ELECTROMAGNETIC RADIATION

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ELECTROMAGNETIC RADIATION

AND THE MECHANICAL REACTIONS ARISING FROM IT

BEING AN ADAMS PRIZE ESSAY IN THE UNIVERSITY OF CAMBRIDGE

by

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PREFACE

THE subject proposed for the Adams Prize in 1908 was "The Radiation from Electric Systems or Ions in Accelerated Motion and the Mechanical Reactions on their Motion which arise from it." This also happened to be the subject-matter of several investigations which I had carried out from time to time as preliminaries to a number of papers published in the *Philosophical Magazine* and other journals; advantage was taken of the opportunity then afforded to collect these investigations together and to extend them. The time available for this purpose however proved to be too short for a full treatment of the whole subject; while Radiation was discussed pretty fully, the Mechanical Reactions had to be dismissed very briefly. During the interval which has elapsed since the award of the prize in 1909, the treatment of the latter part of the subject has been extended so as to complete the original program; I venture to hope that there has been sufficient improvement in the essay to justify the delay in publication which has arisen.

In order to avoid any great change in the form of the essay most of the additional matter has been introduced in seven Appendices, and the alterations in the original text have been confined to extensions of two problems already included in it, but somewhat scantily treated. The first problem deals with the motion of a β-particle in a uniform electric field, Ch. XI, §§ 151—154, and the second with the electromagnetic field generated by this motion, Ch. V, §§ 43—60. Room has been made for the additional matter by slightly curtailing the treatment of two problems dealing with the field generated by a uniformly accelerated electric charge, Ch. V, §§ 25—42. These changes have been deemed advisable because the second problem admits of a simple solution in a finite form and thus affords an exceptionally good illustration of the method of the point potentials. All these additions to the text, as well as a few others of slight extent, have been enclosed in square brackets in order to distinguish them from the older portions of the essay.

In order to avoid misunderstandings it will be well to say a few words as to the scope and method of the present investigation. In consequence of the discoveries of the last few years in the fields of radioactivity, vacuum-tube phenomena, magnetism and radiation, a great need has arisen for a comprehensive Electron Theory of Matter, which shall systematize the results already achieved, as well as serve as a guide in future researches. A beginning has been made by J. J. Thomson in his well known paper on the Structure of the Atom and in his books on Electricity and Matter and the Corpuscular Theory of Matter, but these investigations have been carried out under the restriction, no doubt introduced for the sake of simplicity, that the negative electrons in the atomic model move with velocities so small compared with the velocity of light, that they can be treated like the particles of ordinary mechanics. Moving electric charges however do not behave like the particles of ordinary mechanics: they do not obey the Law of Action and Reaction, their mass varies with their speed, and they generate a magnetic field which reacts on their motion in various ways. When the speeds of the charges are small compared with that of light, all these effects are small, but we have no right to regard them as negligible even in that case. For example, Ritz's well known theory of the production of Spectrum Series rests on the effect of a magnetic field in the atom on the motion of negative electrons; a theory of this type is not provided for by an Electron Theory of Matter, which assimilates the electron to the particle of ordinary mechanics.

Moreover it is by no means certain that all the electrons inside the atom are moving with speeds small compared with that of light; we know that β-particles are expelled from comparatively stable atoms, like that of Radium, with speeds differing from that of light by only 2 %. The kinetic energy of such a β-particle amounts to three millionths of an erg, which is five times the mutual electrostatic energy of two negative electrons in contact. It is not easy to imagine an arrangement of negative and positive charges in equilibrium, or in slow stationary motion, which shall be sufficiently permanent and stable to serve as a model of the Radium atom, and at the same time be capable of setting free sufficient potential energy to supply the kinetic energy of a β-particle and also overcome the attraction of the positive charges. If on the other hand we suppose the β-particle to be already moving inside the Radium atom with a speed comparable with that of light, this difficulty does not arise. A supposition of this kind no doubt has difficulties of its own, but it would be unwise to ignore it altogether.

For these reasons it is desirable, before attempting to frame any comprehensive Electron Theory of Matter, to develope the theory of moving electric charges with as few restrictions respecting their structure and motion as possible. When expressions for the electromagnetic field due to them have been obtained, the radiation from them and the mechanical reaction on them can be calculated; the former depends on the field at a great distance, the latter on that inside them. The object of this essay is to carry out these preliminary investigations, to provide, as it were, prolegomena to any future Electron Theory of Matter.

Accordingly the essay is in method deductive and mathematical, rather than constructive and physical; but since there is considerable danger, in a purely mathematical investigation, of losing touch with reality, the methods developed have been freely illustrated by concrete examples of their application. The solutions of the problems dealt with in this way have not as a rule been carried to the point of numerical calculation; this could not have been done without increasing the size of the book unduly, even if there had been sufficient time available. Many of the computations would have meant a great expenditure of labour, as a large number of the integràls and series have been very little studied. In this connection it is worth noting that the series principally involved in the expressions found for the radiation and the mechanical reaction are of the type known as Kapteyn Series of Bessel Functions; a simple example is afforded by Bessel's well known solution of Kepler's equation. The study of the convergence and methods of summation of these series offers many problems to the pure mathematician and is very necessary for progress in the Electron Theory of Matter.

Let us now consider the fundamental assumptions on which the present investigation is based, and also their relation to the Theory of Relativity and to the Unitary Theory of Radiation (Quantentheorie), which have arisen since it was begun and have been very widely discussed in the interval. In choosing the fundamental assumptions I have throughout aimed at securing the greatest generality consistent with thoroughly well established experimental results. Additional assumptions have only been introduced when further progress seemed impossible without them, or when a comparison of results already obtained with experiment clearly indicated that further restrictions were desirable. For these reasons I have refrained from making any use, either of the Postulate of Relativity, or of the Aether Hypothesis, which by some are regarded as inconsistent with each other. Some of the results obtained in this essay are consistent with the Postulate of Relativity; others cannot be reconciled with it, at least when it is used in the strictest possible sense, and find their natural explanation in terms of the aether. All however are deduced quite independently of either hypothesis. It does not appear to me that either of the new theories is so well established and so generally accepted yet as to be properly made the basis of an investigation in which the utmost generality is aimed at.

The following are the assumptions which have been made:—

(1) The electromagnetic equations employed are those of Maxwell and Hertz, in the form used by Larmor and Lorentz. These equations already imply the existence of a system of axes to which they are referred; it is immaterial for our purpose whether these axes be regarded as fixed in space, relative to a fixed aether, or as only fixed relative to the observer.

(2) The expressions for the potentials and the electric and magnetic forces deduced from the electromagnetic equations are assumed to be continuous functions of the time and coordinates in general. They may indeed,

at any given instant, be discontinuous at certain surfaces, or, at any given point, they may undergo sudden changes at certain times; these discontinuities however will be supposed to be exceptional.

(3) The electric charges, whether positive or negative, are assumed to be distributed throughout finite, though small regions of space, separated from each other by regions free from charge. It will be found that, for the purpose of calculating the radiation by means of the expressions relating to the distant field, a knowledge of the size and structure of the electric charges is not needed; only the motion of each charge as a whole must be given. Hence for this purpose they may be treated as point charges moving in a prescribed way.

(4) The mechanical force on an element of electric charge, so far as it depends on the electromagnetic field, is assumed to be given by the formula of Maxwell for a current-element, as adapted by Larmor and Lorentz to the case of a moving element of charge. This formula gives the mechanical force in terms of the velocity of the element and of the electric and magnetic forces at the place where it is for the moment, whether the field be due to other elements of the same charge, or to external charges, or to both.

(5) For the sake of generality we shall admit the possible presence of a mechanical force of non-electromagnetic origin, but we shall suppose that the resultant of all these forces for the several elements of an electric charge vanishes identically, or at any rate is proportional to the acceleration of the charge as a whole. The last alternative amounts to postulating that each element of charge has associated with it, is as it were loaded with a certain amount of mass of non-electromagnetic origin; we shall accordingly make this last assumption as well as the first.

(6) The motion of each separate element of charge is to be found from the mechanical forces (4) and (5) by applying Newton's Second Law of Motion. In virtue of (5) the motion of the charge as a whole only depends upon the resultant of all the electromagnetic forces, internal as well as external, which act on the several elements, and on the total non-electromagnetic mass, if there be any.

Little need be said as to assumptions (1) and (4); they are generally accepted as representing the results of experiments on matter in bulk. Their extension to the case of an isolated charge is of course hypothetical, but is justified by the agreement of the conclusions obtained with experiment; for instance the path calculated for a single Lorentz electron moving in a given electromagnetic field agrees with that found by experiment for a thin pencil of β-rays. These assumptions also form the basis of the Theory of Relativity.

Assumption (2) appears to be a natural, if not a necessary consequence of (1); nevertheless it may lead to difficulties if some forms of the Unitary Theory of Radiation be adopted. It is well known that one form of this theory postulates an atomic structure of radiant energy in order to explain the production of secondary β-rays from X-rays. In order to obtain the necessary concentration of energy, it is assumed that the space occupied by a unit of energy is exceedingly small, so that the distribution of energy in the field is practically discontinuous. In the case of periodic disturbances we can represent a discontinuous field of the required kind by means of series of Fourier's type, built up of periodic solutions of the electromagnetic equations, and thus are able to explain interference and polarisation. But on this view a truly homogeneous radiation would be impossible, because the fundamental vibration would always be accompanied by upper partials, whose intensities would be comparable with its own. This difficulty does not occur in a later form of the Unitary Theory, in which the emission of energy by the source takes place discontinuously, but the radiant energy itself is not discontinuous, not concentrated in isolated regions of the field. A theory of this later type appears to be sufficient for the statistical theory of radiation and does not conflict with our assumption (2).

Assumption (3) is fundamental in the treatment of the mechanical reactions adopted in the present essay. The only alternative is to assume that each charge is concentrated in a mathematical point, that is, that it is a mere centre of force. The two assumptions are equivalent as regards the determination of the field produced by the moving charge; for since the point charge is a singular point of the field, we may exclude it by means of a small closed surface surrounding it, and treat it as equivalent, at all outside points, to a suitable distribution of charge on the surface. This amounts to regarding it as an extended charge for all purposes which do not require us to enquire into its internal structure.

But when we desire to determine the motion of the charge under the action of a given external field, the difference between the two assumptions becomes apparent. For a point charge we must, in the first place, assign a formula expressing the connection between the mechanical force acting upon it and the electric and magnetic forces of the field, for instance the formula of Larmor and Lorentz given by assumption (4). Secondly, we must endow the point charge with a proper amount of energy and mass, and assume formulae expressing the variation of these quantities with the speed, for example those given by the Postulate of Relativity. Lastly, we must determine the reaction on the charge produced by its radiation, for instance by the method used by Abraham in §15 of his *Elektromagnetische Theorie der Strahlung,* where he deduces it from the expressions for the momentum and energy radiated from the moving charge. Thus the intrinsic energy and the mass are irreducible properties for a point charge, but there is nothing in its nature to determine the laws according to which they depend upon its speed; these laws have to be determined by considerations having nothing to do with the character of the charge itself.

For the extended charge on the other hand we must assume in the first place a definite size and structure when it is at rest, and secondly a formula expressing the mechanical force on each element of the charge, for instance the formula of Larmor and Lorentz adopted in assumption (4). The motion of each element having been assumed provisionally, we are able to determine the energy, the momentum, the mass and the mechanical reaction of the radiation, all by one and the same process. No assumption is needed which does not follow naturally from the notion of an extended charge; in this respect this hypothesis is more satisfactory than the last.

In making the assumption of an extended charge we however encounter a problem which compels our attention. When we assume that a charge has parts which repel one another, we are bound to explain how it happens that the charge can exist in spite of the mutual repulsions of its elements. This problem is meaningless for a point charge, which is *ipso facto* indivisible. Its solution for an extended charge requires us to postulate the existence of forces of non-electromagnetic origin, in accordance with assumption (5). These, together with the mechanical forces due to the electromagnetic field, act upon each element of the charge and produce the actual state of rest, or of motion, as the case may be, in accordance with some assigned law, such as Newton's Second Law of Motion, adopted in assumption (6). We see that in this way assumptions (5) and (6) are necessary corollaries of assumption (3).

Thus there are two alternative methods of finding the mechanical reactions on an electric charge: one based on the hypothesis of the extended, and the other on that of the point charge. So far as can be judged *a priori*, each, by the help of suitable subsidiary assumptions, can be made to give a consistent account of the phenomena to be explained, and there seems to be no decisive reason compelling us to choose one rather than the other. The choice is largely a matter of personal preference; I have selected the method of the extended charge because it affords more scope for illustration by means of mechanical models.

The scheme of the essay may be summarized as follows:—

Ch. I gives a brief discussion of the fundamental equations of the electron theory in the form established by Maxwell and Hertz, and further developed by Larmor and Lorentz; it involves nothing materially new.

Chs. II and III give the transformation of the Lorentz expressions for the retarded potentials into integrals, which constitute an extension of Fourier's type. They are shown to lead to Sommerfeld's integrals and to the point potentials of Liénard and Wiechert.

In Chs. IV—VI the point potentials are discussed fully and applied to the determination of the electromagnetic field in special cases of the motion of a point charge. The most important problem here considered concerns the field generated by a Lorentz electron, which moves parallel to the lines of force of a uniform electrostatic field. When the very small reaction due to the radiation is neglected, this problem admits of an elegant solution in a finite form; it is, I believe, the only case of an accelerated motion for which the field has been completely determined. Attention is also drawn to the discrepancy between the result obtained in Problem 3 of Ch. V and that found by Descoudres for the uniform motion of a point charge with a speed greater than that of light. This discrepancy indicates that there is a difference in kind between the field due to a uniform motion which has been acquired from rest, and one which has existed for all past time.

The next four chapters deal with the field generated by motions of electric charges with any speeds not exceeding that of light, the treatment throughout being based upon the integral solutions obtained in Ch. II.

Chs. VII and VIII are devoted to the consideration of the electromagnetic fields due to periodic motions of systems of electric charges. Ch. VII gives the case of a monoperiodic motion, a motion involving only one period. The results are applied to the uniform motion of a circular ring of charges and to the calculation of the radiation from it.

Ch. VIII deals with the fields generated by polyperiodic motions, that is motions involving several incommensurable periods. General expressions are obtained for the electric and magnetic forces at a great distance from the system of charges producing the field and are applied to the cases of a uniform circular motion, a simple harmonic rectilinear motion, elliptic motion about the centre, disturbed circular motion, motion in an epitrochoid and the precessional motion of a vibrating system of charges. The last problem is of some importance for the theory of the Zeeman effect developed by Ritz; the results show that precession of suitable amount will account for resolutions of a line into as many as nine components, symmetrical as to position but not necessarily symmetrical as to intensity.

Ch. IX deals with motions which are not strictly periodic, or are aperiodic, for instance damped vibrations. Motions of this kind involve discontinuities occurring at one or more periods of their existence; it is shown that the effect of one of these discontinuities is confined to a greater or less interval of time near the instant at which the discontinuity occurred, and that at times other than this the potentials and forces of the field can be expanded in Power Series of Lagrange's type.

Ch. X concerns the electromagnetic field at points on or close to the orbit of a moving charge, but distant from the charge itself by several diameters at least. The expressions obtained for the electric and magnetic forces are applied to the case of a circular ring of electrons in uniform motion.

Ch. XI gives a brief account of the equations of motion of a single charge, with applications to uniform circular motion, the same motion when slightly disturbed, and the motion of a Lorentz electron in a uniform electrostatic field. The last problem has been already referred to in Ch. V, Problems 4 and 5.

Ch. XII deals with the motion of a group of electric charges, particularly with the steady motion of a circular ring of electrons. Owing to the loss of energy by radiation, a strictly uniform motion is impossible if the charges be absolutely invariable, unless the loss be supplied from some external source, which produces the tangential force needed to keep the speed uniform. Moreover, a system of charges of this kind is essentially indeterminate in structure, because there is but one equation of motion, the radial one, while there are two independent variables, the speed and the radius of the orbit. One of the two variables may have its value arbitrarily assigned, and then the other is completely determined by the equation of motion; there is however no condition which shall enable us to fix upon any particular value of the first variable, and consequently the structure of the ring is only determined within certain limits necessary to secure stability.

The determinateness of structure of the ring, as well as a high degree of permanence in spite of radiation, can be secured, if the mass of each charge be allowed to diminish at an exceedingly slow rate, which would occur in consequence of a very slow expansion of the charge. Then the loss of energy due to radiation takes place at the expense of the internal electric energy of the charges; the condition that this compensation may occur gives a second equation of motion, the tangential one, and thus we have the two equations necessary to fix the values of the speed and radius of the ring. They are not absolutely constant, but are subject to small secular changes; the ring is not absolutely permanent, but it changes very much more slowly than it would do if its radiation were not compensated. It has definite periods of vibration, but they undergo small secular changes.

It is possible that the same result may be secured in another way, namely by means of an asymmetric charge. It is shown in Appendix D that a tangential force component is required, as well as the radial component, in order to keep an asymmetric charge moving in a circle; it may happen that the drag due to radiation just supplies the force that is needed.

These twelve chapters constitute the essay practically as it was submitted to the examiners; it has been stated already that the treatment of the motion of electric charges given in Chs. XI and XII is altogether inadequate, owing to the lack of time to complete the essay in accordance with the original plan. The rest of the book, with the exception of Appendix A, is devoted to remedying this defect.

Appendix A gives an application of the results of Chs. VIII and IX to the theory of the Doppler effect.

Appendix B is devoted to an investigation of the disturbed motion of a circular ring of charges. In Problem 1 the methods of Ch. X are employed to find the electric and magnetic forces produced by an electric charge, when slightly disturbed from its uniform motion in a circle, at points close to the circle but at a distance from the charge itself. In Problem 2 the field due to a ring of charges is calculated, and the mechanical force on one of them due to all the rest is found. In Problem 3 the methods of Chs. XI and XII and the results of the two preceding problems are applied to find the equations of motion of a circular ring of charges when it is disturbed from its steady motion.

Appendix C deals with the electromagnetic field close to a moving point charge. The Lagrangian Series obtained for the potentials in Ch. IX are transformed into expansions proceeding according to the accelerations of increasing order. These expansions are required in the investigations of Appendices D and E.

The object of Appendix D is to supply a proof from first principles of the equations of motion of a single charge, which have been already given in Ch. XI. The resultant of all the mechanical forces exerted on one element of the charge by all the rest is calculated, and the mechanical force exerted by the charge on itself is found by summation over all its elements. On account of the failure of Newton's Third Law of Motion in the case of electric charges, the total mechanical force on a charge generally differs from zero; an expression is found for it, which leads to definitions of the electromagnetic momentum and mass of the charge, as well as of the reaction on it due to its radiation; this we already know from the investigations of Abraham, Sommerfeld and others. For the extended electron the electromagnetic momentum and mass depend upon its size and structure; the reaction due to radiation however does not, but has the same value as for an equal point charge. It is here shown that the electromagnetic momentum is not in the direction of the mean motion of the charge unless the charge be symmetrical with respect to a plane perpendicular to the direction of the mean motion, or to two planes parallel to the direction of the mean motion and perpendicular to each other.

Moreover the investigation of this Appendix gives definite information concerning an objection raised by Abraham to the Theory of Relativity in the second edition of his *Elektromagnetische Theorie der Strahlung*, § 49, which came to my notice too late to be considered in the text. Abraham points out that, according to the Theory of Relativity, the mass, m_0, and the electric energy, W_0, of a slowly moving electron satisfy the relation $m_0 = W_0/c^2$, while the mass of the Lorentz electron is 4/3 of the amount determined by this equation. The equations (350), § 229 of the text, show that even for a symmetrical electron m_0 always exceeds W_0/c^2, whatever the configuration of the electron may be when it is at rest. Thus Abraham's objection appears to be valid, namely that the electromagnetic dynamics of the electron is inconsistent with the Postulate of Relativity; at any rate this is so if the hypothesis of the extended electron be accepted.

The results of this Appendix are applied to the particular cases of the electrons of Abraham, Bucherer and Lorentz, and also to that of an asymmetrical electron derived from the Lorentz electron by a small deformation symmetrical about an axis which is inclined to the direction of the mean motion.

Appendix E deals with the mechanical explanation of the electron, a problem which compels our attention since we have accepted the hypothesis of the extended electron, as we have already pointed out. Poincaré's explanation of the Lorentz electron by means of a uniform surface pressure is here found to have a much wider application. It is shown that the hypothesis that the electron is subjected to a uniform surface pressure, whatever its motion may be, is of itself sufficient to lead to the Lorentz mass-formula, and to a configuration of the electron in which the surfaces of equal density are Heaviside ellipsoids, so long as the speed remains constant. When the speed changes, small redistributions of the charge inside the electron occur, as well as slight deformations of its bounding surface. By means of these changes an observer moving with the electron could in theory detect the acceleration, so that the Postulate of Relativity cannot hold for accelerated motion.

Moreover it is found that an extended electron, whatever the distribution of its charge may be, cannot exist unless it is subjected to a suitable pressure on its outer surface. Hence we must either postulate the existence of some external medium which shall produce the required surface pressure, or admit that the elements of charge of the electron exert on each other actions at a distance, which are not electromagnetic and follow quite different laws. The first hypothesis, that of an external medium, appears to be the more reasonable of the two and amounts to admitting the existence of the electromagnetic aether.

Thus it appears that the acceptance of the Postulate of Relativity in its strictest form almost necessitates the adoption of the hypothesis of the point charge, while the hypothesis of the extended charge leads naturally to the adoption of the aether hypothesis.

Appendix F gives a brief sketch of the mechanics of the Lorentz electron; its object is to show that the adoption of the Lorentz mass-formula leads to a workable system of equations of motion, not indeed as simple as those of ordinary mechanics, yet simple enough to allow of our obtaining complete solutions of many important problems. It is found possible to write down the equations of Lagrange and Hamilton in the most general case of the motion of a Lorentz electron under the influence of a variable electromagnetic field. When the external field is steady, an energy integral exists; when it is symmetrical about an axis, an integral of angular momentum about that axis can be found. These integrals can be used as in ordinary mechanics to reduce the order of the system of differential equations of motion; all the necessary eliminations can be actually performed owing to the simple form of the Lorentz mass-formula. It is possible to deduce Lagrangian and Hamiltonian equations of motion for other mass-formulae also, at any rate in theory, but in practice they are useless, because they are too unwieldy on account of the complications introduced by the forms of the mass-formulae. In this respect the Lorentz mass-formula possesses an overwhelming advantage over all others.

Appendix G gives applications of the methods of Appendix F to particular cases, namely the motion of a Lorentz electron in a steady and uniform electromagnetic field, including the theory of the experiments of Kauffmann and Bucherer, and its motion in a steady field, in which the electric force is central and a function of the radius alone, while the magnetic force is uniform and of small intensity. The last problem is important for the precessional theory of the Zeeman effect, which is due to Ritz and has already been considered in Problem 6, Ch. VIII. It is here found that the angular velocity of precession, μ in Problem 6, is equal to the usual value, $eH/2cm$, assumed for the Zeeman effect, but the mass of the electron, m, is that appropriate to its speed at the moment considered. The angular velocity of the precessing system about its own axis, n in Problem 6, depends on the constitution of the system under consideration; in the particular example studied in the present Appendix the angular velocity n is variable, and has no connection with the angular velocity of precession, μ, so that Runge's rule remains unexplained.

It is obvious that in a long-continued investigation like the present one, there must be many points of contact with the work of other writers, and accordingly many of the results here obtained have been anticipated. In cases of this kind I have not made any change in the text, because the proofs here given are as a rule different from those used by my predecessors in publication and may very well serve as verifications of their methods. I hope that I have taken note of most cases of anticipation of this kind, but no doubt there are others which have escaped my notice altogether, or have been seen too late to receive due mention. I cannot however close this preface without expressing the very great debt I owe to Sir J. Larmor and to Prof. H. A. Lorentz, whose writings have furnished the theoretical foundations of this essay, and particularly to my former teacher, Sir J. J. Thomson, to whose paper on Cathode Rays this investigation owes its inception.

I have to thank my friend and colleague, Mr J. S. G. Thomas, for his kindness in reading the proofs of the essay and for his valuable criticisms and suggestions.

My thanks are also due to the officials of the University Press for the exactness of their printing and the care shown in the reproduction of the diagrams, as well as for the obliging manner in which they have received many corrections.

G. A. SCHOTT.

ABERYSTWYTH,
March 30, 1912.

CONTENTS

CORRIGENDA

Page 5, line 33 }
„ 6, „ 29 } *for* (XIII) *read* (XII).
„ 7, lines 9, 11, 13 }
„ 9, line 16, *for* (344), § 227 *read* (354), § 231.
„ 38, „ 4, *for* $t = v\tau - x$ *read* $0 = v\tau - x$.
„ 38, Fig. 9, the Roman numerals I and II are to be interchanged.
„ 43, line 29, *before* continues *insert* and.
„ 45, at the foot, *for* C *read* c.
„ 47, line 2, *for* (*c*) *read* (*b*).
„ 48, „ 2 from the foot, *for* (50_2) *read* (50_4).
„ 49, „ 10, *for* [*a*] *read* **a**.
„ 49, last line, *for* $t = c/f$ *read* $z = c^2/f^2$.
„ 57, Fig. 19 } the letter G has been used to denote the position of the
„ 58, lines 13, 14, 19 } charge at the time U by inadvertence, although it is used elsewhere to denote the initial wave GDG'.
„ 70, line 4, *for* § 43 *read* § 44.
„ 74, „ 7, *for* R^2 *read* R_1^2.
„ 74, „ 27, *for* Fig. 25 *read* Fig. 24.
„ 78, „ 11, *for* § 43 *read* § 44.
„ 83, „ 33, *for* $\beta = \omega/c$ *read* $\beta = v/c$.
„ 84, „ 8, *for* $R = \Delta$ *read* $R = f\Delta$.
„ 85, lines 14, 15, 38, 39, the values of ϕ and **a** must be divided by f.
„ 109, line 19, *for* latitude *read* longitude.
„ 130, „ 18, *after* (146) *insert* (159) and (160).
„ 132, „ 25, *after* as before *insert* except that s_1 is now equal to s.
„ 137, „ 3, *eq.* (*g*), the sign of the second member must be changed.
„ 137, „ 15, *for* $(m - l\beta)^2$ *read* $(m - l\beta^2)^2$.
„ 145, „ 16, *for* Π *read* Π_i.
„ 146, „ 23, *for* $A = B$ *read* A or B be zero.
Pages 147 and 148, *for* θ *read* Θ throughout.
Page 147, line 24, *for* D *read* D^2.
„ 151, at the foot, *for* $\delta\mu$ *read* $d\mu$.
„ 154, line 12, *delete* the second Σ *from eq.* (179).
„ 155, lines 1 and 12, *for* (177) *read* (178).
„ 165, line 4, *for* Problem 2, Appendix A *read* Appendix C.
„ 170, „ 15, *after* $\sin j\psi$ *insert* j.
„ 180, „ 30, *for* $-\iota\frac{\dot{\mu}}{\omega}$ *read* $+\iota\frac{\dot{\mu}}{\omega}$.

Page 185, line 4 from the foot, *after* generally *insert* continue to.

„ 187, at the foot, *the lower limit of the sum* $\mathfrak{V}$ *should be* 0.

„ 194, line 23, *before* $(\mathbf{r}_1\mathbf{v}')$ *insert* $=$.

„ 201, lines 5 and 6, *interchange* $(x+\delta x, y+\delta y, z+\delta z)$ *and* $(\xi+\delta\xi, \eta+\delta\eta, \zeta+\delta\zeta)$.

„ 202, line 13, PU is the forward tangent to the circle at P.

„ 209, „ 18, *for* $\frac{\partial}{\partial\psi}$ *in the first term read* $\frac{\partial}{\partial\psi}+\iota\sigma$.

„ 218, „ 30, *for* $F_z=Q$ *read* $F_z=-Q$.

„ 223, „ 9, *for* b *read* β.

„ 223, lines 12 and 19, *for* h_ρ *read* $-h_\varpi$.

„ 223, line 22, *for* $\iota\left(\frac{e^2}{\rho^2}D+\frac{1}{2}c\dot{m}\beta\right)\sigma$ *read* $\iota\left(\frac{e^2}{\rho^2}D+\frac{1}{2}c\dot{m}\beta\sigma\right)$.

„ 225, „ 17, *delete* f_{13}, f_{31}.

„ 230, „ 30, *for* $(-v/c)^s$ *read* $(v/c)^s$.

„ 231, „ 6, *for* x' *read* ξ.

„ 264, at the foot, *for* M *read* $\frac{M}{e}$.

„ 265, line 4, non-electromagnetic momentum is really produced at the rate $\mu(\dot{v}+\dot{u})$ per unit volume, but the term $\mu\dot{u}$ is of the order zero and therefore is to be neglected.

„ 268, *add footnote*, Liapounoff has proved that the sphere is the only stable equilibrium figure for homogeneous gravitating liquid at rest (cf. Poincaré, *Figures d'Équilibre d'une Masse Fluide*, pp. 15 and 22, 1903).

„ 282, *add footnote*, cf. M^c^Laren, *Phil. Mag.* [6], XXI. p. 15, XXII. p. 66, 1911. Wassmuth, *Ber. d. Deutsch. Phys. Ges.*, p. 76, 1912.

„ 286, line 17, *after* (459) *insert* and (460).

CHAPTER I

FUNDAMENTAL EQUATIONS OF THE ELECTRON THEORY

1. THE most satisfactory foundation for the following investigation at the present day is furnished by the electromagnetic equations for the free aether, in the form developed by Maxwell and Hertz, together with the additional equations representing the effect of electric charges (electrons, ions) due in the main to Larmor and Lorentz. The former set of equations is sufficiently firmly established as a result of the experiments of Hertz, as well as of all subsequent experience. The latter set rests, not only on the fact of the existence of discrete electric charges capable of motion relative to the aether, or if we prefer, relative to a system of axes fixed with respect to the observer, but also on certain assumptions, expressed or implied, as to the connection between the aether and the electric charges.

In the first place it is assumed that the connection is a mechanical one, in so far as the equations expressing its effect are developed by an application of the principle of Least Action. Secondly it is assumed that electric charges move freely through the aether, without disturbing it or altering its properties to any appreciable extent.

That these two assumptions are mutually consistent is a distinct postulate, in favour of which not much evidence has yet been offered, and which is made mainly for the sake of simplicity.

Any disturbing effect of the electric charge on the aether may be expected to be limited to its immediate vicinity, and probably the details of structure of the charge will have no appreciable effect on the nature of the disturbance produced by it in the aether at a great distance away: this will practically be that due to a point charge moving with the mean velocity of the actual charge. Thus the electron theory in the form developed by Larmor and Lorentz affords a sufficient basis for the calculation of the radiation due to electric charges moving in prescribed ways.

But when we wish to determine the motion of an electric charge under the influence of a prescribed system of electric and magnetic forces the bases of the calculation are not so certain. The mechanical forces acting on the charge due to the impressed field, and the mechanical reaction on it due to its own radiation, both depend on the nature of the connection assumed to exist between the charge and the aether [or on the law assumed for the action at a distance between it and other charges, if we prefer to discard the idea of the aether]. Any uncertainty as to the mutual consistency of the two fundamental hypotheses of the electron theory will affect the conclusions arrived at concerning the motion of the electric charge and the reaction due to its radiation.

Experiments on the variation of the mass of a moving charge with its velocity however lead to results in substantial agreement with calculations based on the electron theory. We conclude that the equations developed by Larmor and Lorentz represent the effect of the connection between the charge and the aether, whatever it be, sufficiently closely to serve as a basis for the calculation of its motion, even though there may be some uncertainty as to the assumptions from which those equations have been derived.

2. The equations of the field. For the sake of uniformity we shall use the notation of H. A. Lorentz, employed by him in his [*Theory of Electrons,* which is in better agreement with the practice of English writers than that used in his] article in the *Encyklopädie der Mathematischen Wissenschaften,* "Elektronentheorie," Vol. v. Sect. 14, except that for the sake of brevity we shall use electrostatic units in place of rational (Heaviside) units for electric quantities, and magnetic units for magnetic quantities.

Scalar quantities will be denoted by plain letters, vectors by Latin letters of Clarendon type, their tensors by plain letters, and their components by plain letters with suffixes attached. Scalar products will be enclosed in round brackets (...) when necessary to avoid misunderstanding, vector products in square ones [...].

The equations of the Maxwell-Hertz theory are:

$$\text{div. } \mathbf{d} = 0 \text{ (free aether)} \quad \text{(I)},$$

$$\text{div. } \mathbf{d} = 4\pi\rho \text{ (electric charge)} \quad \text{(I a)},$$

$$\frac{\partial \rho}{\partial t} + \text{div. } \rho\mathbf{v} = 0 \quad \text{(II)},$$

$$c \text{ curl } \mathbf{h} = \dot{\mathbf{d}} + 4\pi\rho\mathbf{v} \quad \text{(III)},$$

$$c \text{ curl } \mathbf{d} = -\dot{\mathbf{h}} \quad \text{(IV)},$$

$$\text{div. } \mathbf{h} = 0 \quad \text{(V)}.$$

These equations are either based on experience, or are equations of definition.

The additional equation of Larmor and Lorentz is:

$$\mathbf{f} = \mathbf{d} + [\mathbf{v} \,.\, \mathbf{h}]/c \quad \text{(VI)}.$$

This expresses the mechanical force per unit charge due to the assumed connection between charge and aether. It rests on the assumption that there is no resultant mechanical force acting on a finite charge, due to its connection with the aether, of any kind, elastic or hydrodynamic, whatsoever it be, other than that directly due to the electric and magnetic force in the free aether. [It does not require that there be no stresses between neighbouring elements of charge, or between elements of aether to which they are attached, provided that these stresses balance for an isolated charge, or electron. The question of the existence of such stresses will be considered in Appendix E below.]

In these equations ρ denotes the volume density of the electric charge at the point (x, y, z) and at time t, and $\mathbf{v}$ its velocity relative to the stagnant aether, or, if we prefer, relative to a system of axes fixed with respect to the observer, $\mathbf{f}$ is the mechanical force per unit charge, $\mathbf{d}$ and $\mathbf{h}$ are the electric and magnetic forces in the free aether, and c is the velocity of light in the free aether.

3. Potentials. Equations (IV) and (V) enable us to express the electric and magnetic forces in terms of a scalar potential ϕ and vector potential $\mathbf{a}$; this is generally the best course, as the potentials are more easily determined than the forces themselves. In virtue of (V) we may put

$$\mathbf{h} = \text{curl}\,\mathbf{a} \qquad \text{(VII)}.$$

Substituting this expression in (IV) we get

$$\text{curl}\,(\mathbf{d} + \dot{\mathbf{a}}/c) = 0,$$

whence

$$\mathbf{d} = -\nabla\phi - \dot{\mathbf{a}}/c \qquad \text{(VIII)}.$$

Obviously only two of the three equations for h_x, h_y, h_z, derived from (VII), are independent. Thus $\mathbf{a}$ is not completely determined, and a third condition may be imposed upon it. Thus a singly infinite series of different functions $\mathbf{a}$ are possible, but only two have been used.

(1) Suppose that we impose the condition div. $\mathbf{a} = 0$.

Then (I a) gives $\qquad \nabla^2\phi = -4\pi\rho.$

And (III) gives $\qquad \nabla^2\mathbf{a} = -(\dot{\mathbf{d}} + 4\pi\rho\mathbf{v})/c.$

Hence we find in the usual way

$$\phi = \int \rho\, d\Omega/R,$$

$$c\mathbf{a} = \int (\rho\mathbf{v} + \dot{\mathbf{d}}/4\pi)\, d\Omega/R,$$

where R is the distance of the fieldpoint (Aufpunkt), that is of the point (x, y, z) of the field, where the potentials and forces are required, from the position (ξ, η, ζ) of the volume element $d\Omega$ at the time t. The volume integration is to be extended over all space, the density ρ being zero at points where there is no charge at the time t.

These expressions for the potentials have been used by Maxwell* and Larmor†. They are convenient in so far as the quantities ρ and $\mathbf{v}$ are to be

* *Electricity and Magnetism*, Vol. II. p. 237.

† *Aether and Matter*, pp. 92, 111.

taken for the actual time t, and therefore are known explicitly. When there is motion this convenience is more than counterbalanced by the fact that the expression for $\mathbf{a}$ involves the *total* current, that is, the displacement current $\dot{\mathbf{d}}/4\pi$ as well as the convection current $\rho\mathbf{v}$. Generally the quantities ρ and $\mathbf{v}$ are prescribed functions of t and (ξ, η, ζ), but $\dot{\mathbf{d}}$ itself remains to be determined. If however we content ourselves with a first approximation, in which v is neglected in comparison with c, the term involving $\dot{\mathbf{d}}$ may be neglected. This is done by Larmor in calculating the field due to a vibrating doublet (*loc. cit.* p. 223), but the resulting expression is incomplete. This approximation is insufficient for our purpose.

(2) Suppose that we impose the condition

$$\text{div.}\,\mathbf{a} + \dot{\phi}/c = 0 \qquad \text{(IX).}$$

Then (I a) gives $\qquad \nabla^2\phi - \ddot{\phi}/c^2 = -4\pi\rho.$

And (III) gives $\qquad \nabla^2\mathbf{a} - \ddot{\mathbf{a}}/c^2 = -4\pi\rho\mathbf{v}/c.$

Lorentz* gives the following particular integrals of these equations:

$$\phi = \int [\rho]\, d\Omega/R \qquad \text{(X),}$$

$$\mathbf{a} = \int [\rho\mathbf{v}/c]\, d\Omega/R \qquad \text{(XI).}$$

The brackets [...] here mean that the values of ρ and $\mathbf{v}$ are to be taken for the time $t - R/c$, and not for the actual time t; that is, the values of ρ and $\mathbf{v}$ correspond to the time of emission of a disturbance, which leaves the element $d\Omega$ at (ξ, η, ζ) so as to reach the fieldpoint (x, y, z) at time t. Thus the quantities $[\rho]$ and $[\rho\mathbf{v}]$ are prescribed functions of $t - R/c$ and (ξ, η, ζ). They involve (ξ, η, ζ) explicitly in the parameters of the functions, and also implicitly through R; they also involve (x, y, z) implicitly through R. The seven variables $t, \xi, \eta, \zeta, x, y, z$ in these integrals are all independent.

The integrals (X) and (XI) satisfy the condition (IX) in virtue of (II).

4. Complementary functions. The Lorentz integrals (X) and (XI) generally do not satisfy the prescribed boundary and initial conditions, and it is necessary to add complementary functions, ϕ_0 and $\mathbf{a}_0$, in order to complete the solution. We wish to determine the nature of these complementary functions.

They obviously satisfy equations of the form $\nabla^2\psi - \ddot{\psi}/c^2 = 0$. By a theorem due to Kirchhoff† we have

$$4\pi\psi = \int \left\{ \frac{f(t - R/c)}{R} - \frac{\partial}{\partial n} \frac{\psi(t - R/c)}{R} \right\} dS,$$

where $f(t) = \dfrac{\partial\psi(t)}{\partial n}$, dS is an element of the boundary at (ξ, η, ζ), n its outward normal, and R its distance from the fieldpoint (x, y, z) as before.

* *Enc. Mat.* v. 14, p. 157. [*Theory of Electrons*, p. 19.]

† *Optik*, p. 27, equation (12).

This expression for ψ shows that the complementary functions merely represent the effect of charges on and outside the boundary, so that the Lorentz potentials represent the required effect of the inside charges completely. Thus complementary functions are unnecessary for our purpose.

Since the integrals (X) and (XI) give the complete solution of our problem, and moreover only involve the prescribed quantities ρ and $\mathbf{v}$, we shall make them the bases of our investigation.

5. Equation of energy. We get the equation of energy in the usual way: multiply (III) scalarly by $\mathbf{d}/4\pi$, (IV) by $-\mathbf{h}/4\pi$, add, use (VI) and the identity $(\mathbf{h} \text{ curl } \mathbf{d}) - (\mathbf{d} \text{ curl } \mathbf{h}) = \text{div.} [\mathbf{d} . \mathbf{h}]$. Multiply the resulting equation by the volume element $d\Omega$ and integrate throughout the space enclosed by any *fixed* surface S; then we get

$$\int (\mathbf{f} . \mathbf{v}) \rho \, d\Omega + \frac{\partial}{\partial t} \int \frac{d^2 + h^2}{8\pi} d\Omega + \int \frac{c \, [\mathbf{d} . \mathbf{h}]_n}{4\pi} dS = 0 \quad \text{......(XII)}.$$

The first term represents the rate at which the field does work in moving the charges inside S; the second is generally interpreted to mean the rate at which the electromagnetic energy of the aether inside increases; hence the third term represents the rate at which electromagnetic energy flows out across the surface. Since the surface is quite arbitrary, the Poynting vector

$$\mathbf{s} = c \, [\mathbf{d} . \mathbf{h}]/4\pi \quad \text{..........................(XIII)}$$

must necessarily be identified with the rate of flow of electromagnetic energy per unit area.

This interpretation however is not free from difficulty if we accept a mechanical theory of the aether. In deriving the electromagnetic equation (VI) by means of the Principle of Least Action it has been assumed generally, whether necessarily or not we will not determine here, that the potential energy of the free aether is to be taken as $d^2/8\pi$ per unit volume, but the kinetic energy as $(\mathbf{a} \, \{\dot{\mathbf{d}} + 4\pi\rho\mathbf{v}\})/8\pi c$, and not as $h^2/8\pi$, per unit volume (Lorentz*, Larmor†, Macdonald‡). Macdonald§ however points out that the two expressions for the kinetic energy cannot really be identified; indeed we find by means of (VII) and (III)

$$h^2 = (\mathbf{h} \text{ curl } \mathbf{a}) = \text{div.} [\mathbf{a} . \mathbf{h}] + (\mathbf{a} \, \{\dot{\mathbf{d}} + 4\pi\rho\mathbf{v}\})/c.$$

When this value is substituted in the equation of energy we get an additional surface integral on the left-hand side of (XIII), namely

$$\frac{\partial}{\partial t} \int [\mathbf{a} . \mathbf{h}]_n \, dS/8\pi,$$

which requires interpretation.

* *Enc. Mat.* v. 13, p. 165. [*Theory of Electrons*, p. 23.]

† *Aether and Matter*, p. 94.

‡ *Electric Waves*, p. 160.

§ *Loc. cit.* p. 32.

We see at once that in all optical applications, where we wish to find the mean rate of flow of energy across an element of surface, the value of the vector $[\mathbf{a}\,.\,\mathbf{h}]$ is stationary on the average for long periods of time. Hence the additional surface integral contributes nothing to the average radiation from the system of moving charges, so that we may identify the radiation vector with the Poynting flux without fear of error.

But when we wish to determine the radiation from a system of moving charges which are not in stationary motion, such as a stream of β-particles and the like, the difficulty signalised above subsists.

[In order to get a clear understanding of these energy relations let us consider the case of a single unit charge moving in any manner. We shall take as our space Ω the space enclosed between two infinitely close fixed spheres S and S', which coincide with the instantaneous positions at time t of two consecutive spherical waves emitted by the charge at the times τ and τ', when it was in the positions E and E'.

The radii, R and R', of these spheres are given by the characteristic equation (28), § 13, *infra*, so that $R = c(t-\tau)$, and $R' = c(t-\tau')$. The field due to the charge is given by equations such as (26) and (27), or (30)—(33), § 13. By choosing R and R' very large, we gain three advantages: (1) we can treat the charge as if it were concentrated in one point; (2) in the expressions for the forces, (32) and (33), we can neglect all but the lowest powers of $1/R$; and (3), in differentiating these and similar expressions with respect to the time we can treat $\mathbf{R}$ as constant, because its variation only gives rise to terms of higher order. We must bear in mind that $\frac{\partial\tau}{\partial t} = 1/K$, where, by (29), $K = 1 - (\mathbf{v}\mathbf{R}_1)/c$; and that $\frac{\partial K}{\partial \tau} = -(\dot{\mathbf{v}}\mathbf{R}_1)/c$.

To this approximation (32) and (33), § 13, give

$$\mathbf{h} = [\mathbf{R}_1\mathbf{d}], \quad \mathbf{d} = -\frac{\dot{\mathbf{v}}}{c^2K^2R} + \frac{(\mathbf{R}_1 - \mathbf{v}/c)(\dot{\mathbf{v}}\mathbf{R}_1)}{c^2K^3R}.$$

Thus we have $(\mathbf{R}_1\mathbf{h}) = (\mathbf{R}_1\mathbf{d}) = 0$, $d^2 = h^2$, $[\mathbf{dh}] = \mathbf{R}_1 d^2$.

Let us begin by verifying equation (XIII) for the shell Ω enclosed between the two spheres S and S'.

The first integral vanishes identically, because the electric volume density ρ vanishes everywhere inside the shell.

As for the second, $d\Omega$ is a fixed element of volume; hence the integral is equal to $\int \frac{d\Omega}{8\pi}\frac{\partial}{\partial t}(d^2 + h^2)$.

Now the thickness of the shell is easily seen to be $c(\tau' - \tau)K$; hence $d\Omega = c(\tau' - \tau)K\,dS$. Also $\frac{\partial}{\partial t} = \frac{1}{K}\frac{\partial}{\partial \tau}$, and $d^2 = h^2$, and τ has the same value for every point of the sphere S, since it coincides with the wave emitted at the time τ. Thus the volume integral reduces to $\frac{c(\tau'-\tau)}{4\pi}\int dS\frac{\partial}{\partial\tau}d^2$, that is, to $(\tau'-\tau)\frac{\partial}{\partial\tau}\int\frac{cd^2dS}{4\pi}$, or to $\int\frac{cd'^2dS'}{4\pi}-\int\frac{cd^2dS}{4\pi}$.

Lastly, the surface integral has to be taken over the two spheres S and S'. For the sphere S the outward normal n is along $\mathbf{R}$, so that $[\mathbf{dh}]_n=d^2$; for the sphere S' the outward normal n' is along $-\mathbf{R}'$, so that $[\mathbf{d'h'}]_{n'}=-d'^2$. Hence the surface integral gives, for both spheres together,

$$\int\frac{cd^2dS}{4\pi}-\int\frac{cd'^2dS'}{4\pi},$$

that is, it just cancels the second volume integral, so that (XIII) is verified.

Again, let us consider the modified energy equation. In the second volume integral of (XIII) h^2 is to be replaced by $(\mathbf{a}\{\dot{\mathbf{d}}+4\pi\rho\mathbf{v}\})/c$, that is, by $(\mathbf{a}\dot{\mathbf{d}})/c$ in the present case, where $\rho=0$. That is, we must add a volume integral $\frac{\partial}{\partial t}\frac{1}{8\pi}\int\left\{\frac{(\mathbf{a}\dot{\mathbf{d}})}{c}-h^2\right\}d\Omega$ to the left-hand side of (XIII).

We have already found that we must add an additional surface integral $\frac{\partial}{\partial t}\int\frac{[\mathbf{ah}]_n\,dS}{8\pi}$. Thus the modified equation will be verified if we prove that the sum, $\int\left\{\frac{(\mathbf{a}\dot{\mathbf{d}})}{c}-h^2\right\}d\Omega+\int[\mathbf{ah}]_n\,.\,dS$, is zero, or independent of t.

In the first place, we have to our approximation in the shell Ω, $h^2=d^2$ and $[\mathbf{ah}]_n=[\mathbf{a}[\mathbf{R}_1\mathbf{d}]]_{\mathbf{R}_1}=(\mathbf{ad})$. Now $(\mathbf{a}\dot{\mathbf{d}})=\frac{\partial}{\partial t}(\mathbf{ad})-(\dot{\mathbf{a}}\mathbf{d})$; and from (27) we find

$$\dot{\mathbf{a}}=\frac{\partial\mathbf{a}}{K\partial t}=\frac{\dot{\mathbf{v}}}{cK^2R}+\frac{\mathbf{v}(\dot{\mathbf{v}}\mathbf{R}_1)}{c^2K^3R}=\frac{\mathbf{R}_1(\dot{\mathbf{v}}\mathbf{R}_1)}{cK^3R}-c\mathbf{d}.$$

Hence, since $(\mathbf{R}_1\mathbf{d})=0$, we have $(\dot{\mathbf{a}}\mathbf{d})=-cd^2$. Therefore

$$\frac{(\mathbf{a}\dot{\mathbf{d}})}{c}-h^2=\frac{\partial}{\partial t}\frac{(\mathbf{ad})}{c}.$$

As before we have

$$\frac{\partial}{\partial t}(\mathbf{ad})=\frac{\partial(\mathbf{ad})}{K\partial\tau},\quad d\Omega=c(\tau'-\tau)K\,dS.$$

Hence the volume integral reduces to $(\tau'-\tau)\int\frac{\partial(\mathbf{ad})}{\partial\tau}dS$, that is, to $\int(\mathbf{a'd'})\,dS'-\int(\mathbf{ad})\,dS$. As before, this just cancels the surface integral extended to the two spheres S and S'. Thus the modified energy equation has also been verified.

In the same way we can verify either form of the energy equation for a region bounded by two spherical waves at any distance apart, by integration from the case of the infinitely thin shell, provided only that the radius of the inner wave is sufficiently great. We must now consider the values of the several terms in the equation.

In the first place, the electromagnetic energy of the thin shell Ω according to the usual definition is equal to $\int (d^2 + h^2)\, d\Omega/8\pi$, that is, just as above, to $c(\tau' - \tau)\int d^2 K\, dS/4\pi$.

Take polar coordinates (R, θ, ϕ) with origin at E, polar axis along EE' ($\mathbf{v}$), and initial meridian plane through $\dot{\mathbf{v}}$. Writing β for v/c as usual, and α for the angle between $\mathbf{v}$ and $\dot{\mathbf{v}}$, we have $K = 1 - \beta\cos\theta$, and

$$(\mathbf{v}\mathbf{R}_1)/c = 1 - K, \quad (\dot{\mathbf{v}}\mathbf{R}_1) = |\dot{\mathbf{v}}|(\cos\alpha\cos\theta + \sin\alpha\sin\theta\cos\phi).$$

Also
$$dS = R^2 \sin\theta\, d\theta\, d\phi\,;$$

hence using the approximate value of $\mathbf{d}$ found above, we get for the energy

$$\frac{\tau' - \tau}{4\pi c^3}\int_0^\pi\int_0^{2\pi}\left\{\frac{\dot{\mathbf{v}}^2}{K^3} + 2\frac{(\mathbf{v}\dot{\mathbf{v}})(\dot{\mathbf{v}}\mathbf{R}_1)}{cK^4} - \frac{(1-\beta^2)(\dot{\mathbf{v}}\mathbf{R}_1)^2}{K^5}\right\}\sin\theta\, d\theta\, d\phi.$$

Substitute for $(\dot{\mathbf{v}}\mathbf{R}_1)$ its value and integrate with respect to ϕ, bearing in mind that

$$\int_0^{2\pi}(\dot{\mathbf{v}}\mathbf{R}_1)\, d\phi = 2\pi|\dot{\mathbf{v}}|\cos\alpha\cos\theta,$$

and
$$\int_0^{2\pi}(\dot{\mathbf{v}}\mathbf{R}_1)^2\, d\phi = 2\pi\dot{\mathbf{v}}^2(\cos^2\alpha\cos^2\theta + \tfrac{1}{2}\sin^2\alpha\sin^2\theta).$$

Since $(\mathbf{v}\dot{\mathbf{v}}) = |\dot{\mathbf{v}}|\, v\cos\alpha$, we get

$$\frac{(\tau' - \tau)\dot{\mathbf{v}}^2}{2c^3}\int_0^\pi\left\{\frac{1}{K^3} - \frac{(1-\beta^2)\sin^2\theta}{2K^5}\right\}\sin\theta\, d\theta$$
$$+\frac{(\tau' - \tau)(\mathbf{v}\dot{\mathbf{v}})^2}{2c^3}\int_0^\pi\left\{\frac{2\cos\theta}{cvK^4} - \frac{(1-\beta^2)(3\cos^2\theta - 1)}{2v^2K^5}\right\}\sin\theta\, d\theta.$$

Remembering that $K = 1 - \beta\cos\theta$, we get, most easily by changing the variable from θ to K,

$$\frac{2c(\tau' - \tau)}{3(c^2 - v^2)^2}\left\{\dot{\mathbf{v}}^2 + \frac{(\mathbf{v}\dot{\mathbf{v}})^2}{c^2 - v^2}\right\}.$$

The coefficient of $\tau' - \tau$ is precisely the expression found by Liénard for the rate of loss of energy from the moving charge owing to radiation. Thus his definition of the radiation rests on the assumption that the electromagnetic energy of the aether is equal to $(d^2 + h^2)/8\pi$ per unit volume.

Again, we found that on the mechanical theory of the aether it is necessary to replace the term $h^2/8\pi$ by $(\mathbf{a}\dot{\mathbf{d}})/8\pi c$, that is, we must add to the energy of the shell Ω the supplementary integral $\int\{(\mathbf{a}\dot{\mathbf{d}})/c - h^2\}\, d\Omega/8\pi$, and this we found to be equal to $(\tau' - \tau)\dfrac{\partial}{\partial\tau}\int(\mathbf{a}\mathbf{d})\, dS/8\pi$.

Now by (27), § 13, we have $\mathbf{a} = \mathbf{v}/cKR$. Using the approximate value of $\mathbf{d}$ as above we find that the supplementary term is equal to

$$\frac{\tau' - \tau}{8\pi c^3} \frac{\partial}{\partial \tau} \int_0^{\pi} \int_0^{2\pi} \left\{ -\frac{(\mathbf{v}\dot{\mathbf{v}})}{K^3} + c \frac{(1 - K - \beta^2)(\dot{\mathbf{v}}\mathbf{R}_1)}{K^4} \right\} \sin\theta \, d\theta \, d\phi.$$

In the same way as before we find that this expression reduces to

$$-(\tau' - \tau) \frac{\partial}{\partial \tau} \frac{2c(\mathbf{v}\dot{\mathbf{v}})}{3(c^2 - v^2)^2}.$$

Hence, when we adopt a mechanical theory of the aether, and accordingly reckon its kinetic energy as $(\mathbf{a}\{\dot{\mathbf{d}} + 4\pi\rho\mathbf{v}\})/8\pi c$ per unit volume, we must supplement Liénard's expression for the radiation from a moving unit charge by adding the term $-\frac{\partial}{\partial \tau} \frac{2c(\mathbf{v}\dot{\mathbf{v}})}{3(c^2 - v^2)^2}$, where τ is the time of emission of the radiation, and $\mathbf{v}$ and $\dot{\mathbf{v}}$ are the velocity and acceleration of the charge at time τ.

In Chapter XI and Appendix D we shall deduce the equations of motion and energy of a small extended charge on the basis of the equations (I)—(VI) of § 2, without making any assumptions other than those implied in these equations. We shall find that the resulting energy equation, (211), § 145, and (344), § 227, involves not merely the Liénard radiation, but also the supplementary term just found.]

6. Resultant mechanical forcive. The resultant mechanical force acting on all the charges inside a fixed surface S due to the field is on account of (VI) given by

$$\mathbf{F} = \int \{\mathbf{d} + [\mathbf{v} \cdot \mathbf{h}]/c\} \rho \, d\Omega.$$

Using (I a), (III) and (IV), and integrating by parts we get*

$$\left.\begin{aligned} \mathbf{F} &= \mathbf{F}_1 + \mathbf{F}_2 \\ \mathbf{F}_1 &= \int \{2d_n \mathbf{d} + 2h_n \mathbf{h} - (d^2 + h^2)\,\mathbf{n}_1\} \, dS/8\pi \\ \mathbf{F}_2 &= -\frac{\partial}{\partial t} \int \mathbf{s} \, d\Omega/c^2 \end{aligned}\right\} \quad \text{where} \quad \ldots\ldots\ldots\ldots \text{(XIV)}.$$

Here $\mathbf{n}_1$ denotes a unit vector in the direction of the outward normal to the surface element dS, and $\mathbf{s}$ is the Poynting flux as before.

$\mathbf{F}_2$ is usually interpreted as the rate of decrease of electromagnetic momentum; hence $\mathbf{F}_1$ may be interpreted as the resultant reaction due to radiation, or as we may call it, the resultant radiation pressure. It is clearly calculated as if the surface S were subject to the Maxwell stress.

In calculating $\mathbf{F}_1$ and $\mathbf{F}_2$ for a finite system of charges it is however assumed that the integrals occurring in (XIV) approach determinate limiting values as the surface S increases in size, and that these are independent of

* Lorentz, *Enc. Mat.* v. 14, p. 161. [*Theory of Electrons*, p. 26.]

the shape of the surface. Otherwise we could not use the terms electromagnetic momentum and radiation pressure for the system at all. Now in any case where charges are moving with acceleration, the parts of the forces **d** and **h**, which depend on the acceleration, are ultimately of the order $1/R$, and therefore the quantities under the signs of integration in (XIV) are ultimately of the order $1/R^2$. On the other hand the element of surface dS is ultimately of the order $R^2 d\omega$, and that of volume $d\Omega$ of the order $R^2 dR\, d\omega$, where $d\omega$ is an element of solid angle. Hence it may happen that the parts of space at a great distance may contribute largely to the values of the two integrals; therefore it is not legitimate to ignore their influence without examination.

Apart altogether from these considerations it seems a somewhat circuitous method to evaluate the forcive acting on a *finite* system of charges from the actions that take place in *infinite* space surrounding them, so far as that forcive is due to the system itself. It is surely more appropriate to express it by means of integrals extending to the regions occupied by the charges themselves. This can be done easily by making use of the potentials.

Substitute from (VII) and (VIII) in (VI); we get

$$\mathbf{f} = -\nabla\phi - \frac{\partial \mathbf{a}}{c\partial t} + [\mathbf{v}\,.\,\mathrm{curl}\,\mathbf{a}]/c, \quad \text{..................(XV)},$$

where all differential coefficients are partial. Now

$$[\mathbf{v}\,.\,\mathrm{curl}\,\mathbf{a}] = [\mathbf{v}\,[\nabla\mathbf{a}]] = \nabla_a(\mathbf{va}) - (\mathbf{v}\nabla)\,\mathbf{a},$$

where ∇_a operates only on **a**, not on **v**.

When we are dealing with an extended distribution of charge we can always regard it as continuous, by using the artifice of transition layers. Then the velocity of elements such as de at (x, y, z) varies continuously from point to point; that is, it is a continuous function of (x, y, z) as well as of t. It is convenient to let the operation ∇ refer to all the variable quantities which follow it. If we do this we must write $\frac{\partial}{\partial x}(\mathbf{va}) - a_x\frac{\partial v_x}{\partial x} - a_y\frac{\partial v_y}{\partial x} - a_z\frac{\partial v_z}{\partial x}$, and so on, in place of the components of $\nabla_a(\mathbf{va})$. Let

$$\sigma_{11} = \frac{\partial v_x}{\partial x}, \quad \sigma_{21} = \frac{\partial v_y}{\partial x}, \quad \sigma_{31} = \frac{\partial v_z}{\partial x}, \quad \sigma_{12} = \frac{\partial v_x}{\partial y}, \text{ etc. ......(XVI)}.$$

σ_{11}, σ_{22} and σ_{33} are the velocities of elongation in the directions of the three axes; $\sigma_{23} + \sigma_{32}$, $\sigma_{31} + \sigma_{13}$ and $\sigma_{12} + \sigma_{21}$ are the velocities of shear parallel to the coordinate planes; and $\frac{1}{2}(\sigma_{32} - \sigma_{23})$, $\frac{1}{2}(\sigma_{13} - \sigma_{31})$ and $\frac{1}{2}(\sigma_{21} - \sigma_{12})$ are the components of molecular rotation about the three axes. The resulting displacement of the point (x, y, z) increases at the rate

$$(\sigma_{11}x + \sigma_{12}y + \sigma_{13}z,\ \sigma_{21}x + \sigma_{22}y + \sigma_{23}z,\ \sigma_{31}x + \sigma_{32}y + \sigma_{33}z).$$

We shall write this vectorially as $\sigma\mathbf{r}$, so that σ denotes the operator of the rotational strain. We may also write

$$\sigma\mathbf{r} = \chi\mathbf{r} + [\omega\mathbf{r}] \quad \text{(XVII)},$$

where χ is the operator of pure strain, and ω the molecular rotation. We shall denote the conjugate strain by $\bar{\sigma}$, so that

$$\bar{\sigma}\mathbf{r} = \chi\mathbf{r} - [\omega\mathbf{r}] \quad \text{(XVII')}.$$

With this notation we get

$$\nabla_a(\mathbf{va}) = \nabla(\mathbf{va}) - \bar{\sigma}\mathbf{a} = \nabla(\mathbf{va}) - \chi\mathbf{a} + [\omega\mathbf{a}],$$

whence

$$[\mathbf{v}\,\mathrm{curl}\,\mathbf{a}] = \nabla(\mathbf{va}) - \bar{\sigma}\mathbf{a} - (\mathbf{v}\nabla)\,\mathbf{a}.$$

Again, the differential coefficient $\frac{\partial\mathbf{a}}{\partial t}$ in the expression for $\mathbf{f}$ denotes the local rate of increase of $\mathbf{a}$. Let $\frac{d\mathbf{a}}{dt}$ denote its substantial rate of increase, that is, the rate of increase at a given element of charge de, as noted by an observer moving with it. Then we have $\frac{d\mathbf{a}}{dt} = \frac{\partial\mathbf{a}}{\partial t} + (\mathbf{v}\nabla)\,\mathbf{a}$.

Substituting in the equation for $\mathbf{f}$ we get

$$\left.\begin{aligned}\mathbf{f} &= -\nabla\left\{\phi - \frac{(\mathbf{va})}{c}\right\} - \frac{d\mathbf{a}}{c\,dt} - \frac{\bar{\sigma}\mathbf{a}}{c}\\ &= -\nabla\left\{\phi - \frac{(\mathbf{va})}{c}\right\} - \frac{d\mathbf{a}}{c\,dt} - \frac{\chi\mathbf{a}}{c} + \frac{[\omega\mathbf{a}]}{c}\end{aligned}\right\} \quad \text{(XVIII)}.$$

This is the mechanical force on a unit charge at (x, y, z) which moves with velocity $\mathbf{v}$, and is undergoing strain at the rate $\sigma\mathbf{r}$.

[It is worthy of note that the expressions (XV) and (XVIII) also hold with but slight modification when the origin is moving with the velocity $\mathbf{w}$, and the axes are rotating with angular velocity $\boldsymbol{\theta}$ about themselves. In this case the symbols $\mathbf{v}$, σ, χ, ω, $\frac{\partial}{\partial t}$ and $\frac{d}{dt}$ are all to be taken *relative* to the moving axes, ϕ is to be replaced by $\phi - (\mathbf{wa})/c - (\boldsymbol{\theta}\mathbf{ra})/c$, and both ϕ and $\mathbf{a}$ are to be calculated from the absolute velocity, relative to the aether, or if we prefer, relative to the observer.]

In order to find the total mechanical force $\mathbf{F}$ on a system of charges we must multiply either of the expressions (XV) or (XVIII) for $\mathbf{f}$ by the element of charge de and integrate over the whole system. In order to get the total couple $\mathbf{N}$ we must multiply vectorially by $\mathbf{r}\,de$ and integrate. That is, we get

$$\mathbf{F} = \int \mathbf{f}\,de, \quad \mathbf{N} = \int [\mathbf{rf}]\,de.$$

In virtue of the principal of conservation of electric charge, de does not alter as it moves, so that $\frac{d}{dt}de = 0$. Thus the integration with respect to the charge is commutative with the substantial differentiation, $\frac{d}{dt}$, not with the local differentiation, $\frac{\partial}{\partial t}$. Thus, we have

$$\int \frac{d\mathbf{a}}{dt}\, de = \frac{d}{dt}\int \mathbf{a}\, de, \text{ but } \int \frac{\partial \mathbf{a}}{\partial t}\, de = \frac{d}{dt}\int \mathbf{a}\, de - \int (\mathbf{v}\nabla)\, \mathbf{a}\, de.$$

The symbol $\frac{\partial}{\partial t}\int \mathbf{a}\, de$ has no meaning when there is relative motion of the parts of the system, and the operations $\frac{\partial}{\partial t}$ and $\int de$ are not commutative. For this reason it is convenient to use the expression (XVIII) for **f**, rather than (XV). Hence we get

$$\mathbf{F} = -\frac{d}{dt}\int \mathbf{a}\, de/c - \int \{\nabla \{\phi - (\mathbf{va})/c\} + \bar{\sigma}\,\mathbf{a}/c\}\, de \quad \text{......(XIX).}$$

This expression can be transformed as follows. In the second term write $de = \rho\, d\Omega$, where $d\Omega$ is an element of volume and ρ the corresponding volume density of electric charge.

Integrating the first term by parts we get

$$\int \nabla \{\phi - (\mathbf{va})/c\}\, \rho\, d\Omega = \int \{\phi - (\mathbf{va})/c\}\, \rho\, d\mathbf{S} - \int \{\phi - (\mathbf{va})/c\}\, \nabla\rho\, d\Omega,$$

where $d\mathbf{S}$ denotes a vector element of surface with the positive normal outwards. Hence we get

$$\mathbf{F} = -\frac{d}{dt}\int \mathbf{a}\, de/c - \int \{\phi - (\mathbf{va})/c\}\, \rho\, d\mathbf{S}$$
$$+ \int (\{\phi - (\mathbf{va})/c\}\, \nabla \log \rho - \bar{\sigma}\,\mathbf{a}/c)\, \rho\, d\Omega \quad \text{..............(XX).}$$

The first term in the first line on the right represents rate of decrease of momentum, not necessarily the whole; for the second line may, and generally does, contribute a term of the same form.

The second term in the first line includes not merely a part due to the boundary of the system, but also parts due to surfaces of discontinuity, whenever such exist. But if we regard these surfaces as limits of thin layers of transition in the usual way, their effect will be included in the terms in the second line, and need not be considered any further. Then the term in question represents the resultant of a distribution of pressure on the boundary, amounting to $\{\phi - (\mathbf{va})/c\}\, \rho$ per unit area, and we may regard the corresponding part of the mechanical force as due to a hydrostatic pressure throughout the charged parts of the system.

[This resolution of the mechanical force is of importance in connection with the possibility of a mechanical explanation of the electron to be considered in Appendix E below. The volume integral in the second line of (XX) generally does not vanish, nor reduce to terms of the type of those in the first line. For example, the condition that it should vanish is that $\nabla \log \rho = \bar{\sigma}\mathbf{a}/\{c\phi - (\mathbf{va})\}$; but in virtue of the equation of continuity for ρ, (II), this leads to a condition limiting the character of the motion.

Just as we derived the expression (XX) for the resultant mechanical force $\mathbf{F}$, so we may derive the following expression for the resultant couple:

$$\mathbf{N} = -\frac{d}{dt}\int [\mathbf{ra}]\, de/c - \int \{\phi - (\mathbf{va})/c\}\, \rho\, [\mathbf{r}\, dS] + \int [\mathbf{va}]\, de/c$$
$$+ \int [\mathbf{r}\, (\{\phi - (\mathbf{va})/c\}\, \nabla \log \rho - \bar{\sigma}\mathbf{a}/c)]\, \rho\, d\Omega \quad \text{..............(XXI)}.$$

The first term in the first line represents rate of decrease of angular momentum, and the second the moment of the surface pressure, but the third has no analogue in (XX). The second line, like that in (XX), only vanishes for particular types of motion. We shall make no use of (XXI); it is only added here for the sake of completeness.]

CHAPTER II

TRANSFORMATION OF THE POTENTIALS

7. In order to coordinate the facts of our experience we may make the following assumptions:

(1) Electric charges of two kinds exist, negative and positive. Their properties and modes of occurrence are so different that it is best to treat them as separate entities.

(2) Electrification of either kind by itself is indestructible. Although negative and positive charges occasionally perhaps occupy the same space and neutralize each other's action to a greater or less extent, yet we are able to separate them by the application of a sufficiently powerful external field. Thus they do not annul one another.

(3) Neither kind of electrification is distributed continuously throughout all space; each is confined to limited regions, separated from each other. These aggregations of charge are capable of an individual existence and can move about in space; they constitute the electrons and ions. Whether their inertia is entirely electromagnetic, so that they may be regarded as free from matter, as in the case of the negative electrons, or not, as for positive ions, is immaterial for our purpose.

In our problem of determining the radiation due to electric charges in motion, we are given the position and configuration of each electron or ion as functions of the time. But the integrals (X) and (XI), § 3, involve the values of the density ρ and velocity $\mathbf{v}$ of the electric charge which is at each point (ξ, η, ζ) of space at the time of emission $t - R/c$. It is desirable to transform them so as to put in evidence the data actually given, that is to say, the coordinates and velocities of every element of charge at the time t. Every such element de can be identified by means of three parameters, which may be its initial coordinates, or any three independent functions of them; and its coordinates (ξ, η, ζ) at time t will be certain prescribed functions of these parameters and of the time t. Our transformation corresponds in some sort to the passage from Eulerian to Lagrangian coordinates in hydrodynamics.

Regarded as functions of t, the density ρ and current $\rho\mathbf{v}$ for a given point of space may in the limit be discontinuous, when the boundary of an electron crosses that point; or they may be infinite, when a charged surface passes. But from the nature of the case the number of such discontinuities or infinities occurring during a finite interval of time is necessarily finite and for each infinity $\int [\rho]\, d\tau$ is finite. Hence the quantities $[\rho]$ and $[\rho\mathbf{v}]$, which occur in the integrals (X) and (XI), can always be expressed as Fourier Integrals. If we suppose the values of the density and current at the point (ξ, η, ζ) to be given for all values of the time, we may write

$$[\rho] = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \epsilon^{\iota\mu(t-R/c-\tau)}\, \rho\, d\tau\, d\mu \qquad (1),$$

$$[\rho\mathbf{v}] = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \epsilon^{\iota\mu(t-R/c-\tau)}\, \rho\mathbf{v}\, d\tau\, d\mu \qquad (2).$$

Here ρ and $\rho\mathbf{v}$ are prescribed functions of τ involving ξ, η, ζ as parameters, while

$$R = \sqrt{\{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2\}} \qquad (3),$$

and is a function of $x, y, z, \xi, \eta, \zeta$ but not of τ. It must be particularly noted that *the positive sign is always to be taken for the root*, since the time of emission of the disturbance from (ξ, η, ζ), namely $t - R/c$, is always anterior to the time t.

If however we prescribe the values of the density and current only for times anterior to the actual time t, we must use the Cosine and Sine Integrals, and write

$$[\rho] = \frac{2}{\pi}\int_0^{\infty}\int_{-\infty}^{t} \cos\mu(t - R/c)\cos\mu\tau\,.\,\rho\, d\tau\, d\mu$$

or

$$[\rho] = \frac{2}{\pi}\int_0^{\infty}\int_{-\infty}^{t} \sin\mu(t - R/c)\sin\mu\tau\,.\,\rho\, d\tau\, d\mu,$$

with similar expressions for $\rho\mathbf{v}$.

In this case it is simpler to change the variable from τ to $\sigma = t - \tau$, so that σ is the time reckoned back from t to all past time. Thus we may write

$$[\rho] = \frac{2}{\pi}\int_0^{\infty}\int_0^{t+\infty} \cos\frac{\mu R}{c}\cos\mu\sigma\,.\,\rho_{t-\sigma}\,.\, d\sigma\, d\mu \qquad (4),$$

$$[\rho\mathbf{v}] = \frac{2}{\pi}\int_0^{\infty}\int_0^{t+\infty} \cos\frac{\mu R}{c}\cos\mu\sigma\,.\,(\rho\mathbf{v})_{t-\sigma}\,.\, d\sigma\, d\mu \qquad (5),$$

or

$$[\rho] = \frac{2}{\pi}\int_0^{\infty}\int_0^{t+\infty} \sin\frac{\mu R}{c}\sin\mu\sigma\,.\,\rho_{t-\sigma}\,.\, d\sigma\, d\mu \qquad (6),$$

$$[\rho\mathbf{v}] = \frac{2}{\pi}\int_0^{\infty}\int_0^{t+\infty} \sin\frac{\mu R}{c}\sin\mu\sigma\,.\,(\rho\mathbf{v})_{t-\sigma}\,.\, d\sigma\, d\mu \qquad (7).$$

The upper limit for σ is written $t + \infty$ to indicate that, although independent of σ and μ, it is not an absolute constant, but a function of t.

The importance of this fact was pointed out by Lindemann* in his criticism of Sommerfeld's calculation of the mass of the electron.

8. Thus we have the choice of three methods of representation, either by means of (1) and (2), or (4) and (5), or (6) and (7). The last two equations, as we shall see later, lead to Sommerfeld's well known expressions for the potentials of a spherical electron. They have the disadvantage of failing at the limit $\sigma = 0$, that is, $\tau = t$; but this is of importance only when $R = 0$, that is, for points close to the element of charge de. This however does not matter very much, since the elements of charge close to the fieldpoint (x, y, z) contribute only a vanishing amount to the potentials, as in the ordinary theory of the potential. Nor does it affect the values of the forces due to volume distributions for the same reason; but it may very well lead to error in estimating the forces due to surface distributions, where the elements of charge contribute largely to the forces in their immediate neighbourhoods.

It is interesting to note that Sommerfeld† comes to the conclusion that while the motion of a spherical volume charge with velocity exceeding that of light is possible, the motion of a spherical surface charge with such a velocity is impossible, because it involves an infinitely great mechanical force. It would be interesting to enquire, although beyond the scope of the present investigation, whether this difference may not after all be due to the failure of the integrals (6) and (7) to represent the motion completely at the time $\tau = t$.

Again, the deduction of the electric and magnetic forces from the potentials involves differentiations with respect to t, x, y, z; we must therefore enquire how far it is allowable to differentiate the Fourier integrals which we have obtained.

As regards (1) and (2) differentiation is allowable provided ρ and $\rho\mathbf{v}$ vanish at both limits, conditions which are generally satisfied in all physical applications. These conditions however are not sufficient for the cosine and sine integrals (4)—(7), which involve t in the upper limit, as well as in the functions $\rho_{t-\sigma}$ and $(\rho\mathbf{v})_{t-\sigma}$. The difficulties arising from these causes have been completely discussed by Lindemann in the memoir already referred to, but they may be completely evaded by the use of the integrals (1) and (2).

At first sight it might appear that these latter integrals require more data to be given than the others, since they involve the values of ρ and $\mathbf{v}$ for future as well as for past time. This difficulty however is only apparent; for we may choose the values of ρ and $\mathbf{v}$ quite arbitrarily so far as future time is concerned. But when these values have been so chosen, the values

* *Abhandlungen der K. Bayer. Akademie der Wiss.* II. Kl. XXIII. Bd. II. Abt. p. 320, 1907.

† *Göttinger Nachrichten*, 1904, p. 387.

of $[\rho]$ and $[\rho\mathbf{v}]$ are quite determinate for all time, and this is the important point—they have the proper values for all past time, whatever values of ρ and $\mathbf{v}$ be selected for the future. If experience should show that at some future time the values of $[\rho]$ and $[\rho\mathbf{v}]$ ceased to be correctly represented by the integrals (1) and (2), because the charges had not moved in the way that was anticipated, that would only be in accordance with our limited power of foretelling the course of future events. For these reasons the integrals (1) and (2) constitute the best foundation for our investigation. They lead to the expressions for the potentials given by Schott*.

9. Schott's solutions. Substitute the values of $[\rho]$ and $[\rho\mathbf{v}]$ from (1) and (2), § 7, in (X) and (XI), § 3; we get

$$\phi = \frac{1}{2\pi}\int d\Omega \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \epsilon^{\iota\mu(t-R/c-\tau)} \frac{\rho\, d\tau\, d\mu}{R} \quad \text{..............(8)},$$

$$\mathbf{a} = \frac{1}{2\pi c}\int d\Omega \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \epsilon^{\iota\mu(t-R/c-\tau)} \frac{\rho\mathbf{v}\, d\tau\, d\mu}{R} \quad \text{...........(9)}.$$

Here $R=\sqrt{\{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2\}}$; it is a function of the six independent variables $x, y, z, \xi, \eta, \zeta$, but not of τ. ρ and $\rho\mathbf{v}$ are prescribed functions of ξ, η, ζ and τ. $d\Omega = d\xi d\eta d\zeta$ and the integration is for all parts of space for which ρ is different from zero. Thus the variables (ξ, η, ζ, τ) are analogous to Eulerian coordinates in hydrodynamics.

It has already been pointed out that the data usually available are the coordinates of each element of charge for every value of the time τ. Any element of charge de pursues an individual existence, and therefore can be identified by means of its initial coordinates (ξ_0, η_0, ζ_0) for any selected standard time τ_0. Its coordinates (ξ, η, ζ) at the time τ are given by equations of the type

$$\xi = f(\xi_0, \eta_0, \zeta_0, \tau), \quad \eta = g(\xi_0, \eta_0, \zeta_0, \tau), \quad \zeta = h(\xi_0, \eta_0, \zeta_0, \tau) \text{...(10)},$$

where the form of each of the functions f, g, h is known.

The set of variables $(\xi_0, \eta_0, \zeta_0, \tau)$ constitute, as it were, Lagrangian coordinates of the element.

Since the charge is indestructible we have

$$de = \rho\, d\xi d\eta d\zeta = \rho_0 d\xi_0 d\eta_0 d\zeta_0 \quad \text{....................(11)},$$

where ρ_0 is the density at time τ_0 and therefore independent of τ.

We may in our integrals choose the element of volume $d\Omega$ arbitrarily for each time τ, since this only amounts to a rearrangement of the terms of a triple sum which is for physical reasons known to be absolutely convergent. Selecting any value of τ, choose $d\Omega$ to be the volume of the element of charge de at the time τ. Then $\rho d\Omega = de$. But for this element de is the

* *Ann. der Phys.* 24, p. 637, 1907.

same for all times τ; hence we can, if we please, change the order of integration and sum with respect to τ first.

Thus we get from (8) and (9)

$$\phi = \frac{1}{2\pi}\int de \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \epsilon^{\iota\mu(t-R/c-\tau)}\frac{d\tau\, d\mu}{R} \quad\text{..............(12)},$$

$$\mathbf{a} = \frac{1}{2\pi c}\int de \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \epsilon^{\iota\mu(t-R/c-\tau)}\frac{\mathbf{v}\, d\tau\, d\mu}{R} \quad\text{...........(13)}.$$

Here R is a function of x, y, z and τ, as well as of the parameters ξ_0, η_0, ζ_0—briefly of the charge de—in virtue of the equations (10). The expressions (12) and (13) are essentially different in character from (8) and (9); for whereas the latter, regarded as functions of τ, are Fourier integrals, because R does not involve τ, the former are so no longer, since in virtue of the transformation (10) R has become a function of τ.

The remaining integrals (4)—(7), § 7, lead to analogous expressions:

$$\phi = \frac{2}{\pi}\int de \int_0^{\infty}\int_0^{t+\infty} \cos\frac{\mu R}{c}\cos\mu\sigma\,\frac{d\sigma\, d\mu}{R} \quad\text{..............(14)},$$

$$\mathbf{a} = \frac{2}{\pi c}\int de \int_0^{\infty}\int_0^{t+\infty} \cos\frac{\mu R}{c}\cos\mu\sigma\,\frac{\mathbf{v}\, d\sigma\, d\mu}{R} \quad\text{...........(15)},$$

from (4) and (5); and

$$\phi = \frac{2}{\pi}\int de \int_0^{\infty}\int_0^{t+\infty} \sin\frac{\mu R}{c}\sin\mu\sigma\,\frac{d\sigma\, d\mu}{R} \quad\text{..............(16)},$$

$$\mathbf{a} = \frac{2}{\pi c}\int de \int_0^{\infty}\int_0^{t+\infty} \sin\frac{\mu R}{c}\sin\mu\sigma\,\frac{\mathbf{v}\, d\sigma\, d\mu}{R} \quad\text{...........(17)},$$

from (6) and (7).

Here σ is time measured backwards from the variable instant t, and $\xi, \eta, \zeta, R, \mathbf{v}$ are therefore functions of $t-\sigma$. The solutions (16) and (17) will be found later to lead to Sommerfeld's solutions for a spherical electron.

10. Electric and magnetic forces. The equations (VII) and (VIII), § 3, at once lead to the expressions for these forces. Remembering that

$$\nabla R = \mathbf{R}_1,$$

where $\mathbf{R}_1$ denotes the unit vector in the direction of R, we get from (12) and (13), § 9,

$$\mathbf{h} = \frac{1}{2\pi c^2}\frac{\partial}{\partial t}\int de \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \epsilon^{\iota\mu(t-R/c-\tau)}\frac{[\mathbf{v}\mathbf{R}_1]\, d\tau\, d\mu}{R}$$
$$+ \frac{1}{2\pi c}\int de \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \epsilon^{\iota\mu(t-R/c-\tau)}\frac{[\mathbf{v}\mathbf{R}_1]\, d\tau\, d\mu}{R^2} \quad\text{...(18)},$$

$$\mathbf{d} = \frac{1}{2\pi c}\frac{\partial}{\partial t}\int de \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \epsilon^{\iota\mu(t-R/c-\tau)}\frac{(\mathbf{R}_1 - \mathbf{v}/c)\, d\tau\, d\mu}{R}$$
$$+ \frac{1}{2\pi}\int de \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \epsilon^{\iota\mu(t-R/c-\tau)}\frac{\mathbf{R}_1\, d\tau\, d\mu}{R^2} \quad\text{...(19)}.$$

The first integral in each case, being of order $1/R$, is effective at large distances from the charge, the second at small ones. It is easily seen that for a finite system of charges, and at a fieldpoint at a distance so large compared with the dimensions of the system, that quantities of order $1/R^2$ may be neglected and $\mathbf{R}_1$ treated as a constant vector, we have $\mathbf{h} = [\mathbf{R}_1 . \mathbf{d}]$ and $0 = (\mathbf{R}_1 . \mathbf{d})$. That is, at great distances $\mathbf{h}$ and $\mathbf{d}$ are at right angles to each other and to the radius vector, just as in the case of a Hertz vibrator.

It is to be noted that in (18) and (19) the differentiation $\frac{\partial}{\partial t}$ is partial, (x, y, z) being kept constant. This is of importance in calculating the mechanical force exerted by the system on a moving charge.

In the same way other expressions may be deduced from (14)—(17), § 9, but as they are more complicated and will not be required in our investigation, we refrain from writing them down.

CHAPTER III

OTHER TYPES OF SOLUTION

11. We shall now compare our solutions with other expressions which have been given by various authors. It will be found that these can all be deduced from the integrals (12)—(17), § 9. The proofs may serve as a verification of them.

Sommerfeld's solutions*. Sommerfeld has given expressions for the potentials due to spherical electrons carrying uniform surface or volume charges. These may be deduced as follows from the expressions (16) and (17). Consider first the case of an electron of radius a charged with e units of electricity uniformly distributed over its surface.

Take the centre as origin of polar coordinates; let the fieldpoint be at (r, θ, ϕ) and the element de at (a, θ', ϕ'). We may write

$$R = \sqrt{(r^2 + a^2 - 2ra\cos\gamma)},\quad de = \frac{e\sin\theta'\, d\theta'\, d\phi'}{4\pi},$$

where $$\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - \phi').$$

Write $\mu = cs$; we get from (16)

$$\phi = \frac{ec}{2\pi^2}\int_0^\pi\int_0^{2\pi}\sin\theta'\, d\theta'\, d\phi'\int_0^\infty\int_0^{t+\infty}\sin Rs\,.\,\sin cs\sigma\,.\,\frac{ds\, d\sigma}{R}.$$

Now $\sin Rs = \sqrt{(\frac{1}{2}\pi Rs)}\, J_{\frac{1}{2}}(Rs)$, and by a theorem due to Gegenbauer†

$$J_{\frac{1}{2}}(Rs) = \sqrt{\left(\frac{2\pi R}{sar}\right)}\sum_{n=0}^{n=\infty}(n+\tfrac{1}{2})\, J_{n+\frac{1}{2}}(rs)\, J_{n+\frac{1}{2}}(as)\, P_n(\cos\gamma) \quad \text{...(20)},$$

in the usual notation of Bessel Functions and Zonal Harmonics.

Substituting this in the expression for ϕ and integrating over the surface of the sphere, we find that, owing to the presence of the zonal harmonic, every term vanishes except the term $n = 0$, and we get

$$\phi = \frac{2ec}{\pi ar}\int_0^\infty\int_0^{t+\infty}\sin as\,.\,\sin cs\sigma\,.\,\sin rs\,\frac{ds\, d\sigma}{s} \quad \text{............(21)},$$

* *Gött. Nach.* 1904, pp. 107—110.

† Gray and Mathews, *Bessel Functions*, p. 239, Ex. 50.

which is identical with Sommerfeld's expression (16), but for slight differences of notation. The chief difference is that the upper limit for the τ integration is expressed as $t+\infty$, and not simply as ∞, in accordance with the contention of Lindemann already referred to, that the upper limit must be treated as variable in differentiating to derive the electric force. The reader who is desirous of following up the controversy between these two authors will find their papers in the *Abhandlungen* and in the *Sitzungsberichte der K. Bayer. Akademie* for 1907 and 1908, and a paper by Schott in the *Annalen der Physik*, 1908, 25, p. 63.

Sommerfeld's expression for a volume charge is got at once from (21) by putting r' for a, $3er'^2dr'/a^3$ for e, and integrating with respect to r' from 0 to a. We get

$$\phi = \frac{6ec}{\pi ar}\int_0^\infty \int_0^{t+\infty} \frac{\sin sa - sa\cos sa}{(sa)^2} \sin cs\sigma \,.\, \sin sr \frac{ds\,d\sigma}{s} \quad \ldots\ldots(22),$$

agreeing with Sommerfeld's expression (18).

The expressions for the vector potential are got from (17) in precisely the same way.

12. New solution for the sphere. Proceeding by the method just used we can get a new solution from the integrals (14) and (15), § 9. We have an expansion corresponding to (20),

$$\begin{aligned}\cos Rs &= \sqrt{(\tfrac{1}{2}\pi Rs)}\,.\,Y_{\frac{1}{2}}(Rs)\\ &= -\frac{\pi R}{\sqrt{(ar)}}\sum_{n=0}^{n=\infty}(n+\tfrac{1}{2})\,J_{n+\frac{1}{2}}(rs)\,Y_{n+\frac{1}{2}}(as)\,P_n(\cos\gamma)\ldots(23),\end{aligned}$$

provided $r < a$. If $r > a$, r and a must be interchanged.

Using this in (14) we get at once for a surface charge

$$\left.\begin{aligned}\phi &= \frac{ec}{\pi ar}\int_{-\infty}^{\infty} d\tau \int_0^\infty \cos s\,\{c(t-\tau)-a\}\sin sr\frac{ds}{s},\quad (r<a)\\ &= \frac{ec}{\pi ar}\int_{-\infty}^{\infty} d\tau \int_0^\infty \cos s\,\{c(t-\tau)-r\}\sin sa\frac{ds}{s},\quad (r>a)\end{aligned}\right\}\ldots(24).$$

As before we get for a volume charge

$$\phi = \frac{3ec}{2\pi a}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \epsilon^{\iota sc(t-\tau)}\left\{\frac{\sin rs}{rs}(1+\iota sa)\,e^{-\iota sa}-1\right\}\frac{ds\,d\tau}{s^2a^2}\ldots\ldots(25).$$

In these expressions r is the distance of the *fixed* point (x, y, z) from the moving centre of the electron. Thus it is a function of x, y, z and τ, but not of t. In Sommerfeld's integrals, where τ is replaced by $\sigma = t-\tau$, on the other hand r is a function of x, y, z and $t-\sigma$, that is, a function of t. This is the essential difference between the two types of solution.

13. **The point laws of Liénard* and Wiechert†.** These writers have given the following expressions for the potentials due to a unit electric charge of small dimensions at a fieldpoint so distant that the charge may be treated as if concentrated at a point:

$$\phi = \frac{1}{[KR]} \quad \text{.......................(26)},$$

$$\mathbf{a} = \frac{[\mathbf{v}]}{c[KR]} \quad \text{.......................(27)}.$$

The square brackets mean that the enclosed function of the time is to be taken for a time τ given in terms of the actual time t by the equation

$$t = \tau + R/c \quad \text{.......................(28)}.$$

In other words the disturbance which reaches the fieldpoint at time t was emitted by the unit charge at time τ. [We shall call (28) the characteristic equation of the motion.]

K is the so-called Doppler factor; it is given by

$$K = \frac{\partial t}{\partial \tau} = 1 + \frac{1}{c}\frac{\partial R}{\partial \tau} = 1 - (\mathbf{v}\,.\,\mathbf{R}_1)/c \quad \text{..................(29)}.$$

There has been some controversy as to the proof and the proper expression for the point law‡. We shall now show that, provided the velocity of the electron be less than that of light, the expressions (26) and (27) can be deduced without ambiguity from the general expressions (12) and (13), § 9, so that there can be little doubt as to their correctness, at any rate so long as the velocity of the point charge is less than that of light. The proof however can be extended so as to give the proper expressions when the velocity of the charge exceeds that of light.

Change the variable in (12), § 9, from τ to t', where

$$t' = \tau + \frac{R}{c} \quad \text{.......................(28')}.$$

We get
$$\phi = \frac{1}{2\pi}\int de \int_{-\infty}^{\infty} d\mu \int \epsilon^{\iota\mu(t-t')}\frac{dt'}{K'R'},$$

where K' and R' are the values of K and R when t' is substituted for τ by means of (28'). The limits remain to be determined.

In the first place suppose the velocity of the electron to remain constantly less than that of light.

Since $K' = 1 - (\mathbf{v}\,.\,\mathbf{R}_1)/c$, and the tensor of $\mathbf{v} < c$, while that of $\mathbf{R}_1$ is unity, it follows that K', that is, $\frac{dt'}{d\tau}$, is always positive. Hence as τ increases from

* *L'Éclairage électrique*, July, 1898.

† *Archives néerlandaises*, 1900, p. 549.

‡ De la Rive, *Archives de Genève*, 1907, p. 433.

$-\infty$ to $+\infty$, t' constantly increases between the same limits. Thus the new limits for ϕ are also $-\infty$ and $+\infty$.

But the double integral for ϕ, namely

$$\int_{-\infty}^{\infty} d\mu \int_{-\infty}^{\infty} \epsilon^{\iota\mu(t-t')}\, dt'/K'R',$$

is a Fourier integral, and has the value $2\pi/[KR]$. For the equation (28′) reduces to (28), since the function $1/K'R'$ is to be taken for $t'=t$. Hence

$$\phi = \int \frac{de}{[KR]} \quad \text{.................................(30).}$$

In the same way (13), § 9, leads to

$$\mathbf{a} = \int \frac{[\mathbf{v}]\, de}{c\,[KR]} \quad \text{.............................(31).}$$

These equations express the point laws in another form; in fact when R is large compared with the linear dimensions of the charge, so that the latter may be regarded as concentrated at one point, and when the total charge is unity, (30) and (31) reduce to (26) and (27) respectively. When however the charge is extended we must use the former equations.

[Similarly we can get expressions for the electric and magnetic forces by differentiating (30) and (31), or, for a point charge, (26) and (27). We may also deduce them from (18) and (19), § 10, as we deduced (30) from (12), § 9. The latter method gives for a unit point charge

$$\mathbf{h} = \frac{\partial}{\partial t}\left[\frac{[\mathbf{v}\mathbf{R}_1]}{c^2KR}\right] + \left[\frac{[\mathbf{v}\mathbf{R}_1]}{cKR^2}\right],$$

$$\mathbf{d} = \frac{\partial}{\partial t}\left[\frac{\mathbf{R}_1 - \mathbf{v}/c}{cKR}\right] + \left[\frac{\mathbf{R}_1}{KR^2}\right].$$

We must bear in mind that the quantities in the square brackets are functions of τ, where $\tau = t - R/c$, by (28), as well as of (x, y, z) explicitly. By (29) we have $\frac{\partial\tau}{\partial t} = 1/K$; hence we find

$$\frac{\partial KR}{c\,\partial t} = 1 - \{(\dot{\mathbf{v}}\mathbf{R}) + c^2 - v^2\}/c^2K,$$

and

$$\frac{\partial \mathbf{R}_1}{c\,\partial t} = \{\mathbf{R}_1(1-K) - \mathbf{v}/c\}/KR.$$

Using these results we find

$$\mathbf{h} = \left[\frac{[\dot{\mathbf{v}}\mathbf{R}_1]}{c^2K^2R} + \frac{[\mathbf{v}\mathbf{R}_1]\,\{(\dot{\mathbf{v}}\mathbf{R}) + c^2 - v^2\}}{c^3K^3R^2}\right] \quad \text{..................(32),}$$

$$\mathbf{d} = \left[-\frac{\dot{\mathbf{v}}}{c^2K^2R} + \frac{(\mathbf{R}_1 - \mathbf{v}/c)\,\{(\dot{\mathbf{v}}\mathbf{R}) + c^2 - v^2\}}{c^2K^3R^2}\right] \quad \text{...........(33).}$$

These equations agree with those found by Liénard (*loc. cit.* p. 4). They show that the terms involving the acceleration $\dot{\mathbf{v}}$ are of the order R^{-1}, while those involving the velocity only are of the order R^{-2}. The former preponderate at great distances, the latter near the charge.

We see that $\mathbf{h} = [\mathbf{R}_1\mathbf{d}]$, that is, the magnetic force is perpendicular to the radius vector and the electric force everywhere. Also $(\mathbf{R}_1\mathbf{d}) = (c^2 - v^2)/c^2K^2R^2$; thus the electric force is not transverse to the radius vector, except when the velocity of the charge equals that of light, but the deviation from transversality becomes smaller and smaller as the distance increases.]

14. Point law for velocities greater than that of light. The preceding investigation obviously applies so long as K', that is $dt'/d\tau$, does not vanish. Now by (29) we have

$$K' = 1 - \beta' \cos\theta',$$

where $\beta' = v'/c$ and θ' is the angle between the directions of the velocity and the radius vector drawn from the point charge to the fieldpoint, all for the time t'.

It may very well happen that even when $\beta' > 1$, yet $\beta' \cos\theta' < 1$, particularly in directions approximately transverse to the direction of motion, for example, in the case where the path is a plane curve of small dimensions, and the fieldpoint is in a direction nearly perpendicular to its plane and far away. In this case (30)—(33) still hold.

This ceases to be the case when K' vanishes at any time; this certainly occurs when $\beta' > 1$ and the fieldpoint lies in the direction of motion, so that $\theta' = 0$. Thus (30)—(33) fail for some points of the field, whenever the velocity exceeds that of light.

Now we have $t' = \tau + R/c$, while R is essentially positive; hence $t' = \infty$ when $\tau = \infty$, whether R be finite or not, so that for large positive values of τ K' is certainly positive. Its behaviour for other values is best seen from a diagram.

Plot a graph of t' against τ*. The line AB is given by $t' = \tau$, and the graph lies above it everywhere.

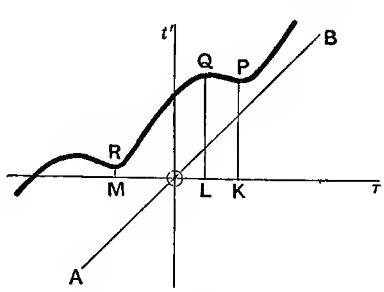

Fig. 1.

Let P be the minimum corresponding to the greatest value of τ which makes K' vanish, Q the next maximum (if it exist), R the next minimum (if it exist), and so on; and let $OK, OL, OM, \ldots$ be the corresponding abscissae. In the interval $K\infty$ we have K' positive; denote this interval by a suffix 1, so that $K_1' > 0$. In the interval LK, K'

* [We shall call the curve PQR the characteristic curve of the motion.]

is negative; denoting this interval by suffix 2 we have $K_2' < 0$. Similarly in ML we have $K_3' > 0$, and so on. It is obvious that for the last interval extending to $\tau = -\infty$, K' is positive or negative according as t' is $-\infty$ or $+\infty$. In the former case the number of intervals is odd, and the number of maxima and minima together even, being one less; in the latter case the reverse is true.

Denote $OK, OL, OM, \ldots$ by $\tau_{12}, \tau_{23}, \tau_{34}, \ldots$, and $KP, LQ, MR, \ldots$ by $t'_{12}, t'_{23}, t'_{34}, \ldots$, the pair of indices referring to the index numbers of the adjacent intervals. Then we have obviously

$$\tau_{12} > \tau_{23} > \tau_{34} > \ldots,$$

$$t'_{12} < t'_{23},\ t'_{23} > t'_{34},\ \ldots,$$

whilst t'_{12} may be greater or less than t'_{34} according to circumstances.

Let the number of intervals be p, so that the last pair of critical values are $\tau_{p-1,\,p}$ and $t'_{p-1,\,p}$. When p is even we have a minimum, and $K_p' < 0$; when it is odd, we have a maximum, and $K_p' > 0$.

We may write (12), (§ 9), in the form

$$\phi = \frac{1}{2\pi}\int de \int_{-\infty}^{\infty} d\mu \left(\int_{\tau_{12}}^{\infty} + \int_{\tau_{23}}^{\tau_{12}} + \int_{\tau_{34}}^{\tau_{23}} + \ldots + \int_{-\infty}^{\tau_{p-1,p}}\right) \epsilon^{\iota\mu(t-R/c-\tau)}\frac{d\tau}{R},$$

where the limits for the several integrals have been put in evidence explicitly.

Transform from τ to t'; we must write $dt = dt'/K'$ and arrange each integral so that the new variable t' increases throughout its range. Thus we must put

$$\int_{\tau_{12}}^{\infty} d\tau = \int_{t'_{12}}^{\infty}\frac{dt'}{K'},\qquad \int_{\tau_{23}}^{\tau_{12}} d\tau = -\int_{t'_{12}}^{t'_{23}}\frac{dt'}{K'},\qquad \int_{\tau_{34}}^{\tau_{23}} d\tau = \int_{t'_{34}}^{t'_{23}}\frac{dt'}{K'},$$

and so on, and finally,

$$\int_{-\infty}^{\tau_{p-1,p}} d\tau = \int_{-\infty}^{t'_{p-1,p}}\frac{dt'}{K'},\quad \text{or} \quad -\int_{t'_{p-1,p}}^{\infty}\frac{dt'}{K'},$$

the first or second alternative being chosen in the last case, according as K' is positive or negative, that is, according as p is odd or even.

Hence we get

$$\phi = \frac{1}{2\pi}\int de \int_{-\infty}^{\infty} d\mu \left(\int_{t'_{12}}^{\infty} - \int_{t'_{12}}^{t'_{23}} + \int_{t'_{34}}^{t'_{23}} - \ldots + \int_{-\infty}^{t'_{p-1,p}},\ \text{or}\ -\int_{t'_{p-1,p}}^{\infty}\right)\epsilon^{\iota\mu(t-t')}\frac{dt'}{K'R'}.$$

Each of the integrals on the right-hand side is a Fourier integral of the usual type, and its value is $\frac{1}{[KR]}$, or zero, according as t lies between, or outside, the limits of the integral. The square brackets are used as usual to denote that in the function KR the variable τ is to be eliminated and t substituted by means of the characteristic equation (28), § 13.

Thus we get

$$\phi = \int de \left(\frac{1}{[K_1 R_1]} - \frac{1}{[K_2 R_2]} + \frac{1}{[K_3 R_3]} - \dots \pm \frac{1}{[K_p R_p]}\right) \quad \text{......(34).}$$

The suffixes have been added to indicate the various intervals for t. It is to be understood that each term is present when t lies within the interval belonging to it; whether it lies within other intervals at the same time or not is immaterial. When t lies outside any interval the corresponding term is absent. Since K is positive in the first, third, ..., in fact in all odd intervals, and negative in the even ones, it is obvious that every term of (34) is essentially positive, and that no two terms can ever annul one another.

A precisely similar expression is got for $\mathbf{a}$, viz.

$$\mathbf{a} = \int de \left(\frac{[\mathbf{v}_1/c]}{[K_1 R_1]} - \frac{[\mathbf{v}_2/c]}{[K_2 R_2]} + \frac{[\mathbf{v}_3/c]}{[K_3 R_3]} - \dots \pm \frac{[\mathbf{v}_p/c]}{[K_p R_p]}\right) \quad \text{......(35).}$$

15. For example, suppose that $p = 2$, so that we have but one stationary value of t, which is necessarily a minimum, since t and τ become $+\infty$ together. We get from (34),

$$\phi = \int de \left(\frac{1}{[K_1 R_1]} - \frac{1}{[K_2 R_2]}\right), \text{ when } t > t'_{12},$$

$$= 0, \text{ when } t < t'_{12}.$$

In other words, until $t = t'_{12}$ there is no disturbance at the given fieldpoint at all, not even an electrostatic one. The character of the field changes once, at the time t'_{12}; the change is discontinuous in this sense, that the *form* of the potentials changes suddenly. The *values* of the potentials and of their differential coefficients up to a certain order are continuous, but the differential coefficients from some finite order onwards are discontinuous when $t = t'_{12}$; otherwise the potentials would always be zero, for $t > t'_{12}$, as well as for $t < t'_{12}$.

Since the critical value t'_{12} is determined as the single real root less than t of the equation $K' = 0$, regarded as an equation in t', it follows that when $t = t'_{12}$, each of the two terms in the expression for ϕ becomes infinite. As we have already pointed out, K_1 is positive and K_2 negative, whilst R_1 and R_2 must both be taken positively. Hence both terms are essentially positive, and cannot balance each other. Apparently ϕ becomes infinite for $t = t'_{12}$.

This however is not really the case; the infinity affects only one element of charge de at a time, and this element contributes only an infinitesimal amount to the value of ϕ in its neighbourhood or elsewhere. Hence the value of ϕ, and similarly that of $\mathbf{a}$, due to an *extended* distribution of charge, remains finite and continuous everywhere, although that due to a finite charge concentrated in a point, if that were physically possible, would become infinitely great, not only at the charge itself, but at all points for which t had the critical value t'_{12}.

The case just considered occurs for a charge moving with uniform velocity in a straight line when its velocity exceeds that of light, as we shall see later.

CHAPTER IV

PHYSICAL INTERPRETATION OF THE SOLUTIONS OBTAINED

16. WE have obtained two types of solution of the problem of finding the potentials due to an electric charge moving in a prescribed way. The first type, given by the equations (12)—(17), § 9, requires the evaluation of double integrals, which may be regarded as generalisations of Fourier's integrals. The second, given by the equations (30)—(35), §§ 13, 14, constitutes a generalisation of the ordinary "point-law" expression for the electrostatic potential. It requires the solution of the characteristic equation, $t = \tau + R/c$, where R is given as an explicit function of τ. The value of τ found must be substituted in the function $1/KR$ occurring in the integrals. The solution of the equation in finite terms is rarely possible; moreover its solution by means of infinite series, by the use of Lagrange's Theorem or some equivalent expansion, has only been effected in the case when the characteristic equation has only a single real root less than t, that is, when the velocity of the charge has never at any time exceeded the velocity of light.

It is true that from a physical point of view this restriction, at any rate at present, is of no moment, since no experiment has yet been made which indicates the occurrence of such large velocities; whilst useful physical theories such as the "*Relatiftheorie*" of Lorentz and Einstein are incompatible with their existence. Nevertheless from the standpoint of complete mathematical generality this limitation is undesirable. In so far as the use of the integrals (12)—(17) does not require it, these integral solutions are preferable.

Even when velocities exceeding that of light are excluded and the solution of the characteristic equation can be effected by means of infinite series, these series can often be obtained more directly from the integrals. But the integrals can also be used directly for the calculation of the field due to the system of charges and of the mechanical forcive acting on the system. In fact they have been so applied by Sommerfeld* and by Lindemann† to the

* *Loc. cit. Gött. Nach.* 1904, pp. 99, 363; 1905, p. 201.

† *Abh. der K. Bay. Akad.* II. Kl. 1907, p. 235 and p. 339. See also the discussion in the *Sitzungsberichte der K. Bay. Akad.* II. Kl. 1907.

problem of the uniform rectilinear motion of a spherical charge and also to that of its quasi-stationary motion. Even in these simple cases the investigation is very exacting and liable to error, because the integrals are discontinuous. For this reason the safest, though perhaps a circuitous, method is to develope the integrals in series and determine the electric and magnetic forces from them, and to calculate the mechanical reaction on the system by direct integration over it. This is the method we shall follow later.

17. Although the point laws are unsuitable for the actual calculation of the forces of the field, they give us the best picture of the physical processes involved in the propagation of the disturbance from the moving charge to the observer. For this reason we shall consider them a little in detail.

The characteristic equation, $t = \tau + R/c$, may be interpreted as follows :

R is the distance from the fieldpoint (x, y, z) to the point (ξ, η, ζ), where the element of charge was at τ, the time of emission of the disturbance which reaches (x, y, z) at time t. The equation simply expresses the fact that the disturbance traverses this distance with the velocity c.

If (x, y, z) be regarded as a variable point, then the equation represents a sphere, whose centre is at (ξ, η, ζ) and whose radius is $c(t - \tau)$. The sphere is the position at time t of that wave which was emitted by the moving charge when it occupied the position (ξ, η, ζ), at the time τ.

If t be regarded as a variable parameter, the equation determines a family of concentric spheres, namely the successive positions of the particular wave emitted at the time τ.

If τ be regarded as a variable parameter, it determines a family of spheres, whose centres lie on the path of the charge, namely the positions at any time t of all the waves emitted up to that time.

So long as the velocity of the charge is less than that of light the oldest wave contains all the subsequent ones, and every wave equally contains all

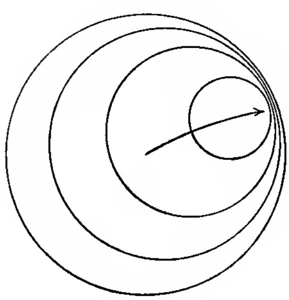

Fig. 2.

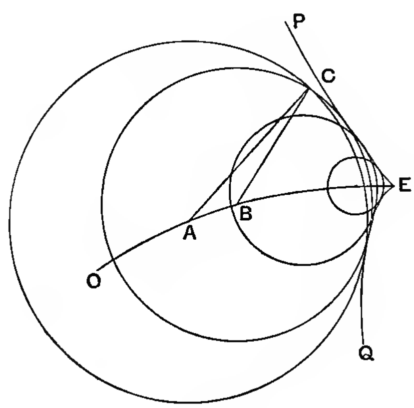

Fig. 3.

the later ones, and therefore also the whole of that portion of the path described subsequent to its time of emission, as shown in Fig. 2. In this case no two waves intersect.

When the velocity of the charge exceeds that of light, so that the charge outruns the waves emitted by it, some of the waves cut each other as shown in Fig. 3. Thus the family of spheres has an envelope PEQ, with a cusp at E, the position of the charge at time t. The envelope may have several sheets, but only one is shown.

If both t and τ be regarded as variable parameters, the characteristic equation represents a doubly infinite system of spheres, namely all the positions of all the waves emitted by the charge. We get this system by supposing either family of spheres just considered to take up all possible positions; for instance, the second family may be supposed to expand, each sphere with the velocity c, while new spheres are being constantly generated as the charge moves along, and the envelope is constantly being added to at the cusp and is moving outwards elsewhere.

All these families of spheres can be constructed quite uniquely when the path of the charge and its law of description are given; and the construction thus supplies a complete solution of the problem of determining the disturbance at every point of the field. [See note at the end of this chapter.]

18. In practice however the problem is different; we fix upon a definite fieldpoint (x, y, z) and a definite time t, and wish to determine the values of the potentials for these particular values without having to solve the whole problem at the same time.

So long as the velocity is less than that of light there is no difficulty, beyond the labour of computation. If the fieldpoint be outside the oldest wave emitted by the moving charge, there is no disturbance there, for as we have seen, this wave includes the whole region of disturbance.

If it be inside it is disturbed, but only by a single wave at a time; for no two waves, corresponding to the same value of t, but to different values of τ, ever intersect. Thus t being given, τ is determined uniquely; the characteristic equation gives but one real value of τ less than t. The corresponding values of K and R can be calculated, if not directly, at all events by means of Lagrange's Theorem, and the values of the potentials are then given by (30) and (31), § 13.

When the velocity exceeds that of light, difficulties arise for points inside the envelope corresponding to the time t. Every such point is the intersection of at least two waves emitted at different times; for instance, in Fig. 3 the point C is the intersection of the spherical waves whose centres are at the points A and B. For points on the envelope the two waves touch, and their centres coalesce. For points outside it the waves are imaginary.

In this case the characteristic equation, regarded as an equation giving τ when x, y, z and t are prescribed, has two or more real roots less than t, one corresponding to each of the waves which reach the point (x, y, z) at the time t. When (x, y, z) passes the envelope, two of these roots become equal and then imaginary. Thus the envelope is the locus of points for which two roots of the characteristic equation are equal and is given by the equation

$$\frac{\partial t}{\partial \tau} \equiv K = 1 + \frac{1}{c}\frac{\partial R}{\partial \tau} = 0 \quad \text{.......................(29).}$$

The roots of this equation are just the critical values τ_{12}, τ_{23}, τ_{34}, ... used in § 14; and the corresponding values of t, given by the characteristic equation, are identical with the maximum and minimum values of t', denoted by t'_{12}, t'_{23}, t'_{34}, ..., both sets of quantities being taken for the prescribed values of (x, y, z). They are perfectly determinate functions of (x, y, z); therefore an equation, such as $t'_{12} = t$, represents a family of surfaces, namely the successive positions of one of the sheets of the envelope, such as PEQ in Fig. 3. When the characteristic equation leads to $p-1$ double roots τ_{12}, τ_{23}, ... $\tau_{p-1,\,p}$, the set of $p-1$ equations

$$t'_{12} = t, \quad t'_{23} = t, \quad \ldots \quad t'_{p-1,\,p} = t \quad \text{....................(36)}$$

determines $p-1$ families of surfaces, each of which represents the successive positions of one of the $p-1$ sheets of the envelope. These sheets separate space into regions, differing as regards the number of waves which reach a point simultaneously. As we pass from one region to the next, this number changes by two, either because two waves coalesce and become imaginary, or because two new waves appear.

The interpretation of expressions (34) and (35), § 14, is obvious; fixing our attention on the particular fieldpoint (x, y, z), let us draw the (t, τ) curve corresponding to it, as in Fig. 1, § 14. Two cases occur: in the one t is $-\infty$ when τ is $-\infty$, and the curve has an even number of maxima and minima of t; in the other $t = +\infty$ when $\tau = -\infty$, and their number is odd.

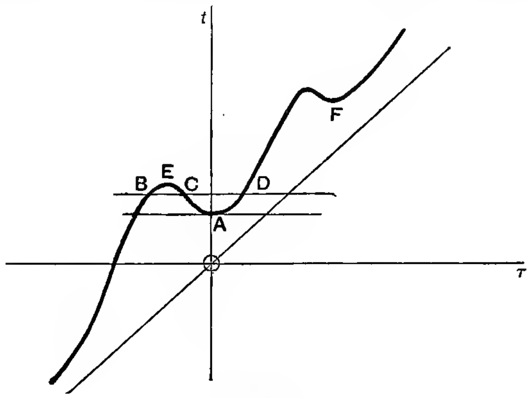

Fig. 4.

In the first case, to begin with there is one wave, until the time corresponding to the least of all the minima is reached, say at A. Until this moment each of the expressions (34) and (35) has but one term.

At the time A the first sheet of the envelope reaches (x, y, z), and thereafter this point is disturbed by three waves simultaneously, corresponding to the three points in which the horizontal BCD cuts the characteristic curve. Each of the expressions (34) and (35) now has three terms, one representing the effect of each wave.

This continues until another sheet of the envelope reaches the point (x, y, z); if it correspond to a maximum, as at E, the oldest wave coalesces with one of the later pair, and only one wave remains; accordingly the expressions (34) and (35) are again reduced to single terms. But if the sheet of the envelope correspond to a second minimum, as at F, two new waves appear, so that (x, y, z) is disturbed by five waves at once, and the expressions (34) and (35) each have five terms. It is easy to understand what occurs in other possible cases: whenever the passing envelope sheet corresponds to a minimum value of t, two new waves appear, but for a maximum two waves already present coalesce and disappear. The number of waves disturbing (x, y, z) simultaneously is always *odd*. (Fig. 4.)

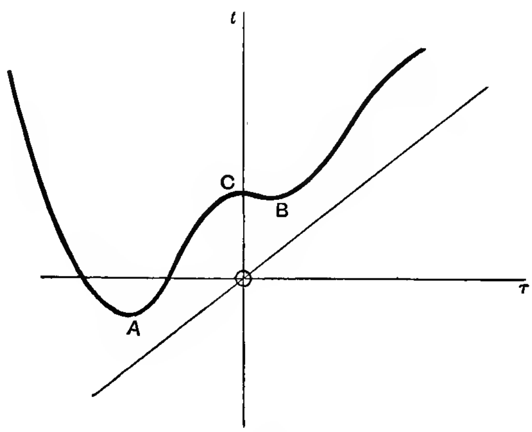

Fig. 5.

In the second case there is no disturbance whatever to begin with. The first envelope sheet to reach (x, y, z) corresponds to a minimum, as at A, and brings with it two waves; accordingly the expressions (34) and (35) have two terms. The second envelope sheet again corresponds to a minimum, as at B, bringing two new waves, so that now the point (x, y, z) is disturbed by four waves at once, and the expressions (34) and (35) have four terms. If the next envelope sheet correspond to a maximum, as at C, two out of the four waves coalesce and disappear, otherwise we may get six. In every case we have an *even* number of waves. (Fig. 5.)

An example of the first case is, as we shall see presently, afforded by the problem of the uniform rectilinear motion of a charge when its velocity is less than that of light; and one of the second by the same problem when the velocity exceeds that of light.

When the moving charge starts from rest, or when its motion is suddenly changed by some impressed force, these results are modified to this extent, that the disturbance at the point (x, y, z) only begins when the first spherical wave emitted by the charge reaches it, or that the character of the disturbance changes suddenly on the arrival of the wave emitted by the charge when its motion was suddenly changed.

19. Graphic representation of the potentials. The characteristic curve for the fieldpoint (x, y, z) affords a graphic representation of the potentials, which is sufficiently accurate to allow us to draw conclusions as to the relative importance of the terms due to the several disturbing waves.

The diagram represents a characteristic curve with two minima and one maximum. The horizontal $ABCDK$ is shown cutting it in four points

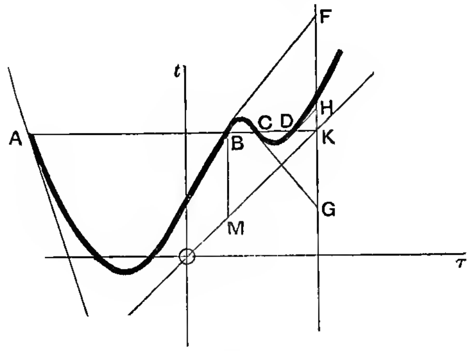

Fig. 6.

A, B, C and D, corresponding to four disturbing waves. Let it cut the line $t=\tau$ in the point K, and draw the ordinate $EGKHF$, cutting the tangents to the curve at A, B, C and D respectively in E, F, G and H (the intersection E is not shown in the diagram).

Consider the effect of one of the waves, say that of B. The term due to it in the potential ϕ is by (34)

$$\int \frac{de}{[K_3 R_3]},$$

where it is to be remembered that the square bracket denotes that the value of τ for B is to be substituted in K and R. The term is numbered 3 because B lies in the third interval of τ, reckoning from $\tau=+\infty$. The value of K_3 is positive, because B is on an ascending branch of the (t, τ) curve, and $K \equiv \frac{\partial t}{\partial \tau}$. Thus $[K_3] = \tan KBF$.

Again, $R_3 = c(t - \tau)$ by definition (characteristic equation); thus $[R_3] = c$ times the intercept MB. But $MB = BK$; hence

$$[K_3R_3] = c \,.\, BK \tan KBF = c \,.\, KF.$$

In precisely the same way we have

$$[K_1R_1] = c \,.\, KH, \quad [K_2R_2] = -c \,.\, KG, \quad [K_4R_4] = -c \,.\, KE,$$

where KE, ... denote distances without regard to sign.

Now for the present case (34) gives

$$\phi = \int de \left\{ \frac{1}{[K_1R_1]} - \frac{1}{[K_2R_2]} + \frac{1}{[K_3R_3]} - \frac{1}{[K_4R_4]} \right\}.$$

Hence we get $$\phi = \frac{1}{c} \int de \left\{ \frac{1}{KE} + \frac{1}{KF} + \frac{1}{KG} + \frac{1}{KH} \right\} \quad \text{...............(37)}.$$

Similarly $$\mathbf{a} = \frac{1}{c^2} \int de \left\{ \frac{\mathbf{v_A}}{KE} + \frac{\mathbf{v_B}}{KF} + \frac{\mathbf{v_C}}{KG} + \frac{\mathbf{v_D}}{KH} \right\} \quad \text{...............(38)},$$

where $\mathbf{v_A}$, ... denote the velocities of the element of charge at the times of emission τ_A, ... of the waves A, ...

These expressions show that generally the most important waves are those which have been emitted most recently, e.g. the wave D. But two waves rise into exceptional importance as they approach to coalescence, since in that case the inclinations of the corresponding tangents of the curve to the horizontal approach the limit zero, and their intercepts on the ordinate tend to vanish.

20. [**Note. Geometrical constructions.** The relations between the path of the charge, the waves emitted by it and their envelope are rendered much clearer by the following geometrical constructions.

We saw on p. 29 that the spherical wave emitted at time τ from the point (ξ, η, ζ) is represented in its position at time t by the characteristic equation

$$(a) \qquad t = \tau + R/c \quad \text{.................................(28)},$$

where $R = \sqrt{\{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2\}}$.

The equation of the wave-envelope is got by eliminating τ between this equation and its first differential

$$(b) \qquad R' = -c \quad \text{...................................(39)},$$

where R' is the total differential coefficient with respect to τ, (ξ, η, ζ) being treated as given functions of τ. We shall denote the resulting eliminant by (ab).

The geometrical interpretation of (39) is simple. Any two waves (a), emitted in positions E_1 and E_2 of the charge, intersect in a circle. As t changes, E_1 and E_2 remaining the same, this circle describes a hyperboloid of revolution with E_1 and E_2 as foci and real semiaxis equal to $\frac{1}{2}c(\tau_2 - \tau_1)$. If E_2 and E_1 be allowed to coincide the hyperboloid reduces to the asymptotic cone, and if its semivertical angle be θ, we obviously have $\cos\theta = c/v$, where v is the velocity of the charge at the time of emission τ. The front half of this cone belongs to times t greater than τ. Now we have identically

$$R' = -\frac{(x-\xi)\dot{\xi} + (y-\eta)\dot{\eta} + (z-\zeta)\dot{\zeta}}{R} = -v\cos\theta \quad \text{.........(40).}$$

Thus (39) is the equation of the front half of the asymptotic cone.

The semicone (39) obviously cuts the spherical wave and the wave-envelope perpendicularly. The circle of intersection is a line of curvature of the envelope, the vertex is the corresponding centre of curvature, and the path of the charge represents the corresponding sheet of the surface of centres.

The condition that three successive waves (a) touch is

(c) $$R'' = 0 \quad \text{..................................(41).}$$

Eliminating τ between (b) and (c) we get an equation (bc), involving neither t nor τ, but only (x, y, z). It represents a fixed surface along which three successive waves (a), and therefore also two sheets of the wave-envelope (ab) touch. Thus the wave-envelope has a cuspidal ridge, which as it moves generates the surface (bc).

But (c) also represents the condition that two successive semicones (b) touch, and therefore the surface (bc) is also the envelope of the family of semicones. Since each cone is normal to the corresponding wave, and to the envelope, the surface (bc) is also the second sheet of the surface of centres of the family of envelopes, which are parallel surfaces.

Again, differentiating (40) we get

$$RR'' = \dot{\xi}^2 + \dot{\eta}^2 + \dot{\zeta}^2 - \frac{\{(x-\xi)\dot{\xi} + (y-\eta)\dot{\eta} + (z-\zeta)\dot{\zeta}\}^2}{R^2}$$
$$- (x-\xi)\ddot{\xi} - (y-\eta)\ddot{\eta} - (z-\zeta)\ddot{\zeta}.$$

Let f be the resultant acceleration of the charge at time τ, and ψ the angle it makes with the radius vector R. Then using (40) and (41) we get

$$fR\cos\psi = v^2 - c^2 \quad \text{..........................(42).}$$

This equation, together with (39) and (40), enables us to construct the cusplocus (bc), when we are given the path of the charge and its mode of description.

Since the cusplocus forms one sheet of the surface of centres of the envelope we can by means of it construct the envelope for any time t.

In Fig. 7 E is the position of the charge at the time τ, ET the direction in which it is moving, and EF a length equal to $(v^2 - c^2)/f$, drawn in the direction of the resultant acceleration. The plane of the paper is the osculating plane at E.

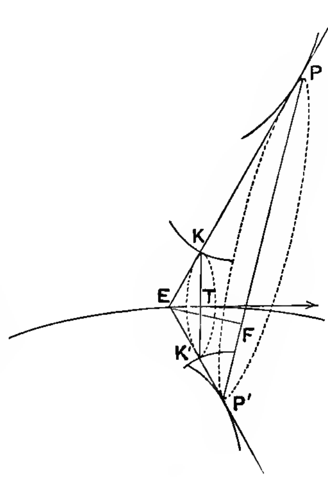

Fig. 7.

PEP' is the trace of the semicone (b), so that $PET = P'ET = \theta = \cos^{-1}(c/v)$.

PFP' is perpendicular to EF.

The dotted curve PP' is the conic in which the cone (b) is cut by the plane (42), which passes through PP' and is perpendicular to the osculating plane TEF. The cusplocus is generated by this conic, and is indicated by its traces at P and P'. These traces touch PE and $P'E$ respectively.

Again, to construct the envelope, we get from (39), (40), and (28)

$$R\cos\theta = -\frac{RR'}{v} = \frac{cR}{v} = \frac{c^2(t-\tau)}{v} \quad \text{..................(43)}.$$

Make $ET = c^2(t-\tau)/v$, and draw $K'TK$ perpendicular to ET. The plane through KK' perpendicular to the osculating plane is given by (43), and cuts the cone (39) in the circle KK'. The envelope corresponding to E in its position at time t is generated by this circle, and is indicated by its traces at K and K', which are perpendicular to EP and EP' respectively.

As the charge moves, K describes a line of curvature of the envelope and P its evolute, both of which are generally tortuous curves. The line of curvature is limited, either by its intersection with the path, or with its evolute. In the former case all the lines of curvature of the same family cut the path in the same point, which represents the limit of the circular lines of curvature of the second family, is a conical point of the envelope, and corresponds to the position of the charge at the time t. In the latter case the intersection of line of curvature and evolute is the point of contact of three successive spherical waves, and therefore forms the starting-point of a line of curvature belonging to a new sheet of the envelope.]

CHAPTER V

ILLUSTRATIVE EXAMPLES

21. We shall now investigate some problems in illustration of the methods developed in the last chapter. For this purpose it will be sufficient to confine ourselves to the case of a single point charge, for the field due to several such charges is merely the geometric sum of those due to each taken singly; in particular that due to an extended charge can be obtained by integration. As we have seen, the potentials, and therefore also the electric and magnetic forces, due to a point charge moving with a velocity greater than that of light, become infinitely great at points on the envelope of the spherical waves, as well as at the charge itself. It has already been pointed out that this difficulty does not exist in the case of extended distributions, whether on surfaces or in space, provided that the surface, or volume, density, as the case may be, remains finite. A good example will be met with in our first problem, that of a point charge moving with uniform velocity in a straight line; the difficulty has in this case been completely resolved by Sommerfeld. The general case may be treated in a similar way.

22. Problem 1. Uniform rectilinear motion. This problem has been treated by several writers, particularly by Sommerfeld*, but as the solution is simple, it will afford a good illustration of the general method.

Take the line of motion as the x-axis of a system of cylindrical coordinates of (x, ϖ, ϕ); since there is symmetry about this axis, we need only consider a fieldpoint in the plane $\phi = 0$, say the point (x, ϖ).

The position of the charge is given by

$$\xi = v\tau, \quad \eta = \zeta = 0 \quad \text{from} \quad \tau = -\infty \quad \text{to} \quad \tau = +\infty.$$

Hence we get

$$\left.\begin{aligned} R &= \sqrt{\{(v\tau - x)^2 + \varpi^2\}} \\ t &= \tau + \sqrt{\{(v\tau - x)^2 + \varpi^2\}}/c \\ K &\equiv \frac{\partial t}{\partial \tau} = 1 + v(v\tau - x)/cR \\ KR &= R + v(v\tau - x)/c \end{aligned}\right\} \quad \text{.....................(44).}$$

* *Amsterdam Proceedings*, 1904, p. 357.

We see from (44_2) that the characteristic curve is the upper branch of the hyperbola

$$c^2(t-\tau)^2 - (v\tau - x)^2 = \varpi^2.$$

The lines $t = \tau$ and $t = v\tau - x$ are conjugate diameters. The lines

$$c(t-\tau) = \pm (v\tau - x)$$

are asymptotes, and obviously furnish the characteristic curve for a point on the line of motion, for which $\varpi = 0$. The characteristic curves for the cases of a velocity less than that of light, and one greater, are essentially distinct and are shown in Fig. 8 and Fig. 9 respectively.

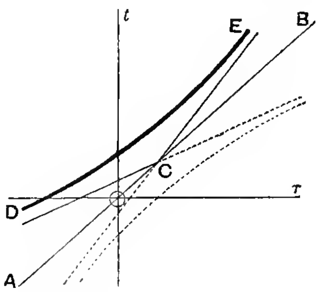

Fig. 8. For $v=\frac{1}{2}c$, $x=\frac{1}{2}c$, $\varpi=c$.

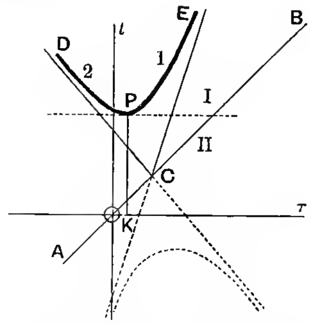

Fig. 9. For $v=2c$, $x=2c$, $\varpi=2c$.

In the first case the curve ascends continually, and a line $t=$ constant cuts it once and only once. In the second case the curve has a minimum at P, and a line $t=$ constant cuts it either twice or not at all. The minimum value of t is t_{12}, which is equal to KP, and is given by $vt_{12} = x + \varpi\sqrt{(v^2/c^2-1)}$; thus $t_{12} = 2{\cdot}73$ in Fig. 9. The corresponding critical value of τ is τ_{12}, which is equal to OK and is given by $v\tau_{12} = x - \varpi/\sqrt{(v^2/c^2-1)}$; thus $\tau_{12} = 0{\cdot}42$ in Fig. 9. When $t < t_{12}$, the characteristic equation has no roots; when $t > t_{12}$, it has two, τ_1 and τ_2, of which $\tau_1 > \tau_{12}$ and $\tau_2 < \tau_{12}$. One corresponds to the branch (1), PE, the other to the branch (2), DP. When $t = t_{12}$ we have $\tau_1 = \tau_2 = \tau_{12}$.

In the present problem the roots of the characteristic equation are easily found analytically. Rationalizing (44_2) we get

$$(v^2-c^2)\tau^2 - 2(vx - c^2t)\tau + x^2 + \varpi^2 - c^2t^2 = 0,$$

whence

$$\tau = \frac{vx - c^2t}{v^2-c^2} \pm \frac{\sqrt{\{c^2(vt-x)^2 - (v^2-c^2)\varpi^2\}}}{v^2-c^2} \quad \text{............(45).}$$

This gives

$$t - \tau = \frac{v(vt-x)}{v^2-c^2} \mp \frac{\sqrt{\{c^2(vt-x)^2 - (v^2-c^2)\varpi^2\}}}{v^2-c^2} \quad \text{.........(46).}$$

Upper signs belong together in the two expressions.

Roots, which make $t > \tau$, give available solutions, that is, roots of the characteristic equation. There are two cases:

23. (*a*) *Velocity less than that of light:* $v < c$, *Fig.* 8.

The square root in (45) and (46) never vanishes, and is always greater than $c(vt - x)$ in absolute value, therefore still greater than $v(vt - x)$. (46) shows that the upper sign must be taken in order to make $t > \tau$. This corresponds to the point in which the horizontal line at level t cuts the upper branch DE of the hyperbola in Fig. 8. The second non-available root corresponds to its intersection with the lower branch. Thus there is always one, and never more than one solution; (44_4) gives, with (46), and (30), § 13,

$$[KR] = \sqrt{\{(vt - x)^2 + (1 - v^2/c^2)\,\varpi^2\}},$$

$$\left.\begin{aligned} \phi &= \frac{1}{\sqrt{\{(vt - x)^2 + (1 - v^2/c^2)\,\varpi^2\}}} \\ a_x &= \frac{v}{c\sqrt{\{(vt - x)^2 + (1 - v^2/c^2)\,\varpi^2\}}} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots(47),$$

which is the well-known solution for this case.

24. (*b*) *Velocity greater than that of light:* $v > c$, *Fig.* 9.

The square root in (45) and (46) is real or imaginary according as $(vt - x)^2 \gtrless (v^2/c^2 - 1)\,\varpi^2$. The equation $(vt - x)^2 = (v^2/c^2 - 1)\,\varpi^2$ represents a cone, whose vertex is at the charge and which moves through space with the charge. Its semivertical angle is equal to $\sin^{-1}(c/v)$. Inside it the roots of (45) are real, outside imaginary.

When the square root is real it is obviously less than $c(vt - x)$ in absolute value, therefore still less than $v(vt - x)$. It follows from (46) that both signs make $t \gtrless \tau$ according as $vt - x \gtrless 0$. Thus the characteristic equation (44_2) has two solutions, or none at all, according as $vt - x \gtrless \varpi\sqrt{(v^2/c^2 - 1)}$. The equation

$$vt - x = \varpi\sqrt{(v^2/c^2 - 1)} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(48)$$

represents the back half of the cone just considered. At the surface of this semicone the two roots, τ_1 and τ_2, are both equal to the critical value τ_{12}, which is less than the corresponding minimum, t_{12}, of the characteristic curve. Inside, behind the semicone, they are both real and less than t; outside it, between it and the front half of the complete cone, they are imaginary; and inside, in front of the front half they are both real but greater than t, and therefore not available. For a given fieldpoint (x, ϖ), the semicone (48) corresponds to the time $t = t_{12}$, where as before $vt_{12} = x + \varpi\sqrt{(v^2/c^2 - 1)}$; that is, t_{12} is the time at which the moving semicone reaches the given fieldpoint. When $t < t_{12}$, that is, before the semicone reaches the point, there are no solutions, and there is no disturbance. When $t > t_{12}$, that is, after it has passed the point, there are two solutions, and the disturbance is due to two waves at once. The semicone corresponds to the minimum P of the characteristic curve DPE of Fig. 9.

Fig. 10 gives a meridian section of the semicone for the case $v = 2c$. $RQEQ'$ is the trace of the semicone, and E is the charge, both in their positions for time t. P is the fieldpoint (x, ϖ), inside the semicone. E_1 and E_2 are the centres of the two spherical waves which disturb P at time t. The roots of (45), τ_1 and τ_2, are the times at which the charge passed through E_1 and E_2. As P moves up to the semicone, E_1 and E_2 obviously approach and ultimately coincide, and so also do Q and R. Thus the semicone is the envelope, in its position at time t, of all the waves emitted by the charge at earlier times. Since $E_1Q = c(t-\tau_1)$ and $E_1E = v(t-\tau_1)$, we have

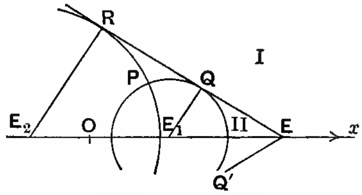

Fig. 10. For $v = 2c$.

$$\sin E_1EQ = c/v.$$

The semicone at any time divides space into two regions: (I) in front of it, where there is no disturbance; and (II) behind it, where the disturbance is due to two waves. These regions are indicated in Figs. 9 and 10.

The potentials are easily found. In region (I) they are of course zero. In region (II) they are given by (34) and (35), § 14, together with (44_4), (45) and (46). The value of τ_1 corresponds to the upper sign in (45); hence we get

$$[K_1R_1] = +\sqrt{\{(vt-x)^2 - (v^2/c^2 - 1)\varpi^2\}}, \quad [K_2R_2] = -\sqrt{\{(vt-x)^2 - (v^2/c^2-1)\varpi^2\}}.$$

With these values we find, for (II),

$$\phi = \frac{2}{\sqrt{\{(vt-x)^2 - (v^2/c^2-1)\varpi^2\}}}, \quad a_x = \frac{2v}{c\sqrt{\{(vt-x)^2-(v^2/c^2-1)\varpi^2\}}} \quad \ldots(49),$$

which is the well-known result for this case.

We notice that as the envelope passes the point, the potentials due to each of the two waves become infinite, equal and of the same sign, so that the resultant potentials themselves are infinite. It has already been pointed out that this infinity is merely due to the supposed concentration of a finite charge at a point, but does not exist in any actually realizable case, as Sommerfeld has exhaustively proved in the investigation already referred to.

25. Problem 2. Uniformly accelerated rectilinear motion.

This problem leads to biquadratic equations and thus can be completely solved in theory, though in practice the expressions obtained are too unwieldly to work with.

Using cylindrical coordinates (x, ϖ) we write for the coordinates of the moving charge

$$\xi = \tfrac{1}{2}f\tau^2, \quad \eta = \zeta = 0 \text{ from } \tau = -\infty \text{ to } \tau = +\infty.$$

The charge moves in the negative direction with uniform retardation from an infinite distance up to the origin, reaches it at time zero, reverses its motion there and moves away again with uniform acceleration. Thus the motion for positive times is exactly the reverse of that for negative ones; nevertheless the field produced is not reversed, nor is it symmetrical with respect to the time zero.

We get

$$\left.\begin{aligned} R &= \sqrt{\{(\tfrac{1}{2}f\tau^2 - x)^2 + \varpi^2\}} \\ t &= \tau + \frac{1}{c}\sqrt{\{(\tfrac{1}{2}f\tau^2 - x)^2 + \varpi^2\}} \\ K &\equiv \frac{\partial t}{\partial \tau} = 1 + f\tau\,(\tfrac{1}{2}f\tau^2 - x)/cR \\ KR &= R + f\tau\,(\tfrac{1}{2}f\tau^2 - x)/c \end{aligned}\right\} \quad \text{......................(50).}$$

Rationalizing (50_2) we get

$$F(\tau) \equiv f^2\tau^4 - 4\,(fx + c^2)\,\tau^2 + 8c^2t\tau + 4\,(r^2 - c^2t^2) = 0 \quad \text{......(51).}$$

Only those roots of (51) which are less than t give solutions of our problem. Putting $\tau = t$, and $\tau = -\infty$, in succession, we see that their number is always even.

Using the same notation as before we denote the roots of (51) by

$$\tau_1 > \tau_2 > \tau_3 > \tau_4,$$

when all are real. Now (51) may obviously be written in the form

$$c^2\,(t - \tau)^2 - R^2 = 0.$$

Thus it represents, not only the characteristic equation $t = \tau + R/c$, whose roots, when real, are less than t, but also the derived equation $t = \tau - R/c$, whose roots, when real, are greater than t.

Hence when the four real roots of (51) are distributed between the two equations, we must take τ_3 and τ_4 as the roots of the characteristic, and τ_1 and τ_2 as the roots of the derived equation.

When the equation (51) has two real roots we must, on account of the principle of continuity, choose them to be τ_3 and τ_4, when they belong to the characteristic, and τ_1 and τ_2, when they belong to the derived equation.

In our study of the problem we shall begin by determining the conditions under which the characteristic equation may have four, or two, or no real roots. This amounts to determining the critical values of τ, which are double roots of (51) less than t and determine the minima and maxima of the characteristic curve, as explained in Ch. III, § 14. We can then determine the position of the various sheets of the envelope at the time t, and so obtain a general idea of the nature of the field.

We shall then consider the characteristic curve, and explain how to calculate the roots of the characteristic equation and obtain expressions for the potentials in the various parts of the field. In this way we shall get some idea of the way in which the field developes itself, both for velocities less and greater than that of light, in the case of the most general motion in a straight line; for the present problem obviously gives a second approximation to every motion of this kind.

26. Critical values. The critical values of τ are found by eliminating t between the characteristic equation and its first differential with respect to τ. We may use (51) in place of the characteristic equation, provided we bear in mind that values of τ less than t are alone relevant to our problem. In this way we find

$$(f\tau^2 - 2x)^2 (f^2\tau^2 - c^2) - 4c^2\varpi^2 = 0 \quad \text{.................(52).}$$

Using (50_2) and (50_3) we get

$$t = \tau + f\tau\,(x - \tfrac{1}{2}f\tau^2)/c^2 \quad \text{.......................(53).}$$

The only available roots of (52) are those which are real and positive, and at the same time make $t > \tau$. That is, we must choose the sign of τ, which is at our disposal, so that τ and $x - \frac{1}{2}f\tau^2$ may have the same sign. This gives us three solutions. The remaining three roots of (52) correspond to the derived equation. Thus we get the following expressions:

(*a*) $2fx - c^2 < 0$ – *one real root.*

$$\left.\begin{aligned} f\tau_{34} &= -\sqrt{\{c^2 + \tfrac{4}{3}(c^2 - 2fx)\sinh^2\theta\}} \\ \sinh^2 3\theta &= 27c^2f^2\varpi^2/(c^2 - 2fx)^3 \end{aligned}\right\} \text{..............(54),}$$

(*b*) $27c^2f^2\varpi^2 > (2fx - c^2)^3 > 0$ – *one real root.*

$$\left.\begin{aligned} f\tau_{34} &= -\sqrt{\{c^2 + \tfrac{4}{3}(2fx - c^2)\cosh^2\theta\}} \\ \cosh^2 3\theta &= 27c^2f^2\varpi^2/(2fx - c^2)^3 \end{aligned}\right\} \text{..............(55),}$$

(*c*) $(2fx - c^2)^3 > 27c^2f^2\varpi^2$ – *three real roots.*

$$\left.\begin{aligned} f\tau_{34} &= -\sqrt{\{c^2 + \tfrac{4}{3}(2fx - c^2)\sin^2(\pi/3 + \theta)\}} \\ f\tau_{23} &= +\sqrt{\{c^2 + \tfrac{4}{3}(2fx - c^2)\sin^2\theta\}} \\ f\tau_{12} &= +\sqrt{\{c^2 + \tfrac{4}{3}(2fx - c^2)\sin^2(\pi/3 - \theta)\}} \\ \sin^2 3\theta &= 27c^2f^2\varpi^2/(2fx - c^2)^3, \quad 0 < \theta < \pi/6 \end{aligned}\right\} \text{...........(56).}$$

It is easy to see that the values of τ_{34} given by these expressions are continuous.

The notation has been chosen to fit our conventions as to the roots of (51).

27. Wave-envelope. The several sheets of the wave-envelope are found by putting $t = t_{34}$, $t = t_{23}$ and $t = t_{12}$, where t_{34}, ... are minimum and maximum values of t given by (53) with τ_{34}, ... in place of τ.

Both t and t/τ are real when, and only when, τ is real. Hence the sheet (t_{34}) exists for all points, but the two sheets (t_{12}) and (t_{23}) exist only for points for which τ_{12} and τ_{23} are real, that is, by (56), inside the semicubical paraboloid

$$27c^2f^2\varpi^2 = (2fx - c^2)^3 \text{..........................(57).}$$

[This is the cusplocus of p. 35.]

From (53) we see that the ratio t/τ has the sign of the quantity $c^2 + fx - \frac{1}{2}f\tau^2$. Thus t_{12}/τ_{12} and t_{23}/τ_{23} are positive on account of (56), and, as τ_{12} and τ_{23} are also positive, t_{12} and t_{23} are both positive. Again, t_{34}/τ_{34} is positive (negative), and, as τ_{34} is negative, t_{34} is negative (positive) according as the fieldpoint (x, ϖ) is inside (outside) the paraboloid

$$f^2\varpi^2 = c^2(2fx + c^2) \quad\quad (58).$$

t_{34} vanishes at all points of this paraboloid, which is therefore the envelope sheet (t_{34}) in its position at time zero.

In order to be able to construct the wave-envelope we require to know its intersections with the x-axis and with the cusplocus.

On the x-axis we have $\varpi = 0$; this gives $\theta = 0$ in cases (*a*) and (*c*), but cannot occur in case (*b*). We find, by (53)—(56),

(*a*) *outside the cusplocus* (57): $x < c^2/2f$,

$$f\tau_{34} = -c, \quad ft_{34} = -(2fx + c^2)/2c, \quad x = -ct_{34} - c^2/2f \quad\quad (59),$$

(*c*) *inside the cusplocus*: $x > c^2/2f$,

$$\left.\begin{array}{lll} f\tau_{34} = -\sqrt{(2fx)}, & ft_{34} = -\sqrt{(2fx)}, & x = \frac{1}{2}ft^2_{34} \\ f\tau_{23} = +c, & ft_{23} = (2fx + c^2)/2c, & x = ct_{23} - c^2/2f \\ f\tau_{12} = +\sqrt{(2fx)}, & ft_{12} = +\sqrt{(2fx)}, & x = \frac{1}{2}ft^2_{12} \end{array}\right\} \quad\quad (60).$$

Putting $t = t_{34}$, and remembering that the x-coordinate of the charge is $\frac{1}{2}ft^2$ at time t, we see that the vertex of the envelope sheet (t_{34}) coincides with the charge until $t = -c/f$, that is, so long as the charge is moving towards the origin with velocity greater than that of light. It leaves it just as the velocity becomes equal to that of light, and ever afterwards moves ahead of it with that velocity, so that at time t its distance from the charge is equal to $\frac{1}{2}f(t + c/f)^2$.

Putting $t = t_{23}$ we see that the envelope sheet (t_{23}) arises at the time $t = c/f$, and at that time its vertex coincides with the charge, which is moving with the velocity of light in the positive direction. The vertex at once falls behind the charge, continues to travel with the velocity of light in the positive direction, so that at time t its distance from the charge is equal to $\frac{1}{2}f(t - c/f)^2$.

Putting $t = t_{12}$ we see that the envelope sheet (t_{12}) arises at the time $t = c/f$, and that its vertex coincides with the charge ever afterwards.

The intersection of the envelope with the cusplocus is got by making $\theta = \pi/6$ in (56); the value of τ_{34} is of no interest, but for the other critical values we get

$$f\tau_{23} = f\tau_{12} = +\sqrt{\tfrac{2}{3}}\,(fx + c^2),$$

whence, using (53) and (57), we find for this intersection

$$x = \tfrac{3}{2} f\tau^2 - c^2/f, \quad \varpi = (f^2\tau^2 - c^2)^{\frac{3}{2}}/cf, \quad t = f^2\tau^3/c^2 \quad \text{.........(61)}.$$

If s be the arc of the semicubical parabola (57), measured from its vertex to the intersection with the envelope, we easily find $\frac{ds}{d\tau} = 3f^2\tau^2/c = c\frac{dt}{d\tau}$: whence $s = c(t - c/f)$. This result verifies that the intersection travels along the semicubical paraboloid with the velocity of light away from the vertex.

Again, we find that $\sqrt{\{(x - \frac{1}{2}f\tau^2)^2 + \varpi^2\}} = c(t - \tau)$; this verifies that τ is the time of emission of the three coincident spherical waves, which touch the sheets (t_{12}) and (t_{23}) along the circle (61).

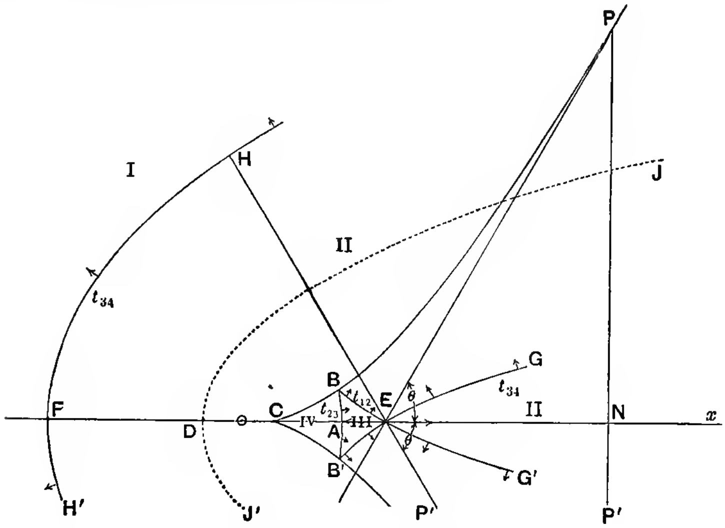

Fig. 11. For $t = \mp 2c/f$.

[28. **Construction of the wave-envelope and its cusplocus for the case of uniformly accelerated motion.** In the present problem there is symmetry about Ox, so that it is sufficient to construct the traces of the various surfaces on a meridian plane of (x, ϖ). In Fig. 11 C is the point $(c^2/2f, 0)$, through which the charge passes at the times $t = \mp c/f$, when its velocity is just equal to $\mp c$. E is the position of the charge at times $\mp 2c/f$, when its velocity is equal to $\mp 2c$, so that $OE = 2c^2/f$. The envelopes for these two times are shown in the figure.

PEP' is the semicone (b) for both times $\mp 2c/f$ (cf. Fig. 7, § 20).

PNP' represents the plane PFP' for both times; EF reduces to EN, because f is always directed along Ox, to the right. The intersection of the semicone and plane of course reduces to a circle indicated by PP' (P' is the intersection of HEP' with PNP' and is below the diagram).

The locus of the circle PP' is the semicubical paraboloid (57), of which the trace is $PBCB'P'$; it is the envelope of the semicone PEP', and the cusplocus of the wave-envelope. BB' indicates the cuspidal ridge for the time $t = +2c/f$.

EP touches the arc CBP at P; if t' be the time at which the cuspidal ridge of the wave-envelope passes through P, we find, by (61), $EP = c(t' - t)$. This merely expresses the fact that the disturbance emitted at time t from E in the direction EP reaches P at the time t'. Further, arc $CB = c(t - c/f)$, and arc $CP = c(t' - c/f)$, since the cuspidal ridge starts from C at time $t = +c/f$, and travels along the cusplocus with the velocity of light. Hence $EP = \text{arc } BP$.

(Parenthetically we may notice a simple construction for P; we have $OC = c^2/2f$, $OE = \frac{1}{2}ft^2$, so that $CE = \frac{1}{2}ft^2 - c^2/2f$. Again, by (61),

$$ON = \tfrac{3}{2}ft^2 - c^2/f,$$

so that $CN = \frac{3}{2}ft^2 - 3c^2/2f$. Hence we have $CN = 3CE$.)

Again, we found, in (60_2), that $OA = ct - c^2/2f$, so that

$$CA = c(t - c/f) = \text{arc } CB.$$

Lastly, we found in (59), that $OF = ct + c^2/2f$, so that $CF = c(t + c/f)$. Thus $FA = 2ct$.

Since the trace of the wave-envelope is an involute of the trace of the cusplocus (§ 20) we can construct the envelope as follows:

From E, the position of the charge at times $\pm t$, draw EP to touch the semicubical parabola CBP at P (by making $CN = 3CE$, and drawing NP vertically).

Unwrap PE from the infinite arc beyond P; the end E describes the arc EG', which generates the envelope sheet (t_{34}), that is, the complete envelope for time $-t$.

Wrap PE on to the finite arc PB; the end E describes the arc EB belonging to the envelope sheet (t_{12}) for time $+t$.

Unwrap a length CB from the arc CB; the end B describes the arc BA belonging to the envelope sheet (t_{23}) for time $+t$.

Lastly, choose a length greater than PE by the length $2ct$, and unwrap it from the complete arc CBP; the free end describes the arc FH' belonging to the envelope sheet (t_{34}) for time $+t$.

It is evident from the construction that the envelope sheet GEG' for time $-t$, and the sheet (t_{12}) of the envelope for time $+t$, are continuous with each other at E, and have equal and opposite conical points there, provided $t > c/f$. The semivertical angle at E is $\sin^{-1}(c/v)$; as E approaches the point C, and its velocity v approaches to equality with the velocity of light C, the semivertical angle approaches to 90°, and the conical point disappears. The envelope GEG' then assumes the shape HFH', and retains it for all times later than $-c/f$. At the time zero it takes up the position JDJ', which represents the trace of the paraboloid (58).]

29. Characteristic curve. The biquadratic (51), § 25, may be represented by a quartic curve with two infinite branches. The upper branch, shown as a thick full line in Figs. 12—15, is the characteristic curve, the lower, shown as a broken line, represents the derived equation got by changing the sign of R. Thus the line $t=\tau$ always bisects the vertical distance between the two branches.

In addition, in the present problem R is an even function of τ, so that the derived is got from the characteristic equation by changing the sign both of t and of τ. Hence the two branches of the quartic are symmetrical with respect to the origin.

There are four cases: (a) fieldpoint inside the paraboloid (58), but outside the cusplocus (57), Fig. 12; (b) fieldpoint inside both surfaces, Fig. 13;

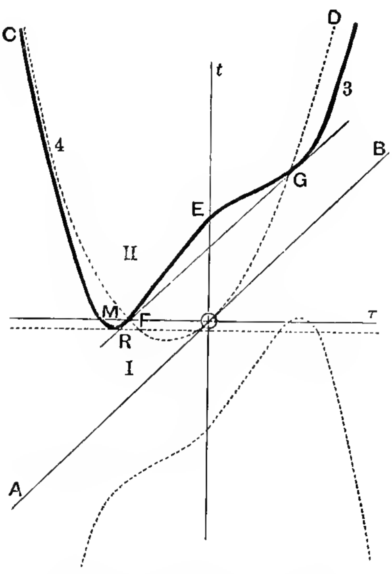

Fig. 12. For $x=2c^2/f$, $\varpi=2c^2/f$.

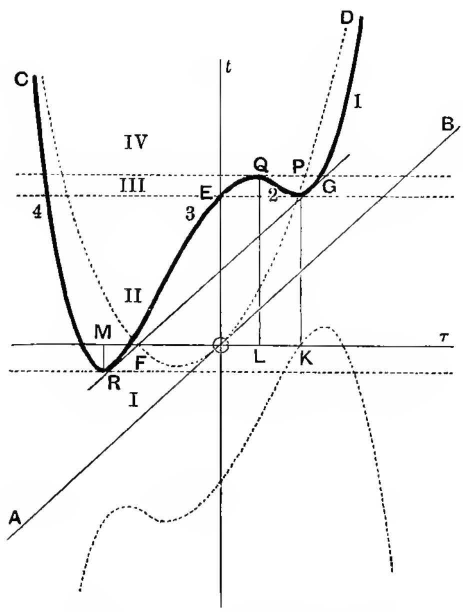

Fig. 13. For $x=7c^2/2f$, $\varpi=2c^2/f$.

(c) fieldpoint outside both surfaces, Fig. 14; (d) fieldpoint outside the paraboloid (58), but inside the cusplocus (57), Fig. 15.

In cases (a) and (b) we have $t_{34}<0$, and both branches cut the axis $t=0$; in (c) and (d) we have $t_{34}>0$, and neither branch cuts the axis $t=0$.

In cases (*a*) and (*c*) the critical value τ_{34} alone exists, and the characteristic equation has at most two real roots; in (*c*) and (*d*) τ_{12} and τ_{23} are real as well as τ_{34}, and the characteristic equation may have as many as four real roots.

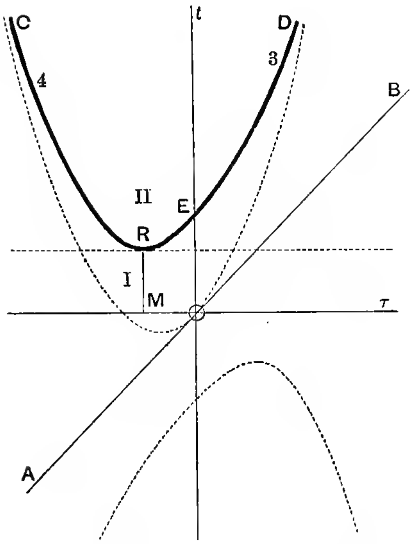

Fig. 14. For $x = -c^2/f$, $\varpi = 2c^2/f$.

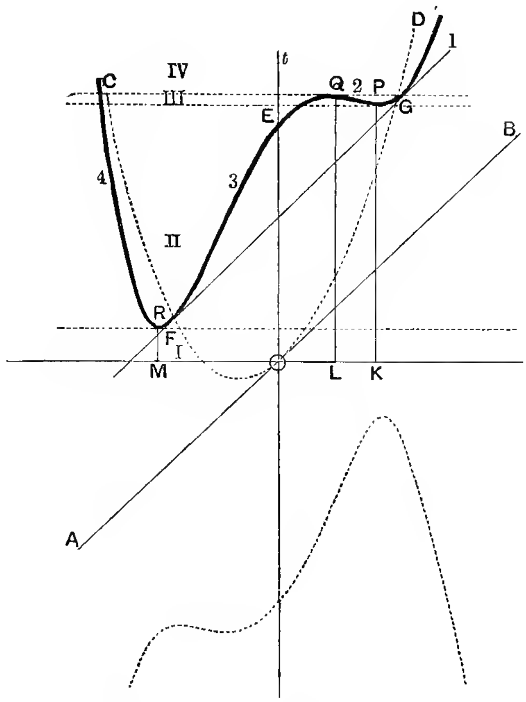

Fig. 15. For $x = 5c^2/f$, $\varpi = 4c^2/f$.

All the curves show the minimum *R*, which corresponds to the envelope sheet (t_{34}); but only the curves in Figs. 13 and 15 have the second minimum *P* and maximum *Q*, corresponding respectively to the envelope sheets (t_{12}) and (t_{23}).

The dotted curve is the parabola $t = \tau + \frac{1}{2} f\tau^2/2c$, which is a curvilinear asymptote.

When $x > 0$, the curve has a double tangent *FG*, which is parallel to the line $t = \tau$ at a vertical height ϖ/c above it, and touches the curve where $\tau = \pm\sqrt{(2x/f)}$ and is below it elsewhere.

30. Progress of events in the field. A study of Figs. 11—15 leads to the following results:

At all times previous to $+c/f$ the envelope has but one sheet (t_{34}), which has the conical shape *GEG′* when $t < -c/f$, and the paraboloidal shape *HFH′* when $-c/f < t < +c/f$ (cf. Fig. 11). This sheet comes from the right, expanding with the velocity of light, and reaches the fieldpoint when $t = t_{34}$. This time is positive (negative) according as the point is outside (inside) the paraboloid (58) (JDJ' in Fig. 11). The two cases are illustrated by Figs. 14 and 15 (12 and 13) respectively. Times earlier than t_{34} correspond to levels below the minimum R in these diagrams, where the horizontal line t does not cut the characteristic curve CED, although it may or may not cut the symmetrical curve. Equation (51), § 25, may, or may not, have real roots, but if it has, they are greater than t, and do not belong to the characteristic equation. Thus no waves, properly speaking no diverging waves, reach the fieldpoint, and it is undisturbed. This region of no disturbance, outside the envelope sheet (t_{34}), we call region (I).

For times immediately after the time t_{34}, when the sheet (t_{34}) has already passed the fieldpoint, the horizontal line t cuts the characteristic curve CED twice, and may also cut the symmetrical curve twice, but not more than twice. The characteristic equation has two real roots, but (51) may have four, in which case two of them are greater than t and belong to the derived equation. The fieldpoint is now disturbed by two divergent waves simultaneously, which according to our notation correspond to roots τ_3 and τ_4 of (51), of which τ_4 is negative, while τ_3 is negative (positive) according as $t \lesseqgtr r/c$. This region of two-wave disturbance (34) we call region (II). For points outside the cusplocus it lasts for ever, as shown in Figs. 12 and 14.

For points inside the cusplocus however, where the second minimum (t_{12}) and the maximum (t_{23}) exist, this type of disturbance lasts only until $t = t_{12}$, that is, until the conical envelope sheet (t_{12}), BEB' in Fig. 11, reaches the point. Immediately afterwards the horizontal line t cuts the characteristic curve CED four times, when it lies between P and Q in Figs. 13 and 15. The characteristic equation has four real roots, $\tau_1 > \tau_2 > \tau_3 > \tau_4$, and the point is disturbed by four divergent waves simultaneously. The region of four-wave disturbance, lying between the envelope sheets (t_{12}) and (t_{23}), $BEB'A$ in Fig. 11, we call region (III).

The four-wave disturbance is only temporary, and disappears as soon as $t = t_{23}$, that is, when the envelope sheet (t_{23}), BAB', passes the fieldpoint. The two intermediate roots, τ_2 and τ_3, of (51) become imaginary, while the extreme roots, τ_1 and τ_4, remain real. The point is again disturbed by two waves simultaneously, but these are the waves (τ_1) and (τ_4). This type of disturbance lasts for ever, but is not continuous with the type (34), existing on the other side of the cusplocus. The region, where it exists, lying between the envelope sheet (t_{23}) and the cusplocus, we call region (IV).

It remains to consider the roots of the characteristic equation in detail, and to find expressions for the potentials of the several types of disturbance.

31. The potentials. Their values are given by (34) and (35), § 14. The values of $[KR]$ for the various waves can be got from (50_2), § 25, by substituting the values of the roots τ_1, τ_2, τ_3 and τ_4, as the case may be.

Since $R = c(t-\tau)$ we easily find by help of (51), § 25, $KR = F'(\tau)/8c$. Putting $t = \tau_1, \ldots$ we get

$$[K_1 R_1] = f^2(\tau_1 - \tau_2)(\tau_1 - \tau_3)(\tau_1 - \tau_4)/8c \quad \text{..............(62)},$$

with similar expressions for the other roots.

Again, since the coordinates of the charge are given by $\xi = \frac{1}{2} f \tau^2$, $\eta = \zeta = 0$, we find for the velocity at time τ_1

$$[v_{1x}] = f\tau_1, \ [v_{1y}] = [v_{1z}] = 0 \quad \text{......................(63)},$$

with similar expressions for the other roots.

It is of course obvious that the y and z components of the vector potential $[a]$ vanish identically, so that we have only to deal with the components $a_{1x}, \ldots$, besides the scalar potential $\phi_1, \ldots$.

It is difficult to express the potentials explicitly in terms of t, x and ϖ, but it is easy to find expressions in terms of the roots of the resolving cubic of (51).

Putting in the usual way

$$\tau_1 = u + v - w, \ \tau_2 = u - v + w, \ \tau_3 = -u + v + w, \ \tau_4 = -u - v - w \ldots\text{(64)},$$

we satisfy the necessary condition $\tau_1 + \tau_2 + \tau_3 + \tau_4 = 0$, and find that u^2, v^2, w^2 are the roots of the cubic

$$f^4 z^3 - 2(fx + c^2) f^2 z^2 + \{(fx + c^2)^2 - f^2(r^2 - c^2t^2)\} z - c^4 t^2 = 0 \ldots\text{(65)},$$

provided only that $$uvw = c^2 t/f^2 \quad \text{..............................(66)}.$$

Our object is to study the variations of the potentials at a given fieldpoint as the time changes; hence we must determine the roots of (51) and (65) as functions of t when x and ϖ are given.

When we plot t as a function of z by means of (65) we obtain a graph of the type shown in Fig. 16, for the case where the fieldpoint lies inside both the cusplocus (57) and the paraboloid (58).

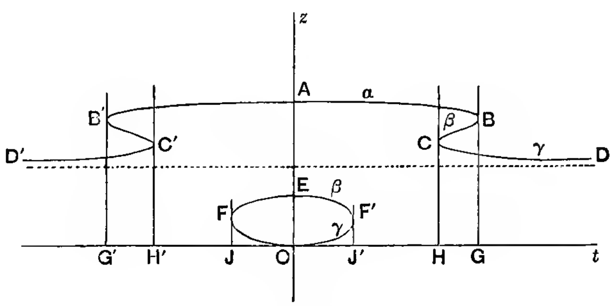

Fig. 16.

The graph consists of the infinite branch $D'C'B'ABCD$, which lies above its asymptote $t = c/f$, and of the oval $EFOF'$, which touches the axis $z = 0$ at the origin. Since (65) involves only the square of t, the graph is symmetrical with respect to the axis $t=0$. It has maxima and minima for t at B, C, F and B', C', F', corresponding respectively to Q, P, R on the characteristic, and to their symmetrical points on the symmetrical dotted curve of Fig. 13. The points B $(t=t_{23})$, C $(t=t_{12})$, and F $(t=t_{34})$ are, the first two on the positive, and the third on the negative side, because t_{23} and t_{12} are positive, and, in the present case, t_{34} is negative.

The graph for Fig. 12 differs in so far as the folds BC and $B'C'$ are absent, and those for Figs. 14 and 15 differ in so far as the oval $EFOF'$ lies below the axis $z=0$ instead of lying above it, as in the former figures.

The intersections of the ordinate t with the infinite branch give real and positive roots of (65); those with the oval give real roots, which are positive (negative) according as the oval lies above (below) the axis $t=0$, that is, according as $t_{34} \lesseqgtr 0$. This is easily verified by a study of (65), but follows directly from our previous results. We shall denote the roots of (65) by α, β and γ.

When the roots are all real we shall choose them so that $\alpha > \beta > \gamma$. When two are conjugate imaginaries, they will be denoted by α and β, or by β and γ, as may be necessary for continuity.

When α, β and γ are all real and positive, u, v and w are real, and so also are the roots τ_1, τ_2, τ_3 and τ_4 of (51). When β and γ are real and negative, v and w are pure imaginaries and unequal, so that τ_1, τ_2, τ_3 and τ_4 are all imaginary. When β and γ are conjugate imaginaries, v and w are so also, and two of the roots τ_1, τ_2, τ_3 and τ_4 are real, and the remaining two imaginary.

The proper expressions for the roots τ_1, τ_2, τ_3 and τ_4 are best got by a study of Fig. 16, and the corresponding graphs for the other cases; but we cannot in this way determine which of them are less and which greater than t, that is to say, which belong to the characteristic equation of our problem and which do not. This is best determined by a study of Figs. 12—15.

32. For example let us study the case corresponding to Figs. 13 and 16.

When t lies between $\pm t_{34}$, the ordinate t cuts the graph of Fig. 16 between FJ and $F'J'$, therefore in three points. Thus α, β and γ are all real and positive, and the roots τ_1, τ_2, τ_3 and τ_4 are all real. The horizontal line t cuts the characteristic curve of Fig. 13 twice, above R, and the symmetrical, broken curve also twice. We take the first two intersections to correspond to τ_4 and τ_3, the others to τ_2 and τ_1. This choice makes τ_4 and τ_3 less than t, so that they belong to the characteristic equation, and τ_2 and τ_1

greater than t, so that they belong to the derived equation. Further we choose $u = +\sqrt{\alpha}$, $v = +\sqrt{\beta}$ and $w = +\sqrt{\gamma}$, so that (64) gives

$$\tau_1 = \sqrt{\alpha} + \sqrt{\beta} - \sqrt{\gamma},\ \tau_2 = \sqrt{\alpha} - \sqrt{\beta} + \sqrt{\gamma},$$
$$\tau_3 = -\sqrt{\alpha} + \sqrt{\beta} + \sqrt{\gamma},\ \tau_4 = -\sqrt{\alpha} - \sqrt{\beta} - \sqrt{\gamma} \quad \ldots(67).$$

This choice makes $\tau_1 > \tau_2 > \tau_3 > \tau_4$ in virtue of the inequalities $\alpha > \beta > \gamma$ provided that $\sqrt{\alpha}$ and $\sqrt{\beta}$ are taken to mean the absolute values of the roots. In order to satisfy (66) we must choose the positive (negative) sign for $\sqrt{\gamma}$, according as $t \gtrless 0$.

As t increases, the ordinate in Fig. 16 moves to the right; when $t = -t_{34}$ (which is positive in the present case) the ordinate reaches $F'J'$, and $\beta = \gamma$, becoming conjugate imaginaries immediately afterwards. By (67) τ_1 and τ_2 become imaginary, while τ_3 and τ_4 remain real. This agrees with Fig. 13, where the horizontal line t just touches the broken curve at its highest point, and thereafter ceases to cut it.

So long as $t < t_{12}$, the ordinate in Fig. 16 lies to the left of CH and cuts the graph only once, so that the characteristic equation has only the two real roots τ_3 and τ_4. Up to this time the disturbance is of the two-wave type (34) and the fieldpoint lies in the region (II) of § 30.

When $t = t_{12}$ the roots β and γ of (65) are again equal, and become real. Between t_{12} and t_{23} the ordinate in Fig. 16 cuts the graph three times, between CH and BG, and the roots τ_1, τ_2, τ_3 and τ_4 are again all real. But now they are all less than t, for the horizontal line t in Fig. 13 cuts the characteristic curve four times, between P and Q, and does not cut the symmetrical curve at all. The disturbance is of the four-wave type, and the fieldpoint is in region (III).

When $t = t_{23}$, the ordinate in Fig. 16 coincides with BG, the roots α and β are equal and become conjugate imaginaries, while γ remains real and positive. From (67) we see that τ_1 and τ_4 remain real, while τ_2 and τ_3 become imaginary. The horizontal line t in Fig. 13 touches the characteristic curve at Q, and thereafter cuts it twice, on the arcs RC and PD. The fieldpoint lies in region (IV).

Thus we see that with our conventions the root α is represented by the arc $B'AB$ of the graph in Fig. 16, the root β by the arcs FEF', BC and $B'C'$, and the root γ by FOF', CD and $C'D'$. This is in agreement with the fact that $\sqrt{\alpha}$ and $\sqrt{\beta}$ are always positive, while $\sqrt{\gamma}$ changes sign with t, and therefore vanishes when $t = 0$, at O.

33. The remaining cases can be treated in precisely the same way, and the expressions (67), with the conventions made, are quite general. Substituting the values of $\tau_1, \ldots$ in (62) and (63) and using (34) and (35), § 14, we easily get the following expressions for the potentials.

(I) *τ_1, τ_2, τ_3 and τ_4 all imaginary, or greater than t.*

Here β and γ are negative, as in Figs. 14 and 15 for t between $\pm t_{34}$; otherwise the horizontal line t cuts the symmetrical curve alone.

$$\phi = a_x = 0 \quad\text{.................................}(68).$$

(II) *τ_1 and τ_2 are imaginary, or greater than t, while τ_3 and τ_4 are real and less than t.*

Here β and γ are real and positive, or are conjugate imaginaries; the horizontal line t cuts the characteristic curve twice and only twice, and the symmetrical curve twice, or not at all.

$$\left.\begin{aligned} \phi &= \frac{2c(\alpha + \sqrt{\beta\gamma})}{f^2(\alpha-\beta)(\alpha-\gamma)(\sqrt{\beta}+\sqrt{\gamma})} \\ a_x &= \frac{2(\beta+\gamma-\alpha+\sqrt{\beta\gamma})\sqrt{\alpha}}{f(\alpha-\beta)(\alpha-\gamma)(\sqrt{\beta}+\sqrt{\gamma})} \end{aligned}\right\} \quad\text{..................}(69).$$

(III) *τ_1, τ_2, τ_3 and τ_4 are all real and less than t.*

Here β and γ are real and positive, and besides the horizontal line t cuts the characteristic curve four times, and the symmetrical curve not at all.

$$\left.\begin{aligned} \phi &= \frac{4c\sqrt{\beta}}{f^2(\alpha-\beta)(\beta-\gamma)} \\ a_x &= \frac{4\sqrt{\alpha\gamma}}{f(\alpha-\beta)(\beta-\gamma)} \end{aligned}\right\} \quad\text{........................}(70).$$

(IV) *τ_1 and τ_4 are real and less than t, while τ_2 and τ_3 are imaginary.*

Here α and β are conjugate imaginaries, and the horizontal line t cuts the characteristic curve twice, and the symmetrical curve not at all.

$$\left.\begin{aligned} \phi &= \frac{2c(\gamma + \sqrt{\alpha\beta})}{f^2(\alpha-\gamma)(\beta-\gamma)(\sqrt{\alpha}+\sqrt{\beta})} \\ a_x &= \frac{2(\alpha+\beta-\gamma+\sqrt{\alpha\beta})\sqrt{\gamma}}{f(\alpha-\gamma)(\beta-\gamma)(\sqrt{\alpha}+\sqrt{\beta})} \end{aligned}\right\} \quad\text{..................}(71).$$

A comparison of (71) with (69) shows that the disturbance (IV), of two-wave type (14), differs from the disturbance (II), of two-wave type (34), merely by the interchange of α and γ, that is by a change of notation. The expressions for the potentials in fact involve the single real root, α or γ, as the case may be, in precisely the same way.

The solution of our problem is now completed, so far as this is possible when the position of the fieldpoint and the time are not actually specified. It represents the solution of the most general problem of accelerated motion with unlimited velocities in all its essentials, for the development of the envelope and of the various possible types of disturbance will take place in a similar manner.

34. Problem 3. A point charge, initially at rest, is subject to a constant acceleration for a given interval of time, and thereafter continues moving uniformly in a straight line with the acquired velocity. Required the resulting electromagnetic field.

This problem illustrates the mode of combining the solutions of two known problems, so as to give the solution of a new one. Moreover it has a special interest in so far as it illustrates, crudely it may be, the process of the generation of X-rays, or of γ-rays, according to the commonly accepted views as to their nature.

We have
$$\left.\begin{aligned} &\xi = 0,\ v_x = 0 \text{ from } \tau = -\infty \text{ to } \tau = 0 \\ &\xi = \tfrac{1}{2} f\tau^2,\ v_x = f\tau \text{ from } \tau = 0 \text{ to } \tau = T \\ &\xi = fT(\tau - \tfrac{1}{2}T),\ v_x = fT \text{ from } \tau = T \text{ to } \tau = \infty \end{aligned}\right\} \quad \ldots\ldots\ldots(72).$$

Further $y = z = 0$, $v_y = v_z = 0$, $a_y = a_z = 0$ always.

There are two cases, according as the final velocity is less or greater than that of light.

35. (*a*) *Final velocity less than that of light: $T < c/f$.*

The equations (54)—(56), § 26, show that the critical value τ_{34} is negative in every case. But in our present problem negative values of τ, the time of emission, correspond to times when the charge was at rest, and its field everywhere electrostatic. Hence the critical value τ_{34} does not enter into consideration at all. This means that the field is everywhere electrostatic until the first wave of disturbance reaches the fieldpoint. This is obviously always true, so that we can leave the envelope sheet (t_{34}) out of account, whether the final velocity be less than that of light or greater.

The discussion is much simplified by a study of the forms and positions of the several critical surfaces separating the various regions of space in which the potentials are represented by expressions of different form.

The diagram, Fig. 17, shows the traces of these surfaces on the meridian plane.

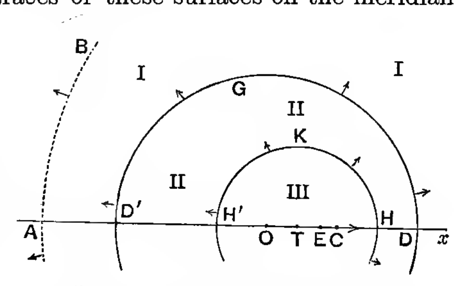

Fig. 17. For $T = 2c/3f$, and $t = c/f$.

O is the origin, where the charge was at rest until $t = 0$, T the point $(\frac{1}{2} f T^2, 0)$, where the uniform motion began, E the position of the charge at the time t. The circle DGD' is the initial spherical wave, emitted at the moment of starting at O, and HKH' is the spherical wave (T), emitted at time T, when the motion changed, both in their positions at the actual time t. The dotted curve AB is the position of the (absent) envelope sheet (t_{34}); it is absent because it is completely outside the initial wave, corresponding as it does to negative times of emission. The several surfaces move normally outward with the velocity of light, c, as indicated by the arrows, while the charge moves onward along Ox. This diagram holds for all time after the steady state has been reached.

36. The sequence of events at a given fieldpoint is best discussed with the help of the characteristic curve, whose equation is

$$t = \tau + R/c \qquad \text{.....(28).}$$

From (72) we see that, since in the present problem

$$R = \sqrt{\{(x-\xi)^2 + \varpi^2\}},$$

R has three different expressions according to the interval for τ which is used, say R_1, R_2 and R_3.

Then we have

$$\left.\begin{aligned} R = R_1 &= \sqrt{\{x^2+\varpi^2\}} = r, \text{ from } \tau = -\infty \text{ to } \tau = 0 \\ R = R_2 &= \sqrt{\{(x-\tfrac{1}{2}f\tau^2)^2+\varpi^2\}}, \text{ from } \tau = 0 \text{ to } \tau = T \\ R = R_3 &= \sqrt{\{(x+\tfrac{1}{2}fT^2-fT\tau)^2+\varpi^2\}}, \text{ from } \tau = T \text{ to } \tau = \infty \end{aligned}\right\} \text{...(73).}$$

There are three corresponding characteristic curves, which together make up the complete curve. Now ξ, R and the velocity v are all continuous throughout the whole range; hence the three sections of the curve must form a continuous line, without corners, though in general with sudden changes of curvature. A break in the curve would imply a sudden jump of the charge from one position to another, a corner a sudden change in its velocity, neither of which is physically possible. The only difficulty that can conceivably occur is when $\dfrac{\partial t}{\partial \tau} = 0$; but this, as we have seen in Ch. IV, requires the velocity to be greater than that of light, a case which will be discussed presently.

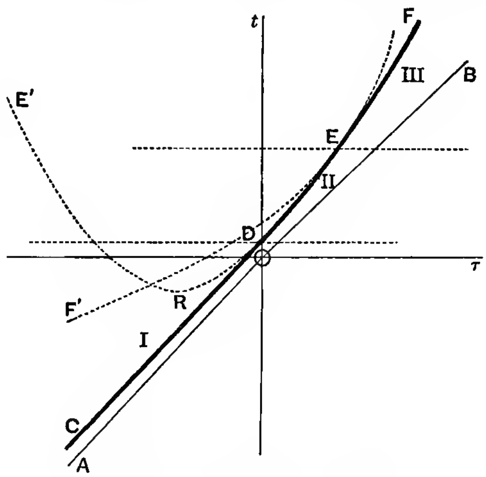

Fig. 18. For $x=0$, $\varpi = c^2/2f$.

The first section of the curve, CD, is given by $t = \tau + r/c$.

It is a straight line parallel to the line $t = \tau$, AB, and at a height r/c above it.

The section, DE, is an arc of the curve of the type shown in Fig. 14, § 29, for the case of uniformly accelerated motion, but extending only from $\tau = 0$ to $\tau = T$. It touches the straight line CD at D.

The last section, EF, is an arc of the hyperbola, of the type shown in Fig. 8, § 22, for the case of uniform motion with a velocity less than that of light.

The remaining portions of these curves are indicated by dotted lines. It is obvious that these portions must be ignored. In fact t is a uniquely determined function of τ and the coordinates of the fieldpoint (x, ϖ); for the charge emits but one wave at a time, and this has no double points. Each wave corresponds to a determinate ordinate in the (t, τ) diagram, which ordinate can cut the characteristic curve only in a single point. Since the points of the full line curve do give actual waves, they are the only ones that do so.

It should be noted that on the present view the state of rest, corresponding to the section CD, is to be regarded as the limiting case of an infinitely slow motion, during which waves are being constantly emitted, but from positions of the charge which are infinitely close together, so that the variations of R become infinitely small.

37. The sequence of events at the selected fieldpoint is now seen to be as follows:

(I) Until $t = r/c$, corresponding to $\tau = 0$, the state is that corresponding to the line CD; the graphic method of representing potentials given in § 19 shows that during this interval we have an electrostatic field given by

$$\left.\begin{aligned} \phi &= \frac{1}{r} \\ a_x &= 0 \end{aligned}\right\} \quad \text{..............................(74).}$$

(II) At the time $t = r/c$, corresponding to the point D, the initial spherical wave arrives, that is, the wave DGD' of Fig. 17. The form of the solution changes to that belonging to region II of the last problem; but the root τ_4, being less than the critical value τ_{34} of τ, which gives the minimum of the curve II in Fig. 18, and is negative, is itself negative, so that the corresponding wave does not exist. Thus we have merely the one wave corresponding to the root τ_3, given by the intersection of the horizontal line $t = \text{constant}$

with the arc DE of the characteristic curve. Thus we have, by (62), § 31,

$$\left.\begin{aligned} \phi &= \frac{1}{[K_3R_3]} = \frac{8c}{f^2(\tau_1-\tau_3)(\tau_2-\tau_3)(\tau_3-\tau_4)} \\ a_x &= \frac{[\mathbf{v}_3]}{c[K_3R_3]} = \frac{8\tau_3}{f(\tau_1-\tau_3)(\tau_2-\tau_3)(\tau_3-\tau_4)} \end{aligned}\right\} \quad \text{............(75)},$$

where τ_1 and τ_2 are the two imaginary roots, τ_3 is the real positive and τ_4 the real negative root of (51), § 25.

(III) The last expressions hold until t reaches the value corresponding to the point E on the characteristic curve, and the spherical wave (T), HKH' of Fig. 17, reaches the fieldpoint. The equation of this wave is

$$R_2 = c(t-T),$$

where in R_2 τ is to be put equal to T. Hence we have for E

$$t = T + \sqrt{\{(x-\tfrac{1}{2}fT^2)^2 + \varpi^2\}}/c.$$

At this moment the forms of the potentials change to those corresponding to rectilinear motion with the uniform velocity $v = fT$, which the electron had at the time T, when the wave was emitted.

The potentials are henceforth given by (47), § 23; thus,

$$\left.\begin{aligned} \phi &= \frac{1}{\sqrt{\{(x+\tfrac{1}{2}fT^2-fTt)^2+(1-f^2T^2/c^2)\varpi^2\}}} \\ a_x &= \frac{fT}{c\sqrt{\{(x+\tfrac{1}{2}fT^2-fTt)^2+(1-f^2T^2/c^2)\varpi^2\}}} \end{aligned}\right\} \quad \text{.........(76)}.$$

It has been remarked already that the continuity of motion of the charge, and the uniqueness of the disturbance due to it, require the potentials to be continuous in value, though discontinuous in form, at each time of transition; but this continuity does not in general extend to the electric and magnetic forces. These are given by (VII) and (VIII), § 3, and involve differentiations with respect to coordinates and time. Since τ is an implicit function of both, in virtue of (28), § 13, these differentiations involve differentiation with respect to τ, and therefore give rise to terms involving the acceleration, which is discontinuous.

38. (*b*) *Final velocity greater than that of light:* $T > c/f$.

So long as the velocity of the charge is still less than that of light, that is for $t < c/f$, the circumstances are much the same as in case (*a*). The field outside the initial spherical wave is electrostatic and given by (74), § 37; the field inside is due to the accelerated motion and given by (75), § 37. There is yet no inner boundary, since the state of uniform motion has not been reached.

What occurs when the velocity of the charge exceeds that of light will be best understood by reference to Fig. 19, which represents the state of affairs at the time $3c/f$, for the case where $T = 2c/f$, that is, where the final velocity is twice that of light.

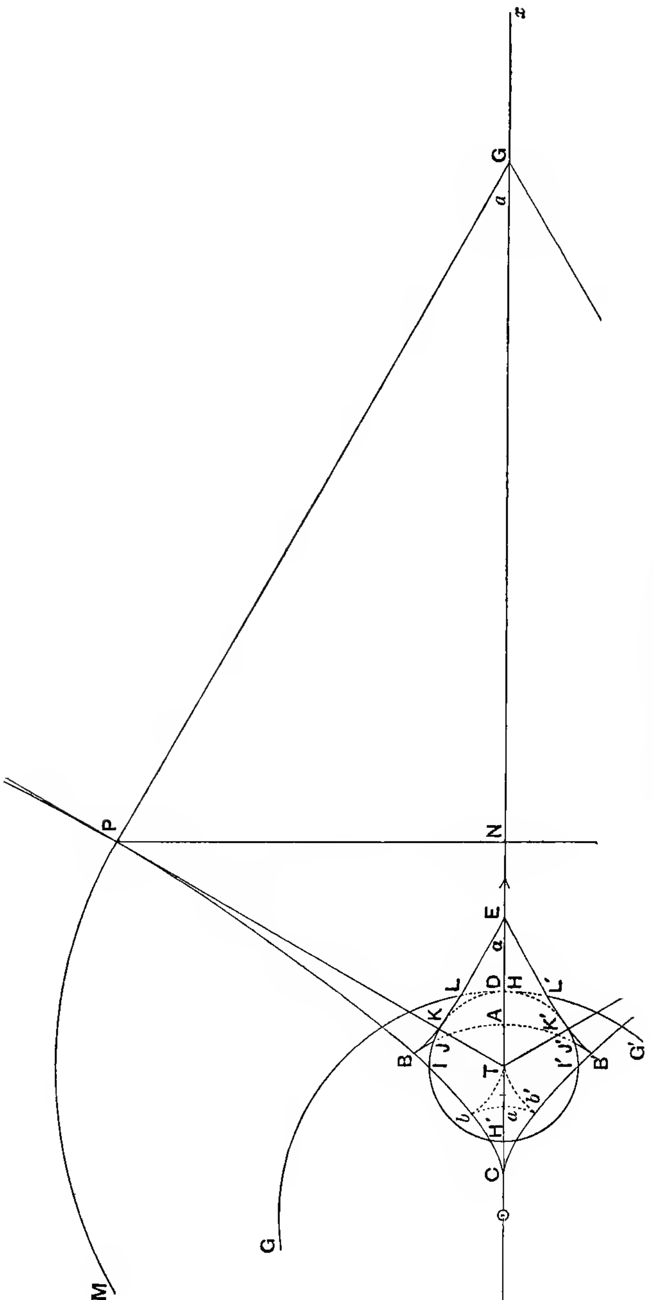

Fig. 19. $T = 2c/f$, $t = 3c/f$.

$B'CBP$ is the semicubical parabola (57), § 27, of which the equation is $(2fx - c^2)^3 = 27c^2f^2\varpi^2$. Thus $OC = c^2/2f$. T is the point reached by the charge at time T, when the motion becomes uniform. Thus $OT = \frac{1}{2}fT^2$.

As the charge passes through C, where its velocity is equal to that of light, the envelope sheets (t_{12}) and (t_{23}) first appear (*vide* § 28 and Fig. 11). At time T they occupy the positions bTb' and bab' respectively. The semi-vertical angle at T is equal to α, where $\sin\alpha = c/v$, and $v = fT$ is the velocity in the uniform motion. The two sheets touch each other along the circle bb', and cut the paraboloid (57) perpendicularly. The line TP touches the semi-cubical parabola at P, which is the position of the cusp at some later time $t = U$. This time is given by (61), § 27, so that

$$U = f^2T^3/c^2 \quad\text{.................................(77).}$$

The charge then is at G, the last vestiges of the envelope sheet $BKK'J'J$ have disappeared, and the envelope is now represented by the semicone PGP' belonging to uniform motion. The spherical wave (T) has expanded into the sphere PM, and extends far beyond the point C, for its radius TP is greater than TN, which is itself equal to twice CT. Thus the region, where the disturbance is due to acceleration, is now limited to the region (II) of the last problem, which lies between the cone PGP' and sphere PM and the initial wave. The disturbance here is of the same one-wave type (75), and has acquired as it were a uniform character. Hence the quantity U may be taken as a measure of the duration of the variable state during which the uniform motion is established.

The field during the variable state is of no great interest; its character will be sufficiently understood by reference to the last problem, when it is stated that it is of type (III) in the ring-shaped region BKJ, between the envelope sheets (t_{12}) and (t_{23}) and the spherical wave (T), of the type (IV) in the ring-shaped region BJI, between the sheet (t_{23}), the spherical wave (T) and the cusplocus (57), and of the type (II) everywhere else between the initial wave GDG', the spherical wave (T) and the semicone KEK'.

39. Since the charge acquires a velocity greater than that of light, whilst the initial wave only expands with the velocity of light, the charge will sooner or later catch it up and penetrate into the region outside. If $T < 2c/f$, this happens when $t = \frac{1}{2}fT^2/(fT - c)$, after the state of uniform motion has been reached; but if $T > 2c/f$, it happens when $t = 2c/f$, during the state of accelerated motion. If $T < (\sqrt{3} + 1)\,c/2f$, it happens at a time $t > U$, that is, after the variable state has ceased, otherwise before. Until the charge pierces the initial wave the disturbance is entirely confined to the space inside it; afterwards the disturbed region is bounded only partially by the initial wave, the rest of the boundary being formed by the portion of the conical envelope sheet which projects beyond the initial wave.

Fig. 20 gives the positions at time $t = 49c/8f$ of the two spherical waves and the conical envelope sheet for the case where $T = 7c/4f$, after the variable *regime* is over and when the charge E has already pierced the initial wave GLD.

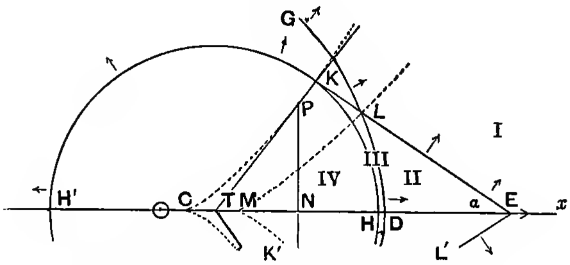

Fig. 20. For $T=7c/4f$, and $t=49c/8f$.

As before HKH' is the wave (T), and $KLEL'$ is the semiconical envelope sheet belonging to the steady motion.

The intersection of the wave (T) with the semicone is a circle through K, which travels along the cone KTK'; the intersection of the initial wave GLD with the semicone is a circle through L, which travels along a surface of revolution generated by the parabolic arc ML.

40. It would lead us too far to examine in detail all the cases which may occur; in order to understand the changes that take place in the field it will be sufficient if we study the progress of events in one particular case, for instance, for a fieldpoint lying within, in front of, the surface of revolution LML'. For this purpose we use the characteristic curve as a guide; it is represented in Fig. 21.

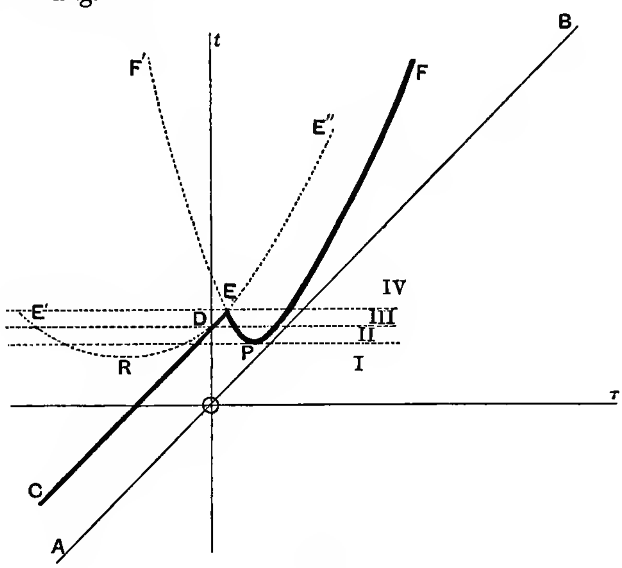

Fig. 21.

Just as in the case of Fig. 18 the characteristic curve consists of three sections.

The first section, CD, is the straight line $t = \tau + r/c$, which is due to the initial electrostatic field.

The second section, DE, is an arc of the curve $E'RDEE''$, of the type of Fig. 14, § 29, due to the accelerated motion.

The third section, EPF, is an arc of the hyperbolic curve, $F'EPF$, of the type of Fig. 9, § 22, due to the uniform motion. Its minimum, P, as before corresponds to the time when the conical envelope sheet KEK' passes the fieldpoint.

41. A consideration of Figs. 20 and 21 leads to the following conclusions:

(I) So long as the conical envelope sheet KEK' has not yet reached the fieldpoint, the horizontal line t cuts the characteristic curve below P, on the section CD. The field is electrostatic, and the potentials are given by (74), § 37. The point is in region (I).

(II) As soon as the envelope sheet has passed across the fieldpoint, it is disturbed simultaneously by two waves emitted during the uniform motion. The horizontal line t cuts the characteristic curve three times, namely the straight line CD once, and the hyperbola EPF twice. The potentials consist of three terms; one due to the electrostatic field and given by (74), the others due to the uniform motion and given by (49), § 24. Hence we get

$$\left.\begin{aligned} \phi &= \frac{1}{r} + \frac{2}{\sqrt{\{(vt - x - \frac{1}{2}fT^2)^2 - (v^2/c^2 - 1)\,\varpi^2\}}} \\ a_x &= \frac{2v}{c\sqrt{\{(vt - x - \frac{1}{2}fT^2)^2 - (v^2/c^2 - 1)\,\varpi^2\}}} \end{aligned}\right\} \quad \text{............(78).}$$

(III) When the initial wave, GLD, reaches the fieldpoint, the electrostatic field disappears and is replaced by the disturbance due to the accelerated motion. The horizontal line t cuts the characteristic curve three times, namely the arc DE, due to the accelerated motion, once, and the hyperbola, EPF, due to the uniform motion, twice. The potentials again consist of three terms, and the only difference from the last case is that the electrostatic term is replaced by a term due to the wave (τ_3), which belongs to the ascending arc $RDEE''$, and is got by means of (62), § 31. We find

$$\left.\begin{aligned} \phi &= \frac{8c}{f^2(\tau_1 - \tau_3)(\tau_2 - \tau_3)(\tau_3 - \tau_4)} + \frac{2}{\sqrt{\{(vt - x - \frac{1}{2}fT^2)^2 - (v^2/c^2 - 1)\,\varpi^2\}}} \\ a_x &= \frac{8\tau_3}{f(\tau_1 - \tau_3)(\tau_2 - \tau_3)(\tau_3 - \tau_4)} + \frac{2v}{c\sqrt{\{(vt - x - \frac{1}{2}fT^2)^2 - (v^2/c^2 - 1)\,\varpi^2\}}} \end{aligned}\right\} \text{(79).}$$

(IV) When the spherical wave (T), $H'KHK'$, crosses the fieldpoint the disturbance due to the accelerated motion as well as one of the waves due to the uniform motion disappear, and only a single wave, due to the uniform motion, reaches the fieldpoint. The horizontal line t cuts the characteristic curve only once, on the ascending hyperbolic arc PF. The potentials have only one term each, due to a single wave, and these are obviously just one half of the values given in (49), § 24. Hence we get

$$\left.\begin{aligned}\phi &= \frac{1}{\sqrt{\{(vt - x - \frac{1}{2}fT^2)^2 - (v^2/c^2 - 1)\,\varpi^2\}}}\\ a_x &= \frac{v}{c\sqrt{\{(vt - x - \frac{1}{2}fT^2)^2 - (v^2/c^2 - 1)\,\varpi^2\}}}\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots(80).$$

42. General considerations. These results complete all that we shall give for the present problem; the other cases lead to analogous expressions, and the differences which occur only concern matters of detail. It only remains to draw attention to some important conclusions which follow from them. These problems, though in themselves very special ones, are typical of large classes of problems, and therefore difficulties which arise in them may be expected to occur generally. The reason why they have been selected for treatment is because they appear to be the only cases which can be treated by comparatively simple methods; others require the solution of algebraic equations of higher degree than the fourth, or lead to transcendental equations.

Attention must be called to a discrepancy between the results of problems 1 and 3 for the case of a velocity greater than that of light.

The usual treatment of the problem of uniform motion due to Heaviside, Descoudres, Sommerfeld and others is that adopted in problem 1. When the velocity exceeds that of light it leads to the expressions (49), § 24.

These expressions hold inside the conical envelope sheet, that is, inside the shadow of the motion, to use Sommerfeld's term. Outside the potentials are zero; at its surface they are infinite (for a point charge).

The result of problem 3 is however quite different. Inside the shadow of motion, at some distance behind the charge, in region (IV), we have the expressions (80), § 41.

The coordinates must be changed for purposes of comparison with problem 1, so that the new origin is the point T, where the uniform motion just begins, and the time is reckoned from the instant at which it begins.

Expressions (80) are just one half of (49), because in the present case only *one* wave is effective, whilst in problem 1 there are *two*.

Inside the shadow of motion, just behind the charge, in region (III), we have the expressions (79), § 41.

These differ from (49) by the addition of the first term, due to the accelerated motion; thus here there are *three* effective waves in place of *two*.

Outside the shadow of motion, in the region (I), we have the expressions (74), § 37.

Thus the field is electrostatic, not zero as in problem 1.

At the surface separating regions (III) and (IV), which is the limiting spherical wave emitted at the instant when the uniform motion begins, we pass from (79) to (80); the potentials, though finite, are discontinuous.

At the surface separating regions (I) and (II), which is the conical sheet of the wave-envelope, we pass from (74) to (78); here the potentials are infinite.

At the surface separating regions (II) and (III) the potentials are finite and continuous.

The first discontinuity is absent from problem 1; the second is common to both problems.

At first sight the discrepancy between the two sets of results would appear to be due to errors of calculation, but a very little examination shows that it is fundamental, and that the two problems are essentially different in kind.

In problem 1 we have assumed that the motion has been going on in precisely the same way for ever, and that there never has been any electrostatic field at all. It is therefore not surprising that outside the shadow of the motion there should be no field.

When however we ask ourselves what is the effect of the initial conditions, we implicitly assume that the motion has started from rest. (In cases where the motion has started from some previously existing motion of a different kind similar considerations obviously apply.) Hence we are dealing with a new problem altogether, in fact problem 3. We cannot without further enquiry apply the results obtained for problem 1 to this new problem, but must begin the investigation *ab initio*.

The result of the investigation proves that the initial circumstances of the motion do have a determining influence for all future time, in the sense that there is a fundamental difference between aperiodic motions which have been going on for ever and which necessarily take place in unclosed orbits extending both ways to infinity, and motions which have started from rest. The latter are necessarily discontinuous, in the sense that the *form* of the analytical expressions for the coordinates of the moving charge as functions of the time is not the same for all time. The former may be continuous.

[I have shown elsewhere* that the discontinuity of form has no effect on the expressions for the transverse and longitudinal masses of an electron calculated from the field due to its motion, and that its effect is confined to the terms of higher order in the reaction due to the radiation.]

* Schott, *Ann. der Phys.* 25, p. 63, 1908.

43. Problem 4. An electron moves in a uniform electric field in a direction parallel to the lines of force. Required the resulting electromagnetic field.

This problem is of interest for two reasons: apart from the case of uniform rectilinear motion, it is the only case which has been solved in finite terms; moreover the motion is almost exactly realisable.

We make two assumptions:

(1) The mass of the electron varies according to the Lorentz mass formula, that is, $m_v = cm/\sqrt{(c^2 - v^2)}$, where m is the mass for zero velocity, and m_v that for velocity v. This means that the momentum is equal to $cm\,.\,\mathbf{v}/\sqrt{(c^2 - v^2)}$, and changes in accordance with Newton's Second Law.

(2) In calculating the mechanical force acting on the electron we neglect the reaction due to radiation, that is, we assume that the motion is quasi-stationary. Our solution nevertheless holds so long as the velocity differs appreciably from the velocity of light, in fact far beyond the range of velocity actually realised with β-particles.

We shall treat the electron as a point charge, and as before calculate the potentials per unit charge.

44. We choose the line of motion as x-axis of cylindrical coordinates as before. With the assumptions made the motion of the charge is given by the equations*

$$\left.\begin{array}{ll} & \left.\begin{array}{l}\xi = \sqrt{(k^2 + c^2\tau^2)} \\ v = c^2\tau/\sqrt{(k^2 + c^2\tau^2)}\end{array}\right\} \text{ from } \tau = -\infty \text{ to } \tau = +\infty \\ \text{where} & k = c^2 m/eX \end{array}\right\} \qquad (81).$$

X is the electric force, and e the charge, both measured in electrostatic units.

When $\tau = 0$ we have $\xi = k$, $v = 0$. The charge starts from positive infinity with the velocity of light, moves against the field, is brought to rest by it, and moves away again to positive infinity, ultimately acquiring the velocity of light. Actually, on account of radiation, it would be brought to rest somewhat sooner, and in the reversed motion would not regain its initial velocity. Of course only a portion of the motion can be realised, so that any real motion is necessarily discontinuous; but the method of the last problem enables us to extend our results to any case of this kind, e.g. the case of problem 5 below.

* See the problem, Ch. XI, § 153 below.

From (81) we deduce the following equations of our problem:

$$\left.\begin{aligned} R &= \sqrt{\{(\sqrt{(k^2+c^2\tau^2)}-x)^2+\varpi^2\}} \\ t &= \tau + R/c \\ K \equiv \frac{\partial t}{\partial \tau} &= 1 + \frac{\{\sqrt{(k^2+c^2\tau^2)}-x\}\, c\tau}{R\sqrt{(k^2+c^2\tau^2)}} \\ KR &= ct - \frac{c\tau x}{\sqrt{(k^2+c^2\tau^2)}} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots(82).$$

Rationalizing the second equation and using the first we get a quadratic for τ, which gives

$$\left.\begin{aligned} c\tau &= \{ct\,(k^2+\varpi^2+x^2-c^2t^2) \pm xs\}/2\,(x^2-c^2t^2) \\ \sqrt{(k^2+c^2\tau^2)} &= \{x\,(k^2+\varpi^2+x^2-c^2t^2) \pm cts\}/2\,(x^2-c^2t^2) \\ R &= -\{ct\,(k^2+\varpi^2-x^2+c^2t^2) \pm xs\}/2\,(x^2-c^2t^2) \\ \text{where}\qquad s &= \sqrt{\{(k^2+\varpi^2+x^2-c^2t^2)^2-4k^2\,(x^2-c^2t^2)\}} \\ &= \sqrt{\{(k^2+\varpi^2-x^2+c^2t^2)^2+4\varpi^2\,(x^2-c^2t^2)\}} \\ &= \sqrt{\{(x^2+\varpi^2-c^2t^2-k^2)^2+4k^2\varpi^2\}} \end{aligned}\right\} \quad \ldots\ldots(83).$$

We must choose the sign of s, the square root in which we take as essentially positive, so that R and $\sqrt{(k^2+c^2\tau^2)}$ are both positive, the former because we must have τ less than t as before, the latter because the electron moves on the positive half of the x-axis.

From the last two equations we get at once

$$x^2s^2 - c^2t^2\,(k^2+\varpi^2-x^2+c^2t^2)^2 = (x^2-c^2t^2)\,\{(k^2+\varpi^2-x^2+c^2t^2)^2+4\varpi^2x^2\},$$

$$x^2\,(k^2+\varpi^2+x^2-c^2t^2)^2 - c^2t^2s^2 = (x^2-c^2t^2)\,\{(k^2+\varpi^2+x^2-c^2t^2)^2+4k^2c^2t^2\}.$$

Thus we see that xs is numerically greater (less) than $ct\,(k^2+\varpi^2-x^2+c^2t^2)$, and $x\,(k^2+\varpi^2+x^2-c^2t^2)$ is numerically greater (less) than cts, according as x is numerically greater (less) than ct.

In the first case the lower sign alone makes R positive, and both signs make $\sqrt{(k^2+c^2\tau^2)}$ positive (negative), according as x is positive (negative).

In the second case the lower sign alone makes $\sqrt{(k^2+c^2\tau^2)}$ positive, and both signs make R positive (negative), according as t is positive (negative).

Thus we must use the lower sign alone in every case.

Moreover the disturbance is confined to the space on the positive side of the plane $x+ct=0$.

45. [**Characteristic curve.** These conclusions can be verified from the characteristic curve, which is of the type shown in Fig. 22. It has two asymptotes, namely the lines $t = -x/c$, and $t = 2\tau - x/c$, and is everywhere ascending. The broken line as before is the curve for R negative.]

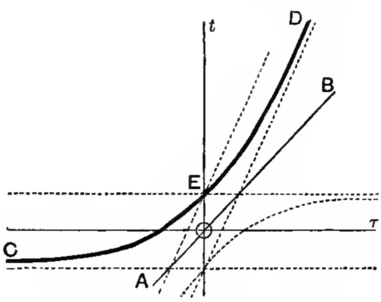

Fig. 22. For $x = 2k$, $\varpi = 2k$.

46. Potentials. The potentials are given by (26) and (27), § 13, where in accordance with the above we must substitute from (83) using the negative sign.

We get by (82), § 44,

$$\frac{v}{c} = \frac{c\tau}{\sqrt{(k^2 + c^2\tau^2)}} = \frac{ct\,(k^2 + \varpi^2 + x^2 - c^2t^2) - xs}{x\,(k^2 + \varpi^2 + x^2 - c^2t^2) - cts};$$

$$KR = ct - \frac{vx}{c} = \frac{(x^2 - c^2t^2)\,s}{x\,(k^2 + \varpi^2 + x^2 - c^2t^2) - cts}.$$

KR is always positive, as it should be. We get

$$\left.\begin{aligned} \phi &= \frac{x\,(k^2 + \varpi^2 + x^2 - c^2t^2)}{(x^2 - c^2t^2)\,s} - \frac{ct}{x^2 - c^2t^2} \\ a_x &= \frac{ct\,(k^2 + \varpi^2 + x^2 - c^2t^2)}{(x^2 - c^2t^2)\,s} - \frac{x}{x^2 - c^2t^2} \\ a_y &= a_z = 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (84).$$

The values of ϕ and a_x are apparently infinite when $x^2 = c^2t^2$, but this is really only the case for $x + ct = 0$.

For instance, ϕ may be written in the form

$$\{(k^2 + \varpi^2 + x^2 - c^2t^2)^2 + 4k^2c^2t^2\}/s\,\{x\,(k^2 + \varpi^2 + x^2 - c^2t^2) + cts\},$$

and this reduces to

$$\{(k^2 + \varpi^2)^2 + 4k^2c^2t^2\}/(x + ct)\,(k^2 + \varpi^2)^2$$

when $x^2 = c^2t^2$; in the same way a_x reduces to

$$\{4k^2c^2t^2 - (k^2 + \varpi^2)^2\}/(x + ct)\,(k^2 + \varpi^2)^2.$$

Thus both ϕ and a_x become infinite at the plane $x + ct = 0$, which, as we have seen, bounds the field on the negative side, but they remain finite at the plane $x - ct = 0$.

Since the characteristic curve CED, Fig. 22 (§ 45), has the line $x + ct = 0$ for an asymptote at $\tau = -\infty$, it is evident that the boundary is a plane of concentration of all the disturbances emitted at times infinitely long past, and thus does not exist in any realisable problem. The infinity there is due to the supposed concentration of a finite charge at a point, and the boundary really represents a sheet of relatively intense disturbance, the thickness of which is of the order of the diameter of the electron, somewhat like the conical envelope sheet due to motion with velocity greater than that of light (§ 17).

At infinity the potentials become infinitely small of the order $1/x$, for s obviously becomes infinitely great of order r^2.

Near the charge, and also near its image in the plane $x = 0$, they again become infinitely great. In fact, write $x = \pm \xi + \rho \cos \theta$, $\varpi = \rho \sin \theta$, where as before $\xi \equiv \sqrt{(k^2 + c^2t^2)}$ is the coordinate of the charge, and ρ is the distance of the fieldpoint from it. Neglecting higher powers of ρ we find by (83), § 44,

$$s = 2\rho \sqrt{(\xi^2 \cos^2 \theta + k^2 \sin^2 \theta)} = 2\rho\xi \sqrt{(1 - \beta^2 \sin^2 \theta)},$$

where $\beta = v/c$, so that $k = \xi \sqrt{(1 - \beta^2)}$ by (81). Hence

$$\phi = \pm \frac{1}{\rho \sqrt{(1 - \beta^2 \sin^2 \theta)}}, \quad a_x = \beta\phi.$$

Thus near the charge $(\xi, 0)$ the potentials are the same as those due to an equal charge moving with *uniform* velocity equal to the instantaneous velocity actually possessed by the charge, as was to be expected. In addition, near its image $(-\xi, 0)$ in the plane $x = 0$, they are the same as those due to an equal charge of opposite sign moving with the same uniform velocity. The image, however, is outside the field, beyond the boundary $x + ct = 0$.

The potentials become infinite of order $1/\rho$, and the equipotential surfaces are similar Heaviside ellipsoids.

47. Electric and magnetic forces. Substituting from (83) and (84) in (VII) and (VIII) (§ 3), we find

$$d_x = \frac{4k^2 (x^2 - \varpi^2 - \xi^2)}{s^3}, \quad d_\varpi = \frac{8k^2 x\varpi}{s^3}, \quad h_\phi = \frac{8k^2 ct\varpi}{s^3} \quad \text{......(85).}$$

The remaining force components vanish.

At the boundary $x + ct = 0$ we get

$$d_x = -\frac{4k^2}{(k^2 + \varpi^2)^2}, \quad d_\varpi = -h_\phi = \frac{8k^2 x\varpi}{(k^2 + \varpi^2)^3}.$$

Thus the forces, unlike the potentials, remain finite at the boundary, but are discontinuous there.

At infinity the electric and magnetic forces become infinitely small, of the orders $1/r^4$ and $1/r^5$ respectively, although the potentials only vanish to the order $1/x$. Hence the difficulty signalised in § 15 does not exist in this problem.

48. [Geometrical representation of the field. The forces can be expressed very simply by means of ring coordinates.

Write $$x = \frac{\xi \sinh \psi}{\cosh \psi - \cos \chi}, \quad \varpi = \frac{\xi \sin \chi}{\cosh \psi - \cos \chi},$$

where as before ξ is the distance of the charge from the origin at time t, so that by (81), § 44,

$$\xi = \sqrt{(k^2 + c^2t^2)}, \quad \beta = v/c = ct/\xi, \quad k = \xi\sqrt{(1-\beta^2)}.$$

The circles χ = constant form a pencil of circles through the charge and its image in the equatorial plane $x = 0$, and the circles ψ = constant form the family of orthogonal circles with the charge and its image for limiting points. The part of the axis between these points is given by $\chi = \pi$, the remainder by $\chi = 0$; the equatorial plane is given by $\psi = 0$, and the charge and its image by $\psi = \pm\infty$ respectively.

Using (83) and (85) we easily deduce the following results:

(1) The electric and magnetic forces are given by

$$d = \frac{(1-\beta^2)(\cosh\psi - \cos\chi)^2}{\xi^2(1-\beta^2\sin^2\chi)^{3/2}}, \quad h = \beta\sin\chi \cdot d \quad \text{...........(86).}$$

Near the charge, and near its image, these reduce to the well-known Heaviside expressions for positive, and negative, unit charges respectively, both moving with the instantaneous velocity v in the positive direction.

(2) The lines of electric force are the circles χ = constant.

The lines of magnetic force are the parallels of latitude.

The electric force acts from the charge to the image.

The magnetic force acts right-handedly with respect to the simultaneous velocity of the charge.

(3) Let the electric induction across the segment of the sphere ψ = constant, from the positive x-axis up to the parallel of latitude χ = constant, be I; and let the magnetic induction across the part of the meridian plane, which lies between the circles (χ) and (ψ), and the axes $\chi = 0$, $\psi = 0$, be J. Then we get

$$I = 2\pi\left\{1 - \frac{\cos\chi}{\sqrt{(1-\beta^2\sin^2\chi)}}\right\}, \quad J = \frac{\beta I\psi}{2\pi}.$$

The total electric induction is given by $\chi = \pi$, and is equal to 4π, as it ought to be for unit charge.

The total magnetic induction is infinite, owing to the supposed, but not realisable, concentration of a finite charge at a point; practically the whole is contributed by the region immediately surrounding the charge, where $\psi = \infty$. The image lies beyond the boundary, $x + ct = 0$, that is, it is virtual and contributes nothing.

(4) The Poynting vector is given by

$$s = \frac{c\beta(1-\beta^2)^2(\cosh\psi - \cos\chi)^4\sin\chi}{4\pi\xi^4(1-\beta^2\sin^2\chi)^3}.$$

The lines of flow of energy are the circles ψ = constant, for they are perpendicular both to the lines of electric and to those of magnetic force.

The flow of energy vanishes at the axis of x; it is away from it between the charge and its image, and towards it beyond them, when the applied electric force aids the motion, but is reversed when it opposes the motion.

Fig. 23 is a meridian section for the case where the velocity is four-fifths of that of light and positive, that is in the direction of the applied electric force.

E is the position of the charge at time t, and F is its image in $x = 0$. Thus $OE = OF = \xi$.

A is the turning-point, so that $OA = k$.

BD is the trace of the boundary, so that $OB = ct$.

$EPCDF$ is the line of electric force through P, so that $\chi = \angle EPF$.

GPJ is the line of flow of energy through P.

K and H are the centres of these circles, so that

$$OK = \xi \cot \chi, \quad KE = \xi \operatorname{cosec} \chi, \quad OH = \xi \coth \psi, \quad GH = \xi \operatorname{cosech} \psi.$$

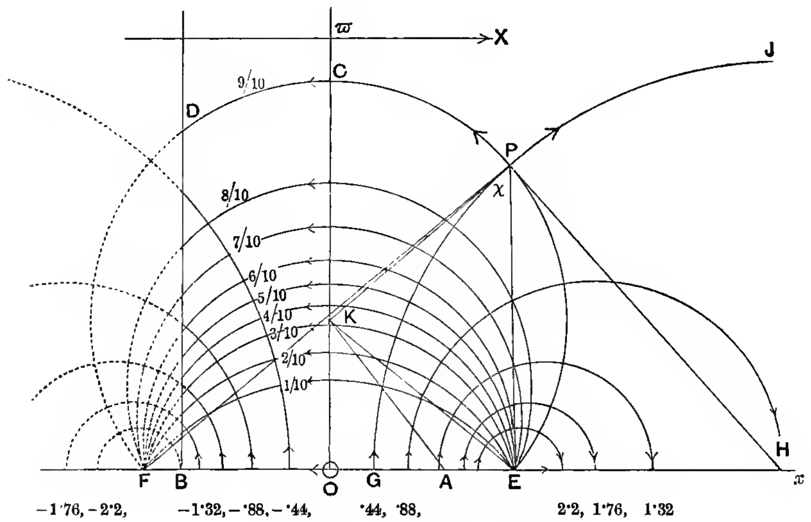

Fig. 23. $ct = \frac{4}{3}k$, $\xi = \frac{5}{3}k$, $\beta = \frac{4}{5}$.

It is easily proved that $I = 2\pi\,(1 - OK/AK)$.

With the aid of this relation the lines of force have been constructed for the values $I = 4\pi$ times 1/10, 2/10, 3/10, ... and $\psi = 0{\cdot}44$ times 1, 2, 3, 4, 5.

Thus the electric induction through each of the 10 zones generated by the revolution of the lines enclosed between consecutive electric lines of force (χ) is the same, namely $4\pi/10$. And the magnetic induction through each of the curvilinear quadrilaterals is equal to $\beta \times 0{\cdot}088$, the two smallest semi-circular areas surrounding E and F being excluded. The values of ψ are chosen so that one of the circles bounding these areas, and given by $\psi = -2{\cdot}20$, just touches the boundary BD.

We have generally $\psi = \log_e (FP/PE)$, so that $OG = \xi \tanh (\psi/2)$. Hence as the circle (ψ) approaches E, the magnetic induction through the enclosed area becomes logarithmically infinite. This is due to the supposed, but physically unrealisable, concentration of unit charge at the point E. Thus we must exclude a very small space around E from consideration; this excluded space represents the volume occupied by the electron.

In order to secure agreement with our fundamental assumptions we must suppose the surface of the electron to be an oblate spheroid of revolution about Ox with its centre at E, its eccentricity equal to β, and its equatorial radius equal to a constant, say a. At this spheroid we have

$$\cosh \psi = \xi \sqrt{(1 - \beta^2 \sin^2 \chi)}/a \sqrt{(1 - \beta^2)},$$

to the lowest order in the small quantity a/ξ. The x-momentum per unit volume is s_x/c^2; hence we find by means of (86) that the x-momentum contained in the part of the field, which lies between the spheroid and the boundary $x + ct = 0$, is, to the lowest order, equal to $2\beta/3ca \sqrt{(1 - \beta^2)}$, which is the value given by the Lorentz mass formula for unit surface charge, as it ought to be.]

49. The electric and magnetic forces are discontinuous at the boundary $x + ct = 0$, for they are finite inside and vanish outside. No difficulty arises on this account, for in the first place the boundary is really a layer of transition, of thickness comparable with the radius of the electron, in which the forces vary rapidly, but continuously, from their zero values outside to the values given by (86) inside. Secondly, this boundary is never realised; it requires the motion of the electron to have continued for an infinitely long time according to the same law, whilst the necessary uniform external electric field can in reality only be of finite duration.

Thus it becomes necessary to consider cases of discontinuous motion; for instance, the case where the uniform field is confined to the space between two planes perpendicular to the x-axis, and the electron is projected along the axis with a given initial velocity.

50. [Problem 5. An electron, moving with uniform velocity in a straight line, is subjected to a parallel uniform electric force for a certain interval. Required to find the electromagnetic field due to the motion.

After what has been said above we shall content ourselves with a very brief statement as to the generation of the pulse, and shall then give an account of its ultimate nature; for it is this alone which is of importance for the theory of X- and γ-rays.

In addition to the initial velocity and the strength of the applied electric field, we must be given either the final velocity generated, or the time during which the field is applied, or the space through which it extends, it is quite immaterial which; the equations (81), § 43, in any case determine the remaining quantities. To fix the ideas we shall suppose that the electron moves with uniform velocity v_1, until it reaches the point G $(g, 0)$ at time t_1. The uniform electric field X then commences to act, and accelerates the electron according to (81), until the point H $(h, 0)$ is reached at time t_2. Then the field ceases to act, and the electron continues to move with the acquired velocity v_2. The origins of space and time are supposed to be taken as in problem 4, and must be determined by means of (81).

The disturbance emitted during the initial and final uniform motions is determined by (47), § 23, but we must bear in mind that the origin is the position occupied by the charge at time t in the particular uniform motion under consideration.

For the initial motion the origin is a point E_1, of which the x-coordinate is given by $\xi_1 = g + v_1(t - t_1)$. Hence in (47) we must replace v by v_1, and $vt - x$ by $v_1 t - x + g - v_1 t_1$.

This disturbance is confined to a region (I), which is unlimited externally, but is bounded internally by a sphere (t_1), the position at time t of the wave emitted at time t_1. Its centre is G, and its radius is given by $R_1 = c(t - t_1)$. It cuts Ox in two points B and B', whose x-coordinates are $g \pm R_1$.

For the final motion the origin is E_2, for which $\xi_2 = h + v_2(t - t_2)$, so that in (47) we replace v by v_2, and $vt - x$ by $v_2 t - x + h - v_2 t_2$.

This disturbance is confined to a region (II), which is bounded externally by a sphere (t_2), the position at time t of the wave emitted at time t_2. Its centre is H, and its radius is given by $R_2 = c(t - t_2)$. It cuts Ox in two points C and C', whose x-coordinates are $h \pm R_2$.

The disturbance emitted during the variable motion is determined by (84)—(86) of problem 4. We use the same notation; in particular we denote the position at time t of the electron in the variable motion by E, its x-coordinate by ξ, and its velocity by v, where

$$\xi = \sqrt{(k^2 + c^2 t^2)}, \text{ and } v = c^2 t/\sqrt{(k^2 + c^2 t^2)}.$$

This disturbance is confined to a region (III)—the pulse—bounded by the spheres (t_1) and (t_2).

The electron coincides with E_1 for $t < t_1$, with E for $t_1 < t < t_2$, and with E_2 for $t > t_2$; we shall however find it necessary to use these points for values of t outside these intervals, in which case they may be regarded as virtual electrons.

51. We shall for convenience collect together below all the data and equations which we shall require later; they all follow readily from (81). We write

$$\left.\begin{aligned}
&v/c=\beta, \quad v_1/c=\beta_1, \quad v_2/c=\beta_2.\\
\text{Then}\quad &\xi\sqrt{(1-\beta^2)}=g\sqrt{(1-\beta_1^2)}=h\sqrt{(1-\beta_2^2)}=k\\
&t\sqrt{(1-\beta^2)}/\beta=t_1\sqrt{(1-\beta_1^2)}/\beta_1=t_2\sqrt{(1-\beta_2^2)}/\beta_2=k/c\\
&\beta=ct/\xi, \quad \beta_1=ct_1/g, \quad \beta_2=ct_2/h\\
&R_1=c(t-t_1)=\beta\xi-\beta_1 g, \quad R_2=c(t-t_2)=\beta\xi-\beta_2 h\\
&\xi_1=g+v_1(t-t_1)=g+\beta_1 R_1, \quad \xi_2=h+v_2(t-t_2)=h+\beta_2 R_2
\end{aligned}\right\}\ldots(87).$$

The time T, for which the field is applied, is given by

$$T=t_2-t_1=\frac{k}{c}\left\{\frac{\beta_2}{\sqrt{(1-\beta_2^2)}}-\frac{\beta_1}{\sqrt{(1-\beta_1^2)}}\right\}.$$

The distance between the points G and H, the space over which the applied field extends, is given by

$$L=h-g=k\left\{\frac{1}{\sqrt{(1-\beta_2^2)}}-\frac{1}{\sqrt{(1-\beta_1^2)}}\right\}.$$

Either T, or L, may be given in addition to v_1 in place of v_2.

The thickness of the pulse varies; at the positive pole it is BC (Fig. 24 below), which is given by

$$BC=g+R_1-h-R_2=(1-\beta_1)g-(1-\beta_2)h=\left(\sqrt{\frac{1-\beta_1}{1+\beta_1}}-\sqrt{\frac{1-\beta_2}{1+\beta_2}}\right)k.$$

At the negative pole it is $B'C'$, which is given by

$$B'C'=h-R_2-g+R_1=(1+\beta_2)h-(1+\beta_1)g=\left(\sqrt{\frac{1+\beta_2}{1-\beta_2}}-\sqrt{\frac{1+\beta_1}{1-\beta_1}}\right)k.$$

Which of these thicknesses is the greater depends on the signs of β_1 and β_2, but is of no great moment. It is important to note that both are independent of t, and of order k, in all practical cases at all events. Now we have, by (81), $k=c^2m/eX$; with $e/cm=1{\cdot}77\,.\,10^7$ we find $k=1700/X$. Thus with fields such as can be reached in the open air, k may amount to several cms.; but in the case of X-rays, diffraction experiments seem to show that k is at most of the order 10^{-8} cm. Hence in experiments on X-rays the thickness of the pulse is always exceedingly small compared with distances, such as R_1 and R_2, at which observations are made. In these cases it is the ultimate nature of the pulse that is important.

Fig. 24 illustrates the construction of the lines of force, the notation being that used above.

$PQRE_2$ is a line of electric force, consisting of three sections:

(1) The part QP of the radius vector E_1QP; (2) the arc QR of the circle (χ), $ERQF$, through E and its image F in the equatorial plane $x=0$, whose centre is at K $(0, \xi\cot\chi)$; and (3) the radius vector E_2R.

The flow of energy at Q is along QS_1 in region (I), and along KQS_3 in region (III); at R it is along RS_2 in region (II), and along KRS_3 in region (III), where QS_1 and RS_2 are respectively perpendicular to E_1Q and E_2R.

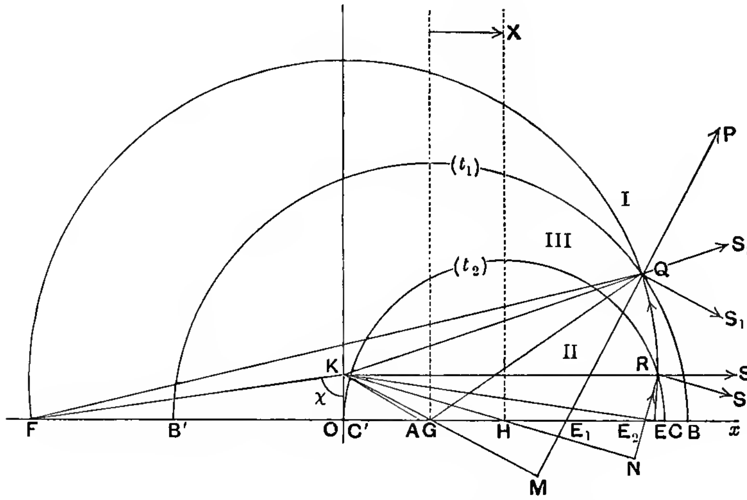

Fig. 24. For $e=k$, $\beta_1=\cdot 5$, $\beta_2=\cdot 886$, $ct=4k$.

It should be noticed that both the electric force and flow of energy are discontinuous at the spheres (t_1) and (t_2), and the same is true of the magnetic force.

It is proved below that KG is perpendicular to E_1Q, and KH to E_2R; these facts afford a simple construction for the electric line of force, when the part of it in any one of the three regions is given.

52. We shall find it convenient to transform our coordinates by referring everything to the spherical wave which passes through the fieldpoint P at the time t. This wave was emitted at some time t', when the electron was at a point E', and moving with velocity $\mathbf{v}'$ along $E'T$. Write $\mathbf{R}$ for the vector $\overline{E'P}$, θ for the angle it makes with $\overline{E'T}$ or $\mathbf{v}'$. Let $v'/c=\beta'$, and as usual write $K=1-\beta'\cos\theta$.

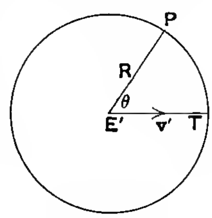

Fig. 25.

The point laws are

$$\phi=\frac{1}{KR},\quad \mathbf{a}=\frac{\mathbf{v}'}{cKR}.$$

They give by means of (VII) and (VIII), § 3, as in § 13, (32) and (33),

$$\mathbf{d}=-\frac{\dot{\mathbf{v}}'}{c^2K^2R}+\frac{\mathbf{R}_1-\mathbf{v}'/c}{K^3R^2}\left\{1-\beta'^2+\frac{(\dot{\mathbf{v}}'\mathbf{R})}{c^2}\right\},\quad \mathbf{h}=[\mathbf{R}_1\,.\,\mathbf{d}].$$

For the uniform motions we must put $\beta'=\beta_1$, or β_2, as the case may be, and $\dot{\mathbf{v}}'=0$.

For the variable motion we have $v' = c^2t'/\sqrt{(k^2 + c^2t'^2)}$, so that

$$\dot{v}'/c^2 = k^2/(k^2 + c^2t'^2)^{\frac{3}{2}} = (1 - \beta'^2)^{\frac{3}{2}}/k.$$

Of course in any case $\mathbf{v}'$ and $\dot{\mathbf{v}}'$ are along Ox. Hence we find for the

$$\left.\begin{array}{ll} \text{Uniform motions:} & d_r = \dfrac{1 - \beta'^2}{R^2(1 - \beta' \cos\theta)^2} \\ & d_\theta = h_\phi = \dfrac{\beta'(1 - \beta'^2)\sin\theta}{R^2(1 - \beta'\cos\theta)^3} \\ \text{Variable motion:} & d_r = \dfrac{1 - \beta'^2}{R^2(1 - \beta'\cos\theta)^2} \\ & d_\theta = h_\phi = \left(1 + \dfrac{\beta'\xi'}{R}\right) \dfrac{(1 - \beta'^2)^{\frac{3}{2}}\sin\theta}{kR(1 - \beta'\cos\theta)^3} \end{array}\right\} \quad \text{.........(88).}$$

In the last equation we have made use of (87), which give for the x-coordinate of E' the value $\xi' = k/\sqrt{(1 - \beta'^2)}$. They also give $R = \beta\xi - \beta'\xi'$, a relation which allows us to replace the factor $1 + \dfrac{\beta'\xi'}{R}$ in this equation by $\beta\xi/R$.

By help of (88) we get the following results:

(1) At the wave (t_1), where $\beta' = \beta_1$, we have

$$d_{r3} = d_{r1},$$
$$d_{\theta 3} : d_{\theta 1} = \beta\xi : \beta_1\xi_1 = \tan\alpha : \tan\alpha_1,$$
$$h_{\theta 3} : h_{\theta 1} = \beta\xi : \beta_1\xi_1 = \tan\alpha : \tan\alpha_1,$$

where the suffixes refer to the regions (I) and (III), and

$$\tan\alpha = \beta\xi/k = \beta/\sqrt{(1 - \beta^2)}, \quad \tan\alpha_1 = \beta_1/\sqrt{(1 - \beta_1^2)}.$$

Thus the normal electric force is continuous, but the tangential electric force and the magnetic force are discontinuous. The lines of electric induction are refracted according to the tangent law, as they would be from a dielectric of specific inductive capacity $\tan\alpha$ to one of capacity $\tan\alpha_1$.

(2) Further, we have, for the normal Poynting vector,

$$s_{3n} : s_{1n} = \tan^2\alpha : \tan^2\alpha_1.$$

This inequality implies no infraction of the Principle of the Conservation of Energy. For since the wave (t_1) is expanding with velocity c, region (I) gains energy at the rate s_{1n} per unit area of surface of the wave on account of the flow of energy, but loses it at the rate cE_1 per unit area on account of the expansion, where E_1 is the energy per unit volume of region (I) close to the wave (t_1). Using (88) we find for the total rate of gain of energy of region (I) per unit area of the wave (t_1) the value $s_{1n} - cE_1 = -cd_{r1}^2/8\pi$. In the same way we find the equal, but opposite, value $cd_{r3}^2/8\pi$ for the rate of gain of energy of region (III) per unit area of the wave (t_1). Similar results hold for the wave (t_2), so that the principle is satisfied for each surface element of each wave separately.

(3) If Q be the point (x, ϖ) the equation of the circle (χ) through it gives

$$2\varpi\xi \cot \chi = x^2 + \varpi^2 - \xi^2.$$

Also Fig. 24 gives

$$x^2 + \varpi^2 = OQ^2 = R_1^2 + g^2 + 2g(x - g);$$

and from (87) we get

$$\beta\xi = R_1 + \beta_1 g, \text{ and } \xi\sqrt{(1-\beta^2)} = g\sqrt{(1-\beta_1^2)},$$

whence

$$\xi^2 = R^2 + g^2 + 2g\beta_1 R_1.$$

Hence we get

$$x^2 + \varpi^2 - \xi^2 = 2g(x - g - \beta_1 R_1) = 2g(x - \xi_1),$$

and $\xi \cot \chi = g(x - \xi_1)/\varpi$, or $OK = OG \cot QE_1x = OG \cot GE_1M$.

Thus the triangles KOG and E_1MG are similar, and M is a right angle. Similarly, N is a right angle.

Hence the construction given above for the line of force has been verified.

(4) As time increases, ξ, R_1 and R_2 become very nearly equal and infinitely large compared with k, g, h, ξ_1 and ξ_2, while β approaches the limit unity. Hence for all points of the pulse d_θ ultimately is infinitely large compared with d_r, except just at the poles, and the electric force is ultimately tangential, though quite near to the poles it is radial, from the charge E, or its image F. The limiting value of the electric force is the same as that of the magnetic force and equal to $(1-\beta'^2)^{\frac{3}{2}} \sin\theta/kR(1-\beta'\cos\theta)^3$. We shall study its variations inside the pulse below.

53. The behaviour of the pulse in the several possible cases will be evident from Figs. 26—29. They are drawn for two typical cases, the first pair for the stoppage of a moderately fast cathode ray particle, the second for the starting of a very fast β-particle. The electric lines of force are drawn by the construction of Fig. 25, so as to correspond to 10 equal increments of induction, as in Fig. 23, § 48; the orthogonal lines of flow of energy are omitted, as they are of no particular interest, and in any case can be easily supplied.

Retarded motion.

In Fig. 26 we have

$$\xi_1 = 1\cdot005k, \quad \xi_2 = k, \quad \xi = 4\cdot12k, \quad \xi' = 0\cdot60k, \quad \xi'' = k.$$

E_2 and E'' coincide with A exactly, while E_1 is practically indistinguishable from it. The waves (t_1) and (t_2) are very nearly concentric, and the pulse is very nearly of uniform thickness.

Fig. 27 represents the same pulse after an infinite time, practically for any time greater than say $100k/c$. On the scale used A, E_1, E_2, and E'' are all indistinguishable from the origin, the pulse is infinitely thin, $OE' = -OB/10$, and E practically coincides with B, and F with B'.

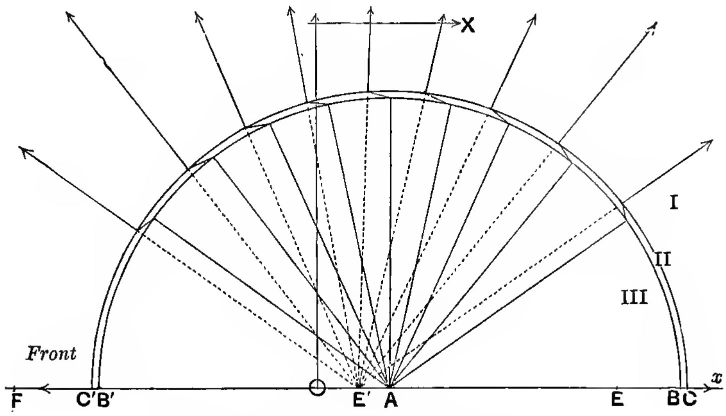

Fig. 26. For $\beta_1 = -\cdot 10$, $\beta_2 = 0$, $ct = 4k$.

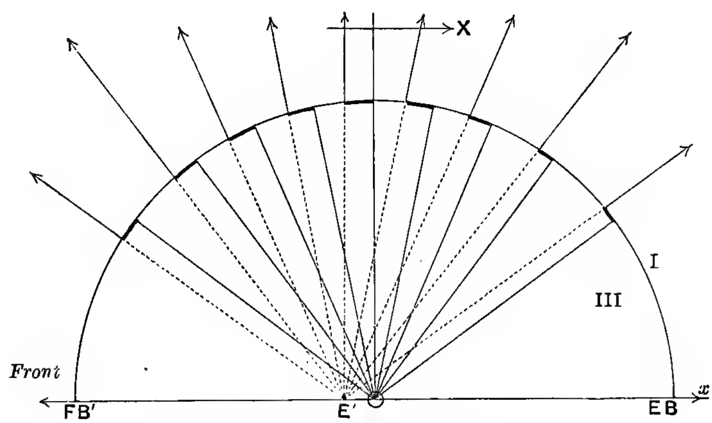

Fig. 27. For $\beta_1 = -\cdot 10$, $\beta_2 = 0$, $t = \infty$.

Accelerated motion.

In Fig. 28 we have

$$\xi_1 = k, \quad \xi_2 = 1{\cdot}67k, \quad \xi = 4{\cdot}12k, \quad \xi' = k, \quad \xi'' = 3{\cdot}80k.$$

E and E' coincide with A.

Fig. 29 represents the same pulse after an infinite time.

On the scale used A, E_1, E_2 and E' are all indistinguishable from the origin, the pulse is infinitely thin, $OE'' = 8OB/10$, and E practically coincides with B, and F with B'.

Owing to the overlapping of the lines of force in the pulse, some care is necessary in associating the corresponding lines in the regions (I) and (III). For instance, the second line in Fig. 29, corresponding to $I = 8\pi/10$, is $E''PQR$, and the curved portion PQ of this line overlaps the lines 3, 4, 5 and 6 in region (III).

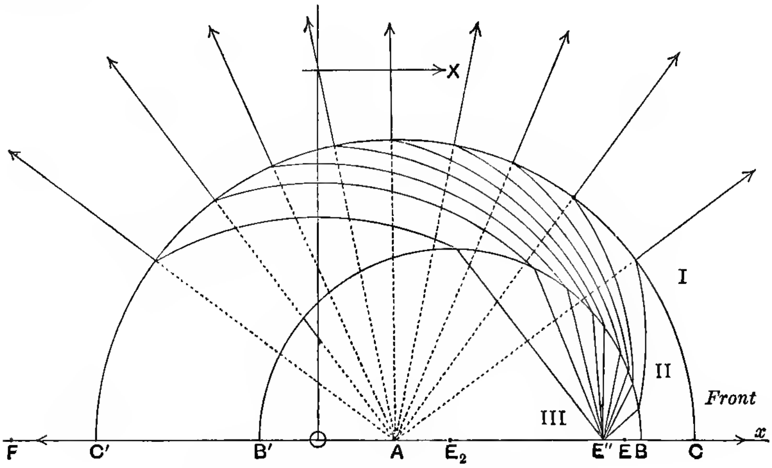

Fig. 28. For $\beta_1 = 0$, $\beta_2 = \cdot 80$, $ct = 4k$.

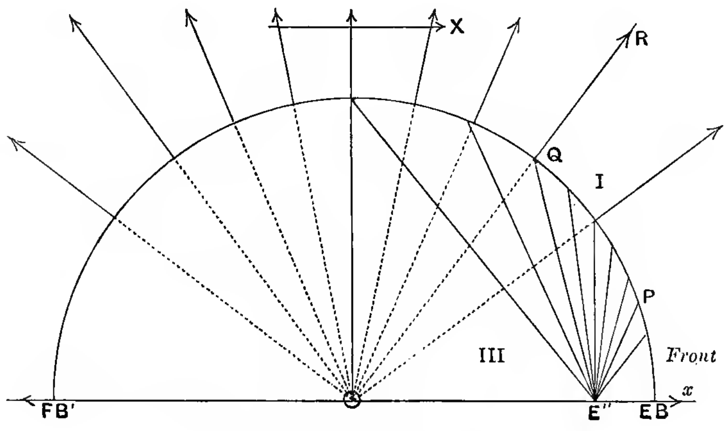

Fig. 29. For $\beta_1 = 0$, $\beta_2 = \cdot 80$, $t = \infty$.

More complicated cases, where neither the initial nor the final velocity vanishes, can be illustrated by a combination of the two types of diagram. For instance, if $\beta_1 = -\cdot 10$, $\beta_2 = \cdot 80$, we must superpose Fig. 26 on Fig. 28, and Fig. 27 on Fig. 29, so that region (I) in the first figure represents region (I), and region (III) in the second represents region (III) finally.

54. The ultimate nature of the pulse. Let E' be the position of the electron at the time t', so that $OE' = \xi'$. Let BPB' be the wave emitted at time t' in its position at the time t, and P a point on it, so that $\angle PE'x = \theta$, and

$$E'P = R = c(t - t') = ct - \beta'\xi',$$

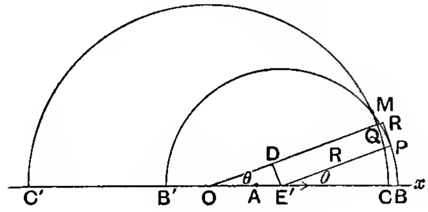

Fig. 30.

by (87). From O as centre describe a sphere CQC' of radius ct, draw OQ parallel to $E'P$, and PR perpendicular to it. Then $QR = \xi'(\cos\theta - \beta') = n$ say. Thus n is the distance of P from the sphere CQC' measured outwards along the radius OQ. By (87) we get

$$n = \frac{k(\cos\theta - \beta')}{\sqrt{(1 - \beta'^2)}} \quad \text{.........................(89).}$$

As time increases and the pulse expands, the distances QR and PR remain constant, while OQ and $E'P$ increase indefinitely. Hence ultimately the points P, Q and R may be regarded as coincident, and the radii OP and OQ as parallel and equal to R, or ct, indifferently, for all purposes for which we may neglect the quantity k/R. We know from experiments on the diffraction of X-rays that the thickness of the pulse is of the order 10^{-9} cm., and this is also the order of the length k, except in very particular cases. Thus our approximation is sufficient, even when R is as small as ten or a hundred times the radius of an atom.

From (88) and (89) we see that the radial electric force may be neglected, and the electric and magnetic forces regarded as equal and given by one or other of the following equivalent expressions:

$$d = h = \frac{e(1 - \beta'^2)^{\frac{3}{2}} \sin\theta}{kR(1 - \beta'\cos\theta)^3} = \frac{ek^2 \sin\theta}{R(n^2 + k^2\sin^2\theta)^{\frac{3}{2}}} \text{...........(90),}$$

where the charge of the generating electron is taken as e electrostatic units instead of unity as before.

The first expression shows how the force varies in different parts of the same wave, given by $\beta' =$ constant, the second how it varies along the same radius of the pulse, given by $\theta =$ constant.

The electric and magnetic forces are ultimately perpendicular to the radius vector, as well as to each other. The *electric force* is along the meridian, from the positive to the negative pole, that is, always in the reverse direction to the *acceleration* of the generating electron at the instant of emission of the wave. The *magnetic force* is along the parallel, always right-handed with reference to the *acceleration* of the generating electron at the instant of emission. Their directions are reversed only when the acceleration of the generating electron is reversed. Thus when a pulse is produced by a sudden change in the motion of an electron, the electric and magnetic forces have the same direction throughout the pulse.

The pulse is completely polarised at every point in a plane perpendicular to the meridian, except at the poles, where however the forces vanish.

55. **Small velocities.** When the velocities β' throughout the pulse are so small that their squares may be neglected, we find by (89) that the thickness of the pulse is uniform and given by $d = k(\beta_2 - \beta_1)$, where β_1 and β_2 correspond respectively to the outer and inner surfaces of the pulse. The electric force which must be applied to the electron to produce this pulse is, by (81), § 43, given by $X = c^2 m/ek = c^2 m(\beta_2 - \beta_1)/ed$. Taking $e/cm = 1{\cdot}77 \,.\, 10^7$, we get $X = 1700(\beta_2 - \beta_1)/d$. The force at the surface of an electron is of the order 10^{16}; taking this as an upper limit to the electric force X, we see that the least thickness is given by $d = 1{\cdot}7 \,.\, 10^{-13}(\beta_2 - \beta_1)$. Since β_2 and β_1 are supposed to be small compared with unity, this is well below the maximum, 10^{-9} cm., assigned by the diffraction of X-rays, even allowing for the probability that the force X is much less than the upper limit which we have assigned. When β_1 and β_2 are not small, the thickness of the pulse is variable, but its mean value is given by $d = k\left\{\frac{\beta_2}{\sqrt{(1-\beta_2^2)}} - \frac{\beta_1}{\sqrt{(1-\beta_1^2)}}\right\}$. Even when β_1 and β_2 are of opposite signs and as great as $\cdot 97$—the value given by Kaufmann for his fastest β-particles—d is only eight times k, and about 10^{-12} cm., still leaving a sufficiently wide margin between it and the maximum assigned by diffraction experiments.

As regards the distribution of force inside the pulse in the particular case of small velocities, we find, by (90), $d = h = e\sin\theta/kR$.

This law is the same as that in the wave emitted by a Hertzian vibrator, apart from the periodic factor present in that case. This was to be expected, seeing that the Hertzian vibrator is equivalent to an electron vibrating with very small velocity. The force is symmetrical, as regards magnitude, with respect to the equatorial plane, has a maximum there and vanishes at the poles. Its maximum, e/kR, is R/k times as great as the electrostatic force at the same distance R from an equal charge at rest. Owing to the smallness of k, it is enormously greater for all but extremely small distances, showing that the pulse is a region of exceptionally intense force.

56. **General case.** We see from (90) that as the velocity β' of a wave increases, the wave loses its symmetry with respect to the equatorial plane. The maximum of the force is given by $\cos\theta = \{\sqrt{(1 + 24\beta'^2)} - 1\}/4\beta'$ or, what is the same thing, by $\beta' = \cos\theta/(1 + 2\sin^2\theta)$, and its value is easily found to be $e(5 + 4\sin^2\theta)^{\frac{3}{2}}/27kR\sin^2\theta$. Thus θ is less, or greater, than $\frac{1}{2}\pi$, according as β' is positive, or negative, and the maximum increases as β' increases

numerically. For example, when $\beta' = \cdot 5$, the maximum is given by $\theta = 36°$, and its value is $1\cdot 8e/kR$; when $\beta' = \cdot 97$, it is given by $\theta = 6° \, 23'$, and its value is $20e/kR$. Hence as β' approaches unity, the maximum approaches very closely to either pole, and becomes very high and steep.

Since the extreme values of β' are the values β_1 and β_2, the greatest of all the maxima is found at one or other of the two surfaces of the pulse.

57. Variation of the force across the pulse. The second expression (90) shows that as we pass across the pulse along a radius vector $\theta =$ constant, the force increases to a maximum and then diminishes again. The value of the maximum is $e/kR \sin^2 \theta$, and it occurs when $n = 0$, that is, at the point Q on the circle CQC' of Fig. 30, § 54. The law of variation of the force is the same for every radius vector, when the normal distance n is expressed in terms of the length $k \sin \theta$ as unit. But we must remember that the pulse is limited by the values $n = n_1$ and $n = n_2$, given by putting $\beta' = \beta_1$ and $\beta' = \beta_2$ respectively in (89), so that only a part of the whole range of the second expression (90) is available for the pulse. When $\cos \theta = \beta_1$, the maximum of the force just falls on the outer surface, and when $\cos \theta = \beta_2$ on the inner. When θ lies between the two limits thus assigned, the maximum falls inside the pulse, otherwise outside. In the latter case the extreme values of the force are found at the two surfaces.

58. The energy of the pulse. The energy per unit area of a layer of thickness dn is equal to $(d^2 + h^2)\, dn/8\pi$. By (89) we find

$$dn = -k(1 - \beta' \cos \theta)\, d\beta'/(1 - \beta'^2)^{\frac{3}{2}};$$

hence by (90) we get for the energy density per unit area of the pulse

$$\left.\begin{aligned} E &= \frac{e^2 \sin^2 \theta}{4\pi k R^2} \int_{\beta_1}^{\beta_2} \frac{(1 - \beta'^2)^{\frac{3}{2}}\, d\beta'}{(1 - \beta' \cos \theta)^5} \\ &= \frac{e^2}{32\pi k R^2 \sin^3 \theta} \left\{3\phi - 2 \sin 2\phi + \tfrac{1}{4} \sin 4\phi\right\}_1^2 \\ \text{with} \qquad \tan \phi &= \frac{\sqrt{(1 - \beta'^2)} \sin \theta}{\cos \theta - \beta'} \end{aligned}\right\} \quad \ldots\ldots(91).$$

The indices denote that the values ϕ_1 and ϕ_2, given by $\beta' = \beta_1$ and $\beta' = \beta_2$, are to be put for ϕ in succession in the bracket, and the first result subtracted from the second.

When β' is small, $\tan \phi = \tan \theta (1 + \beta' \sec \theta)$, or $\phi = \theta + \beta' \sin \theta$. The value of the bracket then is $8(\beta_2 - \beta_1) \sin^5 \theta$, so that

$$E = (\beta_2 - \beta_1)\, e^2 \sin^2 \theta / 4\pi k R^2.$$

Thus for small velocities the energy is distributed in the same way as it is for a Hertzian vibrator, as was to be expected, symmetrically with respect to the equatorial plane, with a maximum there and with zeros at the poles.

As β' increases, the energy density behaves like the force. Its maximum approaches nearer to either pole, and increases in height and steepness as β' approaches unity.

The total energy of the pulse is found by multiplying E by $2\pi R^2 \sin\theta\, d\theta$, and integrating with respect to θ from 0 to π, most easily by performing the θ-integration first. In this way we find for the total energy,

$$V = \frac{2e^2}{3k}\left\{\frac{\beta'}{\sqrt{(1-\beta'^2)}}\right\}_1^2 \quad \text{.......................(92).}$$

This expression is easily got by direct integration of the well-known expression for the radiation from an electron moving with acceleration which is due to Liénard and Abraham.

Since the mean thickness of the pulse is given by $d = k\{\beta'/\sqrt{(1-\beta'^2)}\}_1^2$, (92) may also be written

$$V = \frac{2e^2}{3d}\left(\left\{\frac{\beta'}{\sqrt{(1-\beta'^2)}}\right\}_1^2\right)^2.$$

Taking $e = 4{\cdot}7\,.\,10^{-10}$, $d = 10^{-9}$ cm., $\beta_1 = -{\cdot}97$, and $\beta_2 = +{\cdot}97$ as extreme values, we find that $V = {\cdot}94\,.\,10^{-8}$ erg, which may be regarded as an upper limit for the thickness assumed.

The increase in the kinetic energy of the generating electron according to the Lorentz formula is given by

$$K = c^2 m\left\{\frac{1}{\sqrt{(1-\beta'^2)}}\right\}_1^2.$$

With $m = 8{\cdot}9\,.\,10^{-28}$ gram., and $\beta_1 = 0$, $\beta_2 = {\cdot}97$, we get $K = 2{\cdot}4\,.\,10^{-6}$ erg. In this case $V = 2{\cdot}4\,.\,10^{-9}$ erg, and thus is one thousand times smaller.

When $\beta_1 = 0$ and β_2 is very small, the ratio V/K approaches the limit $4e^2/3c^2md$; for $d = 10^{-9}$ cm. its value is 1/2700.

Hence for a pulse as thick as 10^{-9} cm., our original assumption, that the reaction due to radiation could be neglected in comparison with the force generating the motion of the electron, is amply justified even for the fastest β-particles known. But it is clear that if the pulse were much thinner the reaction would have to be taken into account.

59. The strength of the pulse. Both the force and the energy of the pulse involve the constant k, or if we prefer, the mean thickness d, as well as the terminal velocities β_1 and β_2, and to this extent depend on the field which generated the pulse. For many purposes, however, what is required is not so much the force, nor the energy, but the integral of the force across the pulse, and this we shall find to be independent of k, or d. Remembering that by (89) $dn = -k(1-\beta'\cos\theta)\,d\beta'/(1-\beta'^2)^{\frac{3}{2}}$, we get by (90)

$$I = \int d\,.\,dn = \frac{e\sin\theta}{R}\int_{\beta_1}^{\beta_2}\frac{d\beta'}{(1-\beta'\cos\theta)^2} = \frac{e}{R}\left\{\frac{\beta'\sin\theta}{1-\beta'\cos\theta}\right\}_1^2 \quad \text{...(93).}$$

We may call I the strength of the pulse at the particular place.

As the pulse expands and passes across a fixed point the time-integral of the force there is obviously equal to I/c.

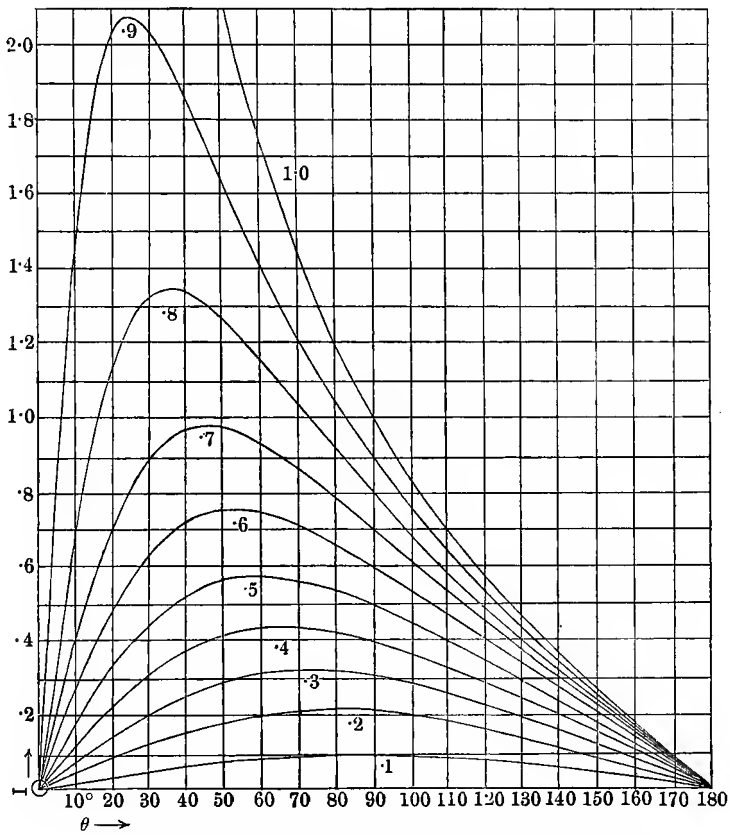

Fig. 31. Variation of time-integral of field in the pulse.
The numbers attached to the curves are the values of the velocity β'.

Fig. 31 gives curves for I as a function of θ for a pulse which is bounded by waves β' and $\beta' = 0$. If the pulse be bounded by β_1 and β_2, we need only take the algebraical difference of the two corresponding curves $\beta' = \beta_1$ and $\beta' = \beta_2$.

The curves are given for $\beta' = \cdot 1, \cdot 2, \ldots, \cdot 9, 1\cdot 0$; those for negative values are got by putting $\pi - \theta$ in place of θ. The curve for $\beta' = 0$ is the horizontal axis, that for $\beta' = 1$ is $I = \cot \tfrac{1}{2}\theta$. The unit throughout is e/R.

The maximum is given by $\cos \theta = \beta'$, and its value by

$$I = \cot \theta = \beta'/\sqrt{(1 - \beta'^2)}.$$

When β' is very small the maximum lies on the equator; as β' increases it approaches either pole, and at the same time increases in height and steepness. It always lies at M, the parallel in which the particular wave (β') cuts the sphere CMC' of Fig. 30, § 54.

An important result follows from the fact that I does not involve the constant k, that is, does not directly involve the strength of the generating field X, but only the initial and final velocities of the electron.

Let $\mathbf{D} = \int \mathbf{d} \,.\, dn$ and $\mathbf{H} = \int \mathbf{h} \,.\, dn$, so that $\mathbf{D}$ and $\mathbf{H}$ are the integrals across the pulse of the electric and magnetic forces regarded as vectors.

The vector $\mathbf{D}$ is perpendicular to the vector $\mathbf{R}_1$, a unit vector along the radius, and to $\mathbf{v}'$, the velocity of the electron, and lies in the meridian plane; the vector $\mathbf{H}$ is perpendicular to the meridian plane; and the magnitude of each vector is I. Hence we get from (93)

$$\left.\begin{aligned} \mathbf{H} &= \frac{e}{R}\left\{\frac{[\mathbf{v}' \,.\, \mathbf{R}_1]}{c - v'\cos\theta}\right\}_1^2 \\ \mathbf{D} &= \frac{e}{R}\left\{\frac{c\mathbf{R}_1 - \mathbf{v}'}{c - v'\cos\theta}\right\}_1^2 \end{aligned}\right\} \quad \text{.......................(94).}$$

Since these equations involve merely the terminal velocities and the angles they make with the radius vector $\mathbf{R}$, but do not involve the generating field directly, they may be applied to any pulse whatever, provided only that the distance through which the generating electron is moved in the process is negligible in comparison with R.

For any pulse, however complex, may be supposed to be broken up into a succession of elementary pulses, to each of which (94) may be applied in succession; the form of the expressions shows that the final terms alone appear in the result.

The expressions (94) could have been obtained by direct integration of the general expressions for the forces given on p. 23, and derived from (VII) and (VIII), but the present process is instructive, and will serve as a verification of the work.

60. Effect of the pulse. Let us suppose the pulse to pass across a second electron in static, or kinetic, equilibrium under the action of a given system of forces. We shall suppose the time of passage of the pulse to be so small that the effect on the given system of forces of the displacement and change of velocity of the electron due to the pulse may be neglected. This approximation suffices whenever the time of passage is small compared with the free periods of the electron.

On this assumption the increase in the momentum of the electron due to the pulse is given by

$$\Delta(m\mathbf{v}) = e\int \{\mathbf{d} + [\mathbf{v} \,.\, \mathbf{h}]/c\}\, dt = \frac{e}{c}\{\mathbf{D} + [\mathbf{v} \,.\, \mathbf{H}]/c\}.$$

It follows, by (94), that the increase in velocity, $\Delta\mathbf{v}$, is of the order e^2/cmR, which with the usual value for e^2/cm, and for $R=1$ cm., is ·008 cm./sec.

The maximum velocity of an electron, necessary to account for the radiation in a spectrum line, is of the order 10^6 cm./sec.; this follows readily from Wien's measurement of the energy radiated by a single canal ray ion in the blue line of hydrogen, namely 10^{-6} erg/sec., on the assumption that only one electron takes part in its production. If all the electrons of the hydrogen atom shared equally, which is a most improbable assumption, the velocity would still be as great as $2 . 10^4$ cm./sec.

The comparative smallness of the velocity produced by the pulse indicates that the ionisation produced by X- and γ-rays in stable substances like hydrogen is to be accounted for on the aether-pulse theory by the cumulative effect of a large number of successive pulses, and that something in the nature of resonance is involved in the process. The results of Barkla respecting the selective absorption of secondary rays support this view, but it would lead us too far to enquire into the matter here. Such an enquiry presupposes an investigation into the motion of systems of electrons, which can only be given later.

Note. Since the above was written, Sommerfeld (*Phys. Zeitsch.* 10, p. 969, 1909) has given a theory of aether pulses, which agrees with what has been given above.]

61. Problem 6. A point-charge moves in a circle with uniform velocity; to find the resulting field.

We shall content ourselves with investigating the general character of the field, for the calculation of the potentials requires the solution of a transcendental equation, which cannot be effected in finite terms.

Take the centre of the circle as origin and its axis as axis of x of a system of cylindrical coordinates of (x, ϖ, ϕ).

Let the coordinates of the charge be given by

$$\xi=\rho\cos\omega\tau,\quad \eta=\rho\sin\omega\tau,\quad \zeta=0,\quad \text{from}\quad \tau=-\infty \quad\text{to}\quad \tau=+\infty.$$

The radius of the orbit is ρ, the angular velocity ω, the period of revolution $2\pi/\omega$, the linear velocity of the charge $v\equiv\omega\rho$. As usual write $\beta=\omega/c$. We get at once

$$\left.\begin{aligned} R&=\sqrt{\{x^2+\varpi^2+\rho^2-2\varpi\rho\cos(\omega\tau-\phi)\}}\\ t&=\tau+\sqrt{\{x^2+\varpi^2+\rho^2-2\varpi\rho\cos(\omega\tau-\phi)\}}/c\\ K&\equiv\frac{\partial t}{\partial\tau}=1+\beta\varpi\sin(\omega\tau-\phi)/R\\ KR&=R+\beta\varpi\sin(\omega\tau-\phi)\end{aligned}\right\}\quad\text{......(95).}$$

It is convenient to write

$$\psi = \tfrac{1}{2}(\omega t - \phi + \pi), \quad \chi = \tfrac{1}{2}(\omega\tau - \phi + \pi),$$
$$f = \sqrt{\{x^2 + (\varpi + \rho)^2\}}, \quad g = \sqrt{\{x^2 + (\varpi - \rho)^2\}},$$
$$k = \sqrt{(f^2 - g^2)}/f, \quad k' = g/f, \quad \Delta = \sqrt{(1 - k^2 \sin^2\chi)}, \quad \gamma = \omega f/2c.$$

Thus f and g are respectively the greatest and least distances of the fieldpoint (x, ϖ, ϕ) from the orbit.

We get from (95)

$$R = \Delta, \quad K = 1 - \gamma k^2 \sin\chi \cos\chi/\Delta.$$

The multiple roots of (95_2) are given by $K = 0$, that is by

$$\gamma = \Delta/\sqrt{(1 - \Delta^2)(\Delta^2 - k'^2)}.$$

This equation gives on solution

$$\Delta = \tfrac{1}{2}\sqrt{\{(1 + k')^2 - 1/\gamma^2\}} \pm \tfrac{1}{2}\sqrt{\{(1 - k')^2 - 1/\gamma^2\}} \quad \text{.........(96).}$$

The roots are real when $\gamma > 1/(1 - k')$.

They are equal when $\gamma = 1/(1 - k')$.

They are imaginary when $\gamma < 1/(1 - k')$.

Since $\gamma = \omega f/2c$, the limiting case is given by

$$f - g = 2c/\omega = 2\rho/\beta \quad \text{..........................(97).}$$

This equation represents a hyperboloid of revolution of one sheet. Its focal line is the orbit, its real semi-axis is ρ/β, and its eccentricity is β. Thus the hyperboloid is real when, and only when, $\beta > 1$, that is, when the velocity of the charge is greater than that of light.

Inside the hyperboloid, that is, on the same side of it as the orbit, the roots of (96) are real, outside it they are imaginary.

62. The characteristic curve. In the present notation the characteristic equation (95_2) becomes

$$\psi = \chi + \gamma\Delta.$$

This equation is more convenient to deal with than the original one, because the coordinates of the fieldpoint enter into it only through f and g, that is, through γ and k.

The least value of Δ is k', and it occurs for $\chi = \tfrac{1}{2}\pi$. Its greatest value is 1, and it occurs for $\chi = 0$.

These values repeat whenever χ increases by π, so that ψ is a periodic function of χ of period π.

The chief types of the characteristic curve are the following:

(*a*) *Velocity less than that of light: $v < c$.*

The maxima of Δ, and of R, correspond to points such as D, and the minima to points such as C. At both sets of points the curve is parallel to AB.

Obviously, for every value of ψ, and also of t, there is but one value of χ, and therefore also of τ, that is, but one wave.

Since the curve only involves the two parameters $k' = g/f$, and $\gamma = \omega f/2c$, the same curve holds for all points on the anchor ring $g/f = \cdot 707$, provided that the appropriate value of ω be selected.

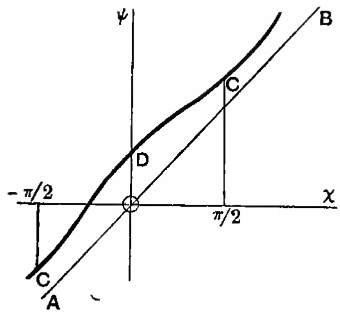

Fig. 32.
For $k = k' = \cdot 707$, and $\gamma = \cdot 785$.

The values of the potentials are

$$\phi = \frac{1}{\Delta - \gamma k^2 \sin\chi \cos\chi}, \quad \text{by (26), § 13,}$$

$$\mathbf{a} = \frac{\mathbf{v}}{c(\Delta - \gamma k^2 \sin\chi\cos\chi)}, \quad \text{by (27), § 13.}$$

The magnitude of the vector $\mathbf{v}/c$ is β; its direction is that of the motion of the charge at the time of emission τ, and makes with the initial line the angle $\omega\tau + \frac{1}{2}\pi$, that is, $2\chi + \phi - \frac{1}{2}\pi$.

(*b*) *Velocity greater than that of light: $v > c$.*

(1) As the velocity increases, while the fieldpoint remains fixed, γ increases while f, g, k and k' remain constant. In the present case γ may increase until it equals the quantity $1/(1 - k')$, that is, $\beta = 2\rho/(f - g)$. The hyperboloid (97) is real, because $\beta > 1$, and it expands until it passes through the given fieldpoint. So long however as the fieldpoint is still on its outside, that is, on the opposite side to the orbit, the roots of (96) are imaginary, which means that the characteristic curve is continually ascending, without maxima or minima, and of the type shown in Fig. 32. The potentials are given by (26) and (27) as before.

(2) When the velocity is so great that the fieldpoint lies inside the hyperboloid (97), the roots of (96) are real, and the characteristic curve has maxima (Q) and minima (P), infinite in number. For all values of ψ between KP and LQ there are three values of χ, and therefore three simultaneous waves. For all other values within the range from C to C, there is but one.

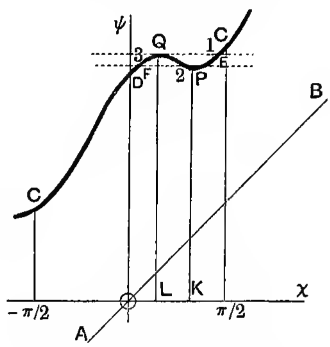

Fig. 33.
For $k = k' = \cdot 707$, and $\gamma = 3 \cdot 93$.

The potentials are given by (34) and (35), § 14,

$$\phi = \Sigma \frac{1}{\Delta - \gamma k^2 \sin\chi\cos\chi},$$

$$\mathbf{a} = \Sigma \frac{\mathbf{v}}{c(\Delta - \gamma k^2 \sin\chi\cos\chi)},$$

where the sum is to be taken for three values of χ, χ_1, χ_2 and χ_3, corresponding respectively to the arcs EP, PQ and QF in Fig. 33.

(3) As the velocity increases still further, it may happen that the maxima increase and the minima diminish so much that a maximum Q is not only above the next minimum P, but above the next but one as well. In this case horizontal lines can be drawn below Q, but above the second minimum, and these cut the characteristic curve five times and correspond to five simultaneous waves. The potentials then have five terms each, of the same form as before. In the same way, for still greater velocities, and suitable positions of the fieldpoint, we may have seven simultaneous waves, and so on. We shall not trouble about these more complicated cases, but merely note that the number of simultaneous waves is always odd.

63. [**Note. Geometrical representation.** The present problem affords an interesting example of the construction of p. 36 for finding the cusplocus and the envelope. In Fig. 34, E is the position of the electron at time t, and ET its line of motion. PEP' is the trace of the semi-cone $\cos\theta = c/v = 1/\beta$, and CF that of the plane $fR\cos\psi = v^2 - c^2$, that is,

$$R\cos\psi = (v^2 - c^2)/f = \rho(\beta^2 - 1)/\beta^2.$$

Thus $EF = \rho\sin^2\theta$, and $OF = \rho\cos^2\theta$.

Obviously CF and EP intersect at C, so that $OC = \rho\cos\theta$, and $\angle C = 90°$. Thus EP touches the circle BCD, of centre O and radius $\rho\cos\theta$.

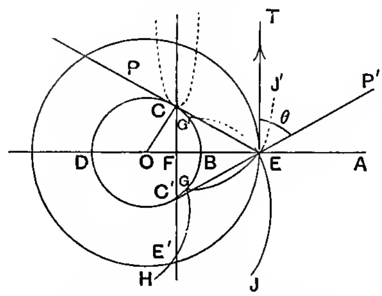

Fig. 34. For $\beta=2$, $\theta=60°$.

The cusplocus is generated by the intersection of the plane CF and semi-cone PEP', which is obviously a hyperbola, and is indicated by a dotted line. Its equation is easily found. For if the angle made by OE with a fixed radius be ωt as before, the equation of the plane CF is $\rho = \beta^2\varpi\cos(\omega t - \phi)$, and that of the semi-cone is $\sqrt{\{x^2 + \varpi^2 + \rho^2 - 2\varpi\rho\cos(\omega t - \phi)\}} = \beta\varpi\sin(\omega t - \phi)$. Hence eliminating t we find that the equation of the cusplocus is $x^2 + \varpi^2 + \rho^2 = \beta^2\varpi^2 + \rho^2/\beta^2$; this easily reduces to $f - g = 2\rho/\beta$, that is, to (97). Thus the cusplocus is the hyperboloid (97), as was to be expected.

64. **The wave-envelope.** The wave-envelope has the cusplocus and the orbit for the two sheets of its surface of centres. Its trace on the plane of the diagram is readily found, for it must be generated by the wrapping of an inextensible and flexible thread on the circle $BCDG$, which is the intersection of the cusplocus by the plane of the diagram. Thus it consists of arcs of involutes of this circle, namely the arc GE, got by wrapping CE on to

the arc CBG of the circle; the arc EJ, got by unwrapping $C'E$; and lastly, the arc GH, parallel to the last. At G there is a cusp, at E a conical point. The remaining parts of the arcs GE and JE, namely, the broken lines EJ' and EG', obviously belong to the reversed motion.

From Fig. 34 it is easy to find the velocity for which we can get five, seven,... simultaneous waves. Obviously we cannot get five waves, for a point in the plane of the diagram, until the traces GE' and GE of the wave-envelope intersect again between the two circles, that is, until E' and E coincide, and $\angle BOG = \pi$. It is easy to show that this occurs when $\tan \sqrt{(\beta^2 - 1)} = \sqrt{(\beta^2 - 1)}$, that is, for the values $\beta^2 = 1,\ 1 + (1{\cdot}43 . \pi)^2,\ 1 + (2{\cdot}45 . \pi)^2, \ldots$. These values correspond to three, five, seven, ... waves respectively.]

CHAPTER VI

REMARKS ON THE SOLUTIONS OBTAINED, AND ON THE METHODS OF CALCULATING THE POTENTIALS IN GENERAL

65. PROBLEMS concerning the field due to a moving charge may be grouped in three classes according to the nature of the relation between the time of emission of a wave of disturbance, τ, and its time of action at a given fieldpoint, t. This relation is illustrated by the characteristic curve for that fieldpoint as previously explained.

(1) The curve for the first class is *ascending* both when τ has an infinitely great negative and an infinitely great positive value. Fig. 8, § 22, for uniform motion with velocity less than that of light (problem 1 *a*), Figs. 18, § 36, and 21, § 40, for uniform motion with any velocity generated from rest with uniform acceleration (problem 3), and Fig. 22, § 45 (problem 4), show curves of this type.

Their characteristic is that a line $t =$ constant cuts the curve an odd number of times, so that any fieldpoint can be disturbed by an odd, but never by an even number of waves simultaneously. Ultimately there is but one such wave.

(2) The curve for the second class is *descending* for infinitely great negative values of τ, *ascending* for infinitely great positive values. Fig. 9, § 22, for uniform motion with velocity greater than that of light (problem 1 *b*), and Figs. 12, 13, 14 and 15, § 29, for uniformly retarded motion turning later into a uniformly accelerated one (problem 2), show curves of this type.

The line $t =$ constant cuts them an even number of times, or not at all. Any fieldpoint is disturbed by an even number of waves at once, or not at all. Ultimately there are two such waves.

(3) The third class includes periodic motions, such as uniform circular motion (problem 6). The curve is either continually *ascending* (velocity less than that of light), or alternately *ascending* and *descending*, but of the same character however great the value of τ may be, whether positive or negative. The line $t =$ constant cuts the curve an odd number of times; any fieldpoint is disturbed by an odd number of simultaneous waves.

It will now be understood why the results obtained for uniform motion with velocity greater than that of light in problem 3 show no agreement with those obtained here, as well as by all previous writers, for problem 1 *b*. The two problems are essentially different in kind, and rest on entirely different assumptions. Before we can identify two given motions, which for a given interval of time appear to be the same, we must enquire into their previous history; if this has been different, the electromagnetic field will generally be different. We cannot avoid the consideration of the initial conditions in a given problem merely by supposing the commencement of the motion to have been infinitely long past.

Our conclusions are in this respect in agreement with the criticism of Lindemann* on the work of Sommerfeld, so far as motions with velocities greater than that of light are concerned. But there seems to be no doubt as to the correctness of Sommerfeld's results for quasi-stationary motions with velocities less than that of light. This question has also been discussed from a different point of view by Schott† with the same result.

66. We must next draw attention to the difference in character of the field due to motions with velocities less and greater than that of light.

When the velocities are less than that of light the potentials are everywhere finite and continuous, and therefore the forces are everywhere finite, except at a point-charge, just as for the ordinary electrostatic potential and force.

In consequence the discrepancy just signalised for velocities greater than that of light between problems of the type of problems 1 and 2, and those of the type of problem 3, does not exist in this case.

Moreover, the existence of these discontinuities and infinities of the potentials has a bearing on the physical question whether charges moving with velocities greater than that of light really occur in nature.

It may be urged that this question is of no real importance, seeing that all phenomena hitherto observed can be explained without assuming the existence of such charges.

We must at once admit this, so far as our knowledge of the properties of free electric charges goes. Indeed the recent experiments of Bucherer‡ are decidedly in favour of Lorentz's formula for the mass of a β-particle, which makes the mass vary as $1/\sqrt{c^2 - v^2}$, and therefore leads to an imaginary expression for velocities greater than that of light. If this formula were universally true, velocities greater than that of light would be impossible.

The experiments however have only been made with β-particles having velocities considerably less than that of light; thus the extension of the

* *K. Bay. Akad.*, II. Kl., XXIII. Bd., II. Abt., 1907, p. 235.

† *Ann. der Phys.* 25, 1908, p. 63.

‡ *Phys. Zeitsch.* 1908, p. 755.

formula up to the velocity of light involves an extrapolation much beyond the range of experiment, and can only be justified by theoretical reasons.

The Lorentz mass formula is required by the so-called Lorentz-Einstein Relatiftheorie*, which has the great advantage of explaining phenomena of aberration as well as the null effect of the earth's motion on optical and electrical phenomena to every order of approximation.

Unfortunately however this theory takes no account of the loss of energy from an electric charge moving with acceleration; this loss is very small so long as the velocity remains small, but becomes very important when it approaches near to that of light.

In consequence the idea of mass itself becomes ill-defined for velocities closely approaching that of light; this has been proved conclusively by Sommerfeld†, precisely on the basis of the Lorentz theory. It is obviously difficult to reconcile this result with the results of the Relatiftheorie.

We must therefore be careful not to allow theoretical views such as these unduly to influence our notions as to the possibility, or otherwise, of velocities greater than that of light.

67. On the other hand there are a number of phenomena which we have hitherto been utterly unable to account for on any of the current mechanical theories, namely those afforded by spectrum series.

When we review the experimental evidence, quite apart from theoretical considerations of any kind, we can hardly avoid the conclusion that a series actually contains an infinite number of lines. It may fairly be said that this is the view of those best qualified to judge, namely the observers themselves.

But when we try to account for these series by means of finite systems of discrete electric charges we are at once met by the difficulty that the number of their degrees of freedom is finite.

This difficulty is however only apparent; it is not allowable to treat electric charges as bodies which obey the ordinary laws of mechanics, for they influence the surrounding aether and are indissolubly linked with it. If we treat the aether as a continuum, the system of aether and charges has an infinite number of degrees of freedom.

Accordingly it has been proved by Herglotz‡ and Sommerfeld§ that a spherical electric charge can execute an infinite number of free rotational, to and fro, oscillations about an axis, and therefore can emit a series of an infinite number of lines.

* Einstein, *Ann. der Phys.* 17, p. 892, 1905. Planck, *K. Preus. Akad.* XXIX. p. 542.

† *Gött. Nach.* 1905, p. 201.

‡ *Gött. Nach.* 1903, p. 375.

§ *Gött. Nach.* 1904, p. 431.

These series it is true are not like the spectrum series to be accounted for; in particular their wave-lengths are much too small, being of the order of the dimensions of the charge. But they show conclusively that when we do not limit ourselves to small velocities, and thus alter the form of the solution from the commencement, infinite series are possible. Whether motions of charges can be found, which shall give series like those known in nature, cannot be decided beforehand; a detailed investigation of the problem is necessary, and it can only be carried out on the assumption that velocities of all magnitudes are possible.

68. From this digression it appears that anything which may throw light on the question of the existence of charges moving with velocities greater than that of light is of importance. If such charges exist they must be looked for inside the atom.

We may expect evidence of their presence in two ways: when the atom is disturbed by outside influences it is no doubt set in vibration and its charges certainly emit waves, for no accelerated motion is possible without radiation to some extent. It is generally taken for granted that these waves constitute light. But so far as I am aware no vibrating system of *atomic* dimensions has yet been imagined, which can give a sufficient number of waves as long as those of the spectrum. Every mechanical or electrical system, made up of separate particles, such as the Saturnian systems of Nagaoka and J. J. Thomson, so far constructed, has most of its wave-lengths very little greater than the diameter of the atom. At present we cannot gain much information as to the motion of the charges inside the atom from spectrum observations.

But a second way suggests itself. When the disturbance becomes so large that the charges inside the atom acquire very great accelerations, aperiodic waves, or pulses, of electric and magnetic force are emitted. It is usual to regard these as constituting X-rays and γ-rays. We have seen in problems 3 and 5 that the duration of passage of the pulse across a field-point is of the order of the time that the acceleration lasts, and its thickness of the order of the distance moved through by the emitting charge while its velocity is changing.

In the case of a disintegrating atom the latter distance may be expected to be of atomic dimensions, let us say of order 10^{-8} cm.; the thickness of the pulse will be of the same order.

When a series of such pulses falls on a dispersing or diffracting apparatus, this apparatus acts as a harmonic analyser, and breaks up the disturbance into Fourier components. The fundamental, which usually predominates, has a wave-length comparable with the thickness of the pulse, and is therefore of the order 10^{-8} cm. All this is sufficiently well known; it is in agreement with the few experiments made regarding the diffraction of X-rays, for instance those of Haga and Wind*.

When the charge causing the pulse is moving with a velocity greater than that of light an additional effect may be expected, due to the discontinuity of the potential.

For an extended charge the surfaces of discontinuity, which occur only for a single element at a time, become shells of transition, whose thickness is of the same order as the linear dimension of the charge; for a β-particle this is of the order 10^{-13} cm.

The potentials in these shells vary rapidly and remain finite; the same is true of the electric and magnetic forces. Sommerfeld† finds that in the shell, which replaces the conical envelope sheet in problem 1, the potential is of the order $a^{-\frac{1}{2}}$, where a is the radius of the charge, supposed spherical. The electric and magnetic forces will be of order a^{-} . In the rest of the pulse they are of much smaller order. Thus the shells of transition are places of exceptionally intense forces, and constitute exceptionally thin and intense aether pulses.

When they fall on the analysing apparatus we therefore expect to get a fundamental wave of great intensity, but of exceptionally short wave-length, only of the order 10^{-13} cm.

Thus these pulses may be expected to have special properties, for instance, to be extremely penetrating. Their existence would prove conclusively the presence of charges in the atom possessing velocities greater than that of light, and would require us to very materially alter our views as to its constitution.

I am not acquainted with any experimental result, which cannot be explained without them, and the theoretical investigation of their motions would be extremely difficult.

69. We shall therefore during the remainder of this investigation restrict ourselves to the case of motions less than that of light, except when special mention is made to the contrary.

Hitherto we have mentioned two methods of calculating the potentials.

(1) The method of Sommerfeld depending on the direct evaluation of the integral expressions (21) and (22), § 11.

We have already seen that the use of these integrals offers difficulties on account of the variability of their upper limits, so that it is doubtful how far

* *Wied. Ann.* 1899, 68, p. 884. Also Sommerfeld, *Phys. Zeitsch.* 1901, p. 55. [B. Walter u. R. Pohl, *Ann. der Phys.* 1909, 29, p. 331.]

† *Amst. Proc.* 1904, p. 362.

differentiation is allowable. They have been the subject of the polemic between Sommerfeld and Lindemann already referred to on p. 21; this in itself is sufficient evidence that their evaluation is very difficult and intricate.

The method has only been used for a spherical charge; it is doubtful how far it can be adapted to other cases, for instance, the case of a Heaviside ellipsoid, that is, a spheroid of revolution with its short axis in the line of motion, and of eccentricity v/c.

On the other hand it is not limited by considerations regarding the magnitude of the velocity; it is equally applicable to velocities less and greater than that of light, but of course is more complicated for the latter.

Since however for our purposes it is of importance to have a method, which shall apply equally well whatever the arrangement of the charge may be, whether it applies to velocities greater than that of light or not, Sommerfeld's method is inconvenient.

(2) The method of the point potentials, due originally to Liénard and to Wiechert.

We have used this method hitherto in our illustrative problems. It is very convenient and suggestive as long as we do not require explicit and calculable expressions for the potentials, and only wish to get a general idea of the nature of the field due to a given motion.

Its main disadvantage is that the point potentials are explicit functions of the time of emission of the disturbing wave, τ, and not of its time of action, t. In order to express them explicitly as functions of t we must solve an equation, such as 28, § 13. This solution can only be effected in finite terms in exceptional cases; in the most interesting cases, like that of uniform circular motion (problem 6), the equation is transcendental, and a finite solution unobtainable.

Its solution can be effected in series by means of Lagrange's Theorem whenever there is but one root, that is to say in all problems of motion with a velocity less than that of light, and for some cases where the velocity is greater than that of light, but then only for restricted portions of space. Since for many purposes, such as the calculation of the radiation from the moving charge, and of the mechanical reaction of the resulting field on it, we require the potentials for *all* points of space, this method may be said to be restricted to problems where the velocity is less than that of light.

In these cases it gives us a complete solution of the problem so long as the series remain convergent.

When the equation has more than one root Lagrange's Theorem fails. In this case there seems to be no known general method of obtaining a solution, but as it can only occur when the velocity is greater than that of light this failure is not of much importance for our purpose.

This method of expansion though available is not convenient; for the series can be obtained more easily directly from the integrals themselves, especially from (12) and (13), § 9, and that too in the form most convenient for the problem in hand.

Besides, these integrals for periodic motions lead at once to the appropriate expansions of the potentials in Fourier series. For instance, in problem 6, where the equation giving τ in terms of t is practically Kepler's equation, the integrals give the series corresponding to Bessel's solution of that equation.

We shall first treat of the problem of determining the field due to given motions of charges at distant points, with a view to calculating the radiation from them.

We shall then consider the problem of determining the field in the neighbourhood and inside the charge, and thence show how to calculate the mechanical reaction of the field upon it.

Since our object in this investigation is to discover new results rather than to establish them with absolute mathematical rigour, we shall make use of the simplest methods available, for instance of symbolical methods, leaving the expressions obtained to be justified *a posteriori*. Such a justification will generally consist in a determination of the range of validity of the expressions obtained, but this is usually sufficiently obvious in a physical problem without detailed examination.

CHAPTER VII

PERIODIC MOTIONS

70. It is convenient to classify the motions to be considered according to the number of different, that is, incommensurable, periods or frequencies involved. Thus we may speak of monoperiodic motions, for instance, uniform circular motion or elliptic motion; of diperiodic motions, for instance, motion in a nonreentrant epicycloid; and of polyperiodic motions, for instance, simple harmonic oscillations about uniform circular motion.

We shall begin with the simplest case, where there is only a single period.

Monoperiodic motion. We start from the integral expressions (12) and (13), § 9:

$$\phi = \frac{1}{2\pi}\int de \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \epsilon^{\iota\mu(t-R/c-\tau)} \frac{d\tau\, d\mu}{R} \quad \text{..............(12)},$$

$$\mathbf{a} = \frac{1}{2\pi c}\int de \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \epsilon^{\iota\mu(t-R/c-\tau)} \frac{\mathbf{v}\, d\tau\, d\mu}{R} \quad \text{...........(13)},$$

where

$$R = \sqrt{\{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2\}}.$$

In these expressions the vector ρ, whose components are (ξ, η, ζ), is a given function of τ with the single period T, and $\mathbf{v}$, that is to say $\dot{\rho}$, has the same period.

Separate the integral with respect to τ into the sum of an infinite number of component integrals, that is write

$$\int_{-\infty}^{\infty} d\tau = \sum_{j=-N}^{j=N} \int_{jT}^{(j+1)T} d\tau,$$

where the integer N is to be taken infinitely large ultimately.

In the type integral written down write $jT + \tau$ for τ, thus reducing the limits from jT and $(j+1)T$ to 0 and T. R and $\mathbf{v}$ are unaltered by this change on account of the periodicity. Thus we get, for example,

$$\begin{aligned}
\phi &= \underset{N=\infty}{\text{Limit}} \frac{1}{2\pi} \int de \int_{-\infty}^{\infty} \int_0^T \sum_{j=-N}^{j=N} \epsilon^{\iota\mu(t-jT-R/c-\tau)} \frac{d\tau\, d\mu}{R} \\
&= \underset{N=\infty}{\text{Limit}} \frac{1}{\pi} \int de \int_0^{\infty} \int_0^T \sum_{j=-N}^{j=N} \frac{\cos\mu\,(t - R/c - \tau - jT)\, d\tau\, d\mu}{R} \\
&= \underset{N=\infty}{\text{Limit}} \frac{1}{\pi} \int de \int_0^{\infty} \int_0^T \frac{\sin(N+\frac{1}{2})\mu T}{\sin\frac{1}{2}\mu T} \frac{\cos\mu\,(t - R/c - \tau)\, d\tau\, d\mu}{R} \\
&= \sum_{j=0}^{j=\infty}{}' \frac{2}{T} \int de \int_0^T \cos\frac{2\pi j}{T}(t - R/c - \tau) \frac{d\tau}{R} \qquad \text{.....(98)},
\end{aligned}$$

where Σ' as usual denotes the sum written but with one-half of the first term. Similarly

$$\mathbf{a} = \sum_{j=0}^{j=\infty}{}' \frac{2}{cT} \int de \int_0^T \cos\frac{2\pi j}{T}(t - R/c - \tau) \frac{\mathbf{v}\, d\tau}{R} \qquad \text{.....(99)}.$$

The interchanges of summations and integrations are allowed because we know that the integrals are convergent for physical reasons. It would in any case be quite easy to introduce an appropriate factor to ensure convergence of the integrals.

By expanding the circular functions we may write (98) and (99) in the forms

$$\left.\begin{aligned}
\phi &= \sum_{j=0}^{j=\infty}{}' 2\left(\Phi_j \cos\frac{2\pi jt}{T} + \Phi_j' \sin\frac{2\pi jt}{T}\right) \\
\Phi_j &= \frac{1}{T} \int de \int_0^T \cos\frac{2\pi j}{T}\left(\frac{R}{c} + \tau\right) \frac{d\tau}{R} \\
\Phi_j' &= \frac{1}{T} \int de \int_0^T \sin\frac{2\pi j}{T}\left(\frac{R}{c} + \tau\right) \frac{d\tau}{R}
\end{aligned}\right\} \qquad \text{.....(100)},$$

$$\left.\begin{aligned}
\mathbf{a} &= \sum_{j=0}^{j=\infty}{}' 2\left(\mathbf{A}_j \cos\frac{2\pi jt}{T} + \mathbf{A}_j' \sin\frac{2\pi jt}{T}\right) \\
\mathbf{A}_j &= \frac{1}{cT} \int de \int_0^T \cos\frac{2\pi j}{T}\left(\frac{R}{c} + \tau\right) \frac{\mathbf{v}\, d\tau}{R} \\
\mathbf{A}_j' &= \frac{1}{cT} \int de \int_0^T \sin\frac{2\pi j}{T}\left(\frac{R}{c} + \tau\right) \frac{\mathbf{v}\, d\tau}{R}
\end{aligned}\right\} \qquad \text{.....(101)}.$$

These equations give the expansions of the potentials in series of harmonics; the functions $\Phi_j \ldots$ may be regarded as normal functions for the given motion. They are independent of the time t, but involve the coordinates (x, y, z) of the fieldpoint implicitly, since R is a function of (x, y, z). The time of emission, τ, occurs as a variable parameter.

When the distance R is large compared with the linear dimensions of the charge, though it may be comparable with those of the orbit, the integration with respect to the element of charge, de, may be dispensed with, and the whole charge, e, introduced as a factor.

71. The magnetic and electric forces are easily found by means of the equations (VII) and (VIII), § 3,

$$\mathbf{h} = \text{curl}\,\mathbf{a} \quad \text{.................................(VII)},$$

$$\mathbf{d} = -\,\text{grad.}\,\phi - \frac{\partial \mathbf{a}}{c\partial t} \quad \text{.......................(VIII)}.$$

The series (100) and (101) may be differentiated term by term for the motion is continuous, being periodic.

In finding grad. ϕ and curl $\mathbf{a}$ we must remember that (x, y, z) occur only in the quantity R, which enters into the functions $\Phi_j \ldots$ under the sign of integration.

Now for any scalar function $f(R)$, we have

$$\text{grad.}\, f(R) = \mathbf{R}_1 . f'(R),$$

where $\mathbf{R}_1$ is the unit vector in the direction $\mathbf{R}$.

Similarly for a vector function $\mathbf{g}(\mathbf{R})$, we get

$$\text{curl}\; \mathbf{g}(\mathbf{R}) = [\mathbf{R}_1 . \mathbf{g}'(\mathbf{R})].$$

Applying (VII) and (VIII) to (100) and (101), we get

$$\left.\begin{aligned} \mathbf{d} &= \sum_{j=0}^{j=\infty}{}' \, 2\left(\Theta_j \cos\frac{2\pi jt}{T} + \Theta_j' \sin\frac{2\pi jt}{T}\right) \\ \Theta_j &= -\,\text{grad.}\,\Phi_j - \frac{2\pi j}{\lambda}\mathbf{A}_j' \\ \Theta_j' &= -\,\text{grad.}\,\Phi_j' + \frac{2\pi j}{\lambda}\mathbf{A}_j \end{aligned}\right\} \quad \text{..............(102)},$$

where $\lambda = cT$; it is the wave-length of the fundamental wave of period T.

Performing the differentiations, we get

$$\left.\begin{aligned} \Theta_j &= \frac{2\pi j}{\lambda T}\int de \int_0^T \sin\frac{2\pi j}{T}\left(\frac{R}{c} + \tau\right)\frac{(\mathbf{R}_1 - \mathbf{v}/c)\,d\tau}{R} \\ &\quad + \frac{1}{T}\int d\dot{e}\int_0^T \cos\frac{2\pi j}{T}\left(\frac{R}{c} + \tau\right)\frac{\mathbf{R}_1\,d\tau}{R^2} \\ \Theta_j' &= -\frac{2\pi j}{\lambda T}\int de \int_0^T \cos\frac{2\pi j}{T}\left(\frac{R}{c} + \tau\right)\frac{(\mathbf{R}_1 - \mathbf{v}/c)\,d\tau}{R} \\ &\quad + \frac{1}{T}\int de \int_0^T \sin\frac{2\pi j}{T}\left(\frac{R}{c} + \tau\right)\frac{\mathbf{R}_1\,d\tau}{R^2} \end{aligned}\right\} \quad \text{.........(103)}.$$

In the same way we get, from (VII) and (101),

$$\left.\begin{aligned}\mathbf{h}_j &= \sum_{j=0}^{j=\infty}{}' \, 2\left(\mathbf{H}_j \cos\frac{2\pi j t}{T} + \mathbf{H}_j' \sin\frac{2\pi j t}{T}\right)\\ \mathbf{H}_j &= -\frac{2\pi j}{\lambda^2}\int de\int_0^T \sin\frac{2\pi j}{T}\left(\frac{R}{c}+\tau\right)\frac{[\mathbf{R}_1\,\mathbf{v}]\,d\tau}{R}\\ &\quad -\frac{1}{\lambda}\int de\int_0^T \cos\frac{2\pi j}{T}\left(\frac{R}{c}+\tau\right)\frac{[\mathbf{R}_1\,\mathbf{v}]\,d\tau}{R^2}\\ \mathbf{H}_j' &= \frac{2\pi j}{\lambda^2}\int de\int_0^T \cos\frac{2\pi j}{T}\left(\frac{R}{c}+\tau\right)\frac{[\mathbf{R}_1\,\mathbf{v}]\,d\tau}{R}\\ &\quad -\frac{1}{\lambda}\int de\int_0^T \sin\frac{2\pi j}{T}\left(\frac{R}{c}+\tau\right)\frac{[\mathbf{R}_1\,\mathbf{v}]\,d\tau}{R^2}\end{aligned}\right\} \quad \ldots\ldots\ldots(104).$$

The expressions (102)—(104) give the values of the electric and magnetic forces expanded in series of harmonic terms.

Thus we see that the monoperiodic motion of an electric charge always gives rise to an infinite series of simple harmonic waves, whose frequencies are integral multiples of the frequency of the motion itself.

The amplitudes of the several harmonics are given by the equations (103) and (104); in general they diminish very rapidly as the order of the harmonic increases.

It is to be noted that each amplitude involves terms which at large distances are of different orders. The first line in each equation represents a vibration, which is of order $1/\lambda r$ ultimately, r being as usual the distance of the fieldpoint from the origin supposed to be taken near the orbit; and this component is transverse. The second line represents a vibration ultimately of order $1/r^2$, and therefore negligible in comparison with the first; for the electric force this term is ultimately radial, for the magnetic force transverse. The magnetic force in each order is ultimately perpendicular to the electric force.

72. Hence at large distances from the orbit we may make the following approximations:

(1) We replace the integration $\int de$ by the factor e.

(2) We retain only terms of order $1/\lambda r$, that is, only the first lines of (103) and (104).

(3) We replace the quantities R, $\mathbf{R}_1$, *where they occur outside the circular functions*, by r, $\mathbf{r}_1$.

(4) In the circular functions we cannot replace R by r simply, because we should then neglect a difference of phase of finite amount. We must proceed as follows:

We have

$$R = \sqrt{\{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2\}} = r - \frac{x\xi + y\eta + z\zeta}{r},$$

where terms of order $1/r$, and higher, have been neglected.

Denoting the vector (x, y, z) by $\mathbf{r}$ as before, and its unit vector by $\mathbf{r}_1$, and the vector (ξ, η, ζ) by ρ, we thus have

$$R = r - (\mathbf{r}_1 . \rho)$$
$$= r - p,$$

where p is the projection of the radius vector of the charge, ρ, on the vector $\mathbf{r}_1$ at time τ.

With this approximate value of R, we get

$$\cos\frac{2\pi j}{T}(R/c + \tau) = \cos\frac{2\pi j}{T}(r/c + \tau - p/c),$$

$$\sin\frac{2\pi j}{T}(R/c + \tau) = \sin\frac{2\pi j}{T}(r/c + \tau - p/c).$$

When these expressions are substituted in terms such as the general term of (100), § 70,

$$\Phi_j \cos\frac{2\pi jt}{T} + \Phi_j' \sin\frac{2\pi tj}{T},$$

we get an expression such as

$$\cos\frac{2\pi j}{T}(t - r/c)\int de \int_0^T \cos\frac{2\pi j}{T}\left(\tau - \frac{p}{c}\right)\frac{d\tau}{R}$$
$$+ \sin\frac{2\pi j}{T}(t - r/c)\int de \int_0^T \sin\frac{2\pi j}{T}\left(\tau - \frac{p}{c}\right)\frac{d\tau}{R}.$$

73. Field at a great distance. Making the approximations mentioned above, in (1), (2) and (3), and changing the notation slightly, we get the following expressions for a distant point:

$$\left.\begin{aligned}\phi &= \frac{2e}{r}\sum_{j=0}^{j=\infty}{}' \left(\Phi_j \cos\frac{2\pi j}{T}(t - r/c) + \Phi_j' \sin\frac{2\pi j}{T}(t - r/c)\right)\\ \Phi_j &= \frac{1}{T}\int_0^T \cos\frac{2\pi j}{T}(\tau - p/c)\, d\tau\\ \Phi_j' &= \frac{1}{T}\int_0^T \sin\frac{2\pi j}{T}(\tau - p/c)\, d\tau\end{aligned}\right\}\quad\ldots(105),$$

$$\left.\begin{aligned}\mathbf{a} &= \frac{2e}{r}\sum_{j=0}^{j=\infty}{}' \left(\mathbf{A}_j \cos\frac{2\pi j}{T}(t - r/c) + \mathbf{A}_j' \sin\frac{2\pi j}{T}(t - r/c)\right)\\ \mathbf{A}_j &= \frac{1}{\lambda}\int_0^T \cos\frac{2\pi j}{T}(\tau - p/c)\,\mathbf{v}\, d\tau\\ \mathbf{A}_j' &= \frac{1}{\lambda}\int_0^T \sin\frac{2\pi j}{T}(\tau - p/c)\,\mathbf{v}\, d\tau\end{aligned}\right\}\quad\ldots(106).$$

By direct differentiation, or from (102)—(104), we get

$$\left.\begin{aligned} \mathbf{d} &= \frac{4\pi e}{\lambda r}\sum_{j=0}^{j=\infty}{}' j\left(\Theta_j \cos\frac{2\pi j}{T}(t - r/c) + \Theta_j' \sin\frac{2\pi j}{T}(t - r/c)\right) \\ \Theta_j &= \quad \Phi_j' . \mathbf{r}_1 - \mathbf{A}_j' \\ \Theta_j' &= - \Phi_j . \mathbf{r}_1 + \mathbf{A}_j \end{aligned}\right\} \quad (107),$$

$$\left.\begin{aligned} \mathbf{h} &= \frac{4\pi e}{\lambda r}\sum_{j=0}^{j=\infty}{}' j\left(\mathbf{H}_j \cos\frac{2\pi j}{T}(t - r/c) + \mathbf{H}_j' \sin\frac{2\pi j}{T}(t - r/c)\right) \\ \mathbf{H}_j &= - [\mathbf{r}_1 . \mathbf{A}_j'] \\ \mathbf{H}_j' &= \quad [\mathbf{r}_1 . \mathbf{A}_j] \end{aligned}\right\} \quad (108).$$

Expressions of this type were first given by Schott*.

74. By means of the expressions (107) and (108) we can easily verify that, to the approximation used, the electric and magnetic forces are perpendicular to the radius vector $\mathbf{r}_1$ and to each other.

In the first place, form the scalar product of $\mathbf{r}_1$ and Θ_j, we get

$$(\mathbf{r}_1 . \Theta_j) = \frac{1}{T}\int_0^T \sin\frac{2\pi j}{T}(\tau - p/c) . \left\{1 - \left(\mathbf{r}_1 . \frac{\mathbf{v}}{c}\right)\right\} d\tau.$$

Now since we have by definition

$$p = (\mathbf{r}_1 . \rho) \text{ and } \dot{\rho} = \mathbf{v},$$

we get
$$\frac{\partial}{\partial t}\left(\frac{p}{c}\right) = \frac{(\mathbf{r}_1 . \mathbf{v})}{c},$$

so that
$$1 - \frac{(\mathbf{r}_1 . \mathbf{v})}{c} = \frac{\partial}{\partial t}\left(\tau - \frac{p}{c}\right).$$

Thus the function under the sign of integration is a perfect differential, and the integral vanishes when taken between the limits 0 and T on account of the periodicity. Hence

$$(\mathbf{r}_1 . \Theta_j) = 0,$$

and in the same way

$$(\mathbf{r}_1 . \Theta_j') = 0,$$

thus showing that each component of the electric force is perpendicular to the radius vector $\mathbf{r}_1$†.

The form of $\mathbf{H}_j$ and $\mathbf{H}_j'$ shows that the magnetic force is perpendicular to $\mathbf{r}_1$.

By forming the vector product of $\mathbf{r}_1$ and Θ_j we find at once $[\mathbf{r}_1 . \Theta_j] = \mathbf{H}_j$; and similarly $[\mathbf{r}_1 . \Theta_j'] = \mathbf{H}_j'$. Hence each harmonic component of the magnetic force is perpendicular and numerically equal to the corresponding component of the electric force, and in the same phase.

* *Ann. der Phys.* 1907, 24, p. 635.

† This gives, by (107), $\Phi_j = (\mathbf{r}_1 . \mathbf{A}_j)$, $\Phi_j' = (\mathbf{r}_1 . \mathbf{A}_j')$.

75. An examination of the equations (107) shows that each harmonic component of the electric force is the resultant of two components:

(1) A purely radial component proportional to, but one quarter of its own period in advance of, the corresponding harmonic of the scalar potential, ϕ.

(2) A component proportional to and in the direction of, but one quarter of its own period later than, the corresponding harmonic of the vector potential, **a**. The radial component of this term is just sufficient to neutralize the first term.

Similarly equations (108) show that each harmonic component of the magnetic force is proportional to, but one quarter of its own period later than, the corresponding harmonic of the vector potential, **a**; and its direction perpendicular to both the radius vector and the harmonic of the vector potential.

It follows that each harmonic vibration is elliptically polarized. In order that we may understand the nature of this polarization, let us consider Fig. 35. It is taken in the wavefront at the fieldpoint P, looking along the radius vector **r**, but inwards, just as an observer would view the incident vibration.

$P\theta$ and $P\phi$ are in the directions in which the polar distance θ, and the longitude ϕ increase. These directions form a right-handed system with **r**.

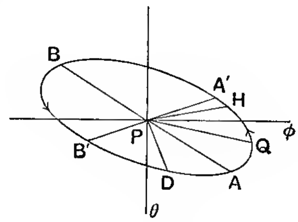

Fig. 35.

Let PA represent the projection on the wavefront of the vector $\mathbf{A}_j$, PA' that of the vector $\mathbf{A}_j'$. These lines are conjugate radii of the ellipse described by the radius vector PQ, the projection of the harmonic component $\mathbf{a}_j$ of the vector potential on the wavefront.

It is obvious that PB' represents the vector Θ_j, and that PA represents Θ_j'; this follows at once from the geometrical interpretation of (107).

Hence PD, the radius conjugate to PQ and behind it, represents the corresponding harmonic; and the harmonic component, $\mathbf{d}_j$, of the electric force is represented by $4\pi ej/r\lambda$ times PD.

It follows that the harmonic component, $\mathbf{h}_j$, of the magnetic force is represented by $4\pi ej/r\lambda$ times PH, where PH is a right angle in front of PD, or behind it, according as the rotation of all the vectors is right-handed, as shown, or left-handed.

Thus we see that a knowledge of the vectors $\mathbf{A}_j$ and $\mathbf{A}_j'$ is sufficient for a complete description of the distant field.

76. The Poynting vector. The Poynting vector is given by equation (XIII), § 5,

$$\mathbf{s} = \frac{c\,[\mathbf{d}\,.\,\mathbf{h}]}{4\pi}.$$

It is obvious that in finding the mean value of $\mathbf{s}$ for a long time, or for a whole number of periods of the motion, the terms in $\mathbf{s}$ due to products of different harmonics disappear identically on integration. Hence we may without loss of generality confine our investigation to terms involving only a single harmonic, such as

$$\mathbf{s}_j = \frac{c\,[\mathbf{d}_j\,.\,\mathbf{h}_j]}{4\pi}.$$

The only terms of this type which contribute to the mean radiation are those which involve $\cos^2 \frac{2\pi j}{T}(t - r/c)$ and $\sin^2 \frac{2\pi j}{T}(t - r/c)$; and the mean value of each of these factors is $\frac{1}{2}$.

Thus, omitting the time factors, we get for the mean radiation vector due to the harmonic j

$$\begin{aligned}\mathbf{s}_j &= \frac{c}{8\pi}\left(\frac{4\pi e j}{\lambda r}\right)^2 [\Theta_j\,.\,\mathbf{H}_j] + [\Theta_j'\,.\,\mathbf{H}_j'] \\ &= \frac{c}{2\pi}\left(\frac{2\pi e j}{\lambda r}\right)^2 (H_j^2 + H_j'^2)\,\mathbf{r}_1 \\ &= \frac{c}{2\pi}\left(\frac{2\pi e j}{\lambda r}\right)^2 \{A_j^2 \sin^2 (r\,.\,A_j) + A_j'^2 \sin^2 (r\,.\,A_j')\}\,\mathbf{r}_1 \quad \text{.........(109)}.\end{aligned}$$

This equation together with (106), § 73, enables us to determine the total loss of energy due to radiation from the moving charge.

77. Group of charges. The expressions obtained above can easily be generalized for the case where we have a set of n similar charges moving in regular succession round the same orbit in the same way.

Let (ξ_i, η_i, ζ_i) be the coordinates of the ith charge; and suppose that for all values of i from 0 to $n-1$ we are given equations of the following forms:

$$\xi_i = f\left(\tau + \frac{iT}{n}\right), \quad \eta_i = g\left(\tau + \frac{iT}{n}\right), \quad \zeta_i = h\left(\tau + \frac{iT}{n}\right) \quad \text{...(110)},$$

where f, g, h are three periodic functions of the same period T.

A set of charges satisfying these conditions may conveniently be called a "group" of charges; an example is afforded by n equidistant electrons describing the same circle with the same uniform velocity. Thus corresponding elements of all the charges describe the same curve in succession, passing each point at intervals T/n.

Hence the quantities R_i and $\mathbf{v}_i$ for the ith charge at time τ have the same values as the quantities R and $\mathbf{v}$ for the 0th charge at the time

$$\tau + iT/n.$$

The contribution of the ith charge to ϕ is, by (98), § 70,

$$\sum_{j=0}^{j=\infty}{}' \frac{2}{T} \int de \int_0^T \cos \frac{2\pi j}{T} (t - R_i/c - \tau)\, d\tau / R_i.$$

Change τ into $\tau - iT/n$; then, by what has just been said, R_i takes the value of R for the time τ. The limits become iT/n and $T + iT/n$, but may be changed back to 0 and T on account of the periodicity of the motion. Thus the contribution of the ith charge becomes

$$\sum_{j=0}^{j=\infty}{}' \frac{2}{T} \int de \int_0^T \cos \frac{2\pi j}{T} (t + iT/n - R/c - \tau)\, d\tau / R$$

$$= \sum_{j=0}^{j=\infty}{}' 2 \left\{ \Phi_j \cos \left(\frac{2\pi j}{T} t + \frac{2\pi i j}{n} \right) + \Phi_j' \sin \left(\frac{2\pi j}{T} t + \frac{2\pi i j}{n} \right) \right\}.$$

Summing for all the charges, that is for i from $i = 0$ to $i = n - 1$, we get

$$\phi = n \sum_{s=0}^{s=\infty}{}' 2 \left(\Phi_{sn} \cos \frac{2\pi snt}{T} + \Phi'_{sn} \sin \frac{2\pi snt}{T} \right) \quad \text{.........(111).}$$

Similarly

$$\mathbf{a} = n \sum_{s=0}^{s=\infty}{}' 2 \left(\mathbf{A}_{sn} \cos \frac{2\pi snt}{T} + \mathbf{A}'_{sn} \sin \frac{2\pi snt}{T} \right) \quad \text{.........(112).}$$

Thus all the harmonics disappear except those corresponding to multiples of n.

Whereas for a single charge the mean radiation vector is given by

$$\mathbf{s} = \sum_{j=0}^{j=\infty}{}' \mathbf{s}_j \quad \text{.................................(113),}$$

in the present case of a group it is given by

$$\mathbf{s} = n^2 \sum_{s=0}^{s=\infty}{}' \mathbf{s}_{sn} \quad \text{..............................(114),}$$

for by (112) each harmonic component which is left is n times as great as before, and therefore each component of $\mathbf{s}$ is n^2 times as great.

We shall find that the component harmonics $j = 0$ and $s = 0$ contribute nothing to the radiation, while successive harmonics diminish very rapidly in magnitude. Thus the absence from (114) of the components $j = 1, 2, \ldots n-1, n+1, n+2, \ldots 2n-1, \ldots$ is much more than sufficient to make up for one factor n in (114). It follows that the radiation *per charge* from a group of charges is much less than that from a single charge.

The first example of interference of this kind was given by J. J. Thomson*.

78. Illustrative Problems. We shall now treat of some problems in illustration of the methods just developed.

Problem 1. n equidistant electrons move in a circle with uniform velocity. To find the electro-magnetic field produced. This problem

* *Phil. Mag.* [6], Vol. VI. p. 681.

has been treated by Schott* for the case of a fieldpoint at a great distance. The approximate solution for infinitely small velocities was first given by J. J. Thomson†.

Let the radius of the circle be ρ and the angular velocity ω. If we confine ourselves to the case of a fieldpoint not very close to any one of the electrons, we may treat them as points with charge e.

Take the centre of the circle as origin, and its axis as axis of z in a system of cylindrical coordinates (z, ϖ, ϕ). Sometimes we shall use also polar coordinates (r, θ, ϕ).

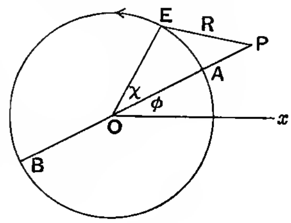

Fig. 36.

In Fig. 36, P is the projection of the fieldpoint (z, ϖ, ϕ) on the plane of the circle, A and B the intersections of the meridian plane through it with the circle. E is the position of the 0th charge at the time τ.

xOP is the angle ϕ, OP is equal to ϖ, and the height of P above the plane of the diagram is z.

Let the angle xOE be equal to $\omega\tau + \delta$, and let $\chi = \omega\tau + \delta - \phi$. By choosing the zero of time at the instant when the electron E crosses OP, we can make $\delta = \phi$; but it is convenient to retain ϕ explicitly in the equations. This we can do by replacing t wherever it occurs by $t + (\delta - \phi)/\omega$.

ω is equal to the angular velocity, so that the period T is equal to $2\pi/\omega$.

We have, from Fig. 36,

$$R = \sqrt{\{z^2 + \varpi^2 + \rho^2 - 2\varpi\rho \cos\chi\}} = \sqrt{2\varpi\rho\,(C - \cos\chi)},$$

where
$$C = (z^2 + \varpi^2 + \rho^2)/2\varpi\rho.$$

On the circle, where $z = 0$ and $\varpi = \rho$, we have $C = 1$; elsewhere it is greater than 1; at an infinite distance it is infinitely great, and we have

$$R = \sqrt{(2\varpi\rho C)} - \cos\chi \sqrt{\frac{\varpi\rho}{2C}} = r - \rho \sin\theta \cos\chi$$

very approximately; that is, in equations (105) and (106), § 73, we must put

$$p = \rho \sin\theta \cos\chi.$$

When we replace t in equations (100) and (101), § 70, by $t + (\delta - \phi)/\omega$, and therefore also τ by $\tau + (\delta - \phi)/\omega$, and change the variable from τ to χ, we get equations such as

$$\Phi_j = \frac{1}{2\pi} \int de \int_0^{2\pi} \frac{\cos j\,\{\chi + \alpha\sqrt{(C - \cos\chi)}\}\, d\chi}{\sqrt{2\varpi\rho\,(C - \cos\chi)}}.$$

* *l. c. Ann. der Phys.* 1907, 24, p. 641; *Phil. Mag.* [6], Vol. XIII. p. 189.
† *Phil. Mag.* [6], Vol. VI. p. 681.

Dividing the integral into two, $\int_0^\pi + \int_\pi^{2\pi}$, and replacing χ in the second by $2\pi - \chi$, we get

$$\Phi_j = \frac{1}{\pi}\int de \int_0^\pi \frac{\cos j\alpha \sqrt{(C-\cos\chi)}\cos j\chi\, d\chi}{\sqrt{2\varpi\rho}\,(C-\cos\chi)}.$$

In this equation α stands for $\omega\sqrt{(2\varpi\rho)}/C$.

Each of the vectors $\mathbf{A}_j$ and $\mathbf{A}_j'$ may be divided into a radial component $A_{\varpi j}$ and $A'_{\varpi j}$, and a component along the parallel of latitude, $A_{\phi j}$ and $A'_{\phi j}$. The components of $\mathbf{v}/c$ in these directions are given by

$$\frac{v_\varpi}{c} = -\beta\sin\chi, \qquad \frac{v_\phi}{c} = \beta\cos\chi,$$

where $\beta = \omega\rho/c$, so that β is the ratio of the velocity of the electron to that of light.

79. In this way we get

$$\left.\begin{aligned}
\phi &= \sum_{j=0}^{j=\infty}{}' \, 2\{\Phi_j \cos j(\omega t+\delta-\phi) + \Phi_j' \sin j(\omega t+\delta-\phi)\}\\
\Phi_j &= \frac{1}{\pi}\int de \int_0^\pi \frac{\cos j\alpha\sqrt{(C-\cos\chi)}\cos j\chi\, d\chi}{\sqrt{2\varpi\rho}\,(C-\cos\chi)}\\
\Phi_j' &= \frac{1}{\pi}\int de \int_0^\pi \frac{\sin j\alpha\sqrt{(C-\cos\chi)}\cos j\chi\, d\chi}{\sqrt{2\varpi\rho}\,(C-\cos\chi)}
\end{aligned}\right\} \quad \ldots(115).$$

$$\left.\begin{aligned}
a_\phi &= \sum_{j=0}^{j=\infty}{}' \, 2\{A_{\phi j} \cos j(\omega t+\delta-\phi) + A'_{\phi j} \sin j(\omega t+\delta-\phi)\}\\
A_{\phi j} &= \frac{\beta}{\pi}\int de \int_0^\pi \frac{\cos j\alpha\sqrt{(C-\cos\chi)}\cos j\chi\cos\chi\, d\chi}{\sqrt{2\varpi\rho}\,(C-\cos\chi)}\\
A'_{\phi j} &= \frac{\beta}{\pi}\int de \int_0^\pi \frac{\sin j\alpha\sqrt{(C-\cos\chi)}\cos j\chi\cos\chi\, d\chi}{\sqrt{2\varpi\rho}\,(C-\cos\chi)}
\end{aligned}\right\} \quad \ldots(116).$$

$$\left.\begin{aligned}
a_\varpi &= \sum_{j=0}^{j=\infty}{}' \, 2\{A_{\varpi j} \cos j(\omega t+\delta-\phi) + A'_{\varpi j} \sin j(\omega t+\delta-\phi)\}\\
A_{\varpi j} &= \frac{\beta}{\pi}\int de \int_0^\pi \frac{\sin j\alpha\sqrt{(C-\cos\chi)}\sin j\chi\sin\chi\, d\chi}{\sqrt{2\varpi\rho}\,(C-\cos\chi)}\\
A'_{\varpi j} &= -\frac{\beta}{\pi}\int de \int_0^\pi \frac{\cos j\alpha\sqrt{(C-\cos\chi)}\sin j\chi\sin\chi\, d\chi}{\sqrt{2\varpi\rho}\,(C-\cos\chi)}
\end{aligned}\right\} \quad (117).$$

The values of the electric and magnetic forces can easily be written down from (103) and (104), § 71, but we refrain from doing so, as they are of no great interest. It suffices to say that they have both radial and transverse components, and may be resolved into series of harmonics, the frequency of the fundamental being ω, the angular velocity of the charge.

The values of the potentials for the group of n electrons are at once given by equations (111) and (112), that is to say, by equations (115)—(117) with terms of orders $j = sn$ alone.

The functions occurring in these equations are all of the same type. When $\alpha = 0$, that is, $\omega = 0$, they reduce to toroidal functions; thus we may regard them in some sort as generalized toroidal functions.

We shall not stop to discuss their properties, but pass on to the consideration of an important particular case, namely that of a distant fieldpoint, for which C is very large.

80. Distant field. We have seen that for very large values of C we may write very approximately,

$$R = r - \rho \sin\theta \cos\chi, \qquad \frac{\omega R}{c} = \frac{\omega r}{c} - \beta \sin\theta\cos\chi, \qquad p = \rho\sin\theta\cos\chi.$$

Introducing these values into (116) and (117), where $\alpha\sqrt{(C - \cos\chi)} = \omega R/c$, and $\sqrt{(2\varpi\rho)}\sqrt{(C - \cos\chi)} = R$, and proceeding as in § 73, we get without any difficulty

$$\left.\begin{aligned} a_\phi &= \frac{2e\beta}{r}\sum_{j=0}^{j=\infty}{}' \{A_{\phi j}\cos j(\omega.\overline{t - r/c} + \delta - \phi) + A'_{\phi j}\sin j(\omega.\overline{t - r/c} + \delta - \phi)\} \\ A_{\phi j} &= \frac{1}{\pi}\int_0^\pi \cos(j\beta\sin\theta\cos\chi)\cos\chi\cos j\chi\, d\chi \\ A'_{\phi j} &= -\frac{1}{\pi}\int_0^\pi \sin(j\beta\sin\theta\cos\chi)\cos\chi\cos j\chi\, d\chi \end{aligned}\right\} \quad (118).$$

$$\left.\begin{aligned} a_\varpi &= \frac{2e\beta}{r}\sum_{j=0}^{j=\infty}{}' \{A_{\varpi j}\cos j(\omega.\overline{t - r/c} + \delta - \phi) + A'_{\varpi j}\sin j(\omega.\overline{t - r/c} + \delta - \phi)\} \\ A_{\varpi j} &= -\frac{1}{\pi}\int_0^\pi \sin(j\beta\sin\theta\cos\chi)\sin\chi\sin j\chi\, d\chi \\ A'_{\varpi j} &= -\frac{1}{\pi}\int_0^\pi \cos(j\beta\sin\theta\cos\chi)\sin\chi\sin j\chi\, d\chi \end{aligned}\right\} \quad \ldots\ldots(119).$$

81. The normal functions $A_{\phi j}\ldots$ are easily expressed as Bessel Functions. We get at once from the well-known expansion

$$\epsilon^{\iota x\cos\chi} = \sum_{j=0}^{j=\infty}{}' 2\iota^j J_j(x)\cos j\chi$$

the following integrals which occur in (118) and (119):

$$\begin{aligned} \frac{1}{\pi}\int_0^\pi \cos(x\cos\chi)\cos j\chi\, d\chi &= \cos\tfrac{1}{2}j\pi \, . \, J_j(x), && \text{when } j \text{ is even,} \\ &= 0, && \text{when } j \text{ is odd,} \\ \frac{1}{\pi}\int_0^\pi \sin(x\cos\chi)\cos j\chi\, d\chi &= 0, && \text{when } j \text{ is even,} \\ &= \sin\tfrac{1}{2}j\pi \, . \, J_j(x), && \text{when } j \text{ is odd.} \end{aligned}$$

Using these results, and also the relations

$$J_{j-1}(x) + J_{j+1}(x) = \frac{2j\,J_j(x)}{x},$$

$$J_{j-1}(x) - J_{j+1}(x) = 2J_j'(x),$$

and putting $x = j\beta \sin\theta$, we get the following results:

$$\left.\begin{array}{lll} A_{\phi j} = 0, & A'_{\phi j} = \cos\tfrac{1}{2}j\pi\,.\,J_j'(j\beta\sin\theta), & j \text{ even} \\ A_{\phi j} = \sin\tfrac{1}{2}j\pi\,.\,J_j'(j\beta\sin\theta), & A'_{\phi j} = 0, & j \text{ odd} \\ A_{\varpi j} = \cos\tfrac{1}{2}j\pi\,.\,\dfrac{J_j(j\beta\sin\theta)}{\beta\sin\theta}, & A'_{\varpi j} = 0, & j \text{ even} \\ A_{\varpi j} = 0, & A'_{\varpi j} = -\sin\tfrac{1}{2}j\pi\,.\,\dfrac{J_j(j\beta\sin\theta)}{\beta\sin\theta}, & j \text{ odd} \end{array}\right\} \quad \ldots\ldots(120).$$

Substituting in (118) and (119), we get

$$a_\phi = \frac{2e\beta}{r}\sum_{j=0}^{j=\infty}{}' J_j'(j\beta\sin\theta)\,.\sin j(\omega\,.\,\overline{t - r/c} + \delta - \phi + \tfrac{1}{2}\pi) \quad \ldots(121),$$

$$a_\varpi = \frac{2e}{r\sin\theta}\sum_{j=0}^{j=\infty}{}' J_j(j\beta\sin\theta)\,.\cos j(\omega\,.\,\overline{t - r/c} + \delta - \phi + \tfrac{1}{2}\pi) \quad (122).$$

82. Electric and magnetic forces. It is easy to deduce the expressions for the forces from equations (107) and (108), § 73. Since we know that the electric force is equal to the magnetic force, perpendicular to it and to the radius vector, we have

$$d_\theta = h_\phi, \quad d_\phi = -h_\theta.$$

Thus it is sufficient to calculate **h**.

Equations (108) give at once

$$H_{\phi j} = -A'_{\theta j} = -\cos\theta A'_{\varpi j}, \quad H'_{\phi j} = \cos\theta A_{\varpi j},$$

$$H_{\theta j} = A'_{\phi j}, \quad H'_{\theta j} = -A_{\phi j},$$

for the angle between the directions of θ and ϖ is equal to θ. Hence we get

$$\begin{aligned} &H_{\phi j}\cos j(\omega\,.\,\overline{t - r/c} + \delta - \phi) + H'_{\phi j}\sin j(\omega\,.\,\overline{t - r/c} + \delta - \phi) \\ &\quad = \cos\theta\,.\,\{A_{\varpi j}\sin j(\omega\,.\,\overline{t - r/c} + \delta - \phi) - A'_{\varpi j}\cos j(\omega\,.\,\overline{t - r/c} + \delta - \phi)\} \\ &\quad = \cos\theta\,.\,\{A_{\varpi j}\cos[j(\omega\,.\,\overline{t - r/c} + \delta - \phi) - \tfrac{1}{2}\pi] \\ &\qquad\qquad + A'_{\varpi j}\sin[j(\omega\,.\,\overline{t - r/c} + \delta - \phi) - \tfrac{1}{2}\pi]\}, \end{aligned}$$

$$\begin{aligned} &H_{\theta j}\cos j(\omega\,.\,\overline{t - r/c} + \delta - \phi) + H'_{\theta j}\sin j(\omega\,.\,\overline{t - r/c} + \delta - \phi) \\ &\quad = -\{A_{\phi j}\cos[j(\omega\,.\,\overline{t - r/c} + \delta - \phi) - \tfrac{1}{2}\pi] \\ &\qquad\qquad + A'_{\phi j}\sin[j(\omega\,.\,\overline{t - r/c} + \delta - \phi) - \tfrac{1}{2}\pi]\}, \end{aligned}$$

showing, as was pointed out before, that every harmonic of **d** and **h** is in quadrature with the corresponding harmonic of **a**. Thus we get $d_\theta \equiv h_\phi$ from a_ϖ by multiplying by $2\pi j\cos\theta/\lambda$, that is, by $j\beta\cos\theta/\rho$, and diminishing the phase of each harmonic by $\frac{1}{2}\pi$, and $d_\phi \equiv -h_\theta$ from a_ϕ by multiplying by $2\pi j/\lambda$, that is, by $j\beta/\rho$, and diminishing the phase by $\frac{1}{2}\pi$.

Writing ψ for $\omega(t-r/c)+\delta-\phi+\frac{1}{2}\pi$, we get from (121) and (122)

$$d_\theta = h_\phi = \frac{2e\beta \cot\theta}{\rho r}\sum_{j=1}^{j=\infty} j\,J_j(j\beta\sin\theta)\sin j\psi \quad\text{.........(123)},$$

$$d_\phi = -h_\theta = -\frac{2e\beta^2}{\rho r}\sum_{j=1}^{j=\infty} j\,J_j'(j\beta\sin\theta)\cos j\psi \quad\text{.........(124)}.$$

The forces due to the group of n electrons are found as before by replacing the constant δ by $\delta + 2\pi i/n$ for the ith electron and summing for i from $i=0$ to $i=n-1$. The result is that all harmonics disappear except those for which $j=sn$, and these are multiplied by n. Hence for the group

$$d_\theta = h_\phi = \frac{2e\beta \cot\theta n^2}{\rho r}\sum_{s=1}^{s=\infty} s\,J_{sn}(sn\beta\sin\theta)\sin sn\psi \quad\text{.........(125)},$$

$$d_\phi = -h_\theta = -\frac{2e\beta^2 n^2}{\rho r}\sum_{s=1}^{s=\infty} s\,J'_{sn}(sn\beta\sin\theta)\cos sn\psi \quad\text{.........(126)}.$$

These agree with the values given by Schott* (*loc. cit.*).

83. Character of the field. The forces consist of series of harmonic terms, all vibrating transversely to the radius vector. The term of order zero obviously vanishes identically on account of the presence of the factor j in (123) and (124), and s in (125) and (126). The amplitudes of the successive harmonics are of the order $j\,J_j(j\beta\sin\theta)$ or $j\,J_j'(j\beta\sin\theta)$. When β is small, that is the velocity very small compared with that of light, they diminish with very great rapidity as the order of the harmonic, j, increases. In particular, even the harmonic, $s=2$, in (125) and (126) may be quite inappreciable, particularly near the poles, where the factor $\sin\theta$ makes the argument of the Bessel Function small. For by Duhamel's formula we have, when j is very large†,

$$J_j(j\sin\Theta) = \frac{(e^{\cos\Theta}\tan\frac{1}{2}\Theta)^j}{\sqrt{2j\pi\cos\Theta}} \quad\text{.....................(127)}.$$

This formula shows that the series always converge absolutely when $\beta<1$, that is the velocity less than that of light. When however $\beta=1$, and $\sin\theta=1$, so that $\sin\Theta=1$, the Bessel Function is ultimately of the order $j^{-\frac{1}{3}}$, and the series are no longer convergent.

The polarization of each harmonic vibration is generally elliptic; the ratio of the amplitudes of the two components in and perpendicular to the meridian is equal to

$$\frac{d_{\phi j}}{d_{\theta j}} = \beta\tan\theta\frac{J_j'(j\beta\sin\theta)}{J_j(j\beta\sin\theta)} = \sec\theta\,\frac{J_{j-1}(j\beta\sin\theta)-J_{j+1}(j\beta\sin\theta)}{J_{j-1}(j\beta\sin\theta)+J_{j+1}(j\beta\sin\theta)}.$$

* *Phil. Mag.* [6], Vol. XIII. p. 194.

† Graf and Gubler, *Besselsche Funktionen*, Bern, Pt. 1, p. 102, 1898.

At the equator, where $\theta = \frac{1}{2}\pi$, the polarization is linear, the electric force being along the equator and the magnetic force perpendicular to it. Near the axis, where θ is very small, the polarization is approximately circular, but the harmonic $j = 1$ alone has any appreciable value. The direction of rotation is that of the electron in its orbit. As θ increases in passing from the axis to the equator, the polarization becomes elliptic, the axes of the vibration ellipse being in and perpendicular to the meridian; and the perpendicular axis predominates more and more.

84. Problem 2. To calculate the radiation from the ring. Using (123) and (124), § 82, we find for the mean value of the Poynting vector at the point (r, θ, ϕ) the expression

$$\bar{s}_j = \frac{ce^2\beta^2 j^2}{2\pi\rho^2 r^2}\left[\cot^2\theta \,.\, \{J_j(j\beta\sin\theta)\}^2 + \beta^2\{J_j'(j\beta\sin\theta)\}^2\right] \quad \text{......(128).}$$

Now we have, by Neumann's Addition Theorem*,

$$\{J_j(x)\}^2 = \frac{1}{\pi}\int_0^\pi J_0(2x\sin\phi)\cos 2j\phi\, d\phi,$$

whence also

$$\{J_j'(x)\}^2 = \frac{1}{\pi}\int_0^\pi J_0(2x\sin\phi)\left(\cos 2\phi - \frac{j^2}{x^2}\right)\cos 2j\phi\, d\phi.$$

Putting $x = j\beta\sin\theta$, we get from (128)

$$\bar{s}_j = \frac{ce^2\beta^2 j^2}{2\pi^2\rho^2 r^2}\int_0^\pi J_0(2j\beta\sin\phi\sin\theta)(\beta^2\cos 2\phi - 1)\cos 2j\phi\, d\phi.$$

Since $\bar{s}_j$ is independent of the latitude, we must multiply this by $2\pi r^2\sin\theta\, d\theta$ and integrate over the whole sphere of radius r, that is from $\theta = 0$ to $\theta = \pi$.

Now we have†

$$\int_0^{\frac{\pi}{2}} J_0(r\sin\theta)\sin\theta\, d\theta = \sqrt{\frac{\pi}{2r}}\, J_{\frac{1}{2}}(r) = \frac{\sin r}{r}.$$

Put $r = 2j\beta\sin\phi$; then we get for the rate of radiation of energy on account of the jth harmonic

$$\begin{aligned} R_j &= \int_0^\pi \bar{s}_j \,.\, 2\pi r^2 \sin\theta\, d\theta \\ &= \frac{2ce^2\beta^2 j^2}{\pi\rho^2}\int_0^\pi \frac{\sin(2j\beta\sin\phi)}{2j\beta\sin\phi}(\beta^2\cos 2\phi - 1)\cos 2j\phi\, d\phi \\ &= \frac{2ce^2\beta}{\rho^2}\left[-\frac{\beta^2 j}{\pi}\int_0^\pi \sin(2j\beta\sin\phi)\cos 2j\phi\sin\phi\, d\phi \right. \\ &\qquad \left. - \frac{(1-\beta^2)j^2}{\pi}\int_0^\pi \frac{\sin 2j\beta\sin\phi}{2j\sin\phi}\cos 2j\phi\, d\phi\right]. \end{aligned}$$

* Gray and Mathews, *Bessel Functions*, p. 28.

† *ibid.* p. 240, Ex. 55.

The first integral is easily seen to be $-\pi J'_{2j}(2j\beta)$.

The second may be written

$$\int_0^\beta dx \int_0^\pi \cos(2jx \sin\phi) \cos 2j\phi \, d\phi,$$

and is therefore equal to $\pi \int_0^\beta J_{2j}(2jx)\,dx$.

Hence we get

$$R_j = \frac{2ce^2\beta}{\rho^2} \left[j\beta^2 J'_{2j}(2j\beta) - j^2(1-\beta^2) \int_0^\beta J_{2j}(2jx)\,dx \right].$$

The total rate of radiation from the ring of n electrons is found by taking $j = sn$, multiplying by n^2 and summing for s from $s = 1$ to $s = n$; we get

$$R = \frac{2ce^2\beta n^2}{\rho^2} \sum_{s=1}^{s=\infty} \left[sn\beta^2 J'_{2sn}(2sn\beta) - s^2n^2(1-\beta^2) \int_0^\beta J_{2sn}(2snx)\,dx \right] \quad \text{...(129)}.$$

85. The series is easily shown to be convergent when $\beta < 1$. Its terms are of necessity positive, since every term such as R_j is by (128) the sum of two squares. Hence R is less than n^2 times the value for the case $n = 1$. But in this case the series can be summed.

In fact we have*

$$\sum_{s=1}^{s=\infty} \frac{J_{2s}(2sx)}{s^2} = \tfrac{1}{2}x^2.$$

Using the differential equation for the Bessel Function we get by differentiation in succession

$$\sum_{s=1}^{s=\infty} s\,J'_{2s}(2sx) = \frac{x}{2(1-x^2)^2}, \qquad \sum_{s=1}^{s=\infty} s^2 J_{2s}(2sx) = \frac{x^2(1+x^2)}{2(1-x^2)^4},$$

$$\sum_{s=1}^{s=\infty} s^2 \int_0^\beta J_{2s}(2sx)\,dx = \frac{\beta^3}{6(1-\beta^2)^3}.$$

Putting $n = 1$ and substituting in (129), we get for a single electron

$$R = \frac{2ce^2\beta^4}{3\rho^2(1-\beta^2)^2} \quad \text{.........................(130)}.$$

Thus the series (129) has a finite sum, certainly less than n^2 times this amount; actually it is very much smaller.

The expression (130) agrees with the value of the radiation given by Liénard†.

[In strictness the sum of the Kapteyn series used above has only been found when $x < 0{\cdot}659\ldots$, but the agreement of the resulting expression (130) with Liénard's expression leaves hardly any doubt as to its correctness up to the limit $x = 1$.]

* Nielsen, *Cylinderfunktionen*, p. 303, 1904.

† *L'Éclairage électrique*, July, 1898, eq. (21).

CHAPTER VIII

ON THE DISTANT FIELD DUE TO A MOVING CHARGE

86. THE number of cases in which tractable expressions can be obtained for the potentials is exceedingly limited, unless the generality of the problem is circumscribed. If however we limit ourselves to the calculation of the field at a distance from the orbit, which is large compared with its dimensions, all the expressions become very much simplified. In this way we may, for instance, calculate the field for polyperiodic and even for some aperiodic motions. As it happens the only direct experimental means we have of investigating the motion of charges in the atom is by means of the spectroscope, which enables us to resolve the distant field due to these charges into its components and to a certain extent to determine their relative intensities. From these experimental results it is required to infer the nature of the motions which produce them. Thus the study of the distant field due to prescribed motions of charges of the most general character is one of the most important problems we have to attack.

We commence with the general expressions of § 9,

$$\phi = \frac{e}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \epsilon^{\iota\mu(t-R/c-\tau)}\frac{d\tau\, d\mu}{R} \quad\text{..................(12)},$$

$$\mathbf{a} = \frac{e}{2\pi c}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \epsilon^{\iota\mu(t-R/c-\tau)}\frac{\mathbf{v}\, d\tau\, d\mu}{R} \quad\text{..............(13)},$$

where we have replaced $\int de$ by e, since the linear dimensions of the charge are exceedingly small compared with the distance R.

We now introduce the approximation of § 72. We write

$$R = r - p,$$

where $p = (\mathbf{r}_1 . \rho)$, and is the projection on the radius vector, $\mathbf{r}$, to the fieldpoint, of the radius vector, ρ, to the charge. Thus, neglecting p in the denominator R, *but not in the exponential* (since it modifies the phase to a finite extent, but the amplitude inappreciably), we get

$$\phi = \frac{e}{2\pi r}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \epsilon^{\iota\mu(t-r/c+p/c-\tau)}\, d\tau\, d\mu \quad\text{...........(131)},$$

$$\mathbf{a} = \frac{e}{2\pi c r}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \epsilon^{\iota\mu(t-r/c+p/c-\tau)}\,\mathbf{v}\, d\tau\, d\mu \quad\text{...........(132)}.$$

87. Using equations (VII) and (VIII), § 3, we get the electric and magnetic forces. We must remember that we need only differentiate the exponential, with respect to t or r, because the differentiation of the factor $1/r$ leads only to terms of order $1/r^2$, which we have already neglected. Hence we may write

$$\text{grad.} = \mathbf{r}_1 \cdot \frac{\partial}{\partial r} = -\mathbf{r}_1 \cdot \frac{\partial}{c\partial t},$$

$$\text{curl} = \left[\mathbf{r}_1 \cdot \frac{\partial}{\partial r}\right] = -\left[\mathbf{r}_1 \cdot \frac{\partial}{c\partial t}\right].$$

Thus we get

$$\mathbf{d} = -\text{grad.}\,\phi - \frac{\partial \mathbf{a}}{c\partial t}$$

$$= \frac{e}{2\pi c r}\frac{\partial}{\partial t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \epsilon^{\iota\mu(t-r/c+p/c-\tau)}\left(\mathbf{r}_1 - \frac{\mathbf{v}}{c}\right) d\tau\, d\mu \quad \text{.........(133)},$$

$$\mathbf{h} = \text{curl}\,\mathbf{a}$$

$$= \frac{e}{2\pi c^2 r}\frac{\partial}{\partial t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \epsilon^{\iota\mu(t-r/c+p/c-\tau)}\,[\mathbf{v}\,\mathbf{r}_1]\, d\tau\, d\mu \quad \text{............(134)}.$$

It is to be noticed that these expressions for **d** and **h** are based merely on our knowledge that the integrals involved in (131) and (132) are functions of t and r only in the combination $t - r/c$. They do not assume that differentiation under the sign of integration is allowable, for the integrals are to be evaluated before the integration is performed.

It is well known that if the integrand is a discontinuous function of τ, differentiation under the sign of integration is not generally allowable*. In our problem the integrand is a function of τ explicitly and implicitly, the latter through p and $\mathbf{v}$. If the motion be continuous in *form*, that is to say, if the differential coefficients of *all* orders of the radius vector ρ be finite and continuous, as in the case of periodic motions, differentiation under the sign of integration is allowable to every order. If however the acceleration, or any differential coefficient of higher order, be discontinuous, as in the case of a motion starting from rest, unlimited differentiation is generally not permissible, and the discontinuity produces an effect which must be allowed for. In our case, where the radius vector ρ and the velocity $\mathbf{v}$ are both continuous of necessity, the integrand itself is always continuous. Hence only one differentiation under the sign of integration would appear to be allowable.

88. Polyperiodic motions. We shall begin with the case where the motion, though not periodic in the ordinary sense, is the geometrical resultant of a set of simply-periodic component motions. The periods cannot all be commensurable, otherwise the motion would be monoperiodic.

* Stokes, "Critical values of the sums of periodic series," Sec. II. *Collected Papers*, Vol. I. p. 271.

We may write
$$\left.\begin{aligned}\xi &= \Sigma a_i \sin(\omega_i\tau + \alpha_i)\\ \eta &= \Sigma b_i \sin(\omega_i\tau + \beta_i)\\ \zeta &= \Sigma c_i \sin(\omega_i\tau + \gamma_i)\end{aligned}\right\} \quad\text{.......................(135)},$$

where (ξ, η, ζ) are the coordinates of the charge at the time τ, and therefore the components of the vector ρ.

Let (l, m, n) be the direction cosines of the unit radius vector $\mathbf{r}_1$ drawn in the direction of the fieldpoint. Then

$$p = (\mathbf{r}_1 . \rho) = l\xi + m\eta + n\zeta = \Sigma p_i \sin(\omega_i\tau + \delta_i)$$

where

$$\left.\begin{aligned}p_i \cos\delta_i &= la_i \cos\alpha_i + mb_i \cos\beta_i + nc_i \cos\gamma_i\\ p_i \sin\delta_i &= la_i \sin\alpha_i + mb_i \sin\beta_i + nc_i \sin\gamma_i\end{aligned}\right\} \quad\text{.........(136)}.$$

Also we have for the components of $\mathbf{v}$

$$\left.\begin{aligned}\dot\xi &= \Sigma \omega_i a_i \cos(\omega_i\tau + \alpha_i)\\ \dot\eta &= \Sigma \omega_i b_i \cos(\omega_i\tau + \beta_i)\\ \dot\zeta &= \Sigma \omega_i c_i \cos(\omega_i\tau + \gamma_i)\end{aligned}\right\} \quad\text{.....................(137)}.$$

These expressions are to be substituted in equations (131)—(134), §§ 86, 87, and the resulting terms developed. As absolute mathematical rigour is not of prime importance we shall employ symbolic methods of development, leaving it to the future to supply a more rigorous treatment.

89. Potentials. We have to develop exp. $\iota\mu(t - r/c + p/c - \tau)$.

By Taylor's Theorem we may write this in the form

$$\epsilon^{pD} \exp. \iota\mu(t - r/c - \tau),$$

where D is written for the operator $\frac{\partial}{c\partial t}$, and is to be treated as if it were an algebraic constant quantity.

Now we have the well-known expansion

$$\epsilon^{\iota x \sin\phi} = \sum_{s=-\infty}^{s=\infty} J_s(x)\, \epsilon^{\iota s\phi}.$$

Hence we find, using (136),

$$\epsilon^{pD} = \Pi\, \epsilon^{p_i D . \sin(\omega_i\tau + \delta_i)} = \Pi \sum_{s_i=-\infty}^{s_i=\infty} J_{s_i}(-\iota p_i D)\, \epsilon^{\iota s_i(\omega_i\tau + \delta_i)}$$

$$= \Sigma_{s_1} \sum_{-\infty}^{\infty}{}_{s_2} \ldots \Sigma_{s_i} J_{s_1}(-\iota p_1 D)\, J_{s_2}(-\iota p_2 D) \ldots J_{s_i}(-\iota p_i D)\, \epsilon^{\iota(\Omega\tau + \Delta)},$$

where $\quad \Omega = s_1\omega_1 + s_2\omega_2 + \ldots + s_i\omega_i, \quad \Delta = s_1\delta_1 + s_2\delta_2 + \ldots + s_i\delta_i$ ......(138).

Substituting this symbolic expression for ϵ^{pD} in (131), we get

$$\phi = \frac{e}{2\pi r} \Sigma_{s_1} \sum_{-\infty}^{\infty}{}_{s_2} \ldots \Sigma_{s_i} J_{s_1}(-\iota p_1 D)\, J_{s_2}(-\iota p_2 D) \ldots$$
$$J_{s_i}(-\iota p_i D) \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \epsilon^{\iota\{\mu(t - r/c - \tau) + \Omega\tau + \Delta\}}\, d\tau\, d\mu.$$

The justification lies in the fact that we may develop the Bessel Function operator in a series of powers of D, that is of $\frac{\partial}{c\partial t}$, and then interchange the order of the integration and of the differentiations to any order; for since the integrand is a continuous function, differentiation to any order under the sign of integration is permissible as we have already explained.

The integral which occurs in ϕ is a Fourier Integral of the usual type, and its value is $2\pi\epsilon^{\iota\{\Omega(t-r/c)+\Delta\}}$.

Hence the differentiation can be performed by the substitution of the quantity $\iota\Omega/c$ in place of the symbol D; thus we get

$$\phi = \frac{e}{r}\Sigma^{s_1}\sum_{-\infty}^{\infty}{}^{s_2}\dots\Sigma^{s_i} J_{s_1}\left(\frac{\Omega p_1}{c}\right) J_{s_2}\left(\frac{\Omega p_2}{c}\right)\dots J_{s_i}\left(\frac{\Omega p_i}{c}\right)\epsilon^{\iota\{\Omega(t-r/c)+\Delta\}} \quad (139).$$

The i summations with respect to the i indices $s_1, s_2, \dots s_i$ are each to be taken from $-\infty$ to $+\infty$. The value of ϕ can easily be written in a real form; for if $(s_1, s_2, \dots s_i)$ denote any set of values of the indices, the sum also contains a term $(-s_1, -s_2, \dots -s_i)$, and we may group each such pair of terms together. The values of Ω and δ for the second term are equal but of opposite sign to those for the first, that is they are $-\Omega$ and $-\delta$. Any Bessel Function, such as $J_{s_i}(\Omega p_i/c)$, in the first term is replaced by $J_{-s_i}(-\Omega p_i/c)$ in the second; since both the order and the argument are merely changed in sign, the value of the Bessel Function is unaltered. Thus the factors of the exponentials in the two terms are equal, and the sum of the two terms has merely $2\cos\{\Omega(t-r/c)+\Delta\}$ in place of the exponential. Hence we may also write

$$\left.\begin{aligned} \phi &= \frac{e}{r}\Sigma^{s_1}\sum_{-\infty}^{\infty}{}^{s_2}\dots\Sigma^{s_i} F(\Omega)\cos\{\Omega(t-r/c)+\Delta\} \\ F(\Omega) &= J_{s_1}\left(\frac{\Omega p_1}{c}\right) J_{s_2}\left(\frac{\Omega p_2}{c}\right)\dots J_{s_i}\left(\frac{\Omega p_i}{c}\right) \end{aligned}\right\} \quad \dots\dots\dots(140),$$

where for convenience of notation the summations from $-\infty$ to $+\infty$ are still retained.

The expression for the vector potential is easily found in the same way. For instance the component a_x involves the factor $\dot{\xi}$ under the sign of integration; but by (137) this may be written

$$\dot{\xi} = \Sigma^i \tfrac{1}{2}\omega_i a_i \{\epsilon^{\iota(\omega_i\tau+\alpha_i)} + \epsilon^{-\iota(\omega_i\tau+\alpha_i)}\}.$$

Hence in the present case the symbolic operator

$$J_{s_1}(-\iota p_1 D)\, J_{s_2}(-\iota p_2 D)\dots J_{s_i}(-\iota p_i D)$$

operates on the function

$$\Sigma^i \frac{\omega_i a_i}{2c}\{\epsilon^{\iota[(\Omega+\omega_i)(t-r/c)+\Delta+\alpha_i]} + \epsilon^{\iota[(\Omega-\omega_i)(t-r/c)+\Delta-\alpha_i]}\}.$$

Thus we get in place of (139)

$$a_x = \frac{e}{2cr} \Sigma^i \omega_i a_i \{\Sigma^{s_1} \sum_{-\infty}^{\infty}{}^{s_2} \ldots \Sigma^{s_i} F(\Omega + \omega_i)\, \epsilon^{\iota[(\Omega+\omega_i)(t-r/c)+\Delta+a_i]}$$
$$+ \Sigma^{s_1} \sum_{-\infty}^{\infty}{}^{s_2} \ldots \Sigma^{s_i} F(\Omega - \omega_i)\, \epsilon^{\iota[(\Omega-\omega_i)(t-r/c)+\Delta-a_i]}\}.$$

This expression may be simplified.

In the first sum replace s_i by $s_i - 1$.

This does not alter the limits of the summation with respect to s_i, which still extends from $-\infty$ to $+\infty$.

The value of $\quad \Omega + \omega_i = s_1\omega_1 + s_2\omega_2 + \ldots + (s_i + 1)\,\omega_i$
is changed into $\quad \Omega = s_1\omega_1 + s_2\omega_2 + \ldots + s_i\omega_i.$

The value of Δ is changed into $\Delta - \delta_i$.

In the function $F(\Omega + \omega_i)$ the factor $J_{s_i}\{(\Omega + \omega_i)\,p_i/c\}$ occurs; it is changed into $J_{s_i-1}(\Omega p_i/c)$; every other factor, such as $J_{s_1}\{(\Omega + \omega_i)\,p_1/c\}$, is changed into $J_{s_1}(\Omega p_1/c)$, its index being unaltered.

Thus the first sum in the bracket becomes

$$\Sigma^{s_1} \sum_{-\infty}^{\infty}{}^{s_2} \ldots \Sigma^{s_i} J_{s_1}\left(\frac{\Omega p_1}{c}\right) J_{s_2}\left(\frac{\Omega p_2}{c}\right) \ldots J_{s_i-1}\left(\frac{\Omega p_i}{c}\right) \epsilon^{\iota[\Omega(t-r/c)+\Delta+a_i-\delta_i]}.$$

Similarly, in the second sum we change s_i into $s_i + 1$, and thus get in the same way

$$\Sigma^{s_1} \sum_{-}^{\infty}{}^{s_2} \ldots \Sigma^{s_i} J_{s_1}\left(\frac{\Omega p_1}{c}\right) J_{s_2}\left(\frac{\Omega p_2}{c}\right) \ldots J_{s_i+1}\left(\frac{\Omega p_i}{c}\right) \epsilon^{\iota[\Omega(t-r/c)+\Delta-a_i+\delta_i]}.$$

The sum of these two terms involves the factor

$$J_{s_i-1}\left(\frac{\Omega p_i}{c}\right) \epsilon^{\iota(a_i-\delta_i)} + J_{s_i+1}\left(\frac{\Omega p_i}{c}\right) \epsilon^{-\iota(a_i-\delta_i)}.$$

This by the properties of the Bessel Function reduces to

$$\frac{2cs_i}{\Omega p_i} J_{s_i}\left(\frac{\Omega p_i}{c}\right) \cos(a_i - \delta_i) + 2\iota J'_{s_i}\left(\frac{\Omega p_i}{c}\right) \sin(a_i - \delta_i).$$

Substituting in the expression found for a_x we get

$$a_x = \frac{e}{r} \Sigma^{s_1} \sum_{-\infty}^{\infty}{}^{s_2} \ldots \Sigma^{s_i} F(\Omega)\, \epsilon^{\iota\{\Omega(t-r/c)+\Delta\}} \left\{\frac{s_i\omega_i a_i \cos(a_i - \delta_i)}{\Omega p_i} + \iota\, \frac{\omega_i a_i J'_{s_i}(\Omega p_i/c) \sin(a_i - \delta_i)}{c J_{s_i}(\Omega p_i/c)}\right\}.$$

By taking together the terms with the set of indices $(s_1, s_2, \ldots s_i)$ and $(-s_1, -s_2, \ldots -s_i)$, and remembering that $J_{-s_i}(-\Omega p_i/c) = J_{s_i}(\Omega p_i/c)$, while $J'_{-s_i}(-\Omega p_i/c) = -J'_{s_i}(\Omega p_i/c)$, we get

$$a_x = \frac{e}{r} \Sigma^{s_1} \sum_{-\infty}^{\infty}{}^{s_2} \ldots \Sigma^{s_i} F(\Omega) \left[\frac{s_i\omega_i a_i \cos(a_i - \delta_i) \cos\{\Omega(t - r/c) + \Delta\}}{\Omega p_i} - \frac{\omega_i a_i J'_{s_i}(\Omega p_i/c) \sin(a_i - \delta_i) \sin\{\Omega(t - r/c) + \Delta\}}{c J_{s_i}(\Omega p_i/c)}\right] \ldots(141).$$

Similar expressions hold for a_y and a_z, with b_i, β_i and c_i, γ_i in place of a_i, α_i.

90. Electric and magnetic forces. The expressions for the electric forces may be found in the same way from (133) and (134), or more simply from (140) and (141), by the equations

$$\mathbf{d} = \frac{\partial}{c\partial t}(\mathbf{r}_1 . \phi - \mathbf{a}),$$

$$\mathbf{h} = \frac{\partial}{c\partial t}[\mathbf{a} . \mathbf{r}_1].$$

Remembering that the components of $\mathbf{r}_1$ are (l, m, n), we get at once from (138), (140) and (141)

$$d_x = \frac{e}{r}\sum_{s_1}\sum_{-\infty}^{\infty} {}_{s_2} \ldots \sum_{s_i} F(\Omega) \left[\frac{s_i\omega_i\{a_i\cos(\alpha_i - \delta_i) - lp_i\}\sin\{\Omega(t - r/c) + \Delta\}}{cp_i} + \frac{\Omega\omega_i J'_{s_i}(\Omega p_i/c)\, a_i \sin(\alpha_i - \delta_i)\cos\{\Omega(t - r/c) + \Delta\}}{c^2 J_{s_i}(\Omega p_i/c)}\right] \quad \ldots(142),$$

$$h_x = \frac{e}{r}\sum_{s_1}\sum_{-\infty}^{\infty} {}_{s_2} \ldots \sum_{s_i} F(\Omega)$$
$$\times \left[\frac{s_i\omega_i\{mc_i\cos(\gamma_i - \delta_i) - nb_i\cos(\beta_i - \delta_i)\}\sin\{\Omega(t - r/c) + \Delta\}}{cp_i} + \frac{\Omega\omega_i J'_{s_i}(\Omega p_i/c)\{mc_i\sin(\gamma_i - \delta_i) - nb_i\sin(\beta_i - \delta_i)\}\cos\{\Omega(t - r/c) + \Delta\}}{c^2 J_{s_i}(\Omega p_i/c)}\right]$$
$$\ldots\ldots\ldots(143),$$

with similar expressions for the remaining components.

The process employed in deducing these absolutely general expressions for the field due to any polyperiodic motion whatsoever cannot claim to be rigorous; but it is extremely difficult to deduce them in any other way. There is however no reason to doubt their substantial correctness, so long as the series remain convergent. We shall verify below that they give the expressions (123) and (124), § 82, for the forces found by a rigorous method in problem 1 of Chap. VII.

91. Character of the distant field. We shall now discuss the character of the field represented by the expressions (142) and (143). We easily deduce the following conclusions:

(1) Each force component consists of an infinity, of multiplicity i, of simple harmonic vibrations.

The type of their argument is $\Omega(t - r/c) + \Delta$, where

$$\Omega = s_1\omega_1 + s_2\omega_2 + \ldots + s_i\omega_i, \quad \Delta = s_1\delta_1 + s_2\delta_2 + \ldots + s_i\delta_i.$$

Thus the frequency of each harmonic is a linear function with integral coefficients of i fundamental frequencies $\omega_1, \omega_2, \ldots \omega_i$. The fundamental frequencies are those of the harmonic constituents of the polyperiodic motion which emits the vibrations; the vibrations are as it were sum and difference harmonics of the fundamentals.

(2) Each harmonic component vibration, e.g. that of frequency Ω, consists of i component vibrations, each of which can be resolved into two components proportional to $\begin{smallmatrix}\sin\\ \cos\end{smallmatrix}\{\Omega(t-r/c)+\Delta\}$.

In d_x the amplitude of the sine component is of the type

$$\frac{e}{r}F(\Omega)\frac{s_i\omega_i\{a_i\cos(\alpha_i-\delta_i)-lp_i\}}{cp_i},$$

and that of the cosine component is of the type

$$\frac{e}{r}F(\Omega)\frac{\Omega\omega_i J_s'(\Omega p_i/c)\,a_i\sin(\alpha_i-\delta_i)}{c^2 J_{s_i}(\Omega p_i/c)}.$$

Now we easily find from (136), § 88,

$$p_i = la_i\cos(\alpha_i-\delta_i)+mb_i\cos(\beta_i-\delta_i)+nc_i\cos(\gamma_i-\delta_i),$$
$$0 = la_i\sin(\alpha_i-\delta_i)+mb_i\sin(\beta_i-\delta_i)+nc_i\sin(\gamma_i-\delta_i).$$

Hence we see at once that each of the two components is perpendicular to the radius vector (l, m, n).

In the same way each component of h_x is perpendicular to the radius vector (l, m, n), as well as to the component of the same phase in d_x.

Thus each component harmonic vibration is transverse, with electric and magnetic forces at right angles.

(3) Choose (l', m', n') and (l'', m'', n''), any two directions at right angles to each other and to the radius vector (l, m, n) and forming with it a right-handed system. Let the ith components of the electric and magnetic forces of the harmonic Ω be d_i', h_i' and d_i'', h_i'' in these two directions. We easily find from (142) and (143)

$$d_i' = h_i'' = \frac{2e}{r}F(\Omega)\frac{s_i\omega_i}{cp_i}[C_i'\sin\{\Omega(t-r/c)+\Delta\}$$
$$+k_iS_i'\cos\{\Omega(t-r/c)+\Delta\}] \quad \ldots(144),$$

$$d_i'' = -h_i' = \frac{2e}{r}F(\Omega)\frac{s_i\omega_i}{cp_i}[C_i''\sin\{\Omega(t-r/c)+\Delta\}$$
$$+k_iS_i''\cos\{\Omega(t-r/c)+\Delta\}] \quad \ldots(145),$$

where

$$\left.\begin{aligned} C_i' &= S\,l'a_i\cos(\alpha_i-\delta_i), \quad S_i' = S\,l'a_i\sin(\alpha_i-\delta_i)\\ C_i'' &= S\,l''a_i\cos(\alpha_i-\delta_i), \quad S_i'' = S\,l''a_i\sin(\alpha_i-\delta_i)\\ k_i &= \frac{\Omega p_i J'_{s_i}(\Omega p_i/c)}{cs_iJ_{s_i}(\Omega p_i/c)} \end{aligned}\right\} \quad \ldots\ldots(146).$$

As before we have

$$F(\Omega) = J_{s_1}\left(\frac{\Omega p_1}{c}\right) J_{s_2}\left(\frac{\Omega p_2}{c}\right) \dots J_{s_i}\left(\frac{\Omega p_i}{c}\right) \quad \text{............(140)},$$

$$p_i = S\, la_i \cos(\alpha_i - \delta_i),\ \ 0 = S\, la_i \sin(\alpha_i - \delta_i),\ \text{by (136)},$$

$$\Omega = s_1\omega_1 + s_2\omega_2 + \dots + s_i\omega_i,\ \ \Delta = s_1\delta_1 + s_2\delta_2 + \dots + s_i\delta_i \text{......(138)}.$$

S denotes a sum of three terms, one for each coordinate.

The factor 2 is introduced in order to take account of the fact that to each term of (142), or (143), with the set of indices $(s_1, s_2, \dots s_i)$ there corresponds an equal term with each index equal, but of opposite sign to $(s_1, s_2, \dots s_i)$.

There are i terms of the type (144), or (145), one for each of the i component simple harmonic motions which constitute the motion of the charge. These together give the harmonic vibration of frequency Ω under consideration.

They have a common factor $2eF(\Omega)/r$, involving the product of i Bessel Functions given in (140).

Now the wave-length λ of the harmonic is equal to $2\pi c/\Omega$, so that the arguments of the Bessel Functions are, in order, $2\pi p_1/\lambda$, $2\pi p_2/\lambda$, ... $2\pi p_i/\lambda$. Since $p_1, p_2, \dots p_i$ are of the order of the linear dimensions of the orbit, we see that if λ correspond to any observable line of the spectrum, each of the arguments is exceedingly small. Thus the harmonic is very weak unless the indices $s_1, s_2, \dots s_i$ are small.

We cannot expect a visible line to be produced unless this be the case.

(4) Eq. (144) shows that each component of the harmonic is an elliptic vibration. By the ordinary formulae we see that the axes of the ellipse are equal to

$$\left.\begin{aligned} &\frac{2eF(\Omega)\, s_i\omega_i \sqrt{(C_i'^2 + k_i^2 S_i'^2)}}{rcp_i} \\ \text{and}\quad &\frac{2eF(\Omega)\, s_i\omega_i \sqrt{(C_i''^2 + k_i^2 S_i''^2)}}{rcp_i} \end{aligned}\right\} \text{..................(147)},$$

and that the first axis is reached from (l', m', n') by rotation towards (l'', m'', n'') through the angle θ_i, where

$$\tan 2\theta_i = \frac{2\,(C_i' C_i'' + k_i^2 S_i' S_i'')}{C_i''^2 - C_i'^2 + k_i^2 (S_i''^2 - S_i'^2)} \text{..............(148)}.$$

The rotation in the ellipse is right or left-handed according as

$$C_i'' S_i' - C_i' S_i'' > \text{or} < 0.$$

(5) It is worthy of note that the vibration cannot be linear unless either

$$C_i' = S_i' = 0,\ \text{ or }\ C_i'' = S_i'' = 0,\ \text{ or }\ k_i = 0.$$

The first case gives

$$l' : m' : n' = b_i c_i \sin(\beta_i - \gamma_i) : c_i a_i \sin(\gamma_i - \alpha_i) : a_i b_i \sin(\alpha_i - \beta_i),$$

which is consistent with the condition $ll' + mm' + nn' = 0$ only if

$$l\,\frac{\sin(\beta_i - \gamma_i)}{a_i} + m\,\frac{\sin(\gamma_i - \alpha_i)}{b_i} + n\,\frac{\sin(\alpha_i - \beta_i)}{c_i} = 0 \quad \text{......(149).}$$

The second case is the same as this, merely with (l'', m'', n'') in place of (l', m', n'), and thus is not essentially distinct.

(149) is the equation of a plane through the origin; for every direction (l, m, n) lying in this plane the elliptic vibration becomes linear.

This plane of linear polarization is generally different for each component of the harmonic.

The third case occurs whenever $\Omega p_i/c$, that is, $2\pi p_i/\lambda$, is one of the roots of the equation

$$J'_{s_i}(x) = 0 \quad \text{..............................(150).}$$

Now by (136) we have

$$l^2 a_i^2 + m^2 b_i^2 + n^2 c_i^2 + 2mn\, b_i c_i \cos(\beta_i - \gamma_i) + 2nl\, c_i a_i \cos(\gamma_i - \alpha_i)$$
$$+ 2lm\, a_i b_i \cos(\alpha_i - \beta_i) = p_i^2 = \left(\frac{\lambda x}{2\pi}\right)^2 \quad \text{...(151).}$$

This is the equation of a quadric cone on which the radius vector (l, m, n) must lie in order that the root x of (150) may correspond to linear polarization.

Since the least root of $J'_{s_i}(x) = 0$ is of the order $\frac{1}{4}(2s_i + 1)\pi$, the condition in this case cannot be satisfied for light waves, for which $2\pi p_i/\lambda$ is always very small. Hence this case is of no practical importance.

We shall not consider the general case any further, but proceed to the study of some illustrative problems.

92. Problem 1. Uniform circular motion. With the notation of the last chapter we get by (135) and (136), § 88,

$$\xi = \rho\cos(\omega\tau + \delta), \quad \eta = \rho\sin(\omega\tau + \delta), \quad \zeta = 0,$$

giving $\quad \omega_1 = \omega, \quad a_1 = b_1 = \rho, \quad c_1 = 0, \quad \alpha_1 = \delta + \frac{1}{2}\pi, \quad \beta_1 = \delta.$

Also $\quad l = \sin\theta\cos\phi, \quad m = \sin\theta\sin\phi, \quad n = \cos\theta.$

Hence $\quad p_1 = \rho\sin\theta, \quad \delta_1 = \delta - \phi + \frac{1}{2}\pi.$

Thus there is only a single component vibration.

In the equations (138)—(140), § 89, we have

$$\Omega = s\omega, \quad \Delta = s(\delta - \phi + \tfrac{1}{2}\pi),$$

$$F(\Omega) = J_s(s\beta\sin\theta), \quad kF(\Omega) = \beta\sin\theta J_s'(s\beta\sin\theta).$$

We shall choose (l', m', n') in the direction of increasing θ and (l'', m'', n'') in the direction of increasing ϕ; hence

$$l' = \cos\theta\cos\phi,\quad m' = \cos\theta\sin\phi,\quad n' = -\sin\theta,$$
$$l'' = \quad -\sin\phi,\quad m'' = \quad \cos\phi,\quad n'' = 0.$$

With these values we get from (146)

$$C' = \rho\cos\theta,\quad S' = 0,\quad C'' = 0,\quad S'' = -\rho.$$

Substituting in (144) and (145), writing $\psi = \omega(t - r/c) + \delta - \phi + \frac{1}{2}\pi$, and summing for s, we get

$$d_\theta = h_\phi = \frac{2e\beta\cot\theta}{\rho r}\sum_{s=1}^{s=\infty} sJ_s(s\beta\sin\theta)\sin s\psi,$$
$$d_\phi = -h_\theta = -\frac{2e\beta^2}{\rho r}\sum_{s=1}^{s=\infty} sJ_s'(s\beta\sin\theta)\cos s\psi,$$

which are identical with (123) and (124), § 82. This will serve as a verification of the general formulae.

93. Problem 2. Simple harmonic rectilinear motion.

Suppose Oz to be the line of motion, and let

$$\zeta = a\sin(\omega\tau + \alpha).$$

With the usual notation we get by (136), § 88,

$$p_1 = a\cos\theta,\quad \delta_1 = \alpha = \gamma_1.$$

By (138) and (140), § 89, we have

$$\Omega = s\omega,\quad \Delta = s\alpha,\quad F(\Omega) = J_s(s\beta\cos\theta),$$

where $\beta = a\omega/c$, so that β is the ratio of the maximum velocity to that of light.

Take (l', m', n') and (l'', m'', n'') in the directions of increasing θ and ϕ as before; then

$$l' = \cos\theta\cos\phi,\quad m' = \cos\theta\sin\phi,\quad n' = -\sin\theta,$$
$$l'' = \quad -\sin\phi,\quad m'' = \quad \cos\phi,\quad n'' = 0.$$

By (146)

$$C' = -a\sin\theta,\quad S' = 0,\quad C'' = 0,\quad S'' = 0.$$

Hence we get from (144) and (145)

$$\left.\begin{aligned} d_\theta &= h_\phi = -\frac{2e\beta\tan\theta}{ar}\sum_{s=1}^{s=\infty} sJ_s(s\beta\cos\theta)\sin s[\omega(t - r/c) + \alpha] \\ d_\phi &= -h_\theta = 0 \end{aligned}\right\} \quad \ldots(152).$$

Thus the electric force is in the meridian, the magnetic force along the parallel. The vibration is linearly polarized at right angles to the meridian.

The amplitude vanishes on the axis for every harmonic.

For all harmonics, except the first, it also vanishes at the equator; but when $s = 1$ its value there is $e\beta^2/ra$, and is a maximum.

Since $\beta < 1$, $J_s(s\beta\cos\theta)$ can never vanish, and is constantly positive or negative according as θ lies between 0 and $\frac{1}{2}\pi$, or between $\frac{1}{2}\pi$ and π, when s is odd, while it is always positive when s is even. Thus the product $\tan\theta J_s(s\beta\cos\theta)$ is always positive when s is odd, but is positive or negative according as θ lies between 0 and $\frac{1}{2}\pi$, or between $\frac{1}{2}\pi$ and π, when s is even.

Hence for even values of s the electric and magnetic forces are of opposite sign on opposite sides of the equator, the electric force in particular being towards or away from both poles at the same time. For odd values of s they are of the same sign, so that the electric force is from pole to pole, in the same direction on both sides of the equator, but reversing every half-period. The only node occurs at the equator.

94. Radiation. The average value of the Poynting vector due to the harmonic s is given by

$$\bar{\mathbf{s}}_s = \frac{ce^2\beta^2\tan^2\theta}{2\pi a^2 r^2} s^2 \{J_s(s\beta\cos\theta)\}^2 . \mathbf{r}_1 .$$

The radiation due to it is equal to

$$R_s = \frac{ce^2\beta^2 s^2}{a^2}\int_0^\pi \{J_s(s\beta\cos\theta)\}^2 \tan^2\theta \sin\theta\, d\theta$$

$$= \frac{ce^2\beta^2 s^2}{a^2}\int_{-1}^1 \frac{\{J_s(s\beta\mu)\}^2(1-\mu^2)\,d\mu}{\mu^2} .$$

It does not appear to be possible to find this integral in finite terms, but it is possible to find the total radiation R. We get

$$R = \frac{ce^2\beta^2}{a^2}\int_{-1}^1\left(\sum_{s=1}^{s=\infty} s^2\{J_s(s\beta\mu)\}^2\right)\frac{(1-\mu^2)\,d\mu}{\mu^2} \quad \text{.........(153).}$$

We require the series of Bessel Functions.

Nielsen gives the following series*

$$\sum_{s=1}^{s=\infty}\frac{\{J_s(sx)\}^2}{s^2} = \tfrac{1}{4}x^2 .$$

By differentiation we get

$$\sum_{s=1}^{s=\infty}\frac{J_s(sx)J_s'(sx)}{s} = \tfrac{1}{4}x, \quad \sum_{s=1}^{s=\infty}[J_s(sx)J_s''(sx) + \{J_s'(sx)\}^2] = \tfrac{1}{4}.$$

Now by the differential equation for the Bessel Function we have

$$\frac{(1-x^2)J_s(sx)}{x^2} = J_s''(sx) + \frac{J_s'(sx)}{sx} .$$

Hence, multiplying by $J_s(sx)$ and summing, we get

$$\frac{1-x^2}{x^2}\sum_{s=1}^{s=\infty}\{J_s(sx)\}^2 = \sum_{s=1}^{s=\infty} J_s(sx)J_s''(sx) + \sum_{s=1}^{s=\infty}\frac{J_s(sx)J_s'(sx)}{sx}$$

$$= \sum_{s=1}^{s=\infty} J_s(sx)J_s''(sx) + \tfrac{1}{4},$$

* *Cylinderfunktionen*, p. 307, eq. (8) (with $\nu = 0$, $p = 1$).

by the second equation above. Again, multiplying by $2sx^2 J_s'(sx)$ and summing, we get

$$(1-x^2)\frac{d}{dx}\sum_{s=1}^{s=\infty}\{J_s(sx)\}^2 = \frac{d}{dx}x^2\sum_{s=1}^{s=\infty}\{J_s'(sx)\}^2.$$

Multiplying the previous equation by x^2, differentiating and adding to the last, we get

$$2(1-x^2)\frac{d}{dx}\sum_{s=1}^{s=\infty}\{J_s(sx)\}^2 - 2x\sum_{s=1}^{s=\infty}\{J_s(sx)\}^2 = x,$$

by the third equation above.

Integrating the last equation and remembering that the sum vanishes identically for $x=0$, we get

$$\sum_{s=1}^{s=\infty}\{J_s(sx)\}^2 = \frac{1}{2\sqrt{(1-x^2)}} - \frac{1}{2}.$$

The same process gives

$$\sum_{s=1}^{s=\infty} s^2\{J_s(sx)\}^2 = \frac{x^2(4+x^2)}{16(1-x^2)^{7/2}}.$$

Substituting in (153), we get

$$R = \frac{ce^2\beta^4}{16a^2}\int_{-1}^{1}\frac{(4+\beta^2\mu^2)(1-\mu^2)\,d\mu}{(1-\beta^2\mu^2)^{7/2}}$$

$$= \frac{ce^2\beta^4(4-3\beta^2)}{12a^2(1-\beta^2)^{3/2}} \quad \text{..................................(154).}$$

Remembering that $\beta = \omega a/c = 2\pi a/\lambda$, we find that this reduces to $16\pi^4 ce^2a^2/3\lambda^4$ when β is very small. This is the value given by Larmor for this case*

The value (154) may easily be obtained direct from the general formula of Liénard, which for rectilinear motion gives for the rate of radiation

$$\dot{R} = \frac{2ce^2\dot{v}^2}{3c^4(1-v^2/c^2)^3}.$$

Substituting $v/c = \beta\cos(\omega t+\alpha)$, $\dot{v}/c^2 = -\beta^2\sin(\omega t+\alpha)/a$, and integrating with respect to t, we get (154). This will serve as another verification of the general formulae (144) and (145).

95. **Problem 3. Elliptic motion about the centre.** We may write

$$\xi = a\cos(\omega\tau+\alpha),\quad \eta = b\sin(\omega\tau+\alpha),\quad \zeta = 0.$$

Hence there is but one period, so that

$$\omega_1=\omega,\quad a_1=a,\quad b_1=b,\quad c_1=0,\quad \alpha_1=\alpha+\tfrac{1}{2}\pi,\quad \beta_1=\alpha.$$

* *Aether and Matter*, p. 226.

By (136), § 88, we get, using polar coordinates (r, θ, ϕ),

$$p_1 \cos\delta_1 = \sin\theta\,(-a\cos\phi\sin\alpha + b\sin\phi\cos\alpha),$$
$$p_1 \sin\delta_1 = \sin\theta\,(\;\;a\cos\phi\cos\alpha + b\sin\phi\sin\alpha).$$

Write $$a\cos\phi = \rho\cos\psi,\quad b\sin\phi = \rho\sin\psi,$$

so that $$\rho = \sqrt{(a^2\cos^2\phi + b^2\sin^2\phi)},\quad \tan\psi = \frac{b\tan\phi}{a}.$$

Then we get $$p_1 = \rho\sin\theta,\quad \delta_1 = \alpha - \psi + \tfrac{1}{2}\pi.$$

By (138), § 89, we have

$$\Omega = s\omega,\quad \Delta = s\delta_1 = s\,(\alpha - \psi + \tfrac{1}{2}\pi).$$

Choose (l', m', n') and (l'', m'', n'') in the directions of increasing θ and ϕ; then we find by (146)

$$C_1' = \rho\cos\theta,\quad S_1' = 0,\quad C_1'' = -\frac{(a^2 - b^2)\sin\phi\cos\phi}{\rho},\quad S_1'' = -\frac{ab}{\rho},$$

$$k_1 = \frac{\omega\rho\sin\theta\, J_s'(s\omega\rho\sin\theta/c)}{cJ_s(s\omega\rho\sin\theta/c)}.$$

With these values we get from (144) and (145)

$$d_\theta = h_\phi = \frac{2e\omega\cot\theta}{cr}\sum_{s=1}^{s=\infty} sJ_s(s\omega\rho\sin\theta/c)\sin s\,[\omega(t - r/c) + \alpha - \psi + \tfrac{1}{2}\pi]\ldots(155),$$

$$d_\phi = -h_\theta = -\frac{2e\omega^2 ab}{c^2\rho r}\sum_{s=1}^{s=\infty} sJ_s'(s\omega\rho\sin\theta/c)\cos s\,[\omega(t - r/c) + \alpha - \psi + \tfrac{1}{2}\pi]$$
$$-\frac{2e\omega(a^2 - b^2)\sin\phi\cos\phi}{c\rho^2 r\sin\theta}\sum_{s=1}^{s=\infty} sJ_s(s\omega\rho\sin\theta/c)\sin s\,[\omega(t - r/c) + \alpha - \psi + \tfrac{1}{2}\pi]$$
$$\ldots\ldots\ldots(156).$$

It is easily seen that, when $a = b = \rho$, these expressions reduce to those for the circle, (123) and (124), § 82; when $b = 0$ they reduce to (152), § 93, provided we make allowance for the difference in the coordinates used.

In the diagram $ABA'B'$ is the ellipse and $ACA'C'$ its eccentric circle. P is the projection of the fieldpoint (r, θ, ϕ) on the plane of the ellipse, and Q is the point for which the eccentric angle is ϕ.

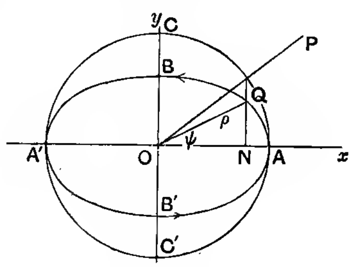

Fig. 37.

Then ρ is the radius OQ, and ψ is the angle xOQ, which OQ makes with Ox.

In (155) and (156) we may if we wish express everything in terms of ψ; we easily find

$$\rho = \frac{ab}{\sqrt{(a^2\sin^2\psi + b^2\cos^2\psi)}},$$

$$\frac{\sin\phi\cos\phi}{\rho^2} = \frac{\sin\psi\cos\psi}{ab}.$$

We notice that along the axis, where $\theta = 0$ or π, we have

$$d_\theta = h_\phi = \frac{e\omega^2\rho}{c^2 r}\sin\left[\omega\left(t - r/c\right) + \alpha - \psi + \tfrac{1}{2}\pi\right],$$

$$d_\phi = -\,h_\theta = -\frac{e\omega^2 ab}{c^2\rho r}\cos\left[\omega\left(t - r/c\right) + \alpha - \psi + \tfrac{1}{2}\pi\right]$$
$$-\frac{e\omega^2\left(a^2 - b^2\right)\sin\phi\cos\phi}{c^2\rho r}\sin\left[\omega\left(t - r/c\right) + \alpha - \psi + \tfrac{1}{2}\pi\right].$$

Apparently the forces depend on ϕ, which would be absurd; but we easily find that

$$d_x = d_\theta\cos\phi - d_\phi\sin\phi = \frac{e\omega^2 a}{c^2 r}\sin\left[\omega\left(t - r/c\right) + \alpha + \tfrac{1}{2}\pi\right],$$

$$d_y = d_\theta\sin\phi + d_\phi\cos\phi = -\frac{e\omega^2 b}{c^2 r}\cos\left[\omega\left(t - r/c\right) + \alpha + \tfrac{1}{2}\pi\right],$$

that is to say

$$d_x = \frac{e\omega^2\xi\left(t - r/c\right)}{c^2 r},\quad d_y = \frac{e\omega^2\eta\left(t - r/c\right)}{c^2 r},$$

where $\xi\left(t - r/c\right)$ and $\eta\left(t - r/c\right)$ denote the coordinates of the charge at the time of emission, $t - r/c$.

Thus on the axis the electric force is always in the meridian plane drawn through the charge in its position at the time of emission. The change of the electric force is exactly proportional to the change in the radius vector to the charge.

This explains the apparent difficulty.

On the equator, where $\theta = \frac{1}{2}\pi$, we get

$$d_\theta = h_\phi = 0,\ \ d_\phi = -\,h_\theta = -\frac{2e\omega^2 ab}{c^2\rho r}\sum_{s=1}^{s=\infty} sJ_s'\left(\frac{s\omega\rho}{c}\right)\cos s\left[\omega\left(t - r/c\right) + \alpha - \psi + \tfrac{1}{2}\pi\right]$$
$$-\frac{2e\omega\left(a^2 - b^2\right)\sin\phi\cos\phi}{c\rho^2 r}\sum_{s=1}^{s=\infty} sJ_s\left(\frac{s\omega\rho}{c}\right)\sin s\left[\omega\left(t - r/c\right) + \alpha - \psi + \tfrac{1}{2}\pi\right].$$

The vibration is linear and polarized in the meridian.

For every other direction the vibration is elliptic.

96. **Group of charges.** A number, n, of charges describing the ellipse in succession form a group, that is, partially absorb each other's radiation, provided the epoch α differs by the angle $2\pi/n$ from one charge to the next. For in this case, on summing for all the charges, all the circular functions in (155) and (156) disappear, except those for which s is a multiple of n.

Thus the forces due to the group of n charges are given by (155) and (156), provided s be replaced by sn, and the expressions be multiplied by n, just as in § 82.

Now the angle $\omega\tau + \alpha$ is the eccentric angle of the charge at the time τ; hence n charges moving in an ellipse form a group provided their eccentric angles are always in arithmetical progression.

97. Radiation. We see from (155) and (156) that the average value of the Poynting vector due to the harmonic of order s is given by

$$\bar{\mathbf{S}}_s = \frac{c\,(\overline{d_\theta^2} + \overline{d_\phi^2})}{4\pi}$$

$$= \frac{e^2\omega^2 s^2}{2\pi c r^2}\left[\left\{\cot^2\theta + \frac{(a^2-b^2)^2\sin^2\phi\cos^2\phi}{\sin^2\theta\,(a^2\cos^2\phi + b^2\sin^2\phi)^2}\right\}\left\{J_s\left(\frac{s\omega\rho\sin\theta}{c}\right)\right\}^2 + \frac{\omega^2a^2b^2}{c^2\rho^2}\left\{J_s'\left(\frac{s\omega\rho\sin\theta}{c}\right)\right\}^2\right]\mathbf{r}_1,$$

$$R_s = \frac{e^2\omega^2s^2}{2\pi c}\int_0^\pi\int_0^{2\pi}\left[\left\{\cot^2\theta + \frac{(a^2-b^2)^2\sin^2\phi\cos^2\phi}{\sin^2\theta\,(a^2\cos^2\phi + b^2\sin^2\phi)^2}\right\}\left\{J_s\left(\frac{s\omega\rho\sin\theta}{c}\right)\right\}^2 + \frac{\omega^2a^2b^2}{c^2(a^2\cos^2\phi+b^2\sin^2\phi)}\left\{J_s'\left(\frac{s\omega\rho\sin\theta}{c}\right)\right\}^2\right]\sin\theta\,d\theta\,d\phi \quad\ldots(157).$$

There seems little hope of evaluating this integral; but we can sum the series for s from 1 to ∞ and thus get an integral for the total radiation R.

In § 94, p. 122, we found

$$\sum_{s=1}^{s=\infty} s^2\{J_s(sx)\}^2 = \frac{x^2(4+x^2)}{16\,(1-x^2)^{7/2}}.$$

Similarly we find

$$\sum_{s=1}^{s=\infty} s^2\{J_s'(sx)\}^2 = \frac{4+3x^2}{16\,(1-x^2)^{5/2}}.$$

Put $x = \omega\rho\sin\theta/c$ and substitute in (157); we get

$$R = \sum_{s=1}^{s=\infty} R_s$$

$$= \frac{e^2\omega^4}{32\pi c^3}\int_0^\pi\int_0^{2\pi}\left[\left\{\rho^2\cos^2\theta + \frac{(a^2-b^2)^2\sin^2\phi\cos^2\phi}{\rho^2}\right\}\frac{4+\dfrac{\omega^2\rho^2\sin^2\theta}{c^2}}{\left(1-\dfrac{\omega^2\rho^2\sin^2\theta}{c^2}\right)^{7/2}} + \frac{a^2b^2}{\rho^2}\,\frac{4+3\dfrac{\omega^2\rho^2\sin^2\theta}{c^2}}{\left(1-\dfrac{\omega^2\rho^2}{c^2}\sin^2\theta\right)^{5/2}}\right]\sin\theta\,d\theta\,d\phi,$$

where $$\rho^2 = a^2\cos^2\phi + b^2\sin^2\phi.$$

By means of the substitution $\omega\rho\cos\theta = \tan\chi\sqrt{(c^2 - \omega^2\rho^2)}$, we get after some transformation

$$R = \frac{e^2\omega^2}{16\pi c}\int_0^{\frac{\pi}{2}} \left[2\frac{3 - \frac{\omega^2(a^2+b^2)}{c^2} + 7\frac{\omega^4 a^2 b^2}{c^4}}{\left(1 - \frac{\omega^2\rho^2}{c^2}\right)^2} - \frac{6 - 7\frac{\omega^2(a^2+b^2)}{c^2}}{1 - \frac{\omega^2\rho^2}{c^2}} \right] d\phi.$$

This gives on integration

$$R = \frac{e^2\omega^2\left\{4\frac{\omega^2(a^2+b^2)}{c^2} + 4\frac{\omega^4 a^2 b^2}{c^4} - 3\frac{\omega^4(a^2+b^2)^2}{c^4}\right\}}{12c\left(1 - \frac{\omega^2 a^2}{c^2}\right)^{3/2}\left(1 - \frac{\omega^2 b^2}{c^2}\right)^{3/2}} \quad \text{......(158)}.$$

When we put $a = b = \rho$, $\omega a/c = \omega b/c = \beta$, this reduces to the value for the circle, (130), § 85.

When we put $b = 0$, it reduces to the value given above for the rectilinear vibration.

Thus there can be no doubt as to the correctness of the result, which may indeed be deduced directly from Liénard's general expression for the rate of radiation*

$$\dot{R} = \tfrac{2}{3}ce^2\left\{\frac{v^4/c^4}{\rho^2(1 - v^2/c^2)^2} + \frac{\dot{v}^2/c^4}{(1 - v^2/c^2)^3}\right\},$$

where ρ is the radius of curvature of the orbit.

Thus we have another verification of the formulae (144) and (145), § 91.

98. Problem 4. An electron is disturbed from its state of uniform circular motion. To find the field produced.

We shall use the notation of problem 1, § 78.

We shall use the method of representation of Maxwell, invented for the problem of Saturn's Rings†.

Write
$$\xi = \rho(1 - \mu)\cos(\omega\tau + \delta + \lambda),$$
$$\eta = \rho(1 - \mu)\sin(\omega\tau + \delta + \lambda),$$
$$\zeta = \rho\nu.$$

Thus (λ, μ, ν) measure the component displacements of the electron from its position in steady motion, in the direction of the tangent, inward radius and axis respectively.

* *L'Éclairage électrique*, July, 1898. See also Searle, *Phys. Zeitsch.* 9, p. 887, 1908.

† *Collected Papers*, vol. I. pp. 321—323.

We shall suppose them to be small and shall neglect their squares and products in the usual way. Then we get

$$\xi = \rho\,\{(1-\mu)\cos(\omega\tau+\delta) - \lambda\sin(\omega\tau+\delta)\},$$
$$\eta = \rho\,\{(1-\mu)\sin(\omega\tau+\delta) + \lambda\cos(\omega\tau+\delta)\},$$
$$\zeta = \rho\nu.$$

We shall suppose (λ, μ, ν) to be sums of periodic functions of τ, in accordance with our general assumptions for the present chapter. Thus we write

$$\lambda = \Sigma A\sin(q\tau+\alpha),$$
$$\mu = \Sigma B\sin(q\tau+\beta),$$
$$\nu = \Sigma C\sin(q'\tau+\gamma).$$

The frequency is relative to the ring, not absolute.

With these values we get

$$\xi = \rho\sin(\omega\tau+\delta+\tfrac{1}{2}\pi) + \Sigma\tfrac{1}{2}\rho\sqrt{\{A^2+B^2+2AB\sin(\alpha-\beta)\}}\sin\{(q+\omega)\tau+\delta+\varpi+\tfrac{1}{2}\pi\}$$
$$+\Sigma\tfrac{1}{2}\rho\sqrt{\{A^2+B^2-2AB\sin(\alpha-\beta)\}}\sin\{(q-\omega)\tau-\delta+\upsilon-\tfrac{1}{2}\pi\},$$
$$\eta = \rho\sin(\omega\tau+\delta) + \Sigma\tfrac{1}{2}\rho\sqrt{\{A^2+B^2+2AB\sin(\alpha-\beta)\}}\sin\{(q+\omega)\tau+\delta+\varpi\}$$
$$+\Sigma\tfrac{1}{2}\rho\sqrt{\{A^2+B^2-2AB\sin(\alpha-\beta)\}}\sin\{(q-\omega)\tau-\delta+\upsilon\},$$
$$\zeta = \Sigma\rho C\sin(q'\tau+\gamma),$$

where $$\tan\varpi = \frac{A\sin\alpha+B\cos\beta}{A\cos\alpha-B\sin\beta},\quad \tan\upsilon = \frac{A\sin\alpha-B\cos\beta}{A\cos\alpha+B\sin\beta}.$$

These expressions are of the form (135), § 88, where we must write

$$\left.\begin{array}{l}\omega_1=\omega,\ a_1=b_1=\rho,\ c_1=0,\ \alpha_1=\delta+\tfrac{1}{2}\pi,\ \beta_1=\delta\\ \omega_2=q+\omega,\ a_2=b_2=\tfrac{1}{2}\rho\sqrt{\{A^2+B^2+2AB\sin(\alpha-\beta)\}},\ c_2=0,\\ \qquad\qquad\alpha_2=\delta+\varpi+\tfrac{1}{2}\pi,\ \beta_2=\delta+\varpi\\ \omega_3=q-\omega,\ a_3=b_3=\tfrac{1}{2}\rho\sqrt{\{A^2+B^2-2AB\sin(\alpha-\beta)\}},\ c_3=0,\\ \qquad\qquad\alpha_3=-\delta+\upsilon-\tfrac{1}{2}\pi,\ \beta_3=-\delta+\upsilon\\ \omega_4=q',\ a_4=b_4=0,\ c_4=\rho C,\ \gamma_4=\gamma\end{array}\right\}\ldots(159)$$

Thus we have to deal with a preponderant uniform circular motion (index 1), compounded with a number of triplets of small harmonic motions, in general of incommensurable periods.

Two of the constituent motions of each triplet are in the plane of the undisturbed motion, and are sum and difference motions of the uniform circular motion and the given relative disturbance. Their amplitudes and phases are different, the amplitude B of the radial disturbance appearing with opposite sign in the two.

The third constituent of the triplet constitutes an independent motion of the same frequency as the original relative disturbance parallel to the axis.

In general the quantities q and ω are incommensurable, so that the same is true of the periods of the constituent motions.

By means of (136), § 88, and (159), we find

$$\left.\begin{array}{ll} p_1 = \rho\sin\theta, & \delta_1 = \alpha_1 - \phi \\ p_2 = a_2\sin\theta, & \delta_2 = \alpha_2 - \phi \\ p_3 = a_3\sin\theta, & \delta_3 = \alpha_3 + \phi \\ p_4 = c_4\cos\theta, & \delta_4 = \gamma \end{array}\right\} \quad \text{......................(160).}$$

By (138), § 89, (159) and (160), we get

$$\left.\begin{array}{l} \Omega = (s_1 + s_2 - s_3)\,\omega + (s_2 + s_3)\,q + s_4 q' + \dots \\ \Delta = -(s_1 + s_2 - s_3)\,\phi + s_1\alpha_1 + s_2\alpha_2 + s_3\alpha_3 + s_4\gamma + \dots \end{array}\right\} \quad \text{......(161).}$$

99. Electric and magnetic forces. We must now consider the values of the electric and magnetic forces given by (144) and (145), § 91.

In the first place we must examine the function

$$F(\Omega) = J_{s_1}(\Omega p_1/c)\,J_{s_2}(\Omega p_2/c)\,J_{s_3}(\Omega p_3/c)\,J_{s_4}(\Omega p_4/c)\,\dots.$$

The value of $J_{s_1}(\Omega p_1/c)$, that is, $J_{s_1}(\Omega\rho\sin\theta/c)$, is for the present to be regarded as finite, not very small, since we wish to limit ourselves in no wise as to the possible values of the velocity of the charge.

The values of $J_{s_2}(\Omega p_2/c)$, ... on the other hand are small quantities of the orders $p_2^{s_2}$, ..., since p_2, ... are all of the order of the amplitude of the disturbance. Hence we must reject all values of s_2, ... greater than unity, and replace functions like $J_0(\Omega p_2/c)$ by unity, and functions like $J_1(\Omega p_2/c)$ by $\Omega p_2/2c$, neglecting higher terms in each case.

We shall for convenience omit the index 1, writing s instead of s_1 in future.

Thus we must consider terms of two types in (144) and (145):

(1) Terms for which s_2, s_3, ... are all zero, while s_1 takes any value from 1 to ∞.

In this case $F(\Omega) = J_s(s\beta\sin\theta)$, where $\beta = \omega\rho/c$ as before. These zero order terms represent the field due to the undisturbed uniform circular motion, and have been fully considered in problem 1.

(2) Terms for which one of the indices s_2, s_3, s_4, ... is either plus or minus unity, all the rest being zero, while s_1 takes every value from $-\infty$ to ∞. These terms represent the first order effect of the disturbance, and are the only ones to be taken into account.

The terms corresponding to $s_2 = \pm 1$, $s_3 = s_4 = \dots = 0$, and to $s_3 = \pm 1$, $s_2 = s_4 = \dots = 0$, represent the effect due to orbital oscillations, that is, oscillations in the plane of the orbit. Those corresponding to $s_2 = s_3 = 0$, $s_4 = \pm 1$, represent the effect due to oscillations parallel to the axis, i.e. axial oscillations. These are quite independent of the orbital oscillations, so far as the field due to them is concerned. We shall consider the two types of disturbance separately.

100. Orbital oscillations. In equations (142) and (143), § 90, terms occur for which $s_2 = \pm 1$ and $s_1 = \pm$ integer, and for which $s_3 = \pm 1$, $s_1 = \pm$ integer. These group themselves in two sets, as may be seen from the following table, which gives the frequencies Ω and epochs Δ for the typical terms of each set.

s_1	s_2	s_3	Ω	Δ
$+(s-1)$	$+1$	0	$+(q+s\omega)$	$+\{s(a_1-\phi)-a_1+a_2\}$
$-(s-1)$	-1	0	$-(q+s\omega)$	$-\{s(a_1-\phi)-a_1+a_2\}$
$+(s+1)$	0	$+1$	$+(q+s\omega)$	$+\{s(a_1-\phi)+a_1+a_3\}$
$-(s+1)$	0	-1	$-(q+s\omega)$	$-\{s(a_1-\phi)+a_1+a_3\}$
$-(s+1)$	$+1$	0	$+(q-s\omega)$	$-\{s(a_1-\phi)+a_1-a_2\}$
$+(s+1)$	-1	0	$-(q-s\omega)$	$+\{s(a_1-\phi)+a_1-a_2\}$
$-(s-1)$	0	$+1$	$+(q-s\omega)$	$-\{s(a_1-\phi)-a_1-a_3\}$
$+(s-1)$	0	-1	$-(q-s\omega)$	$+\{s(a_1-\phi)-a_1-a_3\}$

The values of Ω and of Δ are given by (161).

An inspection of this table shows that there are four terms in the sums (142) and (143) which give the same frequency $q + s\omega$, though with different epochs, namely those corresponding to the upper half of the table. Similarly there are four terms giving the frequency $q - s\omega$ belonging to the lower half of the table. When $s = 0$, these terms all give the frequency q, but the two halves of the table being identical for this case, we get four terms for this value also. Hence to each frequency of the form $q + s\omega$, four terms correspond for all values of s, positive, negative or zero.

Let us consider the terms of frequency $q + s\omega$.

In equations (144) and (145) the terms $s_1 = s - 1$, $s_2 = 1$, and $s_1 = -(s-1)$, $s_2 = -1$ are taken together, as also the terms $s_1 = s + 1$, $s_3 = 1$, and $s_1 = -(s+1)$, $s_3 = -1$, this in fact being the reason for introducing the factor 2, which does not appear in (142) and (143).

Using the notation of (144) and (145) we get the following terms:

$$d_{2\theta} = h_{2\phi} = \frac{2e}{r} F(\Omega) \left[\left(\frac{s_1\omega_1}{cp_1} C_1' + \frac{s_2\omega_2}{cp_2} C_2' \right) \sin \{\Omega (t - r/c) + \Delta\} \right.$$
$$\left. + \left(\frac{s_1\omega_1 k_1}{cp_1} S_1' + \frac{s_2\omega_2 k_2}{cp_2} S_2' \right) \cos \{\Omega (t - r/c) + \Delta\} \right],$$

$$d_{2\phi} = -h_{2\theta} = \frac{2e}{r} F(\Omega) \left[\left(\frac{s_1\omega_1}{cp_1} C_1'' + \frac{s_2\omega_2}{cp_2} C_2'' \right) \sin \{\Omega (t - r/c) + \Delta\} \right.$$
$$\left. + \left(\frac{s_1\omega_1 k_1}{cp_1} S_1'' + \frac{s_2\omega_2 k_2}{cp_2} S_2'' \right) \cos \{\Omega (t - r/c) + \Delta\} \right].$$

These arise from the sets of indices $s_1 = s - 1$, $s_2 = 1$, and $s_1 = -(s-1)$, $s_2 = -1$. There is an exactly similar set of terms arising from the sets of indices $s_1 = s + 1$, $s_3 = 1$, and $s_1 = -(s+1)$, $s_3 = -1$, which are got by merely interchanging the indices 2 and 3, and changing $s - 1$ into $s + 1$.

By (140)

$$F(\Omega) = J_{s-1}(\Omega p_1/c)\, J_1(\Omega p_2/c)\, J_0(\Omega p_3/c) \ldots .$$

To our degree of approximation we may write

$$J_1(\Omega p_2/c) = \Omega p_2/2c, \quad J_0(\Omega p_3/c) = 1, \ \ldots,$$

terms involving squares and higher powers of p_2, p_3, ..., but not of p_1, being neglected. Hence

$$F(\Omega) = \frac{\Omega p_2}{2c} J_{s-1}\left(\frac{\Omega p_1}{c}\right).$$

Again by (146)

$$k_1 = \frac{\Omega p_1 J'_{s-1}(\Omega p_1/c)}{c(s-1) J_{s-1}(\Omega p_1/c)}, \qquad k_2 = \frac{\Omega p_2 J_1'(\Omega p_2/c)}{c J_1(\Omega p_2/c)} = 1.$$

Equations (159) and (160) give with (138)

$$\Delta = s(\delta - \phi + \tfrac{1}{2}\pi) - \alpha_1 + \alpha_2.$$

Hence writing $\qquad \chi = \Omega(t - r/c) + s(\delta - \phi + \tfrac{1}{2}\pi)$,

we get $\qquad \Omega(t - r/c) + \Delta = \chi - \alpha_1 + \alpha_2$.

Further, we get by (146),

$$\begin{array}{llll} C_1' = \rho\cos\theta, & S_1' = 0, & C_2' = a_2\cos\theta, & S_2' = 0, \\ C_1'' = 0, & S_1'' = -\rho, & C_2'' = 0, & S_2'' = -a_2, \\ p_1 = \rho\sin\theta, & & p_2 = a_2\sin\theta, & \\ \omega_1 = \omega, & & \omega_2 = q + \omega. & \end{array}$$

Using these results we get

$$d_{2\theta} = h_{2\phi} = \frac{e\Omega^2\cos\theta}{c^2 r} J_{s-1}\left(\frac{\Omega p_1}{c}\right) a_2 \sin(\chi - \alpha_1 + \alpha_2),$$

$$d_{2\phi} = -h_{2\theta} = -\frac{e\Omega}{c^2 r}\left\{\frac{\omega\Omega p_1}{c} J'_{s-1}\left(\frac{\Omega p_1}{c}\right) + (q+\omega) J_{s-1}\left(\frac{\Omega p_1}{c}\right)\right\} a_2\cos(\chi - \alpha_1 + \alpha_2).$$

In precisely the same way we get

$$d_{3\theta} = h_{3\phi} = \frac{e\Omega^2\cos\theta}{c^2 r} J_{s+1}\left(\frac{\Omega p_1}{c}\right) a_3 \sin(\chi + \alpha_1 + \alpha_3),$$

$$d_{3\phi} = -h_{3\theta} = -\frac{e\Omega}{c^2 r}\left\{\frac{\omega\Omega p_1}{c} J'_{s+1}\left(\frac{\Omega p_1}{c}\right) - (q-\omega) J_{s+1}\left(\frac{\Omega p_1}{c}\right)\right\} a_3\cos(\chi + \alpha_1 + \alpha_3).$$

The negative sign of the second term in the bracket of $d_{3\phi}$ arises from the fact that $S_3'' = +a_3$, while $S_2'' = -a_2$.

The terms we require are the sums of these two sets of terms, and the complete expressions for the forces are obtained by summing with respect to s from $-\infty$ to ∞; for as we see from the table above we also get a similar set of terms for which $\Omega = q - s\omega$.

Now we get by (159)

$$a_2 = \tfrac{1}{2}\rho\sqrt{\{A^2 + B^2 + 2AB\sin(\alpha - \beta)\}}, \quad \tan(\alpha_2 - \alpha_1) = \frac{A\sin\alpha + B\cos\beta}{A\cos\alpha - B\sin\beta}.$$

Hence

$$a_2\sin(\chi - \alpha_1 + \alpha_2) = \tfrac{1}{2}\rho\{A\sin(\chi + \alpha) + B\cos(\chi + \beta)\},$$
$$a_2\cos(\chi - \alpha_1 + \alpha_2) = \tfrac{1}{2}\rho\{A\cos(\chi + \alpha) - B\sin(\chi + \beta)\}.$$

Similarly by changing the sign of B we get

$$a_3\sin(\chi + \alpha_1 + \alpha_3) = \tfrac{1}{2}\rho\{A\sin(\chi + \alpha) - B\cos(\chi + \beta)\},$$
$$a_3\cos(\chi + \alpha_1 + \alpha_3) = \tfrac{1}{2}\rho\{A\cos(\chi + \alpha) + B\sin(\chi + \beta)\}.$$

When we substitute these expressions in the values got for $d_{2\theta}$ and $d_{3\theta}$, we find for the factor in $d_{2\theta} + d_{3\theta}$ of

$$A\sin(\chi + \alpha): \qquad \frac{e\Omega^2\cos\theta\rho}{2c^2r}\left\{J_{s-1}\left(\frac{\Omega p_1}{c}\right) + J_{s+1}\left(\frac{\Omega p_1}{c}\right)\right\},$$

$$B\cos(\chi + \beta): \qquad \frac{e\Omega^2\cos\theta\rho}{2c^2r}\left\{J_{s-1}\left(\frac{\Omega p_1}{c}\right) - J_{s+1}\left(\frac{\Omega p_1}{c}\right)\right\}.$$

Similarly we find for the factor in $d_{2\phi} + d_{3\phi}$ of

$$A\cos(\chi + \alpha): \qquad -\frac{e\Omega\rho}{2c^2r}\left[\frac{\omega\Omega p_1}{c}\left\{J'_{s-1}\left(\frac{\Omega p_1}{c}\right) + J'_{s+1}\left(\frac{\Omega p_1}{c}\right)\right\}\right.$$
$$\left. + (q + \omega)J_{s-1}\left(\frac{\Omega p_1}{c}\right) - (q - \omega)J_{s+1}\left(\frac{\Omega p_1}{c}\right)\right],$$

$$B\sin(\chi + \beta): \qquad \frac{e\Omega\rho}{2c^2r}\left[\frac{\omega\Omega p_1}{c}\left\{J'_{s-1}\left(\frac{\Omega p_1}{c}\right) - J'_{s+1}\left(\frac{\Omega p_1}{c}\right)\right\}\right.$$
$$\left. + (q + \omega)J_{s-1}\left(\frac{\Omega p_1}{c}\right) + (q - \omega)J_{s+1}\left(\frac{\Omega p_1}{c}\right)\right].$$

We use the well known equations

$$J_{s-1}\left(\frac{\Omega p_1}{c}\right) + J_{s+1}\left(\frac{\Omega p_1}{c}\right) = \frac{2csJ_s(\Omega p_1/c)}{\Omega p_1},$$

$$J_{s-1}\left(\frac{\Omega p_1}{c}\right) - J_{s+1}\left(\frac{\Omega p_1}{c}\right) = 2J_s'\left(\frac{\Omega p_1}{c}\right),$$

$$\left(\frac{\Omega p_1}{c}\right)^2 J_s''\left(\frac{\Omega p_1}{c}\right) + \frac{\Omega p_1}{c}J_s'\left(\frac{\Omega p_1}{c}\right) = \left\{s^2 - \left(\frac{\Omega p_1}{c}\right)^2\right\}J_s\left(\frac{\Omega p_1}{c}\right),$$

and write

$$\beta = \omega\rho/c, \quad \Omega = q + s\omega = l\omega,$$

so that

$$\frac{\Omega p_1}{c} = \frac{\Omega\rho\sin\theta}{c} = l\beta\sin\theta.$$

Then we easily get, on summation from $s=-\infty$ to $s=+\infty$.

$$d_\theta = h_\phi = \frac{e\cot\theta}{\rho r} A \sum_{s=-\infty}^{s=\infty} sl\beta J_s(l\beta\sin\theta).\sin(\chi+\alpha)$$
$$+\frac{e\cos\theta}{\rho r} B \sum_{s=-\infty}^{s=\infty} l^2\beta^2 J_s'(l\beta\sin\theta).\cos(\chi+\beta) \quad\ldots(162),$$

$$d_\phi = -h_\theta = -\frac{e}{\rho r} A \sum_{s=-\infty}^{s=\infty} l^2\beta^2 J_s'(l\beta\sin\theta).\cos(\chi+\alpha)$$
$$+\frac{e}{\rho r} B \sum_{s=-\infty}^{s=\infty}\left(\frac{s}{\sin\theta} - l\beta^2\sin\theta\right) l\beta J_s(l\beta\sin\theta).\sin(\chi+\beta) \quad\ldots(163),$$

where $\chi = \Omega(t-r/c) + s(\delta-\phi+\frac{1}{2}\pi) = q(t-r/c)+s\psi$,

and $\psi = \omega(t-r/c)+\delta-\phi+\frac{1}{2}\pi$.

101. Axial oscillations. The field due to these oscillations is easily obtained by using (159) and (160) in (144) and (145).

As before we neglect higher powers of C, that is, of p_4, than the first, but not higher powers of p_1.

Hence in (144) and (145) we have to consider terms of the two types:

$$s_1 = s, \quad s_2 = s_3 = 0, \quad s_4 = \pm 1.$$

The upper sign gives by (138)

$$\Omega = s\omega + \omega_4 = q + s\omega, \qquad \Delta = s\delta_1 + \delta_4 = s(\delta-\phi+\tfrac{1}{2}\pi)+\gamma.$$

The lower sign gives

$$\Omega = s\omega - \omega_4 = -(q - s\omega), \quad \Delta = s\delta_1 - \delta_4 = s(\delta-\phi+\tfrac{1}{2}\pi)-\gamma.$$

Both may be included by summing for negative, as well as positive, values of s; thus we need only consider the upper sign.

As before we get from (144) and (145)

$$d_{4\theta} = \frac{2e}{r} F(\Omega)\left[\left(\frac{s_1\omega_1}{cp_1}C_1' + \frac{s_4\omega_4}{cp_4}C_4'\right)\sin\{\Omega(t-r/c)+\Delta\}\right.$$
$$\left.+\left(\frac{s_1\omega_1 k_1}{cp_1}S_1' + \frac{s_4\omega_4 k_4}{cp_4}S_4'\right)\cos\{\Omega(t-r/c)+\Delta\}\right],$$

$$d_{4\phi} = \frac{2e}{r} F(\Omega)\left[\left(\frac{s_1\omega_1}{cp_1}C_1'' + \frac{s_4\omega_4}{cp_4}C_4''\right)\sin\{\Omega(t-r/c)+\Delta\}\right.$$
$$\left.+\left(\frac{s_1\omega_1 k_1}{cp_1}S_1'' + \frac{s_4\omega_4 k_4}{cp_4}S_4''\right)\cos\{\Omega(t-r/c)+\Delta\}\right].$$

The values of $\omega_1, p_1, k_1, C_1', S_1', C_1'', S_1''$ are the same as before. The other quantities are given by the following equations, got just as before:

$$F(\Omega) = \frac{\Omega p_4}{2c} J_s\left(\frac{\Omega p_1}{c}\right),$$
$$k_4 = 1,$$
$$\Omega(t-r/c)+\Delta = \chi+\gamma,$$
$$C_4' = -c_4\sin\theta, \quad S_4' = 0, \quad C_4'' = 0, \quad S_4'' = 0,$$
$$p_4 = c_4\cos\theta, \qquad \omega_4 = q, \quad c_4 = \rho C.$$

We easily find

$$d_{4\theta} = h_{4\phi} = \frac{e}{\rho r} C \sum_{s=-\infty}^{s=\infty} \left(\frac{s\beta}{\sin\theta} - l\beta\sin\theta\right) l\beta J_s(l\beta\sin\theta)\sin(\chi+\gamma) \ldots(164).$$

$$d_{4\phi} = -h_{4\theta} = -\frac{e\beta\cos\theta}{\rho r} C \sum_{s=-\infty}^{s=\infty} l^2\beta^2 J_s'(l\beta\sin\theta)\cos(\chi+\gamma) \;\ldots\ldots\ldots(165).$$

102. Group of n electrons. To get the forces due to a group of n electrons describing the same circle we need only write for the ith electron

$$\xi = \rho\cos\left(\omega\tau + \delta + \frac{2\pi i}{n}\right), \qquad \eta = \rho\sin\left(\omega\tau + \delta + \frac{2\pi i}{n}\right), \quad \zeta = 0,$$

$$\lambda = \Sigma A\sin\left(q\tau + \alpha - k\frac{2\pi i}{n}\right), \quad \mu = \Sigma B\sin\left(q\tau + \beta - k\frac{2\pi i}{n}\right),$$

$$\nu = \Sigma C\sin\left(q\tau + \gamma - k\frac{2\pi i}{n}\right).$$

The first three equations represent the steady motion of the group as in problem 1. They show that the electrons are equidistant.

The last three equations represent sets of waves of disturbance travelling round the circle with a velocity q/k relative to the rotating electrons. For if i be increased by unity, τ must be increased by $2\pi k/qn$ in order that λ, μ, ν may be unaltered. In other words the disturbance requires the time $2\pi k/qn$ to travel from one electron to the next, through the intervening angle $2\pi/n$.

The substitutions only affect the values of the three arguments $\chi+\alpha$, $\chi+\beta$, $\chi+\gamma$ in (162)—(165).

Remembering that $\chi = \Omega(t - r/c) + s(\delta - \phi + \frac{1}{2}\pi)$, the substitutions of $\delta + 2\pi i/n$ for δ, $\alpha - k2\pi i/n$ for α, $\beta - k2\pi i/n$ for β, and $\gamma - k2\pi i/n$ for γ, change $\chi+\alpha$, $\chi+\beta$, $\chi+\gamma$ respectively into

$$\chi + \alpha + (s-k)\,2\pi i/n, \quad \chi + \beta + (s-k)\,2\pi i/n, \quad \chi + \gamma + (s-k)\,2\pi i/n.$$

When the equations are now written down for all the n electrons and summed from $i = 0$ to $i = n-1$, all the circular functions disappear except those for which s is of the form $k + sn$, s being an integer, and these are multiplied by n.

Write $\qquad m = k + sn, \quad \psi = \omega(t - r/c) + \delta - \phi + \frac{1}{2}\pi.$

Then $\qquad \Omega = q + k\omega + sn\omega, \quad l = k + sn + q/\omega, \quad \chi = q(t - r/c) + m\psi.$

Then we get finally, from (162)—(165),

$$d_\theta = h_\phi = \frac{ne\cot\theta}{\rho r} A \sum_{s=-\infty}^{s=\infty} ml\beta J_m(l\beta\sin\theta)\sin\{m\psi + q(t-r/c) + \alpha\}$$
$$+ \frac{ne\cos\theta}{\rho r} B \sum_{s=-\infty}^{s=\infty} l^2\beta^2 J_m'(l\beta\sin\theta)\cos\{m\psi + q(t-r/c) + \beta\}$$
$$+ \frac{ne}{\rho r} C \sum_{s=-\infty}^{s=\infty} \left(\frac{m\beta}{\sin\theta} - l\beta\sin\theta\right) l\beta J_m(l\beta\sin\theta)\sin\{m\psi + q(t-r/c) + \gamma\}$$
$$\ldots\ldots\ldots(166).$$

$$d_\phi = -h_\theta = -\frac{ne}{\rho r} A \sum_{s=-\infty}^{s=\infty} l^2\beta^2 J_m'(l\beta\sin\theta)\cos\{m\psi + q(t-r/c)+\alpha\}$$
$$+\frac{ne}{\rho r} B \sum_{s=-\infty}^{s=\infty}\left(\frac{m}{\sin\theta} - l\beta^2\sin\theta\right) l\beta J_m(l\beta\sin\theta)$$
$$\sin\{m\psi + q(t-r/c)+\beta\}$$
$$-\frac{ne\beta\cos\theta}{\rho r} C \sum_{s=-\infty}^{s=\infty} l^2\beta^2 J_m'(l\beta\sin\theta)\cos\{m\psi+q(t-r/c)+\gamma\}$$
$$\qquad\qquad \text{.........(167).}$$

These equations were first given by Schott* in a slightly different form (there is a misprint in the coefficient of B in d_ϕ in the first article referred to —the first β in $\frac{m\beta}{\sin\theta} - l\beta^2\sin\theta$ is to be deleted, as above). The above alternative proof has been selected in order to test the formulae (144) and (145) for the case of a polyperiodic motion.

103. Character of the field. The argument in (166) and (167) is of the form $m\psi + q(t-r/c)+\alpha$, ...

Now $$\psi = \omega(t-r/c)+\delta-\phi+\tfrac{1}{2}\pi.$$

It is the argument of the fundamental wave due to the steady circular motion, just as in problem 1.

On the other hand the remaining part of the argument, $q(t-r/c)+\alpha$, ..., is the argument of the corresponding component λ, ... of the oscillation relative to the rotating ring which generates the harmonic under discussion.

Every harmonic of the field is therefore a sum or difference vibration of the fundamental due to the steady motion, and of one of the fundamentals due to the disturbance, as was to be expected.

It is to be noted that the important harmonics as usual are those for which the order, m, of the Bessel Function involved is least, that is to say, one of the two harmonics of order k, or of order $n-k$, whichever is least.

We can include all the possible disturbances of the ring in our scheme by giving to k all integral values from 0 to $n-1$, or if we prefer, from $-\frac{1}{2}n$ to $+\frac{1}{2}n$. The latter is generally the most convenient arrangement for our purpose; it makes the orders of the most important harmonics positive or negative, according as k is positive or negative, that is, according as the wave of disturbance travels round the ring in a forward, or backward, direction relative to the direction of rotation of the ring.

The polarization of the harmonic m is as follows:

On the axis, for $\theta=0$ and $\theta=\pi$, the forces vanish (except for the case of a single electron).

* *Phil. Mag.* [6], Vol. XIII. p. 197. *Ann. der Phys.* 24, p. 653, 1907.

At the equator, for $\theta = \frac{1}{2}\pi$, we get

$$d_\theta = \quad h_\phi = -\frac{ne}{\rho r} C \sum_{s=-\infty}^{s=\infty} (l-m)\, l\beta^2 J_m(l\beta) \sin\{m\psi + q(t - r/c) + \gamma\},$$

$$d_\phi = -h_\theta = -\frac{ne}{\rho r} A \sum_{s=-\infty}^{s=} l^2\beta^2 J_m'(l\beta) \cos\{m\psi + q(t - r/c) + \alpha\}$$
$$+ \frac{ne}{\rho r} B \sum_{s=-\infty}^{s=\infty} (m - l\beta^2)\, l\beta J_m(l\beta) \cos\{m\psi + q(t - r/c) + \beta\}.$$

In other words, at the equator the waves due to the axial disturbance are completely polarized in the plane of the equator, those due to the orbital disturbance are completely polarized in the perpendicular plane. Thus the plane of the equator is a plane of complete polarization for each type of disturbance separately.

In every other direction each type of disturbance emits waves with elliptic polarization.

104. Radiation. The calculation of the radiation has been given by Schott in his paper in the *Annalen* (*l. c.* p. 653). [But he only gives it for the case where none of the waves due to the disturbance interfere with waves due to the steady motion. On reference to (125) and (126), § 82, and (166) and (167), § 102, we notice that the arguments of the periodic functions in the field due to steady motion are of the form $sn\psi$, and those in the field due to the disturbance of the form $m\psi + q(t - r/c) +$ constant, where $m = k + sn$, and $\psi = \omega(t - r/c) + \delta - \phi + \frac{1}{2}\pi$. The radiation is given by the integral

$$R = \frac{c}{4\pi} \int_0^\pi \int_0^{2\pi} (\overline{d_\theta^2} + \overline{d_\phi^2})\, r^2 \sin\theta\, d\theta\, d\phi,$$

where $\overline{d_\theta^2}$ and $\overline{d_\phi^2}$ denote mean values of d_θ^2 and d_ϕ^2 for a long interval of time, or, what comes to the same thing, for a whole period.

The integration with respect to ϕ, and the averaging with respect to time, cause all products of different harmonics to disappear, except those for which both factors have the same coefficients in their arguments both for ϕ and for t.

Clearly this cannot happen for harmonics, both of which are due to steady motion, or both of which are due to disturbance, but it can happen when one harmonic is due to steady motion and the other to disturbance, but only when the disturbance is such that $q = 0$ and $k = 0$. Thus we have to deal with terms of three types:

(1) Terms due to steady motion alone; these have been calculated in § 84.

(2) Terms due to disturbance alone; these will be calculated below.

(3) Terms resulting from interaction between waves due to steady motion and those due to a disturbance, if any, for which $q = 0$ and $k = 0$; these will also be calculated below.

105. Radiation due to disturbance alone. From what has been stated above it follows that this part of the radiation is given by a series of the form $R = \Sigma R_s$, where R_s is the part of the radiation due to the harmonics of order s in (166) and (167), and is the same as if these harmonics existed by themselves.

On averaging with respect to t and integrating with respect to ϕ, we easily see that the remaining integral with respect to θ is of such a form that terms involving the first power of $\cos\theta$ as a factor vanish identically; such are the terms which involve the products AC and BC. The remaining ones give

$$R_s = \frac{ce^2n^2l^2\beta^2}{2\rho^2}\left[A^2I_1 + B^2I_2 + 2AB\sin(\alpha-\beta)\,I_3 + C^2I_4\right],$$

where $I_1, \ldots$ are integrals defined as follows:

$$I_1 = \int_0^{\frac{\pi}{2}} \left[m^2\cot^2\theta\,.\,\{J_m(l\beta\sin\theta)\}^2 + l^2\beta^2\{J_m'(l\beta\sin\theta)\}^2\right]\sin\theta\,d\theta,$$

$$I_2 = \int_0^{\frac{\pi}{2}} \left[l^2\beta^2\cos^2\theta\,.\,\{J_m'(l\beta\sin\theta)\}^2 + \left(\frac{m}{\sin\theta} - l\beta^2\sin\theta\right)^2\{J_m(l\beta\sin\theta)\}^2\right]\sin\theta\,d\theta,$$

$$I_3 = l\beta\int_0^{\frac{\pi}{2}} \left[m\frac{\cos^2\theta}{\sin\theta} + \frac{m}{\sin\theta} - l\beta^2\sin\theta\right] J_m(l\beta\sin\theta)\,J_m'(l\beta\sin\theta)\sin\theta\,d\theta,$$

$$I_4 = \beta^2\int_0^{\frac{\pi}{2}} \left[\left(\frac{m}{\sin\theta} - l\sin\theta\right)^2.\{J_m(l\beta\sin\theta)\}^2 + l^2\beta^2\cos^2\theta\,\{J_m'(l\beta\sin\theta)\}^2\right]\sin\theta\,d\theta.$$

We proceed just as in problem 2, § 84; for this purpose we require the following results, which follow without much difficulty from the Addition Theorem and the known integral expressions for the Bessel Function:

(a) $$\pi\{J_m(l\beta\sin\theta)\}^2 = \int_0^\pi J_0(2l\beta\sin\theta\sin\phi)\cos 2m\phi\,d\phi,$$

(b) $$\pi\{J_m'(l\beta\sin\theta)\}^2 = \int_0^\pi J_0(2l\beta\sin\theta\sin\phi) \times\left(1 - \frac{m^2}{l^2\beta^2\sin^2\theta} - 2\sin^2\phi\right)\cos 2m\phi\,d\phi,$$

(c) $$\pi J_m(l\beta\sin\theta)\,J_m'(l\beta\sin\theta) = \int_0^\pi J_0'(2l\beta\sin\theta\sin\phi)\sin\phi\cos 2m\phi\,d\phi,$$

(d) $$\int_0^{\frac{\pi}{2}} J_0(2l\beta\sin\theta\sin\phi)\sin\theta\,d\theta = \frac{\sin(2l\beta\sin\phi)}{2l\beta\sin\phi} = \frac{1}{\beta}\int_0^\beta \cos(2lx\sin\phi)\,dx,$$

(e) $$\int_0^{\frac{\pi}{2}} J_0(2l\beta\sin\theta\sin\phi)\cos^2\theta\sin\theta\,d\theta = \frac{\sin(2l\beta\sin\phi)}{(2l\beta\sin\phi)^3} - \frac{\cos(2l\beta\sin\phi)}{(2l\beta\sin\phi)^2} = \frac{1}{2\beta^3}\int_0^\beta \cos(2lx\sin\phi)(\beta^2 - x^2)\,dx,$$

$$(f)\quad \int_0^{\frac{\pi}{2}} J_0'(2l\beta\sin\theta\sin\phi)\,d\theta = \frac{\cos(2l\beta\sin\phi)}{2l\beta\sin\phi} - \frac{1}{2l\beta\sin\phi}$$
$$= -\frac{1}{\beta}\int_0^{\beta}\sin(2lx\sin\phi)\,dx,$$

$$(g)\quad \int_0^{\frac{\pi}{2}} J_0'(2l\beta\sin\theta\sin\phi)\cos^2\theta\,d\theta = \frac{1}{2l\beta\sin\phi} - \frac{\sin(2l\beta\sin\phi)}{(2l\beta\sin\phi)^2}$$
$$= -\frac{1}{\beta^2}\int_0^{\beta}\sin(2lx\sin\phi)(\beta - x)\,dx,$$

$$(h)\quad \frac{1}{\pi}\int_0^{\pi}\int_0^{\beta}\cos(2lx\sin\phi)\cos 2m\phi\,dx\,d\phi = \int_0^{\beta} J_{2m}(2lx)\,dx,$$

$$(i)\quad \frac{l^2}{\pi}\int_0^{\pi}\int_0^{\beta}\cos(2lx\sin\phi)\cos 2m\phi\,x^2\,dx\,d\phi = l^2\int_0^{\beta} J_{2m}(2lx)\,x^2\,dx$$
$$= -\tfrac{1}{2}l\beta^2 J_{2m}'(2l\beta) + \tfrac{1}{4}\beta J_{2m}(2l\beta) + (m^2 - \tfrac{1}{4})\int_0^{\beta} J_{2m}(2lx)\,dx,$$

$$(k)\quad \frac{l}{\pi}\int_0^{\pi}\int_0^{\beta}\sin(2lx\sin\phi)\sin\phi\cos 2m\phi\,dx\,d\phi = -\tfrac{1}{2}J_{2m}(2l\beta),$$

$$(l)\quad \frac{l}{\pi}\int_0^{\pi}\int_0^{\beta}\sin(2lx\sin\phi)\sin\phi\cos 2m\phi\,x\,dx\,d\phi$$
$$= -\tfrac{1}{2}\beta J_{2m}(2l\beta) + \tfrac{1}{2}\int_0^{\beta} J_{2m}(2lx)\,dx.$$

We get, by means of (a), (b), (d) and (h),

$$I_1 = \frac{1}{\pi}\int_0^{\pi}\int_0^{\pi/2} J_0(2l\beta\sin\theta\sin\phi)(l^2\beta^2 - m^2 - 2l^2\beta^2\sin^2\phi)\cos 2m\phi\sin\theta\,d\theta\,d\phi$$
$$= l\beta J_{2m}'(2l\beta) - \frac{m^2 - l^2\beta^2}{\beta}\int_0^{\beta} J_{2m}(2lx)\,dx.$$

Again, by means of (a), (b), (d), (e), (h) and (i),

$$I_2 = \frac{1}{\pi}\int_0^{\pi}\int_0^{\pi/2} J_0(2l\beta\sin\theta\sin\phi)\{(m - l\beta)^2 + l^2\beta^2\cos^2\theta(1 - \beta^2 - 2\sin^2\phi)\}$$
$$\times\cos 2m\phi\sin\theta\,d\theta\,d\phi = \tfrac{1}{4}(1-\beta^2)l\beta J_{2m}'(2l\beta) + \tfrac{1}{8}(3+\beta^2)J_{2m}(2l\beta)$$
$$+\frac{1}{\beta}\left\{\tfrac{1}{2}(1+\beta^2)(m^2 + l^2\beta^2) - 2ml\beta^2 - \frac{3+\beta^2}{8}\right\}\int_0^{\beta} J_{2m}(2lx)\,dx.$$

Again, by means of (c), (f), (g), (k) and (l),

$$I_3 = \frac{l\beta}{\pi}\int_0^{\pi}\int_0^{\pi/2} J_0'(2l\beta\sin\theta\sin\phi)\{m - l\beta^2 + (m + l\beta^2)\cos^2\theta\}\sin\phi\cos 2m\phi\,d\theta\,d\phi$$
$$= \tfrac{1}{2}(m - l\beta^2)J_{2m}(2l\beta) + \frac{m + l\beta^2}{2\beta}\int_0^{\beta} J_{2m}(2lx)\,dx.$$

Lastly, by means of (a), (b), (d), (e), (h) and (i),

$$I_4 = \frac{\beta^2}{\pi}\int_0^{\pi}\int_0^{\pi/2} J_0(2l\beta\sin\theta\sin\phi)\{(m - l)^2 + l^2\cos^2\theta(\beta^2 - 1 - 2\beta^2\sin^2\phi)\}$$
$$\times\cos 2m\phi\sin\theta\,d\theta\,d\phi = -\tfrac{1}{4}(1-\beta^2)l\beta J_{2m}'(2l\beta) + \tfrac{1}{8}(1+3\beta^2)J_{2m}(2l\beta)$$
$$+\frac{1}{\beta}\left\{\tfrac{1}{2}(1+\beta^2)(m^2 + l^2\beta^2) - 2ml\beta^2 - \frac{1+3\beta^2}{8}\right\}\int_0^{\beta} J_{2m}(2lx)\,dx.$$

Collecting all the terms together] we find for the radiation due to disturbance alone

$$R = \frac{ce^2n^2}{2\rho^2}\Bigg\{A^2\sum_{s=-\infty}^{s=\infty} l\beta\left[l^2\beta^2 J_{2m}'(2l\beta) - (m^2 - l^2\beta^2)\, l\int_0^\beta J_{2m}(2lx)\,dx\right]$$

$$+ B^2\sum_{s=-\infty}^{s=\infty} l\beta\Bigg[\tfrac{1}{4}(1-\beta^2)\,l^2\beta^2 J_{2m}'(2l\beta) + \tfrac{1}{8}(3+\beta^2)\,l\beta J_{2m}(2l\beta)$$

$$+ \{\tfrac{1}{2}(1+\beta^2)(m^2 + l^2\beta^2) - 2ml\beta^2 - \tfrac{1}{8}(3+\beta^2)\}\, l\int_0^\beta J_{2m}(2lx)\,dx\Bigg]$$

$$+ 2AB\sin(\alpha-\beta)\sum_{s=-\infty}^{s=\infty} l\beta\Bigg[\tfrac{1}{2}(m - l\beta^2)\, l\beta J_{2m}(2l\beta)$$

$$+ \tfrac{1}{2}(m + l\beta^2)\, l\int_0^\beta J_{2m}(2lx)\,dx\Bigg]$$

$$+ C^2\sum_{s=-\infty}^{s=\infty} l\beta\Bigg[-\tfrac{1}{4}(1-\beta^2)\,l^2\beta^2 J_{2m}'(2l\beta) + \tfrac{1}{8}(1+3\beta^2)\,l\beta J_{2m}(2l\beta)$$

$$+ \{\tfrac{1}{2}(1+\beta^2)(m^2 + l^2\beta^2) - 2ml\beta^2 - \tfrac{1}{8}(1+3\beta^2)\}\, l\int_0^\beta J_{2m}(2lx)\,dx\Bigg]\Bigg\} \quad \ldots(168).$$

106. [Radiation due to interaction between steady motion and disturbance. Putting $q = 0$ and $k = 0$ in (166) and (167) we see that the parts giving rise to terms of the present type in the radiation reduce to

$$d_\theta = \frac{2e\beta n^3 \cot\theta}{\rho r} A\sin\alpha\sum_{s=1}^{s=\infty} s^2 J_{sn}(sn\beta\sin\theta)\cos sn\psi$$

$$- \frac{2e\beta^2 n^3\cos\theta}{\rho r} B\sin\beta\sum_{s=1}^{s=\infty} s^2 J_{sn}'(sn\beta\sin\theta)\sin sn\psi$$

$$+ \frac{2e\beta^2 n^3\cos^2\theta}{\rho r\sin\theta} C\sin\gamma\sum_{s=1}^{s=\infty} s^2 J_{sn}(sn\beta\sin\theta)\cos sn\psi,$$

$$d_\phi = \frac{2e\beta^2 n^3}{\rho r} A\sin\alpha\sum_{s=1}^{s=\infty} s^2 J_{sn}'(sn\beta\sin\theta)\sin sn\psi$$

$$+ \frac{2e\beta n^3(1-\beta^2\sin^2\theta)}{\rho r\sin\theta} B\sin\beta\sum_{s=1}^{s=\infty} s^2 J_{sn}(sn\beta\sin\theta)\cos sn\psi$$

$$+ \frac{2e\beta^3 n^3\cos\theta}{\rho r} C\sin\gamma\sum_{s=1}^{s=\infty} s^2 J_{sn}'(sn\beta\sin\theta)\sin sn\psi.$$

Comparing these expressions with the corresponding ones in (125) and (126), § 82, we see that products of the two sets of terms involving A and C also involve the product $\sin sn\psi \cos sn\psi$ as a factor, which disappears in averaging for the time, but products involving B also involve either $\cos^2 sn\psi$ or $\sin^2 sn\psi$ as a factor, of which the average value is $\frac{1}{2}$. Hence the required terms in the radiation, after integration with respect to ϕ, reduce to

$$R = -\frac{4ce^2\beta^3 n^5}{\rho^2} B\sin\beta\sum_{s=1}^{s=\infty} s^3\int_0^{\pi/2} J_{sn}(sn\beta\sin\theta)\, J_{sn}'(sn\beta\sin\theta)(1+\cos^2\theta - \beta^2\sin^2\theta)\,d\theta.$$

The integral is at once seen to be simply the integral I_3, defined in § 105, divided by $s^2n^2\beta$, where for l and m are put the value sn, to which they reduce when q and k are zero. Using the value of I_3 given, we find

$$R = -\frac{2ce^2\beta n^4}{\rho^2} B \sin\beta \sum_{s=1}^{s=\infty} s^2 \left[\beta(1-\beta^2) J_{2sn}(2sn\beta) + (1+\beta^2)\int_0^\beta J_{2sn}(2snx)\,dx\right].$$

When we put $\rho(1 - B\sin\beta)$ in place of ρ, and $\beta(1 - B\sin\beta)$ in place of β in expression (129), § 84, for the radiation from a ring in steady motion, we find, on developing in powers of B, that the linear term obtained is precisely the value just found for R. This must be so; for a disturbance, for which q and k both vanish, represents either a rotation of the ring as a whole about its centre proportional to $A \sin\alpha$, or a contraction of the ring as a whole proportional to $B\sin\beta$, or a displacement of the ring as a whole perpendicular to its plane proportional to $C\sin\gamma$. Of these the second alone alters the steady motion radiation, and since $q = 0$, it takes place without altering the angular velocity ω of this rotation, so that β must diminish in proportion as ρ diminishes. This being the character of the exceptional disturbance just considered, it is obviously unnecessary to trouble any further about it, and we shall therefore confine our discussion to the expression (168)].

107. We have $\beta = \omega\rho/c$ and $l = m + q/\omega$; hence $l\beta = (q + m\omega)\,\rho/c$. Now $q + m\omega$ is the frequency of the particular type of wave under consideration relative to a stationary observer; thus $l\beta = 2\pi\rho/\lambda$, where λ is the corresponding wave-length. For spectrum lines it is always small; in fact it is only 1/160 for the extreme ultraviolet lines of Schumann. For such waves we may put $J_{2m}(2l\beta) = (l\beta)^{2\mu}/2\mu\,!$, where μ is the absolute value of m. Using this approximation I have calculated an upper limit to the radiation from a ring for several classes of disturbance, that is for several values of k, namely for $k = \pm 3$, ± 2, ± 1, and 0. [See the paper in the *Phil. Mag.* referred to in § 102, note.] For the classes $k = \pm 2$, the radiation per ring is about $3 \, . \, 10^{-11}$ erg/sec., and for the classes $k = \pm 1$ and $k = 0$ of the order 10^{-9} erg/sec., all for violet light.

Wien* finds that for the canal ray ion in a Geissler tube and for the line $H\beta$ the radiation per ion is of the order 10^{-7} erg/sec. This result shows that if spectrum lines can be ascribed to disturbances of rings of electrons from their steady motion at all, they can only be accounted for by means of disturbances of the classes $k = \pm 1$ and $k = 0$. These disturbances have one pair of nodes and no nodes respectively; the former include to and fro oscillations in the plane of the ring (orbital), and oscillations about a diameter (axial), and the latter, periodic expansions and contractions of the ring (orbital), and to and fro oscillations perpendicular to its plane (axial).

It should be noted that nothing has as yet been proved as to aperiodic motions involving periodic components, such as damped oscillations; for the

* *Ann. der Phys.* [4], 23, p. 415, 1907.

method used fails for such motions. It will however be proved in Ch. IX that our results hold for these cases also, provided that the damping be not too great.

108. Problem 5. An electron moves in an epitrochoidal orbit. To find the field produced. We shall treat this case briefly as an example of a diperiodic motion. We may write

$$\xi = a\cos\omega\tau - b\cos\omega'\tau,$$
$$\eta = a\sin\omega\tau - b\sin\omega'\tau,$$
$$\zeta = 0.$$

Hence, comparing with (135), § 88, we have

$$\omega_1 = \omega,$$
$$a_1 = b_1 = a, \quad c_1 = 0, \quad \alpha_1 = \tfrac{1}{2}\pi, \quad \beta_1 = 0,$$
$$\omega_2 = \omega',$$
$$a_2 = b_2 = b, \quad c_2 = 0, \quad \alpha_2 = \tfrac{3}{2}\pi, \quad \beta_2 = \pi.$$

Hence by (136), § 88, with $l = \sin\theta\cos\phi$, $m = \sin\theta\sin\phi$, $n = \cos\theta$,

$$p_1 = a\sin\theta, \quad \delta_1 = \tfrac{1}{2}\pi - \phi,$$
$$p_2 = b\sin\theta, \quad \delta_2 = \tfrac{3}{2}\pi - \phi.$$

Choose (l', m', n') and (l'', m'', n'') in the direction of increasing θ and ϕ respectively, so that

$$l' = \cos\theta\cos\phi, \quad m' = \cos\theta\sin\phi, \quad n' = -\sin\theta,$$
$$l'' = -\sin\phi, \quad m'' = \cos\phi, \quad n'' = 0.$$

By (146), § 91, we have

$$C_1' = a\cos\theta, \quad S_1' = 0, \quad C_1'' = 0, \quad S_1'' = -a,$$
$$k_1 = \frac{\Omega a \sin\theta\, J_{s_1}'(\Omega a\sin\theta/c)}{c s_1 J_{s_1}(\Omega a \sin\theta/c)},$$
$$C_2' = b\cos\theta, \quad S_2' = 0, \quad C_2'' = 0, \quad S_2'' = -b,$$
$$k_2 = \frac{\Omega b \sin\theta\, J_{s_2}'(\Omega b\sin\theta/c)}{c s_2 J_{s_2}(\Omega b \sin\theta/c)}.$$

By (140), § 89, we have

$$F(\Omega) = J_{s_1}\left(\frac{\Omega a\sin\theta}{c}\right) J_{s_2}\left(\frac{\Omega b\sin\theta}{c}\right).$$

Lastly, by (138), § 89, we have

$$\Omega = s_1\omega + s_2\omega', \quad \Delta = \tfrac{1}{2}(s_1 + 3s_2)\pi - (s_1 + s_2)\phi.$$

Hence we get by (144) and (145), § 91,

$$\left.\begin{aligned} d_\theta = h_\phi &= \frac{e\Omega\cot\theta}{cr}\sum_{-\infty}^{\infty}{}^{s_1}\sum{}^{s_2} J_{s_1}\left(\frac{\Omega a\sin\theta}{c}\right) J_{s_2}\left(\frac{\Omega b\sin\theta}{c}\right)\sin\{\Omega(t - r/c) + \Delta\} \\ d_\phi = -h_\theta &= -\frac{e\Omega}{c^2 r}\sum_{-\infty}^{\infty}{}^{s_1}\sum{}^{s_2}\left[a\omega J_{s_1}'\left(\frac{\Omega a\sin\theta}{c}\right) J_{s_2}\left(\frac{\Omega b\sin\theta}{c}\right)\right. \\ &\left.\quad + b\omega' J_{s_1}\left(\frac{\Omega a\sin\theta}{c}\right) J_{s_2}'\left(\frac{\Omega b\sin\theta}{c}\right)\right]\cos\{\Omega(t - r/c) + \Delta\} \end{aligned}\right\} \quad (169),$$

where we take all values, positive and negative, both of s_1 and s_2, and therefore omit the factor 2 in (144) and (145). We must however bear in mind that there are two terms giving the same frequency, namely those whose indices are (s_1, s_2) and $(-s_1, -s_2)$ respectively.

109. The character of the field is easily understood from (169). On the axis the forces vanish. This is obvious for all harmonics of order 2 or higher. The harmonic $s_1 = s_2 = 0$, of order zero, is absent on account of the vanishing factor Ω; hence the only difficulty is for $s_1 = \pm 1$, $s_2 = 0$ and $s_1 = 0$, $s_2 = \pm 1$. The corresponding terms do not vanish. We easily find that for these two sets together

$$d_x = d_\theta \cos\phi - d_\phi \sin\phi = \frac{ea\omega^2}{c^2 r}\cos\omega(t - r/c) - \frac{eb\omega'^2}{c^2 r}\cos\omega'(t - r/c),$$

$$d_y = d_\theta \sin\phi + d_\phi \cos\phi = \frac{ea\omega^2}{c^2 r}\sin\omega(t - r/c) - \frac{eb\omega'^2}{c^2 r}\sin\omega'(t - r/c),$$

which of course are independent of ϕ. They evidently represent two circularly polarized vibrations.

At the equator, where $\theta = \frac{1}{2}\pi$, we have

$$d_\theta = h_\phi = 0,$$

$$d_\phi = -h_\theta = -\frac{e\Omega}{c^2 r}\sum_{-\infty}^{\infty}\sum\left[a\omega J_{s_1}'\left(\frac{\Omega a}{c}\right)J_{s_2}\left(\frac{\Omega b}{c}\right) + b\omega' J_{s_1}\left(\frac{\Omega a}{c}\right)J_{s_2}'\left(\frac{\Omega b}{c}\right)\right].\cos\{\Omega(t - r/c) + \Delta\},$$

which represents a vibration completely polarized in a plane at right angles to the equator.

In every other direction the vibration is elliptically polarized.

The most intense vibrations are those for which

(1) $s_1 = \pm 1$, $s_2 = 0$: the frequency is ω, that of the first fundamental motion.

(2) $s_1 = 0$, $s_2 = \pm 1$: the frequency is ω', that of the second fundamental motion.

The amplitudes of these vibrations are of the orders

$$(e\omega/cr)\, J_1(\omega a \sin\theta/c)\, J_0(\omega b \sin\theta/c),$$

and

$$(e\omega'/cr)\, J_0(\omega' a \sin\theta/c)\, J_1(\omega' b \sin\theta/c),$$

respectively. If $\omega a/c$ and $\omega' b/c$ be both small, the orders are $e\omega^2 a/c^2 r$, and $e\omega'^2 b/c^2 r$, respectively.

(3) $s_1 = \pm 1$, $s_2 = \pm 1$: frequency $\omega \pm \omega'$, that of the first sum or difference vibration.

$s_1 = \pm 2$, $s_2 = 0$: frequency 2ω.

$s_1 = 0$, $s_2 = \pm 2$: frequency $2\omega'$.

The amplitudes are of the order $e\omega^3 ab/c^3 r$, that is of order higher by unity than the last.

110. **Group of electrons.** It is of interest to enquire whether it is possible to arrange n electrons, moving in the same epitrochoid, so as to interfere. This we found was always possible in the case of a mono-periodic motion.

In order that interference may occur it is necessary that the argument of the circular functions in (169) be of the form $\Omega(t - r/c) + \Delta + k2\pi i/n$ for the ith electron.

This can only be accomplished by adding multiples of $2\pi i/n$ to the phases $\omega\tau$, and $\omega'\tau$, of the two constituent motions; that is we must replace

$$\omega(t - r/c) \text{ by } \omega(t - r/c) + m2\pi i/n,$$

and

$$\omega'(t - r/c) \text{ by } \omega'(t - r/c) + m'2\pi i/n,$$

where m and m' are integers.

Thus the argument in question becomes

$$\Omega(t - r/c) + \Delta + (ms_1 + m's_2)\, 2\pi i/n.$$

The circular functions then disappear for all values of s_1 and s_2, when the expressions (169) are summed from $i = 0$ to $i = n - 1$, except only those which satisfy an equation of the form

$$ms_1 + m's_2 = jn,$$

where j is any integer including zero.

When $m = m' = 1$, the value $j = 0$ gives rise to terms $s_1 = \pm 1$, $s_2 = \mp 1$, which are the second order; these are obviously the terms of lowest order, the terms of the first order having been destroyed by interference.

When $m = 2$, $m' = 1$, the value $j = 0$ gives terms $s_1 = \pm 1$, $s_2 = \mp 2$, of the third order, those of the second, as well as of the first order, having been destroyed by interference.

It is obvious that we may confine ourselves to values of m and m' which are less than n; the only restriction is that they must be so chosen that no two electrons can occupy the same position on the curve. In general large values give greater interference than small ones; it is an interesting problem to find which is the best arrangement for a given number of electrons, that is to say, which makes the smallest value of $|s_1| + |s_2|$ as large as possible.

111. Radiation. It is easy to write down the expression for the Poynting vector corresponding to (169), but it is of little use, since it will involve products of four Bessel Functions. Series of this type have not been summed, nor does it seem likely that integrals involving such products can be evaluated. For this reason we refrain from writing down the expression for the radiation.

112. Problem 6. Precessional motion of a system of vibrating charges. In conclusion we shall discuss the effect on the waves emitted by a vibrating system of the type considered in this chapter, when it is given a precessional motion. There is some reason for supposing that an atom may be likened to a symmetrical magnetic top, of the type studied by Du Bois; such an analogy has been employed by Langevin* in his electron theory of magnetism, and by Ritz† in his paper on Spectrum Series and Atomic Fields. The problem is of some importance, since it has an obvious bearing on the theory of the Zeeman effect.

Let us suppose that a charge is executing oscillations relative to axes (OA, OB, OC), which are in precessional motion relative to the fixed axes (Ox, Oy, Oz).

With the usual notation of rigid dynamics, let Θ be the angle between Oz and OC, ψ that between the planes zOx and zOC, and ϕ that between zOC and AOC; and suppose that

$$\Theta = \text{constant}, \quad \psi = \mu\tau, \quad \phi = n\tau, \quad \text{at the time } \tau.$$

Further suppose that the components parallel to the moving axes of the radius vector to the charge are

$$\Sigma A \sin(q\tau + \alpha), \quad \Sigma B \sin(q\tau + \beta), \quad \Sigma C \sin(q\tau + \gamma).$$

The well known equations for the direction cosines in this case are

$$\begin{aligned}
(XA) &= \cos\Theta\cos\mu\tau\cos n\tau - \sin\mu\tau\sin n\tau,\\
(XB) &= -\cos\Theta\cos\mu\tau\sin n\tau - \sin\mu\tau\cos n\tau,\\
(XC) &= \sin\Theta\cos\mu\tau,\\
(YA) &= \cos\Theta\sin\mu\tau\cos n\tau + \cos\mu\tau\sin n\tau,\\
(YB) &= -\cos\Theta\sin\mu\tau\sin n\tau + \cos\mu\tau\cos n\tau,\\
(YC) &= \sin\Theta\sin\mu\tau,\\
(ZA) &= -\sin\Theta\cos n\tau,\\
(ZB) &= \sin\Theta\sin n\tau,\\
(ZC) &= \cos\Theta,
\end{aligned}$$

* *Journal de Physique*, [4], 4, p. 678, 1905.
† *Ann. der Phys.* 25, p. 660, 1908.

By their means we easily find, for the components of the radius vector parallel to the fixed axes, the expressions

$$\xi = \Sigma\, [\tfrac{1}{4}(1+\cos\Theta)(D\sin\{(q+n+\mu)\tau+\varpi\} + E\sin\{(q-n-\mu)\tau+\upsilon\})$$
$$+ \tfrac{1}{4}(1-\cos\Theta)(D\sin\{(q+n-\mu)\tau+\varpi-\pi\} + E\sin\{(q-n+\mu)\tau+\upsilon+\pi\})$$
$$+ \tfrac{1}{2}\sin\Theta\, C(\sin\{(q+\mu)\tau+\gamma\} + \sin\{(q-\mu)\tau+\gamma\})] \quad\ldots\ldots(170),$$

$$\eta = \Sigma\, [\tfrac{1}{4}(1+\cos\Theta)(D\sin\{(q+n+\mu)\tau+\varpi-\tfrac{1}{2}\pi\} + E\sin\{(q-n-\mu)\tau+\upsilon+\tfrac{1}{2}\pi\})$$
$$+ \tfrac{1}{4}(1-\cos\Theta)(D\sin\{(q+n-\mu)\tau+\varpi-\tfrac{1}{2}\pi\} + E\sin\{(q-n+\mu)\tau+\upsilon+\tfrac{1}{2}\pi\})$$
$$+ \tfrac{1}{2}\sin\Theta\, C(\sin\{(q+\mu)\tau+\gamma-\tfrac{1}{2}\pi\} + \sin\{(q-\mu)\tau+\gamma+\tfrac{1}{2}\pi\})] \quad\ldots(171),$$

$$\zeta = \Sigma\, [\tfrac{1}{2}\sin\Theta\,(D\sin\{(q+n)\tau+\varpi+\pi\} + E\sin\{(q+n)\tau+\upsilon+\pi\})$$
$$+ \cos\Theta\, C\sin(q\tau+\gamma)] \quad\ldots\ldots(172),$$

where

$$D = \sqrt{\{A^2+B^2+2AB\sin(\alpha-\beta)\}}, \quad E = \sqrt{\{A^2+B^2-2AB\sin(\alpha-\beta)\}},$$

$$\tan\varpi = \frac{A\sin\alpha + B\cos\beta}{A\cos\alpha - B\sin\beta}, \quad \tan\upsilon = \frac{A\sin\alpha - B\cos\beta}{A\cos\alpha + B\sin\beta}.$$

We notice that each oscillation, of frequency q, gives rise to eight sum and difference oscillations, of frequencies

$$q+n+\mu,\quad q-n-\mu,\quad q-n+\mu,\quad q+n-\mu,\quad q+\mu,\quad q-\mu,\quad q+n,\quad q-n,$$

in addition to the original vibration of frequency q.

113. Using the notation of (135), § 88, we may classify them as follows:

Circular vibrations.

(1) *Right-handed about Oz.*

$$\omega_1 = q+n+\mu,\quad a_1 = b_1 = \tfrac{1}{4}(1+\cos\Theta)D,\quad c_1 = 0,\quad \alpha_1 = \varpi,\quad \beta_1 = \varpi - \tfrac{1}{2}\pi,$$
$$\omega_2 = q-n+\mu,\quad a_2 = b_2 = \tfrac{1}{4}(1-\cos\Theta)E,\quad c_2 = 0,\quad \alpha_2 = \upsilon+\pi,\quad \beta_2 = \upsilon + \tfrac{1}{2}\pi,$$
$$\omega_3 = q+\mu,\quad a_3 = b_3 = \tfrac{1}{2}\sin\Theta\, C,\quad c_3 = 0,\quad \alpha_3 = \gamma,\quad \beta_3 = \gamma - \tfrac{1}{2}\pi.$$

(2) *Left-handed about Oz.*

$$\omega_4 = q+n-\mu,\quad a_4 = b_4 = \tfrac{1}{4}(1-\cos\Theta)D,\quad c_4 = 0,\quad \alpha_4 = \varpi-\pi,\quad \beta_4 = \varpi - \tfrac{1}{2}\pi,$$
$$\omega_5 = q-n-\mu,\quad a_5 = b_5 = \tfrac{1}{4}(1+\cos\Theta)E,\quad c_5 = 0,\quad \alpha_5 = \upsilon,\quad \beta_5 = \upsilon + \tfrac{1}{2}\pi,$$
$$\omega_6 = q-\mu,\quad a_6 = b_6 = \tfrac{1}{2}\sin\Theta\, C,\quad c_8 = 0,\quad \alpha_8 = \gamma,\quad \beta_6 = \gamma + \tfrac{1}{2}\pi.$$

(3) *Linear vibrations parallel to Oz.*

$$\omega_7 = q+n,\quad a_7 = b_7 = 0,\quad c_7 = \tfrac{1}{2}\sin\Theta\, D,\quad \gamma_7 = \varpi+\pi,$$
$$\omega_8 = q-n,\quad a_8 = b_8 = 0,\quad c_8 = \tfrac{1}{2}\sin\Theta\, E,\quad \gamma_8 = \upsilon+\pi,$$
$$\omega_9 = q,\quad a_9 = b_9 = 0,\quad c_9 = \cos\Theta\, C,\quad \gamma_9 = \gamma.$$

114. Electric and magnetic forces. The waves of electric and magnetic force emitted by the vibrating charge are determined by (144) and (145), § 91, as before.

If we suppose that the amplitudes A, B, C are all small, so that their squares and products may be neglected, we need only consider the fundamental waves for which one of the numbers, s_i, is unity, all the others being zero. In any case these are the most important, and we shall for simplicity consider them alone.

We have, by (136), § 88,

$$p_i \cos \delta_i = \sin \theta \, (a_i \cos \phi \cos \alpha_i + b_i \sin \phi \cos \beta_i) + c_i \cos \theta \cos \gamma_i,$$
$$p_i \sin \delta_i = \sin \theta \, (a_i \cos \phi \sin \alpha_i + b_i \sin \phi \sin \beta_i) + c_i \cos \theta \sin \gamma_i,$$

where we have used polar coordinates (r, θ, ϕ) as before.

Again we have, by (146), § 91,

$$C_i' = \cos \theta \, \{a_i \cos \phi \cos (\alpha_i - \delta_i) + b_i \sin \phi \cos (\beta_i - \delta_i)\} - c_i \sin \theta \cos (\gamma_i - \delta_i),$$
$$S_i' = \cos \theta \, \{a_i \cos \phi \sin (\alpha_i - \delta_i) + b_i \sin \phi \sin (\beta_i - \delta_i)\} - c_i \sin \theta \sin (\gamma_i - \delta_i),$$
$$C_i'' = - a_i \sin \phi \cos (\alpha_i - \delta_i) + b_i \cos \phi \cos (\beta_i - \delta_i),$$
$$S_i'' = - a_i \sin \phi \sin (\alpha_i - \delta_i) + b_i \cos \phi \sin (\beta_i - \delta_i),$$

where the directions (l', m', n') and (l'', m'', n'') have been taken in the directions of increasing θ and ϕ respectively.

Lastly, by (138) and (140), § 89, we get

$$\Omega = \omega_i, \quad \Delta = \delta_i,$$
$$F(\Omega) = J_1 \left(\frac{\omega_i p_i}{c}\right) J_0 \left(\frac{\omega_i p_1}{c}\right) J_0 \left(\frac{\omega_i p_2}{c}\right) \ldots = J_1 \left(\frac{\omega_i p_i}{c}\right) \Pi,$$

say.

Thus we find by means of (144) and (145)

$$d_{i\theta} = h_{i\phi} = \frac{e\omega_i^2}{c^2 r} \Pi_i \left[\frac{2c}{\omega_i p_i} J_1 \left(\frac{\omega_i p_i}{c}\right) C_i' \sin \{\omega_i (t - r/c) + \delta_i\} + 2J_1' \left(\frac{\omega_i p_i}{c}\right) S_i' \cos \{\omega_i (t - r/c) + \delta_i\} \right] \quad \ldots(173),$$

$$d_{i\phi} = - h_{i\theta} = - \frac{e\omega_i^2}{c^2 r} \Pi_i \left[\frac{2c}{\omega_i p_i} J_1 \left(\frac{\omega_i p_i}{c}\right) C_i'' \sin \{\omega_i (t - r/c) + \delta_i\} + 2J_1' \left(\frac{\omega_i p_i}{c}\right) S_i'' \cos \{\omega_i (t - r/c) + \delta_i\} \right] \quad \ldots(174).$$

115. These equations simplify considerably in our problem, where we are dealing with circular and linear vibrations alone.

(1) *Circular vibrations—right-handed.*

In this case $a_i = b_i, \quad c_i = 0, \quad \alpha_i = \beta_i + \tfrac{1}{2}\pi.$

Hence, by (136), $p_i = a_i \sin \theta, \quad \delta_i = \alpha_i - \phi.$

By (146) $C_i' = a_i \cos \theta, \quad S_i' = 0, \quad C_i'' = 0, \quad S_i'' = - a_i.$

(2) *Circular vibrations—left-handed.*

Here $a_i = b_i, \quad c_i = 0, \quad \alpha_i = \beta_i - \tfrac{1}{2}\pi.$

$$p_i = a_i \sin \theta, \quad \delta_i = \alpha_i + \phi.$$
$$C_i' = a_i \cos \theta, \quad S_i' = 0, \quad C_i'' = 0, \quad S_i'' = + a_i.$$

(3) *Axial vibrations.*

$$a_i = b_i = 0, \quad p_i = c_i \cos \theta, \quad \delta_i = \gamma_i.$$
$$C_i' = - c_i \sin \theta, \quad S_i' = 0, \quad C_i'' = 0, \quad S_i'' = 0.$$

The circular vibrations produce elliptically polarized waves, reducing to circularly polarized waves on the axis, and to linearly polarized ones on the equator. The direction of rotation about the axis is the same as that of the generating vibration.

The linear vibrations give waves completely polarized at right angles to the meridian, the vibration in the wave being as nearly parallel to the generating vibration as transversality will permit.

Thus the general character of the waves emitted is the same as that of the generating vibration.

The arrangement of the lines on a scale of frequency is easily seen from the table in § 113.

The right-handed circular lines form a symmetrically placed triplet; so also do the left-handed ones, and the linearly polarized ones. The last are symmetrically placed on either side of the original line, q; the former are symmetrically placed about the lines $q \pm \mu$, which are themselves symmetrically placed about q; thus the whole group of nine lines is symmetrically placed with respect to the centre line, q.

116. As regards intensity there is in general no symmetry, except that the two lines, $q \pm \mu$, are always equally intense. There is symmetry of intensity with respect to the centre provided $D = E$; but the triplets of circular lines can never possess symmetry of intensity with regard to their own centres, $q \pm \mu$. The condition for symmetry of intensity requires, either that $A = B$, or that $\alpha = \beta$; in either case the generating vibration must take place in a plane through the axis of symmetry OC of the system of charges. This condition is both necessary and sufficient.

An examination of the known cases of Zeeman effect shows that the symmetry of intensity is always present*. Hence if the Zeeman effect be due to precessional motion of a symmetrical atom or ion under the influence of the magnetic field, then all vibrations causing lines possessing a Zeemam effect of this type must take place in a meridian plane of the atom or ion.

117. As regards the spacing of the lines, it is to be noticed that the distance between the components of each of the three triplets, (1), (2) and (3), is the same, namely n. This arrangement is the usual one in all cases of Zeeman effect showing triplets.

A considerable variety of arrangements of the nine lines may be got by choosing suitable values for the ratio μ/n; the cases where $\mu/n = 3$, 2, or $\frac{3}{2}$, are all known.

* Since this was written cases of asymmetry have been discovered.

Runge's rule*, that the distances of the several lines from the centre of the group are small multiples of a small sub-multiple of a standard distance, obviously requires that μ/n should be a ratio of small integers. This important result is due to Ritz.

118. We shall now consider the relative intensities of the several lines on the assumption that there is symmetry, that is, that $D=E$. The table in § 113 gives in this case, for the amplitude of lines,

$$\left.\begin{array}{ll} \omega_1,\ \omega_5: & \tfrac{1}{4}(1+\cos\theta)\,D \\ \omega_3,\ \omega_6: & \tfrac{1}{2}\sin\theta\,.\,C \\ \omega_2,\ \omega_4: & \tfrac{1}{4}(1-\cos\theta)\,D \\ \omega_7,\ \omega_8: & \tfrac{1}{2}\sin\theta\,.\,D \\ \omega_9: & \cos\theta\,.\,C \end{array}\right\} \quad\text{.......................(175)},$$

the lines being supposed to be observed along the equator, that is, in a direction at right angles to the axis Oz, or at right angles to the direction of the magnetic force causing the precession.

In this case we see by (173) and (174), § 114, that the factor, by which the amplitude of the generating vibration must be multiplied to give that of the wave emitted, is practically the same for each line. Thus (175) give the amplitudes of the lines emitted, as well as of the generating vibrations.

For this direction we have seen that all the lines are plane polarized; ω_7, ω_9, ω_8 vibrate parallel to the axis, i.e. parallel to the lines of force (p); the others perpendicular to them (s).

In general, the lines (p) are the strongest, and $\omega_9>\omega_7$, showing that $\cos^2\theta\,.\,C^2>\tfrac{1}{4}\sin^2\theta\,.\,D$, by (175).

In general also $\omega_1<\omega_3<\omega_2$, showing that $\theta>\tfrac{1}{2}\pi$, and $C^2<\tfrac{1}{4}\tan^2\dfrac{\theta}{2}\,.\,D^2$.

The two conditions together require that $\theta>\tfrac{2}{3}\pi$.

The case where $\omega_9>\omega_7$, and at the same time $\omega_1>\omega_3>\omega_2$, also occurs; it requires $C^2>\tfrac{1}{4}\tan^2\theta\,.\,D^2$, and $C^2<\tfrac{1}{4}\cot^2\dfrac{\theta}{2}\,.\,D^2$, together with $\theta<\tfrac{1}{2}\pi$. They require that $\theta<\tfrac{1}{3}\pi$.

In certain cases some of the lines are absent.

$D=0$ gives ω_3, ω_9, ω_6 alone, a normal triplet. This can only be due to axial, without orbital, vibrations.

$C=0$ gives the sextet ω_7, ω_8, ω_1, ω_5, ω_2, ω_4; it is equally spaced when $\mu=2n$. It is due to orbital vibrations, i.e. vibrations perpendicular to the axis of symmetry.

$\theta=0$ gives the triplet ω_1, ω_9, ω_5, with abnormally wide spacing. It is due to spinning about the line of force, without precession.

* *Phys. Zeitsch.* 8, p. 232, 1907.

$\theta = \pi$ gives the triplet ω_2, ω_9, ω_4, with abnormally narrow spacing. It is due to spinning about the negative direction of the line of force, without precession.

$\theta = \frac{1}{2}\pi$ gives the octet, with the central line ω_9 missing. It is due to precession with the axis of symmetry perpendicular to the line of force.

We may also get exceptional cases by a suitable choice of the ratio $\mu : n$.

For instance, $n = 0$ gives a triplet; $\mu = n$ gives a quintet, ω_2 and ω_4 coinciding with ω_9, and ω_3, ω_6 respectively with ω_7, ω_8. If at the same time $C = 0$, and $\theta = 0$, only ω_1, ω_5 are left, and we get a doublet.

Thus the hypothesis that the Zeeman effect is due to precessional motion, explains a good many of the observed cases; but there are some cases of resolution into 10 and 15 components which it does not explain very readily*.

This hypothesis appears to be due to Ritz (*loc. cit.* § 112).

* It is unlikely that these lines could be accounted for by means of sum and difference vibrations of higher order. It is true that, for example, a line of frequency $q + \mu + 2n$ could be got by making $s_1 = s_7 = 1$, $s_9 = -1$, giving $\omega_1 + \omega_7 - \omega_9$, which has the right value; but the intensity of such a line would be excessively small.

CHAPTER IX

PSEUDO-PERIODIC AND APERIODIC MOTIONS

119. THE treatment of polyperiodic motions given in the last chapter was restricted to strictly periodic motions, that is, to motions involving no aperiodic components. Such a limitation is however undesirable for the purposes of Physics, because many of the most important physical problems deal with motions which are not strictly periodic, such as damped free vibrations, "pseudo-periodic motions," or with motions which involve no periodic components at all, such as motions in orbits with infinite branches, "aperiodic motions." Many of these motions are discontinuous, that is to say, they do not follow the same law for all time; for instance, damped free vibrations imply a time at which they were started by the application of some disturbing force. The beginning marks a time at which the functions representing the coordinates are discontinuous in form.

In deducing equations (140) and (141), § 89, we made the assumption that the Fourier Integrals of the type

$$I = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{\partial}{\partial t}\right)^n \epsilon^{\iota\mu(t-r/c-\tau)} f(\tau)\,d\tau\,d\mu$$

could be replaced by

$$\left(\frac{\partial}{\partial t}\right)^n \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \epsilon^{\iota\mu(t-r/c-\tau)} f(\tau)\,d\tau\,d\mu,$$

or what is the same thing, by $f^{(n)}(t - r/c)$.

Here $f(\tau)$ is a function of the coordinates and velocity of the moving charge at time τ. The transformation is meaningless unless $f^{(n)}(t)$ be finite, except for singular values of t; but this condition is always satisfied for every value of n in a physical problem.

The transformation is obviously justifiable when $f(t)$ is continuous in *form*, so that all its differential coefficients are continuous, and not merely finite, for all values of t. This occurs for all periodic motions, and indeed for all motions which follow the same law for ever, so that the coordinates of the moving charge are always given by the same expressions. But many cases occur where this is not the case. For instance, when the charge starts moving from rest under the action of a given finite force, the function $f(t)$ is constant, and all its differential coefficients are zero for times before the commencement of the motion. But immediately afterwards those of its differential coefficients which involve the acceleration of the moving charge, assume certain definite values different from zero, but finite. In cases such as these, where the motion is discontinuous, in the sense that the coordinates of the moving charge are given as functions of t by expressions of different *form* before and after the instant of time at which the discontinuity occurs, we cannot apply the transformation without further examination.

The coordinates and velocity components are continuous in *value*, but one at least of the higher derivatives is discontinuous in value, and we must enquire what effect this discontinuity produces.

120. We shall use the method developed by Stokes* in Section II of his memoir on the critical values of the sums of periodic series.

The integral

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \epsilon^{\iota\mu(t-r/c-\tau)} f(\tau)\, d\tau\, d\mu$$

is convergent, because $f(\tau)$ is finite, except for singular values of τ, but some of the integrals to be derived from it are not. For this reason we use the equivalent integral

$$f(t) = \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} \cos\mu(t - r/c - \tau) f(\tau)\, d\tau\, d\mu,$$

which is the limit to which the integral

$$\frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} \epsilon^{-h\mu} \cos\mu(t - r/c - \tau) f(\tau)\, d\tau\, d\mu$$

tends as h is indefinitely diminished. The last integral is absolutely convergent provided that h be positive. This we assume to be the case.

Again the integral under examination, I, may be regarded in the same way as the limit when $h = 0$ of the absolutely convergent integral

$$\frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} \epsilon^{-h\mu}\left(\frac{\partial}{\partial t}\right)^n \cos\mu(t - r/c - \tau) f(\tau)\, d\tau\, d\mu.$$

The inner integral may be written in the form

$$\int_{-\infty}^{\infty} f(\tau)\left(-\frac{\partial}{\partial \tau}\right)^n \cos\mu(t - r/c - \tau)\, d\tau.$$

Integrating by parts and replacing $\frac{\partial}{\partial \tau}$ by $-\frac{\partial}{\partial t}$, we get

$$\int_{-\infty}^{\infty} \cos\mu(t - r/c - \tau) f^{(n)}(\tau)\, d\tau$$

* *Collected papers*, Vol. I. p. 271.

$$+\sum_{i=0}^{i=n-1} S \operatorname{Lim.}_{\tau=t_1} \left(\frac{\partial}{\partial t}\right)^{n-i-1} \left[f^{(i)}(\tau) \cos \mu (t - r/c - \tau) \right]_{t_1-\epsilon}^{t_1+\epsilon}$$
$$-\sum_{i=0}^{i=n-1} \operatorname{Lim.}_{\tau=\infty} \left(\frac{\partial}{\partial t}\right)^{n-i-1} \left[f^{(i)}(\tau) \cos \mu (t - r/c - \tau) \right]_{-\infty}^{\infty}.$$

Here t_1 denotes one of the instants at which a discontinuity occurs, and the summation S is for all these times of discontinuity. The square bracket denotes that the difference of the quantity inside is to be taken for times $t_1 \pm \epsilon$, where ϵ is to be made equal to zero in the limit. It should be noted that the infinite limits together are to be treated like the times of discontinuity. The result obtained is the equivalent of the equations (55) and (63) of Stokes (*loc. cit.*).

Substituting the value just obtained for the inner integral in the expression for I, and noticing that all the single integrals are absolutely convergent on account of the factor $\epsilon^{-h\mu}$, we get

$$I = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty \epsilon^{-h\mu} \cos \mu (t - r/c - \tau) f^{(n)}(\tau)\, d\tau\, d\mu$$
$$+\sum_{i=0}^{i=n-1} S \operatorname{Lim.}_{\tau=t_1} \left(\frac{\partial}{\partial t}\right)^{n-i-1} \frac{1}{\pi}\int_0^\infty \epsilon^{-h\mu} [\cos \mu (t - r/c - \tau) f^{(i)}(\tau)]\, d\mu$$
$$-\sum_{i=0}^{i=n-1} \operatorname{Lim.}_{\tau=\infty} \left(\frac{\partial}{\partial t}\right)^{n-i-1} \frac{1}{\pi}\int_0^\infty \epsilon^{-h\mu} [\cos \mu (t - r/c - \tau) f^{(i)}(\tau)]\, d\mu \quad \ldots(176).$$

The double integral in the first line remains convergent when $h = 0$, and then reduces to the Fourier Integral for $f^{(n)}(t - r/c)$, that is to say, to

$$\left(\frac{\partial}{\partial t}\right)^n \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty \cos \mu (t - r/c - \tau) f(\tau)\, d\tau\, d\mu.$$

If the transformation previously used is allowable, the remaining, or complementary, terms in the expression (176) for I must vanish. We have to determine under what conditions this takes place.

121. It is obvious that it cannot take place at all unless the complementary terms approach a definite limit as h is diminished to zero, and this limit must be zero quite independently of the discontinuities of the functions $f^{(i)}(\tau)$.

If we can prove that the limit is definite, though not zero, then the complementary terms give the correction which is necessary on account of the discontinuity.

Let us consider one of the single integrals in (176) due to the discontinuity t_1. We may put $\tau = t_1 + \epsilon$, and make ϵ equal to zero in the limit. Thus we have to consider the limiting value of a term of the type

$$\left(\frac{\partial}{\partial t}\right)^{n-i-1} \int_0^\infty \epsilon^{-h\mu} \cos \mu (t - r/c - t_1 - \epsilon) f^{(i)}(t_1 + \epsilon)\, \delta\mu.$$

It may be written in the form

$$f^{(i)}(t_1+\epsilon)\left(\frac{\partial}{\partial t}\right)^{n-i-1}\int_0^\infty \epsilon^{-h\mu}\cos\mu(t-r/c-t_1)\,d\mu$$

$$+f^{(i)}(t_1+\epsilon)\left(\frac{\partial}{\partial t}\right)^{n-i-1}\int_0^\infty \epsilon^{-h\mu}\{\cos\mu(t-r/c-t_1-\epsilon)-\cos\mu(t-r/c-t_1)\}\,d\mu.$$

The integral in the last line is equal to

$$\epsilon\int_0^\infty \epsilon^{-h\mu}\sin\mu\,(t-r/c-t_1-\tfrac{1}{2}\epsilon)\frac{\sin\frac{1}{2}\mu\epsilon}{\frac{1}{2}\mu\epsilon}\,\mu\,d\mu,$$

that is, to
$$-\epsilon\frac{\partial}{\partial t}\int_0^\infty e^{-h\mu}\cos\mu\,(t-r/c-t_1-\tfrac{1}{2}\epsilon)\frac{\sin\frac{1}{2}\mu\epsilon}{\frac{1}{2}\mu\epsilon}\,d\mu.$$

It is clear that the factor $\sin\frac{1}{2}\mu\epsilon/\frac{1}{2}\mu\epsilon$ improves the convergence of the last integral for large values of μ, however small ϵ and h may be. Since ϵ occurs as a factor outside the integral, the term is ultimately negligible.

Hence the term in question reduces ultimately to

$$f^{(i)}(t_1+0)\left(\frac{\partial}{\partial t}\right)^{n-i-1}\int_0^\infty \epsilon^{-h\mu}\cos\mu\,(t-r/c-t_1)\,d\mu,$$

and the two complementary terms due to the discontinuity reduce to

$$\left\{\left(\frac{\partial}{\partial t}\right)^{n-i-1}\frac{1}{\pi}\int_0^\infty \epsilon^{-h\mu}\cos\mu\,(t-r/c-t_1)\,d\mu\right\}[f^{(i)}(t_1)],$$

where the square bracket now denotes $f^{(i)}(t_1+0)-f^{(i)}(t_1-0)$.

This complementary term vanishes when either factor vanishes.

The second factor vanishes when $f^{(i)}(t_1+0)=f^{(i)}(t_1-0)$, that is, when there is no discontinuity at t_1. This of course must be so, and justifies the presence of the factor $[f^{(i)}(t_1)]$.

It also vanishes when $f^{(i)}(\tau)$ is zero; this frequently occurs for $\tau=\pm\infty$.

In all other cases it is necessary to consider the first factor

$$\left(\frac{\partial}{\partial t}\right)^{n-i-1}\frac{1}{\pi}\int_0^\infty \epsilon^{-h\mu}\cos\mu\,(t-r/c-t_1)\,d\mu.$$

When h is greater than zero we have

$$\int_0^\infty \epsilon^{-h\mu}\cos\mu\,(t-r/c-t_1)\,d\mu=h/\{h^2+(t-r/c-t_1)^2\},$$

and therefore

$$\left(\frac{\partial}{\partial t}\right)^{n-i-1}\int_0^\infty \epsilon^{-h\mu}\cos\mu\,(t-r/c-t_1)\,d\mu$$
$$=(-1)^{n-i-1}(n-i-1)!\;\frac{\sin\{(n-i)\tan^{-1}(h/(t-r/c-t_1))\}}{\{h^2+(t-r/c-t_1)^2\}^{\frac{1}{2}(n-i)}}.$$

As h approaches zero, $t-r/c-t_1$ remaining constant, the last expression oscillates between finite limits until h becomes less than $(t-r/c-t_1)\tan\pi/2(n-i)$, and thereafter diminishes monotonously to zero. Hence the limit for $h=0$ is definite, and ultimately zero, whenever $t-r/c-t_1$ differs from zero by a finite amount. This is all the more the case when $t_1=\pm\infty$, on account of the infinite denominator.

Thus the complementary terms in (176) due to the two infinite limits of the τ-integral vanish, because $f^{(i)}(\tau)$ is finite.

The terms due to a discontinuity at t_1 vanish practically whenever t differs from t_1+r/c by an appreciable amount.

But when t becomes very nearly equal to t_1+r/c the complementary terms execute large oscillations because the denominator

$$\{h^2+(t-r/c-t_1)^2\}^{\frac{1}{2}(n-i)}$$

becomes very small in the limit. The series in (176) become divergent and useless for purposes of calculation. In this case we must either calculate the complementary term by a different method, or if that prove to be impossible, give up this method of calculating the integral I altogether.

122. In conclusion we may sum up our discussion as follows:

When t differs from any of the values t_1+r/c, ... corresponding to the discontinuities of the motion by a finite amount, we may write

$$\frac{1}{\pi}\int_0^\infty\int_{-\infty}^\infty\frac{\partial^n}{\partial t^n}\cos\mu(t-r/c-\tau)f(\tau)\,d\tau\,d\mu=f^{(n)}(t-r/c)\quad\ldots(177).$$

This is equivalent to taking the operator $\dfrac{\partial^n}{\partial t^n}$ outside the signs of integration.

But when t is nearly equal to t_1+r/c, where t_1 is any one of the times of discontinuity of the motion, (177) ceases to give a sufficient approximation; we must add the complementary term

$$\sum_{i=0}^{i=n-1}\underset{h=0}{\text{Lim.}}\ \underset{\tau=t_1}{\text{Lim.}}\left[f^{(i)}(\tau)\left(\frac{\partial}{\partial t}\right)^{n-i-1}\frac{1}{\pi}\int_0^\infty\epsilon^{-h\mu}\cos\mu(t-r/c-\tau)\,d\mu\right].$$

Unless this complementary term can be proved to converge to the limit zero as h converges to zero, the expression (176) is no longer true.

123. As examples we shall deduce two important expansions for the potentials due to a point charge.

In the first place, let us expand the exponential in (131), § 86, in powers of p by means of Taylor's Theorem; we get, on changing the limits of μ,

$$\phi=\frac{e}{\pi r}\sum_{s=0}^{s=\infty}\frac{c^{-s}}{s\,!}\int_0^\infty\int_{-\infty}^\infty\left(\frac{\partial}{\partial t}\right)^s\cos\mu(t-r/c-\tau)\,.\,p^s\,d\tau\,d\mu.$$

Each term of this expansion is an integral of the type I just considered. Hence we get from (177)

$$\phi = \frac{e}{r}\sum_{s=0}^{s=\infty}\frac{1}{s!}\left(\frac{\partial}{c\partial t}\right)^s \{p(t-r/c)\}^s \quad \text{.................(178).}$$

This expansion is valid, provided that it converges, and that t is not nearly equal to $t_1 + r/c$, where t_1 is one of the times of discontinuity. The first condition may be assumed to be satisfied for physical reasons in all problems to which the expansion would be applied.

When however t is nearly equal to $t_1 + r/c$, we must evaluate the complementary term, which becomes

$$\frac{e}{c\pi r}\sum_{s=1}^{s=\infty}\sum_{i=0}^{i=s-1}\frac{1}{s!}\operatorname*{Lim.}_{h=0}\operatorname*{Lim.}_{\tau=t_1}\left[\left(\frac{\partial}{c\partial\tau}\right)^i\{p(\tau)\}^s\right]\left(\frac{\partial}{c\partial t}\right)^{s-i-1}\int_0^\infty \epsilon^{-h\mu}\cos\mu(t-r/c-\tau)\,d\mu.$$

In precisely the same way we get from (132)

$$\mathbf{a} = \frac{e}{cr}\sum_{s=1}^{s=\infty}\sum_{i=0}^{i=s-1}\frac{1}{s!}\left(\frac{\partial}{c\partial t}\right)^s\left(\{p(t-r/c)\}^s\,\mathbf{v}(t-r/c)\right) \quad \text{......(179),}$$

which holds under the same conditions as (178), and in the case of failure must be corrected by the addition of an analogous complementary term.

124. Again, applying (12), § 9, to a point charge, by omitting the integration with respect to de, and introducing a factor e instead, we get in the same way as before, for a unit charge ($e = 1$),

$$\phi = \sum_{s=0}^{s=\infty}\frac{(-1)^s}{s!}\left(\frac{\partial}{c\partial t}\right)^s R^{s-1} \quad \text{....................(180),}$$

where R is the distance of the fieldpoint from the position of the point charge at the actual time t.

When t differs appreciably from t_1, one of the times of discontinuity of the motion, the expansion (180) is valid whenever it is convergent, that is, whenever R is small enough compared with the radii of curvature and torsion and similar lengths characteristic of the path of the moving charge.

But when t is nearly equal to t_1 we must evaluate a complementary term which is found to be

$$\frac{e}{c\pi r}\sum_{s=1}^{s=\infty}\sum_{i=0}^{i=s-1}\frac{(-1)^s}{s!}\operatorname*{Lim.}_{h=0}\operatorname*{Lim.}_{\tau=t_1}\left[\left(\frac{\partial}{c\partial\tau}\right)^i R^{s-1}\right]\left(\frac{\partial}{c\partial t}\right)^{s-i-1}\int_0^\infty \epsilon^{-h\mu}\cos\mu(t-\tau)\,d\mu,$$

where R is taken for time τ.

In the same way and under the same conditions we get from (13), § 9,

$$c\mathbf{a} = \sum_{s=0}^{s=\infty}\frac{(-1)^s}{s!}\left(\frac{\partial}{c\partial t}\right)^s\{R^{s-1}\mathbf{v}\} \quad \text{.................(181),}$$

where R and $\mathbf{v}$ both correspond to the time t.

125. The fact that all series for the potentials of the types (177)—(181) fail when t is nearly equal to t_1 is susceptible of a simple physical interpretation.

We know that the moving charge is continually emitting waves of disturbance, by means of which the field is propagated to all parts of the surrounding medium with the velocity of light. In particular, at a time t_1, at which its motion changes its form discontinuously, a wave is emitted which separates space into two regions in which the potentials are determined by different mathematical expressions. Such a wave may be called a wave of discontinuity, for the potentials change their form discontinuously there, although their numerical values do not necessarily suffer any change. At points which are not close to a wave of discontinuity the expressions (177)—(181) are valid, and have a physical meaning. But at points on a wave of discontinuity they lose all meaning, unless the complementary terms belonging to the wave can be evaluated. In general this cannot be done, and it is better to resort to the integral expressions from which the series have been derived. This failure is quite analogous to the failure of the point laws under similar circumstances, of which we have met several examples in Ch. V.

Since all distributions of charge met with in nature appear to be extended distributions and not point charges, the failure of our series can only affect an infinitely small element of charge at one and the same time, namely the charge of that part of the system which is enclosed between two infinitely close spheres, whose radii are infinitely nearly equal to r, and whose centres are at the fieldpoint. Any difficulty arising from the discontinuity will disappear when the integration for the charge is extended to the whole system, for the discontinuity is then, as it were, spread over a layer of transition of small but finite thickness.

It should be noted that the series (180) and (181) are the developments by Lagrange's Theorem of the point potentials (26) and (27), § 13. They imply, in the first place, that the characteristic equation $t = \tau + R/c$ has but one root less than t, i.e. that the velocity of the charge is less than that of light; and secondly, that all the differential coefficients of the coordinates with respect to the time are finite.

126. The series (180) and (181) have been discussed by Schott* for the particular case of rectilinear motion. From his discussion the following results follow:

In the equations (178)—(181) the series in the first lines represent the potentials due to the given motion regarded as continuous. For instance, if we are required to find the potentials due to a charge which starts from rest

* *Ann. der Phys.* 25, p. 63, 1908. See also A. W. Conway, *Proc. Roy. Irish Acad.* Vol. XXVIII. A. p. 5, 1910.

at time zero according to a prescribed law, e.g. with uniform acceleration, they give the potentials due, not only to the given motion for all positive time, but in addition that due to a motion taking place for all negative time according to the same law, e.g. with uniform retardation in the reverse direction.

The complementary terms in the second lines of the equations give the correction to be applied in order to neutralize the effect of the motion for negative time, which is continuous with the given motion, but in the case of a discontinuous motion actually does not take place. These terms are due to waves which would be emitted at negative times if the actual motion were really continuous.

In the language of Ch. III, § 14, the first line represents the effect of the whole continuous characteristic curve; the second line represents the correction to be applied in order to replace the negative branch of the continuous characteristic curve, which is not required, by the negative branch of the discontinuous curve, which is actually present.

Thus we see that for discontinuous motions the only general method is to use the integral expressions (12) and (13), § 9, or their equivalents, the point laws, (26)—(27), § 13. At great distances from the charge the former may be replaced by (131) and (132), § 86.

For all continuous motions with a velocity less than that of light we may use the series (178) and (179) at a great distance from the charge, and (180) and (181) close to it, provided that in either case they be convergent. This point requires special investigation.

It is possible that these series may also be used in cases of discontinuous motion, if the complementary terms can be determined. In this case however the value of these terms must be found *before* we make h equal to zero, in order to make sure of the convergence of the integral.

We shall now pass on to the consideration of some problems in illustration of the methods of this chapter.

127. Problem 1. To find the distant field due to a monoperiodic motion of variable amplitude. We assume that the motion involves a single period, but that its amplitude varies, so that it is not strictly periodic. For instance, we may have a charge executing a damped vibration in a straight line, or moving in a logarithmic spiral, or some similar curve. The potentials cannot be worked out by means of equations (140) and (141), § 89, because such a motion must have had a beginning, before which the charge was either held at rest, or was moving according to some different law. We must therefore use (131) and (132), § 86, which we may write in the form

$$\phi = \frac{e}{r\pi}\int_0^\infty \int_{-\infty}^\infty \cos \mu\,(t - r/c - \tau + p/c)\, d\tau\, d\mu,$$

$$\mathbf{a} = \frac{e}{cr\pi}\int_0^\infty \int_{-\infty}^\infty \cos \mu\,(t - r/c - \tau + p/c)\, \mathbf{v}\, d\tau\, d\mu.$$

If we suppose the given pseudo-periodic motion to start at time zero, the part of the integral from $\tau = 0$ to $\tau = \infty$ represents the effect of the required motion, the remaining part the effect of the preceding state of rest, or of the preceding motion, which is discontinuous with the required one in form. The values of the displacement and velocity of the charge are necessarily continuous, but those of the acceleration and higher differential coefficients will generally be discontinuous (one at least must be so). The effect of the discontinuity must be found by Stokes' method. If this be done, we may take the time integral from 0 to ∞, and the result will give the potentials for any time t greater than r/c.

Suppose that $$\frac{p}{c} = f(\tau) \sin(\omega\tau + \alpha) \quad \text{..........................(182)},$$

where $f(\tau)$ is for the present arbitrary. We get at once for the potentials at the times mentioned

$$\phi = \frac{e}{r\pi}\int_0^\infty \int_0^\infty \cos \mu\,\{t - r/c - \tau + f(\tau)\sin(\omega\tau + \alpha)\}\, d\tau\, d\mu,$$

$$\mathbf{a} = \frac{e}{cr\pi}\int_0^\infty \int_0^\infty \cos \mu\,\{t - r/c - \tau + f(\tau)\sin(\omega\tau + \alpha)\}\, \mathbf{v}\, d\tau\, d\mu.$$

Now we have the well known equations*

$$\cos(x \sin\phi) = \sum_{s=0}^{s=\infty}{}' \, 2J_{2s}(x)\cos 2s\phi,$$

$$\sin(x \sin\phi) = \sum_{s=0}^{s=\infty} 2J_{2s+1}(x)\sin(2s+1)\phi.$$

Hence

$$\phi = \frac{2e}{r\pi}\sum_{s=0}^{s=\infty}{}' \int_0^\infty \int_0^\infty J_{2s}\{\mu f(\tau)\}\cos \mu\,(t - r/c - \tau)\cos 2s(\omega\tau + \alpha)\, d\tau\, d\mu$$
$$- \frac{2e}{r\pi}\sum_{s=0}^{s=\infty} \int_0^\infty \int_0^\infty J_{2s+1}\{\mu f(\tau)\}\sin \mu\,(t - r/c - \tau)\sin(2s+1)(\omega\tau + \alpha)\, d\tau\, d\mu,$$

$$\mathbf{a} = \frac{2e}{cr\pi}\sum_{s=0}^{s=\infty}{}' \int_0^\infty \int_0^\infty J_{2s}\{\mu f(\tau)\}\cos \mu\,(t - r/c - \tau)\cos 2s(\omega\tau + \alpha)\, \mathbf{v}\, d\tau\, d\mu$$
$$- \frac{2e}{cr\pi}\sum_{s=0}^{s=\infty} \int_0^\infty \int_0^\infty J_{2s+1}\{\mu f(\tau)\}\sin \mu\,(t - r/c - \tau)\sin(2s+1)(\omega\tau + \alpha)\, \mathbf{v}\, d\tau\, d\mu.$$

* Gray and Mathews, *Bessel Functions*, p. 18.

128. These integrals can be transformed further. For we have*

$$\int_0^\infty J_{2s}(\xi)\cos(\xi x)\,d\xi = \frac{\cos\{2s\sin^{-1}x\}}{\sqrt{(1-x^2)}} \text{ for } 0 < x < 1,$$
$$= 0 \qquad \text{for } x > 1,$$

and
$$\int_0^\infty J_{2s+1}(\xi)\sin(\xi x)\,d\xi = \frac{\sin\{(2s+1)\sin^{-1}x\}}{\sqrt{(1-x^2)}} \text{ for } 0 < x < 1,$$
$$= 0 \qquad \text{for } x > 1.$$

It is obvious that the integrals for negative values of x are got by simply changing its sign. They are continuous at $x = 0$.

Put $\xi = \mu f(\tau), \quad x = (t - r/c - \tau)/f(\tau)$.

The critical values of x, ± 1, obviously correspond to $\tau = \tau_1$ and $\tau = \tau_2$, where

$$\left.\begin{aligned}\tau_1 + f(\tau_1) = t - r/c\\ \tau_2 - f(\tau_2) = t - r/c\end{aligned}\right\} \quad \text{.........................(183).}$$

By (182) we can always arrange matters so that $f(\tau)$ is essentially positive, by allowing p to be negative as well as positive, according to the sign of $\sin(\omega\tau + \alpha)$.

Then we have by (183)

$$\tau_1 < t - r/c < \tau_2.$$

Two cases arise; in the first case τ_1 is negative, and the lower limit of the τ integral becomes zero. In the second τ_1 is positive and itself gives the lower limit of the integral.

We now find on substitution, for case

I. $\tau_1 < 0$.

$$\phi = \frac{2e}{r\pi}\sum_{s=0}^{s=\infty}{}' \int_0^{\tau_2} \frac{\cos 2s(\omega\tau+\alpha)\cos 2s\sin^{-1}[(t-r/c-\tau)/f(\tau)]\,d\tau}{\sqrt{\{[f(\tau)]^2-(t-r/c-\tau)^2\}}}$$
$$-\frac{2e}{r\pi}\sum_{s=0}^{s=\infty} \int_0^{\tau_2} \frac{\sin(2s+1)(\omega\tau+\alpha)\sin(2s+1)\sin^{-1}[(t-r/c-\tau)/f(\tau)]\,d\tau}{\sqrt{\{[f(\tau)]^2-(t-r/c-\tau)^2\}}}$$
$$\text{......(184)},$$

$$\mathbf{a} = \frac{2e}{cr\pi}\sum_{s=0}^{s=\infty}{}' \int_0^{\tau_2} \frac{\cos 2s(\omega\tau+\alpha)\cos 2s\sin^{-1}[(t-r/c-\tau)/f(\tau)]\,\mathbf{v}\,d\tau}{\sqrt{\{[f(\tau)]^2-(t-r/c-\tau)^2\}}}$$
$$-\frac{2e}{cr\pi}\sum_{s=0}^{s=\infty} \int_0^{\tau_2} \frac{\sin(2s+1)(\omega\tau+\alpha)\sin(2s+1)\sin^{-1}[(t-r/c-\tau)/f(\tau)]\,\mathbf{v}\,d\tau}{\sqrt{\{[f(\tau)]^2-(t-r/c-\tau)^2\}}}$$
$$\text{......(185)},$$

with $\tau_2 - f(\tau_2) = t - r/c$.

II. $\tau_1 > 0$.

The same expressions with the lower limit equal to τ_1, where

$$\tau_1 + f(\tau_1) = t - r/c.$$

* Nielsen, *Cylinderfunktionen*, p. 195.

The conditions as to the limits are best seen from the diagram. AB is the line $\tau = t - r/c$, CD is the curve (1) $\tau_1 + f(\tau_1) = t - r/c$, and JFH the curve (2) $\tau_2 - f(\tau_2) = t - r/c$.

JI and IC are each equal to $f(0)$.

When $t - r/c < f(0)$, we have case (I), and the limits are from E to F.

When $t - r/c > f(0)$, we have case (II), and the limits are from G to H.

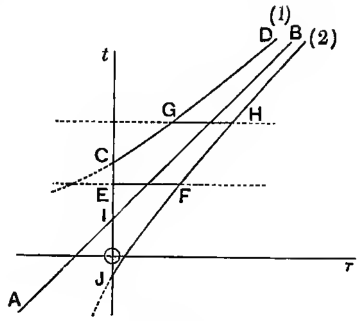

Fig. 38.

The second case represents, as it were, the permanent régime and is obviously the most important; the first case represents a transition period during which the permanent régime is being established. It is clear that this transition period lasts only a very short time, for $f(0)$ is merely the initial value of p/c (by (182)), and thus is of the order of the time required by light to travel a distance of the same order as the initial amplitude of the vibration. For this reason we shall henceforth only consider the permanent régime (II), for which the limits of the integral are τ_1 and τ_2.

129. We can now transform the integrals (184) and (185) into a form more suitable for calculation. Write

$$\frac{t - r/c - \tau}{f(\tau)} = \cos\chi \qquad \text{.........................(186).}$$

We get

$$\frac{d\tau}{\sqrt{\{[f(\tau)]^2 - (t - r/c - \tau)^2\}}} = \frac{d\tau}{d\chi}\,\frac{d\chi}{f(\tau)\sin\chi}.$$

The limits for χ are:

for $\tau = \tau_1$, $\qquad t - r/c - \tau_1 = f(\tau_1), \qquad \chi = 0,$

for $\tau = \tau_2$, $\qquad t - r/c - \tau_2 = -f(\tau_2), \qquad \chi = \pi.$

Thus the limits become independent of t in the permanent régime. Moreover we get

$$\cos 2s \sin^{-1}[(t - r/c - \tau)/f(\tau)] = (-1)^s \cos 2s\chi,$$

$$\sin(2s+1)\sin^{-1}[(t - r/c - \tau)/f(\tau)] = (-1)^s \cos(2s+1)\chi.$$

Lastly we require to express the functions

$$\frac{\cos 2s(\omega\tau + \alpha)}{f(\tau)}\frac{d\tau}{d\chi} \equiv \frac{d}{d\chi}\int_0^\tau \frac{\cos 2s(\omega\tau + \alpha)\,d\tau}{f(\tau)},$$

$$\frac{\sin(2s+1)(\omega\tau + \alpha)}{f(\tau)}\frac{d\tau}{d\chi} \equiv \frac{d}{d\chi}\int_0^\tau \frac{\sin(2s+1)(\omega\tau + \alpha)\,d\tau}{f(\tau)},$$

as functions of χ. This is best effected by means of Lagrange's Theorem. We have by (186)

$$F(\tau) = F(t - r/c) - \frac{\cos\chi}{1!} f(t - r/c) F'(t - r/c) + \dots$$
$$+ (-1)^n \frac{\cos^n \chi}{n!} \frac{\partial^{n-1}}{\partial t^{n-1}} [\{f(t-r/c)\}^n F'(t-r/c)] \dots.$$

Thus we get

$$\int_0^\tau \frac{\cos 2s(\omega\tau + \alpha)\, d\tau}{f(\tau)} = \int_0^{t-r/c} \frac{\cos 2s(\omega\tau + \alpha)\, d\tau}{f(\tau)} - \cos\chi \cos 2s\{\omega(t - r/c) + \alpha\}$$
$$+ \dots + (-1)^n \frac{\cos^n\chi}{n!} \frac{\partial^{n-1}}{\partial t^{n-1}} [\{f(t-r/c)\}^{n-1} \cos 2s\{\omega(t-r/c)+\alpha\}]$$
$$+ \dots$$

with a similar series for the second function. Hence

$$\frac{\cos 2s(\omega\tau+\alpha)}{f(\tau)} \frac{d\tau}{d\chi} = \sum_{n=0}^{n=\infty} (-1)^n \frac{\cos^n\chi \sin\chi}{n!} \frac{\partial^n}{\partial t^n} [\{f(t-r/c)\}^n \cos 2s\{\omega(t-r/c)+\alpha\}]$$

with a corresponding series for the second function.

Substituting in (184) we get first a series of terms of the type

$$\frac{2e}{r\pi} \frac{(-1)^{n+s}}{n!} \frac{\partial^n}{\partial t^n} [\{f(t-r/c)\}^n \cos 2s\{\omega(t-r/c)+\alpha\}] \int_0^\pi \cos^n\chi \cos 2s\chi\, d\chi.$$

The integral vanishes when n is odd.

When n is even, $= 2j$, it is equal to $\dfrac{\pi \,.\, 2j\,!}{2^{2j}\, \overline{j+s}\,!\, \overline{j-s}\,!}$ when $s \leqslant j$, otherwise zero.

Secondly, we get a series of terms of the type

$$\frac{2e}{r\pi} \frac{(-1)^{n+s+1}}{n!} \frac{\partial^n}{\partial t^n} [\{f(t-r/c)\}^n \sin(2s+1)\{\omega(t-r/c)+\alpha\}]$$
$$\int_0^\pi \cos^n\chi \cos(2s+1)\chi\, d\chi.$$

The integral vanishes when n is even.

When n is odd, $= 2j+1$, it is equal to $\dfrac{\pi \,.\, \overline{2j+1}\,!}{2^{2j+1}\, \overline{j+s+1}\,!\, \overline{j-s}\,!}$, provided that $s \leqslant j$, otherwise it is zero.

We put $n = 2s + 2k$, or $2s + 2k + 1$ as the case may be, and sum from $k = 0$ to $k = \infty$. We get a first series

$$\frac{2e}{r}(-1)^s . \sum_{k=0}^{k=\infty} \frac{1}{2^{2s+2k} . \overline{2s+k}\,!\, k\,!} \left(\frac{\partial}{\partial t}\right)^{2s+2k} . [\{f(t-r/c)\}^{2s+2k} . \cos 2s\{\omega(t-r/c)+\alpha\}]$$

and a second

$$\frac{2e}{r}(-1)^s . \sum_{k=0}^{k=\infty} \frac{1}{2^{2s+2k+1} . \overline{2s+k+1}\,!\, k\,!} \left(\frac{\partial}{\partial t}\right)^{2s+2k+1} .$$
$$[\{f(t-r/c)\}^{2s+2k+1} . \sin(2s+1)\{\omega(t-r/c)+\alpha\}],$$

which may be included in a single series by introducing $\alpha - \frac{1}{2}\pi$ for α. Hence we get

$$\phi = \frac{2e}{r} \sum_{s=0}^{s=\infty}{}' \sum_{k=0}^{k=\infty} \frac{1}{2^{s+2k} \cdot \overline{s+k}\,!\; k\,!} \left(\frac{\partial}{\partial t}\right)^{s+2k} \times [\{f(t-r/c)\}^{s+2k} \cdot \cos s\,\{\omega(t-r/c)+\alpha-\tfrac{1}{2}\pi\}] \quad \ldots(187).$$

In the same way we get

$$\mathbf{a} = \frac{2e}{cr} \sum_{s=0}^{s=\infty}{}' \sum_{k=0}^{k=\infty} \frac{1}{2^{s+2k} \cdot \overline{s+k}\,!\; k\,!} \left(\frac{\partial}{\partial t}\right)^{s+2k} \times [\{f(t-r/c)\}^{s+2k} \cdot \cos s\,\{\omega(t-r/c)+\alpha-\tfrac{1}{2}\pi\} \cdot \mathbf{v}(t-r/c)] \quad \ldots(188).$$

These equations could have been deduced from the series (178) and (179), § 123, by neglecting the complementary terms. But the present process is more rigorous, provided that the velocity be less than that of light, and has the further advantage of showing that the effect of the discontinuity, represented by the complementary terms, disappears when the permanent régime is reached.

130. So long as the series (187) and (188) converge, we may differentiate them with respect to t, and derive the electric and magnetic forces by means of the equations

$$\mathbf{d} = \frac{\partial}{c\partial t}(\mathbf{r}_1\phi - \mathbf{a}),$$

$$\mathbf{h} = \frac{\partial}{c\partial t}[\mathbf{a} \cdot \mathbf{r}_1].$$

We get at once

$$\mathbf{d} = \frac{2e}{cr} \sum_{s=0}^{s=\infty}{}' \sum_{k=0}^{k=\infty} \left(\frac{\partial}{\partial t}\right)^{s+2k+1} \cdot \left\{\frac{\{f(t-r/c)\}^{s+2k}}{2^{s+2k} \cdot \overline{s+k}\,!\; k!} \cos s\,\{\omega(t-r/c)+\alpha-\tfrac{1}{2}\pi\} \cdot \left(\mathbf{r}_1 - \frac{\mathbf{v}}{c}\right)\right\} \quad \ldots\ldots(189),$$

$$\mathbf{h} = \frac{2e}{c^2 r} \sum_{s=0}^{s=\infty}{}' \sum_{k=0}^{k=\infty} \left(\frac{\partial}{\partial t}\right)^{s+2k+1} \cdot \left\{\frac{\{f(t-r/c)\}^{s+2k}}{2^{s+2k} \cdot \overline{s+k}\,!\; k!} \cos s\,\{\omega(t-r/c)+\alpha-\tfrac{1}{2}\pi\} \cdot [\mathbf{v} \cdot \mathbf{r}_1]\right\} \quad \ldots\ldots(190).$$

The series for k gives the harmonic of order s, and frequency $s\omega$. It is of order s in the amplitude, and therefore generally small. We shall now consider an example.

131. Example. Damped rectilinear vibration. We write

$$\zeta = a\epsilon^{-\kappa\tau} \sin(\omega\tau + \alpha), \quad \text{for } \tau \geqslant 0.$$

Using polar coordinates (r, θ, ϕ) we get

$$p = a\cos\theta \cdot \epsilon^{-\kappa\tau} \sin(\omega\tau + \alpha),$$

whence by comparison with (182), § 127,

$$f(\tau)=\frac{a\cos\theta}{c}\,\epsilon^{-\kappa\tau}.$$

Substituting in (187) and replacing the cosine by an exponential, of which the real part only is to be taken in the usual way, we get

$$\phi=\frac{2e}{r}\sum_{s=0}^{s=\infty}{}'\sum_{k=0}^{k=\infty}\frac{\left(\dfrac{a\cos\theta}{2c}\right)^{s+2k}}{\overline{s+k}!\,k!}\left(\frac{\partial}{\partial t}\right)^{s+2k}.\,\epsilon^{-(s+2k)\kappa(t-r/c)+\iota s\{\omega(t-r/c)+\alpha-\frac{1}{2}\pi\}}$$

$$=\frac{2e}{r}\sum_{s=0}^{s=\infty}{}'\sum_{k=0}^{k=\infty}\frac{\left\{\dfrac{a\cos\theta}{2c}\epsilon^{-\kappa(t-r/c)}\{\iota s\omega-(s+2k)\kappa\}\right\}^{s+2k}}{\overline{s+k}!\,k!}\epsilon^{\iota s\{\omega(t-r/c)+\alpha-\frac{1}{2}\pi\}}\quad\ldots(191).$$

The modulus of the general term of this series is

$$\frac{2e}{r}\frac{\left\{\dfrac{a\cos\theta}{2c}\epsilon^{-\kappa(t-r/c)}\sqrt{\{s^2\omega^2+(s+2k)^2\kappa^2\}}\right\}^{s+2k}}{\overline{s+k}!\,k!}.$$

Ultimately it behaves like

$$\frac{2e}{r\sqrt{2\pi(s+2k)}}\frac{\overline{s+2k}!}{2^{s+2k}\,\overline{s+k}!\,k!}\left\{\frac{\epsilon\kappa a\cos\theta}{c}\epsilon^{-\kappa(t-r/c)}\sqrt{\left\{1+\frac{s^2\omega^2}{(s+2k)^2\kappa^2}\right\}}\right\}^{s+2k}.$$

It is therefore convergent for all values of θ provided that

$$t-r/c>\frac{\log(\epsilon\kappa a/c)}{\kappa}\quad\ldots\ldots\ldots\ldots\ldots\ldots(192).$$

If l be the logarithmic decrement and λ the wave-length of the vibration, we have $\dfrac{\epsilon\kappa a}{c}=\dfrac{2\epsilon la}{\lambda}$.

Unless the damping be enormous this is very small

Condition (192) may be written

$$\frac{c(t-r/c)}{\epsilon a}>\frac{\log(\epsilon\kappa a/c)}{\epsilon\kappa a/c}.$$

The greatest value of the right-hand member occurs for $\kappa a/c=1$, or $2la/\lambda=1$, and its magnitude is $1/\epsilon$. Hence we must have $c(t-r/c)>a$, which is precisely the condition we obtained on p. 159, for the existence of the permanent régime.

As a matter of fact, in all practical cases the convergence of the series is very rapid indeed, because $\kappa\epsilon a/c\equiv 2\epsilon la/\lambda$ is an exceedingly small quantity.

For this reason the only terms which contribute appreciably to the k-sum are those for which k is small. For these terms we may neglect $2k\kappa$ in the quantity $\iota s\omega - (s + 2k)\kappa$. The k-sum in (191) can then be effected and gives

$$\epsilon^{\frac{1}{2}\iota s\pi} J_s \left\{ \frac{sa\cos\theta\,(\omega + \iota\kappa)}{c}\,\epsilon^{-\kappa(t-r/c)} \right\},$$

and we get

$$\phi = \frac{2e}{r}\sum_{s=0}^{s=\infty}{}' J_s \left\{ \frac{sa\cos\theta\,(\omega + \iota\kappa)}{c}\,\epsilon^{-\kappa(t-r/c)} \right\} \epsilon^{\iota s\{\omega(t-r/c)+\alpha\}} \quad \ldots(193),$$

the real part alone being taken.

132. The Bessel Function is easily developed. Write

$$\kappa/\omega = \tan\epsilon, \quad \omega a/c = \beta.$$

The Bessel Function may be written

$$J_s\{s\beta\sec\epsilon\cos\theta\,.\,\epsilon^{-\kappa(t-r/c)}\,.\,\epsilon^{\iota\epsilon}\} = J_s\{s\beta\sec\epsilon\cos\theta\,.\,\epsilon^{-\kappa(t-r/c)}\}\,\epsilon^{\iota s\epsilon}$$
$$- \iota\,\frac{s\kappa a\cos\theta/c}{1\,!}\,\epsilon^{-\kappa(t-r/c)}\,.\,J_{s+1}\{s\beta\sec\epsilon\cos\theta\,.\,\epsilon^{-\kappa(t-r/c)}\}\,\epsilon^{\iota(s+1)\epsilon}$$
$$- \frac{(s\kappa a\cos\theta/c)^2}{2\,!}\,\epsilon^{-2\kappa(t-r/c)}\,.\,J_{s+2}\{s\beta\sec\epsilon\cos\theta\,.\,\epsilon^{-\kappa(t-r/c)}\}\,\epsilon^{\iota(s+2)\epsilon}$$
$$+ \ldots.$$

On substitution in (193) we get, taking the real part,

$$\phi = \frac{2e}{r}\sum_{s=0}^{s=\infty}{}' \Big\{ J_s\{s\beta\sec\epsilon\cos\theta\,.\,\epsilon^{-\kappa(t-r/c)}\}\,.\cos s\{\omega(t-r/c)+\alpha+\epsilon\}$$
$$+ \frac{s\kappa a\cos\theta/c}{1\,!}\,\epsilon^{-\kappa(t-r/c)}\,.\,J_{s+1}\{s\beta\sec\epsilon\cos\theta\,.\,\epsilon^{-\kappa(t-r/c)}\}$$
$$.\cos[s\{\omega(t-r/c)+\alpha+\epsilon\}+\epsilon-\tfrac{1}{2}\pi]$$
$$+ \frac{(s\kappa a\cos\theta/c)^2}{2\,!}\,\epsilon^{-2\kappa(t-r/c)}\,.\,J_{s+2}\{s\beta\sec\epsilon\cos\theta\,.\,\epsilon^{-\kappa(t-r/c)}\}$$
$$.\cos[s\{\omega(t-r/c)+\alpha+\epsilon\}+2\epsilon-\pi]$$
$$+ \ldots \Big\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(194).$$

The mode of formation of successive terms is now obvious.

If the damping be so small that the terms involving κ as a factor may be neglected, and the principal term only retained, we see that we can take account of damping by giving to the amplitude the value it had at the time $t - r/c$, namely $a\epsilon^{-\kappa(t-r/c)}$, that is, we may use the instantaneous value of the amplitude.

In fact if κ be zero, equation (194), or (193), leads to precisely the values we found previously for the forces in the absence of damping in problem 2, p. 120.

If on the other hand the motion be aperiodic, so that $\omega = 0$ and $\alpha = 0$, we get from (193)

$$\phi = \text{real part of } \frac{2e}{r} \sum_{s=0}^{s=\infty}{}' J_s \left\{ \iota \frac{s\kappa a \cos\theta}{c} \epsilon^{-\kappa(t-r/c)} \right\}$$

$$= \frac{2e}{r} \sum_{s=0}^{s=\infty}{}' (-1)^s I_{2s} \left\{ \frac{2s\kappa a \cos\theta}{c} \epsilon^{-\kappa(t-r/c)} \right\} \quad \text{...............(195)},$$

where $I_s(x) = \iota^{-s} J_s(\iota x)$ in the usual way.

We shall refrain from calculating the vector potential and electric and magnetic forces, for our chief object in this problem is not so much to determine the field in all its details, as to investigate the character of the deviation produced by a small variation from strict periodicity, as for instance, a small amount of damping. This question is sufficiently illustrated by the expression just calculated for the scalar potential ϕ. The chief result is that in the permanent régime simple harmonic components exist, but with slowly varying amplitudes.

CHAPTER X

ON THE FIELD NEAR THE ORBIT OF A MOVING CHARGE OR GROUP

133. [For the purpose of studying the motion of a charge, and more particularly of a group of charges, we shall require to know the field at points on or near the orbit, not necessarily close to any one of the charges. In Problem 2, Appendix A, an investigation is given of the field close to a charge moving in any prescribed manner, but the results obtained there are insufficient for our purpose. Our present object is to calculate the field at a point, which is on or near the orbit of the electron producing it, but at a distance from the latter of at least several of its diameters, so that it may be treated as a point charge.]

The most rigorous and complete method would be to use the equations (18) and (19), § 10. This however would be a complicated process, and is not necessary for our purpose. Nor can we introduce our simplification that the electron is a point charge, directly into these equations, because the integrals with respect to τ would then diverge.

For this reason we shall use a different method.

134. Potentials. We have just stated that in order to find the equations of motion of a group of electrons, not merely in its permanent state, but also when it is slightly disturbed, we require expressions for the field due to the permanent motion and the small disturbances from it at points close to the orbit as well as actually on it. It is convenient to treat the whole problem at once; for this purpose we must develope the potentials as far as small quantities of the second order. We start from equations (12) and (13), § 9; for a unit point charge they may be written

$$\phi = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\epsilon^{\iota\mu(t-R/c-\tau)}}{R}\, d\tau\, d\mu \quad \ldots\ldots\ldots\ldots\ldots\ldots(12),$$

$$\mathbf{a} = \frac{1}{2\pi c}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\epsilon^{\iota\mu(t-R/c-\tau)}}{R}\, \mathbf{v}\, d\tau\, d\mu \quad \ldots\ldots\ldots\ldots\ldots(13).$$

In future we shall suppose ξ, η, ζ, R and $\mathbf{v}$ to refer to the undisturbed motion and (x, y, z) to be a point on the undisturbed path of the charge. We shall denote by δ prefixed to any symbol the variation by which we pass from the permanent to the disturbed motion, and from the point (x, y, z) actually on the undisturbed path to a point close to but not on it. For example, $\delta\mathbf{R}$ represents the vector whose components are $(\delta x - \delta\xi,\ \delta y - \delta\eta,\ \delta z - \delta\zeta)$, and $\delta\mathbf{R}_1$ the unit vector in the direction of $\delta\mathbf{R}$. For shortness write $p = (\mathbf{R}_1 . \delta\mathbf{R})$, so that p is the projection of $\delta\mathbf{R}$ on the vector $\mathbf{R}$ drawn from (ξ, η, ζ) to (x, y, z); and write $q = \sqrt{\{(\delta R)^2 - p^2\}}$, so that q is the length of the perpendicular let fall on $\mathbf{R}$ from the end of the vector $\delta\mathbf{R}$. Then we have

$$(R + \delta R)^2 = (R + p)^2 + q^2,$$

$$R + \delta R = R + p + q^2/2R,$$

$$\frac{1}{R + \delta R} = \frac{1}{R} - \frac{p}{R^2} + \frac{2p^2 - q^2}{2R^3},$$

$$\epsilon^{\iota\mu\{t - (R+\delta R)/c - \tau\}} = \left\{1 - \left(p + \frac{q^2}{2R}\right)\frac{\partial}{c\partial t} + \tfrac{1}{2}p^2\left(\frac{\partial}{c\partial t}\right)^2\right\}\epsilon^{\iota\mu(t - R/c - \tau)}.$$

Hence

$$\delta\frac{\epsilon^{\iota\mu(t - R/c - \tau)}}{R} = \left\{-\frac{p}{R^2} + \frac{2p^2 - q^2}{2R^3} - \left(\frac{p}{R} - \frac{2p^2 - q^2}{2R^2}\right)\frac{\partial}{c\partial t} + \frac{p^2}{2R}\left(\frac{\partial}{c\partial t}\right)^2\right\}\epsilon^{\iota\mu(t - R/c - \tau)}.$$

All these expansions are correct as far as terms of the second order. Putting for $2p^2 - q^2$ its value $3p^2 - (\delta R)^2$, and substituting in (12), we get

$$\begin{aligned}\phi + \delta\phi = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}&\left\{\frac{1}{R} - \frac{p}{R^2} + \frac{3p^2 - (\delta R)^2}{2R^3}\right\}\epsilon^{\iota\mu(t - R/c - \tau)}\,d\tau\,d\mu \\ -\left(\frac{\partial}{c\partial t}\right)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}&\left\{\frac{p}{R} - \frac{3p^2 - (\delta R)^2}{2R^2}\right\}\epsilon^{\iota\mu(t - R/c - \tau)}\,d\tau\,d\mu \\ +\left(\frac{\partial}{c\partial t}\right)^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}&\frac{p^2}{2R}\epsilon^{\iota\mu(t - R/c - \tau)}\,d\tau\,d\mu.\end{aligned}$$

We treat the integrals just in the same way as the integrals (12) and (13), § 9, from which the point laws were deduced. Changing the variable from τ to t', where $t' = \tau + R/c$, we find that

$$\phi + \delta\phi = \left[\frac{1}{KR} - \frac{p}{KR^2} + \frac{3p^2 - (\delta R)^2}{2KR^3}\right] - \frac{\partial}{c\partial t}\left[\frac{p}{KR} - \frac{3p^2 - (\delta R)^2}{2KR^2}\right] + \left(\frac{\partial}{c\partial t}\right)^2\left[\frac{p^2}{2KR}\right] \quad \ldots(196).$$

As before K is the Doppler factor $\dfrac{\partial t}{\partial\tau}$, and the functions of τ within the square brackets are to be transformed from functions of the time of emission, τ, to functions of the time of reception, t, by means of the characteristic equation $t = \tau + R/c$. It is assumed that this equation has but one root τ less than t, which implies that the velocity of the charge is less than that of light. Otherwise we should have to introduce one additional set of terms for each additional root of the characteristic equation, just as in § 14.

It is assumed that the fieldpoint (x, y, z) is at a sufficient distance from the charge to permit of our treating the latter as a point charge; otherwise we must divide the charge into sufficiently small elementary charges and integrate for the whole.

In precisely the same way we get from (13)

$$c(\mathbf{a}+\delta\mathbf{a}) = \left[\frac{\mathbf{v}}{KR} - \frac{p\mathbf{v}}{KR^2} + \frac{3p^2-(\delta R)^2}{2KR^3}\mathbf{v} + \frac{\delta\mathbf{v}}{KR} - \frac{p\,\delta\mathbf{v}}{KR^2}\right]$$
$$-\frac{\partial}{c\partial t}\left[\frac{p\mathbf{v}}{KR} - \frac{3p^2-(\delta R)^2}{2KR^2}\mathbf{v} + \frac{p\,\delta\mathbf{v}}{KR}\right] + \left(\frac{\partial}{c\partial t}\right)^2\left[\frac{p^2\mathbf{v}}{2KR}\right] \quad \ldots(197).$$

135. Electric and magnetic forces. The introduction of the displacement $(\delta x, \delta y, \delta z)$ of the fieldpoint enables us to deduce the forces very simply, so far as they involve differentiations with respect to the coordinates (x, y, z). For now we may treat these coordinates as invariable, and instead differentiate with respect to $(\delta x, \delta y, \delta z)$. The terms of first order in the potentials then give the parts of the forces which depend on the permanent motion, and these for points on the path; the second order terms give the parts of the forces due to the disturbance from the permanent motion, and to displacements of the fieldpoint away from the permanent path.

In differentiating with respect to the time we get the corresponding parts of the forces due to the permanent motion from zero order terms, and the parts due to disturbance and displacement from the first order terms, while we may neglect the second order terms altogether. It is however convenient to delay the differentiations with respect to the time as long as possible.

In the equations giving the forces, that is (VII) and (VIII), § 3,

$$\mathbf{h}+\delta\mathbf{h} = \text{curl}\,(\mathbf{a}+\delta\mathbf{a}), \quad \mathbf{d}+\delta\mathbf{d} = -\,\text{grad.}\,(\phi+\delta\phi) - \frac{\partial(\mathbf{a}+\delta\mathbf{a})}{c\partial t},$$

the operations grad. and curl now refer to $(\delta x, \delta y, \delta z)$ which occur in p and δR, while x, y, z and R must be treated as constants. We easily find that

$$\text{grad.}\,p = \mathbf{R}_1, \quad \text{grad.}\,(\delta R)^2 = 2\delta\mathbf{R}, \quad \text{curl}\,(p\,.\,\mathbf{v}) = [\mathbf{R}_1\,.\,\mathbf{v}],$$
$$\text{curl}\,\{(\delta R)^2.\,\mathbf{v}\} = 2\,[\delta\mathbf{R}\,.\,\mathbf{v}], \quad \text{curl}\,(p\,.\,\delta\mathbf{v}) = [\mathbf{R}_1\,.\,\delta\mathbf{v}],$$

while such quantities as grad. (KR), curl $\dfrac{\delta\mathbf{v}}{KR}$, and so on, are to be considered as vanishing. The terms in the forces which would arise from them are in fact already included in those derived from the first order terms.

In this way we get, from (196) and (197),

$$\mathbf{d}+\delta\mathbf{d} = \left[\frac{\mathbf{R}_1}{KR^2}\right] + \frac{\partial}{c\partial t}\left[\frac{\mathbf{R}_1-\mathbf{v}/c}{KR}\right]$$
$$+\left[\frac{\delta\mathbf{R}-3p\mathbf{R}_1}{KR^3}\right] + \frac{\partial}{c\partial t}\left[\frac{\delta\mathbf{R}-3p\mathbf{R}_1}{KR^2} + \frac{p\mathbf{v}}{cKR^2} - \frac{\delta\mathbf{v}}{cKR}\right]$$
$$-\left(\frac{\partial}{c\partial t}\right)^2\left[\frac{p(\mathbf{R}_1-\mathbf{v}/c)}{KR}\right] \quad \ldots\ldots\ldots\ldots\ldots(198),$$

$$\mathbf{h} + \delta\mathbf{h} = \left[\frac{[\mathbf{v}.\mathbf{R}_1]}{cKR^2}\right] + \frac{\partial}{c\partial t}\left[\frac{[\mathbf{v}.\mathbf{R}_1]}{cKR}\right]$$
$$+ \left[\frac{[\mathbf{v}.(\delta\mathbf{R} - 3p\mathbf{R}_1)]}{cKR^3} + \frac{[\delta\mathbf{v}.\mathbf{R}_1]}{cKR^2}\right]$$
$$+ \frac{\partial}{c\partial t}\left[\frac{[\mathbf{v}.(\delta\mathbf{R} - 3p\mathbf{R}_1)]}{cKR^2} + \frac{[\delta\mathbf{v}.\mathbf{R}_1]}{cKR}\right] - \left(\frac{\partial}{c\partial t}\right)^2\left[\frac{p[\mathbf{v}.\mathbf{R}_1]}{cKR}\right] \quad \text{...(199).}$$

The first line in each equation gives the force due to the permanent motion at a fieldpoint on the permanent path, and the remaining ones give the forces due to disturbance and for a fieldpoint close to but not on the permanent path.

A notable simplification of the work arises from the fact that certain combinations of symbols, such as $[\mathbf{v}.\mathbf{R}_1]$, $\delta\mathbf{R} - 3p\mathbf{R}_1$, and $[\mathbf{v}.(\delta\mathbf{R} - 3p\mathbf{R}_1)]$, occur with varying factors, and affected by a varying number of differentiations with respect to the time t. Thus in the expression for $\mathbf{d} + \delta\mathbf{d}$ the combination $\delta\mathbf{R} - 3p\mathbf{R}_1$ occurs undifferentiated with the factor $1/KR^3$, and differentiated once with the factor $1/KR^2$, and so on.

As an illustration of the method we shall consider the problem of a point charge moving uniformly in a circle.

136. Problem. A point charge moves in a circle with uniform velocity. Required to find the electromagnetic field at a point on the circle. With the notation previously employed we write

$$\xi = \rho\cos(\omega\tau + \delta), \quad \eta = \rho\sin(\omega\tau + \delta), \quad \zeta = 0, \quad \beta = \omega\rho/c.$$

Further, the coordinates of the fieldpoint are

$$x = \rho\cos\phi, \quad y = \rho\sin\phi, \quad z = 0.$$

With these values we get

$$R = \{(x-\xi)^2 + (y-\eta)^2\}^{\frac{1}{2}} = 2\rho\sin\tfrac{1}{2}(\omega\tau + \delta - \phi) = 2\rho\sin\chi,$$

where χ is a new variable defined by the equation

$$\chi = \tfrac{1}{2}(\omega\tau + \delta - \phi).$$

In the same way write

$$\psi = \tfrac{1}{2}(\omega t + \delta - \phi).$$

Then the characteristic equation $t = \tau + R/c$ gives

$$\psi = \chi + \beta\sin\chi \quad \text{.............................(200),}$$

and the Doppler factor is given by

$$K = \frac{\partial t}{\partial\tau} = \frac{\partial\psi}{\partial\chi} = 1 + \beta\cos\chi.$$

The transformation from the time of emission τ to the time of reception t by means of the characteristic equation, indicated by the use of the outer square brackets in (196)—(199), now reduces to a change of variable from χ to ψ by means of (200), which is nothing more than Bessel's well known equation.

For the purposes of the present problem we only require the zero order terms in (196)—(199), which involve the vectors $\mathbf{R}_1$, $\mathbf{v}$ and $[\mathbf{v}.\mathbf{R}_1]$. We require their components along the inward* radius vector PO at the fieldpoint, perpendicular to it along the tangent PT, drawn in the direction of motion, and perpendicular to the plane of the orbit.

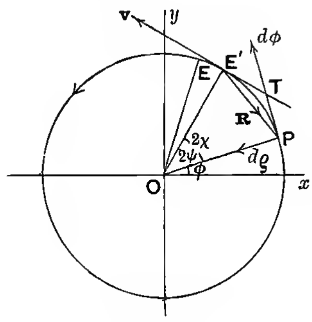

Fig. 39.

In Fig. 39 the orbit is the circle $PE'E$. P is the fieldpoint (x, y), so that $x\hat{O}P = \phi$. E' is the position of the point charge at time τ, so that $P\hat{O}E' = 2\chi$; and E is its position at time t, so that $P\hat{O}E = 2\psi$. The vector $\mathbf{R}_1$ is along $E'P$, $\mathbf{v}$ is along TE', and $[\mathbf{v}.\mathbf{R}_1]$ perpendicular to the paper and upwards, along Oz. We get for the components of these vectors in the directions ρ, (PO), ϕ, (PT) and z

$$R_{1\rho} = -\sin\chi, \quad R_{1\phi} = -\cos\chi, \quad R_{1z} = 0,$$
$$v_\rho = c\beta\sin 2\chi, \quad v_\phi = c\beta\cos 2\chi, \quad v_z = 0,$$
$$[\mathbf{v}.\mathbf{R}_1]_\rho = 0, \quad [\mathbf{v}.\mathbf{R}_1]_\phi = 0, \quad [\mathbf{v}.\mathbf{R}_1]_z = c\beta\sin\chi.$$

Substitute in (196)—(199), replace $\sin 2\chi$ by $2\sin\chi\cos\chi$ and $\cos 2\chi$ by $1 - 2\sin^2\chi$, and notice that the expression $\cos\chi/\sin^2\chi(1+\beta\cos\chi)$, which occurs in d_ϕ, may be written in the form $-\frac{\partial}{\partial\psi}\left(\frac{1}{\sin\chi}\right)$. After a little reduction we get

$$\phi = \frac{1}{2\rho\sin\chi(1+\beta\cos\chi)}, \quad a_\phi = \frac{\beta}{2\rho\sin\chi(1+\beta\cos\chi)} - \frac{\beta\sin\chi}{\rho(1+\beta\cos\chi)},$$
$$a_\rho = \frac{1}{\rho} - \frac{1}{\rho(1+\beta\cos\chi)}, \quad d_\phi = \frac{\partial}{\partial\psi}\left[\frac{1-\beta^2}{4\rho^2\sin\chi(1+\beta\cos\chi)} + \frac{\beta^2\sin\chi}{2\rho^2(1+\beta\cos\chi)}\right],$$
$$d_\rho = -\frac{1}{4\rho^2\sin\chi(1+\beta\cos\chi)} + \frac{\partial}{\partial\psi}\left[\frac{\beta}{4\rho^2(1+\beta\cos\chi)}\right],$$
$$h_z = \frac{\beta}{4\rho^2\sin\chi(1+\beta\cos\chi)} + \frac{\partial}{\partial\psi}\left[\frac{\beta^2}{4\rho^2(1+\beta\cos\chi)}\right] \quad \text{......................(201).}$$

137. It is to be noticed that these expressions only involve three distinct functions. Two of these, $\frac{1}{1+\beta\cos\chi}$ and $\frac{\sin\chi}{1+\beta\cos\chi}$, are at once expansible in Fourier Series with ψ as variable; the third, $\frac{1}{\sin\chi(1+\beta\cos\chi)}$, being

* The inward direction makes the axis Oz right-handed relative to the motion, when ϕ, ρ and z are a right-handed system.

infinite at the limits 0 and π is not so expansible. But its excess above the function $1/\sin\psi$ vanishes at both limits, and may be expanded. Write

$$\frac{1}{\sin\chi\,(1+\beta\cos\chi)}=\frac{1}{\sin\psi}+\sum_{j=1}^{j=\infty}2A_j\sin j\psi,$$

$$\frac{1}{1+\beta\cos\chi}=1+\sum_{j=1}^{j=\infty}2B_j\cos j\psi,\qquad \frac{\sin\chi}{1+\beta\cos\chi}=\sum_{j=1}^{j=\infty}2C_j\sin j\psi.$$

We get, by the usual method, using (200),

$$A_j=\frac{1}{\pi}\int_0^\pi\left\{\frac{1}{\sin\chi\,(1+\beta\cos\chi)}-\frac{1}{\sin\psi}\right\}\sin j\psi\,d\psi.$$

In the first integral change the variable from ψ to χ, remembering (200), and in the second change the notation by writing χ in place of ψ. We get

$$A_j=\frac{1}{\pi}\int_0^\pi\left\{\frac{\sin j\,(\chi+\beta\sin\chi)}{\sin\chi}-\frac{\sin j\chi}{\sin\chi}\right\}d\chi$$

$$=\frac{j}{\pi}\int_0^\beta dx\int_0^\pi\cos j\,(\chi+x\sin\chi)\,d\chi=(-1)^j j\int_0^\beta J_j(jx)\,dx.$$

Similarly $\qquad B_j=(-1)^j J_j(j\beta),\quad C_j=-(-1)^j J_j'(j\beta).$

138. Substitute in the expressions for the potentials and forces, perform the necessary differentiations with respect to ψ (which are obviously permissible), and supply the charge e as a factor. We get

$$\left.\begin{aligned}\phi&=\frac{e}{2\rho\sin\psi}+\frac{e}{\rho}\sum_{j=1}^{j=\infty}(-1)^j\sin j\psi\int_0^\beta J_j(jx)\,dx\\ a_\phi&=\frac{e\beta}{2\rho\sin\psi}+\frac{2e\beta}{\rho}\sum_{j=1}^{j=\infty}(-1)^j\sin j\psi\left[J_j'(j\beta)+\tfrac{1}{2}j\int_0^\beta J_j(jx)\,dx\right]\\ a_\rho&=-\frac{2e}{\rho}\sum_{j=1}^{j=\infty}(-1)^j\cos j\psi\,J_j(j\beta)\end{aligned}\right\}\ldots(202),$$

$$\left.\begin{aligned}d_\phi&=-\frac{e\,(1-\beta^2)\cos\psi}{4\rho^2\sin^2\psi}-\frac{e}{\rho^2}\sum_{j=1}^{j=\infty}(-1)^j\cos j\psi\left[j\beta^2 J_j'(j\beta)\right.\\ &\qquad\left.-\tfrac{1}{2}(1-\beta^2)j^2\int_0^\beta J_j(jx)\,dx\right]\\ d_\rho&=-\frac{e}{4\rho^2\sin\psi}-\frac{e}{2\rho^2}\sum_{j=1}^{j=\infty}(-1)^j\sin j\psi\left[j\beta J_j(j\beta)+j\int_0^\beta J_j(jx)\,dx\right]\\ h_z&=\frac{e\beta}{4\rho^2\sin\psi}-\frac{e\beta}{2\rho^2}\sum_{j=1}^{j=\infty}(-1)^j\sin j\psi\left[j\beta J_j(j\beta)-j\int_0^\beta J_j(jx)\,dx\right]\end{aligned}\right\}(203).$$

The remaining force components vanish identically.

These series are available so long as the distance of the fieldpoint from the charge, $2\rho\sin\psi$, is at least equal to several multiples of the diameter of the charge. This restriction is not serious in problems dealing with the circular motions of electrons, even when the diameter of the circle is much smaller than the atomic diameter, for the latter is many thousand times as great as the diameter of the electron.

139. Group of n charges. When we have n charges moving uniformly round the circle at equal angular distances apart, we may take ψ, which is equal to $\frac{1}{2}(\omega t+\delta-\phi)$, to be one-half of the angular distance from the field-point to any one of the charges, which we may number zero. For the ith charge we must replace δ by $\delta+2\pi i/n$, and ψ by $\psi+\pi i/n$, and in order to get the field due to the whole group at a point on the circle we must sum (203) from $i=0$ to $i=n-1$. We have

$$\begin{aligned}\sum_{i=0}^{i=n-1}\epsilon^{\iota j(\psi+\pi i/n)}&=n\epsilon^{\iota 2sn\psi}, \text{ for } j=2sn,\\ &=0, \text{ for other even values of } j,\\ &=\iota\frac{\epsilon^{\iota(2s+1)(\psi-\pi/2n)}}{\sin(2s+1)\pi/2n}, \text{ for } j=2s+1.\end{aligned}$$

In this way we get

$$\left.\begin{aligned}
d_\phi=&-\frac{e(1-\beta^2)}{4\rho^2}\sum_{i=0}^{i=n-1}\frac{\cos(\psi+i\pi/n)}{\sin^2(\psi+i\pi/n)}\\
&-\frac{2ne}{\rho^2}\sum_{s=1}^{s=\infty}\left[sn\beta^2J'_{2sn}(2sn\beta)-s^2n^2(1-\beta^2)\int_0^\beta J_{2sn}(2snx)\,dx\right]\cos 2sn\psi\\
&-\frac{e}{\rho^2}\sum_{s=0}^{s=\infty}\left[(2s+1)\beta^2J'_{2s+1}\{(2s+1)\beta\}-\tfrac{1}{2}(2s+1)^2(1-\beta^2)\right.\\
&\qquad\left.\times\int_0^\beta J_{2s+1}\{(2s+1)x\}\,dx\right]\frac{\sin(2s+1)\left(\psi-\dfrac{\pi}{2n}\right)}{\sin\dfrac{(2s+1)\pi}{2n}}\\
d_\rho=&-\frac{e}{4\rho^2}\sum_{i=0}^{i=n-1}\frac{1}{\sin(\psi+i\pi/n)}\\
&-\frac{ne}{\rho^2}\sum_{s=1}^{s=\infty}\left[sn\beta J_{2sn}(2sn\beta)+sn\int_0^\beta J_{2sn}(2snx)\,dx\right]\sin 2sn\psi\\
&+\frac{e}{2\rho^2}\sum_{s=0}^{s=\infty}\left[(2s+1)\beta J_{2s+1}\{(2s+1)\beta\}+(2s+1)\int_0^\beta J_{2s+1}\{(2s+1)x\}\,dx\right]\\
&\qquad\times\frac{\cos(2s+1)\left(\psi-\dfrac{\pi}{2n}\right)}{\sin\dfrac{(2s+1)\pi}{2n}}\\
h_z=&\frac{e\beta}{4\rho^2}\sum_{i=0}^{i=n-1}\frac{1}{\sin(\psi+i\pi/n)}\\
&-\frac{ne\beta}{\rho^2}\sum_{s=1}^{s=\infty}\left[sn\beta J_{2sn}(2sn\beta)-sn\int_0^\beta J_{2sn}(2snx)\,dx\right]\sin 2sn\psi\\
&+\frac{e\beta}{2\rho^2}\sum_{s=0}^{s=\infty}\left[(2s+1)\beta J_{2s+1}\{(2s+1)\beta\}-(2s+1)\int_0^\beta J_{2s+1}\{(2s+1)x\}\,dx\right]\\
&\qquad\times\frac{\cos(2s+1)\left(\psi-\dfrac{\pi}{2n}\right)}{\sin\dfrac{(2s+1)\pi}{2n}}
\end{aligned}\right\}\quad\ldots(204).$$

In these equations the variable ψ may be taken to lie between the limits 0 and π/n without any loss of generality, but a very small range at either limit must be excluded, because of its nearness to one of the electrons. If it were not for the necessary exclusion of these portions of the range, we might be tempted to find the limit of the forces for infinitely large values of n. We cannot however use our results for values of n beyond a certain limit, which depends on the ratio of the diameter of the electron to that of the orbit.

For small velocities the harmonic terms vanish, and the remaining ones reduce to the values of the forces for static charges, such as those employed in the theories of Nagaoka and J. J. Thomson.

In general all the odd harmonics are present, as well as even harmonics of order $2sn$. For anything like large values of n the latter have exceedingly small amplitudes, but the amplitudes of the former for small values of s are not small except when β is small.

These equations will be of use later in finding the mechanical force acting on one charge of a group. We shall not consider them further here.

CHAPTER XI

THE MECHANICAL FORCES ACTING ON ELECTRIC CHARGES IN MOTION

140. HITHERTO, in calculating the electromagnetic field due to electric charges moving in an assigned manner, we have only made use of the equations (I)—(V) of Ch. I, § 2, and of the equations which follow directly from them. We have already pointed out that the determination of the mechanical forces acting on a charge involves an additional hypothesis, and have adopted that due to Lorentz. This assumes that the mechanical connection between the charge and the aether is such that the *resultant* mechanical force on the charge is completely determined by the electric and magnetic forces on the one hand, and the velocity of the charge on the other, according to the equation

$$\mathbf{f} = \mathbf{d} + [\mathbf{vh}]/c \quad \text{.............................(VI).}$$

The assumption expressed by this equation implies that, whatever the mechanical connection between the charge and aether may be, that part of the reaction due to it, which is not included in (VI), reduces to a system of stresses which can consume and do work, but contributes nothing to the motive force because it is self-equilibrating.

That such stresses must generally exist is obvious when we consider the case of the deformable electron, for example that of Lorentz. It is well known that, when this electron is accelerated, its gain of kinetic energy is not all accounted for by the work done by the motive force, but is in part derived from work done by the aether in compressing the electron, according to the hypothesis of Poincaré*, a difficulty first pointed out by Abraham.

141. The total force on a charge is given by (XIX), Ch. I, § 6, which may be written in the form

$$\mathbf{F} = -\frac{d}{c\,dt}\int \mathbf{a}\,de - \int\{\nabla\,(\phi - (\mathbf{va})/c) + \chi\mathbf{a}/c - [\omega\mathbf{a}]/c\}\,de \quad \text{...(XIX}'\text{).}$$

Here ω is the angular velocity of rotation at the element de, and $\chi\mathbf{a}$ is the vector whose components are

$$(ea_x + ca_y + ba_z,\;\; ca_x + fa_y + aa_z,\;\; ba_x + aa_y + ga_z),$$

* *Comptes Rendus*, CXI. 1905, p. 1504.

where (e, f, g, a, b, c) are the six components of velocity of pure strain at the element.

For a perfectly rigid electron, such as the sphere of Abraham, the strain is zero. Its angular velocity is found to contribute to the resultant force only terms of higher order in the dimensions of the electron than the remainder, so long as the velocity of the charge is less than that of light.

For a deformable electron the corresponding terms are important. For instance, in the case of the Lorentz electron, which is well known to be an oblate spheroid, with equatorial axis a, and minor axis $a\sqrt{(1 - v^2/c^2)}$, lying in the direction of motion, there is a homogeneous compression parallel to the direction of motion at the rate $-v\dot{v}/(c^2 - v^2)$, together with a rotation about the binormal at the rate v/ρ. Hence the pure strain is given by $e = -\beta\dot{\beta}/(1 - \beta^2)$, with the remaining five components all zero, and the rotation is given by $\omega_z = v/\rho$, with the remaining two components zero. Thus the vector $\chi\mathbf{a}/c - [\omega\mathbf{a}]/c$ has the components

$$\left(-\frac{\beta\dot{\beta}}{c(1-\beta^2)}a_x + \frac{\beta}{\rho}a_y, \quad -\frac{\beta}{\rho}a_x, \quad 0\right),$$

where $\beta = v/c$ as usual.

It follows from (XIX′) that on the hypothesis which we have made, the mechanical force is completely determined when the motion of the charge and the potentials ϕ and $\mathbf{a}$ of the field are given. In practice it is often more convenient to work with the forces $\mathbf{d}$ and $\mathbf{h}$, and use (VI) directly.

142. Whichever method of working be adopted, it is convenient to separate the mechanical force acting on one of a system of discrete charges into three parts:

(1) The first part is the resultant of all the internal forces, due to interactions between the elementary charge under consideration and the remaining elements of the electron; it is got by a double integration over the whole electron, and will be denoted by the symbol $\mathbf{F}_i$.

(2) The second part is the resultant of the actions of the external field on all the elements of the charge, so far as that external field is due to charges outside the particular charge considered but belonging to the same system. We shall denote it by the symbol $\mathbf{F}_1$.

(3) The third part is the resultant of the actions of the external field on all the elements of the charge, so far as it is due to charges outside the system. We shall denote it by the symbol $\mathbf{F}_2$.

The distinction between the second and third parts of the force is somewhat arbitrary, but it is convenient when the system of charges forms a group. When we do not wish to draw any such distinction, we shall use the symbol $\mathbf{F}_e$ to denote the whole external force, whatever its origin, so that $\mathbf{F}_e = \mathbf{F}_1 + \mathbf{F}_2$.

The resultant internal mechanical force $\mathbf{F}_i$ leads to a determination of the electromagnetic momentum and mass of the charge, and of the mechanical reaction due to its radiation. For this reason we shall consider it first. The external force will be studied later.

143. The internal mechanical force. The force exerted by a moving charge on itself has been the subject of a good deal of investigation, first by Abraham*, then by Sommerfeld†, and lastly by Lindemann‡, in each case for the rigid spherical electron, always on the basis of the Lorentz theory with (VI) as a foundation§.

In the case of quasi-stationary motion—where the terms involving the accelerations of higher orders of the electron, such as the torsion of the path, are small compared with those involving the acceleration proper and the curvature—we need only retain the terms of the two lowest orders. We have

(1) a principal term of the form

$$-\frac{d\mathbf{G}}{dt} = -\frac{d(m\mathbf{v})}{dt} \qquad\text{.........................(205)},$$

where $\mathbf{G}$ is the electromagnetic momentum and m the "transverse mass."

For the rigid spherical electron of Abraham of radius a

$$m = \frac{3e^2}{5c^2 a\beta^2}\left(\frac{1+\beta^2}{2\beta}\log\frac{1+\beta}{1-\beta} - 1\right) \qquad\text{..............(206)}.$$

For the deformable spheroidal electron of constant volume of Bucherer, with longitudinal semiaxis $a(1-\beta^2)^{\frac{2}{3}}$, and transverse semiaxes $a(1-\beta^2)^{-\frac{1}{3}}$,

$$m = \frac{4e^2}{5c^2 a\sqrt[3]{(1-\beta^2)}} \qquad\text{.........................(207)}.$$

For the deformable spheroidal electron of Lorentz, with longitudinal semiaxis $a(1-\beta^2)^{\frac{1}{2}}$, and transverse semiaxes a,

$$m = \frac{4e^2}{5c^2 a\sqrt{(1-\beta^2)}} \qquad\text{.........................(208)},$$

where $\beta = v/c$ as usual.

All these values are for uniform volume distributions; for the corresponding surface distributions the values are five-sixths of these. In addition we have

* *Ann. der Phys.* 10, p. 105, 1903. *Theorie der Strahlung*, p. 136, 1905.

† *Gött. Nach.* pp. 99 and 363, 1904; p. 201, 1905.

‡ *K. Bay. Akad.* II. Kl. XXIII. Bd. II. Abt. pp. 235 and 339, 1907.

§ [It is calculated by a different method in Appendix D below, without any restrictions as to the form and structure of the electron, but with the velocity restricted to be less than that of light. The result agrees with that of Abraham and Sommerfeld for the rigid spherical electron, and with that got by Einstein on the basis of the Relative Theory for the Lorentz electron.

Recently Walker (*Phil. Trans.* A, 210, p. 145, 1910) and Nicholson (*Phil. Mag.* [6], 20, p. 610, 1910) obtain a different result for the cases of a conducting and of a dielectric sphere. They however start from fundamentally different assumptions, in so far as the distribution of the charge on their sphere is not rigid, but depends on the motion. See also Livens, *Phil. Mag.* [6], 20, p. 640, 1911, and 21, p. 169, 1911.]

(2) a small term due to radiation, given by

$$\mathbf{K} = \frac{2e^2}{3c^3}\left\{\frac{\ddot{\mathbf{v}}}{1-\beta^2} + \frac{(\mathbf{v}\ddot{\mathbf{v}})\,\mathbf{v}}{c^2(1-\beta^2)^2} + \frac{3(\mathbf{v}\dot{\mathbf{v}})\,\dot{\mathbf{v}}}{c^2(1-\beta^2)^2} + \frac{3(\mathbf{v}\dot{\mathbf{v}})^2\,\mathbf{v}}{c^4(1-\beta^2)^3}\right\} \dots (209).$$

Abraham, in his *Theorie der Strahlung*, deduces this expression from the conditions that the force be equal to the rate of gain of electromagnetic momentum due to radiation, and its activity to the rate of gain of energy due to radiation, both rates of gain being of course negative. [It is deduced by a direct method in Appendix D.]

144. Resolving the two forces (1) and (2) along the tangent (s), the principal normal towards the centre of curvature (ρ), and the binormal (n), we get the following expressions for the components in these directions of the internal mechanical force $\mathbf{F}_i$:

$$\left.\begin{aligned} F_{is} &= -\dot{v}\,\frac{d(m\beta)}{d\beta} + \frac{2e^2}{3}\left\{\frac{\ddot{\beta}}{c^2(1-\beta^2)^2} + \frac{3\beta\dot{\beta}^2}{c^2(1-\beta^2)^3} - \frac{\beta^3}{\rho^2(1-\beta^2)^2}\right\} \\ F_{i\rho} &= -\frac{mv^2}{\rho} + \frac{2e^2}{3}\left\{\frac{3\beta\dot{\beta}}{c\rho(1-\beta^2)^2} - \frac{\beta^2\dot{\rho}}{c\rho^2(1-\beta^2)}\right\} \\ F_{in} &= \frac{2e^2\beta^3}{3\rho\tau(1-\beta^2)} \end{aligned}\right\} \dots (210),$$

where ρ and τ are the radii of curvature and torsion*.

145. Energy relations. The activity of the force $\mathbf{F}_i$ is equal to $(\mathbf{vF}_i)$, that is, to vF_{is}. We get from (210_1)

$$\begin{aligned} (\mathbf{vF}_i) &= -v\dot{v}\,\frac{d(m\beta)}{d\beta} + \tfrac{2}{3}ce^2\left\{\frac{\beta\ddot{\beta}}{c^2(1-\beta^2)^2} + \frac{3\beta^2\dot{\beta}^2}{c^2(1-\beta^2)^3} - \frac{\beta^4}{\rho^2(1-\beta^2)^2}\right\} \\ &= -\left\{\frac{d(c^2m\beta^2)}{d\beta} - c^2m\beta\right\}\dot{\beta} + \frac{d}{dt}\left\{\frac{2e^2\beta\dot{\beta}}{3c(1-\beta^2)^2}\right\} \\ &\quad - \frac{2ce^2}{3(1-\beta^2)^2}\left\{\frac{\dot{\beta}^2}{c^2(1-\beta^2)} + \frac{\beta^4}{\rho^2}\right\} \dots\dots (211). \end{aligned}$$

The behaviour of the terms in the first and second lines of this equation is quite different. Since m is a function of β alone, the first term, like the second, is a complete differential coefficient with respect to the time of

* [Let the direction cosines of the moving axes (s, ρ, n) with respect to the fixed axes (x, y, z) be given by the annexed scheme.

	x	y	z
s	l	m	n
ρ	l'	m'	n'
n	l''	m''	n''

Then we have $\dot{l} = l'v/\rho$, $\dot{l}' = l''v/\tau - lv/\rho$, with similar equations for m and n.

Moreover, we get

$$\dot{x} = lv,\quad \ddot{x} = l\dot{v} + l'v^2/\rho,\quad \dddot{x} = l(\ddot{v} - v^3/\rho^2) + l'(3v\dot{v}/\rho - v^2\dot{\rho}/\rho^2) + l''v^3/\rho\tau,$$

with similar equations for y and z.

From these equations we get

$$\dot{v}_s = \dot{v},\quad \dot{v}_\rho = v^2/\rho,\quad \ddot{v}_s = \ddot{v} - v^3/\rho^2,\quad \ddot{v}_\rho = 3v\dot{v}/\rho - v^2\dot{\rho}/\rho^2,\quad \ddot{v}_n = v^3/\rho\tau.$$

With the help of these results we easily get (210) from (205) and (209).]

a function of the motion; but the third term cannot be expressed in this form.

When we form the time integral of the activity, and so obtain the work done by the internal force during any interval, $\int_{t_1}^{t_2} (\mathbf{vF}_i)\,dt$, the terms in the second line of (211) give a quantity which can be expressed as the difference between the initial and final values of a certain function of the motion, namely,

$$-\left[\int_0^\beta \left\{\frac{d\,(c^2 m\beta^2)}{d\beta} - c^2 m\beta\right\} d\beta - \frac{2e^2\beta\dot{\beta}}{3c^2(1-\beta^2)^2}\right]_1^2 \quad \text{.........(212).}$$

But the terms in the third line give rise to a time integral which cannot be reduced until the motion is completely known, namely

$$-\int_{t_1}^{t_2} \frac{2ce^2}{3(1-\beta^2)^2}\left\{\frac{\dot{\beta}^2}{c^2(1-\beta^2)} + \frac{\beta^4}{\rho^2}\right\} dt \quad \text{..............(213).}$$

The quantity (212) is reversible; by this is meant that it would change sign if the motion were completely reversed, and would vanish if the initial and final motion were identical. It represents work done by the external forces and consumed by the internal force, which is recoverable on reversing the motion, and may be regarded as stored in the charge in virtue of its motion. Therefore the function of the motion inside the bracket in (212) may be regarded as the kinetic energy of the charge.

The function under the sign of integration in (213) is essentially positive, and accordingly the integral never vanishes. It represents work done by the external forces and consumed by the internal force, which is not recoverable. It is not stored in the charge, but radiated away from it. In fact, the function under the sign of integration in (213) is equivalent to Liénard's expression for the rate of loss of energy from a moving charge owing to radiation [in accordance with the result already obtained in § 5, p. 8].

Returning to the kinetic energy we see from (212) that it involves two terms: (1) a principal term, derived from the mass m and depending only on the velocity; (2) a small term, $-2e^2\beta\dot{\beta}/3c^2(1-\beta^2)^2$, depending on the acceleration as well as on the velocity.

146. If we adopt the usual convention that the kinetic energy depends on the velocity only, and not on the acceleration, we must exclude the second small term from the kinetic energy, and regard it as reversible radiant energy. The distinction is not of much importance; but as a matter of convenience we shall define the kinetic energy, T, by the equation

$$T = c^2 m\beta^2 - \int_0^\beta c^2 m\beta\,d\beta = \int_0^\beta c^2 m'\beta\,d\beta \quad \text{..............(214).}$$

From (206)—(208), § 143, we get:

For the Abraham electron

$$T = \frac{6e^2}{5a}\left(\frac{1}{2\beta}\log\frac{1+\beta}{1-\beta} - 1\right) \text{ ....(215).}$$

For the Bucherer electron

$$T = \frac{e^2}{5a}\left(\frac{3+\beta^2}{\sqrt[3]{(1-\beta^2)}} - 3\right) \text{ ....(216).}$$

For the Lorentz electron

$$T = \frac{4e^2}{5a}\left(\frac{1}{\sqrt{(1-\beta^2)}} - 1\right) \text{ ....(217).}$$

The last agrees with the expression given by Einstein on the basis of the Relative Theory.

[The small acceleration term omitted from the kinetic energy is precisely the supplementary term, which must be added to Liénard's expression for the radiation, when we adopt a mechanical theory of the aether, as we found in Ch. I, § 5, p. 9.]

147. In calculating the energy we have completely ignored the rate of working of the internal stresses. The work done by the external force is necessarily the same as before, for it is the work consumed by the internal force $\mathbf{F}_i$, and by the fundamental assumption of § 140, $\mathbf{F}_i$ does not depend on the internal stresses of the electron, which are self-equilibrating.

Nevertheless, if the electron be deformed, work is done and its potential energy is altered without any expenditure of work by the external forces. This difficulty occurs with the Lorentz electron, as was first noticed by Abraham. It can only be resolved by assuming with Poincaré that the electron is held together by a pressure exerted by the aether, which supplies the work done in compressing the electron. This is indicated by the form of equation (XX), § 6.

148. Equations of Motion. The equations of motion are easily written down on the basis of our fundamental assumption, that the resultant mechanical force is completely expressed by the terms $\mathbf{F}_i + \mathbf{F}_e$ derived from (VI), § 140.

By Newton's Laws of Motion the resultant force must be equal to the rate of increase of the ordinary (non-electromagnetic) momentum of the charge, $M\mathbf{v}$. We shall suppose M zero, in accordance with the most recent measurements of the specific charge of the electron, which makes the introduction of any non-electromagnetic mass unnecessary.

Hence we have $\mathbf{F}_e = -\mathbf{F}_i$, and get by means of (205) and (209), § 143,

$$\frac{d(m\mathbf{v})}{dt} - \frac{2e^2}{3c^3}\left\{\frac{\ddot{\mathbf{v}}}{1-\beta^2} + \frac{(\mathbf{v}\ddot{\mathbf{v}})\,\mathbf{v}}{c^2(1-\beta^2)^2} + \frac{3(\mathbf{v}\dot{\mathbf{v}})\,\dot{\mathbf{v}}}{c^2(1-\beta^2)^2} + \frac{3(\mathbf{v}\dot{\mathbf{v}})^2\,\mathbf{v}}{c^4(1-\beta^2)^3}\right\} = \mathbf{F}_e \text{ ...(218).}$$

Or by using (210), § 144, we get

$$\left.\begin{aligned} \frac{d\,(cm\beta)}{dt}+\frac{2e^2\beta^3}{3\rho^2(1-\beta^2)^2}-\frac{2e^2\ddot{\beta}}{3c^2(1-\beta^2)^2}-\frac{2e^2\beta\dot{\beta}^2}{c^2(1-\beta^2)^3}&=F_s \\ \frac{c^2m\beta^2}{\rho}+\frac{2e^2\beta^2\dot{\rho}}{3c\rho^2(1-\beta^2)}-\frac{2e^2\beta\dot{\beta}}{c\rho(1-\beta^2)^2}&=F_\rho \\ -\frac{2e^2\beta^3}{3\rho\tau(1-\beta^2)}&=F_n \end{aligned}\right\}\ldots(219),$$

where the suffix e has been omitted from F as no longer necessary.

We shall now consider some examples in illustration.

149. Example 1. Uniform circular motion. Here we have $\dot{\beta}=0$, $\ddot{\beta}=0$, $\dot{\rho}=0$, $\tau=\infty$. For the sake of generality we shall admit the possibility of a slow secular expansion of the electron, expressed by a change in the parameter a at the rate $\dot{a}$, and leading to a slow change in m at the rate $\dot{m}$, even when β does not alter.

With cylindrical coordinates (z, ϖ, ϕ), the directions (s, ρ, n) in (219) coincide with the directions (ϕ, $-\varpi$, z) respectively. Hence we get

$$\left.\begin{aligned} c\dot{m}\beta+\frac{2e^2\beta^3}{3\rho^2(1-\beta^2)^2}&=F_\phi \\ \frac{c^2m\beta^2}{\rho}&=-F_\varpi \\ 0&=F_z \end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots(220).$$

150. Example 2. Disturbed uniform circular motion. We may represent the rectangular coordinates of the disturbed electron by expressions of the form

$$\left.\begin{aligned} &x=\rho(1-\mu)\cos(\phi+\lambda),\quad y=\rho(1-\mu)\sin(\phi+\lambda),\quad z=\rho\nu \\ &\phi=\omega t+\delta \end{aligned}\right\}\ldots(221),$$

where ρ is the radius of the circular orbit and ω the uniform angular velocity of the undisturbed electron, and (λ, μ, ν) are functions of t, so small that their squares and products may be neglected. Obviously ρ (λ, μ, ν) are the components along the tangent, inward radius and axis of the displacement of the electron from its undisturbed position at the same time t, when its azimuth would have been ϕ.

We shall denote by the prefix δ the change produced by the disturbance in any quantity, whether scalar or vector*, and shall apply the operation δ to both sides of (218).

* [For instance, if $\mathbf{F}$ be the external force in the undisturbed motion at time t, with components (F_ϕ, F_ρ, F_z) along the tangent, inward radius and axis for the undisturbed position of the electron, and if $\mathbf{F}'$ be its value in the disturbed motion at the same time t, with components (F_λ', F_μ', F_ν'), perpendicular to and along the inward radius vector, and parallel to the

We get in the first instance

$$\left.\begin{aligned} \delta\mathbf{v} &= c\beta\left(\frac{\dot\lambda}{\omega}-\mu,\ \frac{\dot\mu}{\omega},\ \frac{\dot\nu}{\omega}\right)\\ \delta\dot{\mathbf{v}} &= \frac{c^2\beta^2}{\rho}\left(\frac{\ddot\lambda}{\omega^2}-2\frac{\dot\mu}{\omega},\ \frac{\ddot\mu}{\omega^2}+2\frac{\dot\lambda}{\omega}-\mu,\ \frac{\ddot\nu}{\omega^2}\right)\\ \delta\ddot{\mathbf{v}} &= \frac{c^3\beta^3}{\rho^2}\left(\frac{\dddot\lambda}{\omega^3}-3\frac{\ddot\mu}{\omega^2}-3\frac{\dot\lambda}{\omega}+\mu,\ \frac{\dddot\mu}{\omega^3}+3\frac{\ddot\lambda}{\omega^2}-3\frac{\dot\mu}{\omega},\ \frac{\dddot\nu}{\omega^3}\right) \end{aligned}\right\}\ \ldots(222)^*.$$

The first triplet of equations gives

$$\delta\beta=\beta\left(\frac{\dot\lambda}{\omega}-\mu\right),\quad \delta m=\beta\frac{dm}{d\beta}\left(\frac{\dot\lambda}{\omega}-\mu\right)\ldots\ldots\ldots\ldots\ldots(223).$$

Denoting by $\dot m$ the secular rate of change in m, with β constant, due to a possible change in the parameter a which determines the size of the electron, we find

$$\delta\frac{d(m\mathbf{v})}{dt}=\dot m\delta\mathbf{v}+\beta\frac{d\dot m}{d\beta}\left(\frac{\dot\lambda}{\omega}-\mu\right)\mathbf{v}$$
$$+m\delta\dot{\mathbf{v}}+\beta\frac{dm}{d\beta}\left(\frac{\dot\lambda}{\omega}-\mu\right)\dot{\mathbf{v}}+\frac{c\beta^2}{\rho}\frac{dm}{d\beta}\left(\frac{\ddot\lambda}{\omega^2}-\frac{\dot\mu}{\omega}\right)\mathbf{v}\quad\ldots(224).$$

Substituting from (222)—(224) in (218), § 148, and resolving in the directions (λ, μ, ν), we get the following equations for the disturbed motion:

$$\left.\begin{aligned} &-\frac{2e^2\beta^3}{3\rho^2(1-\beta^2)^2}\frac{\dddot\lambda}{\omega^3}+\frac{\beta^2}{\rho}\frac{d(c^2m\beta)}{d\beta}\frac{\ddot\lambda}{\omega^2}+\left\{\beta\frac{d(c\dot m\beta)}{d\beta}+\frac{2e^2\beta^3(3+\beta^2)}{3\rho^2(1-\beta^2)^3}\right\}\frac{\dot\lambda}{\omega}\\ &+\frac{2e^2\beta^3}{\rho^2(1-\beta^2)^2}\frac{\ddot\mu}{\omega^2}-\frac{\beta}{\rho}\frac{d(c^2m\beta^2)}{d\beta}\frac{\dot\mu}{\omega}-\left\{\beta\frac{d(c\dot m\beta)}{d\beta}+\frac{2e^2\beta^3(1+3\beta^2)}{3\rho^2(1-\beta^2)^3}\right\}\mu=\delta F_\lambda\\ &-\frac{2e^2\beta^3}{\rho^2(1-\beta^2)^2}\frac{\ddot\lambda}{\omega^2}+\frac{\beta}{\rho}\frac{d(c^2m\beta^2)}{d\beta}\frac{\dot\lambda}{\omega}\\ &-\frac{2e^2\beta^3}{3\rho^2(1-\beta^2)}\frac{\dddot\mu}{\omega^3}+\frac{c^2m\beta^2}{\rho}\frac{\ddot\mu}{\omega^2}+\left\{c\dot m\beta+\frac{2e^2\beta^3(3+\beta^2)}{3\rho^2(1-\beta^2)^2}\right\}\frac{\dot\mu}{\omega}-\frac{\beta^2}{\rho}\frac{d(c^2m\beta)}{d\beta}\mu=\delta F_\mu\\ &-\frac{2e^2\beta^3}{3\rho^2(1-\beta^2)}\frac{\dddot\nu}{\omega^3}+\frac{c^2m\beta^2}{\rho}\frac{\ddot\nu}{\omega^2}+\left\{c\dot m\beta+\frac{2e^2\beta^5}{3\rho^2(1-\beta^2)^2}\right\}\frac{\dot\nu}{\omega}=\delta F_\nu \end{aligned}\right\}$$

$$\ldots\ldots(225).$$

axis for the disturbed position of the electron, then the disturbing force is $\delta\mathbf{F}$, and its components are (δF_λ, δF_μ, δF_ν) in the directions (λ, μ, ν), where $\delta F_\lambda=F_\lambda'-F_\phi$, $\delta F_\mu=F_\mu'-F_\rho$, $\delta F_\nu=F_\nu'-F_z$.]

* [These equations are easily proved as follows:

The direction cosines of the moving axes of (λ, μ, ν) are given by the annexed scheme.

	x	y	z
λ	$-\sin(\phi+\lambda)$	$\cos(\phi+\lambda)$	0
μ	$-\cos(\phi+\lambda)$	$-\sin(\phi+\lambda)$	0
ν	0	0	1

Let $\mathbf{v}'$ denote, for the moment, the velocity in the disturbed motion, and so on. Then we have

$$v_\lambda'+\iota v_\mu'=(\dot y-\iota\dot x)\,\epsilon^{-\iota(\phi+\lambda)},\quad v_\nu'=\dot z,$$

where ($\dot x$, $\dot y$, $\dot z$) are to be got from (221).

We have $y-\iota x=\iota\rho(\mu-1)\,\epsilon^{\iota(\phi+\lambda)}$; differentiating and bearing in mind that $\beta=\omega\rho/c$, we get

$$\dot y-\iota\dot x=c\beta\left\{1+\frac{\dot\lambda}{\omega}-\mu-\iota\frac{\dot\mu}{\omega}\right\}\epsilon^{\iota(\phi+\lambda)}.$$

Substituting in the expression for $v_\lambda'+\iota v_\mu'$, and equating real and imaginary parts, we get the two first equations of the first triplet (222). The remaining equations are got in the same way.]

These equations rest on the assumption that all the quantities λ, $\dot{\lambda}/\omega$, $\ddot{\lambda}/\omega^2$, $\dddot{\lambda}/\omega^3$, ... are small. When the disturbance is periodic, with the period T and wave-length Λ, this assumption requires that $\lambda(2\pi\rho/\Lambda)^3$, ... be all small compared with β^3. When these conditions are satisfied, the disturbed path nowhere has large curvature or torsion, and the change in the acceleration of the electron always remains small.

The equations cannot be applied to orbits with short and steep corrugations, such as the looped and cusped paths described by the vertex of a top, when it is disturbed from steady precessional motion.

It is important to notice that the equations are linear, but of the *third* order, not of the second, as in ordinary dynamics. Hence generally the frequency equations of a ring of electrons are at least of the third degree.

The equations (225) were first given by Schott* with a slightly different notation.

As a further illustration of the use of the vector equation of motion we shall now work out the motion of a β-particle in a uniform electrostatic field.

151. Problem. A β-particle is projected in any manner in a uniform electrostatic field. Required to find the motion.

We shall make the following assumptions:

(1) The mass is entirely electromagnetic, and is connected with the velocity, $v = c\beta$, by the mass formula of Lorentz

$$m_v = m/\sqrt{(1-\beta^2)} \quad \text{.........................(226)},$$

where m is now written for the mass corresponding to zero velocity.

This formula has been verified experimentally to a very high degree of accuracy by Bucherer and his pupil Wolz†. The latest result gives $e/cm = 1{\cdot}77 \,.\, 10^7$ electromagnetic units.

(2) The motion is quasi-stationary, so that the small radiation terms can be neglected. Since the value of $2e^2/3cm$ is about 0·0035, this condition admits of a value for β which is very close to unity, unless the acceleration and curvature of the orbit become abnormally great. The degree of the approximation will be examined *a posteriori*.

* *Phil. Mag.* [6], Vol. XVI. p. 180, 1908.

[In the equations (12) of this paper the longitudinal mass, $\mu \equiv \dfrac{d(m\beta)}{d\beta}$, is used by mistake instead of the transverse mass m in the coefficients of $\dot{\eta}/\omega$ and $\dot{\zeta}/\omega$ in the last two equations.]

† *Ann. der Phys.* [4], 28, p. 513, and 30, p. 273, 1909.

152. Let the electric force be X electrostatic units in the direction of Ox.

At time t let the particle be moving with the velocity $c\beta$ in the plane xOy, and in a direction making an angle θ with Oy.

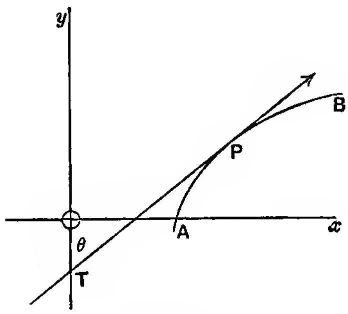

Fig. 40.

Neglecting the small radiation terms for the present, we may write the equations of motion in the form

$$\frac{d(m_v\dot{x})}{dt} = eX, \quad \frac{d(m_v\dot{y})}{dt} = 0, \quad \frac{d(m_v\dot{z})}{dt} = 0 \qquad \text{......(227).}$$

The last two equations give

$$m_v\dot{y} = m_v v \cos\theta = eXa, \quad m_v\dot{z} = 0 \text{...(228),}$$

where a is a constant. Thus the motion continues to take place in the xy plane. The first equation (226) now gives, since $\dot{x} = v\sin\theta$,

$$a\frac{d\tan\theta}{dt} = 1, \text{ whence } t = a\tan\theta \text{.................(229),}$$

where the constant of integration is chosen so that $\theta = 0$ when $t = 0$. Writing

$$b = cm/eX \text{ ....(230),}$$

we find from (226), (228) and (230)

$$\beta = \frac{a}{\sqrt{(a^2 + b^2\cos^2\theta)}} \text{ ....(231).}$$

Hence we get

$$\left.\begin{aligned} x &= \int v\sin\theta\,dt = c\sqrt{(a^2+b^2+t^2)} \\ y &= \int v\cos\theta\,dt = \tfrac{1}{2}ca\log\frac{\sqrt{(a^2+b^2+t^2)}+t}{\sqrt{(a^2+b^2+t^2)}-t} \end{aligned}\right\} \text{.........(232),}$$

where the constants of integration have been chosen so as to make x equal to $c\sqrt{(a^2+b^2)}$ and y equal to zero when $t = 0$. These equations give

$$x = c\sqrt{(a^2+b^2)}\cosh(y/ca), \quad t = \sqrt{(a^2+b^2)}\sinh(y/ca) \text{......(233).}$$

Let us choose the axis of y so that y and t have the same sign, and therefore $\dot{y}$ is positive. Then we see from (228) and (233) that a, $\sqrt{(a^2+b^2)}$ and x all have the sign of e, that is, of b.

Initially we have $x = c\sqrt{(a^2+b^2)}$, $y = \dot{x} = 0$, $\dot{y} = ca/\sqrt{(a^2+b^2)}$. If β_0 be the initial value of β we find that $a = b\beta_0/\sqrt{(1-\beta_0^2)}$, and $\sqrt{(a^2+b^2)} = b/\sqrt{(1-\beta_0^2)}$. Hence we get, by (230) and (233),

$$\left.\begin{aligned} x &= \frac{c^2m}{eX\sqrt{(1-\beta_0^2)}}\cosh\frac{yeX\sqrt{(1-\beta_0^2)}}{c^2m\beta_0} \\ t &= \frac{cm}{eX\sqrt{(1-\beta_0^2)}}\sinh\frac{yeX\sqrt{(1-\beta_0^2)}}{c^2m\beta_0} \end{aligned}\right\} \text{...........(234).}$$

153. The first equation represents the curve known as the "menoclinoid*," of which the two parameters are $c^2m/eX\sqrt{(1-\beta_0^2)}$ and $c^2m\beta_0/eX\sqrt{(1-\beta_0^2)}$, so that their ratio is $1:\beta_0$. The curve is got from a common catenary of parameter $c^2m/eX\sqrt{(1-\beta_0^2)}$ by compressing it at right angles to its axis Ox in the ratio $1:\beta_0$. If the velocity at the vertex were equal to the velocity of light, and at the same time X so large that $X(1-\beta_0^2)$ remained finite, the path would be the common catenary; if the velocity at the vertex were zero, it would be the part of the x-axis beyond the point $x = c^2m/eX$ traversed to and fro. The last case gives $x=\sqrt{(k^2+c^2t^2)}$, where $k = c^2m/eX$, which is precisely the first equation (81) of Problem 4, Ch. V.

[Fig. 41 represents on a scale of 1 : 80 the path of a negative electron, in a field of intensity 50 E.S.U., or 15000 volt/cm., and for the value $\beta_0 = 0{\cdot}508$. Thus the minimum velocity, which occurs at the vertex A, is nearly one half of that of light.

The value of e/cm is taken to be $-1{\cdot}77\,.\,10^7$ E.M.U. This makes the two parameters $-39{\cdot}4$ cm. and -20 cm. respectively. Thus the distance of the vertex from the origin, OA, is $39{\cdot}4$ cm., and the curve lies wholly on the negative side of the x-axis.]

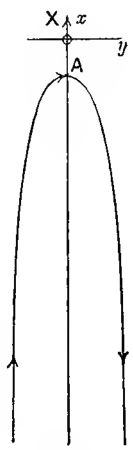

Fig. 41.

154. The error made in neglecting radiation. In order to estimate the error to which the results just obtained [and also the results obtained in Appendix G below] are liable on account of the neglect of the small radiation terms, we return to equations (219), § 148. The terms neglected in the first equation may be written in the form

$$\frac{2e^2}{3c^2\sqrt{(1-\beta^2)}}\left\{\frac{c^2}{\rho^2}\left(\frac{\beta}{\sqrt{(1-\beta^2)}}\right)^3 - \frac{d^2}{dt^2}\left(\frac{\beta}{\sqrt{(1-\beta^2)}}\right)\right\} \quad\text{.........(235)}.$$

Those neglected in the second equation are

$$-\frac{2e^2\sqrt{(1-\beta^2)}}{3c^2\beta}\frac{d}{dt}\left\{\frac{c}{\rho}\left(\frac{\beta}{\sqrt{(1-\beta^2)}}\right)^3\right\} \quad\text{...............(236)}.$$

Those in the third equation are identically zero, because the path is a plane curve.

From (229) and (231), § 152, we find

$$\beta=\sqrt{\frac{a^2+t^2}{a^2+b^2+t^2}},\quad \sqrt{(1-\beta^2)}=\frac{b}{\sqrt{(a^2+b^2+t^2)}},\quad \frac{\beta}{\sqrt{(1-\beta^2)}}=\frac{\sqrt{(a^2+t^2)}}{b},$$

$$\frac{c}{\rho}=\frac{c}{v}\frac{d\theta}{dt}=\frac{a}{\beta(a^2+t^2)},\quad \frac{c}{\rho}\left(\frac{\beta}{\sqrt{(1-\beta^2)}}\right)^3=\frac{a\sqrt{(a^2+b^2+t^2)}}{b^3}.$$

* Loria, *Ebene Kurven*, p. 580, 1902.

Hence the terms (235) become

$$\frac{2e^2a^2}{3c^2b^4\beta} = \frac{2e^4X^2\beta_0^2}{3c^4m^2(1-\beta_0^2)\beta},$$

and the terms (236) become

$$-\frac{2e^2at}{3c^2b^2\sqrt{(a^2+t^2)}\sqrt{(a^2+b^2+t^2)}}.$$

The first quantity is greatest when β has its least value β_0, and then is equal to $2e^4X^2\beta_0/3c^4m^2(1-\beta_0^2)$, that is, its greatest value is numerically equal to $2e^3X\beta_0/3c^4m^2(1-\beta_0^2)$ times the external mechanical force on the electron, eX.

The second quantity has its greatest value when

$$t^2 = a\sqrt{(a^2+b^2)} = b^2\beta_0/(1-\beta_0^2),$$

and then it is equal to $-2e^4X^2\beta_0/3c^4m^2(1+\beta_0)$, that is, its greatest value is equal numerically to $2e^3X\beta_0/3c^4m^2(1+\beta_0)$ times the external mechanical force.

It is clear that both values are of the same order of magnitude, but the first is the greater on account of the factor $1-\beta_0$ in the denominator. Hence we need only consider it in estimating the error.

For a β-particle we have $e/cm = 1{\cdot}77 \,.\, 10^7$ E.M.U. numerically, and $e/c = 1{\cdot}55 \,.\, 10^{-20}$ E.M.U. Hence $2e^3/3c^4m^2 = 1{\cdot}08 \,.\, 10^{-16}$. If further X be equal to F volt/cm., we find that the reaction due to radiation is at most equal to $3{\cdot}6 \,.\, 10^{-19} F\beta_0/(1-\beta_0^2)$ times the external mechanical force. Even if the field were as strong as one million volts per centimetre, and the minimum velocity were as great as 999/1000ths of that of light, the error would be less than two ten thousand millionths.

Hence the effect of the reaction due to radiation is quite inappreciable in this, and probably in all practical cases.

CHAPTER XII

THE MOTION OF GROUPS OF ELECTRIC CHARGES

155. In the last chapter we left over for future study the external mechanical force, $\mathbf{F}_e$, which acts on one of a system of charges, whether it be due to other charges of the same system, or to charges outside it. We shall take up the consideration of this part of the force in the present chapter, and shall then show how to deduce the equations of motion of a group of charges.

In calculating this force from the external field by means of (VI), § 140, we shall assume that the linear dimensions of each charge are small compared with its distance from its nearest neighbours. Then we get

$$\mathbf{F}_e = e\,\{\mathbf{d} + [\mathbf{vh}]/c\} \quad \text{.......................(237)},$$

where $\mathbf{d}$ and $\mathbf{h}$ are the electric and magnetic forces at some conveniently chosen point of the charge, and $\mathbf{v}$ is the velocity of that point relative to the observer*.

Resolving along the tangent, principal normal drawn towards the centre

* [We may obtain an idea of the degree of approximation attained in (237) as follows: let de be any element of the charge, and $\mathbf{f}_e$ the mechanical force per unit charge for the element, so far as it depends on charges other than the charge considered, whether belonging to the same group or not. Then we have $\mathbf{F}_e = \int \mathbf{f}_e\, de$.

Take any origin in or near the charge and moving with it, and let $\mathbf{v}$ be its velocity relative to the observer. Also let $\mathbf{r}$ be the radius vector drawn from it to the element de, and let $\mathbf{u}$ be the velocity of de relative to it. Then $\mathbf{r}$ and $\mathbf{u}$ are small quantities of the order of the linear dimensions of the charge, and $\mathbf{f}_e$ is a function of $\mathbf{r}$ and $\mathbf{u}$, as well as of the position and velocity of the origin. Hence

$$\mathbf{f}_e = \mathbf{f} + (\mathbf{r}\nabla)\, f + (\mathbf{u}D)\, f + \ldots,$$

where $\mathbf{f}$ is the value at the origin, where $\mathbf{r} = \mathbf{u} = 0$, and ∇ and D denote vector differentiation with respect to the coordinates and velocity components respectively.

Let us choose the origin so that $\int \mathbf{r}\, de = 0$; since de is independent of the time we have $\int \mathbf{u}\, de = 0$ also. The point thus found may be called the "electric centre" of the charge, from analogy with the mass centre. It does not generally coincide with any particular element of charge, but is never far from any element. When we treat the charge as a point charge, we may suppose it concentrated at the electric centre. Then (237) holds when $\mathbf{d}$, $\mathbf{h}$ and $\mathbf{v}$ refer to the electric centre, and we neglect the squares of the linear dimensions of the charge.]

of curvature, and the binormal drawn so as to make a right-handed system of moving axes (s, ρ, n), we get

$$\left.\begin{aligned} F_s &= ed_s \\ F_\rho &= e(d_\rho - \beta h_n) \\ F_n &= e(d_n + \beta h_\rho) \end{aligned}\right\} \quad \text{.........................(238)},$$

where the suffix e has been omitted as no longer necessary to distinguish the external force **F**.

We now proceed to the consideration of an illustrative problem.

156. **Problem. A group of n electrons moves uniformly in a circle. Required to find the mechanical force on one of them due to the rest, and the equations of motion.** We shall use the notation of the problem, § 136, and shall take the 0th electron of the group as the one whose motion is required.

We must write down the electric and magnetic forces due to the ith electron at the electric centre of the 0th, as given by (203), § 138, and sum from $i = 1$ to $i = n - 1$ [not from $i = 0$ to $i = n - 1$ as in § 139].

The azimuth of the 0th electron is $\omega t + \delta$; this is to be taken as the value of ϕ.

The azimuth of the ith electron is $\omega t + \delta + 2\pi i/n$, because it forms one of the same group. Hence we get

$$\psi = \tfrac{1}{2}(\omega t + \delta - \phi + 2\pi i/n) = \pi i/n.$$

Substituting this value in (203) we get for the electric and magnetic forces due to charges of the same group

$$d_{1\phi} = -\frac{e(1-\beta^2)}{4\rho^2}\sum_{i=1}^{i=n-1}\frac{\cos \pi i/n}{\sin^2 \pi i/n}$$
$$-\frac{e}{\rho^2}\sum_{i=1}^{i=n-1}\sum_{j=1}^{j=\infty}(-1)^j\left[j\beta^2 J_j'(j\beta) - \tfrac{1}{2}(1-\beta^2)j^2\int_0^\beta J_j(jx)\,dx\right]\cos\frac{j\pi i}{n},$$

$$d_{1\rho} = -\frac{e}{4\rho^2}\sum_{i=1}^{i=n-1}\frac{1}{\sin \pi i/n}$$
$$-\frac{e}{2\rho^2}\sum_{i=1}^{i=n-1}\sum_{j=1}^{j=\infty}(-1)^j\left[j\beta J_j(j\beta) + j\int_0^\beta J_j(jx)\,dx\right]\sin\frac{j\pi i}{n},$$

$$h_{1z} = \frac{e\beta}{4\rho^2}\sum_{i=1}^{i=n-1}\frac{1}{\sin \pi i/n}$$
$$-\frac{e\beta}{2\rho^2}\sum_{i=1}^{i=n-1}\sum_{j=1}^{j=\infty}(-1)^j\left[j\beta J_j(j\beta) - j\int_0^\beta J_j(jx)\,dx\right]\sin\frac{j\pi i}{n}.$$

As regards the sums with respect to i we have

$$\sum_{i=1}^{i=n-1}\frac{\cos \pi i/n}{\sin^2 \pi i/n} = 0, \qquad \sum_{i=1}^{i=n-1}\frac{1}{\sin \pi i/n} = 4K \quad \text{...........(239)},$$

with the notation of Maxwell*. Moreover

$$\sum_{i=1}^{i=n-1} \cos\frac{j\pi i}{n} = n-1, \text{ for } j = 2sn,$$
$$= -1, \text{ for other even values of } j,$$
$$= 0, \text{ for odd values of } j,$$
$$\sum_{i=1}^{i=n-1} \sin\frac{j\pi i}{n} = 0, \text{ for even values of } j,$$
$$= \cot\frac{(2s+1)\pi}{2n}, \text{ for } j = 2s+1.$$

The change of order of the summations is allowable because the series are convergent; hence we get

$$\left.\begin{aligned}
d_{1\phi} &= \frac{2e}{\rho^2}\sum_{s=1}^{s=\infty}\left[s\beta^2 J'_{2s}(2s\beta) - s^2(1-\beta^2)\int_0^\beta J_{2s}(2sx)\,dx\right] \\
&\quad - \frac{2ne}{\rho^2}\sum_{s=1}^{s=\infty}\left[sn\beta^2 J'_{2sn}(2sn\beta) - s^2n^2(1-\beta^2)\int_0^\beta J_{2sn}(2snx)\,dx\right] \\
d_{1\varpi} &= \frac{eK}{\rho^2} - \frac{e}{\rho^2}\sum_{s=0}^{s=\infty}(s+\tfrac{1}{2})\cot\frac{(2s+1)\pi}{2n}\left[\beta J_{2s+1}\{(2s+1)\beta\}\right. \\
&\qquad\qquad \left. + \int_0^\beta J_{2s+1}\{(2s+1)x\}\,dx\right] \\
h_{1z} &= \frac{e\beta K}{\rho^2} + \frac{e\beta}{\rho^2}\sum_{s=0}^{s=\infty}(s+\tfrac{1}{2})\cot\frac{(2s+1)\pi}{2n}\left[\beta J_{2s+1}\{(2s+1)\beta\}\right. \\
&\qquad\qquad \left. - \int_0^\beta J_{2s+1}\{(2s+1)x\}\,dx\right]
\end{aligned}\right\}\ldots(240).$$

157. By comparison with (129), § 84, we see that the first series in $d_{1\phi}$, when multiplied by $2ce^2\beta/\rho^2$, gives the loss of energy due to radiation from a single electron describing the given circle with the given velocity, and the second series, when multiplied by $2ce^2\beta n^2/\rho^2$, gives the total radiation from the group of n electrons.

The radiation from a single electron has been calculated already and is given by (130), § 85; hence the sum of the first series is $\beta^3/3(1-\beta^2)^2$.

The second series has not been summed, nor have the remaining series in (240). For the sake of brevity we shall write

$$U = 2\sum_{s=1}^{s=\infty}\left[sn\beta^2 J'_{2sn}(2sn\beta) - s^2n^2(1-\beta^2)\int_0^\beta J_{2sn}(2snx)\,dx\right]\ldots\ldots\ldots\ldots(241),$$

$$\mathcal{V} = \sum_{s=1}^{s=\infty}(s+\tfrac{1}{2})\cot\frac{(2s+1)\pi}{2n}\left[\beta(1-\beta^2)J_{2s+1}\{(2s+1)\beta\}\right.$$
$$\left. + (1+\beta^2)\int_0^\beta J_{2s+1}\{(2s+1)x\}\,dx\right]\ldots(242).$$

* *Collected Papers*, Vol. I. p. 314.

Remembering that the directions (s, ρ, n) here coincide with $(\phi, -\varpi, z)$, we get from (238) and (240)

$$\left.\begin{aligned} F_{1\phi} &= \frac{2e^2\beta^3}{3\rho^2(1-\beta^2)^2} - \frac{e^2 n U}{\rho^2} \\ F_{1\varpi} &= \frac{e^2 K(1+\beta^2)}{\rho^2} - \frac{e^2 \mathfrak{V}}{\rho^2} \\ F_{1z} &= 0 \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(243).$$

The last equation follows from the fact that d_z and h_ϖ are both zero.

158. The physical interpretation of the first equation is interesting. Comparing it with (210_1), § 144, we see that its first term on the right is precisely the negative of F_{is} in the present case, where $\dot{\beta} = \ddot{\beta} = 0$, that is, it is just sufficient to balance the radiation pressure on the electron caused by its own radiation. In fact, it gives the radiation from the electron when it is multiplied by the velocity $c\beta$. Moreover, the negative of the second term on the right of (243_1), when multiplied by the velocity $c\beta$, is equal to $1/n$th part of R, the radiation from the ring of n electrons, as is seen at once by referring to (129), § 84.

Hence the work done on any one electron by the tangential pull exerted by the remaining electrons of the ring is just equal to the loss of energy due to radiation from the electron less the $1/n$th part of the loss from the whole ring. This result may be expressed in a different way by saying that of all the energy lost by radiation from any one electron, the major part is re-absorbed by the remaining electrons of the ring, and only a small portion is lost to the system by radiation into space. The mechanism of this process acts through the tangential pull exerted on one electron by the rest.

159. Equations of motion of the ring. We apply (220), § 149, where the external force $\mathbf{F}$ is the sum of $\mathbf{F}_1$, due to the rest of the ring and given by (243), and $\mathbf{F}_2$, due to charges outside the ring. We get at once

$$\left.\begin{aligned} c\dot{m}\beta &= -\frac{e^2 n U}{\rho^2} + F_\phi \\ \frac{c^2 m\beta^2}{\rho} &= -\frac{e^2 K(1+\beta^2)}{\rho^2} + \frac{e^2\mathfrak{V}}{\rho^2} - F_\varpi \\ 0 &= F_z \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots(244),$$

where the suffix 2 has been omitted from $\mathbf{F}$ as no longer necessary to distinguish the impressed force external to the ring as a whole.

These equations were first given by Schott*.

* *Phil. Mag.* [6], Vol. XII. p. 21, 1906.

A little consideration shows that in the first equation F_ϕ must vanish. For its magnitude remains the same for all positions of the moving electron, although its direction changes. Thus it is periodic relative to fixed space, and its period is $2\pi/\omega$.

In Ch. VII we saw that a force of this type only arises from one of the harmonic waves emitted by an electron, or group of electrons, describing a circle coaxal with the given one, with uniform angular velocity ω/k, where k is an integer.

Any such electron, or group of electrons, emits waves capable of interfering with those from the given group, and therefore should be reckoned as forming part of that group.

For this reason we shall omit the term F_ϕ from (244_1).

160. Let us now consider this equation more closely, and for the sake of generality let us admit the existence of a small acceleration, or retardation, $\dot\beta$, such that $\dot\beta/\beta$ is of the same order as the ratio of the small radiation terms to the principal terms in (219), § 148. Then the radiation terms in (219_1) involving $\dot\beta$ and $\ddot\beta$ will be of the second order, and may be neglected in comparison with the two first terms on the left. The same thing applies to the radiation terms in (219_2), which only involve the small quantities $\dot\rho$ and $\dot\beta$. Hence to this approximation (244_1) becomes

$$c\dot\beta\,\frac{d\,(m\beta)}{d\beta}+c\dot m\beta=-\,\frac{e^2nU}{\rho^2}\;\dots\dots\dots\dots\dots\dots(245),$$

while (244_2) and (244_3) are unaltered.

We saw in § 157 that U, when multiplied by $ce^2\beta^2n^2/\rho^2$, is equal to the radiation from the ring, and therefore is an essentially positive quantity*. Hence the right-hand member of (245) is always negative. An approximate value of it is easily got by taking only the first term in U, for $s=1$, and only the lowest power of β in it: we get $U=\dfrac{2\,(n+1)\,(n\beta)^{2n+1}}{2n+1\,!}$.

Remembering that $d\,(m\beta)/d\beta$ is what is usually called the longitudinal mass, m', of the electron, we get the following approximate equation from (245)

$$\frac{m'}{m}\frac{\dot\beta}{\beta}+\frac{\dot m}{m}=-\,\frac{2e^2}{cm\rho^2}\,\frac{n^2\,(n+1)\,(n\beta)^{2n}}{2n+1\,!}\;\dots\dots\dots\dots(246).$$

* [A direct proof of this statement may be given. From (241) we get by means of the differential equation of the function J

$$\frac{d}{d\beta}\left(\frac{U}{1-\beta^2}\right)=2\sum_{s=1}^{s=\infty}\left[\frac{sn\beta\,(1+\beta^2)}{(1-\beta^2)^2}\,J'_{2sn}\,(2sn\beta)+s^2n^2\,J_{2sn}\,(2sn\beta)\right].$$

$J_{2sn}\,(2sn\beta)$ and $J'_{2sn}\,(2sn\beta)$ are both positive so long as β is less than unity, and U vanishes with β. Hence as β increases from zero to unity, $U/(1-\beta^2)$, and with it U, increases from zero and remains positive.]

On the left-hand side of this equation the factor of $\dot{\beta}/\beta$, namely m'/m, is unity for very small velocities and increases to infinity as β increases to unity.

The term $\dot{m}/m$ can be replaced by $-\dot{a}/a$, where a is the radius of the electron supposed spherical, or its equatorial radius, if ellipsoidal, or generally, a length determining the size of the electron. We see that this is so from (206)—(208), § 143, and it can be proved generally from a consideration of dimensions.

On the right-hand side the function $n^2(n+1)(n\beta)^{2n}/2n+1!$ diminishes as β diminishes, all the more rapidly the greater n is. The first few values are, for

$$n=2: (2\beta)^4/10; \quad n=3: (3\beta)^6/140; \quad n=4: (4\beta)^8/4536.$$

For large values of n we get by Stirling's formula $(\frac{1}{2}\epsilon\beta)^{2n}.n^{3/2}\epsilon/4\sqrt{\pi}$. For small values of n this formula gives a value about twice too large, but still of the right order.

The numerical factor, $2e^2/cm\rho^2$, is greatest for small velocities, and diminishes to zero as β increases to unity, on account of the increase in m. Choosing for e/cm its greatest value, $1{\cdot}77 . 10^7$ E.M.U., for e the value $4{\cdot}8 . 10^{-10}$ E.S.U., and for ρ the value 10^{-8} cm., we get $2e^2/cm\rho^2 = 0{\cdot}86 . 10^{14}$.

161. Let us begin by assuming that the electron does not change in linear dimensions, that is, that a is constant, and therefore $\dot{m}=0$.

Then (246) shows that $\dot{\beta}$ is necessarily negative. Thus a steady motion without a supply of energy from without is impossible.

But that is not all; under these conditions the size of the ring and its speed of rotation are indeterminate. In fact, the tangential equation of motion, (244_1) or (246), serves only to determine the retardation in speed of the ring; the radial equation, (244_2), supplies but one relation between the speed and the radius. One of the two is quite arbitrary; given the number of electrons in the ring, its properties are indeterminate. In particular, it has no definite periods of oscillation, and an assemblage of such rings would produce a banded, and not a line spectrum, in the absence of any mechanism sufficient to confine its structure within narrow limits.

In order that the ring should be fairly permanent, the acceleration $\dot{\beta}$ must be small. The form of (246) requires that β be small, all the smaller the smaller the value of n. For instance, in order that $\dot{\beta}/\beta$ may be less than 1/100, we must have β less than $9 . 10^{-5}$ for $n=2$, $1{\cdot}7 . 10^{-3}$ for $n=3$, $7{\cdot}3 . 10^{-3}$ for $n=4$, and so on.

162. As an alternative let us admit a change in size of the electron, and consequently also a change in mass. We can then satisfy (246) by supposing $\dot{\beta}$ to be zero, and β to be constant.

Since the right-hand member is always negative, we must suppose $\dot{m}$ negative. Hence the mass of the electron diminishes, and its size increases, quite independently of temporary changes due to temporary changes of speed.

The change of mass need only take place very slowly provided that the velocity of the ring be small enough. In order that it should not contradict our experience as to the constancy of the properties of electrons and atoms, it is sufficient that $\dot{m}/m$ be less than, let us say 10^{-15}, or one thirty millionth part per annum, but of course it might be very much smaller. What is contemplated is a slow secular diminution in the mass of the electron, superposed on all the changes it undergoes when its speed changes.

Replacing $\dot{m}$ by its value $-m\dot{a}/a$ in (245) we get

$$\frac{nU}{\beta} = \frac{cm\rho^2\dot{a}}{e^2a} \quad \text{.............................(247).}$$

Given the value of $\dot{a}/a$ as a constant, or as a slowly varying function of the time, this equation constitutes a second relation between β and ρ, which together with (244_2) determines both completely as functions of n, the number of electrons in the ring, and of the parameters fixing the external field of force in which the ring moves. In addition β and ρ will involve the constant, or slowly varying, parameter $\dot{a}/a$.

The form of U as a function of β and n is such that even for small values of n quite considerable variations in the value of nU/β have little influence in changing the value of β. This is particularly true when n is large; for instance, when $n = 5$, doubling the value of nU/β only increases that of β by 7 per cent.

Hence we may alter the values of $\dot{a}/a$, of m, or of ρ considerably in (247) without changing β very much. Two consequences follow from this fact: in the first place, the value of β depends mainly on that of n, so that we can calculate it even when we have only an approximate value of ρ at our disposal; the radial equation, (244_2), then enables us to calculate ρ exactly, and if necessary we can correct β.

Secondly, any slight changes in the values of $\dot{a}/a$, or of m, produce practically no effect on the value of β; thus it is immaterial whether $\dot{a}/a$ and m be constant, or be undergoing slow secular changes. Any such changes in the values of $\dot{a}/a$, or of m, will produce very much slower changes in the values of β and ρ, that is to say in the properties of the ring of electrons, so that its motion is steady to a very high degree of approximation.

[**163**. It is interesting to examine the mechanism by means of which the expansion of the electron regulates the structure of the ring of electrons. In (246) the first term, $m'\dot{\beta}/m\beta$, represents the reaction which the electron opposes to any change of its speed, in consequence of the electromagnetic actions between its parts, the second, $\dot{m}/m$, the reaction which it opposes in virtue of a diminution in size and consequent increase in mass, and the third the part of the radiation pressure on the electron, which is not balanced by the pull of the rest of the ring. Owing to the mutual repulsion of its parts, the electron tends to expand, and will do so, unless it is prevented by some external system, for instance by the aether on Poincaré's hypothesis. If the mutual repulsions overbalance the pressure of the aether, the electron expands, and its mass diminishes, and then we get a negative second term in (246), which is equivalent to a forward pull on the electron. If this pull exceeds the drag due to radiation, the speed of the ring increases and the drag increases with it, until it is just sufficient to balance the pull due to expansion. If the pull is less than the drag, the reverse process takes place.

If through any cause the speed of the ring be increased beyond the value necessary for steady motion, the drag due to radiation increases beyond the amount which balances the pull due to expansion, and produces a retardation of the ring, and vice versa. Thus the drag due to radiation supplies the mechanism for regulating the speed of the ring; its great effectiveness for the purpose is due to the fact that it increases very greatly for small increases of velocity, particularly when the number of electrons in the ring is large.

When the motion of the ring is steady, the drain of energy due to radiation is supplied through the pull due to expansion. This occurs at the expense of the electromagnetic energy of the electron, which has also to supply the work done in pushing back the aether opposing the expansion.

Another alternative altogether is to assume that the electron is deformed so as to become unsymmetrical. We shall see in Appendix D, § 230, that in this case an additional term arises on the left-hand side of (244), namely $-m_\eta v^2/\rho$, where m_η is of the nature of a transverse mass. If m_η be positive, and the velocity suitably chosen, this term may balance the radiation term on the right-hand side.]

APPENDIX A

ON THE DOPPLER EFFECT

164. A radiating system of charges moves with small uniform velocity and without change of constitution. To find the frequencies and intensities of its radiations relative to a stationary distant observer (Doppler effect).

This problem, with the limitations mentioned, is obviously the problem of finding the Doppler effect due to the motion. That the limitations are necessary will be seen during the course of the investigation.

For a distant observer the potentials are given by the equations (131) and (132), § 86, and the electric and magnetic forces by (133) and (134), § 87. The latter are

$$\mathbf{d} = \frac{e}{2\pi c r}\frac{\partial}{\partial t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \epsilon^{\iota\mu(t-r/c+p/c-\tau)} . (\mathbf{r}_1 - \mathbf{v}/c)\, d\tau\, d\mu \quad \text{......(133)},$$

$$\mathbf{h} = \frac{e}{2\pi c^2 r}\frac{\partial}{\partial t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \epsilon^{\iota\mu(t-r/c+p/c-\tau)} . [\mathbf{v} . \mathbf{r}_1]\, d\tau\, d\mu \quad \text{.........(134)}.$$

Here p is the projection of the displacement of the charge at time τ on the radius vector $\mathbf{r}$ drawn to the observer from some fixed point, whose distance from the charge always remains small compared with r. Thus, while the distance travelled by light in a certain time is comparable with r, the distance travelled by the charge in the same time is of order p, small compared with r. Hence the average velocity of the charge must be assumed to be small compared with that of light when we apply (133) and (134) to the case of a progressive motion of the charge.

In reality this progressive motion cannot have continued for all negative time, but we saw in Ch. IX that the influence of the initial conditions may be disregarded as soon as the permanent régime has been reached. As regards values of τ later than the time of emission $t - r/c$, we have already seen that the corresponding motions have no influence on the field at the time t, because the waves due to them have not yet arrived. Hence we may apply (133) and (134) without fear of error to the case of a motion which is partly progressive, although the condition that p be small compared with r will certainly be violated sooner or later, because the values of τ, for which it is violated, have no effect on the field which we wish to find.

165. We shall suppose the motion of the charge to be compounded of two motions: (1) a uniform progressive motion with velocity $\mathbf{u}$ at an angle θ with the radius vector $\mathbf{r}$; (2) a motion relative to this with velocity $\mathbf{v}$. Choose the origin so that the progressive displacement of the charge vanishes at the time $\tau = 0$; then its value at the time τ is $c\beta\tau$, where $\beta = u/c$. In order to satisfy the condition as to the smallness of the projection p we must have β small, for the part of this projection due to the progressive motion is $c\beta \cos\theta \,.\, \tau$. Let us now change our notation and henceforth denote by p the part of the projection of the whole displacement which is due to the relative motion alone, so that the whole projection is now denoted by $p + c\beta\cos\theta \,.\, \tau$. Then in (133) and (134) we must replace $\mathbf{v}$ by $\mathbf{v} + \mathbf{u}$ and p by $p + c\beta\cos\theta \,.\, \tau$.

Thus we get

$$\mathbf{d} = \frac{e}{2\pi c r}\frac{\partial}{\partial t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \epsilon^{\iota\mu\{t - r/c + p/c - (1-\beta\cos\theta)\tau\}}\left(\mathbf{r}_1 - \frac{\mathbf{v}+\mathbf{u}}{c}\right) d\tau\, d\mu,$$

$$\mathbf{h} = \frac{e}{2\pi c^2 r}\frac{\partial}{\partial t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \epsilon^{\iota\mu\{t - r/c + p/c - (1-\beta\cos\theta)\tau\}}\left[\mathbf{r}_1\,(\mathbf{v}+\mathbf{u})\right] d\tau\, d\mu.$$

These expressions can be reduced to the forms (133) and (134) by means of the substitution

$$\left.\begin{aligned} \mu &= \frac{\mu'}{1-\beta\cos\theta}, & t &= (1-\beta\cos\theta)\,t' \\ r &= (1-\beta\cos\theta)\,r', & p &= (1-\beta\cos\theta)\,p' \end{aligned}\right\} \quad \ldots\ldots\ldots(248).$$

We get $\mathbf{v} = \mathbf{v}'$, because we must still have $\dot{p}' = \frac{\partial p'}{\partial t'}(\mathbf{r}_1\mathbf{v}')$.

Now we find

$$\left.\begin{aligned} \mathbf{d} &= \frac{e}{2\pi c r'(1-\beta\cos\theta)^3}\frac{\partial}{\partial t'}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \epsilon^{\iota\mu'(t' - r'/c + p'/c - \tau)}\left(\mathbf{r}_1 - \frac{\mathbf{v}'+\mathbf{u}}{c}\right) d\tau\, d\mu' \\ \mathbf{h} &= \frac{e}{2\pi c^2 r'(1-\beta\cos\theta)^3}\frac{\partial}{\partial t'}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \epsilon^{\iota\mu'(t' - r'/c + p'/c - \tau)}\left[\mathbf{r}_1\,(\mathbf{v}'+\mathbf{u})\right] d\tau\, d\mu \end{aligned}\right\}$$

$$\ldots\ldots(249).$$

Denote by ϕ', $\mathbf{d}'$, $\mathbf{h}'$ what ϕ, $\mathbf{d}$, $\mathbf{h}$ become respectively, when we replace t and r by t' and r'.

Notice that the integral which multiplies $\mathbf{u}/c$ is simply ϕ'.

166. Then we get at once

$$\mathbf{d} = \frac{\mathbf{d}'}{(1-\beta\cos\theta)^3} - \frac{\mathbf{u}\dot{\phi}'}{c^2(1-\beta\cos\theta)^3} \quad \ldots\ldots\ldots\ldots\ldots(250),$$

$$\mathbf{h} = \frac{\mathbf{h}'}{(1-\beta\cos\theta)^3} - \frac{[\mathbf{r}_1\mathbf{u}]\,\dot{\phi}'}{c^2(1-\beta\cos\theta)^3} \quad \ldots\ldots\ldots\ldots\ldots(251).$$

These equations contain the whole theory of the Doppler effect; they enable us to express the field due to a moving system when we have found that due to a modified resting system, obtained from the given one by the transformation (248).

The direction of the vector $\mathbf{r}'$ is the same as that of $\mathbf{r}$, although its magnitude is changed in the ratio $1-\beta\cos\theta:1$. Hence $\mathbf{d}'$ and $\mathbf{h}'$ are to be calculated by (133) and (134), § 164, for the modified system. It follows that we have

$$(\mathbf{r}_1\mathbf{d}')=0,\quad (\mathbf{r}_1\mathbf{h}')=0,\quad \mathbf{h}'=[\mathbf{r}_1\mathbf{d}'].$$

Thus we get from (250) and (251), since $(\mathbf{r}_1\mathbf{u})=c\beta\cos\theta$,

$$(\mathbf{r}_1\mathbf{d})=-\frac{\beta\cos\theta\dot{\phi}'}{c(1-\beta\cos\theta)^3},\quad (\mathbf{r}_1\mathbf{h})=0,\quad \mathbf{h}=[\mathbf{r}_1\mathbf{d}]\quad \text{......(252).}$$

Hence the magnetic force due to the moving system at a great distance is still transverse and perpendicular to the electric force, but the latter is not transverse except at the equator.

167. The Poynting vector is given by

$$\mathbf{s}=\frac{c\,[\mathbf{d}\mathbf{h}]}{4\pi}=\frac{cd^2}{4\pi}\mathbf{r}_1-\frac{c\,(\mathbf{r}_1\mathbf{d})}{4\pi}\mathbf{d},\quad \text{by (252).}$$

Hence

$$\left.\begin{aligned}(\mathbf{s}\mathbf{r}_1)&=\frac{c\,\{d^2-(\mathbf{r}_1\mathbf{d})^2\}}{4\pi}=\frac{c^2d'^2-2\,(\mathbf{d}'\mathbf{u})\,\dot{\phi}'+\beta^2\sin^2\theta\,.\,\dot{\phi}'^2}{4\pi c\,(1-\beta\cos\theta)^6}\\ [\mathbf{s}\mathbf{r}_1]&=\frac{c\,(\mathbf{r}_1\mathbf{d})}{4\pi}\mathbf{h}=-\frac{\beta\cos\theta\dot{\phi}'}{4\pi\,(1-\beta\cos\theta)^6}\mathbf{h}'+\frac{\beta\cos\theta\dot{\phi}'^2}{4\pi c^3\,(1-\beta\cos\theta)^3}[\mathbf{r}_1\mathbf{u}]\end{aligned}\right\}\quad \text{......(253).}$$

These equations show that the flow of energy is not entirely radial, but that there is a component flow in the direction of the magnetic force $\mathbf{h}$, that is a circulation of energy along the parallels of latitude. This obviously contributes nothing to the radiation and therefore need concern us no further. The rest of the flow is entirely radial, but is of a different form from that due to a system at rest.

On the line of motion we have $\theta=0$, or $\theta=\pi$, according as the motion is towards, or away from the observer. Here $\mathbf{u}$ is along the radius vector, and therefore $(\mathbf{d}'\mathbf{u})=0$; hence, on the line of motion,

$$(\mathbf{s}\mathbf{r}_1)=\frac{cd'^2}{4\pi\,(1\mp\beta)^6}\quad \text{..........................(254),}$$

$\mp$ according as the motion is towards (away from) the observer, On the equator we have $\theta = \frac{1}{2}\pi$, and $\mathbf{d}'$ is parallel to $\mathbf{u}$, so that $(\mathbf{d}'\,.\,\mathbf{u})/c$ is equal to $\pm\beta d'$, according as the directions of $\mathbf{d}'$ and $\mathbf{u}$ are the same, or opposite.

Hence, on the equator,

$$(\mathbf{sr}_1) = \frac{(cd' \mp \beta\dot{\phi}')^2}{4\pi} \quad \text{........................(255).}$$

We must now consider in what way the force $\mathbf{d}'$ differs from its value when the system is at rest. There are two kinds of changes to be investigated: (1) the change of frequency of any constituent vibration; and (2) its change of intensity.

168. Change of frequency. We must remember that $\mathbf{d}'$ is given by (133), § 164, except that in accordance with (248) and (249), § 165, t, r and p are replaced by t', r' and p'.

When the system is polyperiodic, with frequencies ω_1, ω_2, ..., the radiation from it when at rest consists of an infinite number of simple harmonic sum and difference vibrations, given by equations such as (144)—(146), § 91, the argument being of the form $\Omega\,(t - r/c) + \Delta$, where $\Omega = s_1\omega_1 + s_2\omega_2 + \ldots$, and s_1, s_2, ... are integers.

If we suppose the constitution of the system to be unchanged when set in uniform rectilinear motion, we must suppose ω_1, ω_2, ... to be unaltered, and therefore also Ω. But as we have seen, t and r are to be replaced by t' and r', so that the argument of the vibration for a distant stationary observer becomes of the form $\Omega\,(t' - r'/c) + \Delta$, and this by (248) may be written

$$\frac{\Omega}{1 - \beta\cos\theta}(t - r/c) + \Delta,$$

let us say $\Omega'\,(t - r/c) + \Delta$. Thus the frequency is changed from Ω to Ω', where

$$\Omega' = \Omega/(1 - \beta\cos\theta) \quad \text{........................(256).}$$

This equation expresses the well known Doppler effect, as observed in canal rays. The fact that it agrees with experiment shows that the constitution of the canal ray ion is unaltered by its motion, at any rate to the first order. According to the Theory of Relativity we may expect it to be altered to the second order, but such a change, though theoretically determinable by observation of the canal rays in a direction transverse to their motion, would be difficult to establish by experiment*.

169. Intensity of the radiation. We shall for simplicity consider only the most important case, where the radiation is observed in the line of motion, in front of, or behind, the moving system, that is where $\theta = 0$ or π. In this

* Einstein, *Ann. der Phys.* 23, p. 197, 1907. Schott, *Phys. Zeitsch.* Vol. 8, p. 292, 1907.

case, as we have seen in § 167, (254), the radiation $(\mathbf{sr}_1) = cd'^2/4\pi(1 \mp \beta)^6$. In order to have a perfectly definite problem before us, let us suppose that the moving system is a uniformly rotating ring of n equidistant electrons.

The field due to such a ring is given by (125) and (126), § 82. To avoid confusion we shall however replace the polar distance θ of the radius vector $\mathbf{r}_1$ by χ, and the velocity of the rotational motion by $c\gamma$. The general term of order s is given by

$$d_\chi' = \frac{2e\gamma' \cot \chi \,.\, n^2}{r'\rho'} s\, J_{sn}(sn\,\gamma' \sin\chi) \sin sn\,\{\omega(t' - r'/c) + \delta - \phi + \tfrac{1}{2}\pi\},$$

$$d_\phi' = \frac{2e\gamma'^2 n^2}{r'\rho'} s\, J'_{sn}(sn\,\gamma' \sin\chi) \cos sn\,\{\omega(t' - r'/c) + \delta - \phi + \tfrac{1}{2}\pi\},$$

where we have replaced t, r, ρ and γ by their values t', r', ρ' and γ' in the progressive motion of the ring. It must be remembered in the first place that $\rho \sin\chi$ in the arguments of the Bessel Functions represents p, and therefore, when the ring has a progressive motion, must be changed in accordance with (248), § 165, and secondly that $\gamma = \omega\rho/c$, so that γ must also be changed to γ', where $\gamma = (1 - \beta\cos\theta)\gamma'$, because ω is not altered by the progressive motion. Thus $\gamma'/\rho' = \gamma/\rho$ and $\gamma'/r' = \gamma/r$. Hence the changes in amplitude of d_χ' and d_ϕ' depend essentially on the changes of $J_{sn}(sn\,\gamma' \sin\chi)/r'$, and of $J'_{sn}(sn\,\gamma' \sin\chi)$. If we are content with a first approximation, we may assume $J_{sn}(sn\,\gamma' \sin\chi)$ to be proportional to γ'^{sn}. Hence to this approximation the amplitudes both of d_χ' and d_ϕ' are got from those of d_χ and d_ϕ, corresponding to a ring at rest, by dividing by $(1 - \beta\cos\theta)^{sn-1}$.

It now follows from (254), § 167, that for our ring, the radiation in the direction of motion is increased in the ratio

$$1 : (1 - \beta)^{2n+4} \text{ for the lowest harmonic } (s = 1),$$

$$1 : (1 - \beta)^{4n+4} \text{ for the second } (s = 2),$$

and so on. And in the opposite direction it is diminished in the ratio $1 : (1 + \beta)^{2n+4}$ for the lowest harmonic, $1 : (1 + \beta)^{4n+4}$ for the second, and so on.

In other words, the radiation in the direction of the motion is to that in the opposite direction in the ratio $\left(\frac{1+\beta}{1-\beta}\right)^{2n+4}$ for the lowest harmonic, $\left(\frac{1+\beta}{1-\beta}\right)^{4n+4}$ for the second, and so on.

Thus the effect of the motion is to destroy the centro-symmetry of the radiation of the stationary ring, by causing it to radiate more strongly in the direction of motion than in the opposite direction. The asymmetry produced is more pronounced the higher the order and frequency of the harmonic, and the greater the number (n) of electrons in the ring. A more complete investigation shows that the total radiation is increased, but only to the second order*.

170. As a second example, consider the oscillations of our ring, the field of which is given by (166) and (167), § 102. The general term is of a type such as

$$\frac{e \cot \chi \, nA'}{r'\rho'} ml\gamma' J_m (l\gamma' \sin \chi) \sin \{(q + m\omega)(t' - r'/c) + m(\delta - \phi + \pi/2) + \alpha\},$$

where A is the amplitude of the oscillation selected as a type. Since it represents a displacement it is subject to the same transformation,

$$A = (1 - \beta \cos \theta) A',$$

to which t, r and γ are subject. Hence we have $A'/r' = A/r$, and $\gamma'/\rho' = \gamma/\rho$, so that the change of amplitude due to the progressive motion depends on the change in $J_m (l\gamma' \sin \chi)$. To the same approximation as before this changes in the ratio $1 : (1 - \beta \cos \theta)^m$. Now we have $m = k + sn$, where k is the class of the oscillation selected, that is the number of nodes in the wave which disturbs the ring; and the only vibrations which produce appreciable spectrum lines are those for which $s = 0$, and k is a small number. For these m is small, and the change of amplitude due to progressive motion is small in consequence.

171. These two examples are sufficient to show that the vibrations due to systems of electrons fall into two classes as regards the distribution of intensity in the radiation produced when they are in slow progressive motion:

(1) Vibrations due to cyclic motions taking place with a velocity comparable with that of light; for example, the rotation of a circular ring of equidistant electrons, or the motion of an elliptic group, and other motions of the same kind. These vibrations may exhibit considerable asymmetry of intensity of radiation, even for a slow progressive motion, particularly when there are many electrons in the group.

(2) Small oscillations of groups of electrons about their state of permanent motion. The only waves emitted, which are strong enough to be observed, are of an order too low to show any appreciable asymmetry of radiation intensity due to progressive motion. The harmonics of higher order, which possess the asymmetry, are too weak to be observed.

* *Phil. Mag.* [6], Vol. XIII. p. 657, 1907.

Recently, Stark* has examined the canal rays of hydrogen and of mercury. He finds that there is little, if any, asymmetry in the case of the hydrogen line H_β, while there is considerable asymmetry, in the direction required by theory, for the mercury lines λ 4359, λ 4047, λ 3650, but none for the mercury line λ 4078. These experiments are by no means decisive, and it would be premature to draw any very definite conclusions from them, but so far as they go they seem to indicate that the hydrogen line H_β and the mercury line λ 4078 may be due, either to small oscillations of some kind or other, or to motions of large amplitude, but of small groups of electrons, whilst the remaining three mercury lines may be due to motions of large amplitude and of groups of many electrons.

* *Phys. Zeitsch.* Vol. 11, p. 179, 1910.

APPENDIX B

ON THE DISTURBED MOTION OF A RING OF ELECTRONS

172. THE calculation of the motion of a ring of electrons, when it is slightly disturbed from its state of steady motion, affords a good illustration of the methods of Chapters X—XII. The investigation naturally falls under three heads: (1) the calculation of the field due to a single point charge disturbed from its uniform motion in a circle; (2) the deduction of the field due to a ring of electrons, and of the mechanical force on one of them due to the rest of the ring, when a wave of disturbance travels round the ring; (3) the deduction of the equations for the free and forced small oscillations of the ring. We shall consider these three problems in order.

173. **Problem 1. A point charge is slightly disturbed from uniform circular motion. Required to find the field close to the orbit.** We shall use the same notation as in Ch. XI, § 150. In Fig. 42 P is the point (x, y, z) on the circle, and Q is the projection on its plane of the point $(x + \delta x, y + \delta y, z + \delta z)$, at which the field is required.

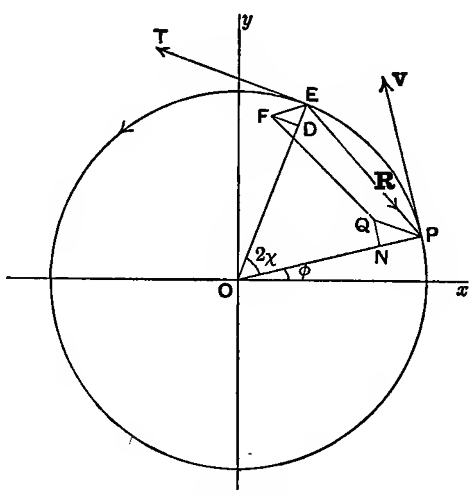

Fig. 42.

E is the position (ξ, η, ζ) of the point charge in the uniform motion at the time of emission τ, and F is the projection of its position at the same time τ in the disturbed motion, that is the projection of the point $(\xi + \delta\xi, \eta + \delta\eta, \zeta + \delta\zeta)$. The vector $\mathbf{R}$ in the undisturbed motion is drawn from E to P, and the vector $\mathbf{R} + \delta\mathbf{R}$ in the disturbed motion from $(x + \delta x, y + \delta y, z + \delta z)$ to $(\xi + \delta\xi, \eta + \delta\eta, \zeta + \delta\zeta)$. Thus FQ is the projection of $\mathbf{R} + \delta\mathbf{R}$.

We shall write as before

$$\left.\begin{aligned} &x + \delta x = \rho(1-\mu)\cos(\phi+\lambda), \quad y + \delta y = \rho(1-\mu)\sin(\phi+\lambda), \quad z + \delta z = \rho\nu \\ &\xi + \delta\xi = \rho(1-\mu')\cos(\omega\tau + \delta + \lambda'), \\ &\eta + \delta\eta = \rho(1-\mu')\sin(\omega\tau + \delta + \lambda'), \quad \zeta + \delta\zeta = \rho\nu' \\ &\psi = \tfrac{1}{2}(\omega t + \delta - \phi), \quad \chi = \tfrac{1}{2}(\omega\tau + \delta - \phi), \quad \psi = \chi + \beta\sin\chi \end{aligned}\right\} \quad \ldots\ldots(257),$$

where (λ, μ, ν) and (λ', μ', ν') are small quantities whose squares and products are to be neglected. Hence we get

$$\left.\begin{aligned} &\delta x = -\rho(\mu\cos\phi + \lambda\sin\phi), \quad \delta y = \rho(-\mu\sin\phi + \lambda\cos\phi), \quad \delta z = \rho\nu \\ &\delta\xi = -\rho\{\mu'\cos(2\chi+\phi) + \lambda'\sin(2\chi+\phi)\}, \\ &\delta\eta = \rho\{-\mu'\sin(2\chi+\phi) + \lambda'\cos(2\chi+\phi)\}, \quad \delta\zeta = \rho\nu' \end{aligned}\right\} \ldots(258).$$

In Fig. 42 NQ is $\rho\lambda$, PN is $\rho\mu$, DF is $\rho\lambda'$, ED is $\rho\mu'$, and $\rho\nu$ and $\rho\nu'$ are the heights of the fieldpoint and disturbed point charge above their respective projections, Q and F.

As before we have $R = 2\rho\sin\chi$, $K = 1 + \beta\cos\chi$, and $\dfrac{\partial}{c\partial t} = \dfrac{\beta}{2\rho}\dfrac{\partial}{\partial\psi}$; hence, substituting these values in (198) and (199), § 135, and taking only the small terms, we get

$$\left.\begin{aligned} \delta\mathbf{d} = &\frac{\delta\mathbf{R} - 3p\mathbf{R}_1}{8\rho^3\sin^3\chi(1+\beta\cos\chi)} + \frac{\partial}{\partial\psi}\left\{\frac{\beta(\delta\mathbf{R} - 3p\mathbf{R}_1)}{8\rho^3\sin^2\chi(1+\beta\cos\chi)}\right\} \\ &+ \frac{\partial}{\partial\psi}\left\{\frac{\beta p\mathbf{v}}{8c\rho^3\sin^2\chi(1+\beta\cos\chi)}\right\} + \frac{\partial^2}{\partial\psi^2}\left\{\frac{\beta^2 p\mathbf{v}}{8c\rho^3\sin\chi(1+\beta\cos\chi)}\right\} \\ &- \frac{\partial}{\partial\psi}\left\{\frac{\beta\delta\mathbf{v}}{4c\rho^2\sin\chi(1+\beta\cos\chi)}\right\} - \frac{\partial^2}{\partial\psi^2}\left\{\frac{\beta^2 p\mathbf{R}_1}{8\rho^3\sin\chi(1+\beta\cos\chi)}\right\} \\ \delta\mathbf{h} = &\frac{[\mathbf{v}\,.\,(\delta\mathbf{R} - 3p\mathbf{R}_1)]}{8c\rho^3\sin^3\chi(1+\beta\cos\chi)} + \frac{\partial}{\partial\psi}\left\{\frac{\beta[\mathbf{v}\,.\,(\delta\mathbf{R} - 3p\mathbf{R}_1)]}{8c\rho^3\sin^2\chi(1+\beta\cos\chi)}\right\} \\ &+ \frac{[\delta\mathbf{v}\,.\,\mathbf{R}_1]}{4c\rho^2\sin^2\chi(1+\beta\cos\chi)} + \frac{\partial}{\partial\psi}\left\{\frac{\beta[\delta\mathbf{v}\,.\,\mathbf{R}_1]}{4c\rho^2\sin\chi(1+\beta\cos\chi)}\right\} \\ &- \frac{\partial^2}{\partial\psi^2}\left\{\frac{\beta^2 p[\mathbf{v}\,.\,\mathbf{R}_1]}{8c\rho^3\sin\chi(1+\beta\cos\chi)}\right\} \end{aligned}\right\} \ldots(259).$$

174. Our object will be to express these forces in series, proceeding according to sines and cosines of multiples of ψ, by means of the characteristic equation $\psi = \chi + \beta \sin \chi$. For this purpose it is best to reduce all the terms to one or other of three fundamental types

$$\frac{f(\chi)}{\sin\chi\,(1+\beta\cos\chi)}, \quad \frac{f(\chi)}{1+\beta\cos\chi}; \quad \text{and} \quad \frac{f(\chi)\sin\chi}{1+\beta\cos\chi};$$

and their first or second differential coefficients with respect to ψ. Each of these three types can be expanded in the required form when $f(\chi)$ is an exponential function; this is the case in our problem, where $f(\chi)$ represents one of the disturbances λ', ... or one of their differential coefficients. The first step in the reduction is the expression of the numerators, such as $(\delta\mathbf{R} - 3p\mathbf{R}_1)$, in powers of $\cos\chi$ and $\sin\chi$.

We shall resolve each vector into rectangular components, respectively in the directions PU, PO and normal to the plane of the circle; these we shall denote by suffixes ϕ, ρ and z as before.

As before we have

$$R_{1\phi} = -\cos\chi, \quad R_{1\rho} = -\sin\chi, \quad R_{1z} = 0,$$
$$v_\phi = c\beta\cos 2\chi, \quad v_\rho = c\beta\sin 2\chi, \quad v_z = 0.$$

Further we get

$$\delta v_\phi = -\delta\dot{\xi}\sin\phi + \delta\dot{\eta}\cos\phi = \rho\,\{(\dot{\lambda}' - \omega\mu')\cos 2\chi - (\dot{\mu}' + \omega\lambda')\sin 2\chi\},$$
$$\delta v_\rho = -\delta\dot{\xi}\cos\phi - \delta\dot{\eta}\sin\phi = \rho\,\{(\dot{\lambda}' - \omega\mu')\sin 2\chi + (\dot{\mu}' + \omega\lambda')\cos 2\chi\},$$
$$\delta v_z = \rho\dot{\nu}'.$$

In order to find the components of $\delta\mathbf{R}$ we notice that $\overline{FQ}$ in Fig. 42 is the projection of $\mathbf{R} + \delta\mathbf{R}$ on the plane of the circle, while $\overline{EP}$ is $\mathbf{R}$; hence the projection of $\delta\mathbf{R}$ on the plane of the circle is the geometric sum of the vectors $\overline{PQ}$ and $\overline{FE}$, that is, of the vectors $\overline{FD}$, $\overline{DE}$, $\overline{PN}$ and $\overline{NQ}$. And its normal component is $\delta z - \delta\zeta$. Thus we find

$$\delta R_\phi = \rho\,(\lambda - \lambda'\cos 2\chi + \mu'\sin 2\chi), \quad \delta R_\rho = \rho\,(\mu - \lambda'\sin 2\chi - \mu'\cos 2\chi),$$
$$\delta R_z = \rho\,(\nu - \nu').$$

Lastly, p is the sum of the projections of the vectors $\overline{FD}$, $\overline{DE}$, $\overline{PN}$ and $\overline{NQ}$ on the vector $\overline{EP}$. Thus

$$p = -\rho\,\{(\lambda - \lambda')\cos\chi + (\mu + \mu')\sin\chi\}.$$

With these values we get the following expressions for the components in the directions ϕ, ρ and z of the several vectors which occur in the numerators of (259), § 173; all are arranged according to ascending powers of $\sin\chi$:

$$
\begin{aligned}
&(\delta R_\phi - 3pR_{1\phi})/\rho = -2(\lambda-\lambda') - (3\mu+\mu')\cos\chi\sin\chi + (3\lambda-\lambda')\sin^2\chi,\\
&(\delta R_\rho - 3pR_{1\rho})/\rho = \mu-\mu' - (3\lambda-\lambda')\cos\chi\sin\chi - (3\mu+\mu')\sin^2\chi,\\
&(\delta R_z - 3pR_{1z})/\rho = \nu-\nu',\\
&pv_\phi/c\rho = -\beta(\lambda-\lambda')\cos\chi - \beta(\mu+\mu')\sin\chi + 2\beta(\lambda-\lambda')\cos\chi\sin^2\chi\\
&\qquad\qquad + 2\beta(\mu+\mu')\sin^3\chi,\\
&pv_\rho/c\rho = -2\beta(\lambda-\lambda')\sin\chi - 2\beta(\mu+\mu')\cos\chi\sin^2\chi + 2\beta(\lambda-\lambda')\sin^3\chi,\\
&pv_z/c\rho = 0,\\
&\delta v_\phi/\rho = \dot\lambda' - \omega\mu' - 2(\dot\mu'+\omega\lambda')\cos\chi\sin\chi - 2(\dot\lambda'-\omega\mu')\sin^2\chi,\\
&\delta v_\rho/\rho = \dot\mu' + \omega\lambda' + 2(\dot\lambda'-\omega\mu')\cos\chi\sin\chi - 2(\dot\mu'+\omega\lambda')\sin^2\chi,\\
&\delta v_z/\rho = \dot\nu',\\
&pR_{1\phi}/\rho = \lambda-\lambda' + (\mu+\mu')\cos\chi\sin\chi - (\lambda-\lambda')\sin^2\chi,\\
&pR_{1\rho}/\rho = (\lambda-\lambda')\cos\chi\sin\chi + (\mu+\mu')\sin^2\chi,\\
&pR_{1z}/\rho = 0,\\
&[\mathbf{v}\,.\,(\delta\mathbf{R} - 3p\mathbf{R}_1)]_\phi/c\rho = 2\beta(\nu-\nu')\cos\chi\sin\chi,\\
&[\mathbf{v}\,.\,(\delta\mathbf{R} - 3p\mathbf{R}_1)]_\rho/c\rho = -\beta(\nu-\nu') + 2\beta(\nu-\nu')\sin^2\chi,\\
&[\mathbf{v}\,.\,(\delta\mathbf{R} - 3p\mathbf{R}_1)]_z/c\rho = \beta(\mu-\mu') + \beta(\lambda-3\lambda')\cos\chi\sin\chi + \beta(\mu+3\mu')\sin^2\chi,\\
&[\delta\mathbf{v}\,.\,\mathbf{R}_1]_\phi/\rho = \dot\nu'\sin\chi,\quad [\delta\mathbf{v}\,.\,\mathbf{R}_1]_\rho/\rho = -\dot\nu'\cos\chi,\\
&\qquad\qquad [\delta\mathbf{v}\,.\,\mathbf{R}_1]_z/\rho = (\dot\mu'+\omega\lambda')\cos\chi + (\dot\lambda'-\omega\mu')\sin\chi,\\
&p[\mathbf{v}\,.\,\mathbf{R}_1]_\phi = 0,\quad p[\mathbf{v}\,.\,\mathbf{R}_1]_\rho = 0,\\
&\qquad\qquad p[\mathbf{v}\,.\,\mathbf{R}_1]_z/c\rho = -\beta(\lambda-\lambda')\cos\chi\sin\chi - \beta(\mu+\mu')\sin^2\chi.
\end{aligned}
$$

175. An examination of (259), § 173, shows that when these values are substituted in the numerators of those equations, all the terms reduce to one or other of the three standard types

$$\frac{f(\chi)}{\sin\chi(1+\beta\cos\chi)},\quad \frac{f(\chi)}{1+\beta\cos\chi},\quad \frac{f(\chi)\sin\chi}{1+\beta\cos\chi}:$$

where $f(\chi)$ is a linear combination of the small displacements $\lambda, \mu, \nu, \lambda', \mu', \nu'$, and the first differential coefficients of the last three, or they reduce to a term of the type $\beta f(\chi)\cos\chi/(1+\beta\cos\chi)$, or lastly they reduce to one or other of the following four combinations:

$$\frac{f}{\sin^3\chi(1+\beta\cos\chi)} + \frac{\partial}{\partial\psi}\left\{\frac{f\beta}{\sin^2\chi(1+\beta\cos\chi)}\right\} \qquad\text{.............}(a),$$

$$\frac{f\cos\chi}{\sin^2\chi(1+\beta\cos\chi)} + \frac{\partial}{\partial\psi}\left\{\frac{f\beta\cos\chi}{\sin\chi(1+\beta\cos\chi)}\right\} \qquad\text{.............}(b)$$

$$\frac{\partial}{\partial\psi}\left\{\frac{f\beta\sin\chi}{1+\beta\cos\chi}\right\} + \frac{\partial^2}{\partial\psi^2}\left\{\frac{f\beta^2\sin^2\chi}{1+\beta\cos\chi}\right\} \qquad\text{.............}(c),$$

$$\frac{\partial}{\partial\psi}\left\{\frac{f\beta\cos\chi}{1+\beta\cos\chi}\right\} + \frac{\partial^2}{\partial\psi^2}\left\{\frac{f\beta^2\sin\chi\cos\chi}{1+\beta\cos\chi}\right\} \qquad\text{.............}(d).$$

The middle type at once reduces to the second standard type, for it may be written in the form $f - f/(1+\beta\cos\chi)$; the last four are reduced by means of the lemmas A, B, C and D, which may be verified by differentiating out the brackets, remembering that $\frac{\partial}{\partial\psi} = \frac{1}{1+\beta\cos\chi}\frac{\partial}{\partial\chi}$. They are

Lemma A.

$$\frac{f}{\sin^3\chi\,(1+\beta\cos\chi)} + \frac{\partial}{\partial\psi}\left\{\frac{f\beta}{\sin^2\chi\,(1+\beta\cos\chi)}\right\}$$
$$= \frac{f+f''}{2\sin\chi\,(1+\beta\cos\chi)} - \frac{\partial}{\partial\psi}\left\{\frac{f'}{\sin\chi\,(1+\beta\cos\chi)} - \frac{f\beta}{2\,(1+\beta\cos\chi)}\right\}$$
$$+ \frac{\partial^2}{\partial\psi^2}\left\{\frac{f(1-\beta^2)}{2\sin\chi\,(1+\beta\cos\chi)} + \frac{f\beta^2\sin\chi}{2\,(1+\beta\cos\chi)}\right\}.$$

Lemma B.

$$\frac{f\cos\chi}{\sin^2\chi\,(1+\beta\cos\chi)} + \frac{\partial}{\partial\psi}\left\{\frac{f\beta\cos\chi}{\sin\chi\,(1+\beta\cos\chi)}\right\}$$
$$= \frac{f'}{\sin\chi\,(1+\beta\cos\chi)} - \frac{\partial}{\partial\psi}\left\{\frac{f}{\sin\chi\,(1+\beta\cos\chi)}\right\}.$$

Lemma C.

$$\frac{\partial}{\partial\psi}\left\{\frac{f\beta\sin\chi}{1+\beta\cos\chi}\right\} + \frac{\partial^2}{\partial\psi^2}\left\{\frac{f\beta^2\sin^2\chi}{1+\beta\cos\chi}\right\}$$
$$= -\frac{f''}{1+\beta\cos\chi} + \frac{\partial}{\partial\psi}\left\{\frac{2f\beta\sin\chi}{1+\beta\cos\chi} + \frac{2f'}{1+\beta\cos\chi}\right\} - \frac{\partial^2}{\partial\psi^2}\left\{\frac{f(1-\beta^2)}{1+\beta\cos\chi}\right\}.$$

Lemma D.

$$\frac{\partial}{\partial\psi}\left\{\frac{f\beta\cos\chi}{1+\beta\cos\chi}\right\} + \frac{\partial^2}{\partial\psi^2}\left\{\frac{f\beta^2\sin\chi\cos\chi}{1+\beta\cos\chi}\right\}$$
$$= \frac{2f'}{1+\beta\cos\chi} + \frac{\partial}{\partial\psi}\left\{\frac{f'\beta\sin\chi}{1+\beta\cos\chi} - \frac{2f}{1+\beta\cos\chi}\right\} - \frac{\partial^2}{\partial\psi^2}\left\{\frac{f\beta\sin\chi}{1+\beta\cos\chi}\right\}.$$

It is important to notice that all the differentiations with respect to ψ occurring in these lemmas, as well as in the equations (259), § 173, represent the partial differentiations with respect to t, which occur in the equations (198) and (199), § 135. During these differentiations the fieldpoint is kept unchanged, so that x, y, z, ρ, ϕ, δx, δy, δz, and therefore also λ, μ and ν, are to be treated as invariable, whilst λ', μ' and ν', which are functions of τ, and therefore of χ, are to be varied. Further, since $\chi = \frac{1}{2}(\omega\tau + \delta - \phi)$, such functions as f', f'', where f stands for λ', or μ', or ν', are equal to $2\dot{f}/\omega$, $4\ddot{f}/\omega^2$, the dots as usual denoting differentiations with respect to the time τ.

176. Substituting the expressions for the numerators in (259), § 173, reducing where necessary to the standard types of terms in the manner indicated, and collecting corresponding terms together, we find the following expressions for the components of the electric and magnetic forces:

$$\left.\begin{aligned}
\delta d_\phi &= \frac{2\ddot{\lambda}' - \omega\dot{\mu}' + \omega^2\lambda}{4\omega^2\rho^2 \sin\chi\,(1+\beta\cos\chi)} \\
&\quad + \frac{\partial}{\partial\psi}\left\{\frac{-4\dot{\lambda}' + (1+\beta^2)\,\omega\mu' + (3-\beta^2)\,\omega\mu}{8\omega\rho^2\sin\chi\,(1+\beta\cos\chi)} + \frac{\beta\dot{\mu}' - \beta\omega\lambda}{4\omega\rho^2(1+\beta\cos\chi)} + \frac{\beta^2\sin\chi\mu}{2\rho^2(1+\beta\cos\chi)}\right\} \\
&\quad - \frac{\partial^2}{\partial\psi^2}\left\{\frac{(1-\beta^2)(\lambda-\lambda')}{8\rho^2\sin\chi\,(1+\beta\cos\chi)} + \frac{\beta(1-\beta^2)(\mu+\mu')}{8\rho^2(1+\beta\cos\chi)} + \frac{\beta^2\sin\chi\,(\lambda-\lambda')}{4\rho^2(1+\beta\cos\chi)}\right\} \\
\delta d_\rho &= -\frac{4\ddot{\mu}' - 4\omega\dot{\lambda}' + \omega^2(3\mu' + 5\mu)}{16\omega^2\rho^2\sin\chi\,(1+\beta\cos\chi)} \\
&\quad + \frac{\partial}{\partial\psi}\left\{\frac{2(1-\beta^2)\,\dot{\mu}' - \omega\lambda' + (3-2\beta^2)\,\omega\lambda}{8\omega\rho^2\sin\chi\,(1+\beta\cos\chi)} + \frac{-4\beta\dot{\lambda}' + 3\beta\omega(\mu-\mu')}{16\omega\rho^2(1+\beta\cos\chi)} + \frac{\beta^2\sin\chi\lambda}{2\rho^2(1+\beta\cos\chi)}\right\} \\
&\quad + \frac{\partial^2}{\partial\psi^2}\left\{\frac{(1-\beta^2)(\mu-\mu')}{16\rho^2\sin\chi\,(1+\beta\cos\chi)} - \frac{\beta(\lambda-\lambda')}{8\rho^2(1+\beta\cos\chi)} + \frac{\beta^2\sin\chi\,(\mu'+3\mu)}{16\rho^2(1+\beta\cos\chi)}\right\} \\
\delta d_z &= \frac{\omega^2(\nu-\nu') - 4\ddot{\nu}'}{16\omega^2\rho^2\sin\chi\,(1+\beta\cos\chi)} \\
&\quad + \frac{\partial}{\partial\psi}\left\{\frac{(1-\beta^2)\,\dot{\nu}'}{4\omega\rho^2\sin\chi\,(1+\beta\cos\chi)} + \frac{\beta(\nu-\nu')}{16\rho^2(1+\beta\cos\chi)}\right\} \\
&\quad + \frac{\partial^2}{\partial\psi^2}\left\{\frac{(1-\beta^2)(\nu-\nu')}{16\rho^2\sin\chi\,(1+\beta\cos\chi)} + \frac{\beta^2\sin\chi\,(\nu-\nu')}{16\rho^2(1+\beta\cos\chi)}\right\} \\
\delta h_\phi &= -\frac{\beta\dot{\nu}'}{4\omega\rho^2\sin\chi\,(1+\beta\cos\chi)} - \frac{\partial}{\partial\psi}\left\{\frac{\beta(\nu-\nu')}{4\rho^2\sin\chi\,(1+\beta\cos\chi)} - \frac{\beta^2\dot{\nu}'}{4\omega\rho^2(1+\beta\cos\chi)}\right\} \\
\delta h_\rho &= \frac{\beta(3\omega^2\nu - 3\omega^2\nu' - 4\ddot{\nu}')}{16\omega^2\rho^2\sin\chi\,(1+\beta\cos\chi)} + \frac{\partial}{\partial\psi}\left\{\frac{3\beta^2(\nu-\nu')}{16\rho^2(1+\beta\cos\chi)}\right\} \\
&\quad - \frac{\partial^2}{\partial\psi^2}\left\{\frac{\beta(1-\beta^2)(\nu-\nu')}{16\rho^2\sin\chi\,(1+\beta\cos\chi)} + \frac{\beta^3\sin\chi\,(\nu-\nu')}{16\rho^2(1+\beta\cos\chi)}\right\} \\
\delta h_z &= \frac{\beta(4\ddot{\mu}' + 3\omega^2\mu + \omega^2\mu')}{16\omega^2\rho^2\sin\chi\,(1+\beta\cos\chi)} - \frac{\partial}{\partial\psi}\left\{\frac{\beta(\lambda-\lambda')}{8\rho^2\sin\chi\,(1+\beta\cos\chi)} - \frac{\beta^2(3\mu+\mu')}{16\rho^2(1+\beta\cos\chi)}\right\} \\
&\quad - \frac{\partial^2}{\partial\psi^2}\left\{\frac{\beta(1-\beta^2)(\mu-\mu')}{16\rho^2\sin\chi\,(1+\beta\cos\chi)} - \frac{\beta^2(\lambda-\lambda')}{8\rho^2(1+\beta\cos\chi)} + \frac{\beta^3\sin\chi\,(3\mu+\mu')}{16\rho^2(1+\beta\cos\chi)}\right\}
\end{aligned}\right\} \quad \ldots\ldots(260).$$

The parts of these expressions which involve the quantities λ, μ, ν can be developed in Fourier Series by the methods used in § 137, for λ, μ, ν are to be treated as independent of χ and ψ; but the development of the remaining terms, which involve λ', μ' and ν', necessarily requires a knowledge of these quantities as functions of the time τ, or what comes to the same thing, as functions of χ. The case which is of importance for a study of the stability of rings of revolving electrons is that in which λ', μ' and ν' are expressible as sums of damped simple harmonic vibrations, each set representing a wave of disturbance, forced or free, travelling round a ring of electrons. What is wanted is the mechanical force on any one electron of the ring due to the rest. Since we are considering only terms of the first order in the disturbance, this force is the sum of the forces due to each wave separately, and there is no loss of generality in supposing the disturbance due to a single wave. Hence we are led to consider the following problem.

177. Problem 2. A group of n electrons in uniform circular motion is disturbed by a wave travelling round the circle. To find the disturbing force on any one electron due to the rest. As in § 98 and § 136 we take the electron under consideration to be the 0th, its azimuth in steady motion to be given by $\phi = \omega t + \delta$, and its disturbance from steady motion to be given by $(\lambda, \mu, \nu) = (\mathscr{A}, \mathscr{B}, \mathscr{C})$ exp. ιpt, real parts as usual being taken. The quantity p is complex of the form $q + \iota\kappa$, so that q is the frequency relative to the revolving ring, and κ is the damping.

We take another electron to be the ith, and assume its azimuth to be $\omega\tau + \delta + 2\pi i/n$, and its disturbance to be given by

$$(\lambda', \mu', \nu') = (\mathscr{A}, \mathscr{B}, \mathscr{C})\, \epsilon^{\iota(p\tau - k2\pi i/n)} \quad \text{..............(261)},$$

where k is an integer, which we may without loss of generality suppose to lie between $\pm \frac{1}{2}n$. The expression gives $(\lambda', \mu', \nu') = (\lambda, \mu, \nu)$ when $i = 0$, as it should do; it shows that (λ', μ', ν') at time $t + k2\pi i/nq$ assume the values which (λ, μ, ν) had, or will have, at the time t, according as k is positive or negative, except for the presence of a damping factor. Hence (261) represents a damped wave of disturbance, of which the phase travels round the rotating ring from the 0th to the ith electron in the time $k2\pi i/nq$. Thus the angular velocity of the wave relative to the rotating ring is q/k; relative to a stationary observer however it is $(q + k\omega)/k$, the velocity in each case being reckoned as positive in the direction of rotation of the ring. Further, (λ', μ', ν') vanish for k azimuths θ, $\theta + 2\pi/k$, $\theta + 4\pi/k$, ... $\theta + (k-1)\, 2\pi/k$, where θ is some angle between 0 and $2\pi/k$, and have maxima for the halfway azimuths; thus the wave has k nodes and k loops travelling round the ring and affecting the electrons in turn. An observer rotating with the ring notices q/k times k, that is q nodes passing per second, a stationary observer $(q + k\omega)/k$ times k, that is $q + k\omega$ nodes; so that the frequency relative to the ring is q, but that relative to a stationary observer is $q + k\omega$. This distinction between relative and absolute frequencies is fundamental in the theory of the forced and free vibrations of a rotating ring and of the waves radiated from it.

178. The angles ψ and χ have been defined by the equations

$$\psi = \tfrac{1}{2}(\omega t + \delta - \phi), \quad \chi = \tfrac{1}{2}(\omega\tau + \delta - \phi),$$

and the characteristic equation consequently reduces to

$$\psi = \chi + \beta \sin \chi \quad \text{.........................(262)}.$$

Writing for the sake of brevity

$$\sigma = 2p/\omega \quad \text{..............................(263)},$$

we get from (261)

$$(\lambda', \mu', \nu') = (\mathscr{A}, \mathscr{B}, \mathscr{C})\, \epsilon^{\iota\{\frac{1}{2}\sigma(\phi-\delta)+\sigma\psi-\sigma\beta\sin\chi-k2\pi i/n\}} \quad \text{........(264)}.$$

This may also be written in the form

$$(\lambda', \mu', \nu') = (\lambda, \mu, \nu)\, \epsilon^{-\iota(\sigma\beta\sin\chi+k2\pi i/n)} \quad \text{..............(265)}.$$

The second form shows that we may pass from (λ', μ', ν') to (λ, μ, ν) by putting $\sigma = 0$ and $k = 0$; but when we wish to differentiate a function of (λ', μ', ν') partially with respect to t, the coordinates of the fieldpoint remaining constant, we must employ form (264), in which ϕ is to be treated as constant while ψ and χ are variable. Now in equations (260), § 176, the differentiations with respect to ψ represent partial differentiations with respect to t, where λ', μ', ν' and χ must all be varied, while ϕ and λ, μ, ν, where expressed explicitly, must be treated as constant, because ϕ and λ, μ, ν, wherever they occur explicitly, represent the fixed coordinates of the fieldpoint, which happens to coincide with the position of the 0th electron at time t. After these differentiations have been performed we may replace (λ', μ', ν') by their expressions (265).

179. Now (264) shows that (λ', μ', ν') are of the form $\epsilon^{\iota\sigma\psi} f(\chi)$; hence by the usual theory of symbolic operators we may write

$$\frac{\partial^q}{\partial\psi^q}(\lambda', \mu', \nu') = \epsilon^{\iota\sigma\psi}\left(\frac{\partial}{\partial\psi} + \iota\sigma\right)^q \{\epsilon^{-\iota\sigma\psi}(\lambda', \mu', \nu')\},$$

and using the second form (265), we get, for any function $F(\chi)$,

$$\frac{\partial^q}{\partial\psi^q}\{F(\chi)(\lambda', \mu', \nu')\} = (\lambda, \mu, \nu)\left(\frac{\partial}{\partial\psi} + \iota\sigma\right)^q \{\epsilon^{-\iota(\sigma\beta\sin\chi + k2\pi i/n)} F(\chi)\} \quad \text{......(266)},$$

which is a very useful formula of transformation.

It may be remarked that λ is a supernumerary coordinate introduced merely for convenience. We have in fact written $\phi + \lambda$ for the azimuth of the 0th electron in its disturbed position, where the single letter ϕ would have sufficed. By doing this we secure the advantage that the angle ψ, which is one half the difference in azimuth of the 0th and ith electrons at time t in the undisturbed motion, may be put equal to $\pi i/n$ without any further correction in the disturbed motion, for the change in ψ is completely represented by $\frac{1}{2}(\lambda' - \lambda)$.

180. The mechanical force. The substitution of the values of (λ', μ', ν') in (260), § 176, and the subsequent development involve some complicated analysis. Since the mechanical force on the 0th electron is all that is of interest, half the labour will be saved by using the expressions for its three components in place of the six equations (260).

The mechanical force is given generally by

$$\mathbf{f} = \mathbf{d} + [\mathbf{v}\,.\,\mathbf{h}]/c \quad \text{.............................(VI)},$$

whence for a small disturbance,

$$\delta\mathbf{f} = \delta\mathbf{d} + [\mathbf{v}\,.\,\delta\mathbf{h}]/c + [\delta\mathbf{v}\,.\,\mathbf{h}]/c.$$

In our problem the directions (ϕ, ρ, z) form a right-handed system to which this formula is directly applicable. The velocity components are

$$v_\phi/c = \beta,\ v_\rho/c = 0,\ v_z/c = 0,$$

$$\delta v_\phi/c = \beta(\dot{\lambda} - \omega\mu)/\omega,\ \delta v_\rho/c = \beta(\dot{\mu} + \omega\lambda)/\omega,\ \delta v_z/c = \beta\dot{\nu}/\omega.$$

The components of $\mathbf{h}$ are $(0, 0, h_z)$, the last being given by (201), § 136, or (203), § 138; those of $\delta\mathbf{d}$ and $\delta\mathbf{h}$ are given by (260), § 176. Hence we get

$$\delta f_\phi = \delta d_\phi + \beta(\dot{\mu} + \omega\lambda)\,h_z/\omega,$$
$$\delta f_\rho = \delta d_\rho - \beta\delta h_z - \beta(\dot{\lambda} - \omega\mu)\,h_z/\omega,$$
$$\delta f_z = \delta d_z + \beta\delta h_\rho.$$

181. A further simplification is possible; instead of resolving the force in the directions (ϕ, ρ, z), which correspond to the undisturbed position of the 0th electron, we may resolve along the tangents to the circle and the perpendicular radius through the electron in its disturbed position. The components f_ϕ and f_ρ of the mechanical force in the steady motion now contribute first order terms in the new tangential and radial directions, namely λf_ρ to the new tangential, and $-\lambda f_\phi$ to the new radial force. Indicating the components in the new system by suffixes λ, μ and ν, ν being parallel to z, we get

$$\delta f_\lambda = \delta d_\phi + \lambda(f_\rho + \beta h_z) + \beta\dot{\mu}h_z/\omega,$$
$$\delta f_\mu = \delta d_\rho - \beta\delta h_z - \lambda f_\phi + \beta h_z\mu - \beta\dot{\lambda}h_z/\omega,$$
$$\delta f_\nu = \delta d_z + \beta\delta h_\rho.$$

182. We notice that $f_\rho + \beta h_z = d_\rho$, and $f_\phi = d_\phi$; these, as well as h_z, are given by (201), § 136, while δd_ϕ, δd_ρ, δd_z, δh_ρ and δh_z are given by (260), § 176.

Further, we have $\dot{\lambda} = \iota p\lambda = \frac{1}{2}\iota\omega\sigma\lambda$, ... and $\ddot{\lambda}' = -p^2\lambda' = -\frac{1}{4}\omega^2\sigma^2\lambda'$, ... With these results we use (266), § 179, and write for shortness

$$\Theta = \sigma\beta\sin\chi + k2\pi i/n \quad \text{............................(267)}.$$

Then we get the following expressions:

$$\delta f_\lambda = \frac{\lambda}{\rho^2}\left[\left(\frac{\partial}{\partial\psi}+\iota\sigma\right)^2\left\{\frac{\beta^2\sin\chi\epsilon^{-\iota\Theta}}{4(1+\beta\cos\chi)}-\frac{\beta^2\epsilon^{-\iota\Theta}}{8\sin\chi(1+\beta\cos\chi)}\right\}\right.$$
$$\left.-\frac{\partial^2}{\partial\psi^2}\left\{\frac{\beta^2\sin\chi}{4(1+\beta\cos\chi)}+\frac{1-\beta^2-\epsilon^{-\iota\Theta}}{8\sin\chi(1+\beta\cos\chi)}\right\}\right]$$
$$+\frac{\mu}{\rho^2}\left[\tfrac{1}{2}\iota\sigma\beta\rho^2h_z+\rho^2 f_\phi+\left(\frac{\partial}{\partial\psi}+\iota\sigma\right)^2\left\{\frac{\beta^3\epsilon^{-\iota\Theta}}{8(1+\beta\cos\chi)}\right\}\right.$$
$$-\frac{\partial}{\partial\psi}\left(\frac{\partial}{\partial\psi}+\iota\sigma\right)\left\{\frac{\beta\epsilon^{-\iota\Theta}}{8(1+\beta\cos\chi)}\right\}-\frac{\partial^2}{\partial\psi^2}\left\{\frac{\beta(1-\beta^2)}{8(1+\beta\cos\chi)}\right\}$$
$$+\left(\frac{\partial}{\partial\psi}+\iota\sigma\right)\left\{\frac{\beta^2\epsilon^{-\iota\Theta}}{8\sin\chi(1+\beta\cos\chi)}\right\}+\frac{\partial}{\partial\psi}\left\{\frac{\epsilon^{-\iota\Theta}}{8\sin\chi(1+\beta\cos\chi)}\right\}$$
$$\left.+\frac{\partial}{\partial\psi}\left\{\frac{1+\beta^2}{8\sin\chi(1+\beta\cos\chi)}\right\}\right]$$

$$\delta f_\mu = \frac{\lambda}{\rho^2}\left[-\tfrac{1}{2}\iota\sigma\beta\rho^2h_z-\left(\frac{\partial}{\partial\psi}+\iota\sigma\right)^2\left\{\frac{\beta^3\epsilon^{-\iota\Theta}}{8(1+\beta\cos\chi)}\right\}\right.$$
$$+\frac{\partial}{\partial\psi}\left(\frac{\partial}{\partial\psi}+\iota\sigma\right)\left\{\frac{\beta\epsilon^{-\iota\Theta}}{8(1+\beta\cos\chi)}\right\}-\frac{\partial^2}{\partial\psi^2}\left\{\frac{\beta(1-\beta^2)}{8(1+\beta\cos\chi)}\right\}$$
$$-\left(\frac{\partial}{\partial\psi}+\iota\sigma\right)\left\{\frac{\beta^2\epsilon^{-\iota\Theta}}{8\sin\chi(1+\beta\cos\chi)}\right\}-\frac{\partial}{\partial\psi}\left\{\frac{\epsilon^{-\iota\Theta}}{8\sin\chi(1+\beta\cos\chi)}\right\}$$
$$\left.+\frac{\partial}{\partial\psi}\left\{\frac{1+\beta^2}{8\sin\chi(1+\beta\cos\chi)}\right\}\right]$$
$$+\frac{\mu}{\rho^2}\left[-\frac{1}{2\sin\chi(1+\beta\cos\chi)}+\frac{\partial^2}{\partial\psi^2}\left\{\frac{\beta^2(1-\beta^2)\sin\chi}{4(1+\beta\cos\chi)}\right\}\right.$$
$$+\left(\frac{\partial}{\partial\psi}+\iota\sigma\right)^2\left\{\frac{\beta^2(1-\beta^2)\sin\chi\epsilon^{-\iota\Theta}}{16(1+\beta\cos\chi)}-\frac{\beta^2(1+\beta^2)\epsilon^{-\iota\Theta}}{16\sin\chi(1+\beta\cos\chi)}\right\}$$
$$-\left(\frac{\partial}{\partial\psi}+\iota\sigma\right)\left\{\frac{\beta(3+\beta^2)\epsilon^{-\iota\Theta}}{16(1+\beta\cos\chi)}\right\}+\frac{\partial}{\partial\psi}\left(\frac{\partial}{\partial\psi}+\iota\sigma\right)\left\{\frac{\beta^2\epsilon^{-\iota\Theta}}{4\sin\chi(1+\beta\cos\chi)}\right\}$$
$$+\frac{\partial^2}{\partial\psi^2}\left\{\frac{(1-\beta^2)^2-(1+\beta^2)\epsilon^{-\iota\Theta}}{16\sin\chi(1+\beta\cos\chi)}-\frac{\beta^2(1-\beta^2)\sin\chi}{16(1+\beta\cos\chi)}\right\}$$
$$\left.+\frac{\partial}{\partial\psi}\left\{\frac{\beta(3+\beta^2)}{16(1+\beta\cos\chi)}\right\}+\frac{(3+\beta^2)(1-\epsilon^{-\iota\Theta})}{16\sin\chi(1+\beta\cos\chi)}\right]$$

$$\delta f_\nu = \frac{\nu}{\rho^2}\left[-\left(\frac{\partial}{\partial\psi}+\iota\sigma\right)^2\left\{\frac{\beta^2(1-\beta^2)\sin\chi\epsilon^{-\iota\Theta}}{16(1+\beta\cos\chi)}+\frac{\beta^2(1+\beta^2)\epsilon^{-\iota\Theta}}{16\sin\chi(1+\beta\cos\chi)}\right\}\right.$$
$$-\frac{\partial}{\partial\psi}\left\{\frac{\beta(1+3\beta^2)\epsilon^{-\iota\Theta}}{16(1+\beta\cos\chi)}\right\}+\frac{\partial}{\partial\psi}\left(\frac{\partial}{\partial\psi}+\iota\sigma\right)\left\{\frac{\beta^2\epsilon^{-\iota\Theta}}{4\sin\chi(1+\beta\cos\chi)}\right\}$$
$$+\frac{\partial^2}{\partial\psi^2}\left\{\frac{(1-\beta^2)^2-(1+\beta^2)\epsilon^{-\iota\Theta}}{16\sin\chi(1+\beta\cos\chi)}+\frac{\beta^2(1-\beta^2)\sin\chi}{16(1+\beta\cos\chi)}\right\}$$
$$\left.+\frac{\partial}{\partial\psi}\left\{\frac{\beta(1+3\beta^2)}{16(1+\beta\cos\chi)}\right\}+\frac{(1+3\beta^2)(1-\epsilon^{-\iota\Theta})}{16\sin\chi(1+\beta\cos\chi)}\right]$$

......(268).

When $\sigma = 0$ and $k = 0$ we have $\Theta = 0$ also, and when $\sigma = 0$ and $\Theta = 0$ these expressions become very greatly simplified, showing that most of the terms which do not involve the exponential $\epsilon^{-\iota\Theta}$ can be derived from those which involve it by means of this substitution, a fact which will be of use later.

183. Expansion in series. The first stage in the process of reduction is to expand the functions of χ on the right-hand sides of (268) in Fourier series proceeding according to exponentials of integral multiples of $\iota\psi$. For this purpose we notice that the terms involving h_z and f_ϕ are already known from equations (203), § 138, because $f_\phi = d_\phi$; or we may use (243), § 157. Further we find by means of the results used in the problem of § 136:

$$-\frac{1}{2\sin\chi(1+\beta\cos\chi)} + \frac{\partial^2}{\partial\psi^2}\left\{\frac{\beta^2(1-\beta^2)\sin\chi}{4(1+\beta\cos\chi)}\right\}$$

$$= -\frac{1}{2\sin\psi} + \sum_{j=1}^{j=\infty}(-1)^j \sin j\psi \left[\tfrac{1}{2}\beta^2(1-\beta^2)\, j^2 J_j'(j\beta) - j\int_0^\beta J_j(jx)\,dx\right] \quad \text{......(269).}$$

This equation gives the expansion of the first line in the coefficient of μ in the equation for δf_μ.

All the remaining terms are of the kind just mentioned, which can be determined when the terms involving Θ are known, so that the latter alone remain to be considered. Since we have $\Theta = \sigma\beta\sin\chi + k2\pi i/n$, by (267), § 182, they depend on the following three functions of χ:

$$\frac{\epsilon^{-\iota\sigma\beta\sin\chi}}{\sin\chi(1+\beta\cos\chi)}\ (a), \qquad \frac{\epsilon^{-\iota\sigma\beta\sin\chi}}{1+\beta\cos\chi}\ (b), \qquad \frac{\sin\chi\,\epsilon^{-\iota\sigma\beta\sin\chi}}{1+\beta\cos\chi}\ (c).$$

Of these (*b*) can be derived from (*a*) by differentiating with respect to the combination $\sigma\beta$, leaving σ and β unaltered wherever they occur separately, and multiplying the result by ι; in the same way we can derive (*c*) from (*b*), so that we need only expand (*a*).

Now (*a*) is infinite at both limits, when $\chi = 0$ and $\psi = 0$, as well as when $\chi = \pi$ and $\psi = \pi$; but $\dfrac{\epsilon^{-\iota\sigma\beta\sin\chi}}{\sin\chi(1+\beta\cos\chi)} - \dfrac{1}{\sin\psi}$ vanishes at both limits and can be expanded in a series of exponentials, differentiable both with respect to $\sigma\beta$ and to ψ. Write

$$\frac{\epsilon^{-\iota\sigma\beta\sin\chi}}{\sin\chi(1+\beta\cos\chi)} - \frac{1}{\sin\psi} = \sum_{j=-\infty}^{j=\infty} A_j \epsilon^{\iota j\psi}.$$

We get in the usual way, since $\psi = \chi + \beta\sin\chi$,

$$A_j = \frac{1}{\pi}\int_0^\pi \left\{\frac{\epsilon^{-\iota\sigma\beta\sin\chi}}{\sin\chi(1+\beta\cos\chi)} - \frac{1}{\sin\psi}\right\}\epsilon^{-\iota j\psi}\,d\psi = -\iota(-1)^j . J_j^{-1}\{(\sigma+j)\beta\}.$$

$J_j\{(\sigma+j)\beta\}$ as usual denotes the Bessel Function of the first kind, and the index -1 means that it is to be integrated once between the limits 0 and $(\sigma+j)\beta$. The functions (*b*) and (*c*) can be derived by differentiating the Bessel Function with respect to its argument and multiplying by ι, once or twice as the case may be. Multiplying by $\epsilon^{-\iota k2\pi i/n}$ we get the following expansions:

$$\left.\begin{aligned}
&(a)\quad \frac{\epsilon^{-\iota\Theta}}{\sin\chi(1+\beta\cos\chi)}=\frac{\epsilon^{-\iota k2\pi i/n}}{\sin\psi}-\iota\sum_{j=-\infty}^{j=\infty}(-1)^j\epsilon^{\iota(j\psi-k2\pi i/n)}J_j^{-1}\{(\sigma+j)\beta\}\\
&(b)\quad \frac{\epsilon^{-\iota\Theta}}{1+\beta\cos\chi}=\sum_{j=-\infty}^{j=\infty}(-1)^j\epsilon^{\iota(j\psi-k2\pi i/n)}J_j\{(\sigma+j)\beta\}\\
&(c)\quad \frac{\sin\chi\,\epsilon^{-\iota\Theta}}{1+\beta\cos\chi}=\iota\sum_{j=-\infty}^{j=\infty}(-1)^j\epsilon^{\iota(j\psi-k2\pi i/n)}J_j'\{(\sigma+j)\beta\}
\end{aligned}\right\} \quad\ldots\ldots(270).$$

When we put $\sigma=0$ and $\Theta=0$ we get the expansions already used in the problem of § 136.

184. Summation of the series. The second stage is to substitute for ψ its value in steady motion, $\pi i/n$, which is allowable since we have introduced the auxiliary quantity λ to represent the displacement from steady motion, and to sum for i from 1 to $n-1$. It is of course necessary first to develope all the differential coefficients with respect to ψ, which obviously offers no difficulty in the case of the exponential series, for we need only replace the operator $\frac{\partial}{\partial\psi}$ by j in the usual way. But the terms derived from the term (*a*), (270), § 183, which involves $1/\sin\psi$, require separate treatment.

An inspection of (268), § 182, shows that there are four functions of Θ to be considered, which we shall denote by Θ_1, Θ_2, Θ_3 and Θ_4 in order. These are:

In the coefficient of λ in δf_λ,

$$\Theta_1=\left(\frac{\partial}{\partial\psi}+\iota\sigma\right)^2\left\{\frac{\beta^2\sin\chi\,\epsilon^{-\iota\Theta}}{4(1+\beta\cos\chi)}-\frac{\beta^2\epsilon^{-\iota\Theta}}{8\sin\chi(1+\beta\cos\chi)}\right\}+\frac{\partial^2}{\partial\psi^2}\left\{\frac{\epsilon^{-\iota\Theta}}{8\sin\chi(1+\beta\cos\chi)}\right\}\quad\ldots\ldots(271).$$

In the coefficient of μ in δf_λ and that of λ in δf_μ,

$$\Theta_2=\left(\frac{\partial}{\partial\psi}+\iota\sigma\right)^2\left\{\frac{\beta^3\epsilon^{-\iota\Theta}}{8(1+\beta\cos\chi)}\right\}-\frac{\partial}{\partial\psi}\left(\frac{\partial}{\partial\psi}+\iota\sigma\right)\left\{\frac{\beta\epsilon^{-\iota\Theta}}{8(1+\beta\cos\chi)}\right\}+\left(\frac{\partial}{\partial\psi}+\iota\sigma\right)\left\{\frac{\beta^2\epsilon^{-\iota\Theta}}{8\sin\chi(1+\beta\cos\chi)}\right\}+\frac{\partial}{\partial\psi}\left\{\frac{\epsilon^{-\iota\Theta}}{8\sin\chi(1+\beta\cos\chi)}\right\}\quad(272).$$

In the coefficient of μ in δf_μ,

$$\Theta_3 = \left(\frac{\partial}{\partial\psi} + \iota\sigma\right)^2 \left\{\frac{\beta^2(1-\beta^2)\sin\chi\epsilon^{-\iota\Theta}}{16(1+\beta\cos\chi)} - \frac{\beta^2(1+\beta^2)\epsilon^{-\iota\Theta}}{16\sin\chi(1+\beta\cos\chi)}\right\}$$
$$- \left(\frac{\partial}{\partial\psi} + \iota\sigma\right)\left\{\frac{\beta(3+\beta^2)\epsilon^{-\iota\Theta}}{16(1+\beta\cos\chi)}\right\} + \frac{\partial}{\partial\psi}\left(\frac{\partial}{\partial\psi} + \iota\sigma\right)\left\{\frac{\beta^2\epsilon^{-\iota\Theta}}{4\sin\chi(1+\beta\cos\chi)}\right\}$$
$$- \frac{\partial^2}{\partial\psi^2}\left\{\frac{(1+\beta^2)\epsilon^{-\iota\Theta}}{16\sin\chi(1+\beta\cos\chi)}\right\} - \frac{(3+\beta^2)\epsilon^{-\iota\Theta}}{16\sin\chi(1+\beta\cos\chi)} \quad \ldots\ldots\ldots\ldots(273).$$

In the coefficient of ν in δf_ν,

$$\Theta_4 = -\left(\frac{\partial}{\partial\psi} + \iota\sigma\right)^2 \left\{\frac{\beta^2(1-\beta^2)\sin\chi\epsilon^{-\iota\Theta}}{16(1+\beta\cos\chi)} + \frac{\beta^2(1+\beta^2)\epsilon^{-\iota\Theta}}{16\sin\chi(1+\beta\cos\chi)}\right\}$$
$$- \left(\frac{\partial}{\partial\psi} + \iota\sigma\right)\left\{\frac{\beta(1+3\beta^2)\epsilon^{-\iota\Theta}}{16(1+\beta\cos\chi)}\right\} + \frac{\partial}{\partial\psi}\left(\frac{\partial}{\partial\psi} + \iota\sigma\right)\left\{\frac{\beta^2\epsilon^{-\iota\Theta}}{4\sin\chi(1+\beta\cos\chi)}\right\}$$
$$- \frac{\partial^2}{\partial\psi^2}\left\{\frac{(1+\beta^2)\epsilon^{-\iota\Theta}}{16\sin\chi(1+\beta\cos\chi)}\right\} - \frac{(1+3\beta^2)\epsilon^{-\iota\Theta}}{16\sin\chi(1+\beta\cos\chi)} \quad \ldots\ldots\ldots\ldots(274).$$

We shall denote by $\Theta_{10}, \ldots$ the functions got by putting $\sigma = 0$ and $\Theta = 0$ in $\Theta_1, \ldots$, which are clearly derived from the parts of the electric and magnetic forces involving (λ, μ, ν). Then the coefficient of

λ in δf_λ involves $\Theta_1 - \Theta_{10}$,
μ in δf_λ involves $\Theta_2 + \Theta_{20}$,
λ in δf_μ involves $-\Theta_2 + \Theta_{20}$,
μ in δf_μ involves $\Theta_3 - \Theta_{30}$,
ν in δf_ν involves $\Theta_4 - \Theta_{40}$.

We shall consider the Bessel Function parts of these functions first, and then the parts derived from $1/\sin\psi$.

185. Bessel Function series. Using (271)—(274), we easily find for the Bessel Function parts of the functions Θ:

$$\left.\begin{aligned}
\Theta_1 &= -\iota \sum_{j=-\infty}^{j=\infty} (-1)^j \epsilon^{\iota(j\psi - k2\pi i/n)} \,.\, a\{j, (\sigma+j)\beta\} \\
\Theta_2 &= \sum_{j=-\infty}^{j=\infty} (-1)^j \epsilon^{\iota(j\psi - k2\pi i/n)} \,.\, b\{j, (\sigma+j)\beta\} \\
\Theta_3 &= -\iota \sum_{j=-\infty}^{j=\infty} (-1)^j \epsilon^{\iota(j\psi - k2\pi i/n)} \,.\, c\{j, (\sigma+j)\beta\} \\
\Theta_4 &= -\iota \sum_{j=-\infty}^{j=\infty} (-1)^j \epsilon^{\iota(j\psi - k2\pi i/n)} \,.\, d\{j, (\sigma+j)\beta\}
\end{aligned}\right\} \quad \ldots\ldots(275),$$

where a, b, c, d are four generalized Bessel Functions defined by the equations

$$\left.\begin{aligned}
a\{m, l\beta\} &= \tfrac{1}{4}l^2\beta^2 J_m'(l\beta) - \tfrac{1}{8}(m^2 - l^2\beta^2) J_m^{-1}(l\beta) \\
b\{m, l\beta\} &= \tfrac{1}{8}(m - l\beta^2) l\beta J_m(l\beta) + \tfrac{1}{8}(m + l\beta^2) J_m^{-1}(l\beta) \\
c\{m, l\beta\} &= \tfrac{1}{16}(1-\beta^2) l^2\beta^2 J_m'(l\beta) + \tfrac{1}{16}(3+\beta^2) l\beta J_m(l\beta) \\
&\quad + \tfrac{1}{16}\{(1+\beta^2)(m^2 + l^2\beta^2) - 4ml\beta^2 - 3 - \beta^2\} J_m^{-1}(l\beta) \\
d\{m, l\beta\} &= -\tfrac{1}{16}(1-\beta^2) l^2\beta^2 J_m'(l\beta) + \tfrac{1}{16}(1+3\beta^2) l\beta J_m(l\beta) \\
&\quad + \tfrac{1}{16}\{(1+\beta^2)(m^2 + l^2\beta^2) - 4ml\beta^2 - 1 - 3\beta^2\} J_m^{-1}(l\beta)
\end{aligned}\right\} \ldots(276).$$

In these equations $l = \sigma + m$, where $\sigma = 2p/\omega$. When $\sigma = 0$ we therefore have $l = m$, and the three functions a, c and d become odd functions of m, while b becomes an even function. We shall denote them by a zero suffix, *e.g.* a_0, b_0, c_0, d_0.

We must now put $\psi = \pi i/n$ in (275) and sum from $i = 1$ to $i = n-1$; we have

$$\sum_{i=1}^{i=n-1} \epsilon^{\iota(j-2k)\pi i/n} = \iota \cot \frac{(2s+1)\pi}{2n}, \text{ for } j = 2k + 2s + 1,$$
$$= -1, \text{ for } j = 2k + 2s, \ s \text{ not a multiple of } n,$$
$$= n - 1, \text{ for } j = 2k + 2sn.$$

Hence we get, for the Bessel Function parts,

$$\left.\begin{aligned}
\sum_{i=1}^{i=n-1} \Theta_1 &= -\mathscr{A} - \iota A + \iota \sum_{s=-\infty}^{s=\infty} a\{2s, (\sigma+2s)\beta\} \\
\sum_{i=1}^{i=n-1} \Theta_2 &= -\iota\mathscr{B} + B - \sum_{s=-\infty}^{s=\infty} b\{2s, (\sigma+2s)\beta\} \\
\sum_{i=1}^{i=n-1} \Theta_3 &= -\mathscr{C} - \iota C + \iota \sum_{s=-\infty}^{s=\infty} c\{2s, (\sigma+2s)\beta\} \\
\sum_{i=1}^{i=n-1} \Theta_4 &= -\mathscr{D} - \iota D + \iota \sum_{s=-\infty}^{s=\infty} d\{2s, (\sigma+2s)\beta\}
\end{aligned}\right\} \quad \ldots\ldots(277),$$

where

$$\left.\begin{aligned}
\mathscr{A} &= \sum_{s=-\infty}^{s=\infty} \cot \frac{(2s+1)\pi}{2n} a\{2k+2s+1, (\sigma+2k+2s+1)\beta\} \\
A &= n \sum_{s=-\infty}^{s=\infty} a\{2k+2sn, (\sigma+2k+2sn)\beta\}
\end{aligned}\right\} \ldots(278),$$

with similar expressions for $\mathscr{B}$, $\mathscr{C}$, $\mathscr{D}$, B, C and D.

Denoting the values of $\mathscr{A}$, ... for $\sigma = 0$ by $\mathscr{A}_0$, ... we see that $\mathscr{B}_0 = 0$ and $A_0 = C_0 = D_0 = 0$, since a_0, c_0 and d_0 are odd functions and b_0 is an even function of m.

186. The third sum in each of equations (277) requires no special symbol, since it can be evaluated as follows. By Taylor's Theorem we have

$$J_{2s}^{-1}\{(\sigma+2s)\beta\} = J_{2s}^{-1}(2s\beta) + \sigma\beta J_{2s}(2s\beta) + \tfrac{1}{2}\sigma^2\beta^2 J'_{2s}(2s\beta) + \ldots,$$

with similar expressions for $J_{2s}\{(\sigma+2s)\beta\}$ and $J'_{2s}\{(\sigma+2s)\beta\}$. In this manner the functions $a_{2s}\{2s, (\sigma+2s)\beta\}$ may be expanded in a series of terms of the types $s^q J_{2s}^{(r)}(2s\beta)$, where $q = 0$, 1 or 2, and $r = -1, 0, 1, \ldots$. The last terms of Θ_1, ... thus reduce to sums of the type $\sum_{s=-\infty}^{s=\infty} s^q J_{2s}^{(r)}(2s\beta)$.

Now all sums of this type vanish identically, unless $q - r$ is even and at the same time $q \geqslant r$, and further $\sum_{s=1}^{s=\infty} J_{2s}(2s\beta)/(2s)^2 = \frac{1}{8}\beta^2$*, as we saw before. Differentiating this equation repeatedly with respect to β, and using the differential equation of the Bessel Function where necessary to eliminate differential coefficients of the higher orders, we get

$$\sum_{s=1}^{s=\infty} J_{2s}(2s\beta) = \beta^2/2(1-\beta^2),$$

whence

$$\sum_{s=-\infty}^{s=\infty} J_{2s}(2s\beta) = 1/(1-\beta^2).$$

By the same process we get

$$\sum_{s=-\infty}^{s=\infty} s^2 J_{2s}(2s\beta) = \beta^2(1+\beta^2)/(1-\beta^2)^4,$$

and by integration with respect to β from 0 to β we get

$$\sum_{s=-\infty}^{s=\infty} s J_{2s}^{-1}(2s\beta) = 2\beta^3/3(1-\beta^2)^3.$$

From these all the necessary sums may be derived by differentiation with respect to β, and we get without difficulty

$$\sum_{s=-\infty}^{s=\infty} J_{2s}^{-1}\{(\sigma+2s)\beta\} = \frac{\sigma\beta}{1-\beta^2},$$

$$\sum_{s=-\infty}^{s=\infty} sJ_{2s}^{-1}\{(\sigma+2s)\beta\} = \frac{2\beta^3}{3(1-\beta^2)^3} + \frac{\sigma^2\beta^3}{2(1-\beta^2)^2},$$

$$\sum_{s=-\infty}^{s=\infty} s^2 J_{2s}^{-1}\{(\sigma+2s)\beta\} = \frac{\sigma\beta^3(1+\beta^2)}{(1-\beta^2)^4} + \frac{\sigma^3\beta^3(1+3\beta^2)}{12(1-\beta^2)^3}.$$

From these the remaining sums required are got by differentiating with respect to σ.

Substituting in (277) and using (276) we get

$$\left.\begin{aligned}
\sum_{i=1}^{i=n-1} \Theta_1 &= -\mathscr{A} - \iota\left\{A - \frac{\beta^3(3+\beta^2)}{3(1-\beta^2)^3}\sigma - \frac{\beta^3}{12(1-\beta^2)^2}\sigma^3\right\}\\
\sum_{i=1}^{i=n-1} \Theta_2 &= -\iota\mathscr{B} + B - \frac{2\beta^3(1+\beta^2)}{3(1-\beta^2)^3} - \frac{\beta^3}{2(1-\beta^2)^2}\sigma^2\\
\sum_{i=1}^{i=n-1} \Theta_3 &= -\mathscr{C} - \iota\left\{C - \frac{\beta^3(3+\beta^2)}{3(1-\beta^2)^2}\sigma - \frac{\beta^3}{12(1-\beta^2)}\sigma^3\right\}\\
\sum_{i=1}^{i=n-1} \Theta_4 &= -\mathscr{D} - \iota\left\{D - \frac{\beta^5}{3(1-\beta^2)^2}\sigma - \frac{\beta^3}{12(1-\beta^2)}\sigma^3\right\}
\end{aligned}\right\} \quad \ldots(279).$$

* Nielsen, *Cylinderfunktionen*, p. 303.

187. Sums derived from $1/\sin\psi$. The terms in the functions $\Theta_1, \ldots$, which are derived from $1/\sin\psi$, by (270), § 183, are seen to be derived from terms of type (a), which involve $\sin\chi$ in the denominator. From (271)—(274), § 184, we see that these terms, apart from the common factor $\epsilon^{-\iota k 2\pi i/n}$, are given by the equations

$$\Theta_1 = \frac{\sigma^2\beta^2}{8\sin\psi} - \frac{\partial}{\partial\psi}\left\{\frac{\iota\sigma\beta^2}{4\sin\psi}\right\} + \frac{\partial^2}{\partial\psi^2}\left\{\frac{1-\beta^2}{8\sin\psi}\right\},$$

$$\Theta_2 = \frac{\iota\sigma\beta^2}{8\sin\psi} + \frac{\partial}{\partial\psi}\left\{\frac{1+\beta^2}{8\sin\psi}\right\}.$$

$$\Theta_3 = \frac{\sigma^2\beta^2(1+\beta^2)}{16\sin\psi} + \frac{\partial}{\partial\psi}\left\{\frac{\iota\sigma\beta^2(1-\beta^2)}{8\sin\psi}\right\} - \frac{\partial^2}{\partial\psi^2}\left\{\frac{(1-\beta^2)^2}{16\sin\psi}\right\} - \frac{3+\beta^2}{16\sin\psi};$$

$$\Theta_4 = \frac{\sigma^2\beta^2(1+\beta^2)}{16\sin\psi} + \frac{\partial}{\partial\psi}\left\{\frac{\iota\sigma\beta^2(1-\beta^2)}{8\sin\psi}\right\} - \frac{\partial^2}{\partial\psi^2}\left\{\frac{(1-\beta^2)^2}{16\sin\psi}\right\} - \frac{1+3\beta^2}{16\sin\psi}.$$

Since

$$\frac{\partial}{\partial\psi}\left\{\frac{1}{\sin\psi}\right\} = -\frac{\cos\psi}{\sin^2\psi}; \quad \text{and} \quad \frac{\partial^2}{\partial\psi^2}\left\{\frac{1}{\sin\psi}\right\} = \frac{1+\cos^2\psi}{\sin^3\psi},$$

we get the following expressions, which occur in (268), § 182, by supplying the factor $\epsilon^{-\iota k 2\pi i/n}$, subtracting, or adding the functions $\Theta_{10}, \ldots$ as may be necessary, putting $\psi = \pi i/n$, and summing from $i=1$ to $i=n-1$:

$$\left.\begin{aligned}
\sum_{i=1}^{i=n-1}(\Theta_1 - \Theta_{10}) &= -(1-\beta^2)N + 2\beta^2 M\sigma + \tfrac{1}{2}\beta^2(K-2H)\sigma^2\\
\sum_{i=1}^{i=n-1}(\Theta_{20} \pm \Theta_2) &= \pm\iota\{(1+\beta^2)M + \tfrac{1}{2}\beta^2(K-2H)\sigma\}\\
\sum_{i=1}^{i=n-1}(\Theta_3 - \Theta_{30}) &= \tfrac{1}{2}(1-\beta^2)^2 N + \tfrac{1}{2}(3+\beta^2)H - \beta^2(1-\beta^2)M\sigma\\
&\qquad + \tfrac{1}{4}\beta^2(1+\beta^2)(K-2H)\sigma^2\\
\sum_{i=1}^{i=n-1}(\Theta_4 - \Theta_{40}) &= \tfrac{1}{2}(1-\beta^2)^2 N + \tfrac{1}{2}(1+3\beta^2)H - \beta^2(1-\beta^2)M\sigma\\
&\qquad + \tfrac{1}{4}\beta^2(1+\beta^2)(K-2H)\sigma^2
\end{aligned}\right\} \quad \ldots\ldots(280),$$

where with Maxwell's notation* we have

$$\left.\begin{aligned}
N &= \sum_{i=1}^{i=n-1}\tfrac{1}{4}\sin^2 k\pi i/n\,.\,(1+\cos^2\pi i/n)\operatorname{cosec}^3\pi i/n\\
M &= \sum_{i=1}^{i=n-1}\tfrac{1}{8}\sin 2k\pi i/n\,.\,\cot\pi i/n\,.\,\operatorname{cosec}\pi i/n\\
\text{as before}\quad K &= \sum_{i=1}^{i=n-1}\tfrac{1}{4}\operatorname{cosec}\pi i/n\\
\text{and in addition}\quad H &= \sum_{i=1}^{i=n-1}\tfrac{1}{4}\sin^2 k\pi i/n\,.\,\operatorname{cosec}\pi i/n
\end{aligned}\right\} \quad \ldots\ldots(281).$$

* *Collected Papers*, Vol. I. p. 314.

The following series are useful for purposes of calculation and may conveniently be noted here:

$$\left.\begin{aligned} 2\pi K &= n\log_e n + n\{\log_e(2/\pi) + \cdot 5772\} - \frac{B_1^2\pi^2}{n\,.\,2\,!} - \frac{(2^3-1)\,B_3^2\pi^4}{n^3\,.\,2\,.\,4\,!} - \ldots \\ N &= 2k^2K - \sum_{s=0}^{s=k-1} \tfrac{1}{4}(2k-2s-1)^2 \cot\frac{(2s+1)\,\pi}{2n} \\ M &= kK - \sum_{s=0}^{s=k-1} \tfrac{1}{4}(2k-2s-1)\cot\frac{(2s+1)\,\pi}{2n} \\ H &= \sum_{s=0}^{s=k-1} \tfrac{1}{4}\cot\frac{(2s+1)\,\pi}{2n} \end{aligned}\right\} \quad \ldots\ldots(282).$$

188. The remaining terms. Summing (269), § 183, from $\psi = \pi/n$ to $\psi = (n-1)\,\pi/n$, we get

$$\sum_{i=1}^{i=n-1}\left[-\frac{1}{2\sin\chi\,(1+\beta\cos\chi)} + \frac{\partial^2}{\partial\psi^2}\left\{\frac{\beta^2(1-\beta^2)\sin\chi}{4\,(1+\beta\cos\chi)}\right\}\right]_{\psi=\pi i/n}$$
$$= -2K + 2\mathcal{V} - \beta\frac{\partial\mathcal{V}}{\partial\beta} \;\ldots\ldots\ldots\ldots\ldots(283),$$

where, as in § 157,

$$\mathcal{V} = \sum_{s=0}^{s=\infty}(s+\tfrac{1}{2})\cot\frac{(2s+1)\,\pi}{2n}\left[\beta(1-\beta^2)\,J_{2s+1}\{(2s+1)\,\beta\} + (1+\beta^2)\int_0^\beta J_{2s+1}\{(2s+1)\,x\}\,dx\right] \ldots(242).$$

Further we get from (203_1), § 138, on summation, for unit charge

$$\rho^2 f_\phi = \frac{2\beta^3}{3\,(1-\beta^2)^2} - nU,$$

where, as in § 157,

$$U = 2\sum_{s=1}^{s=\infty}\left[sn\beta^2 J'_{2sn}(2sn\beta) - s^2n^2(1-\beta^2)\int_0^\beta J_{2sn}(2snx)\,dx\right] \ldots(241).$$

Lastly, we get on summing (203_3), § 138,

$$\rho^2 h_z = \beta\,(K + \mathcal{W}),$$

where

$$\mathcal{W} = \sum_{s=0}^{s=\infty}(s+\tfrac{1}{2})\cot\frac{(2s+1)\,\pi}{2n}\left[\beta J_{2s+1}\{(2s+1)\,\beta\} - \int_0^\beta J_{2s+1}\{(2s+1)\,x\}\,dx\right] \quad \ldots\ldots(284).$$

Substituting from (279) and (280), and the equations just found, in (268), § 182, and supplying the factor e^2 we get for the disturbing force due to the rest of the ring

$$\left.\begin{aligned}
\delta F_\lambda = &-\frac{\lambda e^2}{\rho^2}\Big[(1-\beta^2)N - 2\beta^2 M\sigma - \tfrac{1}{2}\beta^2(K-2H)\sigma^2 + \mathscr{A} - \mathscr{A}_0 \\
&\quad + \iota\left\{A - \frac{\beta^3(3+\beta^2)}{3(1-\beta^2)^3}\sigma - \frac{\beta^3}{12(1-\beta^2)^2}\sigma^3\right\}\Big] \\
&+\frac{\mu e^2}{\rho^2}\Big[\iota\{(1+\beta^2)M + \beta^2(K-H+\tfrac{1}{2}\mathscr{W})\sigma - \mathscr{B}\} \\
&\quad + B + B_0 - nU - \frac{2\beta^3(1+3\beta^2)}{3(1-\beta^2)^3} - \frac{\beta^3}{2(1-\beta^2)^2}\sigma^2\Big] \\
\delta F_\mu = &-\frac{\lambda e^2}{\rho^2}\Big[\iota\{(1+\beta^2)M + \beta^2(K-H+\tfrac{1}{2}\mathscr{W})\sigma - \mathscr{B}\} \\
&\quad + B - B_0 - \frac{\beta^3}{2(1-\beta^2)^2}\sigma^2\Big] \\
&+\frac{\mu e^2}{\rho^2}\Big[\tfrac{1}{2}(1-\beta^2)^2 N + \tfrac{1}{2}(3+\beta^2)H - \beta^2(1-\beta^2)M\sigma \\
&\quad + \tfrac{1}{4}\beta^2(1+\beta^2)(K-2H)\sigma^2 - 2K + 2\mathscr{V} - \beta\frac{\partial\mathscr{V}}{\partial\beta} - \mathscr{C} + \mathscr{C}_0 \\
&\quad - \iota\left\{C - \frac{\beta^3(3+\beta^2)}{3(1-\beta^2)^2}\sigma - \frac{\beta^3}{12(1-\beta^2)}\sigma^3\right\}\Big] \\
\delta F_\nu = &\frac{\nu e^2}{\rho^2}\Big[\tfrac{1}{2}(1-\beta^2)^2 N + \tfrac{1}{2}(1+3\beta^2)H - \beta^2(1-\beta^2)M\sigma \\
&\quad + \tfrac{1}{4}\beta^2(1+\beta^2)(K-2H)\sigma^2 - \mathscr{D} + \mathscr{D}_0 \\
&\quad - \iota\left\{D - \frac{\beta^5}{3(1-\beta^2)^2}\sigma - \frac{\beta^3}{12(1-\beta^2)}\sigma^2\right\}\Big]
\end{aligned}\right\} \qquad \text{......(285).}$$

It is to be remembered that by § 178,

$$\sigma = \frac{2p}{\omega} = \frac{2(q+\iota\kappa)}{\omega} \qquad \text{........................(263),}$$

where q is the frequency and κ the damping relative to the rotating ring.

But for slight differences of notation these equations agree with those given by Schott*, except that in the coefficient of ξ in the second of equations (14) the signs of the last three terms are wrongly given.

189. **Problem 3. A ring of n equidistant electrons is slightly disturbed from its state of steady motion in a circle. Required to find the equations of motion.** We found the equations of disturbed motion for a single electron in Ch. XI, (225), § 150. The quantities $(\delta F_\lambda, \delta F_\mu, \delta F_\nu)$ are the components in the directions (λ, μ, ν) of the disturbing force, which includes a part due to the remaining electrons of the ring, given by (285), § 188, and a part due to electrons outside the ring and constituting the external force proper. Before proceeding to the study of the equations of motion we must consider the external force briefly.

* *Phil. Mag.* [6], Vol. x. p. 181, (14).

190. The external field. We must take into account two parts of the external field: (1) the steady part, already used in Ch. XII, § 159, in considering the steady motion of the ring, and (2) the external disturbing force proper, due to electromagnetic waves generated outside the ring. Both parts produce small mechanical forces disturbing the steady motion of the ring.

We have already seen in § 159 that the steady part of the tangential mechanical force, F_ϕ, may be assumed to vanish in every case that is at all likely to occur. Moreover, the steady part of the radial force, F_ϖ, is constant, the same in every position of any one electron, and the same for all the electrons. Lastly, the axial component, F_z, vanishes.

These conditions require that the external field, so long as it is undisturbed, be steady and symmetrical about the axis of the circle in which the electrons move.

Hence the electric force is derivable from a scalar potential ψ, and the magnetic force from a vector potential a_ϕ, whose direction is along the parallel of latitude; and both ψ and a_ϕ are independent of t and ϕ and functions of z and ϖ alone.

Thus the steady part of the forces of the field are given by

$$d_\phi = 0, \quad d_\varpi = -\frac{\partial \psi}{\partial \varpi}, \quad d_z = -\frac{\partial \psi}{\partial z},$$

$$h_\phi = 0, \quad h_\varpi = -\frac{\partial a_\phi}{\partial z}, \quad h_z = \frac{\partial (\varpi a_\phi)}{\varpi \partial \varpi},$$

$$F_\phi = 0, \quad F_\varpi = e\left\{-\frac{\partial \psi}{\partial \varpi} + \frac{\beta}{\varpi}\frac{\partial (\varpi a_\phi)}{\partial \varpi}\right\}, \quad F_z = e\left\{-\frac{\partial \psi}{\partial z} + \beta\frac{\partial a_\phi}{\partial z}\right\}.$$

Now we have $\beta = \varpi\omega/c$ for steady motion in a circle of radius $\varpi = \rho$ with angular velocity ω. Moreover the differentiations with respect to z and ϖ are performed on the supposition that ϕ is kept constant. Let us regard ω, or what amounts to the same thing, β/ϖ, as constant during these differentiations, and let us write

$$\Phi = e(\psi - \omega\varpi a_\phi/c), \quad \frac{\partial \Phi}{\partial \varpi} = P, \quad \frac{\partial \Phi}{\partial z} = Q \quad \text{............(286)}.$$

Then we get

$$F_\phi = 0, \quad F_\varpi = -P, \quad F_z = Q \text{.....................(287)}.$$

With this notation the equations of steady motion, (244), § 159, may be written in the form

$$\left.\begin{aligned} c\dot{m}\beta + \frac{e^2 n U}{\rho^2} &= 0 \\ \frac{c^2 m\beta^2}{\rho} + \frac{e^2\{(1+\beta^2)K - \mathcal{V}\}}{\rho^2} &= P \\ 0 &= Q \end{aligned}\right\} \text{..................(288)}.$$

In these equations (z, ϖ) are supposed to have been put equal to 0 and ρ respectively. The last two may be regarded as the equations of the steady orbit.

191. In calculating the disturbing forces we must bear in mind that the displacements of the electrons are supposed to be so small that their squares and products may be neglected. Indeed, it is impossible to obtain manageable equations on any other assumption, and thus we are compelled to make it by the necessities of the analysis. It implies that the disturbing forces themselves are small. Accordingly we shall treat both the disturbing forces and the displacements and changes of velocity of the electrons of the ring as small quantities of the first order, and neglect their squares and products. This procedure leads to the following simplifications:

In the first place, in calculating the parts of the disturbing forces due to the displacement and change of velocity of the electron under consideration itself, we may treat the external field as if it were in the condition corresponding to the steady motion. This process gives the first parts of the disturbing forces.

Secondly, in calculating the effect of the disturbances in the external field itself, we may treat the electron as if it were executing its steady motion. In this way we get the second parts of the disturbing forces.

192. (1) As before let the component displacements of the electron considered, the 0th, from its position in steady motion be $\rho(\lambda, \mu, \nu)$, perpendicular to and along the inward radius vector, and parallel to the axis. The electric and magnetic forces at the disturbed position are slightly changed, and they are in the (new) meridian plane, which is the plane of μ and ν. The tangential displacement, $\rho\lambda$, has no effect on the magnitudes, but only changes the direction of the forces, because the steady external field is symmetrical about the axis. The changes in the components of the mechanical force on this account are given by

$$\left.\begin{aligned}\delta F_\lambda &= 0\\ \delta F_\mu &= \rho\mu e\left(\frac{\partial d_\varpi}{\partial \varpi} + \beta\frac{\partial h_z}{\partial \varpi}\right) - \rho\nu e\left(\frac{\partial d_\varpi}{\partial z} + \beta\frac{\partial h_z}{\partial z}\right)\\ \delta F_\nu &= -\rho\mu e\left(\frac{\partial d_z}{\partial \varpi} - \beta\frac{\partial h_\varpi}{\partial \varpi}\right) + \rho\nu e\left(\frac{\partial d_z}{\partial z} - \beta\frac{\partial h_\varpi}{\partial z}\right)\end{aligned}\right\} \quad \ldots\ldots(289),$$

where ϖ is to be put equal to ρ, and z to zero.

In addition we have terms depending on the change in velocity, of which the components are $c\beta\left(\frac{\dot{\lambda}}{\omega} - \mu,\ \frac{\dot{\mu}}{\omega},\ \frac{\dot{\nu}}{\omega}\right)$ in accordance with (222), § 150. By means of (VI), § 140, we find for the corresponding changes in the components of the mechanical force

$$\delta F_\lambda = e\beta\left(\frac{\dot\mu}{\omega}h_z + \frac{\dot\nu}{\omega}h_\varpi\right),\quad \delta F_\mu = -e\beta\left(\frac{\dot\lambda}{\omega} - \mu\right)h_z,\quad \delta F_\nu = -e\beta\left(\frac{\dot\lambda}{\omega} - \mu\right)h_\varpi \quad \text{......(290).}$$

The terms (289) and (290) together give the first parts of the disturbing forces.

Consider the terms in both sets of equations which involve the displacements (λ, μ, ν) themselves.

In the component δF_μ we find terms $\rho\mu e\beta\frac{\partial h_z}{\partial\varpi} + e\beta\mu h_z$; remembering that $\varpi = \rho$, $\beta = \omega\rho/c$, and that ω is to be treated as constant, we may write these terms in the form $\rho\mu e\frac{\partial}{\partial\varpi}(\omega\varpi h_z/c)$, that is, $\rho\mu e\frac{\partial^2(\beta a_\phi)}{\partial\varpi^2}$. Similarly, in the component δF_ν we find terms $\rho\mu e\beta\frac{\partial h_\varpi}{\partial\varpi} + e\beta\mu h_\varpi$, which together may be written in the form $-\rho\mu e\frac{\partial^2(\beta a_\phi)}{\partial z\partial\varpi}$. Similar considerations apply to the terms involving ν. Let us use the function Φ defined by (286), § 190, and write

$$\frac{\partial^2\Phi}{\partial\varpi^2} = R,\quad \frac{\partial^2\Phi}{\partial\varpi\partial z} = S,\quad \frac{\partial^2\Phi}{\partial z^2} = T \quad \text{....................(291),}$$

with the same understanding as before, that ω is to be treated as constant in the differentiations, and that z and ϖ are to be replaced by their steady motion values, 0 and ρ.

Then the first parts of the disturbing forces, depending on the displacement and change of velocity of the electron, are given by

$$\left.\begin{aligned}\delta F_\lambda &= e\beta\left(\frac{\dot\mu}{\omega}h_z + \frac{\dot\nu}{\omega}h_\varpi\right)\\ \delta F_\mu &= -\rho\mu R + \rho\nu S - e\beta\frac{\dot\lambda}{\omega}h_z\\ \delta F_\nu &= \rho\mu S - \rho\nu T - e\beta\frac{\dot\lambda}{\omega}h_\varpi\end{aligned}\right\} \quad \text{.................(292).}$$

193. (2) Let us denote by $(\delta d_\phi, \delta d_\varpi, \delta d_z)$ and $(\delta h_\phi, \delta h_\varpi, \delta h_z)$ the deviations of the electric and magnetic forces of the external field from the values corresponding to steady motion for the point (z, ϖ, ϕ). In calculating the corresponding mechanical forces on the 0th electron, which is at the point $(0, \rho, \omega t + \delta)$ in the steady motion at time t, we are to neglect its displacement and change of velocity, since these would only give rise to terms of the second order. Hence in the general expressions for the deviations of the forces we must put $z=0$, $\varpi=\rho$, $\phi=\omega t+\delta$, and take the velocity of the electron to be $c\beta$ in the direction ϕ. Hence we get by (VI), § 140,

$$\delta F_\lambda = e\delta d_\phi,\quad \delta F_\mu = -e(\delta d_\varpi + \beta\delta h_z),\quad \delta F_\nu = e(\delta d_z - \beta\delta h_\varpi)\ldots(293).$$

These give the second parts of the disturbing forces, that is, the external disturbing forces proper. They are given functions of t.

194. The equations of the disturbed motion. We must now substitute for the forces (δF_λ, δF_μ, δF_ν) on the right-hand sides of (225), § 150, the sums of three sets of terms:

(1) the disturbing forces on the 0th electron, due to the disturbances of the remaining electrons of the ring and given by (285), § 188;

(2) the changes in the forces of the steady field, due to the displacement and change of velocity of the 0th electron from its steady motion and given by (292), § 192;

(3) the external disturbing forces, due to changes in the external field and given by (293), § 193.

The forces under the first head have been calculated for a damped simple harmonic oscillation of the type $(A, B, C)\exp.\iota(pt-k2\pi i/n)$, where (A, B, C) are arbitrary constants, generally complex, i is the number fixing the electron in the ring, $p=q+\iota\kappa$, q is the frequency and κ the damping, both relative to the rotating ring, and k is an integer, the "class," lying between the limits $\pm\frac{1}{2}n$ and determining the mode of the vibration.

The forces under the second head are obviously expressible by means of sums of terms of the same type.

Hence we must express the forces under the third head, the external disturbing forces, in the same way.

They are due to electromagnetic waves of the most general type, and can always be expressed by means of sums of terms of the form $f(z, \varpi, \phi)\exp.\iota Nt$, where N is generally complex with positive imaginary part. Its real and imaginary parts give respectively the frequency and damping of the external disturbing forces relative to an observer outside the rotating ring and fixed with respect to its centre.

The amplitude, $f(z, \varpi, \phi)$, is a periodic function of ϕ, with the period 2π, and can be expanded in a Fourier series of terms of the type $F(z, \varpi)\exp.(-ik\phi)$, where k is an integer.

For the ith electron of the ring we have

$$z=0,\quad \varpi=\rho,\quad\text{and}\quad \phi=\omega t+\delta+2\pi i/n.$$

Hence we can express the external disturbing forces on it as sums of terms of the type $F \exp. \iota(pt - k2\pi i/n)$, where $p = N - k\omega$. We can, if we wish, suppose k to lie between the limits $\pm\frac{1}{2}n$ without any loss of generality, by slightly changing the meanings of the symbols N and p. Thus the external disturbing forces can always be expressed in the required form.

The frequency of any term relative to the rotating ring, which we denoted by q, must be carefully distinguished from that relative to an outside observer, which is $q + k\omega$.

195. Since $\sigma = 2p/\omega = 2(q + \iota\kappa)/\omega$, by (263), § 178, we must replace such quantities as $\dot{\lambda}/\omega$, $\ddot{\lambda}/\omega^2$, $\dddot{\lambda}/\omega^3$, by $\frac{1}{2}\iota\sigma\lambda$, $-\frac{1}{4}\sigma^2\lambda$, $-\frac{1}{8}\iota\sigma^3\lambda$, respectively on the left-hand sides of (225), § 150. When we compare the resulting expressions with the right-hand members of (285), § 188, which determine the parts of the disturbing forces due to the rest of the ring, we see that all those terms, which do not involve the mass m in the former expressions, just cancel equal terms in the latter.

For example, we find on the left-hand side of the third equation (225), the terms

$$\iota\left\{\frac{e^2\beta^5}{3\rho^2(1-\beta^2)^2}\sigma + \frac{e^2\beta^3}{12\rho^2(1-\beta^2)}\sigma^3\right\}\nu.$$

These terms also occur on the right-hand side of the third equation (285), and therefore cancel out. The same thing occurs in the first two equations of motion.

The terms in question have a simple physical meaning: they represent the reactions on the electron due to its own radiation, so far as it is due to the disturbance from steady motion. When the electron forms one of a group, the radiation per electron is much reduced, and accordingly the corresponding reaction is partially balanced by the forces exerted by the remaining electrons of the ring. The small unbalanced part left over corresponds to the reduced radiation from the ring and is represented by the terms involving the Bessel Function Series $A, \ldots$ in (285). The complete cancelling out of the large terms in the reactions due to radiation serves as a verification of the correctness of both (285) and (225).

The terms in (225) which involve m can be expressed more conveniently by means of the longitudinal mass $m' = \dfrac{d(m\beta)}{d\beta}$; for instance, we have $\dfrac{d(\dot{m}\beta)}{d\beta} = m'\dot{m}/m$. We might eliminate m and $\dot{m}$ by means of the equations of steady motion (244), § 159, but the ratio m'/m would still remain, so that there is little advantage in doing so.

196. On substituting the values of the three sets of disturbing forces from (292), (293) and (285) in (225), we find that the equations of disturbance may be written in the following form:

$$\left.\begin{aligned} f_{11}\lambda + f_{12}\mu + f_{13}\nu &= e\,\delta d_\phi \\ f_{21}\lambda + f_{22}\mu + f_{23}\nu &= -e\,(\delta d_\varpi + \beta\delta h_z) \\ f_{31}\lambda + f_{32}\mu + f_{33}\nu &= e\,(\delta d_z - \beta\delta h_\varpi) \end{aligned}\right\} \quad \text{.............(294).}$$

The nine coefficients are given by

$$\left.\begin{aligned}
f_{11} &= \frac{e^2}{\rho^2}\{(1-\beta^2)N - 2\beta^2 M\sigma - \tfrac{1}{2}\beta^2(K-2H)\sigma^2 + \mathcal{A} - \mathcal{A}_0\} - \frac{c^2 m'\beta^2}{4\rho}\sigma^2 \\
&\quad + \iota\left(\frac{e^2}{\rho^2}A + \tfrac{1}{2}c\dot{m}b\frac{m'}{m}\sigma\right) \\
f_{12} &= -\iota\left[\frac{e^2}{\rho^2}\{(1+\beta^2)M + \beta^2(K-H+\tfrac{1}{2}\mathcal{W})\sigma - \mathcal{B}\} + \tfrac{1}{2}\left\{e\beta h_z + \frac{c^2(m'+m)\beta^2}{\rho}\right\}\sigma\right] \\
&\quad - \frac{e^2}{\rho^2}(B + B_0 - nU) - c\dot{m}\beta\frac{m'}{m} \\
f_{13} &= \tfrac{1}{2}\iota e\beta h_\rho\sigma \\
f_{21} &= \iota\left[\frac{e^2}{\rho^2}\{(1+\beta^2)M + \beta^2(K-H+\tfrac{1}{2}\mathcal{W})\sigma - \mathcal{B}\} + \tfrac{1}{2}\left\{e\beta h_z + \frac{c^2(m'+m)\beta^2}{\rho}\right\}\sigma\right] \\
&\quad + \frac{e^2}{\rho^2}(B - B_0) \\
f_{22} &= -\frac{e^2}{\rho^2}\{\tfrac{1}{2}(1-\beta^2)^2 N + \tfrac{1}{2}(3+\beta^2)H - \beta^2(1-\beta^2)M\sigma + \tfrac{1}{4}\beta^2(1+\beta^2)(K-2H)\sigma^2 \\
&\quad - 2K + 2\mathcal{V} - \beta\frac{d\mathcal{V}}{d\beta} - \mathcal{C} + \mathcal{C}_0\} + \rho R - \frac{c^2 m'\beta^2}{\rho} - \frac{c^2 m\beta^2}{4\rho}\sigma^2 \\
&\quad + \iota\left(\frac{e^2}{\rho^2}C + \tfrac{1}{2}c\dot{m}\beta\sigma\right) \\
f_{23} &= -\rho S \\
f_{31} &= -\tfrac{1}{2}\iota e\beta h_\rho\sigma \\
f_{32} &= -\rho S \\
f_{33} &= -\frac{e^2}{\rho^2}\{\tfrac{1}{2}(1-\beta^2)^2 N + \tfrac{1}{2}(1+3\beta^2)H - \beta^2(1-\beta^2)M\sigma + \tfrac{1}{4}\beta^2(1+\beta^2)(K-2H)\sigma^2 \\
&\quad - \mathcal{D} + \mathcal{D}_0\} + \rho T - \frac{c^2 m\beta^2}{4\rho}\sigma^2 + \iota\left(\frac{e^2}{\rho^2}D + \tfrac{1}{2}c\dot{m}\beta\right)\sigma
\end{aligned}\right\} \quad \text{......(295).}$$

The values of the several functions involved in these equations are given by (276) and (278), § 185, (281), or (282), § 187, (241), (242), and (284), § 188. For convenience of reference we collect all these equations here. They are

$$\left.\begin{aligned}
a\{m, l\beta\} &= \tfrac{1}{4} l^2\beta^2 J_m'(l\beta) - \tfrac{1}{8}(m^2 - l^2\beta^2) J_m^{-1}(l\beta) \\
b\{m, l\beta\} &= \tfrac{1}{8}(m - l\beta^2) l\beta J_m(l\beta) + \tfrac{1}{8}(m + l\beta^2) J_m^{-1}(l\beta) \\
c\{m, l\beta\} &= \tfrac{1}{16}(1-\beta^2) l^2\beta^2 J_m'(l\beta) + \tfrac{1}{16}(3+\beta^2) l\beta J_m(l\beta) \\
&\quad + \tfrac{1}{16}\{(1+\beta^2)(m^2 + l^2\beta^2) - 4ml\beta^2 - 3 - \beta^2\} J_m^{-1}(l\beta) \\
d\{m, l\beta\} &= -\tfrac{1}{16}(1-\beta^2) l^2\beta^2 J_m'(l\beta) + \tfrac{1}{16}(1+3\beta^2) l\beta J_m(l\beta) \\
&\quad + \tfrac{1}{16}\{(1+\beta^2)(m^2 + l^2\beta^2) - 4ml\beta^2 - 1 - 3\beta^2\} J_m^{-1}(l\beta)
\end{aligned}\right\} \quad \text{......(276)},$$

$$\left.\begin{aligned}
\mathscr{A} &= \sum_{s=-\infty}^{s=\infty} \cot\frac{(2s+1)\pi}{2n}\, a\{2k+2s+1,\ (\sigma+2k+2s+1)\beta\} \\
A &= n\sum_{s=-\infty}^{s=\infty} a\{2k+2sn,\ (\sigma+2k+2sn)\beta\}
\end{aligned}\right\} \quad \text{......(278)},$$

where $$\sigma = 2p/\omega = 2(q + \iota\kappa)/\omega \text{........................(263)},$$

and $$J_m^{-1} = \int J_m(y)\, dy,$$

with similar expressions for $\mathscr{B}$, $\mathscr{C}$, $\mathscr{D}$, B, C and D. The values of these functions for $\sigma = 0$ are denoted by a suffix 0.

Again, we have

$$U = 2\sum_{s=1}^{s=\infty}\left[sn\beta^2 J'_{2sn}(2sn\beta) - s^2n^2(1-\beta^2)\int_0^\beta J_{2sn}(2snx)\, dx\right] \quad \text{......(241)},$$

$$\mathscr{V} = \sum_{s=0}^{s=\infty}(s+\tfrac{1}{2})\cot\frac{(2s+1)\pi}{2n}\left[\beta(1-\beta^2) J_{2s+1}\{(2s+1)\beta\} + (1+\beta^2)\int_0^\beta J_{2s+1}\{(2s+1)x\}\, dx\right] \text{...(242)},$$

$$\mathscr{W} = \sum_{s=0}^{s=\infty}(s+\tfrac{1}{2})\cot\frac{(2s+1)\pi}{2n}\left[\beta J_{2s+1}\{(2s+1)\beta\} - \int_0^\beta J_{2s+1}\{(2s+1)x\}\, dx\right] \quad \text{......(284)}.$$

Lastly, we have

$$\left.\begin{aligned}
K &= \sum_{i=1}^{i=n-1} \tfrac{1}{4}\operatorname{cosec}\pi i/n \\
H &= \sum_{i=1}^{i=n-1} \tfrac{1}{4}\sin^2 k\pi i/n \,.\, \operatorname{cosec}\pi i/n \\
M &= \sum_{i=1}^{i=n-1} \tfrac{1}{8}\sin 2k\pi i/n \,.\, \cot\pi i/n \,.\, \operatorname{cosec}\pi i/n \\
N &= \sum_{i=1}^{i=n-1} \tfrac{1}{4}\sin^2 k\pi i/n \,.\, (1 + \cos^2\pi i/n)\operatorname{cosec}^3\pi i/n
\end{aligned}\right\} \quad \text{......(281)}.$$

The equations (294) and (295) together are equivalent to the equations given by Schott*. They are however somewhat more general, in so far

* *Phil. Mag.* [6], Vol. xv. pp. 180—182, 1908, eqs. 12—15.

as in the present case the steady field in which the ring moves is not merely electrostatic, as was assumed in the paper just cited, but includes a magnetostatic field as well. Consequently the values of the potential Φ and its differential coefficients are somewhat altered by the addition of terms involving the vector potential a_ϕ of the magnetostatic field, as shown in (286), § 190. Moreover, the equations of the disturbed motion (294) and (295) involve the components h_ϖ and h_z of the steady magnetic field explicitly. The additional terms introduced in this way are gyrostatic terms as usual, that is to say, no work is done on their account*.

197. The following points should be noticed:

(1) The explicitly imaginary parts of the coefficients f_{12} and f_{21} are equal, but of opposite sign; so also those of f_{13} and f_{31}, while those of f_{23} and f_{32} are zero. The corresponding terms in (294) represent gyrostatic motional forces, which do not consume or do work. "Explicitly" imaginary merely refers to the occurrence of ι as a factor; as a matter of fact when σ is complex these terms have real parts.

(2) The explicitly real parts of $f_{12}, f_{21}, f_{13}, f_{31}$, and the explicitly imaginary parts of f_{11}, f_{22} and f_{33}, all involve functions of the type A, together with terms which have $\dot{m}$ for a factor. The first equation of steady motion (244), § 159, enables us to replace $\dot{m}$ by the series U, which is of the same type as A. An examination of the expressions found for the radiation from the ring, namely (129), § 84, and (168), § 105, shows that the radiation due to steady motion is of the order of U, and that due to disturbance is of order $A, \ldots$ Thus we may conclude that the corresponding terms in (294) represent damping forces due to radiation. When the radiation, and consequently the damping, is small, these terms are small.

(3) The effect of the external field is represented by two sets of terms: (*a*) the gyrostatic motional terms involving the magnetic forces h_ϖ and h_z; (*b*) the conservative terms derived from the potential Φ, namely ρR in f_{22}, ρT in f_{33}, and the terms $f_{23} = f_{32} = -\rho S$. When the external field is purely electrostatic, the first set of terms disappears. When the electrostatic field is due to a continuous homogeneous distribution of electric charge, as in the case of J. J. Thomson's well known model of the atom, S vanishes identically. The same thing occurs when the field is due to a central point charge, as in Nagaoka's model. In these cases the third equation (294) is independent of the first two, but this is not generally so.

(4) The functions $\mathscr{A}, \ldots$ $A, \ldots$ are infinite series of Bessel Functions involving σ in their arguments, that is, they are functions of the frequency of infinite degree.

* Slight errors in the equations (12)—(15) just cited have been corrected; namely in the last two of (12) the term $\mu\omega\dot{m}/m$ has been replaced by its proper value $\omega\dot{m}$, and in the second of (14) the sign of $-B+B_0$ has been changed.

Hence, when the external disturbing forces on the right of (294) are absent, so that the ring is executing its free vibrations, the frequency equation got by eliminating (λ, μ, ν) is of infinite degree for each class of vibration. Thus the ring has n sets, each of an infinite number of free periods, and from this point of view must be regarded as possessing an infinite number of degrees of freedom. We need feel no surprise at this result, when we reflect that each electron of the ring is linked with the electromagnetic aether, and so cannot be compared with the particle of ordinary dynamics.

In this connection it is interesting to notice that Herglotz and Sommerfeld* some years ago found that a single spherical rigid electron can execute an infinite number of free rotational vibrations, and therefore may be regarded as possessing an infinite number of degrees of freedom. But these vibrations differ from those of our ring of electrons in so far as their wave-lengths are only of the order of the diameter of the electron, while ours are of the order of the diameter of the ring or larger. Moreover for a surface charge, Sommerfeld's vibrations are undamped and therefore do not radiate, while for a volume charge they are damped and radiate.

The calculation of the free periods of a ring of electrons revolving in a given steady field presupposes a knowledge of the properties of the Kapteyn series of Bessel Functions, which appear in equations (276) and (278), § 196. Series of this type have hardly been studied yet; consequently any detailed discussion of the equations of disturbance would be far beyond the scope of this essay. For this reason we shall content ourselves with the deduction of the equations of steady motion and disturbance; they will form the basis of any theory of atomic structure which is founded on the hypothesis that the atom contains circular rings of electrons in revolution.

* *Gött. Nach.* 1904, p. 434.

APPENDIX C

ON THE FIELD CLOSE TO A POINT CHARGE IN MOTION

198. IN order to calculate the mechanical force exerted by a moving electron upon itself, with a view to deducing expressions for its electromagnetic energy, momentum and mass, we must develope a method of finding the field close to a moving point charge. The mechanical force exerted by one element of the electron upon another can then be found, and hence the mechanical force exerted by the whole electron on one of its elements, and also on itself, can be obtained by integration over the whole charge.

This method has already been used by Schwarzschild*, Sommerfeld (*loc. cit.* § 143), Lindemann (*loc. cit.* § 143), Walker (*loc. cit.* § 143) and other writers, but with certain restrictions as to the form and structure of the electron, imposed so as to render the necessary integrations possible.

For the purpose of our investigation in the following Appendix D it is undesirable to introduce definite assumptions respecting the form and structure of the electron at an earlier stage than absolutely necessary; but on the other hand we shall find it necessary to restrict ourselves to speeds less than that of light, and to times when the motion is permanent in character. For instance, if the motion of the electron changes discontinuously, our results will not apply to the interval embracing the discontinuity, during which the electron is still disturbed by the waves due to it.

What occurs during the establishment of the permanent régime has been the subject of investigations by Lindemann, Schott and Walker. Walker in particular has shown that the effect of the discontinuity mainly consists in small displacements and changes of velocity of the electron, due to strongly damped waves generated by the discontinuity. Owing to the smallness of the electron these effects are practically negligible.

199. For the sake of simplicity we shall take the whole charge of the electron as unity. When it is equal to e electrostatic units we need only supply e as a factor in the potentials and forces of the field due to the electron,

* *Gött. Nach.* 1903, pp. 126, 132 and 245.

and e^2 as a factor in the electromagnetic energy, momentum and mass, and generally in the resultant mechanical force of the electron on itself.

We shall base our investigation on the series (180) and (181), § 124, but shall neglect the complementary terms. The use of the series limits us to velocities less than that of light, and to points near the charge, and the omission of the complementary terms implies that the permanent régime has been established.

The result will be expressed in the form of series, which can be integrated over the electron and converge for points not too far away from it. The conditions of convergence imply certain restrictions on the values of the accelerations of all orders of the electron.

200. It will be convenient to develope the expression for **a**, (181), § 124, so as to put the successive differential coefficients of the velocity **v** in evidence explicitly. Write

$$U^{(k)} = \sum_{s=0}^{s=\infty} \frac{(-1)^s}{s!\,c^s} \frac{\partial^s R^{s+k-1}}{\partial t^s} \quad \text{.....................(296).}$$

Then we get in place of (180) and (181)

$$\phi = U^{(0)} \quad \text{.......................................(297),}$$

$$\mathbf{a} = \sum_{k=0}^{k=\infty} \frac{(-1)^k\, U^{(k)}\, \mathbf{v}^{(k)}}{k!\, c^{k+1}} \quad \text{.....................(298).}$$

Here $\mathbf{v}^{(k)}$ denotes the kth differential coefficient of **v** with respect to t, or as we may call it, the acceleration of the kth order of the charge at the time t.

We shall whenever necessary denote the coordinates of the moving charge by (ξ, η, ζ), and those of the fieldpoint by (x, y, z), both referred to rectangular axes fixed with respect to the observer; also we shall denote the radii drawn from the fixed origin to the moving charge and to the fieldpoint by ρ and $\mathbf{r}$ respectively. Thus we have $\mathbf{v} = \dot{\rho}$, and generally $\mathbf{v}^{(k)} = \rho^{(k+1)}$. These quantities may be supposed to be given quite arbitrarily.

201. Expansions for $\mathbf{U}^{(k)}$. The expansion (296) is inconvenient because it involves partial differentiations with respect to t. We desire to replace it by an expansion involving partial differentiations with respect to (x, y, z), so as to make the velocity and accelerations of various orders appear explicitly in the coefficients. We have

$$R = \{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2\}^{\frac{1}{2}},$$

where (ξ, η, ζ) may for the present be regarded as given functions of t, but independent of (x, y, z), the latter being themselves independent of the operator $\frac{\partial}{\partial t}$.

Let Δ prefixed to any function of t denote the increase of the function when t is changed into $t+\tau$. Then we may write

$$\frac{\partial^s R^{s+k-1}}{\partial t^s} = \underset{\tau=0}{\text{Lim.}} \frac{\partial^s (R+\Delta R)^{s+k-1}}{\partial t^s}.$$

But by the symbolic form of Taylor's Theorem we have

$$(R+\Delta R)^{s+k-1} = \{(x-\xi-\Delta\xi)^2 + (y-\eta-\Delta\eta)^2 + (z-\zeta-\Delta\zeta)^2\}^{\frac{1}{2}(s+k-1)}$$
$$= \epsilon^{-\Delta(\rho\nabla)} R^{s+k-1},$$

where ∇ denotes vector differentiation with respect to (x, y, z), and $(\rho\nabla)$ denotes the scalar product of ρ and ∇ in the usual way. Since ρ is independent of (x, y, z), ∇ does not operate on it, and the order of the factors can be interchanged. The same result holds for all products of the types $(\mathbf{v}^{(k)}\nabla)$, $[\mathbf{v}^{(k)}\nabla]$, scalar as well as vector. Hence we get

$$\frac{\partial^s R^{s+k-1}}{\partial t^s} = \left\{\underset{\tau=0}{\text{Lim.}} \frac{\partial^s}{\partial \tau^s} \epsilon^{-\Delta(\rho\nabla)}\right\} R^{s+k-1} \quad \text{..............(299).}$$

In this equation τ only occurs in the exponent $\Delta(\rho\nabla)$, so that the symbolic operator can be found by developing the exponential in powers of τ, differentiating and proceeding to the limit. Thus we need only find the coefficient of $\tau^s/s!$ in the exponential.

Now we have

$$\epsilon^{-\Delta(\rho\nabla)} = \sum_{n=0}^{n=\infty} \frac{(-1)^n}{n!} \{\Delta(\rho\nabla)\}^n \quad \text{.................(300).}$$

Further we have

$$\Delta(\rho\nabla) = \xi(t+\tau)\frac{\partial}{\partial x} + \eta(t+\tau)\frac{\partial}{\partial y} + \zeta(t+\tau)\frac{\partial}{\partial z} - \xi(t)\frac{\partial}{\partial x} - \eta(t)\frac{\partial}{\partial y} - \zeta(t)\frac{\partial}{\partial z}$$
$$= \tau(\mathbf{v}\nabla) + \tau^2\frac{(\dot{\mathbf{v}}\nabla)}{2!} + \tau^3\frac{(\ddot{\mathbf{v}}\nabla)}{3!} + \dots,$$

where $\mathbf{v}$, $\dot{\mathbf{v}}$, $\ddot{\mathbf{v}}$, ... all refer to the time t.

Substituting in (300), and collecting the several terms, we find that the coefficient of τ^s in $\epsilon^{-\Delta(\rho\nabla)}$ is equal to

$$\frac{(-1)^s}{s!}\left\{(\mathbf{v}\nabla)^s - \frac{s(s-1)}{2!}(\mathbf{v}\nabla)^{s-2}(\dot{\mathbf{v}}\nabla) + \frac{s(s-1)(s-2)}{3!}(\mathbf{v}\nabla)^{s-3}(\ddot{\mathbf{v}}\nabla) - \dots \right.$$
$$\left. + \frac{s(s-1)(s-2)(s-3)}{(2!)^3}(\mathbf{v}\nabla)^{s-4}(\dot{\mathbf{v}}\nabla)^2 - \dots\right\}.$$

Substituting this value for $s!\,\dfrac{\partial^s}{\partial\tau^s}\epsilon^{-\Delta(\rho\nabla)}$ in (299), and using the result in (296), we get,

$$U^{(k)} = \sum_{s=0}^{s=\infty} \frac{1}{s!\,c^s}\left\{(\mathbf{v}\nabla)^s - \frac{s(s-1)}{2!}(\mathbf{v}\nabla)^{s-2}(\dot{\mathbf{v}}\nabla) + \frac{s(s-1)(s-2)}{3!}(\mathbf{v}\nabla)^{s-3}(\ddot{\mathbf{v}}\nabla)\right.$$
$$\left. - \dots + \frac{s(s-1)(s-2)(s-3)}{(2!)^3}(\mathbf{v}\nabla)^{s-4}(\dot{\mathbf{v}}\nabla)^2 - \dots\right\} R^{s+k-1} \quad \text{......(301).}$$

This result may be written in the form

$$U^{(k)} = V^{(k)} - \frac{(\dot{\mathbf{v}}\nabla)}{2c^2} V^{(k+2)} + \frac{(\ddot{\mathbf{v}}\nabla)}{6c^3} V^{(k+3)} + \frac{(\dot{\mathbf{v}}\nabla)^2}{8c^4} V^{(k+4)} + \dots \quad \dots(302),$$

where $V^{(k)} = \sum_{s=0}^{s=\infty} \frac{(\mathbf{v}\nabla)^s R^{s+k-1}}{s!\, c^s}$ ....(303).

The expansion expressed by these equations is particularly convenient because the series $V^{(k)}$ can be summed and expressed in a finite form and the series $U^{(k)}$ reduced to single series. It is easy to see that the symbolic operators in (302) are the terms involving c^{-2}, c^{-3}, c^{-4}, ... and so on, in the scalar part of the expansion of exp. $\left(-\frac{\dot{\mathbf{v}}}{2c^2} + \frac{\ddot{\mathbf{v}}}{6c^3} - \frac{\dddot{\mathbf{v}}}{24c^4} + \dots\right)\nabla$, where ∇ is treated as a vector.

202. Expression for $\mathbf{V}^{(k)}$. $V^{(k)}$ is a function of the coordinates (x, y, z) and (ξ, η, ζ) and of the velocity $\mathbf{v}$, but is independent of the accelerations of all orders. Hence (302) gives as it were an osculating expansion for $U^{(k)}$, and the motion which osculates the given motion at the time t is a uniform rectilinear motion, in which the charge is in its actual position and is moving with its actual velocity. The value of $U^{(k)}$ for this osculating motion is precisely the function $V^{(k)}$. In particular, the value of $U^{(0)}$ for the osculating motion is $V^{(0)}$.

But $U^{(0)}$ is the scalar potential of the osculating motion, by (297), § 200, and we know that this potential is given by (47), § 23, since the velocity is less than that of light in our present problem, otherwise the series (180) and (181), § 124, would not hold.

Hence, when the axis of x is chosen to lie in the direction of the instantaneous motion, and the axes of y and z are any two mutually perpendicular lines normal to the path of the charge, for instance, the principal normal and binormal, we have

$$V^{(0)} = 1/S \quad \dots\dots(304),$$

where $S = \{(x-\xi)^2 + (1 - v^2/c^2)((y-\eta)^2 + (z-\zeta)^2)\}^{\frac{1}{2}}$.

This value may also be deduced from (303); for with the present choice of axes, the symbolic operator $(\mathbf{v}\nabla)$ reduces to $v\frac{\partial}{\partial x}$ and therefore

$$V^{(0)} = \sum_{s=0}^{s=\infty} \frac{(-v/c)^s}{s!} \frac{\partial^s R^{s-1}}{\partial x^s}.$$

Now we have identically

$$\frac{\partial^{2s} R^{2s-1}}{\partial x^{2s}} = \frac{1^2 . 3^2 . \dots (2s-1)^2 \{(y-\eta)^2 + (z-\zeta)^2\}^s}{R^{2s+1}}, \quad \frac{\partial^{2s+1} R^{2s}}{\partial x^{2s+1}} = 0.$$

Hence substituting we get

$$V^{(0)} = \sum_{s=0}^{s=\infty} \frac{1.3\ldots.(2s-1)}{2.4\ldots.(2s)} \frac{v^{2s}\{(y-\eta)^2+(z-\zeta)^2\}^s}{c^{2s}R^{2s+1}} = \frac{1}{S}.$$

The value of $V^{(k)}$ may be deduced as follows.

In the first place, we proceed to find $V^{(1)}$. We have

$$\frac{\partial^{2s}R^{2s}}{\partial x^{2s}} = 2s\,!,$$

$$\frac{\partial^{2s+1}R^{2s+1}}{\partial x^{2s+1}} = 1^2.3^2\ldots.(2s+1)^2\left\{\frac{x-x'}{R} - \frac{s}{1\,!}\frac{(x-x')^3}{3R^3} + \frac{s(s-1)}{2\,!}\frac{(x-x')^5}{5R^5} - \ldots\right\}.$$

Substituting these values in the series (303), we get

$$V^{(1)} = \frac{(v/c)(x-\xi)+S}{(1-v^2/c^2)\,S} \quad\ldots\ldots\ldots\ldots(305).$$

We next proceed to establish an equation of differences for $V^{(k)}$; we have identically

$$\frac{\partial^s R^{s+k-1}}{\partial x^s} = R^2\frac{\partial^s R^{s+k-3}}{\partial x^s} + 2s(x-\xi)\frac{\partial^{s-1}R^{s+k-3}}{\partial x^{s-1}} + s(s-1)\frac{\partial^{s-2}R^{s+k-1}}{\partial x^{s-2}}.$$

Substituting in the series (303), § 201, for $V^{(k)}$ we get

$$V^{(k)} = \frac{2(v/c)(x-\xi)}{1-v^2/c^2}V^{(k-1)} + \frac{R^2}{1-v^2/c^2}V^{(k-2)}.$$

Solving this equation in the usual way we find

$$V^{(k)} = Aa^k + Bb^k,$$

where A and B are arbitrary constants, and a and b are the roots of the quadratic

$$a^2 - 2\frac{(v/c)(x-\xi)}{1-v^2/c^2}a - \frac{R^2}{1-v^2/c^2} = 0.$$

Thus $$a = \frac{(v/c)(x-\xi)+S}{1-v^2/c^2} \quad\text{and}\quad b = \frac{(v/c)(x-\xi)-S}{1-v^2/c^2}.$$

Making $k=0$ and 1 in succession, we find that $A=1/S$ while $B=0$; hence

$$V^{(k)} = \frac{\{(v/c)(x-\xi)+S\}^k}{(1-v^2/c^2)^k\,S} \quad\ldots\ldots\ldots\ldots(306).$$

Without reference to any special system of axes we may write this equation vectorially in the form

$$V^{(k)} = \frac{\{(\mathbf{vR})/c+S\}^k}{(1-v^2/c^2)^k\,S} \quad\ldots\ldots\ldots\ldots(307),$$

where $$S = \{(\mathbf{vR})^2/c^2 + (1-v^2/c^2)R^2\}^{\frac{1}{2}},$$

and $\mathbf{R}$ is the vector drawn from the point (ξ, η, ζ) to the fieldpoint (x, y, z).

If θ be the angle made by $\mathbf{R}$ with the direction of motion, and $\beta = v/c$, we have

$$(\mathbf{vR})/c = R\beta\cos\theta, \quad S = R\sqrt{(1-\beta^2\sin^2\theta)}.$$

Hence
$$V^{(k)} = \frac{\{\beta\cos\theta + \sqrt{(1-\beta^2\sin^2\theta)}\}^k}{(1-\beta^2)^k\sqrt{(1-\beta^2\sin^2\theta)}} R^{k-1} \quad\text{..............(308).}$$

This expression shows that $V^{(k)}$ is of the order $k-1$ in the distance R.

203. First form of the expansions for the potentials. Substituting the expression (302), § 201, for $U^{(k)}$ in (297) and (298), § 200, we get

$$\phi = V^{(0)} - \frac{(\dot{\mathbf{v}}\nabla)}{2c^2} V^{(2)} + \frac{(\ddot{\mathbf{v}}\nabla)}{6c^3} V^{(3)} + \frac{(\dot{\mathbf{v}}\nabla)^2}{8c^4} V^{(4)} + \dots \quad\text{.........(309),}$$

$$\mathbf{a} = \frac{\mathbf{v}}{c}\left\{V^{(0)} - \frac{(\dot{\mathbf{v}}\nabla)}{2c^2} V^{(2)} + \frac{(\ddot{\mathbf{v}}\nabla)}{6c^3} V^{(3)} + \frac{(\dot{\mathbf{v}}\nabla)^2}{8c^4} V^{(4)} + \dots\right\}$$
$$- \frac{\dot{\mathbf{v}}}{c^2}\left\{V^{(1)} - \frac{(\dot{\mathbf{v}}\nabla)}{2c^2} V^{(3)} + \dots\right\} + \frac{\ddot{\mathbf{v}}}{2c^3}\left\{V^{(2)} - \dots\right\} - \dots \quad\text{...(310).}$$

We notice that $V^{(0)}$ is of the order -1 in R, by (307), $V^{(1)}$ is of order 0, $V^{(2)}$ of order 1, $V^{(3)}$ of order 2, $V^{(4)}$ of order 3, and so on.

Further, the operator ∇ reduces the order of its operand by one unit, so that the function $(\dot{\mathbf{v}}\nabla) V^{(2)}$ is of order 0, and the functions $(\dot{\mathbf{v}}\nabla) V^{(3)}$, $(\ddot{\mathbf{v}}\nabla) V^{(3)}$ and $(\dot{\mathbf{v}}\nabla)^2 V^{(4)}$ are all of the order 1 in R, the quantities $\dot{\mathbf{v}}$, $\ddot{\mathbf{v}}$, ... not counting in establishing the order, since they are quite arbitrary. Hence the expansions (309) and (310) are series proceeding according to terms of ascending order in R. The principal terms, depending on $V^{(0)}$, are of order -1, and involve the coordinates and velocity only; the terms next in importance, depending on $V^{(1)}$ and $\nabla V^{(2)}$, are of order 0, and involve the acceleration in addition to the coordinates and velocity; the next, depending on $V^{(2)}$, $\nabla V^{(3)}$ and $\nabla^2 V^{(4)}$, are of order 1, and involve the acceleration of the second order in addition, and so on.

204. Second form of the expansions. The expansions just found are convenient when the expressions for the functions $V^{(k)}$ are comparatively simple, particularly when we are dealing with a point charge. But in other cases, particularly when we are dealing with an extended charge and have to integrate throughout a given volume, the presence of the higher functions $V^{(k)}$ gives rise to troublesome integrations. For instance, this difficulty arises when we wish to find the mechanical force exerted by an electron on itself due to the interaction of its several elements. It is possible to replace the higher functions $V^{(k)}$ by functions of lower order in the following way.

In this second method of development we regard $U^{(k)}$ as a function of the independent variables (x, y, z), (ξ, η, ζ), (v_x, v_y, v_z). $(\dot{v}_x, \dot{v}_y, \dot{v}_z)$, ..., all the components of the velocity and of the accelerations of various orders being treated as independent since the motion is arbitrary.

Let D denote the partial differential operator whose components are $\left(\frac{\partial}{\partial v_x}, \frac{\partial}{\partial v_y}, \frac{\partial}{\partial v_z}\right)$, so that D bears the same relation to $\mathbf{v}$ that ∇ does to the vector $\mathbf{r}$ whose components are (x, y, z). From what has been said we must suppose that D operates only on functions of $\mathbf{v}$, and not at all on (x, y, z), (ξ, η, ζ), $(\dot{v}_x, \dot{v}_y, \dot{v}_z)$, or any component of a higher order acceleration. In particular it does not operate on R, while $D(\mathbf{v}\nabla) = \nabla$.

Operate with D on the series (303), § 201; we obtain

$$DV^{(k)} = \sum_{s=1}^{s=\infty} \frac{\nabla (\mathbf{v}\nabla)^{s-1} R^{s+k-1}}{\underline{s-1}!\, c^s}.$$

Replacing s by $s+1$ in the series and using (303), we get

$$\nabla V^{(k+1)} = cDV^{(k)} \quad \text{..........................(311).}$$

This relation is fundamental; by means of it we can express the function $U^{(k)}$ in terms of functions of lower order. We find from (302), § 201,

$$U^{(k)} = V^{(k)} - \frac{(\dot{\mathbf{v}}D)}{2c} V^{(k+1)} + \frac{(\ddot{\mathbf{v}}D)}{6c^2} V^{(k+2)} + \frac{(\dot{\mathbf{v}}D)^2}{8c^2} V^{(k+2)} + \ldots \quad \text{...(312).}$$

205. Substituting from this equation in (297) and (298), § 200, we get the following expansions for the potentials:

$$\phi = V^{(0)} - \frac{(\dot{\mathbf{v}}D)}{2c} V^{(1)} + \frac{(\ddot{\mathbf{v}}D)}{6c^2} V^{(2)} + \frac{(\dot{\mathbf{v}}D)^2}{8c^2} V^{(2)} + \ldots \quad \text{.........(313).}$$

$$\mathbf{a} = \frac{\mathbf{v}}{c}\left\{V^{(0)} - \frac{(\dot{\mathbf{v}}D)}{2c} V^{(1)} + \frac{(\ddot{\mathbf{v}}D)}{6c^2} V^{(2)} + \frac{(\dot{\mathbf{v}}D)^2}{8c^2} V^{(2)} + \ldots\right\}$$
$$- \frac{\dot{\mathbf{v}}}{c^2}\left\{V^{(1)} - \frac{(\dot{\mathbf{v}}D)}{2c} V^{(2)} + \ldots\right\} + \frac{\ddot{\mathbf{v}}}{2c^3}\left\{V^{(2)} - \ldots\right\} - \ldots \quad \text{......(314).}$$

As before we have

$$V^{(k)} = \frac{\{(\mathbf{vR})/c + S\}^k}{(1 - v^2/c^2)^k S} \quad \text{..................................(307),}$$

where $$S = \{(\mathbf{vR})^2/c^2 + (1 - v^2/c^2) R^2\}^{\frac{1}{2}}.$$

When we regard R as of the first order we must treat $V^{(0)}$ as of order -1, $V^{(1)}$ as of order 0, $V^{(2)}$ as of order 1, and so on. Moreover, the operator D leaves the order of its operand unaltered.

Thus the approximations in (313) and (314) are carried to the order 1. The principal terms are of the order -1 and involve the velocity, but not the accelerations.

The terms next in importance are of order 0 and involve the acceleration as well as the velocity, but not the accelerations of the second and higher orders. The smallest terms written down are of order 1, and involve the accelerations of the two lowest orders as well as the velocity, but not those of the third and higher orders, and so on.

The principal terms give the potentials due to the osculating uniform rectilinear motion, in which we have a point charge occupying the same position, and moving with the same speed and in the same direction as the actual charge at the time considered, but without acceleration. The terms of the two lowest orders together give the potentials due to an osculating motion in the circle of curvature with the actual speed and acceleration, and so on.

The series (309) and (310), § 203, or (313) and (314), are equivalent to the series obtained by Schwarzschild*. They converge provided that R be less than a definite multiple of the distance cT, where T is an upper limit to the interval of time for which the developments of the coordinates of the point charge in powers of t are absolutely convergent. This condition excludes times for which an acceleration, of whatsoever order, becomes infinite; that is to say, it excludes times at which the motion of the charge becomes discontinuous, as was to be expected. Moreover, it assumes that all the functions $V^{(k)}$ remain finite; this requires that the denominators, and therefore also the quantity $1 - v^2/c^2$, be different from zero. In fact convergence requires that the speed of the charge be always less than that of light.

* *loc. cit.* pp. 249 and 250, IV *a* and IV *b*.

APPENDIX D

THE MECHANICAL FORCE EXERTED BY AN ELECTRON ON ITSELF

206. OUR object in this appendix is to calculate the internal mechanical force, $\mathbf{F}_i$, on a moving electron, the expressions for which have been given in § 143, Ch. XI, for several types of electron. This will be done with as few restrictions as possible on the form and structure of the electron, as a preparation for an investigation—to be carried out in Appendix E—of the conditions under which a mechanical explanation of the electron is feasible.

We shall start from the expansions (313) and (314), § 205, for the potentials due to a moving point charge, calculate the mechanical force due to one element of the electron on another by means of the general formula of Lorentz (VI), § 140, and sum for the whole electron, thus obtaining the resultant mechanical force exerted by it on one of its elements. This will be required for the investigation of Appendix E.

For the purpose of verifying the expressions given in § 143, we shall integrate the expression just mentioned over the whole electron and in this way obtain an expression for the internal mechanical force $\mathbf{F}_i$. It need hardly be pointed out that the force $\mathbf{F}_i$ only vanishes for an electron which moves uniformly without any relative motion of its parts, because Newton's Third Law does not hold for two electric charges in relative motion.

207. The convergence of the series (313) and (314) requires that the greatest distance between any two elements of the electron be small compared with such lengths as the radius of curvature of the path of either element, and that their speeds be less than that of light. When there is relative motion, due to rotation or deformation of the electron, these two conditions require that the relative velocity of the two elements be small compared with that of light; consequently the velocities of rotation and of strain must all be finite. For the present we shall make no further assumptions respecting the form or structure of the electron.

208. **Notation.** In order to express the relative motion explicitly we shall use a moving origin, namely the electric centre already introduced in the footnote to § 155. We shall denote its radius vector and velocity relative to axes fixed with respect to the observer by $\mathbf{r}$ and $\mathbf{v}$ respectively.

Let de_1 denote an elementary charge of the electron, and $\mathbf{r}_1$, $\mathbf{v}_1$, $\mathbf{u}_1$ its radius vector and velocity relative to fixed axes, and relative to the centre respectively. Then we have from the definition of the electric centre

$$\left.\begin{aligned} e\mathbf{r} &= \int \mathbf{r}_1 de_1 \\ e\mathbf{v} &= \int \mathbf{v}_1 de_1 \\ 0 &= \int \mathbf{u}_1 de_1 \end{aligned}\right\} \quad \ldots\ldots\ldots(315).$$

If we expand $\mathbf{u}_1$ in powers of the coordinates of de_1 relative to the electric centre and substitute in the last equation (315), the linear terms disappear on integration on account of the first equation (315). Hence the constant term of the expansion must just neutralize the terms of the second and higher degrees. This means that the electric centre moves about in the electron, because its velocity differs by second order quantities from that of the element of charge with which it coincides for the moment.

209. Denote by $\mathbf{R}$ the vector drawn *from* de_1 *to* de_2, so that $\mathbf{R} = \mathbf{r}_2 - \mathbf{r}_1$.

Also denote by $V_1^{(k)}$, or $V_2^{(k)}$, as the case may be, what the function $V^{(k)}$, defined by (307), § 202, becomes when $\mathbf{v}$ is replaced by $\mathbf{v}_1$, or $\mathbf{v}_2$, and (x, y, z) and (ξ, η, ζ) by (x_2, y_2, z_2) and (x_1, y_1, z_1), or (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively. Then we get from (307)

$$\left.\begin{aligned} V_1^{(k)} &= \frac{\{(\mathbf{v}_1\mathbf{R})/c + S_1\}^k}{(1 - v_1^2/c^2)^k S_1}, \qquad S_1 = \{(\mathbf{v}_1\mathbf{R})^2/c^2 + (1 - v_1^2/c^2) R^2\}^{\frac{1}{2}} \\ V_2^{(k)} &= \frac{\{-(\mathbf{v}_2\mathbf{R})/c + S_2\}^k}{(1 - v_2^2/c^2)^k S_2}, \qquad S_2 = \{(\mathbf{v}_2\mathbf{R})^2/c^2 + (1 - v_2^2/c^2) R^2\}^{\frac{1}{2}} \end{aligned}\right\} \ldots(316).$$

Thus $V_1^{(k)}$ and $V_2^{(k)}$ differ in two ways: (1) because the sign of $\mathbf{R}$ is changed; (2) because $\mathbf{v}_1$ and $\mathbf{v}_2$ are interchanged, that is to say, on account of the relative motion.

210. Let ϕ_{12} and $\mathbf{a}_{12}$, or ϕ_{21} and $\mathbf{a}_{21}$, as the case may be, denote the potentials at the position of de_2 due to unit charge at de_1, or at de_1 due to unit charge at de_2. Also let D_1, or D_2, denote vector differentiation with respect to the components of $\mathbf{v}_1$, or $\mathbf{v}_2$, respectively. Then ϕ_{12} and $\mathbf{a}_{12}$ are got from (313) and (314), § 205, in the form

$$\phi_{12} = V_1^{(0)} - \frac{(\dot{\mathbf{v}}_1 D_1)}{2c} V_1^{(1)} + \frac{(\ddot{\mathbf{v}}_1 D_1)}{6c^2} V_1^{(2)} + \frac{(\dot{\mathbf{v}}_1 D_1)^2}{8c^2} V_1^{(2)} + \ldots \quad \ldots\ldots(317),$$

$$\mathbf{a}_{12} = \frac{\mathbf{v}_1}{c}\left\{V_1^{(0)} - \frac{(\dot{\mathbf{v}}_1 D_1)}{2c} V_1^{(1)} + \frac{(\ddot{\mathbf{v}}_1 D_1)}{6c^2} V_1^{(2)} + \frac{(\dot{\mathbf{v}}_1 D_1)^2}{8c^2} V_1^{(2)} + \ldots\right\}$$
$$- \frac{\dot{\mathbf{v}}_1}{c^2}\left\{V_1^{(1)} - \frac{(\dot{\mathbf{v}}_1 D_1)}{2c} V_1^{(2)}\right\} + \frac{\ddot{\mathbf{v}}_1}{2c^3}\left\{V_1^{(2)} - \ldots\right\} \ldots(318).$$

ϕ_{21} and $\mathbf{a}_{21}$ are given by the same expressions with a suffix 2 in place of the suffix 1.

211. Let $\mathbf{f}_{12}$, or $\mathbf{f}_{21}$, denote the mechanical force on a unit charge at de_2 due to a unit charge at de_1, or on a unit charge at de_1 due to a unit charge at de_2, as the case may be. They are found most easily from (XV), § 6,

$$\mathbf{f} = -\nabla\phi + \nabla_a \frac{(\mathbf{va})}{c} - \frac{d\mathbf{a}}{c\,dt},$$

where ∇_a operates only on $\mathbf{a}$, not on $\mathbf{v}$.

Let ∇_1, or ∇_2, denote vector differentiations with respect to the components of $\mathbf{r}_1$, or $\mathbf{r}_2$, as the case may be.

Putting $\phi = \phi_{12}$, $\mathbf{a} = \mathbf{a}_{12}$, $\mathbf{v} = \mathbf{v}_2$ and $\nabla = \nabla_2$, we get by means of (317) and (318)

$$\begin{aligned}\mathbf{f}_{12} = -\left\{1 - \frac{(\mathbf{v}_1\mathbf{v}_2)}{c^2}\right\}\Bigg\{\nabla_2 V_1^{(0)} - \frac{(\dot{\mathbf{v}}_1 D_1)}{2c}\nabla_2 V_1^{(1)} + \frac{(\ddot{\mathbf{v}}_1 D_1)}{6c^2}\nabla_2 V_1^{(2)} \\ + \frac{(\dot{\mathbf{v}}_1 D_1)^2}{8c^2}\nabla_2 V_1^{(2)} + \dots\Bigg\} \\ - \frac{(\dot{\mathbf{v}}_1\mathbf{v}_2)}{c^3}\left\{\nabla_2 V_1^{(1)} - \frac{(\dot{\mathbf{v}}_1 D_1)}{2c}\nabla_2 V_1^{(2)} + \dots\right\} + \frac{(\ddot{\mathbf{v}}_1\mathbf{v}_2)}{2c^4}\left\{\nabla_2 V_1^{(2)} - \dots\right\} - \dots \\ - \frac{d}{dt}\left[\frac{\mathbf{v}_1}{c^2}\left\{V_1^{(0)} - \frac{(\dot{\mathbf{v}}_1 D_1)}{2c}V_1^{(1)} + \dots\right\} - \frac{\dot{\mathbf{v}}_1}{c^3}\left\{V_1^{(1)} - \dots\right\} + \dots\right] \quad \dots(319).\end{aligned}$$

$\mathbf{f}_{21}$ is got by interchanging the suffixes 1 and 2.

Since the operator ∇_2 lowers the order of its operand by one unit, the principal term in $\mathbf{f}_{12}$ is that involving $\nabla_2 V_1^{(0)}$, and it is of order -2. The terms next in importance are those which involve $V_1^{(0)}$ and $\nabla_2 V_1^{(1)}$, and they are of order -1. All the remaining terms are of order 0, and those not written down are of order 1, or higher.

212. The mechanical force exerted on the element de_2 by the rest of the electron may be written in the form $\mathbf{F}_{i2} de_2$, where

$$\mathbf{F}_{i2} = \int \mathbf{f}_{12} de_1 \quad \dots\dots\dots(320).$$

This is the force required for the investigation of Appendix E. As a rule the suffix 2 may be omitted as unnecessary.

The resultant internal force exerted by the electron on itself is given by

$$\mathbf{F}_i = \int \mathbf{F}_{i2} de_2 = \int \mathbf{F}_{i1} de_1 = \tfrac{1}{2}\int (\mathbf{f}_{12} + \mathbf{f}_{21})\, de_1\, de_2 \quad \dots\dots\dots(321).$$

The last, symmetrical, form is the most convenient for use. In it both integrations are extended over the whole electron.

This is the force for which expressions have been given in Ch. XI, § 143. It does not generally vanish, because the relation required by Newton's Third Law, namely, $\mathbf{f}_{12} + \mathbf{f}_{21} = 0$, is only satisfied for two charges which move with the same uniform speed in parallel directions.

The method just outlined for finding the mechanical force, based as it is on the series (319), involves troublesome expansions, but the work is very much simplified owing to the fact that a large number of terms cancel out from (321) by symmetry, and those left in happen to be easily amenable to calculation.

We might have deduced the force by an application of the Principle of Least Action, based on the electrokinetic potential of Schwarzschild, but the calculation of the Action involves more troublesome integrations than the direct method, which for that reason has been preferred here.

213. Before integrating the expression (319) we shall find it convenient to expand it in terms of the relative velocity $\mathbf{u}$, but as a preliminary to this process we shall transform (319) slightly. Since the function $V_1^{(1)}$ only involves $\mathbf{v}_1$ and the differences of the coordinates $(x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1)$ we have

$$\frac{dV_1^{(1)}}{dt} = (\dot{\mathbf{v}}_1 D_1)\, V_1^{(1)} + (\{\mathbf{v}_2 - \mathbf{v}_1\}\, \nabla_2)\, V_1^{(1)}.$$

Also from (311), § 204, we get $\nabla_2 V_1^{(1)} = cD_1 V_1^{(0)}$. Hence

$$\frac{(\dot{\mathbf{v}}_1 D_1)}{2c}\, V_1^{(1)} = \frac{d}{dt}\left(\frac{V_1^{(1)}}{2c}\right) - \tfrac{1}{2}\,(\{\mathbf{v}_2 - \mathbf{v}_1\}\, D_1)\, V_1^{(0)}.$$

A precisely similar relation holds for $\nabla_2 V_1^{(1)}$.

Substituting these results in (319), and using (311), § 204, wherever possible, we find after some simple transformations

$$\begin{aligned}
\mathbf{f}_{12} = &- \left\{1 - \frac{(\mathbf{v}_1\mathbf{v}_2)}{c^2}\right\} \left\{\nabla_2 V_1^{(0)} + \tfrac{1}{2}\,(\{\mathbf{v}_2 - \mathbf{v}_1\}\, D_1)\, \nabla_2 V_1^{(0)} + \frac{(\ddot{\mathbf{v}}_1 D_1)}{6c}\, D_1 V_1^{(1)} \right. \\
&\qquad \left. + \frac{(\dot{\mathbf{v}}_1 D_1)^2}{8c}\, D_1 V_1^{(1)}\right\} \\
&+ \frac{(\mathbf{v}_1\dot{\mathbf{v}}_2) - (\dot{\mathbf{v}}_1\mathbf{v}_2)}{2c^2}\, D_1 V_1^{(0)} + \frac{(\dot{\mathbf{v}}_1\mathbf{v}_2)\,(\dot{\mathbf{v}}_1 D_1)}{2c^3}\, D_1 V_1^{(1)} + \frac{(\ddot{\mathbf{v}}_1\mathbf{v}_2)}{2c^3}\, D_1 V_1^{(1)} \\
&+ \frac{d}{dt}\left[\tfrac{1}{2}\left\{1 - \frac{(\mathbf{v}_1\mathbf{v}_2)}{c^2}\right\} D_1 V_1^{(0)} - \frac{\mathbf{v}_1}{c^2}\{V_1^{(0)} + \tfrac{1}{2}\,(\{\mathbf{v}_2 - \mathbf{v}_1\}\, D_1)\, V_1^{(0)}\} + \frac{\dot{\mathbf{v}}_1}{c^3}\, V_1^{(1)} \right. \\
&\qquad \left. + \frac{\mathbf{v}_1}{2c^3}\, \frac{dV_1^{(1)}}{dt}\right] \quad \ldots(322).
\end{aligned}$$

This form for $\mathbf{f}_{12}$ has the advantage that, by means of (311), all the higher functions $V_1^{(k)}$ have been replaced by the two lowest, $V_1^{(0)}$ and $V_1^{(1)}$, so far as the terms of the three lowest orders are concerned; the functions $V_1^{(2)}$, ... only occur in terms of order higher than zero, which have been neglected in (319) and (322).

214. Expansion in powers of the relative velocity. We must now expand (322) in powers of the relative velocity. In doing this we must bear in mind that, in accordance with § 207, the relative velocities $\mathbf{u}_1$ and $\mathbf{u}_2$ are to be reckoned as small quantities of the first order. To be quite exact, the ratios $\mathbf{u}_1/c$ and $\mathbf{u}_2/c$ are to be treated as of the same order as the ratio of the distance R to the radius of curvature of the path, or any similar length. We shall begin with the expansion of the function $V_1^{(k)}$. Replacing $\mathbf{v}$ by $\mathbf{v}_1$ and ∇ by ∇_2 in the series (303), § 201, we get

$$V_1^{(k)} = \sum_{s=0}^{s=\infty} \frac{(\mathbf{v}_1\nabla_2)^s}{s!\, c^s} R^{s+k-1} = \sum_{s=0}^{s=\infty} \frac{(\{\mathbf{v}+\mathbf{u}_1\}\nabla_2)^s}{s!\, c^s} R^{s+k-1}.$$

Now ∇_2 does not operate on $\mathbf{v}$ or $\mathbf{u}_1$; hence, expanding the symbolic operators, arranging according to powers of $\mathbf{u}_1$, and using (303) and (311), we find

$$V_1^{(k)} = \{1 + (\mathbf{u}_1 D) + \tfrac{1}{2}(\mathbf{u}_1 D)^2 + \ldots\} V^{(k)} \quad \ldots\ldots\ldots\ldots(323),$$

where D may operate on $\mathbf{u}_1$, provided that we interpret $(\mathbf{u}_1 D)^2$ to mean $(\mathbf{u}_1(\mathbf{u}_1 D) D)$, and so on, always taking care to keep the velocity $\mathbf{u}_1$ in front of all the operators D whenever it depends on $\mathbf{v}$.

215. We see from (316), § 209, that the function $V^{(k)}$ is defined by (307), § 202. If we separate the rational and irrational parts of this equation we can write

$$\left.\begin{aligned} V^{(k)} &= V_k + W_k \\ V_k &= \frac{(\mathbf{vR})^k c^{-k} + (k)_2 S^2 (\mathbf{vR})^{k-2} c^{-(k-2)} + \ldots}{(1-v^2/c^2)^k S} \\ W_k &= \frac{(k)_1 (\mathbf{vR})^{k-1} c^{-(k-1)} + (k)_3 S^2 (\mathbf{vR})^{k-3} c^{-(k-3)} + \ldots}{(1-v^2/c^2)^k} \end{aligned}\right\} \ldots(324).$$

V_k is an irrational homogeneous function of the differences of the coordinates $(x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1)$ of the degree $k-1$, and W_{k-1} is an integral rational homogeneous function of the same degree.

The first two pairs of functions are

$$\left.\begin{aligned} V_0 &= \frac{1}{S}, & W_0 &= 0 \\ V_1 &= \frac{(\mathbf{vR})}{c(1-v^2/c^2)S}, & W_1 &= \frac{1}{1-v^2/c^2} \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots(325).$$

We see from (316) that the corresponding expansions for the functions $V_2^{(k)}$ are got from (323)—(325) by interchanging the suffixes 1 and 2, that is, by replacing $\mathbf{u}_1$ in (323) by $\mathbf{u}_2$, and changing the signs of all the odd functions V_k, namely V_1, V_3, ..., and those of all the even functions W_k, namely W_0, W_2, ...

216. Operate on both sides of (323) with ∇_2, bearing in mind that ∇_2 does not operate on $\mathbf{u}_1$, nor on $\mathbf{v}$, and is commutative with D. Then we get

$$\nabla_2 V_1^{(k)} = \{1 + (\mathbf{u}_1 D) + \tfrac{1}{2}(\mathbf{u}_1 D)^2 + \ldots\} \nabla_2 V^{(k)} \quad \text{.........(326).}$$

Remembering (311), § 204, we see that we may replace $\nabla_2 V_1^{(k)}$ by $D_1 V_1^{(k)}$ and $\nabla_2 V^{(k)}$ by $DV^{(k)}$. Multiplying scalarly by $\mathbf{v}_2 - \mathbf{v}_1$, which is the same thing as $\mathbf{u}_2 - \mathbf{u}_1$, we get as far as terms of the second degree,

$$(\{\mathbf{v}_2 - \mathbf{v}_1\} D_1) V_1^{(k)} = \left\{(\{\mathbf{u}_2 - \mathbf{u}_1\} D) + (\{\mathbf{u}_2 - \mathbf{u}_1\} (\mathbf{u}_1 D) D)\right\} V^{(k)} \ldots\text{(327).}$$

Adding one half of this to (323) we get

$$V_1^{(k)} + \tfrac{1}{2}(\{\mathbf{v}_2 - \mathbf{v}_1\} D_1) V_1^{(k)} = \left\{1 + \tfrac{1}{2}(\{\mathbf{u}_1 + \mathbf{u}_2\} D) + \tfrac{1}{2}(\mathbf{u}_2 (\mathbf{u}_1 D) D)\right\} V^{(k)} \quad \text{......(328).}$$

Operating on (327) with ∇_2, remembering that ∇_2 operates on $\mathbf{v}_2$ and $\mathbf{u}_2$, but not on $\mathbf{u}_1$, or $\mathbf{v}$ or $\mathbf{v}_1$, and using (323), we easily see that we may replace $V_1^{(k)}$ in (328) by $\nabla_2 V_1^{(k)}$, provided that we replace $V^{(k)}$ by $\nabla_2 V^{(k)}$.

217. Transformation of the force. We shall now apply the results just obtained in §§ 214—216 to the expansion of the expression (322), § 213, in powers of the relative velocities, beginning with the large terms of orders -2 and -1, which are conveniently taken together. These terms are the first and second in the first line and the first and second in the last. We get by means of (323), (326) and (328)

$$-\left\{1 - \frac{(\mathbf{v}_1\mathbf{v}_2)}{c^2}\right\}\left\{1 + \tfrac{1}{2}(\{\mathbf{u}_1 + \mathbf{u}_2\} D) + \tfrac{1}{2}(\mathbf{u}_2(\mathbf{u}_1 D) D)\right\} \nabla_2 V_0$$
$$+ \frac{d}{dt}\left[\tfrac{1}{2}\left\{1 - \frac{(\mathbf{v}_1\mathbf{v}_2)}{c^2}\right\}\{1 + (\mathbf{u}_1 D)\} D V_0 - \frac{\mathbf{v}_1}{c^2}\left\{1 + \tfrac{1}{2}(\{\mathbf{u}_1 + \mathbf{u}_2\} D)\right\} V_0\right] \quad \text{......(329).}$$

Consider the first line of (329). Since $\mathbf{v}_1 = \mathbf{v} + \mathbf{u}_1$, $\mathbf{v}_2 = \mathbf{v} + \mathbf{u}_2$, we have

$$1 - \frac{(\mathbf{v}_1\mathbf{v}_2)}{c^2} = 1 - \frac{v^2}{c^2} - \frac{(\{\mathbf{u}_1 + \mathbf{u}_2\} \mathbf{v})}{c^2} - \frac{(\mathbf{u}_1\mathbf{u}_2)}{c^2}.$$

Hence retaining only squares and products of $\mathbf{u}_1$ and $\mathbf{u}_2$, we get for the first line of (329)

$$-\left(1 - \frac{v^2}{c^2}\right)\nabla_2 V_0 - \tfrac{1}{2}\left(1 - \frac{v^2}{c^2}\right)(\{\mathbf{u}_1 + \mathbf{u}_2\} D)\nabla_2 V_0 + \frac{(\{\mathbf{u}_1 + \mathbf{u}_2\}\mathbf{v})}{c^2}\nabla_2 V_0$$
$$-\tfrac{1}{2}\left(1 - \frac{v^2}{c^2}\right)(\mathbf{u}_2(\mathbf{u}_1 D) D)\nabla_2 V_0 + \frac{(\{\mathbf{u}_1 + \mathbf{u}_2\}\mathbf{v})}{2c^2}(\{\mathbf{u}_1 + \mathbf{u}_2\} D)\nabla_2 V_0 + \frac{(\mathbf{u}_1\mathbf{u}_2)}{c^2}\nabla_2 V_0,$$

which reduces to

$$-\left\{1 + \tfrac{1}{2}(\{\mathbf{u}_1 + \mathbf{u}_2\} D) + \tfrac{1}{2}(\mathbf{u}_2(\mathbf{u}_1 D) D)\right\}\left(1 - \frac{v^2}{c^2}\right)\nabla_2 V_0$$
$$+ \frac{(\{\mathbf{u}_2 - \mathbf{u}_1\}\mathbf{v})}{2c^2}(\{\mathbf{u}_2 - \mathbf{u}_1\} D)\nabla_2 V_0 \quad \text{............................(330),}$$

because $D(1 - v^2/c^2) = -2\mathbf{v}/c^2$, and consequently

$$(\{\mathbf{u}_1 + \mathbf{u}_2\}\,\mathbf{v})/c^2 = -\tfrac{1}{2}\,(\{\mathbf{u}_1 + \mathbf{u}_2\}\,D)\,(1 - v^2/c^2),$$

and

$$(\mathbf{u}_1\mathbf{u}_2)/c^2 = -\tfrac{1}{2}\,(\mathbf{u}_2\,(\mathbf{u}_1 D)\,D)\,(1 - v^2/c^2).$$

In the same way we get for the second line of (329)

$$\frac{d}{dt}\left[\tfrac{1}{2}\left\{1 + \tfrac{1}{2}\,(\{\mathbf{u}_1 + \mathbf{u}_2\}\,D)\right\} D\left(1 - \frac{v^2}{c^2}\right) V_0 + \frac{\mathbf{u}_2 - \mathbf{u}_1}{2c^2}\,V_0 - \tfrac{1}{4}\left(1 - \frac{v^2}{c^2}\right)(\{\mathbf{u}_2 - \mathbf{u}_1\}\,D)\,D V_0\right] \quad \ldots(331).$$

218. The remaining terms in (322), § 213, are all small of the order 0, and we may use first approximations.

In the first term of the second line we may replace $D_1 V_1^{(0)}$ by its first approximation DV_0, and in the factor neglect the products $(\mathbf{u}_1\dot{\mathbf{u}}_2)$ and $(\dot{\mathbf{u}}_1\mathbf{u}_2)$. Thus we get

$$\frac{(\{\dot{\mathbf{u}}_2 - \dot{\mathbf{u}}_1\}\,\mathbf{v}) - (\{\mathbf{u}_2 - \mathbf{u}_1\}\,\dot{\mathbf{v}})}{2c^2}\,DV_0 \quad \ldots\ldots\ldots\ldots(332).$$

In the remaining terms, which involve $V_1^{(1)}$, we may replace $\mathbf{v}_1, \dot{\mathbf{v}}_1, \ddot{\mathbf{v}}_1, D_1$ by the symbols $\mathbf{v}$, $\dot{\mathbf{v}}$, $\ddot{\mathbf{v}}$ and D, and $V_1^{(1)}$ by its first approximation $V_1 + W_1$, where V_1 and W_1 have the values (325), § 215. In this way we find

$$\left[-\left(1 - \frac{v^2}{c^2}\right)\left\{\frac{(\ddot{\mathbf{v}}D)}{6c} + \frac{(\dot{\mathbf{v}}D)^2}{8c}\right\} + \frac{(\mathbf{v}\dot{\mathbf{v}})(\dot{\mathbf{v}}D)}{2c^3} + \frac{(\mathbf{v}\ddot{\mathbf{v}})}{2c^3}\right] DV_1 + \frac{d}{dt}\left[\frac{\dot{\mathbf{v}}}{c^3}\,V_1 + \frac{\mathbf{v}}{2c^3}\frac{dV_1}{dt}\right]$$
$$+ \frac{2\ddot{\mathbf{v}}}{3c\,(c^2 - v^2)} + \frac{2\,(\mathbf{v}\ddot{\mathbf{v}})\,\mathbf{v}}{3c\,(c^2 - v^2)^2} + \frac{2\,(\mathbf{v}\dot{\mathbf{v}})\,\dot{\mathbf{v}}}{c\,(c^2 - v^2)^2} + \frac{2\,(\mathbf{v}\dot{\mathbf{v}})^2\,\mathbf{v}}{c\,(c^2 - v^2)^3} \quad \ldots(333),$$

where the terms in the last line are derived from W_1, and are got by differentiating the functions in the first two lines, but with V_1 replaced by W_1, or $(1 - v^2/c^2)^{-1}$.

219. Collecting together all the terms found in (330)—(333), we get

$$\begin{aligned}
\mathbf{f}_{12} = &-\left\{1 + \tfrac{1}{2}\,(\{\mathbf{u}_1 + \mathbf{u}_2\}\,D) + \tfrac{1}{2}\,(\mathbf{u}_2\,(\mathbf{u}_1\,D)\,D)\right\}\left(1 - \frac{v^2}{c^2}\right)\nabla_2 V_0 \\
&+ \frac{(\{\mathbf{u}_2 - \mathbf{u}_1\}\,\mathbf{v})}{2c^2}\,(\{\mathbf{u}_2 - \mathbf{u}_1\}\,D)\,\nabla_2 V_0 \\
&+ \frac{d}{dt}\left[\tfrac{1}{2}\left\{1 + \tfrac{1}{2}\,(\{\mathbf{u}_1 + \mathbf{u}_2\}\,D)\right\} D\left(1 - \frac{v^2}{c^2}\right) V_0 \right. \\
&\qquad \left. + \frac{\mathbf{u}_2 - \mathbf{u}_1}{2c^2}\,V_0 - \tfrac{1}{4}\left(1 - \frac{v^2}{c^2}\right)(\{\mathbf{u}_2 - \mathbf{u}_1\}\,D)\,DV_0\right] \\
&+ \frac{(\{\dot{\mathbf{u}}_2 - \dot{\mathbf{u}}_1\}\,\mathbf{v}) - (\{\mathbf{u}_2 - \mathbf{u}_1\}\,\dot{\mathbf{v}})}{2c^2}\,DV_0 \\
&+ \left[-\left(1 - \frac{v^2}{c^2}\right)\left\{\frac{(\ddot{\mathbf{v}}D)}{6c} + \frac{(\dot{\mathbf{v}}D)^2}{8c}\right\} + \frac{(\mathbf{v}\dot{\mathbf{v}})(\dot{\mathbf{v}}D)}{2c^3} + \frac{(\mathbf{v}\ddot{\mathbf{v}})}{2c^3}\right] DV_1 \\
&\qquad + \frac{d}{dt}\left[\frac{\dot{\mathbf{v}}}{c^3}\,V_1 + \frac{\mathbf{v}}{2c^3}\frac{dV_1}{dt}\right] \\
&+ \frac{2\ddot{\mathbf{v}}}{3c\,(c^2 - v^2)} + \frac{2\,(\mathbf{v}\ddot{\mathbf{v}})\,\mathbf{v}}{3c\,(c^2 - v^2)^2} + \frac{2\,(\mathbf{v}\dot{\mathbf{v}})\,\dot{\mathbf{v}}}{c\,(c^2 - v^2)^2} + \frac{2\,(\mathbf{v}\dot{\mathbf{v}})^2\,\mathbf{v}}{c\,(c^2 - v^2)^3} \quad \ldots\ldots\ldots\ldots(334).
\end{aligned}$$

This is the desired expansion in powers of the relative velocities up to and including zero order quantities. The corresponding expression for $\mathbf{f}_{21}$ is got by interchanging the two elements of charge de_1 and de_2. For this purpose we must interchange $\mathbf{u}_1$ and $\mathbf{u}_2$, and change the sign of the vector $\mathbf{R}$, which amounts to leaving V_0 unchanged, but changing the signs of $\nabla_2 V_0$ and V_1. The terms in (334) fall under two heads: (1) terms which are unchanged, namely, those in the third and eighth lines; and (2) terms which merely change sign, including all the remaining lines. The latter terms constitute the reversible, Newtonian, part of the force, to be denoted by $\mathbf{f}_n$ in future, the former the irreversible part, $\mathbf{f}_i$.

Again, as regards the order of magnitude of the terms, we see from (325), § 215, that V_0 is of the order -1, and V_1 of the order 0 in the linear dimensions of the electron. Moreover, the operator ∇_2 lowers the order of its operand by one unit, so that $\nabla_2 V_0$ is of the order -2. Lastly, $\mathbf{u}_1$ and $\mathbf{u}_2$ are of the first order.

Hence it follows that the principal term of $\mathbf{f}_n$ is of the order -2, that of $\mathbf{f}_i$ only of the order -1.

We shall only require the complete expression for $\mathbf{f}_{12}$ for the investigation of Appendix E, and that only as far as the two lowest orders; hence we can neglect zero order terms in $\mathbf{f}_n$, which amounts to keeping only the first two terms in the first line of (334).

On the other hand we shall require the value of $\mathbf{f}_i$ in order to find the resultant mechanical force of the electron on itself, and with it to form the equations of motion. These involve external forces, which are of the order zero; hence we must retain zero order terms in $\mathbf{f}_i$.

220. Working expressions for the forces. Collecting together the necessary terms from (334) we get

$$\left.\begin{aligned}
\mathbf{f}_n &= -\left\{1+\tfrac{1}{2}(\{\mathbf{u}_1+\mathbf{u}_2\}\,D)\right\}\left(1-\frac{v^2}{c^2}\right)\nabla_2 V\\
\mathbf{f}_i &= \frac{d}{dt}\left[\tfrac{1}{2}\left\{1+\tfrac{1}{2}(\{\mathbf{u}_1+\mathbf{u}_2\}\,D)\right\}D\left(1-\frac{v^2}{c^2}\right)V\right]\\
&\quad+\frac{2\ddot{\mathbf{v}}}{3c(c^2-v^2)}+\frac{2(\mathbf{v}\ddot{\mathbf{v}})\mathbf{v}}{3c(c^2-v^2)^2}+\frac{2(\mathbf{v}\dot{\mathbf{v}})\dot{\mathbf{v}}}{c(c^2-v^2)^2}+\frac{2(\mathbf{v}\dot{\mathbf{v}})^2\mathbf{v}}{c(c^2-v^2)^3}\\
\mathbf{f}_{12} &= \mathbf{f}_n+\mathbf{f}_i,\quad \mathbf{f}_{21}=-\mathbf{f}_n+\mathbf{f}_i\\
V &= \frac{1}{S}=\frac{1}{\sqrt{\{(1-v^2/c^2)R^2+(\mathbf{vR})^2/c^2\}}}=\frac{1}{R\sqrt{(1-\beta^2\sin^2\theta)}}
\end{aligned}\right\}\ldots(335).$$

The suffix 0 has been omitted from V_0 as no longer necessary. The last equation follows from (325), § 215, and (308), § 202; $\beta=v/c$ as usual, and θ is the angle between $\mathbf{R}$ and $\mathbf{v}$.

For the purpose of calculating $\mathbf{f}_{12}$ or $\mathbf{f}_{21}$, for use in Appendix E, we need only retain the first term in $\mathbf{f}_i$, neglecting all the remaining zero order terms.

221. The relative velocities. Before proceeding further with our investigation we shall find it useful to consider what expressions are possible for the relative velocities.

Obviously the forces cannot be calculated from the working expressions (335) until the expressions for the relative velocities $\mathbf{u}_1$ and $\mathbf{u}_2$ as functions of the relative coordinates of the two elements of charge and of the time have been assigned. On account of the smallness of the electron we can expand the relative velocity $\mathbf{u}$ of any element in a series of powers of its relative coordinates (x, y, z), as we have already pointed out in § 208. It has been shown there that the constant term in this expansion is small of the second order, because we have taken the electric centre as origin. In our working expressions (335) the relative velocities $\mathbf{u}_1$ and $\mathbf{u}_2$ only occur in the small terms; hence it is sufficient to calculate them to a first approximation and retain only linear terms. Thus we may write

$$\left.\begin{aligned}(u_x, u_y, u_z) = (\sigma_{11}x + \sigma_{12}y + \sigma_{13}z,\ \ \sigma_{21}x + \sigma_{22}y + \sigma_{23}z,\ \ \sigma_{31}x + \sigma_{32}y + \sigma_{33}z)\\ \text{or in vector notation} \qquad \mathbf{u} = \sigma\mathbf{r}\qquad\qquad\end{aligned}\right\} \quad \text{......(336).}$$

The quantities σ are independent of (x, y, z), but generally depend on t; usually they are functions of the velocity $\mathbf{v}$ of the electron and of its derivatives.

Hence to our approximation we may regard the electron as homogeneously strained. The strain operator σ involves nine coefficients, because the strain is generally rotational. The configuration of the electron is completely determined when we are given its configuration in some standard state, and the strain as a function of the time. It is convenient to take the state of rest as the standard.

222. The bounding surface. The bounding surface of the electron at any time is completely determined under these conditions. In fact if its equation be $F(x, y, z, t) = 0$, F must satisfy the equation

$$\frac{\partial F}{\partial t} + u_x\frac{\partial F}{\partial x} + u_y\frac{\partial F}{\partial y} + u_z\frac{\partial F}{\partial z} = 0 \quad \text{..................(337).}$$

Thus, if $\phi(x, y, z, t) = \text{constant}$, $\psi(x, y, z, t) = \text{constant}$, and $\chi(x, y, z, t) = \text{constant}$ be three independent integrals of the system

$$\left.\begin{aligned}\frac{dx}{\sigma_{11}x + \sigma_{12}y + \sigma_{13}z} = \frac{dy}{\sigma_{21}x + \sigma_{22}y + \sigma_{23}z} = \frac{dz}{\sigma_{31}x + \sigma_{32}y + \sigma_{33}z} = dt\\ \text{the equation of the bounding surface is}\qquad\qquad\qquad\\ F(\phi, \psi, \chi) = 0\qquad\qquad\end{aligned}\right\} \text{...(338),}$$

where F is an arbitrary function, and is to be determined so as to satisfy the initial conditions.

Vice versa, if the bounding surface of the electron be given at all times, the strain is no longer arbitrary, but must satisfy (337) at every point of the surface.

So far we have supposed the origin to be moving, but the axes to remain parallel to their original directions. If however the axes be rotating in any assigned way, the equations (336) and (337) still hold, provided only that **u** and σ now denote the velocity and strain relative to the moving axes.

223. For example, to make this quite clear, consider the Lorentz electron. Take Ox in the direction of motion of the electric centre, which is taken as origin as before. Then, if (x_0, y_0, z_0) be the relative coordinates of any element when the electron is at rest, and (x, y, z) its coordinates when the velocity is **v**, we have

$$x = x_0 \sqrt{(c^2 - v^2)}/c, \quad y = y_0, \quad z = z_0.$$

Hence $\quad u_x = -\, v\dot{v}\, x/(c^2 - v^2), \quad u_y = u_z = 0, \quad \sigma_{11} = -\, v\dot{v}/(c^2 - v^2),$

and all the other relative strain coefficients are zero.

The equation (337) becomes

$$\frac{\partial F}{\partial t} - \frac{v\dot{v}}{c^2 - v^2} x \frac{\partial F}{\partial x} = 0.$$

The equations (338) are

$$\frac{dx}{-\dfrac{v\dot{v}}{c^2 - v^2} x} = \frac{dy}{0} = \frac{dz}{0} = dt.$$

Three independent integrals are

$$\frac{cx}{\sqrt{(c^2 - v^2)}} = \text{constant}, \quad y = \text{constant}, \quad z = \text{constant}.$$

Hence the equation of the bounding surface is

$$F\left(\frac{cx}{\sqrt{(c^2 - v^2)}}, y, z\right) = 0,$$

$F(x, y, z) = 0$ being its equation in the state of rest.

For instance, if in the state of rest the bounding surface is the sphere $x^2 + y^2 + z^2 - a^2 = 0$, then when the electron is moving with velocity **v**, the bounding surface is the spheroid $\dfrac{c^2x^2}{c^2 - v^2} + y^2 + z^2 - a^2 = 0$, with axis along the line of motion.

Vice versa, if the bounding surface be assigned as the spheroid given above, the relative strain σ must satisfy the condition (337) at every point of the surface; that is, we must have

$$\frac{2v\dot{v}c^2x^2}{(c^2-v^2)^2} + (\sigma_{11}x + \sigma_{12}y + \sigma_{13}z)\frac{2c^2x}{c^2-v^2} + (\sigma_{21}x + \sigma_{22}y + \sigma_{23}z)\,2y$$
$$+ (\sigma_{31}x + \sigma_{32}y + \sigma_{33}z)\,2z = 0,$$

for all values of x, y, z which satisfy the equation

$$\frac{c^2x^2}{c^2-v^2} + y^2 + z^2 - a^2 = 0.$$

From these equations we get

$$\sigma_{23} + \sigma_{32} = 0, \quad \sigma_{31} + \sigma_{13}\frac{c^2}{c^2-v^2} = 0, \quad \sigma_{21} + \sigma_{12}\frac{c^2}{c^2-v^2} = 0,$$

$$\sigma_{11} + \frac{v\dot{v}}{c^2-v^2} = 0, \quad \sigma_{22} = 0, \quad \sigma_{33} = 0.$$

Thus any possible strain is compounded of the Lorentz contraction, of a shear $\frac{v^2}{2c^2}\sqrt{(\sigma_{21}^2 + \sigma_{31}^2)}$ of arbitrary amount in any plane through the axis, of a determinate rotation $\left(1 - \frac{v^2}{2c^2}\right)\sqrt{(\sigma_{21}^2 + \sigma_{31}^2)}$ in the plane of the shear, and of an arbitrary rotation round the axis.

224. The resultant internal force on the electron. The resultant internal force on the electron, that is, the resultant of all the mutual actions and reactions between the several elements of the electron, is found from (335), § 220, in the form

$$\mathbf{F}_i = \tfrac{1}{2}\iint(\mathbf{f}_{12} + \mathbf{f}_{21})\,de_1\,de_2 = \iint \mathbf{f}_i\,de_1 de_2 \quad \text{..............(339).}$$

The factor 1/2 is introduced for the sake of symmetry, so as to allow us to extend each of the two integrations over the whole electron; $\mathbf{f}_i$ is the irreversible component of the electromagnetic force and is given by the second of equations (335), § 220.

By the principle of the conservation of electric charge we have

$$\frac{d}{dt}\,de_1 = \frac{d}{dt}\,de_2 = 0;$$

hence the double integration of the first line of $\mathbf{f}_i$ can be effected behind the operator $\frac{d}{dt}$.

Again, the terms in the second line of $\mathbf{f}_i$ are completely independent of the coordinates, so that the double integration of this line is simply effected by supplying the factor e^2, where e is the total charge of the electron as before, every element being supposed to have the same sign.

Hence we get

$$\mathbf{F}_i = \frac{d}{dt}\left[\tfrac{1}{2}\iint de_1 de_2 \left\{1 + \tfrac{1}{2}\left(\{\mathbf{u}_1 + \mathbf{u}_2\}\, D\right)\right\} D\left(1 - \frac{v^2}{c^2}\right) V\right]$$
$$+ \frac{2e^2\ddot{\mathbf{v}}}{3c\,(c^2 - v^2)} + \frac{2e^2\,(\mathbf{v}\ddot{\mathbf{v}})\,\mathbf{v}}{3c\,(c^2 - v^2)^2} + \frac{2e^2\,(\mathbf{v}\dot{\mathbf{v}})\,\dot{\mathbf{v}}}{c\,(c^2 - v^2)^2} + \frac{2e^2\,(\mathbf{v}\dot{\mathbf{v}})^2\,\mathbf{v}}{c\,(c^2 - v^2)^3} \quad \ldots(340).$$

225. For the purpose of comparing this equation with the equations given in Ch. XI, § 143, we shall write it in the form

$$\left.\begin{aligned} \mathbf{F}_i &= -\frac{d\mathbf{G}}{dt} + \mathbf{K} \\ \text{where}\quad \mathbf{G} &= -\tfrac{1}{2}\iint de_1 de_2 \left\{1 + \tfrac{1}{2}\left(\{\mathbf{u}_1 + \mathbf{u}_2\}\, D\right)\right\} D\left(1 - \frac{v^2}{c^2}\right) V \\ \mathbf{K} &= \frac{2e^2\ddot{\mathbf{v}}}{3c\,(c^2 - v^2)} + \frac{2e^2\,(\mathbf{v}\ddot{\mathbf{v}})\,\mathbf{v}}{3c\,(c^2 - v^2)^2} + \frac{2e^2\,(\mathbf{v}\dot{\mathbf{v}})\,\dot{\mathbf{v}}}{c\,(c^2 - v^2)^2} + \frac{2e^2\,(\mathbf{v}\dot{\mathbf{v}})^2\,\mathbf{v}}{c\,(c^2 - v^2)^3} \end{aligned}\right\} \ldots(341).$$

Comparing these equations with (205) and (209), § 143, we see that $\mathbf{G}$ represents the electromagnetic momentum of the electron, while $\mathbf{K}$ is the radiation pressure, the expression for which was first published by Abraham*. This agreement between his result, obtained by an indirect method, and the present one, found directly, may serve as a verification of our analysis.

The radiation pressure may be dismissed very briefly. It is a small quantity of the order zero, and depends only on the magnitude of the charge and its mean motion, not at all on its configuration nor on the relative motion of its parts. It is not difficult to prove that the expression for $\mathbf{K}$ which is in question, can be obtained by means of the Lorentz-Einstein transformation from the well known expression for the radiation pressure on an electric charge e, which is vibrating with high frequency but small amplitude, so that its velocity is always very small while its acceleration is finite. The radiation pressure on such a charge is equal to

$$\frac{2e^2}{3c^3}\frac{d^3\mathbf{r}'}{dt'^3},$$

where $\mathbf{r}'$ denotes its radius vector and t' the time. If we regard this system as moving relative to a fixed system with velocity $\mathbf{v}$, considered as constant for the time being, and transform by the method of Lorentz and Einstein, we obtain the expression (341_3) for $\mathbf{K}$, for a charge e moving with the velocity $\mathbf{v}$, now regarded as variable.

226. The electromagnetic momentum. The expression (341_2) for the electromagnetic momentum $\mathbf{G}$ shows that there is a principal term of order -1, namely

$$-\tfrac{1}{2}\iint de_1 de_2\, D\left(1 - \frac{v^2}{c^2}\right) V \quad \ldots\ldots\ldots\ldots\ldots\ldots(342),$$

* *Theorie der Strahlung*, p. 123, eq. 85, 1905.

together with a small term of order 0, namely

$$-\tfrac{1}{4}\iint de_1 de_2 (\{\mathbf{u}_1+\mathbf{u}_2\} D) D \left(1-\frac{v^2}{c^2}\right) V \quad \text{..............(343).}$$

The latter is easily disposed of. For we have by (335_5), § 220,

$$V = \{(1-v^2/c^2) R^2 + (\mathbf{vR})^2/c^2\}^{-1/2},$$

while D denotes vector differentiation with respect to the velocity $\mathbf{v}$. Thus V and all its derivatives DV, D^2V, ... are even functions of $\mathbf{R}$, and therefore do not change sign when the sign of $\mathbf{R}$ is changed.

Moreover $\mathbf{u}_1$ and $\mathbf{u}_2$ are linear vector functions of the radii vectores $\mathbf{r}_1$ and $\mathbf{r}_2$ respectively, and change sign when the signs of these radii vectores are changed. This follows from (336), § 221.

Hence if the configuration of the electron at any time be centro-symmetrical, so that to any element de with radius vector $\mathbf{r}$ there corresponds an equal element with radius vector $-\mathbf{r}$, the electron will always remain centro-symmetrical, at any rate to our approximation where we neglect squares of the radius vector. It follows further that the small relative motion term (343) vanishes on account of the centro-symmetry. In other words, *the electromagnetic momentum of every centro-symmetrical electron depends only on its mean motion and not at all on the relative motion of its parts.*

All the types of electron hitherto suggested, such as the rigid spherical electron of Abraham and the deformable spheroidal electrons of Bucherer and of Lorentz, are centro-symmetrical. There does not appear to be any fact known which requires us to consider types of electron not possessing this kind of symmetry; for this reason, and to avoid needless complications, we shall in what follows neglect the small relative motion term (343).

227. The electromagnetic momentum of the centro-symmetrical electron. Bearing in mind the value of V and the meaning of the operator D, explained at the commencement of § 226, we find

$$-\tfrac{1}{2} D\left(1-\frac{v^2}{c^2}\right) V = \frac{\{2(\mathbf{vR})^2 + (c^2-v^2) R^2\}\,\mathbf{v} + (c^2-v^2)(\mathbf{vR})\,\mathbf{R}}{2c^4\{(1-v^2/c^2) R^2 + (\mathbf{vR})^2/c^2\}^{3/2}}.$$

Substituting this value in the expression (342) we find

$$\mathbf{G} = \iint \frac{\{2(\mathbf{vR})^2 + (c^2-v^2) R^2\}\,\mathbf{v} + (c^2-v^2)(\mathbf{vR})\,\mathbf{R}}{2c^4\{(1-v^2/c^2) R^2 + (\mathbf{vR})^2/c^2\}^{3/2}}\, de_1\, de_2 \quad \text{...(344).}$$

This equation shows that the electromagnetic momentum is the resultant of two components: (1) a component in the direction of the velocity $\mathbf{v}$; (2) a component which is itself the resultant of a large number of radial forces, and does not necessarily lie in the direction of the velocity.

Forming the scalar product of $\mathbf{G}$ by the unit vector $\mathbf{v}_1$ we find for the component in the direction of the motion

$$(\mathbf{v}_1\mathbf{G}) = \frac{v}{2c^4}\iint \frac{(c^2+v^2)(\mathbf{v}_1\mathbf{R})^2 + (c^2-v^2)R^2}{\{(1-v^2/c^2)R^2 + (\mathbf{vR})^2/c^2\}^{3/2}}\,de_1\,de_2 \quad \text{.........(345)}.$$

Again, forming the vector product of $\mathbf{G}$ by $\mathbf{v}_1$ we find for the component perpendicular to the direction of motion

$$[\mathbf{v}_1\mathbf{G}] = \frac{(c^2-v^2)\,v}{2c^4}\iint \frac{(\mathbf{v}_1\mathbf{R})[\mathbf{v}_1\mathbf{R}]}{\{(1-v^2/c^2)R^2 + (\mathbf{vR})^2/c^2\}^{3/2}}\,de_1\,de_2 \quad \text{......(346)}.$$

228. In order to understand clearly the physical meaning of the equations (345) and (346), let us take Ox parallel to the direction of motion for the time being, and apply the transformation

$$x = \xi\sqrt{(1-v^2/c^2)}, \quad y=\eta, \quad z=\zeta \quad \text{..............(347)}.$$

It is obvious that these equations merely express the fact that the actual electron can be derived from the transformed electron, that is the electron in the (ξ, η, ζ) system, by applying the Lorentz-Fitzgerald contraction. Of course we are not bound to identify the transformed electron with the actual electron when at rest; such an identification involves the acceptation of the Lorentz-Fitzgerald contraction hypothesis, to accept which may or may not be desirable on other grounds.

Applying the transformation (347) to (345) and (346) we get

$$\left.\begin{aligned}
G_\xi &= \frac{v}{2c\sqrt{(c^2-v^2)}}\iint\left\{1 + \frac{(\xi_2-\xi_1)^2}{(\xi_2-\xi_1)^2+(\eta_2-\eta_1)^2+(\zeta_2-\zeta_1)^2}\right\} \\
&\qquad\qquad \times \frac{de_1\,de_2}{\{(\xi_2-\xi_1)^2+(\eta_2-\eta_1)^2+(\zeta_2-\zeta_1)^2\}^{1/2}} \\
G_\eta &= \frac{v}{2c^2}\iint \frac{(\xi_2-\xi_1)(\eta_2-\eta_1)\,de_1\,de_2}{\{(\xi_2-\xi_1)^2+(\eta_2-\eta_1)^2+(\zeta_2-\zeta_1)^2\}^{3/2}} \\
G_\zeta &= \frac{v}{2c^2}\iint \frac{(\xi_2-\xi_1)(\zeta_2-\zeta_1)\,de_1\,de_2}{\{(\xi_2-\xi_1)^2+(\eta_2-\eta_1)^2+(\zeta_2-\zeta_1)^2\}^{3/2}}
\end{aligned}\right\} \quad \text{......(348)}.$$

The last two of these equations show that the transverse component of the electromagnetic momentum vanishes when one or other of the following two conditions is satisfied, namely

(1) when the electron is symmetrical fore and aft, or

(2) when it possesses two longitudinal planes of symmetry.

In the first case, to any element de at (ξ, η, ζ) corresponds an equal element at $(-\xi, \eta, \zeta)$.

In the second case, to any element de at (ξ, η, ζ) correspond equal elements at $(\xi, -\eta, \zeta)$, $(\xi, \eta, -\zeta)$ and $(\xi, -\eta, -\zeta)$.

Both conditions are satisfied by the electrons of Abraham, Bucherer and Lorentz; accordingly their electromagnetic momenta have no transverse components.

Both conditions are violated by a spheroidal electron whose axis is inclined to the direction of motion; accordingly its electromagnetic momentum has a transverse component, a result which will be verified in example 4, *infra*.

229. The electromagnetic mass. There is no difficulty in defining the electromagnetic mass in the usual way, as the ratio of the electromagnetic momentum to the velocity, so long as the electron is centro-symmetrical. In the general case this definition is inconvenient, because the ratio is one of two differently directed vectors, and therefore neither a vector nor a scalar, but a quaternion. A more useful way is to define the electromagnetic mass as the ratio which the electromagnetic momentum bears to the speed, that is the tensor of the velocity, by the equation

$$\mathbf{G} = v\mathbf{m} \quad \text{.................................(349).}$$

This definition makes the electromagnetic mass a vector quantity, whose components are given by

$$\left.\begin{aligned} m_\xi &= \frac{1}{2c\sqrt{(c^2-v^2)}}\iint\left\{1 + \frac{(\xi_2-\xi_1)^2}{(\xi_2-\xi_1)^2+(\eta_2-\eta_1)^2+(\zeta_2-\zeta_1)^2}\right\} \\ &\qquad\qquad \times \frac{de_1\,de_2}{\{(\xi_2-\xi_1)^2+(\eta_2-\eta_1)^2+(\zeta_2-\zeta_1)^2\}^{1/2}} \\ m_\eta &= \frac{1}{2c^2}\iint\frac{(\xi_2-\xi_1)(\eta_2-\eta_1)\,de_1\,de_2}{\{(\xi_2-\xi_1)^2+(\eta_2-\eta_1)^2+(\zeta_2-\zeta_1)^2\}^{3/2}} \\ m_\zeta &= \frac{1}{2c^2}\iint\frac{(\xi_2-\xi_1)(\zeta_2-\zeta_1)\,de_1\,de_2}{\{(\xi_2-\xi_1)^2+(\eta_2-\eta_1)^2+(\zeta_2-\zeta_1)^2\}^{3/2}} \end{aligned}\right\} \quad \text{......(350).}$$

In the same way we can define the longitudinal mass $\mathbf{m}'$ by the equation

$$\mathbf{m}' = \frac{\partial \mathbf{G}}{\partial v} = \mathbf{m} + v\frac{\partial \mathbf{m}}{\partial v} \quad \text{........................(351).}$$

This definition makes the longitudinal mass also a vector.

When the electron is unsymmetrical, in the sense that it violates both conditions of § 228, both masses have transverse components. But for the electrons of Abraham, Bucherer and Lorentz, both masses lie in the direction of motion, and their absolute values reduce to the transverse and longitudinal masses as usually defined.

230. We can now write down expressions for the components of the external force acting on the electron, so far as they depend on the electromagnetic momentum. This force is of course equal to $\frac{d\mathbf{G}}{dt}$, where the differentiation is total. Let us take as our axes of (ξ, η, ζ) the tangent, principal normal and binormal to the path of the electric centre of the electron, just as in Ch. XI, § 144; then we have

$$\frac{d\mathbf{G}}{dt} = \frac{\partial \mathbf{G}}{\partial t} + [\omega \mathbf{G}] \quad \text{.........................(352)},$$

where $\frac{\partial}{\partial t}$ denotes differentiation relative to the rotating axes, and ω is their angular velocity about themselves. Thus we have $\frac{\partial \mathbf{G}}{\partial t} = \dot{v}\frac{\partial \mathbf{G}}{\partial v}$, while the components of ω are $(v/\tau, 0, v/\rho)$. Bearing in mind (349) and (351), we find for the components of the part of the external force with which we are concerned, that is for $\frac{d\mathbf{G}}{dt}$, the values

$$m_\xi' \dot{v} - m_\eta \frac{v^2}{\rho}, \quad m_\eta' \dot{v} + m_\xi \frac{v^2}{\rho} - m_\zeta \frac{v^2}{\tau}, \quad m_\zeta' \dot{v} + m_\eta \frac{v^2}{\tau} \quad \text{......(353)}.$$

Thus we have

(1) A component $\mathbf{m}'\dot{v}$, proportional to the longitudinal acceleration $\dot{v}$, and in the direction of the vector $\mathbf{m}'$.

(2) A component $(-m_\eta, m_\xi, 0)\, v^2/\rho$, proportional to the centripetal acceleration v^2/ρ, in the osculating plane and perpendicular to the vector $\mathbf{m}$.

(3) A component $(0, -m_\zeta, m_\eta)\, v^2/\tau$, proportional to the torsion $1/\tau$, in the normal plane and perpendicular to the vector $\mathbf{m}$.

When the electron is symmetrical, either fore and aft, or with respect to any two mutually perpendicular longitudinal coordinate planes, the first and second components reduce to the usual longitudinal and transverse ones, while the third vanishes.

When the electron is asymmetric this is not so. For instance, in order to keep it moving in a straight line with acceleration $\dot{v}$, we require transverse forces, $m_\eta'\dot{v}$ and $m_\zeta'\dot{v}$, in addition to the longitudinal force, $m_\xi'\dot{v}$.

Again, in order to keep it moving in a circle of radius ρ with the uniform speed v, we require a longitudinal force, $-m_\eta v^2/\rho$, in addition to the centripetal force, $m_\xi v^2/\rho$.

This case has been referred to already at the end of Ch. XII. If the speed and radius be so adjusted that

$$\frac{m_\eta v^2}{\rho} = \frac{e^2 n U}{\rho^2},$$

this part of the external force is just supplied by the drag on the electron due to radiation from the ring.

Hence a steady motion is possible provided that m_η be positive, and the speed of the ring be properly adjusted to its size.

231. The energy relations of the electron. For the sake of comparison with (211), § 145, we shall begin by finding the activity, $(\mathbf{vF}_i)$, of the resultant internal force $\mathbf{F}_i$. We find from (341), § 225,

$$(\mathbf{vF}_i) = -\left(\mathbf{v}\frac{d\mathbf{G}}{dt}\right) + (\mathbf{vK}).$$

By means of a few simple algebraic transformations we deduce the following form of this equation

$$\left.\begin{aligned}(\mathbf{vF}_i) &= -\frac{d}{dt}\left\{(\mathbf{vG}) - \frac{2ce^2(\mathbf{v}\dot{\mathbf{v}})}{3(c^2-v^2)^2}\right\} + (\dot{\mathbf{v}}\mathbf{G}) - R\\ \text{where}\quad R &= \frac{2ce^2}{3(c^2-v^2)^2}\left\{\dot{\mathbf{v}}^2 + \frac{(\mathbf{v}\dot{\mathbf{v}})^2}{c^2-v^2}\right\}\end{aligned}\right\} \quad \text{.........(354).}$$

R is Liénard's* value for the rate of loss of energy due to radiation from the electron. When the electron is symmetrical, either fore and aft, or with respect to any two longitudinal planes at right angles, so that $\mathbf{G}$ is in the direction of $\mathbf{v}$, the first equation reduces to (211).

When there is relative motion of the parts of the electron, the quantity $(\mathbf{vF}_i)$ does not represent the whole activity of the internal electromagnetic forces. We shall call the latter the internal activity and denote it by the symbol A_i.

232. In order to calculate it, we notice that the activity of the electromagnetic forces exerted between the elements de_1 and de_2 is equal to

$$\{(\mathbf{v}_2\mathbf{f}_{12}) + (\mathbf{v}_1\mathbf{f}_{21})\}\, de_1\, de_2,$$

on the basis of the usual definition of work. Putting $\mathbf{v}_1 = \mathbf{v} + \mathbf{u}_1$, and $\mathbf{v}_2 = \mathbf{v} + \mathbf{u}_2$, using (335), § 220, neglecting squares and products of $\mathbf{u}_1$ and $\mathbf{u}_2$, and integrating, we find by (339), § 224,

$$\left.\begin{aligned}A_i &= \tfrac{1}{2}\iint\{2(\mathbf{vf}_i) + (\{\mathbf{u}_1+\mathbf{u}_2\}\,\mathbf{f}_i) + (\{\mathbf{u}_2-\mathbf{u}_1\}\,\mathbf{f}_n)\}\,de_1\,de_2\\ &= (\mathbf{vF}_i) + A_r\\ \text{where}\quad & \\ A_r &= \tfrac{1}{2}\iint de_1\,de_2\left\{\left(\{\mathbf{u}_1-\mathbf{u}_2\}\,\{1+\tfrac{1}{2}(\{\mathbf{u}_1+\mathbf{u}_2\}\,D)\}\left(1-\frac{v^2}{c^2}\right)\nabla_2 V\right)\right.\\ &\qquad\left. + \tfrac{1}{2}\left(\{\mathbf{u}_1+\mathbf{u}_2\}\,\frac{d}{dt}\left\{D\left(1-\frac{v^2}{c^2}\right)V\right\}\right)\right\}\end{aligned}\right\} \quad \text{......(355).}$$

The factor 1/2 is supplied because in integrating over the whole electron with respect to both de_1 and de_2 we count the pair of elements de_1 and de_2 twice over.

* *L'Éclairage électrique*, 16, p. 5, 1898.

Thus the internal activity A_i exceeds the activity of the resultant internal force $\mathbf{F}_i$ by the amount A_r, which depends on the relative motion and vanishes with it. We shall call A_r the relative activity.

233. The expression (355_3) for A_r can be simplified.

Since V is a function of the differences of the coordinates of the elements de_1 and de_2, as well as of the velocity $\mathbf{v}$, and $\mathbf{v}_2 - \mathbf{v}_1 = \mathbf{u}_2 - \mathbf{u}_1$, we find

$$\frac{d}{dt}\left\{\left(1-\frac{v^2}{c^2}\right)V\right\} = (\dot{\mathbf{v}}D)\left(1-\frac{v^2}{c^2}\right)V + (\{\mathbf{v}_2 - \mathbf{v}_1\}\nabla_2)\left(1-\frac{v^2}{c^2}\right)V,$$

so that
$$(\{\mathbf{u}_1 - \mathbf{u}_2\}\nabla_2)\left(1-\frac{v^2}{c^2}\right)V = \left\{(\dot{\mathbf{v}}D) - \frac{d}{dt}\right\}\left(1-\frac{v^2}{c^2}\right)V.$$

Similarly

$$\left(\{\mathbf{u}_1 + \mathbf{u}_2\}\frac{d}{dt}\left\{D\left(1-\frac{v^2}{c^2}\right)V\right\}\right) = \left(\{\mathbf{u}_1+\mathbf{u}_2\}(\dot{\mathbf{v}}D)D\right)\left(1-\frac{v^2}{c^2}\right)V$$
$$-\left(\{\mathbf{u}_1+\mathbf{u}_2\}(\{\mathbf{u}_1-\mathbf{u}_2\}\nabla_2)D\right)\left(1-\frac{v^2}{c^2}\right)V.$$

Hence the integrand of (355_3) is equal to

$$\left(\dot{\mathbf{v}}\{1+\tfrac{1}{2}(\{\mathbf{u}_1+\mathbf{u}_2\}D)\}D\left(1-\frac{v^2}{c^2}\right)V\right) - \frac{d}{dt}\left\{\left(1-\frac{v^2}{c^2}\right)V\right\}$$
$$+\tfrac{1}{2}\left(\{\mathbf{u}_1-\mathbf{u}_2\}(\{\mathbf{u}_1+\mathbf{u}_2\}D)\nabla_2 - \{\mathbf{u}_1+\mathbf{u}_2\}(\{\mathbf{u}_1-\mathbf{u}_2\}\nabla_2)D\right)\left(1-\frac{v^2}{c^2}\right)V.$$

When the second line is expanded, care being taken to keep the relative velocities $\mathbf{u}_1$ and $\mathbf{u}_2$ in front of the operators, it is at once seen to vanish identically because D and ∇_2 are commutative with each other. By comparing the first term of the first line with the expression (341_2), § 225, we see that it is equal to the scalar product of $\dot{\mathbf{v}}$ by the integrand of $\mathbf{G}$; the second term however is new.

Multiplying by $\frac{1}{2}de_1 de_2$, integrating and bearing in mind that de_1 and de_2 are invariable, we find

$$\left.\begin{aligned} A_r &= -(\dot{\mathbf{v}}\mathbf{G}) - \frac{d\Phi}{dt} \\ \text{where}\quad \Phi &= \tfrac{1}{2}\left(1-\frac{v^2}{c^2}\right)\iint V de_1 de_2 \end{aligned}\right\} \quad \text{..................(356).}$$

When $\mathbf{G}$ is in the direction of motion, it is easily proved by means of (356) and (344), § 227, that

$$\mathbf{G} = -\mathbf{v}_1\frac{\partial\Phi}{\partial v} \quad \text{..........................(357),}$$

where $\mathbf{v}_1$ denotes the unit vector $\mathbf{v}/v$, but this is not generally true.

234. In order to see precisely what the function Φ represents, let us use the transformation (347), § 228.

Using the expression for V given in (335_5), § 220, we find

$$\Phi = \tfrac{1}{2}\sqrt{(1 - v^2/c^2)} \iint \frac{de_1 de_2}{\{(\xi_2 - \xi_1)^2 + (\eta_2 - \eta_1)^2 + (\zeta_2 - \zeta_1)^2\}^{1/2}} \quad \ldots(358).$$

Thus Φ is equal to $\sqrt{(1 - v^2/c^2)}$ times the electrostatic energy of the transformed electron, of course for the same total charge. This energy generally depends on v, because the limits do so; it is independent of v in the particular case of the Lorentz electron.

235. In the preceding articles we have found three different activity equations, namely

$$(\mathbf{vF}_i) = -\frac{d}{dt}\left\{(\mathbf{vG}) - \frac{2ce^2(\mathbf{v}\dot{\mathbf{v}})}{3(c^2 - v^2)^2}\right\} + (\dot{\mathbf{v}}\mathbf{G}) - R \ldots(354), \ § \ 231,$$

$$A_i = (\mathbf{vF}_i) + A_r \ \ldots\ldots\ldots\ldots(355), \ § \ 232,$$

$$A_r = -(\dot{\mathbf{v}}\mathbf{G}) - \frac{d\Phi}{dt} \ \ldots\ldots\ldots\ldots(356), \ § \ 233.$$

From these we deduce a fourth, namely

$$A_i = -\frac{d}{dt}\left\{(\mathbf{vG}) + \Phi - \frac{2ce^2(\mathbf{v}\dot{\mathbf{v}})}{3(c^2 - v^2)^2}\right\} - R \ \ldots\ldots\ldots(359).$$

These equations are true for any electron, whether it be centro-symmetrical or not. The first of them takes the place of the energy equation (211), § 145, which is only true when the electromagnetic momentum is in the direction of motion, and is merely a particular case of (354).

The external mechanical force impressed on the electron, which balances the resultant internal force $\mathbf{F}_i$ whenever the mass is wholly electromagnetic, does work at the rate $-(\mathbf{vF}_i)$, equal to $A_r - A_i$, by (355). Of the work thus available, the internal electromagnetic forces consume a portion at the rate $-A_i$, part of which is irretrievably lost at the rate R owing to radiation, while the rest is stored in the electron and can be recovered if the motion be completely reversed, in accordance with (359). We may regard the part which is independent of the acceleration $\dot{\mathbf{v}}$ as the electromagnetic energy of the electron; it is given by

$$W = (\mathbf{vG}) + \Phi \ \ldots\ldots\ldots\ldots(360).$$

The remaining small acceleration term, $-2ce^2(\mathbf{v}\dot{\mathbf{v}})/3(c^2 - v^2)^2$, is the supplementary reversible radiation term, just as in § 146.

236. The remainder of the available external work, namely the part A_r, cannot be accounted for by means of the electromagnetic energy or the radiation. The work thus unaccounted for is equal to the work done by the internal electromagnetic forces during the relative motion of the parts of the electron, and must be consumed by forces of other than electromagnetic origin.

Substituting the value of $\mathbf{G}$ from (349), § 229, in (356), we find

$$A_r = -m_\xi v\dot{v} - m_\eta \frac{v^3}{\rho} - \frac{d\Phi}{dt} = -\frac{d}{dt}\{\int m_\xi v dv + \Phi\} - m_\eta \frac{v^3}{\rho} \quad \ldots(361).$$

Thus a part of the work in question is stored as non-electromagnetic energy to the amount

$$E = -\int m_\xi v dv - \Phi \quad \ldots\ldots(362),$$

the rest is lost at the rate $-m_\eta v^3/\rho$.

For the rigid spherical electron of Abraham the non-electromagnetic energy E is zero, because there is no relative motion; for the spheroidal electron of Bucherer E is zero because there is no change of volume; but for the spheroidal electron of Lorentz, E has a finite value different from zero. It is well known that this fact has been urged as an objection to the Lorentz electron by Abraham*, and that an explanation has been given by Poincaré†. We shall see in Appendix E that Abraham's difficulty is already involved in the more fundamental difficulty of accounting for extended electrons at all in view of the mutual repulsions between their parts.

237. When we substitute the value (349) for $\mathbf{G}$ in (354), § 235, we find

$$-(\mathbf{vF}_i) = \frac{d}{dt}\left\{m_\xi v^2 - \int m_\xi v dv - \frac{2ce^2(\mathbf{v}\dot{\mathbf{v}})}{3(c^2-v^2)^2}\right\} - m_\eta \frac{v^3}{\rho} + R.$$

This equation shows that part of the external work done is lost to the electron at the rate $R - m_\eta v^3/\rho$, owing to the combined effect of the drag due to radiation and the pull due to asymmetry. Another small part, depending on the acceleration and derived from the radiation pressure, is stored to the amount $-2ce^2(\mathbf{v}\dot{\mathbf{v}})/3(c^2-v^2)^2$. The remainder depending on the speed is stored as kinetic energy to the amount

$$T = m_\xi v^2 - \int m_\xi v dv \quad \ldots\ldots(363).$$

From (360), (362) and (363) we see that

$$T = W + E \quad \ldots\ldots(364),$$

so that the kinetic energy T is partly electromagnetic, to the amount W, and partly non-electromagnetic, to the amount E. When the electron is symmetrical, so that m_ξ reduces to the ordinary transverse mass m, T reduces to the kinetic energy defined by (214), § 146.

We shall now consider some examples in illustration of the results just obtained.

* *Phys. Zeitsch.* 5, p. 576, 1904.

† *Comptes Rendus*, CXI. p. 1504, 1905.

238. **Example 1. The Abraham electron.** We assume that the electron is a rigid sphere, of radius a and charge e, and of uniform volume density. For uniform surface distribution we must multiply the mass by the factor 5/6.

The transformed electron is bounded by the prolate spheroid

$$(1-\beta^2)\,\xi^2+\eta^2+\zeta^2=a^2,$$

where $\beta=v/c$ as usual. It is of uniform volume density and its total charge is e. Its electrostatic energy gives, by (356), § 233,

$$\Phi=\frac{3e^2(1-\beta^2)}{10a\beta}\log\frac{1+\beta}{1-\beta} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(365).$$

Φ only depends on the speed v and has no reference, explicit or implicit, to the direction of motion of the electron. Hence, by (357), § 233,

$$\mathbf{G}=-\mathbf{v}_1\frac{\partial\Phi}{\partial v}=\mathbf{v}_1\frac{3e^2}{5ca\beta}\left(\frac{1+\beta^2}{2\beta}\log\frac{1+\beta}{1-\beta}-1\right) \quad\ldots\ldots\ldots(366).$$

$\mathbf{G}$ is in the direction of motion, so that $m_\xi=m$, $m_\xi'=m'$, and the other components of the vectors $\mathbf{m}$ and $\mathbf{m}'$ are zero. Hence

$$m=\frac{3e^2}{5c^2a\beta^2}\left(\frac{1+\beta^2}{2\beta}\log\frac{1+\beta}{1-\beta}-1\right)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(367),$$

$$m'=\frac{d(m\beta)}{d\beta}=\frac{6e^2}{5c^2a\beta^2(1-\beta^2)}\left(1-\frac{1-\beta^2}{2\beta}\log\frac{1+\beta}{1-\beta}\right)\ldots\ldots(368).$$

The electromagnetic energy is given by (360), § 235, whence

$$W=mv^2+\Phi=\frac{3e^2}{5a}\left(\frac{1}{\beta}\log\frac{1+\beta}{1-\beta}-1\right)\ldots\ldots\ldots\ldots\ldots(369).$$

In the present case there is no relative motion, so that the relative activity A_r vanishes. Accordingly the non-electromagnetic energy E reduces to a constant, and we get by (363), § 237,

$$T=\int_0^\beta m'\beta\,d\beta=\frac{6e^2}{5a}\left(\frac{1}{2\beta}\log\frac{1+\beta}{1-\beta}-1\right) \quad\ldots\ldots\ldots\ldots(370),$$

which only differs from W by a constant.

The values of m and T agree with those given in Ch. XI, (206), § 143, and (215), § 146.

239. **Example 2. The Bucherer electron.** The electron is a Heaviside ellipsoid with semiaxes $a(1-\beta^2)^{1/3}$, $a(1-\beta^2)^{-1/6}$, $a(1-\beta^2)^{-1/6}$, of uniform volume density and total charge e. If the charge were an equilibrium surface distribution the mass would have to be multiplied by the factor 5/6.

The transformed electron is a sphere of radius $a(1-\beta^2)^{-1/6}$, of uniform volume density and with total charge e.

The usual expression for the electrostatic energy gives by (356), § 233,

$$\Phi = \frac{3e^2(1-\beta^2)^{2/3}}{5a} \quad \text{.........................(371).}$$

Φ involves only the speed v explicitly, but it also involves the direction of motion implicitly, on account of the want of spherical symmetry of the original electron. Hence we must use (348), § 228. We find that the second term in G_ξ is one-third of the first, while G_η and G_ζ vanish identically. Hence we find

$$\mathbf{G} = \frac{4e^2\mathbf{v}}{5c^2a\sqrt[3]{(1-\beta^2)}} \quad \text{.......................(372).}$$

The vectors $\mathbf{m}$ and $\mathbf{m}'$ are in the direction of motion, so that $m_\xi = m$, $m_\xi' = m'$, $m_\eta = m_\zeta = m_\eta' = m_\zeta' = 0$. We obtain

$$m = \frac{4e^2}{5c^2a\sqrt[3]{(1-\beta^2)}} \quad \text{............................(373),}$$

$$m' = \frac{d(m\beta)}{d\beta} = \frac{4e^2(1-\frac{1}{3}\beta^2)}{5c^2a(1-\beta^2)^{4/3}} \quad \text{...................(374).}$$

The electromagnetic energy is given by (360), § 235, whence

$$W = mv^2 + \Phi = \frac{e^2(3+\beta^2)}{5a\sqrt[3]{(1-\beta^2)}} \text{...................(375).}$$

The kinetic energy is given by (363), § 237, whence

$$T = \int_0^v m'v\,dv = \frac{e^2}{5a}\left\{\frac{3+\beta^2}{\sqrt[3]{(1-\beta^2)}} - 3\right\} \quad \text{..............(376).}$$

T differs from W by a constant, so that the non-electromagnetic energy E is constant, agreeing with the fact that the relative activity vanishes. This is easily verified by means of (356), § 233. (371) and (372) give

$$(\dot{\mathbf{v}}\mathbf{G}) = \frac{4e^2\beta\dot{\beta}}{5a\sqrt[3]{(1-\beta^2)}} = -\frac{d\Phi}{dt},$$

so that $A_r = 0$.

Thus the relative activity vanishes although there is relative motion of the parts of the electron on account of its rotation and deformation. Since the strain takes place without change of volume, it would seem that the relative activity depends on changes of volume.

The values of m and T found above agree with those given in Ch. XI, (207), § 143, and (216), § 146.

240. Example 3. The Lorentz electron. The electron is a Heaviside ellipsoid with semiaxes $a(\sqrt{(1-\beta^2)}, 1, 1)$, of uniform volume density and charge e. For an equilibrium surface distribution the mass would have to be multiplied by the factor 5/6.

The transformed electron is a sphere of radius a, of uniform volume density and with total charge e. The usual expression for its electrostatic energy gives by (356), § 233,

$$\Phi = \frac{3e^2\sqrt{(1-\beta^2)}}{5a} \quad\text{..........................(377).}$$

As in the last example we must use (348), § 228, to find $\mathbf{G}$; we get

$$\mathbf{G} = \frac{4e^2\mathbf{v}}{5c^2a\sqrt{(1-\beta^2)}} \quad\text{......................(378).}$$

As before $\mathbf{m}$ and $\mathbf{m}'$ reduce to their tangential components, so that $m_\xi = m$, $m_\xi' = m'$, $m_\eta = m_\zeta = m_\eta' = m_\zeta' = 0$. We find

$$m = \frac{4e^2}{5c^2a\sqrt{(1-\beta^2)}} \quad\text{...............................(379),}$$

$$m' = \frac{d(m\beta)}{d\beta} = \frac{4e^2}{5c^2a(1-\beta^2)^{3/2}} \quad\text{.................(380).}$$

The electromagnetic energy is given by (360), § 235, whence

$$W = mv^2 + \Phi = \frac{e^2(3+\beta^2)}{5a\sqrt{(1-\beta^2)}} \quad\text{.................(381).}$$

The kinetic energy is given by (363), § 237, whence

$$T = \int_0^v m'v\,dv = \frac{4e^2}{5a}\left\{\frac{1}{\sqrt{(1-\beta^2)}} - 1\right\} \quad\text{..............(382).}$$

W and T differ by a quantity which depends on β. Accordingly we find that the relative activity A_r is different from zero. In fact we have from (377) and (378)

$$(\dot{\mathbf{v}}\mathbf{G}) = \frac{4e^2\beta\dot{\beta}}{5a\sqrt{(1-\beta^2)}} = -\frac{4}{3}\frac{d\Phi}{dt}.$$

Hence $A_r = \frac{1}{3}\frac{d\Phi}{dt}$, and a non-electromagnetic energy E exists. By (362), § 236, we find

$$E = -\int_0^v mv\,dv - \Phi = \frac{e^2}{5a}\{\sqrt{(1-\beta^2)} - 4\} \quad\text{...........(383).}$$

The relation $T = W + E$ is easily verified. Of course arbitrary constants may be added to W and E, but since T represents kinetic energy, the constant in T has been chosen so as to make it vanish when the speed vanishes.

The values of m and T just found agree with those given in Ch. XI, (208), § 143, and (217), § 146.

241. Example 4. Deformed Lorentz electron. As an example of an asymmetric electron we shall take the case of an electron, which is got from the Lorentz electron by a slight deformation symmetrical about an axis inclined to the direction of motion.

Take axes of (x, y, ζ), such that Ox makes the angle ψ with $O\xi$, measured towards $O\eta$, so that

$$\xi = x\cos\psi - y\sin\psi,\quad \eta = x\sin\psi + y\cos\psi \quad \text{.........(384).}$$

Let us suppose that the transformed electron is given by

$$r = a + \epsilon P_i(\mu),\quad i > 0 \quad \text{........................(385),}$$

where $\mu = \cos\theta$, and (θ, ϕ) are the colatitude and longitude referred to Ox as polar axis and any convenient initial meridian. We shall assume that ϵ is so small that its square may be neglected. The electron deformed in this way is no longer centro-symmetrical in the case of i an odd integer, and therefore the small relative motion term in the electromagnetic momentum, (343), § 226, does not vanish. As however it is small of zero order in any case, it becomes absolutely negligible when ϵ is small. The volume of the transformed electron is the same as that of a sphere of radius a, and the volume density of the uniform electrification is $3e/4\pi a^3$ as before. Any double integral of a function of the coordinates can be reduced by means of the equation

$$\iint f.\,de_1 de_2 = \left(\iint f.\,de_1 de_2\right)_0 + \frac{9e^2\epsilon}{8\pi^2 a^6}\left(\iint f.\,P_i(\mu_1)\,dS_1 dV_2\right)_0 \quad \text{...(386).}$$

The integrals on the right are taken over the sphere $r = a$; the second integral gives the effect of the thin shell between the sphere and the surface (385). For an element of this shell we must put $de_1 = 3e\epsilon P_i(\mu_1)\,dS_1/4\pi a^3$, where dS_1 is a surface element of the sphere, and $de_2 = 3e\,dV_2/4\pi a^3$, where dV_2 is a volume element. In this way we find one half of the second integral; the other half is got by interchanging de_1 and de_2, and obviously has the same value as the first half.

242. We shall first calculate the scalar integral Φ from (356), § 233; for this purpose we put $f = (1-\beta^2)^{1/2}/2R$, where R is the distance between the elements de_1 and de_2 of the transformed electron. The value of the first integral has been found already in § 240, and is given by (377); hence its value is $3e^2\sqrt{(1-\beta^2)}/5a$. The second integral becomes

$$\frac{9e^2\epsilon\sqrt{(1-\beta^2)}}{16\pi^2 a^6}\iint\frac{P_i(\mu_1)\,dS_1 dV_2}{R}$$

taken for a sphere of radius a. This integral vanishes, for it is the mutual electrostatic energy of the sphere and of a shell whose thickness is $\epsilon P_i(\mu)$. In fact the potential due to the sphere at every point of the shell is the same, and the total volume of the shell is zero. Hence we find

$$\Phi = \frac{3e^2\sqrt{(1-\beta^2)}}{5a} \quad \text{..........................(387).}$$

243. We must now calculate the scalar integral **G**, or, what amounts to the same thing, the vector **m** by means of (350), § 229. For this purpose we require to find the integrals

$$\iint \frac{(\xi_2-\xi_1)^2 de_1 de_2}{R^3}, \quad \iint \frac{(\xi_2-\xi_1)(\eta_2-\eta_1) de_1 de_2}{R^3}, \quad \iint \frac{(\xi_2-\xi_1)(\zeta_2-\zeta_1) de_1 de_2}{R^3}.$$

The last vanishes identically because the axis of symmetry has been taken to lie in the plane $\xi\eta$. In the absence of any deformation the first integral is one third of the integral $\iint \frac{de_1 de_2}{R}$ taken over the spherical transformed electron, that is, it is equal to $2e^2/5a$, and the second integral vanishes by symmetry. Hence we find by (386), § 241,

$$\left.\begin{aligned}\iint \frac{(\xi_2-\xi_1)^2 de_1 de_2}{R^3} &= \frac{2e^2}{5a} + \frac{9e^2\epsilon}{8\pi^2 a^6}\iint \frac{(\xi_2-\xi_1)^2 P_i(\mu_1)\, dS_1 dV_2}{R^3} \\ \iint \frac{(\xi_2-\xi_1)(\eta_2-\eta_1) de_1 de_2}{R^3} &= \frac{9e^2\epsilon}{8\pi^2 a^6}\iint \frac{(\xi_2-\xi_1)(\eta_2-\eta_1) P_i(\mu_1) dS_1 dV_2}{R^3}\end{aligned}\right\} \quad \ldots(388).$$

Both the integrals in (388) can be expressed in terms of a single integral by means of the transformation (384), § 241. For this integral we shall choose the following

$$I = \iint \frac{(y_2-y_1)^2 + (\zeta_2-\zeta_1)^2}{R^3} P_i(\mu_1)\, dS_1 dV_2 \quad \ldots\ldots\ldots\ldots(389).$$

By symmetry we have

$$\iint \frac{(x_2-x_1)^2 P_i(\mu_1)\, dS_1 dV_2}{R^3} = -I,$$

$$\iint \frac{(y_2-y_1)^2 P_i(\mu_1)\, dS_1 dV_2}{R^3} = \iint \frac{(\zeta_2-\zeta_1)^2 P_i(\mu_1)\, dS_1 dV_2}{R^3} = \tfrac{1}{2} I,$$

$$\iint \frac{(x_2-x_1)(y_2-y_1) P_i(\mu_1) dS_1 dV_2}{R^3} = 0.$$

Hence we find by (384)

$$\left.\begin{aligned}\iint \frac{(\xi_2-\xi_1)^2 P_i(\mu_1) dS_1 dV_2}{R^3} &= -\tfrac{1}{2} I (3\cos^2\psi - 1) \\ \iint \frac{(\xi_2-\xi_1)(\eta_2-\eta_1) P_i(\mu_1) dS_1 dV_2}{R^3} &= -\tfrac{3}{2} I \sin\psi\cos\psi\end{aligned}\right\} \quad \ldots\ldots(390).$$

244. Calculation of the integral I. We shall omit the suffix 2 as no longer necessary. We have identically

$$\frac{(y-y_1)^2 + (\zeta-\zeta_1)^2}{R^3} = \frac{\partial}{\partial x}\frac{x-x_1}{R} = \frac{1}{R} + (x-x_1)\frac{\partial}{\partial x}\frac{1}{R}.$$

The first term, $1/R$, gives the integral $\iint P_i(\mu_1)\, dS_1 dV/R$, which vanishes identically, as we saw in § 242.

Since the element dS_1 is on the surface of the sphere, and the element dV is inside, we have $r < a$; hence

$$\frac{1}{R} = \sum_{n=0}^{n=\infty} \frac{r^n P_n(\cos\gamma)}{a^{n+1}}, \quad \gamma = \mu\mu_1 + \sqrt{(1-\mu^2)}\sqrt{(1-\mu_1^2)}\cos(\phi - \phi_1).$$

Put $dS_1 = a^2 d\mu_1 d\phi_1$, $dV = r^2 dr\, d\mu\, d\phi$ in (389), § 243, and notice that

$$x - x_1 = r\mu - a\mu_1, \text{ and } \frac{\partial}{\partial x} = \mu\frac{\partial}{\partial r} + \frac{1-\mu^2}{r}\frac{\partial}{\partial \mu}.$$

Thus the integrand of (389) only involves ϕ and ϕ_1 in the factors $P_n(\cos\gamma)$; hence the integrations with respect to ϕ and ϕ_1 can be performed at once by using the addition theorem for spherical harmonics. We get

$$I = \sum_{n=0}^{n=\infty} \frac{4\pi^2}{a^{n-1}} \int_0^a \int_{-1}^1 \int_{-1}^1 r^2 P_i(\mu_1)\, dr\, d\mu\, d\mu_1 (r\mu - a\mu_1)\left(\mu\frac{\partial}{\partial r} + \frac{1-\mu^2}{r}\frac{\partial}{\partial\mu}\right) \cdot r^n P_n(\mu) P_n(\mu_1).$$

We have $$\left(\mu\frac{\partial}{\partial r} + \frac{1-\mu^2}{r}\frac{\partial}{\partial\mu}\right) r^n P_n(\mu) = n r^{n-1} P_{n-1}(\mu),$$

and $$(2n+1)\mu P_n(\mu) = (n+1) P_{n+1}(\mu) + n P_{n-1}(\mu).$$

Substituting and integrating with respect to r, we find

$$I = \sum_{n=0}^{n=\infty} \frac{4\pi^2 a^4 n}{(n+3)(2n-1)} \int_{-1}^1 \int_{-1}^1 \{nP_n(\mu) + (n-1)P_{n-2}(\mu)\} P_n(\mu_1) P_i(\mu_1)\, d\mu\, d\mu_1$$
$$- \sum_{n=0}^{n=\infty} \frac{4\pi^2 a^4 n}{(n+2)(2n+1)} \int_{-1}^1 \int_{-1}^1 \{(n+1) P_{n+1}(\mu_1) + nP_{n-1}(\mu_1)\} P_{n-1}(\mu) P_i(\mu_1)\, d\mu\, d\mu_1.$$

The only values which yield a result different from zero are given by $n = i = 2$ for the first integral, and $n = 1$, $i = 2$ for the second. Hence we find $I = -64\pi^2 a^4/225$ for $i = 2$, zero for all other values of i.

Substituting these values in (388) and (390), § 243, we find

$$\left.\begin{aligned} \iint \frac{(\xi_2 - \xi_1)^2\, de_1\, de_2}{R^3} &= \frac{2e^2}{5a} + \frac{4e^2\epsilon}{25a^2}(3\cos^2\psi - 1), \text{ for } i = 2 \\ \iint \frac{(\xi_2 - \xi_1)(\eta_2 - \eta_1)\, de_1\, de_2}{R^3} &= \frac{12e^2\epsilon}{25a^2}\sin\psi\cos\psi, \text{ for } i = 2 \end{aligned}\right\} \quad \ldots(391).$$

For other values of i the terms in ϵ are absent.

245. The electromagnetic masses. Substituting from (387), § 242, and (391) in (350), § 229, we get, for $i = 2$,

$$\left.\begin{aligned} m_\xi &= \frac{4e^2}{5c^2 a\sqrt{(1-\beta^2)}}\left\{1 + \frac{\epsilon}{10a}(3\cos^2\psi - 1)\right\} \\ m_\eta &= \frac{6e^2\epsilon}{25c^2a^2}\sin\psi\cos\psi, \quad m_\zeta = 0 \end{aligned}\right\} \quad \ldots\ldots\ldots(392).$$

From (351), § 229, we find

$$m_\xi' = \frac{m_\xi}{1-\beta^2}, \quad m_\eta' = m_\eta, \quad m_\zeta' = 0 \quad \text{.................(393).}$$

For all values of i other than 2 the terms involving ϵ are absent. Thus the only axially symmetrical deformation of the transformed Lorentz electron, which has any effect on the masses, is that of order 2, for which

$$r = a + \tfrac{1}{2}\epsilon\,(3\cos^2\theta - 1).$$

This changes the sphere of radius a into a spheroid of revolution, with semiaxes $(a+\epsilon,\ a-\frac{1}{2}\epsilon,\ a-\frac{1}{2}\epsilon)$, and with its axis of symmetry at an angle ψ to the direction of the motion. The proportional change in m_ξ, and also in m_ξ', is independent of the speed, and equal to one fifth of the proportional change of the radius vector drawn in the direction of motion. It is a maximum when the axis of the deformation coincides with the direction of motion, a minimum when it is at right angles, and zero when it makes an angle $\cos^{-1} 1/\sqrt{3}$ with it.

The values of m_η and m_η' are independent of the speed, vanish when the axis of deformation is along or perpendicular to the direction of motion, and are greatest when it makes an angle 45° with it.

246. The energy. We see from (363), § 237, (392) and (393) that the kinetic energy is proportional to that of the Lorentz electron, and given by

$$T = \frac{4e^2}{5a}\left\{\frac{1}{\sqrt{(1-\beta^2)}} - 1\right\}\left\{1 + \frac{\epsilon}{10a}(3\cos^2\psi - 1)\right\} \quad \text{.........(394).}$$

For the electromagnetic energy we find from (360), § 235,

$$W = \frac{e^2}{5a\sqrt{(1-\beta^2)}}\left\{3 + \beta^2 + \frac{2\epsilon\beta^2}{5a}(3\cos^2\psi - 1)\right\} \quad \text{.........(395).}$$

For the non-electromagnetic energy we get from (362), § 236,

$$E = \frac{e^2}{5a}\left\{\sqrt{(1-\beta^2)} - 4 - \frac{2\epsilon\left\{1-\sqrt{(1-\beta^2)}\right\}}{5a}(3\cos^2\psi - 1)\right\} \text{...(396).}$$

We see from (394) and (395) that for a positive value of ϵ, that is, for a small elongation along the axis, both the kinetic and the electromagnetic energies are greatest when the axis of deformation is along the direction of motion, and least when it is at right angles.

For a negative value of ϵ, that is, for a small compression along the axis, the reverse is the case.

The non-electromagnetic energy behaves just in the reverse way.

The electromagnetic forces tend to produce contraction in the direction of motion beyond the amount, appropriate to the speed, given by the Lorentz ratio $1 : \sqrt{(1-\beta^2)}$, while the non-electromagnetic forces resist it.

APPENDIX E

THE MECHANICAL EXPLANATION OF THE ELECTRON

247. Our investigations of App. C and D respecting the mechanical forces on the electron have been avowedly based on the assumption that the electric charge occupies a finite region of space, whether it be a small volume, a small closed surface, or even a small closed curve. This assumption—that the electron has parts—compels us to face the problem of accounting for its continued existence in spite of the mutual electromagnetic repulsions between those parts. We must seek some system of forces of non-electromagnetic origin*, which shall equilibrate the resultant of all the electromagnetic forces on each element, whether they be due to the remaining elements of the electron, or to an impressed external field. Moreover, since the electron is capable of motion in any arbitrary curve with any arbitrary acceleration under the action of the appropriate external field, the system of non-electromagnetic forces must be adaptable to every possible motion of the electron as a whole and consistent with the results of observations on the electromagnetic momentum and electromagnetic mass.

Two alternatives respecting the nature of the non-electromagnetic forces suggest themselves: they may be actions at a distance, or actions between neighbouring elements, equivalent to stresses in an elastic medium. The first alternative may be dismissed in a few words; if actions at a distance are to be admitted at all, we may as well regard the electron itself as a centre of force, and the electromagnetic force due to it as an action at a distance. On this view the electron is an indivisible whole and has no parts; no explanation of its existence can be given, nor should it be expected. This hypothesis is no doubt feasible, but it has difficulties of its own; it does not appear to afford much scope for explaining the differences between positive and negative charges, or for constructing systems of charges sufficiently stable and permanent to serve as models of the atom.

* During the passage of this book through the press, H. T. Wolff, *Ann. d. Phys.* 36, p. 166, 1911, has suggested a way of attaining the same end by modifying the law of electric force. His additional force, as regards mathematical form, is included in the more general expression found below.

248. We shall adopt the second alternative as more in keeping with the general trend of this essay, and accordingly shall assume that the non-electromagnetic forces are due to stresses in an elastic medium. At the present stage we need not enter into the question whether this elastic medium is the electric charge itself, or is different from it, but mechanically connected with it so that the stresses are transmitted to it practically unchanged. All that we do in fact assume is the formal equivalence between the mathematical expressions for the non-electromagnetic forces of the electron and those of the forces due to stresses in an elastic medium.

In this form the problem is too wide to require any special investigation; for it is possible to find a system of stresses in an elastic medium which shall equilibrate any system of forces whatever, whether applied throughout a certain volume, or to a certain surface. In particular a type of stress can always be found which shall equilibrate the electromagnetic forces of the electron, whatever its motion may be, but its investigation would not be of much interest.

We shall content ourselves with determining the conditions under which the stress system reduces to a distribution of hydrostatic pressure, without concerning ourselves with the physical causes producing it. An investigation of this kind will be useful as a basis for some future physical theory of the electron, and of its connection with the electromagnetic aether, if the aether-hypothesis should be adopted. In the latter case the rotational displacements of the aether would be needed for the explanation of the electromagnetic forces, and only the irrotational displacements, involving hydrostatic pressure, would be available in accounting for the non-electromagnetic forces. Even if the aether-hypothesis were rejected, a distribution of hydrostatic pressure would naturally be chosen as being the simplest stress system available.

249. The equation for the pressure. In order to express our hypothesis of the hydrostatic pressure mathematically, let us denote an element of charge of the electron by de, its coordinates referred to the electric centre by (x, y, z), and its electric volume density by ϵ, so that $de = \epsilon dx dy dz$. We shall treat a surface distribution as a limiting case of a volume distribution.

For the sake of generality we shall assume that the electric charge has associated with it a certain mass of non-electromagnetic origin, which may be intrinsic to it, or may arise from its mechanical connection with the elastic medium producing the hydrostatic pressure. It is only to be expected that one effect of this connection may be equivalent to a loading of the charge, to an increase of its effective inertia as it were, and may as well be provided for. Moreover measurements of the specific charge of the electron do not entirely exclude the presence of non-electromagnetic mass, provided that it be but a small fraction of the electromagnetic mass. We see from § 229 that the latter mass is of the order -1 in the linear dimensions of the electron, so that the former is of the same order, but much smaller. We shall denote its value for the whole electron by M, and the non-electromagnetic mass density by μ.

250. The electromagnetic forces acting on the element de are twofold:

(1) We have the resultant of all the electromagnetic forces due to the remaining elements of the electron. In this Appendix we shall denote it by $\mathbf{F}_i$ per unit charge; although in Appendix D and in Ch. XI this symbol was used for the resultant internal force on the whole electron, no inconvenience will result. In (335), § 220, we are given the value of $\mathbf{f}_{12}$, the electromagnetic force exerted by unit charge of the element de_1 on unit charge of the element de_2; if we omit the suffix 2, multiply by de_1, and integrate over the whole electron, we shall obtain the value of $\mathbf{F}_i$. The principal term of $\mathbf{f}_{12}$ is of the order -2 in the linear dimensions of the electron, and it still occurs in $\mathbf{F}_i$, although it disappears on averaging $\mathbf{F}_i$ for the whole electron. We shall go one step further and retain terms of the order -1, neglecting all terms of the order 0, such as the terms involving the relative velocities $\mathbf{u}_1$ and $\mathbf{u}_2$, and the radiation terms in the expression of $\mathbf{f}_i$, (335_2). Since the operators $\frac{d}{dt}$ and D do not affect the charge de_1 on account of its invariability, we may integrate behind these operators. Thus we find

$$\mathbf{F}_i = -\int de_1 \left\{1 + \tfrac{1}{2}\left(\{\mathbf{u} + \mathbf{u}_1\} D\right)\right\} \left(1 - \frac{v^2}{c^2}\right) \nabla V + \frac{d}{dt} \tfrac{1}{2} D \int de_1 \left(1 - \frac{v^2}{c^2}\right) V \quad \ldots (397).$$

251. (2) We have the electromagnetic forces due to the external field, whose resultant will be denoted by $\mathbf{F}_e$ per unit charge. The integral

$$\int (\mathbf{F}_i + \mathbf{F}_e)\, de$$

taken over the whole electron represents the resultant electromagnetic force acting on it due to all causes, and produces non-electromagnetic momentum at the rate $M\dot{\mathbf{v}}$, because the non-electromagnetic forces are self-equilibrating by the fundamental assumption of Ch. XI, § 140. Hence we have by Newton's Second Law

$$\int \mathbf{F}_e de = -\int \mathbf{F}_i de + M\dot{\mathbf{v}}.$$

The integral $\int \mathbf{F}_i de$ is the resultant internal force on the electron, which in § 224 was denoted by $\mathbf{F}_i$ simply, and is of the order -1. Moreover $M\dot{\mathbf{v}}$ is at most of this order, so that $\mathbf{F}_e$ is of the same order.

Hence the variations of $\mathbf{F}_e$ from point to point of the electron are only of the order 0, and can be neglected, so that the integral $\int \mathbf{F}_e de$ may be replaced by $e\mathbf{F}_e$. By means of (340), § 224, we find, neglecting the small zero order quantities,

$$\mathbf{F}_e = -\frac{d}{dt}\frac{D}{2e}\iint de\, de_1 \left(1 - \frac{v^2}{c^2}\right) V + M\dot{\mathbf{v}} \quad \ldots\ldots\ldots\ldots (398).$$

252. The non-electromagnetic force due to the hydrostatic pressure, p, is equal to $-\nabla p$ per unit volume. This, together with the resultant of the internal and external electromagnetic forces, amounting to $\epsilon(\mathbf{F}_i + \mathbf{F}_e)$ per unit volume, produces non-electromagnetic momentum at the rate $\mu\dot{\mathbf{v}}$ per unit volume. Hence we find by Newton's Second Law

$$\mu\dot{\mathbf{v}} = -\nabla p + \epsilon(\mathbf{F}_i + \mathbf{F}_e).$$

Substituting the values of $\mathbf{F}_i$ and $\mathbf{F}_e$ from (397), § 250, and (398), § 251, respectively, we find

$$\nabla p = -\epsilon \int de_1 \left\{1 + \tfrac{1}{2}(\{\mathbf{u} + \mathbf{u}_1\} D)\right\} \left(1 - \frac{v^2}{c^2}\right) \nabla V$$
$$+ \epsilon \frac{d}{dt} \tfrac{1}{2} D \left\{\int de_1 \left(1 - \frac{v^2}{c^2}\right) V - \frac{1}{e} \iint de\, de_1 \left(1 - \frac{v^2}{c^2}\right) V\right\} + \left(\frac{M}{e}\epsilon - \mu\right) \dot{\mathbf{v}} \quad \text{......(399)},$$

where $V = 1/S = 1/\sqrt{\{(1 - v^2/c^2) R^2 + (\mathbf{vR})^2/c^2\}}$, by (335_4), § 220.

This is the desired relation for the hydrostatic pressure; it will be seen that we have eliminated the external electromagnetic force, $\mathbf{F}_e$, by means of (398), § 251, so that the pressure gradient is expressed solely in terms of the configuration and motion of the electron at the instant considered. The motion of the electric centre, which is determined by the values of the velocity $\mathbf{v}$ and acceleration $\dot{\mathbf{v}}$, can be anything we please, so that $\mathbf{v}$ and $\dot{\mathbf{v}}$ are quite arbitrary. On the other hand the configuration of the electron, the distribution of its electric charge and non-electromagnetic mass, and the relative motions of its parts are at our disposal, subject to the restrictions that the resulting expressions for the electromagnetic momentum and mass shall agree with the results of experiments on the specific charge of the electron, and that the non-electromagnetic mass shall admit of a mechanical interpretation.

Our problem may now be stated as follows:

It is required to find all configurations of the electron which shall make an equation of the form (399) possible for all values of $\mathbf{v}$ and $\dot{\mathbf{v}}$, subject only to the restrictions just indicated.

253. Uniform rectilinear translation. In order to gain a clear notion of the nature of the problem, we shall first consider the particular case where the electron is moving as a whole with uniform speed in a fixed direction. The acceleration, $\dot{\mathbf{v}}$, and the relative velocities, $\mathbf{u}$ and $\mathbf{u}_1$, all vanish, so that all the small terms in (399), § 252, disappear, and only the principal term is left. Hence we have

$$\nabla p = -\epsilon \nabla \int de_1 \left(1 - \frac{v^2}{c^2}\right) V \quad \text{....................(400)}.$$

The operator ∇ has been taken outside the sign of integration because it does not affect either of the quantities v or de_1.

Let us apply the transformation of Lorentz, expressed by (347), § 228; these equations may be written in the form

$$x = \kappa\xi, \quad y = \eta, \quad z = \zeta, \quad \kappa = \sqrt{(1 - v^2/c^2)} \quad \text{.........(401)},$$

with similar equations for (x_1, y_1, z_1), provided that the line of motion of the electric centre be taken as the axis of x.

We shall assume with Lorentz that the elements de at (x, y, z) and de_1 at (x_1, y_1, z_1) in the actual electron correspond to equal elements de at (ξ, η, ζ) and de_1 at (ξ_1, η_1, ζ_1) in the transformed electron.

The corresponding elements of volume are in the ratio $\kappa : 1$, so that the corresponding volume densities are in the inverse ratio $1 : \kappa$; hence the volume densities at (ξ, η, ζ) and (ξ_1, η_1, ζ_1) in the transformed electron are equal to $\kappa\epsilon$ and $\kappa\epsilon_1$ respectively.

By means of (399), § 252, and (401) we find

$$V = 1/\kappa \mathrm{P}, \quad \mathrm{P} = \sqrt{\{(\xi - \xi_1)^2 + (\eta - \eta_1)^2 + (\zeta - \zeta_1)^2\}} \quad \text{......(402)}.$$

P is the distance between the elements de and de_1 of the transformed electron. We now find

$$\left.\begin{aligned} \int de_1 \left(1 - \frac{v^2}{c^2}\right) V &= \kappa \int \frac{de_1}{\mathrm{P}} = \kappa\Psi \\ \text{where} \qquad \Psi &= \int \frac{de_1}{\mathrm{P}} \end{aligned}\right\} \quad \text{.................(403)}.$$

Ψ is the electrostatic potential of the transformed electron at the point (ξ, η, ζ), which corresponds to (x, y, z) in the actual electron. We shall denote its mean value for the whole electron by $\overline{\Psi}$, so that

$$\left.\begin{aligned} \overline{\Psi} &= \frac{1}{e} \iint \frac{de\, de_1}{\mathrm{P}} \\ \text{and} \qquad \frac{1}{e} \iint de\, de_1 \left(1 - \frac{v^2}{c^2}\right) V &= \kappa\overline{\Psi} \end{aligned}\right\} \quad \text{....................(404)}.$$

A reference to (356), § 233, shows that $\kappa\overline{\Psi}$ is equal to $2\Phi/e$, where Φ is the function on which the electromagnetic momentum of the electron and the relative activity of its internal forces depend.

254. Returning to (400), § 253, and applying (403), we find

$$\nabla p = -\kappa\epsilon\nabla\Psi \quad \text{..........................(405)}.$$

This equation requires that

$$[\nabla\kappa\epsilon \,.\, \nabla\Psi] = 0 \quad \text{..........................(406)}.$$

Hence we find

$$p = A - \int \kappa\epsilon\, d\Psi \quad \text{..........................(407)},$$

where A can only depend on the time t.

We see from (406) and (407) that a hydrostatic pressure, p, can only exist when the volume density, $\kappa\epsilon$, of the transformed electron involves the coordinates only in so far as they enter into the expression of the electrostatic potential, Ψ, and then p has the same property. Both $\kappa\epsilon$ and p will generally also depend on the time, usually through the medium of the speed v*.

Hence the equipotential surfaces of the electron are also surfaces of constant volume density and constant hydrostatic pressure.

In particular, if the bounding surface of the electron should happen to be an equipotential surface, it will also be a surface of constant pressure. In this case the electron, whether at rest or in uniform motion in a straight line, can exist provided that a uniform external pressure be applied to its surface.

In general this pressure depends on the speed of the electron; if the speed change so slowly that the effect of the acceleration is small, our present solution may be considered as a first approximation. The external pressure needed to ensure the continued existence of the electron will be variable.

If however the configuration of the transformed electron be invariable, its volume density, $\kappa\epsilon$, and electrostatic potential, Ψ, do not depend on the time, and in (407) the constant of integration, A, may also be taken to be independent of the time, so that the hydrostatic pressure, p, is invariable.

Hence in this particular case the external pressure on the electron is not only uniform over its surface, but does not alter as the speed changes.

255. We see from (350), § 229, that the invariability of the transformed electron ensures that the tangential component, m_ξ, of the electromagnetic mass vector there defined shall follow the Lorentz mass formula. In order that m_ξ shall reduce to the ordinary electromagnetic mass, in consequence of the vanishing of the transverse components, m_η and m_ζ, it is necessary in addition that the electron either be symmetrical fore and aft, that is with respect to the yz plane perpendicular to the direction of motion, or be symmetrical with respect to two longitudinal planes intersecting in the line of motion of the electric centre.

* It is worth noticing that, when $\kappa\epsilon$ is the same everywhere inside the electron, p is a linear function of Ψ and therefore satisfies the equation

$$\left(1-\frac{v^2}{c^2}\right)\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial y^2}+\frac{\partial^2 p}{\partial z^2}=0$$

referred to axes moving with the electron, or the equivalent equation

$$\nabla^2 p=\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2}$$

referred to axes fixed relative to the observer. When $\kappa\epsilon$ varies inside the electron the equation for p is much more complicated.

The conditions as to invariability and symmetry are all satisfied in the case of the Lorentz electron, because the surfaces of constant density, as well as the bounding surfaces, whether external or internal, of the corresponding transformed electron are concentric spheres of invariable radius.

Hence the most general electron of the Lorentz type not only leads to the Lorentz mass formula, but also can exist provided that a uniform and invariable pressure be applied to its external surface, if it be solid, or to each of its surfaces, internal as well as external, if it be hollow.

256. We cannot however assert that the Lorentz electron is the only type satisfying these conditions. From the mathematical point of view the problem of finding an electron, capable of existing under the action of an external pressure uniform over its surface, is the same as the problem of finding an equilibrium form of gravitating uniformly rotating liquid in the limiting case when its angular velocity vanishes. Gravitational attractions, it is true, are replaced by electrostatic repulsions, but since the angular velocity vanishes, this only results in a change of sign of the pressure gradient.

Thus when we have found a series of equilibrium forms of gravitating liquid, including a limiting form for zero angular velocity, this limiting form corresponds to a possible type of electron. For instance the sphere, which is one limiting form of the series of MacLaurin's ellipsoids, corresponds to the Lorentz electron. The other limiting form of the same series, the infinitely long and infinitely thin circular cylinder, obviously does not correspond to a possible type of electron.

Whether, amongst the series of equilibrium forms of gravitating rotating liquid, it is possible to find one, which shall lead to a possible type of electron other than that of Lorentz, or not, it is not easy to say. But it is possible to assert that any type which may be found to satisfy all the necessary conditions as to stability and finiteness, will also lead to the Lorentz mass formula, provided that the surface pressure be invariable. For this ensures the invariability of the transformed electron; and according to Poincaré* every possible equilibrium form of gravitating rotating liquid possesses the necessary symmetry. Hence we deduce the following proposition:

Every type of electron, which can exist under the action of a uniform and invariable pressure applied to its bounding surface, whether it be at rest, or in uniform rectilinear motion, will also lead to the Lorentz mass formula.

The converse proposition is also true.

* *Vide* Tisserand, *Mécanique Céleste*, Vol. 2, p. 168.

257. In order to gain some idea of the magnitudes of the pressures involved, we shall consider briefly the case of the ordinary solid Lorentz electron.

The corresponding transformed electron by § 240 is a sphere of invariable radius a and uniform volume density $\kappa\epsilon \equiv 3e/4\pi a^3$. Its electromagnetic mass for zero speed is given by $m = 4e^2/5c^2a$.

From (403), § 253, and (407), § 254, we find

$$\left.\begin{aligned} \Psi &= \frac{e}{2a^3}(3a^2 - s^2), \quad p = B + \frac{3e^2s^2}{8\pi a^6} \\ \text{where} \quad s^2 &= \xi^2 + \eta^2 + \zeta^2 = \frac{x^2}{\kappa^2} + y^2 + z^2 \end{aligned}\right\} \quad \text{..............(408)},$$

and B is a function of t alone, or constant.

The difference of pressure between the centre and the external surface $(s = a)$ is equal to

$$\frac{3e^2}{8\pi a^4} = \frac{1875c^4}{2048\pi e^2}\left(\frac{cm}{e}\right)^4.$$

With the values $e/c = 1{\cdot}6 \,.\, 10^{-20}$ E. M. U. and $e/cm = 1{\cdot}77 \,.\, 10^7$ E. M. U. we find that the difference is about 10^{31} dynes per sq. cm., or 10^{25} atmospheres.

For Lorentz electrons of variable volume density, and for the Lorentz electron with surface charge, the difference of pressure has a different value, but is of the same order of magnitude.

258. Motion of the most general type. We shall now consider the case of an electron whose electric centre is moving in any prescribed curve in any prescribed manner, and shall determine the effect of the acceleration and motion. Partly in order to obtain definite results, partly for the sake of simplicity, we shall however limit our investigation to an electron of the type considered in § 256, for which the Lorentz mass formula holds. We are all the more justified in doing so, in so far as the Lorentz mass formula agrees with the more recent measurements of the ratio e/m for cathode ray and β particles, and is besides the only formula which allows us to treat the motion of electrons mathematically with a reasonable hope of obtaining simple results, as we shall see below in App. F and G.

We have already seen that the acceleration and relative motion terms in (399), § 252, are small of the order -1. Therefore they will produce small changes in the hydrostatic pressure, which will react on the electron producing small changes in its configuration, generally small changes in its volume density as well as slight deformations of its bounding surface.

We shall denote the small increase of hydrostatic pressure by δp, that of the volume density by $\delta\epsilon$, and that of the electrostatic potential by $\delta\Psi$. The last is the electrostatic potential of the volume distribution $\delta\epsilon$, and of a surface distribution representing the deformation of the bounding surface; the volume and surface distributions must be such as to leave the total charge of the electron unaltered.

We shall try to determine the conditions which must be satisfied in order that a hydrostatic pressure δp may be possible.

259. We notice at once that the small changes in the configuration of the electron will only alter the small acceleration and relative motion terms by small amounts of order 0, which are negligible. Thus in calculating these small terms we may treat the electron as exactly of the type considered in § 256; that is, we may assume the corresponding transformed electron to be invariable, so that the coordinates (ξ, η, ζ) and the potential Ψ are independent of the time and do not involve the velocity $\mathbf{v}$. In order to use these coordinates we must however refer the motion to rotating axes as in Ch. XI, §§ 141 and 144, so that the axes of (x, y, z) are the tangent to the path, the principal normal drawn towards the centre of curvature, and the binormal. Then the angular velocity, ω, of the axes about themselves has the components $(v/\tau, 0, v/\rho)$, where ρ and τ are the radii of curvature and torsion respectively. In (399), § 252, on the other hand the velocities $\mathbf{u}$ and $\mathbf{u}_1$ and the total differentiation $\frac{d}{dt}$ are relative to axes fixed in direction.

In this Appendix we shall denote a unit vector drawn in the direction of the velocity $\mathbf{v}$ by $\mathbf{v}_1$; hence we have

$$\dot{\mathbf{v}} = \dot{v}\mathbf{v}_1 + [\omega\mathbf{v}] \quad \text{.........................(409)},$$

a relation by means of which we can eliminate ω.

By the usual formula of rigid dynamics we have

$$\mathbf{u} = \dot{\mathbf{r}} + [\omega\mathbf{r}] \quad \text{.............................(410)},$$

where $\mathbf{r}$ denotes the radius vector whose components are (x, y, z), or $(\kappa\xi, \eta, \zeta)$ by (401), § 253. Since (ξ, η, ζ) do not change owing to the assumed invariability of the transformed electron, while $\kappa \equiv \sqrt{(1 - v^2/c^2)}$, we find

$$\dot{\mathbf{r}} = -\frac{\dot{v}\,(\mathbf{vr})}{c^2\kappa^2}\mathbf{v}_1, \quad \mathbf{u} = -\frac{\dot{v}\,(\mathbf{vr})}{c^2\kappa^2}\mathbf{v}_1 + [\omega\mathbf{r}] \quad \text{..............(411)},$$

with similar equations for $\dot{\mathbf{r}}_1$ and $\mathbf{u}_1$. By means of (409) we find from (411_2)

$$(\mathbf{uv}) = \frac{(c^2 - 2v^2)\,\dot{v}}{c^2\kappa^2}(\mathbf{v}_1\mathbf{r}) - (\dot{\mathbf{v}}\mathbf{r}) \quad \text{..................(412)}.$$

260. The relative motion term in the first line of (399), § 252, becomes by (402), § 253,

$$-\tfrac{1}{2}\epsilon\int de_1\,(\{\mathbf{u} + \mathbf{u}_1\}\,D)\,\kappa\nabla\frac{1}{\mathrm{P}} = \frac{\epsilon}{2c^2\kappa}\int de_1\,(\{\mathbf{u} + \mathbf{u}_1\}\,\mathbf{v})\,\nabla\frac{1}{\mathrm{P}},$$

because P is independent of $\mathbf{v}$ and therefore not affected by the operator D. Using (412), § 259, we may write this in the form

$$\epsilon\left[\frac{(c^2-2v^2)\,\dot{v}}{2c^4\kappa^3}\int(\mathbf{v}_1\{\mathbf{r}+\mathbf{r}_1\})\,\nabla\frac{de_1}{\mathrm{P}}-\frac{1}{2c^2\kappa}\int(\dot{\mathbf{v}}\{\mathbf{r}+\mathbf{r}_1\})\,\nabla\frac{de_1}{\mathrm{P}}\right]$$

$$=\epsilon\left[\frac{(c^2-2v^2)\,\dot{v}}{2c^4\kappa^3}\{(\mathbf{v}_1\mathbf{r})\,\nabla\Psi+\nabla\,(\mathbf{v}_1\mathbf{A})\}-\frac{1}{2c^2\kappa}\{(\dot{\mathbf{v}}\mathbf{r})\,\nabla\Psi+\nabla\,(\mathbf{v}_1\mathbf{A})\}\right] \quad \text{......(413)},$$

where we have used (403), § 253, and put

$$\mathbf{A}=\int\frac{\mathbf{r}_1\,de_1}{\mathrm{P}} \quad \text{..............................(414)}.$$

$\mathbf{A}$ is of the nature of a vector potential due to current $\mathbf{r}_1 de_1$ in the transformed electron.

Since we have identically

$$\nabla\,(\mathbf{v}_1\mathbf{r})=\mathbf{v}_1,\quad \nabla\,(\dot{\mathbf{v}}\mathbf{r})=\dot{\mathbf{v}},\quad \nabla\overline{\Psi}=0,$$

we can write (413) in the more convenient form

$$\epsilon\nabla\left[\frac{(c^2-2v^2)\,\dot{v}}{2c^4\kappa^3}\{(\mathbf{v}_1\mathbf{r})\,(\Psi-\overline{\Psi})+(\mathbf{v}_1\mathbf{A})\}-\frac{1}{2c^2\kappa}\{(\dot{\mathbf{v}}\mathbf{r})\,(\Psi-\overline{\Psi})+(\dot{\mathbf{v}}\mathbf{A})\}\right]$$

$$-\,\epsilon\left[\frac{(c^2-2v^2)\,\dot{v}\,(\Psi-\overline{\Psi})}{2c^4\kappa^3}\mathbf{v}_1-\frac{\Psi-\overline{\Psi}}{2c^2\kappa}\dot{\mathbf{v}}\right] \quad \text{...(415)}.$$

Again, the first term in the second line of (399), § 252, may be treated in the same way; by means of (403) and (404), § 253, we find that it becomes

$$\epsilon\frac{d}{dt}\tfrac{1}{2}D\kappa\,(\Psi-\overline{\Psi})=-\,\epsilon\frac{d}{dt}\frac{\Psi-\overline{\Psi}}{2c^2\kappa}\mathbf{v}$$

$$=-\,\epsilon\left[\frac{v^2\dot{v}\,(\Psi-\overline{\Psi})}{2c^4\kappa^3}\mathbf{v}_1+\frac{\Psi-\overline{\Psi}}{2c^2\kappa}\dot{\mathbf{v}}\right] \quad \text{...(416)},$$

because Ψ and $\overline{\Psi}$ are to be treated as independent of the time t.

The small changes δp, $\delta\epsilon$ and $\delta\Psi$ will have to be taken into account in calculating the large terms in (399), § 252. The increase of the left-hand member is simply $\nabla\delta p$, and that of the first principal term on the right side is $-\kappa\delta\epsilon\nabla\Psi-\kappa\epsilon\nabla\delta\Psi$. Hence taking account of the small terms (415) and (416) we find

$$\nabla\delta p=-\,\kappa\delta\epsilon\nabla\Psi-\kappa\epsilon\nabla\delta\Psi$$

$$+\,\epsilon\nabla\left[\frac{(c^2-2v^2)\,\dot{v}}{2c^4\kappa^3}\{(\mathbf{v}_1\mathbf{r})\,(\Psi-\overline{\Psi})+(\mathbf{v}_1\mathbf{A})\}-\frac{1}{2c^2\kappa}\{(\dot{\mathbf{v}}\mathbf{r})\,(\Psi-\overline{\Psi})+(\dot{\mathbf{v}}\mathbf{A})\}\right]$$

$$-\,\epsilon\frac{\dot{v}\,(\Psi-\overline{\Psi})}{2c^2\kappa}\mathbf{v}_1+\left(\frac{M}{e}\epsilon-\mu\right)\dot{\mathbf{v}} \quad \text{...(417)}.$$

261. Effect of a linear distribution of volume density. Before discussing the equation (417) just obtained for the pressure δp, we shall prove a lemma relating to linear distributions of volume density, which enables us to transform the equation to a more convenient form.

Lemma. If a volume distribution be given by

$$\delta\epsilon = \epsilon\,(\mathbf{br}) \quad\text{.............................(418)},$$

where **b** is any vector independent of the coordinates, but possibly depending on the time, then

$$\begin{aligned}\delta\epsilon\nabla\Psi + \epsilon\nabla\delta\Psi &= \epsilon\,[(\mathbf{br})\,\nabla\Psi + \nabla\,(\mathbf{bA})] \\ &= \epsilon\nabla\,[(\mathbf{br})\,(\Psi - \overline{\Psi}) + (\mathbf{bA})] - \epsilon\,(\Psi - \overline{\Psi})\,\mathbf{b} \quad\text{...(419)}.\end{aligned}$$

The volume density of the transformed electron, corresponding to $\delta\epsilon$ for the actual electron, is $\kappa\delta\epsilon$ by § 253; hence we find by means of (403), § 253, and (414), § 260,

$$\delta\Psi = \int\frac{\kappa\delta\epsilon_1 d\xi_1 d\eta_1 d\zeta_1}{\mathrm{P}} = \int\frac{\delta\epsilon_1}{\epsilon_1}\frac{de_1}{\mathrm{P}} = \int\frac{(\mathbf{br}_1)\,de_1}{\mathrm{P}} = (\mathbf{bA}) \quad\text{......(420)}.$$

Operating with ∇, multiplying by ϵ, and adding the term

$$\delta\epsilon\nabla\Psi \equiv \epsilon\,(\mathbf{br})\,\nabla\Psi,$$

we obtain (419).

262. The equation for the increase of pressure. In order to simplify (417), § 260, we may use the lemma of § 261 to eliminate either all the quantities in the second line of (417), or the first term in the third line. We prefer the first method as it leads to simpler results. Let us write

$$\left.\begin{aligned}\delta\epsilon = \delta_1\epsilon + \delta_2\epsilon, \quad \delta\Psi = \delta_1\Psi + \delta_2\Psi \\ \delta_1\epsilon = \epsilon\left[\frac{(c^2 - 2v^2)\,\dot{v}}{2c^4\kappa^4}\,(\mathbf{v}_1\mathbf{r}) - \frac{1}{2c^2\kappa^2}\,(\dot{\mathbf{v}}\mathbf{r})\right]\end{aligned}\right\} \quad\text{............(421)},$$

where $\delta_1\Psi$ is the electrostatic potential due to the volume distribution $\delta_1\epsilon$. We saw in § 258 that the total additional charge due to acceleration and relative motion vanishes; that due to $\delta_1\epsilon$ also vanishes, because $\delta_1\epsilon/\epsilon$ is a homogeneous linear function of the coordinates relative to the electric centre. Hence the total charge due to $\delta_2\epsilon$, together with that due to the surface distribution, if any, also vanishes.

Substituting from (421) in (417), § 260, and using the lemma of § 261, we find

$$\begin{aligned}\nabla\delta p = &- \kappa\delta_2\epsilon\nabla\Psi - \kappa\epsilon\nabla\delta_2\Psi \\ &- \frac{v^2\dot{v}\epsilon\,(\Psi - \overline{\Psi})}{2c^4\kappa^3}\,\mathbf{v}_1 + \left\{-\frac{\epsilon\,(\Psi - \overline{\Psi})}{2c^2\kappa} + \frac{M}{e}\,\epsilon - \mu\right\}\dot{\mathbf{v}} \quad\text{...(422)}.\end{aligned}$$

Write

$$\mathbf{d} = \frac{v^2\dot{v}}{2c^4\kappa^4}\,\mathbf{v}_1 + \frac{1}{2c^2\kappa^2}\,\dot{\mathbf{v}} \quad\text{.......................(423)},$$

so that **d** is independent of the coordinates. Adding

$$\nabla\kappa\epsilon\,\{\delta_2\Psi + (\Psi - \overline{\Psi})\,(\mathbf{dr})\}$$

to both sides of (422), and noticing that

$$\nabla\,(\mathbf{dr}) = \mathbf{d}, \quad \nabla\epsilon = \frac{\partial\epsilon}{\partial\Psi}\,\nabla\Psi,$$

on account of (406), § 254, we obtain finally

$$\nabla[\delta p+\kappa\epsilon\{\delta_2\Psi+(\Psi-\overline{\Psi})(\mathbf{dr})\}]=\left[-\kappa\delta_2\epsilon+\kappa\frac{\partial\epsilon}{\partial\Psi}\delta_2\Psi+\kappa\frac{\partial\epsilon(\Psi-\overline{\Psi})}{\partial\Psi}(\mathbf{dr})\right]\nabla\Psi+\left(\frac{M}{e}\epsilon-\mu\right)\dot{\mathbf{v}}\ldots(424).$$

We shall determine the conditions under which an equation of this form is possible, and shall then find an expression for δp and determine the further conditions which are necessary to ensure that it vanishes at the surface of the electron.

263. Let us eliminate δp by taking the curl of (424), § 262, and multiply the resulting equation scalarly by $\nabla\Psi$. The first and second terms on the right both disappear, and we obtain

$$(\dot{\mathbf{v}}\nabla\mu\,.\,\nabla\Psi)=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(425).$$

This equation must be true for all values of $\dot{\mathbf{v}}$, which is quite arbitrary. We see from (403), § 253, that Ψ does not depend on $\dot{\mathbf{v}}$ at all, nor can μ do so, for a non-electromagnetic mass depending on the acceleration cannot be explained mechanically. Hence (425) requires that

$$[\nabla\mu\,.\,\nabla\Psi]=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(426).$$

Thus μ, like ϵ, must be a function of Ψ and t alone.

264. Let us subtract a term

$$\nabla\left(\frac{M}{e}\epsilon-\mu\right)(\dot{\mathbf{v}}\mathbf{r})$$

from each side of (424), § 262, bearing in mind that

$$\nabla\epsilon=\frac{\partial\epsilon}{\partial\Psi}\nabla\Psi,\quad \nabla\mu=\frac{\partial\mu}{\partial\Psi}\nabla\Psi,$$

because ϵ and μ involve the coordinates only in so far as they enter into the expression for Ψ. Then we find

$$\nabla\left[\delta p+\kappa\epsilon\{\delta_2\Psi+(\Psi-\overline{\Psi})(\mathbf{dr})\}-\left(\frac{M}{e}\epsilon-\mu\right)(\dot{\mathbf{v}}\mathbf{r})\right]$$
$$=\left[-\kappa\delta_2\epsilon+\kappa\frac{\partial\epsilon}{\partial\Psi}\delta_2\Psi+\kappa\frac{\partial\epsilon(\Psi-\overline{\Psi})}{\partial\Psi}(\mathbf{dr})-\left(\frac{M}{e}\frac{\partial\epsilon}{\partial\Psi}-\frac{\partial\mu}{\partial\Psi}\right)(\dot{\mathbf{v}}\mathbf{r})\right]\nabla\Psi \quad \ldots\ldots(427).$$

This equation is only possible when each of the functions enclosed in square brackets reduces to a function of Ψ and t alone. Hence we have

$$\delta p=-\kappa\epsilon\delta_2\Psi-\kappa\epsilon(\Psi-\overline{\Psi})(\mathbf{dr})+\left(\frac{M}{e}\epsilon-\mu\right)(\dot{\mathbf{v}}\mathbf{r})+f\ldots\ldots(428),$$

$$-\kappa\delta_2\epsilon+\kappa\frac{\partial\epsilon}{\partial\Psi}\delta_2\Psi=-\kappa\frac{\partial\epsilon(\Psi-\overline{\Psi})}{\partial\Psi}(\mathbf{dr})+\left(\frac{M}{e}\frac{\partial\epsilon}{\partial\Psi}-\frac{\partial\mu}{\partial\Psi}\right)(\dot{\mathbf{v}}\mathbf{r})+\frac{\partial f}{\partial\Psi}\ldots(429),$$

where f is an arbitrary function of Ψ and t alone, and therefore does not involve any terms such as $(\mathbf{dr})$ or $(\dot{\mathbf{v}}\mathbf{r})$. This fact shows that f cannot represent an effect due to acceleration and relative motion, but results from other causes, which are not under consideration. Moreover it is always possible to make it vanish at the surface of the electron owing to the fact that Ψ is constant there. For these reasons we may leave it out of account in what follows.

The volume density $\delta_2\epsilon$ and potential $\delta_2\Psi$ are not independent; since $\kappa\delta_2\epsilon$ is the volume density of the transformed electron which, together with some possible surface distribution, produces the electrostatic potential $\delta_2\Psi$, these two quantities are connected by Poisson's equation

$$\frac{\partial^2\delta_2\Psi}{\partial\xi^2}+\frac{\partial^2\delta_2\Psi}{\partial\eta^2}+\frac{\partial^2\delta_2\Psi}{\partial\zeta^2}+4\pi\kappa\delta_2\epsilon=0 \quad \ldots\ldots\ldots\ldots\ldots(430).$$

From (429) and (430) we find, omitting f,

$$\frac{\partial^2\delta_2\Psi}{\partial\xi^2}+\frac{\partial^2\delta_2\Psi}{\partial\eta^2}+\frac{\partial^2\delta_2\Psi}{\partial\zeta^2}+4\pi\kappa\frac{\partial\epsilon}{\partial\Psi}\delta_2\Psi$$
$$=-4\pi\kappa\frac{\partial\epsilon(\Psi-\overline{\Psi})}{\partial\Psi}(\mathbf{dr})+4\pi\left(\frac{M}{e}\frac{\partial\epsilon}{\partial\Psi}-\frac{\partial\mu}{\partial\Psi}\right)(\dot{\mathbf{v}}\mathbf{r}) \quad \ldots(431),$$

which holds everywhere, both inside the electron, where ϵ and μ have given finite values, and outside it, where they vanish.

When the value of $\delta_2\Psi$ obtained from (431) is substituted in (428) we obtain δp. It generally differs from zero, even at the surface, so that the uniformity and invariability, which were found in §§ 253—257 to be characteristic of the surface pressure of our present type of electron when the motion is a uniform rectilinear translation, no longer exist when the motion is general.

Hence we conclude that for the type of electron in question, which gives the Lorentz mass formula, a hydrostatic pressure exists even when there is acceleration and relative motion, but it is no longer uniform and invariable at the surface in general, that is to say, without further restriction of the type of the electron.

265. Additional conditions for a uniform and invariable surface pressure. Let us examine the potential equation (431), § 264, more closely. The right-hand member involves two scalar products of the radius vector $\mathbf{r}$ by constant vectors, namely $(\mathbf{dr})$ and $(\dot{\mathbf{v}}\mathbf{r})$; they occur in the first degree, and their coefficients are functions of Ψ alone, because ϵ and μ have this property by (406), § 254, and (426), § 263.

We see from (423), § 262, that

$$\mathbf{d}=\frac{v^2\dot{v}}{2c^4\kappa^4}\mathbf{v}_1+\frac{1}{2c^2\kappa^2}\dot{\mathbf{v}}.$$

Thus **d** has a component in the direction of the velocity except in the particular case when the speed v is constant. Since both the velocity and acceleration of the electron are perfectly arbitrary, **d** and $\dot{\mathbf{v}}$ are two perfectly arbitrary and independent vectors. Hence in (428) and (431), § 264, the terms involving (**dr**) and ($\dot{\mathbf{v}}\mathbf{r}$) must be treated separately. Moreover, (431) cannot be satisfied for *every* value of **d** and $\dot{\mathbf{v}}$, unless its left-hand member reduces to the sum of two terms involving (**dr**) and ($\dot{\mathbf{v}}\mathbf{r}$) as factors respectively. Therefore $\delta_2\Psi$ must be of the form

$$\delta_2\Psi = A\,(\mathbf{dr}) + B\,(\dot{\mathbf{v}}\mathbf{r}) + C \quad\text{......................(432)},$$

where A, B and C are functions of the time and coordinates, but do not involve either **d** or $\dot{\mathbf{v}}$. Then $A\,(\mathbf{dr})$ must satisfy (431) when ($\dot{\mathbf{v}}\mathbf{r}$) is put equal to zero, $B\,(\dot{\mathbf{v}}\mathbf{r})$ when (**dr**) is zero, and C when both are zero.

Beginning with the term A (**dr**), we find

$$\left(\frac{\partial^2 A}{\partial\xi^2} + \frac{\partial^2 A}{\partial\eta^2} + \frac{\partial^2 A}{\partial\zeta^2} + 4\pi\kappa\frac{\partial\epsilon}{\partial\Psi}A\right)(\mathbf{dr}) + 2\left(\kappa d_\xi\frac{\partial A}{\partial\xi} + d_\eta\frac{\partial A}{\partial\eta} + d_\zeta\frac{\partial A}{\partial\zeta}\right)$$
$$= -4\pi\kappa\frac{\partial\epsilon\,(\Psi - \overline{\Psi})}{\partial\Psi}(\mathbf{dr}) \quad\text{...(433)}.$$

This equation must be satisfied for every value of **d**, so that the second term in the first line must reduce to the form $F\,(\mathbf{dr})$, where F is some function of the coordinates and time, but does not involve **d**. This occurs when, and only when

$$\frac{\partial A}{\partial\xi} : \frac{\partial A}{\partial\eta} : \frac{\partial A}{\partial\zeta} : \frac{\partial A}{\partial s} = \xi : \eta : \zeta : s \quad\text{..................(434)},$$

where $s = \sqrt{(\xi^2 + \eta^2 + \zeta^2)}$ as before. Then we have

$$\kappa d_\xi\frac{\partial A}{\partial\xi} + d_\eta\frac{\partial A}{\partial\eta} + d_\zeta\frac{\partial A}{\partial\zeta} = \frac{1}{s}\frac{\partial A}{\partial s}(\mathbf{dr}) \quad\text{..............(435)},$$

and (433) gives after reduction

$$\frac{\partial^2 A}{\partial s^2} + \frac{4}{s}\frac{\partial A}{\partial s} + 4\pi\kappa\frac{\partial\epsilon}{\partial\Psi}A = -4\pi\kappa\frac{\partial\epsilon\,(\Psi - \overline{\Psi})}{\partial\Psi} \quad\text{.........(436)}.$$

266. The equation (436), § 265, only involves the quantities s, t and Ψ; hence Ψ must be a function of s and t alone, and ϵ and μ have the same property.

Thus we arrive at a necessary condition to be satisfied by every type of electron for which the surface pressure is uniform and invariable when there is acceleration and relative motion, namely, the volume density, non-electromagnetic mass density and electrostatic potential of the corresponding transformed electron must be functions of the radius s only (they cannot involve the time t because the transformed electron is invariable). The surfaces of constant density of the transformed electron are concentric spheres, and therefore those of the actual electron are concentric Heaviside ellipsoids to a first approximation.

All this may be expressed simply by saying that the transformed electron possesses spherical symmetry, or that the actual electron is a generalized Lorentz electron (of that shape, but not necessarily of uniform density). This agrees with Poincaré's well known solution of the energy difficulty of Abraham.

267. Treating the term in (432), § 265, which involves $(\dot{\mathbf{v}}\mathbf{r})$, in the same way as the first, we find

$$\frac{\partial^2 B}{\partial s^2} + \frac{4}{s}\frac{\partial B}{\partial s} + 4\pi\kappa \frac{\partial \epsilon}{\partial \Psi} B = 4\pi \left(\frac{M}{e}\frac{\partial \epsilon}{\partial \Psi} - \frac{\partial \mu}{\partial \Psi}\right) \quad \text{.........(437).}$$

This equation has the same first member as (436), § 265, but a different second member; hence the particular integrals occurring in A and B are different, but the complementary function is the same. Of course the values of the two arbitrary constants involved in it will be different in the two cases. It follows from the theory of differential equations that the solution of either equation entails that of the other as a consequence.

The third term, C, of (432), § 265, is a function of s and t alone, and when substituted in (428) and (429), § 264, merely gives rise to functions of the form of f, which has been already ignored for reasons mentioned in § 264.

268. Substituting the value of $\delta_2\Psi$ given by (432), § 265, with C omitted, in (430), § 264, and using (435) and (436), § 265, and also (437), § 267, we find

$$\delta_2\epsilon = -\frac{1}{4\pi\kappa}\left\{\left(\frac{\partial^2 A}{\partial s^2} + \frac{4}{s}\frac{\partial A}{\partial s}\right)(\mathbf{dr}) + \left(\frac{\partial^2 B}{\partial s^2} + \frac{4}{s}\frac{\partial B}{\partial s}\right)(\dot{\mathbf{v}}\mathbf{r})\right\}$$

$$= \left\{A\frac{\partial \epsilon}{\partial \Psi} + \frac{\partial \epsilon(\Psi - \overline{\Psi})}{\partial \Psi}\right\}(\mathbf{dr}) + \left\{B\frac{\partial \epsilon}{\partial \Psi} - \frac{1}{\kappa}\left(\frac{M}{\epsilon}\frac{\partial \epsilon}{\partial \Psi} - \frac{\partial \mu}{\partial \Psi}\right)\right\}(\dot{\mathbf{v}}\mathbf{r}) \text{...(438),}$$

which might also have been obtained from (429), § 264.

As we have already pointed out, $\delta_2\epsilon$ does not represent the whole effect on the charge, because the potential $\delta_2\Psi$ is generally due in part to $\delta_2\epsilon$, but also in part to distributions on the surfaces of the electron, both internal and external. In order to find these, we must calculate the part of $\delta_2\Psi$ due to $\delta_2\epsilon$.

The quantities $(\mathbf{dr})$ and $(\dot{\mathbf{v}}\mathbf{r})$ are solid spherical harmonics for the transformed electron, and of the first degree. The theory of these functions gives for the potential due to the part of $\delta_2\epsilon$ involving $(\mathbf{dr})$, by (438),

$$-\tfrac{1}{3}(\mathbf{dr})\left\{\int_s^a \left(s\frac{\partial^2 A}{\partial s^2} + 4\frac{\partial A}{\partial s}\right)ds + \frac{1}{s^3}\int_b^s \left(s^4\frac{\partial^2 A}{\partial s^2} + 4s^3\frac{\partial A}{\partial s}\right)ds\right\}$$

$$= (\mathbf{dr})\left\{A - A_a - \tfrac{1}{3}aA_a' + \frac{b^4}{3s^3}A_b'\right\},$$

where b and a are the internal and external radii of the transformed electron, A_a and A_b the corresponding values of A, and A_a' and A_b' those of $\frac{\partial A}{\partial s}$. In obtaining this expression it must be borne in mind that the density for the transformed electron is $\kappa\delta_2\epsilon$, because $\delta_2\epsilon$ belongs to the actual electron. A similar expression is obtained for $(\dot{\mathbf{v}}\mathbf{r})$; adding the two expressions together and subtracting the result from the expression (432), § 265, we find that the part of the potential $\delta_2\Psi$ due to surface distributions is equal to

$$\left\{A_a + \tfrac{1}{3}\,aA_a' - \frac{b^4 A_b'}{3s^3}\right\}(\mathbf{dr}) + \left\{B_a + \tfrac{1}{3}\,aB_a' - \frac{b^4 B_b'}{3s^3}\right\}(\dot{\mathbf{v}}\mathbf{r}) \quad \text{...(439).}$$

The form of this expression shows that the terms independent of s are due to a distribution, on the outer surface of the transformed electron, of surface density

$$\frac{3}{4\pi a}\{(A_a + \tfrac{1}{3}\,aA_a')(\mathbf{dr}) + (B_a + \tfrac{1}{3}\,aB_a')(\dot{\mathbf{v}}\mathbf{r})\} \quad \text{.........(440).}$$

The terms involving s^{-3} are due to a distribution, on the inner surface, of surface density

$$-\frac{1}{4\pi}\{A_b'(\mathbf{dr}) + B_b'(\dot{\mathbf{v}}\mathbf{r})\} \quad \text{.....................(441).}$$

These surface distributions are equivalent to normal displacements of the two surfaces, obtained by dividing each surface density by the volume density of the undeformed electron at the corresponding surface.

In addition to the volume density $\delta_2\epsilon$ we have the volume density $\delta_1\epsilon$ given by (421), § 262. These two volume densities together are equivalent to a displacement of charge inside the electron.

269. Substituting the value (432), § 265, of $\delta_2\Psi$ in (428), § 264, we find

$$\delta p = -\kappa\epsilon\,(A + \Psi - \overline{\Psi})(\mathbf{dr}) + \left(-\kappa\epsilon B + \frac{M}{e}\,\epsilon - \mu\right)(\dot{\mathbf{v}}\mathbf{r}) \quad \text{...(442).}$$

Since the differential equations for A and B are of the second order, so that A and B each involve two arbitrary constants, we can choose these four constants so as to make each of the coefficients in (442) vanish for any two values of s, for instance, for $s = a$ and $s = b$. In that case δp vanishes at each surface of the electron.

The necessary and sufficient conditions for this to occur are

$$\left.\begin{aligned} A_a &= \overline{\Psi} - \Psi_a, \quad & A_b &= \overline{\Psi} - \Psi_b \\ \kappa B_a &= \frac{M}{e} - \frac{\mu_a}{\epsilon_a}, \quad & \kappa B_b &= \frac{M}{e} - \frac{\mu_b}{\epsilon_b} \end{aligned}\right\} \quad \text{..................(443),}$$

where the suffixes a and b denote values taken at the two surfaces respectively, just as before.

Hence it is always possible to equilibrate the electromagnetic forces inside the electron by means of uniform and invariable pressures applied to its internal and external surfaces, whatever its path, velocity and acceleration may be, provided only that it be of the generalized Lorentz type defined in § 266.

270. The surface pressures and electromagnetic mass. We have found that for the generalized Lorentz electron the surface pressures are unaltered by the motion. From (407), § 254, we see that the pressure involves an arbitrary constant, so that one of the two surface pressures is indeterminate; their difference however is determinate and invariable. Consequently it is characteristic of the particular electron selected, like the charge, e, and electromagnetic mass for zero speed, m. We shall denote the excess of the external above the internal pressure by P.

In studying the dependence of m and P on the structure of the electron we shall find it convenient to use the function E defined by the equation

$$E = \int_b^s 4\pi\kappa\epsilon s^2 ds \quad \text{.........................(444).}$$

Thus E is the total charge inside the sphere of radius s in the transformed electron, or the corresponding Heaviside spheroid in the actual electron. It satisfies the boundary conditions

$$E_b = 0, \quad E_a = e \quad \text{.........................(445).}$$

Consequently it is given at both limits.

271. In order to calculate the electromagnetic mass we must know the function Φ defined by (358), § 234, which, as we saw in § 253, is also given by

$$\Phi = \tfrac{1}{2}\kappa e\overline{\Psi} = \tfrac{1}{2}\kappa \int_0^e \Psi dE.$$

For the generalized Lorentz electron we see from (350), § 229, and (358), § 234, that, on account of the spherical symmetry of the transformed electron, we have

$$m = \frac{4\Phi}{3c^2\kappa} = \frac{2}{3c^2}\int_0^e \Psi dE.$$

Ψ is determined by the conditions

$$\frac{\partial \Psi}{\partial s} = -\frac{E}{s^2}, \quad \Psi_a = \frac{e}{a}.$$

Hence we find by partial integration

$$m = \frac{2e^2}{3c^2 a} + \frac{2}{3c^2}\int_b^a \frac{E^2 ds}{s^2} \quad \text{.......................(446).}$$

Similarly we find from (407), § 254,

$$P = \frac{e^2}{8\pi a^4} + \frac{1}{2\pi}\int_b^a \frac{E^2 ds}{s^5} \quad \text{.........................(447).}$$

Putting $b = a$ in (446) and (447) we obtain the usual expressions for the Lorentz electron with surface charge. Putting $b = 0$ and $E = es^3/a^3$ we obtain those for the solid Lorentz electron of uniform density.

It is not difficult to show that P is stationary for $b = a$, subject to the conditions that e and m be given. It seems probable from the form of (447) that this value of P is an absolute minimum; at any rate it is less than that belonging to the uniform solid electron. Whether this be so, or not, we see from (447) that P is essentially positive; that is, the external pressure is always greater than the internal.

Hence we may suppose that the internal pressure is zero; but the external pressure can never be less than a certain positive minimum value, which is probably that belonging to the surface charge. From (446) and (447) we see that this value is equal to

$$\frac{81 c^8 m^4}{128\pi e^6}.$$

With the values $e/c = 1{\cdot}6 \,.\, 10^{-20}$ E.M.U. and $e/cm = 1{\cdot}77 \,.\, 10^7$ E.M.U., we find that the pressure in question is $7{\cdot}2 \,.\, 10^{30}$ dynes per sq. cm.

272. Example. The solid Lorentz electron. As an example of the determination of the effects of acceleration under the influence of a uniform and invariable surface pressure, we shall take the solid Lorentz electron of uniform density, without any non-electromagnetic mass. With the notation of § 257 we find

$$\Psi - \overline{\Psi} = \frac{e}{2a^3}\left(\tfrac{3}{5}a^2 - s^2\right) \quad \text{.........................(448).}$$

The equation (436), § 265, becomes

$$\frac{\partial^2 A}{\partial s^2} + \frac{4}{s}\frac{\partial A}{\partial s} = -4\pi\kappa\epsilon = -\frac{3e}{a^3}.$$

It gives

$$A = C - \frac{3es^2}{10a^3} \quad \text{.........................(449).}$$

A term of the form D/s^3 cannot be present because it becomes infinite at the centre.

The second function B is not required because M and μ are zero.

Determining C from (443), § 269, for $s = a$, we find

$$C = \frac{e}{2a}, \quad A = \frac{e}{2a^3}\left(a^2 - \tfrac{3}{5}s^2\right) \quad \text{.....................(450).}$$

Substituting from (448) and (450) in (442), § 269, we find for the change of pressure

$$\delta p = -\frac{3e^2}{5\pi a^6}(a^2 - s^2)(\mathbf{dr}) \quad \text{.....................(451).}$$

This vanishes at the surface and at the centre, and is greatest for $s = a/\sqrt{3}$.

In order to find the change of density we must use (421), § 262, and (438), § 268. Since $\frac{\partial \epsilon}{\partial \Psi}$ vanishes, we find

$$\delta_2 \epsilon = \epsilon\,(\mathbf{dr}).$$

Adding this to $\delta_1 \epsilon$ and using (423), § 262, we obtain

$$\delta \epsilon = \frac{\epsilon \dot{v}}{2c^2\kappa^2}(\mathbf{v}_1\mathbf{r}) = \frac{\epsilon \dot{v} x}{2c^2\kappa^2} = \frac{\epsilon \dot{v} \xi}{2c^2\kappa} \quad \text{..................(452).}$$

The surface densities are given by (440) and (441), § 268.

Since by (450) $A_a + \frac{1}{3} a A_a'$ and A_0' both vanish, there is no surface distribution and no deformation of the surface.

Thus we see that the only effects produced by the acceleration in the uniform solid Lorentz electron are the change of pressure given by (451) and the redistribution of charge given by (452). The latter is equivalent to a displacement of part of the charge, on the whole in the direction of motion, or the opposite direction, according as the speed is increasing, or diminishing.

273. Conclusion. Premising that the terms electric density, non-electromagnetic mass and potential in what follows refer to the transformed electron, we may summarize the chief stages and conclusions of our argument thus:

(1) In the case of uniform rectilinear motion the electromagnetic forces can be balanced by a distribution of hydrostatic pressure when, and only when, the electric density is a function of the potential and time (or speed) alone (§ 254).

(2) In the last case the pressure is uniform and invariable at the surface of the electron when, and only when, the potential is constant at the surface, and the transformed electron is invariable (invariable here means independent of the speed). These conditions are equivalent to the condition that the electromagnetic mass of the electron shall obey the Lorentz mass formula (§§ 255 and 256).

(3) When the electron is moving as a whole in any path according to any law, a balancing distribution of hydrostatic pressure exists when, and only when, the non-electromagnetic mass, if any, associated with each element of charge depends only on the potential at the element and the time (§§ 263 and 264).

(4) In the last case the pressure at the surface is uniform and invariable, and has the same value as for uniform rectilinear motion when, and only when, the transformed electron is spherically symmetrical (§§ 265 and 266).

(5) The only effects of the acceleration are to produce (*a*) a small redistribution of the charge inside the electron, and (*b*) small deformations of its surfaces (§§ 268 and 269).

(6) In the type of electron to which we have been led under (4), the external always exceeds the internal pressure, so that the internal pressure may be zero, but the external pressure must have a finite positive value (§ 271).

274. In this Appendix we have neglected the small terms due to squares and products of the relative velocities and to radiation. The corresponding terms in the expression for the electromagnetic force (334), § 219, are very complicated, and the investigation of their effect would be troublesome, even though the problem is very much simplified by the result that the transformed electron must be spherically symmetrical. But this result having been once established, probably a simpler method of treating the whole problem *ab initio* can be found, and it is hardly worth while to carry our approximation to a higher stage.

275. So long as we neglect the effects of acceleration and relative motion, our results are in complete agreement with the Postulate of Relativity. The formula for the mass is the Lorentz mass formula, when the pressure at the surface is uniform and invariable, a result which is to be expected on the Relative Theory, for which the pressure is an invariant, that is independent of the speed.

But when the effects of acceleration and relative motion are taken into account, this agreement is absent. In § 272 we found that even in the case of the ordinary solid Lorentz electron, moving under the influence of a uniform and invariable surface pressure, such as the Postulate of Relativity requires, there is a slight redistribution of the charge, which could hardly fail to be detected by an observer moving with the electron and using the coordinates and time appropriate to its speed at the moment. Thus the electron would not appear to be invariable to this observer; he could detect its acceleration in spite of the Postulate of Relativity.

Thus it appears that the Postulate of Relativity, when applied with absolute strictness, excludes a mechanical explanation of the electron. But we must bear in mind that the Postulate was only advanced in the first instance in connection with uniform relative motions. Its extension to motions involving acceleration goes far beyond what is required by experimental facts.

APPENDIX F

THE MECHANICS OF THE LORENTZ ELECTRON

276. WE have just seen, in Appendix E, that the Lorentz mass formula is superior to any other in two respects, both of which are very important from the theoretical point of view. In the first place, it is in agreement with the Principle of Relativity, which we know affords the simplest explanation of the absence of any effect of the earth's motion on optical and other electromagnetic phenomena.

Secondly, it is consistent with any possible mechanical explanation of the electron which accords with the symmetry observed experimentally.

Moreover, it is in better agreement with recent measurements of the mass of β-particles than any other formula; but even if the experimental evidence were not so much in its favour as it actually is, the theoretical advantages would be decisive.

In the present Appendix we shall deduce the equations of motion of the electron on the basis of the Lorentz mass formula, leaving out of account the terms of higher orders; in other words, we shall confine ourselves to quasi-stationary motions, to use Abraham's expression. This limitation is necessary in order to secure manageable equations yielding definite results. It is of little practical importance except in cases of discontinuous motion. These are cases where the acceleration, or some differential coefficient of the velocity of higher order, becomes infinite, so that the expressions for the coordinates and velocity-components become discontinuous in form, but not in value, at one or more instants of time. An example is that of an electron starting its motion from rest.

The effect of such discontinuities of motion has been examined by Schott* on the basis of the electron theory, and by Walker† on the several assumptions that an electron behaves like a conducting and an insulating sphere respectively. The general result of these investigations is that the

* *Ann. d. Phys.* [4], 25, p. 63, 1908.
† *Phil. Trans.* A, 210, p. 145, 1910.

approximation used in this Appendix fails during an interval of time, which is comparable with the time required by an electromagnetic wave to pass across the electron and includes the instant at which the discontinuity occurs. Further, the subsequent motion is affected to a certain extent, but the relative error committed when we neglect this effect is only of the order of the ratios which the radius of the electron bears to the radius of curvature of its path, and to the distance within which its velocity would have been acquired if its acceleration had been uniform.

277. On the basis of this approximation the equations of motion can be expressed in Lagrange's form whatever mass formula is adopted. When the external electromagnetic field is steady, an energy integral can be deduced in the usual way. When it is symmetrical about an axis, we can deduce an integral which expresses the Principle of Conservation of Areas for the axis of symmetry, whether the field be steady or not. When the field is steady as well as symmetrical about an axis, both integrals exist. To this extent all the various mass formulae are on a par.

When the Lorentz mass formula is used, further progress is possible. When the energy integral exists, it can be used to reduce the equations of motion to a form in which they are little more complicated than those of ordinary particle dynamics. For instance, they may be written in Hamilton's form, so that the whole of Jacobi's theory and its subsequent developments become applicable.

When there is symmetry about an axis, the angular coordinate belonging to that axis may be ignored, and the problem reduced to that of motion in two dimensions.

When the mass formulae of Abraham, or of Thomson, or of Walker, or any of the more complicated ones are used, these reductions are practically impossible, because the necessary eliminations cannot be effected.

The simplifications which can be effected in this way with the Lorentz mass formula are so great as to make numerical results possible in many problems of practical importance. The Lorentz mass formula is immeasurably superior to all other formulae in this respect; this fact alone is sufficient to justify its use as a basis for the mechanics of the electron.

278. The equations of motion of the electron in the Newtonian form. We shall use the symbols m and m_v to denote the mass of the electron for zero velocity and for velocity v respectively, just as before. Neglecting the radiation terms in (218), § 148, we get for the vector equation of motion

$$\frac{d(m_v\mathbf{v})}{dt} = e(\mathbf{d} + [\mathbf{vh}]/c).$$

The Lorentz mass formula gives

$$m_v = \frac{cm}{\sqrt{(c^2 - v^2)}} \quad \text{.............................(453).}$$

Hence we find $$\frac{d}{dt}\left\{\frac{cm\mathbf{v}}{\sqrt{(c^2 - v^2)}}\right\} = e\,(\mathbf{d} + [\mathbf{vh}]/c) \quad \text{..................(454).}$$

This is the first Newtonian form of the vector equation of motion.

A second form can be obtained by expressing the external electric and magnetic forces in terms of the scalar potential ϕ and vector potential $\mathbf{a}$ by means of (VII) and (VIII), § 3. We get

$$\mathbf{d} + [\mathbf{vh}]/c = -\nabla\phi - \frac{\partial \mathbf{a}}{c\partial t} + [\mathbf{v}\,[\nabla\mathbf{a}]]/c.$$

Now $$[\mathbf{v}\,[\nabla\mathbf{a}]] = \nabla\,(\mathbf{va}) - (\mathbf{v}\nabla)\,\mathbf{a},$$

where ∇ does not operate on $\mathbf{v}$, that is to say, the coordinates and velocity components are treated as independent variables, just as in ordinary mechanics. Moreover, since fixed axes are presupposed in (VII) and (VIII), we have $\frac{\partial \mathbf{a}}{\partial t} + (\mathbf{v}\nabla)\,\mathbf{a} = \frac{d\mathbf{a}}{dt}$, where $\frac{d}{dt}$ denotes total differentiation. Substituting in (454), we find

$$\frac{d}{dt}\left\{\frac{cm\mathbf{v}}{\sqrt{(c^2 - v^2)}} + \frac{e\mathbf{a}}{c}\right\} = -\nabla\left\{e\phi - \frac{e\,(\mathbf{va})}{c}\right\} \quad \text{...........(455).}$$

This is the second Newtonian form of the equation of motion.

279. Lagrangian equations. Let (x, y, z) be the coordinates of the electron referred to fixed rectangular axes. Then the components of $\mathbf{v}$ are $(\dot{x}, \dot{y}, \dot{z})$.

The potentials ϕ and $\mathbf{a}$ are functions of x, y, z and t, but not of $\dot{x}, \dot{y}, \dot{z}$; they are not independent, but satisfy the equation

$$\frac{\partial\phi}{c\partial t} + \text{div. } \mathbf{a} = 0.$$

This equation of condition is however of no particular moment for what follows. Write

$$L = -cm\sqrt{(c^2 - v^2)} + \frac{e\,(\mathbf{va})}{c} - e\phi \quad \text{.................(456).}$$

Then we find $$\frac{\partial L}{\partial \dot{x}} = \frac{cm\dot{x}}{\sqrt{(c^2 - v^2)}} + \frac{ea_x}{c} \quad \text{......................(457),}$$

with two similar equations; that is, the three quantities $\frac{\partial L}{\partial \dot{x}}, \frac{\partial L}{\partial \dot{y}}$ and $\frac{\partial L}{\partial \dot{z}}$ are the components of the vector quantity $\frac{cm\mathbf{v}}{\sqrt{(c^2 - v^2)}} + \frac{e\mathbf{a}}{c}$ which occurs on the left-hand side of (455), § 278. Remembering that ∇ does not operate on

$\dot{x}$, $\dot{y}$ and $\dot{z}$, we see that the equations of motion, written in vector form in (455), § 278, may also be written in the form

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x} \quad \text{..............................(458),}$$

with two similar equations.

We may regard the three equations of type (458) as Lagrange's equations for the Lorentz electron, and L as the Lagrangian function. On this view the components of momentum of the electron are given by (457), and the two similar equations. The vector $cm\mathbf{v}/\sqrt{(c^2 - v^2)}$ is the momentum of the electron of the ordinary type, depending on its motion; the term $e\mathbf{a}/c$ represents additional momentum due to the presence of the external field.

280. That a Lagrangian function should exist for the system consisting of the electron and the external field need not surprise us when we remember that Schwarzschild* has already proved it to exist for any system of moving charges. From its existence we may conclude, just as Maxwell did for electric currents, that a mechanical explanation of the system of electron and external field is possible, even though no such explanation has yet been found. This agrees with the results of our last Appendix; but it is to be borne in mind that those results go much further, in so far as they indicate that the Lorentz electron is consistent with that particular type of mechanical explanation, which ascribes the forces acting on the electron to stresses in an elastic medium.

A Lagrangian function exists for other types of electron also, for example for the Abraham electron; and similar conclusions as to the possibility of mechanical explanations follow from the fact of its existence. But for these types of electron any possible mechanical explanation is necessarily more complicated than it is for the Lorentz electron.

The usual consequences follow from the existence of a Lagrangian function. The equations of motion can be deduced by varying the integral $\int_{t_0}^{t_1} L\,dt$, and therefore the Principle of Least Action holds. We can introduce generalized coordinates (q_1, q_2, q_3) in place of rectangular coordinates, and in this way transform our equations as we please. When a coordinate does not occur explicitly in the transformed Lagrangian function, that is to say, when it is a speed coordinate, the corresponding momentum is constant, and the coordinate may be ignored by using a modified Lagrangian function.

* *Enc. Math.* v. 14, p. 160.

281. Field symmetrical about an axis. We shall consider this case a little more fully as an illustration of the ignoration of coordinates.

Take the axis of symmetry as x-axis in a system of cylindrical coordinates (x, ϖ, χ). Let v' be the component velocity in the meridian plane, and let a_χ denote the component of the vector potential **a** perpendicular to the meridian plane. Then we have

$$v^2 = v'^2 + \varpi^2\dot{\chi}^2, \quad (\mathbf{va}) = (\mathbf{v}'\mathbf{a}) + \dot{\chi}\varpi a_\chi \quad \text{..............(459).}$$

The potentials ϕ and **a** are independent of χ; hence the Lagrangian function L does not involve χ explicitly, and χ is a speed coordinate and may be ignored.

The modified Lagrangian function L' is defined by the equation

$$L' = L - \dot{\chi}\frac{\partial L}{\partial \dot{\chi}}.$$

Now from (456), § 279, and (459) we get

$$\frac{\partial L}{\partial \dot{\chi}} = \frac{cm\varpi^2\dot{\chi}}{\sqrt{(c^2 - v^2)}} + \frac{e\varpi a_\chi}{c} = \kappa \quad \text{....................(460),}$$

where κ is the cyclic momentum and is constant. Hence

$$L' = -\frac{cm\,(c^2 - v'^2)}{\sqrt{(c^2 - v^2)}} + \frac{e\,(\mathbf{v}'\mathbf{a})}{c} - e\phi.$$

Equations (459) give in succession

$$\frac{c^2m^2\varpi^2\dot{\chi}^2}{c^2 - v^2} = \left(\frac{\kappa}{\varpi} - \frac{ea_\chi}{c}\right)^2,$$

$$\frac{c^2m^2\,(c^2 - v'^2)}{c^2 - v^2} = c^2m^2 + \left(\frac{\kappa}{\varpi} - \frac{ea_\chi}{c}\right)^2.$$

Write
$$m' = \sqrt{\left\{m^2 + \left(\frac{\kappa}{c\varpi} - \frac{ea_\chi}{c^2}\right)^2\right\}} \quad \text{.................(461).}$$

Then we find
$$L' = -\,cm'\sqrt{(c^2 - v'^2)} + \frac{e\,(\mathbf{v}'\mathbf{a})}{c} - e\phi \quad \text{..............(462).}$$

Comparing (462) with (456), § 279, we see that L' is of the same form as L. It differs in so far as the constant mass factor m in L is replaced by the variable mass factor m' in L'.

The similarity in form of the original and modified Lagrangian functions is due entirely to the fact that the Lorentz mass is proportional to $(c^2 - v^2)^{-\frac{1}{2}}$, and the corresponding first term in the Lagrangian function therefore proportional to $(c^2 - v^2)^{\frac{1}{2}}$. Consequently the elimination of the velocity $\dot{\chi}$ is simple, and leads to the result mentioned. In the case of other mass formulae the elimination is very complicated, and usually impracticable; for example, for the Bucherer mass formula, which makes the mass proportional to $(c^2 - v^2)^{-\frac{1}{3}}$, we have to solve a cubic equation, and in consequence the result of the elimination becomes so complicated as to be useless in practice. Again, for the mass formulae of Abraham, of Thomson and of Walker the equation to be solved is transcendental, and in consequence the elimination cannot be effected in finite terms at all. The superiority of the Lorentz formula in this respect is decisive for all problems in which actual results are aimed at. We shall find below that this is also true for the more general Hamiltonian transformation.

282. Hamiltonian equations. Let us use generalized coordinates (q_1, q_2, q_3), and let p_i be the generalized momentum corresponding to q_i as usual. Then

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \qquad \text{.................................(463).}$$

The Lagrangian equations may be written in the form

$$\frac{dp_i}{dt} = \frac{\partial L}{\partial q_i}.$$

As usual write $$H = p_1\dot{q}_1 + p_2\dot{q}_2 + p_3\dot{q}_3 - L \qquad \text{.....................(464),}$$
where $(\dot{q}_1, \dot{q}_2, \dot{q}_3)$ are to be eliminated by means of (463).

We find in the usual manner

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i} \qquad \text{......................(465).}$$

In order to understand what form the function H takes in our problem, let us consider the matter more closely. The coordinates (x, y, z) are given as functions of q_1, q_2, q_3 and t by the equations of transformation.

Differentiating these equations totally with respect to the time t we find

$$\dot{x} = \frac{\partial x}{\partial t} + \dot{q}_1 \frac{\partial x}{\partial q_1} + \dot{q}_2 \frac{\partial x}{\partial q_2} + \dot{q}_3 \frac{\partial x}{\partial q_3},$$

with two similar equations. Hence we get

$$\left.\begin{aligned} v^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2 = b_0 + 2b_1\dot{q}_1 + \dots + b_{11}\dot{q}_1^2 + 2b_{12}\dot{q}_1\dot{q}_2 + \dots \\ \text{where} \quad b_0 = S\left(\frac{\partial x}{\partial t}\right)^2, \quad b_1 = S\frac{\partial x}{\partial t}\frac{\partial x}{\partial q_1}, \\ b_{11} = S\left(\frac{\partial x}{\partial q_1}\right)^2, \quad b_{12} = S\frac{\partial x}{\partial q_1}\frac{\partial x}{\partial q_2} \end{aligned}\right\} \text{...(466).}$$

The sign S denotes a sum of three terms similar to the one written down. Again,

$$(\mathbf{va}) = \dot{x}a_x + \dot{y}a_y + \dot{z}a_z = Sa_x\frac{\partial x}{\partial t} + \dot{q}_1 Sa_x\frac{\partial x}{\partial q_1} + \dots \qquad \text{......(467)}.$$

The values of v^2 and $(\mathbf{va})$ must be substituted in the expression (456), § 279, for L. Differentiating it partially with respect to $\dot{q}_1$ we obtain

$$p_1 = \frac{cm(b_1 + b_{11}\dot{q}_1 + b_{12}\dot{q}_2 + b_{13}\dot{q}_3)}{\sqrt{(c^2 - v^2)}} + \frac{e}{c}Sa_x\frac{\partial x}{\partial q_1} \qquad \text{............(468)}.$$

Similar expressions hold for p_2 and p_3. Hence

$$p_1\dot{q}_1 + p_2\dot{q}_2 + p_3\dot{q}_3 = \frac{cm(v^2 - b_0 - b_1\dot{q}_1 - b_2\dot{q}_2 - b_3\dot{q}_3)}{\sqrt{(c^2 - v^2)}} + \frac{e(\mathbf{va})}{c} - \frac{e}{c}Sa_x\frac{\partial x}{\partial t}.$$

Substituting in (464) and using (456), § 279, we find

$$H = \frac{cm(c^2 - b_0 - b_1\dot{q}_1 - b_2\dot{q}_2 - b_3\dot{q}_3)}{\sqrt{(c^2 - v^2)}} + e\phi - \frac{e}{c}Sa_x\frac{\partial x}{\partial t} \qquad \text{......(469)}.$$

It remains to eliminate $\dot{q}_1$, $\dot{q}_2$ and $\dot{q}_3$ between (468) and (469). This is most easily carried out in two stages. Firstly, eliminating the three quantities $\dot{q}_1/\sqrt{(c^2 - v^2)}$, ... between (468) and (469) we find

$$\begin{vmatrix} \frac{b_0 - c^2}{\sqrt{(c^2 - v^2)}} + \frac{H - e\phi}{cm} + \frac{e}{c^2 m}Sa_x\frac{\partial x}{\partial t}, & b_1, & b_2, & b_3 \\ \frac{b_1}{\sqrt{(c^2 - v^2)}} - \frac{p_1}{cm} + \frac{e}{c^2 m}Sa_x\frac{\partial x}{\partial q_1}, & b_{11}, & b_{12}, & b_{13} \\ \frac{b_2}{\sqrt{(c^2 - v^2)}} - \frac{p_2}{cm} + \frac{e}{c^2 m}Sa_x\frac{\partial x}{\partial q_2}, & b_{12}, & b_{22}, & b_{23} \\ \frac{b_3}{\sqrt{(c^2 - v^2)}} - \frac{p_3}{cm} + \frac{e}{c^2 m}Sa_x\frac{\partial x}{\partial q_3}, & b_{13}, & b_{23}, & b_{33} \end{vmatrix} = 0.$$

For the sake of brevity we shall write

$$\Phi = \frac{e\phi}{cm} - \frac{e}{c^2 m}Sa_x\frac{\partial x}{\partial t}, \quad P_1 = \frac{p_1}{cm} - \frac{e}{c^2 m}Sa_x\frac{\partial x}{\partial q_1}, \ \dots \qquad \text{......(470)}.$$

Thus the last equation may be written

$$\begin{vmatrix} \frac{b_0 - c^2}{\sqrt{(c^2 - v^2)}} + \frac{H}{cm} - \Phi, & b_1, & b_2, & b_3 \\ \frac{b_1}{\sqrt{(c^2 - v^2)}} - P_1, & b_{11}, & b_{12}, & b_{13} \\ \frac{b_2}{\sqrt{(c^2 - v^2)}} - P_2, & b_{12}, & b_{22}, & b_{23} \\ \frac{b_3}{\sqrt{(c^2 - v^2)}} - P_3, & b_{13}, & b_{23}, & b_{33} \end{vmatrix} = 0 \qquad \text{...........(471)}.$$

Secondly, we must find the value of $\sqrt{(c^2 - v^2)}$ in terms of p_1, p_2, and p_3 and substitute it in the last equation. For this purpose we use (470) to reduce the first equation (466) to a linear equation; with the previous notation we obtain

$$\frac{b_0 - c^2}{c^2 - v^2} + 1 + \frac{\dot{q}_1}{\sqrt{(c^2 - v^2)}} \left\{ \frac{b_1}{\sqrt{(c^2 - v^2)}} + P_1 \right\} + \dots = 0.$$

Eliminating $\dot{q}_1/\sqrt{(c^2 - v^2)}, \dots$ between this equation and the three equations (468), we find

$$\begin{vmatrix} \frac{b_0 - c^2}{c^2 - v^2} + 1, & \frac{b_1}{\sqrt{(c^2 - v^2)}} + P_1, & \frac{b_2}{\sqrt{(c^2 - v^2)}} + P_2, & \frac{b_3}{\sqrt{(c^2 - v^2)}} + P_3 \\ \frac{b_1}{\sqrt{(c^2 - v^2)}} - P_1, & b_{11}, & b_{12}, & b_{13} \\ \frac{b_2}{\sqrt{(c^2 - v^2)}} - P_2, & b_{12}, & b_{22}, & b_{23} \\ \frac{b_3}{\sqrt{(c^2 - v^2)}} - P_3, & b_{13}, & b_{23}, & b_{33} \end{vmatrix} = 0.$$

Let B denote the determinant $\Sigma \pm b_{11} b_{22} b_{33}$, that is the discriminant of the quadratic part of v^2, and let B_{11}, B_{12}, ... be its first minors. Then the last equation gives

$$\frac{1}{c^2 - v^2} = \frac{B + B_{11} P_1^2 + B_{22} P_2^2 + 2B_{12} P_1 P_2 + \dots}{B(c^2 - b_0) + B_{11} b_1^2 + B_{22} b_2^2 + 2B_{12} b_1 b_2 + \dots} \quad \dots\dots(472).$$

With the same notation we find from (471)

$$B\left(\frac{H}{cm} - \Phi\right) + B_{11} b_1 P_1 + B_{12}(b_1 P_2 + b_2 P_1) + \dots$$
$$= \frac{B(c^2 - b_0) + B_{11} b_1^2 + 2B_{12} b_1 b_2 + \dots}{\sqrt{(c^2 - v^2)}}$$
$$= \sqrt{\{B(c^2 - b_0) + B_{11} b_1^2 + 2B_{12} b_1 b_2 + \dots\}\{B + B_{11} P_1^2 + 2B_{12} P_1 P_2 + \dots\}} \quad \dots\dots(473).$$

This equation shows that H is of the form

$$f_1(p_1, p_2, p_3) + \sqrt{f_2(p_1, p_2, p_3)} \quad \dots\dots\dots\dots\dots\dots(474),$$

where f_1 and f_2 are respectively linear and quadratic, not homogeneous, functions of the momenta p_1, p_2 and p_3, with coefficients which are functions of q_1, q_2, q_3 and t. The positive sign of the root must be taken, because $\sqrt{(c^2 - v^2)}$ is essentially positive, representing as it does the variable factor in the mass.

It is obvious that the comparative simplicity of the form (474) which we have obtained for H, is due to the form of the Lorentz mass factor. For the other mass formulae the result is either so complicated as to be useless in practice, or it is impossible to obtain an explicit expression for H at all. Thus the superiority of the Lorentz mass formula is again made manifest.

283. The equation of Hamilton and Jacobi. Just as in ordinary dynamics, so here it is possible to make the solution of the problem of motion depend on the determination of the complete integral of a partial differential equation of the first order in the four variables q_1, q_2, q_3 and t. For this purpose we need only make the usual substitution in (474), § 282:

$$H = -\frac{\partial W}{\partial t}, \quad p_1 = \frac{\partial W}{\partial q_1}, \quad p_2 = \frac{\partial W}{\partial q_2}, \quad p_3 = \frac{\partial W}{\partial q_3}.$$

When the equation is rationalized, we get an equation of the second degree, but it must be borne in mind that this also includes the case where the negative sign is taken in (474), § 282, and will therefore give rise to solutions which do not belong to our problem.

284. Equations of transformation free from the time. In this particular case $\frac{\partial x}{\partial t} = \frac{\partial y}{\partial t} = \frac{\partial z}{\partial t} = 0$; hence by (466), § 282, $b_0 = b_1 = b_2 = b_3 = 0$.

Using these values in (470) and (473), § 282, we find

$$H = e\phi + \sqrt{\left\{c^4m^2 + \frac{B_{11}}{B}\left(cp_1 - eSa_x\frac{\partial x}{\partial q_1}\right)^2 \right.}$$
$$\left. + 2\frac{B_{12}}{B}\left(cp_1 - eSa_x\frac{\partial x}{\partial q_1}\right)\left(cp_2 - eSa_x\frac{\partial x}{\partial q_2}\right) + \dots\right\} \quad \dots(475).$$

The interpretation of this equation is easy when we compare it with (469), § 282; with b_0, b_1, ... all zero the latter equation becomes

$$H = e\phi + \frac{c^3m}{\sqrt{(c^2 - v^2)}} \quad \dots\dots\dots\dots(476).$$

Bearing in mind that $e\phi$ is that part of the energy of the electron which is due to the external electric field, and that by §§ 240 and 278, with our present notation, $c^3m/\sqrt{(c^2 - v^2)}$ is the variable part of the energy of the electron, due to its own field and to its motion, we see that H is the variable part of the total energy of the electron, due both to itself and to the external field.

The expression (475) shows that, when H is expressed in terms of the momenta, it will involve t explicitly unless the potentials ϕ and $\mathbf{a}$ are both independent of t, that is to say, unless the external field is steady. In other words $\frac{\partial H}{\partial t}$ is not zero in general even when the equations of transformation do not involve the time explicitly; in this case it only vanishes for a steady field.

It is an interesting problem whether, in the case where the external field is variable, it is possible to choose the equations of transformation so that $\frac{\partial H}{\partial t} = 0$, but we cannot enter into a discussion of this question here.

285. Steady external field—the energy integral. When the external field is steady we can always choose our equations of transformation so that $\frac{\partial H}{\partial t} = 0$. Then the Hamiltonian equations (465), § 282, give as usual $H = \text{constant}$. By (476), § 284, this integral gives

$$\frac{c^3 m}{\sqrt{(c^2 - v^2)}} + e\phi = h \quad\text{..........................(477)},$$

where h is the arbitrary constant value of the energy.

In this case the Hamilton-Jacobi equation becomes

$$\frac{B_{11}}{B}\left(c\frac{\partial W}{\partial q_1} - eSa_x\frac{\partial x}{\partial q_1}\right)^2 + 2\frac{B_{12}}{B}\left(c\frac{\partial W}{\partial q_1} - eSa_x\frac{\partial x}{\partial q_1}\right)\left(c\frac{\partial W}{\partial q_2} - eSa_x\frac{\partial x}{\partial q_2}\right) + \ldots$$
$$= (h - e\phi)^2 - c^4m^2 \quad\text{...(478)}.$$

In particular for rectangular coordinates referred to fixed axes of (x, y, z) we have

$$q_1 = x, \quad q_2 = y, \quad q_3 = z, \quad B = B_{11} = B_{22} = B_{33} = 1, \quad B_{12} = B_{23} = B_{13} = 0.$$

Hence the equation becomes

$$\left(c\frac{\partial W}{\partial x} - ea_x\right)^2 + \left(c\frac{\partial W}{\partial y} - ea_y\right)^2 + \left(c\frac{\partial W}{\partial z} - ea_z\right)^2 = (h - e\phi)^2 - c^4m^2 \quad\text{...(479)}.$$

A complete integral of the Hamilton-Jacobi equation leads to a solution of our problem just as in ordinary dynamics, but in many cases it is easier to work with the equations of motion themselves. Thus the interest in the Hamilton-Jacobi equation is largely theoretical.

When an energy integral of the Hamiltonian equations of motion exists, it can be used as in ordinary dynamics to change the independent variable from t to one of the coordinates, say q_1, and so reduce the order of the Hamiltonian system by one unit, that is from three to two, so that the system after reduction consists of only four equations in place of six.

This reduction however requires us to solve the energy equation (475), § 284, for the momentum p_1; if we get $p_1 = -K(p_2, p_3, q_1, q_2, q_3, h)$ then the reduced system is

$$\frac{\partial q_2}{\partial q_1} = \frac{\partial K}{\partial p_2}, \quad \frac{\partial q_3}{\partial q_1} = \frac{\partial K}{\partial p_3}, \quad \frac{\partial p_2}{\partial q_1} = -\frac{\partial K}{\partial q_2}, \quad \frac{\partial p_3}{\partial q_1} = -\frac{\partial K}{\partial q_3}.$$

On account of the somewhat complicated form of the function K, and the particular form of the energy equation (477), it is more convenient to proceed as follows.

286. Reduction of the equations of motion when an energy integral exists. It is simplest to return to the second Newtonian form of the vector equation of motion (455), § 278, and change the independent variable from t to τ, where

$$\frac{dt}{d\tau} = \frac{c}{\sqrt{(c^2 - v^2)}} = \frac{h - e\phi}{c^2 m} \quad \text{.......................(480)},$$

by the energy integral.

When the problem has been solved, the coordinates of the electron are known functions of t; hence (480) enables us to find the relation between t and τ by a quadrature, and therefore we may also regard the coordinates as functions of τ, to be found by the solution of the transformed equations of motion. When this has been accomplished we get t from the equation

$$t = \int_0^\tau \frac{c\,d\tau}{\sqrt{(c^2 - v^2)}} = \int_0^\tau \frac{(h - e\phi)\,d\tau}{c^2 m} \quad \text{.................(481)},$$

where the constant of integration has been chosen so that t and τ vanish together.

Let us denote total differential coefficients with respect to τ by dashes, e.g. x', ... Then we have

$$\dot{x} = \frac{\sqrt{(c^2 - v^2)}}{c} x' \ldots \quad \text{.......................(482)},$$

with two similar equations.

Also let $\mathbf{w}$ denote the resultant of x', y', z'; then

$$\mathbf{v} = \frac{\sqrt{(c^2 - v^2)}}{c} \mathbf{w} \quad \text{..........................(483)}.$$

We get $c/\sqrt{(c^2 - v^2)} = \sqrt{(c^2 + w^2)}/c$, and therefore

$$t = \int_0^\tau \frac{\sqrt{(c^2 + w^2)}}{c} d\tau \quad \text{.......................(484)}.$$

Multiplying (455), § 278, by $c/\sqrt{(c^2 - v^2)}$, using (480) and remembering that ∇ does not operate on $\dot{x}$, $\dot{y}$, $\dot{z}$, but only on the coordinates as contained in ϕ and $\mathbf{a}$, we easily find

$$\frac{d}{d\tau}\left\{m\mathbf{w} + \frac{e\mathbf{a}}{c}\right\} = \nabla \left\{\frac{(h - e\phi)^2}{2c^2 m} + \frac{e(\mathbf{wa})}{c}\right\} \quad \text{...........(485)},$$

where ∇ now does not operate on x', y', z', the components of $\mathbf{w}$.

This is the vector equation of motion for a charge e of *constant* mass m, moving in a field of which the scalar potential is equal to $-(h - e\phi)^2/2c^2me$ and the vector potential retains the value $\mathbf{a}$. In other words, the motion of the Lorentz electron in the given steady electromagnetic field is the same as that of an electron of equal charge, but of constant mass equal to that of the Lorentz electron for zero velocity, when the electric field is modified in the way indicated. The change in the electric field only depends on the field itself and not on the motion of the electron.

A similar transformation is theoretically possible for any of the other mass formulae, but the necessary eliminations are impracticable.

Since our problem has been reduced to that of finding the motion of a particle of constant mass in a certain field of force, all the results of ordinary mechanics apply without modification. In particular, the Lagrangian function is given by

$$L = \tfrac{1}{2} m w^2 + \frac{e\,(\mathbf{wa})}{c} + \frac{(h - e\phi)^2}{2c^2 m} \quad \text{.................. (486).}$$

When there is symmetry about Ox, the cyclic coordinate χ is a speed coordinate as in § 281, and the cyclic momentum is constant. We have

$$\frac{\partial L}{\partial \chi'} = m\varpi^2 \chi' + \frac{e\varpi a_\chi}{c} = \kappa \quad \text{........................ (487),}$$

where κ is an arbitrary constant.

The modified Lagrangian function is given by

$$L' = L - \chi' \frac{\partial L}{\partial \chi'} = \tfrac{1}{2} m w'^2 + \frac{(h - e\phi)^2}{2c^2 m} - \tfrac{1}{2} m \varpi^2 \chi'^2$$

$$= \tfrac{1}{2} m w'^2 + \frac{(h - e\phi)^2}{2c^2 m} - \frac{(c\kappa - e\varpi a_\chi)^2}{2c^2 m \varpi^2} \quad \text{... (488),}$$

where w' is the velocity (referred to τ) of the electron in the meridian plane.

The Hamiltonian function in the general case is given by

$$H = \tfrac{1}{2} m w^2 - \frac{(h - e\phi)^2}{2c^2 m} \quad \text{........................ (489).}$$

Since H does not involve τ explicitly, the Hamiltonian equations admit the energy integral in the form $H =$ constant, but in virtue of the definition of τ the constant is not arbitrary. By comparing (489) with (477), § 285, and using (483), we find that the energy integral may be written in the form

$$\tfrac{1}{2} m w^2 - \frac{(h - e\phi)^2}{2c^2 m} = -\tfrac{1}{2} c^2 m \quad \text{.................. (490).}$$

When there is symmetry about Ox we get

$$H = \tfrac{1}{2} m w'^2 - \frac{(h - e\phi)^2}{2c^2 m} + \frac{(c\kappa - e\varpi a_\chi)^2}{2c^2 m \varpi^2} \quad \text{............ (491).}$$

Comparing (491) with (488), we see that the motion in the meridian plane is that of a particle of constant mass unity in a field of force of which the force function is given by

$$U = \frac{(h - e\phi)^2}{2c^2 m^2} - \frac{(c\kappa - e\varpi a_\chi)^2}{2c^2 m^2 \varpi^2} \quad \text{..................... (492).}$$

For instance, with cylindrical coordinates (x, ϖ, χ), we have

$$x'' = \frac{\partial U}{\partial x}, \quad \varpi'' = \frac{\partial U}{\partial \varpi} \quad \text{........................ (493).}$$

Again, with polar coordinates (r, θ, χ), where the axis of symmetry is the initial line from which θ is measured, we have

$$r'' - r\theta'^2 = \frac{\partial U}{\partial r}, \quad 2r'\theta' + r\theta'' = \frac{\partial U}{r\partial \theta} \quad \ldots\ldots\ldots\ldots\ldots(494).$$

Changing to θ as variable in the usual way, and writing u for $1/r$, and h for $r^2\theta'$ (in the present case there need be no confusion with the first meaning of h), we get

$$\left.\begin{aligned} pw' = r^2\theta' = h \\ u^2 h\frac{dh}{d\theta} = \frac{\partial U}{\partial \theta}, \quad h^2\left(\frac{d^2u}{d\theta^2} + u\right) + h\frac{dh}{d\theta}\frac{du}{d\theta} = \frac{\partial U}{\partial u} \end{aligned}\right\} \quad \ldots\ldots\ldots(495).$$

These equations admit the energy integral; by comparing (490) and (492) we see that it can be written in the form

$$h^2\left\{\left(\frac{du}{d\theta}\right)^2 + u^2\right\} = 2U - c^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(496).$$

When u and h have been found as functions of θ, we get τ from the equation

$$\tau = \int \frac{d\theta}{hu^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(497),$$

and then t is found from (480), § 286.

It is worth noting that when there is no magnetic field, and the electric force is a function of the radius only, U is a function of u only, and the problem reduces to that of central forces. For in this case the spherical symmetry of the electric field allows us to choose the axis of x so that the initial motion takes place in the meridian plane, and consequently the cyclic momentum is zero. This reduces the expression (492) for U to its first term. This case includes the important problems of an electron moving in the field due to a fixed point charge, and in that due to a spherically symmetrical distribution of charge, for instance the positive sphere of uniform density in J. J. Thomson's model atom.

APPENDIX G

PROBLEMS ILLUSTRATIVE OF THE MOTION OF THE LORENTZ ELECTRON

287. WE shall now consider some problems of motion of the Lorentz electron, selected partly on account of their intrinsic physical interest, partly because of their suitability as illustrations of the methods developed in Appendix F. At the same time they will serve to show that the mechanics of the Lorentz electron is hardly more complicated than that of the bodies of invariable mass postulated in ordinary mechanics, and at any rate is sufficiently simple to allow of our obtaining definite numerical results. As we have already pointed out, this fact distinguishes the Lorentz mass formula from all others which have been proposed hitherto, and makes it the only one of any use for practical purposes.

We have already considered the case of a Lorentz electron which is projected in a uniform electrostatic field (Ch. XI, §§ 151—154). Its motion in a uniform magnetostatic field takes place with uniform velocity, and therefore differs in no wise from that of a charged body with constant mass. The path is a helix of constant pitch, with its axis parallel to the lines of force, in accordance with the results of ordinary mechanics. We need not consider this problem here, and shall confine our discussion to more general cases of motion.

Our first three problems concern the motion of a Lorentz electron in a steady and uniform electromagnetic field. In the first problem the electric and magnetic forces are parallel; it involves the theory of Kaufmann's experiments on the specific charge of β-particles as a special case. The general case suggests a new form of the experiment, which offers great advantages, provided only that fairly intense electrostatic and magnetostatic fields can be secured throughout a space of moderate extent.

In the second problem the electric and magnetic forces are perpendicular to each other; the particular case in which the electric force, measured in electrostatic units, is numerically less than the magnetic force, measured in magnetic units, involves the theory of Bucherer's experiments on the specific charge of β-particles.

The third problem is the most general, in so far as the electric and magnetic forces are inclined at any angle.

The fourth problem treats of the motion of a Lorentz electron in a steady electromagnetic field possessing an axis of symmetry. It is important because it involves the theory of the Zeeman effect for a single electron for any velocity less than that of light. The particular case, where the electric force is central and proportional to the radius, is worked out in detail.

288. **Problem 1. An electron moves in a steady and uniform electromagnetic field, with electric and magnetic forces parallel to each other. Required to find the motion.** The simplest way of attacking this problem is to notice that the field is symmetrical about every line of force. Hence any convenient line of force can be taken as the axis of symmetry in a system of cylindrical coordinates of (x, ϖ, χ). Since the field is steady, an energy integral exists, and since it is symmetrical about Ox, the total angular momentum about Ox is constant; hence we can use the simplification represented in equations (492) and (493), § 286.

For the electric potential we may write

$$\phi = -d \, . \, x \qquad \text{.............................(498)},$$

where d is the electric force, supposed to act in the positive direction of the x-axis and to be measured in electrostatic units.

By a proper choice of the yz plane we can reduce the energy constant h to zero, and as before write the energy integral in the form

$$\frac{c}{\sqrt{(c^2 - v^2)}} = \frac{ed \, . \, x}{c^2 m} \qquad \text{...........................(499)}.$$

In the present problem the vector potential **a** reduces to the component a_χ perpendicular to the meridian, and it is easily seen that in this, and similar cases where the lines of magnetic force lie in the meridian planes, a_χ is equal to ψ/ϖ, where ψ is the stream function. Hence we have

$$a_\chi = \tfrac{1}{2} h\varpi \qquad \text{.............................(500)},$$

where h is the intensity of the magnetic field. It is reckoned positive, or negative, according as the magnetic force is in the same direction as the electric force, or the opposite, and is measured magnetically.

We now choose our axis of x so as to pass through the point of projection of the electron. Thus ϖ vanishes initially, and therefore the cyclic constant κ is identically zero. Hence equations (492) and (493), § 286, give

$$x'' = \frac{\partial U}{\partial x}, \quad \varpi'' = \frac{\partial U}{\partial \varpi}: \quad U = \frac{e^2 d^2 x^2}{2c^2 m^2} - \frac{e^2 h^2 \varpi^2}{8c^2 m^2}.$$

Therefore

$$x'' = \frac{e^2 d^2}{c^2 m^2} x, \quad \varpi'' = -\frac{e^2 h^2}{4c^2 m^2} \varpi \qquad \text{.................(501)}.$$

Also (487), § 286, gives, by (500),

$$\chi' = -\frac{eh}{2cm} \quad \text{.............................(502).}$$

Lastly, (481), § 286, gives

$$t = \int_0^\tau \frac{ed \,.\, x d\tau}{c^2 m} \quad \text{..........................(503).}$$

289. Equations (501)—(503) determine the motion when the initial conditions are given. In order to simplify the results as much as possible we shall suppose the origins of space and time chosen so that the direction of projection is perpendicular to Ox, and is along the initial line from which χ is measured. Hence when $\tau = 0$ we have $\chi = 0$, $\varpi = 0$, $x' = 0$. Therefore we get from (501) and (502), § 288,

$$x = a \cosh \frac{ed \,.\, \tau}{cm}, \quad \varpi = 2b \sin \frac{eh\tau}{2cm}, \quad \chi = -\frac{eh\tau}{2cm} \text{........(504).}$$

The arbitrary constants a and b remain to be determined. Let v_0 be the initial speed, which is essentially positive. From (499), § 288, we find

$$a = \frac{c^3 m}{ed \sqrt{(c^2 - v_0^2)}} \quad \text{............................(505).}$$

Also $\dot{\varpi}_0 = v_0$; hence we obtain from (480), § 286,

$$\varpi_0' = \dot{\varpi}_0 \left(\frac{dt}{d\tau}\right)_0, \quad \text{or} \quad \varpi_0' = cv_0/\sqrt{(c^2 - v_0^2)};$$

therefore by (504)

$$b = \frac{c^2 m v_0}{eh \sqrt{(c^2 - v_0^2)}} \quad \text{..........................(506).}$$

Lastly, (503), § 288, gives

$$ct = a \sinh \frac{ed \,.\, \tau}{cm} \quad \text{..........................(507).}$$

Eliminating τ between (504) and (507) we get

$$\left.\begin{aligned} x &= a \cosh \left(\frac{2d}{h}\chi\right), \quad \varpi = -2b \sin \chi \\ ct &= -a \sinh \left(\frac{2d}{h}\chi\right) \end{aligned}\right\} \text{..............(508).}$$

The first and third give

$$x = \pm \sqrt{(a^2 + c^2 t^2)} \quad \text{..........................(509),}$$

where the upper, or lower, sign must be taken according as a, that is according as e, is positive, or negative.

The energy equation (499), § 288, gives

$$c^2 - v^2 = \frac{a^2}{x^2}(c^2 - v_0^2) \quad \text{.......................(510).}$$

The equations (508)—(510) determine the path and the mode of its description completely.

290. We notice first that a is positive, or negative, according as e is positive, or negative; thus a positive electron moves entirely on the positive side of the x-axis, a negative one entirely on the negative side. The relation between x and t, namely (509), is precisely the same as in the problem at the end of Ch. XI.

Secondly, b is positive, or negative, and χ has the opposite sign to τ and t, or the same sign, according as eh is positive, or negative.

The second equation (508) represents a circle, of radius $\pm b$, touching the initial line Oy at the origin. Fig. 43 represents it viewed from the positive side of the axis of x, for the case where eh is negative, e.g. for a negative electron when the electric and magnetic forces are in the same direction. In this case χ, starting from zero, increases as t increases, and the circle is described in the positive direction, right-handedly with respect to the magnetic force. If the electron were positive, but the magnetic force in the opposite direction to the electric force, eh would still be negative, and the direction of description of the circle the same, but it would be left-handedly with respect to the magnetic force.

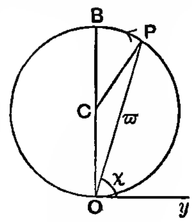

Fig. 43.

It is important to notice that the sign of the product ed alone determines on which side of the yz plane the electron moves, while the sign of the product eh alone determines the direction of description of the path around the lines of force.

Let s denote the arc OP of the circle in Fig. 43; then $s = 2b\chi$. Hence $2d\chi/h = sd/bh = cs/v_0 a$ by (505) and (506), § 289.

Thus the first equation (508), § 289, may be written in the form

$$\left.\begin{aligned} x &= a \cosh \frac{cs}{v_0 a} \\ &= \frac{c^3 m}{ed\sqrt{(c^2 - v_0^2)}} \cosh \frac{ed\sqrt{(c^2 - v_0^2)}}{c^2 m v_0} s \end{aligned}\right\} \quad \text{............(511).}$$

Comparing this with (234), § 152, we see that the path of the electron lies on the right circular cylinder which stands on the circle $\varpi = -2b \sin \chi$ as base, and further that if it be supposed unwrapped from the cylinder it developes precisely into the menoclinoid of the last problem of Ch. XI. In other words, given an electric field, the electron describes a menoclinoid determined by the value of the velocity at the vertex. If the electric field and the velocity at the vertex be kept the same, while a parallel magnetic field is superposed, the menoclinoid is bent round a right circular cylinder, which has the axis of the menoclinoid for a generator, and whose radius is numerically equal to $v_0 d/ch$ times the first parameter a of the menoclinoid.

Each half of the menoclinoid gives rise to a spiral curve. The two spirals are continuous at the vertex, and are images of each other in the diametral plane through the vertex. They cut each other in a series of nodes, given by $\chi = n\pi/2$, where n is an integer. The nodes for which n is even lie on Ox; those for which it is odd lie on the opposite generator of the right circular cylinder. The positions of these nodes are given by

$$x_n = a \cosh (n\pi d/h) \quad \text{.......................(512).}$$

The distances x_n ultimately form a geometrical progression whose ratio is $\pi d/h$. Hence the pitch of the spirals becomes infinitely great ultimately.

In fact we see from (508), § 289, that the rate of revolution of the electron around the cylinder is equal to $ch/2dx$, that is ultimately zero, while the velocity parallel to Ox is ultimately equal to the velocity of light.

As a numerical example let us consider the numerical example of the problem of Ch. XI, with a magnetic field of 100 gauss superposed. As before we have $e/cm = -1{\cdot}77 \,.\, 10^7$,

$$d = 50 \text{ E.S.U.} = 15{,}000 \text{ volt/cm.},$$

and $v_0/c = {\cdot}508$, giving $a = -39{\cdot}4$ cm. Equation (506), § 289, gives $b = 10$ cm., $hb/d = 20$ cm. Fig. 44 gives the projection of the path on the xy plane, on a scale of 1 : 80, the same as that of Fig. 41, § 153.

The point A is the vertex, B, C are the first two nodes. The thick line refers to those portions of the path, which are in front of the right circular cylinder, of which the traces are shown by fine lines, the broken line refers to the parts of the path on the back of the cylinder. The arrows show the direction of motion.

The figure is drawn for the case where h is positive, so that a and b are negative, while $\dot{\chi}$ is positive.

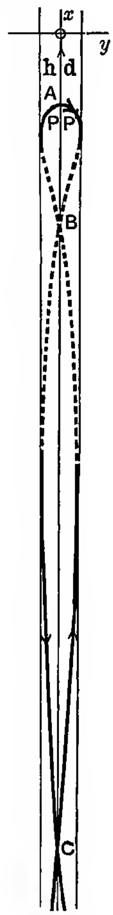

Fig. 44.

291. The circumstances of our present problem are precisely those of Kaufmann's well known experiments on the specific charge of β-rays; in fact the part of the path described by the β-particles in passing through his condenser is a small arc PAP' of the curve of Fig. 44, § 290. Consequently his electric deflection especially was very small, and measurable with difficulty. If however we could realise a considerable portion of the curve, such as the arc $AP'B$ between the vertex A and the first node B, we should have a considerable deflection. Let the coordinates of B, referred to the vertex A as origin, be (ξ, η, ζ); they are given by putting χ equal to $\pm \pi/2$ in (508), § 289, and noticing that

$$\xi = x - a, \quad \eta = y = 0, \quad \zeta = \pm \varpi;$$

hence we get $\xi = 2a \sinh^2(\pi d/2h), \quad \eta = 0, \quad \zeta = 2b$..............(513).

Moreover, if θ denote the inclination of the path at B to the $\xi\zeta$ plane, we get, by (511), § 290,

$$\tan\theta = \frac{ds}{dx} = \frac{v_0}{c\sinh(\pi d/h)} \quad \text{.....................(514).}$$

Eliminating v_0 between (513) and (505) and (506), § 289, we find

$$\frac{e^2 d^2 \xi^2}{4c^4 m^2 \sinh^4(\pi d/2h)} - \frac{e^2 h^2 \zeta^2}{4c^4 m^2} = 1 \quad \text{..................(515).}$$

Thus the locus of the first node B is a hyperbola in the $\xi\zeta$ plane, with its real axis along the line of force at the vertex A. Its real semiaxis is equal to $2c^2 m \sinh^2(\pi d/2h)/ed$, its transverse semiaxis to $2c^2m/eh$, and its eccentricity to $\sqrt{\left(1 + \frac{d^2}{h^2}\operatorname{cosech}^4 \frac{\pi d}{2h}\right)}$.

It is important to notice that the transverse semiaxis can only be reduced to a reasonably small value by using a powerful magnetic field. The magnetomotive force, or fall of magnetic potential, between A and B is constant and equal to $2c^2m/e$.

For a negative electron $2c^2m/e$ is numerically equal to 3430 gauss cm.; a magnetic force equal to 100 gauss would give a transverse semiaxis 34·3 cm.

On the other hand the real semiaxis can be reduced as much as we please by making d small enough. The electromotive force, or fall of electric potential, between A and B is equal to $2c^2m \sinh^2(\pi d/2h)/e$. For a negative electron this is $3430 \sinh^2(\pi d/2h)$ E.S.U., or $1{\cdot}03 . 10^6 \sinh^2(\pi d/2h)$ volts. With $d = \frac{1}{2}h$ we should require 777,000 volts. In this case (514) gives $\tan\theta = {\cdot}43 v_0/c$, so that a particle for which $v_0 = \frac{1}{2}c$, would make an angle of about 11° with the $\xi\zeta$ plane. By increasing h to the utmost and making d/h smaller, say about one fifth, we could diminish the transverse axis, and the difference of electric potential, while the real semiaxis could be reduced to a reasonable value, and the angle θ increased so that the rays should not strike the $\xi\zeta$ plane too obliquely. In this way it should be feasible to realize a considerable arc of the hyperbola (515) on a photographic plate placed in the $\xi\zeta$ plane, without getting a trace too wide to measure accurately, provided a sufficiently fine cylindrical pencil of β-rays were used.

By measuring the trace on the plate and testing how far it agreed with the hyperbola (515) of eccentricity $\sqrt{\left(1 + \frac{d^2}{h^2}\operatorname{cosech}^4 \frac{\pi d}{2h}\right)}$, we could test the accuracy of the Lorentz mass formula; this test would only require us to know the ratio d/h.

Further, by calculating the real semiaxis of the hyperbola and measuring d and h, or by calculating the conjugate semiaxis and measuring h, we could determine the specific charge e/cm.

If the Lorentz mass formula be assumed to be true, so that the curve is assumed to be a hyperbola, the last method only requires an accurate determination of h, not of d.

292. Problem 2. An electron moves in a steady and uniform electromagnetic field with electric and magnetic forces at right angles to one another. Required to find the motion. We shall take Ox in the direction of the electric force **d**, Oy in that of the magnetic force **h**, and Oz in that of the Poynting vector [**dh**]. By a proper choice of the constant involved in the electric potential we may take

$$\phi = -d \,.\, x \quad \text{.............................(516).}$$

The energy integral becomes

$$\frac{c}{\sqrt{(c^2 - v^2)}} = \frac{ed\,(x+k)}{c^2 m} \quad \text{........................(517),}$$

where k is an arbitrary constant to be determined later. Then (481), § 286, gives for the time

$$t = \int_0^\tau \frac{ed\,(x+k)\,d\tau}{c^2 m} \quad \text{........................(518).}$$

Again, we may take for the vector potential **a**

$$a_x = hz, \quad a_y = a_z = 0 \quad \text{.....................(519),}$$

for these values give $h_y = h$, $h_x = h_z = 0$ and satisfy all other necessary conditions.

Using the energy integral for the reduction of the equations of motion to τ as independent variable, we find for the Lagrangian function, by (486), § 286,

$$L = \tfrac{1}{2} m\,(x'^2 + y'^2 + z'^2) + \frac{ehzx'}{c} + \frac{e^2 d^2 (x+k)^2}{2c^2 m}.$$

Then the equations of motion become

$$\left.\begin{aligned} x'' + \frac{eh}{cm} z' &= \frac{e^2 d^2}{c^2 m^2}(x+k) \\ y'' &= 0 \\ z'' &= \frac{eh}{cm} x' \end{aligned}\right\} \quad \text{.....................(520).}$$

The last equation gives

$$z' = z_0' + \frac{eh}{cm}(x - x_0) \quad \text{........................(521),}$$

where the zero suffix denotes initial values. We can express z_0' in terms of the initial velocity $\dot{z}_0$ by means of (482), § 286, which gives

$$z_0' = \frac{c\dot{z}_0}{\sqrt{(c^2 - v_0^2)}} \quad \text{..........................(522),}$$

with similar equations for x_0' and y_0'.

Substituting from (521) in the first of (520) we get

$$x'' + \frac{e^2(h^2-d^2)}{c^2m^2}x = \frac{e^2d^2k}{c^2m^2} + \frac{e^2h^2x_0}{c^2m^2} - \frac{eh}{cm}z_0' \quad \text{............(523).}$$

The constants k and x_0 are not independent, but are connected by the energy equation (517), which gives

$$\frac{ed(k+x_0)}{cm} = \frac{c^2}{\sqrt{(c^2-v_0^2)}}.$$

Subject them to the further condition

$$\frac{e^2d^2k}{c^2m^2} + \frac{e^2h^2x_0}{c^2m^2} = \frac{ehz_0'}{cm} = \frac{eh\dot{z}_0}{m\sqrt{(c^2-v_0^2)}},$$

which makes the right-hand member of (523) vanish. Thus we get

$$k = \frac{c^2mh(\dot{z}_0 d - ch)}{ed(d^2-h^2)\sqrt{(c^2-v_0^2)}}, \quad x_0 = \frac{c^2m(cd-\dot{z}_0 h)}{e(d^2-h^2)\sqrt{(c^2-v_0^2)}} \quad \text{...(524).}$$

Then (521) and (523) give

$$x'' + \frac{e^2(h^2-d^2)}{c^2m^2}x = 0 \quad \text{........................(525),}$$

$$z' = \frac{eh}{cm}x + \frac{cd(\dot{z}_0 d - ch)}{(d^2-h^2)\sqrt{(c^2-v_0^2)}} \quad \text{..................(526).}$$

By means of (482), § 286, we get from the second of (520)

$$y = y_0'\tau = \frac{c\dot{y}_0}{\sqrt{(c^2-v_0^2)}}\tau \quad \text{........................(527),}$$

where we have taken $y_0 = 0$, which implies no limitation of generality.

Lastly, by means of (520) we get from (518)

$$ct = \frac{cm}{ed}x' + \frac{h}{d}z - \frac{c^2m\dot{x}_0}{ed\sqrt{(c^2-v_0^2)}} - \frac{h}{d}z_0 \quad \text{..............(528).}$$

Before proceeding any further we must distinguish two cases according as $d \lesseqgtr h$.

293. Case I. $d < h$.

This case is of importance for the theory of the experiments of Bucherer and Wolz on the measurement of the specific charge of β-rays.

Let us write

$$\left.\begin{aligned} \phi &= \frac{e\sqrt{(h^2-d^2)}}{cm}\tau \\ a &= \frac{c^2m}{e\sqrt{(h^2-d^2)}\sqrt{(c^2-v_0^2)}} \end{aligned}\right\} \quad \text{..................(529).}$$

ϕ is an angle, and a is a time; a is positive, or negative, according as the charge e of the electron is positive, or negative.

Moreover, let us choose the initial values z_0 and t_0 to be zero, which implies no limitation of generality.

We get from (524) and (527), § 292,

$$x_0 = \frac{a\,(\dot{z}_0 h - cd)}{\sqrt{(h^2 - d^2)}}, \quad y_0 = 0, \quad z_0 = 0, \quad t_0 = 0,$$

$$\left(\frac{dx}{d\phi}\right)_0 = a\dot{x}_0, \quad \left(\frac{dy}{d\phi}\right)_0 = a\dot{y}_0, \quad \left(\frac{dz}{d\phi}\right)_0 = a\dot{z}_0, \quad \left(\frac{dt}{d\phi}\right)_0 = a.$$

Hence we get from (525)—(528), § 292, and (529)

$$\left.\begin{aligned} x &= a\left\{\frac{\dot{z}_0 h - cd}{\sqrt{(h^2 - d^2)}}\cos\phi + \dot{x}_0 \sin\phi\right\} \\ y &= a\dot{y}_0\phi \\ z &= \frac{ah}{\sqrt{(h^2 - d^2)}}\left\{\frac{\dot{z}_0 h - cd}{\sqrt{(h^2 - d^2)}}\sin\phi + \dot{x}_0 (1 - \cos\phi) + \frac{d\,(ch - \dot{z}_0 d)}{h\sqrt{(h^2 - d^2)}}\phi\right\} \\ ct &= \frac{ad}{\sqrt{(h^2 - d^2)}}\left\{\frac{\dot{z}_0 h - cd}{\sqrt{(h^2 - d^2)}}\sin\phi + \dot{x}_0 (1 - \cos\phi) + \frac{h\,(ch - \dot{z}_0 d)}{d\sqrt{(h^2 - d^2)}}\phi\right\} \end{aligned}\right\} \quad \text{......(530).}$$

These equations show that the origins of space and time can always be chosen so that $\dot{x}_0 = 0$, except in the special case when $\dot{z}_0 h - cd = 0$. In this special case the choice of $\dot{x}_0 = 0$ would limit the generality of the problem; hence it is best to consider it separately.

294. Special case. $\dot{z}_0 = cd/h$. This is the case of the "compensated rays" in Bucherer's experiment. We get

$$\left.\begin{aligned} x &= a\dot{x}_0 \sin\phi \\ y &= a\dot{y}_0\phi \\ z &= a\left\{\frac{h\dot{x}_0}{\sqrt{(h^2 - d^2)}}(1 - \cos\phi) + c\frac{d}{h}\phi\right\} \\ ct &= a\left\{\frac{d\dot{x}_0}{\sqrt{(h^2 - d^2)}}(1 - \cos\phi) + c\phi\right\} \end{aligned}\right\} \quad \text{............(531).}$$

When the electron is projected at right angles to the electric force, so that $\dot{x}_0 = 0$, its path is the straight line

$$y = z\dot{y}_0 h/cd \quad \text{..............................(532),}$$

which is described with uniform velocity v_0. Since $\dot{z}_0 = cd/h$, while v_0 must be less than c, $\dot{y}_0$ must be less than $c\sqrt{(h^2 - d^2)}/h$ in numerical value. Hence all the compensated rays lie within the sector which is limited by the lines

$$y = \pm z\sqrt{(h^2 - d^2)}/d.$$

The motion is such that $\dot{z}$ is positive, that is in the positive direction of the Poynting vector.

When $\dot{x}_0$ is not zero, the path is a tortuous curve, but for large values of ϕ it approximates to the straight line (532), and $\dot{z}$ is positive. Hence the motion is in the positive direction of the Poynting vector on the whole.

The projection of the path on the plane of xy, the plane of the electric and magnetic forces, is a sine curve, whose base is along Oy, parallel to the magnetic force, amplitude equal to $a\dot{x}_0$, and wave-length to $2\pi a\dot{y}_0$.

The projection on the plane of yz, perpendicular to the electric force, is a sheared cosine curve. Its base is the line $z = a\dot{x}_0 h/\sqrt{(h^2 - d^2)}$, parallel to Oy, the magnetic force, its amplitude is equal to $a\dot{x}_0 h/\sqrt{(h^2 - d^2)}$, and wave-length to $2\pi a\dot{y}_0$. It is sheared parallel to Oz through the angle $\tan^{-1}(cd/h\dot{y}_0)$.

The projection on the plane of xz, perpendicular to the magnetic force, is a curve of the nature of a trochoid. It is derived from a trochoid, whose base is parallel to Oz, that is to the Poynting vector, by reducing all the ordinates in the ratio $h : \sqrt{(h^2 - d^2)}$. The radius of the rolling circle of the trochoid is acd/h, and the distance of the tracing point from its centre $a\dot{x}_0 h/\sqrt{(h^2 - d^2)}$. The trochoid is curtate, or prolate, according as $\dot{x}_0$ is greater, or less, than $cd\sqrt{(h^2 - d^2)}/h^2$, and the projection has loops, or no loops, accordingly. The path itself however has no loops, because there are none in its other two projections.

As a numerical example, let us suppose that $h = 2d$, so that $\dot{z}_0 = \frac{1}{2}c$. Further let $\dot{x}_0 = \frac{1}{2}c$, $\dot{y}_0 = \frac{1}{2}c$. Then

$$x = \tfrac{1}{2}ac\sin\phi, \quad y = \tfrac{1}{2}ac\phi, \quad z = \tfrac{1}{2}ac\left\{\phi + \frac{2}{\sqrt{3}}(1 - \cos\phi)\right\}.$$

The resultant initial velocity is $v_0 = \dfrac{\sqrt{3}}{2}c$. The third projection has loops.

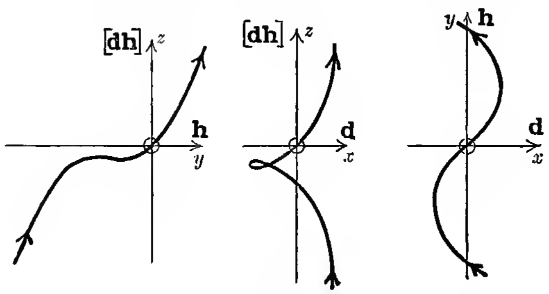

Fig. 45.

The three projections are shown in Fig. 45. The scale of the diagrams is determined by the value of the length $\frac{1}{2}ac$; by (529), § 293, we have

$$\tfrac{1}{2}ac = \frac{c^3 m}{2e\sqrt{(h^2 - d^2)}\sqrt{(c^2 - v_0^2)}} = \frac{c^2 m}{ed\sqrt{3}}.$$

For a negative electron $e/cm = -1{\cdot}77 \,.\, 10^7$ E.S.U. If d be 98 E.S.U., i.e. 29,400 volt/cm., we find that $\frac{1}{2}ac$ is equal to -10 cm.

As a second example, suppose that $\dot{x}_0 = \frac{1}{4}c$, all the other quantities retaining their former values. We get

$$x = \tfrac{1}{4}ac\sin\phi, \quad y = \tfrac{1}{2}ac\,\phi, \quad z = \tfrac{1}{2}ac\left\{\phi + \frac{1}{\sqrt{3}}(1 - \cos\phi)\right\}.$$

The resulting initial velocity is $v_0 = \frac{3}{4}c$. The third projection has no loops. The three projections are shown in Fig. 46.

In all the figures the path is indicated by a thick line, and the direction of motion by arrows. They are all drawn for a positive electron; for a negative one we need only suppose the directions of the vectors **d** and **h** to be reversed.

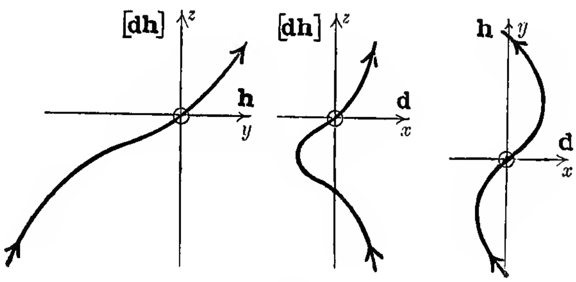

Fig. 46.

295. General case. $\dot{z}_0 \gtrless d/h$. By shifting the origins of space and time we can without any loss of generality ensure that $\dot{x}_0 = 0$. Thus we find from (530), § 293,

$$\left.\begin{aligned} x &= a\,\frac{\dot{z}_0 h - cd}{\sqrt{(h^2 - d^2)}}\cos\phi \\ y &= a\dot{y}_0\phi \\ z &= a\,\frac{d\,(ch - \dot{z}_0 d)}{h^2 - d^2}\left\{\phi + \frac{h\,(\dot{z}_0 h - cd)}{d\,(ch - \dot{z}_0 d)}\sin\phi\right\} \\ ct &= a\,\frac{h\,(ch - \dot{z}_0 d)}{h^2 - d^2}\left\{\phi + \frac{d\,(\dot{z}_0 h - cd)}{h\,(ch - \dot{z}_0 d)}\sin\phi\right\} \end{aligned}\right\} \quad \ldots\ldots\ldots(533).$$

By changing ϕ into $\phi - \pi/2$ and shifting the origins we can write these equations in the same form as (531), § 294, so that the three projections are the same as in the special case, but the constants are different. The constant $\dot{x}_0$ is now replaced by $(\dot{z}_0 h - cd)/\sqrt{(h^2 - d^2)}$, and the constant c, which multiplies ϕ in the bracket in ct, and $d\phi/h$ in the bracket in z, is replaced by $h\,(ch - \dot{z}_0 d)/(h^2 - d^2)$.

Hence the path is of the types shown in Figs. 45 and 46: of the first type with loop, or of the second without loop, according as $h\,(\dot{z}_0 h - cd)$ is greater, or less, than $d\,(ch - \dot{z}_0 d)$ numerically. Now we have

$$h^2\,(\dot{z}_0 h - cd)^2 - d^2\,(ch - \dot{z}_0 d)^2 = (h^4 - d^4)\,\dot{z}_0\left(\dot{z}_0 - c\,\frac{2dh}{d^2 + h^2}\right).$$

Thus the first type occurs when $\dot{z}_0$ is negative, and when it is positive and greater than $2cdh/(d^2 + h^2)$.

296. Case II. $d > h$.

This case has not yet occurred in experimental work and may be dismissed briefly. Write

$$\left.\begin{aligned} \phi &= \frac{e\sqrt{(d^2-h^2)}}{cm}\tau \\ a &= \frac{c^2 m}{e\sqrt{(d^2-h^2)}\sqrt{(c^2-v_0^2)}} \end{aligned}\right\} \quad \text{..................(534).}$$

In the present case we have the same expressions for $x_0 \ldots$ as before, except that $\sqrt{(h^2-d^2)}$ is replaced by $\sqrt{(d^2-h^2)}$. Hence

$$x_0 = \frac{a(cd - \dot{z}_0 h)}{\sqrt{(d^2-h^2)}}, \quad y_0 = 0, \quad z_0 = 0, \quad t_0 = 0,$$

$$\left(\frac{dx}{d\phi}\right)_0 = a\dot{x}_0, \quad \left(\frac{dy}{d\phi}\right)_0 = a\dot{y}_0, \quad \left(\frac{dz}{d\phi}\right)_0 = a\dot{z}_0, \quad \left(\frac{dt}{d\phi}\right)_0 = a.$$

Since $|\dot{z}_0| < c$ and $d > h$, x_0 cannot vanish. Hence we can always shift the origins of space and time so as to make $\dot{x}_0$ vanish. Then we get

$$\left.\begin{aligned} x &= a\frac{cd - \dot{z}_0 h}{\sqrt{(d^2-h^2)}}\cosh\phi \\ y &= a\dot{y}_0\phi \\ z &= a\frac{h(cd-\dot{z}_0 h)}{d^2-h^2}\left\{\sinh\phi + \frac{d(\dot{z}_0 d - ch)}{h(cd-\dot{z}_0 h)}\phi\right\} \\ ct &= a\frac{d(cd-\dot{z}_0 h)}{d^2-h^2}\left\{\sinh\phi + \frac{h(\dot{z}_0 d - ch)}{d(cd-\dot{z}_0 h)}\phi\right\} \end{aligned}\right\} \quad \text{.........(535).}$$

These equations can be found from (533), § 295, by changing ϕ into $\iota\phi$, a into $-\iota a$, and $\sqrt{(h^2-d^2)}$ into $\iota\sqrt{(d^2-h^2)}$.

When ϕ is very large, whether positive, or negative, we get $z = cth/d$, so that the motion is in the positive direction of the Poynting vector on the whole, as in case I.

The projection of the path on the plane of xy, of the electric and magnetic forces, is a menoclinoid, whose axis is Ox, parallel to the electric force, and whose parameters are $a(cd - \dot{z}_0 h)/\sqrt{(d^2-h^2)}$ and $a\dot{y}_0$, the former being the greater of the two. This curve has already occurred in connection with problem 1, § 290.

The projection on the yz plane, perpendicular to the electric force, is got from a trepsiclinoid*, of axis Oz and parameters $ah(cd - \dot{z}_0 h)/(d^2-h^2)$ and $a\dot{y}_0$, by shearing it parallel to Oz through an angle

$$\tan^{-1}\frac{d(\dot{z}_0 d - ch)}{\dot{y}_0(d^2-h^2)}.$$

* Loria, *loc. cit.* p. 580.

The projection on the xz plane, perpendicular to the magnetic force, is similar to the hyperbolic cycloid of Laisant*. It can be generated by a point which moves along one or other branch of the hyperbola

$$x^2 - \frac{d^2 - h^2}{h^2} z^2 = a^2 \frac{(cd - \dot{z}_0 h)^2}{d^2 - h^2},$$

with uniform areal velocity about the centre, while the hyperbola moves parallel to Oz with uniform velocity. When $d(\dot{z}_0 d - ch)$ is negative and numerically greater than $h(cd - \dot{z}_0 h)$, $\dot{z}$ changes sign when

$$\phi = \cosh^{-1} \frac{d(ch - \dot{z}_0 d)}{h(cd - \dot{z}_0 h)},$$

and the projection has a loop. This occurs when $\dot{z}_0$ is negative.

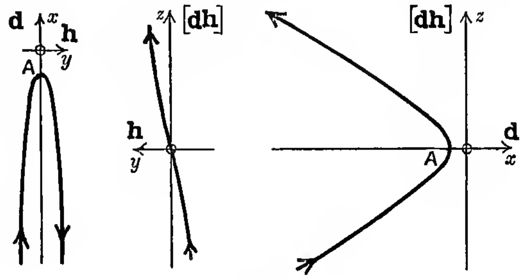

Fig. 47.

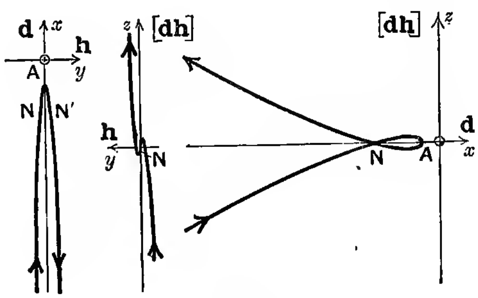

Fig. 48.

As numerical examples we shall take the cases where $d = 2h$, $\dot{y}_0 = \frac{1}{4}c$, and $\dot{z}_0 = \frac{3}{4}c$ and $-\frac{1}{2}c$ respectively. The projections of the path are shown in Figs. 47 and 48; they are given by

$$x = \frac{5ac}{4\sqrt{3}} \cosh\phi, \quad y = \tfrac{1}{4}ac\phi, \quad z = \frac{5ac}{12}(\sinh\phi + \tfrac{4}{5}\phi),$$

and
$$x = \frac{5ac}{2\sqrt{3}} \cosh\phi, \quad y = \tfrac{1}{4}ac\phi, \quad z = \frac{5ac}{6}(\sinh\phi - \tfrac{8}{5}\phi).$$

* Loria, *loc. cit.* p. 476.

As before, the path is shown by a thick line, and the direction of motion by arrows. The charge is negative.

In these figures A is the vertex, and OA is equal to a, which is negative, because the charge is negative.

In Fig. 48 N is the node in the xz projection. It does not occur in the path, and is shown undoubled at NN' in the xy projection.

297. Problem 3. An electron moves in a steady and uniform electromagnetic field. Required to find the motion. This is the general problem, of which those considered so far are particular cases. As before let **d** and **h** be the electric and magnetic forces, measured in electrostatic and magnetic units respectively, and let θ be the angle between them.

We shall take the plane of **d** and **h** as plane of xy, and the positive direction of the Poynting vector [**dh**] as Oz at some arbitrary origin O. Let Ox make an angle α with **d**, measured in the positive direction, that is towards **h**.

Then we may write

$$\phi = - d\,(x \cos \alpha - y \sin \alpha) \qquad \text{.....................(536).}$$

The equation of energy becomes

$$\frac{c}{\sqrt{(c^2 - v^2)}} = \frac{ed}{c^2 m}(x \cos \alpha - y \sin \alpha + k) \qquad \text{...............(537),}$$

where k is an arbitrary constant.

Again, we may write for the vector potential **a**

$$a_x = h \sin(\theta - \alpha)\, z, \quad a_y = - h \cos(\theta - \alpha)\, z, \quad a_z = 0 \qquad \text{......(538).}$$

Hence the Lagrangian function becomes

$$L = \tfrac{1}{2} m\,(x'^2 + y'^2 + z'^2) + \frac{eh}{c}\{x' \sin(\theta - \alpha) - y' \cos(\theta - \alpha)\}\, z + \frac{e^2 d^2 (x \cos \alpha - y \sin \alpha + k)^2}{2c^2 m}.$$

The equations of motion are

$$\left.\begin{aligned} x'' + \frac{eh \sin(\theta - \alpha)}{cm} z' &= \frac{e^2 d^2 \cos \alpha\,(x \cos \alpha - y \sin \alpha + k)}{c^2 m^2} \\ y'' - \frac{eh \cos(\theta - \alpha)}{cm} z' &= - \frac{e^2 d^2 \sin \alpha\,(x \cos \alpha - y \sin \alpha + k)}{c^2 m^2} \\ z'' &= \frac{eh}{cm}\{x' \sin(\theta - \alpha) - y' \cos(\theta - \alpha)\} \end{aligned}\right\} \text{...(539).}$$

The last equation gives

$$\left.\begin{aligned} z' &= \frac{eh}{cm}\{x\sin(\theta-\alpha) - y\cos(\theta-\alpha)\} + \frac{eh}{cm}A \\ \text{where}\qquad \frac{eh}{cm}A &= z_0' - \frac{eh}{cm}\{x_0\sin(\theta-\alpha) - y_0\cos(\theta-\alpha)\} \end{aligned}\right\} \quad\ldots\ldots(540),$$

so that A is arbitrary. Substituting in the first two equations we get two linear differential equations of the second order, both of which contain terms in x and y, as well as constant terms.

The coefficient of y in the first equation and that of x in the first are the same and equal to

$$\frac{e^2\{d^2\sin 2\alpha - h^2\sin 2(\theta-\alpha)\}}{2c^2m^2}.$$

They can be made to vanish by choosing α so that

$$\left.\begin{aligned} \sin 2\alpha &= h^2\sin 2\theta/\sqrt{(d^4+2d^2h^2\cos 2\theta+h^4)} \\ \cos 2\alpha &= (d^2+h^2\cos 2\theta)/\sqrt{(d^4+2d^2h^2\cos 2\theta+h^4)} \end{aligned}\right\} \quad\ldots\ldots(541).$$

With this value of α the coefficient of x in the first equation reduces to $-e^2f^2/c^2m^2$, and that of y in the second to e^2g^2/c^2m^2, where

$$\left.\begin{aligned} 2f^2 &= \sqrt{(d^4+2d^2h^2\cos 2\theta+h^4)} + d^2 - h^2 \\ 2g^2 &= \sqrt{(d^4+2d^2h^2\cos 2\theta+h^4)} - d^2 + h^2 \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots(542).$$

f and g are both real and may be taken to be positive, or negative, as may be most convenient. We shall take f to be positive, and choose the sign of g so that

$$fg = dh\cos\theta \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(543),$$

which obviously agrees with (542). We see at once that

$$f^2+g^2 = \sqrt{(d^4+2d^2h^2\cos 2\theta+h^4)},\quad f^2-g^2 = d^2-h^2 \quad\ldots\ldots(544).$$

Then we easily find from (541)

$$\left.\begin{aligned} \cos\alpha &= \frac{f}{d}\sqrt{\frac{d^2+g^2}{f^2+g^2}},\quad \sin\alpha = \frac{g}{d}\sqrt{\frac{d^2-f^2}{f^2+g^2}} \\ \cos(\theta-\alpha) &= \frac{g}{h}\sqrt{\frac{d^2+g^2}{f^2+g^2}},\quad \sin(\theta-\alpha) = \frac{f}{h}\sqrt{\frac{d^2-f^2}{f^2+g^2}} \end{aligned}\right\} \quad\ldots(545).$$

Again, the constant terms in the equations for x and y vanish when

$$Ah^2\sin(\theta-\alpha) - kd^2\cos\alpha = 0,\quad Ah^2\cos(\theta-\alpha) - kd^2\sin\alpha = 0.$$

The determinant of these two equations in Ah^2 and kd^2 reduces to $\cos\theta$, and only vanishes when $\theta = \pi/2$, a case already treated in problem 2. Hence we may suppose $\cos\theta$ different from zero, and must put A and k equal to zero. Remembering that $z_0' = c\dot{z}_0/\sqrt{(c^2-v_0^2)}$ we get from (540)

$$x_0\sin(\theta-\alpha) - y_0\cos(\theta-\alpha) = \frac{c^2m\dot{z}_0}{eh\sqrt{(c^2-v_0^2)}}.$$

Putting $k=0$, $x=x_0$, $y=y_0$ and $v=v_0$ in the energy equation (537) we find

$$x_0 \cos\alpha - y_0 \sin\alpha = \frac{c^3 m}{ed\sqrt{(c^2-v_0^2)}}.$$

Solving for x_0 and y_0 and using (542)—(545) we get

$$\left.\begin{aligned} x_0 &= \frac{c^2 m\{c\sqrt{(d^2+g^2)} - \dot{z}_0\sqrt{(d^2-f^2)}\}}{ef\sqrt{(f^2+g^2)}\sqrt{(c^2-v_0^2)}} \\ y_0 &= \frac{c^2 m\{c\sqrt{(d^2-f^2)} - \dot{z}_0\sqrt{(d^2+g^2)}\}}{eg\sqrt{(f^2+g^2)}\sqrt{(c^2-v_0^2)}} \end{aligned}\right\} \quad \text{............(546).}$$

The first two equations (539) become

$$x'' - \frac{e^2 f^2}{c^2 m^2} x = 0, \quad y'' + \frac{e^2 g^2}{c^2 m^2} y = 0 \quad \text{..............(547).}$$

Using (545) we find from (540)

$$z' = \frac{ef}{cm}\sqrt{\frac{d^2-f^2}{f^2+g^2}}\, x - \frac{eg}{cm}\sqrt{\frac{d^2+g^2}{f^2+g^2}}\, y \quad \text{............(548).}$$

Lastly, remembering that $\frac{dt}{d\tau} = t' = c/\sqrt{(c^2-v^2)}$, and using (545), we get from (537)

$$ct' = \frac{ef}{cm}\sqrt{\frac{d^2+g^2}{f^2+g^2}}\, x - \frac{eg}{cm}\sqrt{\frac{d^2-f^2}{f^2+g^2}}\, y \quad \text{............(549).}$$

Remembering that $x_0' = c\dot{x}_0/\sqrt{(c^2-v_0^2)}$ and $y_0' = c\dot{y}_0/\sqrt{(c^2-v_0^2)}$, we find from (547)

$$\left.\begin{aligned} x &= x_0 \cosh\frac{ef\tau}{cm} + \frac{c^2 m\dot{x}_0}{ef\sqrt{(c^2-v_0^2)}} \sinh\frac{ef\tau}{cm} \\ y &= y_0 \cos\frac{eg\tau}{cm} + \frac{c^2 m\dot{y}_0}{eg\sqrt{(c^2-v_0^2)}} \sin\frac{eg\tau}{cm} \end{aligned}\right\} \quad \text{............(550).}$$

Substituting these values in (548) and (549), integrating and shifting the origins of space and time, we obtain

$$\left.\begin{aligned} z &= \sqrt{\frac{d^2-f^2}{f^2+g^2}}\left\{x_0\sinh\frac{ef\tau}{cm} + \frac{c^2 m\dot{x}_0}{ef\sqrt{(c^2-v_0^2)}}\cosh\frac{ef\tau}{cm}\right\} \\ &\quad - \sqrt{\frac{d^2+g^2}{f^2+g^2}}\left\{y_0\sin\frac{eg\tau}{cm} - \frac{c^2 m\dot{y}_0}{eg\sqrt{(c^2-v_0^2)}}\cos\frac{eg\tau}{cm}\right\} \\ ct &= \sqrt{\frac{d^2+g^2}{f^2+g^2}}\left\{x_0\sinh\frac{ef\tau}{cm} + \frac{c^2 m\dot{x}_0}{ef\sqrt{(c^2-v_0^2)}}\cosh\frac{ef\tau}{cm}\right\} \\ &\quad - \sqrt{\frac{d^2-f^2}{f^2+g^2}}\left\{y_0\sin\frac{eg\tau}{cm} - \frac{c^2 m\dot{y}_0}{eg\sqrt{(c^2-v_0^2)}}\cos\frac{eg\tau}{cm}\right\} \end{aligned}\right\} \quad \text{...(551).}$$

The equations (550) and (551) completely determine the path and its mode of description in terms of the parameter τ.

298. From (546), § 297, we find

$$x_0^2 - \frac{c^4m^2\dot{x}_0^2}{e^2f^2(c^2-v_0^2)} = \frac{c^4m^2[\{c\sqrt{(d^2+g^2)} - \dot{z}_0\sqrt{(d^2-f^2)}\}^2 - \dot{x}_0^2(f^2+g^2)]}{e^2f^2(f^2+g^2)(c^2-v_0^2)}$$

$$= \frac{c^4m^2[\{c\sqrt{(d^2-f^2)} - \dot{z}_0\sqrt{(d^2+g^2)}\}^2 + (c^2-\dot{x}_0^2-\dot{z}_0^2)(f^2+g^2)]}{e^2f^2(f^2+g^2)(c^2-v_0^2)}$$

$$> 0, \text{ because } c^2 > \dot{x}_0^2 + \dot{z}_0^2.$$

Thus we may write

$$\left.\begin{aligned} x_0^2 - \frac{c^4m^2\dot{x}_0^2}{e^2f^2(c^2-v_0^2)} &= a^2 \\ y_0^2 + \frac{c^4m^2\dot{y}_0^2}{e^2g^2(c^2-v_0^2)} &= b^2 \end{aligned}\right\} \quad\text{.....................(552)},$$

where a and b are both real.

Moreover, write

$$\left.\begin{aligned} \frac{ef\tau}{cm} + \tanh^{-1}\frac{c^2m\dot{x}_0}{ef\sqrt{(c^2-v_0^2)}\,x_0} &= \frac{f}{g}\phi \\ \frac{eg\tau}{cm} - \tan^{-1}\frac{c^2m\dot{y}_0}{eg\sqrt{(c^2-v_0^2)}\,y_0} &= \phi - \epsilon \\ \text{so that}\quad \tan^{-1}\frac{c^2m\dot{y}_0}{eg\sqrt{(c^2-v_0^2)}\,y_0} + \frac{g}{f}\tanh^{-1}\frac{c^2m\dot{x}_0}{ef\sqrt{(c^2-v_0^2)}\,x_0} &= \epsilon \end{aligned}\right\} \quad\text{......(553)}.$$

Then we find, in place of (550) and (551), § 297,

$$\left.\begin{aligned} x &= a\cosh\frac{f}{g}\phi \\ y &= b\cos(\phi-\epsilon) \\ z &= \sqrt{\frac{d^2-f^2}{f^2+g^2}}\,a\sinh\frac{f}{g}\phi - \sqrt{\frac{d^2+g^2}{f^2+g^2}}\,b\sin(\phi-\epsilon) \\ ct &= \sqrt{\frac{d^2+g^2}{f^2+g^2}}\,a\sinh\frac{f}{g}\phi - \sqrt{\frac{d^2-f^2}{f^2+g^2}}\,b\sin(\phi-\epsilon) \end{aligned}\right\} \quad\text{......(554)}.$$

These give $$x^2+y^2+z^2-c^2t^2 = a^2+b^2 \quad\text{.....................(555)}.$$

Equations (554) give the simplest form of the solution.

299. By means of (546), § 297, and (552), § 298, we find

$$f^2a^2 - g^2b^2 = c^4m^2/e^2 \quad\text{.......................(556)},$$

showing that a and b are not independent.

The first equation (554), § 298, shows that x always has the same sign, that of a and x_0; from (546), § 297, we see that x_0 has the sign of e. Hence x and a are positive, or negative, according as the charge e is positive, or negative.

Again, the form of the second equation (554), § 298, shows that b may be taken to be positive without any loss of generality.

For large values of ϕ we get $z = ct\sqrt{(d^2 - f^2)}/\sqrt{(d^2 + g^2)}$; hence z increases, and on the whole the motion is in the positive direction of the Poynting vector, just as it is in the particular cases already examined.

Equations (554), § 298, show that the actual displacement is the resultant of two components:

(1) $x = a\cosh\frac{f}{g}\phi,\ z = \sqrt{\frac{d^2 - f^2}{f^2 + g^2}}\, a\sinh\frac{f}{g}\phi$, whence

$$x^2 - \frac{f^2 + g^2}{d^2 - f^2} z^2 = a^2 \quad \text{..........................(557).}$$

(2) $y = b\cos(\phi - \epsilon),\ z = -\sqrt{\frac{d^2 + g^2}{f^2 + g^2}}\, b\sin(\phi - \epsilon)$, whence

$$y^2 + \frac{f^2 + g^2}{d^2 + g^2} z^2 = b^2 \quad \text{..........................(558).}$$

Equation (557) represents a hyperbola, whose real semiaxis is a, along Ox, conjugate semiaxis $a\sqrt{\frac{d^2 - f^2}{f^2 + g^2}}$, along Oz, and eccentricity $\sqrt{\frac{d^2 + g^2}{f^2 + g^2}}$.

Equation (558) represents an ellipse, whose major semiaxis is $b\sqrt{\frac{d^2 + g^2}{f^2 + g^2}}$, along Oz, minor semiaxis b, along Oy, and eccentricity $\sqrt{\frac{d^2 - f^2}{d^2 + g^2}}$. We find at once

$$y\dot{z} - \dot{y}z = -\sqrt{\frac{d^2 + g^2}{f^2 + g^2}}\, b^2\dot{\phi}.$$

Using (545), § 297, we get $(y\dot{z} - \dot{y}z)\cos(\theta - \alpha) = -\frac{d^2 + g^2}{f^2 + g^2} b^2 \frac{g\dot{\phi}}{h}$, and from (554), § 298, we see that $\dot{\phi}$ has the sign of fa/g. Hence, since f and h are essentially positive, we see that $(y\dot{z} - \dot{y}z)\cos(\theta - \alpha)$ has the opposite sign to a, or e. Thus the direction of revolution of the electron round the positive direction of the magnetic force is positive, right-handed, when the charge is negative, and negative, left-handed, when it is positive.

For a numerical example, take a negative electron, for which

$$e/cm = -1{\cdot}77 \,.\, 10^7 \text{ E.M.U.},$$

moving in a field where $d = 214{\cdot}5$ E.S.U., $h = 214{\cdot}5$ E.M.U. and $\theta = 60°$. We find

$$f = g = 151{\cdot}7,\quad \alpha = 30°,\quad ef/c^2m = eg/c^2m = -0{\cdot}0894.$$

Assume initially $\dot{x}_0 = 0$, $\dot{y}_0 = -{\cdot}544\,c$, $\dot{z}_0 = {\cdot}577\,c$, $v_0 = {\cdot}793\,c$. We get $a = -15$ cm., $b = 10$ cm.

By means of (546), § 297, we find that $y_0 = 0$, so that $\epsilon = \pi/2$. Hence we find from (554), § 298,

$$x = -15 \cosh \phi, \quad y = 10 \sin \phi,$$

$$z = -\frac{15}{\sqrt{2}} \sinh \phi + \frac{10\sqrt{3}}{\sqrt{2}} \cos \phi, \quad ct = -\frac{15\sqrt{3}}{\sqrt{2}} \sinh \phi + \frac{10}{\sqrt{2}} \cos \phi.$$

The path is drawn in Fig. 49 as seen from an infinite distance, looking roughly in a plane through the Poynting vector at 30° to the magnetic force and 120° to the Poynting vector.

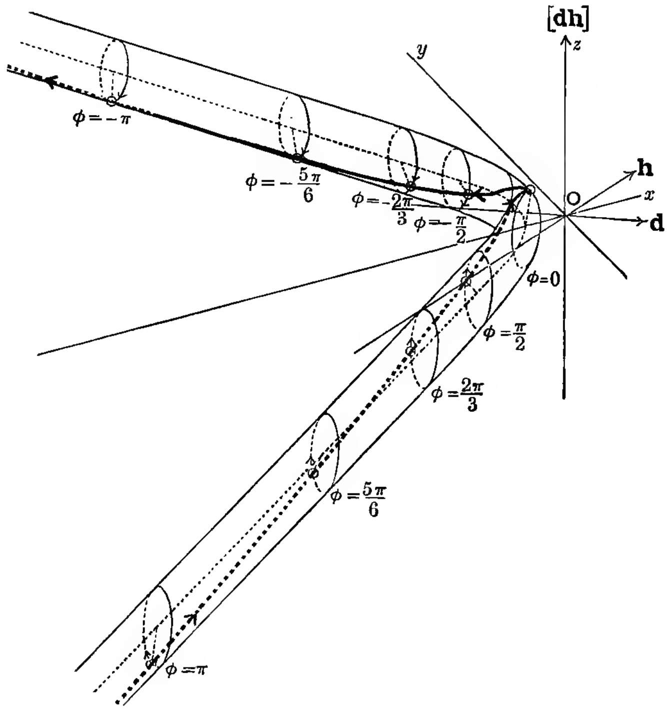

Fig. 49. Path of a negative electron.

Path shown by the thick line, full in front, broken behind. Arrowheads show the direction of motion. Generating ellipse in various positions ($\phi = \pi$ etc.) indicated by fine lines, full in front, dotted behind. Arrowheads show the direction of revolution. Guiding hyperbola shown by dotted line. Boundaries of the tubular surface generated by the ellipse shown by fine full lines. Successive positions of the electron shown by small circles and radii.

300. Problem 4. An electron moves under the combined influence of a central force, which is a function of the distance alone, and of a steady and uniform magnetic field of small intensity. Required to find the motion. This problem has been chosen as an example of the case where an energy integral exists, and the electromagnetic field is symmetrical about an axis. It is particularly interesting in so far as it supplies an extension of the Lorentz theory of the normal Zeeman effect to cases where the motion is not circular, and the velocity need not be small compared with that of light. The Zeeman effect is known to be a linear function of the intensity of the external magnetic field; for this reason we may neglect terms involving the square of the intensity. This produces a very great simplification of the work; indeed, the inclusion of these terms would render the equations well nigh intractable.

With the notation of Appendix F we shall denote by U_0 the force function when the external magnetic field is absent. By hypothesis U_0 is a function of r alone, and therefore it is possible to make the constant of angular momentum, κ, vanish by choosing the axis of x to lie in the plane of projection of the electron through the radius vector.

Hence equation (492), § 286, for the force function, in the absence of the external magnetic field, reduces to

$$U_0 = \frac{(h - e\phi)^2}{2c^2m^2} - \frac{e^2a^2_\chi}{2c^2m^2} \quad \text{.......................(559).}$$

The energy constant, h, is arbitrary, depending only on the circumstances of projection. In order that U_0 may be a function of r alone for *all* values of h, ϕ and a_χ must each be functions of r alone. Since we admit the possibility of a continuous distribution of electric charge in the system to which the electric potential ϕ is due, ϕ can be any function of r alone. But this is not true for the vector potential a_χ; for since the field is steady, there are no electric currents in the system, and therefore a_χ must satisfy the equation

$$\frac{\partial^2 a_\chi}{\partial x^2} + \frac{\partial^2 a_\chi}{\partial \varpi^2} + \frac{1}{\varpi}\frac{\partial a_\chi}{\partial \varpi} - \frac{a_\chi}{\varpi^2} = 0.$$

This equation is incompatible with the requirement that a_χ be a function of r alone other than zero; hence a_χ must vanish in the absence of the external magnetic field.

When the external field is applied, the electromagnetic field as a whole is symmetrical about the line of magnetic force through the centre of force. We take this line as axis of x of our cylindric coordinates (x, ϖ, χ). If the intensity of the field be H, the vector potential is given by

$$a_x = a_\varpi = 0, \quad a_\chi = \tfrac{1}{2}H\varpi \quad \text{.....................(560).}$$

The integral of angular momentum about Ox, (487), § 286, gives

$$\chi' + \frac{eH}{2cm} = \frac{\kappa}{m\varpi^2} = \frac{\kappa}{mr^2\sin^2\theta} \quad \text{..................(561).}$$

Ignoring the coordinate χ we get for the force function U the expression

$$U = U_0 - \frac{(c\kappa - \frac{1}{2}eH\varpi^2)^2}{2c^2m^2\varpi^2} = U_0 - \frac{\kappa^2}{2m^2r^2\sin^2\theta} + \frac{eH\kappa}{2cm^2} \quad \text{......(562),}$$

since $\varpi = r\sin\theta$, and H^2 is neglected.

The equations of motion (494), § 286, give

$$r'' - r\theta'^2 = \frac{\partial U_0}{\partial r} + \frac{\kappa^2}{m^2 r^3 \sin^2\theta}, \quad \frac{d}{d\tau}(r^2\theta') = \frac{\partial U}{\partial \theta} = \frac{\kappa^2 \cos\theta}{m^2 r^2 \sin^3\theta}.$$

We also have the equation of energy, by (496), § 286,

$$r'^2 + r^2\theta'^2 = 2U_0 - \frac{\kappa^2}{m^2 r^2 \sin^2\theta} + \frac{eH\kappa}{cm^2} - c^2 \quad \text{.........(563)},$$

as before; further, the second equation of motion gives

$$r^4\theta'^2 = h^2 - \frac{\kappa^2}{m^2\sin^2\theta} \quad \text{.........(564)},$$

where h is an arbitrary constant. This equation shows that κ^2 must be less than m^2h^2, otherwise the motion could not be real; hence write

$$\kappa = mh\sin\delta \quad \text{.........(565)}.$$

We get
$$r^2\theta' = h\frac{\sqrt{(\sin^2\theta - \sin^2\delta)}}{\sin\theta} \quad \text{.........(566)}.$$

This equation suggests the substitution

$$\cos\theta = \cos\delta\cos\phi \quad \text{.........(567)},$$

whence
$$r^2\phi' = h \quad \text{.........(568)}.$$

Eliminating r^2 between (561) and (566) we find

$$\chi' + \frac{eH}{2cm} = \frac{\theta'\sin\delta}{\sin\theta\sqrt{(\sin^2\theta - \sin^2\delta)}} = \frac{\theta'}{\sin^2\theta\sqrt{(\cot^2\delta - \cot^2\theta)}}.$$

Integrating this equation, we get

$$\cos\left(\chi + \frac{eH\tau}{2cm} - \gamma\right) = \tan\delta\cot\theta \quad \text{.........(569)},$$

where γ is an arbitrary constant.

Eliminating θ' between (563) and (564) we find

$$r'^2 = 2U_0 - \frac{h^2}{r^2} + \frac{eH\kappa}{cm^2} - c^2,$$

whence
$$\tau = \pm\int\frac{dr}{\sqrt{\left(2U_0 - \frac{h^2}{r^2} + \frac{eH\kappa}{cm^2} - c^2\right)}} \quad \text{.........(570)}.$$

Again, combining this equation with (568), we get

$$\phi = \pm\int\frac{h\,dr}{r^2\sqrt{\left(2U_0 - \frac{h^2}{r^2} + \frac{eH\kappa}{cm^2} - c^2\right)}} \quad \text{.........(571)}.$$

Lastly, in order to determine t we must remember that $\frac{dt}{d\tau} = c/\sqrt{(c^2 - v^2)}$, by (480), § 286. Now with the notation of § 286 we have

$$w^2 = r'^2 + r^2\theta'^2 + \varpi^2\chi'^2 = 2U_0 - c^2,$$

by (561) and (563).

But from (483), § 286, we find that $w = cv/\sqrt{(c^2 - v^2)}$; hence

$$\frac{c^2}{\sqrt{(c^2 - v^2)}} = \sqrt{(2U_0)} \qquad \text{(572)},$$

which is another form of the energy integral. Hence we find from (570)

$$ct = \pm \int \frac{\sqrt{(2U_0)}\,.\,dr}{\sqrt{\left(2U_0 - \frac{h^2}{r^2} + \frac{eH\kappa}{cm^2} - c^2\right)}} \qquad \text{(573)}.$$

Equations (567), (569), (570) and (571) determine the orbit, and (573) determines the mode of its description. The two first equations admit of a geometrical interpretation.

301. Let a sphere of unit radius be described with the origin as centre. In Fig. 50 let rectangular axes of (x, y, z) cut it in X, Y, Z, and let the radius vector to the electron cut it in P.

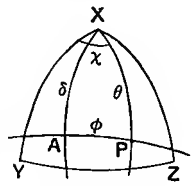

Fig. 50.

Then $\qquad XP = \theta, \quad YXP = \chi.$

With X as centre describe a small circle of radius δ, and draw the great circle PA to touch it at A. Then $XA = \delta$, and A is a right angle.

Hence $\qquad \cos\theta = \cos\delta \cos AP, \quad \cos AXP = \tan\delta \cot\theta.$

Comparing these equations with (567) and (569), § 300, we see that

$$AP = \phi, \quad AXP = \chi + \frac{eH\tau}{2cm} - \gamma,$$

and therefore
$$YXA = \gamma - \frac{eH\tau}{2cm}.$$

Thus (571), § 300, may be regarded as the equation of the orbit, ϕ as the longitude measured from OA as initial line, and the plane OAP as the plane of the orbit. When there is no external magnetic field $YXA = \gamma$, a constant, and the plane of the orbit is fixed in space. But when there is an external magnetic field YXA increases with time at the rate $-\frac{eH}{2cm}\frac{d\tau}{dt}$, while AX retains the constant value δ. Now $\frac{d\tau}{dt} = \frac{\sqrt{(c^2 - v^2)}}{c}$, and the instantaneous mass of the electron is given by the Lorentz mass formula as equal to $m_v = cm/\sqrt{(c^2 - v^2)}$. Hence the angular velocity of the plane of the orbit is equal to $-eH/2cm_v$. In other words the electron describes its orbit relative to the plane OAP, while this plane precesses about Ox with angular velocity $-eH/2cm_v$. This result is analogous to that obtained in the elementary Lorentz theory of the Zeeman effect, where the angular velocity of precession is equal to $-eH/2cm$, and therefore uniform. Here it is variable, because the zero mass m is replaced by the actual mass m_v, which varies as the velocity of the electron varies. This result is important, because it indicates that the normal separation in the Zeeman triplet is to be expected only in those cases, where the electrons generating the spectrum lines are moving with velocities which are small compared with that of light. Otherwise the separation will be less.

302. The two equations of angular momentum, (561) and (568), § 300, remain true even when the electromagnetic field is not steady, that is, during the variable period while the external magnetic field is being established, provided only that the force due to the permanent field remains central. On account of the great importance of this result we shall give a proof from first principles.

With cylindrical coordinates (x, ϖ, χ) we can by (454), § 278, write the equations of motion in the following forms, bearing in mind that centrifugal forces must be added on account of the rotation of the ϖ-axis with angular velocity χ:

$$\frac{d}{dt}\left\{\frac{c\dot{x}}{\sqrt{(c^2-v^2)}}\right\} = -P_0 \cos\theta,$$

$$\frac{d}{dt}\left\{\frac{c\dot{\varpi}}{\sqrt{(c^2-v^2)}}\right\} - \frac{c\varpi\dot{\chi}^2}{\sqrt{(c^2-v^2)}} = -P_0 \sin\theta + \frac{eH}{cm}\varpi\dot{\chi},$$

$$\frac{d}{dt}\left\{\frac{c\varpi\dot{\chi}}{\sqrt{(c^2-v^2)}}\right\} + \frac{c\dot{\varpi}\dot{\chi}}{\sqrt{(c^2-v^2)}} = -\frac{e\dot{H}}{2cm}\varpi - \frac{eH}{cm}\dot{\varpi}.$$

Here the additional terms in the left-hand members of the second and third equations represent the centrifugal forces, P_0 is due to the central attraction of the original field, the terms in H represent the motive force due to the electrodynamic action of the external field, and the term in $\dot{H}$ represents the inductive action due to its establishment.

Multiplying the third equation by ϖ and integrating we find

$$\frac{c\varpi^2\dot{\chi}}{\sqrt{(c^2-v^2)}} + \frac{eH\varpi^2}{2cm} = \frac{\kappa}{m}, \text{ a constant.}$$

Remembering that $\frac{dt}{d\tau} = c/\sqrt{(c^2-v^2)}$, and that $\chi' = \frac{d\chi}{d\tau}$, we get (561), § 300, proved without any use being made of the equation of energy. Multiplying the first two equations of motion by $\frac{dt}{d\tau}$, and using the integral just obtained, we find

$$\frac{d^2x}{d\tau^2} = -\frac{cP_0\cos\theta}{\sqrt{(c^2-v^2)}}.$$

$$\frac{d^2\varpi}{d\tau^2} = -\frac{cP_0\sin\theta}{\sqrt{(c^2-v^2)}} + \frac{\kappa^2}{m^2\varpi^3} - \frac{e^2H^2\varpi}{4c^2m^2},$$

where the last term is to be neglected. Transforming to polar coordinates we get

$$r'' - r\theta'^2 = -\frac{cP_0}{\sqrt{(c^2-v^2)}} + \frac{\kappa^2}{m^2r^3\sin^2\theta}, \quad \frac{d}{d\tau}(r^2\theta') = \frac{\kappa^2\cos\theta}{m^2r^2\sin^3\theta}.$$

The second of these gives (564), § 300, as before. Hence we see that the two quantities κ and h are constant throughout the motion, whether the external field be variable or steady, and therefore the inclination δ of the plane of the orbit to the lines of magnetic force, as well as the angle γ in (569), § 300, are constant. Thus, while the field varies, the inclination of the plane of the orbit to the lines of magnetic force remains the same, but it precesses round Ox with the angular velocity $-eH/2cm_v$, appropriate to the intensity of the external magnetic field and the velocity of the electron at the moment considered.

303. The energy equation cannot be obtained in the form (563), § 300, for although P_0 is a function of r only, when the internal field is steady, yet the factor $c/\sqrt{(c^2-v^2)}$ depends not only on r, but on the time τ, because work is done on the electron by the external electric force of induction, which exists as long as the external magnetic field varies. Therefore the arbitrary constant, which is involved in the energy equation (563), § 300, and included in the force function U_0, generally has a different value when the external magnetic field has become steady, from that which it had before the field was applied.

We shall now use the equations (567)—(573), § 300, to determine the orbit of the electron relative to the precessing plane OAP. For this purpose we must make definite assumptions respecting the form of the force function U_0. We shall only consider the case of an electron moving inside a fixed sphere of uniform density and opposite sign to that of the charge of the electron.

304. Example: The electron moves inside a fixed sphere of uniform density, of charge $-e'$ and radius b.

The charge is taken to be $-e'$, so that ee' may have the positive sign.

There is no magnetic field other than that of intensity H. The electric potential is given by

$$\phi = \frac{e'r^2}{2b^3} \quad\text{..............................(574).}$$

The energy integral (477), § 285, may be written in the form

$$\frac{c}{\sqrt{(c^2-v^2)}} = \frac{ee'(a^2-r^2)}{2c^2mb^3} \quad\text{........................(575),}$$

where a is the arbitrary constant, and is real because ee' is positive.

The force function U_0 is therefore given by

$$U_0 = \frac{e^2 e'^2 (a^2 - r^2)^2}{8c^2 m^2 b^6} \quad \text{.........................(576).}$$

From (570), § 300, we find

$$\tau = \int \frac{dr}{\sqrt{\left\{\frac{e^2 e'^2 (a^2 - r^2)^2}{4c^2 m^2 b^6} - \frac{h^2}{r^2} - c^2 \left(1 - \frac{eH\kappa}{c^3 m^2}\right)\right\}}} \quad \text{.........(577).}$$

The choice of the positive sign for the square root clearly involves no loss of generality; it is merely a question of a convenient choice of the zero from which τ is measured.

The function under the square root is positive for $r^2 = \infty$, negative for $r^2 = a^2$ (unless $eH\kappa/c^3 m^2$ be too large), and negative for $r^2 = 0$. Hence it vanishes for some value of r greater than a. The energy equation (575) however limits r to values less than a. Therefore r^2 must have an upper limit less than a^2, and a lower limit greater, or in the limit, equal to zero, and both of these values must make the function under the square root vanish. Hence this function has three real positive roots; we shall denote them by $a^2 e_1$, $a^2 e_2$, $a^2 e_3$ respectively, where

$$e_1 > 1 > e_2 > e_3 > 0 \quad \text{........................(578).}$$

Writing for brevity $\qquad r^2 = a^2 \xi$ ....(579)

we find

$$\frac{ee' a\tau}{2cmb^3} = \int_{e_3}^{\xi} \frac{d\xi}{\sqrt{4(e_1 - \xi)(e_2 - \xi)(\xi - e_3)}} \quad \text{..............(580),}$$

where $e_2 \geqslant \xi \geqslant e_3$.

Identifying (577) with (580) we find

$$\frac{e^2 e'^2 (a^2 - r^2)^2}{4c^2 m^2 b^6} - \frac{h^2}{r^2} - c^2 \left(1 - \frac{eH\kappa}{c^3 m^2}\right) = \frac{e^2 e'^2 (a^2 e_1 - r^2)(a^2 e_2 - r^2)(r^2 - a^2 e_3)}{4c^2 m^2 b^6 r^2},$$

whence

$$\left.\begin{aligned} e_1 + e_2 + e_3 &= 2 \\ e_2 e_3 + e_3 e_1 + e_1 e_2 &= 1 - \frac{4c^4 m^2 b^6}{e^2 e'^2 a^4}\left(1 - \frac{eH\kappa}{c^3 m^2}\right) \\ e_1 e_2 e_3 &= \frac{4c^2 m^2 b^6 h^2}{e^2 e'^2 a^6} \end{aligned}\right\} \quad \text{.........(581).}$$

We can reduce the elliptic integral to the normal form by writing

$$\xi = e_3 + (e_2 - e_3)\, sn^2 (u, k), \quad k = \sqrt{\frac{e_2 - e_3}{e_1 - e_3}} \quad \text{...........(582).}$$

Then we get from (575), (579) and (580)

$$\left.\begin{aligned} r^2 &= a^2 \xi = a^2 \{e_3 + (e_2 - e_3)\, sn^2 u\} \\ \tau &= \frac{2cmb^3 u}{ee'a\sqrt{(e_1 - e_3)}} \\ \frac{c}{\sqrt{(c^2 - v^2)}} &= \frac{ee'a^2}{2c^2 mb^3}\{1 - e_3 - (e_2 - e_3)\, sn^2 u\} \end{aligned}\right\} \quad \text{.........(583).}$$

In order to find an expression for the time t we use (480), § 286; hence

$$t=\int_0^\tau \frac{cd\tau}{\sqrt{(c^2-v^2)}}=\frac{a}{c\sqrt{(e_1-e_3)}}\int_0^u \{1-e_3-(e_2-e_3)\,sn^2 u\}\,du \quad\text{...(584).}$$

Using Jacobi's notation we get

$$t=\frac{a\sqrt{(e_1-e_3)}}{c}\left\{\left(\frac{E}{K}-\frac{e_1-1}{e_1-e_3}\right)u+Z(u)\right\} \quad\text{...........(585).}$$

The longitude ϕ is determined by (568), § 300; with the help of (583) we find

$$\phi-\alpha=\int_0^\tau \frac{hd\tau}{r^2}=\pm\sqrt{\frac{e_1e_2e_3}{e_1-e_3}}\int_0^u \frac{du}{e_3+(e_2-e_3)\,sn^2 u} \quad\text{......(586),}$$

where α is an arbitrary constant, and the upper, or lower, sign must be taken according as h is positive, or negative. Write

$$sn\,(\iota\eta,\ k)=\iota\sqrt{\frac{e_1-e_3}{e_3}},\ \text{that is,}\ sn\,(\eta,\ k')=\sqrt{\frac{e_1-e_3}{e_1}} \quad\text{...(587).}$$

Reducing in the usual way and using Jacobi's notation we find

$$\begin{aligned}\phi&=\alpha\pm u\sqrt{\frac{e_1e_2}{e_3(e_1-e_3)}}\pm\iota\Pi\,(u,\ \iota\eta,\ k)\\&=\alpha\pm\left\{\sqrt{\frac{e_2e_3}{e_1(e_1-e_3)}}+\frac{\pi\eta}{2KK'}+Z(\eta,\ k')\right\}u\\&\quad\pm\tfrac{1}{2}\iota\log\frac{\Theta(u-\iota\eta,\ k)}{\Theta(u+\iota\eta,\ k)}\quad\text{..............................(588).}\end{aligned}$$

The equations (585)—(588) completely determine the orbit of the electron and its mode of description relative to the precessing plane OAP.

305. Since the Zeta and Theta functions are periodic functions of u of period $2K$, each of the quantities t and ϕ consists of a progressive term, increasing proportionately to u, together with a periodic term. For the sake of brevity we shall write

$$\left.\begin{aligned}\Theta&=\frac{2cmb^3K}{ee'a\sqrt{(e_1-e_3)}}\\T&=\frac{a\{(e_1-e_3)E-(e_1-1)K\}}{c\sqrt{(e_1-e_3)}}\\\Phi&=\left\{\sqrt{\frac{e_2e_3}{e_1(e_1-e_3)}}+Z(\eta,\ k')\right\}K+\frac{\pi\eta}{2K'}\end{aligned}\right\}\quad\text{...........(589).}$$

Then we find from (583), (585) and (588), § 304,

$$\left.\begin{aligned}u-\frac{K}{T}t&=-\frac{a\sqrt{(e_1-e_3)}\,K}{cT}Z(u)\\\tau-\frac{\Theta}{T}t&=-\frac{a\sqrt{(e_1-e_3)}\,\Theta}{cT}Z(u)\\\phi\mp\frac{\Phi}{T}t-\alpha&=\mp\frac{a\sqrt{(e_1-e_3)}\,\Phi}{cT}Z(u)\pm\tfrac{1}{2}\iota\log\frac{\Theta(u-\iota\eta,\ k)}{\Theta(u+\iota\eta,\ k)}\end{aligned}\right\}\quad\text{...(590).}$$

$Z(u)$ is an odd function of u, $\Theta(u)$ is even, and therefore $\log \dfrac{\Theta(u - \iota\eta, k)}{\Theta(u + \iota\eta, k)}$ is odd; all are periodic functions of period $2K$. Therefore the right-hand members of (590) are odd periodic functions of u of period $2K$. The first equation shows that they may also be regarded as odd periodic functions of t of period $2T$, and accordingly can be expanded in series of sines of integral multiples of the argument $\pi t/T$.

Again, we know that $k^2 sn^2 u - (K - E)/K$ can be expanded in a series of cosines of integral multiples of $\pi u/K$. Hence we see from (583), § 304, that $r^2 - a^2 \{e_1 K - (e_1 - e_3) E\}/K$ can be expressed in the same form; therefore it is an even periodic function of t of period $2T$, and can be expanded in a series of cosines of integral multiples of the argument $\pi t/T$. The mean value of r^2 is clearly equal to $a^2 \{e_1 K - (e_1 - e_3) E\}/K$.

Thus the motion relative to the plane OAP is compounded of a uniform rotation in the plane with the angular velocity $\pm \Phi/T$, and an oscillation of the period $2T$. The orbit has an infinite number of apses, the apsidal distances are $a\sqrt{e_2}$ and $a\sqrt{e_3}$, and the apsidal angle is Φ.

The precessional motion of the plane OAP is determined by (567) and (569), § 300. We saw from Fig. 50, § 301, that the angle YXA between the fixed meridian plane XY and the meridian plane XA through the initial line OA, which we shall call ψ, is given by $\psi = \gamma - eH\tau/2cm$. From the second equation (590) we find

$$\psi + \frac{eH\Theta}{2cmT} t - \gamma = \frac{eHa\sqrt{(e_1 - e_3)}\,\Theta}{2c^2mT} Z(u) \quad \text{............(591).}$$

The right-hand member of this equation is an odd periodic function of t of period $2T$, and can be expanded in a series of sines of integral multiples of the argument $\pi t/T$.

Thus the motion of the plane OAP consists of a uniform precession about Ox with angular velocity $-eH\Theta/2cmT$, compounded with an oscillation of period $2T$.

306. In order to get the most convenient representation of the motion, we choose a system of moving axes of (x', y', z'). Let the Eulerian angular coordinates, which define the position of the moving axes relative to the axes of (x, y, z), be (θ', ϕ', ψ'), where

$$\theta' = \tfrac{1}{2}\pi - \delta, \quad \phi' = a \pm \frac{\Phi}{T} t,$$

$$\psi' = \gamma - \frac{eH\Theta}{2cmT} t \quad \text{......(592).}$$

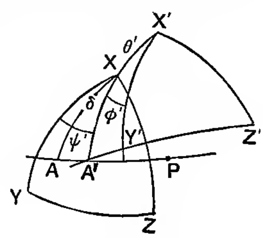

Fig. 51.

Then the axes of (x', y', z') rotate about Ox' with uniform angular velocity $\pm \Phi/T$, and precess about Ox with uniform angular velocity $-eH\Theta/2cmT$. The plane $OY'Z'$ cuts the plane of the orbit OAP in the radius OA', such that $XA' = XA = \delta$. The arc AA' and the angle $PA'Y'$ are periodic functions of t of period $2T$, and vanish whenever t is an integral multiple of T. The coordinates (x', y', z') of the electron at P relative to the moving axes are periodic functions of t of period $2T$, and therefore can be expanded in series of sines and cosines of integral multiples of the argument $\pi t/T$. Thus the motion is a particular case of that studied in problem 6, Ch. VIII, p. 143, such that

$$\theta = \tfrac{1}{2}\pi - \delta, \quad \mu = -\frac{eH\Theta}{2cmT}, \quad n = \pm\frac{\Phi}{T}, \quad q = \frac{s\pi}{T},$$

where s is an integer, which may be supposed positive without loss of generality.

307. Accordingly all the results obtained respecting the distant radiation from the electron apply in the present case. In particular, we see from § 113 that each value of s gives rise to a nonet of vibrations:

(1) A triplet of right-handed circular vibrations about Ox, that is, about the positive direction of the magnetic force, with frequencies

$$(s\pi \pm \Phi - eH\Theta/2cm)/T$$

and

$$(s\pi - eH\Theta/2cm)/T.$$

(2) A triplet of left-handed circular vibrations about Ox, with frequencies $(s\pi \pm \Phi + eH\Theta/2cm)/T$ and $(s\pi + eH\Theta/2cm)/T$.

(3) A triplet of linear vibrations parallel to Ox, with frequencies $(s\pi \pm \Phi)/T$ and $s\pi/T$.

When the external field is absent ($H = 0$) the three triplets coalesce, so as to form three elliptic vibrations with frequencies $(s\pi \pm \Phi_0)/T_0$ and $s\pi/T_0$, where T_0 and Φ_0 correspond to the value $H = 0$.

Thus the effect of the magnetic field is two-fold: (1) it splits up each component of the triplet into three, one linear and two circular vibrations, the separation being $eH\Theta/2cmT$; (2) the central vibration of each triplet is displaced whenever T and Φ differ from T_0 and Φ_0.

When the charge e is negative, the right-handed circular vibrations have the greatest frequencies, when e is positive, the least.

In general, there is symmetry of position in each triplet produced by the field, but there is not symmetry of intensity.

These results are in qualitative agreement with the results of experiments on the Zeeman effect, if the charge e be supposed to be negative in the normal effect. In order that the agreement may be quantitative, corresponding to separation strictly proportional to H, and no shift of the central vibration, it is necessary that T, Φ and Θ be independent of H.

Now equations (589), § 305, determine T, Φ and Θ as functions of the energy constant a, and of the three quantities e_1, e_2 and e_3, which are given by equations (581), § 304, as functions of a, h, κ and H.

We have seen in § 302 that the constants of angular momentum, h and κ, are the same whether the external magnetic field is on or off, but that the energy constant a is generally different, on account of the work done on the electron owing to induction during the variable state.

Hence it becomes necessary to examine under what conditions, if any, the change in the value of a, and the presence of H in the second equation (581), § 304, are without appreciable effect on the quantities T, Φ and Θ.

308. We shall begin by examining the effect of the presence of H in the second equation (581), § 304, where it occurs in the factor $1 - eH\kappa/c^3m^2$ on the right-hand side. Now by (565), § 300, we have $\kappa = mh \sin \delta$; hence from the third equation (581), § 304, we find

$$\frac{eH\kappa}{c^3m^2} = \frac{eH \sin \delta}{c^2m} \frac{ee'a^3 \sqrt{(e_1e_2e_3)}}{2c^2mb^3}.$$

Let r_0 be the least apsidal distance, $a\sqrt{e_3}$, and let v_0 and v_1 be the velocities at the apses; from the first equation (581), § 304, and the last equation (583), § 304,

$$\frac{eH\kappa}{c^3m^2} = r_0 \frac{eH \sin \delta}{c^2m} \left\{\frac{c}{\sqrt{(c^2 - v_0^2)}} + \frac{c}{\sqrt{(c^2 - v_1^2)}}\right\} \sqrt{\frac{e_2}{e_1}}.$$

Here r_0 must be less than the atomic radius, let us say less than 10^{-8} cm.; for $H = 30{,}000$ gauss, and $e/cm = -1{\cdot}77.10^7$ E.M.U., $eH \sin \delta/c^2m$ is numerically less than $17{\cdot}7$; and the last factor is less than unity. Hence $eH\kappa/c^3m^2$ is comparable with 10^{-6}, unless the velocity of the electron at the nearer apse, where it is greatest, is nearly equal to the velocity of light.

Hence the direct effect of the external field on the periods will be less than one millionth, unless the velocity of the electron at some stage of its motion becomes nearly equal to that of light.

As this effect is already near the limit of accuracy of wave-length measurements, we shall neglect the term $eH\kappa/c^3m^2$ in the second equation (581), § 304. Then it gives

$$\frac{ee'a^2}{2c^2mb^3} = (1 - e_2e_3 - e_3e_1 - e_1e_2)^{-\frac{1}{2}} \quad \text{..............(593).}$$

309. Again, in considering the effect of changes in the value of a, we must bear in mind that even when there is no external magnetic field, the energy constant a, as well as the constant of angular momentum h, can have various values. If the motion under consideration is to account for fine spectrum lines, in spite of the fact that waves fall on the optical receiving instrument, which have been emitted by large numbers of electrons, projected in similar fields but under widely different circumstances, one of two conditions must hold: either (1) the periods of the waves emitted by an electron do not depend on the values of the constants a and h to any appreciable extent; or (2) some cause is at work which confines the values of these constants within very narrow limits for all electrons, which are contributing to the particular line under observation.

In the present problem the first condition holds, provided that the velocity of the electron always remains small compared with that of light. In fact, we know that under these circumstances the electron can describe ellipses of various shapes and sizes, but always with the same period.

The third equation (583), § 304, gives for the two apses

$$\frac{c}{\sqrt{(c^2 - v_0^2)}} = \frac{ee'a^2(1 - e_3)}{2c^2mb^3}, \quad \frac{c}{\sqrt{(c^2 - v_1^2)}} = \frac{ee'a^2(1 - e_2)}{2c^2mb^3} \quad \ldots(594).$$

Hence v_0 and v_1 differ appreciably, and one of them at least is comparable with c, unless both e_2 and e_3 are small.

Neglecting squares and products of e_2 and e_3, we find from (581) and (586), § 304, (589), § 305, and (593), § 308,

$$\left.\begin{aligned} \frac{ee'a^2}{2c^2mb^3} &= 1 + e_2 + e_3 \\ \Theta &= \tfrac{1}{2}\pi\sqrt{\frac{mb^3}{ee'}}\{1 - \tfrac{1}{8}(e_2 + e_3)\} \\ T &= \tfrac{1}{2}\pi\sqrt{\frac{mb^3}{ee'}}\{1 + \tfrac{3}{8}(e_2 + e_3)\} \\ \Phi &= \tfrac{1}{2}\pi\{1 + \tfrac{1}{4}\sqrt{(e_2e_3)}\} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(595).$$

Since the apsidal distances are given by $r_0 = a\sqrt{e_3}$, and $r_1 = a\sqrt{e_2}$, we find from (595) approximately

$$e_2 = \frac{\pi^2 r_1^2}{8c^2T^2}, \quad e_3 = \frac{\pi^2 r_0^2}{8c^2T^2} \quad \ldots\ldots\ldots\ldots(596),$$

and from (594)

$$v_0 = c\sqrt{2e_2}, \quad v_1 = c\sqrt{2e_3} \quad \ldots\ldots\ldots\ldots(597).$$

In order that the fundamental period $2T$ should be comparable with that of the D lines of sodium, the third equation (595) requires that $b = 4{\cdot}6 \,.\, 10^{-8}$ cm. for $e' = e$, and that it be larger than this for $e' > e$. In order that r_0 and r_1 may be less than b, it is necessary that e_2 and e_3 be less than $3 \,.\, 10^{-6}$ (596), and v_0/c and v_1/c less than $2{\cdot}5 \,.\, 10^{-3}$ (597).

With these values of e_2 and e_3, we see from (595) that Θ, T and Φ differ by less than one part in five hundred thousand from their mean values, a variation just about small enough to allow of the observed fineness of spectrum lines. To the same approximation the separation in the Zeeman effect then reduces to the normal value $eH/2cm$.

When the velocity of the electron exceeds the limiting value just found, namely one four hundredth of the velocity of light, it is necessary to invoke the second hypothesis in order to account for the observed fineness of spectrum lines.

Moreover, the second hypothesis is required in every case where the potential ϕ is not a quadratic function of the radius vector, for instance, when the field is due to a fixed point charge. A hypothesis tantamount to this was suggested by J. J. Thomson in his book on the *Corpuscular Theory of Matter*, pp. 157 et seq.

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