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Constructive Proof of a Theorem on
the Uniqueness of a Cournot Equilibrium

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A Constructive Proof of a Theorem on
the Uniqueness of a Cournot Equilibrium

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Abstract

The paper presents a constructive proof of sufficient conditions
for uniqueness of a homogeneous product Cournot equilibrium. The
result is among the strongest available; in fact if marginal costs cut
demand from below for a feasible output level (and a few other regu-
larity conditions are met) the result provides necessary and sufficient
conditions for uniqueness.

I. INTRODUCTION

There have been a series of results in recent years giving successively weaker conditions for uniqueness of a Cournot equilibrium (Szidarovszky and Yakowitz, 1977, 1982; Okuguchi, 1983). For the case of differentiable costs and demands, Szidarovszky and Yakowitz (1977) require that costs be convex and demand be concave. This paper serves to extend this result, using a similar method of proof, but considerably weakening requirements on demand and costs. In fact, provided marginal costs cut demand from below for a feasible output level, and a few other regularity conditions are met, our result is a necessary and sufficient condition for uniqueness. The proof presented here is a constructive one in that the proof could be implemented to find Cournot equilibria. A nonconstructive proof of the result, based on degree theory, may be found in Kolstad and Mathiesen (1987).

II. PRELIMINARIES

Let there be N firms each with cost $C_i(q_i)$ for producing output $q_i \geq 0$. The firms face an aggregate inverse demand function, $P(Q)$ (where $Q = \sum_{i=1}^{N} q_i$), defined over $[0, \infty)$. Profit for the ith firm is thus given by

$$\pi_i(q) = q_i P(\sum_{j=1}^{N} q_j) - C_i(q_i) \tag{1}$$

We will assume costs and inverse demand are twice continuously differentiable. Because of differentiability, first order conditions for a profit maximum for each profit function are:

CP1: Find q such that for all i, $1 \leq i \leq N$:

$$- \frac{\partial \pi_i}{\partial q_i} \equiv C_i'(q_i) - P(\sum_{j=1}^{N} q_j) - q_i P'(\sum_{j=1}^{N} q_j) \geq 0, \tag{2a}$$

$$q_i \geq 0 \tag{2b}$$

$$q_i \frac{\partial \pi_i}{\partial q_i} = 0 \tag{2c}$$

If profits are concave, then the first order conditions CP1 are necessary and sufficient for profit maxima and thus solutions to CP1 are precisely the same as the set of Cournot equilibria.

In order to permit us to restrict attention to a bounded set of outputs, we define an upper limit on industry output:

Definition: Industry output is said to be bounded if there is a compact subset, K, of R_+^N, such that for $q \in R_+^N \setminus K$:

$$- \frac{\partial \pi_i}{\partial q_i} (q) > 0 \qquad \text{for all } i \leq N. \tag{3}$$

For example, industry output is bounded if there is an output level for which marginal profits are negative for all firms; and furthermore, these negative marginal profits persist for all greater industry output levels. This is a common condition which would be met, for instance, if when the market is flooded with output, price is driven so low that no firm can maximize profits at a positive output level.

A second requirement is that firms are clearly in the market or out of the market:

Definition: For the case where costs and inverse demand are continuously differentiable, a Cournot equilibrium, q^*, is non-degenerate if for all $i \leq N$,

$$q_i^* = 0 \Rightarrow P\left(\sum_{j=1}^{N} q_j^*\right) - C_i'(0) < 0 \tag{4}$$

With this definition, it becomes possible to restrict attention to a set of positive players in the market and be assured that there is a neighborhood of q* in which all Cournot equilibria involve exactly the same set of players.

III. RESULTS ON UNIQUENESS

We now present the basic results of this paper. We defer a proof of the results until the next section.

Theorem 1:

Assume costs and inverse demand functions are continuously differentiable and further

i) industry output is bounded

ii) there exists an $i \in N$ such that $P(0) > C_i'(0)$;

iii) profits are quasiconcave with respect to own price (eqn. 3).

Then a nontrivial Cournot equilibrium exists.

The conditions of this theorem are relatively weak. Non-differentiable functions can be made differentiable by small perturbations. Condition i) is also weak since in most markets, at some large output level, price necessary to clean the market becomes very small, less than that necessary for any producer. Condition ii) is necessary to eliminate the trivial equilibrium of zero output for all producers. Condition iii) is probably the strongest condition and is necessary to assure existence of an equilibrium. We now turn to uniqueness:

Theorem 2:

Assume costs and inverse demand functions are continuously differen-
tiable and further

i) $P'(Q+x) < C_i''(x)$ for all x, $Q \geq 0$, $i \in N$; and

ii) all Cournot equilibria, q^*, are non-degenerate.

If for all equilibria, q^*,

$$\sum_{j \in M} \frac{P'(\sum_{k=1}^{N} q_k^*) + q_j^* P''(\sum_{k=1}^{N} q_k^*)}{C_j''(q_j^*) - P'(\sum_{k=1}^{N} q_k^*)} < 1, \tag{5}$$

where $M = \{i \mid q_i^* > 0\}$,

then there is at most one nontrivial Cournot equilibrium.

Theorem 3:

Let the assumptions of Theorems 1 and 2 hold. Then, for any
equilibrium, q^*,

a) if eqn. (5) holds, the equilibrium is unique;

b) if the equilibrium is unique then eqn. (5) holds except that
the inequality may be an equality.

These results are significant in a number of ways. For the case of
differentiable profit functions and marginal costs cutting demand from
below, these results subsume all previous uniqueness results. Theorem 4
due to Okuguchi (1983) follows trivially as do the earlier results of
Okuguchi (1976) and the results of Szidarovszky and Yakowitz (1977) and
Murphy et al. (1982). Szidarovszky and Yakowitz's (1977) necessary
condition for uniqueness that inverse demand is downward sloping ($P' < 0$)
and concave ($P'' \leq 0$), is clearly unnecessarily strong.

Perhaps more significant is that our results need only hold at equilibria and not globally. Further, we have given necessary and sufficient conditions for uniqueness (excluding equality in eqn. 5), not just sufficient conditions.

To interpret Theorem 2, consider the simplification that all N firms are identical. If N is large (or the RHS is set to zero instead of one), eqn. (5) basically amounts to

$$\frac{d(MR_j)}{dq_i} = P'(Q) + q_j^* P''(Q) < 0, \quad i \neq j \tag{6}$$

where MR is marginal revenue. When one player (i) increases its output, another player's (j) marginal revenue declines, and thus player i's output will tend to decline.

IV. PROOF OF EXISTENCE AND UNIQUENESS THEOREMS

Since existence of Cournot equilibria for concave profit functions is a known result, derived from the existence of Nash equilibria for concave payoffs, our principal contribution is in characterizing conditions for uniqueness. Our results can be seen as a natural extension of general uniqueness results and in particular, uniqueness results from Cournot equilibrium theory and complementarity theory (CP1 above is a complementarity problem).

For some years, the Gale-Nikaido (1965) theorem has been used to show uniqueness of equilibria, competitive or otherwise. The theorem states that a function f(x) from one rectangular region to another is one-to-one if its Jacobian is everywhere a P-matrix; i.e., has positive principal minors. For the complementarity problem (CP1 above), if the

Jacobian of the marginal profit functions constitutes a P-matrix then at most one solution to CP1 exists (Megiddo and Kojima, 1977).

Recently (Dierker, 1972; Mas-Collel, 1979; Kehoe, 1980) have weakened the P-matrix condition for the general equilibrium problem to the requirement that the Jacobian have a positive determinant at all equilibria. This turns out to be a necessary and sufficient condition for uniqueness of equilibria (subject to some conditions). Since our eqn. (5) is related to the determinant of the Jacobian of the marginal profit function, our development is related to theirs. In fact our result is proved, using degree theory, in Kolstad and Mathiesen (1986).

To sketch our proof, the basic idea is to append to CP1 a constraint that $\sum_{i=1}^{N} q_i = Q$. As Q is varied, the resulting q is a Cournot equilibrium if and only if the Lagrange multiplier (λ) on the constraint is zero (Figure 1). We then show that $\lambda(Q)$ is a continuous function of Q and is positive at $Q = 0$. Thus, if whenever λ cuts the horizontal axis, $\lambda'(Q) < 0$, we can conclude λ cuts the axis at most once. The condition that $\lambda'(Q) < 0$ at an equilibrium is precisely eqn. (5). Non-degeneracy of equilibria implies λ is differentiable at any equilibrium. If we preclude the possibility that $\lambda(Q)$ may be tangent to the horizontal axis, Theorem 3 can be restated that eqn. (5) is a necessary and sufficient condition for the uniqueness of the equilibrium. We now turn to the proof.

Following Murphy et al. (1982) and Szidarovszky and Yakowitz (1977), consider the following mathematical program:

$$\max_{q} \quad g(Q) = P(Q) \sum_{i=1}^{N} q_i + \frac{1}{2} P'(Q) \sum_{i=1}^{N} q_i^2 - \sum_{i=1}^{N} C_i(q_i) \qquad (7a)$$

$$\text{subject to} \quad \sum_{i=1}^{N} q_i = Q \qquad (7b)$$

$$q_i \geq 0 \quad \Psi_i \ \varepsilon \ N \qquad (7c)$$

for which first order conditions for an optimum are

$$CP2(Q): \quad \frac{\partial g}{\partial q_i} \equiv P(Q) + q_i P'(Q) - C_i'(q_i) - \lambda \leq 0, \quad \Psi_i \ \varepsilon \ N \qquad (8a)$$

$$\sum_{i=1}^{N} q_i = Q \qquad (8b)$$

$$\frac{\partial g}{\partial q_i} \ q_i = 0 \qquad (8c)$$

$$q_i \geq 0 \quad \Psi_i \ \varepsilon \ N \qquad (8d)$$

If $P'(Y+x) < C_i''(x)$, for all x, $Y \geq 0$, $i \ \varepsilon \ N$, then $g(Q)$ is concave and

there is a one to one correspondence between solutions to eqn. (7) and

eqn. (8). The complementarity problem $CP2(Q)$ is parameterized by Q and

involves finding $q(Q)$ and $\lambda(Q)$ which satisfies eqn. (8). Note that

$CP2(Q)$ is the same as CP1 with the addition of the constraint (8b). In

the following lemma we show that if Q is chosen so that λ is zero, then

$q(Q)$ which satisfies $CP2(Q)$ satisfies CP1 and vice versa.

Lemma 1: Assume cost and inverse demand functions are twice continuously

differentiable and $P'(Y+x) < C_i''(x)$, for all x, $Y \geq 0$, $i \ \varepsilon \ N$. If q^* is

a Cournot equilibrium then q^* solves $CP2(Q^*)$ where $Q^* = \sum_{i=1}^{N} q_i^*$ and if

$Q* > 0$, $\lambda(Q*) = 0$. Conversely, if for some $\hat{Q}$ the solution to CP2($\hat{Q}$) results in $\lambda(\hat{Q}) = 0$, then $\hat{q}(\hat{Q})$ is unique and is a Cournot equilibrium.

Proof: Suppose q* is a Cournot equilibrium. Then it solves CP1. Clearly, with $\lambda = 0$, q* also solves CP2(Q*) where $Q* = \sum_{i=1}^{N} q_i^*$. Further if $Q* > 0$, no other $\lambda \neq 0$ will work. If $Q* > 0$, then there exists a j such that $q_j^* > 0$ which implies eqn. (2a) holds strictly for that j, implying

$$P(Q*) + q_j^* P'(Q*) - C_j'(q_j^*) = 0. \tag{9}$$

But if q* solves CP2(Q*) and $q_j^* > 0$ then eqn. (8a) must also hold strictly:

$$P(Q*) + q_j^* P'(Q*) - C_j'(q_j^*) - \lambda = 0. \tag{10}$$

Comparing eqn. (9) and eqn. (10) obviously implies $\lambda = 0$.

The converse is trivially true. Since $P'(Y+x) < C_i''(x)$, and eqn. (7b) implies the constraint set is bounded, we know for any Q, there exists a unique solution to CP2(Q): $q(Q)$ and $\lambda(Q)$. If for some $\hat{Q}$, $\lambda(\hat{Q}) = 0$, then eqn. (2) and eqn. (8) are identical.

The next result is an extension of a well known result on the continuity of the optimal solution of a mathematical program (see Fiacco and McCormick, 1968; Hogan, 1973). The following lemma is proved by Murphy et al. (1982), among others.

Lemma 2: Assume cost and inverse demand functions are twice continuously differentiable and $P'(Y+x) < C_i''(x)$, for all x, $Y \geq 0$, $i \in N$. Then the solutions to CP2($\overline{Q}$), $q(\overline{Q})$ and $\lambda(\overline{Q})$ are continuous functions of $\overline{Q}$ over

$(0,\infty)$. Furthermore, if strict complementary slackness holds ($q_i = 0 => P(\overline{Q}) < C_i'(0)$) then q and λ are locally differentiable.

We now turn to proofs of the theorems.

Proof of Theorem 1: Follows directly from result that this concave game has a Nash equilibrium over a compact set of outputs.

Proof of Theorem 2: We show that eqn. (5) is equivalent to requiring $\lambda'(Q^*) < 0$ for all Cournot equilibria, q^*, where $Q^* = \sum_{i=1}^{N} q_i^*$. By Lemma 2, $\lambda(Q)$ is continuous and, because of non-degeneracy of all equilibria, $\lambda(Q)$ is locally differentiable at Q^*. Thus, if λ is always downward sloping when it cuts the $\lambda = 0$ axis, there can be at most one $Q > 0$ for which $\lambda(Q) = 0$. From Lemma 1, this implies at most only one nontrivial q for which $\lambda(\sum_{i=1}^{N} q_i) = 0$ and thus at most one nontrivial Cournot equilibrium. We now show that eqn. (5) implies $\lambda'(Q^*) < 0$.

For a particular Q, let $q(Q)$ solve CP2(Q). Define the index set $M(Q)$ by

$$M(Q) = \{i \in N | q_i(Q) > 0\} \tag{11}$$

For any Cournot equilibrium, q^*, we know that $q^* = q(Q^*)$ where $Q^* = \sum_{i=1}^{N} q_i^*$. Thus from the assumption that all equilibria are non-degenerate, we know from Lemma 2 that there is a small neighborhood, A, about Q^* over which $M(Q)$ does not change and over which $q(Q)$ and $\lambda(Q)$ are continuously differentiable. Thus, over this neighborhood eqn. (8a) holds as an equality if and only if $i \in M(Q^*)$:

$$\lambda(Q) = P(Q) + q_i(Q)P'(Q) - C_i'(q_i(Q)), \quad Q \in A, \ i \in M(Q^*) \tag{12}$$

We can thus totally differentiate eqn. (12) with respect to Q to obtain

$$\lambda'(Q^*) = P'(Q^*) + q_i'(Q^*)P'(Q^*) + q_i^*P''(Q^*) - C_i''(q_i^*)q_i'(Q^*)$$

$$= P'(Q^*) + q_i^*P''(Q^*) + [P'(Q^*) - C_i''(q_i^*)]q_i'(Q^*) \qquad (13)$$

$$\Rightarrow \quad \frac{\lambda'(Q^*)}{C_i''(q_i^*) - P'(Q^*)} = \frac{P'(Q^*) + q_i^*P''(Q^*)}{C_i''(q_i^*) - P'(Q^*)} - q_i'(Q^*) \qquad (14)$$

Eqn. (14) can be summed over all elements of M noting that

$$\sum_{j \in M} q_i'(Q^*) = \frac{d}{dQ^*}[\sum_{j \in M} q_i(Q^*)] = \frac{d}{dQ^*}[Q^*] = 1 \qquad (15)$$

since the index set M remains unchanged over the neighborhood A. Thus, summing eqn. (14):

$$\lambda'(Q^*) \sum_{j \in M} \frac{1}{C_j''(q_j^*) - P'(Q^*)} = \sum_{i \in M} \frac{P'(Q^*) + q_j^*P''(Q^*)}{C_j''(q_j^*) - P'(Q^*)} - 1. \qquad (16)$$

By assumption v), $C_j''(q_j^*) - P'(Q^*) > 0$. Thus $\lambda'(Q^*)$ has the same sign as the right-hand side of eqn. (15), which by assumption is negative.

<u>Proof of Theorem 3</u>: Sufficient conditions for uniqueness of the equilibrium are trivial application of Theorem 1. We thus focus on the necessary conditions for uniqueness.

Suppose there is a unique nontrivial Cournot equilibrium, q*, but that at q*, the left-hand side of eqn. (5) is strictly greater than the right-hand side. We will construct another nontrivial equilibrium, thus proving necessity by contradiction.

From assumption iv), there is a j such that $P(0) > C_j'(0)$. Continuity of P, P' and C_j' implies that there is a positive $\widetilde{Q} < \sum_{i \in N} q_i^*$ such that $P(\widetilde{Q}) - q_j P'(\widetilde{Q}) - C_j'(q_j) > 0$ for all $q_j \leq \widetilde{Q}$. Let $\widetilde{q}_j(\widetilde{Q})$ and $\lambda(\widetilde{Q})$ solve $CP2(\widetilde{Q})$. From eqn. (7a), we know

$$P(\widetilde{Q}) + q_j P'(\widetilde{Q}) - C_j'(\widetilde{q}_j) - \lambda(\widetilde{Q}) \leq 0 \tag{17}$$

which implies $\lambda(\widetilde{Q}) > 0$. By assumption, the left-hand side of eqn. (5) is greater than the right-hand side at $q(Q^*)$. Thus, from the proof of Theorem 2, $\lambda'(Q^*) > 0$. Since $\lambda(Q)$ is differentiable at Q^* and continuous over $(0,\infty)$, this implies that for some Q, where $\widetilde{Q} < \hat{Q} < Q^*$, $\lambda(\hat{Q}) = 0$. From Lemma 1, this yields another distinct Cournot equilibrium, contradicting our assumption of a unique nontrivial Cournot equilibrium.

V. CONCLUSIONS

The main result of this paper is Theorem 3. In that theorem we show the conditions under which a Cournot equilibrium is unique. If the basic market involves quasi-concave profit function and a few other weak restrictions are met, the theorem gives necessary and sufficient conditions for the existence of a unique Cournot equilibrium. Eqn. (5) states those conditions. Roughly interpreted (for the case of identical firms), the condition is that my marginal revenue function declines when you increase your output.

VII. REFERENCES

Dierker, Egbert, "Two Remarks on the Number of Equilibria of an
    Economy," Econometrica, 40:951-953 (1972).

Fiacco, Anthony V. and William P. Hutzler, "Basic Results in the
    Development of Sensitivity and Stability Analysis in Nonlinear
    Programming," Comput. and Ops. Res., 9:9-28 (1982).

Hogan, William W., "Point-to-Set Maps in Mathematical Programming,"
    SIAM Review, 15:591-603 (1973).

Kehoe, Timothy J., "An Index Theorem for General Equilibrium Models
    with Production," Econometrica, 48:1211-32 (1980).

Kolstad, Charles D. and Lars Mathiesen, "Necessary and Sufficient
    Conditions for Uniqueness of a Cournot Equilibrium," Rev. Econ.
    Stud., 54:681-90 (1987).

Mas-Colell, Andreu, "Homeomorphisms of Compact, Convex Sets and the
    Jacobian Matrix," SIAM J. Math. Anal., 10:1105-1109 (1979).

Meggiddo, Nimrod, and Masakazu Kojima, "On the Existence and Unique-
    ness of Solutions in Nonlinear Complementarity Theory," Mathe-
    matical Programming, 12:110-30 (1977).

Murphy, Frederic H., Hanif D. Sherali and Allen L. Soyster, "A Mathe-
    matical Programming Approach for Determining Oligopolistic Market
    Equilibria," Mathematical Programming, 24:92-106 (1982).

Okuguchi, Koji, Expectations and Stability in Oligopoly Models,
    Lecture Notes in Economics and Mathematical Systems No. 138
    (Springer-Verlag, Berlin, 1976).

Okuguchi, Koji, "The Cournot Oligopoly and Competitive Equilibria as
    Solutions to Non-linear Complementarity Problems," Economic Letters,
    12:127-133 (1983).

Szidarovszky, F. and S. Yakowitz, "A New Proof of the Existence and
    Uniqueness of the Cournot Equilibrium," Int'l Economic Rev.,
    18:787-789 (1977).

Szidarovszky, F. and S. Yakowitz, "Contributions to Cournot Oligopoly
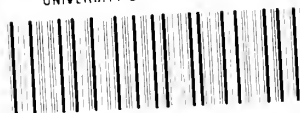   Theory," J. Economic Theory, 28:51-70 (1982).