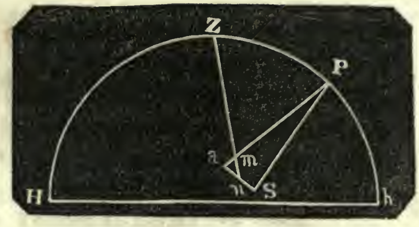
R : 3218 : : sin. 46° 5′ : p

R : 3218 : : cos. 46° 5′ : q

	3.507662		3.507662
sin. 46° 5′	9.857530	cos.	9.841116
p 2918.4	3.465192	q 2232.4	3.348778
Sa 4583.		2490	
1664.6		257.6	

Distance between the centers $=\sqrt{(1664.6)^2+(257.6)^2}$=1674″5.

* This number must be *minus* because cos. 97° is minus, the cosine of any arc over 90°, as far as 270°, is minus.



The sum of the semi-diameters is now 1843″; therefore the sun is still eclipsed, and will be until the apparent motion of the moon passes over 169″, which will require a little over 6 minutes of time, we will therefore compute the apparent distance of the centers for 8 minutes later. In 8m. the angle *ZPS* will increase 2°, and the angle Z*Pm* will increase 2°—(3′+50″) or 1° 56′ 10″. At the last operation Z*Pm* was 97° 38′ 49″, now it must be 99° 34′ 59″, say 99° 35′.

If we take *D* as constant in the last equation, we shall find that all our logarithms will be the same except cos. *P*, and all we have to do is to add to log. —2.946529 the difference between cos. *P* in the last operation and the cosine of 99° 35′, or the sine of 9° 35′ for the log. of the first number composing the natural sine of *A*.

That is, to .... —2.946529
Add .... .097279

Number, —0.110604 .... —1.043808 *

sin. *A*=0.26015—0.110604=.149546. Whence, *A*=8° 36′.†

cos. 8° 36′ : sin. 99° 35′ : : cos. 44° 16′ 33″ : sin. *ZmP*= sin. 45° 33′ 50″.

Log. Horizontal parallax 3.513044
cos. *A* 8° 36′ 9.995089

Approx. val. of *mn* 53′ 42″ 3.508133

☽'s appa. alt. nearly 7° 42′ 18″ cos. 9.996064
3.513044

True value of *mn* 3229″5 .... 3.509108

R : 3229″5 : : sin. 45° 33′ 50″ : *p*

R : 3229″5 : : cos. 45° 33′ 50″ : *q*

* Artifice should be employed to take out the number corresponding to this logarithm, such as is taught in the author's Surveying and Navigation.

† Here we have jumped from one result to another, and did not obtain the difference between one result and another, as we do by the differential method.

	3.509108		3.509108
sin. 45° 33′ 50″....	9.853717	cos.............	9.845168
p 2902″6..........	3.462825	q 2260.5........	3.354276

Before *Sa* was in arc 4916″, increase in 8m. 230″; therefore it is now 5146″ which must be reduced as before.

Log. 5146......3.711470
cos. 21° 12′.....9.969567

Sa in space,...4797″6...3.681037
p....2902″6 *ma* before was......2490
Paral. in R. A....1895 Increase in 8m..... 62
2552
q..........2260.5

Moon apparently north of sun................. 291.5

Distance between the centers $=\sqrt{(1895)^2+(291.5)^2}=1917''3$.

This being greater than 1843″ shows that the eclipse has passed off.

The distance between the centers now is........1917″3
Eight minutes ago the distance was...........167″45
Apparent motion of the moon in 8 minutes......242″8
Corresponding motion for 1 minute 30″35.

From the sum of the semidiameters 1843″, subtract 1674″5, and we obtain 168″5 for the moon to pass over before the end of the eclipse. This at the rate of 30″35 per minute requires 5m. 33s.

Hence, to..................6h. 36m. 3s. appa. time.
Add...................... 5m. 33s.

Eclipse ends.................6h. 41m. 36s. appa. time.

But if we reduce the semidiameters for irradiation and inflection 5″, then we must diminish the time of ending 10 seconds.

We may now observe that the moon's apparent motion across the sun was at the rate of 22″7 per minute at the beginning of the eclipse, 25″15 at the time of nearest approach, and 30″35 per minute at the end. This variability of the apparent motion is owing to the varying effects of the moon's parallax corresponding to the different altitudes, and this makes the problem tedious, and throws over it an air of complexity.

Since six o'clock the rate of the moon's motion from the sun increased very much, and any one can see the *rationale* of this by inspecting the projection on page 293 of Robinson's Astronomy.

Along the mid-day hours the sun and moon have an apparent motion together, but with different velocities. As the time from noon increases, the sun's motion along the ellipse is slower, and the moon appears to run over it faster and faster. After 6 the sun's apparent motion is no longer with the moon's, hence a rapid increase in the moon's apparent motion.

We have made the problem much longer than we should have done, had we simply been in pursuit of results. Our object has been to explain and illustrate the problem to a learner, through each consecutive step, and we have found the following

SUMMARY.

	Appa. time Burlington, Vt.	Mean time.
Beginning of the eclipse,	4h. 17m. 17.6s.	4h. 14m. 2.5s.
Formation of the ring,	5h. 31m. 5.7s.	5h. 27m. 50.6s.
Time of nearest appr. of cen.	5h. 33m. 5.7s.	5h. 29m. 50.6s.
Rupture of the ring,	5h. 35m. 5.3s.	5h. 31m. 50.2s.
End of the eclipse,	6h. 41m. 36 s.	6h. 38m. 21 s.

Duration of the ring 4 minutes nearly; duration of the eclipse 2h. 24m. 19s. When the ring is most perfect, its breadth on the north limb will be 31″, and on the south limb 18″.

Not long since the author received the following request: we extract from the letter.

"One request more. In your Astronomy, page 191, near the bottom, you say, (speaking of the radial force,) '*and the diminution in the one case is double the amount of increase in the other*, and by the *application of the differential calculus*, we learn the mean result for the entire revolution, is a diminution whose analytical expression is $\frac{rS}{2a^3}$; an expression which holds a very prominent place in the lunar theory.'

Now my enquiry is, how can we obtain the expression $\frac{rS}{2a^3}$ for the mean result? What operation in the calculus shall we go through?

Yours, &c., WM. T"

To this we returned the following reply:

On page 193 you will find the following expression,

$$\frac{rS(3\cos.^2x-1)}{a^3}$$

for the radial force corresponding to any angle x from the syzigies.

We already know the value of this force at the syzigies and quadratures, and at these points the result has the same general form; therefore the result for the entire quadrant, that is, the *mean* result for the whole quadrant, will be found by taking the angle $x=45°$, and as the mean result for each quadrant is the same, this will be the *mean* result for the entire revolution.

Whence, $x=45°$, $\sin. x=\cos x$, and $2\cos.^2x=1$, or $\cos.^2x=\frac{1}{2}$.

Or, $(3\cos.^2x-1)=\frac{1}{2}$; whence the above general expression becomes $\frac{rS}{2a^3}$.

To this was returned the following observation:

"I understand your explanation, it is very simple; but why did you not make this explanation in the book,—and more than all, why do you call it an application of the *differential calculus?* *I* can see no calculus in it.

Yours, &c., WM. T."

To this we rejoined as follows:

If the operation I sent you is not calculation, I know not what it is—it may therefore be called calculus; and if in any operation small quantities may be omitted on account of their insignificance in relation to larger quantities, the small difference so omitted constitutes the *differential calculus*, and to obtain that general expression, you will see, by looking on page 193 of the Astronomy, that the powers of r above the first were omitted.

THE CALCULUS.

DIFFERENTIAL CALCULUS.

The differential calculus is a branch of Analytical Geometry. It is a science for computing the *ratio* of small differences.

For example, the side of a square is increased by a very small quantity, what will be the corresponding increase of the square itself?

The side of a cube is increased or diminished by a quantity very small in relation to the side itself,—how much will this increase or diminish the cube?

The arc of a circle is increased or diminished by a quantity very small in comparison with itself, what effect will this have on the sine and cosine of the arc?

The sun's longitude increases a certain distance in 10 minutes, what is its corresponding change in declination? Or, find the law of these corresponding changes, or differences—*called differentials.* These questions explain in part the object of the calculus.

The calls of astronomy gave birth to this science, as we have before remarked.

For the development of this science, see the various works upon it. We confine ourselves in this book to a few difficult or curious operations.

We presume the reader is acquainted with all the *rules* of operation.

EXAMPLES.

(1.) Differentiate the expression $\sqrt{1-x^2}$. *Ans.* $\dfrac{-xdx}{\sqrt{1-x^2}}$.

Put $u=\sqrt{1-x^2}$. Square, $u^2=1-x^2$; then

$$2udu=-2xdx, \text{ or } du=d\cdot\sqrt{1-x^2}=-\frac{xdx}{\sqrt{1-x^2}}.$$

(2.) Find the differential of the equation

$$u=\frac{x}{x+\sqrt{1-x^2}}$$

By the rule for differentiating a fraction, we have

$$du=\frac{dx(x+\sqrt{1-x^2})-xdx+\dfrac{x^2dx}{\sqrt{1-x^2}}}{(x+\sqrt{1-x^2})^2}$$

Or, $$\frac{du}{dx}=\frac{\sqrt{1-x^2}+\dfrac{x^2}{\sqrt{1-x^2}}}{(x+\sqrt{1-x^2})^2}$$

By multiplying numerator and denominator of the second member by $\sqrt{1-x^2}$, then multiplying the equation by dx, we have

$$du=\frac{dx}{(x+\sqrt{1-x^2})^2\sqrt{1-x^2}}$$

(3.) Given $u=\left(a+\sqrt{b-\frac{c}{x^2}}\right)^4$ to find the differential of u.

Put $y=\sqrt{b-\frac{c}{x^2}}$ and extract the 4th root of the original equation; then

$$u^{\frac{1}{4}}=a+y$$

$$\tfrac{1}{4}u^{\frac{1}{4}-1}du=dy$$

$$\frac{du}{u^{\frac{3}{4}}}=4dy$$

$$du=4dy(a+y)^3 \qquad (1)$$

But $y^2=b-\frac{c}{x^2}$ Whence, $ydy=\frac{cdx}{x^3}$. $dy=\dfrac{cdx}{x^3\sqrt{b-\frac{c}{x^2}}}$

Substituting the values of y and dy in (1), we have

$$du=\frac{\dfrac{4c}{x^3}\left(a+\sqrt{b-\frac{c}{x^2}}\right)^3dx}{\sqrt{b-\frac{c}{x^2}}}$$

(4.) Given $u=\frac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}-\sqrt{1-x}}$ to find the differential of u.*

Reduce the second member by multiplying numerator and denominator of the fraction by the numerator; then

$$u=\frac{1+\sqrt{1-x^2}}{x}$$

Apply the rule to differentiate a fraction, and to differentiate the numerator, simply make use of the first example.

$$du=\frac{\frac{-x^2dx}{\sqrt{1-x^2}}-dx(1+\sqrt{1-x^2})}{x^2}$$

Dividing both members by dx, and changing signs, and

$$-\frac{du}{dx}=\frac{1}{\sqrt{1-x^2}}+\frac{1+\sqrt{1-x^2}}{x^2}$$

$$=\frac{x^2+\sqrt{1-x^2}+1-x^2}{x^2\sqrt{1-x^2}}=\frac{1+\sqrt{1-x^2}}{x^2\sqrt{1-x^2}}$$

Whence, $$du=-\left(\frac{1+\sqrt{1-x^2}}{x^2\sqrt{1-x^2}}\right)dx$$

(5.) Given $u=\log.\left(\frac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}-\sqrt{1-x}}\right)$ to find the differential of u.†

The differential of a logarithm is the differential of the quantity divided by the quantity. But we have just obtained the differential of the quantity in the 4th example, therefore all we have to do is to divide that result by the quantity itself.

That is, $-dx\left(\frac{1+\sqrt{1-x^2}}{x^2\sqrt{1-x^2}}\right)\times\frac{x}{(1+\sqrt{1-x^2})}=-\frac{dx}{x\sqrt{1-x^2}}$. *Ans.*

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* These examples are common to all or nearly all the works on the calculus; they are in the works of La Croix, from which they have been extracted into other works.

† This example is worked differently in Davies' Calculus, page 63; the work is extracted from La Croix.

(6.) Given $u=\log.(x+\sqrt{1+x^2})$ to find the differential of u.

By the rule for differentiating a logarithm, we have

$$du=\frac{dx+\frac{xdx}{\sqrt{1+x^2}}}{x+\sqrt{1+x^2}}$$

To simplify the operation, divide both members by dx, then

$$\frac{du}{dx}=\frac{1+\frac{x}{\sqrt{1+x^2}}}{x+\sqrt{1+x^2}}$$

Multiply numerator and denominator of the second member by $\sqrt{1+x^2}$, then

$$\frac{du}{dx}=\frac{\sqrt{1+x^2}+x}{(x+\sqrt{1+x^2})\sqrt{1+x^2}}=\frac{1}{\sqrt{1+x^2}}$$

Whence, $$du=\frac{dx}{\sqrt{1+x^2}}$$

(7.) Given $u=\frac{1}{\sqrt{-1}}\log.(x\sqrt{-1}+\sqrt{1-x^2})$ to find the differential of u.

To avoid the confusion of mind which naturally accompanies the imaginary symbol $\sqrt{-1}$, I put $a=\sqrt{-1}$. Then,

$$au=\log.(ax+\sqrt{1+x^2})$$

$$adu=\frac{adx-\frac{xdx}{\sqrt{1-x^2}}}{ax+\sqrt{1-x^2}}$$

$$\frac{adu}{dx}=\frac{a-\frac{x}{\sqrt{1-x^2}}}{ax+\sqrt{1-x^2}}=\frac{(a\sqrt{1-x^2})-x}{(ax+\sqrt{1-x^2})\sqrt{1-x^2}}$$

Now divide the numerator in the second member by the first factor in the denominator, thus:

$$\sqrt{1-x^2}+ax\;)\;a\sqrt{1-x^2}-x\;(\;a$$
$$a\sqrt{1-x^2}+a^2x$$

There is no remainder because $a^2x=-x$, as will be obvious when we consider $a=\sqrt{-1}$; hence $a^2=-1$.

Whence, $\frac{adu}{dx}=\frac{a}{\sqrt{1-x^2}}$, or $du=\frac{dx}{\sqrt{1-x^2}}$

(8.) Given $u=\log.\left(\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}-x}\right)^{\frac{1}{2}}$ to find the differential of u.

Reduce the fraction by multiplying numerator and denominator by the numerator, then

$$u=\log.\left(\frac{(\sqrt{1+x^2}+x)^2}{1}\right)^{\frac{1}{2}}$$

That is, $u=\log.(x+\sqrt{1+x^2})$, and this is the 6th example.

Therefore, $du=\frac{dx}{\sqrt{1+x^2}}$

(9.) *Differentiate the equation* $u=x^m(\log.x)^n$.

Put $\log.x=z$; then $u=x^mz^n$.

And $du=mx^{m-1}z^ndx+nz^{n-1}x^mdz$.

Because $z=\log.x$, $dz=\frac{dx}{x}$. Now by substitution,

$$du=mx^{m-1}(\log.x)^ndx+n(\log.x)^{n-1}x^{m-1}dx$$

Finally, $\frac{du}{dx}=(m\log.x+n)x^{m-1}(\log.x)^{n-1}$.

(10.) *Differentiate* $u=\frac{x^4(\log.x)^2}{4}-\frac{x^4(\log.x)}{8}+\frac{x^4}{32}$

As before, let $z=\log.x$; then the equation becomes

$$u=\frac{x^4z^2}{4}-\frac{x^4z}{8}+\frac{x^4}{32}$$

Then $du=x^3z^2dx+\frac{x^4zdz}{2}-\frac{x^3zdx}{2}-\frac{x^4dz}{8}+\frac{x^3dx}{8}$

But $dz=\frac{dx}{x}$, and $z=\log.x$; substituting these values, the preceding equation becomes

$$du=x^3(\log.x)^2dx+\frac{x^3(\log.x)dx}{2}-\frac{x^3(\log.x)dx}{2}-\frac{x^3dx}{8}+\frac{x^3dx}{8}$$

Whence, $du=x^3(\log.x)^2dx$.

(11.) *Differentiate* $u=\log.^2 x$: *that is to say, the logarithm of the logarithm of* x.

Put $\log. x=z$. Then $u=\log. z$.

And $du=\frac{dz}{z}$. But $dz=\frac{dx}{x}$.

That is, $$du=\frac{dx}{z\,x}=\frac{dx}{x(\log.x)}$$

(12.) *Differentiate* $u=\log.^5 x$.

By the aid of the previous example we learn that

$$\log.^5 x=\log.(\log.^4 x).$$

Whence, $$du=\frac{d\,(\log.^4 x)}{\log.^4 x} \quad (1)$$

$d\,(\log.^4 x)=\frac{d\,(\log.^3 x)}{\log.^3 x}$, this substituted in (1) reduces it to

$$du=\frac{d\,(\log.^3 x)}{(\log.^4 x)(\log.^3 x)}$$

Another step gives $du=\frac{d\,(\log.^2 x)}{(\log.^4 x)(\log.^3 x)(\log^2 x)}$

Using the result of the previous example, we finally have

$$du=\frac{dx}{(\log.^4 x)(\log.^3 x)(\log.^2 x)(\log.x)x}$$

(13.) *Differentiate* $u=e\,(x-1)$.

Here we must take the logarithm of each member, observing that the log. of e^x is simply x, because e is the base of the Naperian system of logarithms, the system always used in such examples.

$$\log. u=x+\log.(x-1).$$

$$\frac{du}{u}=dx+\frac{dx}{x-1}=\frac{dx}{x-1}$$

Whence, $\frac{du}{dx}=\frac{u}{x-1}=\frac{e^x(x-1)}{x-1}=e^x$. Or, $du=e^x dx$.

(14.) *Differentiate* $u=e^x(x^3-3x^2+6x-6)$.

$$\log. u=x+\log.(x^3-3x^2+6x-6)$$

$$\frac{du}{u}=dx+\frac{3x^2dx-6xdx+6dx}{x^3-3x^2+6x-6}$$

$$\frac{du}{udx}=\frac{x^3}{x^3-3x^2+6x-6}$$

$$\frac{du}{dx}=\frac{x^3u}{x^3-3x^2+6x-6}=e^xx^3. \quad \text{Or, } du=e^xx^3dx.$$

(15.) *Differentiate* $u=\frac{e^xx}{1-x}$.

$$u(1-x)=e^xx. \quad \log.u+\log.(1-x)=x+\log.x.$$

$$\frac{du}{u}-\frac{dx}{1-x}=dx+\frac{dx}{x} \qquad \frac{du}{udx}=\frac{1}{1-x}+\frac{x+1}{x}=\frac{1+x-x^2}{(1-x)x}$$

$$\frac{du}{dx}=\frac{(1+x-x^2)}{(1-x)x}\times\frac{e^xx}{1-x}=\frac{(1+x-x^2)e^x}{(1-x)^2} \text{ Or, } du=\frac{(1+x-x^2)e^xdx}{(1-x)^2}$$

(16.) *Differentiate* $u=e^x\log.x$.

Put $y=\log.x$; then $u=e^xy$.

$$\log.u=x+\log.y. \qquad \frac{du}{u}=dx+\frac{dy}{y}$$

But $dy=\frac{dx}{x}$, and $\frac{dy}{y}=\frac{dx}{x\log.x}$

Whence, $\frac{du}{u}=dx+\frac{dx}{x\log.x} \qquad \frac{du}{udx}=\frac{x\log.x+1}{x\log.x}$

$$\frac{du}{dx}=\frac{(x\log.x+1)u}{x\log.x}=\frac{(x\log.x+1)e^x\log.x}{x\log.x}$$

$$du=\left(\frac{x\log.x+1}{x}\right)e^x dx$$

(17.) *Differentiate* $u=\frac{e^x-1}{e^x+1}$

Put $e^x-1=P$ (1)
And $e^x+1=Q$ (2) Then $u=\frac{P}{Q}$

$$du=\frac{QdP-PdQ}{Q^2} \quad (3)$$

Differentiating (1) gives $e^xdx=dP$. (4)

And (2) gives $e^xdx=dQ$ (5)

Whence. $(e^x+1)e^xdx=QdP$

And, $(e^x-1)e^xdx=PdQ$

By subtraction, $2e^xdx=QdP-PdQ$

Whence, $du=\frac{2e^xdx}{(e^x+1)^2}$.

CIRCULAR FUNCTIONS.

For the sake of reference we will here note down the differential expressions for trigonometrical lines.

Let the radius of a circle be unity. Represent an arc by x, then its differential will be dx.

$$d\ \sin.x=\cos.x\,dx \qquad (1) \qquad d\ \cos.x=-\sin.x\,dx \qquad (2)$$

$$d\ \text{ver.}\sin.x=\sin.x\,dx \qquad (3) \qquad d\ \sec.x=\frac{\tan.x\,dx}{\cos.x} \qquad (4)$$

$$d\ \text{tang.}\,x=\frac{dx}{\cos.^2x} \qquad (5) \qquad d\ \cot.=-\frac{dx}{\sin.^2x} \qquad (6)$$

One great difficulty which troubles and perplexes the student in the calculus, arises from the fact that only the abstract theory of the science has been hitherto brought to our notice, in our elementary books.

All can understand how these equations, (1), (2), (3), &c., are obtained; but what if we can? says the inquiring student. What use are they? What do we learn by them?

It is useless to answer these questions by words only, we must show the answer by the following

EXAMPLES.

Equation (1), for example, shows a general truth. It is true applied any where along the quadrant of a circle. x is any arc that we choose to assume, and dx must be an arc sufficiently small to be considered a straight line.

If $x=20°$, and $dx=1'$, then the difference between the sine of 20° and the sine of 20° 1', is $d\ \sin.x=\cos.20°\times1'$.

cos. 20°	9.972986
Log. sin. or arc of 1'=	6.463726
Sum less 20= log. of 0.0002733	—4.436712

To the natural sine of 20°	.342020
Add the differential	.0002733
Sum is the Nat. sine of 20° 1'	.3422933

Thus we might give examples without end.

Because the differential of a logarithm is the differential of the quantity divided by the quantity, therefore

$$d \log. \sin. x = \frac{d \sin. x}{\sin. x} = \frac{\cos. x}{\sin. x} dx = \cot. x\, dx.$$

This result corresponds to the *modulus* of unity; for the modulus of our common system we must multiply by $0.43429448 = m$.

For example, if we assume $x=25°$, and also assume $dx=1'$, the differential, or the difference between the log. sine of 25° and the log. sine of 25° 1′ is expressed by $m \cot. 25° \times 1'$.

Log. m	—1.637784
cot. 25°	0.331327
Log. sine 1′, less 10	—4.463726
.0002709	—4.432837

To the log. sine of 25°	9.625948
Add the differential	.000271
Log. sine of 25° 1′=	9.626219

We might assume $dx=2'$ as well as 1′, without error as far as six places of decimals; but it would not do to assume $dx=$ any large number of minutes; hence the differential calculus must be applied with judgment.*

To show another example of the utility of the calculus, we will let E represent the obliquity of the ecliptic, L the longitude of the sun at any time whatever, and D its corresponding declination, the radius of the sphere being unity.

Then the following equation is general:

$$\sin. D = \sin. E \sin. L \qquad (1)$$

This equation represents the sine of the sun's declination at any point whatever, along the ecliptic. Because $\sin. L$ is 1 at the points where $L=90°$ or 270°; therefore at these points $\sin. D = \sin. E$, or $D=E$, as it should be.

Now suppose that the sun changes its longitude 10′, which we may call the (dL), or the differential of L, what will be the cor-

* Those who are naturally more nice than wise, are commonly prejudiced against this science, and such frequently say it is no science at all; however, their objections are of no consequence.

responding change in its declination, or what will be the value of (dD)?

To answer this question we must take the differential of each member of the general equation, then we shall have

$$\cos. D\,dD = \sin. E \cos. L\,dL \qquad (2)$$

Now whatever values we may assign to L and dL, equations (1) and (2) will always give D and dD at any point.

For a definite example we give the following:

What will be the differential in declination corresponding to the differential of 10′ *in longitude at* 35° *of longitude?*

In other words, what will be the change in the sun's declination while it passes from longitude 35° to 35° 10′?

From (1) we find D thus:

sin. E	9.599970
sin. L 35°	9.758591
sin. D 13° 12′ 5″	9.358561

From (2), $$dD = \frac{\sin. E \cos. L\,dL}{\cos. D}$$

sin. E		9.599970
cos. L 35°		9.913365
dL=10 log.		1.
cos. D, complement		0.011629
Sum (less 20)	3.343	0.524964

This is 3′ 20″6 nearly, and if the sun's declination is 13° 12′ 5″, when its longitude is 35°, the declination must be 13° 15′ 25″6 at the longitude 35° 10′.

This is *not strictly true,* because the *ratio* of motion in declination changes in a very slight degree between 35° and 35° 10′. But the ratio between the motion in longitude and the motion in declination, is *strictly* as 1 to .3343, at the beginning of the arc between 35° and 35° 10′.

This ratio is constantly changing, but still equation (2) always represents it.

We now require the ratio of motion in declination when the sun's longitude is 90°, that is, $L=90°$; then $\cos. L=0$; and substituting this value in (2) will cause the second member to disappear; and

$$\cos. D\,dD = 0$$

Now one or the other of these factors must be zero; but cos. D is evidently not zero; therefore (dD) must equal 0, showing that there is no motion in declination *exactly* at that point.

On the contrary, we may demand the sun's longitude when the motion in declination becomes zero.

In other words, what will equation (2) show when $(dD)=0$?

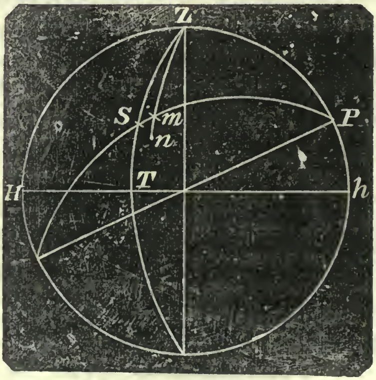
Then, $\sin. E \cos. L\, dL=0.$

One of these three factors must be zero; but $(\sin. E)$ cannot be zero, for it is a known constant quantity; (dL) cannot equal zero in case the least possible time elapses, for the sun's apparent motion never ceases; then $(\cos. L)$ must be zero, and $L=90°$, or $270°$.

To show another example of the utility of the calculus, we present the problem that appears in another shape, on page 229, of our Surveying and Navigation, namely:

Under what circumstances will an error in the altitude of the sun, produce the least possible error in the time deduced therefrom; the declination and latitude being constant quantities.

Let P be the polar point, Hh the horizon, S the position of the sun, and Z the zenith of the observer.



Let $A=$ the altitude of the sun, D its declination, and L the latitude of the observer.

Then by spherical trigonometry (see page 214, Surveying and Navigation) we have

$$\cos. P=\frac{\sin. A-\sin. L \cos. D}{\cos. L \sin. D}$$

The altitude A varies, and P the polar angle, or time from apparent noon, must vary in consequence of the variations of A; and if A is not accurately taken, P will not be accurate.

In short, a differential to A, will produce a differential in P.

Therefore we must differentiate the equation, taking A and P as variables, and L and D constants.

Whence, $$-\sin. P\, dP=\frac{\cos. A\, dA}{\cos. L \sin. D} \qquad (1)$$

But in the triangle PZS, we have

$$\cos. A : \sin. P :: \cos. D : \sin. SZP, \text{ or } \sin. Z$$

Or, $$\cos. A=\frac{\sin. P \cos. D}{\sin. Z} \qquad (2)$$

Substituting this value of $\cos. A$ in (1), and dividing by sine P, we obtain

$$-dP=\frac{\cos. D}{\cos. L \sin. D \sin. Z}\times dA$$

Or, $$-dP=\frac{dA}{\cos. L \tan. D \sin. Z} \qquad (3)$$

Now it is obvious that the numerical value of (dP), or error in time, will be least when the denominator of the fraction in the second member is *greatest*, and that will be greatest when $\sin. Z$ is greatest, which is the case whenever $Z=90°$, which is when the sun is east or west of the observer.

The sign before (dP) being minus, shows that when the altitude A increases, the angle P decreases.

If we make $dA=0$, equation (3) will give $dP=0$, showing that if there be no error in A, there will be none in P.

We give a few more examples in circular functions.

(1.) *Let* Z *be an arc or angle whose radius is unity and cosine* (mx): *we require the differential of the arc in terms of* mx.

In other words, differentiate $Z=\cos.^{-1}(mx)$.

The rules of operation are all comprised in the following equations:

$$d \sin. x=\cos. x\, dx \quad (1) \qquad d \cos. x=-\sin. x\, dx \quad (2)$$

$$d \text{ tang}. x=\frac{dx}{\cos.^2 x} \quad (3) \qquad d \cot. x=-\frac{dx}{\sin.^2 x} \quad (4)$$

in which x represents the arc to radius unity.

Because to all arcs $\sin.^2+\cos.^2=1$. If $\cos.=mx$, the sine $=\sqrt{1-m^2x^2}$.

The differential of the first member of our equation is evidently dZ; that of the second member is $d(\cos.^{-1}mx)$, which by equation (2) gives $dZ=\frac{-mdx}{\sqrt{1-m^2x^2}}$

(2.) *Differentiate* $Z=\sin.^{-1}(mx)$.

If (mx) is the sine of an arc the cosine is $\sqrt{1-m^2x^2}$.

Whence, $dZ=\frac{mdx}{\sqrt{1-m^2x^2}}$

(3.) *Differentiate the equation*

$$Z=\sin.^{-1}\left(\frac{y}{\sqrt{1-y^2}}\right)$$

If $\left(\frac{y}{\sqrt{1-y^2}}\right)$ represents the sine of an arc, the cosine of the same arc must be $\sqrt{\frac{1-2y^2}{1-y^2}}$

We can perform this operation the most clearly, by observing the following proportion:

The differential of any arc, is to the differential of its cosine (taken negatively), as radius (or unity) is to the sine of the same arc.

This proportion is drawn from the consideration that if x represents an arc, its differential is dx and its cosine is $(\cos.x)$ and the differential of $\cos.x$ by (2) is $-\sin.x\,dx$, and taken negatively, it is $\sin.x\,dx$; and obviously

$$dx \;:\; \sin.x\,dx \;::\; 1 \;:\; \sin.x$$

Applying this proportion to the example before us, we have

$$dZ \;:\; -d\sqrt{\frac{1-2y^2}{1-y^2}} \;::\; 1 \;:\; \frac{y}{\sqrt{1-y^2}} \qquad (a)$$

The difficulty, (all there is of difficulty), is in taking the differential of the second term.

Put $\sqrt{\frac{1-2y^2}{1-y^2}}=Q$; then (dQ) will be the differential value of the second term in the proportion.

$$\frac{1-2y^2}{1-y^2}=Q^2.$$

Differentiating the first member as a fraction, we have

$$\frac{-4ydy(1-y^2)+2ydy(1-2y^2)}{(1-y^2)^2}=2QdQ$$

Reducing, $\dfrac{-2(1-y^2)+(1-2y^2)}{(1-y^2)^2}=\dfrac{QdQ}{ydy}$

That is, $\dfrac{-1}{(1-y^2)^2}=\dfrac{QdQ}{ydy}$

Whence, $-dQ=\dfrac{ydy}{(1-y^2)^2Q}=\dfrac{ydy\sqrt{1-y^2}}{(1-y^2)^2\sqrt{1-2y^2}}$

By substituting this value, proportion (a), becomes

$$dZ : \frac{ydy\sqrt{1-y^2}}{(1-y^2)^2\sqrt{1-2y^2}} :: 1 : \frac{y}{\sqrt{1-y^2}}$$

Or, $dZ : \dfrac{dy(1-y^2)}{(1-y^2)^2\sqrt{1-2y^2}} :: 1 : 1$

Whence, $dZ=\dfrac{dy}{(1-y^2)\sqrt{1-2y^2}}$

(4.) *Differentiate the equation* $Z=\sin.^{-1}(2u\sqrt{1-u^2}.)$

If $2u\sqrt{1-u^2}$ is the sine of an arc, the cosine of the same arc must be $(1-2u^2)$.

By the proportion observed in the last example,

$$dZ : -d(1-2u^2) :: 1 : 2u\sqrt{1-u^2}$$

That is, $dZ : 4udu :: 1 : 2u\sqrt{1-u^2}$

Whence $dZ=\dfrac{2du}{\sqrt{1-u^2}}$

(5.) *Differentiate the equation* $u=\cos. x \sin. 2x$.

Regarding the second member as a product, and observing the differentials for sines and cosines, we have

$$du=-\sin. x \sin. 2xdx+2\cos. 2x \cos. xdx$$

Whence, $\dfrac{du}{dx}=(\cos. 2x\cos. x-\sin. 2x\sin. x)+\cos. x\cos. 2x.$

By observing equation (9), page 141, Robinson's Geometry, we perceive that the quantity in parenthesis is the same as cos. $(2x+x)$, or cos. $3x$; therefore,

$$du=(\cos. 3x+\cos 2x\cos. x)dx.$$

(6.) *Differentiate the equation* $u=(\tan. x)^n$.

Put $\tan. x=y$; then $u=y^n$, and $du=ny^{n-1}dy$ (a)

But $y=\tan. x$; therefore $dy=d(\tan. x)=\dfrac{dx}{\cos.^2 x}$, see (3).

Whence (a) becomes $du=\dfrac{n(\tan. x)^{n-1}dx}{\cos.^2 x}$

(7.) *Differentiate the equation* $u=\dfrac{\sin. nx}{(\sin. x)^n}$.

Let $\sin. nx=P$, and $\sin. x=Q$.

Then the equation becomes $u=\dfrac{P}{Q^n}$.

Whence, $du=\dfrac{Q^n\, dP-nQ^{n-1}\, dQ\, P}{Q^{2n}}$

Dividing numerator and denominator by Q^{n-1}, and we have

$$du=\frac{QdP-nP\,dQ}{Q^{n+1}} \qquad (a)$$

Because $P=\sin. nx$, $dP=n\cos. nx\, dx$, and because

$Q=\sin. x$ $\quad dQ=\cos. x\, dx$.

Now by substituting the values of P, Q, dP and dQ, (a) becomes

$$du=\frac{(n\sin. x\cos. nx-n\sin. nx\cos. x)\,dx}{(\sin. x)^{n+1}}$$

That is, by equation (8), page 141, Geometry,

$$du=\frac{n\sin.(nx-x)\,dx}{(\sin. x)^{n+1}}.$$

(8.) *Differentiate the equation* $u=\log.(\cos. x+\sqrt{-1}\sin. x$.

The second member being a logarithm, its differential is the differential of the quantity divided by the quantity. That is,

$$du=\frac{-\sin. x\,dx+\sqrt{-1}\cos. x\,dx.}{\sqrt{-1}\sin. x+\cos. x}$$

Or, $\dfrac{du}{dx}=\dfrac{-\sin. x+\sqrt{-1}\cos. x}{\sqrt{-1}\sin. x+\cos. x}=\sqrt{-1}$

Whence, $du=\sqrt{-1}\,dx$.

(9.) *Differentiate the equation* $u=\sin.^{-1}\dfrac{x}{\sqrt{1+x^2}}$.

If the sine of an arc is $\frac{x}{\sqrt{1+x^2}}$ the cosine of the same arc must be $\frac{1}{\sqrt{1+x^2}}$; and we must have

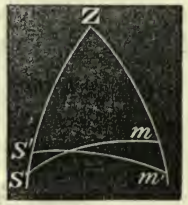
$$du \;:\; -d\left(\frac{1}{\sqrt{1+x^2}}\right) \;::\; 1 \;:\; \frac{x}{\sqrt{1+x^2}}$$

That is, $$du \;:\; \frac{x\,dx\sqrt{1+x^2}}{(1+x^2)^2} \;::\; 1 \;:\; \frac{x}{\sqrt{1+x^2}}$$

Or, $$du \;:\; \frac{dx}{1+x^2} \;::\; 1 \;:\; 1 \qquad du=\frac{dx}{1+x^2}$$

LUNAR OBSERVATIONS.

The differential calculus may be used with great facility and success in clearing lunar distances from the effects of parallax and refraction.



Let $S'm'$ = the apparent central distance between the sun and moon, or the moon and a star, SS' is the refraction of the sun or star, and it is sufficiently small to be taken as the differential of the altitude. Also, $m'm$ is the correction for the moon's apparent altitude, and we may call it the differential of the moon's altitude.

The observed triangle is $ZS'm'$. Let S represent the altitude of the sun or star, m the altitude of the moon, and $x=S'm'$, the observed distance.

Now by the fundamental equation of spherical trigonometry, (see Geometry, page 191), we have

$$\cos. Z=\frac{\cos. x-\sin. S \sin. m}{\cos. S \cos. m} * \qquad (1)$$

We now take the differential of this equation, observing that Z is constant, and that x varies only on account of the variations of m and S.

* Observing that the sine of an altitude is the same as the cosine of the corresponding zenith distance.

First clear of fractions, then differentiate ; then we shall have $-\cos.Z(\sin.S\cos.m\,dS+\cos.S\sin.m\,dm)=-\sin.x\,dx-\cos.S\sin.m\,dS-\cos.m\sin.S\,dm.$

Changing all the signs, and substituting the value of cos. Z from (1), reduces to

$$\left(\frac{\cos.x\sin.S-\sin.^2S\sin.m}{\cos.S}\right)dS+\left(\frac{\cos.x\sin.m-\sin.^2m\sin.S}{\cos.m}\right)dm$$
$$=\sin.x\,dx+\cos.S\sin.m\,dS+\cos.m\sin.S\,dm.$$

Transposing and uniting the coefficients of dS and of dm, will give

$$\frac{(\cos.x\sin.S-\sin.^2S\sin.m-\cos.^2S\sin.m)dS}{\cos.S}+$$
$$\frac{(\cos.x\sin.m-\sin.^2m\sin.S-\cos.^2m\sin.S)dm}{\cos.m}=\sin.x\,dx$$

Observing that $\sin.^2S+\cos.^2S=1$, and $\sin.^2m+\cos.^2m=1$, and changing the order of the terms, we perceive that

$$\sin.x\,dx=\left(\frac{\cos.x\sin.m-\sin.S}{\cos.m}\right)dm+\left(\frac{\cos.x\sin.S-\sin.m}{\cos.S}\right)dS$$

Here we should observe that (dm) is an elevation of the moon's apparent altitude, and (dS) is a depression of the sun or star's apparent altitude, therefore if we take (dm) positive, (dS) must be taken negative. Therefore,

$$dx=\left(\frac{\cos.x\sin m-\sin.S}{\sin.x\cos.m}\right)dm-\left(\frac{\cos.x\sin.S-\sin.m}{\sin.x\cos.S}\right)dS \quad (2)$$

This is the final equation, (dx) representing the quantity between the true and apparent distance.

Sometimes (dx) is positive and sometimes negative, according as the *differential coefficient*, or quantities in parenthesis are positive or negative. When the differential coefficient of (dS) is negative, that term becomes positive, because (dS) is negative, and the product of two negatives is positive.

When the altitude of the sun or star is greater than 60°, the corresponding refraction (dS) will be a very small quantity, which can never be augmented by its coefficient ; therefore in that case the value of (dx) will be sufficiently represented by

$$\left(\frac{\cos.x\sin.m-\sin.S}{\sin.x\cos.m}\right)dm \quad (3)$$

Equation (2) will solve any example that may be prepared. We will solve one or two of those found on page 227, of our Surveying and Navigation.

For the first example there found, in which $S=86^\circ\ 3'$, $m=39^\circ\ 18'$, $x=46^\circ\ 45'$, and the moon's horizontal parallax 53′ 51″, expression (3) will be sufficient.

We must first find (dm), which is the parallax in altitude diminished by the refraction.

53′ 51″=3231″ log	....	3.509337
cos. m 39° 18′	....	9.888651
log. 2499″	....	3.397988
39° 18′ Refraction,...	69″	
	2430″=dm	

For the coefficient, we operate thus : (radius unity.)

	sin. m..—1.801665	cos. m..—1.888651	
	cos. x..—1.835807	sin. x...—1.862353	
	log. 4340 ..—1.637472	—1.751004	
Nat. sin. S	99762	[sub. the upper.]	
	—56362*	log................	—1.750975
			—1.999971
	dm=2430	log..............	3.385606
	dx=40′ 29″=2429″	....	3.385577
x or the app. dis.	46° 45′		
True distance	46° 4′ 31″		

The answer in the book is 46° 4′ 25″. Our omission of the second term in equation (2) might have produced an error of 4″, not more, still making a difference of 2″, but this is of no consequence in itself. Different operators may work the same example by the same or different methods, and they will rarely produce results within 5″ of each other ; and as no observations can be relied on within that limit, such results in a practical point of view are said to agree.

We now take the 6th example from page 227, Surveying and Navigation.

* Because this quantity is minus, (dx) must be minus, and therefore we subtract it from the apparent distance to find the true distance.

Given sun's app. alt. 8° 26′, ☽'s app. alt. 19° 24′, horizontal parallax 57′ 14″. Apparent central distance 120° 18′ 46″, to find the true distance. *Ans.* 120° 1′ 46″.

Here $S=8° 26'$, $m=19° 24'$, and $x=120° 18' 46''$.

Horizontal parallax 57′ 14″=3434″............3.535800
cos. *m* 19° 24′....................9.974614

Parallax in altitude 3239″...................3.510414
Refraction....... 161″
dm=.....3078″ *dS*=6′ 10″=370″.

(Because *x* is greater than 90° its cosine will be *minus*, which will render the differential coefficient of *dm* minus.)

		sin. *m*...—1.521349	cos. *m*...—1.974614	
		cos. *x*...—1.703045	sin. *x*....—1.936152	
	—.16763	—1.224394	—1.910766	den.
sin. *S*	.14666			
	—.31429	log......................	—1.497340	num
			—1.586574	
		dm=3078″ log..........	3.488269	
First term of (*dx*)		.........—1188″.........	3.074843	

---

		sin. *S*...—1.166307	cos. *S*...—1.995278	
		cos. *x*...—1.703045	sin. *x*....—1.936152	
	—.07402	—2.869352	—1.931430	
sin. *m*	.33216			
	—.40618	....	—1.608736	
			—1.677306	
		—*dS*=370″................	2.568202	
Second term of (*dx*)		........+ 176″.......	2.245508	
		—1188″		
		dx=....—1012″=	16′ 52″	

Apparent central distance,..........120° 18′ 46″
True distance,....................120° 1′ 54″

The result differs 8″ from the given answer as determined by other methods, which arises from taking 3078″ as a differential; it is a large arc, rather too large to be taken for a differential arc.

MAXIMA AND MINIMA.

The differential of any quantity is a general expression for a small increase of the quantity; but if the quantity is already a *maximum,* an increase is impossible, and the expression for an increase must be zero.

A decreasing differential is a general expression for a small relative decrease of any quantity; but when the expression is already *a minimum,* it can have no further decrease, and the expression for such decrease must therefore be zero. Hence, in cases of a maximum or minimum, we must put the differenual of the quantity equal to zero, thus forming a new equation, which equation generally gives the results sought.

For example, the following equation always unites the declination of the sun with its longitude:

$$\sin. D = \sin. E \sin. L \qquad (1)$$

Here E is the obliquity of the ecliptic, and is a constant quantity. L is any longitude, and D the corresponding declination. Taking the differential of this equation, we find

$$\cos. D\,dD = \sin. E \cos. L\,dL \qquad (2)$$

If we now assume the condition that D is a maximum, it is the same as to assume that $(dD)=0$, which makes

$$\sin. E \cos. L\,dL = 0$$

Here is the product of three factors, one of which must equal 0. Sine E is of known value, not equal to zero; therefore cos. $L\,dL=0$. If cos. $L=0$, $L=90°$. If $dL=0$, $L=0$. Substituting these values of L in equation (1), we have sin. $D=$sin. E, or $D=E$, when D is a maximum, and sin. $D=0$, or $D=0$, when D is a minimum; which are obvious results.

Again, the general value of the differential of the sine of any arc whose length is x and radius unity, is cos. $x\,dx$.

But if the sine is a maximum, it can have no differential, except in form. That is, cos. $x=0$, or $dx=0$; whence $x=90°$, or $x=0$. Showing that when the arc is 90°, the sine is a maximum, when 0, the sine is 0, a minimum.

For another illustration, *I propose to divide the number* a *into two such parts that the product of the parts shall be the greatest possible.*

At first I will simply get an expression for any indefinite rectangle.—That is, if $x=$ one part, $(a-x)$ will equal the other part, and $(ax-x^2)$ will be an expression for a rectangle which will be larger or smaller according to the relative values of x and a.

Taking the differential of the expression, we have $adx-2xdx$.

Now if I assume $(ax-x^2)$ to be a maximum, it cannot increase, therefore its differential must be zero, or

$$adx-2xdx=0. \quad \text{Or,} \quad a-2x=0, \text{ or, } x=\tfrac{1}{2}a,$$

which is the value of x when the product is a minimum, and may be verified by trial.

EXAMPLES.

(1.) *Required the greatest rectangle that can be inscribed in the quadrant of a given circle.*

It is evident that one extremity of the diagonal of the rectangle must be at the center of the circle, and the other extremity at some point on the arc of the quadrant.

Let $a=$ the diagonal or radius of the circle, and $x=$ the arc from one extremity of the quadrant to the point in which the rectangle meets the curve. Then $a \sin x=$ one side of the rectangle: $a \cos. x=$ the adjacent side, and the area of the rectangle is $a^2 \sin. x \cos. x$.

The problem requires that this expression should be a maximum, which is the same thing as requiring that its differential expression should be zero.

Hence, $\quad a^2 \cos. x\, dx \cos. x - a^2 \sin. x\, dx \sin. x=0.$

Dividing by $a^2 dx$, and $\cos.^2 x - \sin.^2 x=0.$

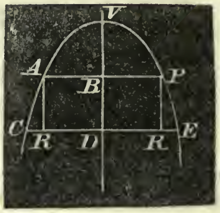
Or, $\quad \cos. x=\sin. x.$

But the arc which has its cosine equal to its sine is $45°$, which shows that the diagonal of the rectangle bisects the quadrant, and the rectangle is in fact a square.

We observe that a^2 disappears in the result, and this shows that the problem is independent of the radius, and equally applies to all circles.

In short, constant factors in a maximum may be cast out by division before we take the differential.

(2.) *Required the greatest possible rectangle that can be inscribed in a given parabola.*



Put $VD=a$, $VB=x$, $PB=y$. Then $BD=a-x$, $2y(a-x)$=maximum, and $y^2=2px$, by the equation of the curve.

Taking the differentials, we have

$$dy(a-x)-ydx=0. \quad \text{Or, } dy=\frac{ydx}{a-x}$$

$$ydy=pdx. \quad \text{Or, } dy=\frac{p\,dx}{y}$$

Whence, $\dfrac{y}{a-x}=\dfrac{p}{y}$. Or, $y^2=ap-px$.

That is, $2px=ap-px$. $2x=a-x$. $x=\frac{1}{3}a$.

This result shows that from the vertex, $\frac{1}{3}$ of the given distance is the point through which to draw one side of the maximum rectangle.

(3.) *Problem 3 on page 253 of this work, is a beautiful example to show the power and utility of the calculus.*

In fact it was the result of the calculus that pointed out the geometrical construction.

Let $AP=a$, $PB=b$, $PD=x$, and call the angle $APD=P$. Then $DH=x\sin.P$, $PH=x\cos.P$, $AH=a-x\cos.P$, $HB=x\cos.P\pm b$, according as H falls between A and B, or between B and P. Corresponding to our figure, $HB=x\cos.P-b$.

In the triangle AHD, we have

$$DH : HA :: 1 : \tan.ADH$$

Or, $x\sin.P : a-x\cos.P :: 1 : \tan.ADH=\dfrac{a-x\cos.P}{x\sin.P}$

Also, $x\sin.P : x\cos.P-b :: 1 : \tan.HDB=\dfrac{x\cos.P-b}{x\sin.P}$

Adding these two tangents according to the mathematical law of the sum of tangents, expressed in equation (28), page 143 of Robinson's Geometry, will give

$$\tan.ADB=\frac{\dfrac{a-b}{x\sin.P}}{1-\dfrac{a-x\cos.P}{x\sin.P}\left(\dfrac{x\cos.P-b}{x\sin.P}\right)}$$

$$=\frac{(a-b)\,x\sin.P}{x^2\sin.^2P-(a-x\cos.P)\,(x\cos.P-b}$$

$$\tan. ADB = \frac{(a-b)\,x \sin. P}{x^2 \sin.^2 P + ab - (a+b)\,x \cos. P + x^2 \cos.^2 P}$$

$$= \frac{(a-b)\,x \sin. P}{x^2 + ab - (a+b)\,x \cos. P}$$

By the requirement of the problem, the angle ADB must be a maximum; but the angle will be a maximum when its tangent is a maximum. Hence we must put the differential of the expression $\frac{(a-b)\,x \sin. P}{x^2+ab-(a+b)\,x \cos. P}$ equal to zero.

By omitting the constant factor $(a-b)$ sin. P, and dividing numerator and denominator by x, we shall have only

$$\frac{1}{x+\frac{ab}{x}-(a+b)\cos. P}$$ to differentiate.

Whence,
$$\frac{-dx+\frac{ab}{x^2}dx}{\left(x+\frac{ab}{x}-(a+b)\cos. P\right)^2}=0.$$

Or,
$$x^2 = ab.$$

This equation directed us to make $PD=\sqrt{500\cdot 200}$ as was done on page 253.

(4.) *The difference of arc between the sun's right ascension and its longitude gives rise to one part of the equation of time. What is the sun's right ascension when this part of the equation is a maximum, and what is the maximum value?*

Let $L=$ the sun's longitude, $x=$ the corresponding right ascension, $E=$ the obliquity of the ecliptic; then by spherical trigonometry we have

$$1\cdot\cos. E = \cot. L \tan. x = \frac{\tan. x}{\tan. L} \qquad (1)$$

(See equation (16) Robinson's Geometry, page 186, also (5), page 139.)

But,
$$\tan. (L-x) = \frac{\tan L - \tan. x}{1+\tan. L \tan. x} \qquad (2)$$

(See page 143, equation (29), Geometry.)

The problem requires that the differential of $(L-x)$, or of tan. $(L-x)$ should be put equal to zero.

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Substituting the value of tan. L taken from (1) in equation (2), we have

$$\tan.(L-x)=\frac{\frac{\tan.x}{\cos.E}-\tan.x}{1+\frac{\tan.^2x}{\cos.E}}=\frac{(1-\cos.E)\tan.x}{\cos.E+\tan.^2x}.$$

Whence, $\frac{\tan.x}{\cos.E+\tan.^2x}$ = maximum, or $\frac{1}{\cos.E\cot.x+\tan.x}$ = maximum.

But this fraction is obviously a maximum when its denominator is a minimum; therefore,

$$\cos.E\cot.x+\tan.x=\text{minimum}.$$

Taking the differential, we find

$$\frac{-\cos.E\,dx}{\sin.^2x}+\frac{dx}{\cos.^2x}=0 \qquad \frac{\sin.^2x}{\cos.^2x}=\cos.E$$

$$\tan.x=\sqrt{\cos.E} \qquad (3)$$

This value of tan. x put in (1) gives

$$\tan.L=\frac{\sqrt{\cos.E}}{\cos.E} \qquad (4)$$

Equations (3) and (4) correspond to radius unity, but we can use the logarithmic table if we add 10 to the index of cos. E before we take the root, thus:

E=23° 27′ 30″ cos. +10................	19.962535 (2
tan. x 43° 45′ 50″..........................	9.981267
	9.962535
tan. L 46° 14″ 10″..........................	10.018732

Diff. of arc 2° 28′ 20″=9m. 54.6s.

(5.) *What must be the inclination of the roof of a building that the water will run off in the least possible time?* *Ans.* 45°.

Let a= the base of a roof, x= its perpendicular altitude: then $\sqrt{a^2+x^2}$ will equal its length, and $\sqrt{\frac{x}{g}}$ will equal the time required for water to fall through the height.

But the time down an inclined plane is to the time through its perpendicular, as the length of the plane is to its height.

Let t= the time down the plane; then

$$t : \sqrt{\frac{x}{g}} :: \sqrt{a^2+x^2} : x$$

Or, $$t = \frac{1}{\sqrt{gx}}(a^2+x^2)^{\frac{1}{2}}$$

But the problem requires that t should be a minimum; its square must therefore be a minimum; hence,

$$\frac{a^2+x^2}{x} = \text{a minimum.}$$

It will be observed that g is the force of gravity, and being a constant factor in the last expression, it was omitted.

Taking the differential, we have

$$\frac{2x^2\,dx - dx(a^2+x^2)}{x^2} = 0$$

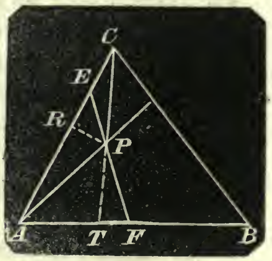
$$2x^2 - a^2 - x^2 = 0$$

$$x^2 = a^2, \quad \text{or} \quad x = a.$$

The perpendicular being equal to the base, shows that the inclination required must be 45°.

(6.) *Within a triangle is a given point* P, *the distance to the nearest angle* A *is given, and the line* AP *divides the angle* A *into two angles* m *and* n, *of which* m *is greater than* n.

It is required to find the line EF *drawn through the point* P, *so that the triangle* AEF *shall be the least possible.*



Let $AP=a$, $AF=x$, $AE=y$. The angle $PAF=m$, $PAE=n$.

The area of the $\triangle\ ABF = ax$ sin. m.

The area of the $\triangle\ APE = ay$ sin. n.

By the conditions,

$$ax \sin. m + ay \sin. n = \text{minimum.}$$

Also, by the conditions,

$$xy \sin. (m+n) = \text{minimum.}$$

From the first minimum,

$$\sin. m dx + \sin. n dy = 0 \qquad (1)$$

From the second, $$xdy + ydx = 0 \qquad (2)$$

From (1), $$dx = -\frac{\sin. n}{\sin. m} dy$$

From (2), $$dx = -\frac{x}{y} dy$$

Whence, $y \sin. n = x \sin. m.$

Or, $ay \sin. n = ax \sin. m.$

This last equation shows that the line EPF must be so drawn that the triangle APF will be equal to the triangle APE.

We presume that the foregoing examples are sufficient to illustrate the power and utility of the calculus in respect to maxima and minima.

INTEGRAL CALCULUS.

The integral calculus is the converse of the differential. In the differential calculus we give the integral and require the differential. In the integral calculus we give the differential quantity, and require the integral from which the differential was derived. Hence all our rules of operation must have reference to the differential rules inversed.

We cannot investigate the rules of operation in this work, but suppose the reader already acquainted with them.

Many persons can operate to some extent in the integral calculus without any distinct idea of what the integral calculus is, and this vagueness can never be fully driven away, except by close attention to the application and utility of the science.

For example.—If we have the differential of a circular arc, by integration, we shall have the arc itself.

If we have the differential of a circular segment, by integration we shall have the segment itself.

If we have the differential quantity of a cone, by integrating that quantity we shall have the solidity of the cone itself.

If we have the differential of a logarithm, by integration we shall have the logarithm itself.

Thus we might go through the chapter.

Because the integral is the opposite of the differential, the inexperienced might conclude that one operation would be obvious from the other.

In some instances the converse is obvious, but not generally so. We must not conclude that it is as easy to move in one direction as in the opposite.—It is not as easy to ascend as to descend a plane—not so easy for a vessel to move against the stream as with it. It is not so easy to find the cube root of a number as it is to cube the root when found.

The sign for the differential is (d). The sign for integration is ($\int$). Hence, $\int du=u$. The two signs destroy one another, and give the quantity u.

If we take the differential of x^4 we shall have $4x^3dx$. Therefore if we must integrate $4x^3dx$, we must frame a rule of operation that will give x^4, which is the following:

Add unity to the index, divide by the index so increased, and take away dx.

The differential of $\frac{1}{\sqrt{1-x^2}}$ is $\frac{xdx}{(1-x^2)^{\frac{3}{2}}}$, conversely then the the integral of $\frac{xdx}{(1-x^2)^{\frac{3}{2}}}$ is $\frac{1}{\sqrt{1-x^2}}$. But to integrate this by the above rule appears at first sight impossible, nevertheless it can be accomplished by the following artifice:

Put $1-x^2=y$. Then $xdx=-\frac{1}{2}dy$.

And, $$\frac{xdx}{(1-x^2)^{\frac{3}{2}}}=-\frac{dy}{2y^{\frac{3}{2}}}=-\frac{1}{2}y^{-\frac{3}{2}}dy$$

To the second member of this equation the rule will apply.

Hence, $$\int\frac{xdx}{(1-x^2)^{\frac{3}{2}}}=y^{-\frac{1}{2}}=\frac{1}{y^{\frac{1}{2}}}=\frac{1}{\sqrt{1-x^2}}. \quad \textit{Ans.}$$

The differential of xy is $(xdy+ydx)$; therefore, $\int(xdy+ydx)=xy$. But the inquiry is, how we shall effect the *integration* under the rule.

Here y is equal to, or greater, or less than x. Therefore we may assume that $y=ax$, a being a constant quantity. Whence, $dy=adx$, $xdy=axdx$.

And, $\int(xdy+ydx)=\int(axdx+axdx)=\int 2axdx=ax^2$.

But $ax^2=ax\cdot x$, and because $y=ax$, $ax^2=xy$. *Ans.*

Then we perceive that the actual integration was performed by the rule; but the reader must not infer that the rule will apply to all cases—*far from it.**

---

* The subject of integration requires the keenest algebraical talent, and few persons are skilful algebraists in the highest sense of that term, who have not been severely disciplined in integration.

We give a few examples under this rule.

(1.) *Given the differential* $(2x^2ydy+2y^2xdx)$, *to find its integral.* *Ans.* x^2y^2.

Here we may assume $y=ax$; then $dy=adx$, $ydy=a^2xdx$.

Whence, $\int(2x^2ydy+2y^2xdx)=\int(2a^2x^3dx+2a^2x^3dx)=\int 4a^2x^3dx$.

To the second member the rule applies; that is,

$$\int 4a^2x^3dx=a^2x^4=a^2x^2\cdot x^2=y^2x^2. \text{ Ans.}$$

(2.) *Integrate* $\dfrac{(b^2+y^2)xdx+(a^2+x^2)ydy}{\sqrt{b^2+y^2}\ \sqrt{a^2+x^2}}$

Ans. $\sqrt{b^2+y^2}\ \sqrt{a^2+x^2}$.

Assume $\sqrt{b^2+y^2}=P$, and $\sqrt{a^2+x^2}=Q$.

Then $ydy=PdP$, and $xdx=QdQ$.

Substituting these values, and the given expression becomes

$$\frac{P^2QdQ+PQ^2dP}{PQ}$$

That is, $PdQ+QdP$.

But, $\int(PdQ+QdP)=PQ$.

That is, $\sqrt{b^2+y^2}\times\sqrt{a^2+x^2}$. *Ans.*

(3.) *Integrate* $\dfrac{-3dy+3dz}{4(a-y+z)^{\frac{1}{4}}}$. *Ans.* $(a-y+z)^{\frac{3}{4}}$

Put $(a-y+z)^{\frac{1}{4}}=P$; then $a-y+z=P^4$.

And $-dy+dz=4P^3dP$.

Whence, $\dfrac{-3dy+3dz}{4(a-y+z)^{\frac{1}{4}}}=3P^2dP$.

And, $\displaystyle\int\frac{-3dy+3dz}{4(a-y+z)^{\frac{1}{4}}}=P^3=(a-y+z)^{\frac{3}{4}}$. *Ans.*

Observation. The differential of $\dfrac{x}{y}$ is $\dfrac{ydx-xdy}{y^2}$; therefore, the integral of this last expression is $\dfrac{x}{y}$. But how shall we integrate this, provided we did not know the integral?

The numerator would be the differential of the product xy, if the sign between the terms in the numerator were plus.

Let us put $y=ax$. Then $ydx=axdx$, and $xdy=axdx$, and the expression $\frac{ydx-xdy}{y^2}$ becomes $\frac{ax\,dx-ax\,dx}{a^2x^2}$ or $\frac{0}{a^2x^2}$, and our effort fails. Now let us examine the cause of the failure. The product xy can represent any magnitude whatever, and if we put $y=ax$, then xy becomes ax^2 ; and because x is variable, ax^2 is still capable of representing any magnitude whatever. But in the fractional expression $\frac{x}{y}$, if $y=ax$, and a be regarded as constant, $\frac{x}{y}$ becomes $\frac{x}{ax}$, or $\frac{1}{a}$, and in that case $\frac{x}{y}$ can only represent the ever constant fraction $\frac{1}{a}$; but $\frac{x}{y}$ must be capable of representing any fraction whatever ; therefore we cannot put $y=ax$, unless we regard a as variable.

Therefore to integrate the expression $\frac{ydx-xdy}{y^2}$, put $y=tx$; both t and x being variable.

Then $ydx=txdx$, $dy=tdx+xdt$.

$$xdy=txdx+x^2dt.$$

Whence, $$ydx-xdy=-x^2\,dt$$

and the expression becomes $-\frac{x^2\,dt}{t^2x^2}$ or $-\frac{dt}{t^2}$

That is, $\int\frac{ydx-xdy}{y^2}=\int-t^{-2}\,dt=t^{-1}$ *by the rule.*

Whence, the required integral is $\frac{1}{t}$; but $y=tx$; therefore, $\frac{1}{t}=\frac{x}{y}$.

This branch of the subject may be treated as follows, provided the operator is cautious, and does not assume too much :

When we differentiate a product as xy, we assume x as constant and y variable ; and then y constant and x variable, and thus we get *two* partial differentials.

Now either one of these integrated on the supposition that the letter which is affixed to (d) is the variable one, and all others constant, will give the true integral. Thus the differential of xy is

$$xdy+ydx.$$

Now integrate xdy on the supposition that x is constant and y variable, and we have xy for the integral. Also, $\int ydx=xy$. It would therefore appear that $2xy$ is the whole integral, provided we did not know, *a priori*, that xy is the integral.

Hence, when we integrate two differential expressions, on the supposition that the letter not affected with the differential sign (d), *is constant, and find two equal integrals, we must take but one of them.*

The same principle holds good in relation to the three or more letters. The differential of xyz is

$$xydz+xzdy+yzdx.$$

Now if we integrate this expression on the supposition that xy is constant in the first term, xz constant in the second, and yz constant in the third, we shall have

$$xyz+xyz+xyz.$$

Here are three equal integrals, but we must take but one of these for the whole integral, because the differential was effected by *three distinct suppositions.*

The differential of $\frac{x}{y}$ is $\frac{ydx-xdy}{y^2}=\frac{dx}{y}-xy^{-2}dy$.

Integrating each of these expressions on the supposition that y is constant in the first, and x constant in the second, we have

$$\frac{x}{y}+\frac{x}{y}$$

but we must only take one of these for the integral, for the same reason as before.

EXAMPLES.

(1.) *Integrate* $(6xy-y^2)dx+(3x^2-2xy)dy$

Ans. $3x^2y-y^2x$.

We integrate the first part on the supposition that y is constant, and the second on the supposition that x is constant, and we obtain

$$3x^2y-y^2x+3x^2y-y^2x,$$

and because we make two distinct suppositions, we divide by 2. Then *test* the result by taking the differential.

(2.) *Integrate* $(2y^2x+3y^3)dx+(2x^2y+9xy^2+8y^3)dy.$

Integrating each term, we obtain

$$y^2x^2+3y^3x+x^2y^2+3xy^3+2y^4.$$

Here we find two terms equal to x^2y^2, and two terms equal to $3xy^3$, and one term $2y^4$; hence I will take

$$x^2y^2+3xy^3+2y^4$$

for the integral sought—which I find to be true by taking the differential.

To integrate the varied expressions in the form

$$x^m(a+bx^m)^p dx,$$

we must resort to the established formulas, explained in elaborate works, which of course we cannot touch upon in a work like this.

Because $d\log. x=\frac{mdx}{x}$. Therefore, $\int\frac{mdx}{x}=\log. x+c$ (*a*)

" $d\sin. x=\cos. xdx$. " $\int\cos. xdx=\sin. x+c$ (*b*)

" $d\cos. x=-\sin. xdx$. " $\int-\sin. xdx=\cos. x+c$ (*c*)

" $d\tan. x=\frac{dx}{\cos.^2 x}$. " $\int\frac{dx}{\cos.^2 x}=\tan. x+c$ (*d*)

Each of these formula is a fundamental rule for integration.

It is not necessary for us to explain the *constant* *c*.

APPLICATION OF THE INTEGRAL CALCULUS.

We shall explain the application and utility of this science by examples.

If x represents an arc of a circle whose radius is unity, and y the sine of the same arc ; then $\sqrt{1-y^2}$ will represent the cosine, and equation (*b*) above will become

$$dx=\frac{dy}{\sqrt{1-y^2}}+c.$$

The integral of the first member will give the *arc*, but it will be *numerically* indefinite, unless we can integrate the second member, and know the value of y corresponding to some definite value of x.

We *cannot* integrate the second member in finite terms, therefore we must develop it in a series, and integrate term by term, and if the series is sufficiently converging, the value of x can be known to any required degree of approximation.

$\frac{dy}{\sqrt{1-y^2}}=(1-y^2)^{-\frac{1}{2}}dy.$ The binomial, expanded by the binomial theorem, produces

$$1+\tfrac{1}{2}y^2+\tfrac{1}{2}\cdot\tfrac{3}{4}y^4+\tfrac{1}{2}\cdot\tfrac{3}{4}\cdot\tfrac{5}{6}y^6+\tfrac{1}{2}\cdot\tfrac{3}{4}\cdot\tfrac{5}{6}\cdot\tfrac{7}{8}y^8+\text{ \&c.}$$

Multiplying each term by dy, and integrating, we obtain

$$x=y+\frac{1\cdot y^3}{2\cdot 3}+\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{y^5}{5}+\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{y^7}{7}+\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{7}{8}\cdot\frac{y^9}{9}+\text{ \&c.}+c.$$

This equation is true for all values of x. It is true then when $x=0$; but if we make $x=0$, y must equal 0 at the same time. Therefore, if we make the supposition that $x=0$, the last equation will become $0=0+c$, or $c=0$. By some such special consideration, the value of c can be determined in almost every problem, although it is indeterminate in the *abstract*.

Now the value of y is known to be $\frac{1}{2}$ when x, the arc, equals 30° ; therefore,

$$\text{The arc of } 30^\circ=\frac{1}{2}+\frac{1.1}{3.2^4}+\frac{1.3.1}{4.5.2^6}+\frac{1.3.5.1}{4.6.7.2^8}+\frac{1.3.5.7.1}{4.6.8.9.2^{10}}$$
$+$ &c.

Multiplying by 6 and taking ten terms of the series, we shall have the value of a semicircle to radius unity. That is, we shall have

$$\pi=3.1415926,$$

which is true to the last figure.

Thus we perceive that one operation in the integral calculus brings a result requiring many operations in common geometry.

SURFACES AND SOLIDS.

In general terms, $aydx$ will represent the differential of any plane surface, and if so, $\int aydx+c$, will represent any surface ;

but we can find the integral only when we know some relation between x and y.

Also, ay^2dx will represent the differential of any solid; therefore $\int ay^2dx+c$ will represent the solid itself; but we can find this integral only when we have some relation between x and y.

EXAMPLES.

Suppose x to represent the perpendicular of any triangle, and y its base; then if x increases downwards by dx, ydx will be the differential of the triangle, the angles remaining constant.

Therefore, $\int ydx$ will be the area of the triangle itself. This integral will require no correction, for when $x=0$, $y=0$.

The area being a triangle, we have a relation between x and y, for no triangle can exist without this numerical relation.

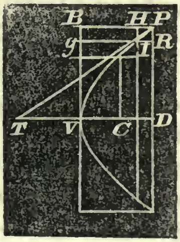
Suppose we measure one unit down the base, and through that point draw a line parallel to the base, and find the length of this to be a units. Then whatever be the magnitudes of x and y, this relation will be constant, and a will be greater or less according to the angles of the triangle.

That is, $x : y :: 1 : a$. Or, $y=ax$.

Consequently, $\int ydx=\int axdx=\frac{ax^2}{2}=\frac{ax}{2}.x=\frac{xy}{2}$

That is, the area of any triangle is half the product of its base and altitude.

Let VCI be any area. $VC=x$, $CI=y$, $CD=dx$, then the space $CDRI=ydx$, the differential of the area.



If VCI represents a portion of a parabola, then $y^2=2px$. Or, $y=(2p)^{\frac{1}{2}}x^{\frac{1}{2}}$.

Whence, $\int ydx=\int(2p)^{\frac{1}{2}}x^{\frac{1}{2}}dx=\frac{2}{3}(2p)^{\frac{1}{2}}x^{\frac{3}{2}}$.

But $xy=(2p)^{\frac{1}{2}}x^{\frac{3}{2}}$; therefore VCI is $\frac{2}{3}$ $VCIg$.

If VCI is a portion of a circular area of which r is the radius,

Then $(r-x)^2+y^2=r^2$, and $y^2=2rx-x^2$.

Or. $y=\sqrt{2rx-x^2}$.

Hence, $\int ydy = \int \sqrt{2rx - x^2}\,dx =$ the area of this semi-segment.

We cannot integrate this in finite terms, we can only approximate to the integral by expanding the binomial, multiplying each term by dx, and then integrating each term separately.

After integration, if we take $x=2r$, the result will be the area of a semicircle whose radius is r.

SOLIDS.

(1.) *To find the solidity of a cone.*

If x represent the perpendicular altitude of an upright cone and y the radius of its base, then πy^2 will equal the area of the base, and if x be increased by dx, $\pi y^2 dx$ will be the differential of the cone. Consequently, $\int \pi y^2 dx$ will be the solidity of the cone.

As any cone whatever must have some constant ratio between its perpendicular and base; therefore,

$$x : y :: 1 : a. \quad \text{Or, } y = ax.$$

Whence, $\int \pi y^2 dx = \int \pi a^2 x^2 dx = \frac{1}{3}\pi a^2 x^3 = \frac{1}{3}x \cdot \pi a^2 x^2 = \frac{1}{3}x \cdot \pi y^2$.

That is, the solidity of a cone is equal to the area of the base multiplied by one-third of the altitude.

N. B. The integral required no correction because x and y vanish together.

(2.) *To find the solidity of a paraboloid.*

Let VCI revolve on the axis VC (see last figure,) it will describe a paraboloid of which $\pi y^2 dx$ is the differential. $y^2=2px$.

Therefore, $\int \pi y^2 dx = \int 2\pi px dx = p\pi x^2$.

To form a correct idea of this solid, we must observe that $\pi y^2 = 2p\pi x$. Consequently $2p\pi x$ is the area of the circle described by the revolution of y, and therefore $2p\pi x^2$ is the solidity of the cylinder which would just circumscribe the paraboloid, *and hence we perceive that the paraboloid is just half of its circumscribing cylinder.*

(3.) *To find the solidity of a sphere.*

Let VCI revolve on the axis VC as before; *now* on the supposition that VCI is the arc of a circle whose radius is r, Then $(r-x)^2+y^2=r^2$. $y^2=2rx-x^2$.

Then $\int \pi y^2 dx=\pi\int(2rx-x^2)dx=\pi\left(rx^2-\frac{x^3}{3}\right)+c$.

This integral requires no correction, because when $x=0$, $y=0$ and then the area equals 0, and $c=0$.

This integral represents the true value of any segment corresponding to any assumed value of x between $x=0$ and $x=2r$.

If $x=2r$ the segment will comprise the whole sphere.

Then $$\pi\left(rx^2-\frac{x^3}{3}\right)=\pi\left(4r^3-\frac{8r^3}{3}\right)=\frac{4\pi r^3}{3}$$

This corresponds to theorem 17, book vii, Geometry.

(4.) *The differential of* $\frac{x^n}{(1+x)^n}$ *is* $\frac{nx^{n-1}dx}{(1+x)^{n+1}}$ *conversely.*

Integrate $\frac{nx^{n-1}dx}{(1+x)^{n+1}}$. *Ans.* $\frac{x^n}{(1+x)^n}$.

Put $n+1=m$, then $n=m-1$, and $n-1=m-2$.

With these substitutions the expression to be integrated is

$$\frac{(m-1)x^{m-2}dx}{(1+x)^m}, \text{ or } (m-1)\int x^{m-2}dx\,(1+x)^{-m}.$$

But $(1+x)^{-m}=1-mx+m\left(\frac{m+1}{2}\right)x^2-m\left(\frac{m+1}{2}\right)\left(\frac{m+2}{3}\right)x^3+$

$$m\left(\frac{m+1}{2}\right)\left(\frac{m+2}{3}\right)\left(\frac{m+3}{4}\right)x^4-\&c.$$

Multiply the second member by $x^{m-2}dx$, then it becomes

$$x^{m-2}dx-mx^{m-1}dx+m\left(\frac{m+1}{2}\right)x^m dx-m\left(\frac{m+1}{2}\right)\left(\frac{m+2}{3}\right)x^{m+1}dx$$

$+$ &c.

Now integrate each term separately, and the result will be

$$\frac{x^{m-1}}{m-1}-x^m+\frac{m}{2}x^{m+1}-\frac{m}{2}\left(\frac{m+1}{3}\right)x^{m+2}+\&c.\ \&c.$$

Multiply each term by $(m-1)$, observing that $m-1$ equals n,

Then $x^n-nx^m+n\left(\frac{m}{2}\right)x^{m+1}-n\left(\frac{m}{2}\right)\left(\frac{m+1}{3}\right)x^{m+1}+\&c., \&c.$

Replacing the value of m, the series becomes

$$x^n - nx^{n+1} + n\left(\frac{n+1}{2}\right)x^{n+2} - n\left(\frac{n+1}{2}\right)\left(\frac{n+2}{3}\right)x^{n+3} + \&c.$$

By factoring, $x^n\left(1 - nx + n\cdot\frac{n+1}{2}x^2 - n\cdot\frac{n+1}{2}\cdot\frac{n+2}{3}x^3 + \&c.\right)$

$= \frac{x^n}{(1+x)^n}$. *Ans.*

We close this volume by giving the two following integrations. which troubled us very much some years ago. They are from *Poisson's Mecanique;* the first on page 223, vol. 1, the second is on page 406 of the same volume.

Poisson gives the equation

$$-b\frac{d^2y}{dx^2} = Py.$$

Then simply says, the integral complete is

$$y = c\sin.\left(x\sqrt{\frac{P}{b}} + f\right),\ c \text{ and } f \text{ being arbitrary constants.}$$

How did he obtain the integral?

Divide both members of the equation by $-b$, and then multiply both by $2dy$. Then we shall have

$$\frac{2dy\cdot d^2y}{dx^2} = -\frac{P}{b}2ydy$$

The first member is the differential of $\frac{dy^2}{dx^2}$, dx^2 being constant.

The second member is easily integrated.

Then $$\frac{dy^2}{dx^2} = \frac{P}{b}c^2 - \frac{P}{b}y^2.$$

We add $\left(\frac{P}{b}c^2\right)$ for the arbitrary constant, for $\left(\frac{P}{b}c^2\right)$ may represent any quantity as well as c alone, and we place it first, because the other term is minus. Taking the square root, we have,

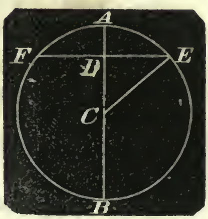
$$\frac{dy}{dx} = \sqrt{\frac{P}{b}}\left(c^2 - y^2\right)^{\frac{1}{2}}$$

$$\frac{dy}{(c^2-y^2)^{\frac{1}{2}}}=\sqrt{\frac{P}{b}}.dx$$

Integrating both sides, and

$$\int\frac{dy}{\sqrt{c^2-y^2}}=\left(x\sqrt{\frac{P}{b}}+f\right)$$

The first member of this equation is an arc of a circle whose sine is $\frac{y}{c}$ and radius unity.



Let AE be that arc; then

$$AE=\left(x\sqrt{\frac{P}{b}}+f\right)$$

$DE=\frac{y}{c}$. $DE=\sin. AE=\sin.\left(x\sqrt{\frac{P}{b}}+f\right)$

Hence, $\quad \frac{y}{c}=\sin.\left(x\sqrt{\frac{P}{b}}+f\right)$

$$y=c\sin.\left(x\sqrt{\frac{P}{b}}+f\right)\ *$$

Integrate $\quad dx-2amxdQ=\frac{2g}{a}\cos. QdQ.$

This corressponds to the general formula,

$$dy+Pydx=Qdx.$$

Assume $\quad y=z.e^{\int-\text{Pdx}} \qquad (1)$

Then theory gives the following formula for the result:

$$y=e^{-\int\text{Pdx}}\left(\int e^{\int\text{Pdx}}.Qdx+c\right) \qquad (F)$$

To apply this formula to our equation, we must make

$x=y$, $P=-2am$, $dx=dQ$, $Q=\frac{2g}{a}\cos. Q$, $Pdx=-2amdQ$, $\int Pdx=-2amQ$.

Differentiating (1), substituting the result in the general formula and reducing, we find $dz=e^{-2amQ}\frac{2g}{a}\cos. QdQ \quad (2)$

To integrate this last equation, we must use the following formula:

$$\int udv=uv-\int vdu \qquad (3)$$

* It will be a good exercise for a learner to differentiate this equation twice, and see if it returns to the original.

$$u=e^{-2amQ},\ dv=\frac{2g}{a}\cos.Q\,dQ,\ v=\frac{2g}{a}\sin.Q$$

These values put in formula (3), give

$$\int e^{-2amQ}\frac{2g}{a}\cos.Q\,dQ=e^{-2amQ}.\frac{2g}{a}\sin.Q+2am\int\left(e^{-2amQ}\frac{2g}{a}\sin.Q\,dQ\right)^{*}\quad(4)$$

By applying the same formula,

$$\int e^{-2am}\frac{2g}{a}\sin.Q\,dQ=-e^{-2amQ}\frac{2g}{a}\cos.Q-2am\int\left(e^{-2amQ}\frac{2g}{a}\cos.Q\,dQ\right)\quad(5)$$

If we compare (2), (4), and (5), we shall perceive that the first member of (4) is z, and the last term of (5) is also z.

Therefore, (4) becomes

$$z=e^{-2amQ}\frac{2g}{a}\sin.Q-e^{2amQ}4gm\cos.Q-4a^2m^2z$$

Or, $$z=\frac{e^{-2amQ}\left(\frac{2g}{a}\cos.Q-4gm\cos.Q\right)}{1+4a^2m^2}$$

This value of z put in the general formula (F), gives

$$x=(z+c)e^{2amQ}=ce^{2amQ}+\frac{2g\sin.Q}{a(1+4a^2m^2)}-\frac{4gm\cos.Q}{(1+4a^2m^2)}.\quad Ans.$$

Let us now reverse the operation and differentiate this equation.

Thus, $dx=d.(ce^{2amQ})+\frac{2g\cos.Q}{a(1+4a^2m^2)}dQ+\frac{4gm\sin Q}{(1+4a^2m^2)}dQ$

To differentiate the first term of the second member, we put

$u=ce^{2amQ}$. Then $\log.u=\log.c+2amQ\log.e$

Observing that $\log.e=1$, and differentiating this last equation we have $\frac{du}{u}=2amdQ$, or $du=2am\,ce^{2amQ}dQ$.

---

* If $u=e^{-2amQ}$ $\log.u=-2amQ$ $\frac{du}{u}=-2amdQ$.

Whence, $du=-2amce^{-2amQ}dQ$.

Whence $\frac{dx}{dQ}=2amce^{2amQ}+\frac{2g\cos. Q}{a(1+4a^2m^2)}+\frac{4gm\sin. Q}{(1+4a^2m^2)}$

Dividing by $2am$,

$$\frac{dx}{2amdQ}=ce^{2amQ}+\frac{2g\cos. Q}{2a^2m(1+4a^2m^2)}+\frac{4gm\sin. Q}{2am(1+4a^2m^2)}$$

But $x=ce^{2amQ}+\frac{2g\sin. Q}{a(1+4a^2m^2)}-\frac{4gm\cos. Q}{(1+4a^2m^2)}$

By subtraction, $\frac{dx}{2amdQ}-x=\frac{2g\cos. Q}{2a^2m(1+4a^2m^2)}+\frac{4gm\cos. Q}{(1+4a^2m^2)}$

That is, $\frac{dx}{2amdQ}-x=\frac{2g\cos. Q}{2a^2m(1+4a^2m^2)}+\frac{8ga^2m^2\cos. Q}{2a^2m(1+4a^2m^2)}$

$$=\frac{(1+4a^2m^2)2g\cos. Q}{(1+4a^2m^2)2a^2m}=\frac{2g\cos. Q}{2a^2m}$$

Therefore, $dx-2amxdQ=\frac{2g}{a}\cos. QdQ$, the original equation.

LOGARITHMIC TABLES;

ALSO A TABLE OF THE

TRIGONOMETRICAL LINES;

AND OTHER NECESSARY TABLES.

LOGARITHMS OF NUMBERS

FROM

1 TO 10000.

N.	Log.	N.	Log.	N.	Log.	N.	Log.
1	0 000000	26	1 414973	51	1 707570	76	1 880814
2	0 301030	27	1 431364	52	1 716003	77	1 886491
3	0 477121	28	1 447158	53	1 724276	78	1 892095
4	0 602060	29	1 462398	54	1 732394	79	1 897627
5	0 698970	30	1 477121	55	1 740363	80	1 903090
6	0 778151	31	1 491362	56	1 748188	81	1 908485
7	0 845098	32	1 505150	57	1 755875	82	1 913814
8	0 903090	33	1 518514	58	1 763428	83	1 919078
9	0 954243	34	1 531479	59	1 770852	84	1 924279
10	1 000000	35	1 544068	60	1 778151	85	1 929419
11	1 041393	36	1 556303	61	1 785330	86	1 934498
12	1 079181	37	1 568202	62	1 792392	87	1 939519
13	1 113943	38	1 579784	63	1 799341	88	1 944483
14	1 146128	39	1 591065	64	1 806180	89	1 949390
15	1 176091	40	1 602060	65	1 812913	90	1 954243
16	1 204120	41	1 612784	66	1 819544	91	1 959041
17	1 230449	42	1 623249	67	1 826075	92	1 963788
18	1 255273	43	1 633468	68	1 832509	93	1 968483
19	1 278754	44	1 643453	69	1 838849	94	1 973128
20	1 301030	45	1 653213	70	1 845098	95	1 977724
21	1 322219	46	1 662758	71	1 851258	96	1 982271
22	1 342423	47	1 672098	72	1 857333	97	1 986772
23	1 361728	48	1 681241	73	1 863323	98	1 991226
24	1 380211	49	1 690196	74	1 869232	99	1 995635
25	1 397940	50	1 698970	75	1 875061	100	2 000000

N. B. In the following table, in the last nine columns of each page, where the first or leading figures change from 9's to 0's, points or dots are now introduced instead of the 0's through the rest of the line, to catch the eye, and to indicate that from thence the corresponding natural numbers in the first column stands in the *next lower line*, and its annexed first two figures of the Logarithms in the second column

N.	0	1	2	3	4	5	6	7	8	9
100	000000	0434	0868	1301	1734	2166	2598	3029	3461	3891
101	4321	4750	5181	5609	6038	6466	6894	7321	7748	8174
102	8600	9026	9451	9876	.300	.724	1147	1570	1993	2415
103	012837	3259	3680	4100	4521	4940	5360	5779	6197	6616
104	7033	7451	7868	8284	8700	9116	9532	9947	.361	.775
105	021189	1603	2016	2428	2841	3252	3664	4075	4486	4896
106	5306	5715	6125	6533	6942	7350	7757	8164	8571	8978
107	9384	9789	.195	.600	1004	1408	1812	2216	2619	3021
108	033424	3826	4227	4628	5029	5430	5830	6230	6629	7028
109	7426	7825	8223	8620	9017	9414	9811	.207	.602	.998
110	041393	1787	2182	2576	2969	3362	3755	4148	4540	4932
111	5323	5714	6105	6495	6885	7275	7664	8053	8442	8830
112	9218	9606	9993	.380	.766	1153	1538	1924	2309	2694
113	053078	3463	3846	4230	4613	4996	5378	5760	6142	6524
114	6905	7286	7666	8046	8426	8805	9185	9563	9942	.320
115	060698	1075	1452	1829	2206	2582	2958	3333	3709	4083
116	4458	4832	5206	5580	5953	6326	6699	7071	7443	7815
117	8186	8557	8928	9298	9668	..38	.407	.776	1145	1514
118	071882	2250	2617	2985	3352	3718	4085	4451	4816	5182
119	5547	5912	6276	6640	7004	7368	7731	8094	8457	8819
120	9181	9543	9904	.266	.626	.987	1347	1707	2067	2426
121	082785	3144	3503	3861	4219	4576	4934	5291	5647	6004
122	6360	6716	7071	7426	7781	8136	8490	8845	9198	9552
123	9905	.258	.611	.963	1315	1667	2018	2370	2721	3071
124	093422	3772	4122	4471	4820	5169	5518	5866	6215	6562
125	6910	7257	7604	7951	8298	8644	8990	9335	9681	0026
126	100371	0715	1059	1403	1747	2091	2434	2777	3119	3462
127	3804	4146	4487	4828	5169	5510	5851	6191	6531	6871
128	7210	7549	7888	8227	8565	8903	9241	9579	9916	.253
129	110590	0926	1263	1599	1934	2270	2605	2940	3275	3609
130	3943	4277	4611	4944	5278	5611	5943	6276	6608	6940
131	7271	7603	7934	8265	8595	8926	9256	9586	9915	0245
132	120574	0903	1231	1560	1888	2216	2544	2871	3198	3525
133	3852	4178	4504	4830	5156	5481	5806	6131	6456	6781
134	7105	7429	7753	8076	8399	8722	9045	9368	9690	..12
135	130334	0655	0977	1298	1619	1939	2260	2580	2900	3219
136	3539	3858	4177	4496	4814	5133	5451	5769	6086	6403
137	6721	7037	7354	7671	7987	8303	8618	8934	9249	9564
138	9879	.194	.508	.822	1136	1450	1763	2076	2389	2702
139	143015	3327	3639	3951	4263	4574	4885	5196	5507	5818
140	6128	6438	6748	7058	7367	7676	7985	8294	8603	8911
141	9219	9527	9835	.142	.449	.756	1063	1370	1676	1982
142	152288	2594	2900	3205	3510	3815	4120	4424	4728	5032
143	5336	5640	5943	6246	6549	6852	7154	7457	7759	8061
144	8362	8664	8965	9266	9567	9868	.168	.469	.769	1068
145	161368	1667	1967	2266	2564	2863	3161	3460	3758	4055
146	4353	4650	4947	5244	5541	5838	6134	6430	6726	7022
147	7317	7613	7908	8203	8497	8792	9086	9380	9674	9968
148	170262	0555	0848	1141	1434	1726	2019	2311	2603	2895
149	3186	3478	3769	4060	4351	4641	4932	5222	5512	5802

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150	176091	6381	6670	6959	7248	7536	7825	8113	8401	8689
151	8977	9264	9552	9839	.126	.413	.699	.985	1272	1558
152	181844	2129	2415	2700	2985	3270	3555	3839	4123	4407
153	4691	4975	5259	5542	5825	6108	6391	6674	6956	7239
154	7521	7803	8084	8366	8647	8928	9209	9490	9771	..51
155	190332	0612	0892	1171	1451	1730	2010	2289	2567	2846
156	3125	3403	3681	3959	4237	4514	4792	5069	5346	5623
157	5899	6176	6453	6729	7005	7281	7556	7832	8107	8382
158	8657	8932	9206	9481	9755	..29	.303	.577	.850	1124
159	201397	1670	1943	2216	2488	2761	3033	3305	3577	3848
160	4120	4391	4663	4934	5204	5475	5746	6016	6286	6556
161	6826	7096	7365	7634	7904	8173	8441	8710	8979	9247
162	9515	9783	..51	.319	.586	.853	1121	1388	1654	1921
163	212188	2454	2720	2986	3252	3518	3783	4049	4314	4579
164	4844	5109	5373	5638	5902	6166	6430	6694	6957	7221
165	7484	7747	8010	8273	8536	8798	9060	9323	9585	9846
166	220108	0370	0631	0892	1153	1414	1675	1936	2196	2456
167	2716	2976	3236	3496	3755	4015	4274	4533	4792	5051
168	5309	5568	5826	6084	6342	6600	6858	7115	7372	7630
169	7887	8144	8400	8657	8913	9170	9426	9682	9938	.193
170	230449	0704	0960	1215	1470	1724	1979	2234	2488	2742
171	2996	3250	3504	3757	4011	4264	4517	4770	5023	5276
172	5528	5781	6033	6285	6537	6789	7041	7292	7544	7795
173	8046	8297	8548	8799	9049	9299	9550	9800	..50	.300
174	240549	0799	1048	1297	1546	1795	2044	2293	2541	2790
175	3038	3286	3534	3782	4030	4277	4525	4772	5019	5266
176	5513	5759	6006	6252	6499	6745	6991	7237	7482	7728
177	7973	8219	8464	8709	8954	9198	9443	9687	9932	.176
178	250420	0664	0908	1151	1395	1638	1881	2125	2368	2610
179	2853	3096	3338	3580	3822	4064	4306	4548	4790	5031
180	5273	5514	5755	5996	6237	6477	6718	6958	7198	7439
181	7679	7918	8158	8398	8637	8877	9116	9355	9594	9833
182	260071	0310	0548	0787	1025	1263	1501	1739	1976	2214
183	2451	2688	2925	3162	3399	3636	3873	4109	4346	4582
184	4818	5054	5290	5525	5761	5996	6232	6467	6702	6937
185	7172	7406	7641	7875	8110	8344	8578	8812	9046	9279
186	9513	9746	9980	.213	.446	.679	.912	1144	1377	1609
187	271842	2074	2306	2538	2770	3001	3233	3464	3696	3927
188	4158	4389	4620	4850	5081	5311	5542	5772	6002	6232
189	6462	6692	6921	7151	7380	7609	7838	8067	8296	8525
190	8754	8982	9211	9439	9667	9895	.123	.351	.578	.806
191	281033	1261	1488	1715	1942	2169	2396	2622	2849	3075
192	3301	3527	3753	3979	4205	4431	4656	4882	5107	5332
193	5557	5782	6007	6232	6456	6681	6905	7130	7354	7578
194	7802	8026	8249	8473	8696	8920	9143	9366	9589	9812
195	290035	0257	0480	0702	0925	1147	1369	1591	1813	2034
196	2256	2478	2699	2920	3141	3363	3584	3804	4025	4246
197	4466	4687	4907	5127	5347	5567	5787	6007	6226	6446
198	6665	6884	7104	7323	7542	7761	7979	8198	8416	8635
199	8853	9071	9289	9507	9725	9943	.161	.378	.595	.813

N.	0	1	2	3	4	5	6	7	8	9
200	301030	1247	1464	1681	1898	2114	2331	2547	2764	2980
201	3196	3412	3628	3844	4059	4275	4491	4706	4921	5136
202	5351	5566	5781	5996	6211	6425	6639	6854	7068	7282
203	7496	7710	7924	8137	8351	8564	8778	8991	9204	9417
204	9630	9843	..56	.268	.481	.693	.906	1118	1330	1542
205	311754	1966	2177	2389	2600	2812	3023	3234	3445	3656
206	3867	4078	4289	4499	4710	4920	5130	5340	5551	5760
207	5970	6180	6390	6599	6809	7018	7227	7436	7646	7854
208	8063	8272	8481	8689	8898	9106	9314	9522	9730	9938
209	320146	0354	0562	0769	0977	1184	1391	1598	1805	2012
210	2219	2426	2633	2839	3046	3252	3458	3665	3871	4077
211	4282	4488	4694	4899	5105	5310	5516	5721	5926	6131
212	6336	6541	6745	6950	7155	7359	7563	7767	7972	8176
213	8380	8583	8787	8991	9194	9398	9601	9805	...8	.211
214	330414	0617	0819	1022	1225	1427	1630	1832	2034	2236
215	2438	2640	2842	3044	3246	3447	3649	3850	4051	4253
216	4454	4655	4856	5057	5257	5458	5658	5859	6059	6260
217	6460	6660	6860	7060	7260	7459	7659	7858	8058	8257
218	8456	8656	8855	9054	9253	9451	9650	9849	..47	.246
219	340444	0642	0841	1039	1237	1435	1632	1830	2028	2225
220	2423	2620	2817	3014	3212	3409	3606	3802	3999	4196
221	4392	4589	4785	4981	5178	5374	5570	5766	5962	6157
222	6353	6549	6744	6939	7135	7330	7525	7720	7915	8110
223	8305	8500	8694	8889	9083	9278	9472	9666	9860	..54
224	350248	0442	0636	0829	1023	1216	1410	1603	1796	1989
225	2183	2375	2568	2761	2954	3147	3339	3532	3724	3916
226	4108	4301	4493	4685	4876	5068	5260	5452	5643	5834
227	6026	6217	6408	6599	6790	6981	7172	7363	7554	7744
228	7935	8125	8316	8506	8696	8886	9076	9266	9456	9646
229	9835	..25	.215	.404	.593	.783	.972	1161	1350	1539
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232	5488	5675	5862	6049	6236	6423	6610	6796	6983	7169
233	7356	7542	7729	7915	8101	8287	8473	8659	8845	9030
234	9216	9401	9587	9772	9958	.143	.328	.513	.698	.883
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237	4748	4932	5115	5298	5481	5664	5846	6029	6212	6394
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242	3815	3995	4174	4353	4533	4712	4891	5070	5249	5428
243	5606	5785	5964	6142	6321	6499	6677	6856	7034	7212
244	7390	7568	7746	7923	8101	8279	8456	8634	8811	8989
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248	4452	4627	4802	4977	5152	5326	5501	5676	5850	6025
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254	4834	5005	5176	5346	5517	5688	5858	6029	6199	6370
255	6540	6710	6881	7051	7221	7391	7561	7731	7901	8070
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257	9933	.102	.271	.440	.609	.777	.946	1114	1283	1451
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260	4973	5140	5307	5474	5641	5808	5974	6141	6308	6474
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266	4882	5045	5208	5371	5534	5697	5860	6023	6186	6349
267	6511	6674	6836	6999	7161	7324	7486	7648	7811	7973
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278	4045	4201	4357	4513	4669	4825	4981	5137	5293	5449
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284	3318	3471	3624	3777	3930	4082	4235	4387	4540	4692
285	4845	4997	5150	5302	5454	5606	5758	5910	6062	6214
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306	5721	5863	6005	6147	6289	6430	6572	6714	6855	6997
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321	6505	6640	6776	6911	7046	7181	7316	7451	7586	7721
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335	5045	5174	5304	5434	5563	5693	5822	5951	6081	6210
336	6339	6469	6598	6727	6856	6985	7114	7243	7372	7501
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382	2063	2177	2291	2404	2518	2631	2745	2858	2972	3085
383	3199	3312	3426	3539	3652	3765	3879	3992	4105	4218
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385	5461	5574	5686	5799	5912	6024	6137	6250	6362	6475
386	6587	6700	6812	6925	7037	7149	7262	7374	7486	7599
387	7711	7823	7935	8047	8160	8272	8384	8496	8608	8720
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392	3286	3397	3508	3618	3729	3840	3950	4061	4171	4282
393	4393	4503	4614	4724	4834	4945	5055	5165	5276	5386
394	5496	5606	5717	5827	5937	6047	6157	6267	6377	6487
395	6597	6707	6817	6927	7037	7146	7256	7366	7476	7586
396	7695	7805	7914	8024	8134	8243	8353	8462	8572	8681
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402	4226	4334	4442	4550	4658	4766	4874	4982	5089	5197
403	5305	5413	5521	5628	5736	5844	5951	6059	6166	6274
404	6381	6489	6596	6704	6811	6919	7026	7133	7241	7348
405	7455	7562	7669	7777	7884	7991	8098	8205	8312	8419
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419	2214	2318	2421	2525	2628	2732	2835	2939	3042	3146
420	3249	3353	3456	3559	3663	3766	3869	3973	4076	4179
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431	4477	4578	4679	4779	4880	4981	5081	5182	5283	5383
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439	2465	2563	2662	2761	2860	2959	3058	3156	3255	3354
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443	6404	6502	6600	6698	6796	6894	6992	7089	7187	7285
444	7383	7481	7579	7676	7774	7872	7969	8067	8165	8262
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447	650308	0405	0502	0599	0696	0793	0890	0987	1084	1181
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452	5138	5235	5331	5427	5526	5619	5715	5810	5906	6002
453	6098	6194	6290	6386	6482	6577	6673	6769	6864	6960
454	7056	7152	7247	7343	7438	7534	7629	7725	7820	7916
455	8011	8107	8202	8298	8393	8488	8584	8679	8774	8870
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460	2758	2852	2947	3041	3135	3230	3324	3418	3512	3607
461	3701	3795	3889	3983	4078	4172	4266	4360	4454	4548
462	4642	4736	4830	4924	5018	5112	5206	5299	5393	5487
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468	670241	0339	0431	0524	0617	0710	0802	0895	0988	1080
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470	2098	2190	2283	2375	2467	2560	2652	2744	2836	2929
471	3021	3113	3205	3297	3390	3482	3574	3666	3758	3850
472	3942	4034	4126	4218	4310	4402	4494	4586	4677	4769
473	4861	4953	5045	5137	5228	5320	5412	5503	5595	5687
474	5778	5870	5962	6053	6145	6236	6328	6419	6511	6602
475	6694	6785	6876	6968	7059	7151	7242	7333	7424	7516
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483	3947	4037	4127	4217	4307	4396	4486	4576	4666	4756
484	4845	4935	5025	5114	5204	5294	5383	5473	5563	5652
485	5742	5831	5921	6010	6100	6189	6279	6368	6458	6547
486	6636	6726	6815	6904	6994	7083	7172	7261	7351	7440
487	7529	7618	7707	7796	7886	7975	8064	8153	8242	8331
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504	2431	2517	2603	2689	2775	2861	2947	3033	3119	3205
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508	5864	5949	6035	6120	6206	6291	6376	6462	6547	6632
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510	7570	7655	7740	7826	7910	7996	8081	8166	8251	8336
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517	3491	3575	3659	3742	3826	3910	3994	4078	4162	4246
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519	5167	5251	5335	5418	5502	5586	5669	5753	5836	5920
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553	2725	2804	2882	2961	3039	3118	3196	3275	3353	3431
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555	4293	4371	4449	4528	4606	4684	4762	4840	4919	4997
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605	1755	1827	1899	1971	2042	2114	2186	2258	2329	2401
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635	2774	2842	2910	2979	3047	3116	3184	3252	3321	3389
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638	4821	4889	4957	5025	5093	5161	5229	5297	5365	5433
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648	1575	1642	1709	1776	1843	1910	1977	2044	2111	2178
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652	4248	4314	4381	4447	4514	4581	4647	4714	4780	4847
653	4913	4980	5046	5113	5179	5246	5312	5378	5445	5511
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685	5691	5754	5817	5881	5944	6007	6071	6134	6197	6261
686	6324	6387	6451	6514	6577	6641	6704	6767	6830	6894
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694	1359	1422	1485	1547	1610	1672	1735	1797	1860	1922
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697	3233	3295	3357	3420	3482	3544	3606	3669	3731	3793
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701	5718	5780	5842	5904	5966	6028	6090	6151	6213	6275
702	6337	6399	6461	6523	6585	6646	6708	6770	6832	6894
703	6955	7017	7079	7141	7202	7264	7326	7388	7449	7511
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705	8189	8251	8312	8374	8435	8497	8559	8620	8682	8743
706	8805	8866	8928	8989	9051	9112	9174	9235	9297	9358
707	9419	9481	9542	9604	9665	9726	9788	9849	9911	9972
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712	2480	2541	2602	2663	2724	2785	2846	2907	2968	3029
713	3090	3150	3211	3272	3333	3394	3455	3516	3577	3637
714	3698	3759	3820	3881	3941	4002	4063	4124	4185	4245
715	4306	4367	4428	4488	4549	4610	4670	4731	4792	4852
716	4913	4974	5034	5095	5156	5216	5277	5337	5398	5459
717	5519	5580	5640	5701	5761	5822	5882	5943	6003	6064
718	6124	6185	6245	6306	6366	6427	6487	6548	6608	6668
719	6729	6789	6850	6910	6970	7031	7091	7152	7212	7272
720	7332	7393	7453	7513	7574	7634	7694	7755	7815	7875
721	7935	7995	8056	8116	8176	8236	8297	8357	8417	8477
722	8537	8597	8657	8718	8778	8838	8898	8958	9018	9078
723	9138	9198	9258	9318	9379	9439	9499	9559	9619	9679
724	9739	9799	9859	9918	9978	..38	..98	.158	.218	.278
725	860338	0398	0458	0518	0578	0637	0697	0757	0817	0877
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727	1534	1594	1654	1714	1773	1833	1893	1952	2012	2072
728	2131	2191	2251	2310	2370	2430	2489	2549	2608	2668
729	2728	2787	2847	2906	2966	3025	3085	3144	3204	3263
730	3323	3382	3442	3501	3561	3620	3680	3739	3799	3858
731	3917	3977	4036	4096	4155	4214	4274	4333	4392	4452
732	4511	4570	4630	4689	4148	4808	4867	4926	4985	5045
733	5104	5163	5222	5282	5341	5400	5459	5519	5578	5637
734	5696	5755	5814	5874	5933	5992	6051	6110	6169	6228
735	6287	6346	6405	6465	6524	6583	6642	6701	6760	6819
736	6878	6937	6996	7055	7114	7173	7232	7291	7350	7409
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738	8056	8115	8174	8233	8292	8350	8409	8468	8527	8586
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752	6218	6276	6333	6391	6449	6507	6564	6622	6680	6737
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754	7371	7429	7487	7544	7602	7659	7717	7774	7832	7889
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756	8522	8579	8637	8694	8752	8809	8866	8924	8981	9039
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762	1955	2012	2069	2126	2183	2240	2297	2354	2411	2468
763	2525	2581	2638	2695	2752	2809	2866	2923	2980	3037
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765	3661	3718	3775	3832	3888	3945	4002	4059	4115	4172
766	4229	4285	4342	4399	4455	4512	4569	4625	4682	4739
767	4795	4852	4909	4965	5022	5078	5135	5192	5248	5305
768	5361	5418	5474	5531	5587	5644	5700	5757	5813	5870
769	5926	5983	6039	6096	6152	6209	6265	6321	6378	6434
770	6491	6547	6604	6660	6716	6773	6829	6885	6942	6998
771	7054	7111	7167	7233	7280	7336	7392	7449	7505	7561
772	7617	7674	7730	7786	7842	7898	7955	8011	8067	8123
773	8179	8236	8292	8348	8404	8460	8516	8573	8629	8655
774	8741	8797	8853	8909	8965	9021	9077	9134	9190	9246
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779	1537	1593	1649	1705	1760	1816	1872	1928	1983	2039
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783	3762	3817	3873	3928	3984	4039	4094	4150	4205	4261
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785	4870	4925	4980	5036	5091	5146	5201	5257	5312	5367
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791	8176	8231	8286	8341	8396	8451	8506	8561	8615	8670
792	8725	8780	8835	8890	8944	8999	9054	9109	9164	9218
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797	1458	1513	1567	1622	1676	1736	1785	1840	1894	1948
798	2003	2057	2112	2166	2221	2275	2329	2384	2438	2492
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802	4174	4229	4283	4337	4391	4445	4499	4553	4607	4661
803	4716	4770	4824	4878	4932	4986	5040	5094	5148	5202
804	5256	5310	5364	5418	5472	5526	5580	5634	5688	5742
805	5796	5850	5904	5958	6012	6066	6119	6173	6227	6281
806	6335	6389	6443	6497	6551	6604	6658	6712	6766	6820
807	6874	6927	6981	7035	7089	7143	7196	7250	7304	7358
808	7411	7465	7519	7573	7626	7680	7734	7787	7841	7895
809	7949	8002	8056	8110	8163	8217	8270	8324	8378	8431
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840	4279	4331	4383	4434	4486	4538	4589	4641	4693	4744
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843	5828	5874	5931	5982	6034	6085	6137	6188	6240	6291
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859	3993	4044	4094	4145	4195	4246	4269	4347	4397	4448
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879	3989	4038	4088	4137	4186	4236	4285	4335	4384	4433
880	4483	4532	4581	4631	4680	4729	4779	4828	4877	4927
881	4976	5025	5074	5124	5173	5222	5272	5321	5370	5419
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885	6943	6992	7041	7090	7140	7189	7238	7287	7336	7385
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902	5207	5255	5303	5351	5399	5447	5495	5543	5592	5640
903	5688	5736	5784	5832	5880	5928	5976	6024	6072	6120
904	6168	6216	6265	6313	6361	6409	6457	6505	6553	6601
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906	7128	7176	7224	7272	7320	7368	7416	7464	7512	7559
907	7607	7655	7703	7751	7799	7847	7894	7942	7990	8038
908	8086	8134	8181	8229	8277	8325	8373	8421	8468	8516
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910	9041	9089	9137	9185	9232	9280	9328	9375	9423	9471
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919	3316	3363	3410	3457	3504	3552	3599	3646	3693	3741
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930	8483	8530	8576	8623	8670	8716	8763	8810	8856	8903
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936	1276	1322	1369	1415	1461	1508	1554	1601	1647	1693
937	1740	1786	1832	1879	1925	1971	2018	2064	2110	2157
938	2203	2249	2295	2342	2388	2434	2481	2527	2573	2619
939	2666	2712	2758	2804	2851	2897	2943	2989	3035	3082
940	3128	3174	3220	3266	3313	3359	3405	3451	3497	3543
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942	4051	4097	4143	4189	4235	4281	4327	4374	4420	4466
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944	4972	5018	5064	5110	5156	5202	5248	5294	5340	5386
945	5432	5478	5524	5570	5616	5662	5707	5753	5799	5845
946	5891	5937	5983	6029	6075	6121	6167	6212	6258	6304
947	6350	6396	6442	6488	6533	6579	6925	6671	6717	6763
948	6803	6854	6900	6946	6992	7037	7083	7129	7175	7220
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951	8181	8226	8272	8317	8363	8409	8454	8500	8546	8591
952	8637	8683	8728	8774	8819	8865	8911	8956	9002	9047
953	9093	9138	9184	9230	9275	9321	9366	9412	9457	9503
954	9548	9594	9639	9685	9730	9776	9821	9867	9912	9958
955	980003	0049	0094	0140	0185	0231	0276	0322	0367	0412
956	0458	0503	0549	0594	0640	0685	0730	0776	0821	0867
957	0912	0957	1003	1048	1093	1139	1184	1229	1275	1320
958	1366	1411	1456	1501	1547	1592	1637	1683	1728	1773
959	1819	1864	1909	1954	2000	2045	2090	2135	2181	2226
960	2271	2316	2362	2407	2452	2497	2543	2588	2633	2678
961	2723	2769	2814	2859	2904	2949	2994	3040	3085	3130
962	3175	3220	3265	3310	3356	3401	3446	3491	3536	3581
963	3626	3671	3716	3762	3807	3852	3897	3942	3987	4032
964	4077	4122	4167	4212	4257	4302	4347	4392	4437	4482
965	4527	4572	4617	4662	4707	4752	4797	4842	4887	4932
966	4977	5022	5067	5112	5157	5202	5247	5292	5337	5382
967	5426	5471	5516	5561	5606	5651	5699	5741	5786	5830
968	5875	5920	5965	6010	6055	6100	6144	6189	6234	6279
969	6324	6369	6413	6458	6503	6548	6593	6637	6682	6727
970	6772	6817	6861	6906	6951	6996	7040	7085	7130	7175
971	7219	7264	7309	7353	7398	7443	7488	7532	7577	7622
972	7666	7711	7756	7800	7845	7890	7934	7979	8024	8068
973	8113	8157	8202	8247	8291	8336	8381	8425	8470	8514
974	8559	8604	8648	8693	8737	8782	8826	8871	8916	8960
975	9005	9049	9093	9138	9183	9227	9272	9316	9361	9405
976	9450	9494	9539	9583	9628	9672	9717	9761	9806	9850
977	9895	9939	9983	..28	..72	.117	.161	.206	.250	.294
978	990339	0383	0428	0472	0516	0561	0605	0650	0694	0738
979	0783	0827	0871	0916	0960	1004	1049	1093	1137	1182
980	1226	1270	1315	1359	1403	1448	1492	1536	1580	1625
981	1669	1713	1758	1802	1846	1890	1935	1979	2023	2067
982	2111	2156	2200	2244	2288	2333	2377	2421	2465	2509
983	2554	2598	2642	2686	2730	2774	2819	2863	2907	2951
984	2995	3039	3083	3127	3172	3216	3260	3304	3348	3392
985	3436	3480	3524	3568	3613	3657	3701	3745	3789	3833
986	3877	3921	3965	4009	4053	4097	4141	4185	4229	4273
987	4317	4361	4405	4449	4493	4537	4581	4625	4669	4713
988	4757	4801	4845	4886	4933	4977	5021	5065	5108	5152
989	5196	5240	5284	5328	5372	5416	5460	5504	5547	5591
990	5635	5679	5723	5767	5811	5854	5898	5942	5986	6030
991	6074	6117	6161	6205	6249	6293	6337	6380	6424	6468
992	6512	6555	6599	6643	6687	6731	6774	6818	6862	6906
993	6949	6993	7037	7080	7124	7168	7212	7255	7299	7343
994	7386	7430	7474	7517	7561	7605	7648	7692	7736	7779
995	7823	7867	7910	7954	7998	8041	8085	8129	8172	8216
996	8259	8303	8347	8390	8434	8477	8521	8564	8608	8652
997	8695	8739	8792	8826	8869	8913	8956	9000	9043	9087
998	9131	9174	9218	9261	9305	9348	9392	9435	9479	9522
999	9565	9609	9652	9696	9739	9783	9826	9870	9913	9957

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	0.000000		10.000000		0.000000		Infinite.	00000	100000	60
1	6.463726		000000		6.463726		13.536274	00029	100000	59
2	764756		000000		764756		235244	00058	100000	58
3	940847		000000		940847		059153	00087	100000	57
4	7.065786		000000		7.065786		12.934214	00116	100000	56
5	162696		000000		162696		837304	00145	100000	55
6	241877		9.999999		241878		758122	00175	100000	54
7	308824		999999		308825		691175	00204	100000	53
8	366816		999999		366817		633183	00233	100000	52
9	417968		999999		417970		582030	00262	100000	51
10	463725		999998		463727		536273	00291	100000	50
11	7.505118		9.999998		7.505120		12.494880	00320	99999	49
12	542906		999997		542909		457091	00349	99999	48
13	577668		999997		577672		422328	00378	99999	47
14	609853		999996		609857		390143	00407	99999	46
15	639816		999996		639820		360180	00436	99999	45
16	667845		999995		667849		332151	00465	99999	44
17	694173		999995		694179		305821	00495	99999	43
18	718997		999994		719003		280997	00524	99999	42
19	742477		999993		742484		257516	00553	99998	41
20	764754		999993		764761		235239	00582	99998	40
21	7.785943		9.999992		7.785951		12.214049	00611	99998	39
22	806146		999991		806155		193845	00640	99998	38
23	825451		999990		825460		174540	00669	99998	37
24	843934		999989		843944		156056	00698	99998	36
25	861663		999988		861674		138326	00727	99997	35
26	878695		999988		878708		121292	00756	99997	34
27	895085		999987		895099		104901	00785	99997	33
28	910879		999986		910894		089106	00814	99997	32
29	926119		999985		926134		073866	00844	99996	31
30	940842		999983		940858		059142	00873	99996	30
31	7.955082	2298	9.999982	0.2	7.955100	2298	12.044900	00902	99996	29
32	968870	2227	999981	0.2	968889	2227	031111	00931	99996	28
33	982233	2161	999980	0.2	982253	2161	017747	00960	99995	27
34	995198	2098	999979	0.2	995219	2098	004781	00989	99995	26
35	8.007787	2039	999977	0.2	8.007809	2039	11.992191	01018	99995	25
36	020021	1983	999976	0.2	020045	1983	979955	01047	99995	24
37	031919	1930	999975	0.2	031945	1930	968055	01076	99994	23
38	043501	1880	999973	0.2	043527	1880	956473	01105	99994	22
39	054781	1832	999972	0.2	054809	1833	945191	01134	99994	21
40	065776	1787	999971	0.2	065806	1787	934194	01164	99993	20
41	8.076500	1744	9.999969	0.2	8.076531	1744	11.923469	01193	99993	19
42	086965	1703	999968	0.2	086997	1703	913003	01222	99993	18
43	097183	1664	999966	0.2	097217	1664	902783	01251	99992	17
44	107167	1626	999964	0.3	107202	1627	892797	01280	99992	16
45	116926	1591	999963	0.3	116963	1591	883037	01309	99991	15
46	126471	1557	999961	0.3	126510	1557	873490	01338	99991	14
47	135810	1524	999959	0.3	135851	1524	864149	01367	99991	13
48	144953	1492	999958	0.3	144996	1493	855004	01396	99990	12
49	153907	1462	999956	0.3	153952	1463	846048	01425	99990	11
50	162681	1433	999954	0.3	162727	1434	837273	01454	99989	10
51	8.171280	1405	9.999952	0.3	8.171328	1406	11.828672	01483	99989	9
52	179713	1379	999950	0.3	179763	1379	820237	01513	99989	8
53	187985	1353	999948	0.3	188036	1353	811964	01542	99988	7
54	196102	1328	999946	0.3	196156	1328	803844	01571	99988	6
55	204070	1304	999944	0.3	204126	1304	795874	01600	99987	5
56	211895	1281	999942	0.4	211953	1281	788047	01629	99987	4
57	219581	1259	999940	0.4	219641	1259	780359	01658	99986	3
58	227134	1237	999938	0.4	227195	1238	772805	01687	99986	2
59	234557	1216	999936	0.4	234621	1217	765379	01716	99985	1
60	241855		999934		241921		758079	01745	99985	0
	Cosine.	D. 10″	Sine.	D. 10″	Cotang.	D. 10″	Tang.	N. cos.	N. sine.	′

89 Degrees.

′	Sine.	D.10″	Cosine.	D.10″	Tang.	D.10″	Cotang.	N. sine.	N. cos.	
0	8.241855	1196	9.999934	0.4	8.241921	1197	11.758079	01742	99985	60
1	249033	1177	999932	0.4	249102	1177	750898	01774	99984	59
2	256094	1158	999929	0.4	256165	1158	743835	01803	99984	58
3	263042	1140	999927	0.4	263115	1140	736885	01832	99983	57
4	269881	1122	999925	0.4	269956	1122	730044	01862	99983	56
5	276614	1105	999922	0.4	276691	1105	723309	01891	99982	55
6	283243	1088	999920	0.4	283323	1089	716677	01920	99982	54
7	289773	1072	999918	0.4	289856	1073	710144	01949	99981	53
8	296207	1056	999915	0.4	296292	1057	703708	01978	99980	52
9	302546	1041	999913	0.4	302634	1042	697366	02007	99980	51
10	308794	1027	999910	0.4	308884	1027	691116	02036	99979	50
11	8.314954	1012	9.999907	0.4	8.315046	1013	11.684954	02065	99979	49
12	321027	998	999905	0.4	321122	999	678878	02094	99978	48
13	327016	985	999902	0.4	327114	985	672886	02123	99977	47
14	332924	971	999899	0.5	333025	972	666975	02152	99977	46
15	338753	959	999897	0.5	333856	959	661144	02181	99976	45
16	344504	946	999894	0.5	344610	946	655390	02211	99976	44
17	350181	934	999891	0.5	350289	934	649711	02240	99975	43
18	355783	922	999888	0.5	355895	922	644105	02269	99974	42
19	361315	910	999885	0.5	361430	911	638570	02298	99974	41
20	366777	899	999882	0.5	366895	899	633105	02327	99973	40
21	8.372171	888	9.999879	0.5	8.372292	888	11.627708	02356	99972	39
22	377499	877	999876	0.5	377622	879	622378	02385	99972	38
23	382762	867	999873	0.5	382889	867	617111	02414	99971	37
24	387962	856	999870	0.5	388092	857	611908	02443	99970	36
25	393101	846	999867	0.5	393234	847	606766	02472	99969	35
26	398179	837	999864	0.5	398315	837	601685	02501	99969	34
27	403199	827	999861	0.5	403338	828	596662	02530	99968	33
28	408161	818	999858	0.5	408304	818	591696	02560	99967	32
29	413068	809	999854	0.5	413213	809	586787	02589	99966	31
30	417919	800	999851	0.6	418068	800	581932	02618	99966	30
31	8.422717	791	9.999848	0.6	8.422869	791	11.577131	02647	99965	29
32	427462	782	999844	0.6	427618	783	572382	02676	99964	28
33	432156	774	999841	0.6	432315	774	567685	02705	99963	27
34	436800	766	999838	0.6	436962	766	563038	02734	99963	26
35	441394	758	999834	0.6	441560	758	558440	02763	99962	25
36	445941	750	999831	0.6	446110	750	553890	02792	99961	24
37	450440	742	999827	0.6	450613	743	549387	02821	99960	23
38	454893	735	999823	0.6	455070	735	544930	02850	99959	22
39	459301	727	999820	0.6	459481	728	540519	02879	99959	21
40	463665	720	999816	0.6	463849	720	536151	02908	99958	20
41	8.467985	712	9.999812	0.6	8.468172	713	11.531828	02938	99957	19
42	472263	706	999809	0.6	472454	707	527546	02967	99956	18
43	476498	699	999805	0.6	476693	700	523307	02996	99955	17
44	480693	692	999801	0.6	480892	693	519108	03025	99954	16
45	484848	686	999797	0.7	485050	686	514950	03054	99953	15
46	488963	679	999793	0.7	489170	680	510830	03083	99952	14
47	493040	673	999790	0.7	493250	674	506750	03112	99952	13
48	497078	667	999786	0.7	497293	668	502707	03141	99951	12
49	501080	661	999782	0.7	501298	661	498702	03170	99950	11
50	505045	655	999778	0.7	505267	655	494733	03199	99949	10
51	8.508974	649	9.999774	0.7	8.509200	650	11.490800	03228	99948	9
52	512867	643	999769	0.7	513098	644	486902	03257	99947	8
53	516726	637	999765	0.7	516961	638	483039	03286	99946	7
54	520551	632	999761	0.7	520790	633	479210	03316	99945	6
55	524343	626	999757	0.7	524586	627	475414	03345	99944	5
56	528102	621	999753	0.7	528349	622	471651	03374	99943	4
57	531828	616	999748	0.7	532080	616	467920	03403	99942	3
58	535523	611	999744	0.7	535779	611	464221	03432	99941	2
59	539186	605	999740	0.7	539447	606	460553	03461	99940	1
60	542819		999735		543084		456916	03490	99939	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N.sine.	′

88 Degrees.

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	8.542819	600	9.999735	0.7	8.543084	602	11.456916	03490	99939	60
1	546422	595	999731	0.7	546691	596	453309	03519	99938	59
2	549995	591	999726	0.7	550268	591	449732	03548	99937	58
3	553539	586	999722	0.8	553817	587	446183	03577	99936	57
4	557054	581	999717	0.8	557336	582	442664	03606	99935	56
5	560540	576	999713	0.8	560828	577	439172	03635	99934	55
6	563999	572	999708	0.8	564291	573	435709	03664	99933	54
7	567431	567	999704	0.8	567727	568	432273	03693	99932	53
8	570836	563	999699	0.8	571137	564	428863	03723	99931	52
9	574214	559	999694	0.8	574520	559	425480	03752	99930	51
10	577566	554	999689	0.8	577877	555	422123	03781	99929	50
11	8.580892	550	9.999685	0.8	8.581208	551	11.418792	03810	99927	49
12	584193	546	999680	0.8	584514	547	415486	03839	99926	48
13	587469	542	999675	0.8	587795	543	412205	03868	99925	47
14	590721	538	999670	0.8	591051	539	408949	03897	99924	46
15	593948	534	999665	0.8	594283	535	405717	03926	99923	45
16	597152	530	999660	0.8	597492	531	402508	03955	99922	44
17	600332	526	999655	0.8	600677	527	399323	03984	99921	43
18	603489	522	999650	0.8	603839	523	396161	04013	99919	42
19	606623	519	999645	0.8	606978	519	393022	04042	99918	41
20	609734	515	999640	0.9	610094	516	389906	04071	99917	40
21	8.612823	511	9.999635	0.9	8.613189	512	11.386811	04100	99916	39
22	615891	508	999629	0.9	616262	508	383738	03129	99915	38
23	618937	504	999324	0.9	619313	505	380687	04159	99913	37
24	621962	501	999619	0.9	622343	501	377657	04188	99912	36
25	624965	497	999614	0.9	625352	498	374648	04217	99911	35
26	627948	494	999608	0.9	628340	495	371660	04246	99910	34
27	630911	490	999603	0.9	631308	491	368692	04275	99909	33
28	633854	487	999597	0.9	634256	488	365744	04304	99907	32
29	636776	484	999592	0.9	637184	485	362816	04333	99906	31
30	639680	481	999586	0.9	640093	482	359907	04362	99905	30
31	8.642563	477	9.999581	0.9	8.642982	478	11.357018	04391	99904	29
32	645428	474	999575	0.9	645853	475	354147	04420	99902	28
33	648274	471	999570	0.9	648704	472	351296	04449	99901	27
34	651102	468	999564	0.9	651537	469	348463	04478	99900	26
35	653911	465	999558	1.0	654352	466	345648	04507	99898	25
36	656702	462	999553	1.0	657149	463	342851	04536	99897	24
37	659475	459	999547	1.0	659928	460	340072	04565	99896	23
38	662230	456	999541	1.0	662689	457	337311	04594	99894	22
39	664968	453	999535	1.0	665433	454	334567	04623	99893	21
40	667689	451	999529	1.0	668160	453	331840	04653	99892	20
41	8.670393	448	9.999524	1.0	8.670870	449	11.329130	04682	99890	19
42	673080	445	999518	1.0	673563	446	326437	04711	99889	18
43	675751	442	999512	1.0	676239	443	323761	04740	99888	17
44	678405	440	999506	1.0	678900	442	321100	04769	99886	16
45	681043	437	999500	1.0	681544	438	318456	04798	99885	15
46	683665	434	999493	1.0	684172	435	315828	04827	99883	14
47	686272	432	999487	1.0	6-6784	433	313216	04856	99882	13
48	688863	429	999481	1.0	689381	430	310619	04885	99881	12
49	691438	427	999475	1.0	691963	428	308037	04914	99879	11
50	693998	424	999469	1.0	694529	425	305471	04943	99878	10
51	8.696543	422	9.999463	1.1	8.697081	423	11.302919	04972	99876	9
52	699073	419	999456	1.1	699617	420	300383	05001	99875	8
53	701589	417	999450	1.1	702139	418	297861	05030	99873	7
54	704090	414	999443	1.1	704246	415	295354	05059	99872	6
55	706577	412	999437	1.1	707140	413	292860	05088	99870	5
56	709049	410	999431	1.1	709618	411	290382	05117	99869	4
57	711507	407	999424	1.1	702083	408	287917	05146	99867	3
58	713952	405	999418	1.1	714534	406	285465	05175	99866	2
59	716383	403	999411	1.1	716972	404	283028	05205	99864	1
60	718800		999404		719396		280604	05234	99863	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N.sine.	′

87 Degrees.

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	8.718800	401	9.999404	1.1	8.719396	402	11.280604	05234	99863	60
1	721204	398	999398	1.1	721806	399	278194	05263	99861	59
2	723595	396	999391	1.1	724204	397	275796	05292	99860	58
3	725972	394	999384	1.1	726588	395	273412	05321	99858	57
4	728337	392	999378	1.1	728959	393	271041	05350	99857	56
5	730688	390	999371	1.1	731317	391	268683	05379	99855	55
6	733027	388	999364	1.2	733663	389	266337	05408	99854	54
7	735354	386	999357	1.2	735996	387	264004	05437	99852	53
8	737667	384	999350	1.2	738317	385	261683	05466	99851	52
9	739969	382	999343	1.2	740626	383	259374	05495	99849	51
10	742259	380	999336	1.2	742922	381	257078	05524	99847	50
11	8.744536	378	9.999329	1.2	8.745207	379	11.254793	05553	99846	49
12	746802	376	999322	1.2	747479	377	252521	05582	99844	48
13	749055	374	999315	1.2	749740	375	250260	05611	99842	47
14	751297	372	999308	1.2	751989	373	248011	05640	99841	46
15	753528	370	999301	1.2	754227	371	245773	05669	99839	45
16	755747	368	999294	1.2	756453	369	243547	05698	99838	44
17	757955	366	999286	1.2	758668	367	241332	05727	99836	43
18	760151	364	999279	1.2	760872	365	239128	05756	99834	42
19	762337	362	999272	1.2	763065	364	236935	05785	99833	41
20	764511	361	999265	1.2	765246	362	234754	05814	99831	40
21	8.766675	359	9.999257	1.2	8.767417	360	11.232583	05844	99829	39
22	768828	357	999250	1.3	769578	358	230422	05873	99827	38
23	770970	355	999242	1.3	771727	356	228273	05902	99826	37
24	773101	353	999235	1.3	773866	355	226134	05931	99824	36
25	775223	352	999227	1.3	775995	353	224005	05960	99822	35
26	777333	350	999220	1.3	778114	351	221886	05989	99821	34
27	779434	348	999212	1.3	780222	350	219778	06018	99819	33
28	781524	347	999205	1.3	782320	348	217680	06047	99817	32
29	783605	345	999197	1.3	784408	346	215592	06076	99815	31
30	785675	343	999189	1.3	786486	345	213514	06105	99813	30
31	8.787736	342	9.999181	1.3	8.788554	343	11.211446	06134	99812	29
32	789787	340	999174	1.3	790613	341	209387	06163	99810	28
33	791828	339	999166	1.3	792662	340	207338	06192	99808	27
34	793859	337	999158	1.3	794701	338	205299	06221	99806	26
35	795881	335	999150	1.3	796731	337	203269	06250	99804	25
36	797894	334	999142	1.3	798752	335	201248	06279	99803	24
37	799897	332	999134	1.3	800763	334	199237	06308	99801	23
38	801892	331	999126	1.3	802765	332	197235	06337	99799	22
39	803876	329	999118	1.3	804858	331	195242	06366	99797	21
40	805852	328	999110	1.3	806742	329	193258	06395	99795	20
41	8.807819	326	9.999102	1.3	8.808717	328	11.191283	06424	99793	19
42	809777	325	999094	1.4	810683	326	189317	06453	99792	18
43	811726	323	999086	1.4	812641	325	187359	06482	99790	17
44	813667	322	999077	1.4	814589	323	185411	06511	99788	16
45	815599	320	999069	1.4	816529	322	183471	06540	99786	15
46	817522	319	999061	1.4	818461	320	181539	06569	99784	14
47	819436	318	999053	1.4	820384	319	179616	06598	99782	13
48	821343	316	999044	1.4	822298	318	177702	06627	99780	12
49	823240	315	999036	1.4	824205	316	175795	06656	99778	11
50	825130	313	999027	1.4	826103	315	173897	06685	99776	10
51	8.827011	312	9.999019	1.4	8.827992	314	11.172008	06714	99774	9
52	828884	311	999010	1.4	829874	312	170126	06743	99772	8
53	830749	309	999002	1.4	831748	311	168252	06773	99770	7
54	832607	308	998993	1.4	833613	310	166387	06802	99768	6
55	834456	307	998984	1.4	835471	308	164529	06831	99766	5
56	836297	306	998976	1.4	837321	307	162679	06860	99764	4
57	838130	304	998967	1.5	839163	306	160837	06889	99762	3
58	839956	303	998958	1.5	840998	304	159002	06918	99760	2
59	841774	302	998950	1.5	842825	303	157175	06947	99758	1
60	843585		998941		844644		155356	06976	99756	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N. sine.	′

86 Degrees.

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	8.843585	300	9.998941	1.5	8.844644	302	11.155356	06976	99756	60
1	845387	299	998932	1.5	846455	301	153545	07005	99754	59
2	847183	298	998923	1.5	848260	299	151740	07034	99752	58
3	848971	297	998914	1.5	850057	298	149943	07063	99750	57
4	850751	295	998905	1.5	851846	297	148154	07092	99748	56
5	852525	294	998896	1.5	853628	296	146372	07121	99746	55
6	854291	293	998887	1.5	855403	295	144597	07150	99744	54
7	856049	292	998878	1.5	857171	293	142829	07179	99742	53
8	857801	291	998869	1.5	858932	292	141068	07208	99740	52
9	859546	290	998860	1.5	860686	291	139314	07237	99738	51
10	861283	288	998851	1.5	862433	290	137567	07266	99736	50
11	8.863014	287	9.998841	1.5	8.864173	289	11.135827	07295	99734	49
12	864738	286	998832	1.5	865906	288	134094	07324	99731	48
13	866455	285	998823	1.6	867632	287	132368	07353	99729	47
14	868165	284	998813	1.6	869351	285	130649	07382	99727	46
15	869868	283	998804	1.6	871064	284	128936	07411	99725	45
16	871565	282	998795	1.6	872770	283	127230	07440	99723	44
17	873255	281	998785	1.6	874469	282	125531	07469	99721	43
18	874938	279	998776	1.6	876162	281	123838	07498	99719	42
19	876615	279	998766	1.6	877849	280	122151	07527	99716	41
20	878285	277	998757	1.6	879529	279	120471	07556	99714	40
21	8.879949	276	9.998747	1.6	8.881202	278	11.118798	07585	99712	39
22	881607	275	998738	1.6	882869	277	117131	07614	99710	38
23	883258	274	998728	1.6	884530	276	115470	07643	99708	37
24	884903	273	998718	1.6	886185	275	113815	07672	99705	36
25	886542	272	998708	1.6	887833	274	112167	07701	99703	35
26	888174	271	998699	1.6	889476	273	110524	07730	99701	34
27	889801	270	998689	1.6	891112	272	108888	07759	99699	33
28	891421	269	998679	1.6	892742	271	107258	07788	99696	32
29	893035	268	998669	1.7	894366	270	105634	07817	99694	31
30	894643	267	998659	1.7	895984	269	104016	07846	99692	30
31	8.896246	266	9.998649	1.7	8.897596	268	11.102404	07875	99689	29
32	897842	265	998639	1.7	899203	267	100797	07904	99687	28
33	899432	264	998629	1.7	900803	266	099197	07933	99685	27
34	901017	263	998619	1.7	902398	265	097602	07962	99683	26
35	902596	262	998609	1.7	903987	264	096013	07991	99680	25
36	904169	261	998599	1.7	905570	263	094430	08020	99678	24
37	905736	260	998589	1.7	907147	262	092853	08049	99676	23
38	907297	259	998578	1.7	908719	261	091281	08078	99673	22
39	908853	258	998568	1.7	910285	260	089715	08107	99671	21
40	910404	257	998558	1.7	911846	259	088154	08136	99668	20
41	8.911949	257	9.998548	1.7	8.913401	258	11.086599	08165	99666	19
42	913488	256	998537	1.7	914951	257	085049	08194	99664	18
43	915022	255	998527	1.7	916495	256	083505	08223	99661	17
44	916550	254	998516	1.8	918034	256	081966	08252	99659	16
45	918073	253	998506	1.8	919568	255	080432	08281	99657	15
46	919591	252	998495	1.8	921096	254	078904	08310	99654	14
47	921103	251	998485	1.8	922619	253	077381	08339	99652	13
48	922610	250	998474	1.8	924136	252	075864	08368	99649	12
49	924112	249	998464	1.8	925649	251	074351	08397	99647	11
50	925609	249	998453	1.8	927156	250	072844	08426	99644	10
51	8.927100	248	9.998442	1.8	8.928658	249	11.071342	08455	99642	9
52	928587	247	998431	1.8	930155	249	069845	08484	99639	8
53	930068	246	998421	1.8	931647	248	068353	08513	99637	7
54	931544	245	998410	1.8	933134	247	066866	08542	99635	6
55	933015	244	998399	1.8	934616	246	065384	08571	99632	5
56	934481	243	998388	1.8	936093	245	063907	08600	99630	4
57	935942	243	998377	1.8	937565	244	062435	08629	99627	3
58	937398	242	998366	1.8	939032	244	060968	08658	99625	2
59	938850	241	998355	1.8	940494	243	059506	08687	99622	1
60	940296		998344		941952		058048	08716	99619	0
	Cosine.		Sine.	D. 10″	Cotang.		Tang.	N. cos.	N. sine.	′

85 Degrees.

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	8.940296	240	9.998344	1.9	8.941952	242	11.058048	08716	99619	60
1	941738	239	998333	1.9	943404	241	056596	08745	99617	59
2	943174	239	998322	1.9	944852	240	055148	08774	99614	58
3	944606	238	998311	1.9	946295	240	053705	08803	99612	57
4	946034	237	998300	1.9	947734	239	052266	08831	99609	56
5	947456	236	998289	1.9	949168	238	050832	08860	99607	55
6	948874	235	998277	1.9	950597	237	049403	08889	99604	54
7	950287	235	998266	1.9	952021	237	047979	08918	99602	53
8	951696	234	998255	1.9	953441	236	046559	08947	99599	52
9	953100	233	998243	1.9	954856	235	045144	08976	99596	51
10	954499	232	998232	1.9	956267	234	043733	09005	99594	50
11	8.955894	232	9.998220	1.9	8.957674	234	11.042326	09034	99591	49
12	957284	231	998209	1.9	959075	233	040925	09063	99588	48
13	958670	230	998197	1.9	960473	232	039527	09092	99586	47
14	960052	229	998186	1.9	961866	231	038134	09121	99583	46
15	961429	229	998174	1.9	963255	231	036745	09150	99580	45
16	962801	228	998163	1.9	964639	230	035361	09179	99578	44
17	964170	227	998151	1.9	966019	229	033981	09208	99575	43
18	965534	227	998139	2.0	967394	229	032606	09237	99572	42
19	966893	226	998128	2.0	968766	228	031234	09266	99570	41
20	968249	225	998116	2.0	970133	227	029867	09295	99567	40
21	8.969600	224	9.998104	2.0	8.971496	226	11.028504	09324	99564	39
22	970947	224	998092	2.0	972855	226	027145	09353	99562	38
23	972289	223	998080	2.0	974209	225	025791	09382	99559	37
24	973628	222	998068	2.0	975560	224	024440	09411	99556	36
25	974962	222	998056	2.0	976906	224	023094	09440	99553	35
26	976293	221	998044	2.0	978248	223	021752	09469	99551	34
27	977619	220	998032	2.0	979586	222	020414	09498	99548	33
28	978941	220	998020	2.0	980921	222	019079	09527	99545	32
29	980259	219	998008	2.0	982251	221	017749	09556	99542	31
30	981573	218	997996	2.0	983577	220	016423	09585	99540	30
31	8.982883	218	9.997984	2.0	8.984899	220	11.015101	09614	99537	29
32	984189	217	997972	2.0	986217	219	013783	09642	99534	28
33	985491	216	997959	2.0	987532	218	012468	09671	99531	27
34	986789	216	997947	2.0	988842	218	011158	09700	99528	26
35	988083	215	997935	2.1	990149	217	009851	09729	99526	25
36	989374	214	997922	2.1	991451	216	008549	06758	99523	24
37	990660	214	997910	2.1	992750	216	007250	09787	99520	23
38	991943	213	997897	2.1	994045	215	005955	09816	99517	22
39	993222	212	997885	2.1	995337	215	004663	09845	99514	21
40	994497	212	997872	2.1	996624	214	003376	09874	99511	20
41	8.995768	211	9.997860	2.1	8.997908	213	11.002092	09903	99508	19
42	997036	211	997847	2.1	999188	213	000812	09932	99505	18
43	998299	210	997835	2.1	9.000465	212	10.999535	09961	99503	17
44	999560	209	997822	2.1	001738	211	998262	09990	99500	16
45	9.000816	209	997809	2.1	003007	211	996993	10019	99497	15
46	002069	208	997797	2.1	004272	210	995728	10048	99494	14
47	003318	208	997784	2.1	005534	210	994466	10077	99491	13
48	004563	207	997771	2.1	006792	209	993208	10106	99488	12
49	005805	206	997758	2.1	008047	208	991953	10135	99485	11
50	007044	206	997745	2.1	009298	208	990702	10164	99482	10
51	9.008278	205	9.997732	2.1	9.010546	207	10.989454	10192	99479	9
52	009510	205	997719	2.1	011790	207	988210	10221	99476	8
53	010737	204	997706	2.1	013031	206	686969	10250	99473	7
54	011962	203	997693	2.2	014268	206	985732	10279	99470	6
55	013182	203	997680	2.2	015502	205	984498	10308	99467	5
56	014400	202	997667	2.2	016732	204	983268	10337	99464	4
57	015613	202	997654	2.2	017959	204	983041	10366	99461	3
58	016824	201	997641	2.2	019183	203	980817	10395	99458	2
59	018031	201	997628	2.2	020403	203	979597	10424	99455	1
60	019235		997614		021620		978380	10453	99452	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N. sine.	′

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	9.019235	200	9.997614	2.2	9.021620	202	10.978380	10453	99452	60
1	020435	199	997601	2.2	022834	202	977166	10482	99449	59
2	021632	199	997588	2.2	024044	201	975956	10511	99446	58
3	022825	198	997574	2.2	025251	201	974749	10540	99443	57
4	024016	198	997561	2.2	026455	200	973545	10569	99440	56
5	025203	197	997547	2.2	027655	199	972345	10597	99437	55
6	026386	197	997534	2.3	028852	199	971148	10626	99434	54
7	027567	196	997520	2.3	030046	198	969954	10655	99431	53
8	028744	196	997507	2.3	031237	198	968763	10684	99428	52
9	029918	195	997493	2.3	032425	197	967575	10713	99424	51
10	031089	195	997480	2.3	033609	197	966391	10742	99421	50
11	9.032257	194	9.997466	2.3	9.034791	196	10.965209	10771	99418	49
12	033421	194	997452	2.3	035969	196	964031	10800	99415	48
13	034582	193	997439	2.3	037144	195	962856	10829	99412	47
14	035741	192	997425	2.3	038316	195	961684	10858	99409	46
15	036896	192	997411	2.3	039485	194	960515	10887	99406	45
16	038048	191	997397	2.3	040651	194	959349	10916	99402	44
17	039197	191	997383	2.3	041813	193	958187	10945	99399	43
18	040342	190	997369	2.3	042973	193	957027	10973	99396	42
19	041485	190	997355	2.3	044130	192	955870	11002	99393	41
20	042625	189	997341	2.3	045284	192	954716	11031	99390	40
21	9.043762	189	9.997327	2.4	9.046434	191	10.953566	11060	99386	39
22	044895	180	997313	2.4	047582	191	952418	11089	99383	38
23	046026	188	997299	2.4	048727	190	951273	11118	99380	37
24	047154	187	997285	2.4	049869	190	950131	11147	99377	36
25	048279	187	997271	2.4	051008	189	948992	11176	99374	35
26	049400	186	997257	2.4	052144	189	947856	11205	99370	34
27	050519	186	997242	2.4	053277	188	946723	11234	99367	33
28	051635	185	997228	2.4	054407	188	945593	11263	99364	32
29	052749	185	997214	2.4	055535	187	944465	11291	99360	31
30	053859	184	997199	2.4	056659	187	943341	11320	99357	30
31	9.054966	184	9.997185	2.4	9.057781	186	10.942219	11349	99354	29
32	056071	184	997170	2.4	058900	186	941100	11378	99351	28
33	057172	183	997156	2.4	060016	185	939984	11407	99347	27
34	058271	183	997141	2.4	061130	185	938870	11436	99344	26
35	059367	182	997127	2.4	062240	185	937760	11465	99341	25
36	060460	182	997112	2.4	063348	184	936652	11494	99337	24
37	061551	181	997098	2.4	064453	184	935547	11523	99334	23
38	062639	181	997083	2.5	065556	183	934444	11552	99331	22
39	063724	180	997068	2.5	066655	183	933345	11580	99327	21
40	064806	180	997053	2.5	067752	182	932248	11609	99324	20
41	9.065885	179	9.997039	2.5	9.068846	182	10.931154	11638	99320	19
42	066962	179	997024	2.5	069038	181	930062	11667	99317	18
43	068036	179	997009	2.5	071027	181	928973	11696	99314	17
44	069107	178	996994	2.5	072113	181	927887	11725	99310	16
45	070176	178	996979	2.5	073197	180	926803	11754	99307	15
46	071242	177	996964	2.5	074278	180	925722	11783	99303	14
47	072306	177	996949	2.5	075356	179	924644	11812	99300	13
48	073366	176	996934	2.5	076432	179	923568	11840	99297	12
49	074424	176	996919	2.5	077505	178	922495	11869	99293	11
50	075480	175	996904	2.5	078576	178	921424	11898	99290	10
51	9.076533	175	9.996889	2.5	9.079644	178	10.920356	11927	99286	9
52	077583	175	996874	2.5	080710	177	919290	11956	99283	8
53	078631	174	996858	2.5	081773	177	918227	11985	99279	7
54	079676	174	996843	2.5	082833	176	917167	12014	99276	6
55	080719	173	996828	2.5	083891	176	916109	12043	99272	5
56	081759	173	996812	2.6	084947	175	915053	12071	99269	4
57	082797	172	996797	2.6	086000	175	914000	12100	99265	3
58	083832	172	996782	2.6	087050	175	912950	12129	99262	2
59	084864	172	996766	2.6	088098	174	911902	12158	99258	1
60	085894		996751		089144		910856	12187	99255	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N. sine.	′

83 Degrees.

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	9.085894	171	9.996751	2.6	9.089144	174	10.910856	12187	99255	60
1	086922	171	996735	2.6	090187	173	909813	12216	99251	59
2	087947	170	996720	2.6	091228	173	908772	12245	99248	58
3	088970	170	996704	2.6	092266	173	907734	12274	99244	57
4	089990	170	996688	2.6	093302	172	906698	12302	99240	56
5	091008	169	996673	2.6	094336	172	905664	12331	99237	55
6	092024	169	996657	2.6	095367	171	904633	12360	99233	54
7	093037	168	996641	2.6	096395	171	903605	12389	99230	53
8	094047	168	996625	2.6	097422	171	902578	12418	99226	52
9	095056	168	996610	2.6	098446	170	901554	12447	99222	51
10	096062	167	996594	2.6	099468	170	900532	12476	99219	50
11	9.097065	167	9.996578	2.7	9.100487	169	10.899513	12504	99215	49
12	098066	166	996562	2.7	101504	169	898496	12533	99211	48
13	099065	166	996546	2.7	102519	169	897481	12562	99208	47
14	100062	165	996530	2.7	103532	168	896468	12591	99204	46
15	101056	165	996514	2.7	104542	168	895458	12620	99200	45
16	102048	165	996498	2.7	105550	168	894450	12649	99197	44
17	103037	164	996482	2.7	106556	167	893444	12678	99193	43
18	104025	164	996465	2.7	107559	167	892441	12706	99189	42
19	105010	164	996449	2.7	108560	166	891440	12735	99186	41
20	105992	163	996433	2.7	109559	166	890441	12764	99182	40
21	9.106973	163	9.996417	2.7	9.110556	166	10.889444	12793	99178	39
22	107951	163	996400	2.7	111551	165	888449	12822	99175	38
23	108927	162	996384	2.7	112543	165	887457	12851	99171	37
24	109901	162	996368	2.7	113533	165	886467	12880	99167	36
25	110873	162	996351	2.7	114521	164	885479	12908	99163	35
26	111842	161	996335	2.7	115507	164	884493	12937	99160	34
27	112809	161	996318	2.7	116491	164	883509	12966	99156	33
28	113774	160	996302	2.8	117472	163	882528	12995	99152	32
29	114737	160	996285	2.8	118452	163	881548	13024	99148	31
30	115698	160	996269	2.8	119429	162	880571	13053	99144	30
31	9.116656	159	9.996252	2.8	9.120404	162	10.879596	13081	99141	29
32	117613	159	996235	2.8	121377	162	878623	13110	99137	28
33	118567	159	996219	2.8	122348	161	877652	13139	99133	27
34	119519	158	996202	2.8	123317	161	876683	13168	99129	26
35	120469	158	996185	2.8	124284	161	875716	13197	99125	25
36	121417	158	996168	2.8	125249	160	874751	13226	99122	24
37	122362	157	996151	2.8	126211	160	873789	13254	99118	23
38	123306	157	996134	2.8	127172	160	872828	13283	99114	22
39	124248	157	996117	2.8	128130	159	871870	13312	99110	21
40	125187	156	996100	2.8	129087	159	870913	13341	99106	20
41	9.126125	156	9.996083	2.9	9.130041	159	10.869959	13370	99102	19
42	127060	156	996066	2.9	130994	158	869006	13399	99098	18
43	127993	155	996049	2.9	131944	158	868056	13427	99094	17
44	128925	155	996032	2.9	132893	158	867107	13456	99091	16
45	129854	154	996015	2.9	133839	157	866161	13485	99087	15
46	130781	154	995998	2.9	134784	157	865216	13514	99083	14
47	131706	154	995980	2.9	135726	157	864274	13543	99079	13
48	132630	153	995963	2.9	136667	156	863333	13572	99075	12
49	133551	153	995946	2.9	137605	156	862395	13600	99071	11
50	134470	153	995928	2.9	138542	156	861458	13629	99067	10
51	9.135387	152	9.995911	2.9	9.139476	155	10.860524	13658	99063	9
52	136303	152	995894	2.9	140409	155	859591	13687	99059	8
53	137216	152	995876	2.9	141340	155	858660	13716	99055	7
54	138128	152	995859	2.9	142269	154	857731	13744	99051	6
55	139037	151	995841	2.9	143196	154	856804	13773	99047	5
56	139944	151	995823	2.9	144121	154	855879	13802	99043	4
57	140850	151	995806	2.9	145044	153	854956	13831	99039	3
58	141754	150	995788	2.9	145966	153	854034	13860	99035	2
59	142655	150	995771	2.9	146885	153	853115	13889	99031	1
60	143555		995753		147803		852197	13917	99027	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N. sine.	′

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	9.143555	150	9.995753	3.0	9.147803	153	10.852197	13917	99027	60
1	144453	149	995735	3.0	148718	152	851282	13946	99023	59
2	145349	149	995717	3.0	149632	152	850368	13975	99019	58
3	146243	149	995699	3.0	150544	152	849456	14004	99015	57
4	147136	148	995681	3.0	151454	151	848546	14033	99011	56
5	148026	148	995664	3.0	152363	151	847637	14061	99006	55
6	148915	148	995646	3.0	153269	151	846731	14090	99002	54
7	149802	147	995628	3.0	154174	150	845826	14119	98998	53
8	150686	147	995610	3.0	155077	150	844923	14148	98994	52
9	151569	147	995591	3.0	155978	150	844022	14177	98990	51
10	152451	147	995573	3.0	156877	150	843123	14205	98986	50
11	9.153330	146	9.995555	3.0	9.157775	149	10.842225	14234	98982	49
12	154208	146	995537	3.0	158671	149	841329	14263	98978	48
13	155083	146	995519	3.0	159565	149	840435	14292	98973	47
14	155957	145	995501	3.1	160457	148	839543	14320	98969	46
15	156830	145	995482	3.1	161347	148	838653	14349	98965	45
16	157700	145	995464	3.1	162236	148	837764	14378	98961	44
17	158569	144	995446	3.1	163123	148	836877	14407	98957	43
18	159435	144	995427	3.1	164008	147	835992	14436	98953	42
19	160301	144	995409	3.1	164892	147	835108	14464	98948	41
20	161164	144	995390	3.1	165774	147	834226	14493	98944	40
21	9.162025	143	9.995372	3.1	9.166654	146	10.833346	14522	98940	39
22	162885	143	995353	3.1	167532	146	832468	14551	98936	38
23	163743	143	995334	3.1	168409	146	831591	14580	98931	37
24	164600	142	995316	3.1	169284	145	830716	14608	98927	36
25	165454	142	995297	3.1	170157	145	829843	14637	98923	35
26	166307	142	995278	3.1	171029	145	828971	14666	98919	34
27	167159	142	995260	3.1	171899	145	828101	14695	98914	33
28	168008	141	995241	3.2	172767	144	827233	14723	98910	32
29	168856	141	995222	3.2	173634	144	826366	14752	98906	31
30	169702	141	995203	3.2	174499	144	825501	14781	98902	30
31	9.170547	140	9.995184	3.2	9.175362	144	10.824638	14810	98897	29
32	171389	140	995165	3.2	176224	143	823776	14838	98893	28
33	172230	140	995146	3.2	177084	143	822916	14867	98889	27
34	173070	140	995127	3.2	177942	143	822058	14896	98884	26
35	173908	139	995108	3.2	178799	142	821201	14925	98880	25
36	174744	139	995089	3.2	179655	142	820345	14954	98876	24
37	175578	139	995070	3.2	180508	142	819492	14982	98871	23
38	176411	139	995051	3.2	181360	142	818640	15011	98867	22
39	177242	138	995032	3.2	182211	141	817789	15040	98863	21
40	178072	138	995013	3.2	183059	141	816941	15069	98858	20
41	9.178900	138	9.994993	3.2	9.183907	141	10.816093	15097	98854	19
42	179726	137	994974	3.2	184752	141	815248	15126	98849	18
43	180551	137	994955	3.2	185597	140	814403	15155	98845	17
44	181374	137	994935	3.2	186439	140	813561	15184	98841	16
45	182196	137	994916	3.3	187280	140	812720	15212	98836	15
46	183016	136	994896	3.3	188120	140	811880	15241	98832	14
47	183834	136	994877	3.3	188958	139	811042	15270	98827	13
48	184651	136	994857	3.3	189794	139	810206	15299	98823	12
49	185466	136	994838	3.3	190629	139	809371	15327	98818	11
50	186280	135	994818	3.3	191462	139	808538	15356	98814	10
51	9.187092	135	9.994798	3.3	9.192294	138	10.807706	15385	98809	9
52	187903	135	994779	3.3	193124	138	806876	15414	98805	8
53	188712	135	994759	3.3	193953	138	806047	15442	98800	7
54	189519	134	994739	3.3	194780	138	805220	15471	98796	6
55	190325	134	994719	3.3	195606	137	804394	15500	98791	5
56	191130	134	994700	3.3	196430	137	803570	15529	98787	4
57	191933	134	994680	3.3	197253	137	802747	15557	98782	3
58	192734	133	994660	3.3	198074	137	801926	15586	98778	2
59	193534	133	994640	3.3	198894	136	801106	15615	98773	1
60	194332		994620		199713		800287	15643	98769	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N.sine	′

81 Degrees.

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	9.194332	133	9.994620	3.3	9.199713	136	10.800287	15643	98769	60
1	195129	133	994600	3.3	200529	136	799471	15672	98764	59
2	195925	132	994580	3.3	201345	136	798655	15701	98760	58
3	196719	132	994560	3.4	202159	135	797841	15730	98755	57
4	197511	132	994540	3.4	202971	135	797029	15758	98751	56
5	198302	132	994519	3.4	203782	135	796218	15787	98746	55
6	199091	131	994499	3.4	204592	135	795408	15816	98741	54
7	199879	131	994479	3.4	205400	134	794600	15845	98737	53
8	200666	131	994459	3.4	206207	134	793793	15873	98732	52
9	201451	131	994438	3.4	207013	134	792987	15902	98728	51
10	202234	130	994418	3.4	207817	134	792183	15931	98723	50
11	9.203017	130	9.994397	3.4	9.208619	133	10.791381	15959	98718	49
12	203797	130	994377	3.4	209420	133	790580	15988	98714	48
13	204577	130	994357	3.4	210220	133	789780	16017	98709	47
14	205354	129	994336	3.4	211018	133	788982	16046	98704	46
15	206131	129	994316	3.4	211815	133	788185	16074	98700	45
16	206906	129	994295	3.4	212611	132	787389	16103	98695	44
17	207679	129	994274	3.5	213405	132	786595	16132	98690	43
18	208452	128	994254	3.5	214198	132	785802	16160	98686	42
19	209222	128	994233	3.5	214989	132	735011	16189	98681	41
20	209992	128	994212	3.5	215780	131	784220	16218	98676	40
21	9.210760	128	9.994191	3.5	9.216568	131	10.783432	16246	98671	39
22	211526	127	994171	3.5	217356	131	782644	16275	98667	38
23	212291	127	994150	3.5	218142	131	781858	16304	98662	37
24	213055	127	994129	3.5	218926	130	781074	16333	98657	36
25	213818	127	994108	3.5	219710	130	780290	16361	98652	35
26	214579	127	994087	3.5	220492	130	779508	16390	98648	34
27	215338	126	994066	3.5	221272	130	778728	16419	98643	33
28	216097	126	994045	3.5	222052	130	777948	16447	98638	32
29	216854	126	994024	3.5	222830	129	777170	16476	98633	31
30	217609	126	994003	3.5	223606	129	776394	16505	98629	30
31	9.218363	125	9.993981	3.5	9.224382	129	10.775618	16533	98624	29
32	219116	125	993960	3.5	225156	129	774844	16562	98619	28
33	219868	125	993939	3.5	225929	129	774071	16591	98614	27
34	220618	125	993918	3.5	226700	128	773300	16620	98609	26
35	221367	125	993896	3.6	227471	128	772529	16648	98604	25
36	222115	124	993875	3.6	228239	128	771761	16677	98600	24
37	222861	124	993854	3.6	229007	128	770993	16706	98595	23
38	223606	124	993832	3.6	229773	127	770227	16734	98590	22
39	224349	124	993811	3.6	230539	127	769461	16763	98585	21
40	225092	123	993789	3.6	231302	127	768698	16792	98580	20
41	9.225833	123	9.993768	3.6	9.232065	127	10.767935	16820	98575	19
42	226573	123	993746	3.6	232826	127	767174	16849	98570	18
43	227311	123	993725	3.6	233586	126	766414	16878	98565	17
44	228048	123	993703	3.6	234345	126	765655	16906	98561	16
45	228784	122	993681	3.6	235103	126	764897	16935	98556	15
46	229518	122	993660	3.6	235859	126	764141	16964	98551	14
47	230252	122	993638	3.6	236614	126	763386	16992	98546	13
48	230984	122	993616	3.6	237368	125	762632	17021	98541	12
49	231714	122	993594	3.7	238120	125	761880	17050	98536	11
50	232444	121	993572	3.7	238872	125	761128	17078	98531	10
51	9.233172	121	9.993550	3.7	9.239622	125	10.760378	17107	98526	9
52	233899	121	994528	3.7	240371	125	759629	17136	98521	8
53	234625	121	993506	3.7	241118	124	758882	17164	98516	7
54	235349	120	993484	3.7	241865	124	758135	17193	98511	6
55	236073	120	993462	3.7	242610	124	757390	17222	98506	5
56	236795	120	993440	3.7	243354	124	756646	17250	98501	4
57	237515	120	993418	3.7	244097	124	755903	17279	98496	3
58	238235	120	993396	3.7	244839	123	755161	17308	98491	2
59	238953	119	993374	3.7	245579	123	754421	17336	98486	1
60	239670		993351		246319		753681	17365	98481	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N. sine.	′

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N.sine.	N. cos.	
0	9.239670	119	9.993351	3.7	9.246319	123	10.753681	17365	98481	60
1	240386	119	993329	3.7	247057	123	752943	17393	98476	59
2	241101	119	993307	3.7	247794	123	752206	17422	98471	58
3	241814	119	993285	3.7	248530	122	751470	17451	98466	57
4	242526	118	993262	3.7	249264	122	750736	17479	98461	56
5	243237	118	993240	3.7	249998	122	750002	17508	98455	55
6	243947	118	993217	3.8	250730	122	749270	17537	98450	54
7	244656	118	993195	3.8	251461	122	748539	17565	98445	53
8	245363	118	993172	3.8	252191	121	747809	17594	98440	52
9	246069	117	993149	3.8	252920	121	747080	17623	98435	51
10	246775	117	993127	3.8	253648	121	746352	17651	98430	50
11	9.247478	117	9.993104	3.8	9.254374	121	10.745626	17680	98425	49
12	248181	117	993081	3.8	255100	121	744900	17708	98420	48
13	248883	117	993059	3.8	255824	120	744176	17737	98414	47
14	249583	116	993036	3.8	256547	120	743453	17766	98409	46
15	250282	116	993013	3.8	257269	120	742731	17794	98404	45
16	250980	116	992990	3.8	257990	120	742010	17823	98399	44
17	251677	116	992967	3.8	258710	120	741290	17852	98394	43
18	252373	116	992944	3.8	259429	120	740571	17880	98389	42
19	253067	116	992921	3.8	260146	119	739854	17909	98383	41
20	253761	115	992898	3.8	260863	119	739137	17937	98378	40
21	9.254453	115	9.992875	3.8	9.261578	119	10.738422	17966	98373	39
22	255144	115	992852	3.8	262292	119	737708	17995	98368	38
23	255834	115	992829	3.9	263005	119	736995	18023	98362	37
24	256523	115	992806	3.9	263717	118	736283	18052	98357	36
25	257211	114	992783	3.9	264428	118	735572	18081	98352	35
26	257898	114	992759	3.9	265138	118	734862	18109	98347	34
27	258583	114	992736	3.9	265847	118	734153	18138	98341	33
28	259268	114	992713	3.9	266555	118	733445	18166	98336	32
29	259951	114	992690	3.9	267261	118	732739	18195	98331	31
30	260633	113	992666	3.9	267967	117	732033	18224	98325	30
31	9.261314	113	9.992643	3.9	9.268671	117	10.731329	18252	98320	29
32	261994	113	992619	3.9	269375	117	730625	18281	98315	28
33	262673	113	992596	3.9	270077	117	729923	18309	98310	27
34	263351	113	992572	3.9	270779	117	729221	18338	98304	26
35	264027	113	992549	3.9	271479	116	728521	18367	98299	25
36	264703	112	992525	3.9	272178	116	727822	18395	98294	24
37	265377	112	992501	3.9	272876	116	727124	18424	98288	23
38	266051	112	992478	4.0	273573	116	726427	18452	98283	22
39	266723	112	992454	4.0	274269	116	725731	18481	98277	21
40	267395	112	992430	4.0	274964	116	725036	18509	98272	20
41	9.268065	111	9.992406	4.0	9.275658	115	10.724342	18538	98267	19
42	268734	111	992382	4.0	276351	115	723649	18567	98261	18
43	269402	111	992359	4.0	277043	115	722957	18595	98256	17
44	270069	111	992335	4.0	277734	115	722266	18624	98250	16
45	270735	111	992311	4.0	278424	115	721576	18652	98245	15
46	271400	111	992287	4.0	279113	115	720887	18681	98240	14
47	272064	110	992263	4.0	279801	114	720199	18710	98234	13
48	272726	110	992239	4.0	280488	114	719512	18738	98229	12
49	273388	110	992214	4.0	281174	114	718826	18767	98223	11
50	274049	110	992190	4.0	281858	114	718142	18795	98218	10
51	9.274708	110	9.992166	4.0	9.282542	114	10.717458	18824	98212	9
52	275367	110	992142	4.0	283225	114	716775	18852	98207	8
53	276024	109	992117	4.1	283907	113	716093	18881	98201	7
54	276681	109	992093	4.1	284588	113	715412	18910	98196	6
55	277337	109	992069	4.1	285268	113	714732	18938	98190	5
56	277991	109	992044	4.1	285947	113	714053	18967	98185	4
57	278644	109	992020	4.1	286624	113	713376	18995	98179	3
58	279297	109	991996	4.1	287301	113	712699	19024	98174	2
59	279948	108	991971	4.1	287977	112	712023	19052	98168	1
60	280599		991947		288652		711348	19081	98163	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N.sine.	′

79 Degrees.

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	9.280599	108	9.991947	4.1	9.288652	112	10.711348	19081	98163	60
1	281248	108	991922	4.1	289326	112	710674	19109	98157	59
2	281897	108	991897	4.1	289999	112	710001	19138	98152	58
3	282544	108	991873	4.1	290671	112	709329	19167	98146	57
4	283190	108	991848	4.1	291342	112	708658	19195	98140	56
5	283836	107	991823	4.1	292013	111	707987	19224	98135	55
6	284480	107	991799	4.1	292682	111	707318	19252	98129	54
7	285124	107	991774	4.2	293350	111	706650	19281	98124	53
8	285766	107	991749	4.2	294017	111	705983	19309	98118	52
9	286408	107	991724	4.2	294684	111	705316	19338	98112	51
10	287048	107	991699	4.2	295349	111	704651	19366	98107	50
11	9.287687	106	9.991674	4.2	9.296013	111	10.703987	19395	98101	49
12	288326	106	991649	4.2	296677	110	703323	19423	98096	48
13	288964	106	991624	4.2	297339	110	702661	19452	98090	47
14	289600	106	991599	4.2	298001	110	701999	19481	98084	46
15	290236	106	991574	4.2	298662	110	701338	19509	98079	45
16	290870	106	991549	4.2	299322	110	700678	19538	98073	44
17	291504	105	991524	4.2	299980	110	700020	19566	98067	43
18	292137	105	991498	4.2	300638	109	699362	19595	98061	42
19	292768	105	991473	4.2	301295	109	698705	19623	98056	41
20	293399	105	991448	4.2	301951	109	698049	19652	98050	40
21	9.294029	105	9.991422	4.2	9.302607	109	10.697393	19680	98044	39
22	294658	105	991397	4.2	303261	109	696739	19709	98039	38
23	295286	104	991372	4.3	303914	109	696086	19737	98033	37
24	295913	104	991346	4.3	304567	109	695433	19766	98027	36
25	296539	104	991321	4.3	305218	108	694782	19794	98021	35
26	297164	104	991295	4.3	305869	108	694131	19823	98016	34
27	297788	104	991270	4.3	306519	108	693481	19851	98010	33
28	298412	104	991244	4.3	307168	108	692832	19880	98004	32
29	299034	104	991218	4.3	307815	108	692185	19908	97998	31
30	299655	103	991193	4.3	308463	108	691537	19937	97992	30
31	9.300276	103	9.991167	4.3	9.309109	107	10.690891	19965	97987	29
32	300895	103	991141	4.3	309754	107	690246	19994	97981	28
33	301514	103	991115	4.3	310398	107	689602	20022	97975	27
34	302132	103	991090	4.3	311042	107	688958	20051	97969	26
35	302748	103	991064	4.3	311685	107	688315	20079	97963	25
36	303364	102	991038	4.3	312327	107	687673	20108	97958	24
37	303979	102	991012	4.3	312967	107	687033	20136	97952	23
38	304593	102	990986	4.3	313608	106	686392	20165	97946	22
39	305207	102	990960	4.3	314247	106	685753	20193	97940	21
40	305819	102	990934	4.4	314885	106	685115	20222	97934	20
41	9.306430	102	9 990908	4.4	9.315523	106	10.684477	20250	97928	19
42	307041	102	990882	4.4	316159	106	683841	20279	97922	18
43	307650	101	990855	4.4	316795	106	683205	20307	97916	17
44	308259	101	990829	4.4	317430	106	682570	20336	97910	16
45	308867	101	990803	4.4	318064	105	681936	20364	97905	15
46	309474	101	990777	4.4	318697	105	681303	20393	97899	14
47	310080	101	990750	4.4	319329	105	680671	20421	97893	13
48	310685	101	990724	4.4	319961	105	680039	20450	97887	12
49	311289	100	990697	4.4	320592	105	679408	20478	97881	11
50	311893	100	990671	4.4	321222	105	678778	20507	97875	10
51	9.312495	100	9.990644	4.4	9.321851	105	10.678149	20535	97869	9
52	313097	100	990618	4.4	322479	104	677521	20563	97863	8
53	313698	100	990591	4.4	323106	104	676894	20592	97857	7
54	314297	100	990565	4.4	323733	104	676267	20620	97851	6
55	314897	100	990538	4.4	324358	104	675642	20649	97845	5
56	315495	100	990511	4.5	324983	104	675017	20677	97839	4
57	316092	99	990485	4.5	325607	104	674393	20706	97833	3
58	316689	99	990458	4.5	326231	104	673769	20734	97827	2
59	317284	99	990431	4.5	326853	104	673147	20763	97821	1
60	317879		990404		327475		672525	20791	97815	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N. sine.	′

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	9.317879	99.0	9.990404	4.5	9.327474	103	10.672526	20791	97815	60
1	318473	98.8	990378	4.5	328095	103	671905	20820	97809	59
2	319066	98.7	990351	4.5	328715	103	671285	20848	97803	58
3	319658	98.6	990324	4.5	329334	103	670666	20877	97797	57
4	320249	98.4	990297	4.5	329953	103	670047	20905	97791	56
5	320840	98.3	990270	4.5	330570	103	669430	20933	97784	55
6	321430	98.2	990243	4.5	331187	103	668813	20962	97778	54
7	322019	98.0	990215	4.5	331803	102	668197	20990	97772	53
8	322607	97.9	990188	4.5	332418	102	667582	21019	97766	52
9	323194	97.7	990161	4.5	333033	102	666967	21047	97760	51
10	323780	97.6	990134	4.5	333646	102	666354	21076	97754	50
11	9.324366	97.5	9.990107	4.6	9.334259	102	10.665741	21104	97748	49
12	324950	97.3	990079	4.6	334871	102	665129	21132	97742	48
13	325534	97.2	990052	4.6	335482	102	664518	21161	97735	47
14	326117	97.0	990025	4.6	336093	102	663907	21189	97729	46
15	326700	96.9	989997	4.6	336702	101	663298	21218	97723	45
16	327281	96.8	989970	4.6	337311	101	662689	21246	97717	44
17	327862	96.6	989942	4.6	337919	101	662081	21275	97711	43
18	328442	96.5	989915	4.6	338527	101	661473	21303	97705	42
19	329021	96.4	989887	4.6	339133	101	660867	21331	97698	41
20	329599	96.2	989860	4.6	339739	101	660261	21360	97692	40
21	9.330176	96.1	9.989832	4.6	9.340344	101	10.659656	21388	97686	39
22	330753	96.0	989804	4.6	340948	101	659052	21417	97680	38
23	331329	95.8	989777	4.6	341552	100	658448	21445	97673	37
24	331903	95.7	989749	4.7	342155	100	657845	21474	97667	36
25	332478	95.6	989721	4.7	342757	100	657243	21502	97661	35
26	333051	95.4	989693	4.7	343358	100	656642	21530	97655	34
27	333624	95.3	989665	4.7	343958	100	656042	21559	97648	33
28	334195	95.2	989637	4.7	344558	100	655442	21587	97642	32
29	334766	95.0	989609	4.7	345157	100	654843	21616	97636	31
30	335337	94.9	989582	4.7	345755	100	654245	21644	97630	30
31	9.335906	94.8	9.989553	4.7	9.346353	99.4	10.653647	21672	97623	29
32	336475	94.6	989525	4.7	346949	99.3	653051	21701	97617	28
33	337043	94.5	989497	4.7	347545	99.2	652455	21729	97611	27
34	337610	94.4	989469	4.7	348141	99.1	651859	21758	97604	26
35	338176	94.3	989441	4.7	348735	99.0	651265	21786	97598	25
36	338742	94.1	989413	4.7	349329	98.8	650671	21814	97592	24
37	339306	94.0	989384	4.7	349922	98.7	650078	21843	97585	23
38	339871	93.9	989356	4.7	350514	98.6	649486	21871	97579	22
39	340434	93.7	989328	4.7	351106	98.5	648894	21899	97573	21
40	340996	93.6	989300	4.7	351697	98.3	648303	21928	97566	20
41	9.341558	93.5	9.989271	4.7	9.352287	98.2	10.647713	21956	97560	19
42	342119	93.4	989243	4.7	352876	98.1	647124	21985	97553	18
43	342679	93.2	989214	4.7	353465	98.0	646535	22013	97547	17
44	343239	93.1	989186	4.7	354053	97.9	645947	22041	97541	16
45	343797	93.0	989157	4.7	354640	97.7	645360	22070	97534	15
46	344355	92.9	989128	4.8	355227	97.6	644773	22098	97528	14
47	344912	92.7	989100	4.8	355813	97.5	644187	22126	97521	13
48	345469	92.6	989071	4.8	356398	97.4	643602	22155	97515	12
49	346024	92.5	989042	4.8	356982	97.3	643018	22183	97508	11
50	346579	92.4	989014	4.8	357566	97.1	642434	22212	97502	10
51	9.347134	92.2	9.988985	4.8	9.358149	97.0	10.641851	22240	97496	9
52	347687	92.1	988956	4.8	358731	96.9	641269	22268	97489	8
53	348240	92.0	988927	4.8	359313	96.8	640687	22297	97483	7
54	348792	91.9	988898	4.8	359893	96.7	640107	22325	97476	6
55	349343	91.7	988869	4.8	360474	96.6	639526	22353	97470	5
56	349893	91.6	988840	4.8	361053	96.5	638947	22382	97463	4
57	350443	91.5	988811	4.9	361632	96.3	638368	22410	97457	3
58	350992	91.4	988782	4.9	362210	96.2	637790	22438	97450	2
59	351540	91.3	988753	4.9	362787	96.1	637213	22467	97444	1
60	352088		988724		363364		636636	22495	97437	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N.sine.	′

77 Degrees.

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine	N. cos.	
0	9.352088	91.1	9.988724	4.9	9.363364	96.0	10.636636	22495	97437	60
1	352635	91.0	988695	4.9	363940	95.9	636060	22523	97430	59
2	353181	90.9	988666	4.9	364515	95.8	635485	22552	97424	58
3	353726	90.8	988636	4.9	365090	95.7	634910	22580	97417	57
4	354271	90.7	988607	4.9	365664	95.5	634336	22608	97411	56
5	354815	90.5	988578	4.9	366237	95.4	633763	22637	97404	55
6	355358	90.4	988548	4.9	366810	95.3	633190	22665	97398	54
7	355901	90.3	988519	4.9	367382	95.2	632618	22693	97391	53
8	356443	90.2	988489	4.9	367953	95.1	632047	22722	97384	52
9	356984	90.1	988460	4.9	368524	95.0	631476	22750	97378	51
10	357524	89.9	988430	4.9	369094	94.9	630906	22778	97371	50
11	9.358064	89.8	9.988401	4.9	9.369663	94.8	10.630337	22807	97365	49
12	358603	89.7	988371	4.9	370232	94.6	629768	22835	97358	48
13	359141	89.6	988342	4.9	370799	94.5	629201	22863	97351	47
14	359678	89.5	988312	5.0	371367	94.4	628633	22892	97345	46
15	360215	89.3	988282	5.0	371933	94.3	628067	22920	97338	45
16	360752	89.2	988252	5.0	372499	94.2	627501	22948	97331	44
17	361287	89.1	988223	5.0	373064	94.1	626936	22977	97325	43
18	361822	89.0	988193	5.0	373629	94.0	626371	23005	97318	42
19	362356	88.9	988163	5.0	374193	93.9	625807	23033	97311	41
20	362889	88.8	988133	5.0	374756	93.8	625244	23062	97304	40
21	9.363422	88.7	9.988103	5.0	9.375319	93.7	10.624681	23090	97298	39
22	363954	88.5	988073	5.0	375881	93.5	624119	23118	97291	38
23	364485	88.4	988043	5.0	376442	93.4	623558	23146	97284	37
24	365016	88.3	988013	5.0	377003	93.3	622997	23175	97278	36
25	365546	88.2	987983	5.0	377563	93.2	622437	23203	97271	35
26	366075	88.1	987953	5.0	378122	93.1	621878	23231	97264	34
27	366604	88.0	987922	5.0	378681	93.0	621319	23260	97257	33
28	367131	87.9	987892	5.0	379239	92.9	620761	23288	97251	32
29	367659	87.7	987862	5.0	379797	92.8	620203	23316	97244	31
30	368185	87.6	987832	5.1	380354	92.7	619646	23345	97237	30
31	9.368711	87.5	9.987801	5.1	9.380910	92.6	10.619090	23373	97230	29
32	369236	87.4	987771	5.1	381466	92.5	618534	23401	97223	28
33	369761	87.3	987740	5.1	382020	92.4	617980	23429	97217	27
34	370285	87.2	987710	5.1	382575	92.3	617425	23458	97210	26
35	370808	87.1	987679	5.1	383129	92.2	616871	23486	97203	25
36	371330	87.0	987649	5.1	383682	93.1	616318	23514	97196	24
37	371852	86.9	987618	5.1	384234	92.0	615766	23542	97189	23
38	372373	86.7	987588	5.1	384786	91.9	615214	23571	97182	22
39	372894	86.6	987557	5.1	385337	91.8	614663	23599	97176	21
40	373414	86.5	987526	5.1	385888	91.7	614112	23627	97169	20
41	9.373933	86.4	9.987496	5.1	9.386438	91.5	10.613562	23656	97162	19
42	374452	86.3	987465	5.1	386987	91.4	613013	23684	97155	18
43	374970	86.2	987434	5.1	387536	91.3	612464	23712	97148	17
44	375487	86.1	987403	5.2	388084	91.2	611916	23740	97141	16
45	376003	86.0	987372	5.2	388631	91.1	611369	23769	97134	15
46	376519	85.9	987341	5.2	389178	91.0	610822	23797	97127	14
47	377035	85.8	987310	5.2	389724	90.9	610276	23825	97120	13
48	377549	85.7	987279	5.2	390270	90.8	609730	23853	97113	12
49	378063	85.6	987248	5.2	390815	90.7	609185	23882	97106	11
50	378577	85.4	987217	5.2	391360	90.6	608640	23910	97100	10
51	9.379089	85.3	9.987186	5.2	9.391903	90.5	10.608097	23938	97093	9
52	379601	85.2	987155	5.2	392447	90.4	607553	23966	97086	8
53	380113	85.1	987124	5.2	392989	90.3	607011	23995	97079	7
54	380624	85.0	987092	5.2	393531	90.2	606469	24023	97072	6
55	381134	84.9	987061	5.2	394073	90.1	605927	24051	97065	5
56	381643	84.8	987030	5.2	394614	90.0	605386	24079	97058	4
57	382152	84.7	986998	5.2	395154	89.9	604846	24108	97051	3
58	382661	84.6	986967	5.2	395694	89.8	604306	24136	97044	2
59	383168	84.5	986936	5.2	396233	89.7	603767	24164	97037	1
60	383675		986904		396771		603229	24192	97030	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N. sine.	′

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	9.383675	84.4	9.986904	5.2	9.396771	89.6	10.603229	24192	97030	60
1	384182	84.3	986873	5.3	397309	89.6	602691	24220	97023	59
2	384687	84.2	986841	5.3	397846	89.5	602154	24249	97015	58
3	385192	84.1	986809	5.3	398383	89.4	601617	24277	97008	57
4	385697	84.0	986778	5.3	398919	89.3	601081	24305	97001	56
5	386201	83.9	986746	5.3	399455	89.2	600545	24333	96994	55
6	386704	83.8	986714	5.3	399990	89.1	600010	24362	96987	54
7	387207	83.7	986683	5.3	400524	89.0	599476	24390	96980	53
8	387709	83.6	986651	5.3	401058	88.9	598942	24418	96973	52
9	388210	83.5	986619	5.3	401591	88.8	598409	24446	96966	51
10	388711	83.4	986587	5.3	402124	88.7	597876	24474	96959	50
11	9.389211	83.3	9.986555	5.3	9.402656	88.6	10.597344	24503	96952	49
12	389711	83.2	986523	5.3	403187	88.5	596813	24531	96945	48
13	390210	83.1	986491	5.3	403718	88.4	596282	24559	96937	47
14	390708	83.0	986459	5.3	404249	88.3	595751	24587	96930	46
15	391206	82.8	986427	5.3	404778	88.2	595222	24615	96923	45
16	391703	82.7	986395	5.3	405308	88.1	594692	24644	96916	44
17	392199	82.6	986363	5.4	405836	88.0	594164	24672	96909	43
18	392695	82.5	986331	5.4	406364	87.9	593636	24700	96902	42
19	393191	82.4	986299	5.4	406892	87.8	593108	24728	96894	41
20	393685	82.3	986266	5.4	407419	87.7	592581	24756	96887	40
21	9.394179	82.2	9.986234	5.4	9.407945	87.6	10.592055	24784	96880	39
22	394673	82.1	986202	5.4	408471	87.5	591529	24813	96873	38
23	395166	82.0	986169	5.4	408997	87.4	591003	24841	96866	37
24	395658	81.9	986137	5.4	409521	87.4	590479	24869	96858	36
25	396150	81.8	986104	5.4	410045	87.3	589955	24897	96851	35
26	396641	81.7	986072	5.4	410569	87.2	589431	24925	96844	34
27	397132	81.7	986039	5.4	411092	87.1	588908	24954	96837	33
28	397621	81.6	986007	5.4	411615	87.0	588385	24982	96829	32
29	398111	81.5	985974	5.4	412137	86.9	587863	25010	96822	31
30	398600	81.4	985942	5.4	412658	86.8	587342	25038	96815	30
31	9.399088	81.3	9.985909	5.5	9.413179	86.7	10.586821	25066	96807	29
32	399575	81.2	985876	5.5	413699	86.6	586301	25094	96800	28
33	400062	81.1	985843	5.5	414219	86.5	585781	25122	96793	27
34	400549	81.0	985811	5.5	414738	86.4	585262	25151	96786	26
35	401035	80.9	985778	5.5	415257	86.4	584743	25179	96778	25
36	401520	80.8	985745	5.5	415775	86.3	584225	25207	96771	24
37	402005	80.7	985712	5.5	416293	86.2	583707	25235	96764	23
38	402489	80.6	985679	5.5	416810	86.1	583190	25263	96756	22
39	402972	80.5	985646	5.5	417326	86.0	582674	25291	96749	21
40	403455	80.4	985613	5.5	417842	85.9	582158	25320	96742	20
41	9.403938	80.3	9.985580	5.5	9.418358	85.8	10.581642	25348	96734	19
42	404420	80.2	985547	5.5	418873	85.7	581127	25376	96727	18
43	404901	80.1	985514	5.5	419387	85.6	580613	25404	96719	17
44	405382	80.0	985480	5.5	419901	85.5	580099	25432	96712	16
45	405862	79.9	985447	5.5	420415	85.5	579585	25460	96705	15
46	406341	79.8	985414	5.6	420927	85.4	579073	25488	96697	14
47	406820	79.7	985380	5.6	421440	85.3	578560	25516	96690	13
48	407299	79.6	985347	5.6	421952	85.2	578048	25545	96682	12
49	407777	79.5	985314	5.6	422463	85.1	577537	25573	96675	11
50	408254	79.4	985280	5.6	422974	85.0	577026	25601	96667	10
51	9.408731	79.4	9.985247	5.6	9.423484	84.9	10.576516	25629	96660	9
52	409207	79.3	985213	5.6	423993	84.8	576007	25657	96653	8
53	409682	79.2	985180	5.6	424503	84.8	575497	25685	96645	7
54	410157	79.1	985146	5.6	425011	84.7	574989	25713	96638	6
55	410632	79.0	985113	5.6	425519	84.6	574481	25741	96630	5
56	411106	78.9	985079	5.6	426027	84.5	573973	25766	96623	4
57	411579	78.8	985045	5.6	426534	84.4	573466	25798	96615	3
58	412052	78.7	985011	5.6	427041	84.3	572959	25826	96608	2
59	412524	78.6	984978	5.6	427547	84.3	572453	25854	96600	1
60	412996		984944		428052		571948	25882	96593	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N. sine.	′

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	9.412996	78.5	9.984944	5.7	9.428052	84.2	10.571948	25882	96593	60
1	413467	78.4	984910	5.7	428557	84.1	571443	25910	96585	59
2	413938	78.3	984876	5.7	429062	84.0	570938	25938	96578	58
3	414408	78.3	984842	5.7	429566	83.9	570434	25966	96570	57
4	414878	78.2	984808	5.7	430070	83.8	569930	25994	96562	56
5	415347	78.1	984774	5.7	430573	83.8	569427	26022	96555	55
6	415815	78.0	984740	5.7	431075	83.7	568925	26050	96547	54
7	416283	77.9	984706	5.7	431577	83.6	568423	26079	96540	53
8	416751	77.8	984672	5.7	432079	83.5	567921	26107	96532	52
9	417217	77.7	984637	5.7	432580	83.4	567420	26135	96524	51
10	417684	77.6	984603	5.7	433080	83.3	566920	26163	96517	50
11	9.418150	77.5	9.984569	5.7	9.433580	83.2	10.566420	26191	96509	49
12	418615	77.4	984535	5.7	434080	83.2	565920	26219	96502	48
13	419079	77.3	984500	5.7	434579	83.1	565421	26247	96494	47
14	419544	77.3	984466	5.7	435078	83.0	564922	26275	96486	46
15	420007	77.2	984432	5.8	435576	82.9	564424	26303	96479	45
16	420470	77.1	984397	5.8	436073	82.8	563927	26331	96471	44
17	420933	77.0	984363	5.8	436570	82.8	563430	26359	96463	43
18	421395	76.9	984328	5.8	437067	82.7	562933	26387	96456	42
19	421857	76.8	984294	5.8	437563	82.6	562437	26415	96448	41
20	422318	76.7	984259	5.8	438059	82.5	561941	26443	96440	40
21	9.422778	76.7	9.984224	5.8	9.438554	82.4	10.561446	26471	96433	39
22	423238	76.6	984190	5.8	439048	82.3	560952	26500	96425	38
23	423697	76.5	984155	5.8	439543	82.3	560457	26528	96417	37
24	424156	76.4	984120	5.8	440036	82.2	559964	26556	96410	36
25	424615	76.3	984085	5.8	440529	82.1	559471	26584	96402	35
26	425073	76.2	984050	5.8	441022	82.0	558978	26612	96394	34
27	425530	76.1	984015	5.8	441514	81.9	558486	26640	96386	33
28	425987	76.0	983981	5.8	442006	81.9	557994	26668	96379	32
29	426443	76.0	983946	5.8	442497	81.8	557503	26696	96371	31
30	426899	75.9	983911	5.8	442988	81.7	557012	26724	96363	30
31	9.427354	75.8	9.983875	5.8	9.443479	81.6	10.556521	26752	96355	29
32	427809	75.7	983840	5.9	443968	81.6	556032	26780	96347	28
33	428263	75.6	983805	5.9	444458	81.5	555542	26808	96340	27
34	428717	75.5	983770	5.9	444947	81.4	555053	26836	96332	26
35	429170	75.4	983735	5.9	445435	81.3	554565	26864	96324	25
36	429623	75.3	983700	5.9	445923	81.2	554077	26892	96316	24
37	430075	75.2	983664	5.9	446411	81.2	553589	26920	96308	23
38	430527	75.2	983629	5.9	446898	81.1	553102	26948	96301	22
39	430978	75.1	983594	5.9	447384	81.0	552616	26976	96293	21
40	431429	75.0	983558	5.9	447870	80.9	552130	27004	96285	20
41	9.431879	74.9	9.983523	5.9	9.448356	80.9	10.551644	27032	96277	19
42	432329	74.9	983487	5.9	448841	80.8	551159	27060	96269	18
43	432778	74.8	983452	5.9	449326	80.7	550674	27088	96261	17
44	433226	74.7	983416	5.9	449810	80.6	550190	27116	96253	16
45	433675	74.6	983381	5.9	450294	80.6	549706	27144	96246	15
46	434122	74.5	983345	5.9	450777	80.5	549223	27172	96238	14
47	434569	74.4	983309	5.9	451260	80.4	548740	27200	96230	13
48	435016	74.4	983273	6.0	451743	80.3	548257	27228	96222	12
49	435462	74.3	983238	6.0	452225	80.2	547775	27256	96214	11
50	435908	74.2	983202	6.0	452706	80.2	547294	27284	96206	10
51	9.436353	74.1	9.983166	6.0	9.453187	80.1	10.546813	27312	96198	9
52	436798	74.0	983130	6.0	453668	80.0	546332	27340	96190	8
53	437242	74.0	983094	6.0	454148	79.9	545852	27368	96182	7
54	437686	73.9	983058	6.0	454628	79.9	545372	27396	96174	6
55	438129	73.8	983022	6.0	455107	79.8	544893	27424	96166	5
56	438572	73.7	982986	6.0	455586	79.7	544414	27452	96158	4
57	439014	73.6	982950	6.0	456064	79.6	543936	27480	96150	3
58	439456	73.6	982914	6.0	456542	79.6	543458	27508	96142	2
59	439897	73.5	982878	6.0	457019	79.5	542981	27536	96134	1
60	440338		982842		457496		542504	27564	96126	0
′	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N. sine.	′

74 Degrees.

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	9.440338	73.4	9.982842	6.0	9.457496	79.4	10.542504	27564	96126	60
1	440778	73.3	982805	6.0	457973	79.3	542027	27592	96118	59
2	441218	73.2	982769	6.1	458449	79.3	541551	27620	96110	58
3	441658	73.1	982733	6.1	458925	79.2	541075	27648	96102	57
4	442096	73.1	982696	6.1	459400	79.1	540600	27676	96094	56
5	442535	73.0	982660	6.1	459875	79.0	540125	27704	96086	55
6	442973	72.9	982624	6.1	460349	79.0	539651	27731	96078	54
7	443410	72.8	982587	6.1	460823	78.9	539177	27759	96070	53
8	443847	72.7	982551	6.1	461297	78.8	538703	27787	96062	52
9	444284	72.7	982514	6.1	461770	78.9	538230	27815	96054	51
10	444720	72.6	982477	6.1	462242	78.7	537758	27843	96046	50
11	9.445155	72.5	9.982441	6.1	9.462714	78.6	10.537286	27871	96037	49
12	445590	72.4	982404	6.1	463186	78.5	536814	27899	96029	48
13	446025	72.3	982367	6.1	463658	78.5	536342	27927	96021	47
14	446459	72.3	982331	6.1	464129	78.4	535871	27955	96013	46
15	446893	72.2	982294	6.1	464599	78.3	535401	27983	96005	45
16	447326	72.1	982257	6.1	465069	78.3	534931	28011	95997	44
17	447759	72.0	982220	6.2	465539	78.2	534461	28039	95989	43
18	448191	72.0	982183	6.2	466008	78.1	533992	28067	95981	42
19	448623	71.9	982146	6.2	466476	78.0	533524	28095	95972	41
20	449054	71.8	982109	6.2	466945	78.0	533055	28123	95964	40
21	9.449485	71.7	9.982072	6.2	9.467413	77.9	10.532587	28150	95956	39
22	449915	71.6	982035	6.2	467880	77.8	532120	28178	95948	38
23	450345	71.6	981998	6.2	468347	77.8	531653	28206	95940	37
24	450775	71.5	981961	6.2	468814	77.7	531186	28234	95931	36
25	451204	71.4	981924	6.2	469280	77.6	530720	28262	95923	35
26	451632	71.3	981886	6.2	469746	77.5	530254	28290	95915	34
27	452060	71.3	981849	6.2	470211	77.5	529789	28318	95907	33
28	452488	71.2	981812	6.2	470676	77.4	529324	28346	95898	32
29	452915	71.1	981774	6.2	471141	77.3	528859	28374	95890	31
30	453342	71.0	981737	6.2	471605	77.3	528395	28402	95882	30
31	9.453768	71.0	9.981699	6.3	9.472068	77.2	10.527932	28429	95874	29
32	454194	70.9	981662	6.3	472532	77.1	527468	28457	95865	28
33	454619	70.8	981625	6.3	472995	77.1	527005	28485	95857	27
34	455044	70.7	981587	6.3	473457	77.0	526543	28513	95849	26
35	455469	70.7	981549	6.3	473919	76.9	526081	28541	95841	25
36	455893	70.6	981512	6.3	474381	76.9	525619	28569	95832	24
37	456316	70.5	981474	6.3	474842	76.8	525158	28597	95824	23
38	456739	70.4	981436	6.3	475303	76.7	524697	28625	95816	22
39	457162	70.4	981399	6.3	475763	76.7	524237	28652	95807	21
40	457584	70.3	981361	6.3	476223	76.6	523777	28680	95799	20
41	9.458006	70.2	9.981323	6.3	9.476683	76.5	10.523317	28708	95791	19
42	458427	70.1	981285	6.3	477142	76.5	522858	28736	95782	18
43	458848	70.1	981247	6.3	477601	76.4	522399	28764	95774	17
44	459268	70.0	981209	6.3	478059	76.3	521941	28792	95766	16
45	459688	69.9	981171	6.3	478517	76.3	521483	28820	95757	15
46	460108	69.8	981133	6.4	478975	76.2	521025	28847	95749	14
47	460527	69.8	981095	6.4	479432	76.1	520568	28875	95740	13
48	460946	69.7	981057	6.4	479889	76.1	520111	28903	95732	12
49	461364	69.6	981019	6.4	480345	76.0	519655	28931	95724	11
50	461782	69.5	980981	6.4	480801	75.9	519199	28959	95715	10
51	9.462199	69.5	9.980942	6.4	9.481257	75.9	10.518743	28987	95707	9
52	462616	69.4	980904	6.4	481712	75.8	518288	29015	95698	8
53	463032	69.3	980866	6.4	482167	75.7	517833	29042	95690	7
54	463448	69.3	980827	6.4	482621	75.7	517379	29070	95681	6
55	463864	69.2	980789	6.4	483075	75.6	516925	29098	95673	5
56	464279	69.1	980750	6.4	483529	75.5	516471	29126	95664	4
57	464694	69.0	980712	6.4	483982	75.5	516018	29154	95656	3
58	465108	69.0	980673	6.4	484435	75.4	515565	29182	95647	2
59	465522	68.9	980635	6.4	484887	75.3	515113	29209	95639	1
60	465935		980596		485339		514661	29247	95630	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N. sine.	′

73 Degrees.

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	9.465935	68.8	9.980596	6.4	9.485339	75.3	10.514661	29237	95630	60
1	466348	68.8	980558	6.4	485791	75.2	514209	29265	95622	59
2	466761	68.7	980519	6.5	486242	75.1	513758	29293	95613	58
3	467173	68.6	980480	6.5	486693	75.1	513307	29321	95605	57
4	467585	68.5	980442	6.5	487143	75.0	512857	29348	95596	56
5	467996	68.5	980403	6.5	487593	74.9	512407	29376	95588	55
6	468407	68.4	980364	6.5	488043	74.9	511957	29404	95579	54
7	468817	68.3	980325	6.5	488492	74.8	511508	29432	95571	53
8	469227	68.3	980286	6.5	488941	74.7	511059	29460	95562	52
9	469637	68.2	980247	6.5	489390	74.7	510610	29487	95554	51
10	470046	68.1	980208	6.5	489838	74.6	510162	29515	95545	50
11	9.470455	68.0	9.980169	6.5	9.490286	74.6	10.509714	29543	95536	49
12	470863	68.0	980130	6.5	490733	74.5	509267	29571	95528	48
13	471271	67.9	980091	6.5	491180	74.4	508820	29599	95519	47
14	471679	67.8	980052	6.5	491627	74.4	508373	29626	95511	46
15	472086	67.8	980012	6.5	492073	74.3	507927	29654	95502	45
16	472492	67.7	979973	6.5	492519	74.3	507481	29682	95493	44
17	472898	67.6	979934	6.6	492965	74.2	507035	29710	95485	43
18	473304	67.6	979895	6.6	493410	74.1	506590	29737	95476	42
19	473710	67.5	979855	6.6	493854	74.0	506146	29765	95467	41
20	474115	67.4	979816	6.6	494299	74.0	505701	29793	95459	40
21	9.474519	67.4	9.979776	6.6	9.494743	74.0	10.505257	29821	95450	39
22	474923	67.3	979737	6.6	495186	73.9	504814	29849	95441	38
23	475327	67.2	979697	6.6	495630	73.8	504370	29876	95433	37
24	475730	67.2	979658	6.6	496073	73.7	503927	29904	95424	36
25	476133	67.1	979618	6.6	496515	73.7	503485	29932	95415	35
26	476536	67.0	979579	6.6	496957	73.6	503043	29960	95407	34
27	476938	66.9	979539	6.6	497399	73.6	502601	29987	95398	33
28	477340	66.9	979499	6.6	497841	73.5	502159	30015	95389	32
29	477741	66.8	979459	6.6	468282	73.4	501718	30043	95380	31
30	478142	66.7	979420	6.6	498722	73.4	501278	30071	95372	30
31	9.478542	66.7	9.979380	6.6	9.499163	73.3	10.500837	30098	95363	29
32	478942	66.6	979340	6.6	499603	73.3	500397	30126	95354	28
33	479342	66.5	979300	6.7	500042	73.2	499958	30154	95345	27
34	479741	66.5	979260	6.7	500481	73.1	499519	30182	95337	26
35	480140	66.4	979220	6.7	500920	73.1	499080	30209	95328	25
36	480539	66.3	979180	6.7	501359	73.0	498641	30237	95319	24
37	480937	66.3	979140	6.7	501797	73.0	498203	30265	95310	23
38	481334	66.2	979100	6.7	502235	72.9	497765	30292	95301	22
39	481731	66.1	979059	6.7	502672	72.8	497328	30320	95293	21
40	482128	66.1	979019	6.7	503109	72.8	496891	30348	95284	20
41	9.482525	66.0	9.978979	6.7	9.503546	72.7	10.496454	30376	95275	19
42	482921	65.9	978939	6.7	503982	72.7	496018	30403	95266	18
43	483316	65.9	978898	6.7	504418	72.6	495582	30431	95257	17
44	483712	65.8	978858	6.7	504854	72.5	495146	30459	95248	16
45	484107	65.7	978817	6.7	505289	72.5	494711	30486	95240	15
46	484501	65.7	978777	6.7	505724	72.4	494276	30514	95231	14
47	484895	65.6	978736	6.7	506159	72.4	493841	30542	95222	13
48	485289	65.5	978696	6.8	506593	72.3	493407	30570	95213	12
49	485682	65.5	978655	6.8	507027	72.2	492973	30597	95204	11
50	486075	65.4	978615	6.8	507460	72.2	492540	30625	95195	10
51	9.486467	66.3	9.978574	6.8	9.507893	72.1	10.492107	30653	95186	9
52	486860	65.3	978533	6.8	508326	72.1	491674	30680	95177	8
53	487251	65.2	978493	6.8	508759	72.0	491241	30708	95168	7
54	487643	65.1	978452	6.8	509191	71.9	490809	30736	95159	6
55	488034	65.1	978411	6.8	509622	71.9	490378	30763	95150	5
56	488424	65.0	978370	6.8	510054	71.8	489946	30791	95142	4
57	488814	65.0	978329	6.8	510485	71.8	489515	30819	95133	3
58	489204	64.9	978288	6.8	510916	71.7	489084	30846	95124	2
59	489593	64.8	978247	6.8	511346	71.6	488654	30874	95115	1
60	489982		978206		511776		488224	30902	95106	0
	Cosine.	D. 10″	Sine.		Cotang.	D. 10″	Tang.	N. cos.	N. sine.	′

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	9.489982	64.8	9.978206	6.8	9.511776	71.6	10.488224	30902	95106	60
1	490371	64.8	978165	6.8	512206	71.6	487794	30929	95097	59
2	490759	64.7	978124	6.8	512635	71.5	487365	30957	95088	58
3	491147	64.6	978083	6.9	513064	71.4	486936	30985	95079	57
4	491535	64.6	978042	6.9	513493	71.4	486507	31012	95070	56
5	491922	64.5	978001	6.9	513921	71.3	486079	31040	95061	55
6	492308	64.4	977959	6.9	514349	71.3	485651	31068	95052	54
7	492695	64.4	977918	6.9	514777	71.2	485223	31095	95043	53
8	493081	64.3	977877	6.9	515204	71.2	484796	31123	95033	52
9	493466	64.2	977835	6.9	515631	71.1	484369	31151	95024	51
10	493851	64.2	977794	6.9	516057	71.0	483943	31178	95015	50
11	9.494236	64.1	9.977752	6.9	9.516484	71.0	10.483516	31206	95006	49
12	494621	64.1	977711	6.9	516910	70.9	483090	31233	94997	48
13	495005	64.0	977669	6.9	517335	70.9	482665	31261	94988	47
14	495388	63.9	977628	6.9	517761	70.8	482239	31289	94979	46
15	495772	63.9	977586	6.9	518185	70.8	481815	31316	94970	45
16	496154	63.8	977544	7.0	518610	70.7	481390	31344	94961	44
17	496537	63.7	977503	7.0	519034	70.6	480966	31372	94952	43
18	496919	63.7	977461	7.0	519458	70.6	480542	31399	94943	42
19	497301	63.6	977419	7.0	519882	70.5	480118	31427	94933	41
20	497682	63.6	977377	7.0	520305	70.5	479695	31454	94924	40
21	9.498064	63.5	9.977335	7.0	9.520728	70.4	10.479272	31482	94915	39
22	498444	63.4	977293	7.0	521151	70.3	478849	31510	94906	38
23	498825	63.4	977251	7.0	521573	70.3	478427	31537	94897	37
24	499204	63.3	977209	7.0	521995	70.3	478005	31565	94888	36
25	499584	63.2	977167	7.0	522417	70.2	477583	31593	94878	35
26	499963	63.2	977125	7.0	522838	70.2	477162	31620	94869	34
27	500342	63.1	977083	7.0	523259	70.1	476741	31648	94860	33
28	500721	63.1	977041	7.0	523680	70.1	476320	31675	94851	32
29	501099	63.0	976999	7.0	524100	70.0	475900	31703	94842	31
30	501476	62.9	976957	7.0	524520	69.9	475480	31730	94832	30
31	9.501854	62.9	9.976914	7.0	9.524939	69.9	10.475061	31758	94823	29
32	502231	62.8	976872	7.1	525359	69.8	474641	31786	94814	28
33	502607	62.8	976830	7.1	525778	69.8	474222	31813	94805	27
34	502984	62.7	976787	7.1	526197	69.7	473803	31841	94795	26
35	503360	62.6	976745	7.1	526615	69.7	473385	31868	94786	25
36	503735	62.6	976702	7.1	527033	69.6	472967	31896	94777	24
37	504110	62.5	976660	7.1	527451	69.6	472549	31923	94768	23
38	504485	62.5	976617	7.1	527868	69.5	472132	31951	94758	22
39	504860	62.4	976574	7.1	528285	69.5	471715	31979	94749	21
40	505234	62.3	976532	7.1	528702	69.4	471298	32006	94740	20
41	9.505608	62.3	9.976489	7.1	9.529119	69.3	10.470881	32034	94730	19
42	505981	62.2	976446	7.1	529535	69.3	470465	32061	94721	18
43	506354	62.2	976404	7.1	529950	69.3	470050	32089	94712	17
44	506727	62.1	976361	7.1	530366	69.2	469634	32116	94702	16
45	507099	62.0	976318	7.1	530781	69.1	469219	32144	94693	15
46	507471	62.0	976275	7.1	531196	69.1	468804	32171	94684	14
47	507843	61.9	976232	7.2	531611	69.0	468389	32199	94674	13
48	508214	61.9	976189	7.2	532025	69.0	467975	32227	94665	12
49	508585	61.8	976146	7.2	532439	68.9	467561	32250	94656	11
50	508956	61.8	976103	7.2	532853	68.9	467147	32282	94646	10
51	9.509326	61.7	9.976060	7.2	9.533266	68.8	10.466734	32309	94637	9
52	509696	61.6	976017	7.2	533679	68.8	466321	32337	94627	8
53	510065	61.6	975974	7.2	534092	68.7	465908	32364	94618	7
54	510434	61.5	975930	7.2	534504	68.7	465496	32392	94609	6
55	510803	61.5	975887	7.2	534916	68.6	465084	32419	94599	5
56	511172	61.4	975844	7.2	535328	68.6	464672	32447	94590	4
57	511540	61.3	975800	7.2	535739	68.5	464261	32474	94580	3
58	511907	61.3	975757	7.2	536150	68.5	463850	32502	94571	2
59	512275	61.2	975714	7.2	536561	68.4	463439	32529	94561	1
60	512642		975670		536972		463028	32557	94552	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N.sine.	′

71 Degrees.

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	9.512642	61.2	9.975670	7.3	9.536972	68.4	10.463028	32557	94552	60
1	513009	61.1	975627	7.3	537382	68.3	462618	32584	94542	59
2	513375	61.1	975583	7.3	537792	68.3	462208	32612	94533	58
3	513741	61.0	975539	7.3	538202	68.2	461798	32639	94523	57
4	514107	60.9	975496	7.3	538611	68.2	461389	32667	94514	56
5	514472	60.9	975452	7.3	539020	68.1	460980	32694	94504	55
6	514837	60.8	975408	7.3	539429	68.1	460571	32722	94495	54
7	515202	60.8	975365	7.3	539837	68.0	460163	32749	94485	53
8	515566	60.7	975321	7.3	540245	68.0	459755	32777	94476	52
9	515930	60.7	975277	7.3	540653	67.9	459347	32804	94466	51
10	516294	60.6	975233	7.3	541061	67.9	458939	32832	94457	50
11	9.516657	60.5	9.975189	7.3	9.541468	67.8	10.458532	32859	94447	49
12	517020	60.5	975145	7.3	541875	67.8	458125	32887	94438	48
13	517382	60.4	975101	7.3	542281	67.7	457719	32914	94428	47
14	517745	60.4	975057	7.3	542688	67.7	457312	32942	94418	46
15	518107	60.3	975013	7.3	543094	67.6	456906	32969	94409	45
16	518468	60.3	974969	7.4	543499	67.6	456501	32997	94399	44
17	518829	60.2	974925	7.4	543905	67.5	456095	33024	94390	43
18	519190	60.1	974880	7.4	544310	67.5	455690	33051	94380	42
19	519551	60.1	974836	7.4	544715	67.4	455285	33079	94370	41
20	519911	60.0	974792	7.4	545119	67.4	454881	33106	94361	40
21	9.520271	60.0	9.974748	7.4	9.545524	67.3	10.454476	33134	94351	39
22	520631	59.9	974703	7.4	545928	67.3	454072	33161	94342	38
23	520990	59.9	974659	7.4	546331	67.2	453669	33189	94332	37
24	521349	59.8	974614	7.4	546735	67.2	453265	33216	94322	36
25	521707	59.8	974570	7.4	547138	67.1	452862	33244	94313	35
26	522066	59.7	974525	7.4	547540	67.1	452460	33271	94303	34
27	522424	59.6	974481	7.4	547943	67.0	452057	33298	94293	33
28	522781	59.6	974436	7.4	548345	67.0	451655	33326	94284	32
29	523138	59.5	974391	7.4	548747	66.9	451253	33353	94274	31
30	523495	59 5	974347	7.5	549149	66.9	450851	33381	94264	30
31	9.523852	59.4	9.974302	7.5	9.549550	66.8	10.450450	33408	94254	29
32	524208	59.4	974257	7.5	549951	66.8	450049	33436	94245	28
33	524564	59.3	974212	7.5	550352	66.7	449648	33463	94235	27
34	524920	59.3	974167	7.5	550752	66.7	449248	33490	94225	26
35	525275	59.2	974122	7.5	551152	66.6	448848	33518	94215	25
36	525630	59.1	974077	7.5	551552	66.6	448448	33545	94206	24
37	525984	59.1	974032	7.5	551952	66.5	448048	33573	94196	23
38	526339	59.0	973987	7.5	552351	66.5	447649	33600	94186	22
39	526693	59.0	973942	7.5	552750	66.5	447250	33627	94176	21
40	527046	58.9	973897	7.5	553149	66.4	446851	33655	94167	20
41	9.527400	58.9	9.973852	7.5	9.553548	66.4	10.446452	33682	94157	19
42	527753	58.8	973807	7.5	553946	66.3	446054	33710	94147	18
43	528105	58.8	973761	7.5	554344	66.3	445656	33737	94137	17
44	528458	58.7	973716	7.6	554741	66.2	445259	33764	94127	16
45	528810	58 7	973671	7.6	555139	66.2	444861	33792	94118	15
46	529161	58.6	973625	7.6	555536	66.1	444464	33819	94108	14
47	529513	58.6	973580	7.6	555933	66.1	444067	33846	94098	13
48	529864	58.5	973535	7.6	556329	66.0	443671	33874	94088	12
49	530215	58.5	973489	7.6	556725	66.0	443275	33901	94078	11
50	530565	58.4	973444	7.6	557121	65.9	442879	33929	94068	10
51	9.530915	58.4	9.973398	7.6	9.557517	65.9	10.442483	33956	94058	9
52	531265	58.3	973352	7.6	557913	65.9	442087	33983	94049	8
53	531614	58.2	973307	7.6	558308	65.8	441692	34011	94039	7
54	531963	58.2	973261	7.6	558702	65.8	441298	34038	94029	6
55	532312	58.1	973215	7.6	559097	65.7	440903	34065	94019	5
56	532661	58.1	973169	7.6	559491	65.7	440509	34093	94009	4
57	533009	58.0	973124	7.6	559885	65.6	440115	34120	93999	3
58	533357	58.0	973078	7.6	560279	65.6	439721	34147	93989	2
59	533704	57.9	973032	7.7	560673	65.5	439327	34175	93979	1
60	534052		972986		561066		438934	34202	93969	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N.sine.	′

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	9.534052	57.8	9.972986	7.7	9.561066	65.5	10.438934	34202	93969	60
1	534399	57.7	972940	7.7	561459	65.4	438541	34229	93959	59
2	534745	57.7	972894	7.7	561851	65.4	438149	34257	93949	58
3	535092	57.7	972848	7.7	562244	65.3	437756	34284	93939	57
4	535438	57.6	972802	7.7	562636	65.3	437364	34311	93929	56
5	535783	57.6	972755	7.7	563028	65.3	436972	34339	93919	55
6	536129	57.5	972709	7.7	563419	65.2	436581	34366	93909	54
7	536474	57.4	972663	7.7	563811	65.2	436189	34393	93899	53
8	536818	57.4	972617	7.7	564202	65.1	435798	34421	93889	52
9	537163	57.3	972570	7.7	564592	65.1	435408	34448	93879	51
10	537507	57.3	972524	7.7	564983	65.0	435017	34475	93869	50
11	9.537851	57.2	9.972478	7.7	9.565373	65.0	10.434627	34503	93859	49
12	538194	57.2	972431	7.8	565763	64.9	434237	34530	93849	48
13	538538	57.1	972385	7.8	566153	64.9	433847	34557	93839	47
14	538880	57.1	972338	7.8	566542	64.9	433458	34584	93829	46
15	539223	57.0	972291	7.8	566932	64.8	433068	34612	93819	45
16	539565	57.0	972245	7.8	567320	64.8	432680	34639	93809	44
17	539907	56.9	972198	7.8	567709	64.7	432291	34666	93799	43
18	540249	56.9	972151	7.8	568098	64.7	431902	34694	93789	42
19	540590	56.8	972105	7.8	568486	64.6	431514	34721	93779	41
20	540931	56.8	972058	7.8	568873	64.6	431127	34748	93769	40
21	9.541272	56.7	9.972011	7.8	9.569261	64.5	10.430739	34775	93759	39
22	541613	56.7	971964	7.8	569648	64.5	430352	34803	93748	38
23	541953	56.6	971917	7.8	570035	64.5	429965	34830	93738	37
24	542293	56.6	971870	7.8	570422	64.4	429578	34857	93728	36
25	542632	56.5	971823	7.8	570809	64.4	429191	34884	93718	35
26	542971	56.5	971776	7.8	571195	64.3	428805	34912	93708	34
27	543310	56.4	971729	7.9	571581	64.3	428419	34939	93698	33
28	543649	56.4	971682	7.9	571967	64.2	428033	34966	93688	32
29	543987	56.3	971635	7.9	572352	64.2	427648	34993	93677	31
30	544325	56.3	971588	7.9	572738	64.2	427262	35021	93667	30
31	9.544663	56.2	9.971540	7.9	9.573123	64.1	10.426877	35048	93657	29
32	545000	56.2	971493	7.9	573507	64.1	426493	35075	93647	28
33	545338	56.1	971446	7.9	573892	64.0	426108	35102	93637	27
34	545674	56.1	971398	7.9	574276	64.0	425724	35130	93626	26
35	546011	56.0	971351	7.9	574660	63.9	425340	35157	93616	25
36	546347	56.0	971303	7.9	575044	63.9	424956	35184	93606	24
37	546683	55.9	971256	7.9	575427	63.9	424573	35211	93596	23
38	547019	55.9	971208	7.9	575810	63.8	424190	35239	93585	22
39	547354	55.8	971161	7.9	576193	63.8	423807	35266	93575	21
40	547689	55.8	971113	7.9	576576	63.7	423424	35293	93565	20
41	9.548024	55.7	9.971066	8.0	9.576958	63.7	10.423041	35320	93555	19
42	548359	55.7	971018	8.0	577341	63.6	422659	35347	93544	18
43	548693	55.6	970970	8.0	577723	63.6	422277	35375	93534	17
44	549027	55.6	970922	8.0	578104	63.6	421896	35402	93524	16
45	549360	55.5	970874	8.0	578486	63.5	421514	35429	93514	15
46	549693	55.5	970827	8.0	578867	63.5	421133	35456	93503	14
47	550026	55.4	970779	8.0	579248	63.4	420752	35484	93493	13
48	550359	55.4	970731	8.0	579629	63.4	420371	35511	93483	12
49	550692	55.3	970683	8.0	580009	63.4	419991	35538	93472	11
50	551024	55.3	970635	8.0	580389	63.3	419611	35565	93462	10
51	9.551356	55.2	9.970586	8.0	9.580769	63.3	10.419231	35592	93452	9
52	551687	55.2	970538	8.0	581149	63.2	418851	35619	93441	8
53	552018	55.2	970490	8.0	581528	63.2	418472	35647	93431	7
54	552349	55.1	970442	8.0	581907	63.2	418093	35674	93420	6
55	552680	55.1	970394	8.0	582286	63.1	417714	35701	93410	5
56	553010	55.0	970345	8.1	582665	63.1	417335	35728	93400	4
57	553341	55.0	970297	8.1	583043	63.0	416957	35755	93389	3
58	553670	54.9	970249	8.1	583422	63.0	416578	35782	93379	2
59	554000	54.9	970200	8.1	583800	62.9	416200	35810	93368	1
60	554329		970152		584177		415823	35837	93358	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N. sine.	′

69 Degrees.

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	9.554329	54.8	9.970152	8.1	9.584177	62.9	10.415823	35837	93358	60
1	554658	54.8	970103	8.1	584555	62.9	415445	35864	93348	59
2	554987	54.7	970055	8.1	584932	62.8	415068	35891	93337	58
3	555315	54.7	970006	8.1	585309	62.8	414691	35918	93327	57
4	555643	54.6	969957	8.1	585686	62.7	414314	35945	93316	56
5	555971	54.6	969909	8.1	586062	62.7	413938	35973	93306	55
6	556299	54.5	969860	8.1	586439	62.7	413561	36000	93295	54
7	556626	54.5	969811	8.1	586815	62.6	413185	36027	93285	53
8	556953	54.4	969762	8.1	587190	62.6	412810	36054	93274	52
9	557280	54.4	969714	8.1	587566	62.5	412434	36081	93264	51
10	557606	54.3	969665	8.1	587941	62.5	412059	36108	93253	50
11	9.557932	54.3	9.969616	8.2	9.588316	62.5	10.411684	36135	93243	49
12	558258	54.3	969567	8.2	588691	62.4	411309	36162	93232	48
13	558583	54.2	969518	8.2	589066	62.4	410934	36190	93222	47
14	558909	54.2	969469	8.2	589440	62.3	410560	36217	93211	46
15	559234	54.1	969420	8.2	589814	62.3	410186	36244	93201	45
16	559558	54.1	969370	8.2	590188	62.3	409812	36271	93190	44
17	559883	54.0	969321	8.2	590562	62.2	409438	36298	93180	43
18	560207	54.0	969272	8.2	590935	62.2	409065	36325	93169	42
19	560531	53.9	969223	8.2	591308	62.2	408692	36352	93159	41
20	560855	53.9	969173	8.2	591681	62.1	408319	36379	93148	40
21	9.561178	53.8	9.969124	8.2	9.592054	62.1	10.407946	36406	93137	39
22	561501	53.8	969075	8.2	592426	62.0	407574	36434	93127	38
23	561824	53.7	969025	8.2	592798	62.0	407202	36461	93116	37
24	562146	53.7	968976	8.2	593170	61.9	406829	36488	93106	36
25	562468	53.6	968926	8.3	593542	61.9	406458	36515	93095	35
26	562790	53.6	968877	8.3	593914	61.8	406086	36542	93084	34
27	563112	53.6	968827	8.3	594285	61.8	405715	36569	93074	33
28	563433	53.5	968777	8.3	594656	61.8	405344	36596	93063	32
29	563755	53.5	968728	8.3	595027	61.7	404973	36623	93052	31
30	564075	53.4	968678	8.3	595398	61.7	404602	36650	93042	30
31	9.564396	53.4	9.968628	8.3	9.595768	61.7	10.404232	36677	93031	29
32	564716	53.3	968578	8.3	596138	61.6	403862	36704	93020	28
33	565036	53.3	968528	8.3	596508	61.6	403492	36731	93010	27
34	565356	53.2	968479	8.3	596878	61.6	403122	36758	92999	26
35	565676	53.2	968429	8.3	597247	61.5	402753	36785	92988	25
36	565995	53.1	968379	8.3	597616	61.5	402384	36812	92978	24
37	566314	53.1	968329	8.3	597985	61.5	402015	36839	92967	23
38	566632	53.1	968278	8.3	598354	61.4	401646	36867	92956	22
39	566951	53.0	968228	8.4	598722	61.4	401278	36894	92945	21
40	567269	53.0	968178	8.4	599091	61.3	400909	36921	92935	20
41	9.567587	52.9	9.968128	8.4	9.599459	61.3	10.400541	36948	92926	19
42	567904	52.9	968078	8.4	599827	61.3	400173	36975	92913	18
43	568222	52.8	968027	8.4	600194	61.2	399806	37002	92902	17
44	568539	52.8	967977	8.4	600562	61.2	399438	37029	92892	16
45	568856	52.8	967927	8.4	600929	61.1	399071	37056	92881	15
46	569172	52.7	967876	8.4	601296	61.1	398704	37083	92870	14
47	569488	52.7	967826	8.4	601662	61.1	398338	37110	92859	13
48	569804	52.6	967775	8.4	602029	61.0	397971	37137	92849	12
49	570120	52.6	967725	8.4	602395	61.0	397605	37164	92838	11
50	570435	52.5	967674	8.4	602761	61.0	397239	37191	92827	10
51	9.570751	52.5	9.967624	8.4	9.603127	60.9	10.396873	37218	92816	9
52	571066	52.4	967573	8.4	603493	60.9	396507	37245	92805	8
53	571380	52.4	967522	8.5	603858	60.9	396142	37272	92794	7
54	571695	52.3	967471	8.5	604223	60.8	395777	37299	92784	6
55	572009	52.3	967421	8.5	604588	60.8	395412	37326	92773	5
56	572323	52.3	967370	8.5	604953	60.7	395047	37353	92762	4
57	572636	52.2	967319	8.5	605317	60.7	394683	37380	92751	3
58	572950	52.2	967268	8.5	605682	60.7	394318	37407	92740	2
59	573263	52.1	967217	8.5	606046	60.6	393954	37434	92729	1
60	573575		967166		606410		393590	37461	92718	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N. Sine	′

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	9.573575	52.1	9.967166	8.5	9.606410	60.6	10.393590	37461	92718	60
1	573888	52.0	967115	8.5	606773	60.6	393227	37488	92707	59
2	574200	52.0	967064	8.5	607137	60.5	392863	37515	92697	58
3	574512	51.9	967013	8.5	607500	60.5	392500	37542	92686	57
4	574824	51.9	966961	8.5	607863	60.4	392137	37569	92675	56
5	575136	51.9	966910	8.5	608225	60.4	391775	37595	92664	55
6	575447	51.8	966859	8.5	608588	60.4	391412	37622	92653	54
7	575758	51.8	966808	8.5	608950	60.3	391050	37649	92642	53
8	576069	51.7	966756	8.6	609312	60.3	390688	37676	92631	52
9	576379	51.7	966705	8.6	609674	60.3	390326	37703	92620	51
10	576689	51.6	966653	8.6	610036	60.2	389964	37730	92609	50
11	9.576999	51.6	9.966602	8.6	9.610397	60.2	10.389603	37757	92598	49
12	577309	51.6	966550	8.6	610759	60.2	389241	37784	92587	48
13	577618	51.5	966499	8.6	611120	60.1	388880	37811	92576	47
14	577927	51.5	966447	8.6	611480	60.1	388520	37838	92565	46
15	578236	51.4	966395	8.6	611841	60.1	388159	37865	92554	45
16	578545	51.4	966344	8.6	612201	60.0	387799	37892	92543	44
17	578853	51.3	966292	8.6	612561	60.0	387439	37919	92532	43
18	579162	51.3	966240	8.6	612921	60.0	387079	37946	92521	42
19	579470	51.3	966188	8.6	613281	59.9	386719	37973	92510	41
20	579777	51.2	966136	8.6	613641	59.9	386359	37999	92499	40
21	9.580085	51.2	9.966085	8.7	9.614000	59.8	10.386000	38026	92488	39
22	580392	51.1	966033	8.7	614359	59.8	385641	38053	92477	38
23	580699	51.1	965981	8.7	614718	59.8	385282	38080	92466	37
24	581005	51.1	965928	8.7	615077	59.7	384923	38107	92455	36
25	581312	51.0	965876	8.7	615435	59.7	384565	38134	92444	35
26	581618	51.0	965824	8.7	615793	59.7	384207	38161	92432	34
27	581924	50.9	965772	8.7	616151	59.6	383849	38188	92421	33
28	582229	50.9	965720	8.7	616509	59.6	383491	38215	92410	32
29	582535	50.9	965668	8.7	616867	59.6	383133	38241	92399	31
30	582840	50.8	965615	8.7	617224	59.5	382776	38268	92388	30
31	9.583145	50.8	9.965563	8.7	9.617582	59.5	10.382418	38295	92377	29
32	583449	50.7	965511	8.7	617939	59.5	382061	38322	92366	28
33	583754	50.7	965458	8.7	618295	59.4	381705	38349	92355	27
34	584058	50.6	965406	8.7	618652	59.4	381348	38376	92343	26
35	584361	50.6	965353	8.8	619008	59.4	380992	38403	92332	25
36	584665	50.6	965301	8.8	619364	59.3	380636	38430	92321	24
37	584968	50.5	965248	8.8	619721	59.3	380279	38456	92310	23
38	585272	50.5	965195	8.8	620076	59.3	379924	38483	92299	22
39	585574	50.4	965143	8.8	620432	59.2	379568	38510	92287	21
40	585877	50.4	965090	8.8	620787	59.2	379213	38537	92276	20
41	9.586179	50.3	9.965037	8.8	9.621142	59.2	10.378858	38564	92265	19
42	586482	50.3	964984	8.8	621497	59.1	378503	38591	92254	18
43	586783	50.3	964931	8.8	621852	59.1	378148	38617	92243	17
44	587085	50.2	964879	8.8	622207	59.0	377793	38644	92231	16
45	587386	50.2	964826	8.8	622561	59.0	377439	38671	92220	15
46	587688	50.1	964773	8.8	622915	59.0	377085	38698	92209	14
47	587989	50.1	964719	8.8	623269	58.9	376731	38725	92198	13
48	588289	50.1	964666	8.9	623623	58.9	376377	38752	92186	12
49	588590	50.0	964613	8.9	623976	58.9	376024	38778	92175	11
50	588890	50.0	964560	8.9	624330	58.8	375670	38805	92164	10
51	9.589190	49.9	9.964507	8.9	9.624683	58.8	10.375317	38832	92152	9
52	589489	49.9	964454	8.9	625036	58.8	374964	38859	92141	8
53	589789	49.9	964400	8.9	625388	58.7	374612	38886	92130	7
54	590088	49.8	964347	8.9	625741	58.7	374259	38912	92119	6
55	590387	49.8	964294	8.9	626093	58.7	373907	38939	92107	5
56	590686	49.7	964240	8.9	626445	58.6	373555	38966	92096	4
57	590984	49.7	964187	8.9	626797	58.6	373203	38993	92085	3
58	591282	49.7	964133	8.9	627149	58.6	372851	39020	92073	2
59	591580	49.6	964080	8.9	627501	58.5	372499	39046	92062	1
60	591878		964026		627852		372148	39073	92050	0
	Cosine.	D. 10″	Sine.		Cotang.		Tang.	N. cos.	N. sine.	′

67 Degrees.

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	9.591878	49.6	9.964026	8.9	9.627852	58.5	10.372148	39073	92050	60
1	592176	49.5	963972	8.9	628203	58.5	371797	39100	92039	59
2	592473	49.5	963919	8.9	628554	58.5	371446	39127	92028	58
3	592770	49.5	963865	9.0	628905	58.4	371095	39153	92016	57
4	593067	49.4	963811	9.0	629255	58.4	370745	39180	92005	56
5	593363	49.4	963757	9.0	629606	58.3	370394	39207	91994	55
6	593659	49.3	963704	9.0	629956	58.3	370044	39234	91982	54
7	593955	49.3	963650	9.0	630306	58.3	369694	39260	91971	53
8	594251	49.3	963596	9.0	630656	58.3	369344	39287	91959	52
9	594547	49.2	963542	9.0	631005	58.2	368995	39314	91948	51
10	594842	49.2	963488	9.0	631355	58.2	368645	39341	91936	50
11	9.595137	49.1	9.963434	9.0	9.631704	58.2	10.368296	39367	91925	49
12	595432	49.1	963379	9.0	632053	58.1	367947	39394	91914	48
13	595727	49.1	963325	9.0	632401	58.1	367599	39421	91902	47
14	596021	49.0	963271	9.0	632750	58.1	367250	39448	91891	46
15	596315	49.0	963217	9.0	633098	58.0	366902	39474	91879	45
16	596609	48.9	963163	9.0	633447	58.0	366553	39501	91868	44
17	596903	48.9	963108	9.1	633795	58.0	366205	39528	91856	43
18	597196	48.9	963054	9.1	634143	57.9	365857	39555	91845	42
19	597490	48.8	962999	9.1	634490	57.9	365510	39581	91833	41
20	597783	48.8	962945	9.1	634838	57.9	365162	39608	91822	40
21	9.598075	48.7	9.962890	9.1	9.635185	57.8	10.364815	39635	91810	39
22	598368	48.7	962836	9.1	635532	57.8	364468	39661	91799	38
23	598660	48.7	962781	9.1	635879	57.8	364121	39688	91787	37
24	598952	48.6	962727	9.1	636226	57.7	363774	39715	91775	36
25	599244	48.6	962672	9.1	636572	57.7	363428	39741	91764	35
26	599536	48.5	962617	9.1	636919	57.7	363081	39768	91752	34
27	599827	48.5	962562	9.1	637265	57.7	362735	39795	91741	33
28	600118	48.5	962508	9.1	637611	57.6	362389	39822	91729	32
29	600409	48.4	962453	9.1	637956	57.6	362044	39848	91718	31
30	600700	48.4	962398	9.2	638302	57.6	361698	39875	91706	30
31	9.600990	48.4	9.962343	9.2	9.638647	57.5	10.361353	39902	91694	29
32	601280	48.3	962288	9.2	638992	57.5	361008	39928	91683	28
33	601570	48.3	962233	9.2	639337	57.5	360663	39955	91671	27
34	601860	48.2	962178	9.2	639682	57.4	360318	39982	91660	26
35	602150	48.2	962123	9.2	640027	57.4	359973	40008	91648	25
36	602439	48.2	962067	9.2	640371	57.4	359629	40035	91636	24
37	602728	48.1	962012	9.2	640716	57.3	359284	40062	91625	23
38	603017	48.1	961957	9.2	641060	57.3	358940	40088	91613	22
39	603305	48.1	961902	9.2	641404	57.3	358596	40115	91601	21
40	603594	48.0	961846	9.2	641747	57.2	358253	40141	91590	20
41	9.603882	48.0	9.961791	9.2	9.642091	57.2	10.357909	40168	91578	19
42	604170	47.9	961735	9.2	642434	57.2	357566	40195	91566	18
43	604457	47.9	961680	9.2	642777	57.2	357223	40221	91555	17
44	604745	47.9	961624	9.3	643120	57.1	356880	40248	91543	16
45	605032	47.8	961569	9.3	643463	57.1	356537	40275	91531	15
46	605319	47.8	961513	9.3	643806	57.1	356194	40301	91519	14
47	605606	47.8	961458	9.3	644148	57.0	355852	40328	91508	13
48	605892	47.7	961402	9.3	644490	57.0	355510	40355	91496	12
49	606179	47.7	961346	9.3	644832	57.0	355168	40381	91484	11
50	606465	47.6	961290	9.3	645174	56.9	354826	40408	91472	10
51	9.606751	47.6	9.961235	9.3	9.645516	56.9	10.354484	40434	91461	9
52	607036	47.6	961179	9.3	645857	56.9	354143	40461	91449	8
53	607322	47.5	961123	9.3	646199	56.9	353801	40488	91437	7
54	607607	47.5	961067	9.3	646540	56.8	353460	40514	91425	6
55	607892	47.4	961011	9.3	646881	56.8	353119	40541	91414	5
56	608177	47.4	960955	9.3	647222	56.8	352778	40567	91402	4
57	608461	47.4	960899	9.3	647562	56.7	352438	40594	91390	3
58	608745	47.3	960843	9.4	647903	56.7	352097	40621	91378	2
59	609029	47.3	960786	9.4	648243	56.7	351757	40647	91366	1
60	609313		960730		648583		351417	40674	91355	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N. sine.	′

66 Degrees.

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	9.609313	47.3	9.960730	9.4	9.648583	56.6	10.351417	40674	91355	60
1	609597	47.2	960674	9.4	648923	56.6	351077	40700	91343	59
2	609880	47.2	960618	9.4	649263	56.6	350737	40727	91331	58
3	610164	47.2	960561	9.4	649602	56.6	350398	40753	91319	57
4	610447	47.1	960505	9.4	649942	56.5	350058	40780	91307	56
5	610729	47.1	960448	9.4	650281	56.5	349719	40806	91295	55
6	611012	47.0	960392	9.4	650620	59.5	349380	40833	91283	54
7	611294	47.0	960335	9.4	650959	56.4	349041	40860	91272	53
8	611576	47.0	960279	9.4	651297	56.4	348703	40886	91260	52
9	611858	46.9	960222	9.4	651636	56.4	348364	40913	91248	51
10	612140	46.9	960165	9.4	651974	56.3	348026	40939	91236	50
11	9.612421	46.9	9.960109	9.5	9.652312	56.3	10.347688	40966	91224	49
12	612702	46.8	960052	9.5	652650	56.3	347350	40992	91212	48
13	612983	46.8	959995	9.5	652988	56.3	347012	41019	91200	47
14	613264	46.7	959938	9.5	653326	56.2	346674	41045	91188	46
15	613545	46.7	959882	9.5	653663	56.2	346337	41072	91176	45
16	613825	46.7	959825	9.5	654000	56.2	346000	41098	91164	44
17	614105	46.6	959768	9.5	654337	56.1	345663	41125	91152	43
18	614385	46.6	959711	9.5	654174	56.1	345326	41151	91140	42
19	614665	46.6	959654	9.5	655011	56.1	344989	41178	91128	41
20	614944	46.5	959596	9.5	655348	56.1	344652	41204	91116	40
21	9.615223	46.5	9.959539	9.5	9.655684	56.0	10.344316	41231	91104	39
22	615502	46.5	959482	9.5	656020	56.0	343980	41257	91092	38
23	615781	46.4	959425	9.5	656356	56.0	343644	41284	91080	37
24	616060	46.4	959368	9.5	656692	55.9	343308	41310	91068	36
25	616338	46.4	959310	9.6	657028	55.9	342972	41337	91056	35
26	616616	46.3	959253	9.6	657364	55.9	342636	41363	91044	34
27	616894	46.3	959195	9.6	657699	55.9	342301	41390	91032	33
28	617172	46.2	959138	9.6	658034	55.8	341966	41416	91020	32
29	617450	46.2	959081	9.6	658369	55.8	341631	41443	91008	31
30	617727	46.2	959023	9.6	658704	55.8	341296	41469	90996	30
31	9.618004	46.1	9.958965	9.6	9.659039	55.8	10.340961	41496	90984	29
32	618281	46.1	958908	9.6	659373	55.7	340627	41522	90972	28
33	618558	46.1	958850	9.6	659708	55.7	340292	41549	90960	27
34	618834	46.0	958792	9.6	660042	55.7	339958	41575	90948	26
35	619110	46.0	958734	9.6	660376	55.7	339624	41602	90936	25
36	619386	46.0	958577	9.6	660710	55.6	339290	41628	90924	24
37	619662	45.9	958619	9.6	661043	55.6	338957	41655	90911	23
38	619938	45.9	958561	9.6	661377	55.6	338623	41681	90899	22
39	620213	45.9	958503	9.7	661710	55.5	338290	41707	90887	21
40	620488	45.8	958445	9.7	662043	55.5	337957	41734	90875	20
41	9.620763	45.8	9.958387	9.7	9.662376	55.5	10.337624	41760	90863	19
42	621038	45.7	958329	9.7	662709	55.4	337291	41787	90851	18
43	621313	45.7	958271	9.7	663042	55.4	336958	41813	90839	17
44	621587	45.7	958213	9.7	663375	55.4	336625	41840	90826	16
45	621861	45.6	958154	9.7	663707	55.4	336293	41866	90814	15
46	622135	45.6	958096	9.7	664039	55.3	335961	41892	90802	14
47	622409	45.6	958038	9.7	664371	55.3	335629	41919	90790	13
48	622682	45.5	957979	9.7	664703	55.3	335297	41945	90778	12
49	622956	45.5	957921	9.7	665035	55.3	334965	41972	90766	11
50	623229	45.5	957863	9.7	665366	55.2	334634	41998	90753	10
51	9.623512	45.4	9.957804	9.7	9.665697	55.2	10.334303	42024	90741	9
52	623774	45.4	957746	9.8	666029	55.2	333971	42051	90729	8
53	624047	45.4	.957687	9.8	666360	55.1	333620	42077	90717	7
54	624319	45.3	957628	9.8	666691	55.1	333309	42104	90704	6
55	624591	45.3	957570	9.8	667021	55.1	332979	42130	90692	5
56	624863	46.3	957511	9.8	667352	55.1	332648	42156	90680	4
57	625135	45.2	957452	9.8	667682	55.0	332318	42183	90668	3
58	625406	45.2	957393	9.8	668013	55.0	331987	42209	90655	2
59	625677	45.2	957335	9.8	668343	55.0	331657	42235	90643	1
60	625948		957276		668672		331328	42262	90631	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N. sine.	′

65 Degrees.

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	9.625948	45.1	9.957276	9.8	9.668673	55.0	10.331327	42262	90631	60
1	626219	45.1	957217	9.8	669002	54.9	330998	42288	90618	59
2	626490	45.1	957158	9.8	669332	54.9	330668	42315	90606	58
3	626760	45.0	957099	9.8	669661	54.9	330339	42341	90594	57
4	627030	45.0	957040	9.8	669991	54.8	330009	42367	90582	56
5	627300	45.0	956981	9.8	670320	54.8	329680	42394	90569	55
6	627570	44.9	956921	9.9	670649	54.8	329351	42420	90557	54
7	627840	44.9	956862	9.9	670977	54.8	329023	42446	90545	53
8	628109	44.9	956803	9.9	671306	54.7	328694	42473	90532	52
9	628378	44.8	956744	9.9	671634	54.7	328366	42499	90520	51
10	628647	44.8	956684	9.9	671963	54.7	328037	42525	90507	50
11	9.628916	44.7	9.956625	9.9	9.672291	54.7	10.327709	42552	90495	49
12	629185	44.7	956566	9.9	672619	54.6	327381	42578	90483	48
13	629453	44.7	956506	9.9	672947	54.6	327053	42604	90470	47
14	629721	44.6	956447	9.9	673274	54.6	326726	42631	90458	46
15	629989	44.6	956387	9.9	673602	54.6	326398	42657	90446	45
16	630257	44.6	956327	9.9	673929	54.5	326071	42683	90433	44
17	630524	44.6	956268	9.9	674257	54.5	325743	42709	90421	43
18	630792	44.5	956208	10.0	674584	54.5	325416	42736	90408	42
19	631059	44.5	956148	10.0	674910	54.4	325090	42762	90396	41
20	631326	44.5	956089	10.0	675237	54.4	324763	42788	90383	40
21	9.631593	44.4	9.956029	10.0	9.675564	54.4	10.324436	42815	90371	39
22	631859	44.4	955969	10.0	675890	54.4	324110	42841	90358	38
23	632125	44.4	955909	10.0	676216	54.3	323784	42867	90346	37
24	632392	44.3	955849	10.0	676543	54.3	323457	42894	90334	36
25	632658	44.3	955789	10.0	676869	54.3	323131	42920	90321	35
26	632923	44.3	955729	10.0	677194	54.3	322806	42946	90309	34
27	633189	44.2	955669	10.0	677520	54.2	322480	42972	90296	33
28	633454	44.2	955609	10.0	677846	54.2	322154	42999	90284	32
29	633719	44.2	955548	10.0	678171	54.2	321829	43025	90271	31
30	633984	44.1	955488	10.0	678496	54.2	321504	43051	90259	30
31	9.634249	44.1	9.955428	10.1	9.678821	54.1	10.321179	43077	90246	29
32	634514	44.0	955368	10.1	679146	54.1	320854	43104	90233	28
33	634778	44.0	955307	10.1	679471	54.1	320529	43130	90221	27
34	635042	44.0	955247	10.1	679795	54.1	320205	43156	90208	26
35	635306	43.9	955186	10.1	680120	54.0	319880	43182	90196	25
36	635570	43.9	955126	10.1	680444	54.0	319556	43209	90183	24
37	635834	43.9	955065	10.1	680768	54.0	319232	43235	90171	23
38	636097	43.8	955005	10.1	681092	54.0	318908	43261	90158	22
39	636360	43.8	954944	10.1	681416	53.9	318584	43287	90146	21
40	636623	43.8	954883	10.1	681740	53.9	318260	43313	90133	20
41	9.636886	43.7	9.954823	10.1	9.682063	53.9	10.317937	43340	90120	19
42	637148	43.7	954762	10.1	682387	53.9	317613	43366	90108	18
43	637411	43.7	954701	10.1	682710	53.8	317290	43392	90095	17
44	637673	43.7	954640	10.1	683033	53.8	316967	43418	90082	16
45	637935	43.6	954579	10.1	683356	53.8	316644	43445	90070	15
46	638197	43.6	954518	10.2	683679	53.8	316321	43471	90057	14
47	638458	43.6	954457	10.2	684001	53.7	315999	43497	90045	13
48	638720	43.5	954396	10.2	684324	53.7	315676	43523	90032	12
49	638981	43.5	954335	10.2	684646	53.7	315354	43549	90019	11
50	639242	43.5	954274	10.2	684968	53.7	315032	43575	90007	10
51	9.639503	43.4	9.954213	10.2	9.685290	53.6	10.314710	43602	89994	9
52	639764	43.4	954152	10.2	685612	53.6	314388	43628	89981	8
53	640024	43.4	954090	10.2	685934	53.6	314066	43654	89968	7
54	640284	43.3	954029	10.2	686255	53.6	313745	43680	89956	6
55	640544	43.3	953968	10.2	686577	53.5	313423	43706	89943	5
56	640804	43.3	953906	10.2	686898	53.5	313102	43733	89930	4
57	641064	43.2	953845	10.2	687219	53.5	312781	43759	89918	3
58	641324	43.2	953783	10.2	687540	53.5	312460	43785	89905	2
59	641584	43.2	953722	10.3	687861	53.4	312139	43811	89892	1
60	641842		953660		688182		311818	43837	89879	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N. sine.	′

'	Sine.	D. 10"	Cosine.	D. 10"	Tang.	D. 10"	Cotang.	N. sine.	N. cos.	
0	9.641842		9.953660		9.688182		10.311818	43837	89879	60
1	642101	43.1	953599	10.3	688502	53.4	311498	43863	89867	59
2	642360	43.1	953537	10.3	688823	53.4	311177	43889	89854	58
3	642618	43.1	953475	10.3	689143	53.4	310857	43916	89841	57
4	642877	43.0	953413	10.3	689463	53.3	310537	43942	89828	56
5	643135	43.0	953352	10.3	689783	53.3	310217	43968	89816	55
6	643393	43.0	953290	10.3	690103	53.3	309897	43994	89803	54
7	643650	43.0	953228	10.3	690423	53.3	309577	44020	89790	53
8	643908	42.9	953166	10.3	690742	53.3	309258	44046	89777	52
9	644165	42.9	953104	10.3	691062	53.2	308938	44072	89764	51
10	644423	42.9	953042	10.3	691381	53.2	308619	44098	89752	50
11	9.644680	42.8	9.952980	10.3	9.691700	53.2	10.308300	44124	89739	49
12	644936	42.8	952918	10.4	692019	53.1	307981	44151	89726	48
13	645193	42.8	952855	10.4	692338	53.1	307662	44177	89713	47
14	645450	42.7	952793	10.4	692656	53.1	307344	44203	89700	46
15	645706	42.7	952731	10.4	692975	53.1	307025	44229	89687	45
16	645962	42.7	952669	10.4	693293	53.1	306707	44255	89674	44
17	646218	42.6	952606	10.4	693612	53.0	306388	44281	89662	43
18	646474	42.6	952544	10.4	693930	53.0	306070	44307	89649	42
19	646729	42.6	952481	10.4	694248	53.0	305752	44333	89636	41
20	646984	42.5	952419	10.4	694566	53.0	305434	44359	89623	40
21	9.647240	42.5	9.952356	10.4	9.694883	52.9	10.305117	44385	89610	39
22	647494	42.5	952294	10.4	695201	52.9	304799	44411	89597	38
23	647749	42.4	952231	10.4	695518	52.9	304482	44437	89584	37
24	648004	42.4	952168	10.4	695836	52.9	304164	44464	89571	36
25	648258	42.4	952106	10.5	696153	52.9	303847	44490	89558	35
26	648512	42.4	952043	10.5	696470	52.8	303530	44516	89545	34
27	648766	42.3	951980	10.5	696787	52.8	303213	44542	89532	33
28	649020	42.3	951917	10.5	697103	52.8	302897	44568	89519	32
29	649274	42.3	951854	10.5	697420	52.8	302580	44594	89506	31
30	649527	42.2	951791	10.5	697736	52.7	302264	44620	89493	30
31	9.649781	42.2	9.951728	10.5	9.698053	52.7	10.301947	44646	89480	29
32	650034	42.2	951665	10.5	698369	52.7	301631	44672	89467	28
33	650287	42.2	951602	10.5	698685	52.7	301315	44698	89454	27
34	650539	42.1	951539	10.5	699001	52.6	300999	44724	89441	26
35	650792	42.1	951476	10.5	699316	52.6	300684	44750	89428	25
36	651044	42.1	951412	10.5	699632	52.6	300368	44776	89415	24
37	651297	42.0	951349	10.5	699947	52.6	300053	44802	89402	23
38	651549	42.0	951286	10.6	700263	52.6	299737	44828	89389	22
39	651800	42.0	951222	10.6	700578	52.5	299422	44854	89376	21
40	652052	41.9	951159	10.6	700893	52.5	299107	44880	89363	20
41	9.652304	41.9	9.951096	10.6	9.701208	52.5	10.298792	44906	89350	19
42	652555	41.9	951032	10.6	701523	52.4	298477	44932	89337	18
43	652806	41.8	950968	10.6	701837	52.4	298163	44958	89324	17
44	653057	41.8	950905	10.6	702152	52.4	297848	44984	89311	16
45	653308	41.8	950841	10.6	702466	52.4	297534	45010	89298	15
46	653558	41.8	950778	10.6	702780	52.4	297220	45036	89285	14
47	653808	41.7	950714	10.6	703095	52.3	296905	45062	89272	13
48	654059	41.7	950650	10.6	703409	52.3	296591	45088	89259	12
49	654309	41.7	950586	10.6	703723	52.3	296277	45114	89245	11
50	654558	41.6	950522	10.6	704036	52.3	295964	45140	89232	10
51	9.654808	41.6	9.950458	10.7	9.704350	52.2	10.295650	45166	89219	9
52	655058	41.6	950394	10.7	704663	52.2	295337	45192	89206	8
53	655307	41.6	950330	10.7	704977	52.2	295023	45218	89193	7
54	655556	41.5	950366	10.7	705290	52.2	294710	45243	89180	6
55	655805	41.5	950202	10.7	705603	52.2	294397	45269	89167	5
56	656054	41.5	950138	10.7	705916	52.1	294084	45295	89153	4
57	656302	41.4	950074	10.7	706228	52.1	293772	45321	89140	3
58	656551	41.4	950010	10.7	706541	52.1	293459	45347	89127	2
59	656799	41.4	949945	10.7	706854	52.1	293146	45373	89114	1
60	657047	41.3	949881	10.7	707166	52.1	292834	45399	89101	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N. sine.	'

63 Degrees.

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	9.657047	41.3	9.949881	10.7	9.707166	52.0	10.292834	45399	89101	60
1	657295	41.3	949816	10.7	707478	52.0	292522	45425	89087	59
2	657542	41.2	949752	10.7	707790	52.0	292210	45451	89074	58
3	657790	41.2	949688	10.8	708102	52.0	291898	45477	89061	57
4	658037	41.2	949623	10.8	708414	51.9	291586	45503	89048	56
5	658284	41.2	949558	10.8	708726	51.9	291274	45529	89035	55
6	658531	41.1	949494	10.8	709037	51.9	290963	45554	89021	54
7	658778	41.1	949429	10.8	709349	51.9	290651	45580	89008	53
8	659025	41.1	949364	10.8	709660	51.9	290340	45606	88995	52
9	659271	41.0	949300	10.8	709971	51.8	290029	45632	88981	51
10	659517	41.0	949235	10.8	710282	51.8	289718	45658	88968	50
11	9.659763	41.0	9.949170	10.8	9.710593	51.8	10.289407	45684	88955	49
12	660009	40.9	949105	10.8	710904	51.8	289096	45710	88942	48
13	660255	40.9	949040	10.8	711215	51.8	288785	45736	88928	47
14	660501	40.9	948975	10.8	711525	51.7	288475	45762	88915	46
15	660746	40.9	948910	10.8	711836	51.7	288164	45787	88902	45
16	660991	40.8	948845	10.8	712146	51.7	287854	45813	88888	44
17	661236	40.8	948780	10.9	712456	51.7	287544	45839	88875	43
18	661481	40.8	948715	10.9	712766	51.6	287234	45865	88862	42
19	661726	40.7	948650	10.9	713076	51.6	286924	45891	88848	41
20	661970	40.7	948584	10.9	713386	51.6	286614	45917	88835	40
21	9.662214	40.7	9.948519	10.9	9.713696	51.6	10.286304	45942	88822	39
22	662459	40.7	948454	10.9	714005	51.6	285995	45968	88808	38
23	662703	40.6	948388	10.9	714314	51.5	285686	45994	88795	37
24	662946	40.6	948323	10.9	714624	51.5	285376	46020	88782	36
25	663190	40.6	948257	10.9	714933	51.5	285067	46046	88768	35
26	663433	40.5	948192	10.9	715242	51.5	284758	46072	88755	34
27	663677	40.5	948126	10.9	715551	51.4	284449	46097	88741	33
28	663920	40.5	948060	10.9	715860	51.4	284140	46123	88728	32
29	664163	40.5	947995	11.0	716168	51.4	283832	46149	88715	31
30	664406	40.4	947929	11.0	716477	51.4	283523	46175	88701	30
31	9.664648	40.4	9.947863	11.0	9.716785	51.4	10.283215	46201	88688	29
32	664891	40.4	947797	11.0	717093	51.3	282907	46226	88674	28
33	665133	40.3	947731	11.0	717401	51.3	282599	46252	88661	27
34	665375	40.3	947665	11.0	717709	51.3	282291	46278	88647	26
35	665617	40.3	947600	11.0	718017	51.3	281983	46304	88634	25
36	665859	40.2	947533	11.0	718325	51.3	281675	46330	88620	24
37	666100	40.2	947467	11.0	718633	51.2	281367	46355	88607	23
38	666342	40.2	947401	11.0	718940	51.2	281060	46381	88593	22
39	666583	40.2	947335	11.0	719248	51.2	280752	46407	88580	21
40	666824	40.1	947269	11.0	719555	51.2	280445	46433	88566	20
41	9.667065	40.1	9.947203	11.0	9.719862	51.2	10.280138	46458	88553	19
42	667305	40.1	947136	11.1	720169	51.1	279831	46484	88539	18
43	667546	40.1	947070	11.1	720476	51.1	279524	46510	88526	17
44	667786	40.0	947004	11.1	720783	51.1	279217	46536	88512	16
45	668027	40 0	946937	11.1	721089	51.1	278911	46561	88499	15
46	668267	40.0	946871	11.1	721396	51.1	278604	46587	88485	14
47	668506	39.9	946804	11.1	721702	51.0	278298	46613	88472	13
48	668746	39.9	946738	11.1	722009	51.0	277991	46639	88458	12
49	668986	39.9	946671	11.1	722315	51.0	277685	46664	88445	11
50	669225	39.9	946604	11.1	722621	51.0	277379	46690	88431	10
51	9.669464	39.8	9.946538	11.1	9.722927	51.0	10.277073	46716	88417	9
52	669703	39.8	946471	11.1	723232	50.9	276768	46742	88404	8
53	669942	39.8	946404	11.1	723538	50.9	276462	46767	88390	7
54	670181	39.7	946337	11.1	723844	50.9	276156	46793	88377	6
55	670419	39.7	946270	11.2	724149	50.9	275851	46819	88363	5
56	670658	39.7	946203	11.2	724454	50.9	275546	46844	88349	4
57	670896	39.7	946136	11.2	724759	50.8	275241	46870	88336	3
58	671134	39.6	946069	11.2	725065	50.8	274935	46896	88322	2
59	671372	39.6	946002	11.2	725369	50.8	274631	46921	88308	1
60	671609		945935		725674		274326	46947	88295	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N. sine.	′

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	9.671609	39.6	9.945935	11.2	9.725674	50.8	10.274326	46947	88295	60
1	671847	39.5	945868	11.2	725979	50.8	274021	46973	88281	59
2	672084	39.5	945800	11.2	726284	50.7	273716	46999	88267	58
3	672321	39.5	945733	11.2	726588	50.7	273412	47024	88254	57
4	672558	39.5	945666	11.2	726892	50.7	273108	47050	88240	56
5	672795	39.4	945598	11.2	727197	50.7	272803	47076	88226	55
6	673032	39.4	945531	11.2	727501	50.7	272499	47101	88213	54
7	673268	39.4	945464	11.3	727805	50.6	272195	47127	88199	53
8	673505	39.4	945396	11.3	728109	50.6	271891	47153	88185	52
9	673741	39.3	945328	11.3	728412	50.6	271588	47178	88172	51
10	673977	39.3	945261	11.3	728716	50.6	271284	47204	88158	50
11	9.674213	39.3	9.945193	11.3	9.729020	50.6	10.270980	47229	88144	49
12	674448	39.2	945125	11.3	729323	50.5	270677	47255	88130	48
13	674684	39.2	945058	11.3	729626	50.5	270374	47281	88117	47
14	674919	39.2	944990	11.3	729929	50.5	270071	47306	88103	46
15	675155	39.2	944922	11.3	730233	50.5	269767	47332	88089	45
16	675390	39.1	944854	11.3	730535	50.5	269465	47358	88075	44
17	675624	39.1	944786	11.3	730838	50.4	269162	47383	88062	43
18	675859	39.1	944718	11.3	731141	50.4	268859	47409	88048	42
19	676094	39.1	944650	11.3	731444	50.4	268556	47434	88034	41
20	676328	39.0	944582	11.4	731746	50.4	268254	47460	88020	40
21	9.676562	39.0	9.944514	11.4	9.732048	50.4	10.267952	47486	88006	39
22	676796	39.0	944446	11.4	732351	50.3	267649	47511	87993	38
23	677030	39.0	944377	11.4	732653	50.3	267347	47537	87979	37
24	677264	38.9	944309	11.4	732955	50.3	267045	47562	87965	36
25	677498	38.9	944241	11.4	733257	50.3	266743	47588	87951	35
26	677731	38.9	944172	11.4	733558	50.3	266442	47614	87937	34
27	677964	38.8	944104	11.4	733860	50.2	266140	47639	87923	33
28	678197	38.8	944036	11.4	734162	50.2	265838	47665	87909	32
29	678430	38.8	943967	11.4	734463	50.2	265537	47690	87896	31
30	678663	38.8	943899	11.4	734764	50.2	265236	47716	87882	30
31	9.678895	38.7	9.943830	11.4	9.735066	50.2	10.264934	47741	87868	29
32	679128	38.7	943761	11.4	735367	50.2	264633	47767	87854	28
33	679360	38.7	943693	11.5	735668	50.1	264332	47793	87840	27
34	679592	38.7	943624	11.5	735969	50.1	264031	47818	87826	26
35	679824	38.6	943555	11.5	736269	50.1	263731	47844	87812	25
36	680056	38.6	943486	11.5	736570	50.1	263430	47869	87798	24
37	680288	38.6	943417	11.5	736871	50.1	263129	47895	87784	23
38	680519	38.5	943348	11.5	737171	50.0	262829	47920	87770	22
39	680750	38.5	943279	11.5	737471	50.0	262529	47946	87756	21
40	680982	38.5	943210	11.5	737771	50.0	262229	47971	87743	20
41	9.681213	38.5	9.943141	11.5	9.738071	50.0	10.261929	47997	87729	19
42	681443	38.4	943072	11.5	738371	50.0	261629	48022	87715	18
43	681674	38.4	943003	11.5	738671	49.9	261329	48048	87701	17
44	681905	38.4	942934	11.5	738971	49.9	261029	48073	87687	16
45	682135	38.4	942864	11.5	739271	49.9	260729	48099	87673	15
46	682365	38.3	942795	11.6	739570	49.9	260430	48124	87659	14
47	682595	38.3	942726	11.6	739870	49.9	260130	48150	87645	13
48	682825	38.3	942656	11.6	740169	49.9	259831	48175	87631	12
49	683055	38.3	942587	11.6	740468	49.8	259532	48201	87617	11
50	683284	38.2	942517	11.6	740767	49.8	259233	48226	87603	10
51	9.683514	38.2	9.942448	11.6	9.741066	49.8	10.258934	48252	87589	9
52	683743	38.2	942378	11.6	741365	49.8	258635	48277	87575	8
53	683972	38.2	942308	11.6	741664	49.8	258336	48303	87561	7
54	684201	38.1	942239	11.6	741962	49.7	258038	48328	87546	6
55	684430	38.1	942169	11.6	742261	49.7	257739	48354	87532	5
56	684658	38.1	942099	11.6	742559	49.7	257441	48379	87518	4
57	684887	38.0	942029	11.6	742858	49.7	257142	48405	87504	3
58	685115	38.0	941959	11.6	743156	49.7	256844	48430	87490	2
59	685343	38.0	941889	11.7	743454	49.7	256546	48456	87476	1
60	685571		941819		743752		256248	48481	87462	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N.sine.	′

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	9.685571	38.0	9.941819	11.7	9.743752	49.6	10.256248	48481	87462	60
1	685799	37.9	941749	11.7	744050	49.6	255950	48506	87448	59
2	686027	37.9	941679	11.7	744348	49.6	255652	48532	87434	58
3	686254	37.9	941609	11.7	744645	49.6	255355	48557	87420	57
4	686482	37.9	941539	11.7	744943	49.6	255057	48583	87406	56
5	686709	37.8	941469	11.7	745240	49.6	254760	48608	87391	55
6	686936	37.8	941398	11.7	745538	49.5	254462	48634	87377	54
7	687163	37.8	941328	11.7	745835	49.5	254165	48659	87363	53
8	687389	37.8	941258	11.7	746132	49.5	253868	48684	87349	52
9	687616	37.7	941187	11.7	746429	49.5	253571	48710	87335	51
10	687843	37.7	941117	11.7	746726	49.5	253274	48735	87321	50
11	9.688069	37.7	9.941046	11.8	9.747023	49.4	10.252977	48761	87306	49
12	688295	37.7	940975	11.8	747319	49.4	252681	48786	87292	48
13	688521	37.6	940905	11.8	747616	49.4	252384	48811	87278	47
14	688747	37.6	940834	11.8	747913	49.4	252087	48837	87264	46
15	688972	37.6	940763	11.8	748209	49.4	251791	48862	87250	45
16	689198	37.6	940693	11.8	748505	49.3	251495	48888	87235	44
17	689423	37.5	940622	11.8	748801	49.3	251199	48913	87221	43
18	689648	37.5	940551	11.8	749097	49.3	250903	48938	87207	42
19	689873	37.5	940480	11.8	749393	49.3	250607	48964	87193	41
20	690098	37.5	940409	11.8	749689	49.3	250311	48989	87178	40
21	9.690323	37.4	9.940338	11.8	9.749985	49.3	10.250015	49014	87164	39
22	690548	37.4	940267	11.8	750281	49.2	249719	49040	87150	38
23	690772	37.4	940196	11.8	750576	49.2	249424	49065	87136	37
24	690996	37.4	940125	11.9	750872	49.2	249128	49090	87121	36
25	691220	37.3	940054	11.9	751167	49.2	248833	49116	87107	35
26	691444	37.3	939982	11.9	751462	49.2	248538	49141	87093	34
27	691668	37.3	939911	11.9	751757	49.2	248243	49166	87079	33
28	691892	37.3	939840	11.9	752052	49.1	247948	49192	87064	32
29	692115	37.2	939768	11.9	752347	49.1	247653	49217	87050	31
30	692339	37.2	939697	11.9	752642	49.1	247358	49242	87036	30
31	9.692562	37.2	9.939625	11.9	9.752937	49.1	10.247063	49268	87021	29
32	692785	37.1	939554	11.9	753231	49.1	246769	49293	87007	28
33	693008	37.1	939482	11.9	753526	49.1	246474	49318	86993	27
34	693231	37.1	939410	11.9	753820	49.0	246180	49344	86978	26
35	693453	37.1	939339	11.9	754115	49.0	245885	49369	86964	25
36	693676	37.0	939267	12.0	754409	49.0	245591	49394	86949	24
37	693898	37.0	939195	12.0	754703	49.0	245297	49419	86935	23
38	694120	37.0	939123	12.0	754997	49.0	245003	49445	86921	22
39	694342	37.0	939052	12.0	755291	49.0	244709	49470	86906	21
40	694564	36.9	938980	12.0	755585	48.9	244415	49495	86892	20
41	9.694786	36.9	9.938908	12.0	9.755878	48.9	10.244122	49521	86878	19
42	695007	36.9	938836	12.0	756172	48.9	243828	49546	86863	18
43	695229	36.9	938763	12.0	756465	48.9	243535	49571	86849	17
44	695450	36.8	938691	12.0	756759	48.9	243241	49596	86834	16
45	695671	36.8	938619	12.0	757052	48.9	242948	49622	86820	15
46	695892	36.8	938547	12.0	757345	48.8	242655	49647	86805	14
47	696113	36.8	938475	12.0	757638	48.8	242362	49672	86791	13
48	696334	36.7	938402	12.1	757931	48.8	242069	49697	86777	12
49	696554	36.7	938330	12.1	758224	48.8	241776	49723	86762	11
50	696775	36.7	938258	12.1	758517	48.8	241483	49748	86748	10
51	9.696995	36.7	9.938185	12.1	9.758810	48.8	10.241190	49773	86733	9
52	697215	36.6	938113	12.1	759102	48.7	240898	49798	86719	8
53	697435	36.6	938040	12.1	759395	48.7	240605	49824	86704	7
54	697654	36.6	937967	12.1	759687	48.7	240313	49849	86690	6
55	697874	36.6	937895	12.1	759979	48.7	240021	49874	86675	5
56	698094	36.5	937822	12.1	760272	48.7	239728	49899	86661	4
57	698313	36.5	937749	12.1	760564	48.7	239436	49924	86646	3
58	698532	36.5	937676	12.1	760856	48.6	239144	49950	86632	2
59	698751	36.5	937604	12.1	761148	48.6	238852	49975	86617	1
60	698970		937531		761439		238561	50000	86603	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N. sine.	′

60 Degrees.

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	9.698970	36.4	9.937531	12.1	9.761439	48.6	10.238561	50000	86603	60
1	699189	36.4	937458	12.2	761731	48.6	238269	50025	86588	59
2	699407	36.4	937385	12.2	762023	48.6	237977	50050	86573	58
3	699626	36.4	937312	12.2	762314	48.6	237686	50076	86559	57
4	699844	36.3	937238	12.2	762606	48.5	237394	50101	86544	56
5	700062	36.3	937165	12.2	762897	48.5	237103	50126	86530	55
6	700280	36.3	937092	12.2	763188	48.5	236812	50151	86515	54
7	700498	36.3	937019	12.2	763479	48.5	236521	50176	86501	53
8	700716	36.3	936946	12.2	763770	48.5	236230	50201	86486	52
9	700933	36.2	936872	12.2	764061	48.5	235939	50227	86471	51
10	701151	36.2	936799	12.2	764352	48.4	235648	50252	86457	50
11	9.701368	36.2	9.936725	12.2	9.764643	48.4	10.235357	50277	86442	49
12	701585	36.2	936652	12.3	764933	48.4	235067	50302	86427	48
13	701802	36.1	936578	12.3	765224	48.4	234776	50327	86413	47
14	702019	36.1	936505	12.3	765514	48.4	234486	50352	86398	46
15	702236	36.1	936431	12.3	765805	48.4	234195	50377	86384	45
16	702452	36.1	936357	12.3	766095	48.4	233905	50403	86369	44
17	702669	36.0	936284	12.3	766385	48.3	233615	50428	86354	43
18	702885	36.0	936210	12.3	766675	48.3	233325	50453	86340	42
19	703101	36.0	936136	12.3	766965	48.3	233035	50478	86325	41
20	703317	36.0	936062	12.3	767255	48.3	232745	50503	86310	40
21	9.703533	35.9	9.935988	12.3	9.767545	48.3	10.232455	50528	86295	39
22	703749	35.9	935914	12.3	767834	48.3	232166	50553	86281	38
23	703964	35.9	935840	12.3	768124	48.2	231876	50578	86266	37
24	704179	35.9	935766	12.4	768413	48.2	231587	50603	86251	36
25	704395	35.9	935692	12.4	768703	48.2	231297	50628	86237	35
26	704610	35.8	935618	12.4	768992	48.2	231008	50654	86222	34
27	704825	35.8	935543	12.4	769281	48.2	230719	50679	86207	33
28	705040	35.8	935469	12.4	769570	48.2	230430	50704	86192	32
29	705254	35.8	935395	12.4	769860	48.1	230140	50729	86178	31
30	705469	35.7	935320	12.4	770148	48.1	229852	50754	86163	30
31	9.705683	35.7	9.935246	12.4	9.770437	48.1	10.229563	50779	86148	29
32	705898	35.7	935171	12.4	770726	48.1	229274	50804	86133	28
33	706112	35.7	935097	12.4	771015	48.1	228985	50829	86119	27
34	706326	35.6	935022	12.4	771303	48.1	228697	50854	86104	26
35	706539	35.6	934948	12.4	771592	48.0	228408	50879	86089	25
36	706753	35.6	934873	12.4	771880	48.0	228120	50904	86074	24
37	706967	35.6	934798	12.5	772168	48.0	227832	50929	86059	23
38	707180	35.5	934723	12.5	772457	48.0	227543	50954	86045	22
39	707393	35.5	934649	12.5	772745	48.0	227255	50979	86030	21
40	707606	35.5	934574	12.5	773033	48.0	226967	51004	86015	20
41	9.707819	35.5	9.934499	12.5	9.773321	48.0	10.226679	51029	86000	19
42	708032	35.4	934424	12.5	773608	47.9	226392	51054	85985	18
43	708245	35.4	934349	12.5	773896	47.9	226104	51079	85970	17
44	708458	35.4	934274	12.5	774184	47.9	225816	51104	85956	16
45	708670	35.4	934199	12.5	774471	47.9	225529	51129	85941	15
46	708882	35.3	934123	12.5	774759	47.9	225241	51154	85926	14
47	709094	35.3	934048	12.5	775046	47.9	224954	51179	85911	13
48	709306	35.3	933973	12.5	775333	47.9	224667	51204	85896	12
49	709518	35.3	933898	12.6	775621	47.8	224379	51229	85881	11
50	709730	35.3	933822	12.6	775908	47.8	224092	51254	85866	10
51	9.709941	35.2	9.933747	12.6	9.776195	47.8	10.223805	51279	85851	9
52	710153	35.2	933671	12.6	776482	47.8	223518	51304	85836	8
53	710364	35.2	933596	12.6	776769	47.8	223231	51329	85821	7
54	710575	35.2	933520	12.6	777055	47.8	222945	51354	85806	6
55	710786	35.1	933445	12.6	777342	47.8	222658	51379	85792	5
56	710967	35.1	933369	12.6	777628	47.7	222372	51404	85777	4
57	711208	35.1	933293	12.6	777915	47.7	222085	51429	85762	3
58	711419	35.1	933217	12.6	778201	47.7	221799	51454	85747	2
59	711629	35.0	933141	12.6	778487	47.7	221512	51479	85732	1
60	711839		933066		778774		221226	51504	85717	0
	Cosine.		Sine		Cotang.		Tang.	N. cos.	N.sine.	′

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N.sine.	N. cos.	
0	9.711839	35.0	9.933066	12.6	9.778774	47.7	10.221226	51504	85717	60
1	712050	35.0	932990	12.7	779060	47.7	220940	51529	85702	59
2	712260	35.0	932914	12.7	779346	47.6	220654	51554	85687	58
3	712469	34.9	932838	12.7	779632	47.6	220368	51579	85672	57
4	712679	34.9	932762	12.7	779918	47.6	220082	51604	85657	56
5	712889	34.9	932685	12.7	780203	47.6	219797	51628	85642	55
6	713098	34.9	932609	12.7	780489	47.6	219511	51653	85627	54
7	713308	34.9	932533	12.7	780775	47.6	219225	51678	85612	53
8	713517	34.8	932457	12.7	781060	47.6	218940	51703	85597	52
9	713726	34.8	932380	12.7	781346	47.5	218654	51728	85582	51
10	713935	34.8	932304	12.7	781631	47.5	218369	51753	85567	50
11	9.714144	34.8	9.932228	12.7	9.781916	47.5	10.218084	51778	85551	49
12	714352	34.7	932151	12.7	782201	47.5	217799	51803	85536	48
13	714561	34.7	932075	12.8	782486	47.5	217514	51828	85521	47
14	714769	34.7	931998	12.8	782771	47.5	217229	51852	85506	46
15	714978	34.7	931921	12.8	783056	47.5	216944	51877	85491	45
16	715186	34.7	931845	12.8	783341	47.5	216659	51902	85476	44
17	715394	34.6	931768	12.8	783626	47.4	216374	51927	85461	43
18	715602	34.6	931691	12.8	783910	47.4	216090	51952	85446	42
19	715809	34.6	931614	12.8	784195	47.4	215805	51977	85431	41
20	716017	34.6	931537	12.8	784479	47.4	215521	52002	85416	40
21	9.716224	34.5	9.931460	12.8	9.784764	47.4	10.215236	52026	85401	39
22	716432	34.5	931383	12.8	785048	47.4	214952	52051	85385	38
23	716639	34.5	931306	12.8	785332	47.3	214668	52076	85370	37
24	716846	34.5	931229	12.9	785616	47.3	214384	52101	85355	36
25	717053	34.5	931152	12.9	785900	47.3	214100	52126	85340	35
26	717259	34.4	931075	12.9	786184	47.3	213816	52151	85325	34
27	717466	34.4	930998	12.9	786468	47.3	213532	52175	85310	33
28	717673	34.4	930921	12.9	786752	47.3	213248	52200	85294	32
29	717879	34.4	930843	12.9	787036	47.3	212964	52225	85279	31
30	718085	34.3	930766	12.9	787319	47.2	212681	52250	85264	30
31	9.718291	34.3	9.930688	12.9	9.787603	47.2	10.212397	52275	85249	29
32	718497	34.3	930611	12.9	787886	47.2	212114	52299	85234	28
33	718703	34.3	930533	12.9	788170	47.2	211830	52324	85218	27
34	718909	34.3	930456	12.9	788453	47.2	211547	52349	85203	26
35	719114	34.2	930378	12.9	788736	47.2	211264	52374	85188	25
36	719320	34.2	930300	13.0	789019	47.2	210981	52399	85173	24
37	719525	34.2	930223	13.0	789302	47.1	210698	52423	85157	23
38	719730	34.2	930145	13.0	789585	47.1	210415	52448	85142	22
39	719935	34.1	930067	13.0	789868	47.1	210132	52473	85127	21
40	720140	34.1	929989	13.0	790151	47.1	209849	52498	85112	20
41	9.720345	34.1	9.929911	13.0	9.790433	47.1	10.209567	52522	85096	19
42	720549	34.1	929833	13.0	790716	47.1	209284	52547	85081	18
43	720754	34.0	929755	13.0	790999	47.1	209001	52572	85066	17
44	720958	34.0	929677	13.0	791281	47.1	208719	52597	85051	16
45	721162	34.0	929599	13.0	791563	47.0	208437	52621	85035	15
46	721366	34.0	929521	13.0	791846	47.0	208154	52646	85020	14
47	721570	34.0	929442	13.0	792128	47.0	207872	52671	85005	13
48	721774	33.9	929364	13.1	792410	47.0	207590	52696	84989	12
49	721978	33.9	929286	13.1	792692	47.0	207308	52720	84974	11
50	722181	33.9	929207	13.1	792974	47.0	207026	52745	84959	10
51	9.722385	33.9	9.929129	13.1	9.793256	47.0	10.206744	52770	84943	9
52	722588	33.9	929050	13.1	793538	46.9	206462	52794	84928	8
53	722791	33.8	928972	13.1	793819	46.9	206181	52819	84913	7
54	722994	33.8	928893	13.1	794101	46.9	205899	52844	84897	6
55	723197	33.8	928815	13.1	794383	46.9	205617	52869	84882	5
56	723400	33.8	928736	13.1	794664	46.9	205336	52893	84866	4
57	723603	33.7	928657	13.1	794945	46.9	205055	52918	84851	3
58	723805	33.7	928578	13.1	795227	46.9	204773	52943	84836	2
59	724007	33.7	928499	13.1	795508	46.8	204492	52967	84820	1
60	724210		928420		795789		204211	52992	84805	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N.sine.	′

58 Degrees.

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	9.724210	33.7	9.928420	13.2	9.795789	46.8	10.204211	52992	84805	60
1	724412	33.7	928342	13.2	796070	46.8	203930	53017	84789	59
2	724614	33.6	928263	13.2	796351	46.8	203649	53041	84774	58
3	724816	33.6	928183	13.2	796632	46.8	203368	53066	84759	57
4	725017	33.6	928104	13.2	796913	46.8	203087	53091	84743	56
5	725219	33.6	928025	13.2	797194	46.8	202806	53115	84728	55
6	725420	33.5	927946	13.2	797475	46.8	202525	53140	84712	54
7	725622	33.5	927867	13.2	797755	46.8	202245	53164	84697	53
8	725823	33.5	927787	13.2	798036	46.7	201964	53189	84681	52
9	726024	33.5	927708	13.2	798316	46.7	201684	53214	84666	51
10	726225	33.5	927629	13.2	798596	46.7	201404	53238	84650	50
11	9.726426	33.4	9.927549	13.2	9.798877	46.7	10.201123	53263	84635	49
12	726626	33.4	927470	13.3	799157	46.7	200843	53288	84619	48
13	726827	33.4	927390	13.3	799437	46.7	200563	53312	84604	47
14	727027	33.4	927310	13.3	799717	46.7	200283	53337	84588	46
15	727228	33.4	927231	13.3	799997	46.6	200003	53361	84573	45
16	727428	33.3	927151	13.3	800277	46.6	199723	53386	84557	44
17	727628	33.3	927071	13.3	800557	46.6	199443	53411	84542	43
18	727828	33.3	926991	13.3	800836	46.6	199164	53435	84526	42
19	728027	33.3	926911	13.3	801116	46.6	198884	53460	84511	41
20	728227	33.3	926831	13.3	801396	46.6	198604	53484	84495	40
21	9.728427	33.2	9.926751	13.3	9.801675	46.6	10.198325	53509	84480	39
22	728626	33.2	926671	13.3	801955	46.6	198045	53534	84464	38
23	728825	33.2	926591	13.3	802234	46.5	197766	53558	84448	37
24	729024	33.2	926511	13.4	802513	46.5	197487	53583	84433	36
25	729223	33.1	926431	13.4	802792	46.5	197208	53607	84417	35
26	729422	33.1	926351	13.4	803072	46.5	196928	53632	84402	34
27	729621	33.1	926270	13.4	803351	46.5	196649	53656	84386	33
28	729820	33.1	926190	13.4	803630	46.5	196370	53681	84370	32
29	730018	33.0	926110	13.4	803908	46.5	196092	53705	84355	31
30	730216	33.0	926029	13.4	804187	46.5	195813	53730	84339	30
31	9.730415	33.0	9.925949	13.4	9.804466	46.4	10.195534	53754	84324	29
32	730613	33.0	925868	13.4	804745	46.4	195255	53779	84308	28
33	730811	33.0	925788	13.4	805023	46.4	194977	53804	84292	27
34	731009	32.9	925707	13.4	805302	46.4	194698	53828	84277	26
35	731206	32.9	925626	13.4	805580	46.4	194420	53853	84261	25
36	731404	32.9	925545	13.5	805859	46.4	194141	53877	84245	24
37	731602	32.9	925465	13.5	806137	46.4	193863	53902	84230	23
38	731799	32.9	925384	13.5	806415	46.3	193585	53926	84214	22
39	731996	32.8	925303	13.5	806693	46.3	193307	53951	84198	21
40	732193	32.8	925222	13.5	806971	46.3	193029	53975	84182	20
41	9.732390	32.8	9.925141	13.5	9.807249	46.3	10.192751	54000	84167	19
42	732587	32.8	925060	13.5	807527	46.3	192473	54024	84151	18
43	732784	32.8	924979	13.5	807805	46.3	192195	54049	84135	17
44	732980	32.7	924897	13.5	808083	46.3	191917	54073	84120	16
45	733177	32.7	924816	13.5	808361	46.3	191639	54097	84104	15
46	733373	32.7	924735	13.6	808638	46.2	191362	54122	84088	14
47	733569	32.7	924654	13.6	808916	46.2	191084	54146	84072	13
48	733765	32.7	924572	13.6	809193	46.2	190807	54171	84057	12
49	733961	32.6	924491	13.6	809471	46.2	190529	54195	84041	11
50	734157	32.6	924409	13.6	809748	46.2	190252	54220	84025	10
51	9.734353	32.6	9.924328	13.6	9.810025	46.2	10.189975	54244	84009	9
52	734549	32.6	924246	13.6	810302	46.2	189698	54269	83994	8
53	734744	32.5	924164	13.6	810580	46.2	189420	54293	83978	7
54	734939	32.5	924083	13.6	810857	46.2	189143	54317	83962	6
55	735135	32.5	924001	13.6	811134	46.1	188866	54342	83946	5
56	735330	32.5	923919	13.6	811410	46.1	188590	54366	83930	4
57	735525	32.5	923837	13.6	811687	46.1	188313	54391	83915	3
58	735719	32.4	923755	13.7	811964	46.1	188036	54415	83899	2
59	735914	32.4	923673	13.7	812241	46.1	187759	54440	83883	1
60	736109		923591		812517		187483	54464	83867	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N. sine.	′

57 Degrees.

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	9.736109	32.4	9.923591	13.7	9.812517	46.1	10.187482	54464	83867	60
1	736303	32.4	923509	13.7	812794	46.1	187206	54488	83851	59
2	736498	32.4	923427	13.7	813070	46.1	186930	54513	83835	58
3	736692	32.3	923345	13.7	813347	46.0	186653	54537	83819	57
4	736886	32.3	923263	13.7	813623	46.0	186377	54561	83804	56
5	737080	32.3	923181	13.7	813899	46.0	186101	54586	83788	55
6	737274	32.3	923098	13.7	814175	46.0	185825	54610	83772	54
7	737467	32.3	923016	13.7	814452	46.0	185548	54635	83756	53
8	737661	32.2	922933	13.7	814728	46.0	185272	54659	83740	52
9	737855	32.2	922851	13.7	815004	46.0	184996	54683	83724	51
10	738048	32.2	922768	13.8	815279	46.0	184721	54708	83708	50
11	9.738241	32.2	9.922686	13.8	9.815555	45.9	10.184445	54732	83692	49
12	738434	32.2	922603	13.8	815831	45.9	184169	54756	83676	48
13	738627	32.1	922520	13.8	816107	45.9	183893	54781	83660	47
14	738820	32.1	922438	13.8	816382	45.9	183618	54805	83645	46
15	739013	32.1	922355	13.8	816658	45.9	183342	54829	83629	45
16	739206	32.1	922272	13.8	816933	45.9	183067	54854	83613	44
17	739398	32.1	922189	13.8	817209	45.9	182791	54878	83597	43
18	739590	32.0	922106	13.8	817484	45.9	182516	54902	83581	42
19	739783	32.0	922023	13.8	817759	45.9	182241	54927	83565	41
20	739975	32.0	921940	13.8	818035	45.8	181965	54951	83549	40
21	9.740167	32.0	9.921857	13.9	9.818310	45.8	10.181690	54975	83533	39
22	740359	32.0	921774	13.9	818585	45.8	181415	54999	83517	38
23	740550	31.9	921691	13.9	818860	45.8	181140	55024	83501	37
24	740742	31.9	921607	13.9	819135	45.8	180865	55048	83485	36
25	740934	31.9	921524	13.9	819410	45.8	180590	55072	83469	35
26	741125	31.9	921441	13.9	819684	45.8	180316	55097	83453	34
27	741316	31.9	921357	13.9	819959	45.8	180041	55121	83437	33
28	741508	31.8	921274	13.9	820234	45.8	179766	55145	83421	32
29	741699	31.8	921190	13.9	820508	45.7	179492	55169	83405	31
30	741889	31.8	921107	13.9	820783	45.7	179217	55194	83389	30
31	9.742080	31.8	9.921023	13.9	9.821057	45.7	10.178943	55218	83373	29
32	742271	31.8	920939	14.0	821332	45.7	178668	55242	83356	28
33	742462	31.7	920856	14.0	821606	45.7	178394	55266	83340	27
34	742652	31.7	920772	14.0	821880	45.7	178120	55291	83324	26
35	742842	31.7	920688	14.0	822154	45.7	177846	55315	83308	25
36	743033	31.7	920604	14.0	822429	45.7	177571	55339	83292	24
37	743223	31.7	920520	14.0	822703	45.7	177297	55363	83276	23
38	743413	31.6	920436	14.0	822977	45.6	177023	55388	83260	22
39	743602	31.6	920352	14.0	823250	45.6	176750	55412	83244	21
40	743792	31.6	920268	14.0	823524	45.6	176476	55436	83228	20
41	9.743982	31.6	9.920184	14.0	9.823798	45.6	10.176202	55460	83212	19
42	744171	31.6	920099	14.0	824072	45.6	175928	55484	83195	18
43	744361	31.5	920015	14.0	824345	45.6	175655	55509	83179	17
44	744550	31.5	919931	14.1	824619	45.6	175381	55533	83163	16
45	744739	31.5	919846	14.1	824893	45.6	175107	55557	83147	15
46	744928	31.5	919762	14.1	825166	45.6	174834	55581	83131	14
47	745117	31.5	919677	14.1	825439	45.5	174561	55605	83115	13
48	745306	31.4	919593	14.1	825713	45.5	174287	55630	83098	12
49	745494	31.4	919508	14.1	825986	45.5	174014	55654	83082	11
50	745683	31.4	919424	14.1	826259	45.5	173741	55678	83066	10
51	9.745871	31.4	9.919339	14.1	9.826532	45.5	10.173468	55702	83050	9
52	746059	31.4	919254	14.1	826805	45.5	173195	55726	83034	8
53	746248	31.3	919169	14.1	827078	45.5	172922	55750	83017	7
54	746436	31.3	919085	14.1	827351	45.5	172649	55775	83001	6
55	746624	31.3	919000	14.2	827624	45.5	172376	55799	82985	5
56	746812	31.3	918915	14.2	827897	45.4	172103	55823	82969	4
57	746999	31.3	918830	14.2	828170	45.4	171830	55847	82953	3
58	747187	31.2	918745	14.2	828442	45.4	171558	55871	82936	2
59	747374	31.2	918659	14.2	828715	45.4	171285	55895	82920	1
60	747562		918574		828987		171013	55919	82904	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N.sine.	′

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine	N. cos.	
0	9.747562	31.2	9.918574	14.2	9.828987	45.4	10.171013	55919	82904	60
1	747749	31.2	918489	14.2	829260	45.4	170740	55943	82887	59
2	747936	31.2	918404	14.2	829532	45.4	170468	55968	82871	58
3	748123	31.1	918318	14.2	829805	45.4	170195	55992	82855	57
4	748310	31.1	918233	14.2	830077	45.4	169923	56016	82839	56
5	748497	31.1	918147	14.2	830349	45.3	169651	56040	82822	55
6	748683	31.1	918062	14.2	830621	45.3	169379	56064	82806	54
7	748870	31.1	917976	14.3	830893	45.3	169107	56088	82790	53
8	749056	31.0	917891	14.3	831165	45.3	168835	56112	82773	52
9	749243	31.0	917805	14.3	831437	45.3	168563	56136	82757	51
10	749426	31.0	917719	14.3	831709	45.3	168291	56160	82741	50
11	9.749615	31.0	9.917634	14.3	9.831981	45.3	10.168019	56184	82724	49
12	749801	31.0	917548	14.3	832253	45.3	167747	56208	82708	48
13	749987	30.9	917462	14.3	832525	45.3	167475	56232	82692	47
14	750172	30.9	917376	14.3	832796	45.3	167204	56256	82675	46
15	750358	30.9	917290	14.3	833068	45.2	166932	56280	82659	45
16	750543	30.9	917204	14.3	833339	45.2	166661	56305	82643	44
17	750729	30.9	917118	14.4	833611	45.2	166389	56329	82626	43
18	750914	30.8	917032	14.4	833882	45.2	166118	56353	82610	42
19	751099	30.8	916946	14.4	834154	45.2	165846	56377	82593	41
20	751284	30.8	916859	14.4	834425	45.2	165575	56401	82577	40
21	9.751469	30.8	9.916773	14.4	9.834696	45.2	10.165304	56425	82561	39
22	751654	30.8	916687	14.4	834967	45.2	165033	56449	82544	38
23	751839	30.8	916600	14.4	835238	45.2	164762	56473	82528	37
24	752023	30.7	916514	14.4	835509	45.2	164491	56497	82511	36
25	752208	30.7	916427	14.4	835780	45.1	164220	56521	82495	35
26	752392	30.7	916341	14.4	836051	45.1	163949	56545	82478	34
27	752576	30.7	916254	14.4	836322	45.1	163678	56569	82462	33
28	752760	30.7	916167	14.5	836593	45.1	163407	56593	82446	32
29	752944	30.6	916081	14.5	836864	45.1	163136	56617	82429	31
30	753128	30.6	915994	14.5	837134	45.1	162866	56641	82413	30
31	9.753312	30.6	9.915907	14.5	9.837405	45.1	10.162595	56665	82396	29
32	753495	30.6	915820	14.5	837675	45.1	162325	56689	82380	28
33	753679	30.6	915733	14.5	837946	45.1	162054	56713	82363	27
34	753862	30.5	915646	14.5	838216	45.1	161784	56736	82347	26
35	754046	30.5	915559	14.5	838487	45.0	161513	56760	82330	25
36	754229	30.5	915472	14.5	838757	45.0	161243	56784	82314	24
37	754412	30.5	915385	14.5	839027	45.0	160973	56808	82297	23
38	754595	30.5	915297	14.5	839297	45.0	160703	56832	82281	22
39	754778	30.4	915210	14.5	839568	45.0	160432	56856	82264	21
40	754960	30.4	915123	14.6	839838	45.0	160162	56880	82248	20
41	9.755143	30.4	9.915035	14.6	9.840108	45.0	10.159892	56904	82231	19
42	755326	30.4	914948	14.6	840378	45.0	159622	56928	82214	18
43	755508	30.4	914860	14.6	840647	45.0	159353	56952	82198	17
44	755690	30.4	914773	14.6	840917	44.9	159083	56976	82181	16
45	755872	30.3	914685	14.6	841187	44.9	158813	57000	82165	15
46	756054	30.3	914598	14.6	841457	44.9	158543	57024	82148	14
47	756236	30.3	914510	14.6	841726	44.9	158274	57047	82132	13
48	756418	30.3	914422	14.6	841996	44.9	158004	57071	82115	12
49	756600	30.3	914334	14.6	842266	44.9	157734	57095	82098	11
50	756782	30.2	914246	14.7	842535	44.9	157465	57119	82082	10
51	9.756963	30.2	9.914158	14.7	9.842805	44.9	10.157195	57143	82065	9
52	757144	30.2	914070	14.7	843074	44.9	156926	57167	82048	8
53	757326	30.2	913982	14.7	843343	44.9	156657	57191	82032	7
54	757507	30.2	913894	14.7	843612	44.9	156388	57215	82015	6
55	757688	30.1	913806	14.7	843882	44.8	156118	57238	81999	5
56	757869	30.1	913718	14.7	844151	44.8	155849	57262	81982	4
57	758050	30.1	913630	14.7	844420	44.8	155580	57286	81965	3
58	758230	30.1	913541	14.7	844689	44.8	155311	57310	81949	2
59	758411	30.1	913453	14.7	844958	44.8	155042	57334	81932	1
60	758591		913365		845227		154773	57358	81915	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N. sine.	′

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	9.758591	30.1	9.913365	14.7	9.845227	44.8	10.154773	57358	81915	60
1	758772	30.0	913276	14.7	845496	44.8	154504	57381	81899	59
2	758952	30.0	913187	14.8	845764	44.8	154236	57405	81882	58
3	759132	30.0	913099	14.8	846033	44.8	153967	57429	81865	57
4	759312	30.0	913010	14.8	846302	44.8	153698	57453	81848	56
5	759492	30.0	912922	14.8	846570	44.7	153430	57477	81832	55
6	759672	29.9	912833	14.8	846839	44.7	153161	57501	81815	54
7	759852	29.9	912744	14.8	847107	44.7	152893	57524	81798	53
8	760031	29.9	912655	14.8	847376	44.7	152624	57548	81782	52
9	760211	29.9	912566	14.8	847644	44.7	152356	57572	81765	51
10	760390	29.9	912477	14.8	847913	44.7	152087	57596	81748	50
11	9.760569	29.8	9.912388	14.8	9.848181	44.7	10.151819	57619	81731	49
12	760748	29.8	912299	14.9	848449	44.7	151551	57643	81714	48
13	760927	29.8	912210	14.9	848717	44.7	151283	57667	81698	47
14	761106	29.8	912121	14.9	848986	44.7	151014	57691	81681	46
15	761285	29.8	912031	14.9	849254	44.7	150746	57715	81664	45
16	761464	29.8	911942	14.9	849522	44.7	150478	57738	81647	44
17	761642	29.7	911853	14.9	849790	44.6	150210	57762	81631	43
18	761821	29.7	911763	14.9	850058	44.6	149942	57786	81614	42
19	761999	29.7	911674	14.9	850325	44.6	149675	57810	81597	41
20	762177	29.7	911584	14.9	850593	44.6	149407	57833	81580	40
21	9.762356	29.7	9.911495	14.9	9.850861	44.6	10.149139	57857	81563	39
22	762534	29.6	911405	14.9	851129	44.6	148871	57881	81546	38
23	762712	29.6	911315	15.0	851396	44.6	148604	57904	81530	37
24	762889	29.6	911226	15.0	851664	44.6	148336	57928	81513	36
25	763067	29.6	911136	15.0	851931	44.6	148069	57952	81496	35
26	763245	29.6	911046	15.0	852199	44.6	147801	57976	81479	34
27	763422	29.6	910956	15.0	852466	44.6	147534	57999	81462	33
28	763600	29.5	910866	15.0	852733	44.5	147267	58023	81445	32
29	763777	29.5	910776	15.0	853001	44.5	146999	58047	81428	31
30	763954	29.5	910686	15.0	853268	44.5	146732	58070	81412	30
31	9.764131	29.5	9.910596	15.0	9.853535	44.5	10.146465	58094	81395	29
32	764308	29.5	910506	15.0	853802	44.5	146198	58118	81378	28
33	764485	29.4	910415	15.0	854069	44.5	145931	58141	81361	27
34	764662	29.4	910325	15.1	854336	44.5	145664	58165	81344	26
35	764838	29.4	910235	15.1	854603	44.5	145397	58189	81327	25
36	765015	29.4	910144	15.1	854870	44.5	145130	58212	81310	24
37	765191	29.4	910054	15.1	855137	44.5	144863	58236	81293	23
38	765367	29.4	909963	15.1	855404	44.5	144596	58260	81276	22
39	765544	29.3	909873	15.1	855671	44.4	144329	58283	81259	21
40	765720	29.3	909782	15.1	855938	44.4	144062	58307	81242	20
41	9.765896	29.3	9.909691	15.1	9.856204	44.4	10.143796	58330	81225	19
42	766072	29.3	909601	15.1	856471	44.4	143529	58354	81208	18
43	766247	29.3	909510	15.1	856737	44.4	143263	58378	81191	17
44	766423	29.3	909419	15.1	857004	44.4	142996	58401	81174	16
45	766598	29.2	909328	15.2	857270	44.4	142730	58425	81157	15
46	766774	29.2	909237	15.2	857537	44.4	142463	58449	81140	14
47	766949	29.2	909146	15.2	857803	44.4	142197	58472	81123	13
48	767124	29.2	909055	15.2	858069	44.4	141931	58496	81106	12
49	767300	29.2	908964	15.2	858336	44.4	141664	58519	81089	11
50	767475	29.1	908873	15.2	858602	44.3	141398	58543	81072	10
51	9.767649	29.1	9.908781	15.2	9.858868	44.3	10.141132	58567	81055	9
52	767824	29.1	908690	15.2	859134	44.3	140866	58590	81038	8
53	767999	29.1	908599	15.2	859400	44.3	140600	58614	81021	7
54	768173	29.1	908507	15.2	859666	44.3	140334	58637	81004	6
55	768348	29.0	908416	15.3	859932	44.3	140068	58661	80987	5
56	768522	29.0	908324	15.3	850198	44.3	139802	58684	80970	4
57	768697	29.0	908233	15.3	850464	44.3	139536	58708	80953	3
58	768871	29.0	908141	15.3	850730	44.3	139270	58731	80936	2
59	769045	29.0	908049	15.3	850995	44.3	139005	58755	80919	1
60	769219		907958		851261		138739	58779	80902	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N. sine.	′

54 Degrees.

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	9.769219	29.0	9.907958	15.3	9.861261	44.3	10.138739	58779	80902	60
1	769393	28.9	907866	15.3	861527	44.3	138473	58802	80885	59
2	769566	28.9	907774	15.3	861792	44.2	138208	58826	80867	58
3	769740	28.9	907682	15.3	862058	44.2	137942	58849	80850	57
4	769913	28.9	907590	15.3	862323	44.2	137677	58873	80833	56
5	770087	28.9	907498	15.3	862589	44.2	137411	58896	80816	55
6	770260	28.8	907406	15.3	862854	44.2	137146	58920	80799	54
7	770433	28.8	907314	15.4	863119	44.2	136881	58943	80782	53
8	770606	28.8	907222	15.4	863385	44.2	136615	58967	80765	52
9	770779	28.8	907129	15.4	863650	44.2	136350	58990	80748	51
10	770952	28.8	907037	15.4	863915	44.2	136085	59014	80730	50
11	9.771125	28.8	9.906945	15.4	9.864180	44.2	10.135820	59037	80713	49
12	771298	28.7	906852	15.4	864445	44.2	135555	59061	80696	48
13	771470	28.7	906760	15.4	864710	44.2	135290	59084	80679	47
14	771643	28.7	906667	15.4	864975	44.1	135025	59108	80662	46
15	771815	28.7	906575	15.4	865240	44.1	134760	59131	80644	45
16	771987	28.7	906482	15.4	865505	44.1	134495	59154	80627	44
17	772159	23.7	906389	15.5	865770	44.1	134230	59178	80610	43
18	772331	28.6	906296	15.5	866035	44.1	133965	59201	80593	42
19	772503	28.6	906204	15.5	866300	44.1	133700	59225	80576	41
20	772675	28.6	906111	15.5	866564	44.1	133436	59248	80558	40
21	9.772847	28.6	9.906018	15.5	9.866829	44.1	10.133171	59272	80541	39
22	773018	28.6	905925	15.5	867094	44.1	132906	59295	80524	38
23	773190	28.6	905832	15.5	867358	44.1	132642	59318	80507	37
24	773361	28.5	905739	15.5	867623	44.1	132377	59342	80489	36
25	773533	28.5	905645	15.5	867887	44.1	132113	59365	80472	35
26	773704	28.5	905552	15.5	868152	44.0	131848	59389	80455	34
27	773875	28.5	905459	15.5	868416	44.0	131584	59412	80438	33
28	774046	28.5	905366	15.6	868680	44.0	131320	59436	80422	32
29	774217	28.5	905272	15.6	868945	44.0	131055	59459	80403	31
30	774388	28.4	905179	15.6	869209	44.0	130791	59482	80386	30
31	9.774558	28.4	9.905085	15.6	9.869473	44.0	10.130527	59506	80368	29
32	774729	28.4	904992	15.6	869737	44.0	130263	59529	80351	28
33	774899	28.4	904898	15.6	870001	44.0	129999	59552	80334	27
34	775070	28.4	904804	15.6	870265	44.0	129735	59576	80316	26
35	775240	28.4	904711	15.6	870529	44.0	129471	59599	80299	25
36	775410	28.3	904617	15.6	870793	44.0	129207	59622	80282	24
37	775580	28.3	904523	15.6	871057	44.0	128943	59646	80264	23
38	775750	28.3	904429	15.7	871321	44.0	128679	59669	80247	22
39	775920	28.3	904335	15.7	871585	44.0	128415	59693	80230	21
40	776090	28.3	904241	15.7	871849	43.9	128151	59716	80212	20
41	9.776259	28.3	9.904147	15.7	9.872112	43.9	10.127888	59739	80195	19
42	776429	28.2	904053	15.7	872376	43.9	127624	59763	80178	18
43	776598	28.2	903959	15.7	872640	43.9	127360	59786	80160	17
44	776768	28.2	903864	15.7	872903	43.9	127097	59809	80143	16
45	776937	28.2	903770	15.7	873167	43.9	126833	59832	80125	15
46	777106	28.2	903676	15.7	873430	43.9	126570	59856	80108	14
47	777275	28.1	903581	15.7	873694	43.9	126306	59879	80091	13
48	777444	28.1	903487	15.7	873957	43.9	126043	59902	80073	12
49	777613	28.1	903392	15.8	874220	43.9	125780	59926	80056	11
50	777781	28.1	903298	15.8	874484	43.9	125516	59949	80038	10
51	9.777950	28.1	9.903202	15.8	9.874747	43.9	10.125253	59972	80021	9
52	778119	28.1	903108	15.8	875010	43.9	124990	59995	80003	8
53	778287	28.0	903014	15.8	875273	43.8	124727	60019	79986	7
54	778455	28.0	902919	15.8	875536	43.8	124464	60042	79968	6
55	778624	28.0	902824	15.8	875800	43.8	124200	60065	79951	5
56	778792	28.0	902729	15.8	876063	43.8	123937	60089	79934	4
57	778960	28.0	902634	15.8	876326	43.8	123674	60112	79916	3
58	779128	28.0	902539	15.9	876589	43.8	123411	60135	79899	2
59	779295	27.9	902444	15.9	876851	43.8	123149	60158	79881	1
60	779463		902349		877114		122886	60182	79864	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N. sine.	′

53 Degrees.

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N.sine.	N. cos.	
0	9.779463	27.9	9.902349	15.9	9.877114	43.8	10.122886	60182	79864	60
1	779631	27.9	902253	15.9	877377	43.8	122623	60205	79846	59
2	779798	27.9	902158	15.9	877640	43.8	122360	60228	79829	58
3	779966	27.9	902063	15.9	877903	43.8	122097	60251	79811	57
4	780133	27.9	901967	15.9	878165	43.8	121835	60274	79793	56
5	780300	27.8	901872	15.9	878428	43.8	121572	60298	79776	55
6	780467	27.8	901776	15.9	878691	43.8	121309	60321	79758	54
7	780634	27.8	901681	15.9	878953	43.7	121047	60344	79741	53
8	780801	27.8	901585	15.9	879216	43.7	120784	60367	79723	52
9	780968	27.8	901490	15.9	879478	43.7	120522	60390	79706	51
10	781134	27.8	901394	16.0	879741	43.7	120259	60414	79688	50
11	9.781301	27.7	9.901298	16.0	9.880003	43.7	10.119997	60437	79671	49
12	781468	27.7	901202	16.0	880265	43.7	119735	60460	79658	48
13	781634	27.7	901106	16.0	880528	43.7	119472	60483	79635	47
14	781800	27.7	901010	16.0	880790	43.7	119210	60506	79618	46
15	781966	27.7	900914	16.0	881052	43.7	118948	60529	79600	45
16	782132	27.7	900818	16.0	881314	43.7	118686	60553	79583	44
17	782298	27.6	900722	16.0	881576	43.7	118424	60576	79565	43
18	782464	27.6	900626	16.0	881839	43.7	118161	60599	79547	42
19	782630	27.6	900529	16.0	882101	43.7	117899	60622	79530	41
20	782796	27.6	900433	16.1	882363	43.6	117637	60645	79512	40
21	9.782961	27.6	9.900337	16.1	9.882625	43.6	10.117375	60668	79494	39
22	783127	27.6	900242	16.1	882887	43.6	117113	60691	79477	38
23	783282	27.5	900144	16.1	883148	43.6	116852	60714	79459	37
24	783458	27.5	900047	16.1	883410	43.6	116590	60738	79441	36
25	783623	27.5	899951	16.1	883672	43.6	116328	60761	79424	35
26	783788	27.5	899854	16.1	883934	43.6	116066	60784	79406	34
27	783953	27.5	899757	16.1	884196	43.6	115804	60807	79388	33
28	784118	27.5	899660	16.1	884457	43.6	115543	60830	79371	32
20	784282	27.4	899564	16.1	884719	43.6	115281	60853	79353	31
39	784447	27.4	899467	16.2	884980	43.6	115020	60876	79335	30
31	9.784612	27.4	9.899370	16.2	9.885242	43.6	10.114758	60899	79318	29
32	784776	27.4	899273	16.2	885503	43.6	114497	60922	79300	28
33	784941	27.4	899176	16.2	885765	43.6	114235	60945	79282	27
34	785105	27.4	899078	16.2	886026	43.6	113974	60968	79264	26
35	785269	27.3	898981	16.2	886288	43.6	113712	60991	79247	25
36	785433	27.3	898884	16.2	886549	43.5	113451	61015	79229	24
37	785597	27.3	898787	16.2	886810	43.5	113190	61038	79211	23
38	785761	27.3	898689	16.2	887072	43.5	112928	61061	79193	22
39	785925	27.3	898592	16.2	887333	43.5	112667	61084	79176	21
40	786089	27.3	898494	16.3	887594	43.5	112406	61107	79158	20
41	9.786252	27.2	9.898397	16.3	9.887855	43.5	10.112145	61130	79140	19
42	786416	27.2	898299	16.3	888116	43.5	111884	61153	79122	18
43	786579	27.2	898202	16.3	888377	43.5	111623	61176	79105	17
44	786742	27.2	898104	16.3	888639	43.5	111361	61199	79087	16
45	786906	27.2	898006	16.3	888900	43.5	111100	61222	79069	15
46	787069	27.2	897908	16.3	889160	43.5	110840	61245	79051	14
47	787232	27.1	897810	16.3	889421	43.5	110579	61268	79033	13
48	787395	27.1	897712	16.3	889682	43.5	110318	61291	79016	12
49	787557	27.1	897614	16.3	889943	43.5	110057	61314	78998	11
50	787720	27.1	897516	16.3	890204	43.4	109796	61337	78980	10
51	9.787883	27.1	9.897418	16.4	9.890465	43.4	10.109535	61360	78962	9
52	788045	27.1	897320	16.4	890725	43.4	109275	61383	78944	8
53	788208	27.1	897222	16.4	890986	43.4	109014	61406	78926	7
54	788370	27.0	897123	16.4	891247	43.4	108753	61429	78908	6
55	788532	27.0	897025	16.4	891507	43.4	108493	61451	78891	5
56	788694	27.0	896926	16.4	891768	43.4	108232	61474	78873	4
57	788856	27.0	896828	16.4	892028	43.4	107972	61497	78855	3
58	789018	27.0	896729	16.4	892289	43.4	107711	61520	78837	2
59	789180	27.0	896631	16.4	892549	43.4	107451	61543	78819	1
60	789342		896532		892810		107190	61566	78801	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N.sine.	′

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	9.789342	26.9	9.896532	16.4	9.892810	43.4	10.107190	61566	78801	60
1	789504	26.9	896433	16.5	893070	43.4	106930	61589	78783	59
2	789665	26.9	896335	16.5	893331	43.4	106669	61612	78765	58
3	789827	26.9	896236	16.5	893591	43.4	106409	61635	78747	57
4	789988	26.9	896137	16.5	893851	43.4	106149	61658	78729	56
5	790149	26.9	896038	16.5	894111	43.4	105889	61681	78711	55
6	790310	26.8	895939	16.5	894371	43.4	105629	61704	78694	54
7	790471	26.8	895840	16.5	894632	43.3	105368	61726	78676	53
8	790632	26.8	895741	16.5	894892	43.3	105108	61749	78658	52
9	790793	26.8	895641	16.5	895152	43.3	104348	61772	78640	51
10	790954	26.8	895542	16.5	895412	43.3	104588	61795	78622	50
11	9.791115	26.8	9.895443	16.6	9.895672	43.3	10.104328	61818	78604	49
12	791275	26.7	895343	16.6	895932	43.3	104068	61841	78586	48
13	791436	26.7	895244	16.6	896192	43.3	103808	61864	78568	47
14	791596	26.7	895145	16.6	896452	43.3	103548	61887	78550	46
15	791757	26.7	895045	16.6	896712	43.3	103288	61909	78532	45
16	791917	26.7	894945	16.6	896971	43.3	103029	61932	78514	44
17	792077	26.7	894846	16.6	897231	43.3	102769	61955	78496	43
18	792237	26.6	894746	16.6	897491	43.3	102509	61978	78478	42
19	792397	26.6	894646	16.6	897751	43.3	102249	62001	78460	41
20	792557	26.6	894546	16.6	898010	43.3	101990	62024	78442	40
21	9.792716	26.6	9.894446	16.7	9.898270	43.3	10.101730	62046	78424	39
22	792876	26.6	894346	16.7	898530	43.3	101470	62069	78405	38
23	793035	26.6	894246	16.7	898789	43.3	101211	62092	78387	37
24	793195	26.5	894146	16.7	899049	43.2	100951	62115	78369	36
25	793354	26.5	894046	16.7	899308	43.2	100692	62138	78351	35
26	793514	26.5	893946	16.7	899568	43.2	100432	62160	78333	34
27	793673	26.5	893846	16.7	899827	43.2	100173	62183	78315	33
28	793832	26.5	893745	16.7	900086	43.2	099914	62206	78297	32
29	793991	26.5	893645	16.7	900346	43.2	099654	62229	78279	31
30	794150	26.4	893544	16.7	900605	43.2	099395	62251	78261	30
31	9.794308	26.4	9.893444	16.8	9.900864	43.2	10.099136	62274	78243	29
32	794467	26.4	893343	16.8	901124	43.2	098876	62297	78225	28
33	794626	26.4	893243	16.8	901383	43.2	098617	62320	78206	27
34	794784	26.4	893142	16.8	901642	43.2	098358	62342	78188	26
35	794942	26.4	893041	16.8	901901	43.2	098099	62365	78170	25
36	795101	26.4	892940	16.8	902160	43.2	097840	62388	78152	24
37	795259	26.3	892839	16.8	902419	43.2	097581	62411	78134	23
38	795417	26.3	892739	16.8	902679	43.2	097321	62433	78116	22
39	795575	26.3	892638	16.8	902938	43.2	097062	62456	78098	21
40	795733	26.3	892536	16.8	903197	43.1	096803	62479	78079	20
41	9.795891	26.3	9.892435	16.9	9.903455	43.1	10.096545	62502	78061	19
42	796049	26.3	892334	16.9	903714	43.1	096286	62524	78043	18
43	796206	26.3	892233	16.9	903973	43.1	096027	62547	78025	17
44	796364	26.2	892132	16.9	904232	43.1	095768	62570	78007	16
45	796521	26.2	892030	16.9	904491	43.1	095509	62592	77988	15
46	796679	26.2	891929	16.9	904750	43.1	095250	62615	77970	14
47	796836	26.2	891827	16.9	905008	43.1	094992	62638	77952	13
48	796993	26.2	891726	16.9	905267	43.1	094733	62660	77934	12
49	797150	26.1	891624	16.9	905526	43.1	094474	62683	77916	11
50	797307	26.1	891523	17.0	905784	43.1	094216	62706	77897	10
51	9.797464	26.1	9.891421	17.0	9.906043	43.1	10.093957	62728	77879	9
52	797621	26.1	891319	17.0	906302	43.1	093698	62751	77861	8
53	797777	26.1	891217	17.0	906560	43.1	093440	62774	77843	7
54	797934	26.1	891115	17.0	906819	43.1	093181	62796	77824	6
55	798091	26.1	891013	17.0	907077	43.1	092923	62819	77806	5
56	798247	26.1	890911	17.0	907336	43.1	092664	62842	77788	4
57	798403	26.0	890809	17.0	907594	43.1	092406	62864	77769	3
58	798560	26.0	890707	17.0	907852	43.1	092148	62887	77751	2
59	798716	26.0	890605	17.0	908111	43.0	091889	62909	77733	1
60	798872		890503		908369		091631	62932	77715	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N. sine.	′

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	9.798772	26.0	9.890503	17.0	9.908369	43.0	10.091631	62932	77715	60
1	799028	26.0	890400	17.1	908628	43.0	091372	62955	77696	59
2	799184	26.0	890298	17.1	908886	43.0	091114	62977	77678	58
3	799339	25.9	890195	17.1	909144	43.0	090856	63000	77660	57
4	799495	25.9	890093	17.1	909402	43.0	090598	63022	77641	56
5	799651	25.9	889990	17.1	909660	43.0	090340	63045	77623	55
6	799806	25.9	889888	17.1	909918	43.0	090082	63068	77605	54
7	799962	25.9	889785	17.1	910177	43.0	089823	63090	77586	53
8	800117	25.9	889682	17.1	910435	43.0	089565	63113	77568	52
9	800272	25.8	889579	17.1	910693	43.0	089307	63135	77550	51
10	800427	25.8	889477	17.1	910951	43.0	089049	63158	77531	50
11	9.800582	25.8	9.889374	17.2	9.911209	43.0	10.088791	93180	77513	49
12	800737	25.8	889271	17.2	911467	43.0	088533	63203	77494	48
13	800892	25.8	889168	17.2	911724	43.0	088276	63225	77476	47
14	801047	25.8	889064	17.2	911982	43.0	088018	63248	77458	46
15	801201	25.8	888961	17.2	912240	43.0	087760	63271	77439	45
16	801356	25.7	888858	17.2	912498	43.0	087502	63293	77421	44
17	801511	25.7	888755	17.2	912756	43.0	087244	63316	77402	43
18	801665	25.7	888651	17.2	913014	42.9	086986	63338	77384	42
19	801819	25.7	888548	17.2	913271	42.9	086729	63361	77366	41
20	801973	25.7	888444	17.2	913529	42.9	086471	63383	77347	40
21	9.802128	25.7	9.888341	17.3	9.913787	42.9	10.086213	63406	77329	39
22	802282	25.6	888237	17.3	914044	42.9	085956	63428	77310	38
23	802436	25.6	888134	17.3	914302	42.9	085698	63451	77292	37
24	802589	25.6	888030	17.3	914560	42.9	085440	63473	77273	36
25	802743	25.6	887926	17.3	914817	42.9	085183	63496	77255	35
26	802897	25.6	887822	17.3	915075	42.9	084925	63518	77236	34
27	803050	25.6	887718	17.3	915332	42.9	084668	63540	77218	33
28	803204	25.6	887614	17.3	915590	42.9	084410	63563	77199	32
29	803357	25.5	887510	17.3	915847	42.9	084153	63585	77181	31
30	803511	25.5	887406	17.3	916104	42.9	083896	63608	77162	30
31	9.803664	25.5	9.887302	17.4	9.916362	42.9	10.083638	63630	77144	29
32	803817	25.5	887198	17.4	916619	42.9	083381	63653	77125	28
33	803970	25.5	887093	17.4	916877	42.9	083123	63675	77107	27
34	804123	25.5	886989	17.4	917134	42.9	082866	63698	77088	26
35	804276	25.4	886885	17.4	917391	42.9	082609	63720	77070	25
36	804428	25.4	886780	17.4	917648	42.9	082352	63742	77051	24
37	804581	25.4	886676	17.4	917905	42.9	082095	63765	77033	23
38	804734	25.4	886571	17.4	918163	42.8	081837	63787	77014	22
39	804886	25.4	886466	17.4	918420	42.8	081580	63810	76996	21
40	805039	25.4	886362	17.5	918677	42.8	081323	63832	76977	20
41	9.805191	25.4	9.886257	17.5	9.918934	42.8	10.081066	63854	76959	19
42	805343	25.3	886152	17.5	919191	42.8	080809	63877	76940	18
43	805495	25.3	886047	17.5	919448	42.8	080552	63899	76921	17
44	805647	25.3	885942	17.5	919705	42.8	080295	63922	76903	16
45	805799	25.3	885837	17.5	919962	42.8	080038	63944	76884	15
46	805951	25.3	885732	17.5	920219	42.8	079781	63966	76866	14
47	806103	25.3	885627	17.5	920476	42.8	079524	63989	76847	13
48	806254	25.3	885522	17.5	920733	42.8	079267	64011	76828	12
49	806406	25.2	885416	17.5	920990	42.8	079010	64033	76810	11
50	806557	25.2	885311	17.6	921247	42.8	078753	64056	76791	10
51	9.806709	25.2	9.885205	17.6	9.921503	42.8	10.078497	64078	76772	9
52	806860	25.2	885100	17.6	921760	42.8	078240	64100	76754	8
53	807011	25.2	884994	17.6	922017	42.8	077983	64123	76735	7
54	807163	25.2	884889	17.6	922274	42.8	077726	64145	76717	6
55	807314	25.2	884783	17.6	922530	42.8	077470	64167	76698	5
56	807465	25.1	884677	17.6	922787	42.8	077213	64190	76679	4
57	807615	25.1	884572	17.6	923044	42.8	076956	64212	76661	3
58	807766	25.1	884466	17.6	923300	42.8	076700	64234	76642	2
59	807917	25.1	884360	17.6	923557	42.7	076443	64256	76623	1
60	808067		884254		923813		076187	64279	76604	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N. sine.	′

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N.sine.	N. cos.	
0	9.808067	25.1	9.884254	17.7	9.923813	42.7	10.076187	64279	76604	60
1	808218	25.1	884148	17.7	924070	42.7	075930	64301	76586	59
2	808368	25.1	884042	17.7	924327	42.7	075673	64323	76567	58
3	808519	25.0	883936	17.7	924583	42.7	075417	64346	76548	57
4	808669	25.0	883829	17.7	924840	42.7	075160	64368	76530	56
5	808819	25.0	883723	17.7	925096	42.7	074904	64390	76511	55
6	808969	25.0	883617	17.7	925352	42.7	074648	64412	76492	54
7	809119	25.0	883510	17.7	925609	42.7	074391	64435	76473	53
8	809269	25.0	883404	17.7	925865	42.7	074135	64457	76455	52
9	809419	24.9	883297	17.8	926122	42.7	073878	64479	76436	51
10	809569	24.9	883191	17.8	926378	42.7	073622	64501	76417	50
11	9.809718	24.9	9.883084	17.8	9.926634	42.7	10.073366	64524	76398	49
12	809868	24.9	882977	17.8	926890	42.7	073110	64546	76380	48
13	810017	24.9	882871	17.8	927147	42.7	072853	64568	76361	47
14	810167	24.9	882764	17.8	927403	42.7	072597	64590	76342	46
15	810316	24.8	882657	17.8	927659	42.7	072341	64612	76323	45
16	810465	24.8	882550	17.8	927915	42.7	072085	64635	76304	44
17	810614	24.8	882443	17.8	928171	42.7	071829	64657	76286	43
18	810763	24.8	882336	17.9	928427	42.7	071573	64679	76267	42
19	810912	24.8	882229	17.9	928683	42.7	071317	64701	76248	41
20	811061	24.8	882121	17.9	928940	42.7	071060	64723	76229	40
21	9.811210	24.8	9.882014	17.9	9.929196	42.7	10.070804	64746	76210	39
22	811358	24.7	881907	17.9	929452	42.7	070548	64768	76192	38
23	811507	24.7	881799	17.9	929708	42.7	070292	64790	76173	37
24	811655	24.7	881692	17.9	929964	42.6	070036	64812	76154	36
25	811804	24.7	881584	17.9	930220	42.6	069780	64834	76135	35
26	811952	24.7	881477	17.9	930475	42.6	069525	64856	76116	34
27	812100	24.7	881369	17.9	930731	42.6	069269	64878	76097	33
28	812248	24.7	881261	18.0	930987	42.6	069013	64901	76078	32
29	812396	24.6	881153	18.0	931243	42.6	068757	64923	76059	31
30	812544	24.6	881046	18.0	931499	42.6	068501	64945	76041	30
31	9.812692	24.6	9.880938	18.0	9.931755	42.6	10.068245	64967	76022	29
32	812840	24.6	880830	18.0	932010	42.6	067990	64989	76003	28
33	812988	24.6	880722	18.0	932266	42.6	067734	65011	75984	27
34	813135	24.6	880613	18.0	932522	42.6	067478	65033	75965	26
35	813283	24.6	880505	18.0	932778	42.6	067222	65055	75946	25
36	813430	24.5	880397	18.0	933033	42.6	066967	65077	75927	24
37	813578	24.5	880289	18.1	933289	42.6	066711	65100	75908	23
38	813725	24.5	880180	18.1	933545	42.6	066455	65122	75889	22
39	813872	24.5	880072	18.1	933800	42.6	066200	65144	75870	21
40	814019	24.5	879963	18.1	934056	42.6	065944	65166	75851	20
41	9.814166	24.5	9.879855	18.1	9.934311	42.6	10.065689	65188	75832	19
42	814313	24.5	879746	18.1	934567	42.6	065433	65210	75813	18
43	814460	24.4	879637	18.1	934823	42.6	065177	65232	75794	17
44	814607	24.4	879529	18.1	935078	42.6	064922	65254	75775	16
45	814753	24.4	879420	18.1	935333	42.6	064667	65276	75756	15
46	814900	24.4	879311	18.1	935589	42.6	064411	65298	75738	14
47	815046	24.4	879202	18.2	935844	42.6	064156	65320	75719	13
48	815193	24.4	879093	18.2	936100	42.6	063900	65342	75700	12
49	815339	24.4	878984	18.2	936355	42.6	063645	65364	75680	11
50	815485	24.3	878875	18.2	936610	42.6	063390	65386	75661	10
51	9.815631	24.3	9.878766	18.2	9.936866	42.5	10.063134	65408	75642	9
52	815778	24.3	878656	18.2	937121	42.5	062879	65430	75623	8
53	815924	24.3	878547	18.2	937376	42.5	062624	65452	75604	7
54	816069	24.3	878438	18.2	937632	42.5	062368	65474	75585	6
55	816215	24.3	878328	18.2	937887	42.5	062113	65496	75566	5
56	816361	24.3	878219	18.3	938142	42.5	061858	65518	75547	4
57	816507	24.2	878.09	18.3	938398	42.5	061602	65540	75528	3
58	816652	24.2	877999	18.3	938653	42.5	061347	65562	75509	2
59	816798	24.2	877890	18.3	938908	42.5	061092	65584	75490	1
60	816943		877780		939163		060837	65606	75471	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N.sine.	′

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	9.816943	24.2	9.877780	18.3	9.939163	42.5	10.060837	65606	75471	60
1	817088	24.2	877670	18.3	939418	42.5	060582	65628	75452	59
2	817233	24.2	877560	18.3	939673	42.5	060327	65650	75433	58
3	817379	24.2	877450	18.3	939928	42.5	060072	65672	75414	57
4	817524	24.1	877340	18.3	940183	42.5	059817	65694	75395	56
5	817668	24.1	877230	18.4	940438	42.5	059562	65716	75375	55
6	817813	24.1	877120	18.4	940694	42.5	059306	65738	75356	54
7	817958	24.1	877010	18.4	940949	42.5	059051	65759	75337	53
8	818103	24.1	876899	18.4	941204	42.5	058796	65781	75318	52
9	818247	24.1	876789	18.4	941458	42.5	058542	65803	75299	51
10	818392	24.1	876678	18.4	941714	42.5	058286	65825	75280	50
11	9.818536	24.0	9.876568	18.4	9.941968	42.5	10.058032	65847	75261	49
12	818681	24.0	876457	18.4	942223	42.5	057777	65869	75241	48
13	818825	24.0	876347	18.4	942478	42.5	057522	65891	75222	47
14	818969	24.0	876236	18.5	942733	42.5	057267	65913	75203	46
15	819113	24.0	876125	18.5	942988	42.5	057012	65935	75184	45
16	819257	24.0	876014	18.5	943243	42.5	056757	65956	75165	44
17	819401	24.0	875904	18.5	943498	42.5	056502	65978	75146	43
18	819545	23.9	875793	18.5	943752	42.5	056248	66000	75126	42
19	819689	23.9	875682	18.5	944007	42.5	055993	66022	75107	41
20	819832	23.9	875571	18.5	944262	42.5	055738	66044	75088	40
21	9.819976	23.9	9.875459	18.5	9.944517	42.5	10.055483	66066	75069	39
22	820120	23.9	875348	18.5	944771	42.4	055229	66088	75050	38
23	820263	23.9	875237	18.5	945026	42.4	054974	66109	75030	37
24	820406	23.9	875126	18.6	945281	42.4	054719	66131	75011	36
25	820550	23.8	875014	18.6	945535	42.4	054465	66153	74992	35
26	820693	23.8	874903	18.6	945790	42.4	054210	66175	74973	34
27	820836	23.8	874791	18.6	946045	42.4	053955	66197	74953	33
28	820979	23.8	874680	18.6	946299	42.4	053701	66218	74934	32
29	821122	23.8	874568	18.6	946554	42.4	053446	66240	74915	31
30	821265	23.8	874456	18.6	946808	42.4	053192	66262	74896	30
31	9.821407	23.8	9.874344	18.6	9.947063	42.4	10.052937	66284	74876	29
32	821550	23.8	874232	18.7	947318	42.4	052682	66306	74857	28
33	821693	23.7	874121	18.7	947572	42.4	052428	66327	74838	27
34	821835	23.7	874009	18.7	947826	42.4	052174	66349	74818	26
35	821977	23.7	873896	18.7	948081	42.4	051919	66371	74799	25
36	822120	23.7	873784	18.7	948336	42.4	051664	66393	74780	24
37	822262	23.7	873672	18.7	948590	42.4	051410	66414	74760	23
38	822404	23.7	873560	18.7	948844	42.4	051156	66436	74741	22
39	822546	23.7	873448	18.7	949099	42.4	050901	66458	74722	21
40	822688	23.6	873335	18.7	949353	42.4	050647	66480	74703	20
41	9.822830	23.6	9.873223	18.7	9.949607	42.4	10.050393	66501	74683	19
42	822972	23.6	873110	18.8	949862	42.4	050138	66523	74663	18
43	823114	23.6	872998	18.8	950116	42.4	049884	66545	74644	17
44	823255	23.6	872885	18.8	950370	42.4	049630	66566	74625	16
45	823397	23.6	872772	18.8	950625	42.4	049375	66588	74606	15
46	823539	23.6	872659	18.8	950879	42.4	049121	66610	74586	14
47	823680	23.5	872547	18.8	951133	42.4	048867	66632	74567	13
48	823821	23.5	872434	18.8	951388	42.4	048612	66653	74548	12
49	823963	23.5	872321	18.8	951642	42.4	048358	66675	74522	11
50	824104	23.5	872208	18.8	951896	42.4	048104	66697	74509	10
51	9.824245	23.5	9.872095	18.9	9.952150	42.4	10.047850	66718	74489	9
52	824386	23.5	871981	18.9	952405	42.4	047595	66740	74470	8
53	824527	23.5	871868	18.9	952659	42.4	047341	66762	74451	7
54	824668	23.4	871755	18.9	952913	42.4	047087	66783	74431	6
55	824808	23.4	871641	18.9	953167	42.3	046833	66805	74412	5
56	824949	23.4	871528	18.9	953421	42.3	046579	66827	74392	4
57	825090	23.4	871414	18.9	953675	42.3	046325	66848	74373	3
58	825230	23.4	871301	18.9	953929	42.3	046071	66870	74353	2
59	825371	23.4	871187	18.9	954183	42.3	045817	66891	74334	1
60	825511		871073		954437		045563	66913	74314	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N.sine.	′

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	9.825511	23.4	9.871073	19.0	9.954437	42.3	10.045563	66913	74314	60
1	825651	23.3	870960	19.0	954691	42.3	045309	66935	74295	59
2	825791	23.3	870846	19.0	954945	42.3	045055	66956	74276	58
3	825931	23.3	870732	19.0	955200	42.3	044800	66978	74256	57
4	826071	23.3	870618	19.0	955454	42.3	044546	66999	74237	56
5	826211	23.3	870504	19.0	955707	42.3	044293	67021	74217	55
6	826351	23.3	870390	19.0	955961	42.3	044039	67043	74198	54
7	826491	23.3	870276	19.0	956215	42.3	043785	67064	74178	53
8	826631	23.3	870161	19.0	956469	42.3	043531	67086	74159	52
9	826770	23.2	870047	19.1	956723	42.3	043277	67107	74139	51
10	826910	23.2	869933	19.1	956977	42.3	043023	67129	74120	50
11	9.827049	23.2	9.869818	19.1	9.957231	42.3	10.042769	67151	74100	49
12	827189	23.2	869704	19.1	957485	42.3	042515	67172	74080	48
13	827328	23.2	869589	19.1	957739	42.3	042261	67194	74061	47
14	827467	23.2	869474	19.1	957993	42.3	042007	67215	74041	46
15	827606	23.2	869360	19.1	958246	42.3	041754	67237	74022	45
16	827745	23.2	869245	19.1	958500	42.3	041500	67258	74002	44
17	827884	23.1	869130	19.1	958754	42.3	041246	67280	73983	43
18	828023	23.1	869015	19.2	959008	42.3	040992	67301	73963	42
19	828162	23.1	868900	19.2	959262	42.3	040738	67323	73944	41
20	828301	23.1	868785	19.2	959516	42.3	040484	67344	73924	40
21	9.828439	23.1	9.868670	19.2	9.959769	42.3	10.040231	67366	73904	39
22	828578	23.1	868555	19.2	960023	42.3	039977	67387	73885	38
23	828716	23.1	868440	19.2	960277	42.3	039723	67409	73865	37
24	828855	23.0	868324	19.2	960531	42.3	039469	67430	73846	36
25	828993	23.0	868209	19.2	960784	42.3	039216	67452	73826	35
26	829131	23.0	868093	19.2	961038	42.3	038962	67473	73806	34
27	829269	23.0	867978	19.3	961291	42.3	038709	67495	73787	33
28	829407	23.0	867862	19.3	961545	42.3	038455	67516	73767	32
29	829545	23.0	867747	19.3	961799	42.3	038201	67538	73747	31
30	829683	23.0	867631	19.3	962052	42.3	037948	67559	73728	30
31	9.829821	22.9	9.867515	19.3	9.962306	42.3	10.037694	67580	73708	29
32	829959	22.9	867399	19.3	962560	42.3	037440	67602	73688	28
33	830097	22.9	867283	19.3	962813	42.3	037187	67623	73669	27
34	830234	22.9	867167	19.3	963067	42.3	036933	67645	73649	26
35	830372	22.9	867051	19.3	963320	42.3	036680	67666	73629	25
36	830509	22.9	866935	19.4	963574	42.3	036426	67688	73610	24
37	830646	22.9	866819	19.4	963827	42.3	036173	67709	73590	23
38	830784	22.9	866703	19.4	964081	42.3	035919	67730	73570	22
39	830921	22.8	866586	19.4	964335	42.3	035665	67752	73551	21
40	831058	22.8	866470	19.4	964588	42.2	035412	67773	73531	20
41	9.831195	22.8	9.866353	19.4	9.964842	42.2	10.035158	67795	73511	19
42	831332	22.8	866237	19.4	965095	42.2	034905	67816	73491	18
43	831469	22.8	866120	19.4	965349	42.2	034651	67837	73472	17
44	831606	22.8	866004	19.5	965602	42.2	034398	67859	73452	16
45	831742	22.8	865887	19.5	965855	42.2	034145	67880	73432	15
46	831879	22.8	865770	19.5	966109	42.2	033891	67901	73413	14
47	832015	22.7	865653	19.5	966362	42.2	033638	67923	73393	13
48	832152	22.7	865536	19.5	966616	42.2	033384	67944	73373	12
49	832288	22.7	865419	19.5	966869	42.2	033131	67965	73353	11
50	832425	22.7	865302	19.5	967123	42.2	032877	67987	73333	10
51	9 832561	22.7	9.865185	19.5	9.967376	42.2	10.032624	68008	73314	9
52	832697	22.7	865068	19.5	967629	42.2	032371	68029	73294	8
53	832833	22.7	864950	19.5	967883	42.2	032117	68051	73274	7
54	832969	22.6	864833	19.6	968136	42.2	031864	68072	73254	6
55	833105	22.6	864716	19.6	968389	42.2	031611	68093	73234	5
56	833241	22.6	864598	19.6	968643	42.2	031357	68115	73215	4
57	833377	22.6	864481	19.6	968896	42.2	031104	68136	73195	3
58	833512	22.6	864363	19.6	969149	42.2	030851	68157	73175	2
59	833648	22.6	864245	19.6	969403	42.2	030597	68179	73155	1
60	833783		864127		969656		030344	68200	73135	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N. sine.	′

47 Degrees.

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N.sine.	N. cos.	
0	9.833783	22.6	9.864127	19.6	9.969656	42.2	10.030344	68200	73135	60
1	833919	22.5	864010	19.6	969909	42.2	030091	68221	73116	59
2	834054	22.5	863892	19.7	970162	42.2	029838	68242	73096	58
3	834189	22.5	863774	19.7	970416	42.2	029584	68264	73076	57
4	834325	22.5	863656	19.7	970669	42.2	029331	68285	73056	56
5	834460	22.5	863538	19.7	970922	42.2	029078	68306	73036	55
6	834595	22.5	863419	19.7	971175	42.2	028825	68327	73016	54
7	834730	22.5	863301	19.7	971429	42.2	028571	68349	72996	53
8	834865	22.5	863183	19.7	971682	42.2	028318	68370	72976	52
9	834999	22.4	863064	19.7	971935	42.2	028065	68391	72957	51
10	835134	22.4	862946	19.8	972188	42.2	027812	68412	72937	50
11	9.835269	22.4	9.862827	19.8	9.972441	42.2	10.027559	68434	72917	49
12	835403	22.4	862709	19.8	972694	42.2	027306	68455	72897	48
13	835538	22.4	862590	19.8	972948	42.2	027052	68476	72877	47
14	835672	22.4	862471	19.8	973201	42.2	026799	68497	72857	46
15	835807	22.4	862353	19.8	973454	42.2	026546	68518	72837	45
16	835941	22.4	862234	19.8	973707	42.2	026293	68539	72817	44
17	836075	22.3	862115	19.8	973960	42.2	026040	68561	72797	43
18	836209	22.3	861996	19.8	974213	42.2	025787	68582	72777	42
19	836343	22.3	861877	19.8	974466	42.2	025534	68603	72757	41
20	836477	22.3	861758	19.9	974719	42.2	025281	68624	72737	40
21	9.836611	22.3	9.861638	19.9	9.974973	42.2	10.025027	68645	72717	39
22	836745	22.3	861519	19.9	975226	42.2	024774	68666	72697	38
23	836878	22.3	861400	19.9	975479	42.2	024521	68688	72677	37
24	837012	22.2	861280	19.9	975732	42.2	024268	68709	72657	36
25	837146	22.2	861161	19.9	975985	42.2	024015	68730	72637	35
26	837279	22.2	861041	19.9	976238	42.2	023762	68751	72617	34
27	837412	22.2	860922	19.9	976491	42.2	023509	68772	72597	33
28	837546	22.2	860802	19.9	976744	42.2	023256	68793	72577	32
29	837679	22.2	860682	20.0	976997	42.2	023003	68814	72557	31
30	837812	22.2	860562	20.0	977250	42.2	022750	68835	72537	30
31	9.837945	22.2	9.860442	20.0	9.977503	42.2	10.022497	68857	72517	29
32	838078	22.1	860322	20.0	977756	42.2	022244	68878	72497	28
33	838211	22.1	860202	20.0	978009	42.2	021991	68899	72477	27
34	838344	22.1	860082	20.0	978262	42.2	021738	68920	72457	26
35	838477	22.1	859962	20.0	978515	42.2	021485	68941	72437	25
36	838610	22.1	859842	20.0	978768	42.2	021232	68962	72417	24
37	838742	22.1	859721	20.1	979021	42.2	020979	68983	72397	23
38	838875	22.1	859601	20.1	979274	42.2	020726	69004	72377	22
39	839007	22.1	859480	20.1	979527	42.2	020473	69025	72357	21
40	839140	22.0	859360	20.1	979780	42.2	020220	69046	72337	20
41	9.839272	22.0	9.859239	20.1	9.980033	42.2	10.019967	69067	72317	19
42	839404	22.0	859119	20.1	980286	42.2	019714	69088	72297	18
43	839536	22.0	858998	20.1	980538	42.2	019462	69109	72277	17
44	839668	22.0	858877	20.1	980791	42.1	019209	69130	72257	16
45	839800	22.0	858756	20.2	981044	42.1	018956	69151	72236	15
46	839932	22.0	858635	20.2	981297	42.1	018703	69172	72216	14
47	840064	21.9	858514	20.2	981550	42.1	018450	69193	72196	13
48	840196	21.9	858393	20.2	981803	42.1	018197	69214	72176	12
49	840328	21.9	858272	20.2	982056	42.1	017944	69235	72156	11
50	840459	21.9	858151	20.2	982309	42.1	017691	69256	72136	10
51	9.840591	21.9	9.858029	20.2	9.982562	42.1	10.017438	69277	72116	9
52	840722	21.9	857908	20.2	982814	42.1	017186	69298	72095	8
53	840854	21.9	857786	20.2	983067	42.1	016933	69319	72075	7
54	840985	21.9	857665	20.3	983320	42.1	016680	69340	72055	6
55	841116	21.8	857543	20.3	983573	42.1	016427	69361	72035	5
56	841247	21.8	857422	20.3	983826	42.1	016174	69382	72015	4
57	841378	21.8	857300	20.3	984079	42.1	015921	69403	71995	3
58	841509	21.8	857178	20.3	984331	42.1	015669	69424	71974	2
59	841640	21.8	857056	20.3	984584	42.1	015416	69445	71954	1
60	841771		856934		984837		015163	69466	71934	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N.sine.	′

46 Degrees.

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	9.841771	21.8	9.856934	20.3	9.984837	42.1	10.015163	69466	71934	60
1	841902	21.8	856812	20.3	985090	42.1	014910	69487	71914	59
2	842033	21.8	856690	20.4	985343	42.1	014657	69508	71894	58
3	842163	21.7	856568	20.4	985596	42.1	014404	69529	71873	57
4	842294	21.7	856446	20.4	985848	42.1	014152	69549	71853	56
5	842424	21.7	856323	20.4	986101	42.1	013899	69570	71833	55
6	842555	21.7	856201	20.4	986354	42.1	013646	69591	71813	54
7	842685	21.7	856078	20.4	986607	42.1	013393	69612	71792	53
8	842815	21.7	855956	20.4	986860	42.1	013140	69633	71772	52
9	842946	21.7	855833	20.4	987112	42.1	012888	69654	71752	51
10	843076	21.7	855711	20.5	987365	42.1	012635	69675	71732	50
11	9.843206	21.6	9.855588	20.5	9.987618	42.1	10.012382	69696	71711	49
12	843336	21.6	855465	20.5	987871	42.1	012129	69717	71691	48
13	843466	21.6	855342	20.5	988123	42.1	011877	69737	71671	47
14	843595	21.6	855219	20.5	988376	42.1	011624	69758	71650	46
15	843725	21.6	855096	20.5	988629	42.1	011371	69779	71630	45
16	843855	21.6	854973	20.5	988882	42.1	011118	69800	71610	44
17	843984	21.6	854850	20.5	989134	42.1	010866	69821	71590	43
18	844114	21.5	854727	20.6	989387	42.1	010613	69842	71569	42
19	844243	21.5	854603	20.6	989640	42.1	010360	69862	71549	41
20	844372	21.5	854480	20.6	989893	42.1	010107	69883	71529	40
21	9.844502	21.5	9.854356	20.6	9.990145	42.1	10.009855	69904	71508	39
22	844631	21.5	854233	20.6	990398	42.1	009602	69925	71488	38
23	844760	21.5	854109	20.6	990651	42.1	009349	69946	71468	37
24	844889	21.5	853986	20.6	990903	42.1	009097	69966	71447	36
25	845018	21.5	853862	20.6	991156	42.1	008844	69987	71427	35
26	845147	21.5	853738	20.6	991409	42.1	008591	70008	71407	34
27	845276	21.4	853614	20.7	991662	42.1	008338	70029	71386	33
28	845405	21.4	853490	20.7	991914	42.1	008086	70049	71366	32
29	845533	21.4	853366	20.7	992167	42.1	007833	70070	71345	31
30	845662	21.4	853242	20.7	992420	42.1	007580	70091	71325	30
31	9.845790	21.4	9.853118	20.7	9.992672	42.1	10.007328	70112	71305	29
32	845919	21.4	852994	20.7	992925	42.1	007075	70132	71284	28
33	846047	21.4	852869	20.7	993178	42.1	006822	70153	71264	27
34	846175	21.4	852745	20.7	993430	42.1	006570	70174	71243	26
35	846304	21.4	852620	20.7	993683	42.1	006317	70195	71223	25
36	846432	21.3	852496	20.8	993936	42.1	006064	70215	71203	24
37	846560	21.3	852371	20.8	994189	42.1	005811	70236	71182	23
38	846688	21.3	852247	20.8	994441	42.1	005559	70257	71162	22
39	846816	21.3	852122	20.8	994694	42.1	005306	70277	71141	21
40	846944	21.3	851997	20.8	994947	42.1	005053	70298	71121	20
41	9.847071	21.3	9.851872	20.8	9.995199	42.1	10.004801	70319	71100	19
42	847199	21.3	851747	20.8	995452	42.1	004548	70339	71080	18
43	847327	21.3	851622	20.8	995705	42.1	004295	70360	71059	17
44	847454	21.2	851497	20.9	995957	42.1	004043	70381	71039	16
45	847582	21.2	851372	20.9	996210	42.1	003790	70401	71019	15
46	847709	21.2	851246	20.9	996463	42.1	003537	70422	70998	14
47	847836	21.2	851121	20.9	996715	42.1	003285	70443	70978	13
48	847964	21.2	850996	20.9	996968	42.1	003032	70463	70957	12
49	848091	21.2	850870	20.9	997221	42.1	002779	70484	70937	11
50	848218	21.2	850745	20.9	997473	42.1	002527	70505	70916	10
51	9.848345	21.2	9.850619	20.9	9.997726	42.1	10.002274	70525	70896	9
52	848472	21.1	850493	21.0	997979	42.1	002021	70546	70875	8
53	848599	21.1	850368	21.0	998231	42.1	001769	70567	70855	7
54	848726	21.1	850242	21.0	998484	42.1	001516	70587	70834	6
55	848852	21.1	850116	21.0	998737	42.1	001263	70608	70813	5
56	848979	21.1	849990	21.0	998989	42.1	001011	70628	70793	4
57	849106	21.1	849864	21.0	999242	42.1	000758	70649	70772	3
58	849232	21.1	849738	21.0	999495	42.1	000505	70670	70752	2
59	849359	21.1	849611	21.0	999748	42.1	000253	70690	70731	1
60	849485		849485		10.000000		000000	70711	70711	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N. sine.	′

45 Degrees.

TABLE III.

LOGARITHMS OF NUMBERS,

FROM 1 TO 110,

INCLUDING TWELVE DECIMAL PLACES.

N.	Log.					N.	Log.				
1	0.	000	000	000	000	36	1.	556	302	500	767
2	0.	301	029	995	644	37	1.	568	201	724	067
3	0.	477	121	254	720	38	1.	579	783	596	617
4	0.	602	059	991	328	39	1.	591	064	607	264
5	0.	698	970	004	336	40	1.	602	059	991	328
6	0.	778	151	250	384	41	1.	612	783	846	720
7	0.	845	098	040	014	42	1.	623	249	290	398
8	0.	903	089	986	992	43	1.	633	468	455	579
9	0.	954	242	509	440	44	1.	643	452	676	486
10	1.	000	000	000	000	45	1.	653	212	513	775
11	1.	041	392	685	158	46	1.	662	757	831	682
12	1.	079	181	246	048	47	1.	672	097	857	936
13	1.	113	943	352	309	48	1.	681	241	237	376
14	1.	146	128	035	678	49	1.	690	196	080	028
15	1.	176	091	259	059	50	1.	698	970	004	336
16	1.	204	119	982	656	51	1.	707	570	176	098
17	1.	230	448	921	378	52	1.	716	003	243	635
18	1.	255	272	505	103	53	1.	724	275	869	601
19	1.	278	753	600	953	54	1.	732	393	759	823
20	1.	301	029	995	664	55	1.	740	362	689	494
21	1.	322	219	294	734	56	1	748	188	027	006
22	1.	342	422	680	822	57	1.	755	874	855	672
23	1.	361	727	836	076	58	1.	763	427	993	563
24	1.	380	211	241	712	59	1.	770	852	011	642
25	1.	397	940	008	672	60	1.	778	151	250	384
26	1.	414	973	347	971	61	1.	785	329	835	011
27	1.	431	363	764	159	62	1.	792	391	689	492
28	1.	447	158	031	342	63	1.	799	340	549	454
29	1.	462	397	997	899	64	1.	806	179	973	984
30	1.	477	121	254	720	65	1.	812	913	356	643
31	1.	491	361	693	834	66	1.	819	543	935	542
32	1.	505	149	978	320	67	1.	826	074	302	701
33	1.	518	513	939	878	68	1.	832	508	912	706
34	1.	531	478	917	042	69	1.	838	849	090	737
35	1.	544	068	044	350	70	1.	845	098	040	014

N.	Log.	N.	Log.
71	1. 851 258 348 719	91	1. 959 041 392 321
72	1. 857 332 496 431	92	1. 963 787 827 346
73	1. 863 322 860 120	93	1. 968 482 948 554
74	1. 869 231 719 731	94	1. 973 127 853 600
75	1. 875 061 263 392	95	1. 977 723 605 289
76	1. 880 813 592 281	96	1. 982 271 233 040
77	1. 886 490 725 172	97	1. 986 771 734 266
78	1. 892 094 602 690	98	1. 991 226 075 692
79	1. 897 627 091 290	99	1. 995 635 194 598
80	1. 903 089 986 992	100	2. 000 000 000 000
81	1. 908 485 018 879	101	2. 004 321 373 783
82	1. 913 813 852 384	102	2. 008 600 171 762
83	1. 919 078 092 376	103	2. 012 837 224 705
84	1. 924 279 286 062	104	2. 017 033 339 299
85	1. 929 418 925 714	105	2. 021 189 299 070
86	1. 934 498 451 244	106	2. 025 305 865 265
87	1. 939 519 252 619	107	2. 029 383 777 685
88	1. 944 482 672 150	108	2. 033 423 755 487
89	1. 949 390 006 645	109	2. 037 426 497 941
90	1. 954 242 509 439	110	2. 041 392 685 158

LOGARITHMS OF THE PRIME NUMBERS

FROM 110 TO 1129.

INCLUDING TWELVE DECIMAL PLACES.

N.	Log.	N.	Log.
113	2. 053 078 443 483	197	2. 294 466 266 162
127	2. 103 803 720 956	199	2. 298 853 076 410
131	2. 117 271 295 656	211	2. 324 282 455 298
137	2. 136 720 567 156	223	2. 348 304 863 222
139	2. 143 014 800 254	227	2. 356 025 857 189
149	3. 173 186 268 412	229	2. 359 835 482 343
151	2. 178 976 947 293	233	2. 367 355 922 471
157	2. 195 899 653 409	239	2. 378 397 902 352
163	2. 212 187 604 404	241	2. 382 017 042 576
167	2. 222 716 471 148	251	2. 399 673 721 509
173	2. 238 046 103 129	257	2. 409 933 123 332
179	2. 252 853 030 980	263	2. 419 955 748 490
181	2. 257 678 574 869	269	2. 429 752 261 993
191	2. 281 033 367 248	271	2. 432 969 290 877
193	2. 285 557 309 008	277	2. 442 479 768 999

N.	Log.					N.	Log.				
281	2.	448	706	319	906	601	2.	778	874	471	998
283	2.	451	786	435	523	607	2.	783	188	691	074
293	2.	466	867	523	562	613	2.	787	460	556	130
307	2.	487	138	375	477	617	2.	790	285	164	033
311	2.	492	760	389	026	619	2.	791	690	648	987
313	2.	495	544	337	550	631	2.	800	029	359	232
317	2.	501	059	267	324	641	2.	806	858	879	634
331	2.	519	827	993	783	643	2.	808	210	973	921
337	2.	527	629	883	034	647	2.	810	904	280	666
347	2.	540	329	475	079	653	2.	814	912	981	274
349	2.	542	826	426	673	659	2.	818	885	490	409
353	2.	547	774	138	015	661	2.	820	201	459	485
359	2.	555	094	447	578	673	2.	828	015	064	225
367	2.	564	666	064	254	677	2.	830	588	667	946
373	2.	571	708	831	809	683	2.	834	420	703	630
379	2.	578	639	209	957	691	2.	839	477	902	551
383	2.	583	198	773	980	701	2.	845	718	017	237
389	2.	589	949	601	323	709	2.	850	646	235	112
397	2.	598	790	506	763	719	2.	856	728	890	383
401	2.	603	144	372	687	727	2.	861	534	410	855
409	2.	611	723	298	019	733	2.	865	103	970	639
419	2.	622	214	022	971	739	2.	868	643	643	162
421	2.	624	282	095	835	743	2.	870	988	813	759
431	2.	634	477	268	999	751	2.	875	639	937	004
433	2.	636	488	015	871	757	2.	879	095	879	497
439	2.	642	464	520	242	761	2.	881	384	656	769
443	2.	646	403	726	235	769	2.	885	926	339	800
449	2.	652	246	388	777	773	2.	888	179	493	917
457	2.	659	916	200	054	787	2.	895	974	732	358
461	2.	663	700	925	389	797	2.	901	458	321	400
463	2.	665	580	991	012	809	2.	977	948	459	773
467	2.	669	317	881	008	811	2.	909	020	854	210
479	2.	680	335	513	415	821	2.	914	343	157	120
487	2.	687	528	961	120	823	2.	915	399	835	203
491	2.	691	081	487	026	827	2.	917	505	509	487
499	2.	698	100	545	623	829	2.	918	554	530	558
503	2.	701	567	985	083	839	2.	923	761	960	830
509	2.	706	717	782	345	853	2.	930	949	031	163
521	2.	716	837	623	304	857	2.	932	980	821	917
523	2.	718	502	688	873	859	2.	933	993	163	838
541	2.	733	197	265	134	863	2.	936	010	794	546
547	2.	737	987	326	358	877	2.	942	999	593	360
557	2.	745	855	195	192	881	2.	944	975	908	412
563	2.	750	508	395	940	883	2.	945	960	703	512
569	2.	755	112	178	598	887	2.	947	923	619	839
571	2.	756	636	108	333	907	2.	957	607	287	059
577	2.	761	175	813	171	911	2.	959	518	376	972
587	2.	768	638	004	455	919	2.	963	315	513	609
593	2.	773	054	693	364	929	2.	968	015	713	997
599	2.	777	427	303	257	937	2.	971	739	590	780

N.	Log.					N.	Log.				
941	2.	973	589	620	234	1039	3.	016	615	547	558
947	2.	976	349	979	055	1049	3.	020	775	488	195
953	2.	979	092	900	639	1051	3.	021	602	716	026
967	2.	985	426	474	084	1061	3.	025	715	383	898
971	2.	987	219	229	907	1063	3.	026	533	264	523
977	2.	989	894	559	717	1069	3.	028	977	705	205
983	2.	992	553	512	733	1087	3.	036	229	513	712
991	2.	996	073	604	003	1091	3.	037	824	749	671
997	2.	998	695	158	313	1093	3.	038	620	157	372
1009	3.	003	891	170	203	1097	3.	040	206	627	571
1013	3.	005	609	445	427	1103	3.	042	575	512	437
1019	3.	008	174	244	007	1109	3.	044	931	546	149
1021	3.	009	025	742	086	1117	3.	048	053	173	103
1031	3.	013	258	660	430	1123	3.	050	379	756	239
1033	3.	014	100	321	518	1129	3.	052	693	942	370

It is not necessary to extend this table, as the logarithm of any one of the higher numbers can be readily computed by the following formula, which may be found in any of the standard works on algebra, namely :

$$\text{Log. }(z+1)=\text{log. }z+0.8685889638\left(\frac{1}{2z+1}\right)$$

The result will be true to ten decimal places for all numbers over 1000, and true to twelve decimals for all numbers over 2000.

The logarithms of composite numbers can be determined by the combination of logarithms already in the table, and the prime numbers from the formula.

Thus, the number 3083 is a prime number, find its logarithm, true to ten places of decimals.

We first find the logarithm of 3082. By factoring this number, we find that it may be composed by the multiplication of 46 into 67.

Log. 46	1.	662 757 8316
Log. 67	1.	826 074 3027
Log. 3082	3.	488 832 1343

Now, $\text{Log. } 3083 = 3.4888321343 + \frac{0.8685889638}{6165}$

We give a few additional prime numbers :

1151	1223	1291	1373	1451	1511
1153	1229	1297	1381	1453	1523
1163	1231	1301	1399	1459	1531
1171	1237	1303	1409	1471	1543
1181	1249	1307	1423	1481	1549
1187	1259	1319	1427	1483	1553
1193	1277	1321	1429	1487	1559
1201	1279	1327	1433	1489	1567
1213	1283	1361	1439	1493	1571
1217	1289	1367	1447	1499	1579

AUXILIARY LOGARITHMS.

N.	Log.		N.	Log.	
1. 009	0. 003 891 170 203	A	1. 0009	0. 000 390 576 304	B
1. 008	0. 003 461 527 188	A	1. 0008	0. 000 347 233 698	B
1. 007	0. 003 030 465 635	A	1. 0007	0. 000 303 836 798	B
1. 006	0. 002 597 985 739	A	1. 0006	0. 000 260 435 561	B
1. 005	0. 002 166 071 750	A	1. 0005	0. 000 217 099 966	B
1. 004	0. 001 733 722 804	A	1. 0004	0. 000 173 690 053	B
1. 003	0. 001 300 943 017	A	1. 0003	0. 000 130 268 803	B
1. 002	0. 000 867 721 529	A	1. 0002	0. 000 086 850 213	B
1. 001	0. 000 434 077 479	A	1. 0001	0. 000 043 427 277	B

N.	Log.	
1. 00009	0. 000 039 084 741	C
1. 00008	0. 000 034 742 166	C
1. 00007	0. 000 030 399 546	C
1. 00006	0. 000 026 056 884	C
1. 00005	0. 000 021 714 178	C
1. 00004	0. 000 017 371 430	C
1. 00003	0. 000 013 028 638	C
1. 00002	0. 000 008 685 802	C
1. 00001	0. 000 004 342 923 (a)	C
1. 000001	0. 000 000 434 294 (b)	C
1. 0000001	0. 000 000 043 429 (c)	C
1. 00000001	0. 000 000 004 343 (d)	C
1. 000000001	0. 000 000 000 434 (e)	C
1. 0000000001	0. 000 000 000 043 (f)	C

Number.	Log.
0. 4342944819	—1. 637 784 298

This decimal number is the modulus of our system of logarithms. Its logarithm is very useful in correcting other logarithms, as may be seen in the Chapter on Logarithms.

TABLE V.
Dip of the Sea Horizon.

Hight of Eye in Ft.	Dip of the Horizon. ' "	Hight of Eye in Ft.	Dip of the Horizon. ' "
1	0 59	38	6 4
2	1 24	41	6 18
3	1 42	44	6 32
4	1 58	47	6 45
5	2 12	50	6 58
6	2 25	53	7 10
7	2 36	56	7 12
8	2 47	59	7 24
9	2 57	62	7 45
10	3 07	65	7 56
11	3 16	68	8 07
12	3 25	71	8 18
13	3 33	74	8 28
14	3 41	77	8 38
15	3 49	80	8 48
16	3 56	83	8 58
17	4 04	86	9 08
18	4 11	89	9 17
19	4 17	92	9 26
20	4 24	95	9 36
21	4 31	98	9 45
22	4 37	101	9 54
23	4 43	104	10 02
24	4 49	107	10 11
25	4 55	110	10 19
26	5 01	113	10 28
27	5 07	116	10 36
28	5 13	119	10 44
29	5 18	122	10 52
30	5 24	125	11 00
31	5 29	128	11 08
32	5 34	131	11 16
33	5 39	134	11 24
34	5 44	137	11 31
35	5 49	140	11 39

TABLE VI.
Dip of the Sea Horizon at different Distances from it.

Dist. in Miles.	Hight of Eye in Ft.					
	5	10	15	20	25	30
	'	'	'	'	'	'
1/4	11	22	34	45	56	68
1/2	6	11	17	22	28	34
3/4	4	8	12	15	19	23
1	4	6	9	12	15	17
1 1/4	3	5	7	9	12	14
1 1/2	3	4	6	8	9	12
2	2	3	5	6	8	10
2 1/2	2	3	5	6	7	8
3	2	3	4	5	6	7
3 1/2	2	3	4	5	6	6
4	2	3	4	4	5	6
5	2	3	4	4	5	5
6	2	3	4	4	5	5

TABLE VII.
Mean Refraction of Celestial Objects.

Alt. ° '	Refr. ' "	Alt. ° '	Refr. ' "	Alt. ° '	Refr. ' "	Alt. ° '	Refr. ' "	Alt. °	Refr. "
0 0	33 0	10 0	5 15	20 0	2 35	32 0	1 30	67	24
10	31 32	10	5 10	10	2 24	40	1 29	68	23
20	29 50	20	5 05	20	2 22	33 0	1 28	69	22
30	28 23	30	5 00	30	2 21	20	1 26	70	21
40	27 00	40	4 56	40	2 29	40	1 25	71	19
50	25 42	50	4 51	50	2 28	34 0	1 24	72	18
1 0	24 29	11 0	4 47	21 0	2 27	20	1 23	73	17
10	23 20	10	4 43	10	2 26	40	1 22	74	16
20	22 15	20	4 39	20	2 25	35 0	1 21	75	15
30	21 15	30	4 34	30	2 24	20	1 20	76	14
40	20 18	40	4 31	40	2 23	40	1 19	77	13
50	19 25	50	4 27	50	2 21	36 0	1 18	78	12
2 0	18 35	12 0	4 23	22 0	2 20	30	1 17	79	11
10	17 48	10	4 20	10	2 19	37 0	1 16	80	10
20	17 04	20	4 16	20	2 18	30	1 14	81	9
30	16 24	30	4 13	30	2 17	38 0	1 13	82	8
40	15 45	40	4 09	40	2 16	30	1 11	83	7
50	15 09	50	4 06	50	2 15	39 0	1 10	34	6
3 0	14 34	13 0	4 03	23 0	2 14	30	1 09	85	5
10	14 04	10	4 00	10	2 13	40 0	1 08	86	4
20	13 34	20	3 57	20	2 12	30	1 07	87	3
30	13 06	30	3 54	30	2 11	41 0	1 05	88	2
40	12 40	49	3 51	40	2 10	30	1 04	89	1
50	12 15	50	3 48	50	2 09	42 0	1 03	90	0
4 0	11 51	14 0	3 45	24 0	2 08	30	1 02		
10	11 29	10	3 43	10	2 07	43 0	1 01		
20	11 08	20	3 40	20	2 06	30	1 00		
30	10 48	30	3 38	30	2 05	44 0	0 59		
40	10 29	40	3 35	40	2 04	30	0 58		
50	10 11	50	3 33	50	2 03	45 0	0 57		
5 0	9 54	15 0	3 30	25 0	2 02	30	0 56		
10	9 38	10	3 28	10	2 01	46 0	0 55		
20	9 23	20	3 26	20	2 00	30	0 54		
30	9 08	30	3 24	30	1 59	47 0	0 53		
40	8 54	40	3 21	40	1 58	30	0 52		
50	8 41	50	3 19	50	1 57	48 0	0 51		
6 0	8 28	16 0	3 17	26 0	1 56	30	0 50		
10	8 15	10	3 15	10	1 55	49 0	0 49		
20	8 03	20	3 12	20	1 55	30	0 49		
30	7 15	30	3 10	30	1 54	50 0	0 48		
40	7 40	40	3 08	40	1 53	30	0 47		
50	7 30	50	3 06	50	1 52	51 0	0 46		
7 0	7 20	17 0	3 04	27 0	1 51	30	0 45		
10	7 11	10	3 03	15	1 50	52 0	0 44		
20	7 02	20	3 01	30	1 49	30	0 44		
30	6 53	30	2 59	45	1 48	53 0	0 43		
40	6 45	40	2 57	28 0	1 47	30	0 42		
50	6 37	50	2 55	15	1 46	54 0	0 41		
8 0	6 29	18 0	2 54	30	1 45	55 0	0 40		
10	6 22	10	2 52	45	1 44	56 0	0 38		
20	6 15	20	2 51	29 0	1 42	57 0	0 37		
30	6 08	30	2 49	20	1 41	58 0	0 35		
40	6 01	40	2 47	40	1 40	59 0	0 34		
50	5 55	50	2 46	30 0	1 38	60 0	0 33		
9 0	5 98	19 0	2 44	20	1 37	61 0	0 32		
10	5 42	10	2 43	40	1 36	62 0	0 30		
20	5 46	20	2 41	31 0	1 35	63 0	0 29		
30	5 41	30	2 40	20	1 33	64 0	0 28		
40	5 25	40	2 38	40	1 32	65 0	0 26		
50	5 20	50	2 37	32 0	1 31	66 0	0 25		