International Library of Technology

323

Arithmetic

For Technical Students

198 Illustrations

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INTERNATIONAL CORRESPONDENCE SCHOOLS

ELEMENTS OF ARITHMETIC
FRACTIONS
DECIMALS
WEIGHTS AND MEASURES
RATIO AND PROPORTION
POWERS AND ROOTS
MENSURATION
FORMULAS
CUBE ROOT
COMMERCIAL CALCULATIONS

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PREFACE

The volumes of the International Library of Technology are made up of Instruction Papers, or Sections, comprising the various courses of instruction for students of the International Correspondence Schools. The original manuscripts are prepared by persons thoroughly qualified both technically and by experience to write with authority, and in many cases they are regularly employed elsewhere in practical work as experts. The manuscripts are then carefully edited to make them suitable for correspondence instruction. The Instruction Papers are written clearly and in the simplest language possible, so as to make them readily understood by all students. Necessary technical expressions are clearly explained when introduced.

The great majority of our students wish to prepare themselves for advancement in their vocations or to qualify for more congenial occupations. Usually they are employed and able to devote only a few hours a day to study. Therefore every effort must be made to give them practical and accurate information in clear and concise form and to make this information include all of the essentials but none of the nonessentials. To make the text clear, illustrations are used freely. These illustrations are especially made by our own Illustrating Department in order to adapt them fully to the requirements of the text.

In the table of contents that immediately follows are given the titles of the Sections included in this volume, and under each title are listed the main topics discussed. At the end of the volume will be found a complete index, so that any subject treated can be quickly found.

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INTRODUCTION

1. Necessity for Calculations.—The worker in almost any branch of industry frequently meets with problems that require figuring. For example, the engineer may wish to figure the horsepower of an engine or the pressure that may be carried safely in a steam boiler. The blacksmith may have to find how long a piece of straight bar must be cut off, so that, when it is bent, it will form a ring of a certain size. The patternmaker may wish to know how to set his dividers so that they will space off a certain number of equal divisions on a circle. The foundryman may need to know the amount of metal required to pour a casting whose dimensions are given on a drawing, and so on in many other occupations. In each of these cases it is necessary to make calculations in order to obtain the desired information. Sometimes the calculations are short and simple, and at other times they are long and difficult.

2. Use of Arithmetic.—Before calculations of any kind can be made, something must be known about figures and numbers, because all calculations bring figures and numbers into use. The study of numbers, or the art of reckoning, is commonly called **arithmetic.** Thus, the various calculations that may be made really depend on an understanding of arithmetic. For this reason, it will be necessary to begin with a study of those arithmetical principles and processes that are to be used later in making engineering and commercial calculations.

3. Fundamental Processes.—The four fundamental processes of arithmetic are *addition, subtraction, multiplication,* and *division.* They are called fundamental processes because all operations in arithmetic are based on them. Every calculation that is made must use one or more of these processes.

4. Unit and Number.—A **unit** is *one,* or a single thing, as *one* inch, *one* dozen. A **number** is one or more units or things. It answers the question "How many?" For example, let the question be, "How many bolt holes are there in that cylinder head?" If the answer is "Eight holes," then *eight* is a number, because it tells how many. A number, however, may be either one or *more* than one, as *one* hour, *six* feet, *ten* dollars.

5. Concrete and Abstract Numbers.—If a number is applied to one particular kind of thing or measure, as three *horses,* five *dollars,* ten *pounds,* it is called a **concrete number.** If a number is not applied to any particular thing or measure, as *three, six, ten,* it is an **abstract number.**

6. Integer and Fraction.—Distinction is made between numbers that indicate one or more *whole* units and those numbers that represent a *portion* of a unit. A whole number is known as an **integer,** or an **integral number.** A number representing a portion, or part, of a unit is called a **fraction.**

7. Systems of Notation and Numeration.—Numbers are expressed by *words,* by *figures,* and by *letters.* **Notation** is the art of *expressing* numbers by figures or letters. **Numeration** is the art of *reading* numbers expressed by figures or letters. Two kinds of notation are in general use, the *Arabic* and the *Roman.*

ARABIC NOTATION

8. Meaning and Arrangement of Figures.—The **Arabic notation** is the method of expressing numbers by figures. This method employs ten characters, called **figures,** which are written and named as follows:

Figures	1	2	3	4	5	6	7	8	9	0
Names	*one*	*two*	*three*	*four*	*five*	*six*	*seven*	*eight*	*nine*	*naught*

The figure **naught** (0) is called also **cipher** and **zero,** and when standing alone means *nothing,* or *no value.* The other nine figures are called **digits,** and when standing alone each has *a definite value.* Ten is written 10.

9. Counting.—One of the first things learned in arithmetic is to *count.* **Counting** is done by naming the numbers successively in the order of their value. The method of counting above ten and up to one thousand, and the names of the various numbers, are given in the following list:

11	*eleven*	21	*twenty-one*	31	*thirty-one* and so	100	*one hundred*
12	*twelve*	22	*twenty-two*		on up to	200	*two hundred*
13	*thirteen*	23	*twenty-three*	40	*forty;* then	300	*three hundred*
14	*fourteen*	24	*twenty-four*	41	*forty-one* and so	400	*four hundred*
15	*fifteen*	25	*twenty-five*		on up to	500	*five hundred*
16	*sixteen*	26	*twenty-six*	50	*fifty;* then to	600	*six hundred*
17	*seventeen*	27	*twenty-seven*	60	*sixty*	700	*seven hundred*
18	*eighteen*	28	*twenty-eight*	70	*seventy*	800	*eight hundred*
19	*nineteen*	29	*twenty-nine*	80	*eighty*	900	*nine hundred*
20	*twenty*	30	*thirty*	90	*ninety*	1,000	*one thousand*

10. Ordinals and Cardinals.—Indicating by means of a number the position of a thing or unit in a row, or series, may also be considered as counting. For instance, *third* house, *seventh* month, etc. When numbers are used in this manner, they are known as **ordinals, or ordinal numbers.** Numbers that simply answer the question "How many?" are sometimes called **cardinals,** or **cardinal numbers.** For example, *one, two, three,* etc. are cardinals, and *first, second, third,* etc. are the corresponding ordinals. In general, an ordinal is formed by adding the letters *th* to a cardinal; as, four*th,* six*th,* ten*th,* fourteen*th,* etc. The ordinals of *one, two,* and *three,* are exceptions to this rule; in the ordinal *fifth* the spelling of five is slightly changed.

11. Simple and Relative Values.—The value represented by a figure depends on its position in relation to other figures; thus, figures may have *simple values* and *relative values.* The **simple value** of a figure is the value it has when it stands alone; for example, the figure 2 standing alone has a value that is one greater than 1 and one less than 3. But if a figure 1 is placed to the right of the 2, making 21, the first figure no longer has the value it had before. This new value that is given to it by placing another figure to the right of it is called its **relative value.** The difference between simple and relative values may be explained as follows:

If the figure 8 stands alone, thus.................. 8
it is simply *eight units,* or **eight.**

Place a 2 to the right of it; thus................... 82

The 2 is now *two units,* but the 8 has moved one place to the left, so that it is no longer *eight units.* Instead, it is *eight* **tens,** or *ten times* 8.

Now place a 5 to the right; thus................... 825

The 8 is now moved another place to the left and its value is again increased ten times, or *ten times eight tens,* making 8 **hundreds.** At the same time the 2 is moved one place to the left and its value is increased to 2 **tens.**

Add a 6 to the right; thus........................ 8256

The 8 is now another place to the left and its value is increased ten times more, or to 8 **thousands.** The 2 is increased to 2 **hundreds,** and the 5 to 5 **tens.**

The last number of four figures is read *eight thousand two hundred fifty-six.*

12. Grouping of Figures.—In writing numbers that contain more than three figures, it is common to divide them into groups of three figures, counting from the *right.* This is called **pointing off,** because a comma (,) is used to point off, or mark, each group of three figures. The object of doing this is to enable the number to be read easily and accurately. The first group at the right is the units group, the next the thousands group, and the next the millions group. This method of pointing off and the naming of the groups is shown in the table at the end of this article.

The illustration given in Art. **11** shows that by moving a figure one place to the left in a number its value is made ten times as great as before. The last position at the right in a number is called the *units* place. Take the number 417,385,926 as an example. The figure 6 is in the units place and is simply *six*. The next place, occupied by the 2, is the *tens of units* place, so that the 2 in this position has a value of 2 tens of units, or *twenty*. The 9 is in the *hundreds of units* place, and its value is 9 hundreds of units, or *nine hundred*. This right-hand group of three figures, therefore, has a value of nine hundreds, two tens, and six units, or *nine hundred twenty-six*, as it would be read.

The next group of three figures, or 385, is at the left of 926; therefore, the figure 5 is in the fourth place from the right end, and its value is ten times as great as it would be in the third position. The third place is the hundreds place, and as the fourth place is ten times as great, it must be the *thousands* place, so that the figure 5 in this position represents *five thousands*. The figure 8 is in the *tens of thousands* place, and its value is 8 tens of thousands, or *eighty thousand*. The figure 3 is in the *hundreds of thousands* place, and its value is 3 hundreds of thousands, or *three hundred thousand*. The middle group of three figures, therefore, has a value of three hundred thousand, eighty thousand, and five thousand, which would be read *three hundred eighty-five thousand*.

The group of figures at the left, or 417, refer to millions, a *million* being ten times as great as a hundred thousand. The 7 is in the *millions* place and has a value of *seven million*. The 1 is in the *tens of millions* place and has a value of *ten million*. The 4 is in the *hundreds of millions* place and has a value of *four hundred million*. This group of figures, therefore, has a value of four hundred million, one ten million, and seven million, or *four hundred seventeen million*, to state it in the way in which it would be read.

The entire number 417,385,926, made up of the three groups of figures, would be read *four hundred seventeen million three hundred eighty-five thousand nine hundred twenty-six.* The following table shows the positions of the figures, the groups, and the name of each of the places, or positions:

Millions			*Thousands*			*Units*		
Hundreds of Millions	Tens of Millions	Millions	Hundreds of Thousands	Tens of Thousands	Thousands	Hundreds of Units	Tens of Units	Units
4	1	7,	3	8	5,	9	2	6

13. Use of the Cipher.—The cipher, 0, has no value in itself, because it represents nothing, or zero, but it is useful in determining the position of other figures. Suppose, for example, that the number *two hundred five* is to be written. It would not be correct to write it 25, because that is *twenty-five.* The 2 must be in the hundreds place and the 5 in the units place, because *two hundred five* means *two hundreds* and *five units;* therefore, it is written by placing a cipher between the 2 and the 5, giving 205. The 2 is then in the hundreds place and the 5 in the units place, as required, and the cipher indicates that there are no *tens.* In the same way, *three thousand twenty-six* would be written 3026, and *six thousand four* would be written 6004. In the last case, two ciphers are needed, because the 6 must be in the thousands place, which is the fourth from the right. If the number to be written is *five thousand nine hundred eighty,* it could *not* be written 598 because that is only *five hundred ninety-eight.* The 5 must be in the fourth, or thousands, place, the 9 in the hundreds place, and the 8 in the tens place; consequently, a cipher is added at the right, giving 5980, which is the correct way of writing *five thousand nine hundred eighty.*

NUMERATION

14. Reading of Numbers.—When reading whole numbers the word *and* should not be used. For instance, the number 205 is read *two hundred five;* not two hundred *and* five.

In numbers that are pointed off, each group, or period, is read in its turn, as though it were a separate number, and then the name of the group is placed after it. For instance, the first group from the left in the number 987,765,432 is read nine hundred eighty-seven, as though standing alone, then *million,* the name of the group, is added. The second group is read seven hundred sixty-five thousand, and the last group is read simply as four hundred thirty-two, the word unit being omitted. In reading numbers, the final *s* in *thousands, millions,* etc. is omitted.

EXAMPLES FOR PRACTICE

Point off and read the following numbers:

(1) 31072; (2) 317020; (3) 1007; (4) 6051; (5) 28970093.

Write the following numbers, using figures:

(6) Seven thousand seventeen; (7) One thousand nine hundred fourteen; (8) Ten million eighty-two thousand thirty-six.

Ans. (1) 31,072 or thirty-one thousand seventy-two; (2) 317,020 or three hundred seventeen thousand twenty; (3) 1,007 or one thousand seven; (4) 6,051 or six thousand fifty-one; (5) 28,970,093 or twenty-eight million nine hundred seventy thousand ninety-three; (6) 7,017; (7) 1,914; (8) 10,082,036.

ROMAN NOTATION

15. Fundamental Letters and Their Combinations.—The method of expressing numbers by means of seven capital letters is known as **Roman notation.** This notation is generally used for numbering chapters in books, tables, rules, formulas, etc. The seven letters and their fixed values, when used singly, are as follows:

Letter	I	V	X	L	C	D	M
Value	1	5	10	50	100	500	1000

When numbers expressed in Roman notation are written by hand, the letters must be formed like the printed capitals, and not like ordinary capitals used in handwriting.

16. The letters used in Roman notation can be combined according to the following principles to represent any number:

I. *If a letter is written before one of greater value, their difference is the value represented;* as, IV, four; IX, nine; XC, ninety.

II. *If a letter is written after one of greater value, their sum is the value represented;* as, VI, six; XI, eleven.

III. *Repeating a letter repeats its value;* thus, II=2, XX=20, CC=200, CCC=300. The letters V, D, and L are never repeated; only I, X, C, and M are ever used more than once in any combination.

17. Some of the combinations in most common use and the values they represent are as follows:

II	two	IX	nine	XVI	sixteen
III	three	XI	eleven	XVII	seventeen
IV	four	XII	twelve	XVIII	eighteen
VI	six	XIII	thirteen	XIX	nineteen
VII	seven	XIV	fourteen	XX	twenty
VIII	eight	XV	fifteen	XXI	twenty-one

ADDITION

VERTICAL ADDITION

18. Definitions.—Addition is the process of *finding a number that is equal to two or more numbers taken together.* The number so obtained is called the **sum,** or the **total.**

19. The sign of addition is +. It is read *plus,* and means *more* or *and.* Thus, 5+6 is read 5 *plus* 6, and means that 5 and 6 are to be added. Or, 5+6 may be read 5 *and* 6.

The sign of equality is =. It is read *equals,* or *is equal to.* Thus, 5+6=11 may be read, *5 plus 6 equals 11.*

In preparing his lessons, the student will have frequent occasion to use the equality sign, and he should therefore clearly understand its meaning. All on the right of the sign should always be equal to all on the left.

Numbers expressed in units of the same kind can be added, but numbers expressed in units of different kinds cannot be added. Thus, 6 dollars can be added to 7 dollars and the sum will be 13 dollars; but 6 dollars cannot be added to 7 feet.

20. Use of Table I.—Table I gives the sum of any two numbers from 1 to 12. This table should be carefully committed to memory. As 0 has no value, the sum of any number and 0 is the number itself; thus, 17 and 0 is 17, or 17+0=17.

21. Method of Vertical Addition.—The term *vertical* means upright, in the direction of a plumb-line. Hence, for **vertical addition,** place the numbers to be added in a vertical row, one below another, taking care to place *units* under *units, tens* under *tens, hundreds* under *hundreds,* and so on. When the numbers are thus written, the *right-hand figure of one number is placed directly under the right-hand figure of the one above it,* thus bringing units under units, tens under tens, etc. In a group of numbers arranged in this manner, each vertical row is said to form a *column.* From the fact that addition takes place up or down these vertical columns, this method of addition is called **vertical addition** to distinguish it from *horizontal addition,* explained later.

The number of columns in a group will depend on the number of figures in each horizontal line. To find the sum of the figures in these columns proceed as in the following rule:

Rule.—I. *When adding numbers, begin at the right, add each column separately, and write the sum, if it is only one figure, under the column added.*

II. *If the sum of any column consists of two or more figures, write down the right-hand figure of the sum under that*

TABLE I

ADDITION TABLE

1 and 1 is 2	2 and 1 is 3	3 and 1 is 4	4 and 1 is 5
1 and 2 is 3	2 and 2 is 4	3 and 2 is 5	4 and 2 is 6
1 and 3 is 4	2 and 3 is 5	3 and 3 is 6	4 and 3 is 7
1 and 4 is 5	2 and 4 is 6	3 and 4 is 7	4 and 4 is 8
1 and 5 is 6	2 and 5 is 7	3 and 5 is 8	4 and 5 is 9
1 and 6 is 7	2 and 6 is 8	3 and 6 is 9	4 and 6 is 10
1 and 7 is 8	2 and 7 is 9	3 and 7 is 10	4 and 7 is 11
1 and 8 is 9	2 and 8 is 10	3 and 8 is 11	4 and 8 is 12
1 and 9 is 10	2 and 9 is 11	3 and 9 is 12	4 and 9 is 13
1 and 10 is 11	2 and 10 is 12	3 and 10 is 13	4 and 10 is 14
1 and 11 is 12	2 and 11 is 13	3 and 11 is 14	4 and 11 is 15
1 and 12 is 13	2 and 12 is 14	3 and 12 is 15	4 and 12 is 16
5 and 1 is 6	6 and 1 is 7	7 and 1 is 8	8 and 1 is 9
5 and 2 is 7	6 and 2 is 8	7 and 2 is 9	8 and 2 is 10
5 and 3 is 8	6 and 3 is 9	7 and 3 is 10	8 and 3 is 11
5 and 4 is 9	6 and 4 is 10	7 and 4 is 11	8 and 4 is 12
5 and 5 is 10	6 and 5 is 11	7 and 5 is 12	8 and 5 is 13
5 and 6 is 11	6 and 6 is 12	7 and 6 is 13	8 and 6 is 14
5 and 7 is 12	6 and 7 is 13	7 and 7 is 14	8 and 7 is 15
5 and 8 is 13	6 and 8 is 14	7 and 8 is 15	8 and 8 is 16
5 and 9 is 14	6 and 9 is 15	7 and 9 is 16	8 and 9 is 17
5 and 10 is 15	6 and 10 is 16	7 and 10 is 17	8 and 10 is 18
5 and 11 is 16	6 and 11 is 17	7 and 11 is 18	8 and 11 is 19
5 and 12 is 17	6 and 12 is 18	7 and 12 is 19	8 and 12 is 20
9 and 1 is 10	10 and 1 is 11	11 and 1 is 12	12 and 1 is 13
9 and 2 is 11	10 and 2 is 12	11 and 2 is 13	12 and 2 is 14
9 and 3 is 12	10 and 3 is 13	11 and 3 is 14	12 and 3 is 15
9 and 4 is 13	10 and 4 is 14	11 and 4 is 15	12 and 4 is 16
9 and 5 is 14	10 and 5 is 15	11 and 5 is 16	12 and 5 is 17
9 and 6 is 15	10 and 6 is 16	11 and 6 is 17	12 and 6 is 18
9 and 7 is 16	10 and 7 is 17	11 and 7 is 18	12 and 7 is 19
9 and 8 is 17	10 and 8 is 18	11 and 8 is 19	12 and 8 is 20
9 and 9 is 18	10 and 9 is 19	11 and 9 is 20	12 and 9 is 21
9 and 10 is 19	10 and 10 is 20	11 and 10 is 21	12 and 10 is 22
9 and 11 is 20	10 and 11 is 21	11 and 11 is 22	12 and 11 is 23
9 and 12 is 21	10 and 12 is 22	11 and 12 is 23	12 and 12 is 24

column and add the remaining figure or figures to the next column.

22. Examples of Addition.—The application of the preceding rule is shown in the following examples:

Example 1.—Find the sum of 131, 222, 21, 2, and 413.

Solution.—After placing the numbers in the proper order, begin at the bottom of the units column and add in accordance with part I of the rule. Thus, 3 and 2 is 5; 5 and 1 is 6; 6 and 2 is 8; 8 and 1 is 9. Write 9 in the units place of the sum. Proceed in a like manner with the second and third columns, thus finding the number 8 for the tens place and 7 for the hundreds place in the sum. The sum is therefore 789.

$$\begin{array}{rrl} & 131 & \\ & 222 & \\ & 21 & \\ & 2 & \\ & 413 & \\ \cline{2-2} \textit{sum} & 789 & \text{Ans.} \end{array}$$

The result obtained in solving an example is called the **answer;** as shown in the preceding example, the word is written in the short form Ans.

Example 2.—Find the sum of 425, 36, 9,215, 4, and 907.

Solution.—Write the numbers as shown, and beginning with the bottom of the right-hand column, add in the following manner, which is shorter than that used in the preceding example, as the successive sums are not repeated. Thus, instead of 7 and 4 is 11; 11 and 5 is 16, etc., we have: 7 and 4 is 11 and 5 is 16 and 6 is 22 and 5 is 27. In accordance with part II of the rule, write 7 in the units place in the sum and carry 2 to the tens column. The term *carry* means in this case that a figure such as 2, is transferred from one column to the next one and added to it. Then, 2 and 1 is 3 and 3 is 6 and 2 is 8. Write 8 in the tens place in the sum, and add the next, or hundreds, column; 9 and 2 is 11 and 4 is 15. Write 5 in the hundreds place in the sum, carry 1 to the thousands place, and add to 9, making 10 to place in the sum; 0 is written in the thousands place and 1 in the tens of thousands place. The sum is therefore 10,587.

$$\begin{array}{rrl} & 425 & \\ & 36 & \\ & 9215 & \\ & 4 & \\ & 907 & \\ \cline{2-2} \textit{sum} & 10587 & \text{Ans.} \end{array}$$

23. Method of Checking Addition.—A good plan, especially when adding several large numbers, is to record the sum of each column plus the number carried, so that the addition can be readily tested, or *checked,* by adding again.

Example.—Add 7,329, 8,564, 9,238, 76,563, 6,417, 36,849, and 58,796.

Solution.—The columns are added as before explained, and each sum is recorded in some convenient place for reference. 46 is the sum of the numbers in the units column, 35 the sum of the numbers in the tens column plus the 4 tens carried, and so on. The final sum is given by writing the last sum, 20, and following it with the last figure of each of the other sums, 3, 7, 5, 6, giving the total sum 203,756.

	7329		
	8564	46	
	9238	35	
	76563	37	
	6417	53	
	36849	20	
	58796		
sum	203756		Ans.

These records of the sums of separate columns are often useful in proving the correctness of the addition.

24. Proof.—*To prove addition, add each column from top to bottom. If the same result is obtained as by adding from bottom to top, the work is probably correct.*

25. Addition of Long Columns.—The following method of addition is useful when adding long columns:

Example.—Find the sum of 425, 36, 9,215, 4, and 907.

Solution.—Beginning with the bottom of the right-hand column, add as follows: 7 and 4 is 11 and 5 is 16 and 6 is 22 and 5 is 27. The first partial sum is 27 and is written as shown. The sum of the numbers in the second, or tens, column is 6 tens, or 60, which is the second partial sum. Write 60 underneath 27, as shown. The sum of the numbers in the third, or hundreds, column, is 1,500, which is the third partial sum. Write 1500 under the two preceding partial sums as shown. There is only one number in the fourth, or thousands, column, 9, which represents 9,000. Write 9000 as the fourth partial sum under the preceding sums. Adding the four partial sums, the sum is 10,587, which is the sum of 425, 36, 9,215, 4, and 907.

	425	
	36	
	9215	
	4	
	907	
first partial sum	27	
second partial sum	60	
third partial sum	1500	
fourth partial sum	9000	
sum	10587	Ans.

26. Mental Addition.—In adding a column containing, for instance, the figures 5, 6, 1, 9, 7, 5, 2, 4, 8, 9 it is permissible, before one becomes familiar with the process of addition, to proceed as follows: 5 and 6 is 11; 11 and 1 is 12; 12 and 9 is 21; and so forth, until all the figures in the column are included in the sum.

After one becomes more experienced the addition should be performed, mentally, in such a manner that only the successive sums are recorded in the mind, as the eye momentarily rests on one figure after another. Thus, the addition of the preceding figures would be: 5, 11, 12, 21, 28, 33, 35, 39, 47, 56.

This abbreviated method of adding may appear more difficult in the beginning, but it will pay to persevere until one becomes fully familiar with it. Adding aloud should be avoided under all circumstances.

EXAMPLES FOR PRACTICE

1. Find the sums of the following numbers:

(*a*) 104+203+613+214.	Ans.	(*a*) 1,134
(*b*) 1,875+3,143+5,826+10,832.		(*b*) 21,676
(*c*) 4,865+2,145+8,173+40,084.		(*c*) 55,267
(*d*) 14,204+8,173+1,065+10,042.		(*d*) 33,484

2. Four castings have the following weights: 3,265, 1,092, 748, and 2,587 pounds (abbreviated lb.). What is their combined weight? Ans. 7,692 lb.

3. The monthly output of a shop manufacturing a line of small tools was as follows: January, 8,502; February, 8,748; March, 9,215; April, 9,770; May, 10,269; June, 12,184. What was the total output in the six months? Ans. 58,688

4. The number of pounds of coal burned in a power plant each day during a week was as follows: Monday, 1,800; Tuesday, 1,655; Wednesday, 1,725; Thursday, 1,690; Friday, 1,648; and Saturday, 1,020. How much coal was burned during the week? Ans. 9,538 lb.

5. A piece of land has three sides, which are respectively 375 feet (abbreviated ft.), 980 feet, and 760 feet long. What is the length of the fence that will be needed to enclose it? Ans. 2,115 ft.

6. During the first week of the month a mill received supplies valued at 3,475 dollars; the supplies furnished during the second, third, and fourth weeks were worth 2,950, 4,380, and 4,895 dollars, respectively. What was the total value of the supplies received for the month? Ans. 15,700 dollars

7. One day's report of a small creamery operating four stations shows the weight, in pounds, of milk and cream received, as follows:

	Sta. 1	Sta. 2	Sta. 3	Sta. 4
Milk	2,833	2,718	3,054	2,967
Cream	1,376	1,271	1,515	1,334

How many pounds of milk and how many pounds of cream were received by the creamery on that day?

Ans. $\begin{cases}\text{11,572 lb. of milk}\\ \text{5,496 lb. of cream}\end{cases}$

8. On a three-day trip a ship covers 360 miles on the first day, 362 miles on the second day, and 359 miles on the third day; what was the total distance covered? Ans. 1,081 miles

9. A pump operated 2 hours and 45 minutes to empty a tank filled with water. The meter readings showed that 4,200 gallons (abbreviated gal.) were removed during the first hour, 5,420 during the second hour, and 3,600 during the last 45 minutes. How many gallons of water did the tank contain originally? Ans. 13,220 gal.

10. A heating, ventilating, and plumbing firm completes three contracts. The payment received on the first contract was 2,560 dollars, on the second 3,125 dollars, and on the third 2,850 dollars. How much was received in all? Ans. 8,535 dollars

HORIZONTAL ADDITION

27. Explanation of Method.—Very often examples are met with that require crosswise addition of numbers standing in different columns or on a line with one another, as well as the ordinary up-and-down addition of numbers arranged in vertical columns. Because this involves picking out from each number in succession the single figure desired, while retaining in mind the partial total already obtained, besides being a method that is unfamiliar, this kind of addition is likely to cause difficulty unless practice is given to it and the closest attention paid to accuracy.

To illustrate the process required, take the following numbers to be added as they stand, without arrangement in columns: $123+567+792+221+546=2{,}249$.

First take the right-hand figure of each number in turn. beginning at the right and ignoring all the other figures. Add as in ordinary addition, thus obtaining 7, 9, 16, 19. Place the 9 as the units figure in the sum and add the tens. Be very careful about picking out the tens figure of each number without losing sight of the partial total in mind. Thus, 1 (carried), 5, 7, 16, 22, 24. Put down 4 and carry 2. Pick out the hundreds and add them; 2 (carried), 7, 9, 16, 21, 22.

Care must be taken to keep in mind the kind of figure that is being added, whether tens, hundreds, etc., as it is easy to err and add the tens figure of one column, for instance, to the hundreds figure of the next, if care is not exercised. Practice will make it easier to choose the right figures.

The following example shows the usual grouping of numbers that require horizontal addition:

EXAMPLE.—Add the following numbers crosswise, then add the results:

			TOTALS
29,680	56,318	73,267	159,265
9,297	89,219	54,298	152,814
76,351	34,876	47,695	158,922
2,987	73,187	47,187	123,361
29,864	69,785	39,284	138,933
37,279	11,567	36,684	85,530
59,812	71,091	29,345	160,248
67,677	64,597	55,641	187,915
45,328	99,873	67,298	212,499
87,875	62,144	76,541	226,560
		Grand total,	1,606,047

28. Proof.—To prove results in horizontal addition, add each column vertically, then add the sums of the vertical columns. The result should be the same as the sum of the horizontal totals. In the example just given, the total of the first column is 446,150, that of the second 632,657, and that of the third 527,240. The sum of these three totals is 1,606,047, the same as that of the horizontal totals.

Since it is so easy to make mistakes in examples of this kind it is well worth while to verify results in every case; in fact, the student should accustom himself in his practice work to prove his results in this manner.

29. Practical Example.—The advantages derived from the ability to perform horizontal addition may be seen from the succeeding table, which is an example of a great number of similar ones required in government statistical work. The table is supposed to give, by days, the grain exports, in bushels, of a certain city for 1 week; it is desired to find the totals in both directions and the grand total. These totals represent the amount of grain exported each day, the total amount of each kind of grain exported during the week, and, finally, the total amount of grain exported during the week.

GRAIN EXPORT OF A CITY FOR 1 WEEK, IN BUSHELS

	Mon.	Tues.	Wed.	Thur.	Fri.	Sat.	Total
Corn	28,325	15,236	35,715	29,128	75,183	46,217	******
Wheat	35,719	41,719	50,108	32,546	59,275	81,126	******
Oats	12,136	9,237	18,265	7,268	6,950	17,230	******
Barley	18,230	15,738	21,375	15,928	19,263	13,637	******
Rye	5,275	6,829	7,201	11,325	7,825	13,261	******
Totals	*****	*****	*****	*****	*****	******	******

The student should find the totals, and prove that the results are correct by adding the totals in the right-hand column, and then adding the totals in the bottom row; the two results should be the same, or 757,270 bushels. The other results are: Corn, 229,804; wheat, 300,493; oats, 71,086; barley, 104,171; rye, 51,716; Mon., 99,685; Tues., 88,759; Wed., 132,664; Thurs., 96,195; Fri., 168,496; Sat., 171,471.

EXAMPLES FOR PRACTICE

1. Find the sum of each of the following columns, then add them crosswise and at last add the results of the horizontal addition:

(*a*)	(*b*)	(*c*)
4568	15431	7386
7391	29685	45371
7854	73648	13764
53469	34519	9887
13470	78234	64348
58143	7843	14627

Ans.: (*a*) 144,895; (*b*) 239,360; (*c*) 155,383; grand total, 539,638.

2. Add the following numbers crosswise, then add the results:

(*a*)	(*b*)	(*c*)	
49850	6542	62165	
17370	63834	16732	
68429	76343	85696	
23156	80931	71883	
21017	79883	50149	
67154	83578	31572	
64353	35647	76844	

Grand total, 1,133,128

SUBTRACTION

30. Definitions.—The process of **subtraction** is just the opposite of that of addition. Instead of combining two numbers to find their value taken together, to *subtract* is to take away a specified number of units from a given number and find out how many units remain. Thus, 9 and 7 taken together make 16. Now, if the operation is reversed to find how many units are left after taking 9 units from 16 units, the process is subtraction. As it is known that to add 9 to 7 gives 16, it is clear that to take the 9 away from 16 leaves 7.

In subtraction but two numbers can be used at a time, and the smaller number is taken from the larger in every case.

The number to be reduced is called the **minuend;** the one to be taken away, the **subtrahend;** the number left after the subtraction is performed, the **difference.** The sign of subtraction is −, read *minus.* 12−7 is read *twelve minus seven,* and means that 7 is to be taken from 12.

31. Method of Subtraction.—The manner in which subtraction is carried out is illustrated in the following solutions:

EXAMPLE.—From 7,568 take 3,425.

SOLUTION.—The larger number is written above the smaller number, and a line is drawn below them. The remainder is placed below this line, thus:

minuend 7568
subtrahend 3425
difference 4143 Ans.

EXPLANATION.—Begin at the right-hand, or units, column and subtract in succession each figure in the subtrahend from the one directly above it in the minuend, and write the remainders below the line. The result is the entire remainder.

32. When there are more figures in the minuend than in the subtrahend, and when some figures in the minuend are less than the figures directly under them in the subtrahend, proceed as in the following example:

EXAMPLE 1.—From 8,453 take 844.

SOLUTION.—

minuend 8453
subtrahend 844
remainder 7609 Ans.

EXPLANATION.—Begin at the right-hand, or units, column to subtract. As 4 cannot be taken from 3, it is necessary to take 1 from 5 in the tens column and prefix it to the 3 in the units column. The 1 taken from the tens column is equal to 10 units, which added to the 3 in the units column gives a sum of 10+3=13 units. 4 from 13=9, which is the first, or units, figure in the remainder.

As 1 was taken from 5, only 4 remains; then, 4 from 4=0, which is the second, or tens, figure. As 8 cannot be taken from 4, it is necessary to take 1 from 8 in the thousands column. Since 1 thousand=10 hundreds, it follows that 10 hundreds+4 hundreds=14 hundreds, and that 8 from 14 leaves 6 as the third, or hundreds, figure in the remainder.

As 1 was taken from 8, only 7 remains, from which there is nothing to subtract; therefore, 7 is the next figure in the remainder, thus completing the answer.

The operation of taking 1 away from one of the figures in the minuend is performed by mentally placing 1 before the figure following the one from which it is taken. For example, the 1 taken from 5 is placed before 3, making it 13, from which 4 is subtracted. The 1 taken from 8 is placed before 4, making 14, from which 8 is subtracted.

EXAMPLE 2.—Out of 306 pieces of work inspected, 14 were rejected as being too small. How many pieces were passed by the inspector?

SOLUTION.—The number of pieces passed by the inspector is the difference between 306 and 14. The subtraction is as follows:

306
14
292 Ans.

EXPLANATION.—The first step is to subtract 4 from 6, leaving 2 as a remainder, which is set down. Now, 1 cannot be taken from 0; so, 1 is taken from the 3 and brought over to add to the 0. But, the 1 taken from 3 has a value 10 times as great as it would have in the position of the cipher, according to Arts. **11** and **12**. Therefore, 10 is added to the cipher, making 10, and the 3 becomes decreased to 2. The minuend might then be written as in the following:

$$\begin{array}{r} 2\overset{10}{}6 \\ 14 \\ \hline 292 \end{array}$$

The remainder of the solution is easy; 1 from 10 leaves 9, and nothing from 2 leaves 2. Hence, 292 pieces were passed by the inspector. It is not customary to rewrite the minuend as here shown; instead, the taking of 1 from the 3 is done mentally.

EXAMPLE 3.—From a stock of 20,000 small machine parts 8,763 were used. How many remained?

SOLUTION.—To find how many remained, subtract the number used from the total stock, thus:

$$\begin{array}{r} 20000 \\ 8763 \\ \hline 11237 \end{array}\ \text{Ans.}$$

EXPLANATION.—As 3 cannot be taken from 0 in the units column, the attempt is made to borrow 1 from the tens column, so as to obtain 10 units. But there are no tens, and on going, successively, to the hundreds column and to the thousands column, a cipher is found in each case. It is not until the ten-thousands place is reached that a number greater than zero is found—2 in this case. It is now necessary to take 1 ten-thousand, or 10 thousands, from the 2 and add it to the 0 thousands, giving 10+0=10 thousands; but it is necessary to continue the borrowing and to take 1 thousand, or 10 hundred, away from the 10 thousands and add it to the hundreds column, leaving a remainder of 10−1=9 thousands. Adding 10 hundreds to 0 hundreds gives a sum of 10 hundreds. But 1 hundred, or 10 tens, must be taken away from the hundreds to add to 0 tens, leaving 10−1=9 hundreds.

Adding 1 hundred, or 10 tens, to 0 tens, the sum is 10 tens. Finally, 1 ten, or 10 units, must be taken from the 10 tens to be added to the 0 unit, giving 10 units as the sum. The minuend might now be written as in the following:

$$\begin{array}{r} 1999\overset{10}{} \\ 8763 \\ \hline 11237 \end{array}$$

Then, 3 from 10 leaves 7; 6 from 9 leaves 3; 7 from 9 leaves 2; 8 from 9 leaves 1; and nothing from 1 leaves 1. Usually the taking of

1 unit from the several places is done mentally, and is not written down as shown.

33. Rule for Subtraction.—Using the preceding explanations as a basis the following rule may be formed:

Rule.—*When subtracting, write the subtrahend under the minuend with units under units, tens under tens, etc., and draw a horizontal line beneath the subtrahend.*

Subtract units of the subtrahend from units of the minuend, tens from tens, etc., writing the remainders beneath the line in the order in which they are obtained.

If any figure in the minuend represents a smaller number than the figure beneath, borrow 1 from the preceding number and add 10 to the smaller number; then subtract the lower number from the sum.

34. Proof.—*To prove an example in subtraction, add the remainder to the subtrahend. The sum should equal the minuend. If it does not, a mistake has been made, and the work should be done over.*

Proof of example 3, Art. **32.**

$$\begin{array}{r} 8763 \\ 11237 \\ \hline 20000 \end{array}$$

35. General Remarks on Subtraction.—The student should practice finding the differences between small numbers, until he is able to name with ease and rapidity the difference between any two numbers less than ten. A special subtraction table is not required for this purpose, as the addition table can be referred to in case of necessity. For instance, according to the table the sum of 7 and 6 is 13. It follows, that if either one of the numbers 7 and 6 is subtracted from 13, the other number must be equal to the remainder. Thus, on subtracting 6 from 13, the remainder is 7. If 7 is subtracted from 13, then 6 is the remainder. In the same manner, it is found that 8 from 16 leaves 8, 9 from 13 leaves 4.

It will be a great help to the student if he at odd moments, while walking or working, will mentally reverse the addition table and use it as a subtraction table. For instance, instead of saying to himself, 6 and 5 is 11, let him ask, 6 from 11 is how much? 9 from 17 is what? etc., and in a short time the right answer will present itself without any mental effort whatever.

EXAMPLES FOR PRACTICE

1. From:

(*a*) 94,278 take 62,574.
(*b*) 53,714 take 25,824.
(*c*) 71,832 take 58,109.
(*d*) 20,804 take 10,408.

Ans. (*a*) 31,704
(*b*) 27,890
(*c*) 13,723
(*d*) 10,396

2. A shop employing 3,214 hands was forced to lay off 736 of them. What number was left? Ans. 2,478

3. A casting weighing 2,785 pounds was poured from a ladle containing 4,210 pounds of metal. How many pounds remained in the ladle to be used for other castings? Ans. 1,425 lb.

4. If an electric meter registers 7,968 watt-hours at one reading, and 10,430 at the next reading, how many watt-hours were used by the customer between the two readings? Ans. 2,462 watt-hours

NOTE.—A watt-hour is the unit by which electric power is measured.

5. On January 1 a power plant had 19,860 tons of coal on hand. During the entire month 3,100 tons were consumed; how much coal was on hand February 1? Ans. 16,760 tons

6. The total weight of a mine car loaded with coal is 4,326 pounds. If the empty car weighs 1,564 pounds, what is the weight of the coal? Ans. 2,762 lb.

7. In setting out the boundary line of a piece of land, it is found that at one point of the line the distance from the starting point is 385 feet. At another point of the same line the distance from the starting point is 1,065 feet. What is the distance between the two points? Ans. 680 ft.

8. On completing a piece of work it was found that out of a stock of 1,037 pounds of pipe fittings there remained a quantity of 259 pounds. How many pounds were used? Ans. 778 lb.

9. During a certain month, a textile mill purchased wool valued at 5,642 dollars and cotton costing 3,834 dollars. How much greater was the expenditure for wool than for cotton? Ans. 1,808 dollars

10. The distance between New York City and Liverpool, England, is 3,166 nautical miles. If a ship sailing for England has covered 1,275 miles of this distance, how far is it from Liverpool?

Ans. 1,891 miles

NOTE.—A nautical mile differs from a statute mile. The former contains 6,080 feet, the latter only 5,280 feet.

MULTIPLICATION

36. Definitions.—Multiplication is merely a shortened process of addition, used when the sum of a group of equal numbers is to be found. To **multiply** a number is to *add* it to itself a certain number of times. For instance, suppose the sum is required that results from using the number 4 three times in the process of addition. By the method of addition it is easy to see that $4+4+4=12$. But, suppose it is desired to take the number 13,976 ninety-nine times. It would be a long, tedious process to find the desired result by addition, so the need of a shorter method is imperative. Multiplication meets this need. By its means it is possible to find the result of taking any number as many times as desired.

37. The number that is to be added to itself, or the number to be multiplied, is called the **multiplicand.**

The number that shows how many times the *multiplicand* is to be taken, or the number by which we *multiply,* is called the **multiplier.**

The sign of multiplication is ×. It is read *times* or *multiplied by.* Thus 9×6 is read *9 times 6,* or *9 multiplied by 6.*

38. The result obtained by multiplying is called the **product.** Thus, $40\times8=320$ is read *40 multiplied by 8 equals 320,* or *40 times 8 equals 320.* In this example, 40 is the multiplicand, 8 the multiplier, and 320 the product.

The product of several numbers is the same, whatever may be the order in which they are multiplied; thus, $4\times3\times2=24$, $4\times2\times3=24$, and $3\times4\times2=24$, etc.

The two or more numbers that, when multiplied together produce a product, are said to be the **factors** of that product.

Thus, 2, 3, and 4 are factors of 24; 5, 6, and 3 are factors of 90; and so on.

39. Practicing Multiplication.—In Table II, the product of any two numbers (neither of which exceeds 12) may be found. The table should be carefully committed to memory. This is especially important, as successful future work depends largely on the student's ability to name promptly these products in any order. Study one section at a time until the product of any pair of numbers can be given at once; then go to the next section. Review the sections repeatedly.

Any number multiplied by 1 gives a product equal to the number itself, as indicated in the first line of each section of the table.

40. Rule for Multiplication.—To multiply numbers larger than those given in Table II proceed according to the following rule and examples:

Rule.—I. *When multiplying, write the multiplier under the multiplicand, so that units are under units, tens under tens, etc.*

II. *Begin at the right, and multiply the multiplicand by the figures of the multiplier, taken in succession, placing the right-hand figure of each partial product directly under the figure used as a multiplier.*

III. *The sum of the partial products will be the required product.*

41. To Multiply by a Number of One Figure Only. The application of the preceding rule to multiplication by one figure will be shown by the following example and solution:

Example.—Multiply 425 by 5.

Solution.—According to part I of the rule, the multiplier 5 is written under the figure 5 of the multiplicand. On looking in the multiplication table, it is seen that 5×5=25, of which the figure 5 is written in the product directly under 5 in the multiplier and 2 is *carried*, or added to the product of 5×2; thus, 5×2+2=12, of which the figure 2 is written in the product and 1 is carried, that is,

multiplicand	425	
multiplier	5	
product	2125	Ans.

added to the product of 5×4. Thus, 5×4=20 and 20+1=21, both figures being written in the product because all the figures of the multiplicand have been used. The work is now complete and the product is 2,125.

This result can also be obtained by adding 425 five times; thus,

	425	
	425	
	425	
	425	
	425	
sum	2125	Ans.

42. To Multiply by a Number of Two or More Figures.—The following example and solution shows the application of the rule, Art. **40,** to cases in which the multiplier has two or more figures:

Example.—Multiply 475 by 234.

Solution.—Proceeding according to part II of the rule, 4×5=20, of which 0 is written in the first partial product and 2 is carried; 4×7=28 and 28+2=30, of which 0 is written as the next figure of the first partial product and 3 is carried; 4×4=16 and 16+3=19, which completes the first partial product, 1,900. Using the next figure of the multiplier, 3×5=15, of which 5 is written directly under 3 as the first figure of the second partial product and 1 is carried; 3×7=21 and 21+1=22, of which 2 is written in the second partial product and 2 is carried; 3×4=12 and 12+2=14, which completes the second partial product, 1,425. Using the last figure, 2, of the multiplier, 2×5=10, of which 0 is written directly under 2 as the first figure of the third partial product and 1 is carried; 2×7=14 and 14+1=15, of which 5 is written as the second figure of the third partial product and 1 is carried, or added, to the product of 2×4, or 8, making 9, which figure completes the third partial product, 950. According to part III of the rule, the sum of the three partial products gives the total product, 111,150.

multiplicand	475	
multiplier	234	
first partial product	1900	
second partial product	1425	
third partial product	950	
product	111150	Ans.

43. Proof.—*To prove the correctness of multiplication, review the work carefully, or multiply the multiplier by the multiplicand. If the same product is obtained, the work is probably correct.*

TABLE II

MULTIPLICATION TABLE

1 times 1 is 1	2 times 1 is 2	3 times 1 is 3	4 times 1 is 4
1 times 2 is 2	2 times 2 is 4	3 times 2 is 6	4 times 2 is 8
1 times 3 is 3	2 times 3 is 6	3 times 3 is 9	4 times 3 is 12
1 times 4 is 4	2 times 4 is 8	3 times 4 is 12	4 times 4 is 16
1 times 5 is 5	2 times 5 is 10	3 times 5 is 15	4 times 5 is 20
1 times 6 is 6	2 times 6 is 12	3 times 6 is 18	4 times 6 is 24
1 times 7 is 7	2 times 7 is 14	3 times 7 is 21	4 times 7 is 28
1 times 8 is 8	2 times 8 is 16	3 times 8 is 24	4 times 8 is 32
1 times 9 is 9	2 times 9 is 18	3 times 9 is 27	4 times 9 is 36
1 times 10 is 10	2 times 10 is 20	3 times 10 is 30	4 times 10 is 40
1 times 11 is 11	2 times 11 is 22	3 times 11 is 33	4 times 11 is 44
1 times 12 is 12	2 times 12 is 24	3 times 12 is 36	4 times 12 is 48
5 times 1 is 5	6 times 1 is 6	7 times 1 is 7	8 times 1 is 8
5 times 2 is 10	6 times 2 is 12	7 times 2 is 14	8 times 2 is 16
5 times 3 is 15	6 times 3 is 18	7 times 3 is 21	8 times 3 is 24
5 times 4 is 20	6 times 4 is 24	7 times 4 is 28	8 times 4 is 32
5 times 5 is 25	6 times 5 is 30	7 times 5 is 35	8 times 5 is 40
5 times 6 is 30	6 times 6 is 36	7 times 6 is 42	8 times 6 is 48
5 times 7 is 35	6 times 7 is 42	7 times 7 is 49	8 times 7 is 56
5 times 8 is 40	6 times 8 is 48	7 times 8 is 56	8 times 8 is 64
5 times 9 is 45	6 times 9 is 54	7 times 9 is 63	8 times 9 is 72
5 times 10 is 50	6 times 10 is 60	7 times 10 is 70	8 times 10 is 80
5 times 11 is 55	6 times 11 is 66	7 times 11 is 77	8 times 11 is 88
5 times 12 is 60	6 times 12 is 72	7 times 12 is 84	8 times 12 is 96
9 times 1 is 9	10 times 1 is 10	11 times 1 is 11	12 times 1 is 12
9 times 2 is 18	10 times 2 is 20	11 times 2 is 22	12 times 2 is 24
9 times 3 is 27	10 times 3 is 30	11 times 3 is 33	12 times 3 is 36
9 times 4 is 36	10 times 4 is 40	11 times 4 is 44	12 times 4 is 48
9 times 5 is 45	10 times 5 is 50	11 times 5 is 55	12 times 5 is 60
9 times 6 is 54	10 times 6 is 60	11 times 6 is 66	12 times 6 is 72
9 times 7 is 63	10 times 7 is 70	11 times 7 is 77	12 times 7 is 84
9 times 8 is 72	10 times 8 is 80	11 times 8 is 88	12 times 8 is 96
9 times 9 is 81	10 times 9 is 90	11 times 9 is 99	12 times 9 is 108
9 times 10 is 90	10 times 10 is 100	11 times 10 is 110	12 times 10 is 120
9 times 11 is 99	10 times 11 is 110	11 times 11 is 121	12 times 11 is 132
9 times 12 is 108	10 times 12 is 120	11 times 12 is 132	12 times 12 is 144

44. Ciphers as Multipliers.—When there is a cipher in the multiplier, multiply by it the same as with the other figures. Thus,

$$\begin{array}{r} (a) \\ 0 \\ \times 0 \\ \hline 0 \end{array}\text{ Ans.}\qquad \begin{array}{r} (b) \\ 2 \\ \times 0 \\ \hline 0 \end{array}\text{ Ans.}\qquad \begin{array}{r} (c) \\ 15 \\ \times\ 0 \\ \hline 00 \end{array}\text{ Ans.}\qquad \begin{array}{r} (d) \\ 708 \\ \times\quad 0 \\ \hline 000 \end{array}\text{ Ans.}$$

$$\begin{array}{r} (e) \\ 3114 \\ 203 \\ \hline 9342 \\ 0000 \\ 6228 \\ \hline 632142 \end{array}\text{ Ans.}\qquad \begin{array}{r} (f) \\ 4008 \\ 305 \\ \hline 20040 \\ 0000 \\ 12024 \\ \hline 1222440 \end{array}\text{ Ans.}\qquad \begin{array}{r} (g) \\ 31264 \\ 1002 \\ \hline 62528 \\ 00000 \\ 00000 \\ 31264 \\ \hline 31326528 \end{array}\text{ Ans.}$$

45. When multiplying by a number containing one or more ciphers, the work may be shortened by simply writing the first cipher of a partial product in its proper place and the next partial product alongside it. For example, the solutions to examples (*e*) and (*g*) in the preceding article may be written as follows:

$$\begin{array}{r} 3114 \\ 203 \\ \hline 9342 \\ 62280 \\ \hline 632142 \end{array}\text{ Ans.}\qquad \begin{array}{r} 31264 \\ 1002 \\ \hline 62528 \\ 3126400 \\ \hline 31326528 \end{array}\text{ Ans.}$$

46. If there are ciphers at the right-hand end of the multiplier, they need not be used in the multiplication, but may be carried down to the product.

EXAMPLE.—Multiply 2,675 by 3,900.

SOLUTION.—

$$\begin{array}{r} 2675 \\ 3900 \\ \hline 24075 \\ 8025 \\ \hline 10432500 \end{array}\text{ Ans.}$$

EXPLANATION.—In a case like this, the multiplier is written so that the ciphers at its right extend to the right of the units figure in the multiplicand. These two ciphers are brought down vertically to the end of the product of 2,675 and 39.

47. Ciphers in the Multiplicand.—If there are ciphers at the end of the multiplicand, the procedure is similar in all respects to that explained in Art. **46;** thus, to multiply 4,907,600 by 487 proceed as follows:

```
   4907600
     487
----------
  343532
 392608
196304
----------
2390001200  Ans.
```

If both multiplicand and multiplier end in ciphers, place the right-hand digits, or figures that are not ciphers, under each other, as above, and add to the product a number of ciphers equal to the sum of the ciphers contained in the multiplicand and the multiplier on the right of their right-hand digits.

Example.—Multiply 590,000 by 420.

Solution.—

```
  590000
  420
---------
 118
236
---------
247800000  Ans.
```

Explanation.—In this case there are, in all, 5 ciphers on the right of the right-hand digits in the multiplicand and the multiplier. Therefore, there must be 5 ciphers on the right of the right-hand digit in the product.

EXAMPLES FOR PRACTICE

1. Find the product of the following:

	Ans.
(*a*) 61,483×6.	(*a*) 368,898
(*b*) 12,375×5.	(*b*) 61,875
(*c*) 4,836×47.	(*c*) 227,292
(*d*) 3,257×246.	(*d*) 801,222
(*e*) 2,875×302.	(*e*) 868,250
(*f*) 17,819×1,004.	(*f*) 17,890,276

2. A certain machine is capable of turning out 48 finished pieces of work in a day. At this rate, find the output of the machine in one year of 296 working days. Ans. 14,208

3. What is the total weight of cast iron in 649 pumps, if the amount of iron in each pump weighs 37 pounds? Ans. 24,013 pounds

4. A ranchman owns 945 head of cattle, valued at 117 dollars a head; how much are they all worth? Ans. 110,565 dollars

5. A certain mill contains 1,830 looms, each of which produces 175 yards (abbreviated yd.) of cloth per week; what is the total production? Ans. 320,250 yd.

6. A machine turns out 50 finished pieces of work in a day. What would be the output of the machine in one year of 306 working days? Ans. 15,300 pieces

7. In a month of 30 days, how many miles can be run by a ship that averages 432 miles per day? Ans. 12,960 miles

8. A water-tube boiler containing 144 tubes needs to be fitted with new tubes. Assuming the cost for each tube to be 9 dollars, what will be the total cost for the 144 tubes? Ans. 1,296 dollars

NOTE.—A *water-tube* boiler is one in which the water is *in* the tubes, which are surrounded by the hot gases of combustion. In a *fire-tube* boiler the water *surrounds* the tubes, and the hot gases pass through the interior of the latter.

9. If 40,635,000 acres of wheat produce, on an average, 13 bushels to the acre, what is the total yield? Ans. 528,255,000 bushels

10. A power station has on hand 38 barrels of lubricating oil; the average contents of a barrel is 50 gallons. What is the total quantity on hand? Ans. 1,900 gallons

DIVISION

48. Definitions.—Division is the process of finding how many times one number is contained in another; or, it is the process of separating a number into a given number of equal parts.

For instance, a brass rod, 32 inches long, is to be divided into short pieces, 2 inches in length. This is an example of finding how many times one number, in this case 2, is contained in another number, here 32. Or, suppose that 150 dollars is to be divided equally among 5 men; how much will each man receive? Here it is a question of separating a number, as 150, into a number of equal parts, corresponding to the number of men.

49. The **dividend** is the number to be divided, or to be separated into equal parts.

The **divisor** is the number by which the dividend is divided.

The **quotient** is the number showing how many times the dividend contains the divisor.

The **sign of division** is ÷. It is read *divided by.* Thus, 54÷9 denotes that 54 is to be divided by 9. Another way to write 54 divided by 9 is $\frac{54}{9}$. Thus, 54÷9=6, or $\frac{54}{9}=6$.

In both of these cases 54 is the *dividend,* 9 is the *divisor,* and 6 is the *quotient.*

Division is the reverse of multiplication, as the latter process, instead of *separating* a number into a number of equal parts, or factors, *combines* a number of equal parts into a complete whole.

NOTE.—The student should study Table II again until when any product given therein and one of its factors are named he can immediately name the other factor. A good plan is to cover the second column of factors with a strip of paper or cardboard and recall them from memory. In carrying out these tests the dividends should not be taken in order, but selected at random; otherwise the answer will be known without testing one's ability in division.

50. Rule for Division.—To divide numbers larger than those given in Table II, proceed according to the following rule and examples:

Rule.—I. *Write the divisor at the left of the dividend, with a curved line between them.*

II. *Begin at the left-hand end of the dividend and find how many times the divisor is contained in the least number of figures that will contain it, and write the result for the first figure of the quotient, separating it from the dividend by a curved line.*

III. *Multiply the divisor by this quotient; subtract the product from the partial dividend used, and to the remainder annex the next figure of the dividend. Divide as before, and thus continue until all the figures of the dividend have been used.*

IV. *If any partial dividend will not contain the divisor, write a cipher in the quotient, annex the next figure of the dividend, and proceed as before.*

V. *If there is a remainder after the last figure of the dividend has been used, draw a straight line after the quotient and write this remainder above the line and the divisor below.*

51. To Divide When the Divisor Consists of But One Figure.—Proceed as in the following example:

EXAMPLE.—What is the quotient of 861÷7?

SOLUTION.—The dividend and the divisor are written according to part I of the rule in Art. **50**, and a place is provided for the quotient at the right of the dividend. On application of part II of the rule, it is found that of the figures at the left-hand end of the dividend, the one figure 8 is sufficient to contain the divisor 7. As 7 is contained 1 time in the figure 8, the figure 1 is written as the first figure of the quotient. According to part III of the rule, 1×7=7, which product is written under 8 and subtracted, leaving 1, to which is annexed the next figure, 6, of the dividend, giving the new partial dividend 16. In this dividend 7 is contained 2 times, and 2 is written after 1 in the quotient. The product 2×7=14 is written under 16 and subtracted, leaving 2, to which is annexed the next figure, 1, of the dividend, giving the second partial dividend 21. In this dividend 7 is contained 3 times, and 3 is written after 2 in the quotient. 3×7=21, which is written under 21 and subtracted, leaving 0. All the figures of the dividend have now been used and the division is complete, the quotient being 123.

$$\begin{array}{rrll}
 & \textit{divisor} & \textit{dividend} & \textit{quotient}\\
 & 7) & 861 & (123\quad\text{Ans.}\\
 & & \underline{7} & \\
\textit{partial dividend} & & 16 & \\
 & & \underline{14} & \\
\textit{partial dividend} & & 21 & \\
 & & \underline{21} & \\
 & & 0 &
\end{array}$$

52. To Divide When the Divisor Consists of Two or More Figures.—The rule in Art. **50** applies also in this case, its application being illustrated by means of the following example:

EXAMPLE.—Divide 2,702,839 by 63.

SOLUTION.—Applying part II of the rule, Art. **50**, it is necessary to find the least number of figures of the dividend that will contain the divisor. As 63 is not contained in 2 nor in 27, the first three figures of the dividend, that is, 270, must be used. To determine how many times 63 is contained in 270, it is noted that 6 is contained in 27 4 times with a remainder, and 4 is tried as the first figure of the quotient. As the product 4×63=252 is found to be smaller than 270, it is retained and subtracted from the latter, giving the remainder 18. Annexing the next figure, 2, of the dividend to this remainder, the first partial dividend 182 is obtained. It is found that 63 is contained in 182 2 times with a remainder; so 2 is written as the second figure of the quotient. At first thought, it might seem that this figure should be 3, because 18÷6=3; but 3×63=189, a number larger than 182; so 3 is too large for the second figure of the quotient. 2×63=126, which is written under 182 and subtracted. To the remainder, 56, the next figure, 8, of the dividend is annexed, giving 568 as a new partial dividend. As 6 is contained in 56 9 times with a remainder, 63 may be assumed to be contained 9 times in 568, and 9 is accordingly written as the third figure of the quotient. 9×63=567, which is written under 568 and subtracted. To the remainder, 1, the next figure, 3, of the dividend is annexed, giving 13, but 63 is not contained in 13. According to part IV of the rule, Art. **50,** 0 is written in the quotient, and the next figure, 9, of the dividend is brought down, making 139 as a new partial dividend. 139÷63=2, with 13 remaining. Since this is the last remainder, all figures of the dividend having been used, 13 is written above a straight line after the quotient with the divisor underneath, according to part V of the rule.

$$
\begin{array}{rl}
 & \textit{divisor}\ \ \textit{dividend}\qquad\textit{quotient}\\
 & 63)2702839(42902\tfrac{13}{63}\quad\text{Ans.}\\
 & \underline{252}\\
\textit{partial dividend} & 182\\
 & \underline{126}\\
\textit{partial dividend} & 568\\
 & \underline{567}\\
\textit{partial dividend} & 139\\
 & \underline{126}\\
\textit{remainder} & 13
\end{array}
$$

53. Long Division and Short Division.—When the process of division is carried out in full, as in the preceding examples, it is called **long division;** when shortened, or abridged, by performing some of the work mentally instead of writing it, the process is called **short division.** Short division is preferable when the divisor is not greater than 12, but should not be attempted before one is fully familiar with long division.

In short division, only the divisor, the dividend, and the quotient are written, the operations being performed mentally. The method followed will be explained by the succeeding example.

Example.—Divide 992 by 8.

Solution.—The divisor is placed to the left of the dividend, as in long division, but the quotient is placed under the dividend, and separated from it by a line, as shown. The mental operation is performed as follows: The divisor 8 is contained in 9 1 time with 1 remaining. This remainder is conceived to be written immediately before the next figure, 9, of the dividend, as shown by 1 in small type, and 8 is contained in 19 2 times with 3 remaining. This remainder is conceived to be written before the next figure, 2, of the dividend, and 8 is contained in 32 4 times with no remainder. This completes the division, giving the quotient 124.

$$\begin{array}{rl} \textit{divisor} & \textit{dividend} \\ 8\,) & 9\,{}^{1}9\,{}^{3}2 \\ \cline{2-2} \textit{quotient} & 124 \quad \text{Ans.} \end{array}$$

54. Proof.—*To prove division, multiply the quotient by the divisor, and add the remainder, if there is any, to the product. The result will be the dividend.* Thus, the correctness of the work in the example, Art. **52**, can be proved, as shown, by multiplying 42,902 by 63 and adding 13. The result is equal to the dividend.

$$\begin{array}{rr} \textit{quotient} & 42902 \\ \textit{divisor} & 63 \\ \cline{2-2} & 128706 \\ & 257412 \\ \cline{2-2} & 2702826 \\ \textit{remainder} & 13 \\ \cline{2-2} \textit{dividend} & 2702839 \end{array}$$

EXAMPLES FOR PRACTICE

1. Divide the following:

(*a*) 126,498 by 58.
(*b*) 3,207,594 by 767.
(*c*) 11,408,202 by 234.
(*d*) 2,100,315 by 581.

Ans. (*a*) 2,181; (*b*) 4,182; (*c*) 48,753; (*d*) 3,615

2. A lot of castings weigh 11,060 pounds. If they are alike, and one weighs 28 pounds, how many are there in the lot? Ans. 395

3. If the driving shaft of a machine makes 9,730 turns in 35 minutes, how often does it turn in 1 minute? Ans. 278 times

4. If sound, under certain conditions, travels at the rate of 1,118 feet per second (abbreviated sec.), what time will be required for it to travel 23,478 feet? Ans. 21 sec.

5. A steamship company pays 6,120 dollars monthly wages to 85 firemen, each receiving the same amount. How much does each fireman receive per month? Ans. 72 dollars

6. There are 5,280 feet in a mile. How many rails would it take to lay a double row of rails, each row 1 mile long, if the length of a rail is 30 feet? Ans. 352 rails

7. How many electric lamps at 30 cents each can be purchased for 90 cents? Ans. 3 lamps

8. If 16 carloads of coal weigh 46,336 pounds, what is the weight per carload, assuming each car to contain the same quantity? Ans. 2,896 lb.

9. A mill purchased cotton to the value of 12,600 dollars, at 42 dollars per bale; how many bales were purchased? Ans. 300 bales

10. If 30 water tanks of equal size contain 25,500 gallons, when all are filled, how many gallons does each tank contain? Ans. 850 gal.

COMBINATIONS OF ARITHMETICAL OPERATIONS

55. Order of Operations.—The various signs used in arithmetical operations, such as +, −, ×, and ÷, are known as **symbols.** Several numbers may be connected by various symbols, so as to constitute a row, or a series, as, for instance, $4+8-3\times2+10\div5$. Such a combination of numbers and signs is called an **expression.** This expression indicates that the product obtained by multiplying 3 by 2 is to be subtracted from the sum found by adding 8 to 4. To the resulting difference is to be added the quotient obtained by dividing 10 by 5.

To insure correct results it is necessary that the operations indicated by the symbols in this and similar expressions should be performed in the following order:

1. Multiplication.
2. Division.
3. Addition or subtraction.

Applying this method to the preceding series, the process of multiplication is performed first; then, division. Thus, $3\times2=6$, and $10\div5=2$. The series may now be written $4+8$

−6+2. Performing the operations indicated by the symbols from left to right the final result is 8.

56. Symbol of Aggregation.—The ordinary parenthesis, ( ), is used also in arithmetical operations. It is then called a **symbol of aggregation,** the word aggregation meaning combination.

A parenthesis indicates that the numbers enclosed within it are to be considered as a whole with respect to any preceding or following arithmetical symbol. It is important to note that the operations indicated by the sign or signs within the parenthesis should be performed first. Thus, in the expression 13×(8−3) the numbers contained within the parenthesis are considered as a whole, and the process of subtraction, indicated by the symbol, is to be performed before multiplication takes place. That is, 3 is to be taken from 8 before multiplying by 13; thus, 13×(8−3)=13×5=65.

When the parenthesis is not used, then the multiplication is to be performed before the subtraction, as explained in Art. **55.** Thus, 13×8−3=104−3=101.

Again, 2×(8−3) means 2×5=10. On removing the parenthesis, thus, 2×8−3, multiplication precedes subtraction. Hence, 2×8=16, and 16−3=13.

57. Examples.—The application of the preceding rules will be further illustrated by the following examples:

Example 1.—Find the value of the expression 4×24−8+17.

Solution.—Performing the operations indicated according to Art. **55,** 4×24=96; 96−8=88; 88+17=105. Ans.

Example 2.—Find the value of the expression 1,296÷12+160−22×3.

Solution.—Multiplication and division must be performed before subtraction can take place. Thus, 22×3=66; 1,296÷12=108. The expression may now be written 108+160−66. Performing the remaining operations from left to right: 108+160=268; 268−66=202. Ans.

Example 3.—Solve the following expression: (26−4)×(16+4).

Solution.—First perform the operations indicated within the parenthesis. Thus, (26−4)=22, and (16+4)=20; then, 22×20=440. Ans.

Example 4.—Solve the following expression: (26−4)×16+4.

Solution.—First remove the parenthesis by performing the operation indicated within; thus, (26−4)=22. Next remove the multiplication sign by performing the required operation; thus, $22\times16=352$. Finally, $352+4=356$. Ans.

NOTE.—Comparison should be made between the answers obtained in the two preceding examples. The only difference in the examples is the omission of a parenthesis in example 4; yet, there is a difference in the results of $440-356=84$.

EXAMPLE 5.—Solve the following expression: $5\times4-21\div7-(15-5+4)$.

SOLUTION.—The parenthesis is removed first by carrying out the operation indicated by the signs within it. Thus, subtracting 5 from 15 leaves 10, to which is added 4, thus obtaining 14. The same result may be obtained by adding 4 to 15, thus obtaining 19, and subtracting 5. Next, the signs of multiplication and division must be removed by performing the required operations. Thus, $5\times4=20$; $21\div7=3$. The expression now has the following form: $20-3-14$. Performing the two subtractions, $20-3=17$, and $17-14=3$. Ans.

EXAMPLES FOR PRACTICE

Find the values of the following expressions:

		Ans.
(*a*)	$(8+5-1)\div4$.	(*a*) 3
(*b*)	$5\times24-32$.	(*b*) 88
(*c*)	$5\times24\div15$.	(*c*) 8
(*d*)	$144-5\times24$.	(*d*) 24
(*e*)	$2{,}080+120-80\times4-1{,}670$.	(*e*) 210
(*f*)	$(90+60)\div(2\times5)$.	(*f*) 15

SIMPLE ARITHMETICAL EQUATIONS

58. Definitions.—An **equation** is an expression in which two equal *quantities* are connected by an equality sign $(=)$. The term **quantity** in this case refers to a single number or to several numbers connected by arithmetical symbols. For example, the number 8 is a quantity, and so is $5+3$. These two quantities are equal, and if they are connected by an equality sign, as $5+3=8$, an equation is obtained. Examples of other equations are: $6+2+1=9$; $4+5+7=10+6$. They are all called **simple equations.**

59. Solution of Equations.—In an equation there is usually a missing number or quantity the value of which has to be found. The position of the missing quantity may be indicated by a question mark (?) or by a letter; for this purpose the letter x is generally used. Following are some examples of simple equations and the methods followed in finding the missing quantity:

EXAMPLE 1.—Find the value of the missing number belonging on the right side of the equality sign in the equation $5+2=?$

SOLUTION.—The sum of $5+2$ is equal to 7; therefore, 7 is the value of the missing number.

EXAMPLE 2.—What is the value of x in the equation $9+x=15$.

SOLUTION.—It is required to find a number, which added to 9 gives a sum equal to 15. If 9 and the unknown number make a sum of 15, it is evident that by subtracting 9 from 15 the difference will give the number required. Thus, $15-9=6$ and $x=6$. Ans.

EXAMPLE 3.—Find the value of x in the equation $5\times x=40$.

SOLUTION.—It is required to find a number, that multiplied by 5 gives a product equal to 40. If the latter product is obtained by multiplying a number by 5, it is evident that the quotient found by dividing 40 by 5 must be the number required. Thus, $40\div5=8$, and $5\times8=40$; therefore, $x=8$. Ans.

EXAMPLE 4.—Find the value of x in the equation $x\div5=6$.

SOLUTION.—It is required to find a number which, when divided by 5, will give the quotient 6. It follows that the product of 5 and 6, which is 30, must be the value of x. Thus, $5\times6=30$, and $30\div5=6$. The value of x is 30. Ans.

60. Incorrect Use of the Equality Sign.—In showing the several steps of a calculation, the equality sign is sometimes used in an incorrect manner, because it is made to connect expressions that are not of equal value. For instance, it is incorrect to use an equality sign as in the following example:

$$200\times4=800\div25=32$$

The expression is apparently an equation, but on closer examination it is found that this is not the case. The product $200\times4=800$, found on one side of the equality sign, is not equal to the quotient 32, obtained by dividing 800 by 25. Hence, the equality sign is out of place and is misleading. The two operations indicated should be written separately, as, for example:

$$200\times4=800$$
$$800\div25=32$$

EXAMPLES FOR PRACTICE

Find the value of x in the following equations:

	Ans.		Ans.
(*a*) $4\times x=12$	(*a*) 3	(*e*) $x-5=12$	(*e*) 17
(*b*) $x+12=16$	(*b*) 4	(*f*) $16\div x=4$	(*f*) 4
(*c*) $7+x=14$	(*c*) 7	(*g*) $36\div x=3$	(*g*) 12
(*d*) $9-x=7$	(*d*) 2	(*h*) $x\div 8=5$	(*h*) 40

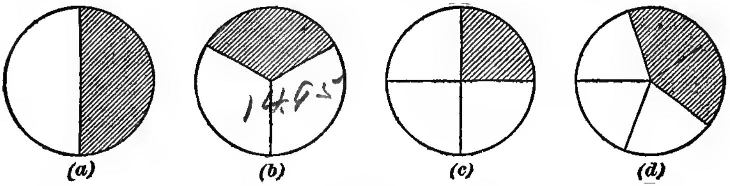
FRACTIONS

FUNDAMENTAL PROCESSES

REDUCTION OF FRACTIONS

INTRODUCTION

1. Fractions in General.—A **fraction** is one or more of the equal parts into which a whole thing, or unit, is divided. The unit may be anything, as a circle, a dollar, a mile, a plot of land, etc. In Fig. 1 (*a*) is shown a circle divided into two equal parts by a straight line, with one of the parts shaded. Each of these parts is a half of the whole circle; that is, each is a fraction of a circle. When the circle is divided into two equal parts, as shown, each part, or fraction, is *one-half*, which is written $\frac{1}{2}$.



14.95

Fig. 1

Another circle is shown in (*b*), divided into three equal parts. Each of these equal parts is a third, or *one-third*, of the entire circle. This fraction is written $\frac{1}{3}$. The circle in (*c*) is divided into four equal parts, each of which is a fourth, or *one-fourth*, of the whole circle. The fraction *one-fourth* is written $\frac{1}{4}$. The circle in (*d*) is divided into five equal parts, each of which is *one-fifth* of the whole circle. This fraction is written $\frac{1}{5}$. Two of the equal parts in (*d*) are shaded; consequently, the part of the whole circle that is shaded is *two-fifths*, which is written $\frac{2}{5}$. Similarly, in (*b*), there are two parts not shaded, so that the part or fraction not shaded is *two-thirds*, written $\frac{2}{3}$. In (*c*), three of the equal parts are not shaded; consequently, the fraction of the circle that is not shaded is *three-fourths*, or $\frac{3}{4}$.

2. Numerator and Denominator.—From what has just been stated it must be plain that two numbers are needed to write a fraction. The numbers are written one above the other, with a line between them, and each has its own name. A fraction expressed by means of two numbers, one over the other, separated by a line is known as a **common fraction,** to distinguish it from a *decimal fraction,* which will be described in a succeeding Section. The number above the line is called the **numerator** of the fraction and the number below the line is called the **denominator** of the fraction. Every common fraction must have a numerator and a denominator. The denominator shows how many equal parts a thing is divided into, and the numerator shows how many of those equal parts are taken, or considered.

For example, in Fig. 1 (*a*) the shaded part of the circle is $\frac{1}{2}$ of the circle. In the fraction $\frac{1}{2}$, the numerator is 1 and the denominator is 2. The denominator 2 shows that the circle is divided into two equal parts, and the numerator 1 shows that one of those parts is considered. In (*d*), the fraction of the circle that is shaded is $\frac{2}{5}$. The denominator 5 of the fraction shows that the circle is divided into five equal parts, and the numerator 2 shows that the two parts that are shaded are considered. If the unshaded part had been considered, the numerator would have been 3, and the fraction would have been $\frac{3}{5}$, because three of the five equal parts are not shaded.

The numerator and the denominator of a fraction are called the **terms** of a fraction.

3. Effect of Denominator on Value of Fraction.—The larger the denominator of a fraction, the smaller is the fraction, the numerator being the same. This may easily be shown by referring to Fig. 1 (*b*) and (*d*). The fraction represented by one of the equal parts in (*b*) is $\frac{1}{3}$ and that represented by one of the equal parts in (*d*) is $\frac{1}{5}$. The denominator 5 is greater than the denominator 3, but the fraction $\frac{1}{5}$ is smaller than the fraction $\frac{1}{3}$. This can be seen by comparing one of the equal parts in (*d*) with one in (*b*). One of the parts in (*d*), or $\frac{1}{5}$ of the circle, is much smaller than one of the parts in (*b*), or $\frac{1}{3}$ of the circle. Hence, if the numerators of two fractions are equal, the fraction with the smaller denominator is the greater. Thus, of the two fractions $\frac{3}{10}$ and $\frac{3}{4}$, the latter is the greater. But if the denominators are equal, the one with the larger numerator is the greater. Take $\frac{3}{5}$ and $\frac{4}{5}$, for example; in this case the denominators are equal, and $\frac{4}{5}$ is greater than $\frac{3}{5}$ because 4 is greater than 3.

To illustrate the point more fully, it may be supposed that the circle represents some object, such as a cheese or a large cake weighing 60 pounds. Then,

One-half weighs $60 \div 2 = 30$ pounds
One-third weighs $60 \div 3 = 20$ pounds
One-fourth weighs $60 \div 4 = 15$ pounds
One-fifth weighs $60 \div 5 = 12$ pounds
One-sixth weighs $60 \div 6 = 10$ pounds
One-tenth weighs $60 \div 10 = 6$ pounds

Therefore, the larger the denominator of the fraction the smaller is its value. It was said before that $\frac{3}{10}$ is less than $\frac{3}{4}$. In the case of the object weighing 60 pounds, one-tenth weighs 6 pounds, and $\frac{3}{10}$, which is three times $\frac{1}{10}$, weighs $3 \times 6 = 18$ pounds; one-fourth weighs 15 pounds, and $\frac{3}{4}$ weighs $3 \times 15 = 45$ pounds. We thus see in another way that $\frac{3}{10}$ weighs less than $\frac{3}{4}$, or $\frac{3}{10}$ is less than $\frac{3}{4}$.

4. Division Expressed by a Fraction.—A fraction may also be used to express division; for example, $4 \div 5$ may be written $\frac{4}{5}$, which is a fraction; and similarly, $3 \div 16$ may be written $\frac{3}{16}$. The **value** of a fraction is the result obtained by dividing the numerator by the denominator.

5. Proper Fractions and Improper Fractions.—If the numerator of a fraction is less than the denominator, the fraction is called a **proper fraction**; thus, $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, $\frac{1}{16}$, $\frac{3}{10}$ are proper fractions. If the numerator is equal to or greater than the denominator, the fraction is an **improper fraction**; thus, $\frac{2}{2}$, $\frac{5}{5}$, $\frac{16}{12}$, $\frac{12}{4}$, $\frac{28}{7}$ are improper fractions.

6. Prime Numbers and Composite Numbers.—A **prime number** is one that cannot be divided by any number except itself and 1, without a remainder; thus, 1, 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, etc. are prime numbers.

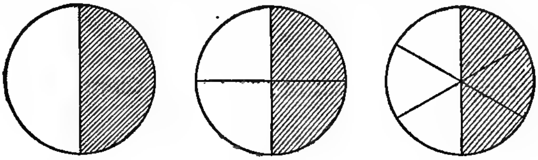
Any number not a prime number is called a **composite number,** and may be considered as the product of two or more prime numbers. Thus, 60 is a composite number, and is equal to the product of the prime numbers 2, 2, 3, and 5; or, $2\times2\times3\times5=60$.

7. Whole Numbers and Mixed Numbers.—A **whole number** is a number that does not contain a fraction. For example, 2, 36, 185, 4,063 are whole numbers. A **mixed number** is a number composed of a whole number and a fraction united. For example, $3\frac{3}{8}$ is a mixed number, being composed of a whole number 3 and a fraction $\frac{3}{8}$. This number is read *three and three-eighths*. It is equal to $3+\frac{3}{8}$, but for convenience the plus sign is omitted in writing it and it appears simply as $3\frac{3}{8}$. The mixed number $10\frac{5}{16}$ is read *ten and five-sixteenths*. A whole number is very frequently called an **integer.** It is also occasionally referred to as an *integral number*.

8. Fractions Smaller or Greater Than 1.—A fraction whose numerator and denominator are equal has a value of 1; thus, $\frac{4}{4}=1$, $\frac{2}{2}=1$, $\frac{8}{8}=1$. If the numerator is less than the denominator, the value of the fraction is less than 1; thus, the value of each of the fractions $\frac{1}{2}$, $\frac{3}{4}$, $\frac{5}{16}$ is less than 1. If the numerator of the fraction is larger than the denominator, the value of the fraction is more than 1; thus, $\frac{12}{2}=6$, $\frac{10}{5}=2$, $\frac{3}{2}=1\frac{1}{2}$.

SIMPLE REDUCTION

9. Same Fraction in Different Forms.—The form of a fraction may be changed without changing its value; or, in other words, a fraction may be expressed in several ways. The form chosen depends on the number of equal parts into which a thing is divided. This may easily be understood by referring to Fig. 2, which shows three circles of equal size. It is supposed that one-half of each circle is to be taken, but that the circles



(a) (b) (c)

Fig. 2

are not divided into the same number of equal parts. The circle in (*a*) is divided into two equal parts, one of which is shaded; that is, the part shaded is $\frac{1}{2}$ of the whole circle. The circle in (*b*) is divided into four equal parts, two of which are shaded; that is, the shaded portion is $\frac{2}{4}$ of the whole. The circle in (*c*) is divided into six equal parts, three of which are shaded; that is, the shaded portion is $\frac{3}{6}$ of the whole. But, as the three circles are of the same size, and the same amount is shaded in each, it follows that $\frac{1}{2}$, $\frac{2}{4}$, and $\frac{3}{6}$ must be equal to one another, because each one represents the same amount, or one-half of the circle.

10. That fractions of equal value may be expressed in different forms may be further illustrated by Fig. 3, which is an oblong divided into equal spaces. There are 4 squares in a row and there are 6 rows, making 24 squares in all. Hence, each square is 1 twenty-fourth of the whole. As there are 6 rows, one row is 1 sixth of the whole. From this it is seen that 4 twenty-fourths equals 1 sixth, or writing this in figures,

$$\frac{4}{24}=\frac{1}{6}$$

Since 4 twenty-fourths equals 1 sixth, 8 twenty-fourths equals 2 sixths, 12 twenty-fourths equals 3 sixths, and 20 twenty-fourths equals 5 sixths. It is also seen that 3 rows are equal to 3 sixths; but 3 rows are equal to 3×4 = 12 squares, which is equal to one-half of the oblong. It follows, then, that

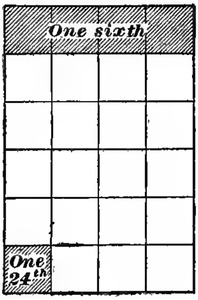


Fig. 3

$$\frac{3}{6}=\frac{1}{2}$$

In each strip running up and down the figure there are 6 squares; but, as there are 4 strips, each strip is 1 fourth of the figure; hence, 6 twenty-fourths equals 1 fourth, or

$$\frac{6}{24}=\frac{1}{4}$$

In two strips, or one-half of the figure, there are 12 squares; hence,

$$\frac{12}{24}=\frac{2}{4}=\frac{1}{2}$$

If the figure is divided into 12 equal parts there will be 2 squares in each part; then,

$$\frac{2}{24}=\frac{1}{12}$$

11. General Principle of Reducing Fractions.—If both terms of a fraction, that is, both numerator and denominator, are multiplied or divided by the same number, the value of the fraction is not changed. For example, suppose that in the fraction $\frac{1}{2}$, both terms are multiplied by 2. Then, $\frac{1\times 2}{2\times 2}=\frac{2}{4}$. The fraction $\frac{2}{4}$ has the same value as $\frac{1}{2}$, as was shown in the preceding article. Again, take $\frac{1}{2}$ and multiply both terms by 3. Then, $\frac{1\times 3}{2\times 3}=\frac{3}{6}$, which has the same value as $\frac{1}{2}$, according to the preceding article. Now take the fraction $\frac{3}{6}$ and divide both terms by 3. Then, $\frac{3\div 3}{6\div 3}=\frac{1}{2}$. But it was shown that $\frac{1}{2}=\frac{3}{6}$, in the preceding article; therefore, by dividing both terms by the same number, the value of the fraction is not changed. This process of changing the form of fractions without changing their values is called **reduction of fractions.**

12. Reducing a Fraction to Higher Terms.—A fraction is reduced to *higher terms* by *multiplying both terms of the fraction by the same number.* For example, $\frac{1}{6}$ is reduced to $\frac{4}{24}$ by multiplying both terms of the fraction by 4. The operation may be written as follows:

$$\frac{1}{6}=\frac{1\times 4}{6\times 4}=\frac{4}{24}$$

That these fractions are equal may be seen from Fig. 3, in which one row is equal to 1 sixth of the whole. Each row contains 4 twenty-fourths; hence, $\frac{1}{6}=\frac{4}{24}$.

As another example, let $\frac{2}{3}$ be changed to $\frac{16}{24}$ by multiplying each term by 8; thus,

$$\frac{2}{3}=\frac{2\times 8}{3\times 8}=\frac{16}{24}$$

If Fig. 3 is divided into three equal parts, and two of the parts are considered, there will be 16 squares in the parts considered, or $\frac{16}{24}$ of the whole figure. This shows that the result just obtained is correct.

13. Reducing a Fraction to Lower Terms.—A fraction is reduced to *lower terms* by *dividing both terms by the same number*. For example, $\frac{20}{24}$ is reduced to $\frac{5}{6}$ by dividing both terms by 4; thus,

$$\frac{20}{24}=\frac{20\div4}{24\div4}=\frac{5}{6}$$

In Art. **10** it was shown that 5 sixths was equal to 20 twenty-fourths, thus proving that the preceding reduction is correct.

14. A fraction is reduced to its *lowest terms* when both its numerator and its denominator cannot be divided by the *same* number without a remainder; for example, $\frac{3}{4}$, $\frac{2}{3}$, $\frac{11}{24}$, $\frac{8}{15}$ are fractions reduced to their lowest terms.

Example 1.—Reduce $\frac{12}{16}$ to its lowest terms.

Solution.—By trial find the greatest number that will divide 12 and 16 without a remainder. This number is 4.

$$\frac{12}{16}=\frac{12\div4}{16\div4}=\frac{3}{4}. \quad \text{Ans.}$$

Example 2.—Reduce $\frac{8}{10}$ to its lowest terms.

Solution.—The number that will divide 8 and 10 without a remainder is 2.

$$\frac{8}{10}=\frac{8\div2}{10\div2}=\frac{4}{5}. \quad \text{Ans.}$$

15. Reducing a Fraction to One of Equal Value and With a Given Denominator.—In practice, it is often required to change a given fraction to another one of equal value, but with a different denominator. In such cases the following rule applies:

Rule.—*Divide the given denominator by the denominator of the given fraction, and multiply both terms of the fraction by the result.*

Example 1.—Reduce $\frac{7}{8}$ to an equal fraction having 96 for a denominator.

Solution.—Both the numerator and the denominator must be multiplied by the same number in order not to change the value of the fraction. The denominator must be multiplied by some number which will, in this case, make the product 96; this number is evidently $96\div8=12$, because $8\times12=96$. Hence,

$$\frac{7\times12}{8\times12}=\frac{84}{96}. \quad \text{Ans.}$$

Example 2.—Reduce $\frac{3}{4}$ to 100ths; that is, to a fraction having 100 for a denominator.

Solution.— 100÷4=25; hence,

$$\frac{3\times25}{4\times25}=\frac{75}{100}. \text{ Ans.}$$

EXAMPLES FOR PRACTICE

Reduce the following:

(*a*) $\frac{7}{16}$ to 128ths.	Ans.	(*a*) $\frac{56}{128}$
(*b*) $\frac{24}{132}$ to its lowest terms.		(*b*) $\frac{2}{11}$
(*c*) $\frac{64}{1000}$ to its lowest terms.		(*c*) $\frac{8}{125}$
(*d*) $\frac{5}{7}$ to 49ths.		(*d*) $\frac{35}{49}$
(*e*) $\frac{13}{16}$ to 10,000ths.		(*e*) $\frac{8125}{10000}$

16. Reducing a Whole Number or a Mixed Number to an Improper Fraction.—The process of reducing a whole number or a mixed number to an improper fraction is of importance as it is used to a great extent in the multiplication and division of fractions. The following rule applies:

Rule.—*To reduce a mixed number to an improper fraction, multiply the whole number by the denominator of the fraction, add the numerator to the product, and place the denominator under the result. If it is desired to reduce a whole number to a fraction, multiply the whole number by the denominator of the given fraction, and write the result over the denominator.*

Example 1.—Reduce 5 to an improper fraction having 4 as a denominator.

Solution.—There are 4 fourths in 1, because $\frac{4}{4}=1$, and in 5 there will be 5×4 fourths, or 20 fourths; that is, $5\times\frac{4}{4}=\frac{20}{4}$. Ans.

Example 2.—Reduce $8\frac{3}{4}$ to an improper fraction.

Solution.—According to the rule,

$$8\tfrac{3}{4}=\frac{8\times4+3}{4}=\frac{32+3}{4}=\frac{35}{4}. \text{ Ans.}$$

EXAMPLES FOR PRACTICE

Reduce to improper fractions:

(*a*) $4\frac{1}{8}$.
(*b*) $5\frac{3}{16}$.
(*c*) $10\frac{2}{10}$.
(*d*) $37\frac{3}{4}$.
(*e*) $50\frac{4}{5}$.
(*f*) Reduce 7 to a fraction whose denominator is 16.

Ans.
(*a*) $\frac{33}{8}$
(*b*) $\frac{83}{16}$
(*c*) $\frac{102}{10}$
(*d*) $\frac{151}{4}$
(*e*) $\frac{254}{5}$
(*f*) $\frac{112}{16}$

---

17. Reducing an Improper Fraction to a Whole or to a Mixed Number.—The result obtained in the multiplication or division of fractions is in many cases in the form of an improper fraction. Preferably, the answer should be in the form of a mixed number. An improper fraction is reduced to a mixed number by applying the following rule:

Rule.—*To reduce an improper fraction to a whole or a mixed number, divide the numerator by the denominator and write the result as in ordinary division.*

EXAMPLE.—Reduce $\frac{21}{4}$ to a mixed number.

SOLUTION.— 4 is contained in 21, 5 times with 1 as a remainder. The latter number is written as the numerator of a fraction with 4 as a denominator. This fraction is added to the whole number. Therefore, $5+\frac{1}{4}$, or $5\frac{1}{4}$, is the mixed number required. Ans.

EXAMPLES FOR PRACTICE

Reduce to whole or mixed numbers:

(*a*) $\frac{145}{6}$.
(*b*) $\frac{185}{8}$.
(*c*) $\frac{701}{6}$.
(*d*) $\frac{149}{3}$.
(*e*) $\frac{76}{19}$.
(*f*) $\frac{125}{25}$.

Ans.
(*a*) $24\frac{1}{6}$
(*b*) $61\frac{2}{3}$
(*c*) $116\frac{5}{6}$
(*d*) $49\frac{2}{3}$
(*e*) 4
(*f*) 5

PROCESSES PREPARATORY TO ADDITION AND SUBTRACTION OF FRACTIONS

FINDING LEAST COMMON DENOMINATOR

18. Adding Fractions With Different Denominators. Fractions cannot be added unless they have the same denominator. If the denominators are not the same the fractions must be changed into such forms that they have the same denominator. For instance, $\frac{3}{4}$ may be added to $\frac{1}{4}$, but $\frac{3}{4}$ cannot be added to $\frac{7}{8}$ as the fractions now stand, because the denominators are different; fourths cannot be added to eighths. The reason for this may be seen from the following example:

Suppose that an apple is divided into 4 equal parts, and that 2 of these are each divided into 2 equal parts. It is evident that there are 2 one-fourths and 4 one-eighths. If these parts are added, the sum is 6. But what does this sum represent? It cannot be fourths, for 6 fourths is equal to $\frac{6}{4}=1\frac{1}{2}$, and only 1 apple was divided; neither can it be eighths; for $\frac{6}{8}=\frac{3}{4}$ is less than 1 apple. By reducing the quarters to eighths, there are $\frac{2}{4}=\frac{4}{8}$; on adding these to the other 4 eighths, there are $4+4=8$ eighths. This result must be correct, since $\frac{8}{8}=1$. Another way would be to combine the eighths into quarters. Thus, $\frac{4}{8}=\frac{2}{4}$; then, adding the quarters, there are $2+2=4$ quarters. This result is also correct, since $\frac{4}{4}=1$.

19. Common Denominator.—Several fractions are said to have a **common denominator** when all their denominators are the same, as $\frac{1}{11}$, $\frac{4}{11}$, $\frac{7}{11}$, $\frac{10}{11}$. Two or more fractions having different denominators can be reduced to others having a common denominator by reducing each fraction to higher or lower terms. For example, $\frac{1}{2}$ and $\frac{1}{3}$ can each be reduced to sixths; thus, $\frac{1}{2}=\frac{3}{6}$, because $\frac{1\times3}{2\times3}=\frac{3}{6}$; also $\frac{1}{3}=\frac{2}{6}$, because $\frac{1\times2}{3\times2}=\frac{2}{6}$. Likewise, $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{3}{4}$ can each be reduced to twelfths; thus, $\frac{1}{2}=\frac{6}{12}$, $\frac{1}{3}=\frac{4}{12}$, and $\frac{3}{4}=\frac{9}{12}$.

20. Least Common Denominator.—Fractions with different denominators may have as a common denominator any number that will contain each of the denominators *without a* *remainder;* for example, the fractions $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$ can be reduced to twelfths, twenty-fourths, thirty-sixths, or to fractions having as a common denominator any number that will contain 2, 3, and 4 without a remainder. But the **least common denominator** is the *least* number that may be divided by each denominator of the given fractions without a remainder. The following example will illustrate the meaning of this definition:

Let it be supposed that the fractions $\frac{1}{2}$, $\frac{2}{3}$, and $\frac{3}{4}$ are to be reduced to fractions that have the same denominator. As 2, 3, and 4 can be divided into 12 without a remainder, 12 is taken as the common denominator. Thus, $12 \div 2 = 6$, then $\frac{1 \times 6}{2 \times 6} = \frac{6}{12}$; similarly, $\frac{2}{3} = \frac{8}{12}$, and $\frac{3}{4} = \frac{9}{12}$, and the fractions $\frac{1}{2}$, $\frac{2}{3}$, and $\frac{3}{4}$ therefore become $\frac{6}{12}$, $\frac{8}{12}$, and $\frac{9}{12}$. The number 24 could also be used as a common denominator, because it can be divided by 2, 3, or 4 without a remainder. The three fractions would then become $\frac{12}{24}$, $\frac{16}{24}$, and $\frac{18}{24}$, and these values would be the same as $\frac{6}{12}$, $\frac{8}{12}$, and $\frac{9}{12}$. But the point to be noticed is that although 12 and 24 can both be used as common denominators for these three fractions, 12 is the *least* common denominator, because it is the *smallest* number that can be divided by 2, 3, and 4 without leaving a remainder. A series of fractions may have a great many different common denominators, but they can have only *one* least common denominator. In adding or subtracting, fractions are generally reduced to the *least* common denominator. The methods by which fractions are reduced to the least common denominator will be explained in detail further on.

21. Finding the Least Common Denominator by Inspection.—The least common denominator of several fractions may be found either by inspection or by calculation. The method of finding it by inspection will be explained by means of an example taken from practice. For this purpose the foot rule, generally used in English-speaking countries for taking measurements of length, will be considered.

A foot rule is divided into equal parts called *inches*, and each inch is further divided into equal parts called fractions of an inch. A part of a rule showing a common way of dividing the inch is illustrated in Fig. 4. The long line *a* at the middle of the inch divides it into halves. At the middle of the halves are shorter lines *b* and *c* that divide the half inches into halves, making quarter inches, and the quarter inches are still further divided by the lines *d*, *e*, *f*, and *g* into eighths. Finally, the shortest lines divide the eighths, so that there are sixteen equal divisions in 1 inch, and these are called sixteenths. Suppose that the rule is used to measure the length of a block *h*. The end of the rule is put in line with the end of the block, and the other end of the block comes just to the line *a*, which is the half-inch mark; therefore, the block is said to be $\frac{1}{2}$ inch long.

22. Now, suppose that the length of the block *h*, Fig. 4, is to be found in quarters, eighths, or sixteenths of an inch. By looking at the rule, it is seen that there are just two quarter inches between the line *a* and the end of the rule; so the length of the block is two quarter inches or two-fourths of an inch, which is written $\frac{2}{4}$ inch. This is exactly equal to $\frac{1}{2}$ inch, because $\frac{1\times 2}{2\times 2}=\frac{2}{4}$. If the length of the block is expressed in eighths of an inch, it is $\frac{4}{8}$ inch long, because there are four eighth-inch divisions between the line *a* and the end of the rule. Also, the block is $\frac{8}{16}$ inch long, because there are eight sixteenth-inch divisions from the line *a* to the end of the rule. This simply shows that $\frac{1}{2}=\frac{2}{4}=\frac{4}{8}=\frac{8}{16}$. Sometimes rules are divided into thirty-seconds and sixty-fourths of an inch.

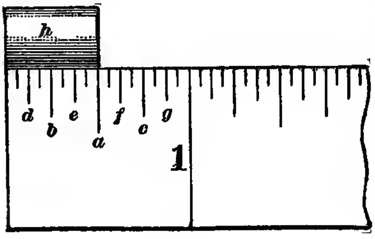


FIG. 4

If a rule similar to Fig. 4 is also divided into thirty-seconds and sixty-fourths of an inch, the fractions considered will be halves, fourths, eighths, sixteenths, thirty-seconds, and sixty-fourths. When a common denominator is to be found for several such fractions, all that is necessary is to take the greatest denominator in the group. For example, in the fractions $\frac{5}{16}$, $\frac{3}{4}$, $\frac{7}{8}$, and $\frac{1}{2}$ the largest denominator is 16, and this is chosen as the least common denominator. Then, $\frac{3}{4}=\frac{12}{16}$, $\frac{7}{8}=\frac{14}{16}$, and $\frac{1}{2}=\frac{8}{16}$. Again, suppose that the fractions are $\frac{27}{64}$, $\frac{3}{8}$, $\frac{7}{32}$, and $\frac{1}{4}$. The largest of these denominators is 64, which is taken as the least common denominator, and then $\frac{3}{8}=\frac{24}{64}$, $\frac{7}{32}=\frac{14}{64}$, and $\frac{1}{4}=\frac{16}{64}$. It is therefore a very simple matter to choose the least common denominator for fractions of an inch as found by a rule divided in the manner explained. This method of finding the least common denominator by merely looking at the denominators and finding the common denominator mentally at once, or after a few trials, is called finding the least common denominator by **inspection.**

23. Finding the Least Common Denominator by Calculation.—It is not always possible to find the least common denominator by inspection as in the preceding article. For this reason, the following method is given, showing how to find by calculation the least common denominator of *any* set of fractions. It is important that this method should be thoroughly understood. The method will be explained by applying it directly to the following examples:

EXAMPLE 1.—Find the least common denominator of $\frac{1}{4}$, $\frac{1}{3}$, $\frac{1}{9}$, and $\frac{1}{16}$.

SOLUTION.—The denominators of the fractions are placed in a line, as in short division, and separated by commas. In this case they are 4, 3, 9, and 16, as shown. The denominators are now divided by some prime number other than 1 that will be contained in at least two of the numbers without a remainder. The number 2 is contained in 4 and 16; hence, dividing by 2 and placing the quotients in the second line, under the numbers divided, the quotients 2 and 8 are obtained. The numbers 3 and 9, which will not contain the divisor without a remainder, are transferred to the second line, as shown.

$$\begin{array}{r|rrrr}
2\,) & 4, & 3, & 9, & 16 \\ \hline
2\,) & 2, & 3, & 9, & 8 \\ \hline
3\,) & 1, & 3, & 9, & 4 \\ \hline
 & 1, & 1, & 3, & 4
\end{array}$$

$2\times2\times3\times3\times4=144$, the least common denominator. Ans.

Next, another prime number is found that will be contained in at least two of the numbers in the second line. The divisor 2 is again selected, as being contained in the numbers 2 and 8. The quotients 1 and 4 are written in the third line, as well as the numbers not divisible by 2, in this case 3 and 9. The divisor 3 is chosen for the third line, giving the quotients 1 and 3. In the fourth line there will then appear the numbers 1, 1, 3, 4, as shown.

These operations are repeated until the last line contains no two numbers that can be divided by the same prime number. In this case the numbers 3 and 4 remain, which cannot both be divided by any one prime number. Finally, the divisors and the remaining numbers in the last line are multiplied together, the product being the least common denominator. In the example the product is $2\times2\times3\times1\times1\times3\times4=144$, which is the least common denominator. Ordinarily the prime numbers **1** are omitted as being without effect on the final result.

Example 2.—Find the least common denominator of $\frac{4}{9}$, $\frac{5}{12}$, $\frac{7}{18}$.

Solution.—

$$\begin{array}{r|l}
3\,) & 9,\ 12,\ 18 \\ \hline
3\,) & 3,\ \ 4,\ \ 6 \\ \hline
2\,) & 1,\ \ 4,\ \ 2 \\ \hline
 & 1,\ \ 2,\ \ 1 \\ \hline
\end{array}$$

$3\times3\times2\times2=36$. Ans.

24. Reducing Fractions to Equivalent Fractions Having the Least Common Denominator.—The preceding article explained the method of finding the least common denominator of several fractions. There remains yet to be shown how a number of given fractions may be reduced to equivalent fractions having the least common denominator. Reductions of this kind are performed in accordance with the following rule:

Rule.—*To reduce fractions to the least common denominator, divide the least common denominator by the denominator of the given fraction, and multiply both terms of the fraction by the quotient.*

Example 1.—Reduce $\frac{2}{3}$, $\frac{3}{4}$, and $\frac{1}{2}$ to fractions having the least common denominator.

Solution.—The least common denominator is first found as follows:

$$\begin{array}{r|l}
2\,) & 3,\ \ 4,\ \ 2 \\ \hline
 & 3,\ \ 2,\ \ 1 \\
\end{array}$$

Then, $2\times3\times2=12$, the least common denominator

In this example, each of the three fractions must be changed to a form in which it will have 12 for its denominator. In the first fraction, $\frac{2}{3}$, the denominator is 3. By the rule, the least common denominator is to be divided by the denominator of the fraction; or, $12\div3=4$. Both terms of the fraction are then to be multiplied by this quotient, 4; that is, $\frac{2\times4}{3\times4}=\frac{8}{12}$. The fraction to be considered next is $\frac{3}{4}$, whose denominator is 4.

Following the same method, 12÷4=3, and $\frac{3\times3}{4\times3}=\frac{9}{12}$. The third fraction, $\frac{1}{2}$, has 2 for a denominator. Then, 12÷2=6, and $\frac{1\times6}{2\times6}=\frac{6}{12}$. Hence, the fractions $\frac{2}{3}$, $\frac{3}{4}$, and $\frac{1}{2}$, when reduced to the least common denominator, become $\frac{8}{12}$, $\frac{9}{12}$, and $\frac{6}{12}$. Ans.

Example 2.—Reduce $\frac{13}{16}$, $\frac{5}{7}$, and $\frac{11}{12}$ to the least common denominator.

Solution.—The least common denominator is found first, as follows:

$$\begin{array}{l} 2\,)\,\underline{16,\ \ 7,\ 12} \\ 2\,)\,\underline{\ \ 8,\ \ 7,\ \ \ 6} \\ \qquad 4,\ \ 7,\ \ \ 3 \end{array}$$

Least common denominator $=2\times2\times4\times7\times3=336$

In the case of the fraction $\frac{13}{16}$, 336÷16=21, and $\frac{13\times21}{16\times21}=\frac{273}{336}$. In the case of $\frac{5}{7}$, 336÷7=48, and $\frac{5\times48}{7\times48}=\frac{240}{336}$. Finally, in the case of $\frac{11}{12}$, 336÷12 =28, and $\frac{11\times28}{12\times28}=\frac{308}{336}$. Therefore, the three fractions reduced to the least common denominator are $\frac{273}{336}$, $\frac{240}{336}$, and $\frac{308}{336}$. Ans.

EXAMPLES FOR PRACTICE

Reduce to fractions having the least common denominator:

(*a*) $\frac{3}{4}$, $\frac{5}{8}$, $\frac{7}{8}$.	Ans.	(*a*) $\frac{6}{8}$, $\frac{5}{8}$, $\frac{7}{8}$
(*b*) $\frac{3}{16}$, $\frac{3}{4}$, $\frac{7}{32}$.		(*b*) $\frac{6}{32}$, $\frac{24}{32}$, $\frac{7}{32}$
(*c*) $\frac{7}{8}$, $\frac{7}{88}$, $\frac{10}{11}$.		(*c*) $\frac{77}{88}$, $\frac{7}{88}$, $\frac{80}{88}$
(*d*) $\frac{3}{5}$, $\frac{5}{8}$, $\frac{11}{40}$.		(*d*) $\frac{24}{40}$, $\frac{25}{40}$, $\frac{11}{40}$
(*e*) $\frac{4}{10}$, $\frac{6}{40}$, $\frac{9}{20}$.		(*e*) $\frac{16}{40}$, $\frac{6}{40}$, $\frac{18}{40}$
(*f*) $\frac{7}{15}$, $\frac{17}{30}$, $\frac{7}{10}$.		(*f*) $\frac{14}{30}$, $\frac{17}{30}$, $\frac{21}{30}$

ADDITION OF FRACTIONS

25. Rule for Addition of Fractions.—The preceding rules and exercises referring to finding the least common denominator are to be considered as preparatory processes that must be performed before addition of fractions can take place. The reasons for providing fractions with the same denominator before adding them have been given in Art. 18, but it may be well to emphasize the point again at this place. For instance, if a workman measures off the thicknesses of three pieces of material as $\frac{1}{16}$ inch, $\frac{5}{16}$ inch, and $\frac{7}{16}$ inch, respectively, he knows that the total thickness is $\frac{13}{16}$ inch. This result is obtained by adding the numerators, thus, 1+5+7=13, and placing that sum over the common denominator, 16, giving $\frac{13}{16}$. But if the thicknesses measured off are, for instance, $\frac{1}{4}$ inch, $\frac{5}{8}$ inch, and $\frac{7}{16}$ inch, the total thickness cannot be found by adding the numerators, because the fractions have different denominators. Before the fractions can be added they must be changed to forms in which they have the same denominator, that is, a common denominator. After they have been reduced to this form the process of addition is a very simple one. The following rule applies to the addition of common fractions:

Rule.—I. *To add fractions, first reduce them to a common denominator. Then add the numerators, and write their sum over the common denominator. If the result is an improper fraction, reduce it to a mixed number and reduce the fractional part to its lowest terms.*

II. *To add whole and mixed numbers, add the whole numbers and the fractions separately, and then add the results.*

The application of this rule will be shown in the following examples:

EXAMPLE 1.—Add $\frac{1}{2}$, $\frac{3}{4}$, and $\frac{5}{12}$.

SOLUTION.— $\frac{1}{2}+\frac{3}{4}+\frac{5}{12}=\frac{6}{12}+\frac{9}{12}+\frac{5}{12}=\frac{6+9+5}{12}=\frac{20}{12}=1\frac{8}{12}=1\frac{2}{3}$. Ans.

Part I of the rule applies. The least common denominator is readily seen to be 12, the new numerators are 6, 9, and 5, and their sum is 20. The rest of the work is reduction, as previously explained. The sum of the fractions is $\frac{20}{12}$, but as the latter is an improper fraction, it must, according to part I of the rule, be reduced to a mixed number, that is, $1\frac{8}{12}$. The fractional part, $\frac{8}{12}$, should, according to the same part of the rule, be reduced to its lowest terms. It is seen that it may be so reduced by dividing numerator and denominator by 4, the reduced fraction being $\frac{2}{3}$. The final form of the mixed number is $1\frac{2}{3}$, as shown.

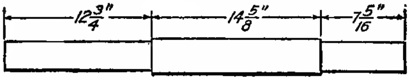


FIG. 5

EXAMPLE 2.—What is the total length, or overall length, of the piece of shafting shown dimensioned in Fig. 5?

NOTE.—The marks (″) used on the drawing, mean *inches;* thus, $12\frac{3}{4}$″ means $12\frac{3}{4}$ inches.

SOLUTION.—The total length is the sum of the three parts, measuring $12\frac{3}{4}$ inches, $14\frac{5}{8}$ inches, and $7\frac{5}{16}$ inches. Then, applying part I of the rule, the fractions are reduced to a common denominator. The least common denominator of the three fractions is 16. Part II of the rule is now applied, and the whole numbers and the fractions are added separately. The sum of the fractions is $\frac{27}{16}=1\frac{11}{16}$; the sum of the whole numbers is 33. The sum of both, which is the total, or overall, length of the shaft, is $34\frac{11}{16}$ inches.

$$12\tfrac{3}{4}=12\tfrac{12}{16}$$
$$14\tfrac{5}{8}=14\tfrac{10}{16}$$
$$7\tfrac{5}{16}=\ 7\tfrac{5}{16}$$
$$\textit{sum}=33+\tfrac{27}{16}=33+1\tfrac{11}{16}=34\tfrac{11}{16}.\ \text{Ans.}$$

Example 3.—A workman has five pieces of belt whose lengths are $9\frac{1}{2}$, $6\frac{1}{4}$, $3\frac{3}{4}$, $3\frac{1}{3}$, and $1\frac{5}{6}$ feet, respectively, and he needs a belt 24 feet long for some special work. Can he make up a belt of the required length by lacing the pieces together, end to end?

Solution.—The least common denominator of the fractions is 12. The fractions reduced to the least common denominator are as follows: $\frac{1}{2}=\frac{6}{12}$; $\frac{1}{4}=\frac{3}{12}$; $\frac{3}{4}=\frac{9}{12}$; $\frac{1}{3}=\frac{4}{12}$; $\frac{5}{6}=\frac{10}{12}$. Therefore, the mixed numbers $9\frac{1}{2}+6\frac{1}{4}+3\frac{3}{4}+3\frac{1}{3}+1\frac{5}{6}$ are equal to $9\frac{6}{12}+6\frac{3}{12}+3\frac{9}{12}+3\frac{4}{12}+1\frac{10}{12}$. The sum of the latter numbers is $22\frac{32}{12}$. Reducing the improper fraction $\frac{32}{12}$ to a mixed number, it becomes $2\frac{8}{12}$, in which the fraction may be reduced to $\frac{2}{3}$; thus, $2\frac{8}{12}=2\frac{2}{3}$. The sum $22+2\frac{2}{3}=24\frac{2}{3}$ is the total length, in feet, of the joined pieces. As the length required is only 24 feet, the pieces can be joined to form the required belt. Ans.

26. Adding Fractions by Means of a Foot Rule.—In practice, it is often required to add dimensions expressed in inches and fractions of an inch. This may be done either by the methods described for the addition of fractions, or by means of a foot rule. To illustrate, let it be supposed that the dimensions $1\frac{1}{2}$, $\frac{5}{8}$, $2\frac{3}{16}$, $\frac{3}{4}$, $\frac{15}{16}$, and $1\frac{3}{8}$ inches are to be added. The several fractions $\frac{1}{2}$, $\frac{5}{8}$, $\frac{3}{16}$, $\frac{3}{4}$, $\frac{15}{16}$, and $\frac{3}{8}$ are first added, by using a rule graduated in sixteenths of an inch, as shown in Fig. 6. Beginning at the end *a*, count off $\frac{1}{2}$ inch, to the point *b*. From *b*

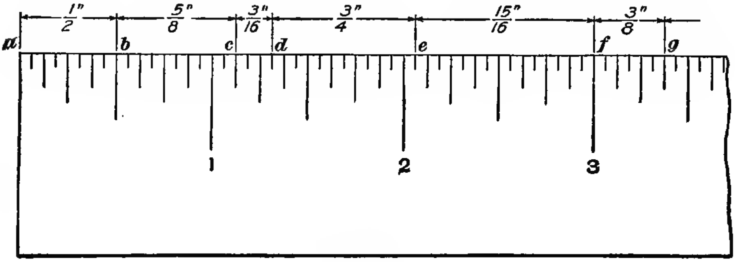


Fig. 6

count off five $\frac{1}{8}$-inch divisions, or $\frac{5}{8}$ inch, to *c*. Next count three $\frac{1}{16}$-inch divisions, or $\frac{3}{16}$ inch, to *d*. From *d* count off three $\frac{1}{4}$-inch spaces, or $\frac{3}{4}$ inch, which is the same as twelve $\frac{1}{16}$-inch divisions, because $\frac{3}{4}=\frac{12}{16}$. This locates the mark *e*. From *e* count off $\frac{15}{16}$ inch, or fifteen $\frac{1}{16}$-inch spaces, to *f*. Finally, from *f* count off three $\frac{1}{8}$-inch divisions, to *g*. The point *g* thus found is at a distance of $3\frac{3}{8}$ inches from *a*; hence, $\frac{1}{2}+\frac{5}{8}+\frac{3}{16}+\frac{3}{4}+\frac{15}{16}+\frac{3}{8}=3\frac{3}{8}$ inches. To this add the whole numbers of inches, 1, 2, and 1, and the total is $3\frac{3}{8}+1+2+1=7\frac{3}{8}$ inches; that is, $1\frac{1}{2}+\frac{5}{8}+2\frac{3}{16}+\frac{3}{4}+\frac{15}{16}+1\frac{3}{8}=7\frac{3}{8}$ inches.

EXAMPLES FOR PRACTICE

1. Find the sum of the following:

(*a*) $\frac{4}{6}$, $\frac{7}{24}$, $\frac{5}{8}$.

(*b*) $\frac{2}{3}$, $\frac{5}{15}$, $\frac{24}{45}$.

Ans. $\begin{cases} (a)\ 1\frac{7}{12} \\ (b)\ 1\frac{8}{15} \end{cases}$

2. The bolt shown in Fig. 7 has the end threaded for a distance of

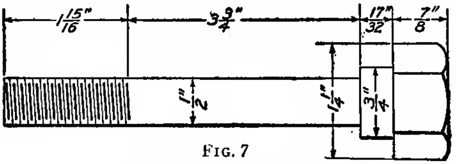


FIG. 7

$1\frac{15}{16}$ inches, is plain for $3\frac{3}{4}$ inches, has a collar $\frac{17}{32}$ inch long, and has a head $\frac{7}{8}$ inch thick. What is the overall length of the bolt? Ans. $7\frac{3}{32}$ inches

3. A helper works $2\frac{3}{4}$ hours on one job, $1\frac{1}{3}$ hours on another, and $\frac{5}{12}$ hour on a third. What is the total time that he worked? Ans. $4\frac{1}{2}$ hours

4. The weights of a number of castings are as follows: $412\frac{3}{4}$ pounds, $270\frac{1}{2}$ pounds, 1,020 pounds, $75\frac{1}{4}$ pounds, and $68\frac{1}{2}$ pounds. What is their total weight? Ans. 1,847 lb.

5. Find the total length of five cuts of cloth, the lengths of the cuts being as follows: $51\frac{1}{4}$ yards, $51\frac{3}{8}$ yards, $55\frac{1}{4}$ yards, $53\frac{5}{8}$ yards, and $48\frac{1}{2}$ yards. Ans. 260 yd.

6. The beam shown in Fig. 8 has rivet holes spaced from left to right at the distances indicated in the drawing. The distances are as follows: $1\frac{15}{16}$ inches, $9\frac{3}{4}$ inches, $6\frac{1}{2}$ inches, $5\frac{3}{8}$ inches, and $2\frac{7}{16}$ inches. What is the length of the beam? Ans. 26 inches

7. A piece of lead pipe work is made up of the following lengths: $39\frac{1}{8}$, $46\frac{1}{4}$, $3\frac{3}{8}$, $9\frac{3}{4}$, and $12\frac{1}{2}$ feet, respectively. What is the total length of the pipe? Ans. 111 ft.

8. During a journey of three days a man travels the following distances: $50\frac{5}{12}$ miles, $56\frac{3}{4}$ miles, and $85\frac{5}{6}$ miles. What is the total length of the journey? Ans. 193 miles

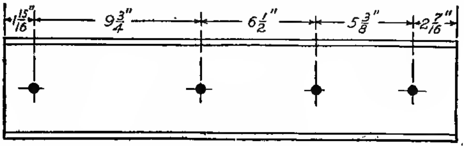


FIG. 8

9. Four castings of the following weights are ordered from a brass foundry: $6\frac{1}{4}$ pounds, $3\frac{1}{2}$ pounds, $5\frac{3}{4}$ pounds, and $8\frac{7}{8}$ pounds. What is the total weight of the castings? Ans. $24\frac{3}{8}$ lb.

10. The stock required for three bolts must be cut from a bar of iron. The finished lengths of the bolts are to be $2\frac{3}{16}$, $1\frac{13}{16}$, and $3\frac{1}{4}$ inches, respectively. Allowing, in all, $\frac{3}{4}$ of an inch of stock for cutting off, for forming heads, and for finishing the ends of the bolts, how long must be the piece of stock? Ans. 8 inches

SUBTRACTION OF FRACTIONS

27. Rule for Subtraction of Fractions.—Reasons were given in Arts. 18 and 25 for the necessity of reducing fractions to a common denominator, before addition can take place. The same reasons hold good in the case of subtraction. For instance, if the fraction $\frac{6}{8}$ were subtracted from $\frac{9}{12}$ by simply finding the differences between the numerators and the denominators, the difference would be $\frac{9}{12}-\frac{6}{8}=\frac{3}{4}$. But $\frac{9}{12}=\frac{3}{4}$ and $\frac{6}{8}=\frac{3}{4}$; therefore, the difference between them cannot be $\frac{3}{4}$, but is instead $\frac{3}{4}-\frac{3}{4}=0$. In other words, *six eighths* cannot be subtracted from *nine twelfths* any more than *six apples* can be subtracted from *nine oranges*. The two fractions must be reduced to a common denominator before one can be subtracted from the other.

The following rule applies to subtraction of fractions:

Rule.—I. *Reduce the fractions to fractions having a common denominator. Subtract one numerator from the other and place the remainder over the common denominator.*

II. *When there are mixed numbers, subtract the fractions and whole numbers separately, and place the remainders side by side.*

III. *When the fraction in the subtrahend is greater than the fraction in the minuend, take 1 from the whole number in the minuend and add it to the fraction in the minuend, from which subtract the fraction in the subtrahend.*

IV. *When the minuend is a whole number, take 1 from it, reduce the 1 to a fraction whose denominator is the same as the denominator of the fraction in the subtrahend, and place it over that fraction for subtraction.*

EXAMPLE 1.—Subtract $\frac{3}{8}$ from $\frac{13}{16}$.

SOLUTION.—The least common denominator is 16, and $\frac{3}{8}=\frac{6}{16}$. Then, according to part I of the rule,

$$\tfrac{13}{16}-\tfrac{3}{8}=\tfrac{13}{16}-\tfrac{6}{16}=\frac{13-6}{16}=\tfrac{7}{16}. \quad \text{Ans.}$$

EXAMPLE 2.—From 7 take $\frac{5}{8}$.

SOLUTION.—There is no fraction in the minuend 7, so it is necessary to take 1 from 7 to form a fraction whose denominator is the same as that of the fraction to be subtracted, or 8. As $1=\frac{8}{8}$, it follows that instead of 7, it is possible to write $6+1$, or $6+\frac{8}{8}$, or simply $6\frac{8}{8}$. Then, $6\frac{8}{8}-\frac{5}{8}=6\frac{3}{8}$; that is, $7-\frac{5}{8}=6\frac{3}{8}$.

$$\begin{array}{lr} \textit{minnend} & 7=6\frac{8}{8} \\ \textit{subtrahend} & \frac{5}{8} \\ \hline \textit{remainder} & 6\frac{3}{8} \end{array} \quad \text{Ans.}$$

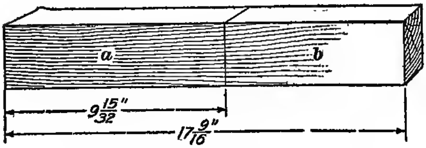


FIG. 9

EXAMPLE 3.—A block of wood $17\frac{9}{16}$ inches long, as shown in Fig. 9, has a piece *a* $9\frac{15}{32}$ inches long sawed off. What is the length of the remaining piece *b* neglecting the thickness of the saw cut?

SOLUTION.—The common denominator of the fractions is 32. $17\frac{9}{16}=17\frac{18}{32}$.

$$\begin{array}{lr} \textit{minuend} & 17\frac{18}{32} \\ \textit{subtrahend} & 9\frac{15}{32} \\ \hline \textit{difference} & 8\frac{3}{32} \end{array} \quad \text{Ans.}$$

That is, the remaining portion *b*, Fig. 9, is $8\frac{3}{32}$ inches long.

EXAMPLE 4.—A piece $4\frac{7}{16}$ inches long is cut from a bar $9\frac{1}{4}$ inches long. What is the length of the remaining piece?

SOLUTION.—The common denominator of the fractions is 16. $9\frac{1}{4}=9\frac{4}{16}$. As the fraction in the subtrahend is greater than the fraction in the minuend, it cannot be subtracted. Therefore, 1, or $\frac{16}{16}$, is taken from the 9 in the minuend and added to the $\frac{4}{16}$; thus, $\frac{4}{16}+\frac{16}{16}=\frac{20}{16}$ Subtracting $\frac{7}{16}$ from $\frac{20}{16}$ leaves a remainder of $\frac{13}{16}$. Since 1 was taken from 9, 8 remains. 4 from $8=4$; $4+\frac{13}{16}=4\frac{13}{16}$, the rerequired answer.

$$\begin{array}{rll} \textit{minuend} & 9\frac{4}{16} \text{ or } & 8\frac{20}{16} \\ \textit{subtrahend} & 4\frac{7}{16} & 4\frac{7}{16} \\ \hline \textit{difference} & 4\frac{13}{16} & 4\frac{13}{16} \text{ Ans.} \end{array}$$

EXAMPLE 5.—From 9 take $8\frac{3}{16}$.

SOLUTION.—There is no fraction in the minuend from which to subtract the fraction in the subtrahend, so 1, or $\frac{16}{16}$, is taken from 9. Thus, $9=8\frac{16}{16}$, and $\frac{16}{16}-\frac{3}{16}=\frac{13}{16}$. Since 1 was taken from 9, only 8 is left, and 8 from 8 $=0$.

$$\begin{array}{rll} \textit{minuend} & 9 \quad \text{or} & 8\frac{16}{16} \\ \textit{subtrahend} & 8\frac{3}{16} & 8\frac{3}{16} \\ \hline \textit{difference} & \frac{13}{16} & \frac{13}{16} \text{ Ans.} \end{array}$$

EXAMPLES FOR PRACTICE

1. In the following examples, subtract:

(a) $\frac{4}{30}$ from $\frac{5}{10}$.
(b) $13\frac{1}{4}$ from $30\frac{1}{2}$.
(c) $12\frac{1}{8}$ from 27.

Ans. (a) $\frac{11}{30}$
(b) $17\frac{1}{4}$
(c) $14\frac{7}{8}$

2. A bar of iron $22\frac{1}{8}$ inches long, as shown in Fig. 10, has three pieces cut from it, measuring $6\frac{1}{2}$ inches, $4\frac{7}{8}$ inches, and $2\frac{5}{32}$ inches in length. What length of the original bar remains? Ans. $8\frac{19}{32}$ inches

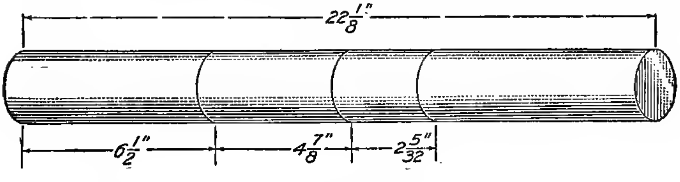


FIG. 10

SUGGESTION.—Add the lengths of the three pieces that are cut off and subtract their sum from the total length.

3. A piece of cast iron $4\frac{1}{8}$ inches thick was planed down to a thickness of $3\frac{31}{32}$ inches. What thickness of metal was removed? Ans. $\frac{5}{32}$ inch

4. At the beginning of a week an engineer had on hand $48\frac{5}{6}$ gallons of cylinder oil. During the week the following quantities were used: $2\frac{1}{4}$ gal-

lons during the first three days, $\frac{3}{4}$ gallon on the fourth day, 1 gallon on the fifth day, $\frac{1}{2}$ gallon on the sixth day, and not any on the seventh day. How much oil remained at the end of the week? Ans. $44\frac{1}{3}$ gal.

5. The main line shaft in a manufacturing plant is driven by an engine capable of developing $250\frac{2}{3}$ horsepower. To operate all the machinery requires $210\frac{3}{4}$ horsepower, and to overcome the friction of the shafting and bearings requires $25\frac{1}{8}$ horsepower. What surplus power is the steam engine capable of developing in case it is required? Ans. $14\frac{19}{24}$ horsepower

6. On a certain day a power plant had on hand $284\frac{1}{3}$ tons of coal. During the day $25\frac{1}{2}$ tons of coal were received, and $50\frac{1}{6}$ tons were consumed. How many tons of coal remained at the end of the day? Ans. $259\frac{2}{3}$ tons

7. A filling box in a textile mill contains $10\frac{3}{4}$ pounds of yarn. When $7\frac{13}{16}$ pounds of the yarn is removed from the box, how many pounds remain? Ans. $2\frac{15}{16}$ lb.

8. At the beginning of a voyage of a large steamship, an engineer's storekeeper had on hand $50\frac{7}{12}$ pounds of various sizes of rod packing. During a period of 6 days, all the hoisting engines, anchor windlasses, steering engines, and other auxiliary machinery were repacked, the quantity used each day being as follows:

Monday	$6\frac{1}{8}$ lb.	Thursday	$6\frac{5}{8}$ lb.
Tuesday	$5\frac{3}{4}$ lb.	Friday	$5\frac{7}{8}$ lb.
Wednesday	$6\frac{1}{4}$ lb.	Saturday	$4\frac{1}{2}$ lb.

How many pounds of packing remained on hand? Ans. $15\frac{11}{24}$ lb.

9. A steamship has to run a distance of $498\frac{3}{4}$ miles. If during the first day she made $374\frac{2}{5}$ miles, how many miles remained to complete the voyage? Ans. $124\frac{7}{20}$ miles

10. A certain factory requires a total motor installation of $27\frac{1}{2}$ horsepower. Three motors have been purchased, one rated at 5 horsepower, a second at $\frac{1}{4}$ horsepower, and a third at $\frac{2}{3}$ horsepower. What must be the combined horsepower of the motors that still remain to be purchased, to give the required total of $27\frac{1}{2}$ horsepower? Ans. $21\frac{7}{12}$ horsepower

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MULTIPLICATION OF FRACTIONS

28. **Principle of Multiplication Explained.**—Fractions can be multiplied without reducing them to a common denominator. A fraction is multiplied by a whole number when the numerator of the fraction is multiplied by that number. For example, the expression $\frac{3}{4} \times 3$ means add the fraction $\frac{3}{4}$ three times; thus, $\frac{3}{4}+\frac{3}{4}+\frac{3}{4}$. As these fractions have a common denominator their sum is found, according to the rule in Art. 25,

by adding the numerators and writing the sum over the common denominator. Hence, $\frac{3}{4}\times 3=\frac{9}{4}$, or $2\frac{1}{4}$.

The word "of" when placed between two fractions, or between a fraction and a whole number, means the same as ×, or *times*. For instance, $\frac{1}{2}$ of $\frac{6}{8}$ is equal to $\frac{1}{2}\times\frac{6}{8}$ and means that one-half of $\frac{6}{8}$ shall be taken. As $\frac{6}{8}=\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}$, it follows that one-half of these fractions is $\frac{1}{8}+\frac{1}{8}+\frac{1}{8}=\frac{3}{8}$. Hence, $\frac{1}{2}\times\frac{6}{8}=\frac{3}{8}$. This example shows the *results* obtained by multiplying $\frac{1}{2}$ by $\frac{6}{8}$, but it does not show *how* the fractions are multiplied to obtain the results. According to the succeeding rule the numerator of one fraction is multiplied by the numerator of the other to obtain the numerator of the product, and the denominator of the one is multiplied by the denominator of the other to obtain the denominator of the product. According to this rule $\frac{1}{2}\times\frac{6}{8}=\frac{6}{16}$, which product reduced to its lowest terms is equal to $\frac{3}{8}$, as before.

29. Rule for Multiplication.—The following rule applies to multiplication of fractions:

Rule.—I. *To multiply either proper or improper fractions, multiply the numerators together for a new numerator, and the denominators together for a new denominator.*

II. *To multiply one mixed number by another, reduce them both to improper fractions and then multiply them as in part I of the rule.*

III. *To multiply a fraction by a whole or by a mixed number, proceed as in part I of the rule, treating the whole number as a numerator or first reducing the mixed number to an improper fraction.*

IV. *To multiply a mixed number by a whole number, first reduce the mixed number to an improper fraction; then multiply the numerator of this fraction by the whole number, and divide the product by the denominator. Or, multiply the whole-number part of the mixed number by the whole-number multiplier, and then the fractional part by the whole-number multiplier, and add the products.*

In the preceding rule reference is made to the multiplication of two numbers only. This was done so as to keep the rule as simple as possible. But it is to be understood that the rule also applies when more than two numbers are to be multiplied together. The application of the rule is illustrated by the following examples:

EXAMPLE 1.—Multiply $\frac{5}{8}$ by $\frac{3}{16}$.

SOLUTION.—Applying part I of the rule, $\frac{5}{8}\times\frac{3}{16}=\frac{15}{128}$. Ans.

EXAMPLE 2.—Multiply $2\frac{3}{4}$ by $4\frac{1}{8}$.

SOLUTION.—Applying part II of the rule, the mixed numbers are reduced to improper fractions. Thus, $2\frac{3}{4}=\frac{11}{4}$, and $4\frac{1}{8}=\frac{33}{8}$; then, $\frac{11}{4}\times\frac{33}{8}=\frac{363}{32}$ $=11\frac{11}{32}$. Ans.

EXAMPLE 3.—Multiply $\frac{3}{8}$ by 2.

SOLUTION.—Part III of the rule is applied. Thus, $\frac{3}{8}\times 2=\frac{3\times 2}{8}=\frac{6}{8}=\frac{3}{4}$. Ans.

EXAMPLE 4.—Multiply $3\frac{4}{5}$ by 5.

SOLUTION.—By the first method in part IV of the rule, $3\frac{4}{5}$ is reduced to an improper fraction and becomes equal to $\frac{19}{5}$; then, $\frac{19}{5}\times 5=\frac{95}{5}=19$. Ans.

By the second method, $5\times 3\frac{4}{5}=5\times 3+5\times\frac{4}{5}=15+\frac{20}{5}=15+4=19$. Ans.

EXAMPLE 5.—Multiply $\frac{3}{8}$ by $\frac{2}{3}$ and by 5.

SOLUTION.—Parts I and III of the rule are applied. Thus, $\frac{3}{8}\times\frac{2}{3}\times 5$ is equal to $\frac{3}{8}\times\frac{2}{3}\times\frac{5}{1}=\frac{30}{24}=1\frac{6}{24}=1\frac{1}{4}$. Ans.

30. Cancelation of Fractions.—It is seen from the preceding rules and examples that when several fractions are multiplied together, a product is found of the numerators and of the denominators, these products representing, respectively, the numerator and the denominator in a new fraction. It follows that the product $\frac{3}{4}\times\frac{8}{9}\times\frac{10}{16}$ may be represented in this form: $\frac{3\times 8\times 10}{4\times 9\times 16}$. That is, if a number of fractions are to be multiplied together, they may be represented as *one* fraction in which all the numerators are written above the line and connected by multiplication signs, and all the denominators are written below the line, likewise connected by multiplication signs.

31. By arranging the fractions in the manner just described, the operation of multiplication may often be shortened. Thus, instead of multiplying the numerators together and then multiplying the denominators together, it may be possible to divide the numerators and the denominators by the *same* number, just as in reducing a fraction to its lowest terms. Doing this will change the *form* of the fraction, but not its *value*, as was proved in Art. **11.** In the example $\frac{3\times8\times10}{4\times9\times16}$ there is a 3 in the numerator and a 9 in the denominator; it is therefore possible to divide by 3. That this division has been done is indicated by drawing a line through the respective numbers and writing the quotient of each division near the dividend. This first step will then be

$$\frac{\overset{1}{\cancel{3}}\times8\times10}{4\times\underset{3}{\cancel{9}}\times16}$$

showing that the divisor 3 is contained once in 3 and 3 times in 9. Again, 8 is above the line and 16 below it, and both of these can be divided by 8. The next step is

$$\frac{\overset{1}{\cancel{3}}\times\overset{1}{\cancel{8}}\times10}{4\times\underset{3}{\cancel{9}}\times\underset{2}{\cancel{16}}}$$

Now, there is a 10 above the line and a 4 below it, and both of these can be divided by 2 without any remainder. The next step then is

$$\frac{\overset{1}{\cancel{3}}\times\overset{1}{\cancel{8}}\times\overset{5}{\cancel{10}}}{\underset{2}{\cancel{4}}\times\underset{3}{\cancel{9}}\times\underset{2}{\cancel{16}}}$$

This process cannot be carried any farther, because there are no numbers above and below the line that can be divided by the same number without a remainder. Therefore, all the uncrossed numbers in the numerator are multiplied together, or $1\times1\times5=5$; next, all the uncrossed numbers in the denominator are multiplied together, or $2\times3\times2=12$. The first product is then written over the second, giving $\frac{5}{12}$ as the reduced fraction. In other words, $\frac{3}{4}\times\frac{8}{9}\times\frac{10}{16}=\frac{5}{12}$. This can be proved very simply; for example, $\frac{3\times8\times10}{4\times9\times16}=\frac{240}{576}$, which can be reduced to its lowest terms by dividing both terms by 48. Thus, $\frac{240\div48}{576\div48}=\frac{5}{12}$. This shows that the process of dividing numbers above and below the line of a fraction by the *same* number will give the same result as taking the product of all the numerators and dividing it by the product of all the denominators.

The operation of dividing the numbers above and below the line by the same number is called **cancelation,** and when a line is drawn through one of the numbers, as shown, the number is said to be *canceled.* In cancelation, if the quotient is 1, it is not written at all, but it is understood. The preceding example, therefore, would commonly be written

$$\frac{\cancel{3}\times\cancel{8}\times\overset{5}{\cancel{10}}}{\underset{2}{\cancel{4}}\times\underset{3}{\cancel{9}}\times\underset{2}{\cancel{16}}}=\tfrac{5}{12}$$

32. It is important to remember that cancelation can be used only in cases where the numbers in both numerator and denominator are *multiplied* together. Where addition or subtraction is indicated in either term of the fraction, cancelation cannot be used.

The following examples show the advantage of employing cancelation in the multiplication of fractions:

Example 1.—What is the product of $\frac{4}{16}$ and $\frac{7}{8}$?

Solution.— $\frac{4}{16}\times\frac{7}{8}=\dfrac{4\times7}{16\times8}=\frac{28}{128}=\frac{7}{32}$. Ans.

Or, by cancelation, $\dfrac{\cancel{4}\times7}{\underset{4}{\cancel{16}}\times8}=\dfrac{7}{4\times8}=\frac{7}{32}$. Ans.

Example 2.—What is $\frac{4}{8}$ of $\frac{3}{4}$ of $\frac{16}{32}$?

Solution.— $\dfrac{\cancel{4}\times3\times\cancel{16}}{8\times\cancel{4}\times\underset{2}{\cancel{32}}}=\dfrac{3}{8\times2}=\frac{3}{16}$. Ans.

Example 3.—What is the product of $32\frac{5}{8}$ and 24?

Solution.—The mixed number $32\frac{5}{8}$ when reduced to an improper fraction becomes $\dfrac{32\times8+5}{8}=\frac{261}{8}$. Applying the first half of part IV of the rule, Art. **29,**

$$\tfrac{261}{8}\times24=\frac{261\times24}{8}=\frac{261\times\overset{3}{\cancel{24}}}{\cancel{8}}=261\times3=783.\quad\text{Ans.}$$

The solution of this example may also be found by means of the method given in the second half of part IV of the rule, Art. **29**, as follows: $32\times24=768$, and $\frac{5}{8}\times24=\frac{5\times\overset{3}{\cancel{24}}}{\cancel{8}}=15$. Then, according to the rule, it is necessary to add the two products, giving $768+15=783$. That is,

$$32\tfrac{5}{8}\times24=783.\quad \text{Ans.}$$

EXAMPLES FOR PRACTICE

1. Find the product of:

(*a*) $7\times\frac{3}{19}$.	Ans. (*a*) $1\frac{2}{19}$
(*b*) $14\times\frac{5}{16}$.	(*b*) $4\frac{3}{8}$
(*c*) $\frac{21}{32}\times\frac{5}{14}\times\frac{8}{3}$.	(*c*) $\frac{5}{8}$

2. How long must a piece of stock be to make 4 bolts, if each bolt requires a piece of stock $3\frac{7}{8}$ inches long, including the allowance for cutting?
Ans. $15\frac{1}{2}$ inches

3. Eleven holes are spaced equally in a straight line, as in Fig. 11.

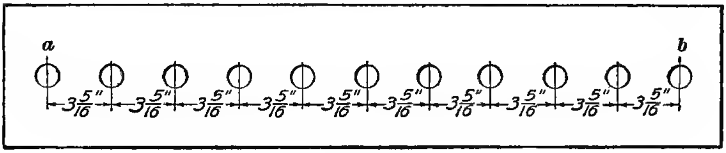


FIG. 11

What is the distance between the centers of the end holes *a* and *b*, if the spacing from center to center of two adjacent holes is $3\frac{5}{16}$ inches?
Ans. $33\frac{1}{8}$ inches

NOTE.—The number of *spaces* is one less than the number of holes.

4. A single belt can transmit $107\frac{2}{3}$ horsepower, but as it is desired to use more power, a double belt of the same width is substituted for it. Suppose that the double belt is capable of transmitting $1\frac{3}{7}$ times as much power as the single belt; how many horsepower can be used after the change?
Ans. $153\frac{17}{21}$ horsepower

5. At $303\frac{3}{5}$ pounds per mile, what is the weight of $2\frac{5}{8}$ miles of copper wire?
Ans. $796\frac{19}{20}$ lb.

6. The grate of a steam boiler contains $20\frac{1}{2}$ square feet. If the boiler burns $8\frac{3}{10}$ pounds of coal an hour per square foot of grate area and can evaporate $7\frac{1}{2}$ pounds of water an hour per pound of coal burned, how many pounds of water are evaporated by the boiler in 1 hour? Ans. $1,276\frac{1}{8}$ lb.

7. If an electrician receiving 70 cents an hour works on a job at different times $8\frac{1}{3}$ hours, $7\frac{1}{4}$ hours, $6\frac{2}{3}$ hours, and $5\frac{1}{2}$ hours, how much does he earn in dollars and cents, 100 cents equaling 1 dollar?
Ans. 19 dollars $42\frac{1}{2}$ cents

8. A mechanic works 56 hours in one week at $50\frac{1}{2}$ cents per hour. How much pay does he receive for that week? Ans. 28 dollars 28 cents

9. Condensed water is discharged by a return pipe at the rate of $73\frac{2}{5}$ gallons per hour. If the discharge takes place during a period of $10\frac{1}{5}$ hours per day, how many gallons will be discharged in $9\frac{3}{4}$ days?
Ans. $7{,}299\frac{63}{100}$ gal.

NOTE.—A return pipe conveys the condensed steam in a heating system back to the steam boiler in which the steam was generated.

10. A foundry turns out 85 cast-iron pulleys weighing $62\frac{5}{8}$ pounds apiece. What is the total amount of metal used? Ans. $5{,}323\frac{1}{8}$ lb.

DIVISION OF FRACTIONS

33. Compound Fractions.—The sign of division used for whole numbers is also used for fractions, as $\frac{1}{2} \div \frac{3}{4}$, or the division may be indicated by writing the dividend above the divisor with a heavy line between them, as $\dfrac{\frac{1}{2}}{\frac{3}{4}}$.

A fraction whose terms are whole numbers, as $\frac{1}{2}$ or $\frac{3}{4}$, is called a *simple fraction* in distinction from a **compound fraction,** either or both of the terms of which are simple fractions. The fractions $\dfrac{\frac{1}{2}}{\frac{3}{4}}$, $\dfrac{1}{\frac{3}{4}}$, and $\dfrac{\frac{1}{2}}{3}$ are compound fractions.

34. Reciprocal of a Fraction.—Any whole number may be considered as a fraction with 1 as its denominator; thus, 16 may be written $\frac{16}{1}$. The **reciprocal** (pronounced *ree-sip'-ro-kal*) of a fraction is obtained by inverting the fraction; that is, turning it upside down by writing the denominator *above* the line and the numerator *below*. Thus, the reciprocal of 12, which may be written $\frac{12}{1}$, is $\frac{1}{12}$; the reciprocal of 2 is $\frac{1}{2}$, etc. The reciprocal of the fraction $\frac{3}{4}$ is $\frac{4}{3}$, and of $\frac{11}{16}$ is $\frac{16}{11}$.

35. Preliminary Remarks on Division of Fractions. At first it may seem difficult to understand what is meant by dividing a number by a fraction or dividing one fraction by another. An attempt will therefore be made in the following to make the subject clearer. The case of dividing by a whole number will be dealt with first, as, for instance, $12 \div 3$. The question to be answered by this expression is: How many times is the number 3 contained in the number 12? Or, into how many parts may 12 be divided, if each part is to consist of 3 units? Or, by what must 3 be multiplied in order to make the product 12? The answer to all three questions is 4.

As the divisor decreases in value, the quotient will increase, until finally, with a divisor equal to 1, the dividend and the quotient will be equal. For instance, $12 \div 1 = 12$, showing that the number 12 may be divided into 12 parts, each equal to 1. Judging from this fact, one may draw the conclusion that on dividing by a number *smaller* than 1, as, for instance, with a proper fraction, the quotient will become greater than the dividend, and this is found to be the case.

36. Proceeding now to division by fractions, the first example chosen for further consideration is $9 \div \frac{1}{3}$. The problem indicated by this expression is to ascertain how many times the fraction $\frac{1}{3}$ is contained in the number 9. Or, by what number must $\frac{1}{3}$ be multiplied in order to make the product 9? As the number 1 contains 3 parts, each equal to one-third of the number, it follows, that the number of one-thirds contained in 9 must be equal to $9 \times 3 = 27$. In other words, $\frac{1}{3}$ must be multiplied by 27 to make the product 9. This example shows that when dividing by a fraction having 1 as a numerator, the quotient is found by multiplying the dividend by the denominator of the divisor. That is, the divisor is *inverted*, or turned upside down, and used as a multiplier. Thus, $9 \div \frac{1}{3} = 9 \times \frac{3}{1} = 27$. Multiplying by the inverted fraction is equivalent to multiplying by the reciprocal; the latter term was defined in Art. **34.**

The same reasoning holds good in the case of dividing one fraction by another. For instance, in the expression $\frac{4}{5} \div \frac{1}{5}$ it is required to find how many times $\frac{1}{5}$ is contained in $\frac{4}{5}$. As the fraction $\frac{4}{5}$ is 4 times as great as $\frac{1}{5}$, it is evident that $\frac{1}{5}$ is contained 4 times in $\frac{4}{5}$, and that the quotient must be 4. The same result is obtained by inverting the divisor and multiplying; thus, $\frac{4}{5} \div \frac{1}{5} = \frac{4}{5} \times \frac{5}{1} = \frac{20}{5} = 4$.

In this case, also, the quotient is greater than either the divisor or the dividend.

If, in the preceding expression, the divisor $\frac{1}{5}$ is replaced by $\frac{2}{5}$, as $\frac{4}{5}\div\frac{2}{5}$, it is evident that as the divisor has been doubled in size, the number of times it is contained in the dividend is correspondingly decreased. In this case the dividend must be divided by 2; or, $4\div2=2$, instead of 4, as in the preceding example. This result is also obtained by multiplying by the inverted divisor, or $\frac{4}{5}\div\frac{2}{5}=\frac{4}{5}\times\frac{5}{2}=\frac{20}{10}=2$.

After these preliminary explanations the subject of dividing fractions will be considered more in detail.

37. Rule for Dividing Fractions.—One fraction can be divided by another fraction without reducing them to a common denominator.

In dividing fractions the following rule applies:

Rule.—*To divide by a fraction, multiply by its reciprocal. Reduce mixed numbers to improper fractions before dividing.*

Example 1.—Divide $\frac{3}{4}$ by $\frac{5}{16}$.

Solution.—In this case, the divisor is $\frac{5}{16}$, and its reciprocal is $\frac{16}{5}$. Then, according to the rule,

$$\frac{3}{4}\div\frac{5}{16}=\frac{3}{4}\times\frac{16}{5}=\frac{3\times\overset{4}{\cancel{16}}}{\cancel{4}\times5}=\frac{12}{5}=2\frac{2}{5}.\quad\text{Ans.}$$

Example 2.—Divide $\frac{3}{4}$ by 3.

Solution.—The divisor is 3, and its reciprocal is $\frac{1}{3}$. According to the rule, the fraction is to be multiplied by this reciprocal; then,

$$\frac{3}{4}\div3=\frac{3}{4}\times\frac{1}{3}=\frac{\cancel{3}\times1}{4\times\cancel{3}}=\frac{1}{4}.\quad\text{Ans.}$$

Example 3.—Divide 48 by $\frac{3}{16}$.

Solution.—The divisor is $\frac{3}{16}$ and its reciprocal is $\frac{16}{3}$. Then,

$$48\div\frac{3}{16}=48\times\frac{16}{3}=\frac{\overset{16}{\cancel{48}}\times16}{\cancel{3}}=256.\quad\text{Ans.}$$

Example 4.—Divide $13\frac{3}{4}$ by $2\frac{1}{2}$.

Solution.—First reduce both mixed numbers to improper fractions; thus, $13\frac{3}{4}=\frac{55}{4}$ and $2\frac{1}{2}=\frac{5}{2}$. Then

$$13\frac{3}{4}\div2\frac{1}{2}=\frac{55}{4}\div\frac{5}{2}=\frac{55}{4}\times\frac{2}{5}=\frac{\overset{11}{\cancel{55}}\times\cancel{2}}{\underset{2}{\cancel{4}}\times\cancel{5}}=\frac{11}{2}=5\frac{1}{2}.\quad\text{Ans.}$$

EXAMPLE 5.—A bar of steel $23\frac{5}{8}$ inches long is divided into 7 equal parts. What is the length of each part?

SOLUTION.—The length of each part must be $23\frac{5}{8} \div 7$. Now, $23\frac{5}{8} = \frac{189}{8}$. The reciprocal of 7 is $\frac{1}{7}$. Then, applying the rule,

$$\frac{189}{8} \div 7 = \frac{189}{8} \times \frac{1}{7} = \frac{\overset{27}{\cancel{189}} \times 1}{8 \times \cancel{7}} = \frac{27}{8} = 3\frac{3}{8}$$

Therefore, the length of each part is $3\frac{3}{8}$ inches. Ans.

EXAMPLES FOR PRACTICE

1. In the following examples, divide:

   (*a*) 15 by $6\frac{3}{7}$.
   (*b*) 172 by $\frac{4}{5}$.
   (*c*) $\frac{103}{6}$ by $14\frac{2}{3}$.

   Ans. (*a*) $2\frac{1}{3}$; (*b*) 215; (*c*) $1\frac{15}{88}$

2. How many gears, each $1\frac{1}{8}$ inches thick, can be set side by side in a space 45 inches long? Ans. 40

3. A distance measuring $34\frac{7}{32}$ inches is divided into 15 equal parts. What is the length of each part? Ans. $2\frac{9}{32}$ inches

4. A boiler plate containing 24 square feet of surface weighs $362\frac{4}{10}$ pounds. What is its weight per square foot? Ans. $15\frac{1}{10}$ lb.

5. A certain boiler has $927\frac{1}{2}$ square feet of heating surface, which is equal to 35 times the area of the grate. What is the area of the grate in square feet? Ans. $26\frac{1}{2}$

6. If the distance around the rim of a locomotive driving wheel is $13\frac{1}{12}$ feet, how many revolutions will the wheel make in traveling 682 feet? Ans. $52\frac{20}{157}$ rev.

7. A tank containing 1,830 gallons of water supplies a family whose daily consumption, according to a meter, averages $521\frac{1}{2}$ gallons. How many days' supply does the tank contain? Ans. $3\frac{531}{1043}$ days

8. If one horsepower will drive $3\frac{3}{4}$ cotton looms of a certain kind, how much power will be required for a weave room containing 600 looms? Ans. 160 horsepower

9. How many bolts can be obtained from a steel bar $15\frac{3}{4}$ inches in length, if each bolt requires a piece $3\frac{15}{16}$ inches long? Ans. 4 bolts

10. A piece of land having an area of $12\frac{3}{4}$ acres is to be divided into plots, each plot containing $\frac{3}{8}$ of an acre. How many plots will there be? Ans. 34 plots

COMBINED PROCESSES

MULTIPLICATION AND DIVISION COMBINED WITH ADDITION AND SUBTRACTION

38. Order of Operations.—When the numerator or the denominator of a fraction contains several numbers or fractions connected by signs that indicate different processes, such processes must be performed in the order used in the case of whole numbers; that is, unless otherwise indicated, multiplication and division must precede addition and subtraction. The method to be used is shown in the following examples:

EXAMPLE 1.—Find the value of $\frac{\frac{1}{4}+\frac{2}{3}\times 9}{10\frac{15}{16}-\frac{3}{4}\times 12}$.

SOLUTION.—As multiplication must precede addition, first find the product of $\frac{2}{3}\times 9$. Add this product to $\frac{1}{4}$ and divide the sum by the difference between $10\frac{15}{16}$ and the product of $\frac{3}{4}\times 12$. The numerator, or dividend, is $\frac{2}{\cancel{3}}\times\overset{3}{\cancel{9}}=6$; $\frac{1}{4}+6=6\frac{1}{4}=\frac{25}{4}$. The denominator, or divisor, is

$$\frac{3}{\cancel{4}}\times\overset{3}{\cancel{12}}=9;\ 10\frac{15}{16}-9=1\frac{15}{16}=\frac{31}{16}$$

Therefore,

$$\frac{\frac{1}{4}+\frac{2}{3}\times 9}{10\frac{15}{16}-\frac{3}{4}\times 12}=\frac{25}{4}\div\frac{31}{16}=\frac{25}{\cancel{4}}\times\frac{\overset{4}{\cancel{16}}}{31}=\frac{25\times 4}{31}=\frac{100}{31}=3\frac{7}{31}.\quad \text{Ans.}$$

EXAMPLE 2.—Find the value of $\frac{1}{\frac{1}{4}+\frac{1}{5}+\frac{1}{7}+\frac{1}{10}}$.

SOLUTION.—The denominator $\frac{1}{4}+\frac{1}{5}+\frac{1}{7}+\frac{1}{10}=\frac{35}{140}+\frac{28}{140}+\frac{20}{140}+\frac{14}{140}=\frac{97}{140}$. Therefore, the whole fraction

$$\frac{1}{\frac{1}{4}+\frac{1}{5}+\frac{1}{7}+\frac{1}{10}}=\frac{1}{\frac{97}{140}}=1\div\frac{97}{140}=1\times\frac{140}{97}=\frac{140}{97}=1\frac{43}{97}.\quad \text{Ans.}$$

EXAMPLES FOR PRACTICE

Find the value of the following expressions:

(*a*) $\dfrac{2}{\frac{1}{3}+\frac{1}{8}+\frac{1}{12}}$.

(*b*) $4+\dfrac{1}{\frac{1}{9}+\frac{1}{18}+\frac{1}{36}+\frac{1}{72}}$.

(*c*) $\dfrac{1}{\frac{1}{4}+\frac{1}{6}+\frac{1}{12}+\frac{1}{18}}+\frac{2}{3}\times3\frac{3}{8}$.

Ans. $\left\{\begin{array}{l}(a)\ 3\frac{9}{13}\\ (b)\ 8\frac{4}{5}\\ (c)\ 4\frac{1}{20}\end{array}\right.$

FACTORS AND AVERAGES

39. Factors.—In some calculations it is necessary to *factor* a fraction or a whole number, that is, to separate the fraction or the number into its factors. The **factors** of a number are simply those numbers which, when multiplied together, will equal the number. Thus, 2 and 3 are factors of 6, because $2\times3=6$; 2 and 5 are factors of 10, because $2\times5=10$; 3 and 4 are factors of 12, because $3\times4=12$. But 6 and 2 are also factors of 12, because $6\times2=12$; also, 3, 2, and 2 are factors of 12, because $3\times2\times2=12$.

A fraction is factored by finding the factors of its numerator and denominator. Thus, the factors of $\frac{10}{16}$ are $\frac{5}{4}$ and $\frac{2}{4}$, because $\frac{10}{16}=\dfrac{5\times2}{4\times4}=\frac{5}{4}\times\frac{2}{4}$.

Some numbers have no factors except themselves and 1; thus, the factors of 5 are 1 and 5, because $1\times5=5$. The factors of 7 are 1 and 7; of 11, 1 and 11; of 13, 1 and 13; and so on. Such numbers are called prime numbers, as explained in Art. **6.**

The same number may often be factored in several different ways; thus, $24=6\times4$, or $3\times2\times4$, or 3×8, or $6\times2\times2$, or $3\times2\times2\times2$, because the product of each of these groups of numbers is 24. Similarly, the fraction $\frac{24}{30}$ has the factors $\frac{4}{6}\times\frac{6}{5}$, $\frac{3}{10}\times\frac{8}{3}$, $\frac{2}{3}\times\frac{12}{10}$, and many others.

40. Averages.—It often becomes necessary to find the average of several numbers; for instance, a number of men will require different lengths of time in which to do a piece of work, and it may be required to find the average time.

The **average** of several terms is equal to their sum divided by the number of terms.

EXAMPLE 1.—Find the average of the numbers 6, 12, 8, 10, 11, and 7.

SOLUTION.—There are six numbers, and their sum is $6+12+8+10+11+7=54$. Then, the average is $54\div 6=9$. Ans.

EXAMPLE 2.—If five gear-wheels weigh 28, 12, 36, 25, and 14 pounds, respectively, what is the average weight per wheel?

SOLUTION.—There are five wheels, and the sum of their weights is $28+12+36+25+14=115$ pounds. The average is therefore $115\div 5=23$ pounds. Ans.

It will be noted that if the average is multiplied by the number of things it will give the sum. Thus, in example 1 the average of six numbers is 9; hence, the sum of the six numbers is $9\times 6=54$. In example 2, the average weight of the gears is 23 pounds; hence, the weight of the five gears is $23\times 5=115$ pounds.

EXAMPLES FOR PRACTICE

1. A workman finishes 23 pieces of work on Monday, 26 on Tuesday, 19 on Wednesday, 27 on Thursday, and 20 on Friday. What is the average per day? Ans. 23

2. Six workmen engaged on the same class of work require 48, 42, 56, 50, 45, and 51 minutes, respectively. What is the average time for the work? Ans. $48\frac{2}{3}$ minutes

3. If the average weight of eight castings is $334\frac{1}{2}$ pounds each, what is the total weight of the castings? Ans. 2,676 lb.

4. Six rivets measure, respectively, $2\frac{1}{4}$, $2\frac{15}{16}$, $2\frac{7}{8}$, $2\frac{3}{8}$, $2\frac{7}{16}$, and $2\frac{1}{2}$ inches in length. What is the average length? Ans. $2\frac{9}{16}$ inches

5. What prime numbers are factors of: (*a*) 210? (*b*) 1,001?

Ans. { (*a*) 2, 3, 5, and 7
(*b*) 7, 11, and 13

DECIMALS

FUNDAMENTAL OPERATIONS

DEFINITIONS, NUMERATION, AND NOTATION

DEFINITIONS OF TERMS AND NUMBERS

1. Decimal Fractions.—In many branches of industry and in nearly all books of a scientific nature, it is found advantageous to use a form of fraction known as a *decimal,* or *decimal fraction,* as by its use the main fundamental processes of arithmetic are greatly simplified.

Decimals are *tenth* fractions; that is, they indicate *tenth* parts of a unit or their subdivisions into *hundredths, thousandths,* etc. The denominator of a decimal consists, therefore, always of the figure 1 followed by one or more ciphers, as 10, 100, 1,000, etc.; for this reason it is possible to use an abbreviated method in writing decimal fractions and to dispense entirely with the method adopted in writing common fractions. Thus, instead of drawing a line below the numerator and writing the denominator below it, the line and the denominator are omitted in a decimal fraction and replaced by a point.

2. Decimal Point.—The **decimal point** (.) used in decimals is written at the left of the number expressing the numerator. For example, .3 is a decimal, and it is read *three tenths.* Its value is the same as the fraction $\frac{3}{10}$; that is, $.3=\frac{3}{10}$. Every decimal consists of a decimal point followed by one or more figures. For instance, .25, .625, .73056, etc.

NOTE.—In English technical literature the decimal point is above the base line and before the middle of the figure, as ·32. On the European continent the decimal point is, in general, replaced by a comma placed on the base line in the usual position.

3. Decimal Places.—Each figure to the right of a decimal point occupies a **decimal place.** These decimal places are of great importance in decimals, as the place occupied by a figure serves as a means for indicating the value of its denominator. When referring to the number of figures after a decimal point, it is customary to state the number of *places* occupied by them. Thus, in the decimals .47 and .372, there are two and three places, respectively.

4. Numerator and Denominator.—The **numerator** of a decimal fraction is the number that follows the decimal point, and the **denominator** is the figure 1 followed by as many ciphers as there are places in the decimal. To illustrate, take the decimal .42. The number following the decimal point is 42, which is the numerator; the figures after the decimal point occupy two places, that is, the decimal has *two places.* Hence, the denominator is 1 followed by two ciphers, or 100. Then, the value of .42 is $\frac{42}{100}$, and it is read *forty-two hundredths.*

5. Changing Value of Denominator.—The value of the denominator of a decimal fraction depends on the number of decimal places. Thus:

One decimal place expresses *tenths*
Two decimal places express *hundredths*
Three decimal places express *thousandths*
Four decimal places express *ten-thousandths*
Five decimal places express *hundred-thousandths*
Six decimal places express *millionths*

The following series of decimals shows how the number of places in a decimal fraction affects the value of its denominator, the same numerator, 3, being used in each decimal:

$.3 = \frac{3}{10} =$ 3 tenths
$.03 = \frac{3}{100} =$ 3 hundredths
$.003 = \frac{3}{1000} =$ 3 thousandths
$.0003 = \frac{3}{10000} =$ 3 ten-thousandths
$.00003 = \frac{3}{100000} =$ 3 hundred-thousandths
$.000003 = \frac{3}{1000000} =$ 3 millionths

From these examples it is seen that to change a decimal to a common fraction, the number in the decimal is written as the *numerator,* and the numeral 1 followed by as many ciphers as there are decimal places in the decimal is written as the *denominator.* It is important to remember these facts. For example, let it be required to write the decimal .5625 as a common fraction. The number 5625 is the numerator of the fraction; there are four decimal places, hence the denominator is 1 and four ciphers, or 10,000, and the complete fraction becomes $\frac{5625}{10000}$.

6. Mixed Numbers.—Very often there are figures written to the left of the decimal point as well as to the right of it, as, for example, 12.5, 7.25, and 18.125. These are called **mixed numbers,** because each consists of a whole number and a fraction combined. The decimal point simply separates the whole number at its left from the decimal at its right. The number 12.5 is read *twelve and five tenths;* 7.25 is read *seven and twenty-five hundredths;* 18.125 is read *eighteen and one hundred twenty-five thousandths.* Another method of reading these numbers is as follows: The number 12.5 is read *twelve point five;* 7.25 is read *seven point twenty-five;* 18.125 is read *eighteen point one hundred twenty-five.*

Sometimes decimals are written with a cipher to the *left* of the decimal point, as 0.6. The cipher in this case merely shows that there is no whole number. A decimal thus written has the same value, and is read in the same way, as though the cipher were omitted.

NUMERATION AND NOTATION

7. Numeration of Decimals.—The relation of decimals and whole numbers is clearly shown by the following table:

hundreds of millions	tens of millions	millions	hundreds of thousands	tens of thousands	thousands	hundreds	tens	units	decimal point	tenths	hundredths	thousandths	ten-thousandths	hundred-thousandths	millionths	ten-millionths	hundred-millionths
9	8	7	6	5	4	3	2	1	.	2	3	4	5	6	7	8	9

As stated in Art. **1**, decimals are *tenth* fractions, that is, they are based on the scale of ten, each division and subdivision being divided in tenths. In the preceding table the starting point is the *units* place in both decimals and whole numbers. The decimals *decrease* on the scale of ten to the *right,* and the whole numbers *increase* on the scale of ten to the *left.* The first figure to the left of units is *tens,* and the first figure to the right of units is *tenths.* The second figure to the left of units is *hundreds,* and the second figure to the right is *hundredths.* The third figure to the left is *thousands,* and the third to the right is *thousandths,* and so on. The figures equally distant from units place correspond in name, the decimals having the ending *ths,* to distinguish them from the whole numbers.

8. In **reading** a decimal, the number indicating the numerator is read first as if it were a whole number, and then the denominator is stated. Thus, .725 is read *seven hundred twenty-five thousandths;* .0175 is read *one hundred seventy-five ten-thousandths.*

In reading mixed numbers, the whole number is read first, followed by the word *and* and then the decimal. For example, the mixed number in the table of Art. **7** is read *nine hundred eighty-seven million six hundred fifty-four thousand three hundred twenty-one and twenty-three million four hundred fifty-six thousand seven hundred eighty-nine hundred-millionths.*

9. Notation of Decimals.—In **writing** a decimal, the number indicating the numerator is written first and the position of the decimal point is then located, so as to give as many decimal places as there are ciphers in the denominator. The places are counted from the right-hand figure of the numerator toward the left, and ciphers are prefixed to the numerator if this is necessary to obtain the required number of places before placing the decimal point. Locating the decimal point is called **pointing off the decimal.**

Example.—Express seventeen thousandths as a decimal.

Solution.—The numerator is 17; the denominator is 1,000, in which there are 3 ciphers. From Art. 4 and the preceding rule there should be as many places in the decimal as there are ciphers in the denominator; that is, there must be three places. As the number 17 requires only two places, it must be preceded by a cipher to obtain the required three decimal places, making it .017. Ans.

10. In writing decimals the following points should be remembered:

1. *Annexing ciphers to the right of a decimal or removing ciphers from the right of a decimal does not change its value.*

For example, .5=.50, because $\frac{5}{10}=\frac{50}{100}$, also .1200=.12, because $\frac{1200}{10000}=\frac{12}{100}$.

2. *Inserting a cipher between the decimal point and the first figure of the decimal divides the decimal by 10.*

For example, .5÷10=.05, because $.5=\frac{5}{10}$ and $\frac{5}{10}\div 10=\frac{5}{10}\times\frac{1}{10}=\frac{5}{100}$, or .05. Likewise, .05÷10=.005; .005÷10=.0005; etc. The last statement may, therefore, also be made in the following form:

Moving the decimal point to the left in a number divides the number by 10 for each place that the point is moved.

3. *Taking away a cipher from the left of a decimal multiplies the decimal by 10.* For example, .0005×10=.005, because $.0005=\frac{5}{10000}$ and $\frac{5}{10000}\times 10=\frac{50}{10000}=\frac{5}{1000}=.005$. Likewise, .005×10=.05; .05×10=.5; etc.

The last statement may also be given the following wording:

Moving the decimal point to the right in a number multiplies the number by 10 for each place the point is moved.

11. In some cases, a mixed number containing a decimal may be more conveniently expressed in the form of a common improper fraction. To do so, it is only necessary to write the entire number, omitting the decimal point, as the numerator of the fraction, and the denominator of the decimal part as the denominator of the fraction. Thus, $127.483=\frac{127483}{1000}$; for, $127.483=127\frac{483}{1000}=\frac{127000}{1000}+\frac{483}{1000}=\frac{127000+483}{1000}=\frac{127483}{1000}$.

EXAMPLES FOR PRACTICE

Express decimally:

(*a*)	Fourteen ten-thousandths.	Ans. (*a*) .0014
(*b*)	Forty-seven millionths.	(*b*) .000047
(*c*)	Four and two tenths.	(*c*) 4.2
(*d*)	Seven hundred twenty-five hundred-thousandths.	(*d*) .00725

ADDITION OF DECIMALS

12. Arranging the Numbers.—In the addition of decimals, tenths are placed under tenths, hundredths under hundredths, etc.; this, of course, brings the decimal points in a *vertical* line, that is, one directly under another. Then addition is performed exactly as in the case of whole numbers. Hence, in placing the numbers to be added, it is only necessary to take care that the *decimal points are in the same vertical line.* In adding whole numbers, the right-hand figures are always in line; but in adding decimals, the right-hand figures will not be in line unless the decimals contain the same number of figures. To insure that the right-hand figures of whole and mixed numbers are always placed in a vertical line, a whole number is sometimes represented as a mixed number by placing a decimal point at its right end, it being understood that the decimal places are occupied by ciphers. In example 1, Art. **13,** the numbers 242 and 6 are treated in this manner.

The similarity in the arrangement of whole numbers and decimals is seen from the three succeeding examples, of which the first shows the addition of whole numbers, the second that of decimals, and the third the addition of mixed numbers. In the first example the right-hand figures are all in line, but this is not the case with the examples containing decimals.

whole numbers	*decimals*	*mixed numbers*
342	.342	342.032
4234	.4234	4234.5
26	.26	26.6782
3	.03	3.06
sum 4605 Ans.	*sum* 1.0554 Ans.	*sum* 4606.2702 Ans.

13. Rule for Adding Decimals.—To add decimals, proceed according to the following rule:

Rule.—*Place the numbers to be added so that the decimal points will be directly under one another. Add as in whole numbers, and place the decimal point in the sum directly under the decimal points above.*

The application of the rule is shown in the succeeding examples.

Example 1.—What is the sum of 242, .36, 118.725, 1.005, 6, and 100.1?

Solution.—

$$\begin{array}{r@{}l}
242. & \\
 .& 36\\
118.& 725\\
1.& 005\\
6. & \\
100.& 1\\
\hline
468.& 190 \quad \text{Ans.}
\end{array}$$

Example 2.—A bar is marked off into 8 parts measuring 1.25, 4.3125, 2.305, 7.6, 10.4375, 5.5625, .875, and 3.0625 inches in length. What is the total length of the bar?

Solution.—The length of the bar is the sum of the lengths of the parts. The numbers are arranged with the decimal points in a vertical line, and added. Thus:

$$\begin{array}{r@{}l}
1.& 25\\
4.& 3125\\
2.& 305\\
7.& 6\\
10.& 4375\\
5.& 5625\\
.& 875\\
3.& 0625\\
\hline
35.& 4050 \quad \text{Ans.}
\end{array}$$

The total length, therefore, is 35.405 inches.

EXAMPLES FOR PRACTICE

1. In the following examples, find the sum of:

(*a*) .2143, .105, 2.3042, and 1.1417.
(*b*) 783.5, 21.473, .2101, and .7816.

Ans. (*a*) 3.7652
Ans. (*b*) 805.9647

2. Four round pieces of bar steel, when measured accurately, are found to have lengths of 11.25, 7.625, 1.3125, and 5.4375 inches (abbreviated in.). What is their total length when placed end to end?
Ans. 25.625 in.

3. The estimated weights of the parts of a return-tubular boiler were as follows: Shell, 3,626 pounds; tubes, 3,564.5 pounds; manhole cover, ring, and yoke, 270.34 pounds; stays, etc., 1,089.4 pounds; steam nozzles, 236.07 pounds; handhole covers and yokes, 120.25 pounds; feedpipe, 34.75 pounds; boiler supports, 350.6 pounds. What was the total estimated weight of the boiler? Ans. 9,291.91 lb.

4. The five sides of a field measure, respectively, 8.13 rods, 4.63 rods, 7.88 rods, 4.76 rods, and 9.29 rods. What is the total length of the sides? Ans. 34.69 rods

5. There are four electric lamps connected to a circuit. The lamps require current as follows: 1 ampere, .5 ampere, .3 ampere, and .25 ampere, respectively. What is the total current in amperes required by the four lamps? Ans. 2.05 amperes

6. A sample of coke was burned to test its quality. The coke was found to contain 5.79 pounds of ash, .597 pound of sulphur, and 93.613 pounds of carbon. What was the weight of the sample? Ans. 100 lb.

7. The inside width of a square steel box is 6.065 inches, and the thickness of the walls is .280 inch. What is the outside width of the box? Ans. 6.625 in.

NOTE.—*Two* opposite walls of the box must be considered in this example.

8. During a five-day voyage a steamer passes over the following distances: 384.75 miles, 372.825 miles, 356.5 miles, 392.625 miles, and 345.25 miles. What was the total distance covered? Ans. 1,851.95 miles

9. A farmer has 10.5 acres in one field, 8.75 acres in another, and 30.25 acres in a third field. How many acres does he possess in all?
Ans. 49.5 acres

10. A merchant sold to four customers the following quantities of cloth: 5.125 yards, 15.5 yards, 9.3125 yards, and 6.625 yards. What is the total length of the cloth that was sold? Ans. 36.5625 yd.

11. By carefully measuring the six sides of a tract of land it was found that the first side measured 537.68 feet, the second 87.36 feet, the third 836.39 feet, the fourth 732.12 feet, the fifth 237.26 feet, and the sixth 523.69 feet; what is the exact distance around the property?
Ans. 2,954.50 ft.

SUBTRACTION OF DECIMALS

14. Rule for Subtracting Decimals.—In the subtraction of decimals, tenths are placed under tenths, hundredths under hundredths, etc., bringing the decimal points in a vertical line, as in addition of decimals.

Rule.—*Place the subtrahend under the minuend, so that the decimal point of the subtrahend will be directly under that of the minuend. Subtract as in whole numbers, and place the decimal point in the remainder directly under the decimal points above.*

When the figures in the decimal part of the subtrahend extend beyond those in the minuend, place ciphers in the minuend above them, and subtract as in the case of whole numbers.

EXAMPLE 1.—Subtract .132 from .3063.

SOLUTION.—

minuend .3063
subtrahend .132
difference .1743 Ans.

It is understood that the fourth decimal place of the subtrahend is occupied by a cipher.

EXAMPLE 2.—A bar 7.895 inches long has a piece .725 inch long cut away from one end. What length remains?

SOLUTION.—The length remaining must be the difference between 7.895 inches and .725 inch.

minuend 7.895
subtrahend .725
difference 7.170 or 7.17 Ans.

That is, the length remaining is 7.17 inches.

EXAMPLE 3.—A block 11 inches thick has .625 inch planed from it. What is its thickness then?

SOLUTION.—The resulting thickness is the difference between 11 and .625 inches.

minuend 11.000
subtrahend .625
difference 10.375 Ans.

That is, the thickness is then 10.375 inches.

EXAMPLES FOR PRACTICE

1. In the following examples, from:

(*a*) 407.385 take 235.0004.
(*b*) 22.718 take 1.7042.

Ans. { (*a*) 172.3846
(*b*) 21.0138 }

2. A block of steel 1.0625 inches thick is cut down to a thickness of .9375 inch. What thickness of metal is removed? Ans. .125 in.

3. If a bar 3.25 inches long has a piece 1.625 inches long cut away, what is the length of the remainder? Ans. 1.625 in.

4. If the temperature of steam at 5 pounds pressure is 227.964 degrees and at 100 pounds pressure is 337.874 degrees, how many degrees hotter is the steam at the higher pressure? Ans. 109.91

5. In a cistern that will hold 326.5 barrels of water there are 178.625 barrels. How much does it lack of being full? Ans. 147.875 barrels

6. A man left to his son and to his brother portions of his estate amounting to .33 of the whole. If the portion received by the son was .25 of the estate, what portion did the brother receive? Ans. .08

7. One bar is 12.0013 inches and another bar 5.9938 inches long. What is the difference in length? Ans. 6.0075 in.

8. A liquid-measure quart has 57.75 cubic inches, and a dry-measure quart has 67.200625 cubic inches. Find the difference in cubic inches between the two kinds of quart measures. Ans. 9.450625 cubic in.

9. One cubic foot of water weighs 62.37 pounds, and one cubic foot of ice weighs 57.5 pounds. How much heavier is the water than the ice? Ans. 4.87 lb.

10. In testing a sample of coke weighing 100 pounds it was found to contain 5.78 pounds of ash and .586 pound of sulphur; the remainder was carbon. How much carbon did the sample contain?

Ans. 93.634 lb.

MULTIPLICATION OF DECIMALS

15. Rule for Multiplication.—Multiplication of decimals is similar to multiplication of whole numbers. The *right-hand* figure of the multiplier is placed under the *right-hand* figure of the multiplicand, irrespective of the relative positions of the two decimal points; in other words, the points need not be in a vertical line. The number of decimal places in the product depends on the total number of decimal places found in the multiplier and the multiplicand. The preceding statement may be formulated as a rule, as follows:

Rule.—*Place the multiplier under the multiplicand, disregarding the position of the decimal points. Multiply as with whole numbers, and in the product point off as many decimal places as there are decimal places in both multiplier and multiplicand, beginning at the right, and prefixing ciphers if necessary.*

16. Application of Rule.—The application of the preceding rule is shown in the solutions of the following examples:

Example 1.—Multiply .825 by 13.

Solution.—In this example, there are three decimal places in the multiplicand and none in the multiplier; therefore, three decimal places are pointed off in the product, beginning to count at the *right*.

multiplicand	.825	
multiplier	13	
	2475	
	825	
product	10.725	Ans.

Example 2.—What is the product of 426 and .005?

Solution.—In this example there are three decimal places in the multiplier and none in the multiplicand; therefore, three decimal places are pointed off in the product, counting from the right.

multiplicand	426	
multiplier	.005	
product	2.130 or 2.13	Ans.

17. When a decimal, serving as a multiplicand or as a multiplier, contains one or more ciphers next to the decimal point, as .0049, such ciphers have no effect on the figures in the product, but they do affect the position of its decimal point.

When multiplying decimals such as .3700, the ciphers at the right-hand end may be dropped, as they only make more figures to deal with and do not change the value of the product in any way. The following examples show the application of these statements:

Example 1.—Multiply 1.205 by 1.15.

Solution.—In the multiplicand there are three decimal places and in the multiplier two places, making in all five places to point off in the product.

multiplicand	1.205	
multiplier	1.15	
	6025	
	1205	
	1205	
product	1.38575	Ans.

Example 2.—Multiply .232 by .001.

Solution.—In this example the multiplicand is multiplied by the figure 1 of the multiplier, which gives 232 as the product. The sum of the numbers of decimal places in the multiplicand and multiplier is six, and six places must be pointed off in the product. But, as there are only three figures in the product, it is necessary to prefix three ciphers, as shown, to obtain the required number of decimal places.

multiplicand	.232	
multiplier	.001	
product	.000232	Ans.

Example 3.—Multiply .92500 by .313.

Solution.—According to the preceding statement the last ciphers in the decimal .92500 could be omitted without affecting the product. The correctness of this statement will be proved by first multiplying the decimals in their present forms and then multiplying without the ciphers in the multiplicand, the products being the same in each case.

In the solution (*a*) there must be eight decimal places in the product, eight being the sum of the decimal places in the multiplicand and multiplier. If the two ciphers are omitted from the multiplicand, as in (*b*), only six decimal places are required in the product. The two products are equal, because the two ciphers to the right of the decimal in the solution (*a*) have no value.

	(*a*)			(*b*)	
multiplicand	.92500		*multiplicand*	.925	
multiplier	.313		*multiplier*	.313	
	277500			2775	
	92500			925	
	277500			2775	
product	.28952500	Ans.	*product*	.289525	Ans.

EXAMPLES FOR PRACTICE

1. In the following examples, find the product of:

(*a*) .000492×4.1418.
(*b*) 4,003.2×1.2

Ans. { (*a*) .0020377656
(*b*) 4,803.84

2. Four equal distances of 2.375 inches are marked off, end to end, on a piece of work. What is the total distance marked off?

Ans. 9.5 in.

3. If 1 cubic inch of cast brass weighs .295 pound, what is the weight of a brass casting containing 768 cubic inches? Ans. 226.56 lb.

4. If a steam pump delivers 2.38 gallons of water per stroke and runs at 51 strokes a minute, how many gallons of water will it pump in 58½ minutes? Ans. 7,100.73 gallons

5. Wishing to obtain the weight of a connecting-rod from a drawing it was calculated that the rod contained 294.8 cubic inches of wrought iron, 63.5 cubic inches of brass, and 10.4 cubic inches of Babbitt. Assuming the weight of wrought iron to be .278 pound per cubic inch, of brass .303 pound, and of Babbitt .264 pound, what was the weight of the rod? Ans. 103.94 lb.

NOTE.—Two decimal places in the answer is sufficiently accurate in this case.

6. Three men *A*, *B*, and *C* bought a tract of land for 42,685 dollars. *A* paid .35 of this price, *B* .44, and *C* .21. How much money did each invest?

Ans. { *A*, 14,939.75 dollars
B, 18,781.40 dollars
C, 8,963.85 dollars

7. A tank contains 86 gallons of a solution of nitric acid. If in making each gallon of solution .012 gallon of nitric acid was used, how much acid was required for the 86 gallons? Ans. 1.032 gallons

8. The cost of manufacturing a certain article is 132 dollars. Of this amount .6 was spent in the foundry, .3 on drilling, and the remainder on filing. What is the cost of each operation?

Ans. { Foundry, 79.20 dollars
Drilling, 39.60 dollars
Filing, 13.20 dollars

9. If a man can shovel 1.7 cubic yards of clay into a cart in 1 hour, how many cubic yards of clay will he shovel in 4.75 days of 8 hours each? Ans. 64.6 cubic yards

10. How many bricks will be laid by a bricklayer in 13.75 days, if he can lay 1,240 bricks per day? Ans. 17,050 bricks

DIVISION OF DECIMALS

18. Rule for Division.—Division of decimals is very similar to division of whole numbers. The numbers are written and divided in the same way, the decimal point being disregarded while dividing. Before dividing, however, it may be necessary to annex ciphers to the decimal part of the dividend so as to make the number of decimal places in the dividend equal to or greater than the number in the divisor. Sometimes it is necessary to annex ciphers to the decimal part of the dividend to make it large enough to contain the divisor. The rule to be used is as follows:

Rule.—I. *Place the divisor to the left of the dividend, and proceed as in division of whole numbers; in the quotient, point off as many decimal places as the number of decimal places in the dividend exceed those in the divisor, prefixing ciphers to the quotient, if necessary.*

II. *If in dividing one number by another there is a remainder, the remainder can be placed as the numerator in a fraction with the divisor as its denominator and considered as a fractional part of the quotient, but it is generally better to annex ciphers to the remainder, and continue dividing until there are 3 or 4 decimal places in the quotient, and then if there still is a remainder, terminate the quotient by the plus sign (+), which shows that it can be carried farther.*

19. Division Without a Remainder.—The application of the preceding rule is shown in the solutions of the following examples in which the divisor is contained in the dividend without a remainder. The quotient obtained by such division is known as an *exact* decimal.

EXAMPLE 1.—Divide .625 by 25.

SOLUTION.—The dividend has 3 decimal places and the divisor none; therefore there are 3—0=3 decimal places in the quotient. One cipher has to be prefixed to the number in the quotient, to make the 3 decimal places required.

```
      divisor dividend quotient
          25).625(.025   Ans.
              50
              ——
              125
              125
              ———
  remainder     0
```

EXAMPLE 2.—Divide 6.035 by .05.

SOLUTION.—In this example the dividend is divided by 5 in the divisor, as if the cipher were not before it. The dividend has 3 decimal places and the divisor 2; the difference being 1, one decimal place is pointed off in the quotient.

```
 divisor dividend quotient
  .0 5 ) 6.0 3 5 ( 1 2 0.7   Ans.
         5
         -
         1 0
         1 0
         ---
            3 5
            3 5
remainder   0
```

EXAMPLE 3.—Divide .125 by .005.

SOLUTION.—In this example there are the same number of decimal places in the dividend as in the divisor; therefore, no decimal places are pointed off in the quotient.

```
 divisor dividend quotient
  .0 0 5 ) .1 2 5 ( 2 5   Ans.
             1 0
             ---
               2 5
               2 5
               ---
remainder      0
```

EXAMPLE 4.—Divide 326 by .25.

SOLUTION.—As 326 is a whole number, it must have annexed to it a decimal portion consisting of two ciphers in order to make the number of decimal places in the dividend and divisor equal. As a consequence, there are no decimal places in the quotient.

```
.2 5 ) 3 2 6.0 0 ( 1 3 0 4   Ans.
       2 5
       ---
         7 6
         7 5
         ---
           1 0 0
           1 0 0
           -----
               0
```

EXAMPLE 5.—Divide .0025 by 1.25.

SOLUTION.—According to Art. **18**, the divisor and dividend may be considered as whole numbers in dividing. When considered as such, it is seen that the dividend 25 will not contain the divisor 125; a cipher must therefore be annexed to the number 25, thus making the dividend 250. As 125 is contained twice in 250, the quotient is 2. In its present form the dividend contains 5 decimal places; there are 2 decimal places in the divisor. The difference, 5−2=3, is the number of the required decimal places in the quotient. In order to point off 3 decimal places, two ciphers must be prefixed to the figure 2, thereby making .002 the quotient.

```
1.2 5 ) .0 0 2 5 0 ( .0 0 2   Ans.
             2 5 0
             -----
                 0
```

20. Short division should be used not only for whole numbers, but also for decimals, whenever it is advantageous to apply it. For instance, examples 2 and 3 in the preceding series can easily be solved by short division. Example 3 would be solved as follows:

Example.—Divide .125 by .005.

Solution.—The explanation accompanying the solution of example 3, Art. **19**, applies also here.

$$\begin{array}{rl} \textit{divisor} & \textit{dividend} \\ .005) & \underline{.125} \\ \textit{quotient} & 25 \quad \text{Ans.} \end{array}$$

21. Division With a Remainder.—In the foregoing examples, the divisor was contained in the dividend without any remainder. The method to follow in case there is a remainder is shown in the following examples:

Example.—What is the quotient of 199 divided by 15?

Solution.—In solution (*a*) the dividend and the divisor are treated as whole numbers, and the remainder, 4, is used as the numerator in a fraction where the divisor, 15, is the denominator. The quotient is a mixed number consisting of the whole number 13 and the fraction $\frac{4}{15}$, or $13\frac{4}{15}$.

$$\begin{array}{rl} & \quad (a) \\ 15) & 199(13+\frac{4}{15}, \text{or } 13\frac{4}{15} \quad \text{Ans.} \\ & \underline{15} \\ & \ \ 49 \\ & \ \ \underline{45} \\ \textit{remainder} & \quad 4 \end{array}$$

$$\begin{array}{rl} & \quad (b) \\ 15) & 199.000(13.266+ \quad \text{Ans.} \\ & \underline{15} \\ & \ \ 49 \\ & \ \ \underline{45} \\ & \quad 40 \\ & \quad \underline{30} \\ & \quad 100 \\ & \quad \ \ \underline{90} \\ & \qquad 100 \\ & \qquad \ \ \underline{90} \\ \textit{remainder} & \qquad 10 \end{array}$$

In solution (*b*) the dividend is treated as a decimal, a decimal point being placed to the right of the dividend and a number of ciphers added corresponding to the number of decimal places required in the quotient. In this case 3 ciphers are added, but it is noted that after including these in the division there is still a remainder of 10; in fact, this example belongs to one of those cases in which the division does not end. If the fraction $\frac{4}{15}$ in solution (*a*) is reduced to a decimal (by the method to be explained later), it becomes .266. This shows that the quotients obtained in solutions (*a*) and (*b*) are equal, or, $13\frac{4}{15}=13.266$.

22. In many cases of decimal division there is a remainder, so that the quotient can be continued without a limit. Decimals of this kind are known as **interminate.** In the case of an interminate decimal it is necessary to decide how many decimal places are required in the quotient. Three decimal places are sufficient for most practical purposes, and more than four or five are rarely needed. After the number of places required has been decided on, the work of division is carried one place farther. This extra figure, which is subsequently omitted, serves as a means for a closer adjustment of the last figure in the required decimal. Thus, if the extra figure is 5 or a greater number, the preceding figure is increased by 1, a minus (−) sign being annexed to it to show that the quotient is not quite as large as indicated.

For instance, if in the last example in Art. **21** it had been required to obtain the answer correct to 4 decimal places, the work should have been carried to 5 places, obtaining 13.26666. Here, the fifth, or extra, figure is larger than 5, hence in omitting it, the fourth figure is increased by 1, and the required answer is 13.2667−.

If the extra figure in the decimal is less than 5, the preceding figure is left unaltered, when omitting the extra figure, but a plus sign (+) is written in its place to indicate that the true value of the quotient is slightly greater than that given. For example, if it is desired to retain 4 decimal places in the number 73.41823 it would be expressed as 73.4182+, the plus sign indicating that the true result is slightly larger than that shown.

These remarks apply to any other calculation involving decimals, when it is desired to omit some of the figures in the decimal. Thus, if it is desired to retain 3 decimal places in the number .2471253, it would be expressed as .247+; if it was desired to retain 5 decimal places, it would be expressed as .24713−. Both the + and − signs are frequently omitted; they are seldom used outside of arithmetic, except in exact calculations, when it is desired to call particular attention to the fact that the result obtained is not *quite* exact.

23. In works of a scientific nature and in calculations dealing with minute quantities, where it is desirable to make a distinction between exact and interminate decimals, it is customary to annex a cipher to the last place of the exact decimal. Thus, by adding a cipher to the decimal .375 and writing .3750 it is intended to convey the information that the decimal is without a remainder.

In tables consisting of exact and interminate decimals it is sometimes necessary, for the sake of uniformity, to add several ciphers to an exact decimal. Thus, in the following column there are added two ciphers to the exact decimal 5, so as to obtain the same number of places as in the adjoining decimals.

14.392
12.500
27.183

24. Repeating Decimals.—When in dividing decimals the divisor is not contained in the dividend without a remainder, the division may be continued indefinitely; that is, the quotient is interminate. In such cases one figure of the quotient may continue to repeat itself. Thus, in dividing 1 by 6 the quotient is .1666..., the figure 6 continuously repeating itself. The decimal is known as a **repeating decimal,** or **repeater.** The fact that a figure is a repeater is indicated by placing a dot over it; thus, the preceding quotient is written $.1\dot{6}$. An instance of a repeating decimal was found in the example of Art. **21.**

Instead of only one figure repeating itself, it is possible for two or more to do so. For instance, on finding the quotient of 1÷7 by short division, as $\begin{array}{r} 7\,)\underline{1.000000000000} \\ .142857142857 \end{array}$, it is found that a whole group of figures keeps on repeating, the group consisting of the figures 1,4,2,8,5,7. This repeating decimal, which is sometimes called a **circulating decimal,** is written $.\dot{1}4285\dot{7}$, the two dots indicating the first and the last figure of the repeating group of figures.

EXAMPLES FOR PRACTICE

1. In the following examples, divide:

(*a*) 101.6688 by 2.36.
(*b*) 187.12264 by 123.107.

Ans. $\begin{cases} (a)\ 43.08 \\ (b)\ 1.52 \end{cases}$

2. A bar 24.375 inches long is divided into equal parts measuring 1.625 inches in length. How many parts are there? Ans. 15

3. In the manufacture of a number of machines 612 pounds of bronze was required. If each machine used 12.75 pounds, how many machines were there? Ans. 48

4. The cost of 18.75 tons of coal was 60.75 dollars. What was the cost per ton? Ans. 3.24 dollars

5. A keg of boiler rivets weighs 100 pounds and contains 595 rivets. What is the weight of one of the rivets? Ans. .168+ lb.

6. A bar, shown on a drawing, measures 18.75 inches in length. If it has to be divided into 6 equal parts, what will be the length of each part? Ans. 3.125 in.

7. A spool contains 75.6 pounds of copper wire. How many coils, weighing 18.9 pounds each, can be made from the wire on the spool? Ans. 4 coils

8. Five gallons of ready-mixed paint are required to cover a surface measuring 1,377.5 square feet. What will be the size of the surface, measured in square feet, that may be covered by 1 gallon of paint? Ans. 275.5 square feet

9. If a carpenter can lay 5,750 shingles in 5.75 days, how many shingles can he lay in 1 day? Ans. 1,000 shingles

REDUCTION OF FRACTIONS AND DECIMALS

REDUCING FRACTIONS TO DECIMALS

25. Rule for Reduction.—Common fractions must frequently be changed to decimals; for example, fractions of an inch must be changed to equivalent decimals. The following rule is used:

Rule.—*To reduce a common fraction to a decimal, place a decimal point after the numerator, annex ciphers to the right of the point, and divide by the denominator. Point off as many decimal places in the quotient as there are ciphers annexed to the numerator.*

EXAMPLE 1.—Find the decimal that is equivalent to the fraction $\frac{3}{4}$.

Solution.—Using short division the solution is as shown. As two ciphers were annexed to the numerator, it follows from the rule that two decimal places must be pointed off in the quotient.

$$\begin{array}{r} 4\,\underline{)\,3.00} \\ .75 \end{array}$$

or $\frac{3}{4}=.75$ Ans.

Example 2.—What decimal is equivalent to $\frac{7}{8}$?

Solution.—

$$\begin{array}{l} 8\,)\,7.000\,(.875 \\ \underline{6\,4} \\ 6\,0 \\ \underline{5\,6} \\ 4\,0 \\ \underline{4\,0} \\ 0 \end{array}$$

or $\frac{7}{8}=.875$ Ans.

Example 3.—What decimal is equivalent to $\frac{4}{11}$?

Solution.—Performing the division as shown, it is found that there is a remainder and that the division may be carried on indefinitely. It is also seen that the quotient is a repeating decimal. Complying with the rules given in Art. **22**, the answer may be written in either of the following forms: .3636+, or .364.

$$\begin{array}{l} 11\,)\,4.0000\,(.3636 \\ \underline{3\,3} \\ 7\,0 \\ \underline{6\,6} \\ 4\,0 \\ \underline{3\,3} \\ 7\,0 \\ \underline{6\,6} \\ 4 \end{array}$$

or $\frac{4}{11}=.3636$ Ans.

26. Table of Decimal Equivalents.—The decimal equivalents given in Table I should be calculated by applying the rule in Art. **25.** In each case the numerator should contain a number of ciphers sufficient to allow the division to be completed without a remainder, since none of the decimals are repeating decimals. The table is convenient for reference, but it is well to memorize the first portion marked "8ths."

TABLE I

DECIMAL EQUIVALENTS

8ths

$\frac{1}{8}=.125$
$\frac{1}{4}=.25$
$\frac{3}{8}=.375$
$\frac{1}{2}=.5$
$\frac{5}{8}=.625$
$\frac{3}{4}=.75$
$\frac{7}{8}=.875$

16ths

$\frac{1}{16}=.0625$
$\frac{3}{16}=.1875$
$\frac{5}{16}=.3125$
$\frac{7}{16}=.4375$
$\frac{9}{16}=.5625$
$\frac{11}{16}=.6875$
$\frac{13}{16}=.8125$
$\frac{15}{16}=.9375$

32ds

$\frac{1}{32}=.03125$
$\frac{3}{32}=.09375$
$\frac{5}{32}=.15625$
$\frac{7}{32}=.21875$
$\frac{9}{32}=.28125$
$\frac{11}{32}=.34375$
$\frac{13}{32}=.40625$
$\frac{15}{32}=.46875$
$\frac{17}{32}=.53125$
$\frac{19}{32}=.59375$
$\frac{21}{32}=.65625$
$\frac{23}{32}=.71875$
$\frac{25}{32}=.78125$
$\frac{27}{32}=.84375$
$\frac{29}{32}=.90625$
$\frac{31}{32}=.96875$

EXAMPLES FOR PRACTICE

Reduce the following common fractions to decimals:

		Ans.
(*a*)	$\frac{4}{5}$	(*a*) .8
(*b*)	$\frac{7}{9}$	(*b*) .778–
(*c*)	$\frac{8}{15}$	(*c*) .533+
(*d*)	$\frac{23}{40}$	(*d*) .575
(*e*)	$\frac{19}{26}$	(*e*) .731–

REDUCING DECIMALS TO FRACTIONS

27. Rules for Reduction.—To reduce a decimal to a common fraction of the same value, the following rule is used:

Rule.—*To reduce a decimal to a common fraction, place under the figures of the decimal the figure 1 with as many ciphers at its right as there are decimal places in the decimal, and reduce the resulting fraction to its lowest terms by dividing both numerator and denominator by the same number.*

EXAMPLE 1.—Reduce .125 to a fraction.

SOLUTION.— $.125=\frac{125}{1000}=\frac{25}{200}=\frac{5}{40}=\frac{1}{8}$. Ans.

Example 2.—Reduce .875 to a fraction.

Solution.— $.875 = \frac{875}{1000} = \frac{175}{200} = \frac{35}{40} = \frac{7}{8}$. Ans.

28. In practice it often happens that a decimal must be reduced to a specified fractional part of a unit, as, for instance, to sixteenths or thirty-seconds of an inch. When, as in this case, the denominator of the fraction is given, the following rule is applied:

Rule.—*To reduce a decimal to a fraction with a given denominator, multiply the decimal by the given denominator and beneath the product place that denominator.*

It will be noted that the method outlined in the preceding rule is equivalent to that of dividing a unit into any number of parts desired, and of these parts taking a number of parts equivalent to the decimal fraction. For instance, if it is required to find the equivalent of .75 inch in sixteenths of an inch, the unit is divided into sixteenths, as $\frac{16}{16}$; $\frac{16}{16} \times .75 = \frac{16 \times .75}{16} = \frac{12}{16}$. Thus, .75 inch is equal to $\frac{12}{16}$ inch.

Example 1.—If a steel plate plate is .5827 inch thick, what is its equivalent thickness in 64ths of an inch?

Solution.—According to the example, the decimal .5827 is to be reduced to a fraction having 64 as its denominator. Applying the rule,

$$.5827 \times \frac{64}{64} = \frac{.5827 \times 64}{64} = \frac{37.2928}{64}$$

As the numerator in the latter fraction is a mixed number, it must be replaced by the nearest equivalent whole number. 37.2928 is nearer to 37 than to 38 because the difference between 37.2928 and 37, or .2928, is smaller than the difference between 38 and 37.2928, which is .7072. So 37 is chosen and the decimal is omitted. It follows that .5827 inch $= \frac{37}{64}$ inch, nearly. Ans.

Example 2.—Change .3917 to 12ths.

Solution.—According to the requirements the decimal .3917 is to be changed to an equivalent proper fraction having 12 as a denominator. Applying the rule,

$$.3917 \times \frac{12}{12} = \frac{.3917 \times 12}{12} = \frac{4.7004}{12}$$

The number 4.7004 being nearer 5 than 4, the number 5 is chosen as the nearest equivalent. That is, $.3917 = \frac{5}{12}$, nearly. Ans.

EXAMPLES FOR PRACTICE

1. Reduce the following to common fractions:

(*a*) .25.	Ans.	(*a*) $\frac{1}{4}$
(*b*) .625		(*b*) $\frac{5}{8}$
(*c*) .3125.		(*c*) $\frac{5}{16}$
(*d*) .04.		(*d*) $\frac{1}{25}$
(*e*) .06.		(*e*) $\frac{3}{50}$

2. Express:

(*a*) .625 in 8ths.	Ans.	(*a*) $\frac{5}{8}$
(*b*) .3125 in 16ths.		(*b*) $\frac{5}{16}$
(*c*) .15625 in 32ds.		(*c*) $\frac{5}{32}$
(*d*) .77 in 64ths.		(*d*) $\frac{49}{64}$
(*e*) .81 in 48ths.		(*e*) $\frac{39}{48}$
(*f*) .923 in 96ths.		(*f*) $\frac{89}{96}$

DECIMAL CURRENCY

29. Examples of Calculations.—In the United States and some other countries the **dollar** is the unit in which money values are expressed. The dollar is 100 **cents**; that is, 1 cent is one-hundredth of a dollar. Represented as a decimal 1 cent =.01 dollar, and 25 cents=.25 dollar. The sign $ is the *dollar mark,* and is usually placed in front of a number representing dollars or cents, when the latter number is given as a decimal part of a dollar. For example, $12 is read *twelve dollars,* and $.25, or $0.25, is read *twenty-five cents.* The expression $12.25 is read *twelve dollars and twenty-five cents.* The decimal point is used to separate the dollars from the cents. All to the left of the decimal point represents dollars, and the two figures to the right of the decimal point represent cents, the cents being hundredths of a dollar.

Calculations in which dollars and cents are used are made in the same way, and according to the same rules, as calculations using decimals, because an expression like $1.15 is a decimal, and represents $1\frac{15}{100}$ dollars. The following examples, with their solutions, will serve to illustrate the methods of using dollars and cents in calculations:

Example 1.—A workman's wages are $2.85 a day. How much does he earn in 26 days?

Solution.—In 26 days he will earn 26 times as much as in 1 day, or 26×$2.85=$74.10. Ans.

Example 2.—The total cost of making 95 pieces of work was $235.60. What was the cost per piece?

Solution.—The cost per piece is equal to the total cost divided by the number of pieces, or $235.60÷95=$2.48. Ans.

Example 3.—If a man receives $36.94 on pay day and immediately pays bills amounting to $19.67, how much has he left?

Solution.—He will have left the difference between what he received and what he paid out, or $36.94−$19.67=$17.27. Ans.

Example 4.—A certain piece of work requires four men to complete it. If these men are paid $12.80, $21.16, $13.54, and $6.15, what is the cost of the labor?

Solution.—The labor cost must be the sum of the four amounts, or $12.80+$21.16+$13.54+$6.15=$53.65. Ans.

EXAMPLES FOR PRACTICE

1. A workman is paid $27.60 for 6 days' work. How much does he earn per day? Ans. $4.60

2. If a certain article costs $.45, what will 48 of these cost? Ans. $21.60

3. Three different grades of pig iron were purchased at $14.25, $19.65, and $17.40 a ton. What was the average cost per ton? Ans. $17.10

4. A man works 112 hours at $.46 an hour, and draws $28.75 on his pay. How much is still due him? Ans. $22.77

5. How much remains when 67 cents is taken from $10.36? Ans. $9.69

6. If a man receives $3.20 per day of 8 hours, how much does he receive after working 16½ hours? Ans. $6.60

7. If a certain product costs 8¾ cents per pound, what will be the cost of 350 pounds? Ans. $30.62½

8. An ice-making plant has an output of 50 tons daily. If the ice costs $3.60 per ton, what is the value of the output for 320 days? Ans. $57,600

9. If lath costs \$.55 per bundle, what must be paid for 45 bundles?
Ans. \$24.75

10. If a carpenter earns \$350 in 12.5 weeks, what are his average wages per week?
Ans. \$28

PER CENT. AND PERCENTAGE

DEFINITIONS

30. Explanation of Term Per Cent.—The term **per cent.** is frequently used in connection with calculations of various kinds, as, for example, in expressing the composition of metals and other substances, profits on investments, etc. The term is an abbreviation of the Latin words *per centum,* and it means *by the hundred,* or *in the hundred.* The meaning of the term may be seen more clearly from the following explanations and examples.

If a man has \$100 and spends \$50 of it, then he has spent $\frac{50}{100}$ or 50 per cent. of his money. If he spends \$20 more, then he has spent $\frac{50+20}{100}=\frac{70}{100}$ or 70 per cent. of the money. Supposing that he spends \$10 more, he has spent 80 per cent. and \$20 is left, which amounts to $\frac{20}{100}$, or 20 per cent., of the money he had.

Again, if it is stated that the population of Scranton, Pa., increased 6 per cent. in the period from 1910 to 1920, it is equivalent to saying that the increase was 6 in every hundred; that is, for every 100 in 1910, there were 6 more, or 106, in 1920.

31. As another example let it be supposed that the composition of a certain specimen of solder is such that a bar of solder, weighing 10 pounds, contains 6 pounds of lead. Then the weight of the lead is $\frac{6}{10}$ of the weight of the bar. But, $\frac{6}{10}=\frac{60}{100}$ and $\frac{60}{100}$ is 60 per cent. of the bar. That is, the bar contains 60 per cent. of lead.

EXAMPLE 1.—Find the weight of tin in 150 pounds of solder, if there is 40 per cent. of tin, by weight, in the solder.

SOLUTION.—The weight of tin is 40 per cent., or 40 hundredths, of the whole weight of solder. Therefore, the weight of the tin is $\frac{40}{100}$ of 150 pounds, or $\frac{40}{100}\times 150$ pounds$=\frac{40\times 150}{100}=60$ pounds. Ans.

Example 2.—A tank contains 75 gallons of a sulphuric-acid solution. If there is 47 per cent. of sulphuric acid in the solution, how many gallons of acid were used?

Solution.— 47 per cent. of 75 gallons is $75 \times \frac{47}{100} = 75 \times .47 = 35.25$ gallons of sulphuric acid. Ans.

32. Sign of Per Cent.—The **sign** of per cent. is %; thus, 2% means 2 per cent. and is read *two per cent.* The figure before the sign % indicates how many per cent. is meant, or the number of hundredths. Thus, 6% is read *6 per cent.*, and means 6 hundredths; 12½% is read *twelve and a half per cent.*, and means 12½ hundredths.

1% of a number means $\frac{1}{100}$ of that number; 5% of a number is $\frac{5}{100}$ of the number; and so on. Thus, 1% of 200 is 2, because $\frac{1}{100}$ of 200 is $\frac{1}{100} \times 200 = 2$. If a man is receiving 60 cents an hour for his work and he gets an increase of 10% in his wages, his increase amounts to 10% of 60, or $\frac{10}{100} \times 60 = 6$ cents; then, his new rate of pay is $60+6=66$ cents an hour.

33. Omission of Per Cent. Sign.—The per cent. sign, %, may be dropped and the number of per cent. expressed in several other ways. For example, 12% may be written 12 per cent. Or, the per cent. may be changed to a common fraction, thus, $6\% = \frac{6}{100}$; $25\% = \frac{25}{100}$; $12\frac{1}{2}\% = \frac{12\frac{1}{2}}{100} = \frac{12.5}{100} = \frac{125}{1000}$; $51.6\% = \frac{51.6}{100} = \frac{516}{1000}$. In all of these examples the per cent. sign is omitted.

34. Per Cent. Expressed as a Decimal.—If the per cent. is to be changed to a decimal instead of to a common fraction, the % sign is dropped and the number of decimal places is increased by two by moving the decimal point two places to the left. For example, suppose that 25% is to be changed to a decimal having the same value. Mentally, this number may be supposed to be written 25.0%. The sign % is dropped and the number of decimal places in the number is increased by two by moving the decimal point two places to the left, giving .250, or .25; that is, 25% = .25. That the method just given is correct may be proved by means of Art. **33.**

According to the latter, $25\% = \frac{25}{100}$; but, $\frac{25}{100} = .25$. Therefore, $25\% = \frac{25}{100} = .25$.

Suppose that 2.5% is to be changed to a decimal. There is already one decimal place in the number, and the two more that must be prefixed when the sign % is dropped make a total of three decimal places; then, $2.5\% = .025$, a cipher being used to give the necessary number of places. Again, $.25\% = .0025$. The number of decimal places to be pointed off in this case is $2+2=4$, and so two ciphers must be prefixed.

35. When a per cent. is used in a calculation, it is always changed to a fraction or to a decimal. It is necessary, therefore, that the preceding rules, explaining the method of changing per cent. to a fraction and to a decimal, should be thoroughly understood. Table II shows various per cents. expressed as decimals and as common fractions.

TABLE II

PER CENTS. EXPRESSED AS DECIMALS AND FRACTIONS

Per Cent.	Decimal	Fraction	Per Cent.	Decimal	Fraction
1%	.01	$\frac{1}{100}$	150%	1.50	$\frac{150}{100}$ or $1\frac{1}{2}$
2%	.02	$\frac{2}{100}$ or $\frac{1}{50}$	500%	5.00	$\frac{500}{100}$ or 5
5%	.05	$\frac{5}{100}$ or $\frac{1}{20}$	$\frac{1}{4}\%$	.0025	$\frac{\frac{1}{4}}{100}$ or $\frac{1}{400}$
10%	.10	$\frac{10}{100}$ or $\frac{1}{10}$	$\frac{1}{2}\%$	.005	$\frac{\frac{1}{2}}{100}$ or $\frac{1}{200}$
25%	.25	$\frac{25}{100}$ or $\frac{1}{4}$	$1\frac{1}{2}\%$	.015	$\frac{1\frac{1}{2}}{100}$ or $\frac{3}{200}$
50%	.50	$\frac{50}{100}$ or $\frac{1}{2}$	$8\frac{1}{3}\%$	$.08\frac{1}{3}$	$\frac{8\frac{1}{3}}{100}$ or $\frac{1}{12}$
75%	.75	$\frac{75}{100}$ or $\frac{3}{4}$	$12\frac{1}{2}\%$	.125	$\frac{12\frac{1}{2}}{100}$ or $\frac{1}{8}$
100%	1.00	$\frac{100}{100}$ or 1	$16\frac{2}{3}\%$	$.16\frac{2}{3}$	$\frac{16\frac{2}{3}}{100}$ or $\frac{1}{6}$
125%	1.25	$\frac{125}{100}$ or $1\frac{1}{4}$	$62\frac{1}{2}\%$	.625	$\frac{62\frac{1}{2}}{100}$ or $\frac{5}{8}$

36. Percentage.—By the term **percentage** is meant the product obtained by multiplying a number by a per cent., expressed as a decimal. For instance, if it is desired to find a number that is 5% of 45, the per cent. is expressed as a decimal, becoming .05. The product $45 \times .05 = 2.25$ is the percentage. Note that *per cent.* and *percentage* do not mean the same thing.

CALCULATIONS INVOLVING PER CENT. AND PERCENTAGE

37. Finding Percentage, When Number and Per Cent. Are Known.—Calculations of percentage are often required in cases where the number is known and also the per cent. These calculations are performed according to the following rule:

Rule.—*To find the percentage of a number, multiply the number by the per cent. expressed as a decimal.*

EXAMPLE 1.—What percentage is obtained by taking 36% of 125?

SOLUTION.—According to the rule, 36% is first changed to a decimal, thus becoming .36. Then, 125×.36=45. Therefore, 36% of 125 gives a percentage of 45. Ans.

EXAMPLE 2.—From a stock of 300 machines, 76% was sold. What was the percentage sold?

SOLUTION.—The number sold was 76% of 300. Expressed as a decimal, 76%=.76. Then, 300×.76=228. The percentage sold was 228 machines. Ans.

EXAMPLE 3.—A brass casting weighs 348 pounds. The brass is made up of 65% of copper and 35% of zinc. (*a*) What is the percentage of copper in the casting? (*b*) What is the percentage of zinc in the casting, both percentages being expressed in pounds?

SOLUTION.—(*a*) The weight of copper is 65% of the whole, or 65% of 348. Expressed as a decimal, 65%=.65. The percentage of copper is, then,

$$348\times.65=226.2\text{ lb.}\quad\text{Ans.}$$

(*b*) The percentage of zinc is

$$348\times.35=121.8\text{ lb.}\quad\text{Ans.}$$

Another way of finding the percentage of zinc is to subtract the percentage of copper from the weight of the casting, or 348—226.2 =121.8 pounds, as before.

38. Finding Per Cent., When Number and Percentage Are Known.—When several numbers are combined into one whole, it is frequently required to find what per cent. each number is of the whole number. These numbers represent, in reality, percentages and the question is to ascertain what per cent. of the whole is represented by each percentage. For example, if a certain weight of copper and a certain weight of tin are melted together to form bronze, it may be necessary to know the per cent. of copper and the per cent. of tin. The constituent parts of copper and tin represent percentages, and the problem amounts, in reality, to this: Given the weight of the casting and the percentages of copper and of tin to find what per cent. of copper and of tin there are in the casting. The following rule applies to problems of this class:

Rule.—*To find what per cent. of the whole is represented by a given percentage, divide the percentage by the whole, and multiply the result by 100.*

EXAMPLE 1.—What per cent. is employed to obtain 16 as a percentage of 64?

SOLUTION.—In this example, 64 is the whole and 16 the percentage. Then, applying the rule, 16÷64=.25. Finally, .25×100=25. Therefore, the percentage 16 is 25% of 64. Ans.

EXAMPLE 2.—In a bronze casting weighing 98 pounds there is a percentage of copper equal to 83.3 pounds. What per cent. of copper is there in the casting?

SOLUTION.—Applying the rule, 83.3÷98=.85, and .85×100=85%. Ans.

EXAMPLE 3.—From a stock of 300 motors a percentage of 228 was sold. What per cent. was sold?

SOLUTION.—Following the rule, 228÷300=.76, and .76×100=76; that is, 76% was sold. Ans.

EXAMPLE 4.—A certain alloy contains percentages of copper, tin, and zinc, equal to 144 pounds, 27 pounds, and 9 pounds, respectively. Find (*a*) the per cent. of copper, (*b*) the per cent. of tin, and (*c*) the per cent. of zinc in the alloy.

NOTE.—An *alloy* is a solidified combination of two or more metals, mixed while in a molten condition.

SOLUTION.—The total weight of the alloy is equal to the sum of the percentages, or 144+27+9=180 pounds.

(*a*) The percentage of copper being 144 pounds, by the rule, 144÷180 =.8, and .8×100=80%; that is, there is 80% of copper in the alloy. Ans.

(*b*) The percentage of tin being 27 pounds, 27÷180=.15, and .15×100=15%; that is, there is 15% of tin in the alloy. Ans.

(*c*) The percentage of zinc is 9 pounds. Hence, by the rule, 9÷180=.05, and .05×100=5%; that is, there is 5% of zinc in the alloy. Ans.

39. Finding Number, When Percentage and Per Cent. Are Known.—Sometimes a number is given which is known to represent a certain percentage of another number, not known. It is desired to find the latter, the per cent. employed being known. The number is found according to the following rule:

Rule.—*To find a number of which the percentage and the per cent. are known, divide the percentage by the per cent. expressed as a decimal.*

Example 1.—At 68 per cent. the percentage of a certain number is 101. What is the number?

Solution.—According to the rule, 68% is first changed to a decimal, thus becoming .68. Then, $101 \div .68 = 148\frac{9}{17}$. Therefore, the number, of which the percentage is 101 at 68%, is $148\frac{9}{17}$. Ans.

Example 2.—If the percentage of a certain number is 784 at $83\frac{1}{3}\%$, what is the number?

Solution.—The per cent. is first expressed as a decimal. For this purpose it is convenient to reduce $83\frac{1}{3}\%$ to an improper fraction; thus, $83\frac{1}{3}\% = \frac{83\frac{1}{3}}{100} = \frac{\frac{250}{3}}{100} = \frac{250}{3} \times \frac{1}{100} = \frac{250}{300}$. According to the rule, 784 is to be divided by $\frac{250}{300}$, or $784 \div \frac{250}{300} = 784 \times \frac{300}{250} = 940.8$. The required number is, therefore, 940.8. Ans.

40. Sum of Percentages Equal to Whole Number. It should be noted that when the composition of a body is given in per cents. of its parts, the sum of the per cents. is equal to 100. For instance, in example 3, Art. **37**, there is 65% of copper and 35% of zinc, or a total of $65+35=100\%$. Also, it is important to note that the *sum* of the numbers that represent the percentages of a thing or quantity must be equal to the number that represents the whole thing or quantity under consideration. For instance, in example 3, Art. **37**, there are percentages of copper and zinc weighing 226.2 and 121.8 pounds, respectively. The sum of these percentages, or $226.2 + 121.8 = 348$ pounds, is equal to the weight of the brass casting. Also, if a certain percentage of a thing or quantity is taken away, the remaining percentage or percentages must be equal to the difference between the whole quantity and the percentage removed. Thus, supposing a water tank holding 8,000 gallons has lost 12% by leakage; the amount remaining is $100-12=88\%$. This can easily be proved. If 12% leaks away, the percentage lost is, according to Art. **37**, 8,000×.12 =960 gallons, and the percentage left in the tank is 8,000−960 =7,040 gallons. Applying the rule in Art. **38**, 7,040÷8,000 =.88, or 88%. In other words, the water remaining is 88% of the original amount, which is exactly the same result as was obtained by subtracting 12% from 100%.

EXAMPLES FOR PRACTICE

1. What per cent. of
(*a*) 360 is 90?
(*b*) 125 is 25?
(*c*) 47 is 94?

Ans. (*a*) 25%
(*b*) 20%
(*c*) 200%

2. In a manufacturing plant an average of 3,640 pounds of coal per day was consumed. Alterations were made that resulted in a saving of 250 pounds per day. What was the per cent. of coal saved?

Ans. 6.9%, nearly

3. If the speed of an engine running at 128 revolutions per minute should be increased 6¼%, how many revolutions would it then make?

Ans. 136 revolutions

NOTE.—Calculate the number of revolutions corresponding to 6¼% and add these to the present number.

4. Mr. A borrows $1,100 from his neighbor, Mr. B. This sum is 18½% of the total amount that Mr. B has on interest in the bank. How much money has Mr. B?

Ans. $5,945.95

NOTE.—The $1,100 borrowed is a percentage and the total amount is found by the rule, Art. **39**.

5. If an electric generator is rated at 250 kilowatts and is guaranteed to be capable of developing 25% overload, (*a*) what is this overload? (*b*) What is the total output of the generator while developing this overload?

Ans. (*a*) 62.5 kilowatts
(*b*) 312.5 kilowatts

NOTE.—A kilowatt, abbreviated K. W., is a unit of electric power.

6. A railway system has 540 cars in operation, out of which 45 must be replaced by new ones. (*a*) What per cent. of cars must be replaced? (*b*) What per cent. of them are still serviceable?

Ans. (*a*) 8⅓%
(*b*) 91⅔%

7. A tank containing 10,000 gallons of water loses in a given time 160 gallons by leakage. (*a*) What per cent. is lost by leakage? (*b*) What per cent. remains in the tank?

Ans. (*a*) 1.6%
(*b*) 98.4%

8. One cubic yard of a certain concrete mixture requires 4 bags of cement, each bag containing 100 pounds. A certain quantity of lime is

to be added to this mixture, amounting to 3 per cent. of the weight of cement used. How many pounds of lime will be required for each cubic yard of concrete made? Ans. 12 lb.

9. The gas bill for last month was 25% higher than the bill for this month, which amounts to $6.40. What was the amount of last month's bill? Ans. $8.00

Note.—25% of $6.40 is to be added to $6.40.

ACCURACY OF CALCULATIONS

REQUIRED NUMBER OF SIGNIFICANT FIGURES

41. Accuracy of Measurements.—Results may be obtained by arithmetical operations that are absolutely correct. But, absolute accuracy in the practical application of the results is in many cases an impossibility. Measurements are liable to error, however carefully made, owing to imperfections of the measuring instruments, erroneous methods of applying them, and unavoidable errors in reading their indications. For example, the measurement of a distance between two given points would appear to be a simple operation, but on attempting to set off a distance of, say, 35.12 inches, thus measuring the length to the hundredth part of an inch, one will appreciate the attending difficulties. The distance, as first set off, may appear as absolutely correct, but let the operation be repeated, taking every precaution to avoid errors, and the results may differ by several hundredths of an inch.

Results based on the average of a series of measurements cannot be expected to be of a greater degree of accuracy than that of the instruments with which such measurements were made. Conversely, it is a waste of time and labor to use very accurate and sensitive instruments to obtain results that are required to be only approximately correct.

42. Significant Figures.—The preceding remarks lead to the consideration of the number of *significant figures* that should be retained in the readings, calculations, and results in order to obtain results within the accuracy of the measurements used. By **significant figures** are meant the figures of a number between the first digit at the left and the last digit at the right, it being understood that a cipher is considered as a figure but not as a digit.

For example, the numbers 20,467, 28.321, 45.673, and .00010569 each have five significant figures. If it were necessary to use only four significant figures, these numbers would be written 20,470, 28.32, 45.67, and .0001057, respectively; that is, if the figure dropped is 5 or greater, the next figure to the left is increased by 1. If the dropped figure is less than 5, the next figure to the left is not changed.

A cipher may be a significant figure. For example, the numbers 405.63, 300.64, and .12005 each have five significant figures. When one or more ciphers are located between two digits, as in the preceding examples, the ciphers become significant figures.

43. The part of a number containing significant figures is called the **significant part** of the number. Thus, in the number .00812, the significant part is 812, and in the number 170.3, the significant part is 1703.

In speaking of the significant figures or of the significant part of a number, the figures are considered in their proper order, from the first digit at the left to the last digit at the right, but no attention is paid to the decimal point. Hence, all numbers that differ only in the position of the decimal point have the same significant part. For example, .002103, 21.03, 2,103 have the same significant figures 2, 1, 0, and 3, and the same significant part 2103.

44. Number of Significant Figures Required.—The degree of accuracy with which calculations should be determined, either in whole numbers or in decimals, depends on the accuracy of the means employed for applying the results to practice or for obtaining the data employed in the calculations. It is evidently useless to employ great accuracy in calculating a practical problem, when the measuring instrument, by which the result is to be applied to practice, is incapable of a sufficiently accurate reading. For instance, it would be absurd to give a dimension in thousandths of an inch, when the rule employed in that particular case did not measure even thirty-seconds; or, to calculate air pressure to hundredths of a pound, when the pressure gauge cannot be read with certainty to tenths of a pound.

45. Generally speaking, in most practical problems, it is not necessary to use more than five significant figures, and in most cases only three to four figures are required. For instance, in a length of 812.75 feet, it would not be expected that the measurement would be accurate to within an inch. Hence, in the fraction .75, which means 9 inches, the last figure could be dropped and the figure 7 be replaced by 8. The length would now be indicated by the number 812.8 feet, which would be sufficiently accurate. As another example, let it be supposed that some chemical is weighed on an ordinary scale and is found to weigh 8 pounds 9 ounces = 8.5625 pounds. Under the circumstances one would not expect the reading to be accurate within an ounce; hence, the figure 5 could be dropped, and the weight be given as 8.563 pounds, which is sufficiently accurate. Again, in calculating the thickness of a steel plate as .81875 inch, this number would contain more decimals than could well be applied in practice, as the measuring instrument usually employed for measuring such dimensions cannot be read to more than 4 decimal places; thus, .81875 could be reduced to .8188 and be accurate enough for all practical purposes.

Again, to take an example referring to greater magnitudes, the average distance of the moon from the earth is 238,000 miles, a number with three significant figures. Considering the great distance and the difficulties involved in obtaining accurate results, the number is considered to be sufficiently accurate.

WEIGHTS AND MEASURES

DENOMINATE NUMBERS

ENGLISH MEASURES

DEFINITIONS

1. Varieties of Measures.—A **measure** is a standard unit, established by law or custom, by means of which a quantity of any kind may be measured. For example, the inch and the mile are *measures of length;* the pint and the gallon are *measures of capacity,* as used for liquids; the ounce and the ton are *measures of weight;* the second and the month are *measures of time,* and so on.

2. Denominate Numbers.—When a number is used in connection with measures, it becomes a **denominate number,** that is, a *named* number. Numbers are said to be of the *same denomination* when they are combined with similarly named units of measure, as 2 pounds and 7 pounds. A pound is a *unit of measure;* a foot, an hour are also units of measure, but of different kinds.

If a denominate number consists of units of measure of but one denomination, it is called a **simple denominate number** or a **simple number.** For example, 14 inches is a simple number, as it contains units of measure of but one denomination, namely, inches. The denominate numbers 16 cents, 10 hours, 6 gallons are all *simple* numbers, but they are not of the same *kind.*

If simple numbers of different denominations but of the same kind of measure are combined, the combination is called a compound denominate number, or a compound number. For example, 3 yards 2 feet 7 inches is a compound number, as it is a combination of the different denominations, yards, feet, and inches; these denominations are of the *same* kind, being all unit measures of length. Other examples of compound numbers are: 2 pounds 3 ounces 10 grains; 5 gallons 3 quarts 1 pint; 10 hours 14 minutes 32 seconds.

A unit measure that is larger than another unit measure is said to be of a **higher denomination;** if smaller than another unit measure, it is of a **lower denomination.** Thus, a *foot* is of a higher denomination than an *inch* and of a lower denomination than a *yard.*

3. Systems of Measures.—In modern practice two systems of measures are employed: the *English system* and the *metric system.* The **English system** is in general use in the United States, Great Britain, and Canada. The **metric system** is used on the European continent and to some extent in the United States, as, for instance, in chemistry and pharmacy. This system is a decimal system; that is, the values of the different units of the same kind increase or decrease by tens, 10 units of each denomination making 1 unit of the next higher denomination.

ABBREVIATIONS

4. Abbreviations of Units.—In writing denominate numbers, it is convenient to use *abbreviations* instead of writing out the name of the unit in full. For example, the names *inch* and *inches* may be abbreviated to *in.;* thus, write 5 *in.* instead of 5 *inches.* Similarly, 8 *ft.* means 8 *feet.* Sometimes, the marks (′) and (″) are used for feet and inches, respectively, in particular on drawings; thus 6′ 2″ means 6 feet 2 inches. The abbreviations commonly used for the various units are given in the following tables, each table showing the relation between various units of measure of the same kind. These tables should be memorized, including the abbreviations. The practice of writing the tables from memory will be found helpful.

MEASURES OF EXTENSION

LINEAR MEASURE

5. Definitions.—Measures of extension are used to measure lengths of lines, areas of surfaces, and contents of solids. When speaking of lines it is to be understood that a *mathematical line* has no breadth or thickness and merely indicates extension in one direction; that is, *length*. A *straight line* is frequently defined as the shortest distance between two points. The lengths of lines are expressed in **linear measures.**

TABLE OF LINEAR MEASURE

		Abbreviations
12 inches (in.)	=1 foot	ft.
3 feet	=1 yard	yd.
5½ yards 16½ feet	=1 rod	rd.
320 rods 5,280 feet	=1 statute mile	mi.
6,080 feet	=1 nautical mile	naut. mi.
3 nautical miles	=1 nautical league	naut. l.

Note.—The statute mile is used for measuring distances on land, lakes, and rivers. The nautical mile is used for measuring distances on the ocean.

Example 1.—A piece of shafting is 7 feet long. What is its length in inches?

Solution.—According to the table, 1 ft.=12 in.; therefore, 7 ft. must be equal to

$$7\times12=84 \text{ in. Ans.}$$

Example 2.—The width of a door is 48 inches. What is the width in feet?

Solution.—According to the table, 12 in.=1 ft.; therefore, the number of feet in 48 in. is

$$48\div12=4 \text{ ft. Ans.}$$

Example 3.—How many feet are there in 3 statute miles?

Solution.—According to the table, 5,280 ft.=1 mi.; therefore, 3 mi. must be equal to

$$3\times5{,}280=15{,}840 \text{ ft. Ans.}$$

LAND MEASURE

6. Surveyor's Linear Measure.—**Surveyor's measure,** also known as **land measure,** is used for measuring land, as in locating and laying out railways, roads, and tracts or building plots.

TABLE OF SURVEYOR'S LINEAR MEASURE

		ABBREVIATIONS
7.92 inches	=1 link	li.
100 links 66 feet	=1 chain	ch.
80 chains	=1 statute mile	mi.

The surveyor's chain (also called Gunter's chain) of 100 links, each 7.92 inches long, has been used very extensively in the past and is still common, but its use is decreasing. Chains or steel tapes 50 or 100 feet in length, graduated in feet and decimals of a foot and sometimes in feet and inches, are being employed to a large extent at the present time.

NOTE.—In Mexico, and in those parts of the United States that belonged to Mexico previous to 1845, the old Mexican measures of length are sometimes used, and referred to in early surveys. The principal units are:

1 vara=2.75 feet=33 inches, English measure
5,000 vara=1 league=2.604 miles, English measure

EXAMPLE 1.—How many inches are there in 50 links?

SOLUTION.—According to the table 1 li.=7.92 in.; therefore, 50 links must be equal to

50×7.92=396 in. Ans.

EXAMPLE 2.—A line set out by a surveyor is found to measure 40 chains in length. What is the equivalent length in feet?

SOLUTION.—According to the table, 1 ch.=66 ft.; therefore, 40 chains must be equal to

40×66=2,640 ft. Ans.

EXAMPLE 3.—A boundary line between two estates is found to measure 440 chains in length. What is the equivalent length in statute miles?

SOLUTION.—According to the table, 1 mi.=80 ch. The problem consists of finding how many times 80 ch. is contained in 440 ch.; or,

440÷80=5½ mi. Ans.

SQUARE MEASURE

7. Definitions.—The term **area** means the extent of a surface within its boundary lines. For example, the area of a floor is the extent of the visible surface limited by the four surrounding walls. As area refers to extent of surface, the unit of measurement must also be a surface. The **unit of area** is a *square,* which is a surface of the form shown in Fig. 1.

By the term **square** is meant a four-sided figure in which the sides are of equal length and at *right angles* to one another; that is, any two adjoining sides meet each other squarely. A **square inch** is a square that is 1 inch long on each side. A **square foot** is a square that measures 1 foot on each side, as in Fig. 1. This may be made to represent square inches by dividing each side into twelve equal parts, 1 foot being equal to 12 inches, and drawing lines across the square from the points of division on one side to those on the opposite side. The square will then appear as in Fig. 2, divided into a number of small squares. As each side of the square was divided into twelve equal parts, each part is $\frac{1}{12}$ foot long, or $\frac{1}{12}\times 12 = 1$ inch long, and each little square therefore measures 1 inch on each side; that is, each little square is 1 square inch. Now, if the total number of little squares is counted, it will be found to be 144. In other words, there are 144 square inches in 1 square foot.

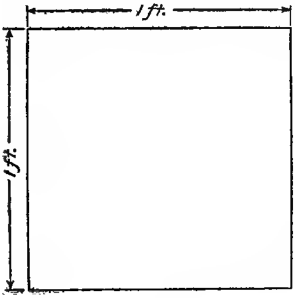


FIG. 1

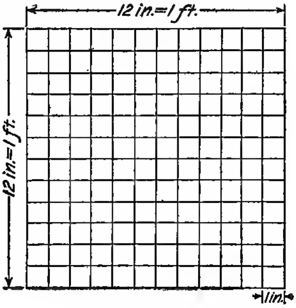


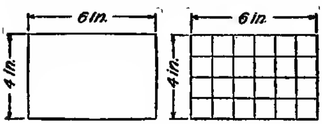
FIG. 2

8. Application of Square Measure.—A square inch, a square foot, etc. belong to the units of **square measure,** which is employed for measuring the extent of areas, such as floors, building lots, estates, etc. Square measure is also used to determine the surface areas of objects, such as engine cylinders, condensers, pipes, etc. Small areas are measured in square inches or square feet, and larger areas in square yards, square rods, acres, and square miles. The accompanying table gives the principal units of square measure.

TABLE OF SQUARE MEASURE

			Abbreviations
144	square inches (sq. in.)	=1 square foot	sq. ft.
9	square feet	=1 square yard	sq. yd.
30¼	square yards	=1 square rod	sq. rd.
160	square rods	=1 acre	A.
640	acres	=1 square mile	sq. mi.

9. Finding Areas of Squares and Rectangles. Instead of finding the area of a square by counting the number of small squares contained by it, as in Art. **7,** it is possible to find the area more easily by simply multiplying the length of the square by its width. Thus, referring to Fig. 2, there are 12 rows of the little squares and 12 squares in each row or 12×12=144 square inches, which is the same result as obtained by counting. When this method of multiplying the length by the width is employed it is important to note that the length and the width must be in the *same* units. In the case just mentioned, both the length and the width were in inches, and the area as a result is in square inches. With the length and the width in feet, the area will be 1 foot×1 foot=1 square foot.



(a) (b)

FIG. 3

Sometimes the area to be measured is a **rectangle;** that is, a figure similar to that shown in Fig. 3 (*a*). It differs from a square in so far that not *all* the sides are of equal length, but only those sides that are opposite each other. The area of a rectangle is found in the same way as that of a square; hence, the following rule applies to both figures:

Rule.—*To find the area of a square or a rectangle, multiply the length by the width, both being expressed in the same units.*

Thus, if the rectangle in Fig. 3 (*a*) is 6 inches long and 4 inches wide, its area is 6×4=24 square inches. This can be proved as shown in (*b*), by dividing each side into inches, drawing lines across the figure from the points of division, and counting the number of squares thus formed. Each of these squares is 1 inch on each side, or 1 square inch in area, and there are 24 in all, or 24 square inches.

10. Difference Between the Terms Square Feet and Feet Square.—Distinction must be made between expressions, such as "4 square feet" and "4 feet square," as they do not mean the same thing. When a square is referred to as being equal to *4 square feet,* reference is had to its *area.* It follows from the preceding rule that each of its sides must be of a length equal to 2 feet, in order that the area may be 2×2=4 square feet.

The expression "4 feet square" does not refer to the *area* of the square, but to *length of the sides,* and means that each side of the square is equal to 4 feet in length. The area of the latter square is 4×4=16 square feet.

11. Application of Rule for Finding Areas of Squares and Rectangles.—The application of the rule in Art. **9** is illustrated by means of the following examples:

Example 1.—How many square inches are there in a rectangle 28 inches long and 13 inches wide?

Solution.—According to the rule, the area is

28×13=364 sq. in. Ans.

Example 2.—A sheet of copper is 4 feet long and 3 inches wide. What is the area of its surface?

Solution.—Both dimensions must be expressed in the same units; reducing 4 ft. to inches, the result is 12×4=48 in. Applying the rule, the area is

48×3=144 sq. in., or 1 sq. ft. Ans.

Example 3.—The base of an office cabinet is 6¼ feet long and 2½ feet wide. What area of floor space does it cover?

SOLUTION.—According to the rule, the area is

$$6\tfrac{1}{4}\times2\tfrac{1}{2}=\frac{25}{4}\times\frac{5}{2}=\frac{125}{8}=15\tfrac{5}{8}\text{ sq. ft.}\quad\text{Ans.}$$

12. Surveyor's Square Measure.—The square measure used by surveyors differs somewhat from that given in Art. **8.** The various units employed are included in the following table:

TABLE OF SURVEYOR'S SQUARE MEASURE

		Abbreviations
10,000 square links (sq. li.)	=1 square chain	sq. ch.
10 square chains	=1 acre	A.
640 acres	=1 square mile	sq. mi.
36 square miles	=1 township	Tp.

NOTE.—The acre contains 4,840 square yards or 43,560 square feet, and it is equal to the area of a square measuring 208.71 feet on each one of the sides.

EXAMPLE 1.—How many square links are there in 11 square chains?

SOLUTION.—According to the table, 1 sq. ch. is equal to 10,000 sq. li.; therefore, 11 sq. ch. must be equal to

11×10,000=110,000 sq. li. Ans.

EXAMPLE 2.—A piece of land contains 85 square chains. What is the equivalent value in acres?

SOLUTION.—There are 10 sq. ch. in an acre; therefore, the number of acres contained in 85 sq. ch. must be

85÷10=8.5 A. Ans.

EXAMPLE 3.—How many acres are there in a township?

SOLUTION.—As one township contains 36 sq. mi. and each square mile contains 640 A., it follows that the number of acres contained in one township must be

36×640=23,040 A. Ans.

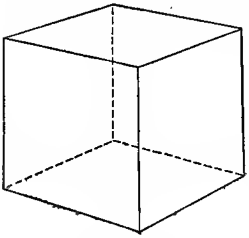


FIG. 4

CUBIC MEASURE

13. Definitions.—A **solid** is a figure that has length, breadth, and thickness. The boundaries of a solid are surfaces. A solid bounded by six squares, as Fig. 4, is a **cube.** If the solid is bounded by six rectangular surfaces it is a **rectangular solid.**

The space included between the bounding surfaces of a solid is called the **cubical contents, capacity,** or **volume** of the solid.

The **unit of volume** is a cube, the edges of which are of a length equal to that of the corresponding unit of length. Thus, 1 **cubic inch** measures 1 inch on each edge; 1 **cubic foot** measures 1 foot on each edge, etc.

TABLE OF CUBIC MEASURE

		ABBREVIATIONS
1,728 cubic inches (cu. in.)	=1 cubic foot	cu. ft.
27 cubic feet	=1 cubic yard	cu. yd.
128 cubic feet	=1 cord	cd.
16½ to 25 cubic feet	=1 perch	P.

14. The **standard cord** is a measure of wood; it is a pile 8 feet long, 4 feet wide, and 4 feet high. In some localities it is customary to consider the cord as a pile of wood 8 feet long and 4 feet high, the length of the wood, that is, the width of the pile, being left out of consideration.

The **perch,** a measure of stone and masonry, is but rarely used at present. Its cubical contents varies, according to locality, from 16½ to 25 cubic feet. The dimensions will vary accordingly; if the contents is 24¾ cubic feet, the perch would be 16½ feet, or 1 rod, long, 1½ feet wide, and 1 foot high.

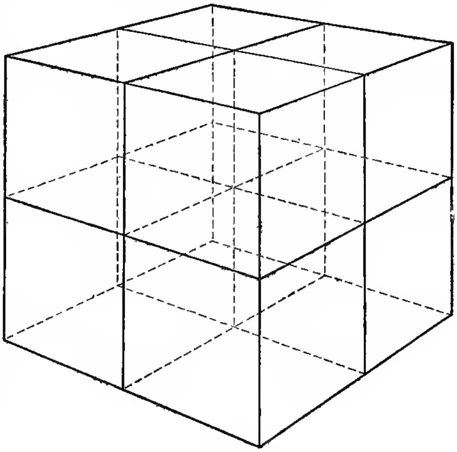


Fig. 5

15. Rule for Calculating Volume.—When writing the three dimensions of a solid, as 3 feet wide by 2 feet deep by 6 feet long, the word "by" is frequently represented by the symbol ×, and the preceding expression is written: 3 ft. ×2 ft.×6 ft.

The volume of any rectangular solid is the product of its three dimensions. For example, a box with inside dimensions 2 ft.×4 ft.×8 ft. contains 2×4×8=64 cubic feet; an iron bar 10 feet (or 10×12 inches) long, and with end surfaces that are each 1 inch square contains 10×12×1×1=120 cubic inches.

A cube 2 feet on each edge is a 2-foot cube, and contains 2×2×2=8 cubic feet, as shown in Fig. 5. A cube 12 inches, or 1 foot, on each edge is 1 cubic foot, or a 12-inch cube, and contains 12×12×12=1,728 cubic inches. A cube 3 feet, or 1 yard, on each edge is 1 cubic yard, or a 3-foot cube, and contains 3×3×3=27 cubic feet. Tables of cubic measure can be calculated from tables of linear measure in this manner.

The preceding method of calculating the volume of a rectangular solid may be stated as a rule, as follows:

Rule.—*To find the volume of a cube or a rectangular solid, multiply together the length, breadth, and depth, all expressed in the same units.*

If the dimensions are stated in inches, the volume is given in cubic inches; if in feet, the volume is given in cubic feet, and so on.

EXAMPLE 1.—A sand bin is 14 feet long, 8 feet wide, and 6 feet high. How many cubic feet does it contain?

SOLUTION.—According to the rule, it contains

$$14\times8\times6=672 \text{ cu. ft.} \quad \text{Ans.}$$

EXAMPLE 2.—How many cubic inches of metal are there in a block 12½ inches long, 8½ inches wide, and 4 inches thick?

SOLUTION.—Applying the rule, the volume is

$$12\tfrac{1}{2}\times8\tfrac{1}{2}\times4=\frac{25}{2}\times\frac{17}{2}\times\frac{4}{1}=425 \text{ cu. in.} \quad \text{Ans.}$$

EXAMPLE 3.—A box contains 86,400 cubic inches. What is the contents expressed in cubic feet?

SOLUTION.—According to the table, 1 cu. ft. contains 1,728 cu. in. The problem is, therefore, to ascertain how many times 1,728 is contained in 86,400 or,

$$86{,}400\div1{,}728=50 \text{ cu. ft.} \quad \text{Ans.}$$

16. Board Measure.—A special branch of cubic measure is known as **board measure.** In measuring sawed lumber, the unit of measure is the **board foot,** which is equal to the contents of a board 1 foot square and 1 inch thick. A board foot is, therefore, equal to one-twelfth of a cubic foot. Boards less than 1 inch thick are usually reckoned as though the thickness were 1 inch; but in buying and selling, the actual thickness is considered in fixing the price.

The following rule may be used for finding the number of feet, board measure (B. M.), in a piece of lumber:

Rule.—*To find the number of feet, board measure, in a piece of lumber, multiply together the length, in feet, the width, in inches, and the thickness, in inches, and divide the product by 12, thicknesses less than 1 inch being considered as 1 inch.*

Example 1.—How many feet, B. M., are there in a board 14 feet long, 8 inches wide, and 1 inch thick?

Solution.—According to the rule the number of feet, B. M., is equal to $\frac{14\times 8\times 1}{12}=9\frac{1}{3}$ ft. B. M. Ans.

Example 2.—How many feet, B. M., are there in a plank 10 feet 6 inches long, 15 inches wide, and 3 inches thick?

Solution.—Length = 10 ft. 6 in. = $10\frac{1}{2}$ ft. = 10.5 ft. The number of feet, B. M., is

$$\frac{10.5\times 15\times 3}{12}=39.375=39\frac{3}{8}\text{ ft. B. M.}$$

Example 3.—Find the number of feet, B. M., in 20 pieces of siding $\frac{1}{2}$ inch thick, $5\frac{1}{2}$ inches wide, and 10 feet 9 inches long.

Solution.—Since the thickness is less than 1 in., the boards are considered to be 1 in. thick when finding the amount of material in them. Length = $10\frac{9}{12}$ ft. = 10.75 ft., width = $5\frac{1}{2}$ in. = 5.5 in. The required length is

$$\frac{20\times 10.75\times 5.5\times 1}{12}=\frac{1{,}182.5}{12}=98.5\text{ ft. B. M. Ans.}$$

Example 4.—How many feet, B. M., of 2-inch planking are needed for a barn floor 18 ft. × 25 ft.?

Solution.—The planks are considered as 25 ft. long. Reducing the width to inches and considering the whole floor as one plank, its width is equal to 18×12=216 in. Proceeding as in the rule, the number of feet required is

$$\frac{25\times216\times2}{12}=900 \text{ ft. B. M. Ans.}$$

Example 5.—A board is 9½ ft.×4 in.×1¼ in. How many feet, B. M., does it contain?

Solution.—Applying the rule,

$$\frac{9\frac{1}{2}\times4\times1\frac{1}{4}}{12}=3.96 \text{ ft. B. M., nearly. Ans.}$$

MEASURES OF CAPACITY

LIQUID MEASURES AND DRY MEASURES

17. Liquid Measure.—The term **capacity** means cubical contents and is used here in connection with measures to indicate the relative amount of space they possess for measuring purposes.

Liquid measure is used for measuring liquids. Liquids in small quantities are measured in pints and quarts; in larger quantities, the gallon is the more common unit, though quantities are sometimes stated in barrels, but rarely in hogsheads. The standard barrel, as used in measuring capacity, contains 31½ gallons, but the barrels ordinarily used vary greatly in size.

One United States standard liquid gallon, known as the *Winchester* or wine gallon, contains 231 cubic inches. One gallon of pure water weighs 8.355, or approximately 8⅓ pounds. One cubic foot contains 7.481, or approximately 7.5, gallons, and 1 cubic foot of water, at 62° Fahrenheit, weighs 62½ pounds, nearly.

In some English-speaking countries the beer or ale gallon of 282 cubic inches' capacity is used for measuring the liquids mentioned.

The British imperial gallon, used in Canada, contains 277.463 cubic inches, and the weight of such a gallon of pure water is 10 pounds. One imperial gallon equals approximately 1.2 United States liquid gallons.

TABLE OF LIQUID MEASURE

English System

		Abbreviations
4 gills (gi.)	=1 pint	pt.
2 pints	=1 quart	qt.
4 quarts	=1 gallon	gal.
31½ gallons	=1 barrel	bbl.
2 barrels	=1 hogshead	hhd.

18. Dry Measure.—Dry articles, such as fruit, grain, vegetables, etc. are measured by **dry measure.** The standard unit is the United States bushel, otherwise known as the *Winchester bushel,* which contains 2,150.42 cubic inches; and ½ peck (dry gallon) contains 268.8 cubic inches.

One British imperial bushel contains 2,219.704 cubic inches, or 1.0322 United States bushels.

The quart used in dry measure is not the same as the quart used in liquid measure. The dry quart contains 67.2 cubic inches, and the liquid quart only 57¾ cubic inches.

TABLE OF DRY MEASURE

		Abbreviations
2 pints (pt.)	=1 quart	qt.
8 quarts	=1 peck	pk.
4 pecks	=1 bushel	bu.

19. The application of the preceding tables is shown in the following examples:

Example 1.—How many quarts are there in 2 barrels?

Solution.—According to the table of liquid measure there are 4 qt. in 1 gal., and 31½ gal. in 1 bbl. Therefore, in 2 bbl. there are

$$2\times31.5=63.0 \text{ gal.}$$

and

$$63\times4=252 \text{ qt. Ans.}$$

Example 2.—How many hogsheads will be required for 315 gallons of molasses?

Solution.—According to the table of liquid measure, 1 hhd.=2 bbl., and 1 bbl.=31½ gal. In 2 bbl. there are 2×31½=63 gal. Therefore, 315 gal. require

$$315\div63=5 \text{ hhd. Ans.}$$

Example 3.—A man bought 10 bushels of potatoes and intends to sell them by the quart. How many quarts of potatoes has he on hand?

Solution.—According to the table of dry measure, 1 bu.=4 pk., and 1 pk.=8 qt., so that in 1 bu. there are 4×8=32 qt.; in 10 bu. there are

10×32=320 qt. Ans.

MEASURES OF WEIGHT

AVOIRDUPOIS AND TROY MEASURES

20. Standard Units of Weight.—There are two English measures of weight in general use, the **avoirdupois** (av.) **weight,** for coarse, heavy substances, such as coal, iron, copper, hay, and grain, and the **Troy weight,** for finer and more valuable substances, such as gold, silver, and jewels. Besides there is the **apothecaries' weight,** which is used by physicians in prescribing and by druggists in compounding medicines. Medicines are bought and sold by avoirdupois weight.

TABLE OF AVOIRDUPOIS WEIGHT

		Abbreviations
27.3438 grains (gr.)	=1 dram	dr.
16 drams	=1 ounce	oz.
16 ounces (oz.)	=1 pound	lb.
100 pounds	=1 hundredweight	cwt.
20 hundredweight 2,000 pounds	=1 ton	T.

TABLE OF TROY WEIGHT

		Abbreviations
24 grains (gr.)	=1 pennyweight	dwt.
20 pennyweights	=1 ounce	oz.
12 ounces	=1 pound	lb.

TABLE OF APOTHECARIES' WEIGHT

		Abbreviations
20 grains (gr.)	=1 scruple	sc.
3 scruples	=1 dram	dr.
8 drams	=1 ounce	oz.
12 ounces	=1 pound	lb.

21. One **avoirdupois pound** equals approximately 1.2153 **Troy pounds,** the former containing 7,000 Troy grains and the latter 5,760 grains. One avoirdupois ounce contains 7,000÷16=437.5 grains, and 1 Troy ounce contains 24×20=480 grains. It is to be noted that 1 avoirdupois grain is equal to 1 Troy grain and to 1 apothecaries' grain.

22. The **ton** containing 2,000 pounds is sometimes called the *short ton* to distinguish it from the *long ton,* containing 2,240 pounds, or 20 hundredweights of 112 pounds each. The short ton is in general use except in the following cases: The *long* ton is used in the United States custom houses, and at the mines for weighing anthracite. It is also used by wholesale dealers in weighing steel rails and a few other iron products, when shipped in bulk. The denomination *hundredweight* is rarely used in practical work, especially with short-ton measures.

23. Gross Weight and Net Weight.—The **gross weight** of an article means the total weight, including the package, crate, or containing vessel; the **net weight** means the weight of the article alone. For example, the gross weight of a reel of wire is the weight of the wire and reel together, and the net weight of the wire is the gross weight less the weight of the reel.

24. Application of Tables of Weight.—The application of the preceding tables of weights is illustrated by the following examples:

EXAMPLE 1.—If a short ton of coal costs $6.00, how much is the cost per pound?

SOLUTION.—According to the table there are 2,000 lb. in 1 T. As 600 cents is the price of 2,000 lb., it follows that the price of 1 lb. is $\frac{1}{2000}$ of 600, or 600÷2,000=.3, or $\frac{3}{10}$ cent.

If the price is expressed in terms of dollars, the operation and answer will be: 6.00÷2,000=.003 dollar. Ans.

EXAMPLE 2.—If a bag of sugar weighs 800 ounces, what is its weight in pounds?

SOLUTION.—According to the table, 1 lb.=16 oz.; therefore, dividing 800 by 16 will give the equivalent number of pounds; thus,

800÷16=50 lb. Ans.

EXAMPLE 3.—How many grains are there in 1 pound of gold?

Solution.—According to the table 1 lb. Troy=12 oz., and 1 oz. is equal to 20 dwt. of 24 gr.; hence, 1 lb. is equal to

$$12\times20\times24=5{,}760 \text{ gr.} \quad \text{Ans.}$$

MEASURES OF TIME

UNITS AND STANDARD TIME

25. Units of Time.—The principal unit of time is the **second.** A minute contains 60 seconds, and a day contains 24 hours, or 86,400 seconds. The time required for the earth to make one complete revolution around the sun is called 1 **solar year,** and is exactly 365 days 5 hours 48 minutes 49.7 seconds.

TABLE OF MEASURES OF TIME

		Abbreviations
60 seconds (sec.)	=1 minute	min.
60 minutes	=1 hour	hr.
24 hours	=1 day	da.
7 days	=1 week	wk.
365 days 12 months	=1 common year	yr.
366 days	=1 leap year	
100 years	=1 century	C.

26. As the solar year contains more than 365 days, the fractional part is omitted in the **common year,** the latter containing 365 days. To compensate for this omission the fractional parts of several years are added periodically to one year, which is then known as a **leap year.** It contains 366 days, the extra day being added to the month of February.

When the number expressing a year is divisible by 4 and is not divisible by 100, that year is a leap year. A centennial year is a leap year, if the number expressing the year is divisible by 400. Thus, 1912, 1916, and 1920 are leap years, because each is exactly divisible by 4 and is not divisible by 100. The centennial year 1900 was not a leap year, because the number 1900 is not divisible by 400. The year 2000 will be a leap year, because the number is divisible by 400.

27. Standard Time.—Five **standard time divisions** are in use in the United States, Canada, and Mexico, and each division is bounded by lines running north and south. These divisions named from east to west are *Atlantic, Colonial,* or *Intercolonial; Eastern; Central; Mountain;* and *Pacific.* The United States lies almost wholly in the four divisions last named.

Time of day is usually stated as A. M., meaning *ante meridiem,* or *before noon,* and P. M., meaning *post meridiem,* or *after noon,* followed by the name of the division when this might otherwise be uncertain. Thus, the time of day may be stated as 10:45 A. M., Central time, or 11:45 A. M., Eastern time. The difference in time in adjacent divisions is exactly 1 hour, and is earlier in each westward division. Thus, when it is noon in Washington, D. C., the time is 11 A. M. in Chicago, 10 A. M. in Denver, and 9 A. M. in San Francisco.

28. Months and Number of Days.—The numbers and names of the months and the number of days in each are as follows:

No.	Name	Number of Days	No.	Name	Number of Days
1.	January (Jan.)	31	7.	July	31
2.	February (Feb.)	28 or 29	8.	August (Aug.)	31
3.	March (Mar.)	31	9.	September (Sept.)	30
4.	April (Apr.)	30	10.	October (Oct.)	31
5.	May	31	11.	November (Nov.)	30
6.	June	30	12.	December (Dec.)	31

For many business purposes, 30 days is considered to be 1 month.

MEASURES OF CURRENCY

AMERICAN AND ENGLISH

29. Units of Money.—The term **currency** refers to money, such as coins, bank notes, etc. The United States measure of money being based on the decimal system, it is possible to write fractions of dollars as decimals. The number of dollars is written as a whole number followed by a decimal fraction representing the number of cents. Thus, the expression 7 dollars and 25 cents is written $7.25. Sometimes, the decimal contains a third place representing *mills;* but the mill is not a coin. It is simply used as a convenient means for representing a tenth part of a cent. For instance, the expression $.065 is read *6 cents and 5 mills.*

Numbers representing pounds, English money, are preceded by the sign £ and followed by numbers representing other denominations each with its abbreviation; thus, £4 12s. 6d. is read *4 pounds 12 shillings 6 pence,* the word *pence* being plural of *penny.*

£1=$4.866½ A crown=5s.
A guinea=21s. A florin =2s.

The following tables give measures of currency as used in the United States and Great Britain. In Canada the units of money are the same as in the United States.

TABLE OF UNITED STATES MONEY

		Abbreviations
10 mills (m.)	=1 cent	c. or ct.
10 cents	=1 dime	d.
10 dimes	=1 dollar	$
10 dollars	=1 eagle	E.

TABLE OF ENGLISH MONEY

		Abbreviations
4 farthings (far.)	=1 penny	d.
12 pence	=1 shilling	s.
20 shillings	=1 pound, or sovereign	£

MEASURES OF ARCS AND ANGLES

CIRCULAR AND ANGULAR MEASURES

30. Definitions.—The measure of arcs and angles is called **circular,** or **angular, measure,** and is applied to the measurement of arcs, angles, and circles. A **circle** is a plane figure bounded by a curved line every point of which is equally distant from a point within called the **center.** The line that bounds a circle is called its **circumference.** Any straight line drawn from the center to the circumference, as *c a, c b,* or *c d,* Fig. 6, is called a **radius**; if a radius is extended to meet the circumference also on the opposite side of the center, as *b c e,* it is called a **diameter.** Any portion of the circumference of a circle, as *b d,* is called a **circular arc,** or simply an **arc.**

The whole circumference of any circle is supposed to be divided into 360 equal parts, and each of these equal parts is called a **degree.** Since 1 degree is $\frac{1}{360}$ of any circumference, it follows that the length of an arc of 1 degree will be different in circles of different diameters, but the length of the circumference of any circle will always be 360 times as large as the length of the arc of 1 degree. Or, the arc of 1 degree is always $\frac{1}{360}$ of the circumference, whatever the size of the circle.

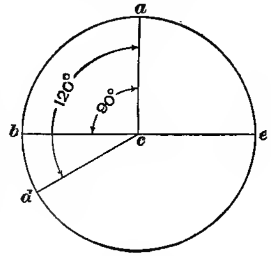


Fig. 6

31. An **angle** is the opening between two straight lines that meet at a point. Thus, in Fig. 6, the lines *b c* and *d c* form an angle at the point *c*. These lines are the **sides** of the angle and the point *c* at which they meet is called the **vertex** of the angle.

If any circle is drawn with its center at the vertex of an angle, that portion of the circumference included between the sides of the angle is known as the intercepted arc. Such an angle is measured by its intercepted arc. For example, if the arc *a d* contains 120 degrees, the angle *a c d* is an angle of 120 degrees. For the same reason the angle *a c b* is equal to 90 degrees if the arc *a b* contains 90 degrees.

If a circumference is divided into four equal parts, each of the parts is called a **quadrant.** The arc *a b* is a quadrant and it contains 90 degrees.

A degree and its subdivisions, minutes and seconds, are indicated by the signs (°), (′), (″), both for angles and arcs; thus, 32° 15′ 10″ is read *thirty-two degrees fifteen minutes ten seconds.*

TABLE OF ANGULAR MEASURE

ABBREVIATIONS

60 seconds (″) .... = 1 minute .... ′
60 minutes .... = 1 degree .... °
90 degrees .... = 1 right angle or quadrant, ∟ or rt. ∠
360 degrees .... = 1 circle .... cir.

MEASURES OF TEMPERATURE

FAHRENHEIT AND CENTIGRADE SCALES

32. Thermometer.—The **temperature** of a body is the measure of its degree of sensible heat; hot bodies are said to have *high* temperatures and cold bodies *low* temperatures. Temperature is usually measured by means of an instrument called a **thermometer.** It consists of a glass tube closed at the upper end and having at the lower end a bulb filled with mercury. When the thermometer is placed in contact with a hot body, the heat causes the mercury to expand, or take up more space, and so it rises in the hollow tube. If the body is cold, the mercury contracts, and occupies less space than before, and the column then descends or grows shorter. The greater the temperature, the higher the mercury rises in the tube; and the lower the temperature, the lower the mercury descends in the tube.

33. Thermometer Scales.—Two thermometric scales are in common use; one is the *Fahrenheit scale,* and the other the *centigrade scale.* A thermometer with these two scales, one on each side of the tube, is shown in Fig. 7. The one on the left, with *F* at the top, is the Fahrenheit scale, and the one at the right, with *C* above it, is the centigrade scale. The Fahrenheit thermometer, named after its inventor, is the one most commonly used in the United States. The centigrade thermometer is used largely in scientific work. The difference between the two is that the boiling point of water is marked 212 on the Fahrenheit scale and 100 on the centigrade; and the freezing point is 32 on the Fahrenheit scale and 0 on the centigrade. Each small division on each scale is a *degree* of temperature. The abbreviation for Fahrenheit is Fahr. or F., and the abbreviation for centigrade is Cent. or C.; thus, 180° F. means 180 degrees on the Fahrenheit scale, and 65° C. means 65 degrees on the centigrade scale.

34. In graduating thermometers, that is, making the scales of values on them, two fixed points of temperature are almost universally employed. These are the temperatures of melting ice, and of the steam of water boiling in an uncovered vessel. In graduating the centigrade scale the distance between these two points is divided into 100 equal graduations or degrees. The freezing point is called zero, or 0°, and the boiling point 100°. Degrees of the same value are carried above and below the boiling and freezing points, respectively. Distinction is made between the degrees above zero and those below by placing a minus sign before the latter; thus, 10° below zero is marked −10° and is read *minus ten degrees*.

35. In graduating the Fahrenheit scale, the distance between the freezing and the boiling point is divided into 180 equal parts, and degrees of the same size are carried above and below the boiling and freezing points. Fahrenheit, who proposed this scale, assumed that the greatest cold obtainable was 32° below the freezing point, and accordingly took that point as the zero and reckoned from it upwards. The freezing point thus became 32° F. and the boiling point 32°+180°=212°. Degrees below the Fahrenheit zero are provided with a minus sign, as are similarly placed degrees on the centigrade scale. On many Fahrenheit thermometers, the space between the freezing point and boiling point is divided into 90 equal parts, so that the division marks will not be so close together. In such a case, each division corresponds to 2 degrees, but each main division is numbered as in Fig. 7.

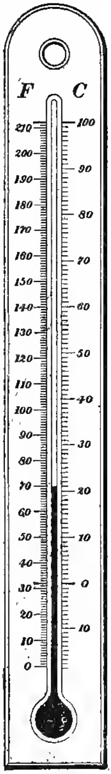


Fig. 7

36. Conversion of Thermometer Readings.—From the fact that 180° F. = 100° C., it follows that 9° F. = 5° C. and 1° F. = $\frac{5}{9}$° C., or 1° C. = $\frac{9}{5}$° F. These relative values form the basis of the following rules by which Fahrenheit temperatures may be changed into their corresponding centigrade values, or vice versa:

Rule I.—*To find the Fahrenheit temperature, multiply the centigrade temperature by $\frac{9}{5}$ and add 32 to the product.*

Rule II.—*To find the centigrade temperature, subtract 32 from the Fahrenheit temperature and multiply the remainder by $\frac{5}{9}$.*

EXAMPLE 1.—What is the equivalent of 85° C. on the Fahrenheit scale?

SOLUTION.—Applying rule I, the temperature is

$$(85\times\tfrac{9}{5})+32=185° \text{ F. Ans.}$$

EXAMPLE 2.—If a temperature is 68° F., what is the equivalent on the centigrade scale?

SOLUTION.—According to rule II, 32 is subtracted from 68 giving the remainder 36; then,

$$36\times\tfrac{5}{9}=20° \text{ C. Ans.}$$

37. Converting Variations of Temperature.—Distinction must be made between a *stationary* temperature and a *variation* of temperature when converting from one scale into the other. For example, suppose that the temperature of an electric generator rises 40° C.; since, according to Art. **36**, 1° C. = $\frac{9}{5}$° F., it follows that 40° C. = $\frac{9}{5}\times 40$ = 72° F., the equivalent change in Fahrenheit degrees. Again, suppose that the temperature of a room decreases from 180° F. to 99° F., or 81° F.; since 1° F. = $\frac{5}{9}$° C., 81° F. = $\frac{5}{9}\times 81$ = 45° C., the equivalent variation in centigrade degrees. In other words, when converting a variation of temperature, the height of the freezing point above 0° F., or 32°, is not considered.

MISCELLANEOUS MEASURES

38. Collective Terms.—The terms *pair, couple, gross, great gross,* and *score* are used in connection with some articles or things; for example, a pair of shoes, a dozen screws, a gross of shears, and a score of years. Paper is usually bought and sold by the *quire, ream,* or *bundle.*

TABLE OF MISCELLANEOUS MEASURES

		ABBREVIATIONS
2 things	=1 couple, or 1 pair	pr.
12 things	=1 dozen	doz.
12 dozen	=1 gross	gr.
12 gross	=1 great gross	g. gr.
20 things	=1 score	
24 sheets	=1 quire	qr.
20 quires	=1 ream	rm.
2 reams	=1 bundle	bdl.
5 bundles	=1 bale	B.

METRIC MEASURES

LINEAR MEASURE

39. Principal Units.—In the **metric system** a uniform scale of 10 is used throughout, as in the ordinary scale of numbers and in United States money. The name of this system is derived from the *meter,* the unit from which all the other units are derived. The metric system of measures is in general use on the continent of Europe and is used to some extent in the United States.

The unit of length is the **meter,** which has a length equal to 39.37 inches, nearly. One hundredth of a meter is called a **centimeter;** and one thousandth of a meter, a **millimeter.**

In the United States, engineers and mechanics are likely to meet with metric units in constructing machinery that has been designed abroad and is being built to order in this country. The units used on the drawings are usually the centimeter and the millimeter.

40. The relative lengths of centimeters and inches may be seen from Fig. 8 in which the upper scale is divided into inches and subdivisions, and the lower scale into centimeters and millimeters. The portion of the scale shown contains about 9 centimeters; each centimeter divided into 10 equal parts to represent millimeters. It requires 100 centimeters, or more than 10 times the length shown in the illustration, to make 1 meter. A millimeter is equal to .03937 inch and a centimeter is equal to .3937 inch; that is, 1 inch is equal to $\frac{1}{.03937}$ =25.4 millimeters, or 2.54 centimeters. The **kilometer** is 1,000 meters and is used in measuring great distances. It is equal to about $\frac{5}{8}$ mile.

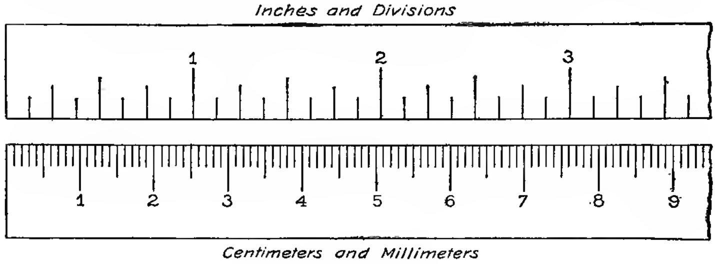


FIG. 8

The following equivalents, or conversion factors, are in most cases approximate: 1 inch=25.4 millimeters=2.54 centimeters; 1 foot=30.5 centimeters=.305 meter; 1 meter=39.37 inches =3.28 feet; 1 mile=1.61 kilometers; 1 kilometer=.6214 mile=3,281 feet.

TABLE OF LINEAR MEASURE

		ABBREVIATIONS
10 millimeters (mm.)	=1 centimeter	cm.
10 centimeters	=1 decimeter	dm.
10 decimeters	=1 meter	m.
1,000 meters	=1 kilometer	km.

SQUARE MEASURE AND CUBIC MEASURE

41. Units of Square Measure.—Areas such as floors, ceilings, and ordinary surfaces are measured in square meters; countries and states, in square kilometers; and small areas, in square centimeters. Land is measured in hectares, 1 *hectare* being equal to 10,000 square meters, which is equivalent to 2.471 acres.

The principal metric units of square measure are the **square centimeter** (sq. cm.), which equals .155 square inch, and the **square meter** (sq. m.), which equals 1.195 square yards. One square inch = 6.45 square centimeters, and 1 square yard = .836 square meter.

TABLE OF SQUARE MEASURE

		ABBREVIATIONS
100 square millimeters (sq. mm.)	= 1 square centimeter	sq. cm.
100 square centimeters	= 1 square decimeter	sq. dm.
100 square decimeters	= 1 square meter	sq. m.

42. Units of Cubic Measure.—The metric units used in practical work are chiefly the **cubic centimeter** (cu. cm.), which equals .06102 cubic inch, and the **cubic meter** (cu. m.), or **stere,** which equals 1.31 cubic yards. One cubic inch = 16.388 cubic centimeters, and 1 cubic yard = .765 cubic meter.

TABLE OF CUBIC MEASURE

		ABBREVIATIONS
1,000 cubic millimeters (cu. mm.)	= 1 cubic centimeter	cu. cm.
1,000 cubic centimeters	= 1 cubic decimeter	cu. dm.
1,000 cubic decimeters	= 1 cubic meter	cu. m.

LIQUID AND DRY MEASURE

43. Units of Capacity.—Metric units of capacity are the same for both liquids and dry substances. The **liter** (l.) is the principal unit of capacity; it is equal in volume to a cube whose edge is 10 centimeters in length. One **hectoliter** equals 100 liters. One liter = 1,000 cubic centimeters = 2.1 pints (liquid) = .91 quart (dry). One hectoliter = 100 liters = 210 pints (liquid) = 91 quarts (dry). One quart (dry) = 1.1 liters; one gallon = 3.785 liters. The volume of 1 liter is equal to 1 cubic decimeter.

TABLE OF LIQUID AND DRY MEASURE

		Abbreviations
10 milliliters (ml.)	=1 centiliter	cl.
10 centiliters	=1 deciliter	dl.
10 deciliters	=1 liter	l.
10 liters	=1 decaliter	Dl.
10 decaliters	=1 hectoliter	Hl.
10 hectoliters	=1 kiloliter	Kl.

MEASURE OF WEIGHT

44. Units of Weight.—The **gram** is the principal metric unit of weight; it is the weight of 1 cubic centimeter of pure distilled water at its temperature of maximum density, or 39.2° F., and is equal to 15.432 grains, Troy. The gram is used in weighing small quantities of gold, silver, etc., also letters (for postage). For greater weights the **kilogram** is used; it is equal to the weight of 1 liter of pure water at a temperature of 39.2° F. One kilogram = 1,000 grams = 2.2046 pounds avoirdupois, and 1 pound (av.) = .4536 kilogram. One metric **ton** = 1,000 kilograms; its weight is equal to that of 1 cubic meter of water at a temperature of 39.2° F.

TABLE OF MEASURE OF WEIGHT

		Abbreviations
10 milligrams (mg.)	=1 centigram	cg.
10 centigrams	=1 decigram	dg.
10 decigrams	=1 gram	g.
10 grams	=1 decagram	Dg.
10 decagrams	=1 hectogram	Hg.
10 hectograms	=1 kilogram	kg.
1,000 kilograms	=1 metric ton	T.

NOTE.—It is to be noted that abbreviations of any units greater than the principal units meter, gram, and liter are capitalized. Thus, decaliter and decagrams are abbreviated *Dl.* and *Dg.*, respectively; and kilometer and kilogram are abbreviated *Km.* and *Kg.*, respectively. However, usage has established *km.* and *kg.* as the most common forms of abbreviation for the latter two units named.

OPERATIONS WITH DENOMINATE NUMBERS

REDUCTION

REDUCTION OF ENGLISH UNITS

45. Reduction to Higher or Lower Denominations.—The **reduction** of denominate numbers is the process of changing their denomination without changing their value. Reduction is divided into two cases. In one case, the number is reduced to units lower than the highest named in the number; in the other case, units of a low denomination are reduced to a higher one. The first is called *reduction descending;* the second, *reduction ascending.*

To change 2 miles 120 rods to feet would require an application of reduction descending, and to change 243,079 pounds to tons would involve reduction ascending.

46. Reduction Descending.—Reduction to a lower denomination, or reduction descending, is made by multiplying the number of the higher denomination by the number of units of the lower denomination contained in one unit of the higher denomination. For example, to reduce 2 hours to minutes, 2 is multiplied by 60, as 1 hour contains 60 minutes. Thus, $2\times60=120$ minutes. In reductions to lower denominations the following rule is applied:

Rule.—*To reduce a compound denominate number to a lower denomination, multiply the number representing the higher denomination by the number of units in the next lower denomination required to make one unit of the higher denomination, and to this product add the number of units of the lower denomination.*

EXAMPLE 1.—Reduce 14 degrees 18 minutes 17 seconds to seconds.

Solution.—In order that the rule may be conveniently applied the numbers are arranged in the manner shown.

	14° 18′ 17″
Multiply degrees by number of minutes in 1 degree	60
	840′
Add minutes given in example	18
	858′
Multiply by number of seconds in 1 minute	60
	51480″
Add seconds given in example	17
	51497″ Ans.

Example 2.—How many seconds in 1 day?

Solution.—

Number of hours in 1 day	24
Multiply by number of minutes in 1 hr.	60
	1440 min.
Multiply by number of seconds in 1 min.	60
	86400 sec. Ans.

47. The preceding examples involve *more* than two different denominations. In practice, as a rule, only two denominations are considered, in which case the method of reduction can be much simplified. The following examples will show the method of procedure:

Example 1.—Reduce 64 quarts 1 pint to pints.

Solution.—Referring to the table of liquid measure, it is found that 1 qt.=2 pt. 64 qt. is equal to 64×2, or 128 pt.; therefore, 64 qt. 1 pt. =128 pt.+1 pt.=129 pt. Ans.

Example 2.—Reduce 6 tons 268 pounds to pounds.

Solution.—One ton contains 2,000 lb. Then, by the rule, 6×2,000 =12,000 lb., and 12,000+268=12,268 lb. Ans.

Example 3.—Change 5 hours 24 minutes to minutes.

Solution.—There are 60 min. in 1 hr. Therefore, according to the rule, 5×60=300 min., and 300+24=324 min. Ans.

Example 4.—How many inches in 3 feet 6½ inches?

Solution.—The number 3 is the higher denomination. According to the rule, this must be multiplied by 12, because it takes 12 of the lower units, or inches, to make one unit of the higher denomination, or 1 ft. Multiplying, 3×12=36 in. To this product is now added the 6½ in. in the given number, making a total of 36+6½=42½ in. Therefore, 3 ft. 6½ in.=42½ in. Ans.

Example 5.—How many square inches in a surface whose area is 3 square feet 28 square inches?

Solution.—In 1 sq. ft. there are 144 sq. in. Then, by the rule, 3×144=432 sq. in., and 432+28=460 sq. in. Ans.

EXAMPLES FOR PRACTICE

1. Reduce:

(*a*) 4 rd. 2 yd. 2 ft. to ft.
(*b*) 4 bu. 3 pk. 2 qt. to qt.

Ans. (*a*) 74 ft.
(*b*) 154 qt.

2. The length of an electric transmission line is 1 mi. 45 rd. 11 ft.; find the length, in feet, of the wire for the complete (two-wire) circuit. Ans. 12,067 ft.

3. A student purchased 8 lb. 5 oz. (av.) of chemicals for experimental work. What was its value at 8 cents an ounce? Ans. $10.64

4. If the bearings of a machine hold 3 pints of oil, how many times can they be filled from 2 barrels? Ans. 168 times

5. The pressure of the atmosphere at sea level is at a given time 14.7 pounds per square inch. What is the pressure per square foot? Ans. 2,116.8 lb.

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48. Reduction Ascending.—Reduction to a higher denomination, or reduction ascending, is made by dividing the number of the lower denomination by the number of units of the lower denomination required to make one unit of the higher denomination. For example, to reduce 32 ounces (av.) to pounds, 32 is divided by 16, because 1 pound contains 16 ounces; thus, 32÷16=2 pounds. In reductions to higher denominations the following rule is applied:

Rule.—*To reduce a denominate number to a higher denomination, divide the number representing the denomination given by the number of units of this denomination required to make one unit of the higher denomination. The remainder will be of the same denomination, but the quotient will be of the higher. The quotient and the remainder together are the required result.*

Example 1.—Reduce 211 inches to yards, feet, and inches.

Solution.—There are 12 in. in 1 ft.; therefore, 211 in. divided by 12=17 ft. and 7 in. remainder. There are 3 ft. in 1 yd.; therefore, 17 ft. divided by 3=5 yd. and 2 ft. remainder. The last quotient and the two remainders constitute the answer, 5 yd. 2 ft. 7 in.

```
12)211(17 ft.
   12
    91
    84
     7 in. rem.
3)17(5 yd.
  15
   2 ft. rem.
```

Example 2.—Reduce 3,557 pints to higher denominations.

Solution.—There are 2 pt. in 1 qt.; therefore, 3,557 pt. divided by 2=1,778 qt. and 1 pt. remainder. There are 4 qt. in 1 gal.; therefore, 1,778 qt. divided by 4=444 gal. and 2 qt. remainder. There are 31.5 gal. in 1 bbl.; therefore, 444.0 gal. divided by 31.5=14 bbl. and 3.0 gal. remainder.

The answer is made up of the last quotient and all the remainders; thus, 14 bbl. 3 gal. 2 qt. 1 pt. Note that the remainder after the last division is not 30 gal. but 3 gal., the decimal point being placed vertically under the point in the dividend.

2) 3557 (1778 qt.
2
15
14
15
14
17
16
1 pt. rem.

4) 1778 (444 gal.
16
17
16
18
16
2 qt. rem.

31.5) 444.0 (14 bbl.
315
1290
1260
3.0 gal. rem.

Example 3.—Reduce 18,000 square inches to square feet.

Solution.—

Divide by the number of square inches in 1 sq. ft. } 144) 18000 (125 sq. ft. Ans.
144
360
288
720
720

Example 4.—How many weeks in 18 days?

Solution.—It takes 7 days to make 1 week, that is, 7 units of the lower denomination to make 1 unit of the higher. By the rule, 18 is to be divided by 7 to get the result; thus,

7) 18 (2 weeks
14
4 days

The result is therefore 2 wk. 4 da. Ans.

Example 5.—Reduce 14,728 pounds to tons.

Solution.—One ton is equal to 2,000 lb. Then, according to the rule, 14,728 is to be divided by 2,000; thus,

2000) 14728 (7 T.
14000
728 lb.

Therefore, 14,728 lb.=7 T. 728 lb. Ans.

Example 6.—Reduce 945 pints to higher denominations.

Solution.—The solution is found in the manner shown in example 2. The final quotient and the various remainders are as follows: 3 bbl.

23.5 gal. 1 pt. The decimal .5 gal. must be reduced to the next lower denomination by multiplying it by the number of quarts in 1 gal., or $.5 \times 4 = 2$ qt. Hence, the final answer is: 3 bbl. 23 gal. 2 qt. 1 pt.

EXAMPLES FOR PRACTICE

1. Reduce the following to units of higher denominations. If fractions appear in the results in any but the lowest denomination, reduce them to lower denominations.

(*a*) 7,460 sq. in.
(*b*) 2,395 pt. (liq.)

Ans. (*a*) 5 sq. yd. 6 sq. ft. 116 sq. in.
Ans. (*b*) 9 bbl. 15 gal. 3 qt. 1 pt.

2. How many tons in 459,875 lb.? Ans. 229.9375 T.

3. If 36 cu. ft. of coal weighs 1 T., how much space is filled by 7 T. 1,590 lb.? Ans. 280.62 cu. ft.

4. How many rails, each 30 feet long, will be required to lay a railroad track 26 miles long? Ans. 9,152

NOTE.—Two lines of rails are to be considered.

REDUCTION OF METRIC UNITS

49. Reduction to Higher or Lower Denominations. In the metric system it is easy to reduce the values of different denominations to the next lower or to the higher denomination, as all that is necessary is to multiply or divide by 10 for linear measures. If one or more of the intermediate denominations are omitted, the factor 10 is increased to 100 or to 1,000, as the case may be. For instance, on reducing meters to centimeters, the number of meters is multiplied by 100, as 1 meter = 100 centimeters. Reducing kilograms to grams, the number of kilograms is multiplied by 1,000, as 1 kilogram = 1,000 grams. A few examples will show the method of procedure.

EXAMPLE 1.—Reduce 8 kilograms to grams.

SOLUTION.—From Art. **44**, one kg. = 1,000 g.; hence,

8 kg. = $8 \times 1,000$ g. = 8,000 g. Ans.

EXAMPLE 2.—Express 7.65 cubic meters in cubic centimeters.

SOLUTION.—From the table of cubic measure, 1 cu. m. = 1,000 cu. dm., and 1 cu. dm. = 1,000 cu. cm. It follows that 7.65 cu. m. = $7.65 \times 1,000 \times 1,000$ cu. cm. = 7,650,000 cu. cm. Ans.

Example 3.—Reduce 97.56 grams to kilograms.

Solution.—From Art. **44,** one kg.=1,000 g. It follows that

$$97.56 \text{ g.} = \frac{97.56}{1{,}000} = .09756 \text{ kg. Ans.}$$

Example 4.—What is the weight of 37 cubic millimeters of water?

Solution.—From the table of cubic measure, 1 cu. cm.=1,000 cu. mm. It follows that

$$37 \text{ cu. mm.} = \tfrac{1}{1000} \times 37 \text{ cu. cm.} = .037 \text{ cu. cm.}$$

From Art. **44,** 1 cu. cm. of water weighs 1 g.; therefore, the weight of 37 cu. mm. of water is

$$.037 \times 1 \text{ g.} = .037 \text{ g. Ans.}$$

REDUCTION FROM ONE SYSTEM TO ANOTHER

50. Conversion of Metric Units to English Units. In changing metric units to English units, the general rule to follow is to multiply the number of metric units by the equivalent of that unit in the desired English units. The method can most easily be illustrated by examples, as follows:

Example 1.—A machine part has a diameter of 115 millimeters. What is its diameter in inches?

Solution.—From Art. **40,** the equivalent of 1 mm. is .03937 in. Therefore, the equivalent of 115 mm. is

$$115 \times .03937 = 4.52755 \text{ in., or } 4\tfrac{17}{32} \text{ in., nearly. Ans.}$$

Example 2.—If a vessel has a volume of 248 cubic centimeters, what is its volume in cubic inches?

Solution.—From Art. **42,** the equivalent of 1 cu. cm. is .06102 cu. in. The volume of the vessel in cubic inches, therefore, is

$$248 \times .06102 = 15.13 \text{ cu. in. Ans.}$$

51. Conversion of English Units to Metric Units. In order to change English units to metric units divide the number of English units by the equivalent of the desired metric unit in English units; or, as multiplication is simpler than division, multiply the number of English units by the number of metric units equivalent to the English unit.

Example 1.—If a bar is $6\frac{3}{8}$ inches long, what is its length in millimeters?

Solution.—According to Art. **40,** 1 in. contains 25.4 mm. Therefore,

$6\frac{3}{8}\times 25.4=162$ mm., very nearly. Ans.

Example 2.—If a tank of oil weighs 280 pounds, what is its weight in kilograms?

Solution.—By Art. **44,** 1 lb.=.4536 kg.; hence,

$280\times .4536=127$ kg. Ans.

EXAMPLES FOR PRACTICE

1. A disk has a diameter of $3\frac{1}{4}$ inches. What is its diameter in millimeters? Ans. 82.55 mm.

2. Find the length in inches of a piece of wire that is 280 millimeters long. Ans. 11 in.

3. A box containing $12\frac{1}{2}$ cubic inches contains how many cubic centimeters? Ans. 205 cu. cm., nearly

4. A bag of flour weighs 12 kilograms. What is its weight in pounds? Ans. $26\frac{1}{2}$ lb., nearly

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52. Conversion Factors.—As a convenient means for reducing the units of one system to those of another a list of the most frequently used *conversion factors* is given herewith. Many of these are factors shown in connection with the preceding tables. The term **conversion factor** means here a multiplier which if used with a given number of one system converts it into an equivalent number of another system. It is stated in Art. **40** that 1 inch=25.4 millimeters. The number 25.4 is a conversion factor by which any number of inches may be converted to millimeters. For example, 7.5 inches are converted to millimeters by using 25.4 as a multiplier; thus, 7.5 inches $=7.5\times 25.4$ millimeters $=190.5$ millimeters.

List of Conversion Factors

Inches×25.4=millimeters; millimeters÷25.4=inches.
Inches×2.54=centimeters; centimeters÷2.54=inches.
Feet×.305=meters; meters×3.28=feet.
Miles×1.61=kilometers; kilometers ×.6214=miles.
Square inches×6.45=square centimeters; square centimeters×.155 =square inches.

Square yards×.836=square meters; square meters×1.195=square yards.

Cubic inches×16.4=cubic centimeters; cubic centimeters×.061=cubic inches.

Cubic yards×.765=cubic meters, or steres; cubic meters×1.31 =cubic yards.

Quarts (dry)×1.1=liters; liters÷1.1=quarts (dry).

Gallons×3.785=liters; liters×.2642=gallons.

Pounds (av.)×.4536=kilograms; kilograms×2.2=pounds.

53. Application of Conversion Factors.—The application of the preceding list to the conversion of numbers from units of one system to units of another system may be seen from the following examples:

Example 1.—How many meters in 4,375 feet?

Solution.—As feet×.305=meters, 4,375 ft.=4,375×.305=1,334.375 m. Ans.

Example 2.—How many pounds in 723 kilograms?

Solution.—As kilograms×2.2=pounds, 723 kg.=723×2.2=1,590.6 lb. Ans.

Example 3.—Reduce 231 kilometers to miles.

Solution.—As kilometers×.6214=miles, 231 kilometers=231×.6214 =143.5 mi. Ans.

REDUCTION OF DENOMINATE NUMBERS TO DECIMALS

54. Rule for Reduction.—In calculations employing denominate numbers, it is often convenient, and in some cases necessary, to reduce a denominate number to a decimal part of the next higher denomination, as inches to a decimal of a foot, minutes to a decimal of an hour, and so forth. The method employed for the reduction of decimal denominate numbers corresponds in its main features with that explained in Arts. **45** to **48.**

Considering first the particular case of reducing 6 inches to a decimal part of a foot, it is to be noted that 1 inch is $\frac{1}{12}$ part of a foot. Therefore, 6 inches is $\frac{6}{12}$ of a foot. Reducing $\frac{6}{12}$ to a decimal, its equivalent value is the quotient of 6÷12=.5. Hence, 6 inches is equivalent to .5 foot. Similarly, as 7 inches is equal to $\frac{7}{12}$ of a foot, and $\frac{7}{12}$ reduced to a decimal is equal to .5833, it follows that $\frac{7}{12}$ foot is equal to .5833 foot.

In a similar manner 15 minutes may be reduced to a decimal of an hour. As 1 hour = 60 minutes, 15 minutes = $\frac{15}{60}$ hour. On reducing $\frac{15}{60}$ to a decimal its equivalent value is found to be 15 ÷ 60 = .25; therefore, 15 minutes = .25 hour.

The method followed in the preceding examples is embodied in the following rule:

Rule.—*To reduce a denominate number to a decimal of a higher denomination divide the given number by the number of units required to make one unit of the higher denomination. The quotient, expressed as a decimal, will be the equivalent decimal value of the given number in terms of the higher denomination.*

55. Application of Rule.—The application of the preceding rule is shown in the following examples:

Example 1.—In the expression 26 feet 9¾ inches reduce the inches to a decimal of a foot and add the decimal to the whole number.

Solution.—According to the rule, the number 9¾ must be divided by 12. This division will be made easier by reducing the number to an improper fraction or to a whole number and a decimal. Following the latter method, it is found that 9¾=9.75. On dividing 9.75 by 12 the quotient is .8125. It follows that 9¾ in.=.8125 ft., and that the complete expression is 26+.8125=26.8125 ft. Ans.

Example 2.—Reduce 36 degrees 27 minutes 48 seconds to degrees and decimal of a degree.

Solution.—Applying the rule, 48″ must be reduced to a decimal of a minute by dividing 48 by 60, the number of seconds in a minute, and expressing the quotient as a decimal. Thus, 48÷60=.8; hence, 48″=.8′. This decimal must be added to the given minutes, the result being 27+.8=27.8′. This number must now be reduced to a decimal of a degree. Applying the rule again, the number 27.8 must be divided by the number of minutes in one degree, or 60. Thus, 27.8÷60=.463; therefore, 27.8′ is equal to .463°. It follows that the expression 36° 27′ 48″ is equivalent to 36.463°. Ans.

Example 3.—Reduce 21 gallons 3¾ quarts 1½ pints to gallons and a decimal part of a gallon.

Solution.—There are 2 pt. in a quart, therefore 1½ pt. reduced to a decimal of a quart is equal to $\frac{1.5}{2}$=.75 qt. Adding this decimal to 3¾ qt., the sum is 4.5 qt. These quarts are reduced to a decimal of a gallon by dividing by the number of quarts in 1 gal., or 4; 4.5÷4

=1.125. Therefore, 4.5 qt. is equal to 1.125 gal. On adding this number to the given number of gallons, the sum is 22.125 gal. It follows that 21 gal. 3¾ qt. 1½ pt. is equal to 22.125 gal. Ans.

EXAMPLES FOR PRACTICE

Reduce each of the following denominate numbers to a decimal part of the next higher denomination.

1. 7½ inches Ans. .625 ft.
2. 13 ounces (av.) Ans. .8125 lb.
3. 46 minutes (time) Ans. .767 hr.
4. 2¾ quarts (liquid) Ans. .6875 gal.
5. 120 square inches Ans. .833 sq. ft.

REDUCTION OF A DECIMAL DENOMINATE NUMBER TO A NUMBER OF A LOWER DENOMINATION

56. Rule for Reduction.—Sometimes it is required to reduce the decimal part of a denominate number to one or more numbers of a lower denomination. For instance, the decimal of a foot is to be reduced to inches, decimal of an hour to minutes and seconds, and so forth.

Since the reduction of a number to a decimal of a higher denomination is made by *division,* the reverse process is effected by multiplication. For example, to reduce .75 foot to inches, .75 must be multiplied by 12, since there are 12 inches in 1 foot; thus, $.75\times 12=9$. Therefore, .75 foot is equivalent to 9 inches. The following rule applies to this kind of reduction:

Rule.—*To reduce a decimal denominate number to a number of a lower denomination, multiply the decimal by the number of units required to make one unit of the higher denomination. The product will be the equivalent value in terms of the lower denomination.*

57. Application of Rule.—The following examples show the application of the preceding rule:

Example 1.—Reduce 36.464 hours to hours, minutes, and seconds.

Solution.—According to the rule, the decimal is to be multiplied by 60, as there are 60 min. in 1 hr.; $.464\times 60=27.84$; therefore, .464 hr.

=27.84 min. The decimal .84 must now be multiplied by 60, since there are 60 sec. in 1 min. Thus, $.84\times60=50.4$; that is, .84 min.=50.4 sec. It follows that 36.464 hr. is equal to 36 hr. 27 min. 50.4 sec. Ans.

Example 2.—Reduce 6.75 feet to feet and inches.

Solution.—The decimal .75 is multiplied by 12, since there are 12 in. in 1 ft., or $.75\times12=9$; that is, .75 ft.=9 in. and 6.75 ft.=6 ft. 9 in. Ans.

Example 3.—Reduce 5.453 feet to feet, inches, and sixteenths of an inch.

Solution.—The decimal .453 is multiplied by 12; thus, $.453\times12=5.436$, showing that .453 ft.=5.436 in. The decimal .436 is reduced to sixteenths by multiplying by $\frac{16}{16}$, since there are 16 sixteenths in 1 inch. Thus, $.436\times\frac{16}{16}=\frac{6.976}{16}$; therefore, .436 in.=6.976 sixteenths of an inch. As it is desirable that the latter number shall be a whole number, the decimal part is omitted, and as the decimal is greater than .5, the whole number is increased by 1; thus $\frac{6.976}{16}=\frac{7}{16}=.436$ in., approximately. It follows that 5.453 ft.=5 ft. $5\frac{7}{16}$ in. Ans.

Example 4.—Reduce 4.29 inches to inches and thirty-seconds of an inch.

Solution.—The decimal .29 is reduced to thirty-seconds by multiplying by $\frac{32}{32}$, there being 32 thirty-seconds in 1 inch. Then, $.29\times\frac{32}{32}=\frac{9.28}{32}=\frac{9}{32}$ in., approximately; the decimal .28 being less than .5, it is simply dropped. Therefore, 4.29 in.=$4\frac{9}{32}$ in. Ans.

EXAMPLES FOR PRACTICE

Reduce the following decimal values to units of lower denominations:

1. .625 feet to inches Ans. $7\frac{1}{2}$ in.
2. .8125 pound (av.) to ounces Ans. 13 oz.
3. .767 hour to minutes Ans. 46 min.
4. .6875 gallon to quarts Ans. $2\frac{3}{4}$ qt.
5. .833 square foot to square inches Ans. 120 sq. in.

ADDITION OF COMPOUND NUMBERS

58. Rule for Adding Compound Numbers.—Compound numbers formed of denominations belonging to the same kind may be added by applying the following rule:

Rule.—*Place the numbers so that like denominations are in the same column. Begin at the right-hand column, and add. Divide the sum by the number of units of this denomination required to make one unit of the higher denomination. Place the remainder under the column added, and carry the quotient to the next column.*

ft.	in.	
1	3½	
2	7¾	
1	11⅛	
3	8⅞	
9	7¼	Ans.

Example 1.—Four bars measure 1 foot 3½ inches, 2 feet 7¾ inches, 1 foot 11⅛ inches, and 3 feet 8⅞ inches, respectively; what is the combined length?

Solution.—The numbers are arranged as shown. On adding the numbers in the first column the sum is found to be 31¼ in., or 2 ft. 7¼ in. The 7¼ in. is put down and the 2 ft. is added to the other column, giving 9 ft. The combined length of the bars is 9 ft. 7¼ in. Ans.

Example 2.—A workman spends 1 hour 40 minutes on one piece of work, 2 hours 35 minutes on another, and 3 hours 10 minutes on a third. What is his time on all three?

Solution.—The sum of the numbers in the right-hand column is 85 min., which is equal to 1 hr. 25 min. The 25 min. is set down and the 1 hr. is carried over and added to the number in the other column. Therefore, the total time is 7 hr. 25 min. Ans.

hr.	min.	
1	40	
2	35	
3	10	
7	25	Ans.

EXAMPLES FOR PRACTICE

1. Find the sum of 7 feet 6 inches, 3 feet 7½ inches, 1 foot 9¾ inches, and 2 feet 2 inches. Ans. 15 ft. 1¼ in.

2. Find the total weight of three lots of scrap iron that weigh 2 tons 765 pounds, 4 tons 1,240 pounds, and 1 ton 800 pounds. Ans. 8 T. 805 lb.

3. A carpenter works 2 hours 45 minutes on one job, 1 hour 5 minutes on another, and 3 hours 25 minutes on another. What is the total time worked? Ans. 7 hr. 15 min.

4. Along the end of a factory, as shown by the diagram, Fig. 9, four machines *a, b, c,* and *d* are set in line. The distances between the machines and between the end machines and the wall, and the lengths of the machines are as marked. What is the distance from one side wall to the other? Ans. 37 ft.

5. Of four oil tanks situated in a power plant, the first tank contains 167 gallons 3 quarts; the second contains 186 gallons 1 quart; the third contains 108 gallons 2 quarts, and the fourth contains 123 gallons 3 quarts. What is the total quantity of oil in the tanks?

Ans. 586 gal. 1 qt.

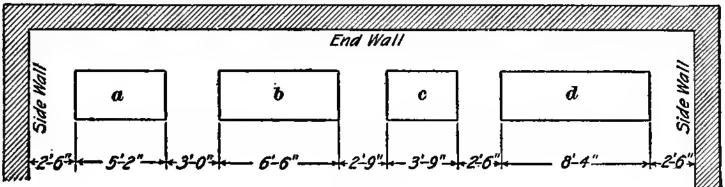


FIG. 9

SUBTRACTION OF COMPOUND NUMBERS

59. Rule for Subtracting Compound Numbers. Compound numbers may be subtracted by applying the following rule:

Rule.—*Place the smaller quantity under the larger quantity, with like denominations in the same column. Beginning at the right, subtract the number in the subtrahend in each denomination from the one above, and place the differences underneath. If the number in the minuend of any denomination is less than the number under it in the subtrahend, one must be taken from the minuend of the next higher denomination, reduced, and added to it.*

EXAMPLE 1.—A pile of coal containing 3 tons 728 pounds has 1 ton 566 pounds taken away. How much remains?

SOLUTION.—Setting down the numbers according to the rule, and subtracting, the remainder is found, as shown.

That is, 2 T. 162 lb. remain. Ans.

	T.	lb.
minuend	3	728
subtrahend	1	566
remainder	2	162

EXAMPLE 2.—A length of pipe 4 feet 6 inches long has a piece 1 foot 8 inches long cut off. What length remains?

ft.	in.
4	6
1	8

SOLUTION.—Arranging the numbers as shown, the remainder is found in the following manner: As it is impossible to subtract 8 from 6, 1 ft. is borrowed from the 4 ft. and added to the 6 in. after converting the foot to inches. Thus, 12+6=18 in. The 1 ft. taken away from the 4 ft. leaves 3 ft.

The numbers will then appear as shown in the second arrangement.

ft.	in.
3	18
1	8
2	10

Arranged in this form, 8 may be taken from 18, leaving 10, and 1 from 3, leaving 2. The remainder is therefore 2 ft. 10 in. Ans.

NOTE.—The operation of taking 1 ft. away from the 4 ft., reducing it to inches, and adding it to the 6 in., would generally be done mentally, and would not be written down as shown.

EXAMPLES FOR PRACTICE

1. From a piece of timber 10 feet 8 inches long two pieces are sawed off, one measuring 2 feet 6 inches and the other 3 feet 8 inches. Neglecting the width of the saw cuts, what length of timber remains?
Ans. 4 ft. 6 in.

2. A glass tube 3 feet 5 inches long, as shown in Fig. 10, is marked to be cut off into two parts, the shorter of which is 1 foot 8 inches. What is the length of the longer part? Ans. 1 ft. 9 in.

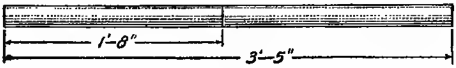


FIG. 10

3. If 2 gallons 3 quarts 1 pint of oil is drawn from a tank containing 8 gallons 1 quart 1½ pint, how much oil remains in the tank?
Ans. 5 gal. 2 qt. ½ pt.

A certain lever may move through an angle of 45° 26′ 43″. If it has moved through an angle of 30° 14′ 55″, through what angle must it yet move to reach its final position? Ans. 15° 11′ 48″

MULTIPLICATION OF COMPOUND NUMBERS

60. Rule for Multiplying Compound Numbers. Compound numbers may be multiplied by other numbers by applying the following rule:

Rule.—*Multiply the number representing the lower denomination by the given multiplier and reduce the product to the higher denomination. Write the remainder under the lower denomination, and add the quotient to the product obtained by multiplying the higher denomination by the multiplier.*

EXAMPLE 1.—What length of stock is needed to make 16 bolts, if each bolt requires a length of 1 foot 3⅛ inches?

Solution 1.—The stock required must be 16 times the length required for one bolt.

Arranging the numbers as shown, $3\frac{1}{8}$ is multiplied by 16; thus, $3\frac{1}{8}\times16=\frac{25}{8}\times16=\frac{400}{8}=50$. But, 50 in. is equal to $\frac{50}{12}=4$ ft. 2 in. The remainder 2 is placed under 16 and the 4 ft. is carried over and added to the next product. Multiplying 1 ft. by 16, the product is 16 ft., to which is added 4 ft., or 16+4=20 ft. The length of stock must therefore be 20 ft. 2 in. Ans.

ft.	in.	
1	$3\frac{1}{8}$	
	16	
20 ft.	2 in.	Ans.

ft.	in.	
1	$3\frac{1}{8}$	
	16	
16 ft.	50 in.	
=20 ft.	2 in.	Ans.

Solution 2.—The problem may also be solved in the following manner: Multiplying 1 ft. $3\frac{1}{8}$ in. by 16 gives 16 ft. and $16\times3\frac{1}{8}$ in.; but $16\times3\frac{1}{8}$ in.$=16\times\frac{25}{8}=\frac{400}{8}=50$ in. But 50 in. =4 ft. 2 in., and the whole product is therefore 16 ft.+4 ft. 2 in.=20 ft. 2 in. Ans.

Example 2.—Multiply 6 lb. 14 oz. (av.) by 12.

Solution.—The numbers 6 pounds and 14 ounces are each multiplied by 12, giving the product, 72 lb. 168 oz. To reduce 168 oz. to pounds and ounces, it is divided by 16, as there are 16 oz. in 1 lb. Thus, 168 oz.÷16=10 lb. 8 oz., and 72 lb. +10 lb.=82 lb., making the whole weight equal to 82 lb. 8 oz.

lb.	oz.	
6	14	
	12	
	28	
	14	
72 lb.	168 oz.	
=82 lb.	8 oz.	Ans.

Example 3.—The circumference of a flywheel, which also serves as a driving pulley, is 47 feet 3 inches. If the belt is in contact with the rim of the flywheel over $\frac{3}{5}$ of its circumference, what is the length of that portion of the belt that makes contact with the pulley?

ft.	in.
47	3
	$\frac{3}{5}$
$28\frac{1}{5}$ ft.	$1\frac{4}{5}$ in.
=28 ft.	$4\frac{1}{5}$ in.

Solution.—Multiplying 47 ft. 3 in. by $\frac{3}{5}$ gives $28\frac{1}{5}$ ft. $1\frac{4}{5}$ in. The fraction $\frac{1}{5}$ ft. must be reduced to inches and added to $1\frac{4}{5}$ in. Thus $12\times\frac{1}{5}=2\frac{2}{5}$ in.; $2\frac{2}{5}+1\frac{4}{5}=3\frac{6}{5}=4\frac{1}{5}$ in. The required length is, then, 28 ft. $4\frac{1}{5}$ in. Ans.

EXAMPLES FOR PRACTICE

1\. Twelve pieces each 1 foot $3\frac{1}{2}$ inches long are sawed from a bar of iron. What is the total length sawed off, neglecting the width of the saw cut? Ans. 15 ft. 6 in.

2\. If 18 boxes each 2 feet $4\frac{1}{2}$ inches wide are set side by side, what will be the total width? Ans. 42 ft. 9 in.

3. Five loads of pig iron, each weighing 4 tons 650 pounds, are purchased. What is the total amount of iron purchased, if the weight is in long tons? Ans. 21 T. 1,010 lb.

4. Each cell of an electric battery requires 2 pounds 12 ounces (av.) of copper sulphate for a charge. How much will 23 cells require? Ans. 63 lb. 4 oz.

5. Multiply 42 degrees 31 minutes 24 seconds by $\frac{5}{7}$. Ans. $30° 22' 25\frac{5}{7}''$

DIVISION OF COMPOUND NUMBERS

61. Rule for Dividing Compound Numbers.—The rule to be used in dividing a compound number by another number is as follows:

Rule.—*Find how many times the divisor is contained in the first or higher denomination of the dividend; reduce the remainder, if any, to the lower denomination, and add to it the number in the given dividend expressing that denomination; divide this new dividend by the divisor, and the quotient will be the next denomination in the quotient required. Or, reduce the dividend to units of the lower denomination and divide by the given divisor.*

EXAMPLE 1.—A piece of bar iron 20 feet 2 inches long is used in making 16 bolts, all of the same length. What is the length used for each bolt?

ft. in.
16) 20 2 (1 ft. $3\frac{1}{8}$ in. Ans.
16
4 ft. rem.
12
48 in.
2 in.
16) 50 in. ($3\frac{1}{8}$ in.
48
$\frac{2}{16}=\frac{1}{8}$

SOLUTION 1.—The numbers are arranged and the division is performed as in the accompanying solution. The total length of the bar, or 20 ft. 2 in., is divided by 16. First, 16 is contained once in 20 ft., with 4 ft. as a remainder. This 4 ft. is now reduced to inches by multiplying it by 12, which gives 48 in. Adding the 2 in. in the original number, gives a sum of 50 in. Then, $50 \div 16 = 3\frac{2}{16} = 3\frac{1}{8}$ in. Thus, the amount used for each bolt is 1 ft. $3\frac{1}{8}$ in. Ans.

SOLUTION 2.—The example may also be solved by reducing the compound number to its lowest denomination and then dividing by the given divisor. Thus, 20 ft. 2 in. is reduced as follows: $20 \times 12 = 240$; $240 + 2 = 242$ in. Then, $242 \div 16 = 15\frac{1}{8}$ in., or 1 ft. $3\frac{1}{8}$ in. Ans.

Example 2.—Divide 28 feet 5 inches by 8.

Solution.— 28 ft. divided by 8 gives a quotient of 3 ft. and 4 ft. as a remainder. The 4 ft. is now reduced to inches by multiplying by 12, which gives 48 in. $48+5=53$ in. and $53\div8=6\frac{5}{8}$ in., making the answer 3 ft. $6\frac{5}{8}$ in.

```
     ft.  in.
8 ) 28    5   ( 3 ft. 6⅝ in.  Ans.
    24
     4 ft. rem.
    12
    48 in.
     5 in.
8 ) 53 in. ( 6⅝ in.
    48
     5/8
```

Example 3.—Divide 85 tons 1,843 pounds by 36.

Solution.— 85 T. divided by $36=2$ T. with a remainder of 13 T.=26,000 lb.; but, 26,000 lb.+1,843 lb.=27,843 lb., and 27,843 lb.÷36=773.4 lb., making the complete answer 2 T. 773.4 lb.

```
36 ) 85 T.     1843  lb. ( 2 T.  773.4 lb. approx.  Ans.
     72
     13 T.
      2000
     26000 lb.
      1843 lb.
36 ) 27843 lb. ( 773.4 lb.
     252
      264
      252
       123
       108
        150
        144
          6
```

EXAMPLES FOR PRACTICE

1. If a bar 5 feet 3 inches long is divided into four equal parts, what is the length of each? Ans. 1 ft. $3\frac{3}{4}$ in.

2. If a man requires 23 hours 50 minutes to make 11 articles of merchandise, how long will it take to make one? Ans. 2 hr. 10 min.

3. How many armature coils weighing 1 lb. 3 oz. each can be made from a reel of copper wire weighing 200 pounds, net? Ans. 168 coils and 8 oz. of wire remaining

4. How long would it take a cannon ball traveling at the rate of 1,950 feet per second to pass over a distance of 10 miles? Ans. 27 sec., nearly

5. How long will 40 gallons of lubricating oil last, if 7 pints are used each day? Ans. $45\frac{5}{7}$ da.

RATIO AND PROPORTION

RATIO

SIMPLE RATIO

EXPRESSING AND FINDING RATIOS

1. Comparison of Numbers.—In practice there are many cases in which it is more useful to know how many times one number is larger than another number, than to know the actual values of these numbers. For example, in calculating the relative speeds of shafts driven by gear wheels, it is more convenient to state how many times more teeth one wheel has than another, than to consider the actual number of teeth on each wheel. These comparative values of the number of teeth are generally stated in the form of a fraction, as $\frac{3}{1}$, the fraction meaning that the number of teeth on the driving wheel is three times as great as that on the driven wheel. The actual number of teeth on the two wheels is left out of consideration for the time being; they may be 90 and 30, 36 and 12, or any other combination in which one wheel has three times as many teeth as the other.

2. Another example showing the advantage of giving the relative values of numbers may be found in government health reports. One of those may state, for instance, that of each 1,000 inhabitants in a certain city 25 persons suffered from influenza, 1.24 from typhoid, etc. Reports given in this form are much more instructive than those which simply state that in the same city 1,250 persons suffered from influenza, 62 from typhoid, etc. By means of the first report it is possible to institute comparisons with other cities, but with reports of the latter kind comparisons are difficult, unless the total number of inhabitants is known in each case.

Similarly, when speed of operation or production is considered, it is much easier to make comparison between various machines, when their relative speeds are known. For example, on comparing the speeds of two railway trains, it is found that a train *A* makes 90 miles in 2 hours 15 minutes and that another train *B* makes 108 miles in 2 hours 24 minutes. From those statements it is difficult quickly to form an idea of the relative speeds of these trains. But, if it is said that the train *A* made 40 miles an hour and the train *B* 45 miles an hour, the comparison is easier, and is still more facilitated if the relative speeds are stated in the form of a fraction, as for instance, the speed of *A* is $\frac{8}{9}$ of the speed of *B*. This expression means that while the train *A* makes 8 miles, the train *B* makes 9 miles.

It is the purpose of this Section to show by what means the relative values of two numbers may be found, and how the existing relation between one pair of numbers may be used for finding another pair, similarly related. This Section deals with one of the most interesting and useful subjects to be found in arithmetic.

3. Finding the Ratio of Two Numbers.—Two numbers may be compared in one of two ways. Suppose, for instance, that the diameters of two belt pulleys are to be compared, one being 30 inches and the other 6 inches. If it is necessary to know how many times 30 is larger than 6, then 30 is divided by 6 giving 5 as the quotient; thus, $30 \div 6 = 5$. Hence, it may be said that 30 inches is 5 times as large as 6 inches, or that the 30-inch diameter contains 5 times as many inches as the 6-inch diameter. Or, the numbers 30 and 6 may be compared by ascertaining what part 6 inches is of 30 inches.

Then, 6 is divided by 30, giving the quotient $\frac{1}{5}$ or .2. Hence, 6 inches is $\frac{1}{5}$, or .2, of 30 inches.

As a practical example of this kind of comparison may be mentioned the case of two pulleys that are to be connected by a belt. It is here necessary to know the relation between the diameters of the two pulleys in order that their relative speeds may be calculated.

4. The two numbers compared must be units of the same denomination. For instance, if one number is given in inches, the other number must also be in inches; thus, 30 inches may be compared with 6 inches, as in the preceding example; but, 30 inches cannot be compared with 6 pounds. From the preceding remarks it will be seen that the term **ratio** means a comparison of two numbers of the same denomination or kind. The operation of comparing two numbers is called *finding the ratio* of the numbers.

5. Expressing a Ratio.—A ratio may be written in two different ways, both of which are correct. Thus, the ratio of 20 to 4, or the value of 20 compared to the value of 4, may be written 20 : 4 or $\frac{20}{4}$. Each of these expressions is read *the ratio of 20 to 4*. The ratio of 4 to 20 would be written either 4 : 20 or $\frac{4}{20}$. The method most commonly used in writing ratios is the first one shown; that is, the two numbers are separated by the sign (:), which is really an abbreviation of ÷, the sign of division, and has the same meaning, division being practically a method of indicating ratio. Hence, 20 : 4 is equal to $20 \div 4 = 5$. In calculations, ratios are frequently written in the form of fractions; thus, the ratio of 20 to 4 may be written $\frac{20}{4}$.

6. Terms of Ratio.—The two numbers to be compared are known as the **terms** of the ratio; thus, in the ratio 30 : 6, 30 and 6 are the two terms. When both terms are considered together they are called a **couplet;** when considered separately, the first term is called the **antecedent,** and the second term the **consequent.** Thus, in the ratio 30 : 6, 30 and 6 form a couplet, in which 30 is the antecedent and 6 the consequent.

7. Simple Ratio.—When a ratio has only one antecedent and one consequent, it is known as a **simple ratio.** Thus, the ratio 30 : 6 is a simple ratio.

8. Direct and Inverse Ratio.—When it is desired to compare two denominate numbers of the same kind, the comparison is usually made by finding the ratio of the first number to the second. This is known as a **direct ratio.** For instance, the direct ratio of 22 feet to 9 feet is 22 : 9. If the given terms are interchanged the ratio becomes an **inverse ratio.** Thus, the inverse ratio of 22 feet to 9 feet is 9 : 22. The direct ratio of 5 pounds to 11 pounds is 5 : 11, and the inverse ratio is 11 : 5.

Every ratio is understood to be a *direct* ratio unless otherwise stated.

9. Value of a Ratio.—Distinction must be made between a *ratio* and its *value.* A ratio is represented by its two terms. The **value** of a ratio is the quotient obtained by dividing the first term by the second. Thus, the ratio of 20 to 4 is 20 : 4; the value of this ratio is $20 \div 4 = 5$.

10. Ratio Considered as a Fraction.—By expressing the ratio in the fractional form, for example, the ratio of 18 to 3 as $\frac{18}{3}$, it follows from the laws of fractions that both terms may be multiplied or both divided by the same number, without altering the value of the ratio. Thus,

$$\frac{18}{3} = \frac{18 \times 4}{3 \times 4} = \frac{72}{12}; \quad \frac{18}{3} = \frac{18 \div 3}{3 \div 3} = \frac{6}{1}$$

In each case the value of the ratio is 6.

In the ratio 6 : 10, in which the antecedent is the smaller one of the two terms, the ratio, when expressed in the fractional form, becomes $\frac{6}{10} = \frac{6 \div 2}{10 \div 2} = \frac{3}{5}$. Hence, the value of the ratio

6 : 10 is $\frac{3}{5}$.

When a ratio is given in the fractional form a direct ratio may be changed to an inverse ratio by simply inverting the fraction. For instance, the direct ratio of 21 to 7 is $\frac{21}{7}$, which has a value of 3. The inverse ratio of 21 to 7 is $\frac{7}{21}$; in this case, the value of the ratio is $\frac{1}{3}$, which is the reciprocal of 3. An inverse ratio is, therefore, also called a **reciprocal ratio,** as its value is the reciprocal of the value of the direct ratio. The *reciprocal* of a number is 1 divided by that number; the reciprocal of a fraction is the fraction inverted.

11. Reducing a Ratio to Its Lowest Terms.—It is preferable that a ratio be reduced to its lowest terms and, if possible, that one of its terms be made unity, or 1. Thus, instead of saying that the ratio of the resistances of two electrical conductors is $\frac{5}{50}$, the ratio may be reduced to $\frac{1}{10}$; such an expression is often used instead of giving the actual values of the respective resistances.

A ratio is reduced to its lowest terms by dividing both terms by the same number until no number except 1 can be found that will divide both terms without a remainder. Thus, the ratio $\frac{84}{48}$, when reduced to its lowest terms, becomes $\frac{7}{4}$, which is obtained by dividing each number by 12. The operation may be written $\frac{84 \div 12}{48 \div 12} = \frac{7}{4}$.

12. Rules for Finding Direct and Indirect Ratio of Two Numbers.—The following rules may be used in cases where the direct or the indirect ratio of two numbers is to be found:

Rule I.—*The direct ratio of two numbers is found by making the first one of the given numbers the antecedent and the second one the consequent of the ratio.*

Rule II.—*The inverse ratio of two numbers is found by making the second number of those given the antecedent and the first one the consequent of the ratio.*

EXAMPLE 1.—(*a*) What is the direct ratio of 9 to 3? (*b*) What is the value of the ratio?

SOLUTION.—(*a*) Applying rule I, the first number, 9, is made the antecedent and the second number, 3, the consequent of the ratio; thus, the direct ratio of 9 to 3 is 9 : 3, or $\frac{9}{3}=\frac{3}{1}$. Ans.

(*b*) From Art. **9** the value of the ratio is $3\div1=3$. Ans.

EXAMPLE 2.—(*a*) What is the inverse ratio of 10 to 5? (*b*) What is the value of the ratio?

SOLUTION.—(*a*) Applying rule II, the second number, 5, is placed as the antecedent of the ratio, and 10 as the consequent; thus, the inverse ratio of 10 to 5 is 5 : 10, or $\frac{5}{10}=\frac{1}{2}$. Ans.

(*b*) From Art. **9** the value of the ratio is $1\div2=.5$, or $\frac{1}{2}$. Ans.

EXAMPLE 3.—(*a*) Reduce the ratio 19 : 76 to its lowest terms. (*b*) What is the value of the ratio?

SOLUTION.—(*a*) According to Art. **11** a ratio is reduced to its lowest terms by dividing both terms by the same number, continuing the process until no number except 1 can be found that will be contained in each term without a remainder. In this case the terms of the ratio may be divided by 19; thus $\frac{19\div19}{76\div19}=\frac{1}{4}$. Ans.

(*b*) The value of the ratio is $\frac{1}{4}$, or .25. Ans.

EXAMPLE 4.—What is the inverse ratio of 8 and 72, and what is the value of the ratio?

SOLUTION.—Applying rule II, the inverse ratio of 8 to 72 is 72 : 8, which reduced to its lowest terms is 9 : 1. The value of the ratio is $9\div1=9$. Ans.

EXAMPLE 5.—What is the direct ratio of 72 to 8, and what is the value of the ratio?

SOLUTION.—Applying rule I, the direct ratio of 72 to 8 is 72 : 8 = 9 : 1. The value of the ratio is $9\div1=9$. Ans.

NOTE.—Examples 4 and 5 show that the direct and the inverse ratio of two numbers may be equal, depending on their relative positions in the statement contained in the example; for instance, whether 8 precedes or follows 72.

EXAMPLE 6.—A pair of gear-wheels contain 60 and 35 teeth, respectively. (*a*) What is the ratio of the number of teeth in the larger to the number of teeth in the smaller? (*b*) What is the ratio of the smaller number to the larger?

SOLUTION.—(*a*) The ratio of the larger number of teeth to the smaller is the ratio of 60 to 35, or 60 : 35, and reducing this to its lowest terms by dividing both terms by 5, the ratio becomes 12 : 7. Ans.

(*b*) The ratio of the smaller number to the larger is 35 : 60, which, reduced to its lowest terms by dividing both terms by 5, is equal to 7 : 12. Ans.

EXAMPLES FOR PRACTICE

1. What is the ratio of 126 to 18, reduced to its lowest terms? Ans. 7 : 1

2. Two gears have 39 and 54 teeth, respectively. What is the value of the ratio of the larger number to the smaller? Ans. $1\frac{5}{13}$

3. What is the value of the ratio of 6.25 to .75? Ans. $8\frac{1}{3}$

4. One pulley is 24 inches in diameter and another is 60 inches in diameter. What is the inverse ratio of the diameter of the smaller pulley to that of the larger? Ans. 5 : 2

5. A man traveled 250 miles, partly by rail and partly by boat. If he traveled 150 miles by rail, state (*a*) the distance traveled by boat; (*b*) the ratio of the distance traveled by rail to that traveled by boat.

$$\text{Ans.}\begin{cases}(a)\ 100\text{ mi.}\\(b)\ 3 : 2\end{cases}$$

6. During one year A and B invest 5 dollars and 4 dollars, respectively, every month. (*a*) What is the ratio of A's investment to that of B? (*b*) What is the ratio of B's investment to that of A? (*c*) What are the values of the two ratios?

$$\text{Ans.}\begin{cases}(a)\ 5 : 4\\(b)\ 4 : 5\\(c)\ 1\frac{1}{4}\text{ and }\frac{4}{5}\end{cases}$$

PROPORTION

SIMPLE PROPORTION

DIRECT AND INVERSE PROPORTIONS

13. Elements of a Proportion.—A **simple proportion** consists of two simple ratios of the same value connected by an equality sign (=) or a double colon (: :). For example, the ratios 8 : 6 and 12 : 9 are of the same value, and a proportion may be formed by them, as 8 : 6 = 12 : 9, or 8 : 6 : : 12 : 9. The equality sign is used more frequently than the double colon, so the proportions in this and other Sections will be written with the equality sign. The proportion 8 : 6 = 12 : 9 is read *8 is to 6 as 12 is to 9*, or *the ratio of 8 to 6 is equal to the ratio of 12 to 9.* This same proportion can also be written $\frac{8}{6}=\frac{12}{9}$, each of the two ratios being given as a fraction.

The term **couplet** used in connection with a ratio is sometimes applied to the elements of a proportion. Each ratio, or couplet, has two terms and the terms of a proportion are called *first, second, third,* and *fourth,* numbering from left to right. The first and fourth terms are the **extremes**; the second and third terms are the **means**.

The following table gives the proportion 25 : 10 = 40 : 16 and the terms applied to its various elements, or parts.

Ratios, or couplets	First				Second		
Terms..............	First		Second		Third		Fourth
Quantities..........	25	:	10	=	40	:	16
	Extreme		*Mean*		*Mean*		*Extreme*

It is seen that 25 : 10 is the first couplet or ratio, and that 40 : 16 is the second couplet or ratio. Numbering from left to right, 25 is the first and 16 the fourth term. It is also seen that the numbers 25 and 16 form the two extremes and the numbers 10 and 40 the two means of the proportion.

14. Use of Proportions.—In practice there are numerous cases in which one ratio is given and it is required to find an equal ratio of which one term is already known; this is the purpose of a proportion, as may be made clearer by an example. Let it be supposed that the top of a table is 6 feet long and 3 feet wide and that it is required to make another table the top of which is 8 feet long, the ratio of length to width being the same as in the smaller table. Here the first ratio is $6 : 3$ and the second ratio $8 : x$, the missing term being indicated by the symbol x. The value of the first ratio is $6 \div 3 = 2$ and if the value of the second ratio is to be 2, also, it is evident that the missing term, x, must be $8 \div 2 = 4$, and the new table top must, therefore, be 4 feet wide. Hence, the complete proportion is $6 : 3 = 8 : 4$.

15. A problem frequently met with in practice may be given as another example. The ratio of the weights of the ingredients in a certain mixture is $7 : 3$. Of the first ingredient there is on hand 91 pounds, and it is required to know what weight of the second ingredient will be required in order that the ratio of the two weights may be as $7 : 3$. The first ratio is $7 : 3$ and the second is $91 : x$. In this case the relation between the two ratios is so simple that the problem may be solved by inspection in the following manner: It is seen that 7 is contained 13 times in 91; according to Art. **10** both terms of a ratio may be multiplied by the same number without altering the value of the ratio. Hence, on multiplying the terms of the ratio $7 : 3$ by 13, the result is $7 \times 13 : 3 \times 13 = 91 : 39$. The numbers 91 and 39 are, therefore, the terms of the second ratio, showing that x, or the weight of the second ingredient, is equal to 39 pounds.

These examples will suffice to show some of the simpler cases in which the principle of proportion is applied in practice. Further explanations will show that the subject of proportion forms one of the most useful sections of arithmetic.

16. Direct and Inverse Proportions.—A **direct proportion** is one in which both couplets are direct ratios. A proportion is always understood to be direct unless the statement of a problem clearly indicates otherwise. An **inverse proportion** is one that requires one of the couplets to be expressed as an inverse ratio. Thus, if 8 is to 4 inversely as 3 is to x, one of the ratios must be reversed (it does not matter which one) and the proportion may be written in either of the following ways: $8 : 4 = x : 3$, or $4 : 8 = 3 : x$. In the first proportion the second couplet is reversed; in the second proportion the first couplet is reversed.

17. Directly Proportional and Inversely Proportional Quantities.—In technical literature, one quantity is said to be *proportional* to another, or to *vary with* it, or to *increase with* it, or to *decrease with* it; any one of these expressions means that a change in one of the quantities causes, or is accompanied by, a corresponding change in the other. If the word *inversely* is used with one of these expressions, the meaning is that a change in one quantity causes an opposite change in the other. Sometimes the word *directly* is used when it is desirable to make sure that the meaning will not be understood to be inverse. For example, the resistance of an electric wire is directly proportional to its length, or varies with, or increases with, its length; that is, the longer the wire, the greater is its resistance. But, if a number of men are engaged in building a fence, the time required to finish the fence is *inversely* proportional to the number of men working on it; that is, the more men, the shorter the time.

18. Rules for Finding Unknown Terms of Proportions.—In any proportion, *the product of the extremes is equal to the product of the means.* For example, in the proportion $17 : 51 = 14 : 42$, the extremes are 17 and 42 and the means are 51 and 14. According to the preceding statement the product 17×42 must be equal to 51×14, which is true because $17 \times 42 = 714$ and $51 \times 14 = 714$. This important principle makes it possible to find an unknown term of a proportion when the three other terms are known. The following rules are based on this principle:

Rule I.—*To find an unknown extreme, divide the product of the means by the given extreme.*

Rule II.—*To find an unknown mean, divide the product of the extremes by the given mean.*

19. Direct Proportion.—In forming the two ratios of a proportion it is important to see that both terms of each ratio are of the *same* kind. For instance, if one term is given in pounds, the other must also be in pounds. But the four terms of a proportion need not be of the same kind; in one ratio both terms may be in pounds, and in the other both in feet, and so forth.

Another point to be noted in forming a direct proportion is that the terms are arranged so that the first term of each ratio refers to *one* of the things compared and that the second term in each ratio refers to the *other* thing compared. For instance, an example states that the weights of two parcels of sugar are 2 and 5 pounds, respectively, and that the cost of the smaller parcel is 12 cents. It is required to find the cost of the larger parcel. Here, one ratio is $2 : 5$, and the other must be $12 : x$, as the term 2 in the first ratio refers to the parcel that costs 12 cents; hence, 12 must be the first term of the second ratio, and the proportion is written

$$2 : 5 = 12 : x$$

20. Arrangement and Solution of Direct Proportions.—In the following method for solving proportions, it is found convenient always to let the unknown term, x, occupy the position of the *fourth* term in the proportion, and for the *third* term to write the number that is of the same kind as the required fourth term.

The conditions given in the example are now examined to ascertain whether the fourth term, x, will be larger or smaller than the third one. If *larger*, the larger number of those that are to form the first ratio is written as its *second* term, and the remaining number is written as the *first* term.

The preceding method of procedure is embodied in the following rule:

Rule.—*Make the unknown term, x, the fourth term of the proportion, and for the third term write the number that is of a* *similar kind. If the fourth term will be larger than the third, the second term must be larger than the first; or if the fourth term will be smaller than the third, the second term must be smaller than the first.*

Example 1.—If 7 pounds of putty costs 56 cents, what will be the cost of 36 pounds?

Solution.—In this example the cost of 36 pounds of putty is the required term, and is, therefore, written as the fourth term, x. The other number of the same kind is 56 cents. So, the second ratio of the proportion must be written $56 : x$.

As 36 pounds of putty must cost more than 7 pounds, it follows that the term x will be larger than the third term, 56. Hence, according to the rule, the second term must be larger than the first. The proportion is, therefore, written

$$7 : 36 = 56 : x$$

In this case one of the extremes is not known. Hence, by rule I, Art. **18**, the unknown extreme, x, is equal to the product of the means divided by the known extreme, or

$$x = \frac{36 \times 56}{7} = 288 \text{ cents, or \$2.88.} \quad \text{Ans.}$$

Example 2.—The weights of two patterns, made of pine, are in the ratio of 2 to 3, and the iron casting made from the smaller pattern weighs 42 pounds. What is the weight of the iron casting made from the larger pattern?

Solution.—The weight of the casting made from the larger pattern is unknown and is therefore indicated by x. The other number of the same kind is 42 pounds. So, the second couplet of the proportion must be written $42 : x$.

The example states that the weights of the patterns are as $2 : 3$, and the weight of the casting made from the lighter pattern is 42 pounds; it follows that the casting made from the other pattern must be heavier. According to the rule, the second term must be larger than the first and the proportion is written

$$2 : 3 = 42 : x$$

By rule I, Art. **18**, the unknown extreme, x, is equal to the product of the means divided by the known extreme, or

$$x = \frac{3 \times 42}{2} = \frac{126}{2} = 63$$

That is, the casting made from the larger pattern weighs 63 lb. Ans.

Example 3.—The ratio of the numbers of teeth in two gear-wheels is as 2 to 5. If the number of teeth in the second or larger wheel is 90, how many teeth are there in the smaller wheel?

SOLUTION.—The unknown term, x, is here the number of teeth on the smaller wheel and is put down as the fourth term in the proportion. The other number of the same kind is 90 teeth, which is made the third term in the proportion. So the second ratio is written

$$90 : x$$

It is seen from the example that the term 2 refers to the unknown number of teeth. As 2 is smaller than 5, it follows that x must be smaller than 90. Hence, the second term must be smaller than the first and the proportion is written

$$5 : 2 = 90 : x$$

By rule I, Art. **18**,

$$x = \frac{2 \times 90}{5} = \frac{180}{5} = 36 \text{ teeth. Ans.}$$

NOTE.—In all problems involving fractions cancelation should be resorted to wherever possible so as to simplify operations. If in these and other examples cancelation is omitted it is for the purpose of concentrating attention on the one process under consideration.

21. Arrangement and Solution of Inverse Proportions.—Many problems in proportion are so stated that the first and third terms in the complete proportion refer to the same thing. This is the case with the preceding examples. For instance, in the proportion given as a solution of example 1, Art. **20**, the first and the third terms, 7 and 56, stand for 7 pounds of putty at a cost of 56 cents. The second and the fourth terms refer to the cost of 36 pounds of putty. Such proportions are known as *direct* proportions.

But the conditions stated in the problems are not always such as to give proportions of this kind. Sometimes, it is found, on examination of the complete proportion, that the first and the third terms do *not* refer to the same thing. Such proportions are known as *inverse* proportions. It must, however, be clearly understood that whether a proportion is direct or inverse does not in any way interfere with the application of the rule in Art. **20**, as will be seen from the following examples in which the solution is in every case found by means of inverse proportion.

EXAMPLE 1.—If 3 men can do a certain job in 20 days, how long will it take 12 men to do a similar job, working at the same rate?

SOLUTION.—The unknown term is the number of days required to do a certain job; it is written as the fourth term. The number of a similar kind is 20 days, which is written as the third term. The second ratio of the proportion is, therefore, $20 : x$.

It is seen from the example that the time required by 12 men must be less than that needed by 3 men. It follows that x must be smaller than 20. Hence, the second term must be smaller than the first and the first ratio is written 12 : 3. The complete proportion is

$$12 : 3 = 20 : x$$

from which
$$x = \frac{3 \times 20}{12} = 5 \text{ days. Ans.}$$

In the complete proportion, 12 : 3 = 20 : 5, the first and the third terms do not refer to the same thing; the term 12 refers to the *larger* number of men, while the term 20 refers to the number of days required by the *smaller* number of men to finish the job. It follows that the proportion is an inverse proportion.

EXAMPLE 2.—A mining camp has provisions enough to sustain 120 men for 28 days. If the working force is increased to 210 men how long will the provisions last?

SOLUTION.—The unknown term is the number of days through which the provisions will last for 210 men; it is written as the fourth term. The number of a similar kind is 28 days, which is written as the third term. The second ratio of the proportion is, therefore, 28 : x.

It is seen from the example that the provisions will last a shorter time with an increase in the number of men. It follows that x must be smaller than 28. According to the rule, the second term of the proportion must be smaller than the first. The proportion is, therefore, written

$$210 : 120 = 28 : x$$

from which according to Art. **18**,

$$x = \frac{120 \times 28}{210} = 16 \text{ days. Ans.}$$

EXAMPLE 3.—It requires 36 hours to fill a tank with water by means of a pump discharging 9 gallons per minute. If the pump is displaced by one that will discharge 16 gallons per minute, how long will it take to fill the tank?

SOLUTION.—The unknown term is the number of hours required to fill the tank; it is written as the fourth term. The number of a similar kind is 36 hours, which is written as the third term. The second ratio of the proportion is, therefore, 36 : x.

It is obvious that the time required by the pump of the greater capacity must be shorter than that required by the pump discharging only 9 gallons per minute. Hence, x must be smaller than 36. As, according to the rule, the second term under the circumstances must be smaller than the first, it follows that the proportion must be written8

$$16 : 9 = 36 : x$$

from which
$$x = \frac{9 \times 36}{16} = 20\tfrac{1}{4} \text{ hours. Ans.}$$

22. Denominate Numbers in Proportions.—It was stated in Art. 4 that the two terms of a ratio must be of the same denomination. Evidently, this requirement must also be complied with in a proportion. When the first and second terms are given in different denominations, they must be reduced to the same denomination. Also, if they are compound numbers, they must be reduced to the same denomination, either the highest or the lowest denomination mentioned in either term. In case the third term is a compound number it must be reduced to one denomination in terms of either its lowest or its highest denomination. The fourth term, x, will be in a denomination corresponding to that of the third term.

For instance, if the first and second terms are 9 inches and 4 feet 3 inches, respectively, the second term may be reduced to inches, as $4 \times 12 = 48$ inches; $48 + 3 = 51$ inches. Or, both terms may be reduced to feet and fractions of a foot. Thus, 9 inches $= \frac{3}{4}$ foot $= .75$ foot, and 4 feet 3 inches $= 4\frac{1}{4}$ feet $= 4.25$ feet. The ratio may, therefore, have one of the following forms: $9 : 51$; $.75 : 4.25$; $\frac{3}{4} : 4\frac{1}{4}$.

EXAMPLE.—If a rod 9 inches long weighs 1 pound 3 ounces, what will be the weight of a rod of the same kind, 3 feet 7 inches long?

SOLUTION.—The unknown term is the weight of the longer bar, and the third term is the weight of the short bar. The second ratio of the proportion will therefore be

$$1 \text{ lb. } 3 \text{ oz} : x$$

As x will be greater than the third term, the second term must be greater than the first, and the complete porportion will be

$$9 \text{ in.} : 3 \text{ ft. } 7 \text{ in.} = 1 \text{ lb. } 3 \text{ oz.} : x$$

As the second term is in feet and inches, it will be reduced to inches to correspond with the first term; thus, 3 ft. 7 in. = 43 in. The third term will be reduced to its lowest denomination; thus, 1 lb. 3 oz. = 19 oz. On substituting the reduced numbers, the proportion will be

$$9 : 43 = 19 : x$$

from which $x = \frac{43 \times 19}{9} = 90.78$ oz. $= 5$ lb. $10\frac{3}{4}$ oz., nearly. Ans.

EXAMPLES FOR PRACTICE

1. If a pump discharging 6 gallons of water per minute can fill a tank in 20 hours, how long will it take a pump discharging 15 gallons per minute to fill it? Ans. 8 hr.

2. If 75 pounds of lead costs $6.75, how much will 125 pounds cost at the same rate? Ans. $11.25

3. The circular seam of a boiler requires 50 rivets when the pitch, or distance between centers of rivets, is $2\frac{1}{2}$ inches; how many would be required if the pitch were $3\frac{1}{8}$ inches? Ans. 40

4. If A does a piece of work in 4 days and B does it in 7 days, how long will it take A to do what B does in 63 days? Ans. 36 da.

5. If an electric car runs 12 miles in 35 minutes, how long will it take to run 30 miles at the same rate?
Ans. $87\frac{1}{2}$ min., or 1 hr. 27 min. 30 sec.

UNIT METHOD

PRINCIPLE AND APPLICATION

23. Solving Problems by the Unit Method.—Problems dealing with two ratios, as in proportion, may also be solved by what is known as the **unit method.** By means of this method it is possible also to solve problems dealing with more than two ratios, as will be shown in some of the succeeding examples. The general procedure in the unit method may be seen from the following example.

It is supposed that 4 bags of cement costs $1.36 and it is required to find the price of 55 bags.

Since the cost of 4 bags of cement = $1.36,

$$\text{the cost of 1 bag of cement} = \frac{\$1.36}{4} = \$.34$$

Therefore, the cost of 55 bags of cement = $.34 × 55 = $18.70.

It is seen that the principal number in the problem, here the price of 4 bags of cement, is reduced to the value of a unit; that is, to the price of 1 bag. This unit price may then serve as a basis for finding the value of any other number of units called for in the problem. In this example the price of 1 bag was multiplied by 55 to find the price of 55 bags.

24. In the preceding example the price of 1 unit is actually calculated, but in practice this is not done. The successive steps, that is, the operations of multiplication and division, are merely indicated by the respective signs, and no multiplication or division is performed until the very last, as then the answer may be obtained more easily by cancelation. This method of procedure should generally be followed in all arithmetical calculations. The following examples show the general method of procedure in solving problems by the unit method.

EXAMPLE 1.—If a pump discharging 6 gallons of water per minute can fill a tank in 18 hours, how long will it take a pump discharging 14 gallons per minute to fill the tank?

SOLUTION.—The problem is solved by the unit method, as follows:

6 gal. per min. requires 18 hr. for filling tank

1 gal. per min. requires 6×18 hr. for filling tank

Therefore, 14 gal. per min. requires $\frac{6\times18}{14}$ hr. for filling tank.

$$\frac{6\times18}{14}=7\tfrac{5}{7}\text{ hr. Ans.}$$

NOTE.—It is seen in this example that the time required for filling the tank at the rate of 1 gal. per min. is not calculated, but simply indicated by the product 6×18 hr. Evidently this time must be *divided* by 14 to find the time required for a rate of discharge of 14 gal. per min., as the time must be one-fourteenth of that required by a 1-gal. discharge.

EXAMPLE 2.—If 4 men earn $65.80 in 7 days, how much can 14 men, paid at the same rate, earn in 12 days?

SOLUTION.— 4 men in 7 days earn $65.80. Therefore, 1 man in 7 days earns $\frac{\$65.80}{4}$, and 1 man in 1 day earns $\frac{\$65.80}{4\times7}$. Therefore, 1 man in 12 days earns $\frac{\$65.80\times12}{4\times7}$, and 14 men in 12 days earn

$$\frac{14\times\$65.80\times12}{4\times7}=\frac{\overset{2}{\cancel{14}}\times\$65.80\times\overset{3}{\cancel{12}}}{\cancel{4}\times\cancel{7}}=\$65.80\times2\times3=\$394.80.\quad\text{Ans.}$$

EXAMPLE 3.—If 3 men can dig a trench in 20 days, how long will it take 12 men to dig a similar trench at the same rate?

SOLUTION.—As 3 men dig a trench in 20 days, 1 man can dig it in 3×20 days, and 12 men can dig it in $\frac{3\times20}{12}$ days $=5$ days. Ans.

EXAMPLE 4.—If a block of granite 8 feet long, 5 feet wide, and 3 feet thick weighs 7,200 pounds, what is the weight of a block of granite 12 feet long, 8 feet wide, and 5 feet thick?

SOLUTION 1.—A block 8 ft. long, 5 ft. wide, and 3 ft. thick weighs 7,200 lb. Therefore, a block 1 ft. long, 5 ft. wide, 3 ft. thick weighs $\frac{7{,}200}{8}$ lb.; and a block 1 ft. long, 1 ft. wide, 3 ft. thick weighs $\frac{7{,}200}{8\times 5}$ lb.; and a block 1 ft. long, 1 ft. wide, 1 ft. thick weighs $\frac{7{,}200}{8\times 5\times 3}$ lb. Therefore, a block 12 ft. long, 8 ft. wide, 5 ft. thick weighs

$$\frac{7{,}200\times 12\times 8\times 5}{8\times 5\times 3}=\frac{7{,}200\times \overset{4}{\cancel{12}}\times \cancel{8}\times \cancel{5}}{\cancel{8}\times \cancel{5}\times \cancel{3}}=28{,}800\text{ lb.}\quad\text{Ans.}$$

SOLUTION 2.—The contents of the blocks, in cubic feet, may be compared.

$$\text{Contents of first block}=8\text{ ft.}\times 5\text{ ft.}\times 3\text{ ft.}=120\text{ cu. ft.}$$
$$\text{Contents of second block}=12\text{ ft.}\times 8\text{ ft.}\times 5\text{ ft.}=480\text{ cu. ft.}$$

If 120 cu. ft. weighs 7,200 lb., then 1 cu. ft. weighs $\frac{7{,}200}{120}$ lb.

Therefore, 480 cu. ft. weighs $\frac{480\times 7{,}200}{120}=28{,}800$ lb. Ans.

EXAMPLE 5.—If 12 horses can plow 96 acres in 6 days, how many horses will be required to plow 64 acres in 8 days?

SOLUTION.—In 6 days 96 acres can be plowed by 12 horses; in 1 day 96 acres can be plowed by 6×12 horses; and in 1 day 1 acre can be plowed by $\frac{6\times 12}{96}$ horses. Therefore, in 8 days, 1 acre can be plowed by $\frac{6\times 12}{96\times 8}$ horses, it being evident that fewer horses are required to do the same work in 8 days than in 1 day. In 8 days 64 acres can be plowed by

$$\frac{64\times 6\times 12}{96\times 8}=6\text{ horses.}\quad\text{Ans.}$$

EXAMPLE 6.—If 7 horses require 35 bushels of oats in 20 days, for how many days will 96 bushels be sufficient for 18 horses?

SOLUTION.— 35 bu. lasts 7 horses for 20 days. Therefore, 1 bu. lasts 7 horses for $\frac{20}{35}$ days, and 1 bu. lasts 1 horse for $\frac{7\times 20}{35}$ days. Therefore, 96 bu. lasts 1 horse for $\frac{96\times 7\times 20}{35}$ days, and 96 bu. lasts 18 horses for

$$\frac{96\times 7\times 20}{35\times 18}=\frac{64}{3}=21\tfrac{1}{3}\text{ days.}\quad\text{Ans.}$$

EXAMPLES FOR PRACTICE

1. If one pump discharges 90,000 gallons of water in 20 hours, in what time will it discharge 144,000 gallons? Ans. 32 hr.

2. If 25 men can finish a certain job in 30 days, how many days will be required by 35 men to do the same job? Ans. $21\frac{3}{7}$ da.

3. If $63 is paid for 15 tons of coal, how much must be paid for 27 tons? Ans. $113.40

4. A pump making 30 strokes per minute discharges 450 gallons of water per minute. If the speed is reduced to 28 strokes per minute, what will be the discharge in gallons per minute? Ans. 420 gal.

5. A wheel makes 248 revolutions in 5 minutes. How many revolutions will it make in 1 hour and 20 minutes? Ans. 3,968 rev.

6. A locomotive runs 2.8 miles in 4 minutes. How far will it go in 1 hour? Ans. 42 mi.

7. If 7 barrels of sugar costs $104.30, what is the cost of 42 barrels? Ans. $625.80

8. If 42 yards of goods are made from 9 pounds of yarn, how many yards can be made from 165 pounds of a similar yarn? Ans. 770 yd.

9. If 17 men, paid at the same rate, receive in all $357 for one week, what is the sum that must be paid to 24 men for 1 week, if paid at the same rate? Ans. $504

10. Twelve men can do a certain piece of work in 15 days. How many days will be required by 36 men to do the same work? Ans. 5 da.

POWERS AND ROOTS

INVOLUTION, OR FINDING POWERS OF NUMBERS

PERFECT AND IMPERFECT POWERS

DEFINITIONS AND RULES

1. Factors.—The **factors** of a number are those numbers which, when multiplied together, will equal that number. Thus, 5 and 3 are the factors of 15, since 5×3=15.

In engineering calculations it is often found necessary to multiply a number by itself one or more times; thus, 5×5×5=125. In this case the number 125 consists of three equal factors, each of which is 5.

2. Powers.—A product obtained from several *equal* factors is called a **power** of the number that is used as the factor. The power is named according to the number of equal factors in the product. Thus, 9 is the **second power,** or **square,** of 3, as 9 is equal to the product of the *two* equal factors 3 and 3. A product is called the **third power,** or **cube,** of a number, if it contains *three* equal factors; thus, 64 is the third power of 4, as 64 is equal to the product of the three factors 4, 4, and 4, or 4×4×4=64.

3. Squares and Cubes.—The term **square** is used for the second power of a number, because the area of a square is equal to the product of two equal numbers, each of which represents the length of one side. Thus, the area of a square is equal to the *second power* of a number that represents the length of one of its sides. For example: The side of a square is 4 feet; its area$=4\times4=16$ square feet.

The term **cube** is used for the third power of a number, because the volume of a cube is equal to the product found by using the length of one edge three times as a factor; that is, its volume is equal to the *third power* of a number representing the length of one of its edges. For example, the edge of a cube is 5 inches; its volume is $5\times5\times5=125$ cubic inches.

4. Involution.—The process of finding powers of quantities is called **involution.** The term *raise* is generally used in connection with this process. Thus, it is said that 7 is *raised* to the third power by using it as a factor three times, or $7\times7\times7=343$.

5. Exponents.—It is not sufficient to say that a *power* of a given number is to be found; one must also know *which* power is required, whether it is to be the second, the third, etc. For the purpose of indicating the required power of a number, a small number, called an **exponent,** is written to the right and near the top of the number. This number indicates the power to which a quantity is to be raised, or the number of times the quantity is to be used as a factor. Thus, in the expression 3^6, the number ^{6} is the exponent, and shows that 3 is to be used as a factor six times, or that 3^6 is a contraction of

$$3\times3\times3\times3\times3\times3$$

When an exponent is attached to a number it is read as in the following examples:

4×4 is written 4^2, and is read **four square,** or *four exponent two;*

$5\times5\times5$ is written 5^3, and is read **five cube,** or *five exponent three;*

$8\times8\times8\times8$ is written 8^4, and is read **eight to the fourth power,** or *eight exponent four.*

6. Use of Parenthesis.—When several numbers are connected by any of the arithmetical signs and the result is to be raised to a given power, the numbers must be written inside a parenthesis and the exponent outside it, as in the following example: $(2+5)^2=7^2=49$. Without the parenthesis the result would be: $2+5^2=2+25=27$.

The same rule applies if a fraction is to be raised to a given power. For instance, if the square of $\frac{3}{4}$ is to be found, it must be written $(\frac{3}{4})^2$. By omitting the parenthesis it will appear as if the exponent refers to the numerator alone, instead of applying to the numerator and the denominator. The effect produced by omitting the parenthesis may be seen from the following example: $\left(\frac{3}{4}\right)^2=\frac{3^2}{4^2}=\frac{9}{16}$. Omitting the parenthesis, the result is $\frac{3^2}{4}=\frac{9}{4}$.

7. Perfect and Imperfect Powers.—There are comparatively few numbers that can be separated into equal factors; these numbers are called **perfect powers.** Thus, 16 is a perfect power of 4, because $16=4\times4$; 216 is a perfect power of 6, because $216=6\times6\times6$. Numbers that cannot be separated into exactly equal factors are called **imperfect powers.** Thus, 10, 12, 15, and 20 are imperfect powers, because none of them is the product of equal factors. In the numbers from 1 to 1,000, inclusive, there are only 50 perfect powers, not counting 1, and of these only 30 are perfect squares and 9 perfect cubes.

TABLE I

PERFECT SQUARES AND CUBES

n	n^2	n^3	n	n^2	n^3
1	1	1	6	36	216
2	4	8	7	49	343
3	9	27	8	64	512
4	16	64	9	81	729
5	25	125	10	100	1,000

Table I contains the squares and cubes of numbers from 1 to 10, inclusive. The column of numbers is headed by the letter *n*, which is an abbreviation of the term *number*. The squares and cubes of the numbers found in the first and fourth columns are given in the other columns, headed by n^2 and n^3, respectively. Thus, the square of 8 is 64 and the cube of 5 is 125.

8. Rule for Raising a Number to Any Power.—To find any power of a number, the following rule should be used:

Rule.—I. *To raise a whole number or a decimal to any power, use the number as a factor as many times as the power requires or as the exponent indicates.*

II. *To raise a fraction to any power, use the numerator and the denominator as factors as many times as the power requires or as the exponent indicates, and write these products as numerator and denominator, respectively.*

Example 1.—What is the third power, or cube, of 35?

Solution.—From Art. **3**, the expression *cube of a number* is equivalent to the number with 3 as an exponent. Applying the rule, $35^3=35\times35\times35$, or

$$\begin{array}{r}
35\\
35\\
\hline
175\\
105\\
\hline
1225\\
35\\
\hline
6125\\
3675\\
\hline
\textit{cube}=42{,}875
\end{array}\quad \text{Ans.}$$

Example 2.—What is the value of 1.2^2?

Solution.—According to the rule, the number must be used as a factor the number of times indicated by the exponent, 2. Hence, $1.2^2=1.2\times1.2=1.44$. Ans.

Example 3.—What is the value of $(\frac{5}{8})^2$?

Solution.—The exponent is 2, so the numerator and the denominator must each be used twice as a factor. Then,

$$\left(\frac{5}{8}\right)^2=\frac{5^2}{8^2}=\frac{5\times5}{8\times8}=\frac{25}{64}. \quad \text{Ans.}$$

Example 4.—What is the fourth power of 15?

Solution.—From Art. **5**, the fourth power of 15 is equivalent to 15^4. Applying the rule, $15^4=15\times15\times15\times15$, or

$$\begin{array}{r} 15\\ 15\\ \hline 75\\ 15\\ \hline 225\\ 15\\ \hline 1125\\ 225\\ \hline 3375\\ 15\\ \hline 16875\\ 3375\\ \hline \textit{fourth power}=50{,}625 \end{array} \quad \text{Ans.}$$

Example 5.—What is the cube of .12?

Solution.—The cube is found by using the number three times as a factor; therefore, the cube of .12 is

$$.12\times.12\times.12=.001728. \quad \text{Ans.}$$

EXAMPLES FOR PRACTICE

Raise the following to the powers indicated:

(*a*) 85^2.	Ans.	(*a*) 7,225
(*b*) $\left(\frac{12}{13}\right)^2$.		(*b*) $\frac{144}{169}$
(*c*) 6.5^2.		(*c*) 42.25
(*d*) 14^4.		(*d*) 38,416
(*e*) $\left(\frac{3}{4}\right)^3$.		(*e*) $\frac{27}{64}$
(*f*) $\left(\frac{5}{6}\right)^3$.		(*f*) $\frac{125}{216}$
(*g*) $\left(\frac{7}{2}\right)^3$.		(*g*) $\frac{343}{8}$
(*h*) 1.4^5.		(*h*) 5.37824

EVOLUTION, OR FINDING ROOTS OF NUMBERS

SQUARE ROOT

DEFINITIONS

9. Roots of Numbers.—It was stated in Art. **2** that the product obtained from a number of equal factors is called a power of the number. If the process is reversed and the number of equal factors into which a number may be separated is found, then any one of these factors is known as the **root** of the number. For instance, if the number 27 is divided into the three factors 3, 3, and 3, then any one of these factors is known as a root of this number, as $27=3\times3\times3$. This process of finding a root of a number is known as **evolution;** it is the reverse of *involution.* The term *extract* is generally used in connection with this process, and it is said that the root of the number is *extracted.*

10. Classification of Roots.—If a number is separated into *two* equal factors, one of these factors is known as the **square root** of the number. Thus, if 25 is separated into two equal factors 5 and 5, then 5 is the square root of 25, because $5\times5=25$. The square root of 49 is 7, because $7\times7=49$; the square root of 1.21 is 1.1, because $1.1\times1.1=1.21$.

If a number is separated into *three* equal factors, one of the factors is known as the **cube root** of the number. Thus, 3 is the cube root of 27, since $3\times3\times3=27$.

The **fourth root** of a number is one of the *four* equal factors into which the number may be separated. Thus, the fourth root of 256 is 4, because $4\times4\times4\times4=256$.

The **fifth root** of a number is one of the *five* equal factors into which the number may be separated. Thus, 7 is the fifth root of 16,807, since $7\times7\times7\times7\times7=16{,}807$.

11. Derivation of Terms Square Root and Cube Root.—The terms *square* and *cube,* when applied to roots, are derived from the same source as similar terms applied to powers of numbers. Reversing the conditions stated in Art. **3,** it follows that, if a number represents the area of a square, the *square root* of the number must be equal to the length of one side of the square. For instance, if the area of a square floor is 81 square yards, the square root of 81, or 9, represents, in yards, the length of one side of the room.

If a given number represents the contents of a cube, the *cube root* of the number must give the length of one of the edges. Thus, if the contents of a cubical box is 27 cubic feet, the cube root of the number, or 3, gives the length of one of the edges, in feet.

12. Radical Sign.—The fact that the root of a number is to be extracted is usually indicated by placing the *radical sign* √ in front of it. The term **radical** is derived from the Latin word *radix,* meaning *root.* A *vinculum* $\overline{}$ is connected with the radical sign and placed over the quantity to which the radical sign applies. Thus, $\sqrt{4,574,300}$ indicates that the root of the number following the radical sign is to be extracted.

13. Index.—Although the radical sign shows that a root is to be extracted, it must also show what root is required, and for this purpose an **index** is used. This is a small figure placed *above* the radical sign; thus, $\sqrt[2]{100}$ indicates the *square root* of 100, and $\sqrt[3]{1,728}$ indicates the *cube root* of 1,728. However, when the square root of a number is to be extracted, the index is usually omitted. Thus, $\sqrt{81}$ means the square root of 81; also, $\sqrt{22.48}$ means the square root of 22.48.

CALCULATION OF SQUARE ROOT

SQUARE ROOTS OF WHOLE NUMBERS AND OF DECIMALS

14. Introduction.—In calculating, or extracting, the square root of a number, it is necessary to perform a number of separate operations in a certain order. If the nature of each operation is kept in mind and if the operations are performed successively in the correct order, the process of extracting a square root should not offer any particular difficulties.

To facilitate the first part of the calculation, it is well to memorize the squares of the first twelve integers, or whole numbers, given herewith. The first line gives the numbers and the second line the corresponding squares.

Integers:	1,	2,	3,	4,	5,	6,	7,	8,	9,	10,	11,	12
Squares:	1,	4,	9,	16,	25,	36,	49,	64,	81,	100,	121,	144

15. Example of Extracting Square Root.—The method of finding the square root can best be explained by using an actual example and describing each step of the work. The first step is to point off, or separate, the number into periods, or parts, each containing two figures. Thus, suppose the square root of the whole number 31,505,769 is to be extracted. Ignore the commas that are used to divide the number and write it without them, thus: 31505769. Now, beginning at the right-hand figure, point off, or separate, the number into periods of two figures each, proceeding toward the left and using the mark ′ to separate the periods. The number will then appear as follows: 31′50′57′69. In case the whole number contains an odd number of figures, there will be only *one* figure in the period at the left. For instance, take the number 53,361. When this is pointed off it becomes 5′33′61, in which there are three periods. Here the 5 forms the first period, even though it consists of only one figure.

The reason for thus pointing off the number is to find out how many figures there will be in the root of the number. It is always true that the number of figures in the root is equal to the number of periods into which the number is divided. Thus, as 31′50′57′69 contains four periods, it is known at once, before any calculations are made, that there will be four figures in the square root of that number. Similarly, as 5′33′61 has three periods, the square root must contain three figures; and this is the case, as the square root of 53,361 is 231. After the number has been properly pointed off, the square root is extracted in the manner shown in the following example:

Example.—Extract the square root of 31,505,769.

Solution.—

	steps	*number*	*root*	
(*a*)	5	3 1′5 0′5 7′6 9(	5 6 1 3	
	2 0	5^2=2 5		
first trial divisor	1 0 0	6 5 0		*first dividend*
	6	6 3 6		
first complete divisor	1 0 6	1 4 5 7		*second dividend*
(*b*)	5 6	1 1 2 1		
	2 0	3 3 6 6 9		*third dividend*
second trial divisor	1 1 2 0	3 3 6 6 9		
	1			
second complete divisor	1 1 2 1			
(*c*)	5 6 1			
	2 0			
third trial divisor	1 1 2 2 0			
	3			
third complete divisor	1 1 2 2 3			

Explanation.—Beginning at the right, the number 31,505,769 is pointed off into periods of two figures each, as already explained. The largest single number whose square is less than 31, the first period, is now found. This is evidently 5, since the square of 6, or 36, is greater than 31. This number, 5, is written to the right of the number, as in long division; it is also written to the left, as at (*a*). The square of this first figure of the root, or 5^2=25, is written under the first period, as shown, and is subtracted from it, leaving 6 as a remainder. The second period of the number is annexed to this remainder, giving 650 as the **first dividend.**

The first figure of the root, written at (*a*), is now multiplied by 20, giving a product 100, which is called the **first trial divisor.** The first dividend, 650, is now divided by this first trial divisor, 100, and the quotient 6 is obtained, which is *probably* the second figure of the root. This figure is written in the root, as snown, and is also added to 100, the first trial divisor, giving the sum 106, which is called the **first complete divisor.**

The first complete divisor, 106, is multiplied by 6, the second figure in the root, giving the product 636, which is subtracted from the first dividend; the remainder is 14, to which the two figures, 57, in the third period of the number are annexed, giving 1457 as the **second dividend.** The two figures of the root, 56, are now written at (*b*) and multiplied by 20, thus giving 1120, which is the **second trial divisor.** Dividing 1457 by the second trial divisor, 1120, the quotient 1 is obtained as the third figure of the root. Adding this figure to the second trial divisor, the result is 1121, which is the **second complete divisor.** Multiplying this divisor by 1, the third figure in the root, gives the product 1121, which is written under the second dividend, 1457. Then, subtracting it from the second dividend, the remainder is 336, to which the fourth period, 69, of the number is annexed, giving 33669 as the **third dividend.**

The three figures 561 in the root are now placed at (*c*) and multiplied by 20, giving the product 11220, which is the **third trial divisor.** Dividing 33669 by the third trial divisor, 11220, the quotient, 3, is obtained as the fourth figure of the root. Adding this figure to 11220, the result is 11223, the **third complete divisor.** Multiplying this divisor by 3, the fourth figure of the root, the product is 33669, which is written under the third dividend and subtracted from it, leaving no remainder. It follows that $\sqrt{31{,}505{,}769}=5{,}613$, and that $31{,}505{,}769=5{,}613 \times 5{,}613$.

16. Choosing Figures for the Root.—In the preceding explanation, where reference was made to finding the second figure, 6, of the root, the statement was made that *probably* the figure 6 would be the one required. The word *probably* is used here, because the various figures that are successively selected for the root are, at first, only trial figures. There can be no certainty that each of these figures will be the correct one, and not too large, until it is multiplied by the corresponding complete divisor, thus allowing the resulting product to be compared with the dividend. Referring to the foregoing example, it is not certain that 6, the second figure in the root, is correct, before it has been multiplied by the first complete divisor, 106, and the product found to be not greater than the first dividend, 650. In this case the product 636 is less than 650; but, suppose that the first dividend were 620, as in the example that follows. The product 636 would be too large, thus indicating that the figure 6 in the root is too large. It would then be necessary to substitute, in place of the 6, the figure 5 as the second figure in the root. The remaining part of the solution is simply a repetition of that already described.

(*a*)	5
	20
first trial divisor	100
	5
first complete divisor	105
(*b*)	55
	20
second trial divisor	1100
	8
second complete divisor	1108
(*c*)	558
	20
third trial divisor	11160
	6
third complete divisor	11166

	root
31'20'33'96 (5586	
$5^2 = 25$	
620	*first dividend*
525	
9533	*second dividend*
8864	
66996	*third dividend*
66996	

17. Main Features of Extracting Square Root.—In order that the main features of calculating a square root may be remembered, it should be noted that the various trial divisors are obtained by multiplying the existing figures in the root by 20. Thus, the *first* figure in the root multiplied by 20 gives the *first* trial divisor; the *first two* figures of the root multiplied by 20 gives the *second* trial divisor, and so forth. Considering this feature as the framework of the process, the remainder will easily be remembered.

18. Method of Procedure When Trial Divisor is Larger Than Its Dividend.—Sometimes, in extracting a root, it may happen that a trial divisor is larger than the corresponding dividend. In such a case the method adopted is similar to that used in long division; that is, a cipher is annexed to the root and to the trial divisor and the next period is annexed to the last remainder to form a new dividend, larger than the new trial divisor. The following example shows how to proceed:

Example.—Find the square root of 255,025.

Solution.—

			root	
(*a*)	5		2 5'5 0'2 5(5 0 5	
	2 0	$5^2=25$		
first trial divisor	1 0 0		5 0 2 5	*second dividend*
second trial divisor	1 0 0 0		5 0 2 5	
	5			
second complete divisor	1 0 0 5			

Explanation.—After the number is separated into periods, it is seen that the root of the first period must be 5. This figure is written to the right of the given number and also at (*a*). The square of 5, or 25, is subtracted from the first period, leaving no remainder. The second period, 50, is brought down as the first dividend and the number 5 at (*a*) is multiplied by 20, giving 100 as the first trial divisor. As this trial divisor, 100, is not contained even once in the first dividend, 50, a cipher is annexed to the root and to the divisor, making the root 50 and the new, or second, trial divisor 1000. The third period of the number, 25, is now brought down and annexed to the remainder 50, making 5025, which is the second dividend. Dividing by the second divisor, 1000, the quotient, 5, is placed as the last figure of the root. Adding 5 to 1000 gives 1005 as the second complete divisor. Multiplying by 5 gives 5025, which is equal to the second dividend; hence, there will be no remainder.

In ordinary practice, instead of writing a separate second trial divisor, as in this example, the cipher would be annexed directly to the first trial divisor. Thus, a cipher could have been annexed to the first trial divisor, 100, giving the second trial divisor, 1000.

19. Extracting Square Roots of Decimals.—The square of any number, wholly decimal, always contains twice as many decimal places as the number squared. For example, $.1^2=.01$, $.13^2=.0169$, $.751^2=.564001$, etc. Conversely, it follows that the square root of a decimal contains only one half the number of decimal places found in the decimal itself. Thus, $\sqrt{12.723489}=3.567$. If the decimal contains an uneven number of figures, a cipher must be annexed to give an even number, as will now be explained.

When the square root of a decimal is to be extracted, the decimal is pointed off into periods of two figures each, *begin-*

ning at the decimal point and going to the *right.* Then, if the last period contains but one figure, a cipher is annexed to complete the period. Thus, the decimal .62371 is pointed off as follows: .62'37'10. Annexing ciphers to the right of a decimal does not change its value, as was explained in a preceding Section.

If the decimal is a portion of a mixed number, as 142.716, the whole-number part is pointed off to the *left* and the decimal part to the *right,* beginning at the decimal point in both cases, and considering the decimal point as a mark of separation. Thus, the number 142.716 is pointed off as follows: 1'42.71'60.

The operation of finding the square root in all these cases is similar to that previously described, except that when the decimal point is reached, a decimal point is placed in the answer. There will be as many decimal places in the root as there are periods in the decimal part of the number.

20. The following examples will show the method of extracting roots of decimals:

EXAMPLE 1.—Find the square root of 606.6369.

SOLUTION.—

			root	
	2		6'06.63'69(24.63	
	20	$2^2 =$	4	
first trial divisor	40		206	*first dividend*
	4		176	
first complete divisor	44		3063	*second dividend*
			2916	
	24			
	20		14769	*third dividend*
			14769	
second trial divisor	480			
	6			
second complete divisor	486			
	246			
	20			
third trial divisor	4920			
	3			
third complete divisor	4923			

Example 2.—What is the square root of .000576?

Solution.—

		root
	2	.0 0'0 5'7 6(.0 2 4
	2 0	$2^2 = 4$
trial divisor	4 0	1 7 6
	4	1 7 6
complete divisor	4 4	

Explanation.—Beginning at the decimal point and separating the decimal into periods of two figures each, it is seen that the first period is composed of ciphers; hence, the first figure in the root must be a cipher. The decimal is now treated as if 5 were the first figure and the calculation is continued in the manner previously explained.

21. Perfect and Imperfect Powers.—Comparatively few numbers are exact squares; consequently, it is only in a small number of cases that the exact square root can be found. A number that has an exact root is called a **perfect power;** in the case of an exact square root the number from which it is extracted is termed a **perfect square.** The factors of the perfect powers are called **rational factors.** An exact square root of a number represents one of the two equal rational factors into which it is possible to separate the perfect square. For instance, $\sqrt{81} = 9$, and $9 \times 9 = 81$; hence, the two numbers 9 are rational factors.

Numbers that cannot be separated into exactly equal factors are called **imperfect powers,** and the factors are called **irrational factors.** Any number, that cannot be divided into as many rational factors as there are units in the index of the root, will have a root with an unending decimal. For example, 20 lies between 16 $(=4^2)$ and 25 $(=5^2)$; hence, the square root of 20, or $\sqrt{20}$, is greater than 4 and less than 5, and is therefore equal to 4 plus an unending decimal. In other words, no matter to how many figures the square root of 20 may be calculated, the root will never be found exactly. Numbers ending in 2, 3, 7, or 8 are imperfect squares.

22. Extracting Root of Imperfect Power.—Although the square root of an imperfect power cannot be found exactly, as close an approximation may be obtained as is desired. The root may be carried to any required number of decimal places by annexing periods of two ciphers each to the number. In practice, five significant figures are all that are likely to be required, and four are generally sufficient.

EXAMPLE 1.—What is the square root of 3? Find the result to five decimal places.

	1		*root*
	20	3.0 0'0 0'0 0'0 0'0 0 (1.7 3 2 0 5+	
		$1^2=1$	
	20		
	7	200	*first dividend*
		189	
first divisor	27		
		1100	*second dividend*
	17	1029	
	20		
		7100	*third dividend*
	340	6924	
	3		
		1760000	*fifth dividend*
second divisor	343	1732025	
	173	27975	
	20		
	3460		
	2		
third divisor	3462		
	1732		
	20		
	346400		
	5		
fifth divisor	346405		

EXPLANATION.—As five decimal places are required, it is necessary to annex five periods of ciphers to the right of the decimal point. The method employed in extracting the square root is the same as explained in the preceding examples. Attention is called only to the omission of the fourth divisor and dividend. After the fourth period is annexed to the remainder, 176, making the fourth dividend 17600, it is found that this dividend does not contain the divisor 34640. A cipher is therefore placed as the next figure in the root and a cipher annexed to the divisor, changing it into a fifth trial divisor. Another period of two ciphers is now annexed to the fourth dividend, changing it into a fifth dividend, and the calculation is continued, 5 being obtained as the next figure of the root. The required five decimal places have now been obtained and the operation could stop at this point; but in the case of an unending decimal, it is advisable to continue the operation one decimal place further, to ascertain whether the next figure in the decimal is 5 or greater. If so, the fifth figure is increased by 1. It will be found that the next figure is a cipher; hence the figure 5 will remain. The square root of 3 is, then, 1.73205+.

PROOF.—*To prove the square root, it is squared. If the number is an exact square, the square of the root will equal it; if it is not an exact square, the square of the root, plus the remainder, will equal it.*

EXAMPLE 2.—What is the square root of .3 to five decimal places?

```
SOLUTION.—   5            .3 0'0 0'0 0'0 0'0 0 (.5 4 7 7 2 2
            20             2 5
           ---             ---
           100               5 0 0
             4               4 1 6
           ---               -----
           104                 8 4 0 0
                               7 6 0 9
            54                 -------
            20                   7 9 1 0 0
          ----                   7 6 6 2 9
          1080                   ---------
             7                     2 4 7 1 0 0
          ----                     2 1 9 0 8 4
          1087                     -----------
                                     2 8 0 1 6 0 0
           547                       2 1 9 0 8 8 4
            20                       -------------
         -----                         6 1 0 7 1 6
         10940
             7
         -----
         10947

          5477
            20
        ------
        109540
             2
        ------
        109542

         54772
            20
       -------
       1095440
             2
       -------
       1095442
```

EXPLANATION.—In this example a cipher is annexed to .3, making the first period .30, since every period in a decimal, as was mentioned in Art. **19**, must have two figures in it. The operation is carried on to six decimal places, to find out whether the sixth figure is greater than 5. It is seen that the sixth figure is 2; hence, the preceding figure is not altered, and the square root of .3 is .54772+ to five decimal places.

23. Abbreviated Method of Extracting Square Root.—When some proficiency is gained in extracting square root, it is possible to perform a portion of the calculation mentally. The preceding solution may then be found as follows:

	.30'00'00'00'00(.54772
	25
100	500
complete divisor 104	416
1080	8400
complete divisor 1087	7609
10940	79100
complete divisor 10947	76629
109540	247100
complete divisor 109542	219084
	28016

Explanation.—The first figure, 5, in the root is squared, as before, and subtracted from 30. The figure 5 in the root is now mentally multiplied by 20 and the product, 100, is written to the left, as before. The next figure in the root, or 4, is mentally added to 100 and the sum, 104, is multiplied by 4; the product 416 is subtracted from 500, leaving 84. The next period is brought down and the two figures 54 in the root are mentally multiplied by 20; the product, 1080, is written in the usual place. Mentally, the quotient 7 is added to 1080 and the sum, 1087, multiplied by 7. The product, 7609, is subtracted. The calculation is continued in this manner until the required number of figures is obtained in the root. After long practice it is possible to omit the writing of products, such as 100, 1080, etc., used to obtain trial divisors and simply write the divisors, as 104, 1087, etc.

24. Rule for Extracting Square Root.—The instructions given in the preceding explanations have been collected and combined in the form of a rule, as follows:

Rule.—*To extract the square root of a number, first point off, or separate, the number into periods of two figures each, commencing at the right; or, if the number contains a decimal, begin at the decimal point and separate the number on either side into periods.*

Find the largest number whose square is contained in the first, or left-hand, period and write this number as the first *figure of the root. Write its square under the first period and subtract. To the remainder annex the next period for a first dividend.*

Place the first figure of the root also to the left of the given number; multiply this figure by 20, and use the product as the first trial divisor.

Divide the dividend by the divisor for the second figure in the root and add this figure to the trial divisor to form the complete divisor. Multiply the latter by the second figure of the root and subtract the product from the dividend. (If this product is larger than the dividend, a smaller number must be tried for the second figure of the root.)

Bring down the third period and annex it to the last remainder. Place the first two figures of the root to the left of the given number and multiply them by 20; the product is the second trial divisor.

Continue in this manner to the last period. Should any additional places in the root be required, annex cipher periods of two figures each to the remainders and continue the operation.

If at any time a trial divisor is larger than the corresponding dividend, place a cipher in the root, annex a cipher to the trial divisor, and bring down another period.

If the root is an unending decimal and it is desired to terminate the operation at some point, as the fourth decimal place, continue the operation to one place further, and if the fifth figure is 5 or greater, increase the fourth figure by 1.

EXAMPLES FOR PRACTICE

Find the square root of:

		Ans.	
(*a*)	186,624	(*a*)	432
(*b*)	2,050,624	(*b*)	1,432
(*c*)	29,855,296	(*c*)	5,464
(*d*)	.0116964 to five decimal places	(*d*)	.10815
(*e*)	198.1369 to four decimal places	(*e*)	14.0761
(*f*)	994,009	(*f*)	997
(*g*)	2.375 to four decimal places	(*g*)	1.5411
(*h*)	.571428 to five decimal places	(*h*)	.75593

SQUARE ROOTS OF FRACTIONS

25. Rule for Extraction.—If the given number is in the form of a fraction, and it is required to find the square root of it, the simplest and most exact method is to reduce the fraction to a decimal and extract the square root of the decimal. If, however, the numerator and denominator of the fraction are perfect squares, extract the square root of each separately, and write the root of the numerator for a new numerator, and the root of the denominator for a new denominator.

Rule.—*Extract the square root of the numerator and denominator separately; or, reduce the fraction to a decimal, and extract the square root of the decimal.*

Example 1.—What is the square root of $\frac{9}{64}$?

Solution.—Applying the rule, the roots of the numerator and the denominator are extracted separately, giving 3 and 8, respectively, as the numerator and denominator in the new fraction. Thus,

$$\sqrt{\frac{9}{64}}=\frac{\sqrt{9}}{\sqrt{64}}=\frac{3}{8}. \text{ Ans.}$$

Example 2.—What is the square root of $\frac{5}{8}$?

Solution.—Reduce the fraction to a decimal; thus, $\frac{5}{8}=.625$. Then the square root of $\frac{5}{8}$ is the square root of .625.

$$\sqrt{.625}=.79057. \text{ Ans.}$$

EXAMPLES FOR PRACTICE

Find the square root of:

		Ans.	
(*a*)	$\frac{25}{256}$	(*a*)	$\frac{5}{16}$
(*b*)	$\frac{9}{16}$	(*b*)	$\frac{3}{4}$
(*c*)	$\frac{3}{4}$ to five decimal places	(*c*)	.86603
(*d*)	$\frac{5}{12}$ to four decimal places	(*d*)	.6455

APPLICATIONS OF SQUARES AND SQUARE ROOTS

INTRODUCTION

26. Plane Surfaces.—The process of finding the square root of a number is very useful in many varieties of calculations, as, for example, in finding certain dimensions of squares, triangles, circles, and other surfaces, when their areas are given. Before entering into a description of the methods to be applied for these purposes, it is necessary to give a short description of the principal properties of these surfaces.

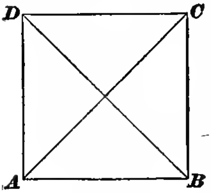


Fig. 1

The following definitions refer only to *plane* surfaces: A **plane surface** is one on which straight lines drawn in any direction lie wholly in the surface. For instance, the surface of a table is a plane surface. A **plane figure** is a plane surface bounded by straight or curved lines. A **polygon** is a plane figure bounded by any number of straight lines. Squares and rectangles, Art. **27**, and triangles, Art. **28**, are examples of polygons. The **area** of a plane figure is the number of square units included within its boundary lines.

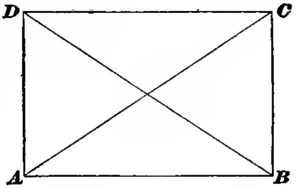


Fig. 2

27. Squares and Rectangles.—A **square,** Fig. 1, is a plane figure bounded by four straight lines *A B, B C, C D,* and *D A* of equal lengths, the adjacent lines being at right angles to each other.

A **rectangle,** Fig. 2, is a plane figure bounded by four straight lines, the adjacent lines being at right angles to each other. Only opposite sides are of equal lengths; thus, $AB = CD$, and $AD = CB$.

28. Triangles.—A **triangle,** Fig. 3, is a plane figure bounded by three straight lines. A **right-angled triangle,**

Fig. 4, also known as a **right triangle,** is a triangle that has one right angle. The side opposite the right angle is known as the **hypotenuse,** and is the longest one of the three sides. The **base** is the side on which the triangle is shown to rest. In the right triangle, Fig. 4, the right angle is at *C* and *A B* is the hypotenuse. The side *A C* is the base, as the triangle is shown resting on this side. The shortest distance between the base and the point at which the other sides meet is known as the **altitude** of a triangle. In Fig. 4 the side *B C* is also the altitude, as *B C* is the shortest distance from the side *A C* to the point *B* at which the sides *A B* and *B C* meet. If one of the two sides that form the right angle serves as a base, the other side is the altitude. Thus, if *B C* is the base, *A C* is the altitude.

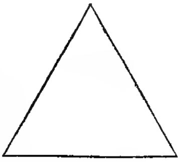


FIG. 3

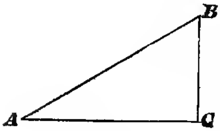


FIG. 4

29. Dividing Surfaces Into Triangles.—A straight line drawn between opposite corners of a square or a rectangle is known as a **diagonal.** In Figs. 1 and 2 the lines *A C* and *B D* are diagonals. Each diagonal divides the square and rectangle into two equal right triangles. For instance, in Figs. 1 and 2, the diagonal *A C* separates each figure into the two triangles *A B C* and *C D A,* and the diagonal *B D* in each produces the triangles *B C D* and *D A B.* When some of the dimensions of rectangles are to be obtained, it often simplifies the calculations to consider the rectangle as composed of two equal right triangles and to calculate the dimensions of one of these, according to the rules given farther on.

NOTE.—In practice, when referring to the diagonal of a square, this dimension is sometimes called the width *across corners.*

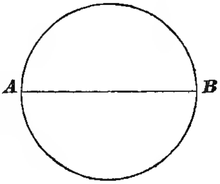


FIG. 5

30. Circles.—A **circle,** Fig. 5, is a plane figure bounded by a curved line called the **circumference.** A **diameter** is a line passing through the center and terminating at the circumference, as *A B.*

CALCULATIONS RELATING TO PLANE SURFACES

31. Finding Side of a Square.—It was shown in Art. **3** that the area of a square is equal to the second power, or square, of the number representing one of its sides. It follows that, if the area of the square is given, the length of one side is found by reversing the process; that is, by extracting the square root of the number that represents the area of the square. In examples of this class the following rule is applied:

Rule.—*To find the length of one side of a square of a given area, extract the square root of the area.*

Example.—The area of a square is 1,764 square inches. What is the length of one side of the square?

Solution.—Applying the rule, the side is equal to $\sqrt{1,764}=42$ in. Ans.

32. Principle of Right Triangle.—If a square is constructed on each of the three sides of a right triangle, as on *A B C*, Fig. 6, it can be shown that the area of the square erected on the hypotenuse *A C* is equal to the combined areas of the squares erected on the sides *A B* and *B C*. For instance, let it be assumed that the sides *A B*, *B C*, and *A C* are 3, 4, and 5 inches long, respectively. If the side *A B* is divided into three parts, each 1 inch long, and the square *A B I K* is erected on *A B*, as one of its sides, the square will contain 9 equal squares, or 9 square inches. Similarly, the square *B C H F* on the side *B C* will contain 16 square inches, and the square *A C E D* will contain 25 square inches, as shown. But, 9+16=25.

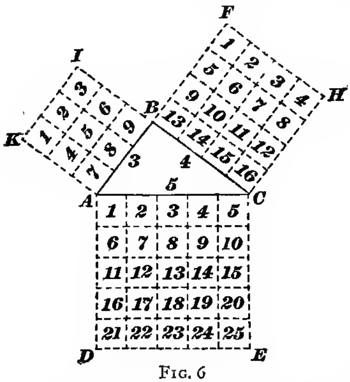


Fig. 6

It was stated in Art. **3** that the area of a square is equal to the square of the number representing the length of one side. The principle illustrated in Fig. 6 may therefore be stated in the following two forms:

I. *The square of the hypotenuse of a right triangle is equal to the sum of the squares of the two sides.*

II. *In a right triangle, the square of one of the sides is equal to the square of the hypotenuse minus the square of the other side.*

On these statements are based the following rules:

Rule I.—*To find the hypotenuse of a right triangle, square the length of each side, add the squares, and take the square root of the sum.*

Rule II.—*To find the length of either side of a right triangle, square the length of the hypotenuse, subtract from it the square of the length of the known side, and take the square root of the remainder.*

33. Calculating Sides of Right Triangles.—The application of the rules in Art. **32** for finding the sides and the hypotenuse of a right triangle is shown in the following examples:

Example 1.—The sides AC and CB, Fig. 4, are 8 and 6 inches long, respectively. What is the length of the hypotenuse AB?

Solution.—By rule I, the lengths of the sides are squared. Thus, $8^2=64$, and $6^2=36$, and $64+36=100$; then, extract the square root, $\sqrt{100}=10$. Therefore, $AB=10$ in. Ans.

Example 2.—If it is assumed that the hypotenuse AB, Fig. 4, is 25 inches long, and that the side AC measures 20 inches, what is the length of the side BC?

Solution.—By rule II, the square of the length of AB is $25^2=625$, and the square of the length of AC is $20^2=400$. $625-400=225$; then, the square root is $\sqrt{225}=15$. Therefore, $BC=15$ in. Ans.

Examplf 3.—In a right triangle the hypotenuse is 12 inches long and one side is 4 inches long. What is the length of the other side?

Solution.—By rule II, the square of the length of the hypotenuse is $12^2=144$, and the square of the length of the known side is $4^2=16$. Then $144-16=128$; and, $\sqrt{128}=11.31$. Therefore, the length of the unknown side is 11.31 in. Ans.

Example 4.—A ladder is leaning against the side of a house with its lower end 8 feet from the side. If the upper end of the ladder is 35 feet above the ground, what is the length of the ladder?

SOLUTION.—The ladder may in this case be considered as the hypotenuse of a right triangle in which the sides are 8 ft. and 35 ft. long, respectively. By rule I, the lengths of the sides are squared. Thus, $35^2=1{,}225$ and $8^2=64$, and $1{,}225+64=1{,}289$. Then $\sqrt{1{,}289}=35.9$. Therefore, the length of the ladder is 35.9 ft. Ans.

34. Finding Diameter of a Circle.—The ratio of the circumference of a circle to its diameter is as 3.1416 to 1. Then if the diameter of a circle is known, its circumference can be found by multiplying the diameter by 3.1416.

The area of a circle is equal to the product 3.1416 $\times\frac{(\text{diameter})^2}{2^2}=3.1416\times\frac{(\text{diameter})^2}{4}=.7854\times(\text{diameter})^2$. Or, the area of a circle is equal to the product obtained by multiplying .7854 by the square of the diameter. Conversely, if the area of a circle is known the diameter may be found by applying the following rule:

Rule.—*To find the diameter of a circle whose area is known, divide the area by .7854 and extract the square root of the quotient.*

EXAMPLE 1.—The area of a circle is 38.4846 square inches. What is its diameter?

SOLUTION.—According to the rule, the value of the area, or 38.4846, is divided by .7854 and the square root of the quotient is extracted. Or,

$$\sqrt{\frac{38.4846}{.7854}}=\sqrt{49}=7$$

Therefore, the diameter is 7 in. Ans.

EXAMPLE 2.—What is the diameter of a circle whose area is 1.7671 square feet?

SOLUTION.—According to the rule, the diameter is

$$\sqrt{\frac{1.7671}{.7854}}=\sqrt{2.25}=1.5$$

The diameter of the circle is 1.5 ft., or $1\frac{1}{2}$ ft. Ans.

EXAMPLES FOR PRACTICE

1. What must be the length of a ladder to reach to the top of a wall 40 feet high, if the foot of the ladder is 9 feet from the wall?

Ans. 41 ft.

2. 5,776 barrels standing on end are to be arranged in the form of a square. How many barrels will there be on each side of this square? Ans. 76 bbl.

3. Two vessels sail from the same port. At a given time one vessel is at a distance of 360 miles due north from the port; the other is at a distance of 450 miles due east from the port. What is the distance between the two vessels? Ans. 576.28 mi.

4. The floor of a room is 24 feet long and 18 feet wide; what is the length of the diagonal? Ans. 30 ft.

5. A light rope, drawn from the top of a flag pole to a point at a distance of 43 feet from the center of the pole support and at the same level as its base, is 110 feet long; what is the height of the pole? Ans. 101.2 ft.

6. The opening of a safety valve on a steam boiler has an area of 12.566 square inches. What is the diameter of the opening? Ans. 4 in.

7. The base of a right triangle, as in Fig. 4, is 20 feet, and the altitude is 18 feet; what is the length of the hypotenuse? Ans. 26.9 ft.

8. The square nut for a bolt is 3⅛ inches on each side. What is the width across corners? Ans. 4.42 in.

9. A circular plate of brass has an area of 100 square inches. What is its diameter? Ans. 11.28 in.

MENSURATION

LINES, SURFACES, AND SOLIDS

LINES AND ANGLES

LINES

1. Geometry.—That branch of mathematics which deals with the relations, properties, and measurements of lines, angles, surfaces, and solids is known as **geometry.** The particular branch of geometry that treats only of the measurement of lines, angles, surfaces, and solids is known as **mensuration.**

In laying out work the mechanic will frequently make use of lines and angles in constructing various surfaces. The general property of these elements, as well as some methods of constructing them, will be explained in this Section, together with the calculation of surface areas and the contents of solids.

2. Points.—In mathematics a **point** is supposed to be without any dimensions; that is, it has no length, breadth, or thickness. In accurate work the position of a point is indicated by two lines crossing each other at the place where the point is intended to be located.

3. Lines.—From a mathematical point of view a **line** has only one dimension—*length;* it serves to connect two points. On paper a line is represented by means of a mark, that may vary in thickness, depending on the means used to make it.

A **straight line,** Fig. 1 (*a*), is one that does not change direction throughout its whole length. A straight line is frequently called a *right line.* Lines shown in illustrations are

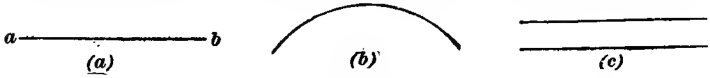


FIG. 1

usually referred to by letters at their ends, as *a b,* Fig. 1 (*a*).

A **curved line,** Fig. 1 (*b*), changes its direction at every point, no part of the line being straight.

4. Lines Named According to Relative Positions. When two straight lines are equally distant from each other throughout their length, as in Fig. 1 (*c*), they are said to be **parallel lines.** Either one of two parallel lines is said to be *parallel to* the other.

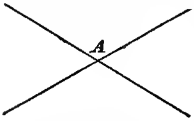


FIG. 2

5. When two lines cross or cut each other, as in Fig. 2, they are said to be **intersecting lines,** and the point *A,* at which they intersect, or cut each other, is called the **point of intersection.**

6. A line is said to be **perpendicular** to another line, when it meets that line so as not to incline toward it on either side. Thus, in Fig. 3 (*a*), the line *A B* is perpendicular to the line *C D*; also, *C D* is perpendicular to *A B.* Either line is also said to be **at right angles** to the other.

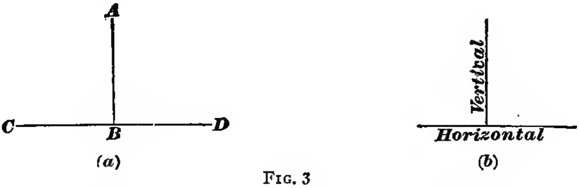


FIG. 3

7. A line that is parallel to the horizon, or water level, as *C D* in Fig. 3 (*a*) or the corresponding line in Fig. 3 (*b*), is known as a **horizontal line.**

8. A line that is perpendicular to a horizontal line, as in Fig. 3 (*b*), is called a **vertical line**; it has the direction of a *plumb-line*.

9. Drawing Perpendicular Lines.—In laying out some classes of work, it is necessary to draw one line perpendicular, or at right angles, to another; therefore, methods of doing this will be explained. In the example, Fig. 4, a line *A B* is drawn, and it is desired to draw a line perpendicular to it at the point *P*. To do this, the legs of a pair of compasses are separated to any convenient distance, the pointed leg is set at the point *P* while with the other leg, which carries a pencil, two marks are made at *C* and *D* on the line *A B*. These marks will be at equal distances from the point *P* and on opposite sides of it. The legs of the compasses are now brought farther apart, so that they will span a distance greater than *P C*, the pointed leg is set at *C*, and a short arc (or part of a circle) is struck with the pencil above *P*, as at *F*. The leg situated at *C* is now removed to *D*, with the legs of the compasses the same distance apart as before, and another arc is struck at *F*. The points at which these arcs intersect is the point required, as a line drawn from it to *P* will be perpendicular to *A B*.

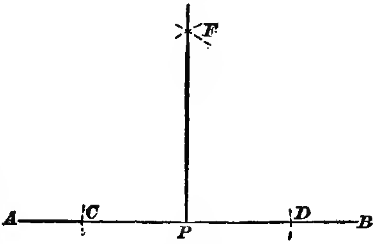


FIG. 4

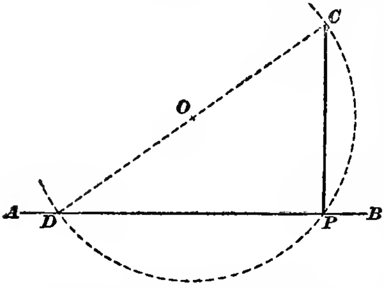


FIG. 5

10. If the point *P*, Fig. 4, at which the perpendicular is to be drawn, lies near the end of the line, as, for instance, at the edge of a piece of work, the foregoing method cannot be used. In such cases, the construction shown in Fig. 5 may be applied, where *A B* is the line and *P* the point at which the perpendicular is to be drawn. The legs of the compasses are separated to some convenient distance, one leg is placed at a point above *A B,* as at *O,* and with the other the greater part of a circle is drawn so as to pass through the point *P,* as shown by the dotted curve. This circular arc will also cross the line *A B* at *D.* From the latter point a line is drawn through the center *O* and continued till it intersects the arc at *C.* A line drawn from the point *C* to the point *P* will then be perpendicular to *A B.*

11. Sometime a perpendicular has to be drawn to a line from a given point outside the line. The method to be employed in this case is shown in Fig. 6, in which *A B* is the line and *P* the given point. One leg of the compasses is placed at *P* and with the other leg an arc is described, so that it will intersect the line *A B* in any two points, *C* and *D,* as shown by dotted curve. One leg of the compasses is placed successively at the points *C* and *D,* and short arcs are described with the other leg at *E.* From their point of intersection, *E,* a line is drawn to *P*; then *E P* is perpendicular to *A B.*

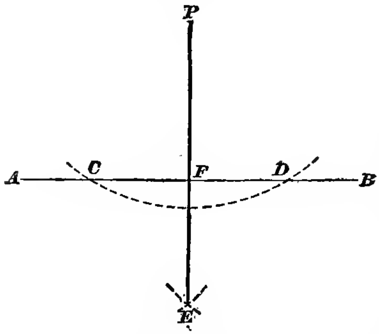


FIG. 6

A perpendicular, drawn from a point outside a straight line, is the **shortest distance** from the point to the line. Thus, in Fig. 6, the perpendicular *P F* is the shortest distance from *P* to *A B.*

ANGLES

12. Definition of an Angle.—When two lines, as *a b* and *b c,* Fig. 7, meet or intersect at a point *b,* the opening between the lines is known as an **angle.** The lines are known as the **sides** and the point *b* as the **vertex** of the angle.

An angle may also be defined as the difference in direction between two straight lines. This may be made clearer by means of a practical illustration. If the lines *a b* and *b c,* Fig. 7, are supposed to represent the legs of a pair of compasses, the arc *d* will represent the path through which a point on the leg *a b* will have to move in order that this point may coincide with another point on the leg *b c,* at the same distance from *b.* The length of the arc *d,* in degrees, etc., will, therefore, serve to indicate the size of the angle.

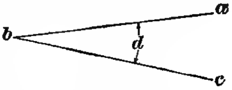
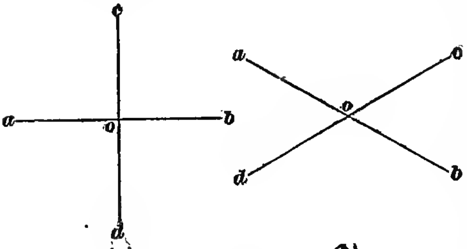


FIG. 7

In referring to an angle, three letters are commonly required, one letter at a point on each leg and one at the vertex. In naming the angle, the letter at the vertex is always placed between the other two. Thus, the angle, Fig. 7, is referred to as the angle *a b c.* If the angle stands by itself, as in this case, it may be referred to simply by the letter at the vertex, as angle *b.* Angles are sometimes referred to by means of a short arc and a letter, as in Fig. 7, where *d* indicates the angle.

13. Classification of Angles.—When two lines cross, as in Fig. 8 (*a*) and (*b*), they form four angles *a o c, c o b, b o d,* and *d o a.* Angles that have the same vertex, and one side in common, are **adjacent angles,** as *a o c* and *c o b,* in which *o* is the vertex of each and the side *o c* is common to both angles.



(*a*) (*b*)

FIG. 8

When two lines intersect, the angles formed on opposite sides of the vertex are **opposite angles.** For instance, the angles made by a pair of scissors blades and by the handles, respectively, are opposite angles. In Fig. 8, *a o c* and *b o d* are opposite angles; so are *a o d* and *b o c.*

14. When one straight line meets another so that the adjacent angles formed are equal, as *a o c* and *c o b,* Fig. 8 (*a*), the angles are called **right angles,** and the lines are said to be **at right angles,** or **normal,** or perpendicular, to each other. When a line is perpendicular to another line, as in Fig. 8 (*a*), the adjacent angles must be right angles.

The sum of all the angles that are on one side of a straight line and have a common vertex is always equal to two right angles. In Fig. 8 (*a*) and (*b*) the sum of the angles *a o c* and *c o b* is equal to two right angles. If additional straight lines are drawn through the vertex *o* the number of angles will increase, but the sum of the angles on the *same* side of the line *a b* will remain equal to two right angles.

If there are two right angles on one side of a straight line, as in the case of the line *a b,* Fig. 8 (*a*), there must also be two right angles on the other side. Hence, the sum of all angles about a common vertex is equal to four right angles.

15. An angle greater than a right angle, as the angle *a o c,* Fig. 8 (*b*), is an **obtuse angle,** and an angle less than a right angle, as the angle *c o b,* is an **acute angle.**

16. Complements and Supplements.—An angle is said to be the complement of another, when the sum of the two angles is *one* right angle. In Fig. 9, if *b o* is at right angles to *a d,* the angle *b o c* is the complement of the angle *c o d,* because the sum of the two angles is a right angle, *b o d.* Two angles that together make one right angle are said to be **complementary.** Thus, *b o c* and *c o d* are complementary angles.

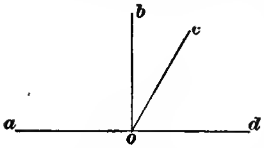


Fig. 9

When the sum of two angles is equal to *two* right angles, the angles are said to be **supplementary,** and each is the **supplement** of the other. In Fig. 9, the angle *c o d* is the supplement of the angle *c o a,* and *c o a* is the supplement of *c o d.* Also, the angles *a o b* and *b o d* are supplementary angles.

17. Angular Measure.—If the length of the arc *d,* Fig. 7, is to serve as a measure of the size of an angle, it is necessary to select a suitable unit. For this purpose the circumference of a circle is divided into 360 equal parts called **degrees,** and each degree is divided into 60 equal parts known as **minutes.** A minute is divided into 60 equal parts called **seconds,** which are used where greater accuracy is required than can be expressed in degrees and minutes.

If two lines are drawn at right angles to each other, as in Fig. 8 (*a*), the point of intersection, *o*, may be used as the center of a circle. This circle will then be divided by the intersecting lines into four equal parts, known as **quadrants,** each representing an angle of $\frac{360}{4}=90$ degrees, usually written 90°, as shown in Fig. 10. One-half of a right angle is $\frac{90}{2}=45^\circ$, one-fourth is 22½°, one-third is 30°, two-thirds is 60°, etc., as indicated.

If the radius of the circle, drawn with the vertex of the angle as a center, is increased in length, the circumference of the circle will also become greater, and, consequently, also the length of each division. But the angle, or amount of opening, included between two intersecting lines will not be affected. Thus, the angle marked 90° in Fig. 10 will include ¼ of 360°, as before.

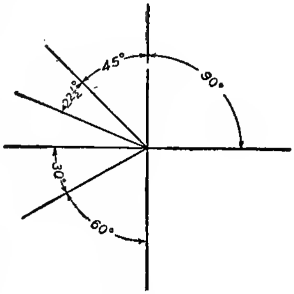


FIG. 10

This system of measuring angles is called **angular measure;** its units and abbreviations are given in the following table. The signs for minutes and seconds are ′ and ″, the same as for feet and inches; but as they usually occur in connection with the sign for degrees, there is little danger of their being misread for feet and inches. The expression 27° 13′ 45″ is read *27 degrees 13 minutes 45 seconds.*

Usually the mechanic will not have to deal with smaller divisions of a degree than minutes; that is, the values of the angles he may have to use in calculations will ordinarily be given in degrees and minutes, and not in degrees, minutes, and seconds.

ANGULAR MEASURE

60 seconds (″) ....=1 minute .... ′
60 minutes ....=1 degree .... °
360 degrees ....=1 circle ....cir.

18. Calculations in Angular Measure.—In some calculations it may be necessary to reduce degrees and minutes to degrees and a fraction of a degree; that is, to reduce a denominate number to a higher denomination, as explained in *Weights and Measures*. To do so, the number of minutes is written as the numerator of a fraction with 60 as the denominator, and added to the whole number of degrees. For example, 12° 45′ is equal to $12\frac{45}{60}$ degrees, or $12\frac{3}{4}$ degrees, which may be expressed decimally as 12.75°. If the expression contains both minutes and seconds, the minutes and seconds are reduced to seconds and divided by 3,600, the number of seconds in one degree, and the fraction is added to the whole number of degrees. For example, if 39° 16′ 15″ is to be reduced to degrees, the expression 16′ 15″, reduced to seconds, becomes $(16\times 60)+15=975$ seconds. Dividing this product by 3,600, the fraction is $\frac{975}{3600}=\frac{13}{48}=.271$. Therefore, 39° 16′ 15″ may be written $39\frac{13}{48}$ degrees, or 39.271°.

19. Angles may be added and subtracted in the same way as other compound numbers. For example, if the angles of 45° and $22\frac{1}{2}$°, Fig. 10, are to be added, their sum is $45°+22\frac{1}{2}° =67\frac{1}{2}°$.

The following examples show how to proceed in calculations dealing with angular measure.

EXAMPLE 1.—Find the sum of 12° 34′, 7° 48′, and 36° 11′.

SOLUTION.—The compound numbers are arranged as shown, with like units in the same column. The sum of the right-hand column is 93′, which is equal to 1° 33′. The number 33 is written under the minutes, the 1° is carried over and added to the degree column, giving a total of 56°. The sum is then 56° 33′.

$$\begin{array}{rr} 12° & 34' \\ 7 & 48 \\ 36 & 11 \\ \hline 56° & 33' \end{array} \quad \text{Ans.}$$

EXAMPLE 2.—Subtract 12° 25′ from 45° 40′.

SOLUTION.—The smaller value is placed under the larger and subtracted in the usual manner.

$$\begin{array}{rr} 45° & 40' \\ 12 & 25 \\ \hline 33° & 15' \end{array} \quad \text{Ans.}$$

EXAMPLE 3.—From 39° 4′ subtract 27° 56′.

SOLUTION.—As 56′ cannot be taken from 4′, 1°, or 60′, is taken from the 39°, leaving 38°, and added to the 4′, making 64′. The minuend will then assume the form as shown below, and the subtraction may now be carried out in the usual manner.

$$\begin{array}{rr} 39^\circ & 4' \\ \underline{27} & \underline{56} \end{array}$$

$$\begin{array}{rrl} 38^\circ & 64' & \\ \underline{27} & \underline{56} & \\ 11^\circ & 8' & \text{Ans.} \end{array}$$

The minuend is not usually rewritten as shown here, but the process of taking 1° from 39° and adding its equivalent, or 60′, to the minutes is done mentally.

20. Means for Measuring Angles.—Angles are commonly measured by the use of a **protractor**, a form of which is shown in Fig. 11. It consists of a piece of celluloid or of metal of a semicircular shape and very thin. Along the curved edge are a number of divisions, the smallest representing $\frac{1}{2}$°, or 30′, the next larger representing 1°, and the larger ones representing 5° and 10°, respectively. To use the protractor, it is laid flat on the angle to be measured, with the point *O* directly on the vertex of the angle and the line *A B* directly over one side of the angle. The point where the other side of the angle crosses the scale shows the size of the angle.

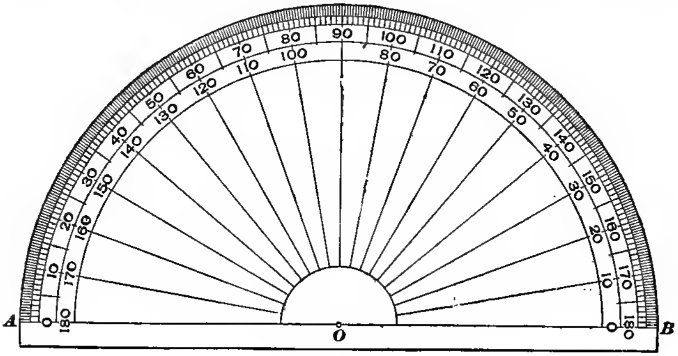


FIG. 11

21. Laying Off an Angle.—If a line is given and it is required to draw another line that inclines toward it at a given angle, the required line may be drawn in the following manner: Let *C F*, Fig. 12, be the given line and *C* the point from which a line is to be drawn at an angle of 54° with the line *C F*. The protractor, Fig. 11, is laid on the line *C F* so that its center *O* comes directly over the vertex *C* of the angle and so that the line *A B* coincides with the line *C F,* Fig. 12. Then 54° is counted off from the lower end *B* of the scale on the protractor. This means 5 large divisions, each of which represents 10°, and 4 small divisions, each of which represents 1°. Opposite the end of the 54° mark the point *D* is located with a sharp pencil or a *scriber,* a pointed steel tool used for *scribing,* or scratching, lines on metal surfaces. Then the protractor is removed and a straight line is drawn or scribed so as to pass through *C* and *D*. This line *C D* will make an angle of 54° with the line *C F*; that is, the angle *D C F* will be 54°.

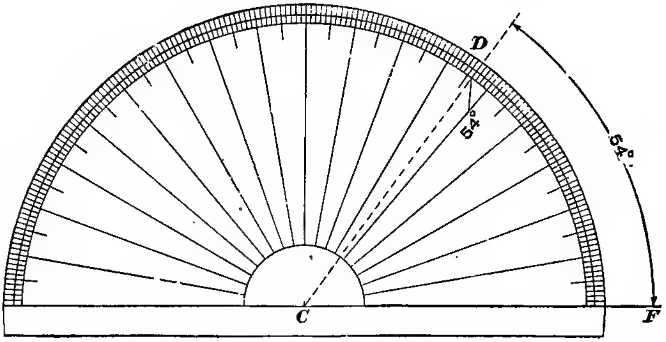


Fig. 12

EXAMPLES FOR PRACTICE

1. Find the sum of 26° 18′ and 18° 42′. Ans. 45°

2. What is the difference between 88° 28′ and 42° 12′? Ans. 46° 16′

3. What is the complement of 36° 32′? Ans. 53° 28′

4. What is the supplement of 87° 29′? Ans. 92° 31′

5. Find the difference between the complement of 18° 30′ and the supplement of the same angle. Ans. 90°

6. Subtract 82° 48′ from 112° 23′. Ans. 29° 35′

7. How many seconds are there in 32° 14′ 6″? Ans. 116,046″

8. How many degrees, minutes, and seconds do 38,582 seconds amount to? Ans. 10° 43′ 2″

9. In a pulley with five arms, what part of a right angle is included between the center lines of any two adjacent arms?

Ans. $\frac{4}{5}$ of a right angle

10. If a number of straight lines meet a given straight line at a given point, all being on the same side of the given line, so as to form six equal angles, how many degrees are there in each angle? Ans. 30°

SURFACES

DEFINITIONS AND CLASSIFICATION

22. Plane Surface.—A **plane surface,** usually called a **plane,** is a surface upon which straight lines may be drawn in any direction. A practical example of a plane is the surface of a *surface plate,* a steel plate ground perfectly flat on its top surface. If a straightedge is laid on its surface, every point along the edge of the straightedge will touch the surface, no matter in what direction it is laid.

23. Plane Figures.—Any part of a plane surface bounded by any number of straight or curved lines or a combination of the two is known as a **plane figure.**

24. Polygons.—When a plane figure is bounded by straight lines only, it is called a **polygon.** The bounding lines are called **sides,** and the combined length of the sides is known as the **perimeter** of the polygon. Polygons include figures with three, four, or more sides, but it is customary to divide polygons into three classes, calling those with three sides *triangles,* those with four sides *quadrilaterals,* and those with more than four sides *polygons.*

TRIANGLES

25. Classification.—A **triangle** is a polygon with three sides. Triangles are named according to the relative lengths of their sides as *isosceles, equilateral,* and *scalene triangles,* and according to the nature of the angles as *right-angled* and *oblique triangles.*

26. When two sides of a triangle are of equal lengths, as in Fig. 13, it is known as an **isosceles** (i-sos′se-leez) **triangle.** This word *isosceles* means *equal legs.*

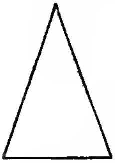


FIG. 13

27. When the three sides of a triangle are of equal lengths, as in Fig. 14, it is called an **equilateral triangle.**

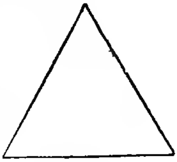


FIG. 14

28. A triangle in which all the sides are of different lengths, as Fig. 15, is called a **scalene triangle.**

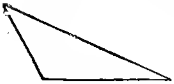


FIG. 15

29. If one of the angles in a triangle is a right angle, the triangle is a **right-angled triangle.** The side opposite the right angle is called the **hypotenuse.** In Fig. 16 the side *A B* is the hypotenuse because it is opposite the right angle *C*. For brevity, a right-angled triangle is generally termed a **right triangle.**

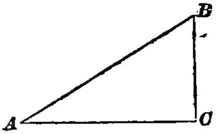


FIG. 16

30. Base and Altitude.—That side of a triangle on which it is supposed to stand is known as the **base;** any side may be considered as the base. In Figs. 16, 17, and 18, *A C* is shown as the base.

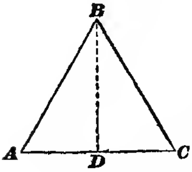


FIG. 17

The **altitude,** or **height,** of any triangle is represented by a line drawn from the vertex of the angle opposite the base perpendicular to the base or to an extension of the base. Thus, in Figs. 17 and 18, *B D* is the altitude of the triangle *A B C*. In Fig. 18 the perpendicular falls outside the triangle, and the base *A C* is shown extended to meet the line *B D*.

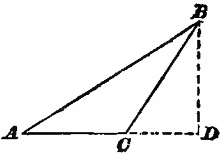


FIG. 18

31. Similar and Equal Triangles.—Two triangles are **similar** when the *angles* of one are equal to the angles of the other. If in the triangle *a b c,* Fig. 19, a line *d e* is drawn parallel to the side *b c,* the triangle *a d e* is similar to the triangle *a b c,* as their angles are equal, each to each.

Two triangles are **equal** when the *sides* of one are equal to the *sides* of the other. It is seen that triangles may be similar without being equal.

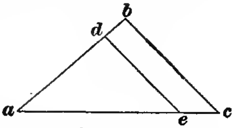


Fig. 19

QUADRILATERALS

32. Parallelograms.—A **parallelogram** is a quadrilateral with opposite sides parallel and opposite angles equal. There are four kinds of parallelograms: the *rectangle,* the *square,* the *rhomboid,* and the *rhombus.*

Fig. 20

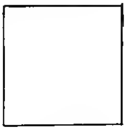


Fig. 21

33. A **rectangle,** Fig. 20, is a parallelogram with sides at right angles to one another. If the sides are equal, as in Fig. 21, the rectangle is known as a **square.**


Fig. 22

34. A **rhomboid,** Fig. 22, is a parallelogram in which there are no right angles and in which the length of one pair of opposite sides differs from that of the other pair.

A **rhombus,** Fig. 23, is a parallelogram without any right angles, and with sides of equal lengths.

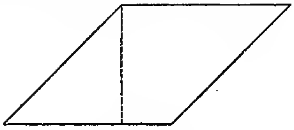


Fig. 23

35. Trapezoid and Trapezium.—A **trapezoid,** Fig. 24, is a quadrilateral which has only two of its sides parallel.

A **trapezium,** Fig. 25, is a quadrilateral without any parallel sides.


Fig. 24

36. Altitudes and Diagonals of Quadrilaterals. Any side of a quadrilateral may be considered as its **base,** but in a trapezoid one of the two parallel sides is usually considered as its base. The **altitude** of a parallelogram or of a trapezoid is the perpendicular distance between the base and the opposite side, as indicated by the dotted line in Figs. 22, 23, and 24.

37. A **diagonal** is a straight line joining two opposite corners of a quadrilateral, as indicated by the dotted line in Fig. 20. Each quadrilateral has two diagonals, and either diagonal divides the quadrilateral into two triangles. In a parallelogram, these triangles are equal; for example, either diagonal of the rectangle, Fig. 20, divides it into two equal triangles. It is important to remember that the two diagonals of any parallelogram **bisect** each other; that is, they divide each other into two equal parts.

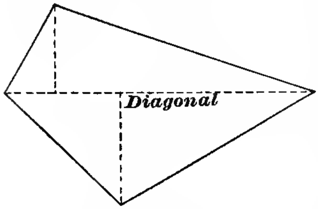


FIG. 25

REGULAR AND IRREGULAR POLYGONS

38. Regular Polygons. — Excluding, according to Art. **24,** any polygons with less than five sides, a **regular polygon** may be defined as one with five or more sides of equal lengths and with equal angles. Fig. 26 shows some common regular polygons with more than four sides. At (*a*) is shown the *pentagon,* or five-sided figure; at (*b*) the *hexagon,* or six-sided figure; at (*c*) the *heptagon,* or seven-sided figure; and at (*d*) the *octagon,* or eight-sided figure.

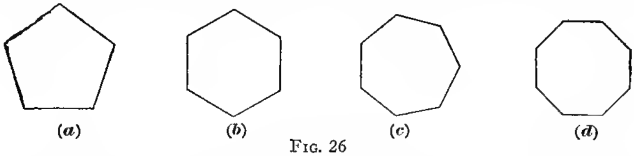

(*a*) (*b*) (*c*) (*d*)

FIG. 26

It is true that equilateral triangles and quadrilaterals are regular polygons of three and four sides, respectively, but they are not commonly called polygons.

39. Constructing Regular Polygons.—If perpendiculars are erected to all the sides of a regular polygon at their middle points, these perpendiculars will meet in a common point, which is called the **center** of the polygon. In the hexagon, Fig. 27, the perpendiculars *P O, Q O,* etc. meet at the center *O* and are of equal lengths. Lines drawn from the vertexes to the center *O,* as *B O* and *C O,* are also equal. It follows that a circle with *O* as center and *O A* as radius will pass through each of the vertexes *A, B, C, D, E,* and *F*; and a circle with *O* as center and *O P* as radius will pass through each of the points *P, Q, R, S, T,* and *U.* A polygon is said to be **inscribed** in a circle when the vertexes of the polygon lie on the circumference of the circle. A polygon is **circumscribed** about a circle when the circumference of the circle touches the center of each side, as at points *P, Q, R,* etc., Fig. 27.

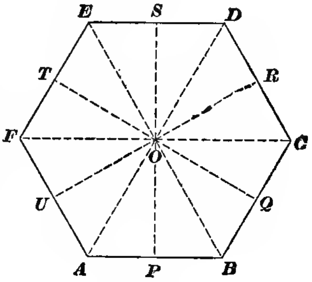


FIG. 27

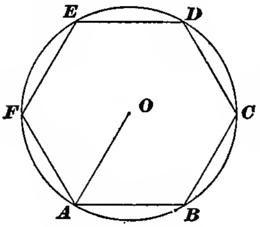


FIG. 28

40. A regular polygon may, therefore, be constructed by inscribing it in a circle. For example, to construct a hexagon, a circle is described from a center *O,* Fig. 28, with a radius *A O* corresponding to the line *A O,* Fig. 27, and divided into six equal parts. The points of division are connected by the straight lines *A B, B C,* etc., thus obtaining the hexagon *A B C D E F.* In a hexagon the sides are equal to the radius of the surrounding circle; hence the length *A O,* Fig. 28, is used for setting out the points of division on the circle.

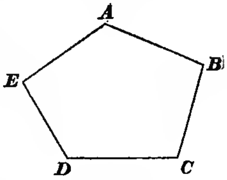


FIG. 29

41. Irregular Polygon.—A polygon in which neither the sides nor the angles are equal, as Fig. 29, is known as an **irregular polygon.**

CIRCLE

42. Arc, Chord, and Radius.—A circle, Fig. 30, is a plane figure bounded by a curved line, called the **circumference,** every point of which is equally distant from a point within, called the **center.**

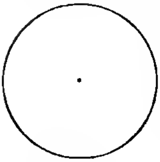


Fig. 30

43. Any part of the circumference, as *a e b,* Fig. 31, is called an **arc.**

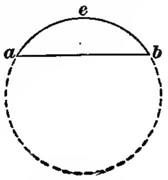


Fig. 31

44. A straight line joining any two points in the circumference is a **chord.** Or, a chord is a straight line joining the extremities of an arc. Thus, in Fig. 31, *a b* is the chord of the arc *a e b.* Every chord divides the circumference into two arcs, and the chord is said to *subtend,* or to be opposite, either arc. If a chord passes through the center, as *A B,* Fig. 32, it is called a **diameter.**

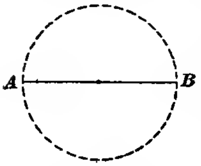


Fig. 32

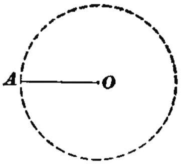


Fig. 33

45. A straight line joining the center with any point in the circumference, as *O A,* Fig. 33, is a **radius** (plural, *radii*) ; a radius is one-half of a diameter.

46. Segment and Semicircle.—A **segment** of a circle is that portion of its area that is included between an arc and its chord. Each chord divides a circle into two segments; thus, in Fig. 31, one segment is between the chord *a b* and the arc *e* in full lines. The other segment is between the chord *a b* and the arc shown in dotted lines.

If the chord of a segment is a diameter, each segment is a **semicircle** and each arc a **semi-circumference.**

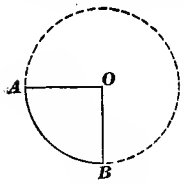


Fig. 34

47. Sector and Quadrant.—A **sector** of a circle is that portion included between an arc and two radii drawn to the ends of the arc. Thus, in Fig. 34, the portion included between the arc $A B$ and the radii $O A$ and $O B$ is a sector of the circle.

48. If two diameters are drawn at right angles to each other, as $A B$, $C D$ in Fig. 35, the circumference, as well as the area, of the circle is divided into four equal parts. Each of the four equal arcs and sectors is called a **quadrant.** Thus, the arcs $A D$, $D B$, $B C$, and $C A$, as well as the sectors $A O D$, $D O B$, $B O C$, and $C O A$, are all known as quadrants.

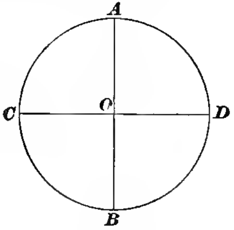


FIG. 35

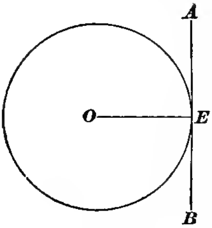


FIG. 36

49. Tangents.—A straight line that meets a circle at one point only is a **tangent** to the circle; thus, in Fig. 36, $A B$ is a tangent touching the circle at the point E. The latter point is called the **point of contact,** or the **point of tangency.**

A tangent to a circle is at right angles to the radius drawn to the point of contact. Thus, if O is the center of the circle, the tangent $A B$ is at right angles to the radius $O E$.

50. Tangent Circles. One circle is said to be **tangent** to another circle if they touch each other at one point only. This point is called the **point of tangency,** or the **point of contact.** In Fig. 37 the points of contact of the circle A and the circles B and C are at a and b, respectively. A straight line passing through the centers of the tangent circles also passes through the point of contact; thus, the point a, at which the circles A and B are tangent, lies on an extension of the line connecting the centers c and c_1. Similarly,

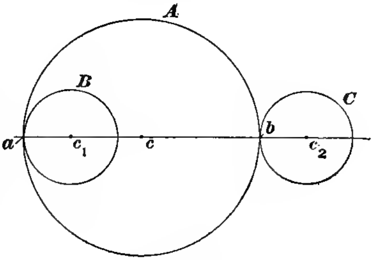


FIG. 37

the point *b*, at which the circles *A* and *C* are tangent, lies on the line $c\ c_2$.

51. Concentric and Eccentric Circles.—When two or more circles are described from the same center, as in Fig. 38, they are called **concentric circles.**

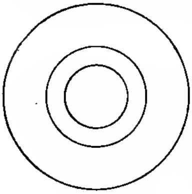


FIG. 38

A circle described inside another circle, but from a different center, is said to be **eccentric** to the larger circle, and the two circles are termed **eccentric circles.** In Fig. 39 the circle *B* is eccentric to the circle *A*, as the centers *b* and *a* are not at the same point.

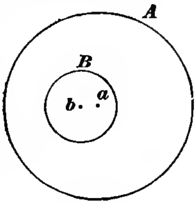


FIG. 39

52. Dividing Circumference of Circle Into Equal Parts.—The usual way of dividing the circumference of a circle into a number of equal parts is to step a pair of dividers around the circle, adjusting the distance between its points until a distance is found that is used the required number of times in going around the circumference. For instance, if the circle, Fig. 40, is to be divided into five equal parts, the dividers are adjusted until they will step off five equal chords *A B*, *B C*, *C D*, *D E*, and *E A*.

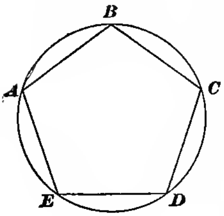


FIG. 40

53. To save the time required to find the exact length of the parts into which a circle is to be divided, Table I is provided, which gives the length of the chord, per inch of diameter, for all divisions from 3 to 100, inclusive. The length of the chord, or the distance between the points of the dividers, is found by multiplying the diameter of the circle by the multiplier corresponding to the number of divisions desired.

The application of the table is shown by means of the following examples:

EXAMPLE 1.—A circle 14 inches in diameter is to have 24 holes spaced equally around it. In setting out the centers of these holes by dividers, what must be the distance between the points of the dividers?

Solution.—According to Table I, the multiplier corresponding to 24 divisions is .13053; but as this applies to a circle 1 in. in diameter, it must be multiplied by 14 for a circle of 14 in. diameter. Therefore, the dividers must be set a distance apart equal to

$$14\times.13053=1.82742, \text{ or } 1.83 \text{ in., nearly.} \quad \text{Ans.}$$

TABLE I

LENGTHS OF CHORDS

Number of Divisions	Multiplier	Number of Divisions	Multiplier	Number of Divisions	Multiplier	Number of Divisions	Multiplier
3	.86603	28	.11197	53	.05924	77	.04079
4	.70711	29	.10812	54	.05815	78	.04027
5	.58779	30	.10453	55	.05709	79	.03976
6	.50000	31	.10117	56	.05607	80	.03926
7	.43388	32	.09802	57	.05509	81	.03878
8	.38268	33	.09506	58	.05414	82	.03830
9	.34202	34	.09227	59	.05322	83	.03784
10	.30902	35	.08964	60	.05234	84	.03739
11	.28173	36	.08716	61	.05148	85	.03695
12	.25882	37	.08480	62	.05065	86	.03652
13	.23932	38	.08258	63	.04985	87	.03610
14	.22252	39	.08047	64	.04907	88	.03569
15	.20791	40	.07846	65	.04831	89	.03529
16	.19509	41	.07655	66	.04758	90	.03490
17	.18375	42	.07473	67	.04687	91	.03452
18	.17365	43	.07300	68	.04618	92	.03414
19	.16460	44	.07134	69	.04552	93	.03377
20	.15643	45	.06976	70	.04487	94	.03342
21	.14904	46	.06824	71	.04423	95	.03306
22	.14232	47	.06679	72	.04362	96	.03272
23	.13617	48	.06540	73	.04302	97	.03238
24	.13053	49	.06407	74	.04244	98	.03205
25	.12533	50	.06279	75	.04188	99	.03173
26	.12054	51	.06156	76	.04133	100	.03141
27	.11609	52	.06038				

Example 2.—A pulley 26 inches in diameter has 5 spokes. What must be the distance between the ends of their center lines, situated along the pulley rim?

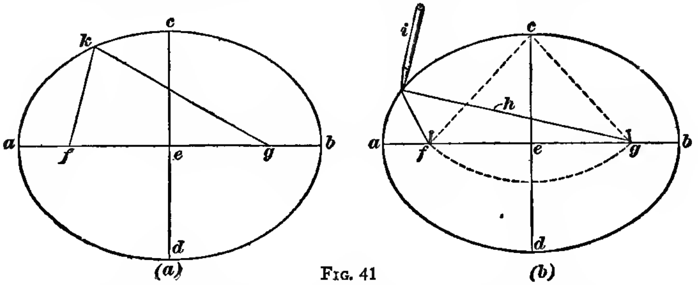
Solution.—According to the table, the multiplier is .58779. Therefore, the dividers must be set to a distance of

$$26 \times .58779 = 15.28 \text{ in. Ans.}$$

ELLIPSE

54. Axes and Foci of Ellipse.—An **ellipse** is a plane figure of oblong outline, as shown in Fig. 41 (*a*). The line *a b* is called the **major axis** or **long diameter** of the ellipse and the line *c d* is the **minor axis** or **short diameter.**

55. Constructing an Ellipse.—The method of constructing an ellipse may be explained by reference to Fig. 41 (*b*). It is assumed that the lengths of the long and short diameters *a b* and *c d* are given. The construction is begun by drawing these diameters at right angles to each other, so as to intersect at the middle point *e* of each diameter. Then, with *c* as a center and a radius equal to *a e*, an arc is described which cuts the major axis at *f* and *g*. Each of these points *f* and *g* is known as a **focus** (plural, *foci*). As will be seen by the method of construction, the distance from each focus to either extremity of the minor axis is equal to one-half of the major axis.



(*a*) Fig. 41 (*b*)

Pins are inserted at the foci, as shown; a pencil *i* is placed at one end of the major axis, the point *a* in this case, and a cord is looped around the pencil point and fastened tautly to the pins *f* and *g* at the foci. The pencil is now moved from the point *a* toward the position *c* and beyond to *b*, keeping the cord stretched taut, thus tracing one-half of an ellipse that will pass through the points *a*, *c*, and *b*. Trace the other half similarly.

It will be noted that the length of cord between the foci equals the length of the major axis, because one position of the pencil is at *c*, Fig. 41 (*b*), and $f\,c+c\,g=a\,b$. Therefore, the sum of the distances from any point in an ellipse to the foci equals the length of the major axis; for example, in Fig. 41 (*a*), $f\,k+k\,g=a\,b$.

MEASUREMENT OF LINES AND AREAS

LENGTHS OF LINES

56. Introduction.—As stated in Art. **1**, calculations dealing with lengths of lines, areas of surfaces, and volumes of solids belong to that branch of mathematics called **mensuration.** The lines whose lengths will be considered in this Section are those that constitute the outlines of triangles, circles, portions of circles, and ellipses.

57. Perimeter of Triangles.—If the lengths of two sides of a right triangle are given it is always possible to find the length of the third side and, thus, the perimeter, or the combined length of the three sides. The following rules, which were given in a preceding Section, apply:

Rule I.—*To find the hypotenuse of a right triangle, extract the square root of the sum of the squares of the other two sides.*

Rule II.—*To find one of the sides about the right angle, subtract from the square of the hypotenuse the square of the other side, and extract the square root of the remainder.*

Example 1.—Find the perimeter of a right triangle, if the sides about the right angle are 9.5 and 7 feet, respectively.

Solution.—By rule I, the hypotenuse is equal to

$$\sqrt{9.5^2+7^2}=\sqrt{90.25+49}=11.8 \text{ ft.}$$

The perimeter is:

$$9.5+7+11.8=28.3 \text{ ft.} \quad \text{Ans.}$$

Example 2.—A piece of land *a b c d*, Fig. 42, has the form of a trapezium, and has right angles at *b* and *d*. A fence is to be built around the land and it is required to know which pair of sides is the longer, $ab+bc$ or $cd+da$. The diagonal *a c* is 65 feet, *b c* is 51 feet, and *a d* is 28.5 feet.

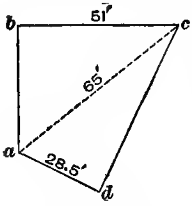


Fig. 42

Solution.—By rule II, $ab=\sqrt{65^2-51^2}=40.3$ ft. and $cd=\sqrt{65^2-28.5^2}=58.4$ ft. It follows that $a\,b+b\,c=40.3+51=91.3$ ft. and that $a\,d+d\,c=28.5+58.4=86.9$ ft. Therefore, there is a difference of $91.3-86.9=4.4$ ft. between the two pairs of sides, the pair $a\,b+b\,c$ being the longer of the two. Ans.

58. Circumference, Diameter, and Radius of Circles.—To find the circumference, the diameter, or the radius of a circle, the following rules are used:

Rule I.—*The circumference of a circle equals the diameter multiplied by 3.1416.*

Rule II.—*The diameter of a circle equals the circumference divided by 3.1416.*

Rule III.—*The radius of a circle equals the circumference divided by 2×3.1416.*

Example 1.—What is the circumference of a circle whose diameter is 15 inches?

Solution.—By rule I, the circumference is $15\times3.1416=47.12$ in. Ans.

Example 2.—(*a*) What is the diameter of a circle whose circumference is 65.9736 inches? (*b*) What is the radius?

Solution.—(*a*) By rule II, the diameter is

$$\frac{65.9736}{3.1416}=21 \text{ in. Ans.}$$

(*b*) By rule III, the radius is

$$\frac{65.9736}{2\times3.1416}=10\tfrac{1}{2} \text{ in. Ans.}$$

Or, the radius is equal to the diameter divided by 2. Thus, $\frac{21}{2}=10\frac{1}{2}$ in.

59. Ratio of Circumference to Diameter.—The number 3.1416 used in the preceding rules is the ratio of the circumference of a circle to its diameter; it is represented very often by the Greek letter π (pronounced "pi"). It is an unending decimal and its value has been calculated to over 700 decimal places, but the value here given is the one most generally used, four decimal places being sufficient for all practical purposes. The values $\frac{1}{4}\pi$, or .7854, and $\frac{1}{6}\pi$, or .5236, are frequently used. In case the calculation does not need to be very accurate, the value of π may be taken as $\frac{22}{7}$, which is equal to 3.1429, a fairly close approximation.

60. Length of Arc of Circle.—The length of an arc of a circle may be found by the following rule:

Rule.—*The length of an arc of a circle equals the circumference of the circle of which the arc is a part, multiplied by the number of degrees in the arc, and the product divided by 360.*

Example.—What is the length of an arc of 24°, the radius of the circle being 18 inches?

Solution.—The diameter of the circle is $2\times18=36$ in., and the circumference is, according to rule I, Art. **58**, 36×3.1416. By the preceding rule, the length of the arc is

$$\frac{36\times3.1416\times24}{360}=7.54 \text{ in. Ans.}$$

61. Approximate Method for Finding Length of Arc.—The following method, if applied to arcs less than 120°, is correct to three figures. It may be used for arcs up to 180°, but when the arc exceeds 120° the results are less accurate.

If the length of the arc ACB, Fig. 43, is to be found, the middle point D of the chord AB is located and DC is drawn at right angles to AB. On drawing AC, this line is the chord of half the arc. After the lengths of these chords are measured, the length of the arc is found by the following rule:

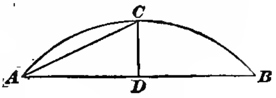


Fig. 43

Rule.—*From eight times the chord of half the arc, subtract the chord of the whole arc, and divide the remainder by 3. The quotient is the length of the arc, approximately.*

Referring to Fig. 43, this rule may also be expressed in the following form:

$$\text{Length of arc } AB=\frac{8\times AC-AB}{3}$$

EXAMPLE.—Find the length of the arc *A C B*, Fig. 44, if the length of the chord *A B* is 72 inches and *D C* is 8 inches.

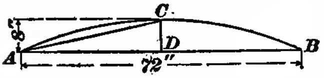


FIG. 44

SOLUTION.—Before the rule can be applied it is necessary to find the length of the chord *A C* of half the arc. It is known that $AD = \frac{1}{2}$ of $AB = \frac{1}{2} \times 72 = 36$ in. From the rules for right triangles, given in Art. **57**, it is known that the hypotenuse $AC = \sqrt{\overline{AD}^2 + \overline{DC}^2} = \sqrt{36^2 + 8^2} = 36.88$ in.

Hence, by the rule,

$$\text{length of arc } AB = \frac{8 \times AC - AB}{3} = \frac{8 \times 36.88 - 72}{3} = 74.35 \text{ in. Ans.}$$

62. Graphic Method for Finding Length of Arc. If the circular arc does not exceed one-sixth of the circumference, its length may be found accurately enough for practical purposes by means of the construction shown in Fig. 45.

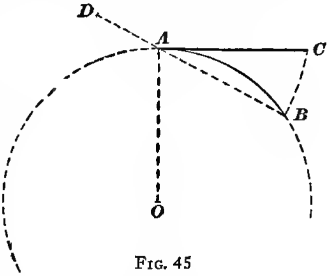


FIG. 45

For example, it is required to find the length of the arc *A B* of a circle described with the radius *O A* from the center *O*. At *A* a line *A C* is drawn at right angles to the radius *O A*; then, the chord *A B* is drawn and extended beyond the point *A* to the point *D*, making *A D* half the length of the chord *A B*. With *D* as a center, and with a radius equal to *D B*, the arc *B C* is described, cutting *A C* at *C*. The length of *A C* will then be very nearly equal to the length of the arc *A B*.

63. Circumference of Ellipse.—The circumference, or length of the curved outline, of an ellipse may be found, approximately, by means of the following rule:

Rule.—*The circumference of an ellipse is approximately equal to the product obtained by multiplying 3.1416 by the square root of half the sum of the squares of the two diameters.*

Or, referring to Fig. 41 (*a*), the rule is stated as follows:

$$\text{Circumference of ellipse} = 3.1416\sqrt{\frac{\overline{ab}^2 + \overline{cd}^2}{2}}$$

Example.—Find the circumference of an ellipse whose diameters are 14 inches and 10 inches, respectively.

Solution.—When the known values are substituted in the formal statement of the rule, the circumference is equal to $3.1416\sqrt{\frac{14^2+10^2}{2}}$ $=3.1416\sqrt{\frac{196+100}{2}}=3.1416\times 12.17=38.233$ in. Therefore, correct to three figures, the circumference is 38.2 in. Ans.

AREAS OF PLANE SURFACES

TRIANGLES, QUADRILATERALS, POLYGONS, CIRCLES, AND ELLIPSES

64. Area of Triangle.—The area of a triangle may be found by the following rule, if the length of the base and the altitude are known:

Rule.—*The area of any triangle equals one-half the product of the base and the altitude.*

If the triangle is a right triangle, one of the short sides may be taken as the base, and the other short side as the altitude; hence, *the area of a right triangle is equal to one-half the product of the two short sides.*

Example.—What is the area of a triangle whose base is 18 inches and whose altitude is 7¾ inches?

Solution.—Applying the rule, the area is

$$\tfrac{1}{2}\times(18\times 7\tfrac{3}{4})=69\tfrac{3}{4} \text{ sq. in.} \quad \text{Ans.}$$

65. The area of any triangle may be found, when the length of each side is known, by means of the following rule:

Rule.—*From half the sum of the three sides, subtract each side separately; find the product of these remainders and of half the sum of the sides. The square root of the product is equal to the area of the triangle.*

If the various values in this rule are expressed by means of symbols it may be given in a more condensed form. First, the sum of the three sides a, b, and c, Fig. 46, is found and divided by 2, as $\frac{a+b+c}{2}$. This quotient is indicated by the letter s; that is, $s=\frac{a+b+c}{2}$. The difference is then found between s and each of the sides, as $s-a$, $s-b$, and $s-c$. These differences are multiplied together and by s, and the square root of the product is extracted.

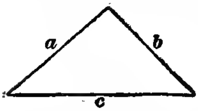


Fig. 46

When these successive calculations are expressed collectively, the whole will appear as follows:

$$\text{Area of triangle} = \sqrt{s\,(s-a)\,(s-b)\,(s-c)}$$

Example.—What is the area of the triangle, Fig. 46, if the sides, a, b, and c, are 18, 20, and 28 feet, respectively?

Solution.—First, it is necessary to find the value of s, or $\frac{a+b+c}{2}$ $=\frac{18+20+28}{2}=33$. Then, the differences between s and each of the sides are found, as,

$$s-a=33-18=15$$
$$s-b=33-20=13$$
$$s-c=33-28=\ 5$$

Then,

$\text{area}=\sqrt{s(s-a)(s-b)(s-c)}=\sqrt{33\times15\times13\times5}=179.37$ sq. ft. Ans.

66. Area of Quadrilateral.—To find the area of a parallelogram the following rule may be used:

Rule.—*The area of any parallelogram equals the product of the base and the altitude.*

Example.—What is the area of a parallelogram whose base is 12 inches and whose altitude is 7½ inches?

Solution.—By the rule, the area is 12×7½=90 sq. in. Ans.

If the area and one dimension are given, the other dimension may be found by dividing the area by the known dimension. If the parallelogram is a square, and its area is given, the length of a side is found by extracting the square root of the area.

67. To find the area of a trapezoid the following rule may be used:

Rule.—*The area of a trapezoid equals one-half the sum of the parallel sides multiplied by the altitude.*

Example.—What is the area of a trapezoid whose parallel sides are 9 inches and 15 inches, and whose altitude is 6 inches?

Solution.—The sum of the parallel sides is $9+15=24$ in., and half of this is 12 in. Then,

$$12\times 6=72 \text{ sq. in.} \quad \text{Ans.}$$

68. Area of Regular Polygon.—The area of a regular polygon may be found by the following rule:

Rule.—*To find the area of a regular polygon, square the length of a side and multiply this by the proper multiplier taken from Table II.*

Example.—A regular octagon has sides 8 inches long. What is the area?

Solution.—The multiplier for an octagon, taken from Table II, is 4.8284. By the rule, the area is

$$8^2\times 4.8284=64\times 4.8284=309.02 \text{ sq. in.} \quad \text{Ans.}$$

TABLE II

MULTIPLIERS FOR FINDING AREAS OF REGULAR POLYGONS

Name	Number of Sides	Multiplier	Name	Number of Sides	Multiplier
Equilateral triangle....	3	0.4330	Octagon...	8	4.8284
Square	4	1.0000	Nonagon ..	9	6.1818
Pentagon....	5	1.7205	Decagon...	10	7.6942
Hexagon.....	6	2.5981	Undecagon	11	9.3656
Heptagon....	7	3.6339	Dodecagon	12	11.1960

69. Area of Irregular Polygon.—The foregoing rules apply for finding areas of such figures as triangles, squares, rectangles, and regular polygons. If the figure or surface is of irregular shape, it can generally be divided by means of diagonals into simpler forms, such as squares, triangles, etc. The areas of these parts may then be calculated by the rules given, and the sum of the various areas will be the total area of the figure.

EXAMPLE.—It is required to find the area of the irregular polygon *A B C D E F*, Fig. 47, the dimensions being as indicated and the angles *F C D* and *C D E* being right angles.

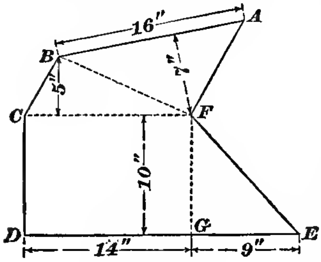


FIG. 47

SOLUTION.—The diagonals *B F* and *C F* are drawn and also the line *F G* perpendicular to *D E*, thus dividing the polygon into the triangles *A B F*, *B C F*, and *F G E*, and the rectangle *F C D G*. With the dimensions indicated in Fig. 47 on the lines representing altitudes, namely, 7, 5, and 10, and on the sides *A B*, *D G*, and *G E*, namely, 16, 14, and 9, the areas are as follows:

$$\text{area } ABF = \frac{16\times7}{2} = 56 \text{ sq. in.}$$

$$\text{area } BCF = \frac{14\times5}{2} = 35 \text{ sq. in.}$$

$$\text{area } FCDG = 14\times10 = 140 \text{ sq. in.}$$

$$\text{area } FGE = \frac{9\times10}{2} = 45 \text{ sq. in.}$$

Total area$=56+35+140+45=276$ sq. in. Ans.

70. Area of Circle.—The area of a circle may be found by the following rule:

Rule.—*The area of a circle is equal to .7854 times the square of the diameter, or 3.1416 times the square of the radius.*

EXAMPLE.—What is the area of a circle whose diameter is 15 inches?

SOLUTION.—By the rule, the area is

$.7854\times15^2=.7854\times225=176.72$ sq. in. Ans.

In practice it is found very convenient to use tables giving the diameters, circumferences, and areas of circles, as they make it unnecessary to perform the calculation just described, and thus save much time.

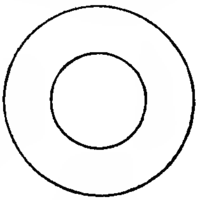


FIG. 48

71. Area of Flat Circular Ring.—The area of a flat circular ring like that in Fig. 48 may be found by the following rule:

Rule.—*The area of a flat circular ring is equal to the area of the outer circle minus the area of the inner circle.*

EXAMPLE.—The diameters of the outer and inner circles of a flat ring are 6½ inches and 4 inches, respectively. What is the area of the ring?

SOLUTION.—The area of the outer circle is $.7854\times(6\frac{1}{2})^2=33.183$ sq. in. and of the inner circle is $.7854\times4^2=12.566$ sq. in. Then, the area of the ring is $33.183-12.566=20.617$ sq. in. Ans.

72. Area of Sector.—The area of a sector may be found by the following rule:

Rule.—*Divide the number of degrees in the arc of the sector by 360, and multiply the result by the area of the circle of which the sector is a part. The product is the area of the sector.*

EXAMPLE.—The angle formed by drawing radii from the center of a circle to the extremities of the arc of the circle is 75° and the diameter of the circle is 12 inches. What is the area of the sector?

SOLUTION.—The area of a circle 12 in. in diameter is $.7854\times12^2=113.1$ sq. in., nearly. By the rule, the area of the sector is

$$\tfrac{75}{360}\times113.1=23.56 \text{ sq. in. Ans.}$$

73. If the length of the arc and the radius of a sector are given, the following rule may be used to find the area:

Rule.—*The area of a sector is equal to one-half the product of the radius and the length of the arc.*

EXAMPLE.—If the radius of an arc is 5 inches, and the length of arc is 4 inches, what is the area of the sector?

SOLUTION.—By the rule, the area is

$$\tfrac{1}{2}\times(5\times4)=10 \text{ sq. in. Ans.}$$

74. Area of Segment.—To find the area of the segment *A B C*, Fig. 49, lines are drawn from the center *D* of the circle to the points *A* and *B*. Then, the area of the segment is equal to the area of the sector *A D B C* minus the area of the triangle *A B D*. Hence, the following rule is used to find the area of a segment of a circle:

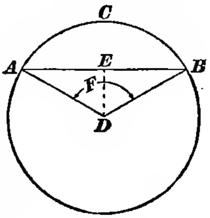


FIG. 49

Rule.—*Draw radii from the center of the circle to the ends of the arc of the segment; find the area of the sector thus* *formed, subtract from this the area of the triangle formed by the radii and the chord of the arc of the segment; the result is the area of the segment.*

In problems where the area of a segment is to be found the angle and the diameter or radius of the circle are usually given. The area of the sector can be found by the rule in Art. **72.** A drawing is then made as in Fig. 49, the known angle *F* being laid off by means of a protractor. The radii are then drawn to scale, as at *D A* and *D B,* the chord is drawn as at *A B,* and the perpendicular distance from the vertex of the angle to the chord is measured, as at *D E.* This distance is the altitude of the triangle *A B D,* and the area of this triangle is one-half the product of the base, or the chord *A B,* and the altitude *D E.*

EXAMPLE.—Find the area of the segment *A B C,* Fig. 49, if the chord *A B* is 8.66 inches, *E D* is $2\frac{1}{2}$ inches, the radius *A D* is 5 inches, and the angle *F* is 120°. (Arc *A C B* is 120°. See Art. **31,** Serial 1978.)

SOLUTION.—By the rule of Art. **72,** the area of the sector is

$$\tfrac{120}{360}\times(3.1416\times5^2)=\tfrac{1}{3}\times78.54=26.18 \text{ sq. in.}$$

The area of the triangle is $\frac{1}{2}\times(8.66\times2\frac{1}{2})=10.83$ sq. in. Therefore, by the foregoing rule, the area of the segment is

$$26.18-10.83=15.35 \text{ sq. in. Ans.}$$

75. Approximate Method for Finding Area of a Segment.—If the conditions are such that the position of a chord, as *A C,* Fig. 50, is known, but not its length, the following rule may be employed for finding the area of a segment. This rule is not exact, but gives results sufficiently correct for practical purposes.

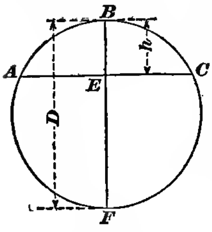


FIG. 50

Rule.—*Divide the diameter by the height of the segment; subtract .608 from the quotient and extract the square root of the remainder. This root is multiplied by 4 and by the square of the height of the segment and then divided by 3. The quotient gives the area, very nearly.*

EXAMPLE.—In the circle shown in Fig. 50, what is the area of the segment *A B C E A,* if the diameter *D* is 54 inches and the height *h* of the segment is 20 inches?

SOLUTION.—The application of the rule is shown by the following operations, which are performed in the order given in the rule:

$$54 \div 20 = 2.7$$
$$2.7 - .608 = 2.092$$
$$\sqrt{2.092} = 1.446$$
$$4 \times 1.446 = 5.784$$
$$5.784 \times 20^2 = 5.784 \times 400 = 2{,}313.6$$
$$2{,}313.6 \div 3 = 771.2 \text{ sq. in.} \quad \text{Ans.}$$

76. The method explained in Art. **75** requires that the diameter be given. Should, however, the chord be given, instead of the diameter, the length of the latter may be found by a method based on the principle that if from any point on the circumference of a circle, as from *A*, Fig. 51, a line *A E* is drawn perpendicular to a diameter, as *B F*, it will divide the diameter into two parts, one of which will be in the same ratio to the perpendicular as the perpendicular is to the other part. That is, the perpendicular will be what is called a *mean proportional* between the two parts of the diameter. For example, in Fig. 51 the perpendicular *A E* divides the diameter into the two parts *B E* and *E F* such that the relation between *A E* and these parts may be stated by the proportion,

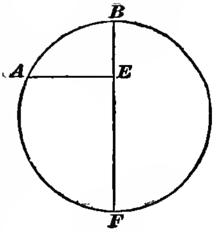


FIG. 51

$$BE : AE = AE : EF$$

The perpendicular *A E*, which is one-half of a chord, is the mean proportional between the two parts of the diameter. The application of this principle will now be shown by means of Fig. 50 and the following example:

EXAMPLE.—If *A C*, Fig. 50, is 30 feet long and the height *B E* is 8 feet, what is the diameter of the circle, the chord *A C* being at right angles to the diameter *B F*?

SOLUTION.—*A E* is one-half of the chord or $30 \div 2 = 15$ ft. According to the preceding explanations the following proportion applies: $BE : AE = AE : EF$; or, $8 : 15 = 15 : EF$.

Therefore, $EF = \frac{15 \times 15}{8} = \frac{225}{8} = 28\frac{1}{8}$ ft. The diameter is equal to $EF + EB = 28\frac{1}{8} + 8 = 36\frac{1}{8}$ ft. Ans.

77. Area of Ellipse.—The area of an ellipse may be found by the following rule:

Rule.—*The area of an ellipse is equal to .7854 times the product of its long and short diameters.*

Example.—The link shown in Fig. 52 has a central part that is elliptical in *cross-section*. That is, if the link is cut in two along the line shown under the section *a*, the end surfaces will each be shaped as shown at *a*. This cross-section, *a*, is an ellipse 4 inches long and 2½ inches wide. What is the area of this cross-section?

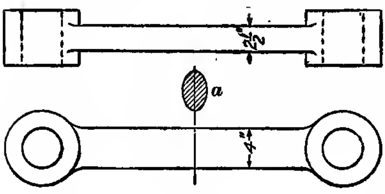


Fig. 52

Solution.—The long diameter is 4 in. and the short diameter is 2½ in. and, by the rule, the area is

$$.7854 \times 4 \times 2\tfrac{1}{2} = 7.854 \text{ sq. in. Ans.}$$

78. Areas of Irregular Curvilinear Figures.—A method was shown in Art. **69** by which the area of an irregular polygon could be calculated. But, if the outline of an irregular figure is made up of curved lines, instead of being composed of straight lines, there is no general rule by which the exact area may be found. In such a case it is necessary to find the area by an approximate method. There are several methods of this kind, all of which will give fairly accurate results. The aim of the following method is to find the dimensions of a rectangle that has an area equivalent to that of the figure.

If the area of an irregular figure, similar to that shown in Fig. 53, is to be found, it is necessary to draw a straight line, as *A B*, below the figure. Then, the perpendiculars *C A* and *D B* are drawn tangent to the curved ends of the figure. The distance *A B*, which is equal to the length of the figure, is

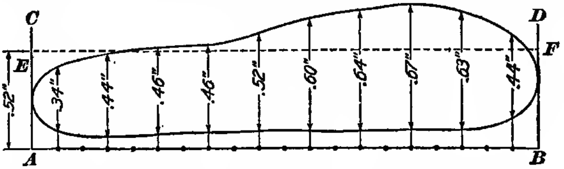


Fig. 53

now divided into a number of equal parts, in this case 20. This can be done by using a pair of dividers, setting their points a certain distance apart, and stepping off from *A* to *B*. The approximate distance between the points of the dividers may be found by measuring the length of *A B* and dividing it by 20. For instance, if the figure is 2.7 inches long, the length of each division is 2.7÷20=.135 inch. The distance between the points should be adjusted at each trial, until a setting is found that will give exactly 20 equal spaces on *A B*. Each division should be marked with a dot made with a sharp pencil point, as shown.

79. Beginning at the dot next to one end of the figure, Fig. 53, a perpendicular is drawn through it, so as to cross the figure. Other perpendiculars are drawn from each alternate point, that is, skipping every second point. There will then be 10 lines drawn across the figure from top to bottom, as shown. The lengths of those portions of the 10 lines that are included between the arrow heads are now measured carefully and the lengths are marked in decimals of an inch. When all the lengths are measured, they are added and their sum divided by the number of lines, in this case 10. The result is the *average* length of the lines, or the average height of the figure. In this example the sum of the lengths is .34+.44+.46+.46+.52+.60+.64+.67+.63+.44=5.2 inches, and 5.2÷10=.52 inch; that is, the average height of the figure is .52 inch. The length of the figure is equal to the length of *A B*, which measures 2.7 inches. The area is equal to the length times the average height, or 2.7×.52=1.4 square inches.

It is seen from Fig. 53 that the area of the figure is assumed to be equal to the area of a rectangle *A E F B* whose length *A B* is equal to the length of the figure and whose height *A E* is equal to the average height of the figure.

NOTE.—It is to be observed that the line *A B* is divided into 20 parts while only 10 perpendiculars are drawn. The purpose of this procedure is to make it possible to locate the first and the last perpendicular at a distance from the respective ends equal to one-half the length of a division. Since the outline of the figure shows the greatest curvature at the ends, greater accuracy in the calculation is secured by placing these perpendiculars nearer the ends.

80. Approximate Area of Any Polygon.—Although the method just described for finding areas is most commonly applied to irregular figures, it can also be used for finding the approximate area of any regular figure, in case the formula for finding such area cannot be remembered. Also, it is not absolutely necessary to divide the base line of the figure into 20 equal parts. Any number of equal parts may be used; but the greater the number, the greater will be the accuracy of the result. Not less than 20 divisions should be used, so that 10 lines of measurement will be obtained, if fairly accurate results are expected.

It is not necessary to measure each of the 10 lines separately, as was done in the preceding article. A narrow strip of paper with one straight edge may be laid across the figure and the lengths of the 10 lines may be marked off on it, end to end, beginning at the left-hand end of the strip. The distance between this end and the last mark made is then measured, which distance is equivalent to the sum of the ten lengths set off. The required calculation is then performed as explained in Art. **79.**

EXAMPLES FOR PRACTICE

1. What is the area in square feet of a parallelogram whose base is 84 inches, and whose altitude is 3 feet? Ans. 21 sq. ft.

2. One side of a shop is 16 feet long. If the floor space is 240 square feet, what is the length of the other side? Ans. 15 ft.

3. How many square feet in one face of a board 12 feet long, 18 inches wide at one end and 12 inches wide at the other end? Ans. 15 sq. ft.

4. What is the area of a triangle whose base is 10 feet 6 inches long, and whose altitude is 18 feet? Ans. 94.5 sq. ft.

5. A wagon wheel is adjusted into a position so that the center lines of two opposite spokes are horizontal. If there are five equally spaced spokes above them, how many degrees are there in the angle between two adjacent spokes? Ans. 30°

6. What is the length of an arc of 64°, the radius of the arc being 30 inches? Ans. 33.51 in.

7. Find the area of a circle 2 feet 3 inches in diameter. Ans. 3.976 sq. ft.

8. From a circle, 42 inches in diameter, a segment is cut off that is 11 inches high. What is the area of the segment? Ans. 289.04 sq. in.

9. Find the area of a flat circular ring whose outside diameter is 12 inches and whose inside diameter is 6 inches. Ans. 84.82 sq. in.

10. The diameter of a flywheel is 18 feet. What is the length of its circumference to the nearest 16th of an inch? Ans. 56 ft. $6\frac{9}{16}$ in.

11. A flat roof, 46 feet by 80 feet in size, is covered by tin roofing weighing one-half pound per square foot; what is the total weight of the roofing? Ans. 1,840 lb.

12. How much would it cost to lay a sidewalk a mile long and 8 feet 6 inches wide, at the rate of 20 cents per square foot? How much at the rate of $1.80 per square yard? Ans. $8,976 in each case

VOLUMES OF SOLIDS

PRISM AND CYLINDER

81. Solids.—A **solid,** or body, has three dimensions: length, breadth, and thickness. The boundaries of a solid are surfaces. If these surfaces are planes, they are known as **faces,** and the lines along which they join or intersect are called **edges.**

82. Classification of Prisms and Cylinders.—A **prism** is a solid whose ends, or bases, are equal and parallel polygons and whose sides are parallelograms. Prisms take their names from the form of their ends. Thus, a *triangular prism* is one whose ends are triangles; a *hexagonal prism* is one whose ends are hexagons, etc. A *rectangular prism,* Fig. 54, is a prism whose ends are rectangles. A *cube,* Fig. 55, is a rectangular prism whose faces and ends are squares.

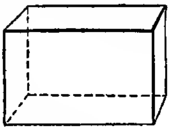


FIG. 54

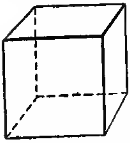


FIG. 55

A **cylinder,** Fig. 56, is a body of uniform diameter whose ends are equal, parallel circles.

The end on which a prism or cylinder is shown to rest is called its **base.**

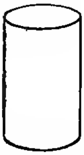


FIG. 56

83. A **right prism** or a **right cylinder** is one whose center line, or axis, is perpendicular to its base. In this Section all of the solids will be considered as having their center lines perpendicular to their bases.

The **altitude** of a prism or a cylinder is the perpendicular distance between its two ends.

84. Volume of Right Prism or Cylinder.—The volume of a prism similar to that shown in Fig. 54 or in Fig. 55 is equal to the product of its length, width, and thickness. But, as the area of the base in either case is equal to the product of two of these dimensions, as, for instance, length and width, it follows that the volume is found by multiplying the area of the base by the altitude of the prism. The three dimensions must be in the same units. The volume of a cylinder also is equal to the product of the area of its base and the altitude. Hence, for finding the volume of a right prism or of a right cylinder the following rule applies:

Rule.—*The volume of any right prism or cylinder equals the area of the base multiplied by the altitude.*

If the given prism is a cube, the three dimensions are all equal, and the volume is equal to the cube of the length of one edge. If the volume and the area of the base are given, the altitude is equal to the volume divided by the area.

Example 1.—What is the volume of a rectangular prism whose base is 6 inches by 4 inches, and whose altitude is 12 inches?

Solution.—As the base of a rectangular prism is a rectangle, its area is equal to $6 \times 4 = 24$ sq. in. By the rule, the volume is $24 \times 12 = 288$ cu. in. Ans.

Example 2.—What is the volume of a cube whose edge is 9 inches?

Solution.—The volume is equal to the cube of one edge, or $9^3 = 9 \times 9 \times 9 = 729$ cu. in. Ans.

Example 3.—What is the volume of a cylinder whose base is 7 inches in diameter, and whose altitude is 11 inches?

Solution.—From Art. **70** the area of a circle is equal to .7854 times the square of the diameter. Hence, the area of the base is $.7854 \times 7^2 = 38.48$ sq. in. By the preceding rule, the volume of the cylinder is $38.48 \times 11 = 423.28$ cu. in. Ans.

EXAMPLE 4.—A cylindrical tank is to contain 140 cubic feet of water. If the area of the base is 28 square feet, what must be the height of the tank?

SOLUTION.—As it was stated that the altitude of a cylinder is equal to the volume divided by the area, it follows that the height of the tank is 140÷28=5 ft. Ans.

85. Volume of Hollow Right Prism or Cylinder. If the cylinder or prism is *hollow,* the volume is equal to the actual area of the base multiplied by the altitude.

EXAMPLE.—The outside diameter of a hollow cylinder, Fig. 57, is 14 inches and the diameter of the interior opening is 12 inches. If the altitude of the cylinder is 9 inches, what is its volume?

SOLUTION.—The base may be considered as a flat circular ring, similar to Fig. 48, and its area is found by using the rule in Art. **71.** The area of the outer circle is $.7854 \times 14^2 = 153.938$ sq. in., and of the inner circle it is $.7854 \times 12^2 = 113.098$ sq. in. Then, the area of the base is $153.938 - 113.098 = 40.84$ sq. in. Then by the rule, the volume is $40.84 \times 9 = 367.56$ cu. in. Ans.

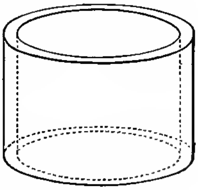


FIG. 57

PYRAMID, CONE, AND SPHERE

86. Pyramid and Cone.—A **pyramid,** Fig. 58, is a solid whose base is a polygon and whose sides are triangles uniting at a common point, called the **vertex.**

A **cone,** Fig. 59, is a solid whose base is a circle and whose convex surface tapers uniformly to a point called the **vertex.**

The **altitude** of a pyramid or a cone is the perpendicular distance from the vertex to the base, as shown by the dotted line in Figs. 58 and 59. If this line passes through the center of the base, the solid is called a *right pyramid* or a *right cone.*

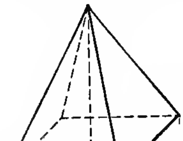


FIG. 58

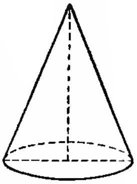


FIG. 59

87. Volume of Right Pyramid or Right Cone. The volume of a right pyramid or of a right cone may be found by the following rule:

Rule.—*The volume of a right pyramid or a right cone equals the area of the base multiplied by one-third of the altitude.*

If the base of the pyramid is a regular polygon, its area may be found by the rule of Art. **68.**

EXAMPLE 1.—What is the volume of a triangular right pyramid, the edges of whose base each measure 6 inches, and whose altitude is 8 inches?

SOLUTION.—The base is an equilateral triangle; hence, by the rule of Art. **68,** its area is $6^2 \times .433 = 15.59$ sq. in. Applying the rule, the volume is

$$15.59 \times \tfrac{8}{3} = 41.57 \text{ cu. in. Ans.}$$

EXAMPLE 2.—What is the volume of a right cone whose altitude is 18 inches, and whose base is 14 inches in diameter?

SOLUTION.— $.7854 \times 14^2 = 153.94$ sq. in., the area of the base. By the rule, the volume is

$$153.94 \times \tfrac{18}{3} = 923.64 \text{ cu. in. Ans.}$$

88. Convex Area of Pyramid, Cone, or Cylinder. The **slant height** of a *pyramid* is a line drawn from the vertex perpendicular to one of the sides of the base. The slant height corresponds, therefore, with the altitude of any one of the triangular faces. The slant height of a *cone* is any straight line drawn from the vertex to a point on the circumference of the base.

To find the convex area of a pyramid or cone (which is the area of the outside surface, not including the base) apply the following rule:

Rule I.—*To find the convex area of a pyramid or of a cone, multiply the perimeter of the base by one-half the slant height.*

The convex area of a cylinder is found by the following rule:

Rule II.—*To find the convex area of a cylinder, multiply the perimeter of the base by the altitude.*

EXAMPLE 1.—What is the convex area of a pentagonal pyramid if one side of the base measures 6 inches and the slant height is 14 inches?

SOLUTION.—The base of a pentagonal pyramid is a pentagon with five equal sides. The perimeter of the base is $6 \times 5 = 30$ in. and, by rule I, the convex area is $30 \times \tfrac{14}{2} = 210$ sq. in. Ans.

EXAMPLE 2.—What is the entire area of a right cone whose slant height is 17 inches and whose base is 8 inches in diameter?

SOLUTION.—The perimeter, or circumference, of the base is $3.1416 \times 8 = 25.1328$ in; the area of the base is $.7854 \times 8^2 = 50.27$ sq. in; the convex area is $25.1328 \times \frac{17}{2} = 213.63$ sq. in., and the total area of cone is $50.27 + 213.63 = 263.90$ sq. in. Ans.

EXAMPLE 3.—What is the convex surface of a cylinder 4 inches in diameter and 75 inches high?

SOLUTION.—The perimeter of the base is $3.1416 \times 4 = 12.57$ in. Hence, by rule II, the convex area is $12.57 \times 75 = 942.75$ sq. in. Ans.

89. Frustums.—If a pyramid is cut off parallel to the base, as in Fig. 60, so as to form two parts, the lower part is called the **frustum** of the pyramid. If a cone is cut in a similar manner, as in Fig. 61, the lower part is called the **frustum** of the cone. The upper end of the frustum of a pyramid or cone is called the **upper base**, and the lower end the **lower base**. The **altitude** of a frustum is the perpendicular distance between the bases, indicated by the dotted lines in Figs. 60 and 61.

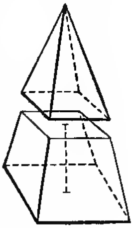


FIG. 60

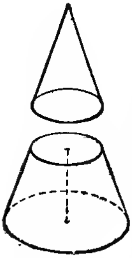


FIG. 61

90. Volume of Frustum of Pyramid Having Regular Polygon for Base.—The volume of a frustum of a pyramid may be found by the following rule:

Rule.—*Multiply together the lengths of one side of the upper and lower bases and the corresponding multiplier taken from Table II, and to the product add the sum of the areas of the upper and lower bases; then multiply the total sum by one-third the altitude of the frustum. The result will be the volume of the frustum.*

EXAMPLE.—In a frustum of a pyramid, whose bases are regular hexagons, each edge of the upper base is 5 inches long and of the lower base is 8 inches long. If the altitude is 14 inches, what is the volume?

SOLUTION.—The area of the upper base, by the rule of Art. **68**, is $5^2 \times 2.5981 = 64.95$ sq. in. By the same rule, the area of the lower base is

$8^2 \times 2.5981 = 166.28$ sq. in. The sum of these areas is $64.95 + 166.28 = 231.23$ sq. in. Then, by the rule, the volume of the frustum is

$$[(5 \times 8 \times 2.5981) + 231.23] \times \frac{14}{3} = (103.92 + 231.23) \times \frac{14}{3}$$
$$= 335.15 \times \frac{14}{3} = 1{,}564 \text{ cu. in. Ans.}$$

91. Volume of Frustum of Cone.—The volume of the frustum of a cone may be found by using the following rule:

Rule.—*Multiply together the diameters of the two bases and .7854 and to the product add the sum of the areas of the upper and lower bases; then multiply the total sum by one-third the altitude of the frustum. The result will be the volume of the conical frustum.*

EXAMPLE.—What is the volume of a frustum of a cone whose upper base is 8 inches in diameter, whose lower base is 12 inches in diameter, and whose altitude is 15 inches?

SOLUTION.—The area of the upper base is $.7854 \times 8^2 = 50.27$ sq. in., and that of the lower base is $.7854 \times 12^2 = 113.1$ sq. in. Their sum is $50.27 + 113.1 = 163.37$ sq. in. By the rule, the volume of the frustum is

$$[(8 \times 12 \times .7854) + 163.37] \times \frac{15}{3} = (75.4 + 163.37) \times \frac{15}{3}$$
$$= 238.77 \times 5 = 1{,}193.85 \text{ cu. in. Ans.}$$

92. Volume of Sphere.—A **sphere,** Fig. 62, is a solid bounded by a uniformly curved surface, every point of which is equally distant from a point within, called the center. The word *ball* is commonly used instead of sphere. The volume of a sphere may be found as follows:

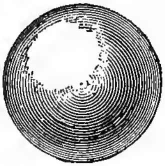


FIG. 62

Rule.—*The volume of a sphere equals the cube of the diameter multiplied by .5236.*

EXAMPLE.—What is the volume of a cannon ball 12 inches in diameter?

SOLUTION.—By the rule, the volume is $12^3 \times .5236 = 12 \times 12 \times 12 \times .5236 = 904.78$ cu. in. Ans.

The volume of a spherical shell, or hollow sphere, is equal to the difference in volume between two spheres having the outer and inner diameters of the shell.

93. Area of Sphere.—The surface area of a sphere is found by the following rule:

Rule.—*Square the diameter and multiply the result by 3.1416.*

EXAMPLE.—What is the surface area of a sphere 14 inches in diameter?

SOLUTION.—According to the rule, the area is

$$14^2 \times 3.1416 = 14 \times 14 \times 3.1416 = 615.75 \text{ sq. in. Ans.}$$

PRISMOIDS.

94. Definition.—A **prismoid** is a solid with two faces, or bases, that are parallel polygons and the remaining faces triangles or quadrilaterals. Thus, the solid shown in Fig. 63 is a prismoid; its bases are the irregular pentagon *A B C D E* and the trapezoid *F G H I*, which are parallel. The other faces of the prismoid are the triangle *G B C* and the quadrilaterals *G C D H*, *H D E I*, *I E A F*, and *F A B G*. The perpendicular distance between the bases of a prismoid is called its **altitude.**

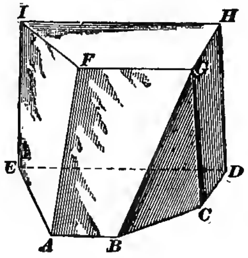


FIG. 63

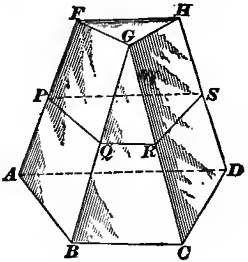


FIG. 64

95. Middle Section of Prismoid.—If an imaginary cut is made through a prismoid, as at *P Q R S*, Fig. 64, parallel to the two bases and at a point equidistant between them, the polygon so formed is known as the **middle section** of the prismoid.

96. Volume of Prismoid.—In calculating the volume of a prismoid the perimeter of the middle section is an important factor. Its length is found by means of the following rule:

Rule I.—*Find one-half the sum of each pair of corresponding sides of the bases. The sum of these results gives the length of the perimeter of the middle section.*

The meaning of this rule will be understood by referring to Fig. 64. Thus, the sides of the middle section are found in the following manner: The side $P\,Q$ is equal to $\frac{A\,B+F\,G}{2}$; the side $Q\,R=\frac{B\,C}{2}$, as the vertex of the triangle has no dimension; the side $R\,S=\frac{G\,H+C\,D}{2}$; and the side $S\,P=\frac{H\,F+D\,A}{2}$. The perimeter of the middle section is equal to the sum of the values found for the sides.

The volume of the prismoid may now be found by means of the following rule and formula, known as the **prismoidal formula:**

Rule II.—*The volume of a prismoid is equal to one-sixth of the product obtained by multiplying the altitude by the sum of the area of the lower base, the area of the upper base, and four times the area of the middle section.*

The latter rule may also be expressed as follows:

$$Volume=\tfrac{1}{6}\ (altitude)\times\left(\begin{matrix}lower\\base\end{matrix}+\begin{matrix}upper\\base\end{matrix}+4\times\begin{matrix}middle\\section\end{matrix}\right).$$

EXAMPLE.—Find the volume of the prismoid shown in Fig. 65, if the altitude is 14 inches and other dimensions are as indicated.

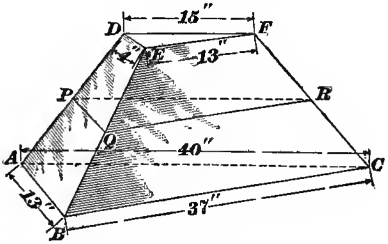


FIG. 65

SOLUTION.—In the illustration the triangle $P\,Q\,R$ is the middle section. The sides of this triangle are found by means of rule I. Thus,

$$PQ = \frac{AB+DE}{2} = \frac{13+4}{2} = 8.5 \text{ in.}$$

$$QR = \frac{BC+EF}{2} = \frac{37+13}{2} = 25 \text{ in.}$$

$$RP = \frac{AC+DF}{2} = \frac{40+15}{2} = 27.5 \text{ in.}$$

The areas of the triangles ABC, DEF, and PQR are calculated by the rule of Art. **65**, which gives area of $ABC=240$ sq. in., area of $DEF=24$ sq. in., and area of $PQR=105.2$ sq. in., nearly. According to rule II of this Article, the volume of the prismoid is

$\frac{1}{6}\times 14\times(240+24+4\times 105.2)=1{,}597.9$ cu. in., nearly. Ans.

97. General Application of Prismoidal Formula. The prismoidal formula explained in Art. **96** is used extensively in practice. It applies not alone to prismoids, but also for calculating the volume of a prism, pyramid, cylinder, cone, frustum of a pyramid, frustum of a cone, sphere, or segments of a sphere, in addition to the irregularly shaped bodies to which it is usually applied.

When the formula is applied to a pyramid and cone the dimensions and area of the upper base are equal to 0; for a sphere the areas of the upper and the lower base are 0; for a prism and a cylinder the areas of the upper and lower bases and of the middle section are all equal.

EXAMPLES FOR PRACTICE

1. A rectangular tank 12 feet long, 8 feet wide, and 10 feet deep, inside measurements, is half full of water. How many cubic feet of water does it contain? Ans. 480 cu. ft.

2. A box filled with sand has the dimensions 4 feet, 3 feet 6 inches, and 2 feet 6 inches on the inside. How many pounds of sand does it hold, if the sand weighs 96 pounds per cubic foot? Ans. 3,360 lb.

3. A forged steel shaft 9 inches in diameter and 10 feet long has a hole 4½ inches in diameter through it from end to end. If forged steel weighs .283 pound per cubic inch, what is the weight of the shaft? Ans. 1,620 lb.

4. How many cubic feet are contained in a foundation wall that is 20 feet long, 8 feet high, and 2 feet thick? Ans. 320 cu. ft.

5. What is the volume of a cylindrical oil tank 3 feet in diameter and 6 feet 6 inches high? Ans. 45 cu. ft. 1,635 cu. in.

6. What is the capacity of a tool box, which, measured on the inside, is 2 feet 3 inches long, 18 inches wide, and 1 foot deep? Ans. 3 cu. ft. 648 cu. in.

7. Considering only its outside dimensions, what is the volume of a ball float with a diameter of 22½ inches? Ans. 5,964 cu. in.

8. The diameters of the upper and lower bases of a conical frustum are 8 and 10 feet, respectively. If the altitude is 12 feet, what is the volume of the frustum? Ans. 766.55 cu. ft.

9. If a tank had the dimensions of the frustum in question 8, how many gallons of water would it hold, one gallon being equivalent to 231 cubic inches? Ans. 5,734.2 gal., nearly

10. The cavity of a mold is 20 inches long, 4 inches wide, and 4 inches deep. If a lead casting is made in this mold, what would be its weight, the weight of lead being .41 pound per cubic inch? Ans. 131.2 lb.

PROBLEMS FOR PRACTICE

CALCULATIONS OF MATERIAL, WEIGHT, AND TAPER

CALCULATING LENGTH OF STOCK

98. Forging Problems.—At times it is difficult to estimate the amount of stock required for a certain job. For instance, a blacksmith may have in stock iron bars of a certain size, that are to be used for making some bars of a smaller size. As the length of stock required will be correspondingly shorter, it will be advantageous to know what length of stock to allow.

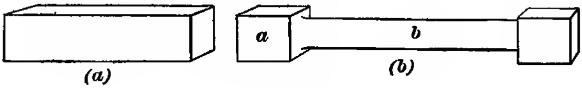


Fig. 66

In Fig. 66 (*a*) is shown a piece of square stock that is to be forged into the form shown at (*b*). The ends *a* of the finished piece are of the same size as the original square bar at (*a*), but the middle part *b* is round, instead of square, and is considerably longer and smaller than the middle part of the square stock. The problem then is to find how much square stock to allow, so that, when it is hammered out into the long, round part *b*, this part will be of the required length. The rule to be used is as follows:

Rule.—*Calculate the volume, in cubic inches, of the part that is to be made from the stock and divide it by the area of the end of the bar stock, in square inches. The quotient is the length of stock required.*

If the finished piece is to be shorter and thicker than the bar stock, the same rule can be used.

EXAMPLE 1.—A forging like that shown in Fig. 66 (*b*) is to be made from a piece of bar stock 3 inches square. If the dimensions of the forging are to be as shown in Fig. 67, what length of stock is required?

SOLUTION.—The ends of the forging are of the same size as the bar stock, that is, 3 in. square, and each is 3½ in. long; therefore, it will take 2×3½ in.=7 in. of stock for the ends. The middle part is a cylinder 2 in. in diameter and 15 in. long. The volume of this part, therefore, is, by the rule of Art. **84**,

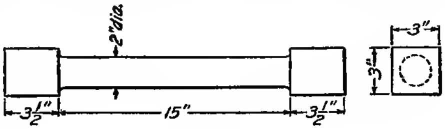


FIG. 67

$$.7854 \times 2^2 \times 15 = 47 \text{ cu. in.}$$

The area of the end of the bar stock, which is 3 in. square, is 3×3=9 sq. in. Then, by the rule of this article, the length of stock required for the middle section of the forging is

$$47 \div 9 = 5.22 \text{ in., or } 5\tfrac{1}{4} \text{ in., very nearly}$$

That is, a piece of 3-in. square stock 5¼ in. long, when hammered down to a round piece 2 in. in diameter, will be 15 in. long. The total length of bar stock required for the forging, therefore, is the sum of that needed for the ends and that needed for the middle part, or

$$7 + 5\tfrac{1}{4} = 12\tfrac{1}{4} \text{ in. Ans.}$$

Example 2.—A forging is to be made, of the shape shown in Fig. 68, from a round 1-inch bar. The central portion, which is 18 inches long, is the same size as the bar, but the two ends are upset until their diameters are increased to 1¼ inches. If each of the upset parts is 6 inches long, how much stock should be allowed for?

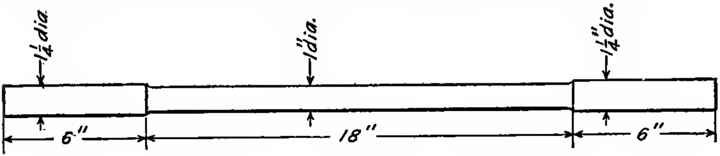


Fig. 68

Note.—When a red-hot bar of iron is hammered on one end in a longitudinal direction, it becomes shorter and thicker, and it is then said to be *upset*.

Solution.—A cylinder 1¼ in. in diameter and 6 inches long has, according to the rule in Art. **84**, a volume of $.7854\times1.25^2\times6=7.363$ cu. in. The volume of the two ends is $2\times7.363=14.726$ cu. in.

The area of the end of the bar stock, which has a diameter of 1 inch, is $.7854\times1^2=.7854$ sq. in. Then, according to the rule of this article, the length of stock required for the end portions is $14.726\div.7854=18.75$ in. The total length of bar stock required is equal to the sum of that needed for the middle part and for the ends, or

$$18+18.75=36\tfrac{3}{4}\text{ in. Ans.}$$

CALCULATION OF WEIGHTS

99. Method of Estimating Weight.—It often becomes necessary to calculate the weight of a body whose dimensions are known but which cannot conveniently be put on scales to be weighed. For instance, suppose that it is desired to estimate the weight of a casting that has not been made. From the drawing it is possible to obtain all the dimensions of the casting. Then, by the principles of mensuration given previously, the volume of the casting can be found. After the volume is known, it is multiplied by the weight of a unit volume of the material, and thus the weight is obtained. Sometimes, castings or forgings have intricate shapes; but usually it is possible to consider them as made up of a combination of cylinders, spheres, rectangular prisms, cones, etc. The volume of each of these is calculated and the total is the approximate volume of the piece.

TABLE III

AVERAGE WEIGHTS OF MATERIALS

Material	Weight per Cubic Foot Pounds	Weight per Cubic Inch Pound
Dry Woods		
Ash	45	.0260
Beech	46	0266
Birch	41	.0237
Box	70	.0405
Cedar	39	.0226
Cherry	41	.0237
Chestnut	35	.0203
Cork	14	.0081
Elm	38	.0220
Fir	37	.0214
Hemlock	24	.0139
Hickory	48	.0278
Lignum vitæ	62	.0359
Mahogany	51	.0287
Maple	42	.0243
Oak, red	46	.0266
Oak, white	48	.0278
Pine, white	28	.0162
Pine, yellow	38	.0220
Poplar	30	.0174
Spruce	28	.0162
Walnut	36	.0208
Willow	34	.0197
Metals		
Aluminum	167	.096
Antimony	420	.243
Babbitt metal	466	.270
Metals—(Cont.)		
Brass castings	510	.295
Brass, rolled	518	.300
Bronze	527	.305
Copper	550	.318
Iron, cast	450	.260
Iron, wrought	480	.278
Lead	710	.411
Steel castings	480	.278
Steel, forged	490	.283
Tin	458	.265
Zinc castings	432	.250
Zinc sheets	450	.260
Miscellaneous		
Cement, loose	92	
Cement in barrel	115	
Concrete	120 to 155	
Earth, loose	72 to 80	
Earth, rammed	90 to 110	
Sand	90 to 110	
Sand, wet	118 to 130	

100. Weights of Materials.—In finding the weight of a machine part, the volume is usually calculated in cubic inches. This volume must then be multiplied by the weight of the material per cubic inch, to obtain the weight of the body. If the volumes are calculated in cubic feet, the weight per cubic foot is used. To enable the weights of bodies to be calculated, the weights of a number of different materials are given in Table III, in pounds per cubic foot and per cubic inch. The method of using the table is shown in the following examples:

EXAMPLE 1.—A hollow box-shaped casting has outside dimensions of 18, 10, and 6 inches, and inside dimensions of 16, 8, and 4 inches. What is its weight, if it is a steel casting?

SOLUTION.—The first step is to calculate the volume of the casting. This is the difference between the volume represented by the outside and inside dimensions. The volume of a solid 18 in. by 10 in. by 6 in. is 18×10×6=1,080 cu. in., and that of a solid 16 in. by 8 in. by 4 in. is 16×8×4=512 cu. in. Then, the volume of the casting is 1,080−512 =568 cu. in. According to Table III, a cubic inch of steel casting weighs .278 lb. Therefore, the weight of the casting is

568×.278=158 lb., nearly. Ans.

EXAMPLE 2.—Find the weight of a cast-iron bracket like that in Fig. 69 (*a*), having the dimensions shown in (*b*).

SOLUTION.—The bracket may be considered as being made up of three parts, two of which are rectangular prisms, and the other of which is a triangular prism. One rectangular prism has a length *a b* of 8 in., a width *c d* of 7½ in., and a thickness *a e* of 1½ in. Its volume, therefore, is 8×7½×1½=90 cu. in. The other rectangular prism has a width *a b* of 8 in., a height *a f* of 9½−1½=8 in., and a thickness *g h* of 1½ in., so its

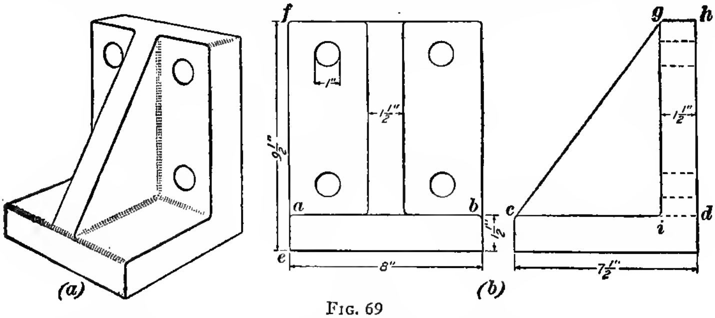


FIG. 69

volume is $8 \times 8 \times 1\frac{1}{2} = 96$ cu. in. The bracing web has the shape of a right triangle whose base *ci* is $7\frac{1}{2} - 1\frac{1}{2} = 6$ in., and whose altitude *ig* is $9\frac{1}{2} - 1\frac{1}{2} = 8$ in. Its area is therefore $\frac{6 \times 8}{2} = 24$ sq. in. As the thickness is $1\frac{1}{2}$ in., the volume of the triangular prism, according to the rule of Art. **84**, is $24 \times 1\frac{1}{2} = 36$ cu. in. There are four holes each 1 in. in diameter and $1\frac{1}{2}$ in. long; so the amount of metal cut out by each is the volume of a cylinder 1 in. in diameter and $1\frac{1}{2}$ in. long. The area of the base is the area of a 1-in. circle, or $.7854 \times 1^2 = .7854$ sq. in. Then, by the rule of Art. **84**, the volume of one cylinder is $.7854 \times 1\frac{1}{2} = 1.1781$ cu. in. As there are four holes, their volume is $4 \times 1.1781 = 4.7$ cu. in., approximately. The actual volume of the bracket, therefore, is $90 + 96 + 36 - 4.7 = 217.3$ cu. in. According to Table III, a cubic inch of cast iron weighs .260 lb. The weight of the bracket, therefore, is

$217.3 \times .260 = 56\frac{1}{2}$ lb., very nearly. Ans.

EXAMPLE 3.—What is the weight of a forged steel shaft of the form and dimensions shown in Fig. 70, the end *d* being square.

SOLUTION.—The shaft may be considered as being made up of a cylindrical piece *a* that is 1 in. in diameter and $2\frac{1}{2}$ in. long, a cylinder *b* $1\frac{1}{2}$ in. in diameter and $9\frac{3}{4}$ in. long, a frustum of a cone with bases $1\frac{1}{4}$ in. and $1\frac{1}{2}$ in. in diameter and 2 in. long, and a prism $\frac{3}{4}$ in. square and $1\frac{1}{2}$ in. long. According to the rule of Art. **84**, volume of part $a = .7854 \times 1^2 \times 2\frac{1}{2} = 1.96$ cu. in., and volume of part $b = .7854 \times (1\frac{1}{2})^2 \times 9\frac{3}{4} = 17.23$ cu. in.

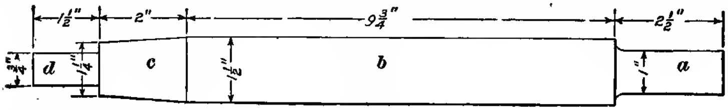


FIG. 70

The part *c* is a frustum of a cone. The area of the smaller base is $.7854 \times (1\frac{1}{4})^2 = 1.23$ and that of the larger base is $.7854 \times (1\frac{1}{2})^2 = 1.77$ cu in. Then, by the rule of Art. **91**, the volume of the frustum is

$$[(1\tfrac{1}{4} \times 1\tfrac{1}{2} \times .7854) + 1.23 + 1.77] \times \tfrac{2}{3} = (1.47 + 1.23 + 1.77) \times \tfrac{2}{3} = 2.98 \text{ cu. in., or}$$
$$3 \text{ cu. in., nearly}$$

The volume of the rectangular solid end *d*, according to the rule of Art. **84**, is $\frac{3}{4} \times \frac{3}{4} \times 1\frac{1}{2} = .8$ cu. in.

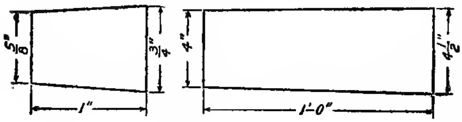
The total volume of the shaft, therefore, is $1.96 + 17.23 + 3 + .8 = 22.99$ cu. in., or 23 cu. in., nearly.

According to Table III, the weight of a cubic inch of forged steel is .283 lb. The weight of the shaft, then, is

$23 \times .283 = 6\frac{1}{2}$ lb., nearly. **Ans.**

CALCULATION OF TAPERS

101. Definition of Taper.—The term *taper* means a gradual and uniform increase or decrease in the width or diameter of a piece from one end to the other. The taper is usually expressed by stating how much the width or diameter of the piece increases or decreases per inch or per foot of length. Thus, if the piece shown in Fig. 71 (*a*) is 1 inch long, $\frac{3}{4}$ inch wide at one end, and $\frac{5}{8}$ inch wide at the other, the taper is $\frac{3}{4}-\frac{5}{8}=\frac{1}{8}$ inch in 1 inch, or $\frac{1}{8}$ inch per inch. If the piece shown in (*b*) is 1 foot long and its diameters at the ends are $4\frac{1}{2}$ inches and 4 inches, the taper is $4\frac{1}{2}-4=\frac{1}{2}$ inch per foot. The taper may be expressed in inches per inch or in inches per foot, as desired.



(*a*) (*b*)

FIG. 71

102. Finding the Taper.—If the length and the end dimensions of a piece are known, the taper may be found by the following rules:

Rule I.—*To find the taper in inches per foot, divide the difference of the end dimensions, in inches, by the length of the piece in feet.*

Rule II.—*To find the taper in inches per inch, divide the difference of the end dimensions, in inches, by the length of the piece in inches.*

EXAMPLE 1.—A tapered bar $2\frac{1}{2}$ feet long is $4\frac{5}{8}$ inches wide at one end and 4 inches wide at the other. What is the taper per foot?

SOLUTION.—By rule I, the taper is

$$(4\tfrac{5}{8}-4)\div 2\tfrac{1}{2}=\tfrac{5}{8}\times\tfrac{2}{5}=\tfrac{1}{4}\text{ in. per ft.}\quad\text{Ans.}$$

EXAMPLE 2.—A round tapered piece 8 inches long has diameters of $1\frac{5}{8}$ and $2\frac{1}{8}$ inches at the ends. What is the taper per inch?

SOLUTION.—By rule II, the taper is

$$(2\tfrac{1}{8}-1\tfrac{5}{8})\div 8=\tfrac{1}{2}\times\tfrac{1}{8}=\tfrac{1}{16}\text{ in. per in.}\quad\text{Ans.}$$

103. Finding Dimensions From Taper.—Sometimes the dimension at one end of the piece and the taper are given and the dimension at the other end is to be calculated. This may be done by the following rules:

Rule I.—*To find the dimension at the large end, multiply the length of the piece in feet (or in inches) by the taper per foot (or per inch) and to the product add the dimension at the small end.*

Rule II.—*To find the dimension at the small end, multiply the length of the piece in feet (or in inches) by the taper per foot (or per inch) and subtract the product from the dimension at the large end.*

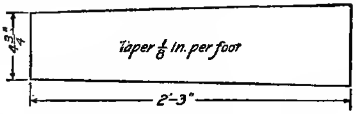


Fig. 72

Example 1.—The piece shown in Fig. 72 has a taper of ⅛ inch per foot, is 4¾ inches wide at the small end, and is 2 feet 3 inches long. What is the width at the large end?

Solution.—By rule I, and remembering that 2 ft. 3 in.=2¼ ft.,

$$2\tfrac{1}{4}\times\tfrac{1}{8}=\tfrac{9}{4}\times\tfrac{1}{8}=\tfrac{9}{32}$$

Then, the width at the large end is

$$\tfrac{9}{32}+4\tfrac{3}{4}=5\tfrac{1}{32}\text{ in. Ans.}$$

Example 2.—A piece 16 inches long has a taper of ¼ inch per inch and is 7¾ inches in diameter at the large end. Find the diameter at the small end.

Solution.—By rule II,

$$16\times\tfrac{1}{4}=4$$

Then, the diameter at the small end is

$$7\tfrac{3}{4}-4=3\tfrac{3}{4}\text{ in. Ans.}$$

FORMULAS

ELEMENTARY OPERATIONS

USE OF SYMBOLS

RULES AND FORMULAS COMPARED

1. Disadvantages of Rules.—In the preceding Sections, and in particular in that one dealing with mensuration, frequent use is made of rules indicating how to use certain numbers for the purpose of finding an unknown length, area, or volume.

When the arithmetical processes are of a simple kind, the rule will be short and easily understood. If, however, the rule deals with several numbers that have to be combined and used in various ways, the rule becomes more complicated. The following rule, which is used for finding the area of a segment, may serve as an example of this kind:

Rule.—*Divide the diameter by the height of the segment; subtract .608 from the quotient and extract the square root of the remainder. This root is multiplied by 4 and by the square of the height of the segment and then divided by 3. The quotient gives the area, very nearly.*

It is difficult to follow the various steps in this rule and it would, therefore, be advantageous to find another method by which the rule could be presented in a more concise form, so that the required treatment of the numbers could be seen at a glance.

2. The great length of a rule is not its only fault. It is also too limited in its application, as it refers to the finding of only one particular value in a given problem. The meaning of this statement will be understood by referring to the rule for finding the volume of the frustum of a cone. In this rule the diameter of the bases and the altitude of the frustum are employed. But, the conditions may be such that the *volume* is given and that the value of one of the other three factors must be found so as to obtain the given volume. It would, therefore, require four rules, so as to make it possible to find any one of the four dimensions, if the three others were given.

3. Formulas.—A rule may be simplified by using a single letter to represent each part, or quantity, instead of the name of the quantity and then joining the letters by signs to indicate the operations to be performed. The relative positions of the numbers will also serve to indicate multiplication or division, as will be explained subsequently.

A rule presented by these means is known as a *formula.* From the preceding explanations it follows that a **formula** may be considered as a shorthand method of expressing a rule. Symbols, such as letters and arithmetical signs, are used instead of words. Any formula can be expressed in words, and when so expressed it becomes a rule.

4. Advantages of Formulas.—A formula shows at a glance all the operations that are to be performed. It does not have to be read three or four times, as is the case with most rules, to enable one to understand its meaning. Also, it takes up much less space and may be adjusted to meet the various requirements of a problem; in short, whenever a rule can be expressed as a formula, the formula is to be preferred.

Before the application of letters and other symbols is explained, it will be necessary to define the meaning of some terms and signs, as well as the principles on which formulas are based.

ARITHMETICAL SIGNS AND THEIR APPLICATION

5. Signs of Aggregation.—The signs used in formulas are the ordinary signs indicative of arithmetical operations and the signs of aggregation. All these signs were explained in preceding Sections, but the signs of *aggregation* will need further explanation, as their application is extended in some directions.

The signs of **aggregation** are four in number, viz., —, ( ), [ ], and { }, respectively called the **vinculum,** the **parenthesis,** the **brackets,** and the **brace.** They are used to a great extent when it is necessary to indicate that all the numbers included by them are to be subjected to the same operation. For example, if the sum of 5 and 8 is to be multiplied by 7, any one of the four signs of aggregation may be used, but it is customary to use the *parenthesis;* thus, $(5+8)\times 7$. As already explained, the vinculum is used extensively in connection with the radical sign to indicate a root.

If *two* signs of aggregation are needed, the brackets and the parenthesis are used, so as to avoid having a parenthesis within a parenthesis, the brackets being placed outside. For example, $[(20-5)\div 3]\times 9$ means that the difference between 20 and 5 is to be divided by 3, and the result multiplied by 9.

If *three* signs of aggregation are required, the brace, brackets, and parenthesis are used, the brace being placed outside, the brackets next, and the parenthesis inside. For example, $\{[(20-5)\div 3]\times 9-21\}\div 8$ means that the quotient obtained by dividing the difference between 20 and 5 by 3 is to be multiplied by 9, and that after 21 has been subtracted from the product thus obtained, the result is to be divided by 8.

6. Order of Operations.—When several quantities are connected by the various signs indicating addition, subtraction, multiplication, and division, the operation indicated by the sign of *multiplication* must always be performed *first; next* in order comes the operation of *division.* Thus, $2+3\times 4$ equals 14, 3 being multiplied by 4 before adding it to 2. Similarly, $10\div 2\times 5$ equals 1, since 2×5 equals 10, and $10\div 10$ equals 1.

If this rule were not followed in the preceding examples, the results would be quite different. For instance, if in the example $2+3\times4$ addition is performed before multiplication, as $2+3=5$, and $5\times4=20$, the result differs from that previously found, which was 14.

In the example $10\div2\times5$, the quotient found by dividing 10 by 2 is 5, and 5 multiplied by 5 is 25. Performed in the correct manner the result was found to be 1.

7. In cases where several numbers are connected by using the sign of division and the plus or minus sign, the operation of division must be performed first. For example, $5-9\div3$ is equal to 2; 9 divided by 3 gives 3 as the quotient, and $5-3=2$.

The signs of addition and subtraction are of equal value; that is, if several quantities are connected by plus and minus signs only, the indicated operations may be performed in the order in which the quantities are placed. Thus, in the example $5+7-4$, the solution may be found as $5+7=12$, and $12-4=8$. Or, $7-4=3$, and $3+5=8$.

EQUATIONS FORMED WITH NUMBERS

FUNDAMENTAL PRINCIPLES

8. Elementary Operations.—The principle on which any formula is based is a condition of equality existing between a combination of numbers arranged in a certain manner. The following example shows a simple combination of numbers of this kind:

$$2+4=5+1$$

The equality sign (=) is placed as the connecting link between the numbers 2 and 4 on one side, and the numbers 5 and 1 on the other side. An expression of this kind is known as an **equation**, because the equality sign connects two combinations of numbers that are equal, the sum of 2 and 4 being equal to the sum of 5 and 1.

This condition of equality is not limited to *sums* of numbers only, but may refer to any combination of numbers, subjected to the arithmetical operations of addition, subtraction, multiplication, division, square root, etc. The following shows an equation in which there is a combination of some of these operations:

$$(9\times8)-2=7\sqrt{64}+\tfrac{42}{3}$$

It is found that if the operations are carried out as indicated, the results on both sides of the equality sign are equal to 70.

9. Quantity.—It is convenient to have a term that may be applied to the whole expression on either side of the equality sign. For this purpose the word *quantity* is in general use. The term **quantity,** as used in mathematics, is applied to any number or combination of numbers on which the ordinary arithmetical operations are to be performed. By using the term quantity one avoids the necessity of specifying what these operations will be. In the last example the numerical combinations $(9\times8)-2$ and $7\sqrt{64}+\tfrac{42}{3}$ are both referred to as quantities.

Employing the term quantity with this meaning, an equation may be said to be an expression of equality between two quantities.

10. First and Second Members.—Distinction is made between the quantities on either side of the equality sign in an equation by applying the term **first member** to the quantity on the left-hand side of the equality sign. The quantity on the other side is called the **second member.** Speaking of both quantities, they are known as *the members of the equation.* In the following example 5×9 is the first member and $60-15$ the second member. Thus,

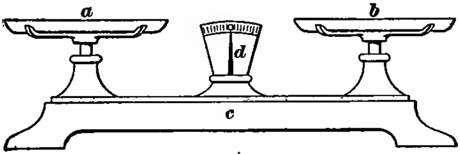


Fig. 1

$$\overset{\textit{first member}}{5\times9} = \overset{\textit{second member}}{60-15}$$

11. Comparison Between Equation and Scale. Equations may be better understood by comparison with the ordinary scale, as used in stores. A scale of this kind, as shown in Fig. 1, has two pans *a* and *b*, which are supported by a lever, pivoted in the stand *c*. When the pointer *d* stands at the zero mark, it indicates that a condition of equality exists between the weights supported by the two pans; that is, the weights are in balance, or equilibrium. In the following examples the scale will be indicated diagrammatically by the balance lever, the two pans, and the weights they contain.

12. A mathematical equation and a balance have this property in common, that a condition of equality cannot exist unless the quantities on both sides of the equality sign, or the weights in both pans of the scale, are equal.

It is impossible to reduce the weights placed in only one pan without disturbing the balance of the scale, just as it is impossible to reduce the value of only one member of the equation without disturbing the equality. But if the same change is made in both weights the scale will remain in balance, and likewise if the same change is made in both members of an equation they will remain equal.

The changes to which the members of an equation may be subjected are conveniently divided into the following four cases:

1. Adding the same quantity to both members of an equation.
2. Subtracting the same quantity from both members.
3. Multiplying both members by the same quantity.
4. Dividing both members by the same quantity.

13. Adding a Quantity to Both Members of an Equation.—It can be proved that if a number is added to one member of an equation, a number of equal value must be added to the other member. For example, Fig. 2 (*a*) represents a scale with two pans *a* and *b*. The pan *a* supports weights of 10 and 8 pounds; and the pan *b* supports weights of 12 and 6 pounds. The total weight on each pan is 18 pounds. If, now, 1 pound is added to the pan *b*, as in Fig. 2 (*b*), the balance is destroyed, but may be regained by placing a similar weight, 1 pound, on the pan *a*, as shown in Fig. 2 (*c*).

The arrangement shown in Fig. 2 (*a*) may be represented by an equation, as follows:

$$10+8=12+6$$

If 1 pound is added as in Fig. 2 (*b*), the resulting change may be shown in the following manner, the vertical dash (|) being used merely to separate the members on the two sides:

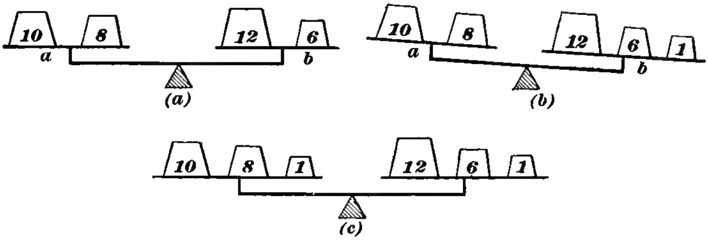


FIG. 2

$$10+8 \mid 12+6+1$$

It is seen that the equality is destroyed and that this expression is not an equation, as the value 18 of the first member is not equal to the value 19 of the second member. To regain equality, a quantity equal to 1 may be added to the left member, making the two members equal, as follows:

$$10+8+1=12+6+1$$

It follows that if a quantity is added to one member of an equation, the same quantity must be added to the other member, to maintain the equality between the members.

14. Subtracting a Quantity From Both Members. If a number is subtracted from one member of an equation, a number of the same value must be subtracted from the other member. For example, the weight on each of the scale pans *a* and *b*, Fig. 3 (*a*), is equal to 27 pounds. If the 2-pound weight is removed from the pan *a* the balance is destroyed,

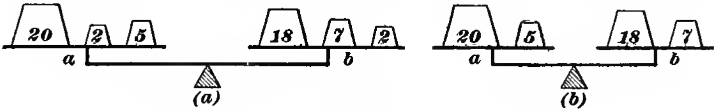


FIG. 3

but it will be regained by removing the 2-pound weight from the pan *b*, as in Fig. 3 (*b*).

The arrangement shown in Fig. 3 (*a*) is represented by the following equation:

$$20+5+2=18+7+2$$

Removing the 2-pound weight from the pan *a*, Fig. 3 (*a*), is equivalent to subtracting 2 from the first member of the equation; thus,

$$20+5+2-2 \mid 18+7+2$$

Performing the subtraction indicated in the first member, the result is $2-2=0$, and the relation will assume the following form:

$$20+5 \mid 18+7+2$$

The equality is seen to be lost, and to regain it, the same value, 2, is subtracted from the second member, as follows:

$$20+5=18+7+2-2=18+7$$

From the preceding examples the following rule is derived:

Rule.—*If a quantity is added to or subtracted from one member of an equation, the same quantity must be added to or subtracted from the other member.*

If this rule is not complied with, the equality ceases to exist; that is, the quantity represented by one member is not equal to the quantity represented by the other member.

15. Multiplying or Dividing Both Members by Same Quantity.—The remarks made about adding a quantity to, or subtracting it from, one member of an equation apply also to the processes of multiplying or dividing a member by a given number. These operations are, in reality, operations of addition or subtraction, repeated a given number of times.

For example, let it be supposed that on the scale pan *a*, Fig. 4 (*a*), there is a weight of 15 pounds, and on the pan *b* the weights of 12 pounds and 3 pounds, also equal to 15 pounds. It is required to multiply the weight on the pan *a* by 3. This is equivalent to increasing the total number of weights to three, or $15+15+15=45$ pounds. The condition will then correspond to that explained in Art. **13**; that is, a weight equal to $45-15=30$ pounds has been added to the pan *a*.

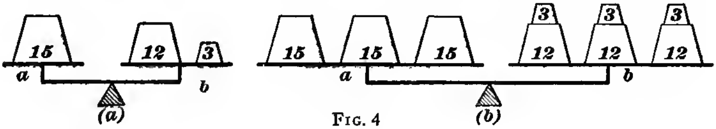


FIG. 4

If equilibrium is to be reestablished the remedy is the same; that is, the weights on the pan *b* must be multiplied by the same number, or 3. These may be as shown in Fig. 4 (*b*), namely, three weights of 12 pounds each and three weights of 3 pounds each. The total weight on the pan *b* is $3\times12+3\times3=45$ pounds.

When this method is applied to an equation the same results are obtained. The arrangement shown in Fig. 4 (*a*) is represented by the equation

$$15=12+3$$

When the first member is multiplied by 3, the relation assumes the following form:

$$15\times3 \mid 12+3$$

This expression shows that the equality is lost, and to regain it, the second member must also be multiplied by 3, thus:

$$15\times3=(12+3)\times3=15\times3$$

The latter equation shows the application of the parenthesis, as explained in Art. **5.** In this case, it is necessary to consider the sum $12+3$ as one number, when multiplying by 3. It is therefore enclosed in parenthesis.

16. Finally, it is necessary to consider the effect produced by dividing one member of an equation by a given number. In Fig. 5 (*a*) there is seen to be a balance between

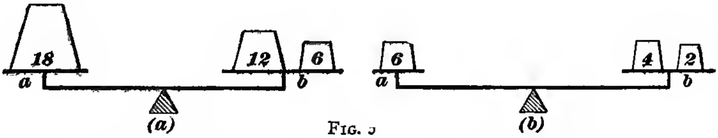


FIG. 5

the weight of 18 pounds, resting on the pan *a*, and the two weights of 12 and 6 pounds, respectively, resting on the pan *b*.

If it is required to divide the weight of 18 pounds by 3, that is, to replace it with a weight one-third as great, or 6 pounds, the evident result will be that the scale is thrown out of balance. To regain equilibrium the weights on the pan *b* must be divided by the same factor, 3. One-third of 12 and of 6 pounds is 4 and 2 pounds, respectively. When the weights on the pan *b* are replaced by these reduced weights, the arrangement will appear as in Fig. 5 (*b*), where it is seen that the pans are again in equilibrium.

On the application of this principle to an equation, similar results are obtained. The arrangement shown in Fig. 5 (*a*) is represented by the equation

$$18=12+6$$

When the first member is divided by 3, the relation assumes the following form:

$$18\div 3 \mid 12+6$$

or,

$$6 \mid 12+6$$

It is seen that, as 6 cannot be equal to 18, the equality is lost, and to regain it, the second member must also be divided by 3, as in the following equation:

$$6=(12+6)\div 3=18\div 3$$

or,

$$6=6$$

Also, in this equation, it is necessary to enclose the second member of the equation, or 12+6, in a parenthesis, before dividing by 3. Otherwise, only the last number, 6, would be divided by 3.

17. Raising a number to a given power is a process of multiplication; that is, of multiplying a number by itself a given number of times. Also, extracting the root of a number is a process of division, as the root must divide into its number, and the successive quotients, as many times as the index of the root indicates. Thus, a number may be divided three times by its cube root. For example, the cube root of 27 is 3; hence, $27\div 3=9$, and $9\div 3=3$, and $3\div 3=1$.

It follows that the rules applying to the multiplication and division of the members of an equation by a given number also apply to raising the members to like powers or extracting like roots of both members. For example, in the equation

$$6=4+2$$

both members may be squared without affecting the equality of the equation. Thus,

$$6^2=(4+2)^2$$

or,

$$36=36$$

Likewise may like roots of both members be extracted. For example, in the equation

$$64=44+20$$

the square root of both members may be extracted; thus,

$$\sqrt{64}=\sqrt{44+20}$$

or,

$$8=8$$

18. From the explanations given in Arts. **13** to **17** the following important rule is derived:

Rule.—*If one member of an equation is subjected to any arithmetical operation, the other member must be treated in the same manner.*

This rule, which is apparently very simple, is of the greatest importance in operations with equations. If not complied with in all cases, serious errors will result. The application of the rule is shown by the following examples:

EXAMPLE 1.—It is required to add 25 to the second member of the equation

$$82+17+14=78+25+10$$

What changes must be made in the equation in order that the equality of the members may be maintained?

SOLUTION.—By the rule, it is necessary to add a number to the first member corresponding with that added to the second one. Or,

$$82+17+14+25=78+25+10+25. \quad \text{Ans.}$$

In its present form the value of each member is 138. In its original form the value was 113; but in both cases there exists a condition of equality.

EXAMPLE 2.—It is desired to subtract 34 from the first member of the following equation: $108+15+23=92+36+18$. What change must be made in the second member?

Solution.—According to the rule, if 34 is subtracted from the first member, a number of equal value must be subtracted from the second member. The equation must, therefore, be written as follows:

$$108+15+23-34=92+36+18-34$$

On subtracting 34 from a larger number in each member, the equation will be as follows:

$$74+15+23=92+2+18. \quad \text{Ans.}$$

In its present form, each member of the equation is equal to 112. In the original equation the value was 146.

Example 3.—If the second member in the equation $94+3=53+44$ is multiplied by 12, what changes must be made in the equation to maintain the equality of the members?

Solution.—According to the rule, it is not allowable to multiply the second member by 12, unless the first member is multiplied by the same factor. The equation must, therefore, be written as follows:

$$(94+3)12=(53+44)12. \quad \text{Ans.}$$

Each member must be enclosed in a parenthesis to insure that both numbers are multiplied by 12, and not only the last one.

Example 4.—The second member of the following equation is to be divided by 6. If the equality is to be maintained, what change is necessary in the equation?

$$81+17-20=64-4+18$$

Solution.—According to the rule, the equation must be written as follows:

$$(81+17-20)\div 6=(64-4+18)\div 6. \quad \text{Ans.}$$

As the whole of each member is to be divided by 6, the members must be enclosed in a parenthesis as shown.

19. Transformation.—In employing equations, it is often necessary to change the position of a number from one member to another; changes of this kind are called **transformations.**

$$15+9+3=20+6+1 \qquad \textbf{(1)}$$

Suppose that in equation **1** it is required to move the number 20 from the right side to the left side of the equality sign. It was explained in Art. **14** that the same number may be subtracted from both members of an equation. Performing this operation by subtracting 20 from both members, the equation will appear as follows:

$$15+9+3-20=20+6+1-20$$

But, in the second member, $20-20=0$. Hence, the equation will be

$$15+9+3-20=6+1 \qquad (\mathbf{2})$$

An examination of equation **2** shows that the equality of the equation is not affected, as, in the first member, $15+9+3-20=7$, and, in the second member, $6+1=7$.

When the equations **1** and **2** are compared,

$$15+9+3=20+6+1$$

and

$$15+9+3-20=6+1$$

it is seen that on transferring 20 from the second to the first member its sign is changed from + to −.

20. It remains to be shown how a number, preceded by a minus sign, may be transposed from one member to the other member of an equation. For example, in the equation

$$105+15-30=40+50$$

it is desired to transfer the number -30 from the first to the second member of the equation. According to the rule in Art. **18,** it is permissible to add the same number to both members. Hence, adding 30 to both members, the equation will appear as follows:

$$105+15-30+30=40+50+30$$

In the first member, the numbers may be written $105+15+30-30$; but $30-30=0$. The equation may, therefore, be written

$$105+15=40+50+30$$

As both members of the equation are equal to 120, the equality of the members has not been affected by the change, and the -30 has been transferred simply by changing its position and its sign.

When no sign precedes a number, a + sign is understood; thus, in the preceding equation, 105 has no sign before it, and therefore it is understood to be +105.

From the two preceding examples the following rule is derived:

Rule.—*If a number or quantity preceded by a plus or a minus sign is transferred from one member of an equation to the other member, its sign must be changed.*

Example 1.—In the equation $32+96+40=100+48+20$, it is required to transfer 32 to the second member.

Solution.—When no sign precedes a number, it is understood that the plus sign is omitted. Hence, the 32 in the first member is considered as $+32$, and by the rule, the equation may be written as follows:

$$96+40=100+48+20-32$$

The sum of the numbers in the first member is 136. In the second member the sum of $100+48+20=168$, and $168-32=136$. Hence, the value of each member is equal to 136.

Example 2.—Transfer the value -46 from the second to the first member in the following equation:

$$96+40+30=212-46$$

By the rule, the equation may be written as follows:

$$96+40+30+46=212$$

When the number -46 is placed in the first member its sign is changed from $-$ to $+$, and each member has a value of 212.

21. Certain numbers may be transferred from one member of an equation to another by means of multiplication or division. For example, if in the equation $\frac{800}{5}=160$ it is required to transfer the denominator 5 to the second member, the rule in Art. **18** is employed and both members are multiplied by 5. Thus,

$$\frac{800\times 5}{5}=160\times 5$$

In the first member the figures 5 in the numerator and denominator cancel each other, and the equation may be written as follows:

$$800=160\times 5$$

It is seen that the equality between the members is not affected.

22. Dividing both members of an equation by the same number is also used as a means for transferring certain numbers or quantities from one member to the other. For example, in the equation

$$450=50\times 9$$

let it be assumed that 50 is to be transferred to the first member, for the purpose of having 9 alone in the second member. This result may be obtained by dividing both members by 50. Thus,

$$\frac{450}{50}=\frac{50\times 9}{50}$$

But, in the second member, the numbers 50, in the numerator and denominator, cancel each other, and the equation may now be written as follows:

$$\frac{450}{50}=9$$

23. The examples in Arts. **21** and **22** show that if the number that requires transposition is a *denominator* of a fraction, both members must be *multiplied* by this number for the purpose of transferring it to the other member. If the number or factor is a *numerator,* both members must be *divided* by the number. If two or more factors are to be transferred, the equation is treated in a similar manner. In all cases the following rule applies:

Rule.—*A factor may be transferred from one member of an equation to the other by multiplying or dividing both members of the equation by the factor that is to be transferred.*

Example 1.—In the equation $325\times 8=2{,}600$ transpose the factor 8 to the second member of the equation.

Solution.—The factor 8 being in the numerator of a fraction having 1 as a denominator, it follows from the preceding explanations and rule that both members must be divided by 8. Thus,

$$\frac{325\times 8}{8}=\frac{2{,}600}{8}$$

Equal factors in the first member may now be canceled and the equation acquires the following form:

$$325=\frac{2{,}600}{8}.\quad \text{Ans.}$$

Example 2.—Transpose the number 5 to the second member of the following equation: $\frac{4{,}500}{5}=900.$

Solution.—By the rule, both members are multiplied by 5, giving the equation the following form:

$$\frac{4{,}500\times 5}{5}=900\times 5$$

The two numbers 5 in the first member cancel each other, and the equation will be

$$4{,}500=900\times5. \quad \text{Ans.}$$

24. In the preceding examples the members of the equations consisted of one or more factors. Equations may besides several factors have single numbers connected by plus or minus signs. For example, $\frac{81}{9}=10+4-5$.

If, in this equation, it is desired to transfer the number 9 from the first to the second member, the latter must be treated as one quantity by enclosing it in a parenthesis. The transposition is then performed in accordance with the rule in Art. **23**, that is, by multiplying both members by 9. Thus,

$$\frac{81\times9}{9}=(10+4-5)9$$

The two 9's in the first member cancel each other and the changed form of the equation is

$$81=(10+4-5)9$$

If, in an equation of the latter kind, the factor that is outside the parenthesis is transferred to the other member, there is no further use for the parenthesis, and it may be removed. The following example will make this clear: In the equation $4\times27=6(25-10+3)$, the factor 6 is to be transferred to the first member. Following the rule in Art. **23**, both members are divided by 6. Thus,

$$\frac{4\times27}{6}=\frac{6(25-10+3)}{6}$$

The factors 6 in the second member cancel each other and the equation may be written

$$\frac{4\times27}{6}=(25-10+3)$$

As the parenthesis is no longer required, it is removed and the final form of the equation is

$$\frac{4\times27}{6}=25-10+3$$

When the operations indicated by the signs are performed, the equation is

$$\frac{108}{6}=25-10+3$$

or,

$$18=18$$

It is seen that the equality of the two members has not been affected by the transposition.

Example.—In the equation $53\times8=4(190-92+8)$ the factor 53 is to be transferred to the second member, so that the first member contains but the one factor 8.

Solution.—By the rule in Art. **23**, both members are divided by 53. Thus,

$$\frac{53\times8}{53}=\frac{4\,(190-92+8)}{53}$$

The two equal factors 53 in the first member cancel and the equation appears as follows:

$$8=\frac{4\,(190-92+8)}{53}$$

That the members still have equal values may be ascertained by performing the required arithmetical operations. Thus, adding and subtracting the numbers in the parenthesis, the result is as shown in the following equation:

$$8=\frac{4\times106}{53}$$

or,

$$8=8$$

EQUATIONS FORMED WITH SYMBOLS

APPLICATION OF SYMBOLS

25. Numbers Represented by Symbols.—The preceding articles explain the fundamental properties of equations by means of numbers, so as to show more clearly the results of the various operations. In the succeeding pages symbols, such as single letters, will be used to represent the various numbers used in the equations. The advantage to be derived from this method will be shown gradually, as the scope of the examples is becoming more extended. No change in method, as regards the treatment of equations, is involved by substituting letters for numbers.

Using letters to represent numbers is not a method limited solely to equations. It is used, for example, to a great extent in stores as a means for indicating prices on articles, so as to prevent the customer from reading the prices. But, the salesman, being furnished with a key, knows the meaning of each letter, and is therefore able to tell the price.

26. Standard Symbols.—Any letter may serve as a symbol for any particular dimension, referring to a surface, a solid, or for dimensions of any part of a machine or structure; also for numbers that represent revolutions, horsepower, speed, etc. A number of the more frequently used symbols are given herewith.

A = area of a surface;
C = circumference of a circle;
d = diameter of a circle;
r = radius of a circle;
h = altitude of a triangle;
H = horsepower;
N = revolutions per minute.

It is important to remember that, while a letter may be given any value whatever, it cannot change its meaning in any particular equation, but must retain the same value throughout.

27. Omitting Multiplication and Division Signs. It was stated in Art. **5** that in formulas and equations the various arithmetical operations are indicated by the same signs as were used in arithmetic. In formulas and equations, however, where brevity is always desired, it is possible in many cases to omit the multiplication and division signs, when such omissions will not cause any misunderstanding. Thus, instead of writing $a \times b$, the sign of multiplication is omitted, and the product of the two letters is indicated by writing them close together, without any intervening sign. Hence, $a \times b$ is written $a\ b$; $d \times e \times f$ is written $d\ e\ f$; $b \times (c+e)$ is written $b\,(c+e)$; and so forth. This simplified method of indicating multiplication cannot, however, be adopted for numbers. For example, 3×7 cannot be written 37, for 37 has been defined, in arithmetic, to mean $30+7$. With the thought constantly in mind that a letter is a symbol for a number, there should be no difficulty in multiplying by letters.

Proceeding to the subject of powers, it follows that, as $3\times3=3^2$, so $a\times a=a^2$. This latter expression is read *a square,* following the rule given in *Powers and Roots.* If there are three similar factors, as $b\times b\times b$, the result is b^3, which is read *b cube;* b^4 is read *b fourth* and is equal to $b\times b\times b\times b$; c^5 is *c fifth* and is equal to $c\times c\times c\times c\times c$; and so on for any other power.

When division of symbols is indicated, as $a\div b$, the sign of division may be omitted by writing the factors as a fraction; thus, $a\div b$ may be written $\frac{a}{b}$. Also, $e\,f\div g$ is written $\frac{e\,f}{g}$; $2\,a\,b\div c\,d$ is written $\frac{2\,a\,b}{c\,d}$; and so forth.

28. Supplementary Symbols.—Only a limited number of letters are employed in the equations dealt with in the succeeding pages. In practice, the letters of the alphabet are, at times, insufficient to meet the requirements, and it is found necessary to have recourse to supplementary marks and symbols, known as *primes* and *subscripts.* The marks ′, ″, ‴, are called **primes;** small letters or numbers, as $_a$, $_b$, $_c$, etc. and $_1$, $_2$, $_3$, etc., printed below the line, are known as **subscripts,** or **subs.**

When capital letters have primes annexed to them, as A', B', A'', C''', they are commonly read *large a prime, large b prime, large a second, large c third,* respectively; but when small letters are used, as a', b', c'', d''', they are read *small a prime, small b prime, small c second, small d third,* respectively. Similarly, such expressions as a_1, D_2, W_3, f_4 are read *small a sub one, large d sub two, large w sub three, small f sub four,* respectively.

The words *large* and *small* are used in reading these expressions only when both capitals *and* small letters occur in the same problem. If all the letters are capitals, or if all the letters are small letters, the words *large* and *small* are dropped. Under these conditions, F', P_2, R'' would be read *f prime, p sub two, r second;* and d_1, g'', a_3 would be read *d sub one, g second, a sub three.*

In writing the different letters, the capital letters are formed like *printed* letters, so that they may be easily distinguished from small letters. On the whole, care should always be taken in writing letters to see that there is no confusion; that, for instance, subs are not mistaken for exponents, and that each letter represents just what is intended.

OPERATIONS WITH THE TERMS OF AN EQUATION

29. Rules Applying to Equations.—All the rules given in the preceding articles apply also to equations composed of symbols and will, therefore, require no further explanations. The application of the preceding rules will be shown by means of examples, and if it is kept in mind that the letters stand for certain given numbers, there should be no difficulty in applying the rules.

The following example will illustrate the application of the rule, in Art. **14**, stating that no quantity can be added to or subtracted from one member of an equation without performing a similar operation with the other member. The given equation is:

$$a+b+c=d+e+f$$

Supposing that the symbol g is to be *added* to the first member, it must also be added to the second member, and the changed equation will assume the following form:

$$a+b+c+g=d+e+f+g$$

These letters may have any value, but the arrangement of the letters in the two members must be such as to comply with the law that the two members must have equal values. In the present case the values of the letters are assumed to be as follows:

$a=1$ $\quad e=4$

$b=3$ $\quad f=7$

$c=9$ $\quad g=8$

$d=2$

When these values are substituted for the letters, the equation assumes the following form:

$$1+3+9+8=2+4+7+8$$

or,

$$21=21$$

If the letter g is *subtracted* from the first member, it must, according to the rule, be subtracted also from the second member. Thus,

$$a+b+c-g=d+e+f-g$$

The substitution of the given values makes the equation

$$1+3+9-8=2+4+7-8$$

or,

$$5=5$$

30. The rule in Art. **18**, according to which both members of an equation may be multiplied or divided by the same number, is illustrated by means of the following examples:

EXAMPLE 1.—The first member of the equation $a=b+c$ is to be multiplied by the factor d.

SOLUTION.—According to the rule both members must be multiplied by d, or

$$a\,d=(b+c)\,d. \text{ Ans.}$$

EXAMPLE 2.—In the equation $a=b+c$, the second member must be divided by d.

SOLUTION.—To comply with the rule both members must be divided by d; hence,

$$\frac{a}{d}=\frac{b+c}{d}. \text{ Ans.}$$

PROOF.—In examples 1 and 2 the letters have the following values:

$$a=12,\ b=9,\ c=3, \text{ and } d=5$$

The substitution of these values in example 1 makes the given equation $12=9+3$, and the solution is $12\times5=(9+3)\,5$. In example 2 the solution is

$$\frac{12}{5}=\frac{9+3}{5}$$

31. Expressions and Terms.—Any number, a single symbol used to represent a mathematical quantity, such as a letter, or a combination of a number and letters is called a **mathematical expression,** or simply an **expression.** Thus, 24 and a, each representing a number, are expressions. Combinations of numbers and letters or of several letters, such as $13\,a$, $a-b$, $a\,b$, a^2+b^2, $\frac{a\,b}{c}$, and $2+5\,a$, are all mathematical expressions, because each may represent a number.

It is seen that the foregoing expressions indicate addition, subtraction, multiplication, and division. In many cases it is preferable to have a special name for single letters and for expressions that refer only to *multiplication* and *division,* such as a, b, $a \times b$, $\frac{a\,b}{c}$, and $\frac{2\,a\,x^2}{e}$, etc. Each of these expressions is called a **term.**

The expression $c\,e - c^2\,r$ may then be said to consist of the two terms $c\,e$ and $c^2\,r$, connected by a minus sign; the expression $a+b-c$ consists of the three terms a, b, and c, connected by a plus and a minus sign.

32. Coefficients and Literals.—When a term is expressed by letters and a number, written before the letters, the number is called the **coefficient.** The coefficient always precedes the letters of the terms; thus in $8\,a\,b$ the figure 8 is a coefficient. The coefficient shows how many times the rest of the term is to be taken. In the term $8\,a\,b$ the coefficient 8 indicates that the quantity $a\,b$ is to be taken eight times.

At times it is convenient to have a special name for the portion of a term that consists of letters, as in $10\,c\,d$. In this and similar terms 10 is the coefficient and $c\,d$ the **literal expression, literal quantity,** or as it may be called, simply the **literal.**

When no coefficient is written in a term, the coefficient 1 is always to be understood. Thus, a means $1\,a$, and $c\,d$ means $1\,c\,d$.

33. Like Terms.—If the sum is to be found of a number of similar coins they are simply counted and the number found is the sum. Thus, counting six single cents, the sum is written 6 cents. Similarly, if the letter c represents 1 cent, the expression $c+c+c+c+c+c$ means that the number of c's is to be found by counting, or addition. The fact that there are six c's is represented by the expression $6\,c$, the coefficient 6 indicating the number of c's.

In counting money, unlike coins are kept separate, as one could not count a combination of cents and dimes, as representing either cents or dimes. Similarly, in the expression

$c+c+c+d+d+d$, one could not add all the letters and state the sum as $6c$ or $6d$, as both would be wrong. Hence, similar letters must be grouped together and then added. In the preceding example, the sum is $3c+3d$.

34. Terms that contain the same letters or combination of letters are called **like terms;** all others are **unlike terms.** The term *like* does not mean, in this case, that the terms are of equal *values,* but that the terms are of the same *kind.* Thus, the terms $3c$ and $5c$ are like terms, because both terms contain the same literals c, but they are not of the same value, as the coefficients 3 and 5 are not equal. Similarly, the expressions 3 cents and 5 cents are like, as both deal with cents, but the values of the two expressions differ.

In the terms $3ab$, $4ab$, and $9ab$ the literals, ab, are the same in all of them; hence, they are like terms. If the letters have the same exponents, they are like terms. Thus, the terms $3cd^2$, $4cd^2$, and $6cd^2$ are like terms, because the literals cd^2 are the same in all of them.

35. Unlike Terms.—Terms are **unlike** when they contain different literals. For example, the terms $2a$, $13b$, and $27c$ are unlike terms, because they contain different letters. Supposing that a, b, and c represent hours, minutes, and seconds, respectively, it is obvious that the sum of the terms $2a$, $13b$, and $27c$ cannot be found by adding the coefficients 2, 13, and 27, as the sum, 42, would not represent either hours, minutes, or seconds. Hence, in adding these terms, the sum must be written $2a+13b+27c$, just as one would write 2 hours 13 minutes 27 seconds, the plus signs being omitted in the latter case, as being understood.

Terms are also classified as unlike if they contain different combinations of letters, or if the exponents of the letters are not alike. Thus, $2ab$, $5ac$, and $3bc$ are unlike terms, as each one contains a different combination of letters. The terms $2a^2b$, $2a^3b$, $3ab$, $7ab^2$, $4ab^3$, and $5a^2b^2$ are all unlike terms, notwithstanding the fact that the terms are composed of the same letters. They are unlike terms, because the exponents of similar letters are different in each term.

Terms as $3a^2b$ and $4a^2b$ are like and so are the terms $5ab^3$, $7ab^3$, and $4ab^3$, as not alone the letters, but also the respective exponents, are like. It is important that this subject of *like* terms be well understood, so as to avoid serious mistakes in addition and subtraction of terms. Remember that *like terms are those which have the same letters affected by the same exponents, though the terms need not have the same coefficients.*

36. Addition of Like Terms.—In a term consisting of a coefficient and one or more letters, as $4ab$, the coefficient, 4, indicates how many times the quantity ab is to be taken; in this case four times, or $ab+ab+ab+ab$. Let it be supposed that $a=2$, and $b=3$; the expression ab is then equal to $2\times3=6$, and the expression $ab+ab+ab+ab$ would be equal to $6+6+6+6$. The term $4ab$ would then be equivalent to four sixes. For similar reasons the term $3ab$ would be equal to three sixes. The expression $4ab+3ab$ would, therefore, be equal to four sixes plus three sixes, or seven sixes. Hence, $4ab+3ab$ may be written $7ab$. Similarly, $5cd+2cd+14cd$ is equal to $21cd$. The principle employed in the preceding explanations may be stated as a rule, as follows:

Rule.—*To add like terms, add the coefficients, and to the sum annex the common letters.*

37. Subtraction of Like Terms.—The explanations referring to the addition of like terms apply also to the subtraction of such terms. Thus, the terms $12cd$ and $9cd$ being like terms, it is possible to find their difference by subtracting the smaller coefficient from the larger, or $12-9=3$. Hence, $12cd-9cd$ is equal to $3cd$, just as 7 inches − 4 inches is equal to 3 inches. For subtracting similar terms the following rule applies:

Rule.—*To subtract like terms, find the difference between the coefficients, and to the difference annex the common letters.*

38. When subtracting like terms it will occasionally happen that the subtrahend is larger than the minuend. For example, in the expression $5-14$, the subtrahend 14 is larger than the minuend 5. In all cases the smaller number is subtracted from the larger, and the difference is given the sign of the larger number; thus $14-5=9$, and $+5-14=-9$.

Similarly, in the expression $12a+14b-20b$, the coefficient 20 is larger than 14 in the like terms $20b$ and $14b$. The difference, $14b-20b$, is equal to $-6b$, and the simplified expression is written $12a-6b$.

In expressions of this kind the following rule applies:

Rule.—*When a subtrahend is larger than the minuend, the difference is found by subtracting the smaller number from the larger and placing a minus sign in front of the result.*

Example.—Combine like terms in the following expression:

$$14ab+3ab-15ab-9bc+14bc-12bc.$$

Solution.—When the like terms having the same sign are combined, the expression becomes $17ab-15ab+14bc-21bc$; $17ab-15ab=2ab$ and $14bc-21bc=-7bc$. The result is $2ab-7bc$. Ans.

39. Application of Rules.—The preceding rules apply to any expressions irrespective of whether they stand alone or form parts of an equation. In the following examples the various expressions are connected by an equality sign; they represent, therefore, equations and the rules in Art. **20** must be applied when a term is transposed from one member of the equation to the other member.

Example 1.—Add like terms in the equation $4a+2b+3a=8c$.

Solution.—The terms $4a$ and $3a$ being like terms, they may be added by adding the coefficients; thus, $4a+3a=7a$. The equation may now be written as follows: $7a+2b=8c$. Ans.

Example 2.—Add or subtract like terms in the equation $16a+4b+9c=8a+2d+7c$.

Solution.—Until sufficient experience is gained it is preferable to have those terms that are to be added or subtracted situated in the same member; but, according to the rule in Art. **20**, a term may be transposed from one member of the equation to the other by changing the sign of the term. Hence, $8a$ and $7c$ may be transposed to the first member, where they become $-8a$ and $-7c$ and the equation is written as follows:

$$16a-8a+4b+9c-7c=2d$$

But $16a-8a=8a$, and $9c-7c=2c$. Hence, the equation is now

$$8a+4b+2c=2d. \text{ Ans.}$$

Example 3.—Add like terms in the equation $a+\frac{3}{10}a+\frac{3}{4}b+\frac{1}{2}b=4c$.

Solution.—Whenever a term is without any coefficient, as a, it is understood that 1 is its coefficient; thus, a means $1a$. Then, $\frac{3}{10}a$ is to be added to $1a$, making $1\frac{3}{10}a$, or $\frac{13}{10}a$.

The term $\frac{1}{2}b$ must be reduced to fourths before it can be added to $\frac{3}{4}b$; thus, $\frac{1}{2}=\frac{2}{4}$, and $\frac{2}{4}+\frac{3}{4}=\frac{5}{4}$. Hence, $\frac{3}{4}b+\frac{1}{2}b=\frac{5}{4}b$.

The whole equation may now be written $\frac{13}{10}a+\frac{5}{4}b=4c$, or, in mixed numbers,

$$1\tfrac{3}{10}a+1\tfrac{1}{4}b=4c. \quad \text{Ans.}$$

Example 4.—Combine like terms as indicated in the equation $5a+\frac{1}{2}a+4b-9b=16c-1\frac{1}{2}c$.

Solution.—In the first member of the equation the sum of $5a$ and $\frac{1}{2}a$ is $5\frac{1}{2}a$, *and* $+4b-9b=-5b$, according to the rule in Art. **38**; in the second member $16c-1\frac{1}{2}c=14\frac{1}{2}c$. The simplified equation may now be written:

$$5\tfrac{1}{2}a-5b=14\tfrac{1}{2}c. \quad \text{Ans.}$$

Example 5.—Combine like terms in the following equation:

$$9.13a-16.42b=11.47c-2.93b-5.25a+8.32a$$

Solution.—When like terms are transposed to the same side of the equation, the equation may be written as follows:

$$9.13a+5.25a-8.32a=16.42b-2.93b+11.47c$$

Note.—At the present stage it does not matter whether the terms are transposed to the first or to the second member, but when the conditions are such as to require one term to be alone in one member, then the remaining terms must all be transposed to the other member.

In the equation the addition and subtraction of like terms may be performed in the following manner: $9.13a+5.25a=14.38a$; $14.38a-8.32a=6.06a$. The difference between $16.42b$ and $2.93b$ is $13.49b$.

The equation in its simplified form is:

$$6.06a=13.49b+11.47c. \quad \text{Ans.}$$

EXAMPLES FOR PRACTICE

Perform the operations indicated in the following expressions:

1. $3c+4c+7c+8c$. Ans. $22c$
2. $ab^2+3ab^2+11ab^2+13ab^2$. Ans. $28ab^2$
3. $7a^2+a^2+13a^2-5a^2$. Ans. $16a^2$

Simplify the following equations:

4. $7a+2a-3a=8b-3b$. Ans. $6a=5b$
5. $9x+4y-3z-6x-6y+4z=14-3$. Ans. $3x-2y+z=11$
6. $9xy+2yz-4xy=15xz-6xz+2yz$. Ans. $5xy=9xz$

40. Multiplication of Terms.—The restriction that prevents addition of unlike terms does not apply to *multiplication* of such terms. Either like or unlike terms can be multiplied. For example, if 6 is multiplied by 3, 6 is to be taken three times, or $6+6+6$. Similarly, if a is multiplied by b, the factor a is to be taken b times, and b may here be of any value. In the first case the product is $6\times 3=18$; in the latter case it is $a\times b=a\,b$.

If a term, consisting of a coefficient and one or more letters, as $24\,a\,b$, is to be multiplied by a number, as 4, the product is found as follows: $24\,a\,b\times 4=96\,a\,b$. That is, the coefficient is multiplied by the number in the usual way and the letters are annexed to the product. If two terms containing literals, or letters, are to be multiplied together, as in the example $14\,a\,b\times 3\,c$, the numbers, or coefficients, are multiplied as before, and the letter c of the second term is annexed to the letters in the first term; thus, $14\,a\,b\times 3\,c=42\,a\,b\,c$.

It is customary to arrange letters and quantities alphabetically in all algebraic expressions; thus, $4\,a\,b\,c$ *rather than* $4\,c\,a\,b$ or $4\,b\,a\,c$, etc. Also, write $4\,a\,b+3\,a\,c+4\,a\,d$ rather than place the term containing b after one containing c or d.

If the same letter is contained in several terms that are to be multiplied together, the letter in the product has an exponent equal to the sum of its exponents in the several terms. That is, the number of times the letter is used as a factor in the product is the sum of its exponents in the given terms. Thus, $11\,c\times 3\,c=33\,c^2$; $6\,b\times 4\,b\times 2\,b=48\,b^3$. If the terms contain several similar letters, the same method is adopted for each letter. For example, $8\,a\,b\,c\times 5\,a\,b\,c=40\,a^2\,b^2\,c^2$. In the following examples, the products are found by applying this method; thus, $9\,a\,b\times 7\,a\,b\,c=63\,a^2\,b^2\,c$; $14\,a\,c\,d\times 2\,c\,d=28\,a\,c^2\,d^2$; $8\,a^2\times 3\,a\,c^3\times 5\,b\,c=120\,a^3\,b\,c^4$.

41. Division of Terms.—The operation of division may be performed also with either like or unlike terms. For example, $16\,a\,b\div c\,d=\frac{16\,a\,b}{c\,d}$; $(2\,a^2\,b+3\,c\,d^2)\div 5\,e=\frac{2\,a^2\,b+3\,c\,d^2}{5\,e}$. Cancelation should be employed, whenever possible, to simplify expressions. A few examples will show the general method to be followed. The expression $2\,a \div 2$ is equal to $\frac{2\,a}{2}$, in which the numbers 2 cancel each other, so that only the letter a remains. Hence, $2\,a \div 2 = a$. Also, $6\,a \div 2\,a$ may be written $\frac{6\,a}{2\,a}$, or $\frac{6\times a}{2\times a}$; the similar letters a are canceled, leaving the fraction $\frac{6}{2}$, of which the quotient is 3. Thus, $6\,a \div 2\,a = 3$. Similarly, $6\,a\,b \div 2\,a = \frac{6\,a\,b}{2\,a} = \frac{6\times a\times b}{2\times a} = \frac{6\,b}{2} = 3\,b$.

The following examples are solved in the same manner:

$$120\,a\,b\,c \div 6\,a\,b = \frac{120\,a\,b\,c}{6\,a\,b} = \frac{120\,c}{6} = 20\,c$$

$$4\,a \div 2\,b = \frac{4\,a}{2\,b} = \frac{2\,a}{b}$$

$$12\,a\,b \div 2\,c = \frac{12\,a\,b}{2\,c} = \frac{6\,a\,b}{c}$$

EXAMPLES FOR PRACTICE

1. Multiply $2\,x$ by $3\,y$. Ans. $6\,x\,y$
2. Multiply $2\,a\,c$ by $4\,a\,b$. Ans. $8\,a^2\,b\,c$
3. Multiply $8\,a\,c$ by $5\,b\,d$. Ans. $40\,a\,b\,c\,d$
4. Multiply $3a^2\,c$ by $4\,b^2\,c^2$. Ans. $12\,a^2\,b^2\,c^3$
5. Divide $2\,x$ by $3\,y$. Ans. $\frac{2\,x}{3\,y}$
6. Divide $12\,a\,b$ by $4\,b$. Ans. $3\,a$
7. Divide $9\,b\,c$ by $3\,a\,d$. Ans. $\frac{3\,b\,c}{a\,d}$
8. Multiply $3\,x$ by $6\,y$ and divide the product by $3\,z$. Ans. $\frac{6\,x\,y}{z}$

PARENTHESIS

42. Application to Symbols.—The application of a parenthesis, brackets, and brace to numbers is explained in Arts. **5** to **7**. The rules given apply also to quantities indicated by letters; thus, $(a+b)\times c$ means that the sum of a and b is to be multiplied by c. The sign of multiplication may be omitted and the letter c written near the parenthesis in the same manner as two letters are combined to indicate multiplication. Thus, $(a+b)\times c$ may be written $(a+b)c$.

The latter expression may be interpreted in two different ways; thus, $(a+b)c$ may be said to mean that the sum of a and b is to be multiplied by c, or the separate letters may be multiplied by c and written thus, $a\,c+b\,c$; that is, $(a+b)c=a\,c+b\,c$. The result is the same in either case, as may be seen from the following example. Letting $a=4$, $b=5$, and $c=3$, it follows that $(a+b)\,c=(4+5)\times 3$, or $9\times 3=27$. If the expression is read in the other way, or $a\,c+b\,c$, then, $4\times 3+5\times 3=12+15=27$, as before.

The expression $(a-b)c$ may also be interpreted in two ways, as, for instance, that b is to be subtracted from a and the remainder multiplied by c; or, that a and b are to be multiplied successively by c and the last product subtracted from the first. That is, $(a-b)c=a\,c-b\,c$.

43. Removal of Parenthesis.—If the numbers 42, 10, and 8 are added, the sum is 60; that is, $42+10+8=60$. If the sum of 10 and 8 is added to 42, the sum is also 60; that is, $42+(10+8)=60$. Hence, $42+(10+8)=42+10+8$. In the same way, $a+(b+c)=a+b+c$.

Again, if 42 and 10 are added together and 8 is subtracted from the sum, the result is 44; that is, $42+10-8=44$. If 8 is subtracted from 10 and the remainder is added to 42, the result is also 44; that is, $42+(10-8)=44$. Hence, $42+(10-8)=42+10-8$. In a similar manner, $a+(b-c)=a+b-c$.

From the preceding examples it is seen that, if a parenthesis is preceded by a *plus sign,* the parenthesis may be removed without changing the value of the expression. From this the following rule is formulated:

Rule.—*If an expression, indicating addition and subtraction only, contains a parenthesis preceded by a plus sign, the parenthesis may be removed without changing any signs.*

44. The preceding rule does not apply, if the parenthesis is preceded by a *minus sign.* For example, in the expression

$42-10-8$, if 10 is subtracted from 42, and then 8 is taken from the remainder, the result is 24; that is, $42-10-8=24$. If the sum of 10 and 8 is subtracted from 42, the result is also 24: that is, $42-(10+8)=24$. Hence,

$$42-(10+8)=42-10-8$$

For similar reasons,

$$a-(b+c)=a-b-c$$

If 10 is subtracted from 42 and 8 is added to the difference, the result is 40; that is, $42-10+8=40$. Again, if the remainder, obtained by subtracting 8 from 10, is subtracted from 42, the result is also 40; that is, $42-(10-8)=40$. Hence,

$$42-(10-8)=42-10+8$$

Similarly,

$$a-(b-c)=a-b+c$$

From these examples the following rule is derived:

Rule.—*If an expression, indicating addition and subtraction only, contains a parenthesis preceded by a minus sign, the parenthesis may be removed, if the sign of every term within the parenthesis is changed.*

EXAMPLE.—Remove the parentheses from the expression $(a+b-2c)-(a-b-4c)$ and perform the required operations of addition and subtraction.

SOLUTION.—As any quantity is supposed to be preceded by a plus sign, unless another sign is indicated, the first parenthesis is supposed to have a plus sign in front of it. The parenthesis may therefore, according to the rule in Art. **43**, be removed without requiring any change of signs of the enclosed terms. The second parenthesis is preceded by a minus sign; hence, on removing the parenthesis, the signs of the enclosed terms must, according to the last rule, be changed. The first term a in the second parenthesis has a plus sign, understood, and when the parenthesis is removed the two quantities will appear as follows: $a+b-2c-a+b+4c$.

It is to be noted that the second letter a has received a minus sign, instead of its previous plus sign, and that the terms b and $4c$ have received plus instead of minus signs.

Addition and subtraction of like terms may now be performed. Thus, $a-a=0$; $b+b=2b$, and $4c-2c=2c$. The simplified expression will have the following form:

$2b+2c$. Ans.

45. The parenthesis *cannot* be removed according to the rules of Arts. **43** and **44** if the expression involves multiplication, unless each of the terms within the parenthesis is multiplied by the factor outside the parenthesis, as explained in Art. **42.** For example, take the expression $(8+7)\times 4$. This means that the sum of 8 and 7, or 15, is to be multiplied by 4, and the result is 60; also, $4\times 8+4\times 7=60$. Now, if the parenthesis were dropped without performing the multiplication, the expression would become $8+7\times 4$, which is equivalent to $8+28=36$, showing that the dropping of the parenthesis without performing the multiplication indicated alters the value of the expression and therefore is incorrect. In other words, the parenthesis cannot be omitted from such an expression as $(a+b)c$ unless both a and b be multiplied by c, because the value would be incorrectly stated if that were done. Similarly, the parenthesis cannot be omitted from the expression $160-4\times(36-28)$. As it stands, this expression indicates $160-4\times 8=160-32=128$. But if the parenthesis were omitted, and the sign changed, the equation would become $160-4\times 36+28=160-144+28=44$, which would be altogether wrong.

Therefore, the rules in Arts. **43** and **44** must not be applied if the expression in parenthesis is multiplied by any term. The rules are applicable only to cases such as have been illustrated in Arts. **43** and **44.**

EXAMPLES FOR PRACTICE

Remove the parentheses from the following expressions and combine like terms:

1. $(a+b-2c)-(a-b-c)$. Ans. $2b-c$
2. $(3e+4f-5)+(2e-5f+1)$. Ans. $5e-f-4$
3. $(2a+b)-(a-b)+(a-2b)$. Ans. $2a$
4. $(14ab-9ab)+(9b-12b)-(13bc-7bc)$. Ans. $5ab-3b-6bc$

EQUATIONS WITH ONE UNKNOWN TERM

46. Preliminaries.—In the preceding pages of this Section the elements of equations have been described in detail, and also the means by which these elements may be combined or transposed to the different parts of an equation. In the equations that have been dealt with hitherto, the terms had known values, and the advantages to be obtained by the use of equations could, therefore, not be displayed. But, having obtained the necessary information by which to manipulate equations, it is now possible to take a step farther and to describe their application.

47. Elementary Equations.—The fundamental parts of an equation are the two members connected by an equality sign, as $2+3=5$.

If one of the terms of an equation is removed or covered, the required conditions of equality will be a means for ascertaining what its value must be. Thus, suppose the black circle in the following equation represents a coin covering one of the terms of the equation:

$$3+5+\bullet=10+7$$

The fundamental requirement of an equation is that both members must be of equal value; therefore, the value of the first member must be equal to that of the second member, or $10+7=17$. The sum of 3 and 5 in the first member is 8, and the number covered must therefore be $17-8=9$. The complete equation is

$$3+5+9=10+7$$

48. In the following equation the second term is hidden and its value is to be found:

$$42-\bullet=18\times3-30$$

The value of the second member is equal to $54-30=24$. As the first member must have the same value, the hidden term must be $42-24=18$, and the complete equation is

$$42-18=18\times3-30$$

In the following equation the denominator of the first member is hidden, and must be found:

$$\frac{8\times4}{\bullet}=56-30-10$$

The value of the second member is $56-40=16$, and the unknown number must be such that when divided into 32 the quotient will be 16; that is, the product of the unknown number and 16 is 32. Therefore, the unknown number must be $32\div16$, or 2, and the complete equation is

$$\frac{8\times4}{2}=56-30-10$$

or,

$$16=16$$

49. Symbol for Unknown Term.—In all equations it is customary to indicate the position of the unknown term by means of a letter, and for this purpose the letter x is in general use. Thus, if the sum is to be found of the numbers 5, 3, 9, and 2, they may be arranged to form an equation by letting the letter x represent the unknown sum. Or,

$$x=5+3+9+2$$

On performing the addition, the sum is found to be 19. Hence,

$$x=19$$

In this example the unknown term x constitutes the *first* member of the equation and the known terms the *second* member. This is the most convenient arrangement for finding the value of an unknown term x, and the various transformations to which an equation is subjected aim to separate x from the other terms.

50. Stating a Problem in the Form of an Equation.—The method to be adopted when a problem is to be stated in the form of an equation is shown by the following examples:

EXAMPLE 1.—What number must be added to 9 to give a sum of 21?

SOLUTION.—Letting x represent the unknown number, the equation may be stated as follows:

$$9+x=21$$

Here, $9+x$ is the first member and 21 the second one. The equation may be solved by transposing the terms so that x stands alone on one side of the equality sign. Thus, transposing 9 to the second member, changing the sign in so doing, the result will be

$$x=21-9$$

or, $$x=12. \text{ Ans.}$$

By reason of the simplicity of the equation, it may also be solved by inspection. In this equation the value of x must be equal to the difference between 21 and 9, or 12, and if x is replaced by 12, the equation is solved, as $9+12=21$. It follows that $x=12$.

Example 2.—What number must be subtracted from 52 to give a difference of 39?

Solution.—If 39 is made the second member of the equation, the equation may be written as follows:

$$52-x=39$$

If 39 is subtracted from 52 the difference is 13, which must be the value of x. Hence, the complete equation is written

$$52-13=39$$

As the requirement of equality between the members is complied with, it follows that $x=13$.

Or, another way to solve this example is to transpose the terms so that x stands alone on one side of the equality sign. Thus, starting with the equation $52-x=39$, transpose $-x$ to the second member and $+39$ to the first member, changing signs in so doing, and the result will be

$$52-39=x$$

or, $$x=52-39=13. \text{ Ans.}$$

Example 3.—What number must be added to the sum of 24 and 9 to give a sum equal to that of 38 and 5?

Solution.—If the first sum and x are placed in the first member and the second sum in the second member, the equation will be as follows:

$$24+9+x=38+5$$

In this example the number of terms is too great to find the value of x conveniently by inspection. It is therefore preferable to transpose all the terms, except x, to the second member. This results in a changing of signs, and the equation will now have the following form:

$$x=38+5-24-9$$

When terms with like signs are combined, $x=43-33$; and when 33 is subtracted from 43, the difference is 10; hence,

$$x=10. \text{ Ans.}$$

That the solution is correct may be proved by substituting 10 for x in the original equation; thus,

$$24+9+10=38+5$$

or, $$43=43$$

Example 4.—There are 41 barrels of oil in storage. If there were three times as many on the preceding day, how many were there? State the equation.

Solution.—The value of x is evidently three times the present number of barrels, or,

$$x=3\times41$$

Hence, $x=123$ bbl. Ans.

Example 5.—A number when decreased by 9 is equal to 3. What is the number?

Solution.—If x is the unknown number, the problem states that if 9 is subtracted from x, the difference is 3. The equation is, therefore,

$$x-9=3$$

or, $x=3+9$

from which $x=12$. Ans.

Example 6.—If 225 tons of coal were shipped on Tuesday, and the quantity was three times as much as that shipped on Monday, how much was shipped on the latter day?

Solution.—The unknown quantity x must be multiplied by 3 to be equal to the known amount, 225 tons. Hence,

$$3\,x=225$$

By the rule in Art. **23**, both members are divided by 3, and

$$\frac{3\,x}{3}=\frac{225}{3}$$

When equal numbers in the first member are canceled, the equation is

$$x=\frac{225}{3}$$

or, $x=75$ tons. Ans.

Example 7.—If 8 is added to the product of 9 and an unknown number, the sum is 71. Find the unknown number.

Solution.—The problem states that if 8 is added to the product $9\,x$, the sum is 71. Hence, the equation is as follows:

$$9\,x+8=71$$

The number 8 is transposed to the second member; thus,

$$9\,x=71-8$$

In order that x may be alone in the first member, both members are divided by 9. Hence,

$$\frac{9\,x}{9}=\frac{71-8}{9}$$

or, $x=\frac{71-8}{9}=\frac{63}{9}=7$. Ans.

Example 8.—The area of a triangle is 92 square inches and its base is 40 inches. What is its altitude?

SOLUTION.—According to the principles of mensuration, the area of a triangle is equal to half the product of its base and altitude. The area is given as 92, the base as 40, and the altitude, which is to be found, may be represented by x. Then $92=\frac{40\,x}{2}$, from which $92=20\,x$ or $20\,x=92$, and $x=\frac{92}{20}=4.6$ in. Ans.

EXAMPLE 9.—A rectangle has an area of 15.2 square inches. If its width is 6 inches, what is its length?

SOLUTION.—From mensuration it is known that the area of a rectangle is equal to the product of its length and width. If A=area, b=width, and x=length, the following equation applies:

$$A=b\,x$$

When both members are divided by b to have x alone in one member, the equation is:

$$\frac{A}{b}=\frac{b\,x}{b}$$

Equal letters in the second member are canceled and the equation will be

$$\frac{A}{b}=x$$

or,

$$x=\frac{A}{b}$$

The given values are then substituted, or $A=15.2$ and $b=6$, and

$$x=\frac{15.2}{6}$$

from which $x=2.53$ in., nearly. Ans.

51. Clearing an Equation of Fractions.—The process of clearing an equation of fractions is most important, and it should be thoroughly understood. The principle on which the process is based will be explained by means of an example. Suppose, for instance, that the value of x is to be found in the equation

$$\frac{x}{2}+\frac{x}{3}=25$$

It has been explained that both members of an equation may be multiplied by the same number without changing the equality of the members. If both members of this equation are multiplied by a number that will contain each denominator without a remainder, as 6, 12, or 18, the denominators can be canceled. If 6 is used as a multiplier, the equation becomes

$$\frac{6x}{2}+\frac{6x}{3}=6\times 25$$

from which by cancelation $3x+2x=150$

$$5x=150$$

$$x=30.\quad \text{Ans.}$$

In this example, 6 is the least common denominator of $\frac{x}{2}$ and $\frac{x}{3}$ and in clearing an equation of fractions the least number that can be used as a multiplier is the least common denominator of all the fractions in the equation.

From this example the following rule is derived:

Rule.—*Any equation can be cleared of fractions by multiplying both members by the least common denominator of the fractions.*

EXAMPLE 1.—Find the value of x in the equation $\frac{2x}{3}+\frac{x}{5}=45$.

SOLUTION.—According to the rule, both members are multiplied by the least common denominator, which is equal to $3\times 5=15$. Thus,

$$\frac{15\times 2x}{3}+\frac{15x}{5}=15\times 45$$

The denominators may now be canceled by dividing the numerators by 3 and 5, respectively; thus,

$$5\times 2x+3x=15\times 45$$

or, $$10x+3x=15\times 45$$

$$13x=15\times 45$$

$$x=\frac{15\times 45}{13}=51\tfrac{12}{13}.\quad \text{Ans.}$$

EXAMPLE 2.—Find the value of x in the equation $\frac{x}{2}+\frac{x}{3}-\frac{x}{7}=63$.

SOLUTION.—The least common denominator is equal to the product $2\times 3\times 7$, or 42. All the terms are therefore multiplied by 42 and the equation becomes

$$\frac{42x}{2}+\frac{42x}{3}-\frac{42x}{7}=42\times 63$$

from which, by cancelation,

$$21x+14x-6x=42\times 63$$

or, $$29x=42\times 63$$

and $$x=\frac{42\times 63}{29}=91\tfrac{7}{22}.\quad \text{Ans.}$$

EXAMPLE 3.—Find the value of x in the equation $\frac{x+1}{3}+\frac{2x}{4}=47-2x$.

SOLUTION.—The least common denominator is 12, and when all the terms are multiplied by 12, the equation is

$$\frac{12(x+1)}{3}+\frac{12\times2x}{4}=12\times47-12\times2x$$

from which, by cancelation,

$$4(x+1)+3\times2x=564-24x$$

When the parenthesis is removed and the necessary multiplications are performed, $4x+4+6x=564-24x$.

The terms $-24x$ and 4 are now transposed and their signs changed; thus:

$$4x+6x+24x=564-4$$

When like terms are combined,

$$34x=560$$

from which $x=\frac{560}{34}=16\frac{8}{17}$. Ans.

EXAMPLE 4.—Solve the equation $2x+\frac{x}{5}=\frac{2}{3}+\frac{3}{5}$ to find the value of x.

SOLUTION.—The equation is multiplied by the least common denominator, which is 15. Thus, $15\times2x+\frac{15x}{5}=15\times\frac{2}{3}+15\times\frac{3}{5}$.

Next, cancel, and $15\times2x+3x=5\times2+3\times3$. Then perform the multiplications, and $30x+3x=10+9$, from which $33x=19$ and $x=\frac{19}{33}$. Ans.

EXAMPLE 5.—Find the value of x in the equation $\frac{32}{x+3}=\frac{18}{x}$.

SOLUTION.—The least common denominator here is the product of the two denominators, or $(x+3)x$. When each term of the equation is multiplied by this quantity the result is

$$\frac{32(x+3)x}{x+3}=\frac{18(x+3)x}{x}$$

The common factor $x+3$ can be canceled from the numerator and denominator of the first member and the common factor x can be similarly canceled from the second member. The equation will then be

$$32x=18(x+3)$$

When the terms in the parenthesis are multiplied by 18, the equation becomes

$$32x=18x+18\times3$$

When the term $+18x$ is transposed to the first member it becomes $-18x$ and the equation then is

$$32x-18x=18\times3$$

from which $14x=54$

and $x=3\frac{12}{14}=3\frac{6}{7}$. Ans.

52. Order of Steps in Solving Equations.—The successive steps to be taken in solving an equation, that is, in finding the value of the unknown quantity, are as follows:

1. Clearing of fractions.
2. Transposing terms.
3. Combining like terms.
4. Dividing both members of the equation by the coefficient of the unknown quantity.

Sometimes the terms are transposed before the clearing of fractions, as in the following example 2.

EXAMPLE 1.—Find the value of x in the equation $9x-9=5x+7$.

SOLUTION.—It was stated in Art. **20** that if a term is without a sign, a plus sign is understood; thus, $5x$ and $9x$ represent $+5x$ and $+9x$.

When the unknown terms are transposed to the first member, the known terms to the second member, and the signs of the transposed terms are changed, then

$$9x-5x=7+9$$

from which $$4x=16$$

and $$x=4.\quad \text{Ans.}$$

EXAMPLE 2.—Find the value of the unknown quantity y in the equation $6+\frac{1}{2}y=8+\frac{1}{3}y$.

SOLUTION.—The unknown term $\frac{1}{3}y$ is first transposed to the first member, and the known term 6 to the second member, both with signs changed, and the equation is then

$$\tfrac{1}{2}y-\tfrac{1}{3}y=8-6=2$$

The members are now multiplied by the least common denominator, which is 6. Thus,

$$\frac{6}{2}y-\frac{6}{3}y=6\times2$$

By cancelation, $$3y-2y=12$$

from which $$y=12.\quad \text{Ans.}$$

EXAMPLE 3.—Find the value of x in the following equation:

$$\frac{x-3}{15}+\frac{x+7}{17}=8$$

SOLUTION.—The equation is cleared of fractions by multiplying the terms by the least common denominator of the fractions, 15×17. Thus,

$$\frac{15\times17\,(x-3)}{15}+\frac{15\times17\,(x+7)}{17}=8\times15\times17$$

The denominators can now be canceled, giving

$$17(x-3)+15(x+7)=8\times15\times17$$

When the multiplications are performed the equation becomes

$$17\,x-51+15\,x+105=2{,}040$$

The known terms are now transposed to the second member,

$$17\,x+15\,x=2{,}040+51-105$$

from which $$32\,x=1{,}986$$

and $$x=\frac{1{,}986}{32}=62\tfrac{1}{16}.\quad \text{Ans.}$$

Example 4.—An iron pipe 42 feet long is cut into two parts, one being 5 times as long as the other. How long is each part?

Solution.—If the shorter part is indicated by x, the other part must be $5\,x$, as it is five times as long as the short part. The sum of these parts, or $5\,x+x$, must be equal to the total length of the pipe, or 42 ft. Hence, the equation is written as follows:

$$5\,x+x=42$$

from which $$6\,x=42$$

and $$x=\frac{42}{6}=7\text{ ft.}$$

The shorter part is therefore 7 ft., and the longer part $5\times 7=35$ ft. Ans.

Example 5.—Find the value of x in the following equation:

$$\frac{46}{x+3}=\frac{23}{x}$$

Solution.—To clear of fractions the members are multiplied by the common denominator $(x+3)\,x$; thus,

$$\frac{46\,(x+3)\,x}{x+3}=\frac{23\,(x+3)\,x}{x}$$

The factor $x+3$ is now canceled from the fraction of the first member and x from the fraction of the second member, giving

$$46\,x=23\,(x+3)=23\,x+69$$

The term $+23\,x$ is now transposed; thus,

$$46\,x-23\,x=69$$

from which $$23\,x=69$$

and $$x=\frac{69}{23}=3.\quad \text{Ans.}$$

Example 6.—Find the value of x in the equation $\frac{x}{2}+\frac{x}{5}+\frac{x}{7}=15$.

Solution.—The least common denominator of all the fractions in this example is equal to the product of the denominators, or $2\times5\times7=70$. When both members are multiplied by 70, the result is;

$$\frac{70\,x}{2}+\frac{70\,x}{5}+\frac{70\,x}{7}=70\times15$$

By cancelation, $35x+14x+10x=70\times 15$.

from which $59x=70\times 15$

and $x=\frac{70\times 15}{59}=17.8$, nearly. Ans.

EXAMPLES FOR PRACTICE

1. Find the value of the unknown quantity x in the following equations:

(*a*) $11x+14=56-3x$.

(*b*) $27x-127=11-19x$.

(*c*) $3x+4x+5x=6x+72$.

(*d*) $x-\frac{x}{3}=\frac{3}{4}+\frac{1}{2}$

(*e*) $\frac{54}{x+4}=\frac{42}{x}$.

$$\text{Ans.}\begin{cases}(a)\ x=3\\(b)\ x=3\\(c)\ x=12\\(d)\ x=1\frac{7}{8}\\(e)\ x=14\end{cases}$$

2. If $x+3a=81$, and $a=17$, what is the value of x? Ans. $x=30$

3. Find the value of x in the equation $\frac{x}{3}+\frac{x}{4}+\frac{x}{5}=47$. Ans. $x=60$

4. When $3x+2a=2a+24$, what is the value of x? Ans. $x=8$

APPLICATION OF FORMULAS

SOLUTIONS OF SIMPLE STANDARD FORMULAS

TRANSFORMATION OF FORMULAS

53. Formulas Requiring Transformation.—Equations which express relations between known and unknown quantities in such a way that the values of the unknown quantities can be determined are very useful in engineering work and in many business and accounting problems. The **formulas** used in mathematical work are equations which express, briefly and concisely, rules or methods which would be difficult to understand and apply if expressed in words. The use of formulas to solve problems has already been illustrated in the solutions of previous examples of equations, but other examples will make still more clear how transformations must sometimes be made.

If a formula always were in the form most suitable for the purpose in view, the application of formulas would be very simple. But the conditions are frequently such that the formula cannot be used in its original form; often one of the quantities that are supposed to be given is unknown and must be found. It is then necessary to transform the formula; that is, a new formula must be found, based on the given formula. It is in such cases that the knowledge of how to transform an equation is particularly advantageous. A simple example will show the necessity and the advantage of such transformations.

54. Example of Transformation.—The following rule is applied in mensuration: "The volume of a right pyramid or a right cone equals the area of the base multiplied by one-third of the altitude." This rule may be expressed by a formula, as follows:

$$V=\frac{a\,b}{3} \qquad \textbf{(1)}$$

in which V = volume, in cubic inches;
a = altitude, in inches;
b = area of base, in square inches.

Or, if a is taken as the altitude in *feet* and b as the area of the base in *square feet,* then V will be the volume in *cubic feet.*

The formula, as stated, gives the *volume* of the pyramid; but, the conditions may be such that the volume and the altitude are known quantities, and that it is necessary to find what *base area* will give the required volume. The formula must then be transformed, so that the symbol b is alone on one side of the equality sign, preferably at the left.

To make this transformation both members of formula **1** are first multiplied by 3; thus,

$$3\,V=\frac{3\,a\,b}{3}$$

The 3 is canceled from the second member, thus:

$$3\,V=a\,b$$

Both members are then divided by a, thus:

$$\frac{3\,V}{a}=\frac{a\,b}{a}$$

and the a is canceled from the second member, giving

$$\frac{3\,V}{a}=b$$

or, $$b=\frac{3\,V}{a} \qquad (\mathbf{2})$$

The formula will now give the required area of the base, if the volume and altitude are known.

Should it, on the other hand, be necessary to calculate the altitude of a pyramid with a given volume and base area, formula **1** must be transformed so that the symbol a constitutes the first member of the formula.

The various steps to be taken are about the same as in the preceding case. Thus, both members are multiplied by 3,

$$3\,V=\frac{3\,a\,b}{3}$$

The 3 is canceled from the second member, thus:

$$3\,V=a\,b$$

Both members are now divided by b, giving

$$\frac{3\,V}{b}=\frac{a\,b}{b}$$

and b is canceled in the second member, thus:

$$\frac{3\,V}{b}=a$$

or, $$a=\frac{3\,V}{b} \qquad (\mathbf{3})$$

Formula **1** is thus used as a basis for developing two new formulas to serve other conditions.

EXAMPLE.—The volume of a pyramid must be equal to 220 cubic inches. If the altitude is 18 inches, what must be the area of the base?

SOLUTION.—By formula **2**, with the given values, $V=220$ and $a=18$, substituted, the formula is:

$$b=\frac{3\times 220}{18}$$

from which $$b=\frac{220}{6}=36.67 \text{ sq. in., nearly. Ans.}$$

FORMULAS RELATING TO ELECTRICAL AND STEAM ENGINEERING

55. Scope of Examples.—A series of examples will now be given showing how transformations are made in some of the formulas that may be encountered in succeeding Sections. It is to be borne in mind, however, that the formulas are chosen merely as examples to show how they and similar formulas may be applied; no knowledge of the practical application is required in connection with the examples.

56. Ohm's Law.—One of the fundamental formulas of electrical engineering is known as **Ohm's law**, which states the relations between the strength of an electric current, its pressure, or electromotive force, and the resistance of the electric circuit.

Let I = current strength, in amperes;
E = electromotive force, or pressure, in volts;
R = resistance, in ohms.

Then,
$$I=\frac{E}{R} \qquad (\mathbf{1})$$

If both members of the equation are multiplied by R, then

$$IR=\frac{ER}{R}$$

By cancelation,
$$IR=E$$

or,
$$E=IR \qquad (\mathbf{2})$$

If the members of formula **2** are divided by I,

$$\frac{E}{I}=\frac{IR}{I}$$

By cancelation,
$$\frac{E}{I}=R$$

or,
$$R=\frac{E}{I} \qquad (\mathbf{3})$$

EXAMPLE 1.—The electromotive force of a current is 3 volts. If the resistance of the electric circuit is 15 ohms, what is the current strength, in amperes?

SOLUTION.—By formula **1**, $I=\frac{E}{R}$ and, with the known values substituted, $I=\frac{3}{15}=.2$ ampere. Ans.

EXAMPLE 2.—What is the resistance of an electric circuit in which a pressure of 110 volts produces a current of 10 amperes?

SOLUTION.—By formula **3**, $R=\frac{E}{I}=\frac{110}{10}=11$ ohms. Ans.

EXAMPLE 3.—In an electric circuit of 8 ohms resistance flows a current of 6 amperes. What is the pressure, or electromotive force?

SOLUTION.—By formula **2**, $E=I\,R=6\times8=48$ volts. Ans.

57. Steam-Engine Formula.—The rule for finding the horsepower of a steam engine is generally given in the following form:

Rule.—*The horsepower of an engine equals the product of the average steam pressure on the piston, in pounds per square inch, the length of the stroke, in feet, the area of the piston, in square inches, and the number of strokes per minute, divided by 33,000.*

This is a very simple rule and easy to remember, yet it is more common to express it as a formula, which is usually written thus:

$$H=\frac{P\,L\,A\,N}{33{,}000} \qquad \textbf{(1)}$$

where H=horsepower of engine;
P=average steam pressure per square inch of piston;
L=length of stroke of piston, in feet;
A=area of piston, in square inches;
N=number of working strokes of piston, per minute.

The number 33,000 is known as a *constant*, because it is a fixed value used in power calculations. Various constants are employed in formulas.

In order that the horsepower of an engine may be found, it is necessary to know the average steam pressure per square inch of the piston; the area of the face of the piston, in square inches; the length of the stroke of the piston, in feet; and the number of working strokes of the piston, per minute. In problems relating to the steam engine, the stroke is often given in inches and then it must be reduced to feet, by dividing by 12, when used in the formula. The number of revolutions per minute is usually given instead of number of strokes, in which case the number of revolutions must be multiplied by the number of working strokes per revolution, which in the following examples is equal to 2. It is also customary to give the diameter of the cylinder instead of its area. As the piston is circular, its area must be found from the rule given in a preceding Section, namely, *the area of a circle is equal to .7854 times the square of the diameter*, which is more easily remembered as the formula,

$$A = .7854\,D^2 \qquad \textbf{(2)}$$

in which A = area of circle, in square inches;
D = diameter, in inches.

Example.—What is the horsepower of an engine, if the piston is 20 inches in diameter, the stroke 36 inches long, the average steam pressure 40 pounds per square inch, and the engine makes 80 revolutions per minute?

Solution.—In order to apply the formula for horsepower, it is first necessary to find the area of the piston. By formula **2**, $A = .7854D^2 = .7854\times20\times20 = 314.16$ sq. in. L, the length of the stroke, must be reduced to feet; hence, $L = 36\div12 = 3$ ft. The number of strokes per minute N is $80\times2 = 160$. $P = 40$ lb. per sq. in. Hence, when these known values are substituted in formula **1**,

$$H = \frac{40\times3\times314.16\times160}{33{,}000} = 182.78 \text{ H. P.} \quad \text{Ans.}$$

58. Transformation of Steam-Engine Formula. Sometimes the horsepower of the engine is given, and likewise the average steam pressure, the length of the stroke, and the number of revolutions per minute, and it is required to find the diameter of the steam cylinder. In that case, formula **1**, Art. **57**, must be transformed so that the area A, which governs the diameter, will be in the first member, or on the left-hand side of the formula, and the other terms on the right-hand side.

Both sides of the formula are multiplied by 33,000, and

$$33{,}000\,H = \frac{P\,L\,A\,N}{33{,}000}\times 33{,}000$$

from which, by cancelation, $33{,}000\,H = P\,L\,A\,N$.

Both members are then divided by $P\,L\,N$, thus:

$$\frac{33{,}000\,H}{P\,L\,N} = \frac{P\,L\,A\,N}{P\,L\,N}$$

By cancelation of $P\,L\,N$ in the second member,

$$\frac{33{,}000\,H}{P\,L\,N}=A$$

or,

$$A=\frac{33{,}000\,H}{P\,L\,N} \qquad (1)$$

The area can now be found from formula **1**, and subsequently the equivalent diameter from formula **2**, Art. **57**; but it is more convenient to substitute the latter formula for A in the formula just found, as these A's represent the same thing.

Thus,

$$.7854\,D^2=\frac{33{,}000\,H}{P\,L\,N}$$

Both sides of the equation may now be divided by .7854; thus,

$$\frac{.7854\,D^2}{.7854}=\frac{33{,}000\,H}{.7854\,P\,L\,N}$$

By cancelation of .7854,

$$D^2=\frac{33{,}000\,H}{.7854\,P\,L\,N}$$

and by dividing .7854 into 33,000,

$$D^2=\frac{42{,}017\,H}{P\,L\,N}$$

The square root of both members can now be extracted according to Arts. **17** and **18**, giving

$$\sqrt{D^2}=\sqrt{\frac{42{,}017\,H}{P\,L\,N}}$$

The term

$$\sqrt{D^2}=D;$$

thus,

$$D=\sqrt{\frac{42{,}017\,H}{P\,L\,N}}$$

The square root of 42,017 is 205, nearly; hence,

$$D=205\sqrt{\frac{H}{P\,L\,N}} \qquad (2)$$

This is read D *equals 205 times the square root of* H *over* $P\,L\,N$.

Example.—What should be the diameter of the cylinder of a 100-horsepower steam engine to work with a mean effective pressure of 42 pounds per square inch, a stroke of 30 inches, and make 100 revolutions per minute?

Solution.—Here $H=100$, $P=42$, $L=\frac{30}{12}=2.5$, and $N=2\times 100=200$. When these values are substituted in formula **2**, it becomes

$$D=205\sqrt{\frac{H}{P\,L\,N}}=205\sqrt{\frac{100}{42\times 2.5\times 200}}$$

The first step is to simplify the fraction under the radical sign.

$$D=205\sqrt{\frac{100}{42\times 2.5\times 200}}=205\sqrt{.004762}$$
$$=205\times .069=14.145 \text{ in., nearly. Ans.}$$

In practice, the nearest common fraction of an inch would be chosen instead of .145, as ⅛, and the answer would be 14⅛ in., nearly.

59. As already stated, it is possible to find the value of the quantity represented by any letter in a formula, if the values represented by the other letters are known. For example, let it be required to transform formula **1**, Art. **57**, so that the value of N may be found, if the other quantities are known.

First multiply both members of the formula $H=\frac{P\,L\,A\,N}{33{,}000}$ by 33,000; thus,

$$33{,}000\,H=\frac{P\,L\,A\,N}{33{,}000}\times 33{,}000$$

By cancelation, $33{,}000\,H=P\,L\,A\,N$

To obtain N alone on one side of the equality sign, divide both members by $P\,L\,A$; thus,

$$\frac{33{,}000\,H}{P\,L\,A}=\frac{P\,L\,A\,N}{P\,L\,A}$$

By cancelation, $$\frac{33{,}000\,H}{P\,L\,A}=N$$

or, $$N=\frac{33{,}000\,H}{P\,L\,A} \qquad \textbf{(1)}$$

Example.—The piston area of a steam engine is 78.54 square inches, the mean effective pressure is 40 pounds per square inch of piston, and the length of stroke is 1¼ feet. If the engine is to develop 60 horsepower, what must be the number of strokes per minute?

Solution.—Here $H=60$, $P=40$, $L=1\frac{1}{4}$, and $A=78.54$, and when these known values are substituted in formula **1**, it becomes

$$N=\frac{33{,}000\times 60}{40\times 1\frac{1}{4}\times 78.54}=504.2 \text{ strokes per minute. Ans.}$$

FORMULAS RELATING TO MENSURATION AND MINE VENTILATION

60. Area of a Triangle.—A good example of a formula using powers and roots of numbers is shown in the following formula for finding the area of a triangle when the three sides are known:

$$A=\frac{b}{2}\sqrt{a^2-\left(\frac{a^2+b^2-c^2}{2b}\right)^2}$$

in which A = area of triangle; a, b, and c = lengths of sides. If a, b, and c are in inches, A will be in square inches; and if a, b, and c are in feet, A will be in square feet.

EXAMPLE.—What is the area of a triangle whose sides are 21 feet, 46 feet, and 50 feet long?

SOLUTION.—It makes no difference which side is represented by a, b, or c; hence, let a represent the side 50 ft. long; b, the side 46 ft. long; and c, the side 21 ft. long. Then, by the formula,

$$A=\frac{b}{2}\sqrt{a^2-\left(\frac{a^2+b^2-c^2}{2b}\right)^2}=\frac{46}{2}\sqrt{50^2-\left(\frac{50^2+46^2-21^2}{2\times 46}\right)^2}$$

$$=\frac{46}{2}\sqrt{2{,}500-\left(\frac{2{,}500+2{,}116-441}{92}\right)^2}=23\sqrt{2{,}500-\left(\frac{4{,}175}{92}\right)^2}$$

$$=23\sqrt{2{,}500-(45.38)^2}=23\sqrt{2{,}500-2{,}059.34}=23\sqrt{440.66}$$

$$=23\times 20.992=482.8 \text{ sq. ft., nearly. Ans.}$$

The operations in this example have been extended much further than is necessary, in order to show every step of the process. It will be found instructive to change the numbers of the different sides so that a represents either the 21- or the 46-foot side and b and c the other sides. The values so taken can then be substituted in the formula and the operations performed as before; the answer will be the same as that already found with a slight variation depending on the number of decimal places to which the results are carried.

61. Mine-Ventilation Formulas. — In calculations relating to mine ventilation it is necessary to ascertain the quantity of air flowing per minute through an airway under certain conditions. The rule to be applied is as follows:

Rule.—*The product of the sectional area of airway, in square feet, and the velocity of air-current, in feet per minute, equals the quantity of air in circulation, in cubic feet per minute.*

The rule written as a formula is,

$$q = a\,v \qquad (\mathbf{1})$$

in which q=quantity of air in circulation, in cubic feet per minute;
a=sectional area of airway, in square feet;
v=velocity of air-current, in feet per minute.

Example.—What quantity of air will be delivered per minute through a 6′×8′ airway, if the air travels at the rate of 500 feet per minute?

Solution.—By the preceding formula, in which the following known values are substituted, namely, $a=6\times 8=48$ sq. ft. and $v=500$, it is found that $q=a\,v=48\times 500=24{,}000$ cu. ft. per min. Ans.

62. If it is required to find the velocity of the air-current, when the sectional area of the airway and the circulating volume of air is known, formula **1**, Art. **61**, must be transformed in the following manner:

Divide both members of the formula $q=a\,v$ by a; thus,

$$\frac{q}{a}=\frac{av}{a}$$

Cancel a in the second member, and

$$\frac{q}{a}=v$$

or,

$$v=\frac{q}{a}$$

Example.—If the quantity of circulating air is 24,000 cubic feet per minute and the airway is 6 ft.×8 ft., what is the velocity of the air-current, in feet per minute?

Solution.—By substituting in the formula a value for $q=24{,}000$ and for $a=6\times 8=48$,

$$v=\frac{24{,}000}{48}=500 \text{ ft. per min.} \quad \text{Ans.}$$

63. The **rubbing surface** of an airway is the entire inner surface of the airway composed of the roof, floor, and sides.

It is estimated in square feet. To find the area of this rubbing surface the following rule is used:

Rule.—*The area of the rubbing surface of the entire airway equals the perimeter, that is, the distance around the cross-section of the airway, multiplied by the length of the airway.*

The formula for this rule is

$$s=lo$$

in which s=area of rubbing surface, in square feet;
l=length of airway, in feet;
o=perimeter, in feet.

Example.—What is the area of the surface of an airway 6 ft.× 8 ft. and 1,250 feet long?

Solution.—Here $o=6+6+8+8=28$ ft.; that is, the distance around the section of the airway equals the sum of the lengths of its four sides. The length $l=1{,}250$ ft. When these values are substituted for o and l in the formula, $s=1{,}250\times 28=35{,}000$ sq. ft. Ans.

64. The pressure necessary to produce a certain velocity of air-current in an airway of given dimensions is as follows:

Rule.—*The pressure, in pounds per square foot, required to produce a given velocity of the air-current, equals the product of the coefficient of friction, the area of the rubbing surface of the airway, and the square of the velocity of the flow, divided by the sectional area of the airway.*

The rule, given as a formula, is

$$p=\frac{k\,s\,v^2}{a}$$

in which p=ventilating pressure, in pounds per square foot;
k=coefficient of friction, which for mine passages may be taken as .00000002;
s=rubbing surface, in square feet;
v=velocity of air-current, in feet per minute;
a=sectional area of airway, in square feet.

It should be noted that s, v, and a of this formula represent the same quantities as similar letters in Arts. **61** and **63.**

Example.—What pressure is required to produce a circulation with a velocity of 500 feet per minute in an airway 6 ft.×8 ft. and 1,250 feet long?

SOLUTION.—Here $k=.00000002$, $s=l\,o=1{,}250\times 28=35{,}000$ ft., $v=500$, $v^2=500\times 500=250{,}000$, and $a=6\times 8=48$ sq. ft Hence, by the formula,

$$p=\frac{.00000002\times 35{,}000\times 250{,}000}{48}=3.646 \text{ lb. per sq. ft. Ans.}$$

65. At times no one formula is in a suitable form for finding a required quantity; that is, the known quantities are not entirely the same as those required by any formula. In such cases it may be necessary to combine two or more formulas to get one in a form suitable for employing the known quantities.

For example, the formula $p=\frac{k\,s\,v^2}{a}$ requires the quantities a, s, and v to be known. If, now, instead of s and v, the quantities l, o, and q are known, it is possible to employ the formulas $v=\frac{q}{a}$ and $s=l\,o$, Arts. **62** and **63**, and by this means obtain the data required by the formula $p=\frac{k\,s\,v^2}{a}$. Thus, instead of using s and v in this formula, the equivalent values may be employed in their places. Hence, instead of s, the term $l\,o$ is inserted and instead of v, the term $\frac{q}{a}$; but as v is squared, the term $\frac{q}{a}$ must likewise be squared and written $\frac{q^2}{a^2}$.

The substitution of these terms is done in the following manner:

$$p=\frac{k\,s\,v^2}{a}=\frac{k\times l\,o\times\frac{q^2}{a^2}}{a}=k\times l\,o\times\frac{q^2}{a^2}\times\frac{1}{a}=\frac{k\,l\,o\,q^2}{a^3}$$

EXAMPLE.—What pressure is required to produce a circulation of 24,000 cubic feet of air per minute in an airway 6 ft.×8 ft. in cross-section and 1,250 feet long?

SOLUTION.—In this problem, $q=24{,}000$, $l=1{,}250$, $o=2(6+8)=28$, $a=6\times 8=48$, $k=.00000002$. Then, by the formula just found,

$$p=\frac{.00000002\times 1{,}250\times 28\times 24{,}000\times 24{,}000}{48\times 48\times 48}=3.646 \text{ lb. per sq. ft. Ans.}$$

FORMULAS RELATING TO BEAMS AND COLUMNS

66. Formula for Wooden Beam.—The following formula is used when it is desired to obtain the size of a rectangular wooden beam that will safely support a given load, uniformly distributed over the beam:

$$\frac{12\,W\,L}{8}=\frac{s}{f}\times\frac{b\,d^2}{6}$$

in which W = total uniform load on the beam, in pounds;

L = length, or span, of beam, in feet;

s = unit fiber stress of the material of which the beam is composed;

f = factor of safety;

b = breadth of beam, in inches;

d = depth of beam, in inches.

Example.—What must be the depth of a rectangular beam of yellow pine to carry a total safe uniformly distributed load of 8,000 pounds, if the span is 15 feet and the width of the beam is 6 inches? It is assumed that the fiber stress s is 7,000 pounds per square inch, and that the factor of safety f is 6.

Solution.—The formula $\frac{12\,W\,L}{8}=\frac{s}{f}\times\frac{b\,d^2}{6}$ may be written $\frac{12\,W\,L}{8}=\frac{b\,d^2\,s}{6\,f}$. As d is the unknown quantity, the formula should be transformed so that d is alone on one side of the equality sign.

Multiply both members by the least common denominator, $24\,f$, to clear of fractions, and obtain.

$$\frac{12\,W\,L\times24\,f}{8}=\frac{b\,d^2\,s\times24\,f}{6\,f}$$

from which $$36\,W\,L\,f=4\,b\,d^2\,s$$

Now divide both members by $4\,b\,s$ in order that d^2 may be alone in the second member.

$$\frac{36\,W\,L\,f}{4\,b\,s}=\frac{4\,b\,d^2\,s}{4\,b\,s}$$

from which $$\frac{9\,W\,L\,f}{b\,s}=d^2$$

or, $$d^2=\frac{9\,W\,L\,f}{b\,s}$$

and $$d=\sqrt{\frac{9\,W\,L\,f}{b\,s}}$$

In the example, $W=8{,}000$, $L=15$, $f=6$, $b=6$, and $s=7{,}000$. When these values are substituted

$$d=\sqrt{\frac{9\times 8{,}000\times 15\times 6}{6\times 7{,}000}}$$

from which $$d=\sqrt{\frac{1{,}080}{7}}=\sqrt{154.29}\doteq 12.4 \text{ in. Ans.}$$

As 12.4 in. is not a stock size of timber, the next larger stock size, which is 14 in., should be chosen.

67. Reinforced-Concrete Columns.—The following formula may be used for designing reinforced-concrete columns:

$$P=\frac{S}{f}(A+n\,a)$$

in which P=total load on the column, in pounds;
S=ultimate unit stress in the concrete, in pounds;
f=factor of safety;
A=area of concrete in column section, in square inches;
n=ratio of modulus of elasticity of steel to modulus of elasticity of concrete, which is about 15;
a=area of steel in column section, in square inches.

Example.—Design a reinforced-concrete column to withstand safely a load of 100,000 pounds. If $S=2{,}000$ pounds, $n=15$, and $f=5$, what must be the sectional area of the concrete and steel, if the sectional area of the steel is 3 per cent. of that of the concrete? That is,

$$a=\frac{3}{100}A$$

Solution 1.—In this problem there are two unknown quantities, A and a; but, as a is equal to $\frac{3}{100}A$, the latter value may be substituted for a in the formula; thus,

$$P=\frac{S}{f}\left(A+\frac{3\,n}{100}A\right)$$

To transform the formula, divide both members by S and multiply by f; thus,

$$\frac{P\,f}{S}=\frac{S\,f}{S\,f}\left(A+\frac{3\,n}{100}A\right)$$

from which $$\frac{P\,f}{S}=A+\frac{3\,n}{100}A$$

To remove the denominator 100, both members are multiplied by 100; thus, $\frac{100\,P\,f}{S}=100\,A+\frac{100\times 3\,n\,A}{100}$.

After cancelation of the equal factors 100 in the last fraction, the equation is written $\frac{100\,P\,f}{S}=100\,A+3n\,A$.

Since the terms in the second member have a common factor A, the terms may be arranged as follows: $A(100+3\,n)$. The formula is now

$$\frac{100\,P\,f}{S}=A\,(100+3n)$$

Now divide the members by $100+3n$ and then cancel, thus:

$$\frac{100\,P\,f}{S\ (100+3n)}=A$$

or,

$$A=\frac{100\,P\,f}{S\ (100+3n)}$$

The given values can now be substituted; then,

$$A=\frac{100\times 100{,}000\times 5}{2{,}000\,(100+3\times 15)}=\frac{5{,}000}{29}=172.41 \text{ sq. in.} \quad \text{Ans.}$$

$$a=\frac{3}{100}\times 172.41=5.17 \text{ sq. in.} \quad \text{Ans.}$$

Solution 2.—At times the solution may be simplified by using the formula in its original form and substituting, at once, the known quantities. In this case, the transformation of the formula is postponed until it is reduced to its simplest form.

For example, the known values may be substituted in the formula as it was first given; thus,

$$100{,}000=\frac{2{,}000}{5}\,(A+15\,a)$$

Since $a=\frac{3}{100}A$, the latter value is substituted in the equation; thus,

$$100{,}000=\frac{2{,}000}{5}\left(A+15\times\frac{3}{100}A\right)$$

or,

$$100{,}000=400\left(A+\frac{45}{100}A\right)$$

Both members can now be divided by 400,

$$250=A+\frac{45}{100}A$$

from which, by reducing the A's to the same denominator and adding,

$$250=\frac{100\,A}{100}+\frac{45\,A}{100}=\frac{145}{100}A=1.45\,A$$

or,

$$250=1.45\ A$$

Hence,

$$A=\frac{250}{1.45}=172.41 \text{ sq. in.} \quad \text{Ans.}$$

EXAMPLES FOR PRACTICE

1. What is the area of a circle 17 feet in diameter? Ans. 227 sq. ft.

2. What is the horsepower of a steam engine having a single cylinder 18 inches in diameter and 30 inches long using steam on both ends, running at 84 revolutions per minute, with a mean effective pressure of 45 pounds per square inch? Ans. 145.74 H. P.

3. If 80,000 cubic feet of air flows through an 8′×8′ airway, what is the velocity of flow? Ans. 1,250 ft. per min.

4. What is the surface of an 8′×8′ airway 1,600 feet long? Ans. 51,200 sq. ft.

5. A current of 20 amperes is flowing under a pressure of 220 volts. What is the resistance, in ohms, of the electric circuit? Ans. 11 ohms

6. In the formula $x=\frac{d+c^2}{d^2-40}$ substitute the following values: $d=10$, $c=2$. What is the value of x? Ans. $x=\frac{7}{30}$

7. The formula for finding the area A of a segment of a circle is $A=\frac{4h^2}{3}\sqrt{\frac{D}{h}-.608}$. If D, the diameter of the circle, is 54 inches, and h, the height of the segment, is 20 inches, what is the area of the segment? Ans. 771.2 sq. in.

8. The formula for finding the surface area of a sphere is $A=D^2\times 3.1416$. If D, the diameter of the sphere, is 14 inches, what is the area? Ans. 615.75 sq. in.

CUBE ROOT

EXTRACTION OF CUBE ROOT

METHODS AVAILABLE

INTRODUCTION

1. Definition.—The **cube root** of a number is one of the three equal factors of that number. For example, 4 is the cube root of 64, because $4\times4\times4=64$; that is, 4 is one of three equal factors that, when multiplied together, produce 64. Similarly, 3 is the cube root of 27, because $3\times3\times3=27$. Again, 2 is the cube root of 8, because $2\times2\times2=8$; 6 is the cube root of 216, because $6\times6\times6=216$; and so on.

2. Indicating Cube Root.—The cube root of any number is indicated by writing the radical sign ($\sqrt{\ }$) in front of it, the vinculum ($\overline{\quad}$) over it, and the index figure ^{3} above and to the left of the radical sign. For example, $\sqrt[3]{125}$ indicates the cube root of 125; $\sqrt[3]{15.625}$ indicates the cube root of 15.625; $\sqrt[3]{\frac{27}{343}}$ indicates the cube root of $\frac{27}{343}$; and so on.

3. Extracting Cube Root.—The operation of finding the cube root of a number is called the *extraction* of the cube root; that is, the cube root of a number is extracted when one of the three equal factors of that number is found. The extraction of the cube roots of numbers must frequently be done in the solution of certain kinds of problems that are met with in engineering work. For this reason it is necessary that instruction be given as to how to find the cube roots of numbers.

4. Methods of Finding Cube Root.—There are a number of different ways in which the cube root of a number may be found, but in this Section only three methods will be described. These will be called (*a*) the trial-and-error method, (*b*) the table method, and (*c*) the exact method, and they will be described in detail in later articles.

5. Significant Figures.—In any number, the figures beginning with the first digit at the left and ending with the last digit at the right, are called the **significant figures** of the number. Thus, the number 405,800 has the four significant figures 4, 0, 5, 8; and the number .000090067 has the five significant figures 9, 0, 0, 6, and 7. A cipher is not a digit. Consequently, if a number begins or ends with ciphers, they are not considered in determining the significant figures of that number. But if ciphers occur between digits, as in the numbers 2,807 and 21.03, they are then considered as significant figures.

The part of a number consisting of its significant figures is called the **significant part** of the number. Thus, in the number 28,070, the significant part is 2807; in the number .00812, the significant part is 812; and in the number 170.3, the significant part is 1703.

The significant figures or the significant part of a number consists of the figures, in their proper order, from the first digit at the left to the last digit at the right, without regard to the position of the decimal point. Hence, *all numbers that differ only in the position of the decimal point have the same significant part.* For example, .002103, 210.3, 21,030, and 210,300 have the same significant figures 2, 1, 0, and 3, and the same significant part 2103.

6. Exact and Approximate Cube Roots.—Only a comparatively few numbers can be separated into exactly equal factors. For example, considering all the numbers from 1 to 1,000, there are only ten of which the exact cube root can be found, namely, 1, 8, 27, 64, 125, 216, 343, 512, 729, and 1,000.

All the other numbers between 1 and 1,000 have only approximate cube roots; that is, their cube roots cannot be found exactly, because those roots will contain unending decimals. For example, suppose that the cube root of 50 is to be found. There are no three equal numbers that, if multiplied together, will produce 50; but 3.6 is an approximate cube root of 50, because $3.6\times3.6\times3.6=46.656$. If the root is carried to another decimal place, or made 3.68, the result is even better, because $3.68\times3.68\times3.68=49.836032$, which is very close to 50. If another decimal place is added and the root is made 3.684, the

TABLE I

PERFECT CUBES AND EXACT CUBE ROOTS

Number	Cube Root	Number	Cube Root	Number	Cube Root	Number	Cube Root
1	1	17,576	26	132,651	51	438,976	76
8	2	19,683	27	140,608	52	456,533	77
27	3	21,952	28	148,877	53	474,552	78
64	4	24,389	29	157,464	54	493,039	79
125	5	27,000	30	166,375	55	512,000	80
216	6	29,791	31	175,616	56	531,441	81
343	7	32,768	32	185,193	57	551,368	82
512	8	35,937	33	195,112	58	571,787	83
729	9	39,304	34	205,379	59	592,704	84
1,000	10	42,875	35	216,000	60	614,125	85
1,331	11	46,656	36	226,981	61	636,056	86
1,728	12	50,653	37	238,328	62	658,503	87
2,197	13	54,872	38	250,047	63	681,472	88
2,744	14	59,319	39	262,144	64	704,969	89
3,375	15	64,000	40	274,625	65	729,000	90
4,096	16	68,921	41	287,496	66	753,571	91
4,913	17	74,088	42	300,763	67	778,688	92
5,832	18	79,507	43	314,432	68	804,357	93
6,859	19	85,184	44	328,509	69	830,584	94
8,000	20	91,125	45	343,000	70	857,375	95
9,261	21	97,336	46	357,911	71	884,736	96
10,648	22	103,823	47	373,248	72	912,673	97
12,167	23	110,592	48	389,017	73	941,192	98
13,824	24	117,649	49	405,224	74	970,299	99
15,625	25	125,000	50	421,875	75	1,000,000	100

accuracy is still further increased, because 3.684×3.684×3.684 =49.998717504. But no matter to how many decimal places the root may be extended, the cube of that root will never be exactly 50, although it will approach nearer and nearer to 50. If an exact root cannot be found, therefore, the approximate root should be carried to as many significant figures as the accuracy of the solution demands.

7. Table I shows all the numbers between 1 and 1,000,000, inclusive, that have exact cube roots, these roots being the whole numbers from 1 to 100, inclusive. The larger numbers are the cubes of the corresponding smaller numbers, and any number that is a perfect cube has an exact cube root.

8. Preliminary Steps in Finding Cube Root.—If the cube root of a number is to be found, the first step is to point off the number into periods of three figures, beginning with the units place if the number is a whole number, and at the decimal point if the number is a decimal. If it is a mixed number, the pointing off is begun at the decimal point and is carried to the right and to the left. These instructions may be made clear by illustrative examples. Suppose, for example, that the cube root of 405,224 is to be found. The number is pointed off into periods, or parts, containing three figures, beginning at the right, or at the units place. When thus pointed off, which may be indicated by prime marks ′ or by commas, the number becomes 405′224. The reason for pointing off in this way is to determine how many figures there are in the root. For every period, or part, there will be one figure in the root; and as 405′224 contains two such periods, the cube root consists of two figures. Table I shows that the cube root of 405,224 is 74, which consists of two figures.

9. It may happen that, when the number whose cube root is to be found is pointed off, the first period at the left will contain one or two figures instead of three. In such a case, the one or two figures are considered as a full period. For example, suppose that the number whose cube root is to be found is 91,125. When pointed off it becomes 91′125, and as the first two figures constitute a period, there are two periods in the number, which indicates that the cube root contains two figures. Table I proves the correctness of this conclusion, as the cube root of 91,125 is 45.

If the number 5,832 is pointed off, it becomes 5′832, in which the first period contains but one figure. However, as there are two periods, the cube root contains two figures. Reference to Table I shows this to be the case, as the cube root of 5,832 is 18.

10. If the number whose cube root is to be found is a decimal, it is divided into periods of three figures each, *beginning at the decimal point.* In case the last period at the right contains only one or two figures, ciphers must be annexed until the period has three figures. The cube root then contains as many figures, following the decimal point, as there are periods in the original number. Annexing ciphers to the right of a decimal, as explained in a preceding Section, does not alter the value of the decimal in any way. It is done in this connection simply to fill out the period that contains less than three figures. For example, if the cube root of .14625 were required, it would be pointed off thus, .146′25, and as the last period has only two figures, a cipher would be annexed, making .146′250. As this number contains two periods, the cube root of it will be a decimal and will contain two figures, one for each period; but as the number .14625 is not a perfect cube, its cube root will not be exact and so the root may be carried to any desired number of decimal places, according to the accuracy required. This is done by annexing extra periods of three ciphers each.

11. If the cube root of a mixed number is to be found, the number is divided into periods of three figures each, beginning at the decimal point and proceeding in each direction. If the last period at the *right* has less than three figures, ciphers are annexed as in the case of a decimal number. For example, suppose that 1,274.4285 is the number to be considered. When pointed off it becomes 1′274.428′5, and ciphers must be annexed to the last period at the right, giving 1′274.428′500. As there are four periods, the root will contain four figures, two preceding the decimal point and two following it. In the number just considered, the decimal point serves to separate the second and third periods, and no other mark of separation is needed.

TRIAL-AND-ERROR METHOD

12. Roots of Whole Numbers and Decimals.—In solving practical problems that require the finding of cube root, it is usually not necessary to obtain the root to more than four significant figures, and in most instances three significant figures will be enough. For such approximate calculations, therefore, a simple way of finding the cube root of a number is by trial and error. As explained in Art. **1,** the cube root of a number is one of the three equal factors that, when multiplied together, will produce the number. The trial-and-error method, therefore, consists simply in assuming a root and cubing it. If the product is greater than the number whose root is to be found, the assumed root is made smaller and the cubing operation is repeated. By making a number of trials in this way, eventually a root is found that, when cubed, will give a product very nearly equal to the given number. That root is then taken as the cube root of the number. The method may be illustrated most clearly by examples and their solutions.

EXAMPLE 1.—Find the cube root of 4,913 by the trial-and-error method.

SOLUTION.—The number is first pointed off into periods, as explained in Art. **8**, which gives 4'913. As there are two periods, the cube root contains two figures. The first period consists of one figure, 4. Now, the first figure of the root cannot be 2, because $2\times2\times2=8$, which is much greater than 4. The smallest root of two figures, beginning with 2, is 20, and $20^3=20\times20\times20=8{,}000$, which is considerably greater than the given number, 4,913. So it is plain that the root must be less than 20. Suppose that 18 is assumed; then, $18\times18\times18=5{,}832$, which is also somewhat larger than 4,913, showing that the root must be less than 18. Hence, 17 is chosen; then $17\times17\times17=4{,}913$, the given number. Therefore, 17 is the exact cube root of 4,913. Ans.

EXAMPLE 2.—By the trial-and-error method find the cube root of 12,764.

Solution.—When pointed off properly, the number becomes 12'764. Reference to Table I shows that the first figure of the root must be 2, because $2^3=8$, which is less than 12, the first period, whereas $3^3=27$, which is greater than the first period. If 12'764 has an exact root, there must be two figures in it, because the number has two periods, and it has just been shown that the first figure is 2. So, let 22 be chosen as a trial root; then, $22\times22\times22=10{,}648$, which is less than 12,764, showing that the root must be greater than 22. Let 23 be chosen next; then, $23\times23\times23=12{,}167$. This is also less than 12,764. So let 24 be chosen; then, $24\times24\times24=13{,}824$. This is considerably larger than the given number, 12,764, and indicates that the cube root cannot be 24. But it was also found that 23 was a little too small. Therefore, the root must be greater than 23 and less than 24. So, take 23.4 as a trial root; then, $23.4\times23.4\times23.4=12{,}812.904$. This is also larger than 12,764, showing that the root must be less than 23.4. Let 23.38 be chosen; then, $23.38\times23.38\times23.38=12{,}780.078472$, which is so close to 12,764 that it may be considered approximately equal to it. Then, 23.38 is the cube root of 12,764, very nearly. Ans.

Example 3.—What is the value of $\sqrt[3]{.0837}$ by the trial-and-error method?

Solution.—The number is first pointed off, and ciphers are annexed to complete the last period, giving .083'700. The given number is wholly decimal and so the root will be wholly decimal. As there are two periods, there will be two figures following the decimal point in the root, if the root is exact, and more if the root is not exact. The first period contains the number 83. Reference to Table I shows that the first figure of the root must be 4, because $4^3=64$, which is less than 83, whereas $5^3=125$, which is greater than 83. Hence, it is known that the root must be a number greater than .40. Let .45 be assumed; then, $.45\times.45\times.45=.091125$, which is greater than .0837, showing that .45 is too large. So .44 is chosen as a trial value; then, $.44\times.44\times.44=.085184$, which is still too large. The root must therefore be less than .44. So .43 is assumed; then, $.43\times.43\times.43=.079507$, which is smaller than .0837, indicating that the root must be greater than .43. Therefore, it is known that the root lies between .43 and .44. Let .437 be tried; then, $.437\times.437\times.437=.083453453$, or .0835, very nearly. This is so close to .0837 that, for all practical purposes, it may be said that $\sqrt[3]{.0837}=.437$. Ans.

Example 4.—Find the value of $\sqrt[3]{442{,}800}$ to four significant figures by trial and error.

Solution.—When pointed off, the number becomes 442'800. The first figure of the root must be 7, because $7^3=343$, which is less than 442, the first period, whereas $8^3=512$, which is larger than 442. As the given number has two periods, the root will have two figures preceding the decimal point. Let 76 be chosen for the first trial; then, $76\times76\times76=438{,}976$, which is less than 442,800, indicating that the root must be greater than 76. If 77 is tried, it is found to be too large, because $77^3=456{,}533$, which exceeds 442,800. Therefore, the root lies somewhere between 76 and 77. As 438,976 is closer to 442,800 than is 456,533, the root is closer to 76 than to 77. So, let 76.3 be chosen as a trial value; then, $76.3^3=444{,}194.947$, which is larger than 442,800, indicating that 76.3 is too large. If 76.2 is tried, it is found that $76.2^3=442{,}450.728$, which is a trifle smaller than 442,800, but very close to it. It is concluded, then, that the root is between 76.2 and 76.3, but not much greater than 76.2. Take 76.21 as a trial value, and $76.21^3=442{,}625$, very nearly. This is still less than the given number, so 76.22 is tried as the root. Then, $76.22^3=442{,}799$, which is so nearly equal to 442,800 that 76.22 may be considered as the required root, to four significant figures. Ans.

13. Roots of Numbers Containing Fractions.—If the number is a fraction or if it contains a fraction, the simplest way of treating it is to reduce the fraction to a decimal and then proceed as in the preceding examples. For example, suppose that the cube root of $\frac{3}{8}$ is to be found by trial and error. Reduce the fraction to a decimal, thus: $\frac{3}{8}=3\div8=.375$. Then, the cube root of .375 is found by the method already explained, and the result thus obtained is the root of $\frac{3}{8}$. If the cube root of a number like $17\frac{26}{47}$ is to be found, the fractional part is reduced to a decimal; thus, $\frac{26}{47}=26\div47=.5532$. The entire number then becomes 17.5532, and the root is found in the manner already explained. The following examples will serve to make the foregoing instructions clearer:

EXAMPLE 1.—Find by trial and error the cube root of $\frac{5}{8}$, to three significant figures.

SOLUTION.—The fraction $\frac{5}{8}$, reduced to a decimal, becomes .625. As .625 is wholly a decimal the cube root will be wholly a decimal. The first period of .625 contains three figures, the remaining periods consisting of ciphers; that is, the number when pointed off may be written .625′000′000. The largest number whose cube is less than .625 is .8, because $.8^3=.512$, whereas $.9^3=.729$. Therefore, the first figure of the root is 8 and the root lies somewhere between .8 and .9. Choose .85 as a trial root; then, $.85^3=.614125$, which is less than .625. So, try .86 as the root; then, $.86^3=.636056$, which is too large, as it exceeds .625. The root of .625 therefore lies between .85 and .86. Choose .855 as a trial root; then, $.855^3=.625026375$, which is so close to .625 that no further calculations are necessary, and .855 may be taken as the cube root of $\frac{5}{8}=.625$. Ans.

Example 2.—Find the value of $\sqrt[3]{470\frac{3}{4}}$ to three significant figures by trial and error.

Solution.—First reduce $\frac{3}{4}$ to a decimal; thus, $\frac{3}{4}=3\div4=.75$. The number whose cube root is to be found is then 470.75. When this number is pointed off, it becomes 470.750, showing that the root has one figure before the decimal point. The largest number whose cube is less than 470 is 7, because $7^3=343$, whereas $8^3=512$. So the first figure of the root is 7 and the required root is between 7 and 8. Let 7.5 be assumed; then, $7.5^3=421.875$, which is less than 470.75. So 7.6 is tried; then, $7.6^3=438.976$, which is still too small. Next, 7.7 is tried; then, $7.7^3=456.533$, which also is less than 470.75. Therefore, 7.8 is tried; then, $7.8^3=474.552$, which is too large. It is now seen that the root is greater than 7.7 but less than 7.8. As the cube of 7.8 is 474.552, which is close to the given number 470.75, it is plain that the root must be close to 7.8, but less than 7.8. So, 7.78 is tried; then, $7.78^3=470.911$, which is so close to 470.75 that no further trials are necessary, and the required root is taken as 7.78. Ans.

EXAMPLES FOR PRACTICE

Find the cube roots of the following numbers, by the trial-and-error method, to three significant figures:

(*a*)	18,610	Ans. (*a*)	26.5
(*b*)	.3505	(*b*)	.705
(*c*)	160,000,000	(*c*)	543
(*d*)	54.5	(*d*)	3.79
(*e*)	$\frac{1}{2}$	(*e*)	.794
(*f*)	$12\frac{5}{16}$	(*f*)	2.31

TABLE METHOD

14. In Table I are given the cubes of all whole numbers from 1 to 100, inclusive; likewise, the table gives the cube roots of all numbers between 1 and 1,000,000 that have exact roots. By means of this table it is possible to find directly the first two significant figures in the cube root of any number and to find the third significant figure by a simple calculation. It will be observed that the numbers in the table are all whole numbers, and that none contains more than seven figures. Therefore, to use the table for finding cube roots, the numbers whose roots are to be found must be altered in such a way as to bring them within the range of the table. The method of doing this will now be described.

15. As already explained, digits only are considered in determining the significant figures of a number; therefore, 5, 50, .5, and .05 have the same significant figure, 5. The cubes of these numbers are 125, 125,000, .125, and .000125, respectively, and thus it will be seen that these cubes also have the same significant figures, 1, 2, and 5, and the same significant part, 125. The numbers 125, 125,000, .125, and .000125, if pointed off preparatory to extracting the cube roots, become 125, 125′000, .125, and .000′125. It will be seen at once that each of these contains the same significant part, 125, and that these three significant figures form a single period in each number. The only difference among the numbers is in the position of the decimal point. Consequently, their cube roots will have the same significant figure, 5, but the position of the decimal point will be different. Thus, as 125 is a whole number of one period, its root is a single figure, 5; as 125′000 has two periods, its root has two figures, 50; as .125 is a decimal and comprises one period, its root is a decimal of one figure, or .5; and as .000′125 is a two-period decimal, its root is a decimal of two figures, or .05, because, when a decimal begins with ciphers, there is a cipher in the root, following the decimal point, for each complete period of *three* ciphers directly following the decimal point in the number.

16. The principle explained in the preceding article is important and should be thoroughly understood, as it forms the basis for the use of Table I for finding cube root. Suppose, for example, that the cube root of .004096 is to be found. The table contains no decimals whatever. But, if .004096 is pointed off, it becomes .004′096, which is a number of two periods with a single figure, 4, in the first period. The significant part of the number is 4096. In the table can be found 4,096, which also contains two periods, with the figure 4 alone in the first period, and which has the same significant figures. Therefore, 4,096 and .004096 will have the same significant figures in their roots, the only difference being in the location of the decimal point. The cube root of 4,096 is 16, according to the table. The cube root of .004096 contains these same significant figures, 16, but as .004′096 has two periods and is wholly a decimal, its root must have two figures and be wholly a decimal. Therefore, the cube root of .004096 must be .16.

17. It has been shown that after a number has been pointed off into full periods of three figures each, by the method explained in Arts. **8** to **11**, the position of the decimal point does not affect the figures of the root. Therefore, after the periods have been determined the number may be considered as a whole number while the figures of the root are being found. If this whole number does not consist of two periods it must be altered, because no number having more than two periods is given in the table and because accurate results cannot usually be obtained by using only one period.

If the number has only one period, three zeros are annexed to form the second period. Thus, if 3 is the number whose cube root is to be found, it is necessary to annex three ciphers to form the second period; the altered number is 3′000. If the number contains more than two periods, ignore the periods after the second, unless the first figure of the third period is 5 or more, in which case increase the last figure in the second period by 1. If the number were 2,743,879, it would be pointed off thus, 2′743′879. As there are three periods, the last period is dropped. Since the first figure of the third period exceeds 5, the last figure of the second period is increased by 1. The altered number is 2744, and is found in the table.

18. If the cube root of the altered number may be found from the table directly it is merely necessary to locate the decimal point in accordance with its position in the original number; the cube root of the original number has the same significant figures as the cube root of the altered number.

It frequently happens that the altered number cannot be found in Table I. In such a case the numbers next greater and smaller than the altered number are found, and then, by the method to be illustrated in subsequent examples, the root is determined.

19. If the number whose root is to be found contains a fraction, reduce the fraction to an equivalent decimal, and then proceed as already explained. For example, if the cube root of $\frac{11}{16}$ is to be found, the fraction is reduced to a decimal; thus, $\frac{11}{16}=11\div16=.6875$. This decimal, when pointed off, becomes .687′500, showing that the root is wholly a decimal, with a digit immediately following the decimal point. The significant figures of the root are exactly the same as those of the cube root of 687,500, which becomes the altered number. If the cube root of $37\frac{1}{8}$ is to be found, the fraction $\frac{1}{8}$ is reduced to a decimal, becoming .125, and the number thus becomes 37.125. The altered number is 37,125, which has the same significant figures for its cube root as has 37.125.

Example 1.—Find the value of $\sqrt[3]{12{,}817}$ by the use of Table I.

Solution.—The given number, 12,817, contains two periods and is a whole number, so that it need not be altered. The number 12,817 does not appear in the table, but the numbers 12,167 and 13,824 appear, and 12,817 lies somewhere between them. The cube root of 12,167 is 23 and the cube root of 13,824 is 24; therefore, the cube root of 12,817 must lie between 23 and 24. It may be assumed that the desired root is at the same point between 23 and 24 as 12,817 is between 12,167 and 13,824. To determine just what this intermediate value of the root may be, the following simple calculation is made: The difference between the two numbers in the table is found, and is called the **first difference**; thus, 13,824−12,167=1,657, which is the first difference. Next the difference between the given number and the *smaller* number in the table is found, and is called the **second difference**; thus, 12,817−12,167=650, which is the second difference. The second difference is then divided by the first difference; thus, 650÷1,657=.39, or .4, nearly. This figure then becomes the third figure of the root. As the first two figures were already found to be 2 and 3, because the root is greater than 23 but less than 24, the required root is 23.4. Ans.

Proof.—The accuracy of any result found in extracting cube root may be tested very readily by simply cubing the value found for the root. The product should be approximately equal to the given number. In the preceding example, for instance, the root was found to be 23.4. By the foregoing test, $23.4^3=23.4\times23.4\times23.4=12{,}813$, very nearly. As this is very close to 12,817, the given number, it is concluded that 23.4 is the cube root, correct to three significant figures.

Example 2.—Find by the table method the cube root of .0732 to three decimal places.

Solution.—When pointed off, the given number becomes .073′200. The significant figures of this number are the same as if the number were 73,200; also, the significant figures of the cube root of .073200 are the same as those of the cube root of 73,200, because each contains two periods composed of like figures. The only difference is that the cube root of .073200 is wholly a decimal, whereas the cube root of 73,200 is a mixed number in which the decimal point occurs after the second figure. All that is necessary, therefore, is to find the cube root of 73,200 from Table I and then to shift the decimal point in that root until the root is wholly a decimal. The result will then be the cube root of .0732. In the table, 73,200 lies between 68,921 and 74,088, the cube roots of which are 41 and 42, respectively. Therefore, the cube root of 73,200 is greater than 41 and less than 42. The third significant figure of the root is found as follows:

$$\begin{array}{rl@{\qquad\qquad}rl} 74{,}088 & & 73{,}200 & \\ \underline{68{,}921} & & \underline{68{,}921} & \\ 5{,}167 & \textit{first difference} & 4{,}279 & \textit{second difference} \end{array}$$

Second difference÷*first difference*=4,279÷5,167=.8

Then, the third significant figure is 8 and the cube root of 73,200 is 41.8. The cube root of .0732 has exactly the same significant figures, but as .0732 is wholly a decimal, its cube root is wholly a decimal; therefore, $\sqrt[3]{.0732}=.418$. Ans.

Example 3.—What is the cube root of .18526?

Solution.—When pointed off, the given number becomes .185′260. The cube root of this number has the same significant figures as the cube root of 185,260, but is wholly a decimal. In the table, 185,260 lies between 185,193 and 195,112, showing that the cube root of 185,260 lies between 57 and 58. The first difference is 195,112−185,193=9,919 and the second difference is 185,260−185,193=67; then, 67÷9,919=.007, nearly. Annex this to 57, and the root is 57.007 or 57.01 to four significant figures. As the cube root of .18526 has the same significant figures, but is wholly a decimal, it must be .5701. Ans.

Example 4.—Find from the table the value of $\sqrt[3]{293{,}475{,}910}$.

Solution.—When pointed off into periods, the given number becomes 293′475′910. The third period is dropped; but, as it begins with 9, the last figure of the second period is increased by 1, as explained in Art. **17.** The altered number thus becomes 293′476. This is not found in the table, but 287,496 and 300,763 are found there. Therefore the required root is greater than 66 and less than 67. The first difference is 300,763−287,496=13,267 and the second difference is 293,476 −287,496=5,980. Then, 5,980÷13,267=.45, showing that the third and fourth significant figures are 4 and 5. The cube root of 287,496 is 66, according to the table, and so the root of 293,476 is 66.45. The original number, however, is a whole number containing three periods; therefore, its root must have three figures ahead of the decimal point. Hence, the root is 664.5. Ans.

EXAMPLE 5.—What is the cube root of .0007425?

SOLUTION.—The given number, pointed off into complete periods, is .000′742′500. According to Art. **17**, this number must be changed to one containing the same significant figures and having two periods. The first period, consisting of ciphers, is therefore dropped, and the altered number is 742,500. It does not appear in the table, but 729,000 and 753,571, the cubes, respectively, of 90 and 91, are given therein. The first difference is 753,571−729,000=24,571, and the second difference is 742,500−729,000=13,500. Then, 13,500÷24,571=.5, approximately. The cube root of 729,000 is 90, and to this must be annexed the .5 just found, giving 90.5 as the approximate cube root of 742,500. The original number, .0007425, has one complete period of ciphers following the decimal point, and so the root must be a decimal having one cipher following the decimal point and in front of the first significant figure. Therefore, the required root is .0905. Ans.

EXAMPLE 6.—Find the value of $\sqrt[3]{\frac{273}{1054}}$.

SOLUTION.—First reduce the fraction to a decimal; thus, $\frac{273}{1054}$=273 ÷1,054=.259013. The problem therefore is to find the cube root of .259013. This number, when pointed off, is .259′013, and the altered number is 259,013. This altered number is not in Table I, but it lies between 250,047 and 262,144, the cube roots of which are 63 and 64, respectively. The first difference is 262,144−250,047=12,097, and the second difference is 259,013−250,047=8,966. Then, 8,966÷12,097=.7, and the cube root of 259,013 is 63.7. The cube root of .259013 has the same significant figures, but is wholly a decimal; therefore, it must be .637. Ans.

EXAMPLE 7.—What is the cube root of $3\frac{2}{3}$?

SOLUTION.—When reduced to a decimal, $\frac{2}{3}$ becomes .6666+ or .667, and the number whose root is to be found is then 3.667. The altered number is 3,667, which lies between 3,375 and 4,096 in the table; and as the roots of 3,375 and 4,096 are 15 and 16, respectively, the root of 3,667 must lie between 15 and 16. The first difference is 4,096−3,375 =721, and the second difference is 3,667−3,375=292. Then, 292÷721 =.4. So the cube root of 3,667 is 15.4. The cube root of 3.667, which has one period preceding the decimal point, has the same significant figures, with one figure preceding the decimal point; therefore, the cube root required is 1.54. Ans.

EXAMPLES FOR PRACTICE

By the use of Table I find the cube roots of the following numbers to three significant figures:

		Ans.
(*a*)	860,000	(*a*) 95.1
(*b*)	27.27	(*b*) 3.01
(*c*)	.09125	(*c*) .450
(*d*)	127.4	(*d*) 5.03
(*e*)	$56\frac{7}{8}$	(*e*) 3.85
(*f*)	$\frac{49}{64}$	(*f*) .915

EXACT METHOD

20. If the cube root of a number is to be found accurately to more than three significant figures, neither of the preceding methods can be used successfully. In such a case, an exact method is required. There are a number of such exact methods, but the one about to be described is simple and may be used for finding the cube root of any number. It is most easily explained by the solution of an example.

21. Cube Roots of Whole Numbers.—Suppose that it is desired to find the cube root of 11,390,625. First separate the number into periods of three figures each, commencing at the right. This gives 11'390'625. Draw a line at the right of the number and find the largest number whose cube is not greater than the left-hand period. Place this number as the first figure of the answer.

The left-hand period is 11 and the largest number whose cube will not be greater than 11 is 2, since $3^3=27$.

Place the 2 as the first figure of the answer and subtract the cube of this number from the left-hand period of the number. To the remainder thus obtained annex the next period of the number. The process thus far is as follows:

$$\begin{array}{l} 11'390'625(2 \\ \underline{8} \\ 3390 \end{array}$$

It is next necessary to find a *trial divisor*. This is obtained by considering the number in the answer as tens, *squaring* it, and multiplying the result by 3. 2 considered as tens is 20, which squared gives 400. 400 multiplied by 3 gives 1,200 as a trial divisor.

The trial divisor is contained 2 times in the remainder 3,390. Place the 2 as the second figure of the answer. This gives the following:

$$\begin{array}{r|l} & 1\,1'3\,9\,0'6\,2\,5\,(\,2\,2 \\ & 8 \\ 3\times20^2=1200 & \bar{3}\,3\,9\,0 \end{array}$$

In order to obtain a *complete divisor,* it is necessary to add together three products.

The *first* product is that obtained for the trial divisor, or, in this case, 1,200.

The *second* product is obtained by considering the first figure of the answer as *tens,* and multiplying it by the *second* figure of the answer, and by 3. In the example being worked, the first figure of the answer considered as tens gives 20, which multiplied by the second figure of the answer, or 2, gives 40, which result multiplied by 3 gives 120 as the second product.

The *third* product is obtained by *squaring* the second figure of the answer, which in this case is $2^2=4$.

Add these three products together and the complete divisor becomes $1{,}200+120+4=1{,}324$. The example worked to this point is as follows:

$$\begin{array}{rr|l} & & 1\,1'3\,9\,0'6\,2\,5\,(\,2\,2 \\ & & 8 \\ 3\times20^2= & 1\,2\,0\,0 & \bar{3}\,3\,9\,0 \\ 3\times(20\times2)\ = & 1\,2\,0 & \\ 2^2= & 4 & \\ \cline{2-2} & 1\,3\,2\,4 & \end{array}$$

Next multiply the complete divisor 1,324 by the second figure of the answer and subtract the result from the remainder previously obtained. To the result annex the next period for a *new remainder.* The process is as follows:

$$
\begin{array}{rr|l}
 & & \multicolumn{1}{l}{11'390'625(22} \\
 & & \multicolumn{1}{l}{8} \\
3\times20^2 = & 1200 & \bar{3}390 \\
3\times(20\times2) = & 120 & \\
2^2 = & 4 & \\
\cline{2-2}
 & 1324 & 2648 \\
\cline{3-3}
 & & 742625
\end{array}
$$

The further processes are but repetitions of those previously described and will not need much explanation.

It is next necessary to find a trial divisor for the remainder. This is done in the same manner as in the first case, the number in the answer being considered as tens, and squared, after which it is multiplied by 3. This will be $220^2 = 48{,}400$, which, multiplied by 3, gives 145,200 for a trial divisor.

The trial divisor divided into the remainder gives 5 as the next figure of the answer.

Next, find the complete divisor by adding together the three products, taking the trial divisor as the first product, 3 times the number already in the answer considered as tens multiplied by the next figure of the answer for the second product, and the square of the third figure of the answer for the third product. Multiply the complete divisor by the third figure of the answer and subtract the result thus obtained from the remainder previously obtained. The complete process is:

$$
\begin{array}{rr|l}
 & & \multicolumn{1}{l}{11'390'625(225 \quad \text{Ans.}} \\
 & & \multicolumn{1}{l}{8} \\
3\times20^2 = & 1200 & \bar{3}390 \\
3\times(20\times2) = & 120 & \\
2^2 = & 4 & \\
\cline{2-2}
 & 1324 & 2648 \\
\cline{3-3}
3\times220^2 = & 145200 & 742625 \\
3\times(220\times5) = & 3300 & \\
5^2 = & 25 & \\
\cline{2-2}
 & 148525 & 742625 \\
\cline{3-3}
\end{array}
$$

Since the product obtained by multiplying the complete divisor by the third term of the answer is just equal to the remainder, the process is complete, and the answer, or 225, is the cube root of the number 11,390,625.

22. Cube Roots of Decimals.—In extracting the cube root of a decimal, proceed as with a whole number, taking care that each period contains *three* figures. Begin the pointing off at the decimal point, going toward the right. If the last period does not contain three figures, annex ciphers until it does.

The number of decimal places in the answer will equal the number of periods in the decimal whose root is to be extracted.

EXAMPLE.—What is the cube root of .009129329?

SOLUTION.—

$$\begin{array}{rr|l}
 & & .009'129'329\,(.209 \quad \text{Ans.} \\
 & & \underline{8} \\
3\times 20^2= & 1200 & 1129 \\
 & & \underline{0000} \\
3\times 200^2= & 120000 & 1129329 \\
3\times(200\times 9)= & 5400 & \\
9^2= & \underline{81} & \\
 & 125481 & \underline{1129329}
\end{array}$$

In this particular example it will be noticed that in the first case the trial divisor 1,200 is greater than the remainder 1,129; consequently, the next figure of the answer is 0 and it is necessary to bring down the next period for a new remainder, after which the regular process is followed.

23. Roots of Mixed Numbers.—One example of extracting the cube root of a mixed number will be given here.

EXAMPLE.—What is the cube root of 47.832147?

SOLUTION.—

$$\begin{array}{rr|l}
 & & 47.832'147\,(3.63 \quad \text{Ans.} \\
 & & \underline{27} \\
3\times 30^2= & 2700 & 20832 \\
3\times(30\times 6)= & 540 & \\
6^2= & \underline{36} & \\
 & 3276 & \underline{19656} \\
3\times 360^2= & 388800 & 1176147 \\
3\times(360\times 3)= & 3240 & \\
3^2= & \underline{9} & \\
 & 392049 & \underline{1176147}
\end{array}$$

24. Rule for Cube Root by Exact Method.—The series of operations that must be performed in order to find the cube root of a number by the method described in the preceding examples may be stated in the form of a rule, as follows:

Rule.—*Separate the number into periods of three figures each, commencing at the right; if the number contains a decimal, begin at the decimal point and work both ways.*

Find the largest number whose cube is not greater than the left-hand period and write this number as the first figure of the answer.

Subtract the cube from the left-hand period and to this result annex the next period of the number.

Square the number in the answer considered as tens, and multiply this result by 3.

Use the result thus obtained for a trial divisor to divide into the remainder, and place the number resulting from this division as the next figure of the answer.

To the trial divisor, add the result obtained by multiplying the first figure of the answer considered as tens by the second figure of the answer and by 3; also add the square of the second figure. The sum thus obtained is the complete divisor.

Multiply the complete divisor by the second figure of the answer and subtract the result thus obtained from the remainder.

To this result annex the next period of the number and proceed in the manner described until all the periods have been used.

25. Cube Roots of Fractions.—If the given number is in the form of a fraction, and it is required to find the cube root of it, the simplest and most exact method is to reduce the fraction to a decimal and extract the cube root of the decimal. If, however, the numerator and denominator of the fraction are perfect powers, extract the cube root of each separately, and write the root of the numerator for a new numerator, and the root of the denominator for a new denominator.

Rule.—*Extract the cube root of the numerator and denominator separately if each is a perfect cube; otherwise, reduce the fraction to a decimal, and extract the root of the decimal.*

EXAMPLE 1.—What is the cube root of $\frac{27}{64}$?

SOLUTION.— $\sqrt[3]{\frac{27}{64}}=\frac{\sqrt[3]{27}}{\sqrt[3]{64}}=\frac{3}{4}$. Ans.

EXAMPLE 2.—What is the cube root of $\frac{1}{4}$?

SOLUTION.—Since $\frac{1}{4}=.25$, $\sqrt[3]{\frac{1}{4}}=\sqrt[3]{.25}=.62996$. Ans.

26. Memorizing the Exact Method.—The extraction of cube root is an operation with which many students have difficulty, particularly when the exact method is used. The trouble is not in performing the several calculations, but in remembering the order in which they must be done and in finding the trial divisor and the complete divisor. Reference to the solutions given in Arts. **21** to **23** shows that, after cubing the first figure of the root and obtaining the first remainder, the same series of operations is repeated until the work is completed. This series of operations is made up of the following steps:

1. Finding the trial divisor.
2. Finding the next figure of the root.
3. Finding the complete divisor.
4. Multiplying the complete divisor by the last figure of the root thus far found.
5. Subtracting and forming a new dividend.

27. The only steps that are likely to cause any trouble are the first and the third, or finding the trial divisor and the complete divisor; but these may easily be remembered by fixing in mind the following expression:

$$3a^2+3ab+b^2$$

The first term, or $3a^2$, represents the trial divisor, and the whole expression, $3\,a^2+3\,a\,b+b^2$ represents the complete divisor. To indicate just how these terms are used to represent the trial and complete divisors, consider again the solution of the example in Art. **21,** which is repeated here for convenience.

			1 1'3 9 0'6 2 5 (2 2 5 Ans.
			8
trial divisor	$=3a^2=$	$3\times20^2=$ 1200	3390
	$3ab=3\times(20\times2)=$	120	
	$b^2=$	$2^2=$ 4	
complete divisor	$=3a^2+3ab+b^2=$	1324	2648
			742625
trial divisor	$=3a^2=$	$3\times220^2=$ 145200	
	$3ab=3\times(220\times5)=$	3300	
	$b^2=$	$5^2=$ 25	
complete divisor	$=3a^2+3ab+b^2=$	148525	742625

28. After the first figure of the root has been cubed and its cube has been subtracted from the first period, the next period is brought down, giving 3,390 as the first dividend, for which a trial divisor must be found. Let a represent 10 times the root *thus far found,* which in this case makes $a=10\times2=20$. Then, the trial divisor can easily be calculated, as it is $3a^2$. As $a=20$, it is readily seen that $3a^2=3\times20^2=3\times20\times20=1,200$. This trial divisor, 1,200, is contained 2 times in the dividend 3,390; so 2 is the next figure of the root, and is written down there. Now let b represent the last figure of the root *thus far found,* which in this case is the second 2; then, $b=2$. To obtain the complete divisor, add $3ab$ and b^2 to the trial divisor $3a^2$, as shown. As $a=20$ and $b=2$, it follows that $3ab=3\times(a\times b)=3\times(20\times2)=3\times40=120$; and $b^2=2^2=2\times2=4$. Then, $3a^2+3ab+b^2=1,200+120+4=1,324$, the first complete divisor. Multiply 1,324 by 2, subtract the product, 2,648, from the first dividend, and bring down the next period, producing the second dividend, or 742,625.

29. The next operation is similar in every way to that just performed. First, a trial divisor must be found. The figures of the root *thus far found* are 22, and a represents 10 times this value; that is, $a=10\times22=220$. Then, the trial divisor is $3a^2=3\times220^2=3\times220\times220=145,200$, as shown. This is contained 5 times in the second dividend, and so 5 is written as the next figure of the root. Then, $b=5$, because b

always represents the last figure of the root *thus far found.* Therefore, $3ab=3\times(a\times b)=3\times(220\times 5)=3\times 1{,}100=3{,}300$; and $b^2=5^2=25$. The second complete divisor is then $3a^2+3ab+b^2=145{,}200+3{,}300+25=148{,}525$, as shown. This is multiplied by the last figure of the root, 5, and the product is written under the second dividend. As there is no remainder, the work is ended and the required cube root is 225.

30. To assist further in fixing in mind the steps just explained, the following restatement of principles is made:

1. The trial divisor at any stage of the operation of finding cube root is always equal to $3a^2$, in which a is 10 times the value of the root *thus far found.*

2. The complete divisor is the sum of the trial divisor, $3a^2$, and the products $3ab$ and b^2, in which b is the value of the last figure of the root *thus far found;* that is, the complete divisor is $3a^2+3ab+b^2$.

If this expression is committed to memory, together with an understanding of what a and b stand for, it should be easy to find the cube root of any number under any circumstances.

EXAMPLES FOR PRACTICE

Find the cube roots of the following numbers by the exact method:

			Ans.
(*a*)	1,860,867	(*a*)	123
(*b*)	.185193	(*b*)	.57
(*c*)	636,056	(*c*)	86
(*d*)	87,528.384	(*d*)	44.4
(*e*)	74,088	(*e*)	42
(*f*)	257,259,456	(*f*)	636
(*g*)	$\frac{125}{4096}$	(*g*)	$\frac{5}{16}$
(*h*)	$\frac{2197}{27000}$	(*h*)	$\frac{13}{30}$
(*i*)	$\frac{87}{125}$	(*i*)	.8862
(*j*)	$\frac{3}{8}$	(*j*)	.7211

APPLICATIONS OF CUBE ROOT

31. Diameter of a Sphere.—There are many sorts of problems whose solutions require the extraction of cube root. One of the most common cases in which cube root must be extracted is in finding the diameter of a sphere. The rule for finding the diameter is as follows:

Rule.—*To find the diameter of a sphere, extract the cube root of the volume and multiply the result by 1.24.*

If the volume is in cubic inches, the diameter will be in inches; if the volume is in cubic feet, the diameter will be in feet; and so on.

Example 1.—A spherical ball contains 2,000 cubic inches. What is the diameter of the ball?

Solution.—According to the rule, the cube root of the volume, or 2,000 cu. in., must first be found. The cube root of 2,000 may be found by any one of the methods previously described. In this case, however, the table method will be used. The number 2,000 does not appear in Table I, but 1,728 and 2,197 are given, their roots being 12 and 13, respectively. Therefore, the cube root of 2,000 is between 12 and 13. The first difference is 2,197−1,728=469 and the second difference is 2,000−1,728=272. Then, 272÷469=.6. Consequently, the cube root of 2,000 is 12.6. To find the diameter, the cube root must be multiplied by 1.24. Therefore, the diameter of the ball is

12.6×1.24=15.624, or 15⅝ in., nearly. Ans.

Example 2.—If a tank in the shape of a sphere must hold 48,000 cubic feet, what must be its inside diameter?

Solution.—First find the cube root of 48,000, using the table method. This value lies between 46,656 and 50,653 in the table. The first difference is 50,653−46,656=3,997, and the second difference is 48,000−46,656 =1,344. Then, 1,344÷3,997=.336, or .34, approximately. The cube root of 46,656 is 36 and so the cube root of 48,000 is 36.34. The diameter of the tank is 1.24 times the root thus found, or

36.34×1.24=45 ft., very closely. Ans.

32. Speed and Coal Consumption of a Steamship. The speed of a steam-driven ship may be increased by forcing the fires, that is, by burning coal more rapidly under the boilers. The amount of coal burned per day, called the coal consumption, is proportional to the cube of the speed. Therefore, if the coal consumption is changed, and it is desired to find what the speed will be at the new rate at which the coal is burned, the following rule may be used:

Rule.—*To find approximately the new speed for a new rate of coal consumption, multiply the new coal consumption by the cube of the known speed, divide the product by the coal consumption corresponding to the known speed, and extract the cube root of the quotient.*

The speeds of vessels are usually expressed in knots, a knot being a speed of 1.15156 miles per hour.

EXAMPLE 1.—A steamer consumes 100 tons of coal a day when running at a speed of 10 knots. What will be the speed if the coal consumption is increased to 200 tons a day?

SOLUTION.—The new rate of coal consumption is 200 tons a day and the known speed is 10 knots. According to the rule, then, 200 is to be multiplied by the cube of 10; thus, $200\times10^3=200\times1{,}000=200{,}000$. This product is then to be divided by the coal consumption at the known speed, or 100 tons a day. Thus, $200{,}000\div100=2{,}000$. The cube root of this quotient, 2,000, is then the new speed; that is, the new speed is equal to $\sqrt[3]{2{,}000}$, which, by the table method or by either of the other methods, is 12.6. Therefore, the new speed will be 12.6 knots. Ans.

EXAMPLE 2.—If a steamer consumes 60 tons of coal a day to produce a speed of 9 knots, how many knots would she make if the coal consumption were decreased to 48 tons a day?

SOLUTION.—The known speed is 9 knots, the coal consumption at that speed is 60 tons a day, and the new coal consumption is 48 tons a day. According to the rule, therefore, the new speed is

$$\sqrt[3]{\frac{48\times9^3}{60}}=\sqrt[3]{\frac{48\times729}{60}}=\sqrt[3]{583.2}=8.35 \text{ knots. Ans.}$$

COMMERCIAL CALCULATIONS
(PART 1)

SPECIAL METHODS IN APPLICATION OF FUNDAMENTAL RULES

1. At this point the student should be able to apply with ease the fundamental rules of arithmetic and to perform all the operations with fractions, both common and decimal. The first essential of an accountant is accuracy. A result obtained in any calculation is worthless if it is not correct. In business life a clerk or accountant whose work shows incompetency in this respect will not be tolerated. Speed ranks in importance next to accuracy. A man's services are valued according to what he can accomplish. A rapid and accurate accountant is, of course, preferred to the man who makes calculations slowly.

We shall now take up some principles which are supplemental to those treated in the preceding Sections, and which, if followed, will be a great aid to one wishing to attain accuracy and rapidity.

RAPID ADDITION

2. There is scarcely anything more useful to the bookkeeper and business man than the ability to add rapidly and correctly; but this can be acquired only by persevering practice. Any time employed in practicing addition will be well spent. If the student will practice addition ten or fifteen minutes daily for a month or so, he will be greatly benefited.

In order to become expert in adding, it is absolutely essential that when two figures are seen or heard pronounced, the student can instantly give their sum. Thus, 15 should suggest itself as soon as 6 and 9, or 8 and 7, are seen or heard pronounced. It should not be necessary to say mentally, 6 and 9 is 15, but 6, 15, the 9 not being pronounced either mentally or orally. (In no case should the student contract the habit of adding aloud; it is not necessary, and it is a very difficult habit to break.)

In adding 5, 6, 1, 9, 7, 5, 2, 4, 8, 9, do not say 5 and 6 is 11, and 1 is 12, and 9 is 21, etc., but *think* 5, 11, 12, 21, 28, 33, 35, 39, 47, 56, repeating the sums about as fast as they can be pronounced.

3. To add rapidly, it is necessary for the student to become accustomed to group the figures of a column and add the sums of the groups. It is most convenient to choose the groups, as far as possible, so that the sum of each group shall be either 10 or 20. An example will show how this is accomplished:

$$\begin{array}{r}3\\9\\2\\5\\8\\2\\6\\4\\3\\5\\0\\9\\1\\2\\\hline 59\end{array}$$

Commencing at the bottom of the column, we see at once that 1 and 9 form a group whose sum is 10; hence we mentally say 2, 12 instead of 2, 3, 12. Now the next two digits, 5 and 3, we group together, and instead of adding 5 and 3 separately, we add the sum, 8. The next two figures, 4 and 6, form a group whose sum is 10, and so do the next two figures, 2 and 8. Looking now at the four figures at the top of the column, we readily see that 5, 2, and 3 form a last group whose sum is 10, leaving only the 9 outside of a group. In adding the column, we would repeat mentally 2, 12, 20, 30, 40, 50, 59. Another grouping would readily appear to the skilful accountant; the 2 at the bottom and the 5 and 3 above the 0 form a group whose sum is 10. Recognizing this group, the mental addition would be: 10, 20, 30, 40, 50, 59.

This process of forming groups may be extended to include those whose sums are 15 or 20. By a judicious selection of the figures composing a group, its sum may usually be made either 10, 15, or 20, and these sums should always be sought in preference to others, since two 15's make 30, and the numbers

$$\begin{array}{rl}5&\\ \left.\begin{array}{r}9\\3\\8\end{array}\right\}&20\\ \left.\begin{array}{r}6\\4\end{array}\right\}&10\\ 7&\\ \left.\begin{array}{r}2\\9\\4\end{array}\right\}&15\\ \left.\begin{array}{r}8\\7\end{array}\right\}&15\\ \hline 72&\end{array}$$

10, 20, and 30 are added with little mental labor. Thus, in the following example, the grouping is shown by the braces. The mental addition is 15, 30, 37, 47, 67, 72. After some practice, the student will be able to recognize a group whose sum is 10 or 20 almost instantly; he should persevere in the solution of examples until he is able to form the groups rapidly and with ease. It is not always possible to get consecutive numbers which will form groups whose sums are 10, 15, or 20. Such groups, however, can often be found by skipping one or more figures in the column. For instance, in the following example, the first two figures, 7 and 3, make a 10, and skipping the fourth figure, 9, the third and fifth figures, 4 and 6, make another 10. Skipping the 8, the 3, 4, and 3 at the top of the column make another 10, thus making three groups of 10 and the figures 9 and 8. In adding, however, we should not leave the figures skipped to be added at the end of the operation, but should add them as they occur. In this example, we would say 10, 20, 29, 39, 47. The order of adding may be shown by the following arrangement:

$$\begin{array}{r}3\\4\\8\\3\\0\\6\\9\\4\\3\\7\\ \hline 47\end{array}$$

$$\overbrace{7+3}^{10}+\overbrace{6+4}^{10}+9+\overbrace{3+4+3}^{10}+8.$$

The columns can be added from the top downwards if this order is preferred.

The following examples may be used for practice:

(1)	(2)	(3)	(4)	(5)
3	13	317	1127	13103
5	27	226	2632	61706
2	92	232	1946	43285
6	35	518	3217	39134
3	64	396	1184	96286
4	22	435	1936	45173
0	30	707	2003	78135
9	97	992	5114	26232
3	18	311	976	19375
1	33	417	5634	8428

The answers are: (1) 36; (2) 431; (3) 4,551; (4) 25,769; (5) 430,857.

4. It is frequently advantageous to be able to add *horizontally*, as it is termed; by this is meant the adding of several numbers as they stand in a horizontal row, without arranging the numbers in vertical columns. Thus,

$$123+567+792+221+546=2{,}249. \quad \text{Ans.}$$

When adding in this manner, straight addition must be employed; i. e., the method of Art. 3 cannot be used. The process would be 6, 7, 9, 16, 19; 5, 7, 16, 22, 24; 7, 9, 16, 21, 22. As an example of this method, consider the accompanying table, which is supposed to give, by days, the grain export, in bushels, of a certain city for one week. It is required to find the amount of grain exported each day, the total amount of each kind of grain exported during the week, and, finally, the total amount of grain exported during the week.

	Mon.	Tues.	Wed.	Thurs.	Fri.	Sat.	Totals
Corn..........	28,325	15,236	35,715	29,128	75,183	46,217	*****
Wheat........	35,719	41,719	50,108	32,546	59,275	81,126	*****
Oats..........	12,136	9,237	18,265	7,268	6,950	17,230	*****
Barley........	18,230	15,738	21,375	15,928	19,263	13,637	*****
Rye..........	5,275	6,829	7,201	11,325	7,825	13,261	*****
Totals........	*****	*****	*****	*****	*****	*****	*****

The student should find the totals, and prove that the results are correct by adding the totals in the right-hand column and then adding the totals in the bottom row; the two results should be the same, namely, 757,270 bushels. The other results are: corn, 229,804; wheat, 300,493; oats, 71,086; barley, 104,171; rye, 51,716; Mon., 99,685; Tues., 88,759; Wed., 132,664; Thurs., 96,195; Fri., 168,496; Sat., 171,471.

SPECIAL METHOD OF SUBTRACTION

5. A method of subtraction which is called the *making change* method is frequently used. Suppose a customer hands the clerk a five-dollar bill in payment of a purchase amounting to $3.76. The clerk does not subtract $3.76 from $5 by the ordinary rules of subtraction but he adds enough to $3.76 to make $5. He would count out 4 cents, two 10-cent pieces, and a dollar, since $3.76 and 4 cents is $3.80, $3.80 and 20 cents is $4, and an additional dollar makes $5. To verify his reckoning he counts as he hands the customer the change—three seventy-six, three eighty, four, five—omitting to say dollars and cents, as these words are, of course, understood. A similar method may be used in written work.

Thus, if the minuend is 8,453 and the subtrahend 844, the remainder is found as follows:

minuend	8453	
subtrahend	844	
remainder	7609	Ans.

Instead of subtracting 4 from 13, we add to 4 a number that will make 13; this number is 9, since 4 and 9 is 13. Write the 9 and carry the 1, as in addition; then $4+1=5$. Since $5+0=5$, for the next figure in the subtrahend we write 0, as shown. There is nothing to carry; hence, since 8 is greater than 4, we consider 4 to be 14, and find what number added to 8 will make 14; this is 6, which we write below the line, as shown. We have 1 to carry, but, as we have no figure in the subtrahend to add it to, we say 1 and 7 is 8, and write the 7 below the line, as shown.

Again, consider the following example:

minuend	10000	
subtrahend	8763	
remainder	1237	Ans.

Here we say 3 and 7 is 10, and write the 7; then, 7 and 3 is 10 (carrying the 1 and adding it to the 6), and write the 3; next, 8 and 2 is 10, and write the 2; finally, 9 and 1 is 10, and write the 1. The student may use whatever method of subtraction he prefers, but it may be stated that the making-change method is becoming more and more popular.

6. The method just described is particularly useful when it is desired to subtract from some number the sum of several numbers, the entire process being performed at one operation. For example, suppose it were desired to subtract the sum of

1,538, 512, 987, 645, 884, and 763 from 7,061. The work might be arranged as follows:

1538	7061	*minuend*
512	1732	Ans.
987		
645		
884		
763		

We add the right-hand column, 3, 7, 12, 19, 29. We cannot subtract 29 from 1 in the minuend, but if 2 is added to 29 the sum will be 31, a number ending in 1. Hence, write 2 under 1 and carry 3. Adding the second column, 3, 9, 17, 21, 30, 33; 3 added to 33 makes 36; hence, write 3 under 6 and carry 3. Adding the third column, 3, 10, 18, 24, 33, 43; 7 added to 43 makes 50; hence, write 7 under 0 and carry 5. Adding 5 to the 1 in the fourth column makes 6, and 6 and 1 is 7; hence, write 1 under 7. The difference between the sum of the numbers and 7,061 is 1,732, the answer.

EXAMPLES FOR PRACTICE

In the following examples use the method just described. The work may be checked by adding the remainder in each case to the numbers added; the sum should be equal to the minuend.

(1)		(2)		(3)	
33604	178884	81220	350915	21824	261132
9775		21413		71737	
11628		37526		24445	
84721		41284		94336	
5109		71733		8972	

Ans.—(1) 34,047; (2) 97,739; (3) 39,818.

SPECIAL METHODS OF MULTIPLICATION

7. A quantity may be multiplied by finding the continued product of the quantity and the factors of the multiplier. For example, a number can be multiplied by 12 by multiplying the number by 3 and the product thus obtained by 4. Multiplying a number and its products in succession by 4, 6, and 8 will produce the same product as multiplying the number by 192, the continued product of 4, 6, and 8.

8. To multiply by factors.

Example.—Multiply 2,972 by 192, taking the factors 4, 6, and 8, and test the result by multiplying directly.

Solution.—

$$\begin{array}{rl} 2972 & \\ 4 & \\ \hline 11888 & \\ 6 & \\ \hline 71328 & \\ 8 & \\ \hline 570624 & \text{Ans.} \end{array} \qquad \begin{array}{rl} 2972 & \\ 192 & \\ \hline 5944 & \\ 26748 & \\ 2972 & \\ \hline 570624 & \text{Ans.} \end{array}$$

In this particular problem no time would be saved by using the factors in multiplication. The principle can often be used, however, to shorten the operation and may be employed to test the accuracy of the product obtained by another method.

EXAMPLES FOR PRACTICE

(*a*) Multiply 536 by 56.
(*b*) Multiply 763 by 24.
(*c*) What will 54 horses cost at $185 each?

Ans.—(*a*) 30,016; (*b*) 18,312; (*c*) $9,990.

---

9. To multiply when one part of the multiplier is a factor of another part.

Example. 1.—Multiply 1,728 by 93.

Solution.—

$$\begin{array}{rrl} & 1728 & \\ & 93 & \\ \hline & 5184 & \\ 3\times5{,}184= & 15552 & \\ \hline & 160704 & \text{Ans.} \end{array}$$

Instead of multiplying 1,728 by 9 in order to obtain the second partial product, multiply 5,184 by 3, as 3×3×1,728 =9×1,728. We do so since it is easier for most people to multiply by 3 than by 9.

Example 2.—Find the cost of 324 acres of land at $28.75 per acre.

Solution.—

$$\begin{array}{rr} & \$28.75 \\ & 324 \\ \hline & 11500 \\ 8\times 11{,}500= & 92000 \\ \hline & \$9315.00 \end{array} \quad \text{Ans.}$$

Observe that 32 equals 8 times 4. The work is shortened by multiplying 11,500 by 8 instead of multiplying 2,875 by 2 and 3 according to the usual way. The example may be solved also by considering 324 as made up of the parts 3 and 24 or 3 and 3×8.

$$\begin{array}{rr} & \$28.75 \\ & 324 \\ \hline & 8625 \\ 8\times 8{,}625= & 69000 \\ \hline & \$9315.00 \end{array} \quad \text{Ans.}$$

In using this method, it is necessary to place the first figure in any partial product directly under the right-hand figure of the corresponding multiplier.

Example 3.—Multiply 5,682 by 2,408.

Solution.—

$$\begin{array}{rr} & 5682 \\ & 2408 \\ \hline & 45456 \\ \left.\begin{array}{r} 3\times 45{,}456 \\ \text{or}\ \ 24\times 5{,}682 \end{array}\right\}= & 136368 \\ \hline & 13682256 \end{array} \quad \text{Ans.}$$

In multiplying, notice that 24=3×8, and that the right-hand figure of the second partial product is written directly under the 4 of the multiplier.

EXAMPLES FOR PRACTICE

Multiply:

(*a*) 3,859 by 567.

(*b*) 6,492 by 364.

(*c*) 3,741 by 824.

(*d*) 9,753 by 864.

Ans.—(*a*) 2,188,053; (*b*) 2,363,088; (*c*) 3,082,584; (*d*) 8,426,592.

1. An agent sells 273 vacuum cleaners for $23.75 apiece. What was the amount of the sales? Ans. $6,483.75

2. What is the cost of 123 textbooks at $1.18 apiece? Ans. $145.14

10. To multiply by a number ending with the figure 1.

EXAMPLE.—Multiply 2,386 by 81.

SOLUTION.—

$$\begin{array}{r} 2386 \\ 19088 \\ \hline 193266 \end{array} \text{ Ans.}$$

The first partial product equals $1\times2{,}386=2{,}386$; hence, it can be set down directly. The next partial product, which is $8\times2{,}386=19{,}088$, can be set down under the first according to the rules of multiplication. It is not even necessary to draw the line below the partial products, but such abbreviated methods should not be used when one expects to have his work reviewed by some one else. In such cases the operations employed should be clearly indicated.

11. To multiply by a number that is nearly a hundred or a thousand.

EXAMPLE 1.—Multiply 2,872 by 499.

SOLUTION.—

$$\begin{array}{rr} 500\times2{,}872= & 1436000 \\ 1\times2{,}872= & 2872 \\ \hline & 1433128 \end{array} \text{ Ans.}$$

We notice that 499 is 1 less than 500, which is a number easy to use as a multiplier. 500 times a number less 1 times that number is evidently 499 times that number.

EXAMPLE 2.—Find the cost of $797\frac{1}{2}$ yards of cloth at $\$2.62\frac{1}{2}$ a yard.

SOLUTION.—We observe that $797\frac{1}{2}$ is $2\frac{1}{2}$ less than 800.

$$\begin{array}{rr} 800\times\$2.625= & \$2100.00 \\ 2\frac{1}{2}\times2.625=\frac{1}{4}\times10\times2.625= & 6.56 \\ \hline & \$2093.44 \end{array} \text{ Ans.}$$

To multiply by $2\frac{1}{2}$, multiply by 10 and divide by 4, since $2\frac{1}{2}$ equals $\frac{10}{4}$.

EXAMPLES FOR PRACTICE

Multiply:

(*a*) 1,234 by 999.	(*c*) 5,246 by 91.
(*b*) 5,937 by 798.	(*d*) 1,234 by 9,995.

Ans.—(*a*) 1,232,766; (*b*) 4,737,726; (*c*) 477,386; (*d*) 12,333,830.

1. A man bought 97 yards of cloth at $1.25 a yard. How much did the cloth cost him? Ans. $121.25

2. How much does a farmer pay for 195 bags of corn at $1.35 a bag? Ans. $263.25

SPECIAL METHODS IN DIVISION

12. In the study of fractions it has been shown that both terms of a fraction may be multiplied by or divided by any number and the value of the fraction will remain the same. Similarly in division we may multiply or divide the dividend and divisor by the same number without changing the value of the quotient. This principle can sometimes be applied with profit, as when the divisor is a mixed number or when the divisor may be factored.

EXAMPLE 1.—A real-estate agent bought $18\frac{1}{3}$ acres of land for $4,000. What was the price paid per acre?

```
SOLUTION.—  18⅓)4000
             3     3
            ───  ─────
            55  )12000.000($218.181 or $218.18  Ans.
                 110
                 ───
                  100
                   55
                  ───
                   450
                   440
                   ───
                    100
                     55
                    ───
                    450
                    440
                    ───
                     100
                      55
                     ───
```

We multiply the divisor and dividend each by 3 in order to clear the divisor of the fraction; $3\times18\frac{1}{3}=55$, and $3\times4{,}000=12{,}000$; $12{,}000\div55=218.18$.

EXAMPLE 2.—Divide 1,000 by 32.

```
SOLUTION.—           4)32   4)1000
                       ──     ────
                        8    ) 250
                             ─────
                               31¼  Ans.
```

32 is divisible by 4; hence, we divide the divisor and dividend each by 4, and obtain 8 and 250. $250 \div 8 = 31\frac{1}{4}$. We could as well divide by 8 and then by 4; thus,

$$\begin{array}{rr} 8)\underline{32} & 8)\underline{1000} \\ 4 &)\underline{\ \ 125\ } \\ & 31\tfrac{1}{4} \end{array}$$

The advantage of this method is that we can reduce the divisor to a smaller number and then possibly use short division. In some cases the division can be performed mentally.

Example 3.—Divide 32,784 by 4,000.

Solution.—

$$\begin{array}{rl} 4.000) & \underline{32.784} \\ & 8.196 \quad \text{Ans.} \end{array}$$

We divide the divisor and dividend each by 1,000 by pointing off three decimal places from the right. The divisor becomes 4.000, or 4, and the dividend 32.784. By short division we find that $32.784 \div 4 = 8.196$.

We observe that when dividing by a number ending in ciphers, the ciphers may be dropped and as many decimal places pointed off in the dividend as there are ciphers dropped.

13. We will give another example to illustrate the use of this principle.

Example.—In a city of 30,000 inhabitants the assessed value of the property is $125,685,000. What is the average value of the property assessed to each person?

Solution.—The divisor ends in four ciphers, which may be dropped if four decimal places are pointed off in the dividend.

$$\begin{array}{rl} 3) & \underline{12568.5000} \\ & \$4189.50 \quad \text{Ans.} \end{array}$$

EXAMPLES FOR PRACTICE

Divide:

(*a*) 2,256 by $5\frac{1}{2}$.

(*b*) 34,678 by 2,000.

(*c*) 3,300 by $3\frac{1}{7}$.

Ans.—(*a*) $410\frac{2}{11}$; (*b*) 17.339; (*c*) 1,050.

14. Short Method of Subtracting a Product.—It is frequently necessary to multiply a number by a single digit and then to subtract the product from another number. This is most conveniently done as in the following example, in which 314,159 is multiplied by 7 and the product is subtracted from 7,208,451.

$$\begin{array}{r@{\qquad}l}
314159 & \underline{7208451}\ \textit{minuend} \\
\underline{7} & 5009338\ \text{Ans.}
\end{array}$$

EXPLANATION.— 7×9=63; 63 cannot be subtracted from 1 in the minuend, so we add the smallest number to 63 that will make the sum end with 1. This number is evidently 8, since 63+8=71; hence, write 8 under 1 and carry 7. Then, 7×5 =35, and 7 is 42; 42 and 3 is 45; write 3 under 5 in the minuend, and carry 4. Then, 7×1=7, and 4 is 11; 11 and 3 is 14. Write 3 under 4 and carry 1. 7×4=28, and 1 is 29; 29 and 9 is 38. Write 9 under 8 and carry 3. 7×1=7 and 3 is 10; 10 and 0 is 10. Write 0 under 0 and carry 1. 7×3=21, and 1 is 22; 22 and 0 is 22. Write 0 under 2 and carry 2. 2 and 5 is 7. Write 5 under 7. The result, 5,009,338, is the answer.

SHORT METHOD OF DIVISION

15. The following method saves about half the figures over the usual method, and we think that there will be fewer mistakes made when using it.

EXAMPLE.—Divide 39,913,910 by 5,494.

SOLUTION.—

$$\begin{array}{r@{}l@{\quad}l}
\textit{dividend} & \quad\textit{divisor} & \\
\underline{39913910}) & \underline{5494} & \\
\underline{14559} & 7265 & \textit{quotient} \\
\underline{35711} & & \\
\underline{27470} & & \\
0000 & &
\end{array}$$

EXPLANATION.—The method of multiplication and subtraction described in the last article is used in this case; the divisor is written on the right of the dividend, and the quotient underneath the divisor. The different figures of the quotient are obtained in exactly the same manner as by the preceding method. Thus, the divisor is contained in the first five figures of the dividend 7 times, and 7 is written for the first figure of the quotient. Now, instead of multiplying the divisor by 7, writing the product under the first five figures of the dividend, and then subtracting, we multiply each figure of the divisor by 7 and subtract from the dividend, writing only the remainder. Thus, 7 times 4 is 28, and 5 is 33; write the 5 under the 3 in the dividend, and carry 3. Then, 7 times 9 is 63 and 3 is 66, and 66 and 5 is 71; write the 5 and carry 7. 7 times 4 is 28 and 7 is 35, 35 and 4 is 39; write the 4 and carry 3. 7 times 5 is 35 and 3 is 38, and 38 and 1 is 39; write the 1. Now bring down the next figure of the dividend, 9, and annex it to the remainder. 14,559÷5,494=2; write 2 as the second figure of the quotient. Then, as above, $2\times4=8$, and $8+1=9$; write the 1 under the 9, as shown. $2\times9=18$, and $18+7=25$; write the 7 and carry the 2. $2\times4=8$, $8+2=10$, and $10+5=15$; write the 5 and carry the 1. $2\times5=10$, $10+1=11$, and $11+3=14$; write the 3. Bringing down 1, the next figure of the dividend, 35,711÷5,494=6, the third figure of the quotient. Proceed in the same way with the remaining figures.

A fast computer would work as follows: In multiplying by 6, he would repeat to himself 6, 24, and 7 is 31 (writing the 7 and carrying the 3). 6, 54, 57, and 4 is 61 (writing the 4 and carrying the 6). 6, 24, 30, and 7 is 37 (writing the 7 and carrying the 3). 6, 30, 33, and 2 is 35 (writing the 2).

The object of writing the divisor on the right is to make it easier to multiply by the figures of the quotient; it also saves space, as may readily be seen. The student is strongly advised to learn this method thoroughly, and always to use it. The process may seem slow and confusing at first, but rapidity can soon be acquired. The best way to attain facility in division is first to practice dividing by small numbers, from 2 to 12, and using the method of short division. After he has become proficient in this, he should practice long division by the method just described.

ALIQUOT PARTS

16. Numbers that form simple fractions of another number are called **aliquot parts** of the latter number. Thus, $12\frac{1}{2}$ cents is an aliquot part of a dollar, since it equals $\frac{1}{8}$ of a dollar.

17. To multiply by a fraction of 100.

EXAMPLE 1.—Find the cost of 2,876 cattle at $25 a head.

SOLUTION.— $100\times 2{,}876 = 287{,}600$; $25\times 2{,}876 = \frac{287{,}600}{4} = \$71{,}900$. Ans.

Since 25 is $\frac{1}{4}$ of 100, we multiply by 100 and divide the product by 4.

EXAMPLE 2.—What is the cost of 250 dozen pencils at 25 cents a dozen?

SOLUTION.—

$$\begin{array}{r} 4\,)\,250.00 \\ \hline \$62.50 \end{array} \quad \text{Ans.}$$

Since the price of each dozen is 25 cents, or $\frac{1}{4}$ dollar, we multiply by $\frac{1}{4}$, or what amounts to the same thing, we divide by 4.

EXAMPLE 3.—What is the cost of 84 penknives at $33\frac{1}{3}$ cents each?

SOLUTION.—

$$\begin{array}{r} 3\,)\;\;84 \\ \hline \$28 \end{array} \quad \text{Ans.}$$

$33\frac{1}{3}$ cents is $\frac{1}{3}$ of a dollar, hence we multiply 84 by $\frac{1}{3}$ by dividing 84 by 3.

18. The fractions of 100 given in the following table are often met with in business transactions and should be memorized.

$2\frac{1}{2} = \frac{1}{40}$ of 100
$3\frac{1}{3} = \frac{1}{30}$ of 100
$4 = \frac{1}{25}$ of 100
$5 = \frac{1}{20}$ of 100
$6\frac{1}{4} = \frac{1}{16}$ of 100
$8\frac{1}{3} = \frac{1}{12}$ of 100
$10 = \frac{1}{10}$ of 100
$12\frac{1}{2} = \frac{1}{8}$ of 100
$16\frac{2}{3} = \frac{1}{6}$ of 100
$20 = \frac{1}{5}$ of 100
$25 = \frac{1}{4}$ of 100
$33\frac{1}{3} = \frac{1}{3}$ of 100
$37\frac{1}{2} = \frac{3}{8}$ of 100
$50 = \frac{1}{2}$ of 100
$62\frac{1}{2} = \frac{5}{8}$ of 100
$66\frac{2}{3} = \frac{2}{3}$ of 100
$75 = \frac{3}{4}$ of 100
$87\frac{1}{2} = \frac{7}{8}$ of 100

Example 1.—Find the cost of 3 dozen textbooks at $87\frac{1}{2}$ cents each.

Solution.— 3 dozen = 36; $87\frac{1}{2}$ cents = $\frac{7}{8}$ dollar.

$$\overset{9}{\cancel{36}}\times\frac{7}{\underset{2}{\cancel{8}}}=\frac{63}{2}=\$31.50.\quad\text{Ans}$$

Example 2.—Find the cost of 2,875 pounds of sugar at $6\frac{1}{4}$ cents a pound.

Solution.— $6\frac{1}{4}$ cents = $\frac{1}{16}$ dollar, and since 16 is 4×4, we may divide by 4 twice.

$$\begin{array}{r} 4\,)\,2875 \\ 4\,)\,718.75 \\ \hline 179.6875 \end{array}\ \text{or } \$179.69.\quad\text{Ans.}$$

Example 3.—What is the cost of 360 baskets of peaches at $\$1.12\frac{1}{2}$ a basket?

Solution.— $\$1.12\frac{1}{2}=\$1\frac{1}{8}$.

$$\begin{array}{r} 8\,)\,360 \\ 45 \\ \hline \$\,405 \end{array}\quad\text{Ans.}$$

At \$1 a basket, the cost would be \$360; but, since the cost is $\$1+\$\frac{1}{8}$, the entire cost is $\$360+\frac{1}{8}$ of \$360.

Example 4.—A merchant bought 225 yards of cloth at $66\frac{2}{3}$ cents a yard. What was the amount of the bill?

Solution.— $66\frac{2}{3}$ cents is $\frac{2}{3}$ of a dollar.

$$\begin{array}{r} 3\,)\,225 \\ 75 \\ \hline \$\,150 \end{array}\quad\text{Ans.}$$

At \$1 a yard the cloth would cost \$225; but, since the cost is $\frac{2}{3}$ of a dollar $=\$1-\$\frac{1}{3}$, the entire cost is $\$225-\frac{1}{3}$ of \$225.

EXAMPLES FOR PRACTICE

1. What is the cost of 455 yards of cloth at 20 cents a yard? Ans. \$91

2. Find the amount received for 32 dozen eggs at $37\frac{1}{2}$ cents a dozen. Ans. \$12

3. A man earning $\$1.87\frac{1}{2}$ a day works 24 days in one month. How much did he receive for the month? Ans. \$45

4. What is the cost of 12 dozen pairs of gloves at $1.25 a pair?
Ans. $180

5. If a boy earns $87\frac{1}{2}$ cents a day, how much will he earn in 96 days?
Ans. $84

6. At $1.33\frac{1}{3}$ each, what will 60 chairs cost? Ans. $80

---

19. Use of Aliquot Parts in Division.—The aliquot parts of 100 can often be used to shorten the work of division.

EXAMPLE 1.—A boy earned $4.75 in a week. His wages are $12\frac{1}{2}$ cents an hour; how many hours did he work?

SOLUTION.— $12\frac{1}{2}$ cents $= \frac{1}{8}$ dollar; $4.75 \div \frac{1}{8} = 4.75 \times 8 = 38$ hours. Ans.

EXAMPLE 2.—Divide 487,634 by 25.

SOLUTION.—Since $487{,}634 \div 25 = 487{,}634 \div \frac{100}{4}$, we may divide by 25 by multiplying by 4 and then dividing by 100, or by dividing by 100 and multiplying the quotient by 4. The work would actually be performed as follows:

$$\begin{array}{r} 4876.34 \\ 4 \\ \hline 19505.36 \end{array} \quad \text{Ans.}$$

487,634 is divided by 100 by pointing off two decimal places.

EXAMPLES FOR PRACTICE

1. How many penknives at $33\frac{1}{3}$ cents each may be purchased for $60?
Ans. 180 penknives

2. The bill for a lot of cotton goods amounted to $6.75. If the goods cost $12\frac{1}{2}$ cents a yard, how many yards were there in the lot?
Ans. 54 yards

3. At a cost of $1.12\frac{1}{2}$ a dozen, how many dozen handkerchiefs can be bought with $90? Ans. 80 dozen

4. How many books costing $1.62\frac{1}{2}$ a piece can be purchased for $71.50? Ans. 44 books

USEFUL APPLICATIONS

20. Methods of Multiplying by a Mixed Number.—In business transactions the multiplication of a mixed number by an integer, an integer by a mixed number, or a mixed number by a mixed number, is of very frequent occurrence. Unless the numbers are very small, which is not usually the case, it is very inconvenient to reduce the mixed numbers to improper fractions, multiply, and then reduce the product to a mixed number. A better way is to use one of the methods given below:

EXAMPLE 1.—Multiply 825 by $29\frac{3}{4}$.

SOLUTION.—*First Method*

$$\begin{array}{r}
825\phantom{\frac{3}{4}} \\
29\frac{3}{4} \\
\hline
4)\,2475\phantom{\frac{3}{4}} \\
\hline
618\frac{3}{4} \\
7425\phantom{0\frac{3}{4}} \\
1650\phantom{00\frac{3}{4}} \\
\hline
24543\frac{3}{4}
\end{array}\quad \text{Ans.}$$

Second Method

$$\begin{array}{r}
825\phantom{\frac{3}{4}} \\
29\frac{3}{4} \\
\hline
206\frac{1}{4} \\
412\frac{2}{4} \\
7425\phantom{0\frac{3}{4}} \\
1650\phantom{00\frac{3}{4}} \\
\hline
24543\frac{3}{4}
\end{array}\quad \text{Ans.}$$

EXPLANATION.—*First Method:* Since $\frac{3}{4}$ of 825 is the same as $\frac{1}{4}$ of 3 times 825, we first find 3 times 825 and take $\frac{1}{4}$ of the product. We then multiply by 29 in the usual way, taking care to put the units place of 7,425 under the units place of 618.

In the *second method* we divide 825 by 4, obtaining $206\frac{1}{4}$. We then multiply $206\frac{1}{4}$ by 2 and write the product, $412\frac{2}{4}$, directly below. The sum of $206\frac{1}{4}$ and $412\frac{2}{4}$ is evidently equal to $\frac{3}{4}$ of 825.

EXAMPLE 2.—Multiply $86\frac{5}{8}$ by $17\frac{3}{4}$.

SOLUTION.—

$$\begin{array}{rr}
 & 86\frac{5}{8} \\
 & 17\frac{3}{4} \\
\cline{2-2}
\frac{3}{4}\times 86 = & 64\frac{1}{2} \\
\frac{5}{8}\times 17 = & 10\frac{5}{8} \\
\frac{5}{8}\times\frac{3}{4} = & \frac{15}{32} \\
17\times 86 = & \left\{\begin{array}{l} 602 \\ 86 \end{array}\right. \\
\hline
 & 1537\frac{19}{32}
\end{array}\quad \text{Ans.}$$

In this case there are four multiplications made; 86 is multiplied by $\frac{3}{4}$; 17 by $\frac{5}{8}$; $\frac{5}{8}$ by $\frac{3}{4}$; and 86 by 17.

21. In actual practice a result is usually given to the nearest cent; hence, an accountant would make it a point to neglect fractions that would not affect the result. Again, calculations must be made rapidly and often mentally without recourse to paper and pencil. The following illustrates a method that would be used in mental reckoning:

EXAMPLE 1.—Find, to the nearest cent, the cost of $17\frac{3}{4}$ pounds of meat at $13\frac{1}{2}$ cents a pound.

SOLUTION.—

$$\begin{array}{rlrl} 17 & \text{lb. @ } 13¢ = & \$\,2.21 & \\ \frac{3}{4} & \text{lb. @ } 13¢ = & .10 & \text{nearly} \\ 17\frac{3}{4} & \text{lb. @ } \frac{1}{2}¢ = & \underline{.09} & \text{nearly} \\ & & \$\,2.40 & \text{Ans.} \end{array}$$

NOTE.—The characters lb., @, and ¢ are, respectively, the symbols for *pounds*, *at*, and *cents*.

We first find the cost of 17 pounds at 13 cents a pound, then of $\frac{3}{4}$ of a pound at the same price. The sum of the two products will equal the cost of $17\frac{3}{4}$ pounds at 13 cents. We next find the cost of $17\frac{3}{4}$ pounds at $\frac{1}{2}$ cent. The entire sum is the cost of $17\frac{3}{4}$ pounds at $13\frac{1}{2}$ cents a pound. A clerk in making the calculations would merely set down the numbers 221, 10, and 9 as he computed them.

EXAMPLE 2.—Find, to the nearest cent, the cost of 12 pounds 7 ounces of chicken at 23 cents a pound.

SOLUTION.—

$$\begin{array}{rrl} 12 \times .23 = & 2.76 & \\ \frac{7}{16} \times .23 = & \underline{.10} & \\ & \$\,2.86 & \text{Ans.} \end{array}$$

The cost of 12 pounds has been found first. An ounce is $\frac{1}{16}$ of a pound. The cost of 7 ounces at 23 cents could be found easily as follows: 8 ounces, or $\frac{1}{2}$ pound, will cost $11\frac{1}{2}$ cents, and 1 ounce will cost about $1\frac{1}{2}$ cents; hence, the cost of 7 ounces will be $1\frac{1}{2}$ cents less than $11\frac{1}{2}$ cents, or 10 cents.

22. A bookkeeper will find that in actual service he will need to use certain prices and quantities more than others. By the familiarity obtained from practice, he will soon learn "by heart" the products of the principal combinations used. He will, also, by a little ingenuity devise means of handling the less frequent cases presenting more than the average difficulty.

EXAMPLES FOR PRACTICE

1. Find the cost of $89\frac{2}{3}$ yards of silk velvet at \$$4\frac{3}{4}$ per yard. Ans. \$$425\frac{1}{12}$

2. How much must be paid for $83\frac{1}{2}$ tons of hay at \$$16\frac{5}{8}$ a ton? Ans. \$$1,388\frac{3}{16}$

3. How far can a man ride on a bicycle in $14\frac{5}{6}$ hours at the rate of $9\frac{3}{4}$ miles per hour? Ans. $144\frac{5}{8}$ miles

4. Find the products of:

   (*a*) $28\frac{7}{8}\times 17\frac{2}{3}$.
   (*b*) $44\frac{7}{9}\times 16\frac{5}{8}$.
   (*c*) $127\frac{4}{5}\times 69\frac{2}{7}$.
   (*d*) $86\frac{5}{12}\times 78\frac{4}{5}$.
   (*e*) $53\frac{3}{4}\times 27\frac{1}{2}$.

Ans.—(*a*) $510\frac{1}{8}$; (*b*) $744\frac{31}{72}$; (*c*) $8,854\frac{5}{7}$; (*d*) $6,809\frac{19}{30}$; (*e*) $1,478\frac{1}{8}$.

---

23. To find the cost of articles sold by the hundred or the thousand.

EXAMPLE.—Find the cost of 1,635 pounds of sugar at \$5.35 a hundred.

SOLUTION.— 1,635 equals $16\frac{35}{100}$ hundreds or 16.35 hundreds. Since each hundred costs \$5.35, the entire cost is equal to the product of 16.35 and \$5.35. The operation is performed as follows:

$$\begin{array}{r}
16.35 \\
5.35 \\
\hline
8175 \\
4905 \\
8175 \\
\hline
87.4725
\end{array}$$

or \$87.47. Ans.

24. Rule.—I. *To find the cost of articles sold by the hundred, point off two decimal places in the quantity and multiply by the price per hundred.*

II. *To find the cost of articles sold by the thousand, point off three decimal places in the quantity and multiply by the price per thousand.*

Example 1.—A builder bought 28,960 feet of cypress lumber at $65.50 a thousand feet. What was the amount of the bill?

Solution.—Point off three decimal places in 28,960 and multiply by the price.

$$\begin{array}{r}
28.960\\
65.50\\
\hline
1448000\\
14480\\
17376\\
\hline
1896.88000
\end{array}\text{ or \$1,896.88. Ans.}$$

To check the result mentally we may find the cost of 30 thousand feet at $65. 30×65=$1,950. This result would indicate that there was no great error in the original solution such as that due to a misplaced decimal point.

The work may be checked, also, as follows:

$$\begin{array}{lr}
\text{29 thousand feet at \$65.50} = & \$1899.50\\
\text{40 feet at \$6.55 per hundred} = & 2.62\\
\cline{2-2}
 & \$1896.88
\end{array}$$

Example 2.—The tax in a certain city is $18.95 on each $1,000 of the valuation of property. What is the tax on a property valued at $11,200?

Solution.— 11.2×$18.95=$212.24. Ans.

25. A bookkeeper is, of course, not expected to use several methods in the same class of problems, yet he should be able to select the best method for his particular class of work.

26. To find the cost of material sold by the ton of 2,000 pounds.

Example 1.—What is the amount of the bill for 48,760 pounds of coal sold at $6.75 a ton?

Solution.— $$48{,}760 \text{ pounds} = \frac{48.760}{2} = 24.38 \text{ tons.}$$

$$24.38 \times 6.75 = 164.565, \text{ or \$164.57. Ans.}$$

The student will observe that pounds can be changed to tons of 2,000 pounds by pointing off three decimal places in the pounds and then dividing by 2.

Example 2.—Potatoes are sold in some localities by the ton. What should a farmer receive for 1,760 pounds of potatoes when the price is $14.75 a ton?

Solution.—The work may be set down as follows:

$$\begin{array}{r} 2\,\underline{)\,1.760} \\ .88 \end{array} \qquad \begin{array}{r} 14.75 \\ \underline{.88} \\ 11800 \\ \underline{11800} \\ 12.9800 \end{array}$$

The required amount is $12.98. Ans.

27. When prices are quoted by the ton of 2,000 pounds, it is sometimes most convenient to reckon the cost on the hundred pounds.

Example.—What is the cost of 740 pounds of poultry feed at $34.40 per ton?

Solution.—The price per hundred is $\$34.40 \div 20 = \$1.72$.

$$\begin{array}{rrl} 7\times\$1.72 = & \$\,12.04, & \text{the cost of 700 lb.} \\ .40\times\$1.72 = & \underline{.69}, & \text{the cost of 40 lb.} \\ & \$\,12.73 & \text{Ans.} \end{array}$$

The result may also be obtained as follows by the method given in Art. **26**.

$$\begin{array}{r} 2\,\underline{)\,.740} \\ .37 \end{array} \qquad \begin{array}{r} 34.40 \\ \underline{.37} \\ 24080 \\ \underline{10320} \\ 12.7280 \end{array}$$

Therefore, the cost of the feed will be $12.73. Ans.

EXAMPLES FOR PRACTICE

1\. At $3.25 per hundred, what must be paid for 3,485 feet of Georgia pine? Ans. $113.26

2\. What will be the cost of 1,400,345 bricks at $7.75 a thousand? Ans. $10,852.67

3\. Find the retail price of 7,384 pounds of coal at $6.75 a ton. Ans. $24.92

CONSTRUCTION OF TABLES

28. A bookkeeper who is required to multiply by the same number frequently should make a table, giving the products of this number and each figure to 10 or more. For example, a coal dealer selling coal at $6, $6.50, and $6.75 per ton would arrange a table as shown below and thus save himself much labor.

Pounds	Price per Ton			Pounds	Price per Ton		
	$6.00	$6.50	$6.75		$6.00	$6.50	$6.75
100	.30	.33	.34	1,100	3.30	3.58	3.71
200	.60	.65	.68	1,200	3.60	3.90	4.05
300	.90	.98	1.01	1,300	3.90	4.23	4.39
400	1.20	1.30	1.35	1,400	4.20	4.55	4.73
500	1.50	1.63	1.69	1,500	4.50	4.88	5.06
600	1.80	1.95	2.03	1,600	4.80	5.20	5.40
700	2.10	2.28	2.36	1,700	5.10	5.53	5.74
800	2.40	2.60	2.70	1,800	5.40	5.85	6.08
900	2.70	2.93	3.04	1,900	5.70	6.18	6.41
1,000	3.00	3.25	3.38	2,000	6.00	6.50	6.75

EXAMPLE.—Find by the table the cost of 1,565 pounds of coal at $6.75 a ton.

SOLUTION.— 1500 pounds cost $5.06
60 pounds ($\frac{1}{10}$ of 600) cost .20
5 pounds ($\frac{1}{100}$ of 500) cost .02
1565 pounds cost $5.28 Ans.

29. By use of the table, a bill may be in excess of the exact amount by a cent, as one-half cent or more is called a whole cent. In case greater precision is required, the tables can be computed to tenths of a cent.

COMPLEX EXPRESSIONS

SYMBOLS OF AGGREGATION

30. The **vinculum** ——, **parenthesis** ( ), **brackets** [ ], and **brace** { } are called **symbols of aggregation,** and are used to include numbers that are to be considered together; thus, $13\times\overline{8-3}$, or $13\times(8-3)$ shows that 13 is to be multiplied by the difference between 8 and 3.

$$13\times(8-3)=13\times5=65$$

$$13\times\overline{8-3}=13\times5=65$$

When the vinculum or parenthesis is not used, we have

$$13\times8-3=104-3=101$$

31. When several numbers are connected by the signs +, −, ×, and ÷, the operations indicated by the multiplication and the division signs are performed first, and the operations indicated by the addition and the subtraction signs are performed in order from left to right.

EXAMPLE 1.—What is the value of $3+8-2+6$?

SOLUTION.— $3+8=11$; $11-2=9$; $9+6=15$. Ans.

EXAMPLE 2.—Simplify $26-5+8\times2$.

SOLUTION.—Here 8 is to be multiplied by 2, and the expression then becomes $26-5+16$.

$$26-5=21;\ 21+16=37.\quad \text{Ans.}$$

EXAMPLE 3.—What is the value of $8\times3\frac{1}{2}-24\div1\frac{1}{2}$?

SOLUTION.—We observe that 8 is to be multiplied by $3\frac{1}{2}$ and that 24 is to be divided by $1\frac{1}{2}$. $8\times3\frac{1}{2}=28$; $24\div1\frac{1}{2}=24\times\frac{2}{3}=16$. The expression may then be written $28-16$, and $28-16=12$. Ans.

The student in explaining a problem may need to use expressions such as are given here and he should follow the rules.

EXAMPLES FOR PRACTICE

Find the values of the following expressions:

(*a*) $(8+5-1)\div4$.

(*b*) $5\times24-32$.

(*c*) $5\times24\div15$.

(*d*) $144-5\times24$.

(*e*) $(1{,}691-540+559)\div3\times57$.

(*f*) $2{,}080+120-80\times4-1{,}670$.

Ans.—(*a*) 3; (*b*) 88; (*c*) 8; (*d*) 24; (*e*) 10; (*f*) 210.

REDUCTION OF COMPLEX FRACTIONS

32. We have learned that a line placed between two numbers indicates that the number above the line is to be divided by the number below it. Thus, $\frac{18}{3}$ denotes that 18 is to be divided by 3. This is also true if a fraction or a fractional expression be placed above or below a line.

$\dfrac{9}{\frac{3}{8}}$ means that 9 is to be divided by $\frac{3}{8}$; $\dfrac{3\times7}{\frac{8+4}{16}}$ means that 3×7 is to be divided by the value of $\dfrac{8+4}{16}$.

$\dfrac{\frac{1}{4}}{\frac{3}{8}}$ is the same as $\frac{1}{4}\div\frac{3}{8}$.

33. It will be noticed that there is a heavy line between the 9 and the $\frac{3}{8}$. This is necessary, since otherwise there would be nothing to show whether 9 is to be divided by $\frac{3}{8}$, or $\frac{9}{3}$ is to be divided by 8. Whenever a heavy line is used, as shown here, it indicates that *all above the line* is to be divided by *all below it.*

34. A fraction whose numerator or denominator is not a simple number is called a **complex fraction.** A complex fraction is reduced to its simplest form by reducing the numerator and the denominator to their simplest forms and then dividing the former by the latter.

EXAMPLE 1.—What is the value of $\dfrac{26\times75\times12}{1-.10}$?

SOLUTION.—The numerator is $26\times75\times12=23{,}400$; the denominator is $1-.10=.90$.

$$\frac{23{,}400}{.90}=26{,}000. \text{ Ans.}$$

EXAMPLE 2.—Simplify

$$\frac{13\frac{1}{3}\times5\frac{1}{4}\times12\frac{3}{8}}{3\frac{3}{8}\times4\frac{1}{2}}$$

SOLUTION.—The numerator becomes

$$\frac{\overset{5}{\cancel{40}}}{\cancel{3}}\times\frac{\overset{7}{\cancel{21}}}{4}\times\frac{99}{\cancel{8}}=\frac{3,465}{4}$$

The denominator becomes $\frac{27}{8}\times\frac{9}{2}=\frac{243}{16}$.

The original expression now reduces to

$$\frac{3,465}{4}\div\frac{243}{16}=\frac{\overset{385}{\cancel{3,465}}}{\cancel{4}}\times\frac{\overset{4}{\cancel{16}}}{\underset{27}{\cancel{243}}}=\frac{1,540}{27}=57\tfrac{1}{27}.\quad\text{Ans.}$$

35. In solutions like the preceding, the work may be shortened by inverting each factor of the denominator instead of the product of the factors. Thus, inverting $\frac{27}{8}$ and $\frac{9}{2}$ and arranging the factors for continued multiplication

$$\frac{\overset{10}{\cancel{40}}}{\cancel{3}}\times\frac{\overset{7}{\cancel{21}}}{\cancel{4}}\times\frac{\overset{11}{\cancel{99}}}{\cancel{8}}\times\frac{\cancel{8}}{27}\times\frac{2}{\cancel{9}}=\frac{1,540}{27},\text{ as before}$$

In solving problems of this kind a person must use his own judgment as to the best method. In some cases it is advisable to reduce common fractions to decimals, and in other cases common fractions can be profitably used throughout the operation.

EXAMPLES FOR PRACTICE

Simplify the following:

(*a*) $\frac{4\frac{2}{3}\times10\frac{2}{5}\times26\frac{1}{4}}{8\frac{2}{5}\times4\frac{7}{8}\times12\frac{3}{5}}$.

(*b*) $\frac{10\frac{2}{3}\times12\frac{4}{7}\times15\frac{3}{4}}{4\frac{7}{12}\times\frac{24}{35}\times8\frac{2}{5}}$.

(*c*) $\frac{20\frac{2}{5}\times34\frac{2}{3}\times8\frac{5}{8}}{14\frac{1}{6}\times3\frac{9}{10}\times18\frac{2}{5}}$.

(*d*) $\frac{15\frac{3}{11}\times32\frac{2}{3}\times25\frac{3}{5}}{28\frac{4}{5}\times38\frac{1}{4}\times17\frac{7}{9}}$.

Ans.—(*a*) $2\frac{38}{81}$; (*b*) 80; (*c*) 6; (*d*) $\frac{5488}{8415}$.

CHECKING RESULTS

FINDING A RESULT APPROXIMATELY

36. The student in making calculations should be able to determine in some simple way whether or not his work is approximately correct. If he should find the amount of a bill of goods to be $500 when he knew from practical experience that such a bill would not be over $50, he would certainly realize that he had made a gross error. A bookkeeper, for example, who figures that 650 pounds of beef amounts to $600 will see that he is reckoning at a price of nearly $1 a pound, and that he has probably misplaced a decimal point. Many problems that involve considerable calculation to obtain the correct result can be checked up more or less roughly by the use of simple figures that are approximately correct. Thus, $26\frac{1}{2}$ pounds of metal at $18\frac{1}{4}$ cents a pound would amount to about $5, for at 20 cents the amount would be $5.30, and at 2 cents less per pound, that is at 18 cents, the amount would be $2\times 26\frac{1}{2} = 53$ cents less than $5.30, or $4.77. With a little further mental calculation the true result could be found. At $\frac{1}{4}$ cent, $26\frac{1}{2}$ pounds will cost 7 cents, to the nearest cent. The entire amount should be $4.77 + .07 = \$4.84$. Many an error on the part of an accountant can be discovered by a little reasoning, which will show that the result is impossible.

EXAMPLE 1.—Find approximately by mental calculation the value of

$$\frac{12\frac{1}{4}\times 3\frac{3}{4}}{2\frac{7}{8}}.$$

SOLUTION.— $12\frac{1}{4}\times 3\frac{3}{4}$ is nearly equal to $12\times 4 = 48$; $48 \div 2\frac{7}{8}$ equals approximately $48 \div 3 = 16$. Ans.

If the result obtained by pencil and paper should differ widely from 16, the fact would indicate a serious error.

EXAMPLE 2.—What will be the approximate cost to lay a sidewalk 239 feet long at 73 cents a foot?

SOLUTION.—Say, for easy figuring, that the length is 240 feet and that the price is 75 cents or $\$\frac{3}{4}$ a foot. Mentally, one would find that $\frac{3}{4}$ of 240 is 180; hence, the approximate cost is \$180. Ans.

PROOFS BY CASTING OUT NINES

37. The remainder obtained by dividing a number by 9 may be found by adding the digits; if the sum contains more than one figure, add the figures of the sum; so continue until a sum is obtained that has but one figure. If this figure is 9, the remainder is 0; otherwise, it is the figure itself. Thus, take the number 6,877; this divided by 9 gives 764 for the quotient and 1 for the remainder. Applying the principle just stated, $6+8+7+7=28$; $2+8=10$; $1+0=1$, the remainder when 6,877 is divided by 9. For 26,478, we have $2+6+4+7+8=27$; $2+7=9$. This result means that the number 26,478 is a multiple of 9; hence, when it is divided by 9, the remainder is 0. This process is called **casting out nines**; it is so called because all the nines are rejected from the number, only the remainder being considered. Thus, in the first case above, 764 nines were rejected (cast out) from 6,877, and the remainder was 1.

If in adding the digits a 9 occurs, it need not be considered; or if two or more figures added together make 9, they may be neglected. For example, consider 364,982; here, since 3 and 6 is 9, we need consider only the figures 4, 8, and 2. For, $3+6+4+9+8+2=32$; $3+2=5$; also, $4+8+2=14$, and $1+4=5$, the same result.

38. The student will find the following test useful in determining whether the answer in multiplication is correct:

Find the sum of the digits in the multiplicand. If the sum consists of more than one figure, add the digits of the sum, and so continue until the sum is one figure. Do the same with the multiplier. Multiply together the final sums thus obtained, and if the result consists of more than one figure, add its digits *until one figure is obtained. If this result is the same as is obtained by adding the digits of the product until one figure is obtained, the work is probably correct.*

To illustrate, multiply 837,295 by 4,631.

SOLUTION.—

$$\begin{array}{rr} \textit{multiplicand} & 837295 \\ \textit{multiplier} & 4631 \\ \hline \textit{product} & 3877513145 \end{array}$$

PROOF.—

$$\begin{array}{r} 8+3+7+2+9+5=34;\ 3+4=\ 7 \\ 4+6+3+1=14;\ 1+4=\underline{\ 5} \\ 35;\ 3+5=8 \\ 3+8+7+7+5+1+3+1+4+5=44;\ 4+4=8 \end{array}$$

The proof given is not absolute, because two or more errors might cause the product to fulfil the conditions of the test. But if, on trial, the final sum of the digits of the product does not agree with that of the product of the final sums of the multiplicand and the multiplier, it is certain that the work is incorrect.

39. The principle of casting out nines may be applied to addition. The number remaining after all the nines are cast out of all of the figures in the terms to be added should be the same as the remainder in the sum. Take the following for an illustration:

$$\begin{array}{r} 5368 \\ 7397 \\ 9243 \\ 2878 \\ \hline 24886 \end{array}$$

The remainder found by casting out the nines in the figures of the four terms to be added is 1. The same remainder is found in the sum; hence, the result is probably correct.

To prove subtraction, cast out the nines in the remainder and subtrahend and see whether the sum of the remainders in these terms equals the remainder in the minuend.

To prove division, cast out the nines in the divisor and the quotient; multiply together the remainders so obtained and add to the remainder in this product the figure left in the remainder after casting out the nines. The result should equal the remainder in the dividend. Observe the following proof.

$$
\begin{array}{r@{}l}
2867)\, & 834000\,(290 \\
 & \underline{5734} \\
 & 26060 \\
 & \underline{25803} \\
 & 2570\ \textit{remainder}
\end{array}
$$

The remainder in 2,867 is 5; the remainder in 290 is 2; 2×5=10; the remainder is 1; the remainder in 2,570 is 5; 1+5=6. The remainder in 834,000 is also 6. Hence, the division is probably correct.

PERCENTAGE

40. One of the most commonly used arithmetical terms is *per cent.* For illustration, the profits of a business, the changes in prices, the proportionate parts of a number, are usually expressed as so many per cent. One per cent. of a number is one hundredth of that number; two per cent. is two hundredths; etc. One per cent. of 100 is 1, one per cent. of 200 is 2. Two per cent. of 100 is 2, of 200 is 4. The symbol for per cent. is %; 5% is read 5 per cent.

41. That part of arithmetic which treats of the computation by hundredths is called **percentage.**

EXAMPLE.—Wages in a mill are to be increased 10%. How much increase will a person receive who is earning $2 a day?

SOLUTION.— 1% or one hundredth part of $2 is 2 ct., and 10% is 10×2=20 ct. The increase is, therefore, 20 ct. a day. Ans.

42. When the symbol % is written after a number and it is desired to drop the symbol, the number should be changed to a fraction whose denominator is 100. For example, 5% may be written either as a common or a decimal fraction, $\frac{5}{100}$ or .05.

As just illustrated, a number of per cent. may be written in any of these ways, and the student is cautioned against making the error of using two methods at the same time; thus, 5% should not be written .05%, as the latter means $\frac{5}{100}$ of 1%.

EXAMPLES FOR PRACTICE

The following examples are to be solved mentally. Write down your answers and when the work is completed compare your answers with those given at the end of the exercise.

Find: (*a*) 1% of 300.
(*b*) 5% of 300.
(*c*) 10% of 300.
(*d*) 25% of 300.
(*e*) 10% of 700.
(*f*) 15% of 400.
(*g*) 1% of 250.
(*h*) 4% of 250.
(*i*) 50% of 1,000.
(*j*) 100% of 600.
(*k*) 3½% of 800.
(*l*) 3¾% of 1,200.
(*m*) 10% of 350.
(*n*) 150% of 400.
(*o*) 89% of 100.
(*p*) 1% of 1,800.
(*q*) 1% of 1,850.
(*r*) 1% of 70,000.

Ans.—(*a*) 3; (*b*) 15; (*c*) 30; (*d*) 75; (*e*) 70; (*f*) 60; (*g*) 2.5; (*h*) 10; (*i*) 500; (*j*) 600; (*k*) 28; (*l*) 45; (*m*) 35; (*n*) 600; (*o*) 89; (*p*) 18; (*q*) 18.5; (*r*) 700.

---

43. The number of which the hundredth is taken is called the **base.** The number denoting the hundredths considered is called the **rate.** The result obtained by taking the portion of the number denoted by the rate is called the **percentage.** Thus, in finding 5% of 200, we get 10 as the result. In this particular example, 200 is the base, 5% is the rate, and 10 is the percentage. The base is always considered to be 100%. One hundredth part of the base is 1% of it, two hundredths of the base is 2% of it, etc.

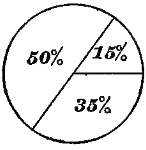


Fig. 1

44. The circle shown in Fig. 1 is divided into three parts. In the large portion there is $\frac{50}{100}$ of the circle, in the next smaller portion $\frac{35}{100}$, and in the smallest portion $\frac{15}{100}$ of the circle. The figure shows to the eye what relation 50%, 35%, and 15% bear, respectively, to the whole, or 100%.

Example.—A manufacturer bought 3,830 pounds of wool, in which there was a loss of 28% due to waste. How many pounds of waste were there in the lot?

Solution.— 1% of 3,830$=\frac{3830}{100}=$38.3 lb.
28% of 3,830=28×38.3=1,072.4 lb. Ans.

Instead of finding 1% first, then 28%, the result can be obtained directly by multiplying 3,830 by .28. Thus:

$$\begin{array}{r} 3830 \\ .28 \\ \hline 30640 \\ 7660 \\ \hline 1072.40 \end{array} \text{ lb. Ans.}$$

45. Rule.—*To find the percentage when the base and rate are given, write the rate as a decimal and then multiply the base by the rate.*

EXAMPLES FOR PRACTICE

What is:

(*a*) 7% of 1,860?
(*b*) 15% of 1,326?
(*c*) $3\frac{3}{4}$% of 1,850?

Ans.—(*a*) 130.2; (*b*) 198.9; (*c*) 69.375.

1. Ten years ago a city had a population of 125,000. The population has increased 12% since that time. How many inhabitants has it now? Ans. 140,000

Note.—Find the increase and add it to the given population.

2. A railroad's receipts in a month were $567,800, of which 72% was for freight. What were the receipts for freight? Ans. $408,816

ALIQUOT PARTS IN PERCENTAGE

46. When the rate per cent. is an aliquot part of 100 per cent., it is usually more convenient to use the equivalent fraction in computations. The principal aliquot parts are shown in the following table:

Per Cent.	Decimal	Fraction	Per Cent.	Decimal	Fraction
2%	.02	$\frac{2}{100}$ or $\frac{1}{50}$	$8\frac{1}{3}\%$......	$.08\frac{1}{3}$	$\frac{8\frac{1}{3}}{100}$ or $\frac{1}{12}$
5%	.05	$\frac{5}{100}$ or $\frac{1}{20}$	$12\frac{1}{2}\%$......	.125	$\frac{12\frac{1}{2}}{100}$ or $\frac{1}{8}$
10%	.10	$\frac{10}{100}$ or $\frac{1}{10}$	$16\frac{2}{3}\%$......	$.16\frac{2}{3}$	$\frac{16\frac{2}{3}}{100}$ or $\frac{1}{6}$
25%	.25	$\frac{25}{100}$ or $\frac{1}{4}$	$33\frac{1}{3}\%$......	$.33\frac{1}{3}$	$\frac{33\frac{1}{3}}{100}$ or $\frac{1}{3}$
50%	.50	$\frac{50}{100}$ or $\frac{1}{2}$	$37\frac{1}{2}\%$......	$.37\frac{1}{2}$	$\frac{37\frac{1}{2}}{100}$ or $\frac{3}{8}$
75 %........	.75	$\frac{75}{100}$ or $\frac{3}{4}$	$62\frac{1}{2}\%$......	.625	$\frac{62\frac{1}{2}}{100}$ or $\frac{5}{8}$
$6\frac{1}{4}\%$	$.06\frac{1}{4}$	$\frac{6\frac{1}{4}}{100}$ or $\frac{1}{16}$	$87\frac{1}{2}\%$......	.875	$\frac{87\frac{1}{2}}{100}$ or $\frac{7}{8}$

Example 1.—What is $16\frac{2}{3}\%$ of 684?

Solution.— $16\frac{2}{3}\%=\frac{1}{6}$. $\frac{1}{6}$ of 684=114. Ans.

Example 2.—A furniture dealer paid $48 for a dozen chairs which he sold at a profit of 25% on the purchase price. What was his profit and what price did he receive?

Solution.— $25\%=\frac{1}{4}$.
$\frac{1}{4}$ of \$48=\$12, the profit. Ans.
Selling price was \$48+\$12=\$60. Ans.

The student should become familiar with the most common aliquot parts of 100, and use the fractions wherever possible in percentage computations.

EXAMPLES FOR PRACTICE

Solve the following by aliquot parts:
(*a*) What is 50% of 1,964?
(*b*) What is $33\frac{1}{3}\%$ of \$630?
(*c*) What is $6\frac{1}{4}\%$ of 1,760?
Ans.—(*a*) 982; (*b*) \$210; (*c*) 110.

AMOUNT AND DIFFERENCE

47. The **amount** is the sum of the base and the percentage.

48. The **difference** is the remainder obtained when the percentage is subtracted from the base.

49. The terms amount and difference are ordinarily used when there is an increase or a decrease in the base. For example, suppose the population of a village is 1,500 and it increases 25 per cent. This means that for every 100 of the original 1,500 there is an increase of 25, or a total increase of 15×25 =375. This increase added to the original population gives the *amount,* or the population after the increase. If, on the other hand, the population decreases 375, the final population is 1,500−375=1,125, and this is the *difference.* The original population, 1,500, is the base on which the percentage is computed; the 25% is the rate, and the increase or decrease, 375, is the percentage. If the base increases, the final value is the amount, and if it decreases, the final value is the difference.

50. To find the relations existing between the amount or the difference and the base and the rate, let us consider an example.

EXAMPLE.—In a factory where 2,100 men are employed, the force is increased 8%. How many new men are employed, and how many men are at work after the increase?

SOLUTION.— 8% of 2,100=2,100×.08=168 men=number of new men employed. Ans. The total number of men after the force is increased is 2,100+168=2,268 men. Ans.

In this example, the original number, 2,100, is the base, and the final number, 2,268, is the amount. For every 100 men originally in the shop, there are 8 more men, or 108 men, after the force is increased.

We may consider the original force as 100%, and, as the increase is 8% of the original, the total force is 100%+8% =108%. 108%=1.08.

Total force is 1.08×2,100=2,268. Ans.

51. If, in the above example, the force had been decreased 8%, the number of men thrown out of work would have been 168, and those left at work, 2,100−168=1,932. As before, the base is 2,100, but the final number, 1,932, is the difference, since the base has decreased. Out of every 100 men formerly at work, 8 have been discharged, leaving 100−8=92 still at work. Since 8% of the force has been thrown out of work, 100%−8%=92% remains. The force that remains equals .92×2,100=1,932, the same result as before.

52. In the case of small fractions of a per cent. care should be observed to read the number correctly. For example, .01% is read $\frac{1}{100}$ of 1%, and equals $\frac{1}{100}$ of $\frac{1}{100}=\frac{1}{10000}$, or .0001. Similarly, .5% or 0.5% is read $\frac{5}{10}$ of 1%. If the number of per cent. is expressed as a common fraction, it is read in a similar manner. Thus, $\frac{1}{8}$% is read $\frac{1}{8}$ of 1%, and equals $\frac{1}{8}$ of $\frac{1}{100}=\frac{1}{800}$, or .00125.

EXAMPLES FOR PRACTICE

Solve the following:
(*a*) What is $12\frac{1}{2}$% of $900?
(*b*) What is $\frac{4}{5}$% of 627?
(*c*) What is $33\frac{1}{3}$% of 54?
Ans.—(*a*) $112.50; (*b*) 5.016; (*c*) 18.

1. If gunpowder contains 75% of saltpeter, 10% of sulphur, 15% of charcoal, how much of each is there in 2,000 pounds of powder?

Ans. { Saltpeter, 1,500 lb.
Sulphur, 200 lb.
Charcoal, 300 lb.

2. A man owning a ship worth $225,000, sells $\frac{1}{4}$ of it to A, 20% of the remainder to B, and 35% of what then remains to C. How much do A, B, and C each pay for their shares?

Ans. { A, $56,250
B, $33,750
C, $47,250

3. A house which cost $4,000 was sold for 7% less than it cost. What was the selling price? Ans. $3,720

4. A school having 200 pupils lost during the year $12\frac{1}{2}$% of the number. How many pupils were left? Ans. 175

5. The receipts of a business in 1 year were $25,346.90. During the next year the receipts were increased 16%. What were the receipts for that year? Ans. $29,402.40

COMMISSION AND BROKERAGE

53. Commission, or **brokerage,** is the sum paid an agent for transacting business for another person; as, for buying or selling merchandise or property, for collecting or investing money, etc.

54. The agent or party who transacts the business is called a **commission merchant,** or **broker;** the party for whom the business is transacted is called the **principal.** The term broker is applied to one who sells and buys stocks, bonds, bills of exchange, and money securities.

55. A **consignment** is a shipment of goods from one party to another; the party that ships the goods is called the **consignor,** or **shipper,** and the party to whom they are shipped is called the **consignee.**

56. When goods are sold on credit, the agent charges an additional amount for guaranteeing the payment of the sale. This extra charge is called **guaranty.**

57. The **gross proceeds** of a sale or collection is the total amount realized by the agent before deducting his commission and other expenses connected with the transaction. The **net proceeds** is the amount due the principal after the commission and all other charges have been deducted.

58. An **account sales** is a detailed statement made by the agent to his principal, showing the goods sold and the prices obtained, giving a list of the charges and expenses, and the net proceeds due the principal. The charges include freight, cartage, storage, insurance, inspection, advertising, commission, and guaranty.

59. The **prime cost** of a purchase is the sum paid by the agent for the goods or property. The **gross cost** is the prime cost plus the commission and expenses incident to the purchase.

60. An **account purchase** is a detailed statement made by the agent to his principal, showing the cost of goods or property bought, the expenses attending the purchase, and the gross cost.

61. The commission, or brokerage, is usually computed at a certain per cent. of the gross proceeds of a sale or the prime cost of a purchase. In some cases, however, it is computed at a certain price per unit of weight or measure; as, so much per ton, per bushel, or per barrel. Examples in commission are solved by the rules of percentage. Either the gross proceeds or the prime cost is the *base;* the net proceeds is the *difference;* the gross cost is the *amount;* the commission is the *percentage;* and the rate of commission is the *rate.* The remittance from the principal to the purchasing agent, including both the investment and the commission, is an *amount.* The following rule is derived directly from the principles of percentage:

62. Rule.—*To find the commission, multiply the prime cost or the selling price by the rate of commission.*

EXAMPLE.—A real estate agent sells a house and lot for $4,375 and receives 2% commission. What is the commission and what is the net proceeds?

SOLUTION.—Commission=selling price×rate=$4,375×.02=$87.50. Ans.

Net proceeds=selling price−commission=$4,375−$87.50=$4,287.50. Ans.

EXAMPLES FOR PRACTICE

What is the commission:

(*a*) If the gross proceeds is $300 and the rate of commission is 3½%?

(*b*) If the gross proceeds is $9,375 and the rate of commission is 2%?

(*c*) If the prime cost is $831.75 and the rate of commission is 1¼%?

(*d*) If the prime cost is $960 and the rate of commission is ⅝%?

Ans.—(*a*) $10.50; (*b*) $187.50; (*c*) $10.40; (*d*) $6.

1. A commission merchant sold a quantity of wool for $4,650. He charged 2½% commission, 2% guaranty, and the transportation, storage, and other expenses amounted to $184. How much should he send his principal? Ans. $4,256.75

INSURANCE

63. **Insurance** is a contract by which one party, the **underwriter,** or **insurer,** agrees, for a consideration, to make good a loss sustained by another party.

64. Insurance is of two kinds, **property insurance** and **personal insurance.** Property insurance includes *fire insurance* (indemnity for loss or damage by fire); *marine insurance* (indemnity for losses at sea); *transit insurance* (indemnity for loss of, or damage to, merchandise during transportation); *stock insurance* (indemnity for loss of live stock); and *accident insurance* (indemnity for breakage of fragile materials, as plate glass, etc.).

Personal insurance includes *life insurance,* which secures the payment of a certain amount to a specified person at the death of the party insured, or after the lapse of a specified time; *accident insurance,* which secures the payment of a certain sum in case of accident to the insured; *health insurance,* which secures the payment of a weekly sum during sickness; and insurance against the dishonesty of employes.

65. A **policy** is a written contract between the insurance company and the party insured; it contains a description of the property insured, the conditions on which the insurance is taken, and the amount to be paid in case of loss.

66. A **premium** is the amount paid to the insurer for assuming the risk of loss or damage. The premium is a certain per cent. of the amount of insurance, as $\frac{5}{8}\%$, $\frac{3}{4}\%$. The rate of premium depends on the nature of the risk and on the length of time the insurance has to run. It is customary to speak of the rate of premium as the cost per $100 of insurance; as 60 cents per $100, $1.20 per $100, etc.

67. In property insurance, all computations are based on the rules of percentage. The amount of insurance is the *base,* the premium is the *percentage,* and the rate of premium is the *rate.*

68. Rule.—*To find the premium, multiply the amount of insurance by the rate of premium.*

Example.—A house and furniture are insured against fire for $3,250, the rate of premium being $\frac{3}{4}\%$, or 75 cents per $100 per year. What is the yearly premium?

Solution.— $\frac{3}{4}\%=.0075$. Using the rule,

Premium=$3,250×.0075=$24.37$\frac{1}{2}$. Ans.

EXAMPLES FOR PRACTICE

1. A store and contents valued at $16,400 are insured for $\frac{3}{5}$ of their value at $1\frac{1}{8}\%$ premium. What is the cost per year of insurance?
Ans. $110.70

2. A boat load of 8,600 bushels of corn, worth 32 cents per bushel, is insured for $\frac{3}{4}$ of its value at $1\frac{5}{8}\%$ premium. If the corn is totally destroyed, what will be the owner's loss? Ans. $721.54

3. A building worth $9,600 is insured for $\frac{3}{4}$ of its value in three companies. The first company takes $\frac{1}{3}$ of the risk at $\frac{1}{2}\%$ premium; the second $\frac{2}{5}$ of it at $\frac{3}{4}\%$; and the third the remainder at $\frac{7}{8}\%$. What is the total premium? Ans. $50.40

4. A fire insurance company took a risk of $42,000 at $\frac{3}{4}\%$ premium, and reinsured $\frac{1}{3}$ of it in another company at $\frac{1}{2}\%$, and $\frac{1}{2}$ of it in a third company at $\frac{5}{8}\%$. What did the company gain by reinsuring?
Ans. $61.25

INTEREST

SIMPLE INTEREST

69. **Interest** is money paid for the use of money belonging to another.

70. The **principal** is the sum for which interest is paid.

71. The **rate of interest** is the per cent. of the principal that is paid for its use for a given time, usually a year.

72. The **amount** is the sum of the principal and interest. If, for example, $100 is loaned for a year at 6 per cent. interest, the principal is $100, the rate is 6 per cent., the interest is 6 per cent. of $100 or $6, and the amount is $100+$6=$106. If the principal is loaned for 2 years, the interest is 2×$6=$12, and the amount $100+$12=$112.

73. The **legal rate** is the rate established by law.

74. **Usury** is a rate that exceeds the legal rate. The penalty for usury is, in some states, the forfeiture of all interest; in others, the forfeiture of both principal and interest. In a number of states, no legal notice is taken of usury.

75. The finding of interest is the direct application of the principle of percentage when the base and rate are given.

EXAMPLE 1.—Find the interest for 1 year on $1,000 at 5%.

SOLUTION.—

$$\begin{array}{r l} \$1000 & \\ .05 & \\ \hline \$50.00 & \text{Ans.} \end{array}$$

EXPLANATION.— 5% expressed as a decimal is .05. The interest is found by multiplying the principal, which is $1,000, by .05, the rate.

EXAMPLE 2.—Find the interest on $1,275.50 for 3 years at 6%.

SOLUTION.—

$$\begin{array}{r l} \$1275.50 & \\ .06 & \\ \hline 76.5300 & =\text{Interest for 1 yr.} \\ 3 & \\ \hline \$229.59 & =\text{Interest for 3 yr. Ans.} \end{array}$$

The example may be solved also by finding the interest on $1 for 1 year, then for 3 years, and multiplying this by the principal. Thus,

$$.06 \times 3 \times 1{,}275.50 = \$229.59. \quad \text{Ans.}$$

EXAMPLES FOR PRACTICE

Find mentally the interest on:

(*a*) $300 for 1 yr. at 5%.
(*b*) $3,000 for 1 yr. at 5%.
(*c*) $700 for 1 yr. at 4%.
(*d*) $700 for 2 yr. at 4%.
(*e*) $2,000 for 4 yr. at 6%.

Solve the following on paper:

Find the interest on:

(*f*) $725 for 1 yr. at 5%.
(*g*) $4,200 for 3 yr. at $4\frac{1}{2}$%.
(*h*) $3,762.87 for 5 yr. at 5%.
(*i*) $2,617.75 for $2\frac{1}{2}$ yr. at 3%.

Ans.—(*a*) $15; (*b*) $150; (*c*) $28; (*d*) $56; (*e*) $480; (*f*) $36.25; (*g*) $567; (*h*) $940.72; (*i*) $196.33.

76. To find the interest when the time is in months.

Example.—What is the interest on $900 at 5% for 7 months?

Solution.—

```
            $900
             .05
       12)45.00=Interest for 1 yr.
            3.75=Interest for 1 mo.
               7
          $26.25=Interest for 7 mo.
```

Explanation.—The interest for 7 months equals 7 times that for 1 month. The interest for 1 month is found by dividing the interest for 1 year by 12.

EXAMPLES FOR PRACTICE

Find the interest on:

(*a*) $825 for 6 mo. at 6%.

(*b*) $930 for 2 mo. at 4½%.

(*c*) $876.27 for 11 mo. at 10%.

(*d*) $572.84 for 1 yr. 5 mo. at 5%.

Ans.—(*a*) $24.75; (*b*) $6.98; (*c*) $80.32; (*d*) $40.58.

77. To find the interest for a number of days.

In computing interest, a year is usually regarded as consisting of 12 months of 30 days each.

Example.—What is the interest on $800 for 21 days at 6%?

Solution.—

```
            $800
             .06
       12)48.00    =Interest for 1 yr.
        30)4.00    =Interest for 1 mo.
             .1333=Interest for 1 da.
                21
              1333
             2666
           $2.7993
```

To the nearest cent, the interest is $2.80. Ans.

78. The interest for a number of days can also be found by considering the days as that many 360ths of a year and finding that fraction of the yearly interest. In the preceding example, the interest for 21 days equals $\frac{21}{360}$ of \$48 = \$2.80. Most persons who do not have occasion to compute interest frequently can check their calculations more readily and feel rather more sure of their results by obtaining the interest for a month and then for the fraction of a month.

Example.—Find the interest on \$248.80 for 5 years 9 months 29 days at 5%.

Solution.— \$248.80
.05
12) 12.4400 = Interest for 1 yr.
30) 1.037 = Interest for 1 mo.
.034 = Interest for 1 da.
5×12.44 = \$62.20 = Interest for 5 yr.
9×1.037 = 9.333 = Interest for 9 mo.
29× .034 = .986 = Interest for 29 da.
\$72.519 or \$72.52. Ans.

The interest is found for 1 year, 1 month, and 1 day, respectively, and the total for 5 years 9 months and 29 days is computed as shown.

EXAMPLES FOR PRACTICE

Find the interest on:
(*a*) \$600 for 1 yr. 4 mo. 15 da. at 6%.
(*b*) \$2,160 for 2 yr. 2 mo. at 5%.
(*c*) \$1,800 for 7 mo. 20 da. at 4½%.
(*d*) \$725.50 for 7 yr. 5 mo. 23 da. at 7½%.
Ans.—(*a*) \$49.50; (*b*) \$234; (*c*) \$51.75; (*d*) \$407.03.

SIXTY-DAY METHOD

79. Among accountants one of the most popular methods of computing interest is known as the sixty-day method. In business dealings generally, loans are made for 30 days, 60 days, or some multiple of these. The sixty-day method meets with favor, as by its use the interest for such periods can be quickly reckoned.

80. Since 60 days is considered as $\frac{1}{6}$ of a year, the interest on \$1 for 60 days at 6% is $\frac{1}{6}$ of 6 cents, or 1 cent. The interest on any sum for 60 days at 6% is, therefore, equal to 1 hundredth of the principal. Hence, *if the decimal point of any sum be moved two places to the left, it will give the interest on that sum for 60 days at 6%.*

Thus, the interest on $3,472.75 for 60 days at 6% is $34.73−, and on $692 it is $6.92.

Example.—Find the interest on $4,268 for 30 days at 6%.

Solution.—The interest on $4,268 for 60 da. at 6% is $42.68. For 30 da. it is one-half of that for 60 da.; $\frac{1}{2}$ of $42.68=$21.34. Ans.

81. In finding the interest by the sixty-day method for any number of days, the work can be shortened and arranged before the eye in an orderly manner by finding the sum or the difference of the aliquot parts of 60 days that will make the given number. For example, to find the interest for 35 days, we would observe that 35 days equals the sum of 30 days and 5 days, each of which is an aliquot part of 60 days, and compute the interest for these separately.

Example 1.—Find the interest on $980 for 35 days.

Solution.—
2) $9.80=Interest for 60 da.
6) 4.90=Interest for 30 da.
.82=Interest for 5 da.
$5.72 Ans.

Explanation.—The interest for 30 days is $\frac{1}{2}$ of that for 60 days, and the interest for 5 days is $\frac{1}{6}$ of that for 30 days.

Example 2.—Find the interest on $8,368 for 99 days at 6%.

Solution.—
$83.68 =Interest for 60 da.
41.84 =Interest for 30 da. = $\frac{1}{2}$ of that for 60 da.
8.368=Interest for 6 da. =$\frac{1}{10}$ of that for 60 da.
4.184=Interest for 3 da.=$\frac{1}{2}$ of that for 6 da.
$138.072=Interest for 99 da. Ans.

82. In case the interest rate is other than 6%, the interest is first calculated at that rate, and then a proportionate part is added or subtracted corresponding to the difference between the given rate and 6%. The following partial table will illustrate:

$$\text{Interest at}\begin{cases} 7\ \% = \text{Interest at } 6\% \text{ plus } \frac{1}{6} \text{ of itself.} \\ 8\ \% = \text{Interest at } 6\% \text{ plus } \frac{1}{3} \text{ of itself.} \\ 9\ \% = \text{Interest at } 6\% \text{ plus } \frac{1}{2} \text{ of itself.} \\ 5\ \% = \text{Interest at } 6\% \text{ less } \frac{1}{6} \text{ of itself.} \\ 4\ \% = \text{Interest at } 6\% \text{ less } \frac{1}{3} \text{ of itself.} \\ 4\frac{1}{2}\% = \text{Interest at } 6\% \text{ less } \frac{1}{4} \text{ of itself.} \end{cases}$$

Example 1.—What is the interest at 9% on $1,264.76 for 49 days?

Solution.—

$12.6476 = Interest for 60 da. at 6%.
6.324 = Interest for 30 da. = $\frac{1}{2}$ of that for 60 da.
3.162 = Interest for 15 da. = $\frac{1}{2}$ of that for 30 da.
.632 = Interest for 3 da. = $\frac{1}{10}$ of that for 30 da.
.211 = Interest for 1 da. = $\frac{1}{3}$ of that for 3 da.
$10.329 = Interest for 49 da. at 6%.
5.165 = Interest for 49 da. at 3%.
$15.494 = Interest for 49 da. at 9%. Ans.

Example 2.—What is the interest at 4½% on $3,000 for 98 days?

Solution.—

$30.00 = Interest for 60 da. at 6%.
15.00 = Interest for 30 da. at 6% = $\frac{1}{2}$ of that for 60 da.
3.00 = Interest for 6 da. at 6% = $\frac{1}{5}$ of that for 30 da.
1.00 = Interest for 2 da. at 6% = $\frac{1}{3}$ of that for 6 da.
$49.00 = Interest for 98 da. at 6%.
12.25 = Interest for 98 da. at 1½% = $\frac{1}{4}$ of that at 6%.
$36.75 = Interest for 98 da. at 4½%. Ans.

Example 3.—Find the interest on $258.50 for 29 days at 6%.

Solution.—

$2.585 = Interest for 60 da.
1.293 = Interest for 30 da. = $\frac{1}{2}$ of that for 60 da.
.043 = Interest for 1 da. = $\frac{1}{30}$ of that for 30 da.
$1.25 = Interest for 29 da. Ans.

EXAMPLES FOR PRACTICE

By the sixty-day method, find the interest on:

1. $8,000 for 87 days at 6%.
2. $6,050 for 96 days at 3%.
3. $875.28 for 77 days at 3½%.

4. $1,468.80 for 123 days at 4%.
5. $23,750 for 108 days at 4½%.
6. $42,690 for 176 days at 3⅝%.
7. $7,200 for 225 days at 5%.
8. $468.24 for 101 days at 5½%.
9. $6,880 for 186 days at 7%.
10. $7,600 for 143 days at 7½%.

Ans.—(1) $116; (2) $48.40; (3) $6.55+; (4) $20.07+; (5) $320.625; (6) $765.26−; (7) $225; (8) $7.23−; (9) $248.83−; (10) $226.42−.

COMMERCIAL CALCULATIONS
(PART 2)

APPLICATIONS OF PERCENTAGE

1. To find the base when the rate and percentage are given.

EXAMPLE 1.—If 3% of a number is 12, what is the number?

SOLUTION.— $3\%=12$; $1\%=\frac{1}{3}$ of $12=12\div 3=4$.
$100\%=100\times 4=400$. Ans.

EXPLANATION.— 1% is evidently $\frac{1}{3}$ as great as 3%. Having found 1% of the number, we find 100%, or the entire number, by multiplying 1% of it by 100.

EXAMPLE 2.—A mechanic received an increase of 30 cents a day in his salary, which was 15% of his former pay. What pay had he been receiving?

SOLUTION.— $15\%=30$ cents; $1\%=\frac{1}{15}$ of 30 cents$=2$ cents.
$100\%=100\times 2$ cents$=\$2.00$. Ans.

In this example 15% is the rate, 30 cents is the percentage, and the base or 100% is to be found.

EXAMPLES FOR PRACTICE

Solve the following mentally:

(*a*) 25 is 5% of what number?
(*b*) 40 is 20% of what number?
(*c*) 4 is $\frac{1}{2}$% of what number?
(*d*) 7 is 2% of what number?
(*e*) 81 is 90% of what number?
(*f*) 50 is 100% of what number?
(*g*) 500 is 25% of what number?

(*h*) 800 is 200% of what number?

(*i*) The rate is 12%, the percentage is 96; what is the base?

Ans.—(*a*) 500; (*b*) 200; (*c*) 800; (*d*) 350; (*e*) 90; (*f*) 50; (*g*) 2,000; (*h*) 400; (*i*) 800.

---

2. The illustrations just given show that when the percentage and rate are known the base can be found by dividing the percentage by the number of hundredths which equals the rate and then multiplying the result thus obtained by 100. The following rule gives a more direct method.

3. Rule.—*To find the base, divide the percentage by the rate expressed decimally.*

EXAMPLE 1.— 45 is 15% of what number?

SOLUTION.— 15%=.15.

.15) 45.00 (300 Ans.
45
00

The division is performed according to the rule for division of decimals.

EXAMPLE 2.—In a certain village there are 308 children, who form 28% of the total population. How many inhabitants are there in the village?

SOLUTION.— 28%=.28.

.28) 308.00 (1100. Ans.
28
28
28
00

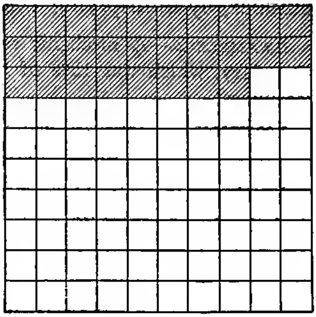


FIG. 1

4. The conditions stated in the last example may be illustrated by the aid of Fig. 1, which is divided into 100 parts. Each small division is 1% of the square. The shaded part covers 28 small divisions, or 28% of the entire square. The entire figure represents the total population and the shaded portion represents the number of children. Since the value of 28 parts is known, the value of 100 parts can be obtained by dividing the known number by .28.

5. Aliquot parts of 100 can often be used to advantage in finding the base when the rate and percentage are known.

Example 1.— 40 is $16\frac{2}{3}\%$ of what number?

Solution.— $16\frac{2}{3}\%=\frac{1}{6}$.
$\frac{1}{6}$ of a number$=40$
$\frac{6}{6}$ of the number$=6\times40=240$. Ans.

Example 2.—In a fire a merchant lost 40% of his stock of goods. If his loss was estimated to be $8,700, what was the estimated value of his entire stock?

Solution.— $40\%=\frac{2}{5}$.
$\frac{2}{5}$ of the stock=$8,700
$\frac{1}{5}$ of the stock=$\frac{1}{2}$ of $8,700=$4,350
$\frac{5}{5}$ of the stock=5×$4,350=$21,750. Ans.

EXAMPLES FOR PRACTICE

1. The percentage is 725 and the rate is 29%. What is the base?
Ans. 2,500

2. The percentage is 1,326 and the rate is 32%. What is the base?
Ans. $4,143.75

3. In the manufacture of certain goods, 43% of the cost is for material. What is the entire cost of the goods if the cost for material is $2,800?
Ans. $6,511.63

4. The receipts of a business place this year were $128,276, which is 92% of the receipts of last year. What were last year's receipts?
Ans. $139,430.43

6. To find the rate when the base and percentage are known.

Example 1.—A farmer owning 200 fowls sold 20 of them. What per cent. did he sell?

Solution.— 1% of 200=2.
20÷2=10%. Ans.

Explanation.—We first find 1% of the entire number to be 2. Since 2=1%, he sold as many per cent. as the number of times that 2 is contained in 20.

Example 2.—What per cent. of 500 is 75?

Solution.— 1% of 500=5.
$75=\frac{75}{5}$, or 15% of 500. Ans.

EXAMPLES FOR PRACTICE

Solve mentally the following examples:
What per cent. of

(*a*) 500 is 200?
(*b*) 500 is 400?
(*c*) 700 is 28?
(*d*) 400 is 30?
(*e*) 100 is 6?
(*f*) 100 is $\frac{1}{4}$?
(*g*) 300 is 600?
(*h*) 200 is 300?

Ans.—(*a*) 40%; (*b*) 80%; (*c*) 4%; (*d*) 7$\frac{1}{2}$%; (*e*) 6%; (*f*) $\frac{1}{4}$%; (*g*) 200%; (*h*) 150%.

---

7. When the base and percentage are given, instead of finding the rate step by step as shown, in actual practice it would be found by the following rule:

8. Rule.—*To find the rate, divide the percentage by the base.*

Example 1.—What per cent. of 240 is 60?

Solution.—

```
240) 60.00 (.25=25%  Ans.
     480
     1200
     1200
```

Explanation.—It is evident that 60 is $\frac{60}{240}$ of 240, since 1 is one 240th part of 240. We find that $\frac{60}{240}$=.25, which equals 25%. In this example, 240 is 100%, or the base; 60 is the percentage; and the rate is to be found.

It must be observed that when the rate is to be expressed with the words per cent. or with the symbol % the decimal point is moved two places to the right.

Example 2.—In a city of 24,367 inhabitants, 2,876 are unable to read. What percentage of the population cannot read?

Solution.—

```
24367) 2876.000 (.118+=11.8%.  Ans.
       24367
        43930
        24367
        195630
        194936
```

Explanation.— 24,367 is the base. This is the number on which the percentage is to be figured; 2,876 is the percentage. The division is made according to the rule for division of decimals.

9. In actual practice the student may be at a loss to know how far to carry the division in problems similar to that given in Art. **8.** This is a question to be determined by the circumstances of the case. In the preceding example, if the information is for a table to show the relative illiteracy in various cities, two decimal places would probably be sufficient and the result would be called 12%. If, however, the percentage is to be used again in some exact calculation, the division should be carried to a greater number of figures.

EXAMPLE.—In shipping freight from one city to another, the freight passes over two different railroads. The distance on the first road is 136 miles, and on the second is 280 miles. What percentage of the freight charges is due to each of the roads?

SOLUTION.— Total mileage is 136+280=416 mi.
136÷416=.327=32.7% for first road. Ans.
280÷416=.673=67.3% for second road. Ans.

EXPLANATION.—The total mileage, or 416 miles, is considered to be 100%. The percentage that each distance is of 416 is then found.

10. The only difficulty that problems in percentage give is in determining what the given quantities stand for or represent.

EXAMPLE 1.—What per cent. of 40 is 30?

SOLUTION.— 30 is $\frac{30}{40}$ of 40.
$\frac{30}{40}=\frac{3}{4}=.75$, or 75%. Ans.

EXPLANATION.—In this case, 30 is to be measured in hundredths of 40. Therefore, 40 is the base. The rate is to be found.

EXAMPLE 2.—What per cent. of 30 is 40?

SOLUTION.— 40 is $\frac{40}{30}$ of 30.
$\frac{40}{30}=\frac{4}{3}=1.33\frac{1}{3}$, or $133\frac{1}{3}$%. Ans.

The unit of measure is one hundredth part of 30; evidently, 30 is the base.

EXAMPLE 3.—The price of a line of goods has advanced 25%. The present price is $6.25 a dozen. What was the former price?

SOLUTION.—The increase of 25% was made on the former price. The former price is 100%; the present price, $6.25, equals the former price plus the advance, or 100%+25%=125%. $6.25 is, therefore, the percentage and 125% is the rate.

125%=1.25; base=$6.25÷1.25=$5. Ans.

To prove the result, we may add to $5, 25% of itself.

25% of $5=$1.25; $5+$1.25=$6.25. Ans.

Example 4.—A fruit dealer bought a lot of oranges. After losing 18% of the lot on account of cold weather, he had 656 boxes left. How many boxes did he buy?

Solution.—The loss is based on the amount purchased, which we shall call 100%. He loses 18%, and he has left 100%−18%=82%. It is now seen that 656 is the percentage and 82% is the rate.

82%=656; 100%=656÷.82=800. Ans.

11. The solutions of the foregoing examples should be studied very carefully, as they illustrate typical cases and as similar problems will be met frequently later in this Section.

In solving any indirect problem in percentage, the student should carefully analyze the statement of the problem to determine what quantity represents 100% and whether this quantity is known or is to be computed.

EXAMPLES FOR PRACTICE

1. A man's salary is $1,800 per year and he saves $225. (*a*) What per cent. of his salary does he save? (*b*) What per cent. of it does he spend?

Ans. $\begin{cases} (a)\ 12\frac{1}{2}\% \\ (b)\ 87\frac{1}{2}\% \end{cases}$

2. A man has 32% of his money invested in stocks, 18% in grain, and the remainder, which is $7,620, in real estate. What is the total value of his property? Ans. $15,240

3. If wool loses 32% of its weight in washing, how many pounds of unwashed wool are required to produce 35,360 pounds of washed wool? Ans. 52,000 lb.

4. In 1910 the population of a city was 85,000, which was 36% more than the population in 1900. What was the population in 1900? Ans. 62,500

5. A man bequeathed to a charity 32% of his estate. To another charity he gave $23,100, which was 23% less than the amount given to the first charity. (*a*) What was the value of the estate? (*b*) What per cent. of the estate was given to the second charity?

Ans. $\begin{cases} (a)\ \$93,750 \\ (b)\ 24,64\% \end{cases}$

PROFIT AND LOSS

12. Profit and loss treats of the gains or losses arising in business transactions.

If the price for which merchandise is sold is greater than the cost of the merchandise, the difference is *profit*, or *gain*. If the selling price is less than the cost, the difference is *loss*.

13. The **gross cost** of merchandise is its first cost plus the expenses of purchase, transportation, and storage. Such expenses are commission, freight, insurance, drayage, etc.

14. The **net selling price** is the gross selling price, less all discounts and expenses of sale.

15. Computation in profit and loss are made according to the rules of percentage. In this Section, unless statement is made to the contrary, the gross cost of the merchandise is considered to be the *base*, upon which the *rate* of profit or loss is computed. The profit or loss is the *percentage*. If the merchandise is sold at a profit, the net selling price is the *amount*; if at a loss, the net selling price is the *difference*.

16. Rule.—*To find the profit or loss, multiply the gross cost by the rate of gain or loss.*

EXAMPLE.—A house costing \$3,000 is sold for 22% above cost. What is the profit?

SOLUTION.—Profit=cost×rate=\$3,000×.22=\$660. Ans.

17. Rule.—*To find the rate of profit or loss, divide the difference between the selling price and gross cost by the gross cost; or divide the profit or loss by the gross cost.*

EXAMPLE.—A merchant sold for \$768 a lot of dry goods for which he paid \$900. What was the per cent. loss?

SOLUTION.— Loss=\$900−\$768=\$132.

Rate of loss=loss÷cost=\$132÷\$900=.14⅔, or 14⅔%. Ans.

18. Rule.—*To find the selling price, the cost and rate of gain or loss being given, multiply the cost by 100 per cent. plus the rate of gain, or by 100 per cent. minus the rate of loss.*

Example.—If hay is bought for $8 per ton, and if baling and shipping cost $5.50 per ton additional, at what price must it be sold to yield a profit of 16%?

Solution.—Gross cost=$8+$5.50=$13.50. The selling price=100% +16%, or 116% of cost. 116% expressed as a decimal becomes 1.16.

Selling price=cost×(100 per cent.+rate)=$13.50×1.16=$15.66. Ans.

19. In business calculations, it is more usual to calculate the profit separately and add it to the cost. The selling price can be found directly, however, by the rule. A process which is the reverse of the application of this rule is given in the next article.

20. Rule.—*To find the cost, the selling price and rate of gain or loss being given, divide the selling price by 100 per cent. plus the rate of gain, or by 100 per cent. minus the rate of loss.*

Example.—A dealer sold drugs for $112 and gained 75%. What was the cost of the drugs, and what was the profit?

Solution.—Selling price=100%+75%=175% of cost. 175%=1.75.

Cost=selling price÷(100%+rate)=$112÷1.75=$64. Ans.

Profit=$112−$64=$48. Ans.

EXAMPLES FOR PRACTICE

What is the profit or loss

(*a*) If the gross cost is $85 and the rate of gain is 32%?

(*b*) If the gross cost is $837.50 and the rate of loss is 12%?

(*c*) If the gross cost is $240 and the rate of gain is $16\frac{2}{3}\%$?

What is the rate of gain or loss

(*d*) If the gross cost is $6.50 and selling price is $9.10?

(*e*) If the gross cost is $14.00 and selling price is $12.50?

(*f*) If the gross cost is $3,500 and profit is $500?

What is the selling price

(*g*) If the cost is $945 and the rate of gain is $33\frac{1}{3}\%$?

(*h*) If the cost is $3.50 and the rate of gain is $12\frac{1}{2}\%$?

(*i*) If the cost is $125 and the rate of loss is 18%?

What is the cost

(*j*) If the selling price is $575 and the rate of gain is 15%?

(*k*) If the selling price is $28 and the rate of loss is $12\frac{1}{2}\%$?

(*l*) If the selling price is $3.50 and the rate of gain is 26%?

Ans.—(*a*) $27.20; (*b*) $100.50; (*c*) $40; (*d*) 40%; (*e*) $10\frac{5}{7}\%$; (*f*) $14\frac{2}{7}\%$; (*g*) $1,260; (*h*) $3.94; (*i*) $102.50: (*j*) $500; (*k*) $32; (*l*) $\$2.77\frac{7}{9}$.

1. A house and lot which cost $3,250 is sold at a profit of 12%. What is (*a*) the profit and (*b*) the selling price? Ans. $\begin{cases}(a)\ \$390\\(b)\ \$3{,}640\end{cases}$

2. What must be the selling price of a suit of clothes which cost $18 in order that the profit may be 33⅓%? Ans. $24

3. A harvesting machine costs a hardware merchant $90 net, and $6 for freight and cartage. If sold for $108, what is the gain per cent.? Ans. 12½%

4. A carload of cattle is sold for $875, which is at a loss of 16%. What was the cost of the cattle? Ans. $1,041.67

5. A sells a steam tug to B, gaining 14%, and B sells it to C for $4,104 and gains 20%. How much did the tug cost A? Ans. $3,000

6. How much must hay sell for per ton, to gain 25%. if when sold for $8.40 per ton, there is a gain of 16⅔%? Ans. $9

7. Six horses were sold at $125 each; three of them at a profit of 25%, and the others at a loss of 25%. What was the net gain or loss? Ans. $50 loss

PROFITS BASED ON SELLING PRICE

21. In marking goods, some merchants reckon the profits and the selling expenses as a percentage of the selling price. This method is claimed to be the logical one, inasmuch as the cost of running a business is always reckoned as a percentage of the amount of sales, not of the cost of goods. Discounts, salesmen's commissions, taxes, etc. are also based on the selling price. Moreover, there is less chance of overestimating the profits when they are based on the selling price. To show that the percentage of profit on the selling price is not the same as that of the cost price, suppose an article costing $1 is sold at an advance of 25% on the cost price. The profit equals 25% of $1=$.25. The selling price equals $1+$.25=$1.25. The profit based on the selling price equals $.25÷$1.25=.20, or 20%. It is thus seen that 25% of the cost price equals only 20% of the selling price.

Example.—An article costing $1 is sold for $2. (*a*) What per cent. of the cost is the gain? (*b*) What per cent. of the selling price is the gain?

SOLUTION.—(*a*) $2—$1=$1, the gain.
$1 is 100% of $1, or 100% of the cost. Ans.
(*b*) $1 is 50% of $2, or 50% of the selling price. Ans.

From the example just solved it has been shown that a profit of 100% on the cost price is equal to a profit of 50% on the selling price. Thus, it is seen that it may make considerable difference what the percentage is based on.

22. Rule.—*To find the selling price, add together the percentage estimated for selling expenses and the percentage of net profit desired and subtract from 100%. Divide the cost price by the remainder, and the result will be the selling price.*

EXAMPLE.—A merchant estimates that his expenses are 20% of his sales. He desires to make a profit of 10% on his receipts. What should he ask for an article that costs him $12.25?

SOLUTION.—The profit and cost of selling=10%+20%=30% of selling price. The purchase price+30% of selling price=entire selling price. The purchase price is, therefore, 100%—30%, or 70%, of selling price.

Selling price is $\$12.25 \div .70 = \$17.50$. Ans.

23. The relation of the purchase price, profit, and cost of selling can be shown by Fig. 2.

FIG. 2

EXAMPLE 1.—A manufacturer allows his salesmen 40% of their sales and proposes to make a profit of 15% on the sales. What will be the selling price of an article that costs $1.95?

SOLUTION.—Cost price=100%—40%—15%=45% of the selling price.
$\$1.95 \div .45 = \$4.33\frac{1}{3}$, selling price. Ans.

EXAMPLE 2.—A merchant estimates that in order to do business at a reasonable profit, the difference between the cost price and the selling price should equal 20% of the latter. In marking his goods, what per cent. should he add to the cost?

SOLUTION.—For goods he sells for $1 he would pay 20% less than $1, or 80 cents. An advance of 20 cents on 80 cents equals $\frac{20}{80}$=25%. The merchant would, therefore, make the marked price 25% more than the cost price. Ans.

24. The student may at first be confused to some extent owing to the different methods of computing the selling price. If, however, he will consider whether profits and selling expenses are to be based on the cost or the selling price he will have no difficulty.

TRADE DISCOUNTS

25. **Trade discounts** are reductions made by manufacturers, jobbers, or merchants from their list or catalog prices.

In many branches of business, manufacturers and dealers list their goods at a fixed price for each article, and allow a rate of discount according to the conditions of the market, the size of the order, and the plan of payment. In fact, two or even three discounts are sometimes allowed on an order. If it becomes necessary to raise or lower the price of the goods, the rate of discount is increased or decreased, the list price remaining the same. The system of discounts thus saves the expense of publishing a new price list every time prices change.

26. Merchandise is frequently sold at *time prices;* that is, payment is to be made in, say, 30, 60, or 90 days after date of sale, and a certain rate of discount is allowed if payment is made at an earlier date. Business houses usually make announcements such as the following upon their bill heads: "Terms: 4 mo., or 5% 60 days"; "Terms: 60 days net; 30 days, 3% off; 10 days, 5% off." Even when no discount is stated in the terms, sellers will usually deduct the legal interest for the time remaining, if the payment is made before it becomes due. Thus, if a payment due in 3 months is made 1 month after the sale, the seller should deduct the interest for the remaining 2 months.

27. Trade discounts are computed by the rules of percentage, the list price of the goods being the base. When several discounts are allowed, the first discount is computed on the list price, the second is computed on the remainder after deducting the first discount, and so on, each remainder being regarded as a base for the computation of the next discount. The several discounts, if there are more than one, form a **discount series.**

28. Rule.—*To find the selling price, multiply the list price by the rate, and subtract the discount thus obtained from the list price. If there is a discount series, compute the second discount, using the first remainder as a base, and subtract the discount from the remainder. Repeat the process, using each successive remainder as a base for computing the next discount. The last remainder is the selling price.*

EXAMPLE 1.—The list price of an article is $62.50 and a discount of 40% is allowed. What is the selling price?

SOLUTION.—

Discount=$62.50×.40=$25.
Selling price=$62.50−$25=$37.50. Ans.

EXAMPLE 2.—On a bill of goods amounting to $720, discounts of 30%, 10%, and 5% are allowed. What is the selling price?

SOLUTION.—

First discount=$720×.30=$216.
Remainder=$720−$216=$504.
Second discount=$504×.10=$50.40.
Remainder=$504−$50.40=$453.60.
Third discount=$453.60×.05=$22.68.
Selling price=$453.60−$22.68=$430.92. Ans.

29. The trade discounts usually allowed are aliquot parts of 100%, and the labor of computation may be shortened by using the fractions corresponding to the rates of discount.

EXAMPLE.—The gross amount of a bill of hardware is $640, and discounts of 25%, 10%, and 5% are allowed. What is the net amount of the bill?

SOLUTION.—

25%=$\frac{1}{4}$, 10%=$\frac{1}{10}$, 5%=$\frac{1}{20}$.

The solution is arranged as shown. To multiply $640 by 25%, or $\frac{1}{4}$, we divide by 4; then the discount, $160, is subtracted and the remainder, $480, is divided by 10. The second discount, $48, is subtracted, and the remainder is divided by 20. The final remainder, $410.40, is the net amount, or selling price.

4)	$640	
	$160	1st discount.
10)	$480	1st remainder.
	$ 48	2d discount.
20)	$432	2d remainder.
	$ 21.60	3d discount.
	$410.40	net amount. Ans.

30. When a discount series is allowed, business men usually reduce the series to an equivalent single discount; if there is a large number of sales, much labor of computation is saved by using the equivalent discount rather than the series.

EXAMPLE.—What single discount on the gross price is equivalent to a discount series of 25%, 20%, and 10%?

SOLUTION.—We will call the list price 100%. A discount of 25% makes the price 100%−25%=75% of the list price. A further discount of 20% makes the price 100%−20%, or 80%, of the 75%; 80% of 75% =.80×75%=60%. A third discount, which is 10%, reduces the price to 100%−10%, or 90% of the 60%. .90×60%=54%. This result can be obtained by continued multiplication as follows:

$$100\%-25\%=75\%=.75;\ 100\%-20\%=80\%=.80$$
$$100\%-10\%=90\%=.90$$
$$.75\times.80\times.90=.54=54\%$$
$$100\%-54\%=46\%,\text{ the discount. Ans.}$$

31. Rule.—*To reduce a discount series to an equivalent single discount, subtract each rate of discount from 100 per cent., and multiply the remainders together. Subtract the product from 100 per cent., and the remainder will be the single discount.*

EXAMPLE 1.—The cost of a line of goods is $350. What must they be marked to give a profit of 20% on the cost, and allow a discount of 30% on the marked price?

SOLUTION.—The profit is $350×.20=$70; therefore, the actual selling price is $350+$70=$420. This is what remains after deducting 30% from the marked price. Since the 30% discount is computed on the marked price, that price must be the base, and the selling price, $420, obtained by subtracting the discount, is 100%−30%=70% of the base. Hence, the base or marked price=$420÷(100%−30%)=$420÷.70=$600. Ans.

EXAMPLE 2.—The cost of manufacturing hats is $36 per dozen. At what price per dozen must they be marked that the manufacturer may realize $16\frac{2}{3}\%$ profit on the cost after allowing the trade discounts of 20% and $12\frac{1}{2}\%$?

SOLUTION.— $16\frac{2}{3}\%=\frac{1}{6}$; profit=$\$36\times\frac{1}{6}=\$6$. Selling price=$36+$6 =$42. 100%−20%=80%=.80. $100\%-12\frac{1}{2}=87\frac{1}{2}\%=.87\frac{1}{2}$. $.80\times.87\frac{1}{2}$ =.70=70%. The equivalent single discount is 100%−70%=30%.

Marked price=$42÷.70=$60 per doz. Ans.

32. Rule.—*To find the price at which goods must be marked to insure a given profit after allowing a discount, or a* *discount series, add to the cost, including the selling expenses, the profit required, and divide the sum by 100% minus the discount series or equivalent single discount.*

33. A bill in which discounts of 10% and 5% have been allowed is shown in Fig. 3.

Scranton, Pa., June 10, 19___

TECHNICAL SUPPLY CO.
Sold to
BROWN & BROS.
Denver, Colo.

10 rolls Blueprint Paper @ $1.50 per roll	$15	00
162 lb. Drawing Paper @ $.15 per lb.	24	30
1 gross Drawing Ink	21	60
	$60	90
Less 10% and 5%	8	83
	$52	07

FIG. 3

EXAMPLES FOR PRACTICE

Reduce the following discount series to equivalent single discounts:
(*a*) 25% and 16%.
(*b*) 30%, 20%, and 5%.
(*c*) 60%, 10%, and 5%.
(*d*) 40%, 20%, 12½%, and 4%.
Ans.—(*a*) 37%; (*b*) 46.8%; (*c*) 65.8%; (*d*) 59.68%.

TAXES

34. Taxes are sums of money levied on persons, properties, or incomes, for public purposes. Thus, taxes are levied to support the state, county, and city governments; to support schools and charities; and to make improvements, such as paved streets and sewers.

35. A **capitation,** or **poll,** tax is a tax levied on persons; a **property** tax is a tax levied on real estate or personal property; an **income** tax is levied on incomes or salaries. The term *real estate* applies to such property as lands and buildings. Other property, such as bonds, notes, goods, and money, is called *personal property.* The poll tax is usually a fixed amount for each citizen over 21 years of age. The property tax is reckoned as a certain per cent. of the assessed valuation of the property subject to taxation.

36. Rule.—*To find a property tax, multiply the assessed value of the property by the rate of taxation.*

EXAMPLE.—What property tax must a person pay who owns real estate assessed at $34,000 and personal property assessed at $12,500, the rate of taxation being 8 mills per $1, that is, rate=.008?

SOLUTION.—Total assessed value=$34,000+$12,500=$46,500.
Tax=$46,500×.008=$372. Ans.

EXAMPLES FOR PRACTICE

What is the tax if the assessed value of the property is
(*a*) $6,300 and the rate of taxation is $1\frac{1}{8}$%?
(*b*) $34,300 and the rate of taxation is $6\frac{1}{2}$ mills per $1?
(*c*) $9,430 and the rate of taxation is 85 cents per $100?
Ans. (*a*) 70.87\frac{1}{2}$; (*b*) $222.95; (*c*) 80.15\frac{1}{2}$.

1. I own real estate worth $8,500 and personal property worth $3,750; both are assessed at $\frac{3}{5}$ of their value. The rate of taxation is 1.2%, but I receive a discount of $2\frac{1}{2}$ per cent. of my taxes for prompt payment. How much do I pay the tax receiver? Ans. $86

2. A has 5 lots worth $1,300 each; B has 4 lots worth $1,000 each; C has 2 lots worth $1,500 each; D has 7 lots worth $800 each, and E has 1 lot worth $1,150. A tax of $1,417.50 for a street improvement is to be divided among them. What should each pay?

NOTE.—To find the rate, divide the amount levied for improvement by the total value of the property.

Ans.—A, $455; B, $280; C, $210; D, $392; E, $80.50.

3. The rate of taxation for a certain state is $3\frac{1}{2}$ mills per $1. How much state tax must be raised by a county whose valuation is fixed by the State Board of Equalization at $13,876,394? Ans. $48,567.38

4. If the income tax is 1% on all income in excess of $4,000 a year, what tax will a person pay whose annual income is $18,230? Ans. $142.30

PROMISSORY NOTES

37. A **promissory note** is a written promise to pay a certain sum at a certain time.

38. The **maker,** or **drawer,** of a note is the person that promises to pay; the **payee** is the person to whom the note is payable; and the **holder** is the person that owns it.

39. The **face** of a note is the sum promised to be paid. This sum should be written both in figures and in words.

40. Notes are of two kinds—notes *bearing interest* and notes *not bearing interest.* When no rate of interest is specified, the legal rate in the state or county where the note is made is to be understood. If a note not bearing interest is not paid when due, it bears interest at the legal rate after that time until paid.

41. The legal rate of interest for money varies in the different states of the Union and depends on two facts:

1. The rate per cent. prescribed by statute in each state must be used when no other rate is agreed upon; this prescribed rate is as low as 5% in some states and as high as 8% in others.
2. If the rate is agreed upon by contract it can be no higher than 6% in some states, and is without any limit in others. Call loans by banks in New York and Pennsylvania—which are loans that may be called in at the pleasure of the bank—may command any rate agreed upon, provided the loan is $5,000 or more.

42. The punishment for usury—which is an interest charge higher than the admissible legal rate—involves various penalties, from the forfeiture of the excess of interest to the forfeiture of both principal and interest.

43. If a debt falls due on a Sunday or on a legal holiday or half holiday, some states require that it shall be paid on the day before, and some on the day following the legal holiday or half holiday.

44. Every state has a statute of limitation according to which, after the expiration of a certain time, a claim for money does not hold in law. This time is different for notes, closed and open accounts, judgment notes, and sealed documents involving claims for money.

45. Rates of interest, penalties for usury, payments with respect to holidays, and statutes of limitation are changed so frequently that any table showing them at a given time is speedily out of date. The student who wishes information relative to any of these matters can get it at any bank, so it is deemed best to omit it here.

46. A note should be so written as to show where it was made and when, the sum promised to be paid, whether it does or does not bear interest, and should bear the words "value received." The law assumes that no one is to be compelled to pay unless he has received what he deems an equivalent. If "value received" is omitted, the holder may have to prove that the maker of the note did actually receive value for the money promised in it.

47. A note usually specifies where it is to be paid—usually at a bank. If no place is designated, the holder's place of business is understood, or his residence.

48. If a note contains the words "or order," it is a **negotiable note,** and may pass like a bank note from one person to another. If the holder of a negotiable note wishes to dispose of it, he is required to guarantee its payment by *indorsing* it—that is, by writing his name across the back of the note. There are several kinds of indorsements. Thus, if the holder is John Smith, he may, on the back of the note, write *John Smith.* This is an indorsement *in blank,* and makes John Smith responsible for the payment of the note.

He may write *Pay to William Jones.* The note is then payable only to William Jones.

If he indorses it *Pay to William Jones, or order,* it is payable to William Jones, or to any one to whom William Jones may order it to be paid.

He may indorse it *Pay to William Jones, or bearer,* and it is payable to any person that presents it.

If it be indorsed *Pay to bearer,* it is payable to the person that presents it for payment.

49. A **joint-and-several** note is a note signed by two or more persons, who become collectively and individually responsible for its payment.

50. A note is **protested** when the holder, or a notary public at his direction, notifies the indorsers in legal form and within the time prescribed by law that the note is unpaid. Unless the note is protested in this manner, the indorsers are not responsible for its payment.

51. Forms of notes used in actual business are given below:

$250. *New York, N. Y., Sept. 17, 1912*
On demand I promise to pay George Camp, or order, Two Hundred and Fifty Dollars, value received.
Howard Gray

$1,000. *Toronto, Ont., July 5, 1913*
Three months after date, for value received, I promise to pay Stephen Girard, or order, One Thousand Dollars, with interest at 5%. *Charles Goldwin*

$3,000. *Philadelphia, Pa., July 5, 1913*
Six months after date, we, or either of us, will pay to George Owen, Three Thousand Dollars, value received, with interest at 6%. *George Kirwin*
Henry Potter
Erastus Kirby
Payable at the First National Bank.

The first two notes above are negotiable.

Responsibility for the payment of a joint-and-several note, such as the last of those above, rests on the first person signing. If he cannot pay, the second person signing is called upon, and so on.

52. In some states, *three days of grace* are allowed before a note must be paid. If, in a state where grace is allowed, a bank discounts a note, interest is charged for days of grace.

EXAMPLES FOR PRACTICE

1. Write a negotiable demand note for $600, with interest at 6%, and make Brown, Jones & Co. the payee.

2. Write a non-negotiable note for $4,000, with interest at 5%, payable in 30 days, yourself being the maker and Howard Crosby the payee.

3. How much will pay the following note when due without grace?

$575 $\frac{50}{100}$. *New York, N. Y., Sept. 19, 1914*

Sixty days after date, for value received, I promise to pay Ralph Newton, or order, Five Hundred Seventy-Five and $\frac{50}{100}$ Dollars, with interest at $4\frac{1}{2}$%. *Henry Miles*

Ans. $579.82

BANK DISCOUNT

53. **Bank discount** is the charge made by a bank for paying a note or other obligation before it is due. This charge is the interest on the amount of the obligation from the time it is discounted until its **maturity;** that is, the date on which it is due. This interest is subtracted from the face of the obligation, and its holder receives for it the remainder, which is called the **proceeds.** Hence, bank discount is inequitable, since interest is charged not only upon the sum actually paid for the obligation, but also upon the discount.

54. If a note for $1,000, with 60 days to run before it is due, is discounted at a bank, the interest on $1,000 is found for 60 days, and is subtracted from $1,000. If the rate of discount is 6%, the holder will receive as proceeds $1,000 − $10 = $990.

55. The **term of discount** is the time from the discounting of the note to its maturity.

56. In the case of an interest-bearing note, the sum discounted is the amount of the note at maturity; that is, the face of the note plus the interest.

57. Banks usually require that a discounted note shall be payable at the bank that discounted it, and ordinarily they do not discount notes having more than 90 days to run.

58. To find the time when a note matures, the term of discount, the discount, and the proceeds.

Example.—Find (*a*) the discount and (*b*) the proceeds of the following note:

$\$484\frac{60}{100}$. *Newark, N. J., Oct. 4, 1913*
Sixty days after date, for value received, I promise to pay William Hall, or order, Four Hundred Eighty-Four and $\frac{60}{100}$ Dollars, at the Ninth National Bank. *Henry Parshall*

Discounted, Oct. 20, 1913, at 6%.

Solution.—
(*a*) Maturity, Dec. 3, 1913.
Term of discount, 44 days.
Discount, $3.55. Ans.
(*b*) Proceeds, $484.60—$3.55=$481.05. Ans.

Explanation.—Sixty days after Oct. 4 is Dec. 3, the date of legal maturity. From the time of discount, Oct. 20, to Dec. 3 is 44 days, for which the interest at 6% is $3.55. Subtracting the discount, $3.55, from the face of the note, $484.60, gives $481.05, the proceeds.

59. The maturity of a note, when grace is allowed, is on the last day of grace. The time of maturity is generally indicated on such notes. Thus, Mar. 7/10, written on a note means that it matures nominally on Mar. 7, and legally on Mar. 10.

Example.—Find (*a*) the discount and (*b*) the proceeds of the following note, allowing for days of grace:

$1,060. *Austin, Tex., August 6, 1913*
For value received, I promise to pay, three months after date, to Ernest Hazard, or order, One Thousand Sixty Dollars, with interest at 8%.
Emil Reeves

Discounted, Sept. 1, 1913, at 6%.

Solution.—

(*a*)	Maturity, Nov. 6/9, 1913.		
	Amount of note at maturity...............	$1081.91	
	Terms of discount, 69 days...............		
	Discount	12.44	Ans.
(*b*)	Proceeds	$1069.47	Ans.

60. Rule.—I. *If the note bears interest, find its amount at the time of legal maturity.*

II. *Find the interest on the face of the note at the given rate of discount, or, if it is an interest-bearing note, on the amount of the note at the date of legal maturity at the given rate of interest, and the result will be the bank discount.*

III. *Subtract the bank discount from the face of the note, or from its amount at maturity, and the remainder will be the proceeds.*

EXAMPLES FOR PRACTICE

1. Find (*a*) the bank discount and (*b*) the proceeds of a note for $5,000, due in 60 days, discounted at 6%.

Ans. $\begin{cases} (a)\ \$50 \\ (b)\ \$4{,}950 \end{cases}$

2. Find (*a*) the bank discount and (*b*) the proceeds of a grace note for $4,000 due in 90 days, discounted at 5%.

Ans. $\begin{cases} (a)\ \$51.67 \\ (b)\ \$3{,}948.33 \end{cases}$

3. $\$8{,}476\frac{00}{100}$. *Scranton, Pa., Jan. 8, 1914*

Six months after date I promise to pay to Charles Brown, or order, Eight Thousand Four Hundred Seventy-Six Dollars, value received, with interest at 6%.

Howard Bristow

Find the proceeds of the foregoing note, if discounted April 24, 1914, at 5%. Ans. $8,639.34

PARTIAL PAYMENTS

61. A debt or obligation may be discharged at one payment; or, from time to time, payments *in part* may be made, and finally at a time of *settlement* the remainder of the debt may be paid. Now, it is obvious that interest should be allowed upon such payments as are made, since interest is charged upon the obligation itself. But, if a payment should be less than the interest upon the debt since a previous payment had been made, to subtract such payment from the debt with accrued interest would result in increasing the principal. This would be a species of compound interest which, in many states, is illegal.

62. When a partial payment of a note is made, the date of payment and its amount are written on the back of the note, and this record of it is called an **indorsement.**

The following rule for partial payments, formulated by the United States Supreme Court, is used in most states:

63. United States Rule.—*Find the amount of the principal to the time when the payment, or the sum of the payments, is greater than the interest then due. From the amount subtract the payment or the sum of the payments, and treat the remainder as a new principal. Proceed in this manner to the date of settlement, and the last amount will be the sum still due.*

Find the difference between dates by putting down the year, the number of the month and the day of the month in the case of both minuend and subtrahend. Then count 30 days to the month, and subtract regularly as in the case of denominate numbers.

EXAMPLE.—

$1,200. *New York, Sept. 16, 1912*

On demand I promise to pay John Crawford, or order, Twelve Hundred Dollars, with interest at 6%, value received.

Edward G. Carson

Indorsements: Jan. 1, 1913, $120; May 7, 1913, $300; Dec. 22, 1913, $16; Sept. 19, 1914, $400. What is due Jan. 1, 1915?

SOLUTION.—

Principal	$1200
Interest from Sept. 16, 1912, to Jan. 1, 1913 (3 mo. 15 da.)	21
Amount	1221
First payment	120
New principal	1101
Interest from Jan. 1, 1913, to May 7, 1913 (4 mo. 6 da.)	23.12
Amount	1124.12
Second payment	300
New principal	824.12
Interest from May 7, 1913, to Sept. 19, 1914 (1 yr. 4 mo. 12 da.)	67.58
Amount	891.70
Sum of third and fourth payments	416.00
New principal	475.70
Interest from Sept. 19, 1914, to Jan. 1, 1915 (3 mo. 12 da.)	8.09
Amount due at time of settlement	$ 483.79

Ans

The third payment of $16 is less than the interest due at the time it was made; hence, according to the rule, it is added to the next payment of $400 and the interest is computed to the time of the fourth payment.

EXAMPLES FOR PRACTICE

Find the amount due at the time of settlement on each of the following:

1. *$2,000. Philadelphia, July 1, 1911*

One year after date, for value received, I promise to pay Wm. Gray, or order, Two Thousand Dollars, with interest at 6%.

Henry G. Brown

Indorsements: Dec. 16, 1911, $350; Mar. 1, 1912, $25; Oct. 25, 1912, $400; June 14, 1913, $275. Time of settlement, July 1, 1914.

2. A note for $3,000 at 6% dated Mar. 5, 1913, bore the following indorsements: Dec. 20, 1913, $400; Mar. 14, 1914, $60; Nov. 30, 1914, $360; July 15, 1915, $600. Time of settlement, Jan. 1, 1916.

Ans.—(1) $1,216.80; (2) $2,024.38.

EXACT INTEREST

64. It is the common practice to consider 360 days as a year in computing interest. This gives a result too great, since 1 day is not as much as $\frac{1}{360}$ of a year but is $\frac{1}{365}$ of a year. The United States government and some banks reckon interest by counting each day as $\frac{1}{365}$ of a year or $\frac{1}{366}$ of a year in case of a leap year. Interest thus computed is called exact interest.

Example.—Find the exact interest on $1,000 for 89 days at 6%.

Solution.—Interest for 1 yr.=6% of $1,000=$60.

Interest for 89 da.=$\frac{89}{365}$ of $60=$14.63. Ans.

65. Should one be obliged to calculate exact interest frequently, he should make a table giving the interest on various sums for various periods; as, for example, on each thousand dollars from one to nine for 1 day to 100 days.

EXAMPLES FOR PRACTICE

Find the exact interest on:

1. $10,000 for 123 days at 6%.
2. $12,800 for 168 days at 6%.
3. $6,400 for 213 days at 5%.
4. $22,800 for 2 yr. 73 da. at 4%.

5. $13,000 from Jan. 17, 1913, to Nov. 29, 1913, at 4½%.

Ans.—(1) $202.19+; (2) $353.49; (3) $186.74−; (4) $2,006.40; (5) $506.47—.

MERCHANTS' RULE

66. When the time from the date of a note or other obligation is less than a year, settlement is usually made by a method called the *merchants' rule.*

67. Rule.—I. *By the method of exact interest, find the amount of each of the several payments from the time each is made to the date of settlement.*

II. *Subtract the sum of these amounts from the amount of the obligation from its date to the time of settlement. The remainder will be the amount still due.*

Example.—Face of note, $2,000; rate, 6%; date of note, Dec. 31, 1914; time of settlement, Nov. 15, 1915. Indorsements: Mar. 10, 1915, $200; June 1, 1915, $300; Aug. 20, 1915, $400; Oct. 1, 1915, $500. What was due at time of settlement?

Solution.—

Principal		$2000
Interest on $2,000 for 319 da.		104.88
Amount		$2104.88
Amount of $200 for 250 da.	$208.22	
Amount of $300 for 167 da.	$308.24	
Amount of $400 for 87 da.	$405.72	
Amount of $500 for 45 da.	$503.70	
Sum of payments, with interest		$1425.88
Amount due at time of settlement		$679.00

The table on pages 31 and 32 can be used to find the difference of time between dates.

COMPOUND INTEREST

68. If the interest on a principal is added to the principal at regular intervals to form by each addition a new principal for the next interval, the resulting interest is called **compound interest.** Thus, if $100 be placed at compound interest at 6%, with the understanding that the interest is to be compounded annually, the principal will be $100 for the first year, $106 for the second year, $112.36 for the third year, etc.

69. Most savings banks allow compound interest, although in most states its payment cannot be legally enforced, even though it be specified in a contract.

The method of compounding interest is used principally in determining the amount a given principal will yield in a given time on the assumption that the interest is invested at the same rate as soon as it is due. Thus, in computing the probable returns on loans made for long periods, this method is used.

70. If the interest is to be compounded annually it is added to the principal at the end of each year. If it be compounded semiannually, one-half of the annual rate is taken as the rate, and the interest is added at the end of each period of 6 months; if quarterly, one-fourth the annual rate is taken; etc.

Example.—Find the amount of $800, at compound interest for 2 years at 6%, interest compounded semiannually.

Solution.—

$800	=prin. 1st 6 mo.
24	=int. 1st 6 mo.=$800×.03
$824	=prin. 2d 6 mo.
24.72	=int. 2d 6 mo.=$824×.03
$848.72	=prin. 3d 6 mo.
25.46	=int. 3d 6 mo.=$848.72×.03
$874.18	=prin. 4th 6 mo.
26.23	=int. 4th 6 mo.
$900.41	=amount for 2 yr. Ans.

71. When the time is given in years, months, and days, the interest is compounded for the greatest number of entire periods included in the time, and the simple interest on the last amount as the principal is found for the remaining time.

Thus, if the time in the preceding example were 1 year 8 months 20 days, $874.18 would be taken as the principal for the last 2 months 20 days, and the simple interest on this principal for 2 months 20 days would be computed. This interest, which is $11.66, added to $874.18 gives $885.84, the amount for 1 year 8 months 20 days.

72. Compound interest is calculated in actual business by means of a table. The accompanying compound-interest table shows the amount of $1 at several different rates, and for

COMPOUND-INTEREST TABLE

Showing the amount of $1, at various rates, compound interest, from 1 to 20 years, interest compounded annually

Yr.	1½ Per Cent.	2 Per Cent.	2½ Per Cent.	3 Per Cent.	3½ Per Cent.	4 Per Cent.
1	1.015000	1.020000	1.025000	1.030000	1.035000	1.040000
2	1.030225	1.040400	1.050625	1.060900	1.071225	1.081600
3	1.045678	1.061208	1.076891	1.092727	1.108718	1.124864
4	1.061364	1.082432	1.103813	1.125509	1.147523	1.169859
5	1.077284	1.104081	1.131408	1.159274	1.187686	1.216653
6	1.093443	1.126162	1.159693	1.194052	1.229255	1.265319
7	1.109845	1.148686	1.188686	1.229874	1.272279	1.315932
8	1.126493	1.171659	1.218403	1.266770	1.316809	1.368569
9	1.143390	1.195093	1.248863	1.304773	1.362897	1.423312
10	1.160541	1.218994	1.280085	1.343916	1.410599	1.480244
11	1.177949	1.243374	1.312087	1.384234	1.459970	1.539454
12	1.195618	1.268242	1.344889	1.425761	1.511069	1.601032
13	1.213552	1.293607	1.378511	1.468534	1.563956	1.665074
14	1.231756	1.319479	1.412974	1.512590	1.618695	1.731676
15	1.250232	1.345868	1.448298	1.557967	1.675349	1.800944
16	1.268986	1.372786	1.484506	1.604706	1.733986	1.872981
17	1.288020	1.400241	1.521618	1.652848	1.794676	1.947901
18	1.307341	1.428246	1.559659	1.702433	1.857489	2.025817
19	1.326951	1.456811	1.598650	1.753506	1.922501	2.106849
20	1.346855	1.485947	1.638616	1.806111	1.989789	2.191123

Yr.	5 Per Cent.	6 Per Cent.	7 Per Cent.	8 Per Cent.	9 Per Cent.	10 Per Cent.
1	1.050000	1.060000	1.070000	1.080000	1.090000	1.100000
2	1.102500	1.123600	1.144900	1.166400	1.188100	1.210000
3	1.157625	1.191016	1.225043	1.259712	1.295029	1.331000
4	1.215506	1.262477	1.310796	1.360489	1.411582	1.464100
5	1.276282	1.338226	1.402552	1.469328	1.538624	1.610510
6	1.340096	1.418519	1.500730	1.586874	1.677100	1.771561
7	1.407100	1.503630	1.605781	1.713824	1.828039	1.948717
8	1.477455	1.593848	1.718186	1.850930	1.992563	2.143589
9	1.551328	1.689479	1.838459	1.999005	2.171893	2.357948
10	1.628895	1.790848	1.967151	2.158925	2.367364	2.593742
11	1.710339	1.898299	2.104852	2.331639	2.580426	2.853117
12	1.795856	2.012197	2.252192	2.518170	2.812665	3.138428
13	1.885649	2.132928	2.409845	2.719624	3.065805	3.452271
14	1.979932	2.260904	2.578534	2.937194	3.341727	3.797498
15	2.078928	2.396558	2.759031	3.172169	3.642482	4.177248
16	2.182875	2.540352	2.952164	3.425943	3.970306	4.594973
17	2.292018	2.692773	3.158815	3.700018	4.327633	5.054470
18	2.406619	2.854339	3.379932	3.996019	4.717120	5.559917
19	2.526950	3.025600	3.616527	4.315701	5.141661	6.115909
20	2.653298	3.207136	3.869684	4.660957	5.604411	6.727500

periods from 1 to 20 years. More extensive tables can be purchased through bookdealers. Having the amount of $1 at any given rate and for any number of periods, we multiply it by the number of dollars in any given principal. The result will be the amount of that sum for the given time. If the original principal be subtracted from this amount, the remainder is the compound interest required.

Example.—What is the amount and the compound interest of $600 for 4 years at 6%, interest compounded annually?

Solution.—The amount of $1 for 4 years at 6% is, according to the table, page 26, $1.262477. The corresponding amount of $600 is

$600×1.262477=$757.49. Ans.

The interest is $757.49−$600=$157.49. Ans.

73. If the interest is computed semiannually, find by the table the amount for double the number of years at half the rate.

Example.—A young man deposited $1,000 in a savings bank. What sum would he have at the end of 10 years if the bank pays 4% interest per annum, and credits the interest semiannually?

Solution.—The rate for each half year is 2%; the number of interest periods is 20. Hence, the amount is equal to that for 20 years at 2%. According to the table, the amount of $1 for 20 years at 2% is $1.485947. Hence, the amount of $1,000 for the same time is $1,000×1.485947 =$1,485.95. Ans.

EXAMPLES FOR PRACTICE

Find the compound interest on the following amounts for the times mentioned, and prove the correctness of your result by using the table. Unless otherwise stated, the interest is to be compounded annually.

1. On $1,600 for 2 years at 8%.
2. On $1,280 for 5 years at 4%.
3. On $480 for 3 years at 8%, interest compounded semiannually.
4. On $3,000 for 2 years 11 months 12 days at 6%, interest compounded semiannually.

Ans.—(1) $266.24; (2) $277.32; (3) $127.35; (4) $571.72.

ADDITIONAL RULES OF INTEREST

74. In the problems in interest so far considered, the principal, the rate, and the time have been given, and the interest or the amount has been found. One may have occasion to compute the principal, the rate, or the time when the other terms necessary for the solution of the problem are known. No new principles are required, as the applications involve those of percentage.

75. Given the principal, the interest, and the time, to find the rate.

The interest for one year is obtained by multiplying the principal by the rate. Then, conversely, the rate is obtained by dividing the interest for 1 year by the principal. The interest for one year is equal to the given interest divided by the time in years.

EXAMPLE.—A pawnbroker charged $8 for a loan of $200 for $2\frac{1}{2}$ months. What rate of interest did he receive?

SOLUTION.— $8 for $2\frac{1}{2}$ months is equivalent to $8÷2.5=$3.20 a mo., or to $3.20×12=$38.40 a year.

Rate=38.40÷200=$.19\frac{2}{10}$, or $19\frac{2}{10}$%. Ans.

76. Given the principal, interest, and rate, to find the time.

It is evident that if the given interest is divided by the interest for 1 year, the time required in years will be obtained. Or if the given interest is divided by the interest for 1 day, the time in days will be obtained.

EXAMPLE 1.—In what time will the interest on $1,000 at 5% be $80?

SOLUTION.—The interest on $1,000 for 1 yr. at 5% is $50. The required time is, therefore, 80÷50=$1\frac{3}{5}$ yr. Ans.

EXAMPLE 2.—If the principal is $978.26 and the interest at 6% is $4.08, what is the time in days?

SOLUTION.—Interest on $978.26 for 60 da. at 6% is $9.78. Interest for 1 day is $9.78÷60=$.163.

Time=4.08÷.163=25 days. Ans.

77. Given the amount, time, and rate, to find the principal.

The amount for any time and rate can be obtained by multiplying the principal by the amount of $1 for the same time and rate. Consequently, the principal can be found by dividing the amount by the amount of $1 for the same time and rate.

Example.—What sum placed at interest for 1 year at 6% will amount to $1,000?

Solution.—The amount of $1 for 1 yr. at 6% is $1.06.

The principal=$1,000÷1.06=$943.396, or $943.40. Ans.

It is seen that the principal is, therefore, $\frac{100}{106}$ of the amount for this rate and time.

PRESENT WORTH

78. The **present worth** of an obligation is a sum such that, if it be put at interest at a specified rate for a given time, it will amount to the obligation.

Thus, if the specified rate is 5%, a debt of $105 due in 1 year is worth $100 *now,* since $100 placed at interest at 5% will in one year amount to $105.

The present worth may be found as shown in the solution of the example in the preceding article. In business practice it is customary when a debt is paid before it is due to deduct the interest; in equity, that is, in justice, the amount paid should be the present worth of the obligation. Usually, however, loans of this character are of such short duration that the discount is reckoned in the ordinary way.

Example.—A debt of $773, which has 9 mo. 20 da. yet to run, is discounted at 5%. What is: (*a*) the present worth? (*b*) the true discount? (*c*) the bank discount?

Solution.—(*a*) 9 mo. 20 da.=290 da., counting 30 days to the month. The interest on $1 for 9 mo. 20 da. at 5% equals $\frac{290}{360}$×.05=$.04028. The amount of $1 for the given time and rate=$1.04028.

Present worth=$773÷1.04028=$743.07. Ans.

(*b*) The true discount is the difference between the present worth and the amount of the obligation.

True discount=$773−$743.07=$29.93. Ans.

(*c*) The bank discount is $773×.04028=$31.14. Ans.

Proof.—Calculate the interest on $743.07 for 9 mo. 20 da. at 5%. It is $29.93; hence, the solution is correct.

79. To find the face of a note when the proceeds, time, and rate are given.

Sometimes one may be required to find what the face of a note should be in order that the proceeds may be equal to a given sum. The face multiplied by the proceeds of $1 equals the proceeds, hence the given proceeds divided by the proceeds of $1 will give the face of a note.

Example.—The proceeds of a note discounted at a bank for 45 days at 6% amounted to $1,488. What was the face of the note?

Solution.—Interest on $1 for 60 da. at 6%=$.01. Interest on $1 for 45 da. at 6%=$.0075.

Proceeds of $1 due in 45 days=$1—$.0075=$.9925. Face of note =$1,488÷$.9925=$1,499.24. Ans.

EXAMPLES FOR PRACTICE

Find the principal, when:

(*a*) Interest=$96, time=2 years, rate=4%.

(*b*) Interest=$131.25, time=2 years 6 months, rate=6%.

Find the time, when:

(*c*) Principal=$4,800, interest=$652, rate=6%.

(*d*) Principal=$680, interest=$163.20, rate=5%.

Find the rate, when:

(*e*) Principal=$2,875, time=4 years 7 months 6 days, interest=$529.

(*f*) Principal=$760, time=3 years 9 months 18 days, interest =$144.40.

Ans.—(*a*) $1,200; (*b*) $875; (*c*) 2 yr. 3 mo. 5 da.; (*d*) 4 yr. 9 mo. 18 da.; (*e*) 4%; (*f*) 5%.

1. A note for 60 days discounted at 4½% yields $81,815.33. What is its face? Ans. $82,433.58

2. Write in proper form a 60-day note payable at the Chemical Bank of New York, which, when discounted when the note is made, will yield at 5%, $7,850 proceeds. Ans. Face, $7,915.97

TABLE FOR FINDING THE NUMBER OF DAYS BETWEEN TWO DATES

80. The table on pages 31 and 32 may be used for finding the time between two dates, also for finding a date that occurs a certain number of days later or earlier than a given date. It will be observed that in the table there are twelve columns, one for each month of the year, and in each column there are as many lines as there are days in the month. We shall now illustrate the use of the table.

In the column headed Feb., we find, opposite 24 in the left-hand column, the numbers 55 and 310. The first number, 55, indicates that Feb. 24 is the 55th day of the year, and the

TABLE FOR FINDING THE NUMBER OF DAYS BETWEEN TWO DATES

Day of Month	Jan.		Feb.		March		April		May		June		Day of Month
1	1	364	32	333	60	305	91	274	121	244	152	213	1
2	2	363	33	332	61	304	92	273	122	243	153	212	2
3	3	362	34	331	62	303	93	272	123	242	154	211	3
4	4	361	35	330	63	302	94	271	124	241	155	210	4
5	5	360	36	329	64	301	95	270	125	240	156	209	5
6	6	359	37	328	65	300	96	269	126	239	157	208	6
7	7	358	38	327	66	299	97	268	127	238	158	207	7
8	8	357	39	326	67	298	98	267	128	237	159	206	8
9	9	356	40	325	68	297	99	266	129	236	160	205	9
10	10	355	41	324	69	296	100	265	130	235	161	204	10
11	11	354	42	323	70	295	101	264	131	234	162	203	11
12	12	353	43	322	71	294	102	263	132	233	163	202	12
13	13	352	44	321	72	293	103	262	133	232	164	201	13
14	14	351	45	320	73	292	104	261	134	231	165	200	14
15	15	350	46	319	74	291	105	260	135	230	166	199	15
16	16	349	47	318	75	290	106	259	136	229	167	198	16
17	17	348	48	317	76	289	107	258	137	228	168	197	17
18	18	347	49	316	77	288	108	257	138	227	169	196	18
19	19	346	50	315	78	287	109	256	139	226	170	195	19
20	20	345	51	314	79	286	110	255	140	225	171	194	20
21	21	344	52	313	80	285	111	254	141	224	172	193	21
22	22	343	53	312	81	284	112	253	142	223	173	192	22
23	23	342	54	311	82	283	113	252	143	222	174	191	23
24	24	341	55	310	83	282	114	251	144	221	175	190	24
25	25	340	56	309	84	281	115	250	145	220	176	189	25
26	26	339	57	308	85	280	116	249	146	219	177	188	26
27	27	338	58	307	86	279	117	248	147	218	178	187	27
28	28	337	59	306	87	278	118	247	148	217	179	186	28
29	29	336			88	277	119	246	149	216	180	185	29
30	30	335			89	276	120	245	150	215	181	184	30
31	31	334			90	275			151	214			31

number 310 indicates that there are 310 more days in the year after Feb. 24. Again, in the tenth line of the Aug. column, we find 222 and 143. These numbers show that Aug. 10 is the 222d day of the year and that there are 143 more days in the year after that date.

TABLE FOR FINDING THE NUMBER OF DAYS BETWEEN TWO DATES—(*Continued*)

Day of Month	July		Aug.		Sept.		Oct.		Nov.		Dec.		Day of Month
1	182	183	213	152	244	121	274	91	305	60	335	30	1
2	183	182	214	151	245	120	275	90	306	59	336	29	2
3	184	181	215	150	246	119	276	89	307	58	337	28	3
4	185	180	216	149	247	118	277	88	308	57	338	27	4
5	186	179	217	148	248	117	278	87	309	56	339	26	5
6	187	178	218	147	249	116	279	86	310	55	340	25	6
7	188	177	219	146	250	115	280	85	311	54	341	24	7
8	189	176	220	145	251	114	281	84	312	53	342	23	8
9	190	175	221	144	252	113	282	83	313	52	343	22	9
10	191	174	222	143	253	112	283	82	314	51	344	21	10
11	192	173	223	142	254	111	284	81	315	50	345	20	11
12	193	172	224	141	255	110	285	80	316	49	346	19	12
13	194	171	225	140	256	109	286	79	317	48	347	18	13
14	195	170	226	139	257	108	287	78	318	47	348	17	14
15	196	169	227	138	258	107	288	77	319	46	349	16	15
16	197	168	228	137	259	106	289	76	320	45	350	15	16
17	198	167	229	136	260	105	290	75	321	44	351	14	17
18	199	166	230	135	261	104	291	74	322	43	352	13	18
19	200	165	231	134	262	103	292	73	323	42	353	12	19
20	201	164	232	133	263	102	293	72	324	41	354	11	20
21	202	163	233	132	264	101	294	71	325	40	355	10	21
22	203	162	234	131	265	100	295	70	326	39	356	9	22
23	204	161	235	130	266	99	296	69	327	38	357	8	23
24	205	160	236	129	267	98	297	68	328	37	358	7	24
25	206	159	237	128	268	97	298	67	329	36	359	6	25
26	207	158	238	127	269	96	299	66	330	35	360	5	26
27	208	157	239	126	270	95	300	65	331	34	361	4	27
28	209	156	240	125	271	94	301	64	332	33	362	3	28
29	210	155	241	124	272	93	302	63	333	32	363	2	29
30	211	154	242	123	273	92	303	62	334	31	364	1	30
31	212	153	243	122			304	61			365	0	31

Example 1.—A note bearing interest was made on Nov. 15 and paid Feb. 1 of the following year. How many days did the note draw interest?

Solution.—In this case we find the number of days in the year after Nov. 15 and add this to the number of days to be counted in the next year. In the fifteenth line in the column headed Nov. the second number is 46. Feb. 1 is the 32d day of the year. Hence, from Nov. 15 to Feb. 1 is 46+32=78 days. Ans.

In case the period considered includes the last day of February of a leap year—that is, February 29—the additional day must be considered.

Example 2.—In a leap year, find the days from Jan. 28 to May 4.

Solution.—Jan. 28 is the 28th day and May 4 is the 124th day of the common year. The difference in time is 124−28=96 days. But as Feb. 29 has not been counted, we add one day more and obtain 96+1 =97 days. Ans.

STOCKS AND BONDS

81. If work involving a large expenditure of money is to be undertaken, it is usual to organize a company and procure a charter under the laws of some state. The chartered company then issues shares, which are sold to persons who have money to invest and are willing to incur the chances of loss. These shares are known as stock.

The advantage of having a charter is that a chartered company can do business just as an individual—that is, it may sue or be sued for debts, enter into contracts, etc. Moreover, the shareholders, in case the business is unprofitable, are, as a rule, liable for the debts of the company only to the amount of their shares. The exception may be noted that owners of stock in national banks are responsible for an amount equal to the face value of their stock in addition to their shares. The members of an unincorporated concern may be compelled to pay all of its debts.

82. The **par value** of shares is the amount—usually $25, $50, or $100—specified in the certificates issued to the subscribers of a corporation.

83. When a company gains in its business, it pays its shareholders part of its profits, called **dividends.**

Dividends are *declared* quarterly, semiannually, or annually, and are usually paid at the general office of the company.

84. If the business is conducted at a loss, the shareholders may be required to make good the loss. Such payments are called **assessments.**

85. When the dividends on stock are considered high, the stock will usually sell above par; if no dividends are paid or if they are small, the stock, as a rule, will sell below par. The price of stock is, of course, influenced by other considerations than present dividends. The stability and the risk of the business, the assets of the company, and the outlook for the future are factors that affect the price.

86. In some corporations the stock is divided into two classes, called **preferred** and **common.** Dividends on preferred stock are payable first and do not exceed a fixed rate. The dividends on common stock depend on the profits of the business after the dividends on the preferred stock are provided for.

87. A **bond** is a written obligation under seal to pay a certain sum at a specified time.

Bonds are issued by national and state governments, cities, counties, towns, and incorporated companies in order to provide money for current or extraordinary expenses, or for such improvements as may be desired. The bonds are secured by the property of those who issue them, and bear interest payable quarterly, semiannually, or annually.

88. Registered bonds are numbered, and the names of their purchasers are recorded. When registered bonds are sold, the transfer must be recorded on the books of the company that issued them. Sometimes bonds have small forms called **coupons** attached, stating the amount of interest due at certain times. These coupons may be cut off and exchanged for money at the general office of the company or deposited in any bank for collection. Owing to the ease with which they may be transferred from one party to another, unregistered bonds usually sell for a slightly higher price than registered ones. Corporation bonds are, as a rule, issued in denominations of $1,000 or multiples thereof. Some bonds are issued in denominations of $100 and of $500.

89. Bonds are usually designated by the interest they bear, or by the time when they are payable. Thus, "U. S. 4's, 1932" are bonds of the United States government bearing interest at 4%, and payable in 1932.

90. A **stock broker** is a person whose business consists in buying and selling bonds or stocks for others. His compensation is a certain per cent. of the *par value* of the stocks bought or sold. The compensation of a broker is called **brokerage.**

91. The price quoted in stock market reports is that for $100 par. For example, when Canadian Pacific is quoted at 125, each $100 par sells for $125. In some localities and in some classes of stock, prices are quoted as the price per share. For example, the par value of a share of Pennsylvania Railroad Company stock is $50. The New York market may quote the stock at 130, which means that $130 is the price for each $100 par, or for two shares. In the Philadelphia market, the price would be quoted at $65, which is the price per share. Information concerning issues of stock and related matters can be obtained from stock brokers.

92. In this Section the *par value* of stocks is to be understood as 100, unless some other value is given. Whatever may be the market price of stocks and bonds, brokerage is calculated on their par value.

EXAMPLE 1.—Find the cost of 480 shares of Canadian Pacific stock bought at $123\frac{1}{2}$, if the brokerage is $\frac{1}{8}$%.

SOLUTION.—The cost of each share is $123.50 and the brokerage is $\frac{1}{8}$% of $100=$\frac{1}{8}$ of $1=$.12$\frac{1}{2}$.

$$\begin{array}{rr} 480\times\$123.50= & \$59280.00 \\ 480\times\$.12\tfrac{1}{2} = & 60.00 \\ \hline & \$59340.00 \text{ Ans.} \end{array}$$

Or, cost per share including brokerage=$123.50+$.12$\frac{1}{2}$=123.62\frac{1}{2}$.

480×123.62\frac{1}{2}$=$59,340.00. Ans.

Example 2.—What will be the cost with accrued interest of a 5% $1,000 bond at 103, including brokerage, if the interest was paid last on July 1 and the bond is sold on October 1?

Solution.—

One $1,000 bond at 103=	$1030.00
Interest on $1,000 for 3 mo. at 5%=	12.50
	$1042.50 Ans.

The former owner would be entitled to the interest of the bond from the date of the last payment of interest until the date of sale. When the quoted price includes the interest due, bonds are said to be sold *flat*.

Example 3.—How many shares of bank stock selling at 56 can be bought for $89,700, if the brokerage is $\frac{1}{8}$%? The par value of the shares is $50.

Solution.—The cost of 1 share of stock at the market price is $56; the brokerage per share is $.00\frac{1}{8}\times\$50=\$0.06\frac{1}{4}$. Therefore, total cost of 1 share=$\$56+\$0.06\frac{1}{4}=\$56.06\frac{1}{4}$, and the number of shares bought=$\$89,700\div\$56.06\frac{1}{4}=1,600$ shares. Ans.

93. Rule.—I. *To find the cost of any number of shares of stock, multiply the sum of the market price per share and the brokerage by the number of shares, and the product will be the cost.*

II. *To find the number of shares that can be bought for a given sum, divide the given sum by the cost of one share, including the brokerage, and the quotient will be the number of shares.*

Example 1.—How much must be invested in railroad stock that pays a quarterly dividend of $2\frac{1}{2}$%, in order to have an income of $4,000, if it is bought at $104\frac{1}{2}$, brokerage being $\frac{1}{8}$%?

Solution.—The expression "a dividend of $2\frac{1}{2}$%" means $2\frac{1}{2}$ per cent. on the *par* value of the stock. Hence, since the *quarterly* dividend is $2\frac{1}{2}$%, the *annual* income per share of $100 will be $10. Consequently, to obtain an annual income of $4,000, there must be bought $4,000\div10=400$ shares; then each share will cost $\$104\frac{1}{2}+\$\frac{1}{8}$, and their total cost will be 400 times as much, or $\$104\frac{5}{8}\times400=\$41,850$. Ans.

Example 2.—What per cent. is realized by buying 4% bonds at $89\frac{7}{8}$, brokerage being $\frac{1}{8}$%?

Solution.—Since a bond of $100 denomination costs $\$89\frac{7}{8}+\$\frac{1}{8}=\$90$, and each $100 bond yields $4 annual income, the per cent. realized will be found by dividing $4 by the entire cost of one $100 bond. Hence,

$\$4 \div (\$89\frac{7}{8} + \$\frac{1}{8}) = .04\frac{4}{9} = 4\frac{4}{9}\%$. Ans.

In the preceding example, consideration has not been taken of the fact that the purchaser of the bond will receive par when the bond becomes due if he holds it until that date. He pays 90 and will receive 100, hence he will realize 10% profit on the par value of the bond in addition to the interest. The 10% profit will not be realized until maturity, so the actual yield depends on the date the bond is due. The method of computing the true yield on bonds is rather complex, but tables are published from which rates may be determined.

94. Rule.—I. *To find the investment that will yield a given income, divide the income by the gain from one share, and the quotient will be the number of shares that must be bought; then multiply the cost of one share by the number of shares, and the product will be the investment.*

II. *To find the rate per cent. of income from money invested in stocks or bonds, divide the gain yielded by one share by the cost of a share, and multiply the quotient by 100.*

EXAMPLES FOR PRACTICE

1. What must be paid for 128 shares of U. S. Steel preferred stock at $128\frac{1}{4}$, brokerage $\frac{1}{8}\%$? Ans. $16,432

2. How many shares of Union Gas Co. stock at $98\frac{3}{4}$ can be bought for $39,550, the brokerage being $\frac{1}{8}\%$? Ans. 400 shares

3. How much will 68 $1,000 U. S. 4% bonds of 1925 cost at $116\frac{1}{2}$, brokerage being $\frac{1}{8}\%$? Ans. $79,305

4. The cost of some railroad stock was $18,150, for which a man paid $\$137\frac{3}{8}$ per share, and $\frac{1}{8}\%$ brokerage. How many shares did he buy? Ans. 132 shares

5. Find the cost of 240 shares of mining stock at $98\frac{3}{8}$, brokerage being $\frac{1}{4}\%$. Ans. $23,670

6. How much must be paid for 5% city bonds to yield an annual income of $1,250, if they cost $104\frac{7}{8}$, brokerage $\frac{1}{8}\%$? Ans. $26,250

7. Bank stock that pays an annual dividend of 10% is bought for $109\frac{7}{8}$, brokerage $\frac{1}{8}\%$. What per cent. is realized by investing in it? Ans. $9\frac{1}{11}\%$

8. How much more is the rate per cent. of income on 8% stock bought at $119\frac{7}{8}$ than on 6% stock bought at $107\frac{7}{8}$, brokerage in each case being $\frac{1}{8}$%? Ans. $1\frac{1}{9}$%

9. How much better is a gain of 20% on an investment at 80 than a gain of 18% on an investment at 90? Ans. 5%

EXCHANGE

95. The subject of **exchange** treats of the methods of paying debts in a distant place without transmitting money. Exchange between different parts of the *same* country is **domestic exchange,** and between *different* countries is **foreign exchange.**

DOMESTIC EXCHANGE

96. The payment of debts in a distant place may be made by means of a post-office money order, an express money order, a personal check, or a draft. At all post offices of importance and at express company offices one may obtain forms giving the rates for money orders and directions for use. If a personal check is used a delay of several days may be caused, as most business houses hold an order for goods until collection has been made on the check. Some business houses will refuse to accept a personal check. The most general method of making payments at a distance is by draft.

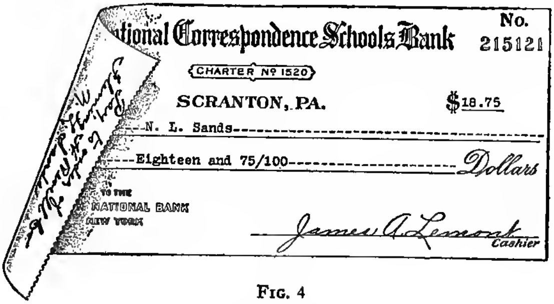
97. The Bank Draft.—A bank draft, which is an order issued on one bank by another, is one convenient means of transmission of credit. Banks in the smaller cities usually have money on deposit in the banks of large cities such as New York or Chicago, subject to draft at any time.

To illustrate the method of using a bank draft, let us suppose that N. L. Sands, of Scranton, Pa., wishes to buy some books from the publishers, Fleming H. Revell Company, New York City, the catalog price of which is $18.75, and decides to send a bank draft. Mr. Sands will apply to his bankers in Scranton for a New York draft, drawn in his own favor for $18.75. They will issue this for a few cents and charge the entire amount, say $18.85, to his bank account. Many banks do not charge their depositors for issuing bank drafts. Mr. Sands indorses the draft to the publishers by writing on the back of it,

Pay to order of
Fleming H. Revell Co.
N. L. Sands

He then incloses it in the letter with his order. Fig. 4 shows the bank draft drawn in Mr. Sands' favor with his indorsement on its back. This is a safe and economical means of paying a bill at a distance.

98. Collection by Draft.—A draft is a written order from one party to another party to pay a certain sum of money to a third party. These three parties are always connected with every draft transaction; namely, the party who makes, or signs, the draft, called the *drawer;* the party who is to pay it, or on whom it is drawn, called the *drawee;* and the party to whom the money is to be paid, called the *payee.* A draft may be drawn *at sight* or *on demand,* and it is then expected that it will be paid by the drawee when it is presented to him; or, it may be written at two, three, or any number of days' sight, which means that the drawee is allowed that many days after presentation in which to pay it. The name of the *drawee* is usually written near the lower left-hand corner of the face of the draft. The *payee* is usually a bank with which the drawer does business. The drawee accepts the draft, that is,



tional Correspondence Schools Bank No. 215121

CHARTER Nº 1520

SCRANTON, PA. $18.75

N. L. Sands

Eighteen and 75/100 Dollars

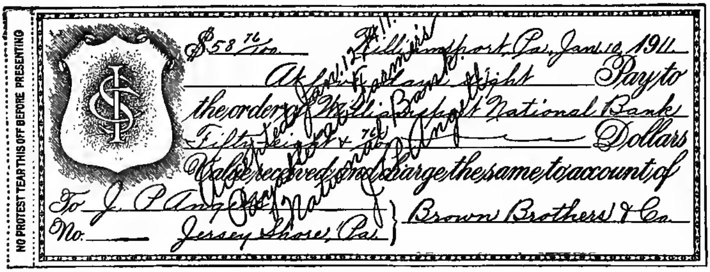
TO THE
NATIONAL BANK
NEW YORK

James A. Lemont
Cashier

FIG. 4

promises to pay it, by writing or stamping on its face the word "Accepted," together with the current date and his signature.

Suppose for example that Brown Brothers & Co., wholesale grocers, of Williamsport, Pa., have shipped $58.76 worth of groceries to J. P. Angell, retail grocer, Jersey Shore, Pa., and that they wish to collect this amount. They go to the Williamsport National Bank, where they do their banking business, and request it to forward a draft on Mr. Angell that they, Brown Brothers & Co., have already filled in. The Williamsport bank indorses the draft to a Jersey Shore bank and forwards it to that bank with instructions to collect it. The Jersey Shore bankers promptly present the draft to Mr. Angell by messenger or notify him by telephone or perhaps by mail that they have received it for collection. Mr. Angell is not compelled to pay his debt in this way; but, if he refuses to do so, he is liable to lower his credit with the banks and business men in his community. Therefore, he *accepts* the draft by writing his name across its face, usually in red ink, pays the money to the Jersey Shore bankers within the five days named, and receives the draft, marked paid, as his receipt. They remit to the Williamsport bankers, who credit to the account of Brown Brothers & Co. the $58.76 less a small commission that they charge for making the collection. In this transaction Brown Brothers & Co. are the *drawers* of the draft, the Williamsport National Bank is the *payee*, and J. P. Angell is the *drawee*. The draft, with Mr. Angell's acceptance written across its face, would appear as in Fig. 5.



NO PROTEST TEAR THIS OFF BEFORE PRESENTING

$58 76/100 Williamsport, Pa., Jan. 10 1911

At five days sight Pay to

the order of Williamsport National Bank

Fifty-eight + 76/100 Dollars

Value received and charge the same to account of

To J. P. Angell } Brown Brothers & Co.

No. —— Jersey Shore, Pa.

Accepted Jan. 12 '11 Farmers National Bank J. P. Angell

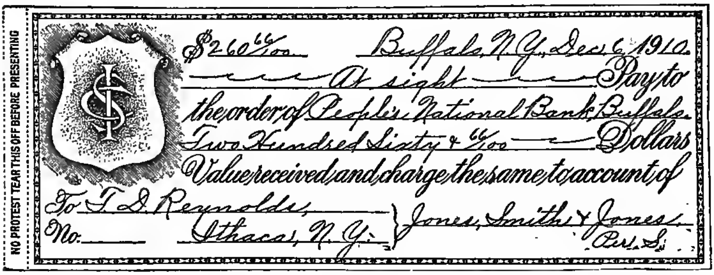
Fig. 5

When a draft is accepted it becomes in effect a promissory note.

The student should carefully observe how the form is filled in. The wording of the last line of the form may be misunderstood. The drawee, Mr. Angell, is directed to charge the amount of the draft against the drawer, Brown Brothers & Co. The name of the drawee follows the word "To" and that of the drawer is written on the right, as shown.

99. The Bill of Lading.—A bill of lading is a printed contract, furnished free by railroad and steamboat companies, on which the shipper, or *consignor,* is required to write the name and address of the person to whom the shipment is to be sent, called the *consignee,* and an itemized statement of the number of pieces, their weight, a brief description of the goods shipped, and his own name as shipper. Always two, and sometimes three, copies of the bill of lading are made at the same time and for each shipment. The first copy is called a *straight bill of lading.* It is signed by the railroad agent and returned to the shipper, who usually mails it to the consignee, so that he may take it to the railroad agent at the point of destination as proof that the goods are for him. The second, or duplicate, copy, called the *shipping order,* is signed by the shipper and retained by the railroad agent as his directions for shipping the goods. The triplicate copy is called a *memorandum* and is retained by the consignor as his receipt.

100. The Sight Draft With Bill of Lading.—Suppose a retail storekeeper, T. D. Reynolds, of Ithaca, N. Y., orders goods from Messrs. Jones, Smith & Jones, wholesalers, of Buffalo, N. Y., amounting to $260.66, and asks them to ship the goods by freight, collect on delivery. The steps in the transaction would be about as follows: Jones, Smith & Jones deliver the goods to the railroad company and receive from the freight agent a *bill of lading,* showing that the goods are consigned to themselves or to their order. The goods are not consigned to Mr. Reynolds. Jones, Smith & Jones make a sight draft on Mr. Reynolds for $260.66, attach it to the bill of lading, and hand both to their bankers in Buffalo, who forward it to a bank in Ithaca with instructions to deliver the bill of lading to Mr. Reynolds when he pays the attached draft. Fig. 6 shows the sight draft on T. D. Reynolds that went to the Ithaca bank with the bill of lading. Messrs. Jones, Smith & Jones promptly mail to Mr. Reynolds an itemized bill for the goods and notify him that their draft for $260.66 has been forwarded through their bankers. Mr. Reynolds calls at the Ithaca bank, pays the draft, receives the bill of lading, presents it to the railroad freight agent, and is allowed to take the goods.



NO PROTEST TEAR THIS OFF BEFORE PRESENTING

$260 66/100 Buffalo, N. Y., Dec 6 1910.
At sight Pay to
the order of People's National Bank, Buffalo.
Two Hundred Sixty & 66/100 Dollars
Value received and charge the same to account of
To T. D. Reynolds, } Jones, Smith & Jones.
No. Ithaca, N. Y. } Per. S.

FIG. 6

101. To find the cost of a bank draft.

The charge, called exchange, made by a bank for a draft is usually stated as a percentage or as the rate per $1,000.

EXAMPLE 1.—What will a sight draft of $2,400 in San Francisco cost in New York when the charge is $\frac{1}{2}$%?

SOLUTION.—Exchange=$\frac{1}{2}$% of $2,400=$\frac{1}{2}$ of $24=$12.

Total cost=$2,400+$12=$2,412. Ans.

EXAMPLE 2.—Mr. Rodriguez, of New Orleans, wishes to pay $24,600 to a New York merchant. What will a draft cost him when exchange is $1 per $1,000?

SOLUTION.—The exchange equals 1 thousandth part of the face value of the draft; hence, the charge may be obtained by pointing off three places in $24,600. The charge is therefore $24.60, and the entire cost of the draft equals $24,600+$24.60=$24,624.60. Ans.

102. To find the proceeds of sight and time drafts.

The drawer of a draft may not wish to wait for the collection to be made and may sell it. The purchaser may or may not give face value for it. If sold below its face value it is said to be sold at a discount, and if above it, at a premium. A time draft is discounted in the same manner as a promissory note. The purchaser may also deduct charges sufficient to cover cost of collection.

Example 1.—A wholesale dealer sold a sight draft on a customer for $5,800 at ¾% discount. What were the proceeds?

Solution.— Face value of draft=$5800.00
Discount=¾% of $5,800= 43.50
Proceeds=$5756.50 Ans.

Example 2.—A furniture manufacturer of Grand Rapids deposited in a bank a 30-day draft for $28,000 on a Memphis dealer. How much credit would he receive for the draft if no charges were made to cover collection? Assume the interest rate to be 6%.

Solution.— Face value of draft=$28000
Discount on $28,000 for 30 da. at 6%= 140
Proceeds=$27860 Ans.

103. A commercial draft may be sold at a premium or a bank draft may be purchased at a discount when there is a tendency of money to move strongly in one direction. Suppose, for example, that the San Francisco banks owe considerably more to the New York banks than the latter owe in return. To adjust the difference, money would be shipped to New York. It is possible that a New York bank would, under such circumstances, sell a draft on San Francisco at a discount, and that in San Francisco a holder of a draft on New York could sell it at a premium. Such drafts would tend to lessen the shipment of currency.

In some states 3 days of grace are allowed on sight and time drafts. If one needs to know at any time whether grace is allowed in any particular state he can obtain the information from a bank.

104. To find the face of a draft when the proceeds are known.

Example.—The proceeds of a sight draft sold at ¾% discount are $2,977.50. What is its face value?

Solution.—The face can be found, when the proceeds are known, in a manner similar to that used in finding the face of a note when the proceeds are known.

Proceeds of $1=$1.00−.00¾=$.99¼
Face=$2,977.50÷.9925=$3,000. Ans.

105. **Rule I.**—*To find the cost of a sight draft, find the premium or the discount. The sum of the face of the draft and the premium, or the difference between the face of the draft and the discount, will be the cost of the draft.*

106. **Rule II.**—*To find the cost of a time draft, find the proceeds of the face of the draft for the time the draft has to run. Find, also, the premium or the discount, on the face of the draft. The sum of the proceeds and the premium, or the difference between the proceeds and the discount, will be the cost of the draft.*

If days of grace are allowed in the state where the draft is payable, find the proceeds for three days more than the time the draft has to run, and then proceed as before.

107. **Rule III.**—*To find the face of a draft when the cost or the proceeds is given, divide the cost of the draft by the cost of $1.*

EXAMPLES FOR PRACTICE

Solve the following examples:

1. Find the cost of a sight draft for $1,876 at: (*a*) $1\frac{1}{4}$% premium; (*b*) $\frac{1}{2}$% discount.
Ans. { (*a*) $1,899.45
(*b*) $1,866.62

2. The face of a sight draft is $7,875.56, and the premium is $\frac{4}{5}$%. Find its cost. Ans. $7,938.56

3. How much will it cost to pay, by a sight draft on San Francisco, a bill of $7,528, when exchange is at $1\frac{1}{2}$% discount? Ans. $7,415.08

4. A man paid $484.72 for a 60-day draft, premium being $1\frac{1}{4}$%, and money worth 6% interest. What was the face of the draft?
Ans. $483.51

5. If a draft payable 30 days after sight costs $2,800 when discount is $\frac{3}{4}$% and money worth 6%, what is its face? Ans. $2,835.44

6. Find the face of a sight draft costing $1,200, when exchange is at $1\frac{3}{8}$% discount. Ans. $1,216.73

FOREIGN EXCHANGE

108. A draft on a foreign country is usually known as a bill of exchange. Foreign bills can be used in the same manner as ordinary drafts. The face of the draft is expressed in the currency of the country on which the draft is drawn. The price of exchange fluctuates according to the balance of trade, the loans and investments in foreign countries, and other conditions that tend to cause a movement of money from one country to another. The Secretary of the Treasury of the United States issues at certain periods a statement showing the value of foreign coinage in dollars and cents.

109. The daily papers of our commercial cities give quotations showing the rates of exchange from day to day. One of these follows:

Sterling exchange was again weak and lower. Continental exchange was also lower. Rates are: Long bills, $\$4.81$ @ $\$4.82\frac{3}{4}$; sight drafts, $\$4.84\frac{3}{4}$ @ $\$4.85$, and cable transfers, $\$4.85\frac{1}{4}$ @ $\$4.85\frac{1}{2}$. Francs are quoted at $5.21\frac{7}{8}$ for long and 5.20 for short; reichsmarks, $94\frac{1}{2}$ @ $94\frac{9}{16}$ for long and $95\frac{1}{8}$ @ $95\frac{3}{16}$ for short; guilders, $39\frac{7}{8}$ @ $39\frac{15}{16}$ for long and 40 @ $40\frac{1}{16}$ for short.

NOTE.—A reichsmark (mark of the empire) is the same as a mark, about $23\frac{3}{4}$ cents. The exchange value of 4 marks, or reichsmarks, is given in commercial quotations in the daily newspapers.

Sterling exchange is Bills of Exchange payable in English money called Pounds Sterling. Long bills are those payable 30, 60, 90, or more days after being received. Short bills are those payable from sight to 30 days after being received. Sight drafts are payable at sight, that is, as soon as received. Long bills will evidently sell for less than those drawn on sight.

Edward Howe wishing to pay a debt in England may buy from Smith, Jones & Co. a draft like the following:

New York, Oct. 1, 1912

Exchange for £820-12-6 sterling.

At sight pay to Edward Howe, or order, the sum of Eight Hundred Twenty Pounds £820-12-6 sterling.

Value received, and charge to the account of

Smith, Jones & Co.

To Baring Bros. & Co.
London, England.

Example 1.—Find the cost of the foregoing draft in New York when exchange on London is \$4.84¾.

Solution.— £820-12-6=£820 12s. 6d.=£820.625, since 6d.=$\frac{6}{12}$=½s., and 12s. 6d. =12½s.=$\frac{12\frac{1}{2}}{20}$=£.625. \$4.8475×820.625=\$3,977.98. Ans.

Example 2.—What must be paid for a draft on Paris of 8,000 francs, when \$1 is quoted at 5.21½?

Solution.— 1 franc=\$1÷5.215;
(\$1÷5.215)×8,000=\$1,534.04. Ans.

110. Rule.—*To find the cost of a draft upon a foreign country, multiply the quoted value of a foreign monetary unit by the given number of such units.*

EXAMPLES FOR PRACTICE

Solve the following examples:

1. Find the cost of a draft on London, at 60 days' sight, for £987 16s., exchange being \$4.82¾. Ans. \$4,768.60

2. When exchange on Paris is quoted at 5.23, what must be paid for a sight draft for 2,800 francs? Ans. \$535.37

3. I bought a long draft on Berlin for 8,425 reichsmarks when exchange was quoted at 94¾ per 4 reichsmarks. What did it cost? Ans. \$1,995.67

4. What must be paid for a draft on Amsterdam for 8,000 guilders, exchange being 40$\frac{1}{16}$? Ans. \$3,205

DUTIES

111. Duties, or **customs,** are taxes levied by governments on imported goods for the purpose of producing revenue and for the protection of home industries.

112. There are two kinds of duties: *ad valorem* and *specific*. An **ad valorem** duty is estimated at a certain per cent. of the market value of the goods in the country from which they are imported; as, silks 50%, musical instruments 15%, etc. The market value of the goods is the invoice value after deducting discounts and before extra charges, such as commission, freight, boxing, etc., are added.

113. A **specific** duty is a duty levied on imported goods according to the weight, measurement, or number of the articles, without reference to their value; as, wheat 15 cents per bushel, coal 75 cents per ton, etc. Some kinds of merchandise are subject to both ad valorem and specific duties. In computing specific duties, the long ton of 2,240 pounds and the hundredweight of 112 pounds are used.

114. An **invoice** is an itemized statement of merchandise shipped. It contains the names of purchaser and seller, a description of the quality and quantity of the goods, prices, and incidental charges. Invoices are made out in the weights and measures and the currency of the country from which the goods are imported. Thus, the price and cost of goods imported from Germany would be given in *marks;* from France, in *francs;* from England in *£ s. d.*

115. The following table gives the monetary units of leading foreign nations and their equivalents in United States money. These rates are proclaimed each year by the Secretary of the Treasury, and are used in Custom House computations:

Country	Monetary Unit	Value in U. S. Gold
Canada	Dollar = 100 cents	$1.00
Great Britain	Pound = 20 shillings	4.8665
France Belgium Switzerland	Franc = 100 centimes	.193
Italy	Lira = 100 centesimi	.193
Spain	Peseta = 100 centimes	.193
Germany	Mark = 100 pfennige	.238
Denmark Norway Sweden	Crown = 100 öre	.268
Japan	Yen = 100 sen	.498

116. Before computing duties the following allowances are made: **Tare,** a deduction for the weight of boxes or crates; **leakage,** an allowance for loss of liquids imported in barrels or casks; and **breakage,** an allowance for loss of liquids imported in bottles. The **net quantity** is what remains after deducting tare, leakage, or breakage.

117. Ad valorem duties are computed by the rules of percentage; the net invoice price is regarded as the *base,* the ad valorem duty as the *percentage,* and the rate of duty as the *rate.*

Duties are not computed on fractions of a dollar; if the cents are less than 50 they are rejected; if more, they are counted as a dollar.

118. Rule.—*To find the ad valorem duty, reduce the net invoice price to United States money, if necessary, deduct allowances, and multiply the remainder (expressed in even dollars) by the rate of ad valorem duty.*

To find the specific duty, multiply the net quantity by the rate of specific duty per unit of quantity.

Example 1.—What is the duty on an invoice of silks valued at 24,360 francs, the ad valorem rate being 60%?

Solution.—According to Art. **115**, 24,360 francs=24,360×.193 =$4,701.48. Duty=$4,701×.60=$2,820.60. Ans.

The 48 cents is rejected, being less than 50 cents. (Art. **117.**)

Example 2.—What is the duty on 820 gallons of brandy at $1.50 per gallon, leakage 3%?

Solution.— Leakage=820×.03=24.6 gal.
Net quantity =820—24.6=795.4 gal.
Duty=795.4×1.50=$1,193.10. Ans.

EXAMPLES FOR PRACTICE

What is the ad valorem duty on an importation invoiced at
(*a*) £430 12s. 4d., allowing 5% breakage, rate of duty 40%?
(*b*) 36,750 lira, allowing 2% for tare, rate of duty 24%?
(*c*) 9,264 marks, rate of duty 85%?

What is the specific duty on an importation of
(*d*) 60 dozen bottles of wine at $3 per dozen, breakage 10%?
(*e*) 125 gross of empty bottles, breakage 6%, duty 10 cents per dozen?
(*f*) 3 T. 6 cwt. of iron castings at $\frac{3}{4}$ cent per pound?

Ans.—(*a*) $796.40; (*b*) $1,668.24; (*c*) $1,874.25; (*d*) $162; (*e*) $141; (*f*) $55.44.

1. An importation of musical instruments from Germany is valued at 13,670 marks. What is the duty at 17½% ad valorem? Ans. $569.28

2. An importer buys French silks at $1.80 per yard and pays a duty of 35% ad valorem, and $.60 per yard specific. At what price per yard must the silk be sold to yield a profit of 25% on the cost? Ans. $3.79

3. What is the duty at 65%, upon a consignment of 1,350 dozen kid gloves invoiced at 115 francs per dozen? Ans. $19,475.95

EQUATION OF PAYMENTS

119. Suppose that Mr. Ellison, a dealer, owes a wholesale house a bill of $100 payable June 10 and another bill of $100 payable June 30. It is evident that he may pay the entire amount on June 20, a date midway between June 10 and June 30, without paying interest or receiving discount, for he will owe on June 20, 10 days' interest on the first bill and he will be entitled to 10 days' discount on the second bill. The process of finding the equitable time when payment of several sums may be made in one payment is called **equation of payments.** The date at which several bills may be equitably paid is called the equated time of payment.

EXAMPLE.—A man owes $250 due March 1, $300 due April 20, $450 due May 5, and $500 due June 25. When is the equated time of payment?

SOLUTION.—We first find the interest that is due on some date, preferably the latest due date. This date is called the *focal date.* The latest due date is June 25. From March 1 to June 25 is 116 da., hence we compute the interest of $250 for 116 da. It does not matter what rate of interest is taken in finding the equated time, but we shall assume the rate to be 6% and use the 60-day method in calculating the interest. The interest on $250 for 60 days is $2.50, and for 116 days is $4.83.

From April 20 to June 25 is 66 da.; the interest on $300 for 66 da. at 6% is $3.30. From May 5 to June 25 is 51 da.; the interest on $450 for 51 da. is $3.83. No interest is computed on $500, since this sum is due on June 25. The total interest is $11.96; the total amount due is $1,500. We now proceed to find at what earlier date $1,500 may be paid so that no interest may be due.

The interest on $1,500 for 1 da. at $6\%=\frac{\$15.00}{60}=\$.25$

The interest on $1,500 will amount to $11.96 in $\frac{11.96}{.25}=48$ da.

The account may be settled 48 da. earlier than June 25, that is, on May 8. May 8 is, therefore, the equated time of payment.

The arithmetical work should be arranged as follows:

DUE DATE	AMOUNT	TIME	INTEREST
Mar. 1,	$250	116 da.	$4.83
Apr. 20,	300	66 da.	3.30
May 5,	450	51 da.	3.83
June 25,	500	no da.	.00
	$1500		$.25) $11.96
			48 da., very nearly

$$\frac{\$15.00}{60} = \$.25$$

Equated time is 48 da. earlier than June 25, or May 8. Ans.

EXAMPLES FOR PRACTICE

Solve the following examples:

1. A owes B $500 due in 8 months, and $900 due in 4 months. When may he equitably pay B both debts in one payment? Ans. 5 mo. 13 da.

2. Find the equated time for paying $400 due May 10, $500 due June 20, $900 due July 30, and $1,000 due Aug. 15. Ans. July 17

3. Tefft, Weller & Co. sold to E. King & Co. goods as follows: on June 15, $2,500 on 30 days' credit, and, on June 30, $3,600 on 20 days' credit. Find the equated date of payment. Ans. July 18

4. What is the equated time for the payment of three notes: one for $600, dated Aug. 9, 1912, for 3 months; the second for $800, dated Oct. 1, 1912, for 2 months; the third for $1,200, dated Dec. 21, 1912, for 6 months? Ans. Feb. 27, 1913

5. On Jan. 1, 1914, a merchant sold a bill of goods amounting to $3,600, payable as follows: $\frac{1}{2}$ in 30 days, $\frac{1}{3}$ in 60 days, and the remainder in 90 days. Find the equated time of payment. Ans. Feb. 20, 1914

EQUATION OF ACCOUNTS

120. The time at which an account having a debit and credit side may be equitably settled can be found in much the same manner as the equated time of payment. The process is called the **equation of accounts.** The method is employed particularly by business houses selling goods on credit, since by this method the balances of accounts may be readily determined at any time.

In finding the equated time of an account some date, called the focal date, is selected and the standing of the account with reference to that date is determined, interest being reckoned to that date on all bills due and on all payments previously made, and discount being computed on items of the account not due at that date. If the balance of interest on the credit side of the account is greater, credit should be extended for the balance of the account to equalize the excess of interest. If, on the other hand, the interest is greater on the debit side, the equated time of payment will be earlier than the focal date. The solutions of the following examples will make the method clear.

EXAMPLE.—Find the equated time for the settlement of the following account:

HENRY WARDELL

1911					1911			
Jan.	20	Mdse., 30 da.,	800		Mar.	1	Cash,	400
Feb.	18	Mdse., 90 da.,	600			20	Cash,	600
Mar.	14	Mdse., 60 da.,	1,000		Apr.	1	Cash,	1,000
Apr.	10	Mdse., 30 da.,	1,200			20	Cash,	500

SOLUTION.—The date of each item and the amount should be shown as in the following table. When credit is given, the date at which the bill is due is set down and not the date of sale. For example, 30 da. credit is given on the bill sold Jan. 20, making the bill due on Feb. 19. The bill of goods sold on Feb. 18 is due May 19, that sold March 14 is due May 13, and that sold April 10 is due May 10. The latest date, which is May 19, will be considered as the focal date. The interest is computed at 6%.

			Interest				Interest
Feb. 19,	$ 800	89 da.	$ 11.87	Mar. 1,	$ 400	79 da.	$ 5.27
May 19,	600	no da.	.00	Mar. 20,	600	60 da.	6.00
May 13,	1000	6 da.	1.00	Apr. 1,	1000	48 da.	8.00
May 10,	1200	9 da.	1.80	Apr. 20,	500	29 da.	2.42
	$3600		$ 14.67		$2500		$21.69
	2500						14.67
	$1100					$.183)	$7.02
							38da.

Interest on $1,100 for 1 da. = $.183.
38 days after May 19 is June 26, the equated time of payment. Ans.

EXPLANATION.—The condition of the account on May 19 is determined. Interest is charged on each bill of goods for the time from the date the bill is due to the focal date, and credit is given for interest on each payment, reckoned from date of payment until the focal date. The total interest on the debit side is $14.67 and on the credit side is $21.69. Mr. Wardell should, therefore, receive credit for $7.02 if the bill is settled on the focal date. To find how far credit may be extended for $1,100, the balance of the account, we divide $7.02 by the interest on $1,100 for 1 day. The interest on $1,100 for 1 day is $.183. In 38 days the interest will amount to $7.02. Mr. Wardell should have credit until 38 days after May 19, or until June 26, for the balance of the account.

The earliest date can be taken as the focal date and the discount on each item computed. Again, the focal date can be taken between the earliest and latest dates. In this case interest must be computed on some items and discount on others, but with the aid of a little thought the method should not present any great difficulty in determining when the balance may be paid without discount or interest.

121. Some accountants prefer to compute the equated time by what is known as the **products method.** The preceding example is solved as follows by that method:

SOLUTION.—

Feb. 19,	$ 800×89=	$71200		Mar. 1,	$ 400×79=	$31600	
May 19,	600×00=	00		Mar. 20,	600×60=	36000	
May 13,	1000× 6=	6000		Apr. 1,	1000×48=	48000	
May 10,	1200× 9=	10800		Apr. 20,	500×29=	14500	
	$3600	$88000			$2500	$130100	
	2500					88000	
	$1100				$1100)	$42100	
						38 da.	

EXPLANATION.—In this method the interest is not actually computed, but the equivalent principal placed at interest for 1 day is found. Taking the first item, the interest on $800 for 89 days is equal to the interest on $800×89=$71,200 for 1 day. There is no interest due on $600 payable May 19. The interest on $1,000 for 6 days is equal to the interest on $1,000×6 =$6,000 for 1 day. Each item is multiplied by the number of days that the due date or date of payment precedes the focal date. When these products are added it is seen that Mr. Wardell owes on the focal date the interest on $88,000 for 1 day. He should, on the other hand, receive credit for the interest on $130,100 for 1 day. The difference in the interest accounts on the focal date is equivalent to the interest on $42,100 for 1 day. Dividing $42,100 by $1,100 gives the number of days in which the interest on $1,100 equals the interest on $42,100 for 1 day. Credit for payment of the balance may be extended to 38 days after the focal date.

It may be observed that this solution gives the same result as that in Art. **120**.

122. In the products method, it is not necessary to calculate the interest on the items. The actual calculation is easier than that by the interest method, but large numbers may be involved. In the interest method, much smaller numbers are used and an error is more likely to be detected. Moreover, by this method the balance of an account, including interest, can be found directly on any date.

EXAMPLE 1.—What will be the equated time of payment of the following account and what will be the balance due: on Jan 2? on Jan. 31? Interest at 5%.

WM. BONNER

1913				1913			
Sept.	1	Mdse., 90 da.,	800	Sept.	12	Cash,	600
	20	Mdse., 60 da.,	900	Oct.	20	Draft, 30 da.,	500
Oct.	25	Mdse., 60 da.,	1,000	Nov.	15	Cash,	800
Nov.	1	Mdse., 30 da.,	2,000	Dec.	12	Cash,	1,000

SOLUTION.—It will be noticed that one of the payments, that of $500 on Oct. 20, is a 30-da. draft. Since this draft can be cashed for its face value only after 30 da., i. e., on Nov. 19, this payment must be considered as having been made on Nov. 19. The latest date entering in the statement of the account is 60 da. after Oct. 25, or Dec. 24.

			INTEREST				INTEREST
Nov. 30,	$800	24 da.	$3.20	Sept. 12,	$600	103 da.	$10.30
Nov. 19,	900	35 da.	5.25	Nov. 19,	500	35 da.	2.92
Dec. 24,	1000	no da.	.00	Nov. 15,	800	39 da.	5.20
Dec. 1,	2000	23 da.	7.67	Dec. 12,	1000	12 da.	2.00
	$4700		$16.12		$2900		$20.42
	2900						16.12
	$1800					$.30)	$4.30
							14 da.

The interest on $1,800 for 1 da. at 6% is $.30.

The equated time is 14 da. after Dec. 24, which is Jan. 7. Ans.

If the account is settled on Jan. 2, a discount for 5 da. should be made.

Interest on $1,800 for 5 da. at 6%=$1.50; at 5% interest=$1.25.

Amount due Jan. 2 is $1,800−$1.25=$1,798.75. Ans.

Should the account be settled on Jan. 31, interest would be due from Jan. 7, or for 24 da. Interest on $1,800 for 24 da. at 5%=$6.

Amount due on Jan. 31 is $1,800+$6=$1,806. Ans.

EXAMPLE 2.—When should interest begin on the balance of the following account?

HENRY WELLINGTON

1914				1914			
June	16	Mdse.,	1,200	July	21	Cash,	800
July	21	Mdse.,	1,000	Aug.	11	Cash,	1,200
Aug.	13	Mdse.,	2,000	Sept.	2	Draft, 30 da.,	1,600
Sept.	15	Mdse.,	3,600		23	Draft, 10 da.,	1,500

Solution.—

			Interest				Interest
June 16,	$1200	109 da.	$21.80	July 21,	$ 800	74 da.	$ 9.87
July 21,	1000	74 da.	12.33	Aug. 11,	1200	53 da.	10.60
Aug. 13,	2000	51 da.	17.00	Oct. 2,	1600	1 da.	.27
Sept. 15,	3600	18 da.	10.80	Oct. 3,	1500	no da.	.00
	$7800		$61.93		$5100		$20.74
	5100		20.74				
	$2700	$.45)	$41.19				
			92 da.				

Interest on $2,700 for 1 da. = $.45.

Equated time is, therefore, 92 da. preceding Oct. 3, or July 3. In this case the interest on the debit side is greater than that on the credit side. This indicates that Mr. Wellington should pay the balance of $2,700 earlier than the focal date. Had the excess of interest been on the credit side the equated time would have been later than the focal date.

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NOTE.—In this volume, each Section is complete in itself and has a number. This number is printed at the top of every page of the Section in the headline opposite the page number, and to distinguish the Section number from the page number, the Section number is preceded by a section mark (§). In order to find a reference, glance along the inside edges of the headlines until the desired Section number is found, then along the page numbers of that Section until the desired page is found. Thus, to find the reference "Altitude, §6, p21," turn to the Section marked §6, then to page 21 of that Section.