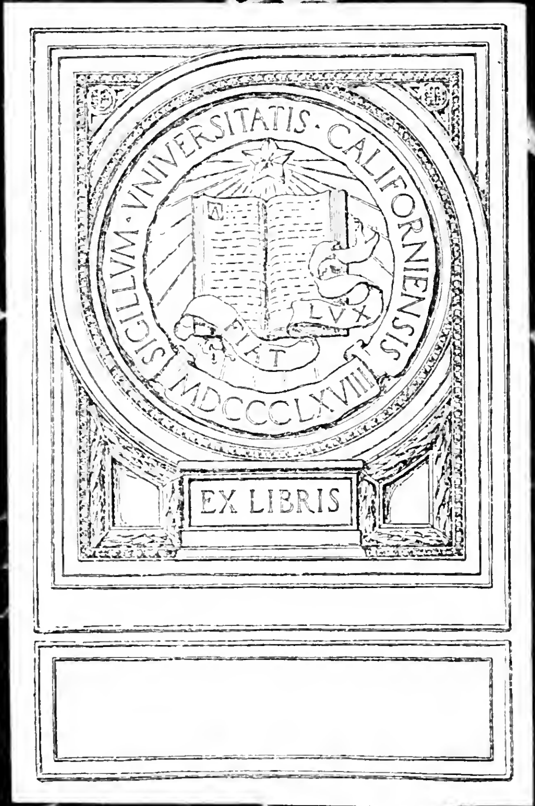
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TECHNICAL DRAWING SERIES

# DESCRIPTIVE GEOMETRY


BY

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AND

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# PREFACE

An extended experience in engineering practice and teaching has emphasized the importance of certain methods for the presentation of principles and problems in graphics which the authors have gradually developed into their present form. Most of the subject-matter which we now publish has been presented in the form of notes and used during the past eight years, and nearly all of the problems have received the critical test of the classroom. Previous to the preparation of the notes, which were made the foundation of this treatise, the third angle of projection had been adopted for problems, and almost exclusively used, as conforming more nearly to engineering practice.

It has been the aim of the authors to make a clear and concise statement of the principles involved, together with a brief analysis and enumeration of the steps to be taken, so that the essentials of each problem shall be clearly set forth. The illustrations have been chosen with care and arranged so as to appear opposite the descriptive text.

Too much stress cannot be laid on the importance of preparing problems with such care that they may clearly illustrate the principles involved. In general it should not be necessary for the student to prescribe the conditions governing the relations of points, lines, and surfaces because of the time consumed and the probable failure to bring out the salient features of the problem. The graphic presentation of the problems to be solved should facilitate the lay-out by the student, and enable the instructor to judge quickly of their character and adaptability to the special cases under consideration. Two sets of problems have been prepared to illustrate most cases, and the number may be further increased by reversing, or inverting, the illustrations given. The unit of measurement may also be changed to adapt the problems to any chosen size of plate, and if the proportion be maintained it will be possible to solve the problems within the given space. If it is desired to change the assignment annually, the even numbers may be used for one year and the odd numbers for the following year. It is hoped that the number of problems is sufficiently great to admit of considerable variety in the course.

Although the authors believe that an elementary course in the orthographic and isometric projections of simple objects should precede the more analytical consideration of line and plane, as herein presented, yet this treatise has been proven adequate to meet the demand of those who have not received such preparation. By thus making it available for students of science and mathematics, as well as for engineers, it is hoped to promote a more general knowledge of the grammatical construction of Graphic Language.

GARDNER C. ANTHONY
GEORGE F. ASHLEY.

TUFTS COLLEGE, January, 1909.

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DESCRIPTIVE GEOMETRY

# DESCRIPTIVE GEOMETRY

## CHAPTER I

### DEFINITIONS AND FIRST PRINCIPLES

1. **Descriptive Geometry** is the art of graphically solving problems involving three dimensions. By its use the form of an object may be graphically defined, and the character, relation, and dimensions of its lines and surfaces determined.

To the student it presents the most admirable training in the use of the imagination, such as the engineer or architect is called upon to exercise for the development of new forms in structure and mechanism, and which must be mentally seen before being graphically expressed.

To the engineer and architect it supplies the principles for the solution of all problems relating to the practical representation of forms, as illustrated by the various types of working drawings which are used by artisans to execute designs.

It is the foundation for the understanding of the different systems of projection such as orthographic, oblique, and perspective.

2. **Projection.** The representation of an object is made on one or more planes by a process known as projection, the picture, drawing, or projection of the object being determined by the intersection of a system of lines with the plane. The lines are known as projectors and are drawn from the object to the plane of projection, or picture plane. If these lines, or projectors, are perpendicular to the plane of projection, the system is known as

**Orthographic Projection**, and at least two projections, or views, are required to fully represent the object. Fig. 1 represents two views of a box by orthographic projection. This system is the one commonly employed for working drawings, and for the solution of problems in Descriptive Geometry.

If the projectors are parallel to each other, and oblique to the plane of projection, the system is known as **Oblique Projection**.* Fig. 2 represents the application of this method to the illustration of the object shown in Fig. 1. This system is used for the purpose of producing a more pictorial effect, but one which is easily executed and susceptible of measurement.

If the projectors converge to a point on the opposite side of the plane of projection, the system is known as **Perspective**. Fig. 3 is an application of this method to the representation of the object previously illustrated. This system is used to produce the pictorial effect obtained by the camera, and is chiefly employed by architects for the representation of buildings as they will appear to the eye of an observer.

3. **Coördinate Planes.** The planes upon which the representations are made are called *coördinate planes*, or *planes of projection*, and are usually conceived to be perpendicular to each other and indefinite in extent. Fig. 4 shows their relative positions, but for convenience of representation they are limited in extent. The plane designated by $H$ is called the *horizontal coördinate plane*, and the representation made thereon is known as the *horizontal projection*, *plan*, or *top view*. The plane designated by $V$ is called the *vertical coördinate plane*, and the representation made thereon is known as the *vertical projection*, *elevation*, or *front view*. The plane designated by $P$ is called the *profile coördinate plane*, and the representation made thereon is known as the *profile projection*, *side elevation*, *end* or *side view*. The line of intersection between the $V$ and $H$ planes is known as the *ground line*.

* For a treatise on Oblique Projection, and that branch of Orthographic Projection known as Isometric Projection, see "Elements of Mechanical Drawing" of this series.

**4. Quadrants or Angles.** The portion of space lying in each of the four diedral angles formed by the vertical and horizontal coördinate planes is designated as follows:

1st quadrant, above $H$ and before $V$.
2nd quadrant, above $H$ and behind $V$.
3rd quadrant, below $H$ and behind $V$.
4th quadrant, below $H$ and before $V$.

**5. Orthographic Projection.** The projection* of a point on any coördinate plane is obtained by letting fall a perpendicular from the point in space to the coördinate plane, its intersection with that plane being the projection of the point. The perpendicular is called the *projector* or *projecting line.* In Fig. 4, $a$ is the point in space and its projections are designated by the same letter with $v$, $h$, or $p$ written above and to the right, as $a^v$, signifying the vertical projection, $a^h$, the horizontal projection, and $a^p$, the profile projection.

For convenience of representation, the vertical coördinate plane is conceived as revolved

*"Projection" used without a qualifying adjective always means orthographic projection.

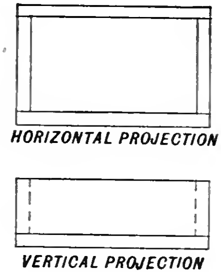


Fig. 1.

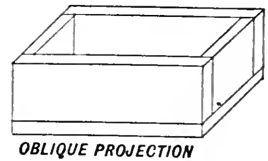


Fig. 2.

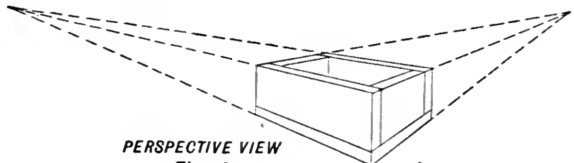


Fig. 3.

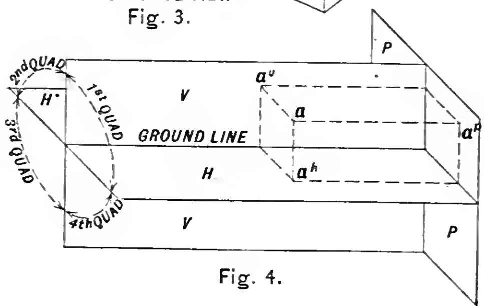
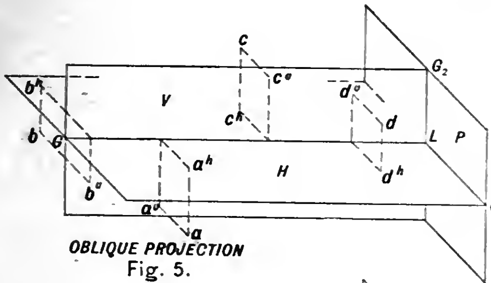


Fig. 4.

about the ground line to coincide with the horizontal coördinate plane in such a way that the first and third quadrants will be opened to 180°, and the second and fourth quadrants will be closed to 0°. In comparing Figs. 5, 6, and 7 it will be observed that any point in the fourth quadrant, as point *a*, has, after the planes have been folded together, both projections below the ground line; any point in the third quadrant, as point *b*, has the vertical projection below, and the horizontal projection above, the ground line; any point in the second quadrant, as point *c*, has both projections above the ground line; and any point in the first quadrant, as point *d*, has the vertical projection above, and the horizontal projection below, the ground line. Thus in Fig. 7 it is evident that the portion of the paper above the ground line represents not only that part of the horizontal coördinate plane which lies behind the vertical coördinate plane, but also that part of the vertical coördinate plane which lies above the horizontal coördinate plane. Likewise the paper below the ground line represents that portion of the horizontal coördinate plane lying in front of the vertical coördinate plane, and also that portion of the vertical coördinate plane lying below the horizontal coördinate plane.

The profile coördinate plane is commonly revolved about $G_2L$ as an axis until it coincides with the vertical coördinate plane, as in Figs. 8 and 9; but it may be revolved about $G_3L$ as an axis until it coincides with the horizontal coördinate plane, as in Figs. 10 and 11. It makes no difference as to the correct solution of the problem whether the profile coördinate plane be revolved to the right or to the left, but care should be used to revolve it into such a position as will cause the least confusion with other lines of the problem.

**6. Notation. Points** in space are designated by small letters, as *a*, *b*, *c*, etc., and their vertical, horizontal, and profile projections by the same letters with *v*, *h*, and *p* placed above and to the right, as $a^v$, $a^h$, $a^p$. When revolved into one of the coördinate planes, the points will be designated by $a'$, $b'$, or $a''$, $b''$, etc.




OBLIQUE PROJECTION
Fig. 5.

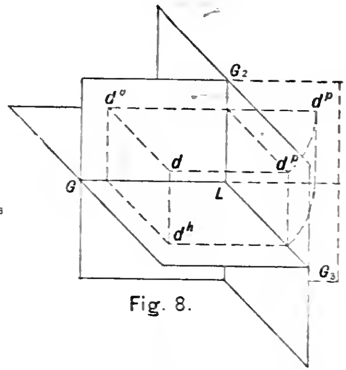



Fig. 8.

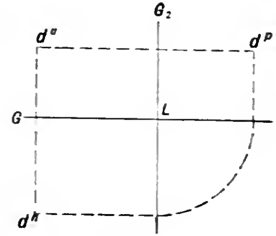



Fig. 9.

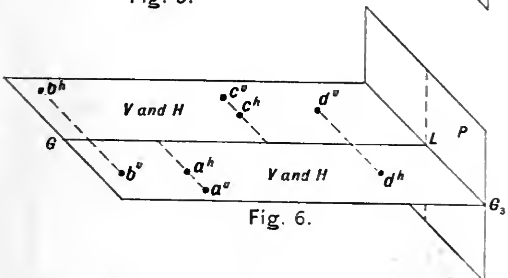



Fig. 6.


ORTHOGRAPHIC PROJECTION
Fig. 7.

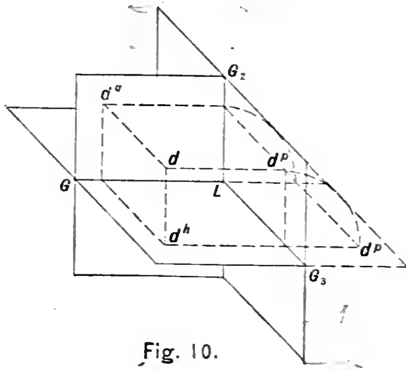



Fig. 10.

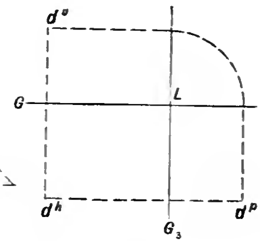



Fig. 11.

A **line** in space is designated by two of its points, as *ab*, or by a capital letter, usually one of the first ten of the alphabet, as *A*, and its projections are designated by $a^v b^v$, $a^h b^h$, $a^p b^p$, or $A^v$, $A^h$, $A^p$. When revolved into one of the coördinate planes, the lines will be designated by $a'b'$, $a''b''$, or $A'$, $A''$, etc. The **trace** of a line, *i.e.* the point in which it pierces a coördinate plane, may be designated by *V-tr*, *H-tr*, or *P-tr*, according as the line pierces the vertical, horizontal, or profile coördinate plane.

A **plane** in space is determined by three points not in the same straight line; by a line and a point; or by two parallel or intersecting lines. In projection a plane is usually designated by its **traces**, *i.e.* the lines in which it pierces the coördinate planes. These traces are designated by the last letters of the alphabet beginning with *M*, as *VM*, *HM*, *PM*, according as the plane *M* intersects the vertical, horizontal, or profile coördinate plane.

**Abbreviations.** The following abbreviations are used in connection with the figures and problems:

*V*, the vertical coördinate plane.
*H*, the horizontal coördinate plane.
*P*, the profile coördinate plane.
*GL*, ground line, the line of intersection between *V* and *H*.
*VP*, the intersection between *V* and *P*.
*HP*, the intersection between *H* and *P*.
*1Q*, *2Q*, *3Q*, *4Q*, the first quadrant, second quadrant, etc.

In general, an object in space is definitely located by two projections only, usually the vertical and the horizontal. Hereafter, only these two projections will be considered, unless otherwise indicated.

**The character of the lines employed** is as follows:

———————— Given and required lines.
- - - - - - - - - - Invisible and projection lines.
———————— All lines not designated above.

**7. Points.** Two projections are necessary to locate a point with reference to *V* and *H*. From Figs. 12 and 13 it is obvious that these two projections must always lie on a perpendicular to the ground line after *V* has been revolved to coincide with *H*. The distance from the point in space to *H* is equal to the distance from its vertical projection to the ground line, and the distance from the point in space to *V* is equal to the distance from its horizontal projection to the ground line. A point on either coördinate plane is its own projection on that plane, and its other projection is in the ground line, as points *c*, *d*, and *e*, Figs. 12 and 13.

The points represented in Figs. 12 and 13 are described as follows:

Point *a*, in *3Q*, 5 units from *V* and 8 units from *H*.

Point *b*, in *2Q*, 4 units from *V* and 7 units from *H*.

Point *f*, in *4Q*, 2 units from *V* and 6 units from *H*.

Point *c*, in *H*, between *1Q* and *4Q*, 4 units from *V*.

Point *d*, in *V*, between *1Q* and *2Q*, 7 units from *H*.

Point *e*, in *GL*.

**8. Lines.** Since a line in space is determined by any two of its points, the projections of these two points determine the projections of the line, Figs. 14 and 15. A line may also be projected by passing planes through it

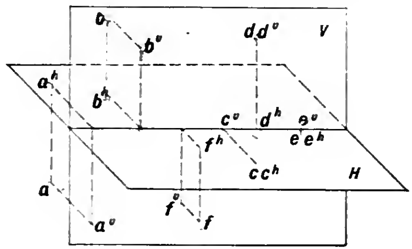



Fig. 12.

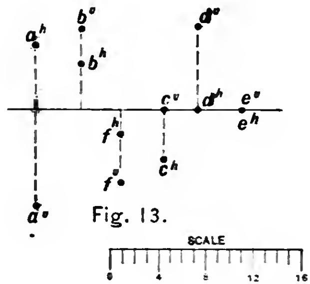



Fig. 13.

perpendicular to the coördinate planes. The intersections of these planes with $V$, $H$, and $P$ will determine respectively the vertical, horizontal, and profile projections of the line. Such auxiliary planes are known as *plane projectors* or *projecting planes* of the lines. In Fig. 14 the plane $abb^va^v$ is the vertical projecting plane of the line $ab$; the plane $abb^ha^h$ is the horizontal projecting plane of the line $ab$, and the plane $abb^pa^p$ is the profile projecting plane of the line $ab$.

**9.** A line parallel to a coördinate plane will have its projection on that plane parallel to the line in space, and its other projection will be parallel to the ground line. Line $A$, Figs. 16 and 17, is a line parallel to $V$. $A^v$ is parallel to $A$ in space, and $A^h$ is parallel to $GL$. Line $B$ is parallel to $H$, line $C$ is parallel to both $V$ and $H$, and line $D$ is parallel to $P$.

**10.** A line perpendicular to a coördinate plane will have for its projection on that plane, a point, and its other projection will be perpendicular to the ground line. Line $E$, Figs. 18 and 19, is a line perpendicular to $V$. $E^v$ is a point and $E^h$ is perpendicular to $GL$. Line $F$ is perpendicular to $H$, and line $K$ is perpendicular to $P$, $K^v$ and $K^h$ being perpendicular to $VP$ and $HP$, respectively, and $K^p$ being a point.

**11.** A line lying in either coördinate plane is its own projection on that plane, and its other projection is in the ground line. In Figs. 20 and 21, line $A$ lies in $H$ and line $B$ lies in $V$.

**12.** A line parallel to one coördinate plane and oblique to the other has its projection on the plane to which it is parallel equal to the true length of the line in space, and the angle which this projection makes with the ground line is the true size of the angle which the line makes with the plane to which it is oblique. Line $A$, Figs. 16 and 17, is seen in its true length in its vertical projection, and it makes an angle of 30° with $H$.

**13.** If two lines are parallel in space, their projections will be parallel, Figs. 22 and 23. Lines $C$ and $D$ are parallel; therefore $C^v$ and $D^v$ are parallel, and $C^h$ and $D^h$ are parallel.

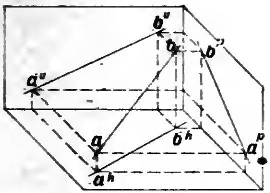



Fig. 14.

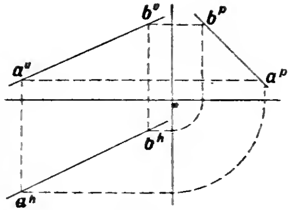



Fig. 15.

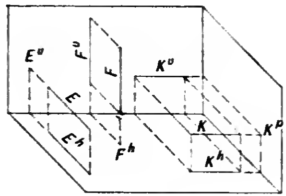



Fig. 18.

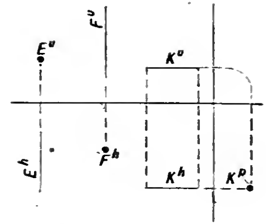



Fig. 19

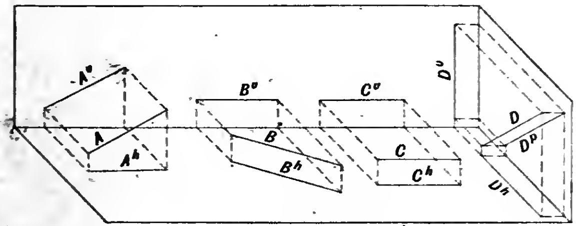
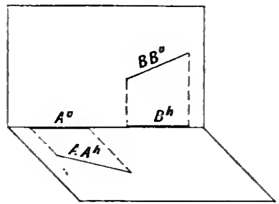



Fig. 16.

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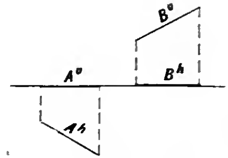


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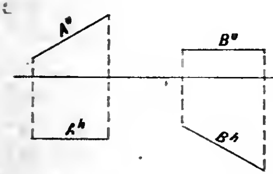


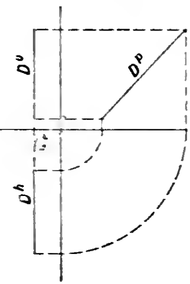




Fig. 17

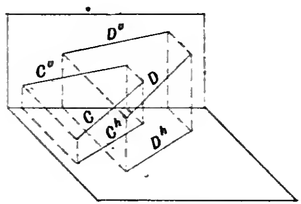



Fig. 22.

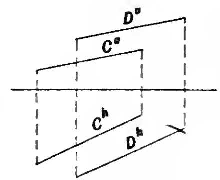



Fig. 23.

**14.** If two lines intersect in space, they have one point in common; hence, the projections of the lines will intersect each other in the projections of the point, as at $a^v$ and $a^h$, Figs. 24 and 25. If the projections of the lines do not intersect on a common perpendicular to the ground line, the lines in space do not intersect. Lines $A$ and $B$, Fig. 26, do not intersect.

**15.** If a line intersects the ground line, as in Fig. 27, its projections will intersect the ground line in the same point.

**16.** The **traces** of a line are the points in which the line pierces the coördinate planes. In Figs. 28 and 29, $a$ is the horizontal trace, and $b$ the vertical trace, of line $C$. The vertical projection of the horizontal trace is in the ground line, as is likewise the horizontal projection of the vertical trace.

**17.** For convenience of expression, it is customary to define the position of a line with respect to the coördinate planes by giving the quadrant in which it lies, together with its inclination with, and distance from, the coördinate planes. Line $C$, Figs. 28 and 29, lies in the first quadrant, and if read from $a$ toward $b$, inclines upward, backward, and toward the right. The vertical projection indicates that the inclination is upward; the horizontal projection indicates that the inclination is, at the same time, backward; while either projection indicates that the inclination is to the right. The angle of inclination will be considered later. If the line be read from $b$ toward $a$, the inclination would be downward, forward, and to the left. Either is correct. Fig. 30 illustrates four other lines as follows:

Line $ab$, in $3Q$, inclined upward, forward, and to the right.

Line $cd$, in $2Q$, inclined downward, backward, and to the right.

Line $ef$, in $4Q$, parallel to $H$, inclined forward and to the right.

Line $gk$, in $3Q$, parallel to $P$, and inclined upward and forward.

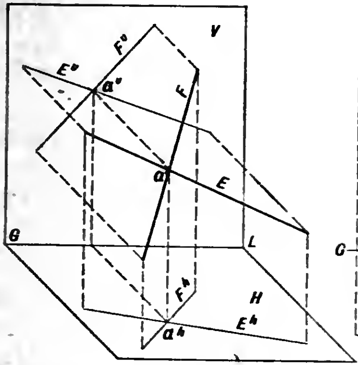



Fig. 24.

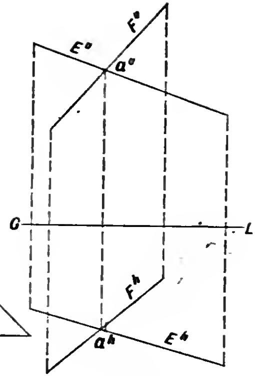



Fig. 25.

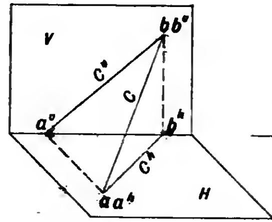



Fig. 28.

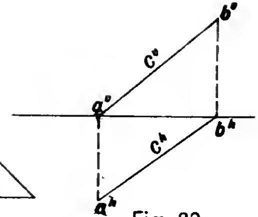



Fig. 29.

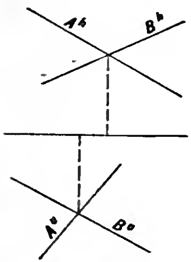



Fig. 26.

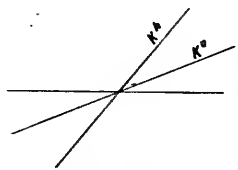



Fig. 27.

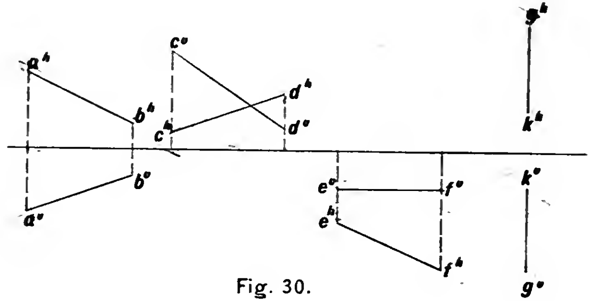



Fig. 30.

**18. Planes.** The position of planes may be represented in projection as follows:

1. By the projections of two intersecting or parallel lines.
2. By the projections of a line and a point.
3. By the projections of three points not in the same straight line.
4. By the lines of intersection with the coördinate planes. (Traces.)

All planes being indefinite in extent must intersect one or both of the coördinate planes. Such lines of intersection are called the **traces of the planes**. Fig. 31 illustrates the intersections of the planes $N$ with $V$ and $H$, the vertical and horizontal traces being lettered $VN$ and $HN$, respectively. The orthographic representation is shown in Fig. 32, save that it is not always customary to continue the traces beyond the ground line, the horizontal trace being drawn on one side and the vertical trace on the other, as in Fig. 40.

Since the vertical trace of a plane is a line lying on $V$, it may also be lettered as the vertical projection of a line. Thus, in Figs. 31 and 32, $VN$ may be lettered $A^v$ and, according to Art. 11, Page 8, $A^h$ must coincide with $GL$. Likewise, $HN$ is a line lying in $H$ and may be lettered $B^h$, while $B^v$ must lie in $GL$.

The following positions of planes are illustrated by Figs. 33 to 48 inclusive:

Perpendicular to $H$ and parallel to $V$, Figs. 33 and 34.

Perpendicular to $V$ and parallel to $H$, Figs. 35 and 36.

Inclined to $V$ and $H$, but parallel to $GL$, Figs. 37 and 38.

Inclined to $H$ and perpendicular to $V$, Figs. 39 and 40.

Inclined to $V$ and perpendicular to $H$, Figs. 41 and 42.

Perpendicular to $V$ and $H$, Figs. 43 and 44.

Inclined to $V$ and $H$, but containing $GL$, Figs. 45 and 46.

Inclined to $V$, $H$, and $GL$, Figs. 47 and 48.

The traces of parallel planes are parallel.

**19.** From the foregoing illustrations it will be observed that the ground line is the horizontal projection of the vertical coördinate plane, and that any point, line, or plane lying in $V$ will have its entire horizontal projection in the ground line. Likewise, it will be observed that the ground line is the vertical projection of the horizontal coördinate plane, and that any point, line, or plane lying in $H$ will have its entire vertical projection in the ground line.

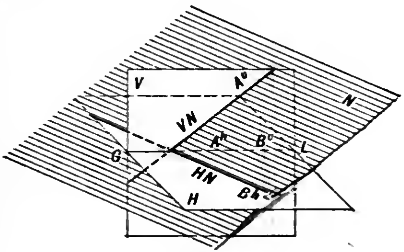


Fig. 31.

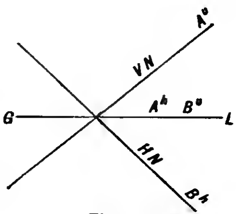


Fig. 32.

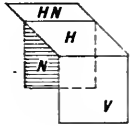


Fig. 33.

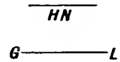


Fig. 34.

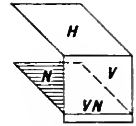


Fig. 35.

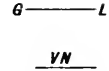


Fig. 36.

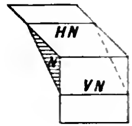


Fig. 37.

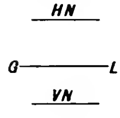


Fig. 38.

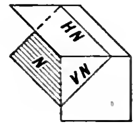


Fig. 39.

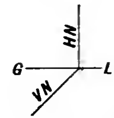


Fig. 40.

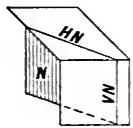


Fig. 41.

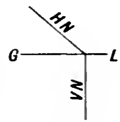


Fig. 42.

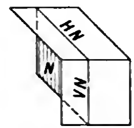


Fig. 43.

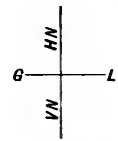


Fig. 44.

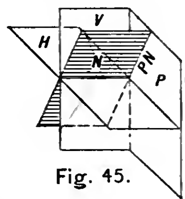


Fig. 45.

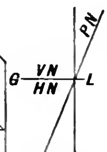


Fig. 46.

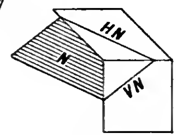


Fig. 47.

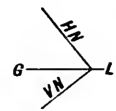


Fig. 48.

# CHAPTER II

## POINTS, LINES, AND PLANES

**20.** Three distinct operations are required for the solution of problems in Descriptive Geometry.

First, a statement of the PRINCIPLES involved.

Second, an outline of the METHOD to be observed, by the enumeration of the steps necessitated.

Third, the graphic CONSTRUCTION of the problem. The first two operations are purely mental, and the last is the mechanical operation of executing the drawing.

**21. To determine three projections of a line.**

PRINCIPLE. The projections of two points of a line determine the projections of the line.

METHOD. 1. Determine the vertical, horizontal, and profile projections of two points of the line. 2. Connect the vertical projections of the points to obtain the vertical projection of the line; connect the horizontal projections of the points to obtain the horizontal projection of the line; and connect the profile projections of the points to obtain the profile projection of the line.

CONSTRUCTION. Figs. 49 and 50. Let it be required to determine the projections of a line passing through point $a$, in $1Q$, 4 units from $V$, 7 units from $H$, and through point $b$, in $4Q$, 16 units from $V$, 9 units from $H$; point $a$ to be 12 units to the left of point $b$. On any perpendicular to $GL$ lay off $b^v$, 9 units below $GL$ and $b^h$, 16 units below $GL$ (Art. 7, page 6). On a second perpendicular, 12 units to the left of $b^v$ and $b^h$, lay off $a^v$, 7 units above $GL$ and $a^h$, 4 units below $GL$. Connect $a^v$ and $b^v$ to obtain the vertical projection of the line, and connect $a^h$ and $b^h$ to obtain the horizontal projection of the line.

To obtain $a^p$ and $b^p$, assume the position of the profile plane, shown in Fig. 49 by its intersection with *V* and *H* as *VP* and *HP*. Revolve *P* about *VP* as an axis to coincide with *V*. Then $a^p$ will lie on a line drawn through $a^v$ parallel to *GL*, and at a distance from *VP* equal to that of $a^h$ from *GL*. This is obtained by projecting $a^h$ to *HP* and revolving *HP* about *L* as a center to coincide with *GL*, and projecting perpendicularly to meet the parallel to *GL* through $a^v$ at $a^p$. Obtain $b^p$ in like manner. Connect $a^p$ and $b^p$ to obtain the profile projection of the line.

**22.** If point *e* in *2Q* be one of the given points, $e^h$ and $e^v$ having been determined, obtain $e^p$ as follows: Project $e^h$ to *HP*, revolve *HP* to *GL*, and project perpendicularly to meet a line drawn parallel to *GL* through $e^v$ at $e^p$; but after $e^h$ has been projected to *HP* it is imperative that *HP* be revolved in the same direction that it was revolved when the profile projections of the other points were determined. If the horizontal projection of one point be revolved, then the horizontal projections of all points must be revolved, and vertical projections must not be revolved.

**23.** *P* may be revolved in either of the directions shown by Figs. 49 and 51. Fig. 51 represents the line when *P* has been revolved about *HP* as an axis to coincide with *H*. Here it will be observed that the vertical projections, and only these, have been revolved.

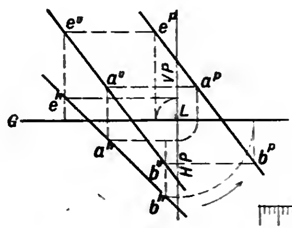



Fig. 49.

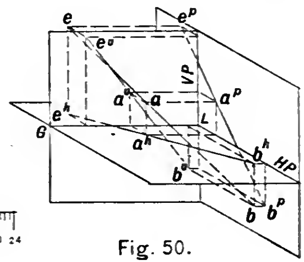




Fig. 50.

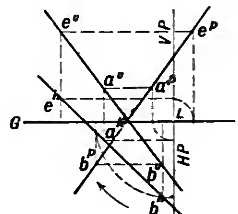



Fig. 51.

**24. To determine the traces of a line.**

PRINCIPLE. The traces of a line are the points in which the line pierces the coördinate planes. The projections of these traces must, therefore, lie in the projections of the line, and one projection of each trace will lie in the ground line (Art. 7, page 6).

METHOD. 1. To obtain the vertical trace of the line, continue the horizontal projection of the line until it intersects the ground line; this will be the horizontal projection of the vertical trace, and its vertical projection will be perpendicularly above or below the ground line in the vertical projection of the given line. 2. To obtain the horizontal trace of the line, continue the vertical projection of the line until it intersects the ground line; this will be the vertical projection of the horizontal trace, and its horizontal projection will be vertically above or below the ground line in the horizontal projection of the given line.

CASE 1. When the line is inclined to $V$, $H$, and $P$.

CONSTRUCTION. Figs. 52 and 53. Let it be required to determine the vertical trace of line $A$. Continue $A^h$ until it intersects $GL$ in $c^h$, which is the horizontal projection of the vertical trace. Next project this point to $A^v$, as at $c^v$, which is the vertical projection of the vertical trace.

The horizontal trace is similarly determined thus: Continue $A^v$ until it intersects $GL$ in $d^v$, which is the vertical projection of the horizontal trace. Next project this point to $A^h$, as at $d^h$, which is the horizontal projection of the horizontal trace.

To determine the profile trace, consider $P$, Fig. 52, to be represented by its intersections with $V$ and $H$, as $VP$ and $HP$, and to be revolved to the right about $VP$ as an axis until it coincides with $V$. Continue $A^v$ until it intersects $VP$ in $f^v$, which is the vertical projection of the profile trace. Continue $A^h$ until it intersects $HP$ in $f^h$, which is the horizontal projection of the profile trace, revolve to $GL$, using the intersection of $HP$ with $GL$ as a center, and project to $f^p$ by a line drawn through $f^v$ parallel to $GL$. This is the profile projection of the profile trace.

Fig. 53 is the oblique projection of the line, and clearly shows that the line passes from one quadrant to another at its vertical and horizontal traces, *i.e.* line $A$ passes from *1Q* to *2Q* at its vertical trace, $c$, and from *1Q* to *4Q* at its horizontal trace, $d$.

NOTE. The vertical projection of the vertical trace of a line is often called the vertical trace of the line, since this trace and its vertical projection are coincident. Likewise the horizontal projection of the horizontal trace is called the horizontal trace of the line.

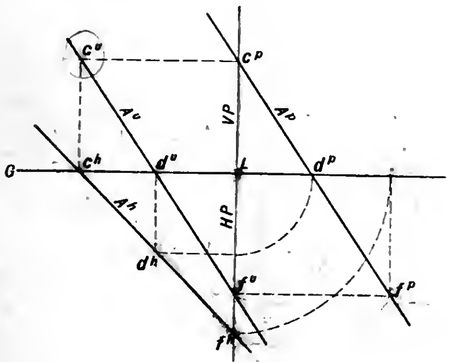


Fig. 52.

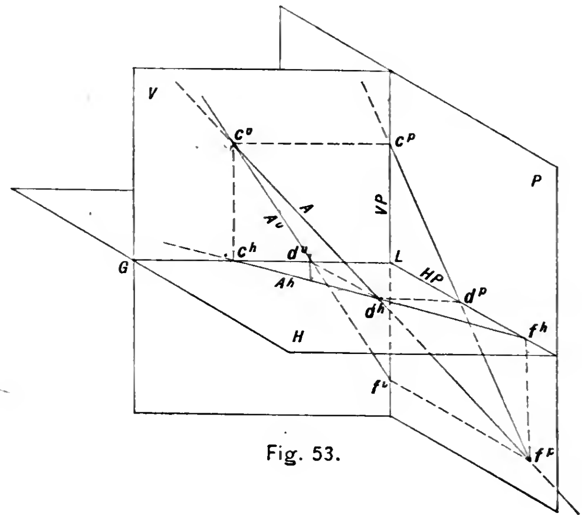


Fig. 53.

**25.** CASE 2. When the line is inclined to *V* and *H* and is parallel to *P*.

CONSTRUCTION. Fig. 54. If the given line is parallel to *P*, its vertical and horizontal traces cannot be determined by the above method, since the projections of the line are perpendicular to the ground line; hence, a profile projection of the line is necessary.

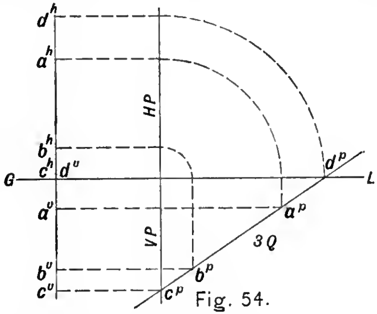



Fig. 54.

Let it be required to determine the vertical and horizontal traces of the line *ab*. *P* is assumed at will and is indicated by *HP* and *VP*. Determine $a^p b^p$, the profile projection of the line (Art. 21, page 14), by revolving *P* about *VP* as an axis until it coincides with *V*. That portion of *P* which before revolution was in *3Q* now falls below *GL* and to the right of *VP*; that portion which was in *4Q* falls below *GL* and to the left of *VP*; that portion which was in *1Q* falls above *GL* and to the left of *HP*. Thus the profile projection of the line, when continued, indicates that the line passes through *3Q* into *2Q* at point *d*, $d^p$ being the profile projection of the horizontal trace. Likewise the line passes from *3Q* into *4Q* at point *c*, $c^p$ being the profile projection of the vertical trace. Counter-revolve *P* to its original position and obtain $d^h$ and $d^v$, which are the horizontal and vertical projections of the horizontal trace, and $c^v$ and $c^h$, which are the vertical and horizontal projections of the vertical trace. Since line *ab* is parallel to *P*, it has no profile trace.

**26. To determine the projections of a line when its traces are given** (Art. 21, page 14).

METHOD. 1. Determine the horizontal projection of the vertical trace, and the vertical projection of the horizontal trace. 2. Connect the horizontal trace with the horizontal projection of the vertical trace to obtain the horizontal projection of the line. Connect the vertical trace with the vertical projection of the horizontal trace to obtain the vertical projection of the line.

**27. Conditions governing lines lying in a plane.** Since the traces of a plane are lines of the plane, they must intersect all other lines of the plane, and conversely, all lines of a plane must have their traces in the traces of the plane. See lines *A* and *B*, Figs. 55 and 56.

28. If a line is parallel to *H*, it intersects *H* at infinity; therefore, its horizontal trace is at infinity and its horizontal projection is parallel to the horizontal trace of the plane in which it lies, its vertical projection being parallel to the ground line. See line *C*, Figs. 57 and 58. Likewise if a line is parallel to *V*, its vertical projection is parallel to the vertical trace of the plane in which it lies, and its horizontal projection is parallel to the ground line.

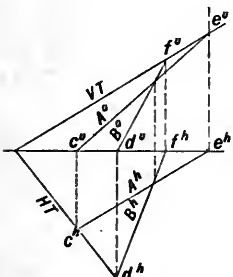


Fig. 55.

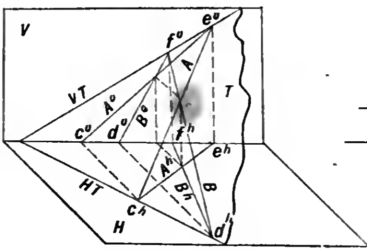


Fig. 56.

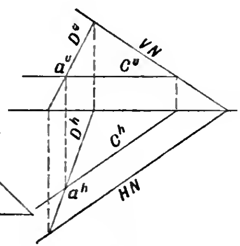


Fig 57.

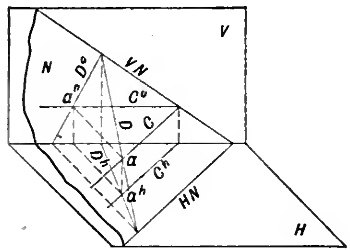


Fig. 58.

**29.** If the traces of any plane be drawn through the traces of a line, the plane must contain the line; therefore, an infinite number of planes may be passed through any line. In Fig. 59, planes *N*, *R*, *T*, and *S* all contain line *E*.

**30. To pass a plane through two intersecting or parallel lines.**

PRINCIPLE. The traces of the plane must contain the traces of the lines (Art. 27, page 19).

CASE 1. METHOD. 1. Determine the traces of the given lines. 2. Connect the two horizontal traces of the lines to obtain the horizontal trace of the plane, and connect the two vertical traces of the lines to obtain the vertical trace of the plane.

CONSTRUCTION. Fig. 60. *A* and *B* are the given intersecting lines. Determine their horizontal traces $c^h$ and $d^h$, and their vertical traces $e^v$ and $f^v$ (Art. 24, page 16). Connect the horizontal traces to determine *HT* and the vertical traces to determine *VT*. Since the traces of the plane must meet in *GL*, only three traces of the lines are necessary.

CHECK. Both traces of the plane must intersect the ground line in the same point.

NOTE. The vertical projection of the vertical trace of a plane will always be spoken of as the vertical trace of the plane, but it must constantly be borne in mind that the vertical trace of a plane is a line lying on *V* and that its horizontal projection is in the ground line. Likewise the horizontal projection of the horizontal trace of a plane will be spoken of as the horizontal trace of that plane, but, as before, the horizontal trace is a line lying in *H* and its vertical projection is in the ground line.

**31.** CASE 2. METHOD. If the traces of the given lines cannot readily be found, new lines intersecting the given lines may be assumed, which, passing through two points of the plane, lie in it, and their traces, therefore, are points in the traces of the required plane.

CONSTRUCTION. Fig. 61. The two given intersecting lines are *A* and *B*, the traces of which cannot be found within the limits of the drawing. Line $C$ is an assumed line joining point $e$ of line $A$ and point $f$ of line $B$, the traces of which are easily located at $g$ and $k$. Line $D$ is a second similar line. $HT$, connecting the horizontal traces of lines $C$ and $D$, is the required horizontal trace of the plane of lines $A$ and $B$. Likewise $VT$, connecting the vertical traces of lines $C$ and $D$, is the required vertical trace of the plane of lines $A$ and $B$.

**32.** Case 3. Method. If one of the given lines is parallel to the ground line, the required plane will be parallel to the ground line, and, therefore (Figs. 37 and 38, page 13), the traces of the plane will be parallel to the ground line. This problem may be solved by Case 2, or by the following method: Determine the profile trace of the required plane by obtaining the profile traces of the given lines. Having found the profile trace of the plane, determine the horizontal and vertical traces.

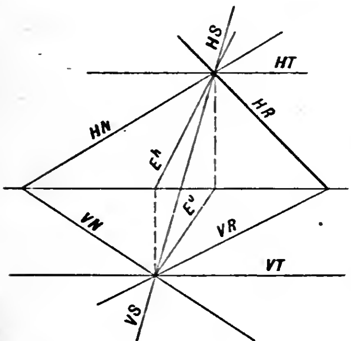


Fig. 59

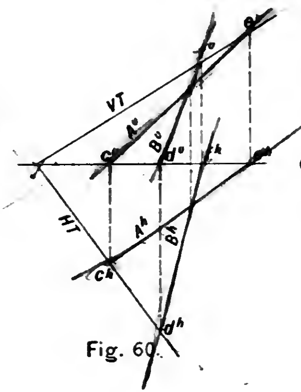


Fig. 60.

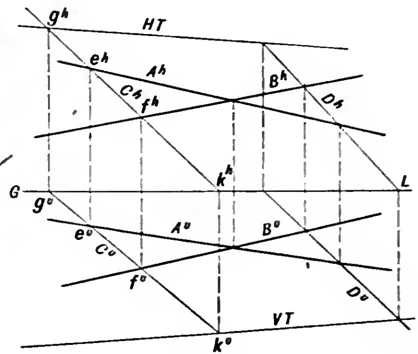


Fig. 61.

CONSTRUCTION. Fig. 62. Let $A$ and $B$ be the given lines parallel to the ground line. Assume $P$ and draw $HP$ and $VP$. Continue the horizontal and vertical projections of lines $A$ and $B$ to intersect $HP$ and $VP$, respectively. The horizontal and vertical projections of the profile trace of line $A$ will lie at $c^h$ and $c^v$, and the profile trace at $c^p$. Similarly determine $d^p$, the profile trace of line $B$. Through $c^p$ and $d^p$ draw $PN$, the profile trace of the required plane. $E^p$ will be the profile projection of the horizontal trace of the required plane, and $K^p$ the profile projection of the vertical trace. By counter-revolution obtain $HN$ and $VN$.

**33. To pass a plane through a line and a point.**

METHOD. Connect the given point with any assumed point of the line and proceed as in Art. 30, page 20.

**34. To pass a plane through three points not in the same straight line.**

METHOD. Connect the three points by auxiliary lines and proceed as in Art. 30, page 20. In Fig. 63, $a$, $b$, and $c$ are the given points.

**35. Given one projection of a line lying on a plane, to determine the other projection.**

PRINCIPLE. The traces of the line must lie in the traces of the plane (Art. 27, page 19).

METHOD. Determine the traces of the line and from them determine the unknown projection of the line.

CONSTRUCTION. Fig. 64. Let $HN$ and $VN$ be the traces of the given plane, and $A^h$ one projection of line $A$ lying in $N$. Continue $A^h$ to meet $HN$ in $a^h$, the horizontal trace of line $A$; $a^v$ is in $GL$ (Art. 24, page 16). Continue $A^h$ to meet $GL$ in $b^h$, the horizontal projection of the vertical trace; $b^v$, the vertical trace, is in $VN$. Connect $a^v$ and $b^v$ to obtain $A^v$, the required vertical projection of line $A$.

If $A^v$ had been the given projection, $A^h$ would have been similarly determined.

If the given projection, $B^v$, Fig. 65, be parallel to $GL$, $B^h$ will be parallel to $HS$, for, *if a line lying in an inclined plane has one projection parallel to the ground line, the other projection is parallel to the trace of the plane;* and conversely, *if a line lying in an inclined plane has one projection parallel to the trace of the plane, the other projection is parallel to the ground line* (Art. 28, page 19). Therefore, to determine $B^h$ continue $B^v$ to meet $VS$ in $c^v$, the vertical trace of the line. Its horizontal projection, $c^h$, will be in $GL$, and $B^h$ will pass through $c^h$ parallel to $HS$.

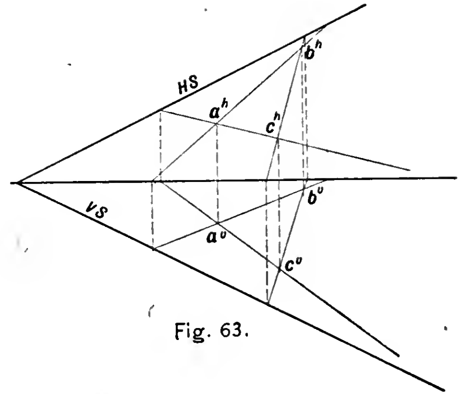


Fig. 63.

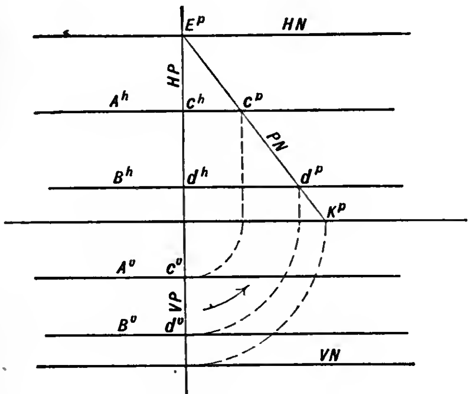


Fig. 62.

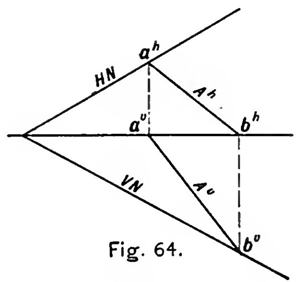


Fig. 64.

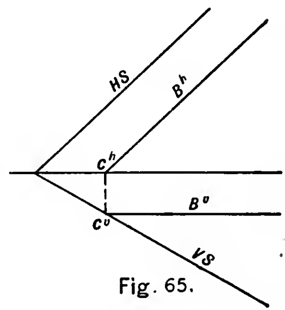


Fig. 65.

**36. Given one projection of a point lying on a plane, to determine the other projection.**

PRINCIPLE. The required projection of the point will lie on the projection of any line of the plane passed through the point.

METHOD. 1. Through the given projection of the point draw the projection of any line lying on the plane. 2. Determine the other projection of the line (Art. 35, page 22). 3. The required projection of the point will lie on this projection of the line.

CONSTRUCTION. Figs. 66 and 67. Let $a^h$ be the given projection of point $a$ on plane $N$. Through $a^h$ draw $B^h$, the horizontal projection of any line of plane $N$ passing through point $a$. Determine $B^v$ (Art. 35, page 22). Then $a^v$ lies at the intersection of $B^v$ and a perpendicular to $GL$ through $a^h$. The same result is obtained by using line $C$ lying in plane $N$ and parallel to $V$, or using line $D$ lying in plane $N$ and parallel to $H$.

If the vertical projection of the point be given, the horizontal projection is determined in a similar manner.

In general, solve the problem by the use of an auxiliary line parallel to $V$ or $H$.

**37. To locate a point on a given plane at a given distance from the coördinate planes.**

PRINCIPLE. The required point lies at the intersection of two lines of the given plane; one line is parallel to, and at the given distance from, $V$, and the other line is parallel to, and at the given distance from, $H$.

METHOD. 1. Draw a line of the plane parallel to, and at the required distance from, one of the coördinate planes (Fig. 65, page 23). 2. Determine a point on this line at the required distance from the other coördinate plane.

CONSTRUCTION. Fig. 68. Let it be required to locate point $b$ on plane $R$, at $x$ distance from $H$ and $y$ distance from $V$. Draw line $A$ in plane $R$, parallel to, and at $y$ distance from, $V$. $A^h$ will be parallel to, and at $y$ distance from, $GL$, and $A^v$ will be parallel to $VR$ (Art. 35, page 22). On $A^v$ determine $b^v$ at $x$ distance from $GL$, and project to $A^h$ to determine $b^h$. Or, point

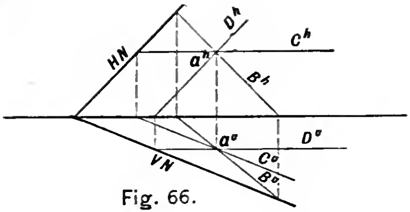



Fig. 66.

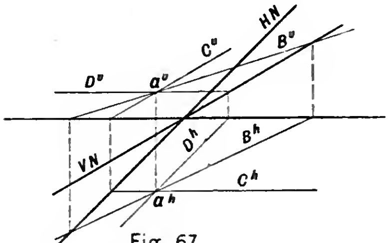



Fig. 67.

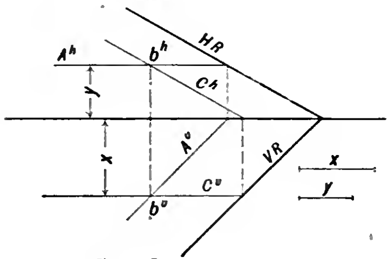



Fig. 68

*b* may be located by passing line *C* in plane *R*, parallel to, and at *x* distance from, *H*.

If the plane be parallel to *GL*, use the following method: 1. Draw on the given plane any line oblique to *V* and *H*. 2. Locate the required point thereon at the required distance from *V* and *H*.

**38. To revolve a point into either coördinate plane.**

Principle. The axis about which the point revolves must lie in the plane into which the point is to be revolved. The revolving point will describe a circle whose plane is perpendicular to the axis, and whose center is in the axis. The intersection of this circle and the coördinate plane is the required revolved position of the point.

Method. 1. Through that projection of the given point on the plane in which the axis lies draw a line perpendicular to the axis. 2. On this perpendicular lay off a point having its distance from the axis equal to the hypotenuse of a right triangle, one leg of which is the distance from the projection of the point to the axis, and the other leg of which is equal to the distance from the second projection of the point to the ground line.

Let $a$, Fig. 69, be the point in space to be revolved into $H$ about $D\,D^h$ lying in $H$ as an axis. If a point be revolved about an axis, its locus will be in a plane perpendicular to the axis. $X$ is this plane, and $HX$, its horizontal trace, is perpendicular to $D^h$. Point $a$, in revolving, will describe a circle with $ab$ as a radius. The points $a'$ and $a''$, in which this circle pierces $H$, are the required revolved positions of the point $a$. Since angle $aa^hb$ is a right angle, $ba$ is equal to the hypotenuse of a right triangle, one leg of which is $a^hb$, the distance from $a^h$ to the axis, and the other leg

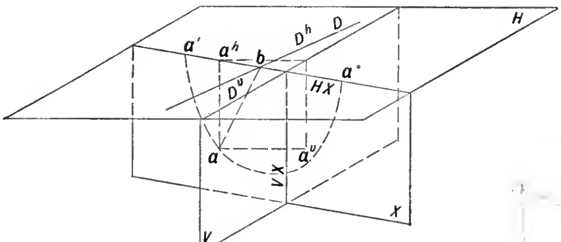


Fig 69.

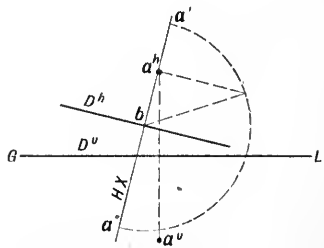


Fig. 70.

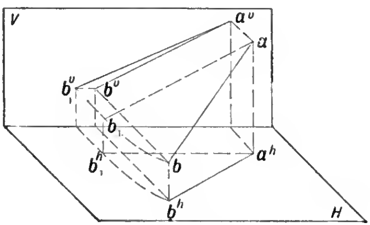


Fig. 71.

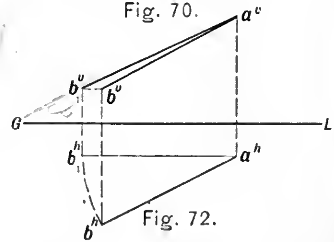


Fig. 72.

is $a^h a$, or the distance from $a^v$ to $GL$.

CONSTRUCTION. Fig. 70. The point $a$ is represented by its two projections, $a^v$ and $a^h$. Through $a^h$ draw $HX$ perpendicular to $D^h$. The revolved position, $a'$ or $a''$, will lie in $HX$ at a distance from $D^h$ equal to the hypotenuse of a right triangle, one leg of which is the distance from $a^h$ to the axis $D^h$, and the other leg is the distance from $a^v$ to $GL$.

If the axis lies in $V$, the revolved position of the point will lie in a line passing through the vertical projection of the point perpendicular to the axis and at a distance from this axis equal to the hypotenuse of a right triangle, one leg of which is the distance from the vertical projection of the point to the axis, while the other leg is the distance from the horizontal projection of the point to the ground line.

**39. To determine the true length of a line.**

PRINCIPLE. A line is seen in its true length on that coördinate plane to which it is parallel, or in which it lies (Figs. 16 to 21, page 9).

METHOD. Revolve the line parallel to, or into either coördinate plane, at which time one of its projections will be parallel to the ground line and the other projection will measure the true length of the line in space.

CASE 1. When the line is revolved parallel to a coördinate plane.

CONSTRUCTION. Fig. 71. Let $ab$ be a line in the first quadrant inclined to both $V$ and $H$. Neither projection will equal the true length of the line in space. Revolve $ab$ about the projecting line $aa^h$ as an axis until it is parallel to $V$. Point $a$ will not move; hence, its projections, $a^v$ and $a^h$, will remain stationary. Point $b$ will revolve in a plane parallel to $H$, to $b_1$; hence, $b^h$ will describe the arc $b^h b_1^h$, and $a^h b_1^h$ will be parallel to $GL$. Then $b^v$ will move parallel to $GL$ to its new position $b_1^v$, and $a^v b_1^v$, the new vertical projection, will equal the true length of line $ab$.

Fig. 72 is the orthographic projection of the problem. It is of interest to note that the angle which this true length makes with $GL$ is the true size of the angle which the line in space makes with $H$.

Figs. 73 and 74 represent the line *ab* revolved parallel to *H*, thus obtaining the same result as to the length of line. Here the angle which the true length of the line makes with *GL* is the true size of the angle which the line in space makes with *V*.

**40.** Case 2. When the line is revolved into a coördinate plane.

Construction. Figs. 75 and 76 represent the line *ab* of the previous figures revolved into *H* about its horizontal projection $a^hb^h$ as an axis. Point *a* revolves in a plane perpendicular to the axis $a^hb^h$ (Art. 38, page 25); therefore, its revolved position, $a'$, will lie in a line perpendicular to $a^hb^h$ and at a distance from $a^h$ equal to $aa^h$, or the distance from $a^v$ to *GL*.* Similarly $b'$ is located.

**41.** From Fig. 76 it will be seen that if the revolved position of the line, $a'b'$, be continued, it will pass through the point where *ab*, continued, intersects its own projection, which is the horizontal trace of the line, and since it is in the axis, it does not move in the revolution. Thus, in every case where a line is revolved into one of the coördinate planes, its revolved position will pass through the trace of the line. Figs. 77 and 78 illustrate a line *cd* piercing *V* in the point *e*. In order to determine its true length it has been revolved into *V* about $c^vd^v$ as an axis. Since the point *e* lies in the axis and does not move, point *c* revolves in one direction while point *d* must revolve in the opposite direction, thus causing the revolved position to pass through the vertical trace of the line.

Again it is of interest to note that when a line is revolved into a coördinate plane to determine its true length, the angle which the revolved position makes with the axis is the true size of the angle which the line in space makes with the coördinate plane into which it has been revolved. Thus in Figs. 77 and 78, angle $c'e^vc^v$ is the true size of the angle which line *cd* makes with *V*.

* This does not contradict Art. 38, page 26, in that one leg of the triangle is equal to zero.

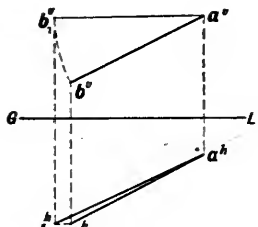


Fig. 73.

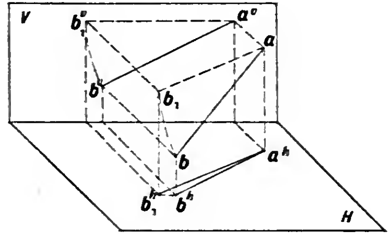


Fig. 74.

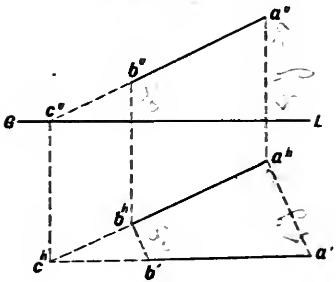


Fig. 75.

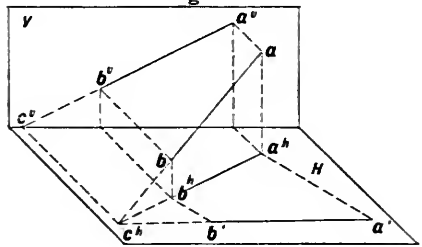


Fig. 76.

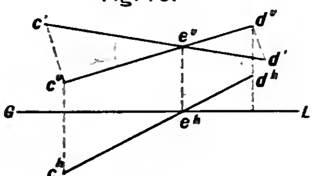


Fig. 77.

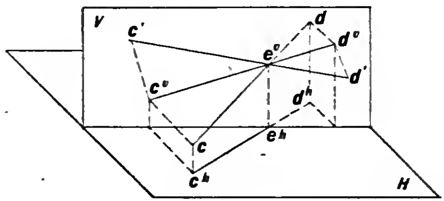


Fig. 78.

**42. Given a point lying on a plane, to determine its position when the plane shall have been revolved about either of its traces as an axis to coincide with a coördinate plane.**

PRINCIPLE. This is identical with the principle of Art. 38, page 25, since either trace of the plane is an axis lying in a coördinate plane, one projection of which is the line itself, and the other projection of which is in the ground line.

METHOD. See Art. 38, page 25.

CONSTRUCTION. Fig. 79. *HN* and *VN* are the traces of the given plane and $b^h$ and $b^v$, the projections of the point. If the plane with point $b$ thereon be revolved into *H* about *HN* as an axis, the point will move in a plane perpendicular to *HN*, and will lie somewhere in $b^h b'$. Its position in this line must be determined by finding the true distance of the point from *HN*. This distance will equal the hypotenuse of a right triangle of which $b^h d$, the horizontal projection of the hypotenuse, is one leg, and $b^v c$, the distance of the point from *H*, is the other leg. By laying off $b^h e$

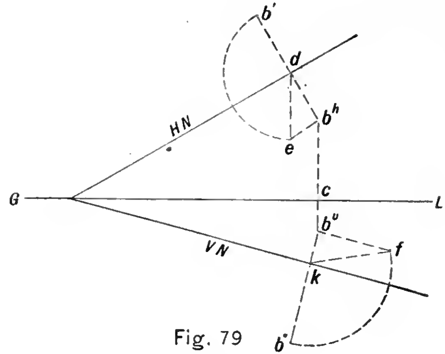



Fig. 79

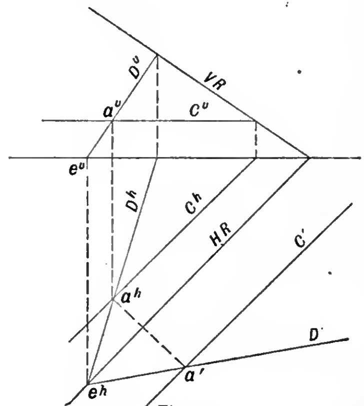



Fig. 80.

equal to $cb^v$, and perpendicular to $b^hd$, the length required, $de$, is determined, and when laid off on $db'$ from $d$, will locate the revolved position of point $b$.

If it were required to revolve the point into $V$ about $VN$ as an axis, it would lie at $b''$ in a perpendicular to $VN$ through $b^v$. The distance $kb''$ will equal $kf$, the hypotenuse of a right triangle, one leg of which is $kb^v$, and the other leg of which is equal to $cb^h$.

**43. The revolved position of a line lying in a plane** is determined by finding the revolved position of two points of the line. If the line of the plane is parallel to the trace which is used as an axis, its revolved position will also be parallel to the trace.

If a line has its trace in the axis, this trace will not move during the revolution; therefore, it will be necessary to determine but one other point in the revolved position.

Fig. 80 represents the lines $C$ and $D$ of the plane $R$ revolved into $H$ about $HR$ as an axis. The revolved position of point $a$ is determined at $a'$, through which $C'$ is drawn parallel to $HR$ (since line $C$ of the plane is parallel to $HR$), and $D'$ through $e^h$, the horizontal trace of line $D$.

**44. To determine the angle between two intersecting lines.**

PRINCIPLE. The angle between intersecting lines may be measured when the plane of the lines is revolved to coincide with one of the coördinate planes.

METHOD. 1. Pass a plane through the two intersecting lines and determine its traces (Art. 30, page 20). 2. Revolve the plane with the lines thereon, about either of its traces as an axis, until it coincides with a coördinate plane (Art. 43, page 31). As it is necessary to determine the revolved position of but one point in each line, let the point be one common to both lines, their point of intersection, and, therefore, the vertex of the angle between them. 3. The angle between the revolved position of the lines is the required angle.

**45. To draw the projections of any polygon having a definite shape and size and occupying a definite position upon a given plane.**

PRINCIPLE. The polygon will appear in its true size and shape, and in its true position on the plane, when the plane has been revolved about one of its traces as an axis to coincide with a coördinate plane.

METHOD. 1. Revolve the given plane into one of the coördinate planes about its trace as an axis. 2. Construct the revolved position of the polygon in its true size and shape, and occupying its correct position on the plane. 3. Counter-revolve the plane, with the polygon thereon, to its original position, thus obtaining the projections of the polygon.

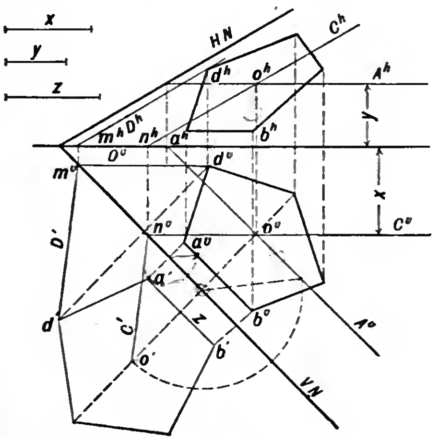
CONSTRUCTION. Fig. 81. Let it be required to determine the projections of a regular pentagon on an oblique plane $N$, the center of the pentagon to be at a distance $x$ from $H$, and $y$ from $V$, and one side of the pentagon to be parallel to $V$, and of a length equal to $z$.

Determine the projections of the center as at $o^v$ and $o^h$ (Art. 37, page 24). Revolve plane $N$ about one of its traces as an axis until it coincides with one of the coördinate planes (in this case about $VN$ until it coincides with $V$), $o'$ being the revolved position of the center (Art. 42, page 30). About $o'$ as a center draw the pentagon in its true size and shape, having one side parallel to $VN$, and of a length equal to $z$. Counter-revolve the plane to obtain the projections of the pentagon.

**46.** COUNTER-REVOLUTION. The counter-revolution may be accomplished in several ways, of which three are here shown.

CONSTRUCTION 1. Through $o'$, Fig. 81, draw any line $C'$ to intersect $VN$ in $n^v$, and connect $n^v$ with $o^v$ to obtain $C^v$. Through $d'$ draw $D'$ parallel to $C'$ to intersect $VN$ in $m^v$, from which point draw $D^v$ parallel to $C^v$. From $d'$ project perpendicularly to $VN$ to intersect $D^v$ in $d^v$, the vertical projection of one point in the required vertical projection of the pentagon. Similarly determine the vertical projections of the other points by drawing lines parallel to $C$.

**47.** Construction 2. A sometimes shorter method of counter-revolution, to obtain the vertical projection of the pentagon, is as follows. Fig. 82. Assume the revolved position of the pentagon to have been drawn as described above. From any point of the pentagon, as $c'$, draw a line through the center $o'$, and continue it to meet the axis of revolution in $m^v$. Then $m^vo^v$ is the vertical projection of this line after counter-revolution (Art. 43, page 31), and $c^v$ will lie on $m^vo^v$ at its intersection with a perpendicular to $VN$ from $c'$ (Art. 42, page 30). Produce $c'b'$ to $VN$, thus determining the line $c^vb^v$, and, therefore, $b^v$. Since $b'a'$ is parallel to $VN$, $a^v$ may be found by drawing a parallel to $VN$ from $b^v$ to meet the perpendicular from $a'$ (Art. 35, page 22). Likewise $d^v$ is determined, since $c'd'$ is parallel to $VN$. Next continue $e'd'$ to the axis at $n^v$; draw $n^vd^v$ to meet $o'o^v$ produced at $e^v$. The horizontal projection of the pentagon is best determined by Art. 35, page 22.



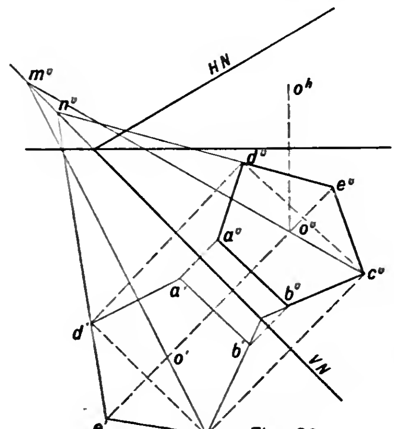



Fig. 82.

**48.** Construction 3. Fig. 83. Let it be assumed that the plane $N$ has been revolved into $V$ about $VN$ as an axis, and that the revolved position of $HN$ has been determined, as $HN'$, by revolving any point $g$ of the horizontal trace of the plane about $VN$ as an axis (Art. 43, page 31). The angle between $HN'$ and $VN$ is the true size of the angle between the traces of the plane; also the area between $HN'$ and $VN$ is the true size of that portion of plane $N$ lying between its traces. Next let it be assumed that the polygon has been drawn in its revolved position, and occupying its correct location with respect to $VN$ and $HN$, $e'$ and $d'$ being two of its vertices. To counter-revolve, continue $e'd'$ to intersect $HN'$ in $k'$, and $VN$ in $n^v$. Since $k$ is a point in the horizontal trace of the plane, its vertical projection will lie in $GL$. Then $k^v$ will lie in $GL$ at the intersection of a perpendicular to $VN$ from $k'$, and $k^h$ is in $HN$. Also the intersection of $e'd'$ with the axis $VN$ is $n^v$, and its horizontal projection will lie in $GL$ at $n^h$. Draw $n^vk^v$, producing it to intersect perpendiculars to $VN$ from $d'$ and $e'$, in $d^v$ and $e^v$. Draw $n^hk^h$, producing it to intersect perpendiculars to $GL$ from $d^v$ and $e^v$, at $d^h$ and $e^h$. Similarly determine the projections of the other points of the pentagon.

A fourth construction for counter-revolution is explained in Art. 91, page 63.

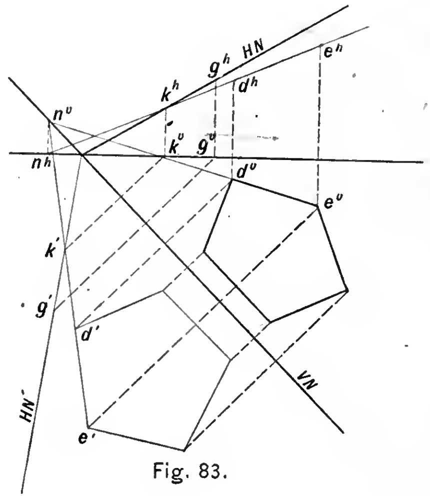



Fig. 83.

**49. To determine the projections of the line of intersection between two planes.**

PRINCIPLE. The line of intersection between two planes is common to each plane; therefore, the traces of this line must lie in the traces of each plane. Hence, the point of intersection of the vertical traces of the planes is the vertical trace of the required line of intersection, and the point of intersection of the horizontal traces of the planes is the horizontal trace of the required line of intersection between the planes.

There may be three cases as follows:

Case 1. When no auxiliary plane is required.

Case 2. When an auxiliary plane parallel to *V* or *H* is used.

Case 3. When an auxiliary plane parallel to *P* is required.

**50.** CASE 1. When like traces of the given planes may be made to intersect, no auxiliary plane is required for the solution of the problem.

METHOD. 1. Determine the points of intersection of like traces of the planes, which points are the traces of the line of intersection between the planes. 2. Draw the projections of the line (Art. 26, page 18).

CONSTRUCTION. Figs. 84 and 85 illustrate the principle and need no explanation, line *cb* being the line of intersection between the given planes *N* and *S*.

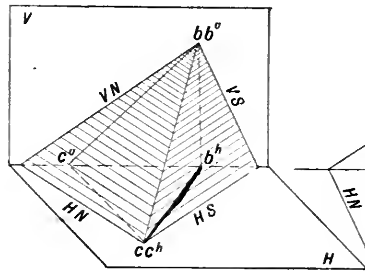



Fig. 84.

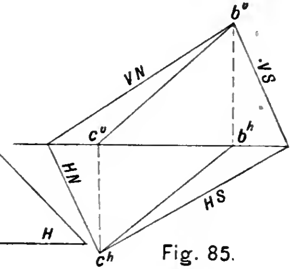



Fig. 85.

Figs. 86 and 87 illustrate a special condition of Case 1, in which one of the planes is parallel to a coördinate plane. Plane *N* is inclined to both *V* and *H*, and plane *R* is parallel to *H*. Since plane *R* is perpendicular to *V*, its vertical trace will be the vertical projection, $e^vf^v$, of the required line of intersection between planes *N* and *R*. As $e^v$ is the vertical trace of line *ef*, the horizontal projection of this line will be $e^hf^h$, which is parallel to *HN* (Art. 35, page 22).

**51.** Case 2. When one or both pairs of like traces of the given planes do not intersect within the limits of the drawing, and are not parallel, or when all the traces meet the ground line in the same point, auxiliary cutting planes parallel to either *V* or *H* may be used for the solution of the problem.

Method. 1. Pass an auxiliary cutting plane parallel to *V* or *H*, and determine the lines of intersection between the auxiliary plane and each of the given planes. Their point of intersection will be common to the given planes, and, therefore, a point in the required line. 2. Determine a second point by passing another auxiliary plane parallel to *V* or *H*, and the required line will be determined.

Construction. Fig. 88 represents two planes, *T* and *M*, with their horizontal traces intersecting, but their vertical traces intersecting beyond the limits of the drawing.

By Case 1, point *d* is one point in the line of intersection between the planes. To obtain a second point an auxiliary plane *X* has been passed parallel to *V*. Then *HX* is parallel to *GL*, and $C^h$, the horizontal projection of the line of intersection between planes *X* and *M*, lies in *HX*, while $C^v$ is parallel to *VM* (Case 1). Likewise $B^h$, the horizontal projection of the line of intersection between planes *X* and *T*, lies in *HX*, while $B^v$ is parallel to *VT*. Then point *a*, the point of intersection between lines *B* and *C*, is a point in the line of intersection between planes *M* and *T*, since point *a* lies in line *C* of plane *M*, and in line *B* of plane *T*. Line *ad* is, therefore, the required line of intersection between the two given planes.

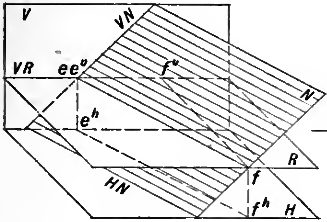



Fig. 86.

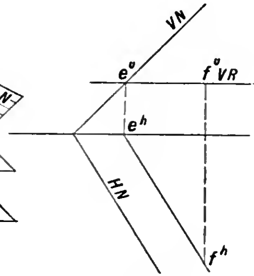



Fig. 87.

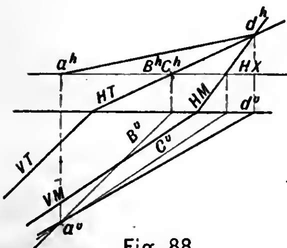



Fig. 88.

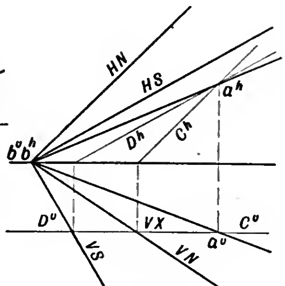



Fig. 89.

If neither the vertical nor horizontal traces of the given planes intersect within the limits of the drawing, it will be necessary to use two auxiliary cutting planes, both of which may be parallel to *V*, both parallel to *H*, or one parallel to *V* and one parallel to *H*.

**52.** If two traces of the given planes are parallel, the line of intersection between them will be parallel to a coördinate plane, and will have one projection parallel to the ground line and the other projection parallel to the parallel traces of the given planes.

**53.** Fig. 89 represents a condition when all four traces of the given planes intersect *GL* in the same point, neither plane containing *GL*. This point of intersection of the traces, *b*, is one point in the required line of intersection between the planes (Case 1). A second point, *a*, is obtained by passing the auxiliary cutting plane *X* parallel to *H*, intersecting plane *N* in line *C* and plane *S* in line *D*. Point *a*, the point of intersection of these lines, is a second point in the required line of intersection, *ab*, between the given planes *N* and *S* (Case 2).

54. CASE 3. When both intersecting planes are parallel to the ground line, or when one of the intersecting planes contains the ground line.

METHOD. 1. Pass an auxiliary cutting plane parallel to $P$. 2. Determine its line of intersection with each of the given planes. 3. The point of intersection of these two lines is one point in the required line of intersection between the two given planes. It is not necessary to determine a second point, for, when both given planes are parallel to the ground line, their line of intersection will be parallel to the ground line.

When one plane contains the ground line, one point in the line of intersection is the point of intersection of all the traces. (Figs. 91, 92.)

CONSTRUCTION. In Fig. 90 the two planes $N$ and $S$ are parallel to $GL$. Pass the auxiliary profile plane $P$ intersecting plane $N$ in the line whose profile projection is $D^p$, and the plane $S$ in the line whose profile projection is $B^p$. $A^p$, the intersection of these two lines, is the profile projection of one point in the line of intersection between planes $N$ and $S$; in fact $A^p$ is the profile projection of the required line, and $A^v$ and $A^h$ are the required projections.

55. In Fig. 91 the plane $S$ contains $GL$ and, therefore, it cannot be definitely located without its profile trace, or its angle with a coördinate plane, and the quadrants through which it passes. Plane $N$ is inclined to $V$, $H$, and $P$, and plane $S$ contains $GL$, passing through $1Q$ and $3Q$ at an angle $\theta$ with $V$. As in the previous example the profile auxiliary plane is required and $PN$, the profile trace of $N$, is determined as before. $PS$, the profile trace of $S$, is next drawn through $1Q$ and $3Q$ and making an angle $\theta$ with $VP$. Then $d^p$, the intersection of $PN$ and $PS$, is the profile projection of one point in the required line of intersection between planes $N$ and $S$, and $d^v$ and $d^h$ are the required projections of this point. The horizontal and vertical traces of this line of intersection are at $b$, for it is here that $VN$ and $VS$ intersect, and also where $HN$ and $HS$ intersect. The projections of two points of

the line having now been determined, the projections of the line, *db*, may be drawn.

Case 3 is applicable to all forms of this problem, and Fig. 92 represents the planes *N* and *S* of Fig. 89 with their line of intersection, *ab*, determined by this method.

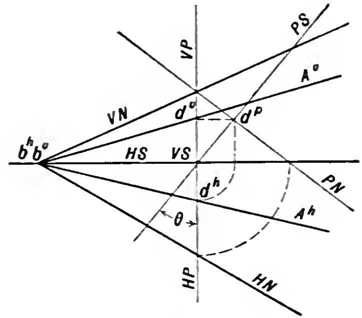



Fig. 91.

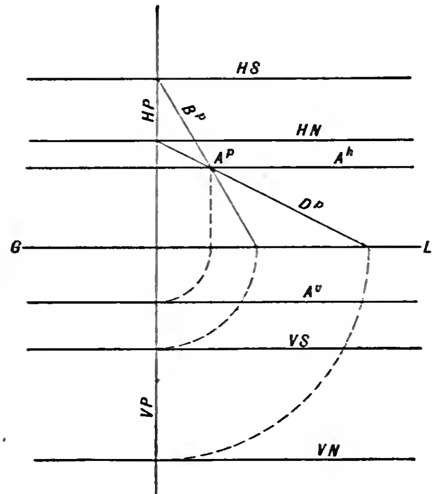



Fig. 90.

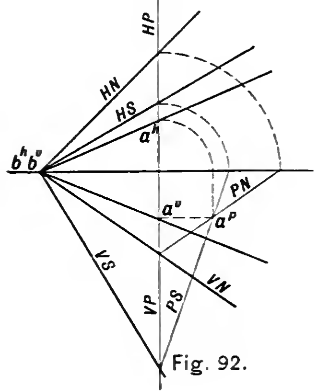



Fig. 92.

**56. Revolution, Quadrants, and Counter-revolution.** Fig. 93. Let it be required to pass a plane $S$ through the first and third quadrants, making an angle $\theta$ with $V$ and intersecting the oblique plane $N$ in the line $A$. The problem is solved by Art. 54, page 38, but this question may arise: In what direction shall $PS$ be drawn and with what line shall it make the given angle $\theta$? Conceive the auxiliary profile plane $P$ to be revolved about $VP$ as an axis until it coincides with $V$, that portion of $P$ which was in front of $V$ to be revolved to the right. Then the line $VP$ $HP$ represents the line of intersection between $V$ and $P$; $GL$ represents the line of intersection between $H$ and $P$; and $PN$, the line of intersection between $N$ and $P$. That portion of the paper lying above $GL$ and to the right of $VP$ will represent that portion of $P$ which, before revolution, was in $1Q$; above $GL$ and to the left of $VP$, in $2Q$; below $GL$ and to the left of $VP$, in $3Q$; and below $GL$ and to the right of $VP$, in $4Q$. Since $S$ is to pass through $1Q$ and $3Q$, making an angle $\theta$ with $V$, the line $PS$, in which $S$ intersects $P$, should be drawn on that portion of $P$ which is in $1Q$ and $3Q$, and should pass through the point of intersection of $VP$ and $GL$, making the angle $\theta$ with $VP$. $PS$ and $PN$ intersect in $d^p$, the profile projection of one point in the required line of intersection. In counter-revolution, that is, revolving $P$ back into its former position perpendicular to both $V$ and $H$, rotation will take place in a direction opposite to that of the first revolution, and $d^v$ and $d^h$ will be as indicated.

Fig. 94 represents the same problem when the auxiliary profile plane $P$ has been revolved in the opposite direction to coincide with $V$.

Figs. 95 and 96 are examples of the same problem when the auxiliary profile plane $P$ has been revolved about $HP$ as an axis until it coincides with $H$. Then $GL$ will represent the revolved position of the line of intersection between $V$ and $P$, and the line $VP$ $HP$ will represent the revolved position of the line of intersection between $H$ and $P$. $PS$ will then make its angle $\theta$ with $GL$, and its direction will be governed by the rotation assumed.

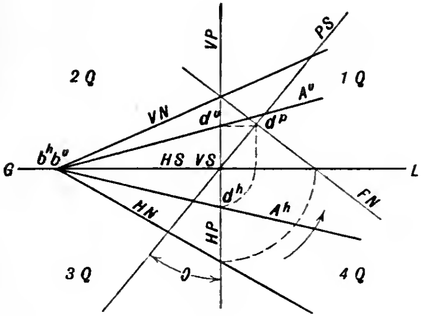



Fig. 93.

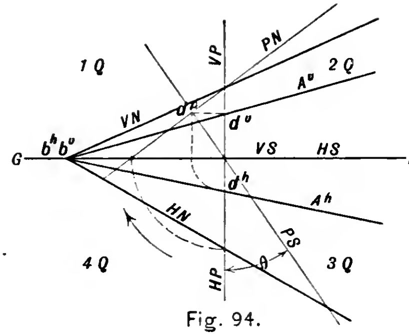



Fig. 94.

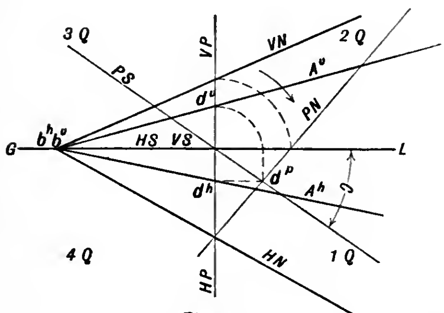



Fig. 95.

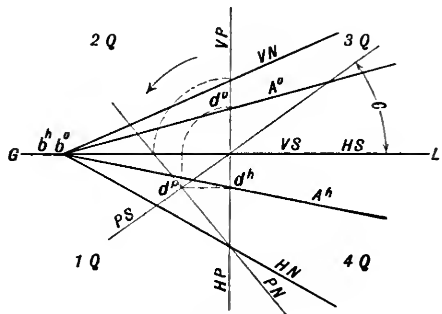



Fig. 96.

**57. To determine the point in which a line pierces a plane.**

METHOD. 1. Pass an auxiliary plane through the line to intersect the given plane. 2. Determine the line of intersection between the given and auxiliary planes. 3. The required point will lie at the intersection of the given line and the line of intersection between the given and auxiliary planes.

There may be four cases as follows:

Case 1. When any auxiliary plane containing the line is used.

Case 2. When the horizontal or vertical projecting plane of the line is used.

Case 3. When the given line is parallel to *P*, thus necessitating the use of an auxiliary profile plane containing the line.

Case 4. When the given plane is defined by two lines which are not the traces of the plane.

**58.** CASE 1. When any auxiliary plane containing the line is used.

CONSTRUCTION. Fig. 97. Let *A* be the given line and *N* the given plane. Through line *A* pass any auxiliary plane *Z* (Art. 29, page 20) intersecting plane *N* in line *C* (Art. 50, page 35). Since lines *A* and *C* lie in plane *Z*, *d* is their point of intersection, and since *C* is a line of plane *N*, point *d* is common to both line *A* and plane *N*; hence, their intersection.

**59.** CASE 2. When the horizontal or vertical projecting plane of the line is used.

CONSTRUCTION. Fig. 98 represents the same line *A* and plane *N* of the previous figure. Pass the horizontal projecting plane *X* of line *A* (Art. 8, page 7), intersecting plane *N* in line *C* (Art. 50, page 35). Lines *A* and *C* intersect in point *d*, the required point of piercing of line *A* and plane *N*.

Fig. 99 is the solution of the same problem by the use of plane *Y*, the vertical projecting plane of line *A*.

**60.** CASE 3. When the given line is parallel to *P*, thus necessitating the use of an auxiliary profile plane containing the line.

CONSTRUCTION. Fig. 100. Let *N* be the given plane and line *ab*, parallel to *P*, the given line. Pass an auxiliary profile plane *P* through the given line *ab*, intersecting plane *N* in the line $C$, having $C^p$ for its profile projection (Art. 54, page 38). Also determine $a^pb^p$, the profile projection of the given line (Art. 21, page 14). $C^p$ and $a^pb^p$ intersect in $d^p$, the profile projection of the required point of piercing of line $ab$ and plane $N$. The vertical and horizontal projections of this point are $d^v$ and $d^h$.

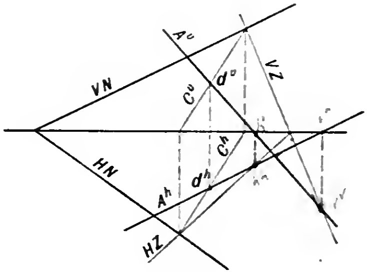



Fig. 97.

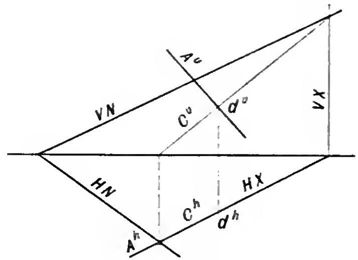



Fig. 98.

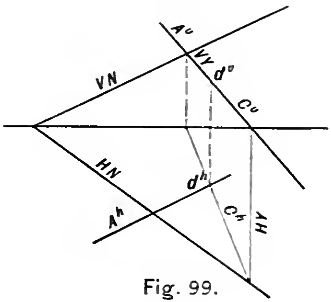



Fig. 99.

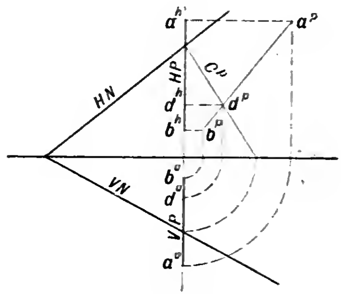



Fig. 100.

**61.** CASE 4. When the given plane is defined by two lines which are not its traces.

CONSTRUCTION. Figs. 101 and 102. Let the given plane be defined by lines $B$ and $D$, and let $A$ be the given line intersecting this plane at some point to be determined. This case should be solved without the use of the traces of any plane. Pass the horizontal projecting plane of line $A$ intersecting line $B$ in point $e$, and line $D$ in point $f$, and consequently intersecting the plane of lines $B$ and $D$ in line $ef$. Because this horizontal projecting plane of line $A$ is perpendicular to $H$ all lines lying in it will have their horizontal projections coinciding, and $e^h f^h$ will be the horizontal projection of the line of intersection between the auxiliary plane and the plane of lines $B$ and $D$. Draw the vertical projection of this line through the vertical projections of points $e$ and $f$; its intersection with $A^v$, at $d^v$, will be the vertical projection of the required point of piercing of line $A$ with the plane of lines $B$ and $D$.

The same result may be obtained by the use of the vertical projecting plane of line $A$.

**62. If a right line is perpendicular to a plane, the projections of that line will be perpendicular to the traces of the plane.** In Figs. 103 and 104 $HN$ and $VN$ are the traces of a plane to which line $A$ is perpendicular. The horizontal projecting plane of line $A$ is perpendicular to $H$ by construction; it is also perpendicular to plane $N$ because it contains a line, $A$, perpendicular to $N$; therefore, being perpendicular to two planes it is perpendicular to their line of intersection, $HN$. But $HN$ is perpendicular to every line in the horizontal projecting plane of line $A$ which intersects it, and, therefore, perpendicular to $A^h$. Q.E.D.

In like manner $VN$ may be proved to be perpendicular to $A$. The converse of this proposition is true.

**63. To project a point on to an oblique plane.**

PRINCIPLE. The projection of a point on any plane is the intersection with that plane of a perpendicular let fall from the point to the plane.

METHOD. 1. From the given point draw

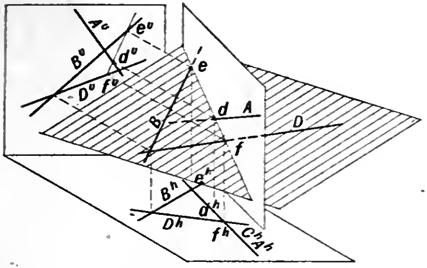


Fig. 101.

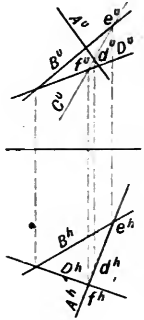


Fig. 102.

a perpendicular to the given plane (Art. 62, page 44). 2. Determine the point of piercing of this perpendicular and the plane (Art. 57, page 42). This point of piercing is the required projection of the given point upon the given oblique plane.

**64. To project a given line on to a given oblique plane.**

METHOD. If the line is a right line, project two of its points (Art. 63, page 44). If the line is curved, project a sufficient number of its points to describe the curve.

**65. To determine the shortest distance from a point to a plane.**

PRINCIPLE. The shortest distance from a point to a plane is the perpendicular distance from the point to the plane.

METHOD. 1. From the given point draw a perpendicular to the given plane (Art. 62, page 44). 2. Determine the point of piercing of the perpendicular and the plane (Art. 57, page 42). 3. Determine the true length of the perpendicular between the point and the plane (Art. 39, page 27).

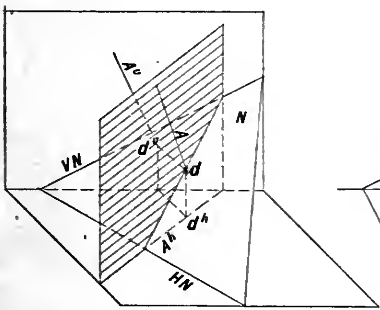


Fig. 103.

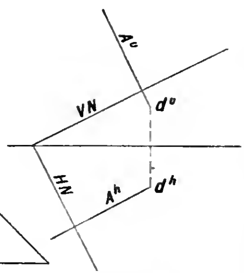


Fig. 104.

**66. Shades and Shadows.** The graphic representation of objects, especially those of an architectural character, may be made more effective and more easily understood by drawing the shadow cast by the object.

When a body is subjected to rays of light, that portion which is turned away from the source of light, and which, therefore, does not receive any of its rays, is said to be in **shade**. See Fig. 105. When a surface is in light and an object is placed between it and the source of light, intercepting thereby some of the rays, that portion of the surface from which light is thus excluded is said to be in **shadow**. That portion of space from which light is excluded is called the **umbra** or invisible shadow.

(*a*) The umbra of a point in space is evidently a line.

(*b*) The umbra of a line is in general a plane.

(*c*) The umbra of a plane is in general a solid.

(*d*) It is also evident from Fig. 105 that the shadow of an object upon another object is the intersection of the umbra of the first object with the surface of the second object.

The line of separation between the portion of an object in light and the portion in shade is called the *shade line*. It is evident from Fig. 105 that the shade line is the boundary of the shade. It is also evident that the shadow of the object is the space inclosed by the shadow of its shade line.

**The source of light** is supposed to be at an infinite distance; therefore, the rays of light will be parallel and will be represented by straight lines. The assumed direction of the conventional ray of light is that of the diagonal of a cube, sloping downward, backward, and to the right, the cube being placed so that its faces are either parallel or perpendicular to *V*, *H*, and *P* (Fig. 106). This ray of light makes an angle of 35° 15′ 52″ with the coördinate planes of projection, but from Figs. 106 and 107 it will be observed that the *projections* of the ray are diagonals of squares, and hence, they make angles of 45° with *GL*.

Since an object is represented by its projections, the ray of light must be represented by its projections.

An object must be situated in the first quadrant to cast a shadow upon both *V* and *H*.

**67. To determine the shadow of a point on a given surface** pass the umbra of the point, or as is generally termed, pass a ray of light through the point and determine its intersection with the given surface by Art. 24, page 16, or by Art. 57, page 42, according as the shadow is required on a coördinate or on an oblique plane.

**68. To determine the shadow of a line upon a given surface** it is necessary to determine the intersection of its umbra with that surface. If the line be a right line, this is generally best accomplished by finding the shadows of each end of the line and joining them. If the line be curved, then the shadows of several points of the line must be obtained.

In Figs. 108 and 109 *cb* is an oblique line having one extremity, *c*, in *H*. By passing the ray of light through *b* and locating its horizontal trace, the shadow of *b* is found to fall upon *H* at $b^{sh}$. Since *c* lies in *H*, it is its own shadow upon *H*; therefore, the shadow of line *cb* upon *H* is $c^h b^{sh}$.

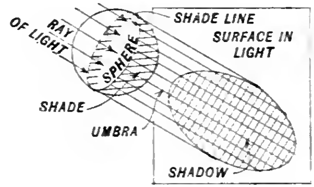


Fig. 105.

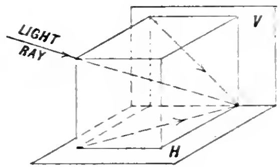


Fig. 106.

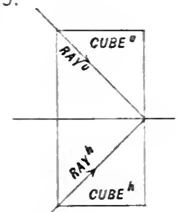


Fig. 107.

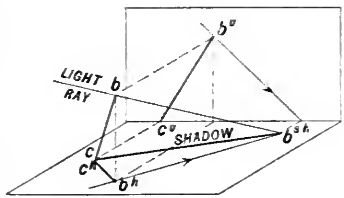


Fig. 108.

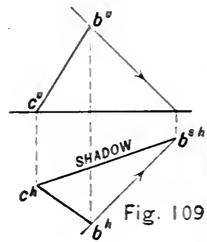


Fig. 109.

**69. To determine the shadow of a solid upon a given surface** it is necessary to determine the shadows of its shade lines. When it is difficult to recognize which lines of the object are its shade lines, it is well to cast the shadow of every line of the object. The outline of these shadows will be the required result.

Fig. 110 represents an hexagonal prism located in the first quadrant with its axis perpendicular to *H*. Its shadow is represented as falling wholly upon *V*, as it would appear if *H* were removed. Fig. 111 represents the prism when the shadow falls wholly upon *H*, as it would appear if *V* were removed. Fig. 112 represents the same prism when both *V* and *H* are in position, a portion of the shadow falling upon *V*, and a portion falling upon *H*. It is now readily observed that the shade lines are *bc*, *cd*, *de*, *ep*, *ps*, *sg*, *gk*, and *kb*, and that the shadow of the prism is the polygon inclosed by the shadows of these shade lines.

When an object is so located that its shadow falls partly upon *V* and partly upon *H*, it is generally best to determine first, its complete shadow upon both *V* and *H*, and to retain only such portions of the shadow as fall upon *V* above *H*, and upon *H* before *V*.

Fig. 113 represents a right hexagonal pyramid resting upon an oblique plane *N* and having its axis perpendicular to that plane. Its shadow has been cast upon *N*, and the construction necessary to determine the shadow of one point, *a*, has been shown (Art. 59, page 42).

From the foregoing figures these facts will be observed:

If a point lies on a plane, it is its own shadow upon that plane. See point *C*, Figs. 108, 109, and the apex of the pyramid, Fig. 113.

If a line is parallel to a plane, its shadow upon that plane will be parallel, and equal in length, to the line in space. Lines *bk* and *ep*, Fig. 110, are parallel to *V*; lines *bc* and *sp*, *cd* and *gs*, *de* and *gk*, Fig. 111, are parallel to *H*, and the sides of the hexagon, Fig. 113, are parallel to *N*.

If two lines are parallel, their shadows are parallel. Observe that the shadows of the opposite sides of the hexagons, and of the lateral edges of the prisms, are parallel.

If a line is perpendicular to a coördinate plane, its shadow on that plane will fall on the projections of the rays of light passing through it. Lines *bk* and *ep*, Fig. 111, fulfill this condition.

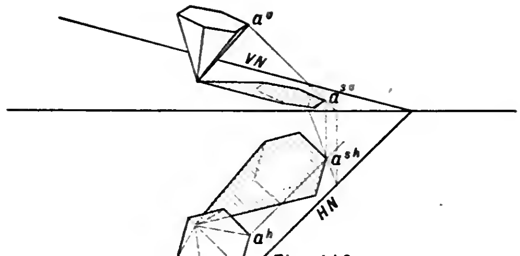


Fig. 113.

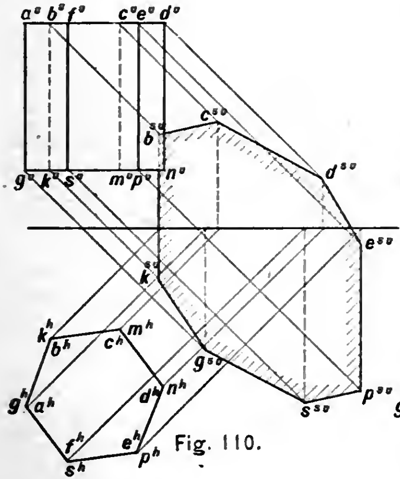


Fig. 110.

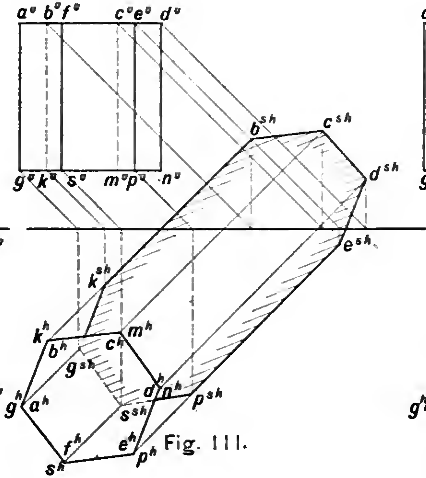


Fig. 111.

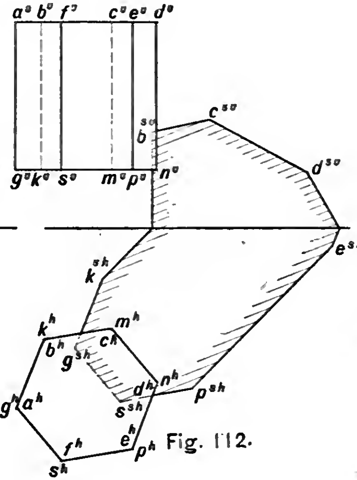


Fig. 112.

**70. Through a point or line to pass a plane having a defined relation to a given line or plane.** There may be five cases, as follows:

Case 1. To pass a plane through a given point parallel to a given plane.

Case 2. To pass a plane through a given point perpendicular to a given line.

Case 3. To pass a plane through a given point parallel to two given lines.

Case 4. To pass a plane through a given line parallel to another given line.

Case 5. To pass a plane through a given line perpendicular to a given plane.

In cases 1 and 2 the directions of the required traces are known. In cases 3, 4, and 5 two lines of the required plane are known.

**71. Case 1. To pass a plane through a given point parallel to a given plane.**

PRINCIPLE. Since the required plane is to be parallel to the given plane, their traces will be parallel, and a line through the given point parallel to either trace will determine the plane.

METHOD. 1. Through the given point pass a line parallel to one of the coördinate planes and lying in the required plane (Art. 9, page 8). 2. Determine the trace of this line, thus determining one point in the required trace of the plane. 3. Draw the traces parallel to those of the given plane.

CONSTRUCTION. Fig. 114. Let $N$ be the given plane and $b$ the given point. Through $b$ pass line $A$ parallel to $V$ and in such a direction that it will lie in the required plane, $A^h$ being parallel to $GL$ and $A^v$ parallel to $VN$ (Art. 35, page 22). Determine $d$, the horizontal trace of line $A$, and through $d^h$ draw $HS$, the horizontal trace of the required plane, parallel to $HN$. $VS$ will be parallel to $A^v$.

**72. Case 2. To pass a plane through a given point perpendicular to a given line.**

PRINCIPLE. Since the required plane is to be perpendicular to the given line, the traces of the plane will be perpendicular to the projections of the line (Art. 62, page 44). Hence, a line drawn through the given point parallel to either coördinate plane, and lying in the required plane, will determine this plane.

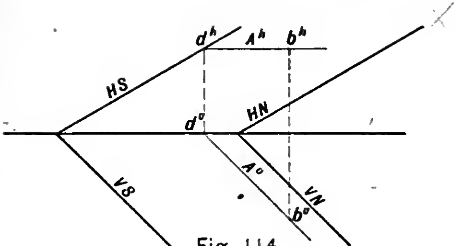



Fig. 114.

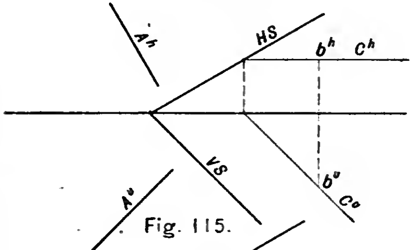



Fig. 115.

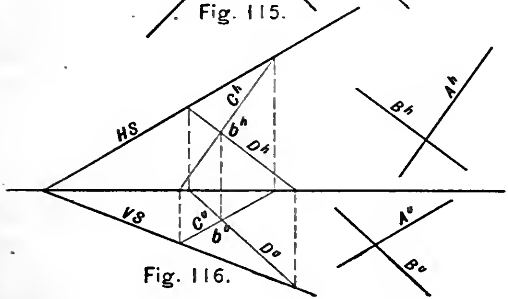



Fig. 116.

Method. 1. Through the given point pass a line parallel to one of the coördinate planes and lying in the required plane. 2. Determine the trace of this line, thus determining one point in the required trace of the plane. 3. Draw the traces perpendicular to the projections of the given line.

Construction. Fig. 115. Through point $b$ draw line $C$ parallel to $H$ and lying in the required plane $S$. $C^v$ will be perpendicular to $A^v$ (Art. 62, page 44). $HS$ and $VS$ are the required traces.

**73. Case 3. To pass a plane through a given point parallel to two given lines.**

Principle. The required plane will contain lines drawn through the given point parallel to the given lines.

Method. 1. Through the given point pass two lines parallel to the two given lines. 2. Determine the plane of these lines (Art. 30, page 20).

Construction. Fig. 116. Given lines $A$ and $B$ and point $b$. Through point $b$ pass lines $C$ and $D$ parallel respectively to lines $A$ and $B$. $S$, the plane of these lines, is the required plane.

**74. Case 4. To pass a plane through a given line parallel to another given line.**

PRINCIPLE. The required plane will contain one of the given lines and a line intersecting it and parallel to the second.

METHOD. 1. Through any point of the given line pass a line parallel to the second given line. 2. Determine the plane of these intersecting lines (Art. 30, page 20).

**75. Case 5. To pass a plane through a given line perpendicular to a given plane.**

PRINCIPLE. The required plane will contain the given line and a line intersecting this line and perpendicular to the given plane.

METHOD. 1. Through any point of the given line pass a line perpendicular to the given plane. 2. Determine the plane of these lines.

CONSTRUCTION. Fig. 117. Given line $A$ and plane $N$. Through any point of line $A$ pass line $C$ perpendicular to plane $N$ (Art. 62, page 44). Determine $S$, the plane of lines $A$ and $C$ (Art. 30, page 20).

**76. Special conditions and methods of Art. 70.**

CASE 1.* To pass a plane through a given point parallel to a given plane.

Fig. 118 illustrates a condition in which the given plane $N$ is parallel to $GL$, which necessitates the use of an auxiliary profile plane. Through the given point $b$ pass the profile plane $P$. Determine $b^p$ and $PN$, and through $b^p$ draw $PS$, the profile trace of the required plane, parallel to $PN$, whence $VS$ and $HS$, the traces of the required plane, are determined.

This condition may also be solved, Fig. 119, by passing a line $A$ through the given point $b$, parallel to *any* line $C$ of the given plane $N$. Through the traces of line $A$ the traces of the required plane $S$ are drawn parallel to $HN$ and $VN$, respectively.

This method is applicable to all conditions of Case 1.

77. CASE 2.* To pass a plane through a given point perpendicular to a given line.

Fig. 120 illustrates a condition in which the projections of the given line are perpendicular to $GL$; hence, the traces of the required plane will be parallel to $GL$. Let $ac$ be the given

* The numbers of the cases are the same as those in Art. 70, page 50.

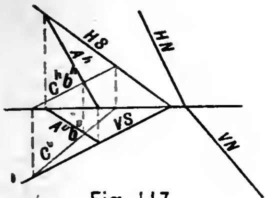


Fig. 117.

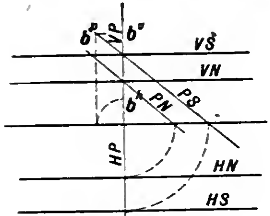


Fig. 118.

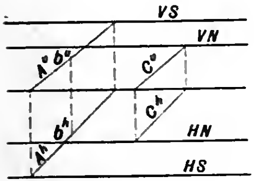


Fig. 119.

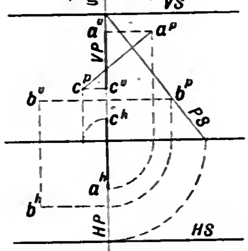


Fig. 120.

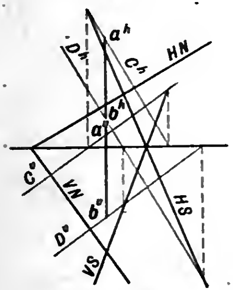


Fig. 121.

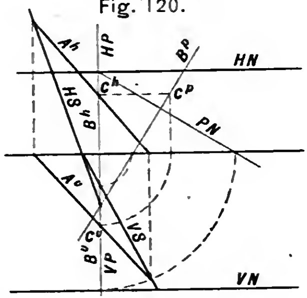


Fig. 122.

line and $b$ the given point. Pass an auxiliary profile plane $P$, and determine $a^p c^p$, the profile projection of the given line, and $b^p$, the profile projection of the given point. Through $b^p$ draw $PS$, the profile trace of the required plane, perpendicular to $a^p c^p$, whence $VS$ and $HS$ are determined.

78. Case 5.* To pass a plane through a given line perpendicular to a given plane.

Fig. 121 illustrates a condition in which the given line $ab$ is parallel to $P$. This may be solved by the use of an auxiliary profile plane, or by the following method: Through two points of the given line $ab$ pass auxiliary lines $C$ and $D$ perpendicular to the given plane $N$, and determine $S$, the plane of these parallel lines. Then $S$ is the required plane containing line $ab$ and perpendicular to plane $N$.

Fig. 122 illustrates a condition in which the given plane $N$ is parallel to $GL$. This solution is identical with that of Art. 75, page 52, save that to determine the traces of the auxiliary line $B$, an auxiliary profile plane $P$ has been used. Since $B$ is to be perpendicular to $N$, $B^p$ must be perpendicular to $PN$.

**79. To determine the projections and true length of the line measuring the shortest distance between two right lines not in the same plane.**

PRINCIPLE. The shortest distance between two right lines not in the same plane is the perpendicular distance between them, and only one perpendicular can be drawn terminating in these two lines.

METHOD. 1. Through one of the given lines pass a plane parallel to the second given line. 2. Project the second line on to the plane passed through the first. 3. At the point of intersection of the first line and the projection of the second erect a perpendicular to the plane. This perpendicular will intersect the second line. 4. Determine the true length of the perpendicular, thus obtaining the required result.

CONSTRUCTION. Figs. 123 and 124. Given lines $A$ and $B$. Pass the plane $S$ through line $A$ parallel to line $B$ (Art. 74, page 52). Project line $B$ on to plane $S$ at $B_1$, using the auxiliary line $D$ perpendicular to plane $S$ and intersecting it at point $k$ (Art. 64, page 45). Since line $B$ is parallel to plane $S$, $B_1$ will be parallel to $B$; hence, their projections are parallel (Art. 13, page 8). $B_1$ intersects $A$ at $n$, at which point erect the required line $E$ perpendicular to plane $S$ (Art. 62, page 44), intersecting line $B$ at $o$. Determine the true length of line $E$ (not shown in the figure) by Art. 39, page 27.

**80. To determine the angle between a line and a plane.**

PRINCIPLE. The angle which a line makes with a plane is the angle which the line makes with its projection on the plane, or the complement of the angle which the line makes with a perpendicular which may project any point of the line on to the plane. The line in space, its projection on the plane, and the projector, form a right-angled triangle.

METHOD. 1. Through any point of the given line drop a perpendicular to the given plane. 2. Determine the angle between this perpendicular and the given line. The angle thus determined is the complement of the required angle.

Construction. Fig. 125. Given line *A* and plane *N*. Through any point *c*, of line *A*, pass line *B* perpendicular to plane *N* (Art. 62, page 44). Determine one of the traces of the plane of lines *A* and *B*, as *VS*. Revolve lines *A* and *B* into *V* about *VS* as an axis, as at *A'* and *B'* (Art. 43, page 31). Then angle $d^v c' e^v$ is the true size of the angle between lines *A* and *B*, and its complement, $e^v c' f$, is the required angle between line *A* and plane *N* (Art. 44, page 31).

Fig. 126 illustrates a condition in which the given plane *N* is parallel with *GL*: hence, the auxiliary line *B*, perpendicular to *N*, is parallel to *P*, thus requiring a profile projection to determine its traces. *VS* is the vertical trace of the plane of lines *A* and *B* (its other trace is not necessary), and angle $e^v c' f$ is the required angle between line *A* and plane *N*.

**81. To determine the angle between a line and the coördinate planes.**

Principle. This is a special case of Art. 80, page 54.

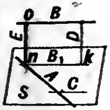


Fig. 123.

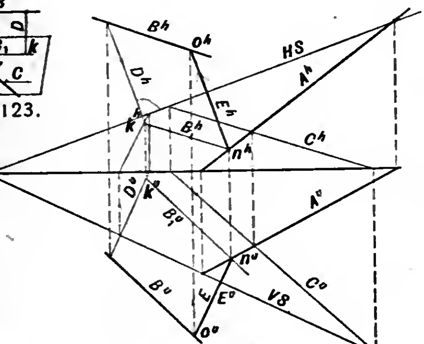


Fig. 124.

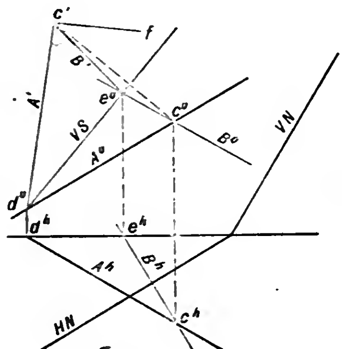


Fig. 125.

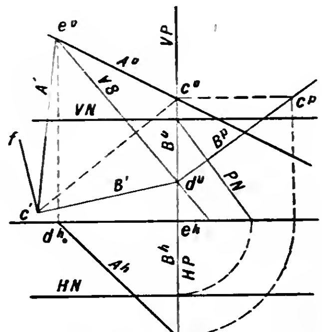


Fig. 126.

1ST METHOD. 1. Revolve the line about its horizontal projection as an axis to obtain the angle which it makes with $H$. 2. Revolve the line about its vertical projection as an axis to obtain the angle which the line makes with $V$.

CONSTRUCTION. Fig. 127 represents the given line $A$ when revolved into $H$ and measures $\alpha$, the true angle which the line makes with $H$. Fig. 127 also represents the line as revolved into $V$ and measuring $\beta$, the true angle which the line makes with $V$ (Art. 41, page 28).

2ND METHOD. 1. Revolve the line parallel to $V$ to obtain the angle which the line makes with $H$. 2. Revolve the line parallel to $H$ to obtain the angle which the line makes with $V$.

CONSTRUCTION. Fig. 128 represents the given line $A$ when revolved parallel to $V$ and measures $\alpha$, the true angle which the line makes with $H$. Fig. 128 also represents the line $A$ as revolved parallel to $H$ and measuring $\beta$, the true angle which the line makes with $V$.

**82. To determine the projections of a line of definite length passing through a given point and making given angles with the coördinate planes.**

PRINCIPLE. This is the converse of Art. 81, page 55, and there may be eight solutions; but the sum of the angles, $\alpha$ and $\beta$, which the line makes with the coördinate planes, cannot be greater than 90°.

CONSTRUCTION. Fig. 128. Let it be required to draw a line through point $b$ having a length equal to $x$ and making angles of $\alpha$ and $\beta$ with $H$ and $V$, respectively. Through $b^v$ draw $b^v c_1^v$ equal to $x$ and making angle $\alpha$ with $GL$; also $b^h c_1^h$ parallel to $GL$. These projections will represent the revolved position of the line, making its required angle $\alpha$ with $H$. Similarly draw the projections $b^h c_2^h$, $b^v c_2^v$ to show the revolved position and required angle $\beta$ with $V$.

In counter-revolution about a vertical axis through $b$, all possible horizontal projections of line $A$ will be drawn from $b^h$ to a line through $c_2^h$ and parallel to $GL$. But all possible horizontal projections must be drawn from $b^h$ to the arc $c_1^h c^h$; hence, $c^h$, the intersection of this arc and the parallel through $c_2^h$, will be the horizontal projection of the other extremity of the required line. Since the vertical projection of this point must also lie in the parallel to $GL$ through $c_1^v$, the line is definitely determined.

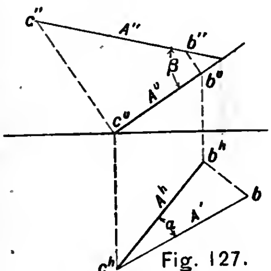



Fig. 127.

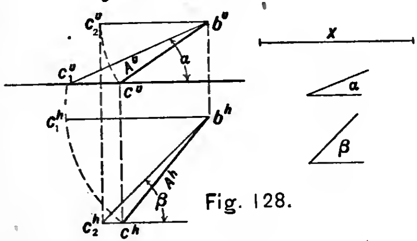



Fig. 128.

**83. To determine the angle between two planes.**

PRINCIPLE. If a plane be passed perpendicular to the edge of a diedral angle, it intersects the planes of the diedral in lines, the angle between which is known as the plane angle of the diedral. The diedral angle between two planes is measured by its plane angle.

METHOD. 1. Determine the plane angle of the given diedral angle by passing an auxiliary plane perpendicular to the line of intersection between the two given planes, and, therefore, perpendicular to each of these given planes. 2. Find the lines of intersection between this auxiliary plane and the given planes. 3. Determine the true size of this plane angle.

**84.** CASE 1. When it is required to determine the angle between any two oblique planes.

CONSTRUCTION. Figs. 129 and 130. The lettering of these two figures is identical although the diedral angles differ.

Pass the auxiliary plane $S$ perpendicular to $A$, the line of intersection between the planes $N$ and $L$. Only one trace of plane $S$ is necessary for the solution of the problem and in this example the horizontal trace has been selected. Then $HS$ must be perpendicular to $A^h$ (Art. 62, page 44). Its intersection with the horizontal traces of the given planes will determine points $f$ and $e$, one in each line of intersection between the auxiliary and given planes. The point $d$, which is common to both lines of intersection, may be obtained as follows: The horizontal projecting plane of $A$ cuts the auxiliary plane in $dc$, a line perpendicular to $A$. By revolving $A$ into $H$ the revolved position of $dc$ may be drawn, as at $d'c$, and by counter-revolution point $d$ obtained. Next revolve $de$ and $df$ into $H$ to measure the angle between them, which is the required diedral angle $ed''f$.

One example of this problem which is commonly met in practice is here shown in solution in Fig. 131. Let $N$ and $L$ be the given planes, $A$ their line of intersection, and plane $S$ the plane passed perpendicular to $A$. Then angle $e^hd''f^h$ is the true size of the angle between planes $N$ and $L$.

**85.** Another method for determining the size of the angle between two planes is illustrated by Fig. 132. This is a pictorial representation of two intersecting planes, $N$ and $R$. From any point in space, as point $a$, two lines are dropped perpendicular, one to each plane. The angle between these perpendiculars is the measurement of the angle between planes $N$ and $R$, or its supplement.

**86.** CASE 2. When it is required to determine the angle between an inclined plane and either coördinate plane.

CONSTRUCTION. Figs. 133 and 134. Let it be required to determine the true size of the angle between planes $N$ and $H$. Pass the auxiliary plane $X$ perpendicular to both $N$ and $H$, and, therefore, perpendicular to their line of intersection, $HN$. Then $HX$ and $VX$ are perpendicular respectively to $HN$ and $GL$.

The angle between the lines $ab$ and $HX$, in which plane $X$ intersects planes $N$ and $H$ respectively, is the required angle, the true size of which, $a$, is determined by revolving the triangle containing it into $V$ about $VX$ as an axis, or into $H$ about $HX$ as an axis.

The line $ab$ is known as the line of *maximum inclination* of plane $N$ with $H$.

Suppose it is required to determine the true size of the angle between planes $N$ and $V$. Determine the line of maximum inclination of plane $N$ with $V$, and its angle with $V$.

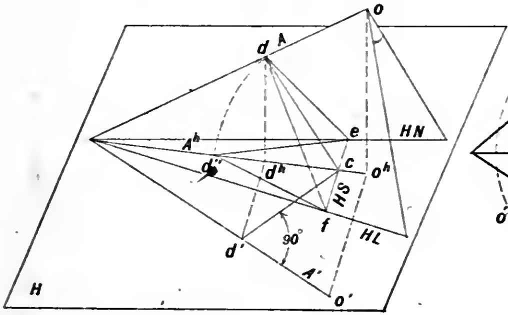



Fig. 129.

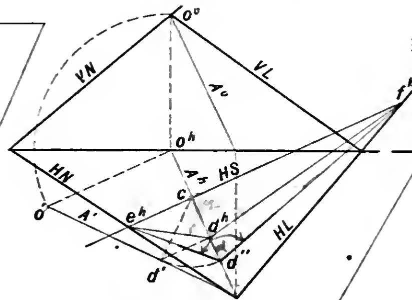



Fig. 130.

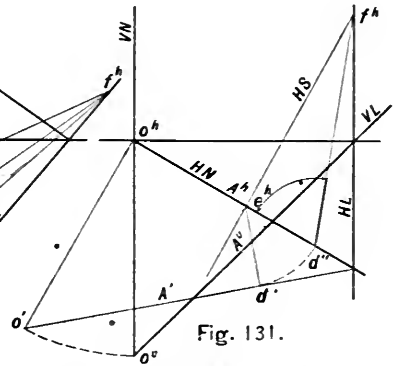



Fig. 131.

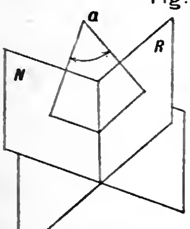



Fig. 132.

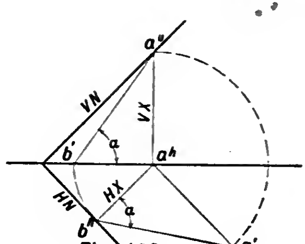



Fig. 133.

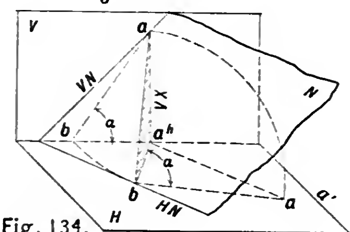



Fig. 134.

**87. To determine the bevels for the correct cuts, the lengths of hip and jack rafters, and the bevels for the purlins for a hip roof.**

Fig. 135 represents an elevation and plan of a common type of hip roof having a pitch equal to $\frac{a^v d}{d^v b^v}$ and a width of $c^v b^v$. *ab* is the center line of the hip rafter; *E* is a cross section of the ridge; *F*, *K*, *L*, *M*, are jack rafters.

Hip. To find the true length of the hip rafter *eb*, and the following angles:

Down cut, *1:* The intersection of the hip with the ridge.

Heel cut, *2:* The intersection of the hip with the plate.

Side cut, *3:* Intersection of the hip with the ridge.

Top bevel of hip, *4*.

The true length of the hip *eb* may be obtained by revolving it parallel to *V*, as at $e_1b^v$, or parallel to *H*, as at $e_2b^h$. The down cut bevel, *3*, is obtained at the same time. The bevel of the top edge of hip is found by passing a plane perpendicular to *ab* intersecting the planes of the side and end roofs. *HZ* is the horizontal trace of this plane, and the bevel, *4*, is obtained as in Art. 84, page 57.

Jack Rafters. The down cut bevel, *5*, and the heel cut bevel, *6*, of the jack rafters are shown in their true values in the elevation; and the side cut, *7*, is shown in the plan. The true lengths of the jack rafters are obtained by extending the planes of their edges to intersect the revolved position of the hip rafter, as at $n_1o^h$.

Purlin. It is required to determine the down cut, *8*, side cut, *9*, and angle between side and end face of purlin, *10*. To obtain the down cut revolve side face parallel to *H* and the true angle, *8*, will be obtained. Similarly, the side cut made on the top or bottom face is obtained by revolving that face parallel to *H*, the true angle being indicated by bevel, *9*. In order to obtain the angle between the side and end faces, the planes of which are indicated in the figure by *S* and *R*, find the intersection between these planes and determine the angle as in Art. 84, page 57.

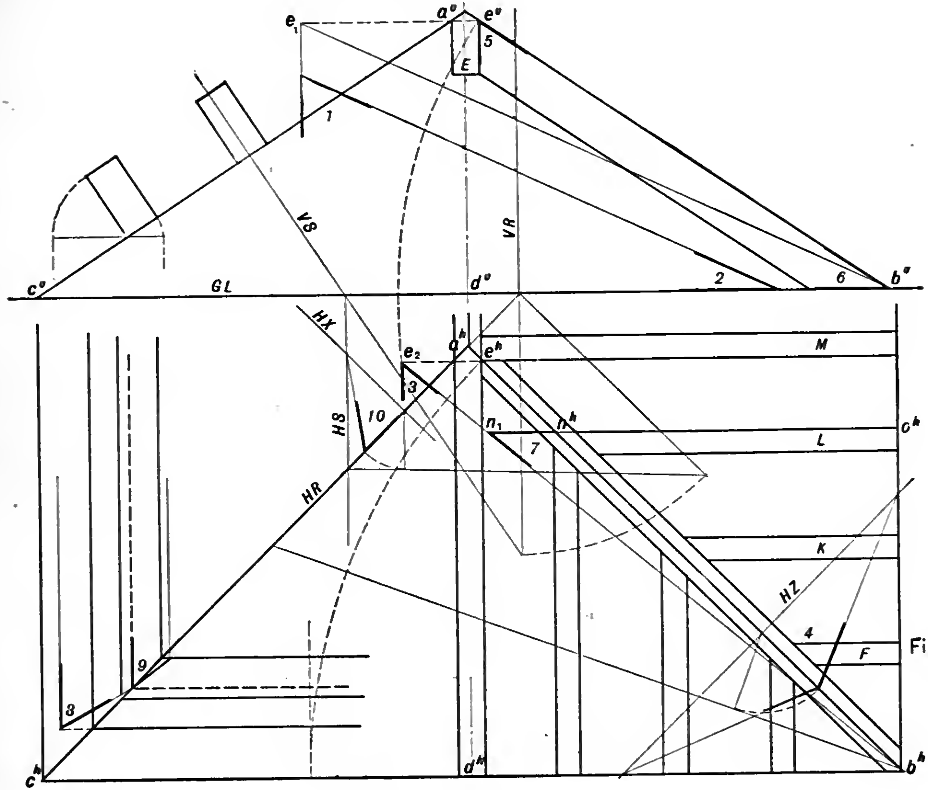

e₁
a^v
e^v
5
E
1
VS
VR
c^v
GL
d^v
2
6
b^v
HX
a^h
e^h
e₂
M
3
HS
10
n₁
n^h
c^h
7
L
HR
K
HZ
4
F
9
8
c^h
d^h
b^h

Fig. 135.

**88. Given one trace of a plane, and the angle between the plane and the coördinate plane, to determine the other trace.**

There must be two cases, as follows:

CASE 1. Given the trace on one coördinate plane and the angle which the plane makes with the *same* coördinate plane.

CONSTRUCTION. Fig. 136. Let $HT$ be the given trace and $a$ the given angle between $T$ and $H$. Draw $A^h$ perpendicular to $HT$, it being the horizontal projection of the line of maximum inclination with $H$ (Art. 86, page 58). If this line be revolved into $H$, it will make the angle $a$ with $A^h$, and $d'$ will be the resolved position of the vertical trace of the line of maximum inclination with $H$. The vertical projection of this trace must lie on the vertical trace of the horizontal projecting plane of $A$ and at a distance from $GL$ equal to $d^h d'$. Therefore, $d^v$ will be a point in $VT$, the required trace.

There may be two solutions, as $VT$ may be above or below $GL$.

89. CASE 2. Given the trace on one coördinate plane and the angle which the plane makes with the *other* coördinate plane.

CONSTRUCTION. Fig. 137. Let $HT$ be the given trace and $\beta$ the given angle between $T$ and $V$.

Consider $B$, the line of maximum inclination with $V$, as revolved about the horizontal trace of its vertical projecting plane and making an angle, $\beta$, with $GL$. $d'$ will be the revolved position of the vertical trace of $B$. From $e$, with radius $ed'$, describe arc $d'd^v$. $VT$, the required trace, will be tangent to this arc. There may be two solutions, as $VT$ may be above or below $GL$.

**90. To determine the traces of a plane, knowing the angles which the plane makes with both coördinate planes.**

CONSTRUCTION. Fig. 138. Let it be required to construct the traces of plane $T$, making an angle $\beta$ with $V$ and angle $a$ with $H$. The sum of $a$ and $\beta$ must not be less than 90° nor more than 180°. Conceive the required plane $T$ as being tangent to a sphere, the center of which, $c$, lies in $GL$. From

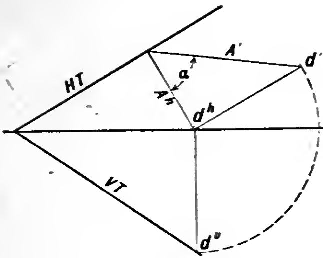



Fig. 136.

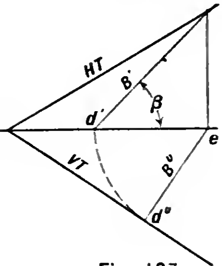



Fig. 137.

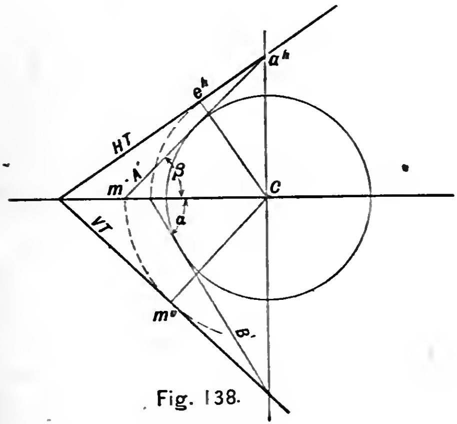



Fig. 138.

the point of tangency of the sphere and plane $T$, conceive to be drawn the lines of maximum inclination with $H$ and $V$. On revolving the line of maximum inclination with $V$, into $H$, about the horizontal axis of the sphere, as an axis, it will continue tangent to the sphere, as at $A'$, making an angle with $GL$ equal to the required angle (between $V$ and $T$), and its horizontal trace will lie in the axis of revolution at $a^h$, its vertical trace lying in the circle described from $c$ as a center, with a radius $cm'$. Similarly revolve the line of maximum inclination with $H$ into $V$, as at $B'$, making an angle with $GL$ equal to the required angle (between $H$ and $T$), and its vertical trace will lie in the axis of revolution at $k^v$, its horizontal trace lying in the circle described from $c$ as a center, with a radius $ce'$. Then $HT$ will contain $a^h$ and be tangent to arc $a^he'$, and $VT$ will contain $k^v$ and be tangent to arc $k^vm'$. The traces must intersect $GL$ in the same point. Both $HT$ and $VT$ may be either above or below $GL$.

**91.** In Art. 48, page 34, reference was made to a fourth construction for counter-revolution. This construction involves the angle of maximum inclination between the oblique and coördinate planes. Let it be required to draw the projections of a regular pentagon lying in plane $N$, Fig. 139, when its revolved position is known. Pass plane $X$ perpendicular to both $N$ and $V$, intersecting plane $N$ in the line of maximum inclination with $V$, shown in revolved position as $gf''$. Since this line shows in its true length, all distances on plane $N$ perpendicular to $VN$ may be laid off on it. Then $ge''$ represents the distance from point $e$ to $VN$, and $e^v$ will lie on a line through $e''$ parallel to $VN$. Likewise other points of the pentagon are determined. Also $e''k$ is the distance from point $e$ in space to $V$; hence, $e^h$ will lie at a distance from $GL$ equal to $e''k$.

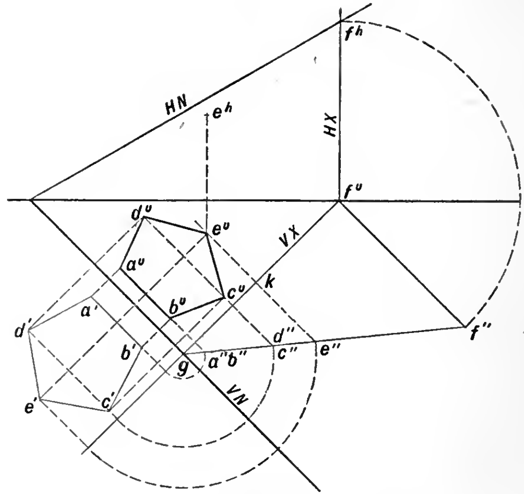


Fig. 139.

# CHAPTER III

## GENERATION AND CLASSIFICATION OF SURFACES

**92.** Every surface may be regarded as having been generated by the motion of a line, which was governed by some definite law. The moving line is called the **generatrix**, and its different positions are called **elements** of the surface. Any two successive positions of the generatrix, having no assignable distance between them, are called consecutive elements. The line which may direct or govern the generatrix is called the **directrix**.

**93. Surfaces are classified** according to the form of the generatrices, viz.:

Ruled Surfaces, or such as may be generated by a rectilinear generatrix.

Double-curved Surfaces, or such as must be generated by a curvilinear generatrix. These have no rectilinear elements.

The ruled surfaces are reclassified as *developable*, and *nondevelopable* or *warped* surfaces.

- Surfaces
  - *Ruled*
    - Plane
    - Single-curved (developable).
      - Cylinder
      - Cone
      - Convolute
    - Warped (nondevelopable)
  - *Double-curved* (nondevelopable)

**94. Ruled Surfaces.** A right line may move so that all of its positions will lie in the same plane; it may move so that any two consecutive elements will lie in the same plane; or it may move so that any two consecutive elements will not lie in the same plane. Thus, ruled surfaces are subdivided into three classes, as follows:

PLANE SURFACES: All the rectilinear elements lie in the same plane.

SINGLE-CURVED SURFACES: Any two consecutive rectilinear elements lie in the same plane, *i.e.* they intersect or are parallel.

WARPED SURFACES: No two consecutive rectilinear elements lie in the same plane, *i.e.* they are neither intersecting nor parallel.

95. **Plane Surfaces** are all alike. The rectilinear generatrix may move so as to touch one rectilinear directrix, remaining always parallel to its first position; so as to touch two rectilinear directrices which are parallel to each other, or which intersect; or it may revolve about another right line to which it is perpendicular.

96. **Single-curved Surfaces** may be divided into three classes, as follows:

CONES: In which all the rectilinear elements intersect in a point, called the *apex*.

CYLINDERS: In which all the rectilinear elements are parallel to each other. The cylinder may be regarded as being a cone with its apex infinitely removed.

CONVOLUTES: In which the successive rectilinear elements intersect two and two, no three having one common point.

97. **Conical Surfaces** are generated by the rectilinear generatrix moving so as always to pass through a fixed point, called the apex, and also to touch a given curve, called the directrix. Since the generatrix is indefinite in length, the surface is divided at the apex into two parts, called *nappes*. The portion of a conical surface usually considered is included between the apex and a plane which cuts all the elements. This plane is called the *base* of the cone and the form of its curve of intersection with the conical surface gives a distinguishing name to the cone, as, circular, elliptical, parabolic, etc. If the base of the cone has a center, the right line passing

through this center and the apex is called the *axis* of the cone (Fig. 140).

A RIGHT CONE is one having its base perpendicular to its axis.

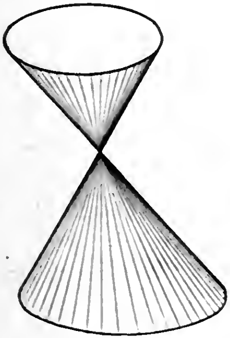


Fig. 140.

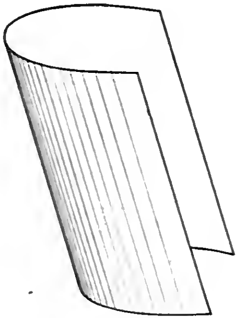


Fig. 141.

**98. Cylindrical Surfaces.** The cylinder is that limiting form of the cone in which the apex is removed to infinity. It may be generated by a rectilinear generatrix which moves so as always to touch a given curved directrix, having all of its positions parallel. A plane cutting all the elements of a cylindrical surface is called its *base*, and the form of its curve of intersection with the surface gives a distinguishing name to the cylinder, as in the case of the cone. If the base has a center, the right line through this center parallel to the elements is called the *axis* (Fig. 141).

A cylinder may also be generated by a curvilinear generatrix, all points of which move in the same direction and with the same velocity.

A RIGHT CYLINDER is one having its base perpendicular to its axis.

**99. Convolute Surfaces** may be generated by a rectilinear generatrix which moves so as always to be tangent to a line of double curvature.* Any two consecutive elements, but no three, will lie in the same plane. Since there is an infinite number of lines of double curvature, a great variety of convolutes may

* A line of double curvature is one of which no four consecutive points lie in the same plane.

exist. One such form which may readily be generated is the **helical convolute** (Fig. 142). It is the surface generated by the hypotenuse of a right triangle under the following conditions: Suppose a right triangle of paper, or some other thin, flexible material, to be wrapped about a right cylinder, one leg of the triangle coinciding with an element of the cylinder. If the triangle be unwrapped, its vertex will describe the involute of the base of the cylinder, and the locus of the points of tangency of its hypotenuse and the cylinder will be a helix, the hypotenuse generating the helical convolute. The convolute may also be regarded as being generated by a rectilinear generatrix moving always in contact with the involute and helix as directrices, and making a definite angle with the plane of the involute.

100. **A Warped Surface** is generated by a rectilinear generatrix moving in such a way that its consecutive positions do not lie in the same plane. Evidently there may be as many warped surfaces as there are distinct laws restricting the motion of the generatrix.

Any warped surface may be generated by a rectilinear generatrix moving so as to touch two linear directrices, and having its consecutive positions parallel either to a given plane, called a **plane director**, or to the consecutive elements of a conical surface, called a **cone director**.

101. The following types, illustrated by Figs. 142 to 148, indicate the characteristic features of warped surfaces :

HYPERBOLIC PARABOLOID, Fig. 143. Two rectilinear directrices and a plane director, or three rectilinear directrices.

CONOID, Fig. 144. One rectilinear and one curvilinear directrix and a plane director.

CYLINDROID, Fig. 145. Two curvilinear directrices and a plane director.

RIGHT HELICOID, Fig. 146. Two curvilinear directrices and a plane director.

OBLIQUE HELICOID, Fig. 147. Two curvilinear directrices and a cone director.

HYPERBOLOID OF REVOLUTION, Fig. 148. Two curvilinear directrices and a cone director, or three rectilinear directices.

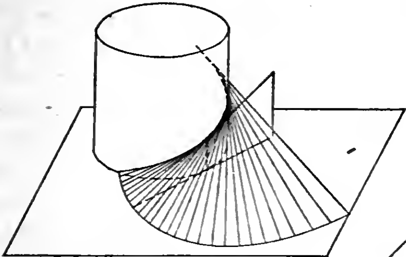


Fig. 142.

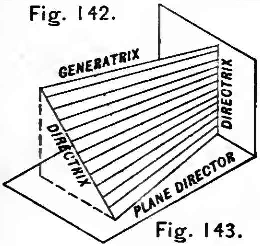



Fig. 143.

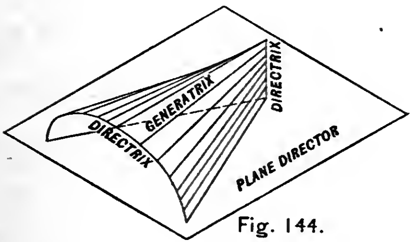



Fig. 144.

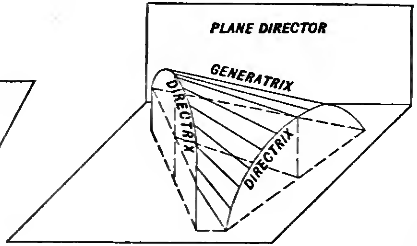



Fig. 145.

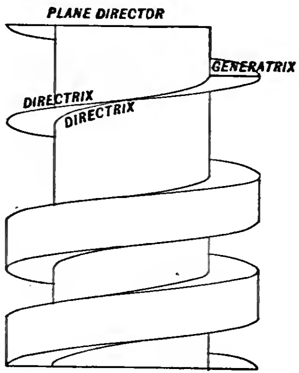



Fig. 146.

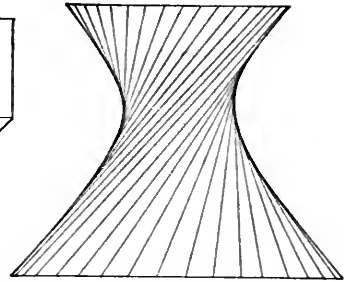


Fig. 148.

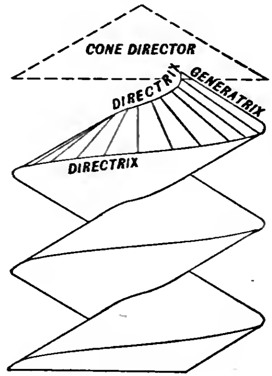



Fig. 147.

102. **A Surface of Revolution**, Fig. 149, is the locus of any line, or generatrix, the position of which remains unaltered with reference to a fixed right line about which it revolves. This fixed right line is called the *axis of revolution.* A circle of the surface generated by any point of the generatrix is called a *parallel*, and planes perpendicular to the axis will cut the surface in parallels. Any plane containing the axis of revolution is called a *meridian plane*, and the line cut from the surface by this plane is called a *meridian line.* All meridian lines of the same surface are obviously identical, and any one of them may be considered as a generatrix. That meridian plane which is parallel to a coördinate plane is called the *principal meridian.*

If the generatrix be a right line lying in the same plane as the axis, it will either be parallel with it or intersect it; in the former case the surface generated will be a cylinder, in the latter, a cone, and these are the only **single-curved surfaces of revolution.**

If the generatrix does not lie in the same plane with the axis, the consecutive positions are neither parallel nor intersecting. The surface must then be warped and its meridian line will be an hyperbola. This is the only warped surface of revolution. It may also be generated by revolving an hyperbola about its conjugate axis, and is known as the hyperboloid of revolution of one nappe (Fig. 148).

103. **Double-curved Surfaces.** With the exception of the cylinder, cone, and hyperboloid of revolution all surfaces of revolution are of double curvature. They are infinite in number and variety. Representative types are:

THE SPHERE: Generated by revolving a circle about its diameter.

THE PROLATE SPHEROID: Generated by revolving an ellipse about its major axis.

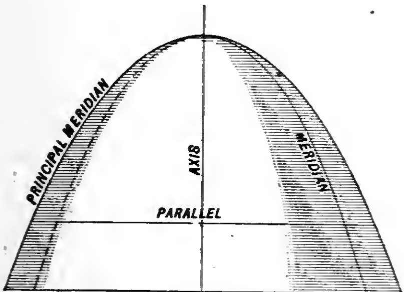
THE OBLATE SPHEROID: Generated by revolving an ellipse about its minor axis.

THE PARABOLOID: Generated by revolving a parabola about its axis (Fig. 149).

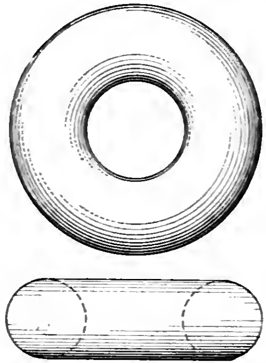
THE HYPERBOLOID OF TWO NAPPES: Generated by revolving an hyperbola about its transverse axis.

THE TORUS (annular or not): Generated by revolving a circle about a line of its plane other than its diameter. Fig. 150 illustrates an annular torus.

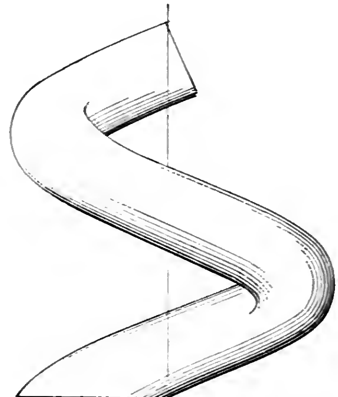
THE DOUBLE-CURVED SURFACE OF TRANSPOSITION — SERPENTINE: Generated by a sphere the center of which moves along an helix (Fig. 151).




PARABOLOID
Fig. 149.



TORUS (ANNULAR)
Fig. 150.



SERPENTINE
Fig. 151.

## CHAPTER IV

### TANGENT PLANES

**104**. A plane is tangent to a single-curved surface when it contains one, and only one element of that surface. The two lines commonly determining a tangent plane are the tangent element and a tangent to the surface at some point in this element. If this second line lies in one of the coördinate planes it will be a trace of the tangent plane.

In Fig. 152 point $d$ is on the surface of a cone to which a tangent plane is to be drawn. The line drawn through point $d$ and the apex of the cone will be the element at which the plane is to be tangent and, therefore, one line of the tangent plane. If a second line, $ck$, be drawn through point $d$, and tangent to any section of the cone containing this point, it will be a second line of the tangent plane. The traces of these lines will determine $HS$ and $VS$, the traces of the required tangent plane. If the base of the single-curved surface coincides with one of the coördinate planes, as in Fig. 152, $bk$ can be used for the tangent line, thus determining the horizontal trace directly.

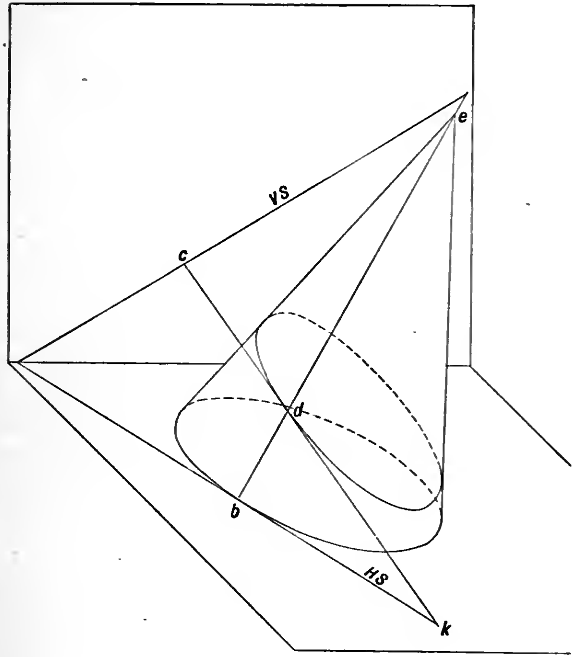


Fig. 152.

**105. One projection of a point on a single-curved surface being given, it is required to pass a plane tangent to the surface at the element containing the given point.**

Principle. The tangent plane will be determined by two intersecting lines, one of which is the element of the surface on which the given point lies, and the second is a line intersecting this element and tangent to the single-curved surface, preferably in the plane of the base.

Method. 1. Draw the projections of the element containing the given point. 2. In the plane of the base draw a second line tangent to the base at the tangent element. 3. Determine the plane of these lines.

If the plane of the base coincides with one of the coördinate planes, the tangent line will be one of the traces of the required tangent plane.

Note. In this and the following problems the base of the single-curved surface is considered as lying on one of the coördinate planes.

Construction. Fig. 153. Let the single-curved surface be a cone which is defined by its projections, and the base of which lies in one of the coördinate planes; in this case in $H$. Let $c^v$ be the vertical projection of the given point. Through $c^v$ draw $E^v$, the vertical projection of the tangent element. The vertical projection of the horizontal trace of this element is $k^v$, which, being a point in the base, will be horizontally projected at $k_1^h$ and $k_2^h$, thus making two possible tangent elements, $E_1$ and $E_2$; also two possible locations for point $c$, and, therefore, two possible tangent planes. Since the base of the cone lies in $H$, the horizontal traces, $HS$ and $HN$ of the tangent planes $S$ and $N$, will be tangent to the base at $k_2^h$ and $k_1^h$. The vertical traces of the tangent elements, $l^v$ and $o^v$, determine the necessary points in $VS$ and $VN$, the required vertical traces of the tangent planes.

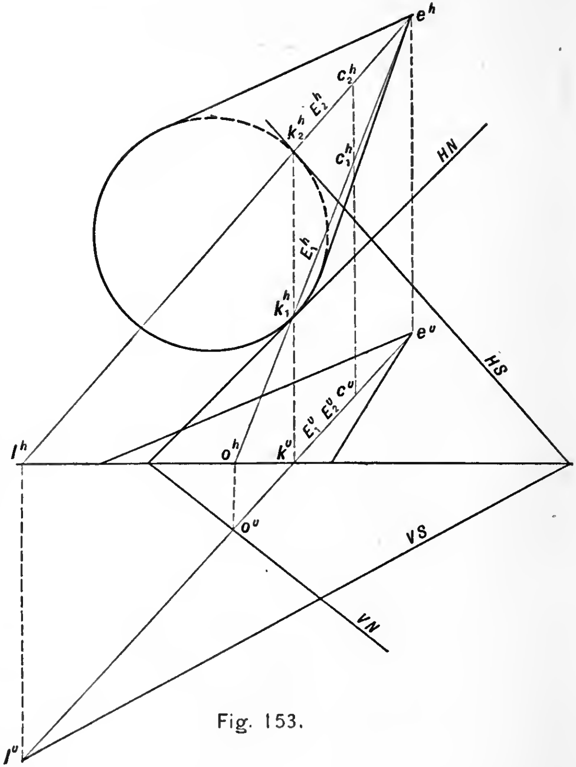


Fig. 153.

**106. To pass a plane tangent to a cone and through a given point outside its surface.**

Principle. Since all tangent planes contain the apex of the cone, the required plane must contain a line drawn through the apex of the cone and the given point. It must also have one of its traces tangent to the base of the cone, since this base is supposed to lie in a coördinate plane.

METHOD. 1. Draw the projections of a line passing through the apex of the cone and the given point. 2. Determine its traces. 3. Through that trace of the auxiliary line which lies in the plane of the base draw the trace of the required plane tangent to the base of the cone. 4. Draw the other trace of the plane through the other trace of the auxiliary line. There are two possible tangent planes.

**107. To pass a plane tangent to a cone and parallel to a given line.**

PRINCIPLE. The tangent plane must contain the apex of the cone and a line through the apex parallel to the given line.

METHOD. 1. Through the apex of the cone draw a line parallel to the given line. 2. Obtain the traces of this auxiliary line. 3. Draw the traces of the required tangent plane through the traces of the auxiliary line, making one of them tangent to the base of the cone. There are two possible tangent planes.

**108. To pass a plane tangent to a cylinder and through a given point outside its surface.**

PRINCIPLE. The tangent plane will be determined by two intersecting lines, one of which is drawn through the given point parallel to the elements of the cylinder, and the second is drawn tangent to the cylinder from some point in the first line, preferably in the plane of the base.

METHOD. 1. Through the given point draw the projections of a line parallel to the elements of the cylinder. 2. Determine the traces of this auxiliary line. 3. Through that trace of the auxiliary line which lies in the plane of the base of the cylinder draw one trace of the required plane tangent to the base. 4. Through the other trace of the auxiliary line draw the second trace of the tangent plane. There are two possible tangent planes.

CONSTRUCTION. Fig. 154. Let $a^v$ and $a^h$ be the projections of a given point through which the plane is to be passed tangent to the cylinder. The base of the cylinder rests on $H$. $B^v$ and $B^h$ are the projections of the auxiliary line drawn through point $a$ parallel to the elements of the cylinder. Through $d^h$, the horizontal trace of this line, $HS$ and $HN$, the horizontal traces of the two possible tangent planes, may be drawn tangent to the base of the cylinder. The vertical traces of these planes, $VS$ and $VN$, must contain $c^v$, the vertical trace of the line $B$. The elements at which planes $N$ and $S$ are tangent are lines $F$ and $K$, respectively.

**109. To pass a plane tangent to a cylinder and parallel to a given line.**

PRINCIPLE. The tangent plane must be parallel to a plane determined by the given line and a line intersecting it, which is parallel to the elements of the cylinder.

METHOD. 1. Through any point of the given line draw a line parallel to the elements

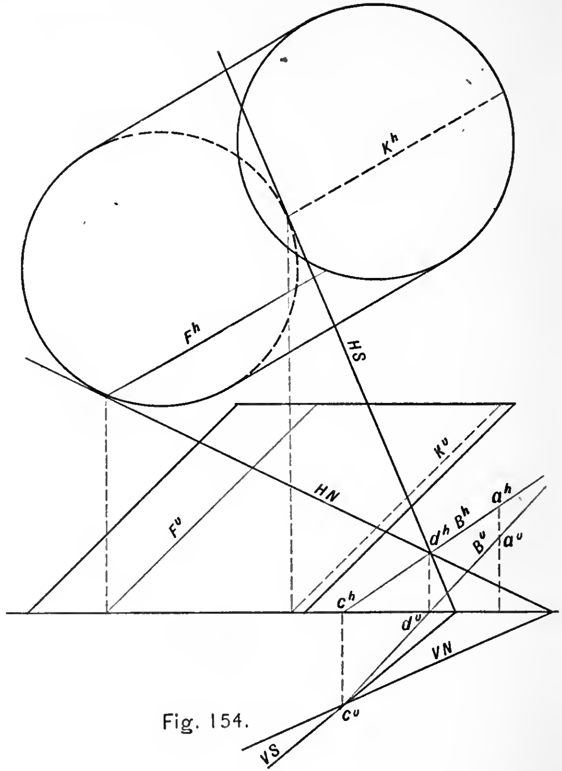


Fig. 154.

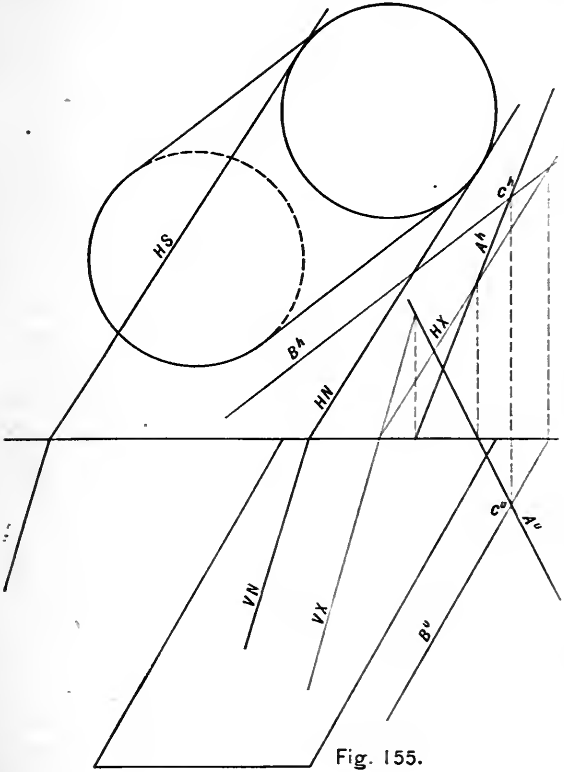


Fig. 155.

of the cylinder. 2. Determine the traces of the plane of these two lines. 3. Draw the traces of the tangent plane parallel to the traces of the auxiliary plane (Art. 18, page 12), one of the traces being tangent to the base of the cylinder. There may be two tangent planes.

CONSTRUCTION. Fig. 155. Through any point, *c*, of the given line *A*, draw line *B* parallel to the elements of the cylinder. The tangent planes will be parallel to *X*, the plane of these lines. *N* and *S* will be the required tangent planes.

**110. A plane is tangent to a double-curved surface** when it contains one, and only one point of that surface. The two lines commonly determining the tangent plane are the lines tangent to the meridian and parallel at the point of tangency (Art. 102, page 70).

**A Normal** is the line perpendicular to the tangent plane at the point of tangency.

**A Normal Plane** is any plane containing the normal.

**III. One projection of a point on the surface of a double-curved surface of revolution being given, it is required to pass a plane tangent to the surface at that point.**

PRINCIPLE. Planes tangent to double-curved surfaces of revolution must contain the tangents to the meridian and parallel at the point of tangency.

There may be two methods.

1ST METHOD. 1. Through the given point draw a meridian and a parallel (Art. 102, page 70). 2. Draw tangents to these curves at the given point. 3. Determine the plane of these tangents.

CONSTRUCTION. Fig. 156. Given the ellipsoid with its axis perpendicular to $H$. A plane is required to be drawn tangent to the surface at the point having $e^h$ for its horizontal projection. With $f^he^h$ as a radius, and $f^h$ as a center, describe the circle which is the horizontal projection of the parallel through $e$. One of the possible vertical projections of the parallel is that portion of $B^v$ lying within the ellipse. Project $e^h$ on to this line to obtain $e^v$, the vertical projection of the given point $e$. Through point $e$ draw line $B$ tangent to the parallel, and, therefore, in the plane of the parallel. That portion of $A^h$ lying within the circle, which is the horizontal projection of the ellipsoid, will be the horizontal projection of the meridian drawn through point $e$. Revolve this meridian line about $fk$ as an axis until it coincides with the principal meridian (Art. 102, page 70), the vertical projection of which will be shown by the ellipse. The revolved position of the vertical projection of point $e$ will now be at $e_1^v$, and a line may be drawn tangent to the meridian at this point, its projections being $A_1^v$, $A_1^h$. Counter-revolve this meridian plane to determine the true position of the tangent line, shown by its projections at $A^v$, $A^h$. The traces of the tangent lines $A$ and $B$ will determine the traces of the required tangent plane. There are two possible tangent planes.

2ND METHOD. 1. Draw the projections of a cone tangent to the double-curved surface of revolution at the parallel passing through

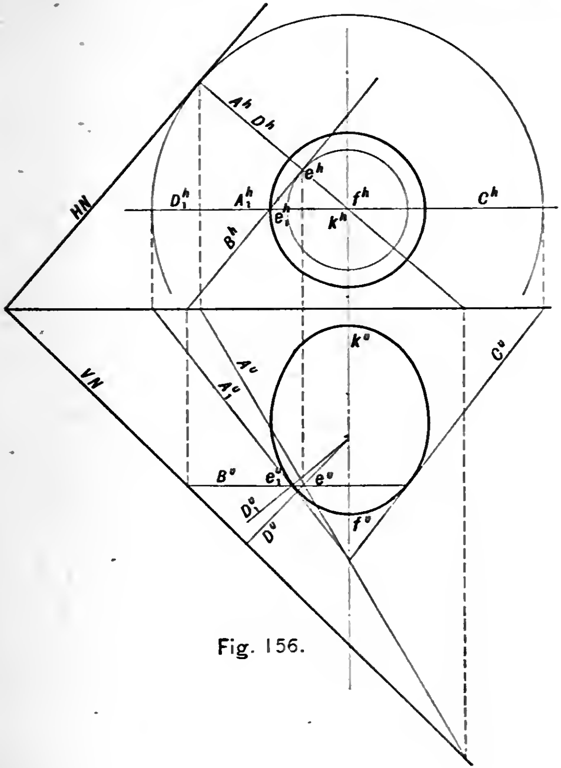



Fig. 156.

the given point. 2. Pass a plane tangent to the auxiliary cone at the element drawn through the given point (Art. 105, page 73).

CONSTRUCTION. Fig. 156. $A^h$ and $A^v$ are the projections of the element of a cone tangent to the ellipsoid and containing point *e*. A plane tangent to the cone at line *A* will be the required plane tangent to the ellipsoid at point *e*. If the vertical trace of line *A* lies beyond the limits of the paper, the direction of *VN* may be determined by observing that the trace of the required plane must be perpendicular to the normal *D* at the point *e* (Art. 110, page 77). There are two possible tangent planes.

**112. Through a point in space to pass a plane tangent to a double-curved surface of revolution at a given parallel.**

METHOD. 1. Draw a cone tangent to the double-curved surface of revolution at the given parallel. 2. Pass a plane through the given point tangent to the cone (Art. 106, page 74). There are two possible tangent planes.

**113. To pass a plane tangent to a sphere at a given point on its surface.**

METHOD. This may be solved as in Art. 111, or by the following method. 1. Through the given point draw a radius of the sphere. 2. Pass a plane through the given point perpendicular to the radius (Art. 72, page 50), and this will be the required plane.

**114. Through a given line to pass planes tangent to a sphere.***

PRINCIPLE. Conceive a plane as passed through the center of the sphere and perpendicular to the given line. It will cut a great circle from the sphere and lines from the required tangent planes. These lines will be tangent to the great circle of the sphere and intersect the given line at the point in which it intersects the auxiliary plane. The planes determined by these tangent lines and the given line will be the required tangent planes.

METHOD. 1. Through the center of the sphere pass a plane perpendicular to the given line (Art. 72, page 50) and determine its traces. 2. Determine the point in which this plane is pierced by the given line (Art. 57, page 42). 3. Into one of the coördinate planes revolve the auxiliary plane containing the center of the sphere, the great circle cut from the sphere, and the point of intersection with the given line. 4. From the latter point draw lines tangent to the revolved position of the great circle of the sphere. These lines will be lines of the required tangent planes. 5. Counter-revolve the auxiliary plane containing the lines of the tangent planes. 6. Determine the planes defined by the given line and each of the tangent lines obtained by 4. These will be the required tangent planes.

CONSTRUCTION. Fig. 157. The given line is $A$, and the center of the sphere is $e$. $HX$ and $VX$ are the traces of the auxiliary plane perpendicular to $A$ and passing through $e$

* For a general solution of problems requiring the drawing of tangent planes to double-curved surfaces of revolution, and through a given line, see Art. 160, page 122.

(Art. 72, page 50). The point $f$ is that of the intersection of line $A$ and plane $X$. Revolve plane $X$ into the vertical coördinate plane and $e'$ will be the revolved position of the center of the sphere, and $f'$ the revolved position of the point $f$. Draw the great circle of the sphere and the tangents $C'$ and $B'$.

In counter-revolution these lines will be at $C$ and $B$, intersecting $A$ at $f$. $C$ and $A$ will be two intersecting lines of one of the required tangent planes, $S$, and the second plane, $N$, will be determined by lines $A$ and $B$.

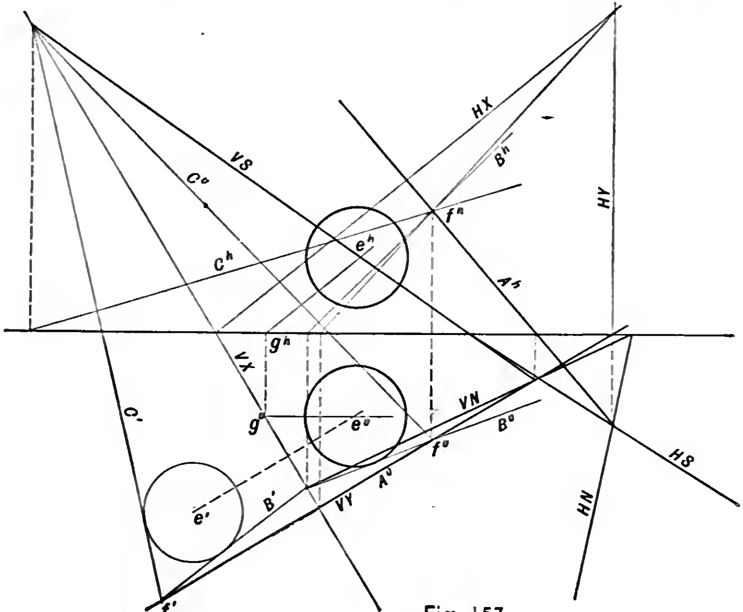


Fig. 157.

# CHAPTER V

## INTERSECTION OF PLANES WITH SURFACES, AND THE DEVELOPMENT OF SURFACES

**115. To determine the intersection of any surface with any secant plane.**

General Method. 1. Pass a series of auxiliary cutting planes which will cut lines, straight or curved, from the surface, and right lines from the secant plane. 2. The intersections of these lines are points in the required curve of intersection.

This method is applicable alike to prisms, pyramids, cylinders, cones, or double-curved surfaces of revolution. The auxiliary cutting planes may be used in any position, but for convenience they should be chosen so as to cut the simplest curves from the surface, that is, straight lines or circles.

With solids such as prisms, pyramids, single-curved, or other ruled surfaces, the above method consists in finding the intersection of each element with the oblique plane by Art. 61, page 44.

116. **A tangent to the curve of intersection** of a plane with a single-curved surface may be drawn by passing a plane tangent to the surface at the point assumed (Art. 111, page 78). The line of intersection of the tangent plane with the secant plane will be the required tangent.

117. **The true size of the cut section** may always be found by revolving it into one of the coördinate planes, about a trace of the secant plane as an axis.

118. **A right section** is the section cut from the surface by a plane perpendicular to the axis.

119. **The development** of a surface is its true size and shape when spread open upon a plane. Only surfaces having two consecutive elements in the same plane can be developed, as only such surfaces can be made to coincide with a plane. Therefore, only single-curved surfaces, and solids bounded by planes, can be developed. Solids bounded by planes are developed by finding the true size and shape of each successive face. Single-curved surfaces are developed by placing one element in contact with the plane and rolling the surface until every element has touched the plane. That portion of the plane covered by the surface in its revolution is the development of the surface.

120. **To determine the intersection of a plane with a pyramid.**

PRINCIPLE. Since a pyramid is a solid bounded by planes, the problem resolves itself into determining the line of intersection between two planes. Again, since the pyramid is represented by its edges, the problem still further resolves itself into determining the points of piercing of these edges with the plane.

METHOD. 1. Determine the points in which the edges of the given pyramid pierce the given plane. 2. Connect the points thus obtained in their order, thereby determining the required intersection between the plane and pyramid.

CONSTRUCTION. Fig. 158. Given the pyramid of which lines $A$, $B$, $C$, $D$, and $E$ are the lateral edges, or elements, intersected by plane $N$. The points of piercing, $a$, $b$, $c$, $d$, and $e$, of the elements with plane $N$ have been determined by the use of the horizontal projecting planes of the elements (Art. 59, page 42), and the lines $ab$, $bc$, $cd$, $de$, and $ea$, joining these points of piercing in their order, are the lines of intersection of plane and pyramid.

Since all the auxiliary planes used contain point $l$, the apex of the pyramid, and are perpendicular to $H$, they must contain a line through $l$ perpendicular to $H$; hence, $o$, the point of piercing of this line and plane $N$, is a point common to all the lines of intersection between plane $N$ and the auxiliary planes. By observing this fact the work of construction can be slightly shortened.

The true size of the cut section is obtained by revolving each of its points into $V$ about $VN$ as an axis (Art 43, page 31).

**121. To develop the pyramid.**

PRINCIPLE. If the pyramid be laid on a plane and be made to turn on its edges until each of its faces in succession has come into contact with the plane, that portion of the plane which has been covered by the pyramid in its revolution will be the development of the pyramid. From the above it is evident that every line and surface of the pyramid will appear in its true size in development.

METHOD. 1. Determine the true length of each line of the pyramid. 2. Construct each face in its true size and in contact with adjacent faces of the pyramid.

CONSTRUCTION. Figs. 158 and 159. The true lengths of the elements and their segments have been determined by revolving them parallel to $V$, as at $A_1$, $B_1$, etc., $a_1$, $b_1$, etc., Fig. 158. The edges $F$, $G$, $I$, $J$, and $K$ are already shown in their true lengths in their horizontal projections, since the plane of these lines is parallel to $H$. Hence, through any point $l$, Fig. 159, representing the apex, draw a line $B$, equal in length to $B_1$. With $l$ as a center and with a radius equal to $C_1$, draw an arc of indefinite length. With the end of $B$ as a center and with a radius equal to $G^h$, draw an arc intersecting the first in point $n$, thus definitely locating lines *C* and *G*. The other faces are determined in like manner.

The development of the cut is obtained by laying off from *l*, on its corresponding element, the true lengths of the elements from the apex of the pyramid to the cut section, and joining the points thus obtained.

The base and cut section may be added to complete the development of the pyramid.

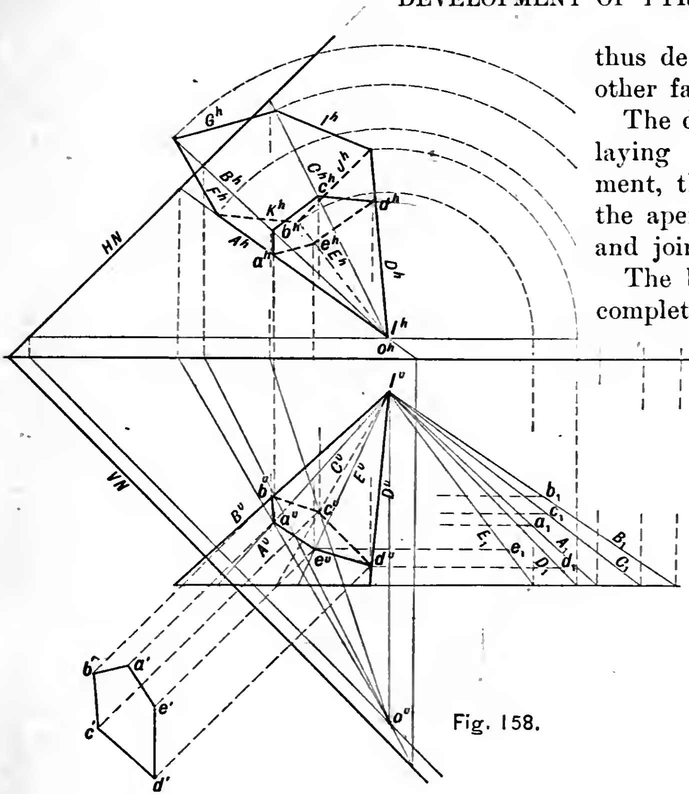



Fig. 158.

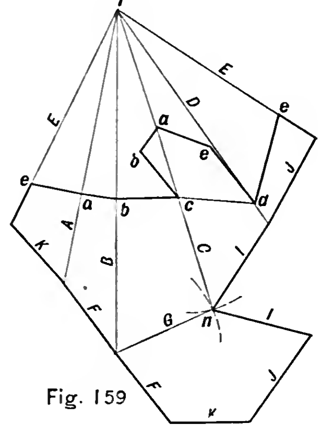



Fig. 159

**122. To determine the curve of intersection between a plane and any cone.**

PRINCIPLE. The problem is identical with that of the intersection of a plane with a pyramid, for a cone may be considered as being a pyramid of an infinite number of faces.

METHOD. 1. Pass a series of auxiliary planes perpendicular to one of the coördinate planes and cutting elements from the cone. 2. Each auxiliary plane, save the tangent planes, will cut two elements from the cone and a right line from the given plane. The intersections of this line with the elements give two points in the required curve.

CONSTRUCTION. Fig. 160. Let it be required to determine the curve of intersection between plane $N$ and the oblique cone with its circular base parallel to $H$. Pass a series of auxiliary cutting planes through the cone, containing its apex, $a$, and perpendicular to $H$. Plane $X$ is one such plane which cuts two elements, $ab$ and $ac$, from the cone, and the line $B$ from plane $N$. $B$ intersects element $ac$ in point $e$ and element $ab$ in point $f$, and these are two points of the required curve of intersection between plane $N$ and the cone. Similarly determine a sufficient number of points to trace a smooth curve. The planes passing through the contour elements of each projection should be among those chosen.

As in the case of the pyramid, the work of construction may be slightly shortened by observing that since all of the auxiliary planes are perpendicular to $H$, and pass through the apex of the cone, they all contain the line passing through the apex and perpendicular to $H$. This line pierces plane $N$ in $d$, a point common to all lines of intersection between $N$ and the auxiliary planes.

The true size of the cut section is determined by revolving each of its points into $V$ about $VN$ as an axis.

**123. To determine the development of any oblique cone.**

PRINCIPLE. When a conical surface is rolled upon a plane, its apex will remain stationary, and the elements will successively roll into contact with the plane, on which they will be seen in their true lengths and at their true distances from each other.

Construction. Since in Fig. 160 the base of the cone is parallel to $H$, the true distances between the elements may be measured upon the circumference of the base; therefore, to develop the conical surface upon a plane, through any point $a$, Fig. 161, representing the apex, draw a line $ac$, equal in length to $a^vc_1$, the revolved position of element $ac$, Fig. 160. With $a$ as a center, and with a radius equal to the true length of the next element, $ak$, draw an arc of indefinite length. With $c$ as a center, and with a radius equal to $c^hk^h$, draw an arc intersecting the first in $k$. This process must be repeated until the complete

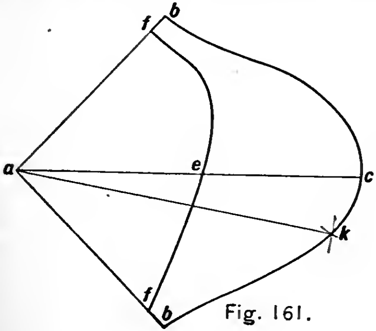


Fig. 161.

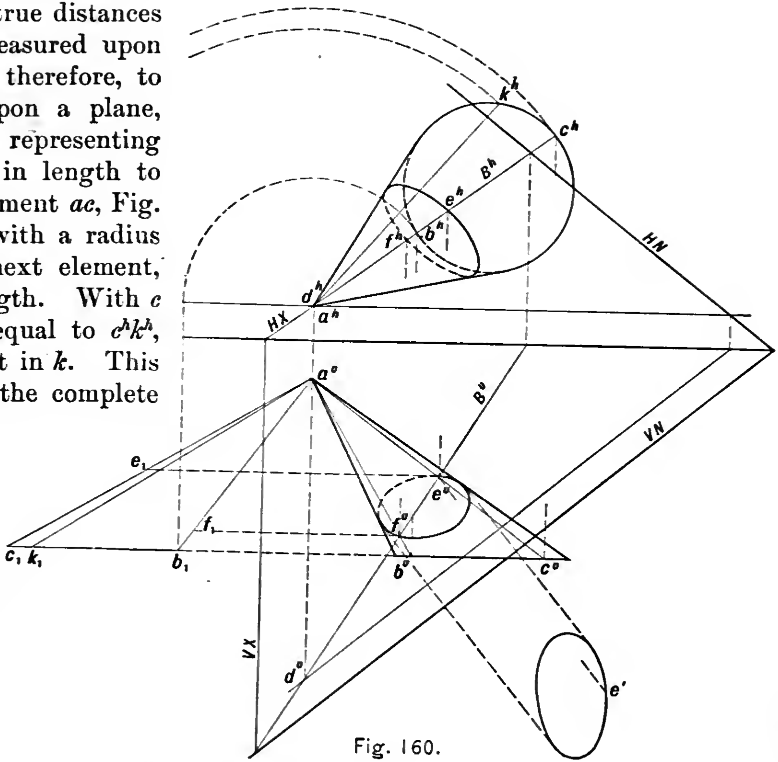


Fig. 160.

development has been found. The accuracy of the development depends upon the number of elements used, the greater number giving greater accuracy.

The development of the curve of intersection is obtained, as in the pyramid, by laying off from the apex, on their corresponding elements, the true lengths of the portions of the elements from the apex to the cut section, and joining the points thus found.

**124. To determine the curve of intersection between a plane and any cylinder.**

PRINCIPLE. A series of auxiliary cutting planes parallel to the axis of the cylinder and perpendicular to one of the coördinate planes will cut elements from the cylinder and right lines from the given plane. The intersections of these elements and lines will determine points in the required curve.

CONSTRUCTION. Fig. 162. Given the oblique elliptical cylinder cut by the plane *N*. Plane *X* is one of a series of auxiliary cutting planes parallel to the axis of the cylinder and perpendicular to *H*. Since it is tangent to the cylinder, it contains but one element, *A*, and intersects the given plane in the line *G*. Since lines *G* and *A* lie in plane *X*, they intersect in *a*, one point in the required curve of intersection between the cylinder and the given plane *N*. Likewise plane *Z* intersects the cylinder in elements *C* and *D*, and the plane *N* in line *K*, the intersections of which with lines *C* and *D* are *c* and *d*, two other points in the required curve.

The true size of the cut section has been determined by revolving it into *V* about *VN* as an axis.

**125. To develop the cylinder.**

PRINCIPLE. When a cylinder is rolled upon a plane to determine its development, all the elements will be shown parallel, in their true lengths, and at their true distances from each other. Since in an oblique cylinder the bases will unroll in curved lines, it is necessary to determine a right section which will develop into a right line, and upon which the true distances between the elements may be laid off. This line will be equal to the periphery of the

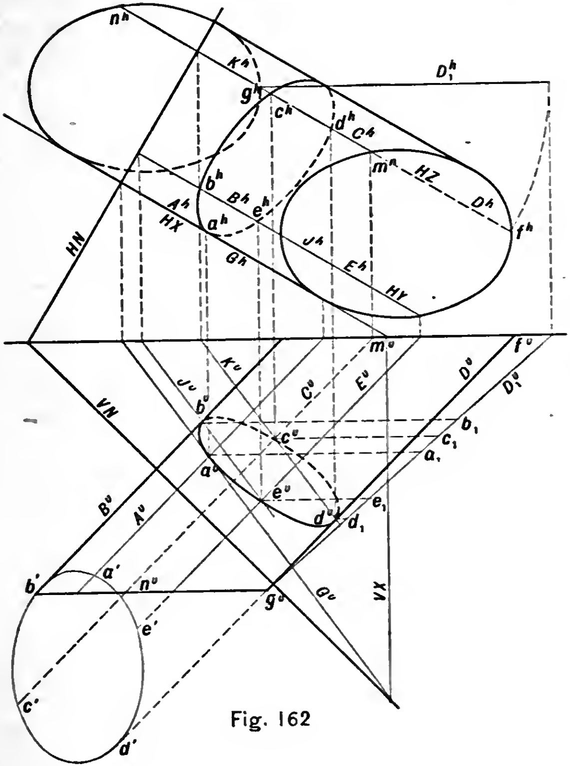



Fig. 162

right section, and the elements will be perpendicular to it. The ends of these perpendiculars will be at a distance from the line equal to the true distances of the ends of the elements from the right section. A smooth curve may then be drawn through the ends of the perpendiculars.

METHOD. 1. Draw a right line equal in length to the periphery of the right section. 2. Upon this right line lay off the true distances between the elements. 3. Through the points thus obtained draw perpendiculars to the right line. 4. On these perpendiculars lay off the true lengths of the corresponding elements, both above and below the right line. 5. Trace a smooth curve through the ends of the perpendiculars.

CONSTRUCTION. Figs. 162 and 163. The secant plane *N* of Fig. 162 has been so chosen as to cut a right section from the cylinder, that is, the traces of the plane are perpendicular to the projections of the axis of the cylinder (Art. 62, page 44). Element *D* has been revolved parallel to *V* to obtain its true

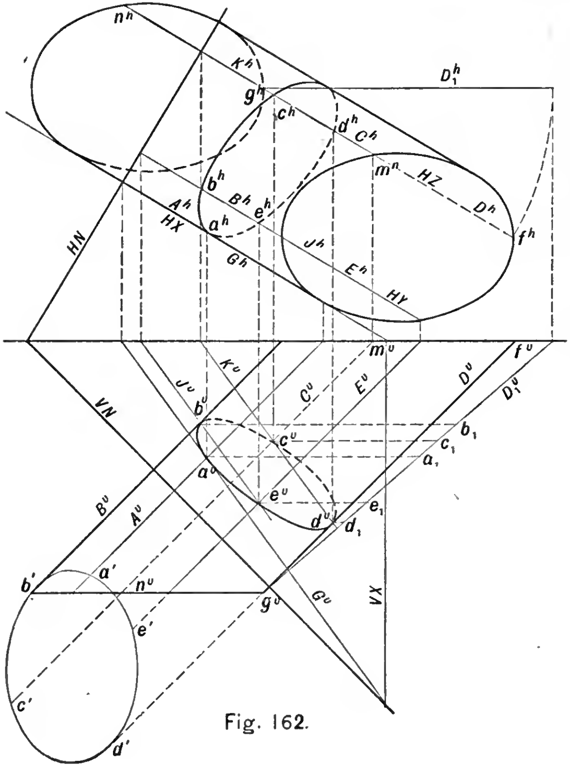



Fig. 162.

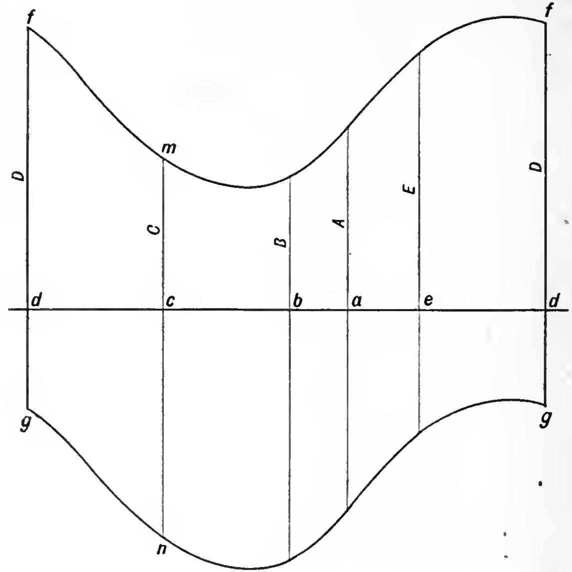



Fig. 163.

length $D_1$, and $d^v$ is projected to $d_1$, thus obtaining the true lengths of each portion of $D$. Since all the elements of a cylinder are of the same length, $D_1$ represents the true length of each element, and their segments are obtained by projecting the various points of the cut section upon it, as at $b_1$, $c_1$, etc. Upon the right line $dd$, Fig. 163, the true distances between the elements, $dc$, $cb$, etc., have been laid off equal to the rectified distances $d'c'$, $c'b'$, etc. Through points $d$, $c$, $b$, etc., perpendiculars to line $dd$ have been drawn equal to the true length of the elements. The portions above and below $dd$ are equal to the lengths of the elements above and below the cut section.

**126. If the axis of the cylinder be parallel to a coördinate plane**, the development may be obtained without the use of a right section.

Fig. 164 represents an oblique cylinder with its axis parallel to $V$ and its bases parallel to $H$. The elements are represented in vertical projection in their true lengths, and in horizontal projection at their true distances apart measured on the periphery of the base. If the cylinder be rolled upon a plane, the ends of the elements will move in planes perpendicular to the elements; therefore, $b'$ will lie on $b^v b'$ perpendicular to $B^v$, and at a distance from $a^v$ equal to $a^h b^h$. Likewise $c'$ will lie on $c^v c'$ perpendicular to $C^v$ and at a distance from $b'$ equal to $b^h c^h$, etc. The figure shows but one half the development.

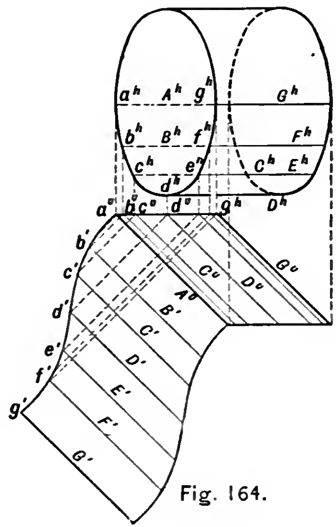


Fig. 164.

**127. To determine the curve of intersection between a plane and a prism.**

PRINCIPLE. A prism may be considered as a pyramid with its apex at infinity; hence, this problem in no wise differs in principle and method from that of the pyramid (Art. 120).

CONSTRUCTION. Fig. 165. Given the prism the elements of which are $A$, $B$, $C$, and $D$ intersected by plane $N$. Points $b$, $c$, and $d$ are the points in which elements $B$, $C$, and $D$, respectively, pierce plane $N$, determined by the use of the vertical projecting planes of the elements (Art. 59, page 42). Element $A$, if extended, will in like manner be found to pierce plane $N$ at point $a$. By connecting $ab$, $bc$, $cd$, and $da$, the required lines of intersection are determined. But, since point $a$ does not lie on the given prism, only the portions of lines $ab$ and $da$ which lie on the prism, *i.e.* $eb$ and $df$, are required, and plane $N$ intersects the top base of the prism in line $ef$.

Points $e$ and $f$ may be obtained by the use of the vertical projecting planes of lines $E$ and $F$, in which case point $a$ is not needed.

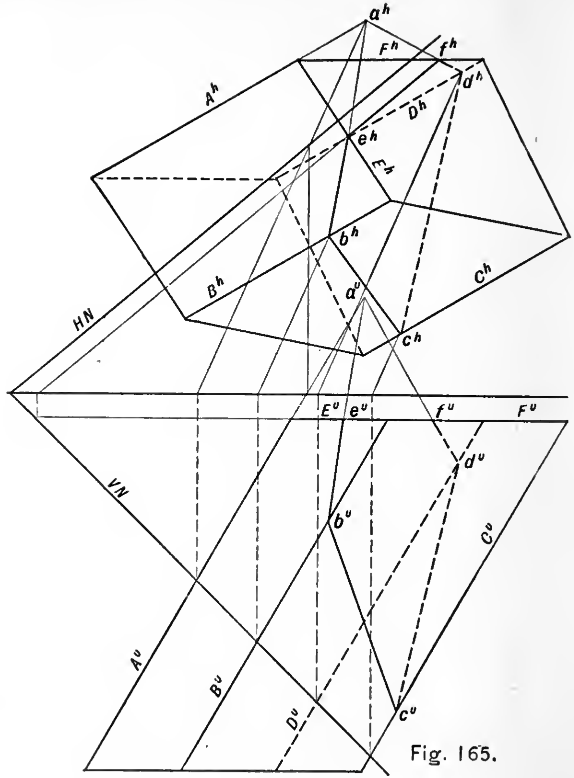


Fig. 165.

The construction may be shortened by observing that since the vertical projecting planes of the elements are parallel, their lines of intersection with the given plane are parallel.

The true size of the cut section is determined by revolving each of its points into $H$ about $HN$ as an axis.

**128. To develop the prism.**

METHOD. Determine the true size of a right section and proceed as in the case of cylinder (Art. 125, page 88), or revolve the prism parallel to a coördinate plane and proceed as for the cylinder (Art. 126, page 91).

**129. The helical convolute.**

Fig. 166 illustrates a plane triangle tangent to a right-circular cylinder, the base of the triangle being equal to the circumference of the base of the cylinder. If the triangular surface be wrapped about the cylinder, the point $c$ will come in contact with the cylinder at $c_1$, $a$ at $a_1$, and the hypotenuse $ac$ will become a helix having $a_1c_1$ for the pitch, and the angle $acd$ will be the pitch angle. If the right line $ac$ be revolved about the cylinder while remaining in contact with, and tangent to, the helix,* and making the constant angle $\theta$ with the plane of the base, it will generate a convolute of two nappes. The lower nappe, which is generated by the variable portion of the line below the point of tangency, will alone be considered.

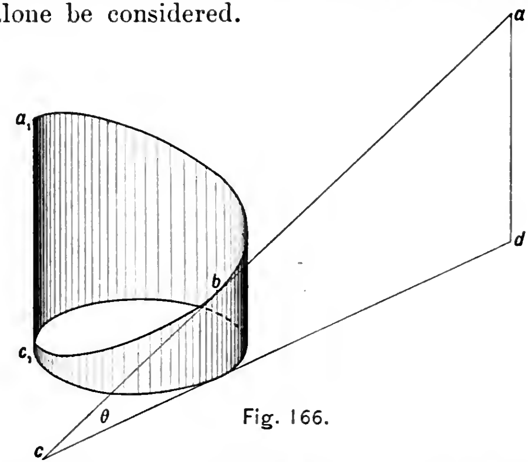



Fig. 166.

* For the theory and construction of the helix and involute curves see page 104 of "Elements of Mechanical Drawing" of this series.

**130. To draw elements of the surface of the helical convolute.**

CONSTRUCTION. Fig. 167. Let the diameter, $a^h l^h$, and the pitch, $a^v c^v$, of the required helical convolute be given. Then will $a^v l^v b^v e^v c^v$ be the vertical projection of the helix, and the tangents to the space helix will be the elements of the convolute. From any point

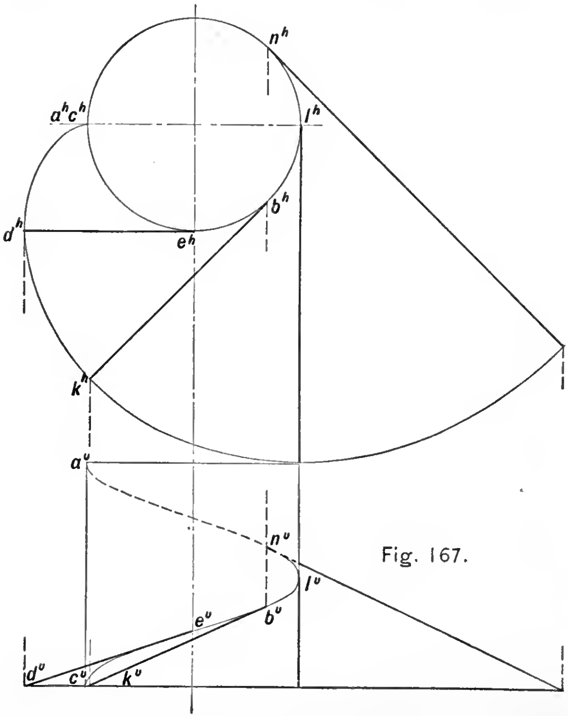



Fig. 167.

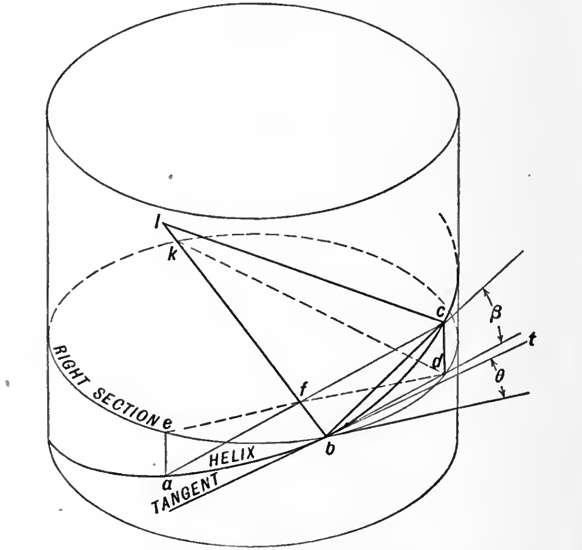



Fig. 168.

$b$ draw a tangent to the helix. Its horizontal projection, $b^h k^h$, will be tangent to the circle of the base at $b^h$, and the length of this projection will equal the arc $b^h e^h c^h$; therefore, $k^h$

will be the horizontal projection of the horizontal trace of the element *bk*, the vertical projection of which will be $b^v k^v$. Similarly traces of other elements may be found, and their locus will be points in the base of the convolute, which curve is an involute of the base of the cylinder on which the helix is described. Any element of the convolute may now be obtained by drawing a tangent to the circle of the base of the cylinder, and limited by the involute of this same circle.

**131. To develop the helical convolute.**

Since this surface is developable (Art. 98, page 67), it may be rolled upon a plane on which the elements will appear in their true lengths as tangents to the developed helix, which, being a curve of constant curvature, will be a circle. The developed surface will be an area bounded by this circle and its involute.

The radius of the circle of the developed helix is determined in the following manner: In Fig. 168 *a*, *b*, and *c* are points in the helix, and *bt* is a tangent to the helix at *b*, which point is equally distant from *a* and *c*. The projection of these points on to the plane of the right section of the cylinder passed through *b* will be at *e* and *d*. *ac* is the chord of a circular arc drawn through *a*, *b*, and *c*, approximating the curvature of the helix. *bl* is the diameter of this circular arc, and *lcb* a triangle inscribed in its semicircumference, and, therefore, a right triangle. Similarly the triangle *bdk*, in the plane of the right section, is a right triangle. In the triangles *bcl* and *bdk*, *bc* will be a mean proportional between *fb* and *bl*, and *bd* will be a mean proportional between *fb* and *bk*. Substituting *R* for the radius of curvature of the helix, $\frac{bl}{2}$, and *r* for the radius of the circle which is the projection of the helix, $\frac{bk}{2}$, we may obtain $\overline{bc}^2 = fb \times 2R$ and $\overline{bd}^2 = fb \times 2r$. Dividing the second by the first, $\frac{\overline{bd}^2}{\overline{bc}^2} = \frac{r}{R}$, but $\frac{bd}{bc} = \cos\beta$, when $\beta$ is the angle which the chord *bc* makes with the horizontal plane; hence, $R = \frac{r}{\cos^2\beta}$. As points *a* and *c* approach each other, the chord $bc$ will approach $bt$, the tangent to the helix at $b$, and at the limit the angle $\beta$ will equal $\theta$, the angle which the tangent makes with the horizontal plane, so that $R = \frac{r}{\cos^2\theta}$.

This value may be graphically obtained in the following simple manner: Fig. 169. Draw a tangent to the helix at $b$. From its intersection with the contour element of the cylinder, at $c$, draw a horizontal line terminated by the perpendicular to the tangent through $b$. Then $dc$ will be the required radius for the developed helix, since $bc = \frac{ac}{\cos\theta}$, and $bc = cd \cos\theta$; hence, $cd = \frac{ac}{\cos^2\theta} = \frac{r}{\cos^2\theta}$.

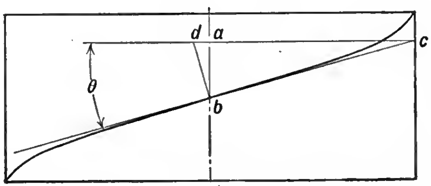



Fig. 169.

Having determined the radius of curvature of the helix, draw the circle and its involute to obtain the development of the convolute. Fig. 170 is the development of the helical convolute shown by its projections in Fig. 167.

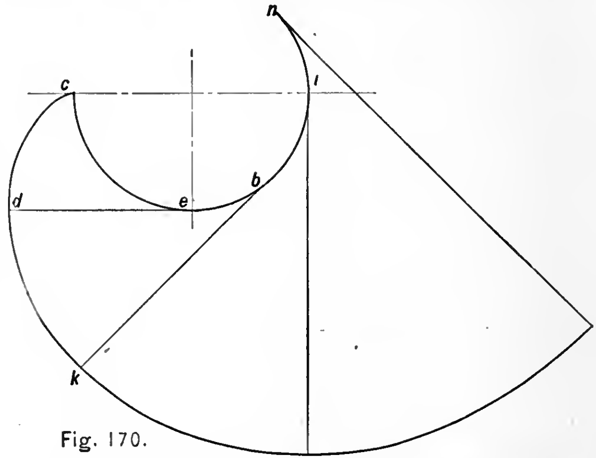



Fig. 170.

**132. To determine the curve of intersection between a plane and a surface of revolution.**

In the following cases the axes of revolution are considered to be perpendicular to one of the coördinate planes.

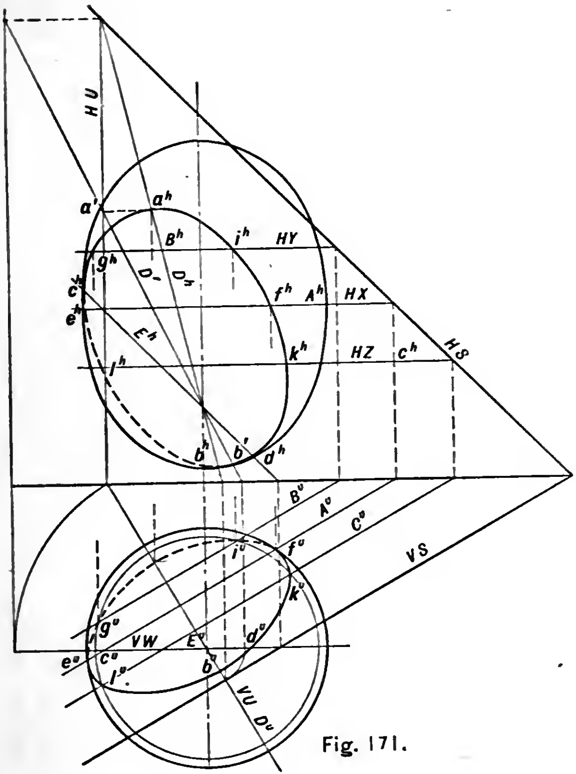



Fig. 171.

METHOD. 1. Pass a series of auxiliary cutting planes perpendicular to the axis of revolution. These planes will cut the surface in circles, and the given plane in right lines. 2. The intersections of the circles and right lines are points in the required curve of intersection.

CONSTRUCTION 1. Fig. 171. Given the ellipsoid with its axis perpendicular to $V$ intersected by the plane $S$. Plane $Y$ is one of a series of auxiliary planes perpendicular to the axis of the ellipsoid. Then $HY$ is parallel to $GL$; it cuts from the ellipsoid a circle which, in horizontal projection, coincides with $HY$, and in vertical projection is a circle, and it cuts from the given plane $S$ the line $B$, intersecting the circle at $i$ and $g$, two points in the required curve of intersection between the ellipsoid and the given plane. Repeat this process until a sufficient number of points is determined to trace a smooth curve.

It is convenient to know at the outset between what limits the auxiliary planes parallel to $H$ should be passed, thus definitely locating the extremities of the curve. This may be accomplished by passing the meridian plane

$U$, perpendicular to $S$, intersecting $S$ in line $D$ upon which the required points $a$ and $b$ are found. It will be noticed that $D^v$ is an axis of symmetry of the vertical projection of the curve.

It is also important that auxiliary planes be so chosen as to determine points of tangency with contour lines. Two such planes, which should be used in the case of the ellipsoid, are the plane $X$, which cuts the surface in its greatest parallel and contour line in vertical projection, and plane $W$, which cuts the surface in a principal meridian. The former of these determines the points of tangency, $e^v$ and $f^v$, in vertical projection, and the latter defines similar points $c^h$ and $d^h$ in horizontal projection. Such points as $a$, $b$, $c$, $d$, $e$, and $f$ are designated as critical points of the curve.

CONSTRUCTION 2. Fig. 172. Given the torus with its axis perpendicular to $H$ intersected by plane $N$. The auxiliary planes are parallel to $H$, and the critical points are $a$, $b$, on plane $U$; $e$, $f$ on plane $X$; and $c$, $d$ on plane $W$. Also the highest points, $g$, $i$, and the lowest points, $l$, $k$, should be determined.

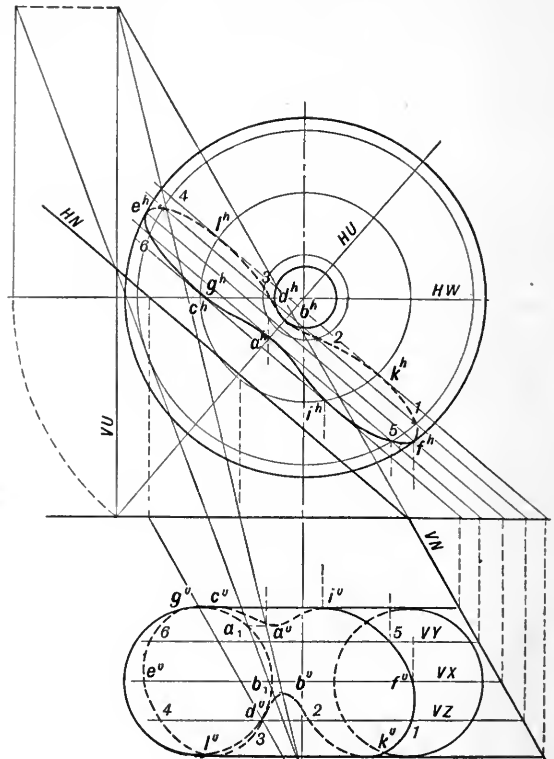



Fig. 172.

## CHAPTER VI

### INTERSECTION OF SURFACES

133. Whenever the surfaces of two bodies intersect, it becomes necessary to determine the line of their intersection in order to illustrate and develop the surfaces. The character of these lines, which are common to both surfaces, is determined by the nature of the surfaces, and by their relative size and position. The principles involved in the determination of these lines, their projections, and the development of the intersecting surfaces, are fully treated in the chapter on the intersection of planes and surfaces; but it is necessary to consider the character of the auxiliary cutting planes and the methods of using them in order to cut elements from two surfaces instead of one.

134. **Character of Auxiliary Cutting Surfaces.** The auxiliary cutting surfaces have been referred to as planes, but cylinders and spheres are also used whenever they will serve to cut the intersecting surfaces in right lines or circular arcs.

The character of the auxiliary plane, or surface, and the method of using it, is dependent upon the nature of the intersecting surfaces, since it is most desirable, and generally possible, to cut lines from the surfaces, which shall be either right lines or circular arcs.

The following cases illustrate the influence of the type of the intersecting surfaces on the character and position of the auxiliary cutting planes or surfaces:

Case 1. Cylinder and cone with axes oblique to the coördinate plane: Use auxiliary planes containing the apex of the cone and parallel to the axis of the cylinder.

Case 2. Two cylinders with axes oblique to the coördinate plane: Use auxiliary planes parallel to the axes of the cylinders.

Case 3. Cone and cone with axes oblique to the coördinate planes: Use auxiliary planes containing the apices of the cones.

Case 4. A single and a double curved surface of revolution with parallel axes which are perpendicular to a coördinate plane: Use auxiliary planes perpendicular to the axes.

Case 5. A single and a double curved surface of revolution, two single-curved surfaces of revolution, or two double-curved surfaces of revolution, with axes oblique to each other but intersecting, and parallel to one of the coördinate planes: Use auxiliary cutting spheres with centers at the intersection of the axes.

Case 6. A double-curved surface of revolution with axis perpendicular to a coördinate plane, and any single-curved surface: If the single-curved surface be a cylinder, use auxiliary cutting cylinders with axes parallel to that of the cylinder, intersecting the axis of the double-curved surface, and cutting circles therefrom. If the single-curved surface be a cone, use auxiliary cones with apices common with that of the given cone and cutting the double-curved surface in circles.

Case 7. A prism may be substituted for the cylinder, or a pyramid for the cone, in each of the above cases, save Case 6.

Case 8. Prisms and pyramids, two of the same or opposite kind: Auxiliary planes not required. Determine the intersection of the edges of each with the faces of the other.

**135. To determine the curve of intersection between a cone and cylinder with axes oblique to the coördinate plane.***

* It is customary to represent these surfaces with one or both of the bases resting on a coördinate plane.

PRINCIPLE. Since the curve of intersection must be a line common to both surfaces, it will be drawn through the points common to intersecting elements. A series of cutting planes passed through the apex of the cone and parallel to the axis of the cylinder will cut elements from both the cone and the cylinder, and since these elements lie in the same plane, they will intersect, their point of intersection being common to both surfaces, and therefore a point in the required curve.

METHOD. 1. Draw a line through the apex of the cone and parallel to the axis of the cylinder. 2. Determine its trace in the plane of the bases of the cylinder and cone. 3. Through this trace draw lines in the plane of the bases of cylinder and cone cutting these bases. These lines will be the traces of the auxiliary cutting planes. 4. From the points of intersection of these traces with the bases of cone and cylinder draw the elements of these surfaces. 5. Draw the required curve of intersection through the points of intersection of the elements of cylinder and cone.

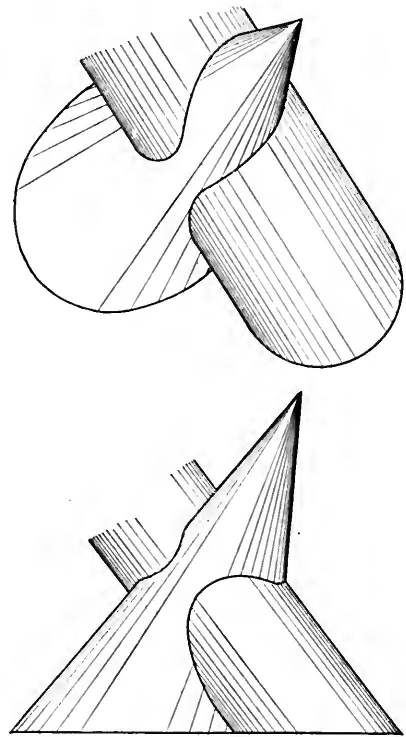


Fig. 173.

Construction. Fig. 174. Through the apex of the cone draw line $A$ parallel to the axis, or elements, of the cylinder. This line will be common to all auxiliary cutting planes, and its horizontal trace, $b^h$, will be a point common to all their horizontal traces. $HN$ is one such trace which cuts, or is tangent to, the base of the cylinder at $c^h$, and cuts the base of the cone in $d^h$ and $e^h$. Since these are points in elements cut from the cylinder and cone by the auxiliary plane $N$, the horizontal and vertical projections of the elements may be drawn. Line $E$ will be the element cut from the cylinder, while $da$ and $ea$ are the elements cut from the cone. The intersection of these elements at $f$ and $g$ will be points common to the cylinder and cone; hence, points in the required curve of intersection.

The planes $N$ and $S$ are tangent planes, the former to the cylinder and the latter to the cone; therefore, they will determine but two points each. Intermediate planes, such as $M$, will cut four elements each, two from each surface, and will determine four points of intersection. Draw as many such planes as may

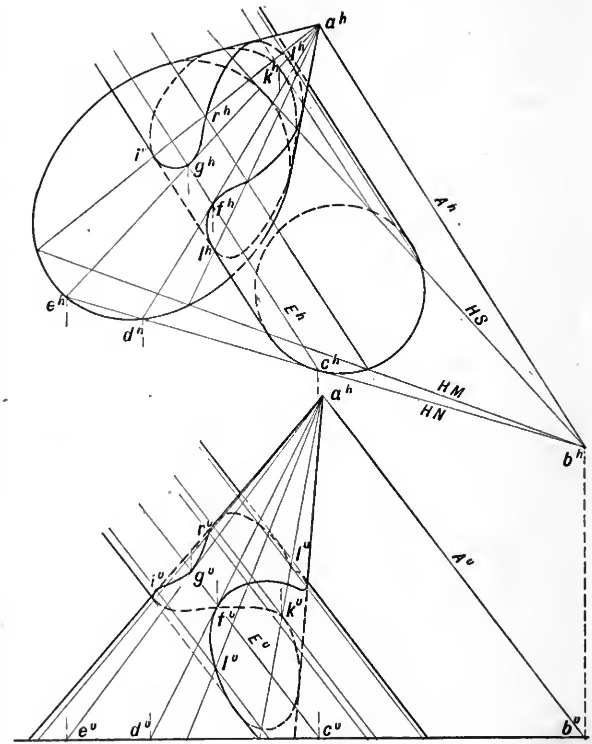


Fig. 174.

be necessary to determine the required curve. Since it is desirable to determine the points of tangency between the curve and contour elements, strive to locate the intermediate planes so that all such elements may be cut by the auxiliary planes.

**136. Order and Choice of Cutting Planes.** The tangent planes should be drawn first, thus determining limiting points in the curve, such as *f*, *g*, *k*, *l*, and, as will be shown in Art. 137, determining whether there be one or two curves of intersection. Next pass planes cutting contour elements in both views. *M* is one such plane cutting a contour element in the horizontal projection of the cylinder and determining points *i* and *l*, limiting points in the horizontal projection of the curve.

**137. To determine if there be one or two curves of intersection.** It is possible to determine the number of curves of intersection between the cylinder and cone previous to finding the required points in the line, or lines, of intersection. In Fig. 174 it will be observed that there is but one continuous curve of intersection, and this is due to the fact that only one of the two tangent auxiliary planes that might be drawn to the cylinder will cut the cone. If the surfaces had been of such size, or so situated, that the two planes tangent to the cylinder had cut the cone, as indicated by Fig. 175, in which only the traces of the cylinder, cone, and planes are drawn,

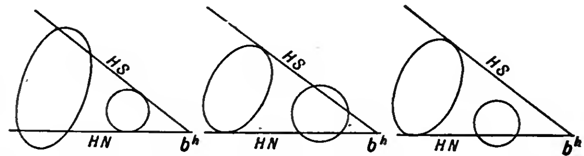



Fig. 175. Fig. 176. Fig. 177.

there would have been two curves, the cylinder passing directly through the cone. Again if two planes tangent to the cone had cut the cylinder, as in Fig. 176, the cone would have pierced the cylinder, making two independent curves of intersection. The condition shown by Fig. 177 is that of the problem solved, and indicates but one curve of intersection.

**138. To determine the visible portions of the curve.** 1. A point in a curve of intersection is visible only when it lies at the intersection of two visible elements. 2. The curve of intersection being common to both surfaces will be visible in vertical projection only as far as it lies on the front portion of both surfaces. 3. The horizontal projection of the curve is visible only as far as it lies on the upper portion of both surfaces. 4. The point of passing from visibility to invisibility is always on a contour element of one of the surfaces. Fig. 173 represents the cone and cylinder with only the visible portions shown.

**139. To determine the curve of intersection between two cylinders, the axes of which are oblique to the coördinate planes.**

PRINCIPLE. Auxiliary planes parallel to the axes of both cylinders will cut elements from each, and their intersections will determine points in the required curve. Since the auxiliary planes are parallel to the axes, they will be parallel to each other, and their traces will be parallel.

METHOD. 1. Through one of the axes pass a plane parallel to the other axis (Art. 74, page 52), and this will be one of the auxiliary planes to which the other cutting planes will be parallel. 2. Determine if there be one or two curves of intersection. 3. Beginning with one of the tangent planes pass auxiliary planes through the contour elements of each view, using such additional planes as may be necessary to determine the requisite number of points in the curve. 4. Draw the curve, determining the visible portions by Art. 138, page 105.

**140. To determine the curve of intersection between two cones, the axes of which are oblique to the coördinate planes.**

PRINCIPLE. Auxiliary planes which contain the apices of the cones will cut elements from each of the surfaces. Hence, all the cutting planes will contain the line joining the apices of the cones, and all the traces of the auxiliary planes will intersect in the traces of this line.

The solution of this problem is similar to that of Art. 135, page 100.

**141. To determine the curve of intersection between an ellipsoid and an oblique cylinder.**

PRINCIPLE. Auxiliary cylinders with axes parallel to that of the oblique cylinder and intersecting the axis of the ellipsoid may be chosen of such section as to cut circles from the ellipsoid and elements from the cylinder.

CONSTRUCTION. Fig. 178. Draw any parallel $b^vc^v$ in vertical projection and conceive it to be the horizontal section of an auxiliary cylinder, the axis of which is $de$. This auxiliary cylinder will intersect the horizontal coördinate plane in a circle having $e^h$ for its center and a diameter equal to $b^vc^v$, since all horizontal sections will be equal. Points $f$ and $k$ lie at the intersection of the bases of the given and auxiliary cylinder, and, therefore, points in elements common to both cylinders. The intersections of these elements with the parallel cut from the ellipsoid, $l$ and $o$, will be points common to the two surfaces and, therefore, points in the required curve. Similarly determine the necessary number of points for the drawing of a smooth curve.

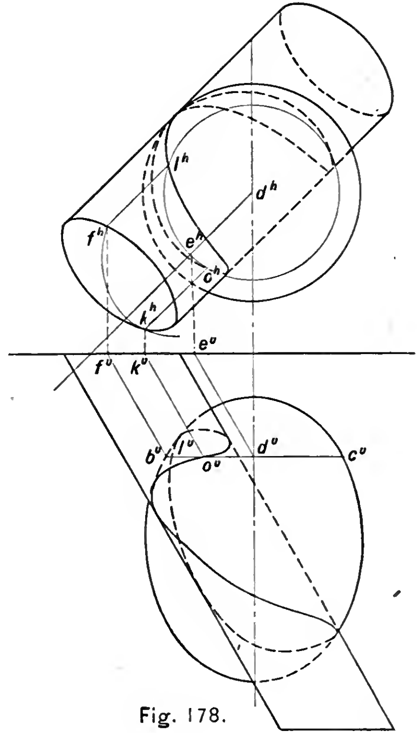



Fig. 178.

**142. To determine the curve of intersection between a torus and a cylinder, the axes of which are perpendicular to the horizontal coördinate plane.**

PRINCIPLE. Auxiliary planes perpendicular to the axes will cut each of the surfaces in circles the intersections of which will determine points common to both surfaces; or, meridian planes of the torus will cut elements from the cylinder and meridians from the torus, the intersections of which will be points in the required curve.

METHOD. 1. Determine the lowest points in the curve on the inner and outer surfaces of the torus by a meridian cutting plane containing the axis of the cylinder. 2. Determine the points of tangency with the contour elements of the cylinder by meridian cutting planes containing said elements. 3. Determine the highest points in the curve by an auxiliary plane cutting the highest parallel. 4. Determine intermediate points by auxiliary planes cutting parallels from the torus and circles from the cylinder.

CONSTRUCTION. Fig. 179. Through the axis of the cylinder pass the meridian plane of the torus, cutting elements from the cylinder and a meridian from the torus. Revolve this plane about the axis of the torus until it is parallel to $V$. In this position the vertical projections of the elements will be $A_1^v$ and $B^1$ and their intersections with the circle cut from the torus will be at $c_1^v$ and $d_1^v$. In counter-revolution these points will fall at $c^v$ and $d^v$, thus determining the lowest points of the curve on the inner and outer surface of the torus. The meridian planes $N$ and $M$ will cut contour elements from the cylinder, and the vertical projections of their points of intersection with the torus at $e^v$ and $f^v$ will be determined as in the previous case. In this manner all the points of intersection may be found; or, planes cutting parallels from the torus may be used, as $R$ and $S$, the former of which is tangent to the upper surface of the torus cutting it and the cylinder in circles which intersect at points $k$ and $l$. The remaining points necessary to the determination of the curve may be similarly found as shown by the auxiliary plane $S$.

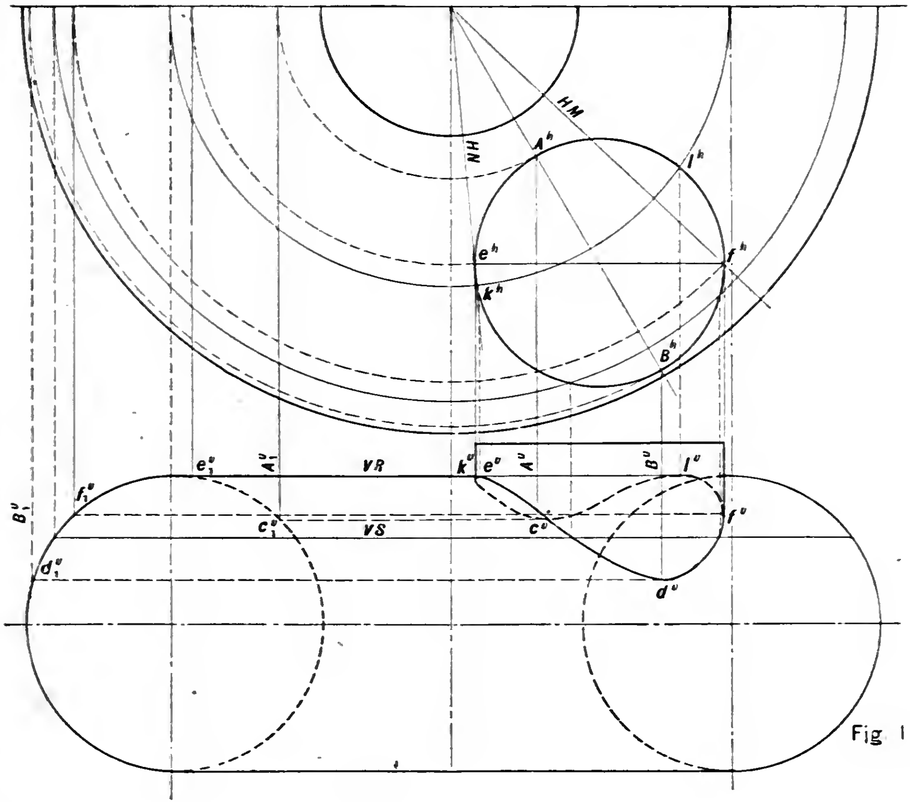

HM
NH
$A^h$
$I^h$
$e^h$
$f^h$
$k^h$
$B^h$
$e_1^v$
$A_1^v$
VR
$k^v$
$e^v$
$A^v$
$B^v$
$I^v$
$f_1^v$
$B_1^v$
$c_1^v$
VS
$c^v$
$f^v$
$d_1^v$
$d^v$

Fig. 179.

**143. To determine the curve of intersection between an ellipsoid and a paraboloid, the axes of which intersect and are parallel to the vertical coördinate planes.**

PRINCIPLE. Auxiliary spheres having their centers at the intersection of the axes of the surfaces of revolution will cut the intersecting surfaces in circles, one projection of which will be right lines.

CONSTRUCTION. Fig. 180 illustrates this case. It will be observed that the horizontal projection of the parabola is omitted since the curve of intersection is completely determined by the vertical projections of the two surfaces and the horizontal projections of the parallels of the ellipsoid.

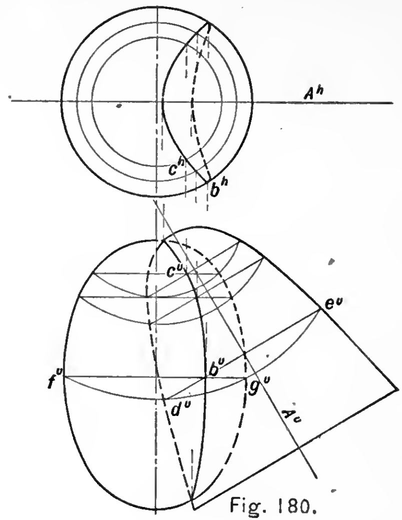



Fig. 180.

# CHAPTER VII

## WARPED SURFACES

**144. Warped Surfaces*** are ruled surfaces, being generated by the motion of a right line, the consecutive positions of which do not lie in the same plane. The right-line generatrix is governed in two distinct ways:

1. By contact with three linear directrices.
2. By contact with two linear directrices while maintaining parallelism with a plane or other type of surface. If the governing surface is a plane, it is called a plane director; if a cone, it is called a cone director, and the generatrix must always be parallel to one of its elements.

* The classification of surfaces considered in Chapter III, and especially that portion of the subject relating to warped surfaces, Arts. 100 and 101, page 68, should be reviewed previous to studying this chapter.

The following types will be considered, the first two being employed to set forth the characteristic features of this class of surfaces:

A surface having its generatrix governed by three curvilinear directrices.

A surface having its generatrix governed by two curvilinear directrices and a plane director.

An hyperbolic paraboloid, illustrating a surface the generatrix of which is governed by two rectilinear directrices and a plane director.

The oblique helicoid, illustrating a surface the generatrix of which is governed by two curvilinear directrices and a cone director.

The hyperboloid of revolution of one nappe, illustrating a surface which may be generated by several methods, and the only warped surface which is a surface of revolution.

**145. Having given three curvilinear directrices and a point on one of them, it is required to determine the two projections of the element of the warped surface passing through the given point.**

Principle. Right lines drawn from a given point on one directrix to assumed points on a second directrix will be elements of a conical surface. If the intersection between this conical surface and a third directrix be obtained, it will be a point of an element of the auxiliary cone, which is also an element of the warped surface, since it will be in contact with the three directrices.

Method. 1. Assume points on one directrix and draw elements of an auxiliary cone to the given point. 2. Determine the intersection between this auxiliary cone and the third directrix. 3. Through the given point and the point of intersection of the curve and directrix draw the required element.

Construction. Fig. 181. $A$, $B$, and $C$ are three curvilinear directrices of a warped surface. It is required to draw an element of the surface passing through point $d$ on $A$. Assume points on one of the other directrices, in this case $C$, and through these points, $e$, $f$, $k$, and $l$, draw lines to $d$. Since $C$ is a curved line, these lines will be elements of a cone. To find the intersection between directrix $B$ and the auxiliary cone, use the auxiliary cylinder which horizontally projects $B$. $B^h$ will be its horizontal trace and the horizontal projection of the curve of intersection between the auxiliary cone and cylinder. Project the horizontal projections of the points of intersection between the elements of the cone and cylinder, $n^h$, $m^h$, $r^h$, and $s^h$, to obtain points in the vertical projection of the curve of intersection, $n^v$, $m^v$, $r^v$, and $s^v$, thereby determining point $o$, the intersection of the directrix $B$ with the auxiliary cone. This point must lie on the auxiliary cone since it lies on the curve of intersection between the cylinder and cone; hence, an element drawn through $o$, will intersect $C$, the directrix of the cone, and it will be an element of the warped surface because it is in contact with the three directrices $A$, $B$, and

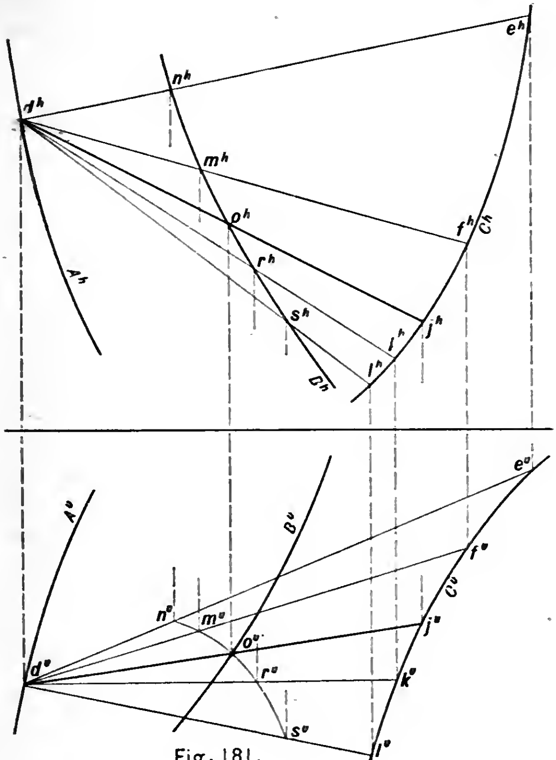


Fig. 181.

$C$; $doj$ is, therefore, the required element of the warped surface.

**146. Having given two curvilinear directrices and a plane director, to draw an element of the warped surface.**

Case 1. In which the element is required to be drawn through a given point on one of the directrices.

Principle. If a plane be passed through the given point parallel to the plane director, it will cut the second directrix in a point which, if connected with the given point, will define an element of the surface, it being parallel to the plane director and in contact with both directrices.

Method. 1. Through any point in the plane director draw divergent lines of the plane. 2. Through the given point of the directrix draw parallels to the assumed lines on the plane director. 3. Determine the intersection between the plane of these lines and the second directrix by use of the projecting cylinder of this directrix. 4. Connect this point with the given point.

CONSTRUCTION. Fig. 182. *A* and *B* are the directrices, *N* the plane director, and *d* the given point. From any point *c*, in the plane director, draw divergent lines *E*, *F*, and *G*. Through point *d* draw *dt*, *ds*, and *dr* parallel to the lines in the plane director, thus defining a plane parallel to *N*. The vertical projection of the curve of intersection between this plane and the horizontal projecting cylinder of *B* will be $r^v s^v t^v$. The intersection between this curve and *B* is at *m*, and *dm* will be the required element of the warped surface.

**147.** CASE 2. In which an element is required to be drawn parallel to a line of the plane director.

PRINCIPLE. If an auxiliary cylinder be used which has one of the curved directrices for its directrix, and its elements parallel to the given line, it will have one element which will intersect the second directrix. Such an element will be parallel to the given line on the plane director, and in contact with both directrices; hence, an element of the warped surface.

METHOD. 1. Through assumed points on one directrix draw lines parallel to the given line in the plane director, thus defining an auxiliary cylinder. 2. Determine the curve of intersection between this cylinder and one of the projecting cylinders of the second directrix. 3. Through the intersection of this curve with the second directrix draw the required element parallel to the given line.

CONSTRUCTION. Fig. 183. *A* and *B* are the directrices, *N* the plane director, and *C* the given line in the plane. *e*, *f*, *k*, and *l* are the assumed points on directrix *A* through which the elements of a cylinder are drawn parallel to line *C*. This auxiliary cylinder through *A* will intersect the horizontal projecting cylinder of *B* in a curve of which $S^v$ is the vertical projection. *m* is a point common to the auxiliary cylinder and the directrix *B*, and *dm* the required element of the warped surface.

**148.** Modifications of the two types of surfaces in Arts. 145 and 146 may be made to include all cases of warped surfaces.

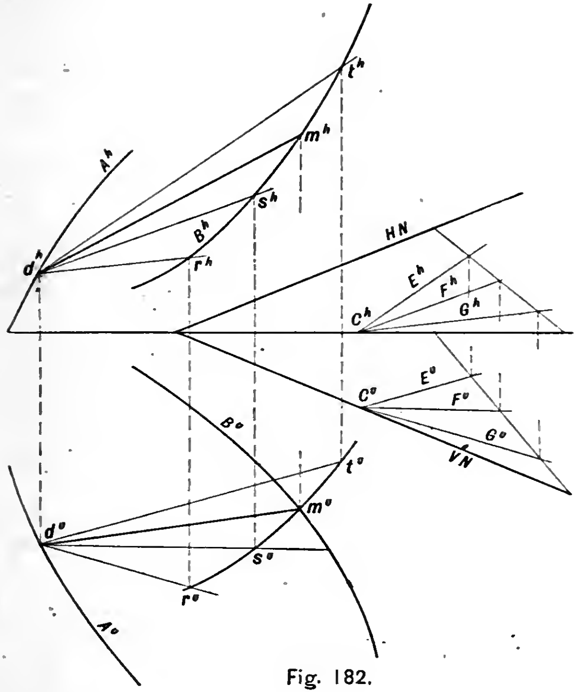



Fig. 182.

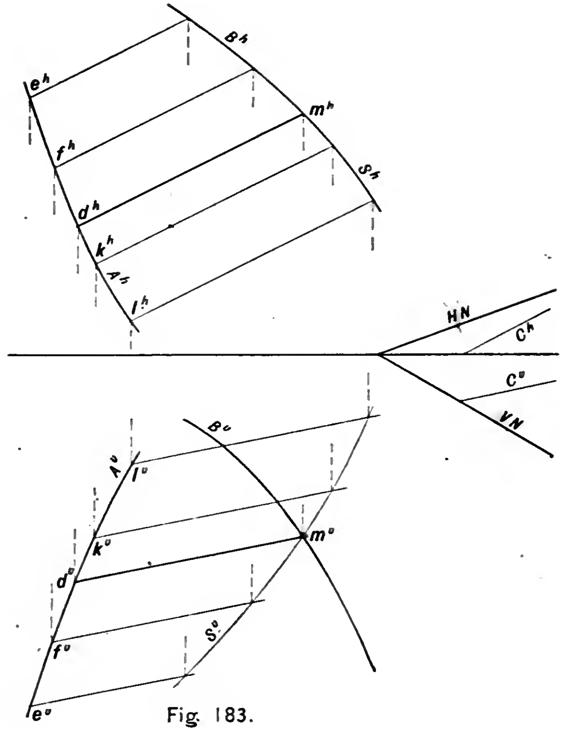



Fig. 183.

In the first type the three linear directrices may be curvilinear, rectilinear, or both curvilinear and rectilinear. In the second type, with two linear directrices and a plane director, the directrices may be curvilinear or rectilinear, and a cone may be substituted for the plane. All conceivable ruled surfaces may be generated under one of these conditions.

**149. The Hyperbolic Paraboloid.** If the case considered in Art. 146 be changed so that the two directrices be rectilinear, while the generatrix continues to be governed by a plane director, the surface will be an hyperbolic paraboloid. It is so called because cutting planes will intersect it in hyperbolas or parabolas. Figs. 184 and 185 illustrate this surface. In Fig. 184, $A$ and $B$ are the directrices, and $H$ the plane director of the surface. The positions of the generatrix, or elements, are shown by the dotted lines. As the elements will divide the directrices proportionally, they may be drawn by dividing the directrices into an equal number of parts, and connecting the points in their numerical order.

This surface is capable of a second generation by conceiving the elements $D$ and $C$ to be directrices, and $P$ to be the plane director. In this case the directrices $C$ and $D$ are divided proportionally by the elements which are now parallel to $P$.

Again, we may consider the lines $A$, $E$, and $B$ to be three rectilinear directrices governing the motion of the generatrix $D$, which is fully constrained and will describe the same surface as before; but if the three directrices were not parallel to the same plane, the character of the surface would be changed, and it would become an hyperboloid of one nappe.

An interesting application of this surface to practice is found in the pilot, or "cow catcher," of a locomotive, which consists of two hyperbolic paraboloids symmetrically placed with respect to a vertical plane through the center of the locomotive. Figs. 186 and 187 illustrate the types which are commonly used. In the former the plane director is vertical and parallel to the rails, and in the latter it is horizontal.

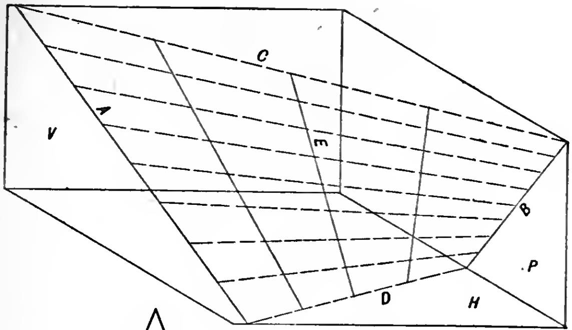


Fig. 184.

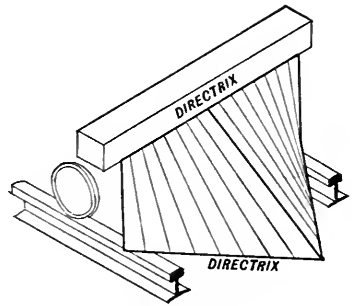


Fig. 186.

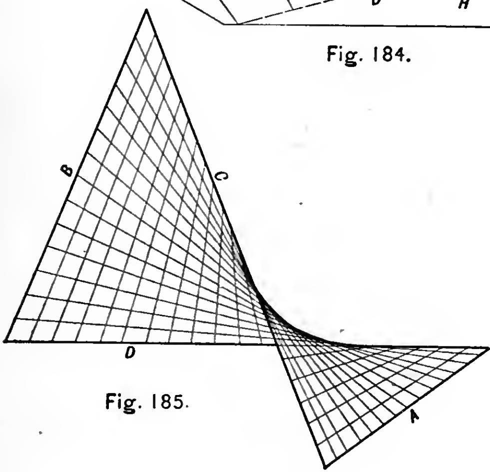


Fig. 185.

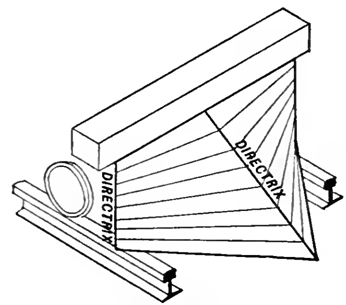


Fig. 187.

**150. Through a given point on a directrix, to draw an element of the hyperbolic paraboloid.**

PRINCIPLE. The required element must lie in a plane containing the given point and parallel to the plane director. A second point in this line will lie at the intersection of the second directrix with the auxiliary plane passed through the given point.

METHOD. 1. Through the given point draw two lines, each of which is parallel to a trace of the plane director (Art. 71, page 50). 2. Determine the point of intersection of the second directrix with the plane of the auxiliary lines (Art. 61, page 44). 3. Connect this point of intersection with the given point.

CONSTRUCTION. Fig. 188. $A$ and $B$ are the directrices, and $n$ the given point. Through $n$ draw $E$ parallel to the horizontal trace of plane $N$, and $F$ parallel to the vertical trace. These lines will determine a plane parallel to $N$. Next obtain the intersection of directrix $B$ with the plane of lines $E$ and $F$. This point is $m$, and $mn$ will be the required element.

**151. Having given one projection of a point on an hyperbolic paraboloid, to determine the other projection, and to pass an element through the point.**

CONSTRUCTION. Fig. 189. $m^h$ is the horizontal projection of the given point which lies on the surface of the hyperbolic paraboloid having $A$ and $B$ for its directrices, and $N$ for the plane director. Through $m$ draw $mg$ perpendicular to $H$, and determine its intersection with the surface, as follows:

Determine two elements, $cd$ and $ef$, near the extremities of the directrices (Art. 150), the work not being shown in the figure. Divide the portion of each directrix limited by the elements $cd$ and $ef$ into an equal number of parts to obtain elements of the surface (Art. 149, page 114). Pass an auxiliary plane $X$ through the perpendicular $mg$. This will intersect the elements at $k$, $l$, and $r$; the curve $S$, connecting these points, will be the line of intersection between the auxiliary plane $X$ and the warped surface. Since the curve $S$ and the perpendicular through $m$ lie in the plane $X$, their intersection will be a point common to the perpendicular and the warped surface. Therefore $m^v$ and $m^h$ will be projections of the required point.

To obtain the required element, pass a plane through this point $m$, parallel to the plane director $N$, and determine its intersection with one of the directrices (Art. 150, page 116). Through this, and the point $m$, draw the required element. The last operation is not illustrated in the figure.

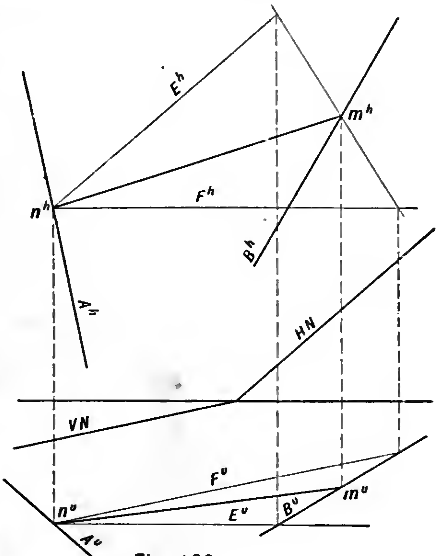



Fig. 188.

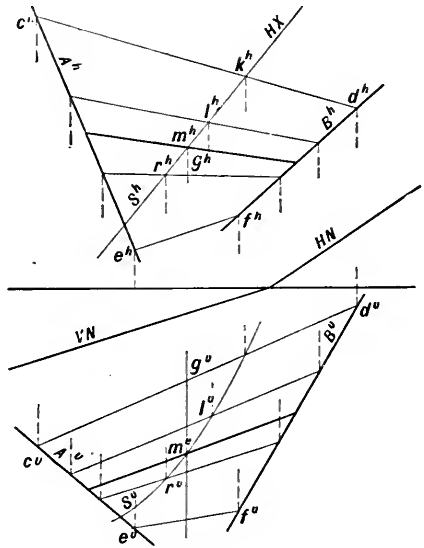



Fig. 189.

**152. Warped Helicoids.** Suppose the line *bc*, of Fig. 190, to be revolved uniformly about the line *cd* as an axis while maintaining the angle $\theta$ constant, and at the same time compelled to move in contact with, and uniformly along, the axis. All points in the line, save that one in contact with the axis, will generate helices of a constant pitch, and the surface generated will be an *oblique helicoid.* The axis, and the helix described by point *b*, may be considered as the directrices, and the generatrix may be governed by a cone which is conaxial with the helicoid. The elements of this cone will make the angle $\theta$ with *H.*

The generatrix may also be governed by two helical directrices and a cone director, as in Fig. 191.

Again, it may be governed by three directrices, which in this case, Fig. 191, may be the two helices and the axis. The V-threaded screw is the most familiar application of the oblique helicoid (Fig. 147, page 69).

**153.** If the generatrix be perpendicular to the axis, as in Fig. 192, it may be governed by directrices similar to the preceding, but the cone director will have become a plane director, and the surface generated will be a *right helicoid.* This type is illustrated by the square-threaded screw (Fig. 146, page 69).

**154.** If the generatrix does not intersect the axis, as in the preceding cases, a more general type will be generated, as shown in Fig. 193. In this case the generatrix is governed by two helical directrices and a cone director, the generatrix being tangent to the cylinder on which the inner helix is described.

**155. Hyperboloid of Revolution of one Nappe.** This is a surface of revolution which may be generated by the revolution of an hyperbola about its conjugate axis, as illustrated in Fig. 194. It is also a warped surface in that it may be generated by the revolution of a right line about an axis which it does not intersect, and to which it is not parallel. Furthermore, it will be shown that the rectilinear generatrix may be governed by three rectilinear directrices, by three curvilinear directrices, or by two curvilinear directrices and a cone director.

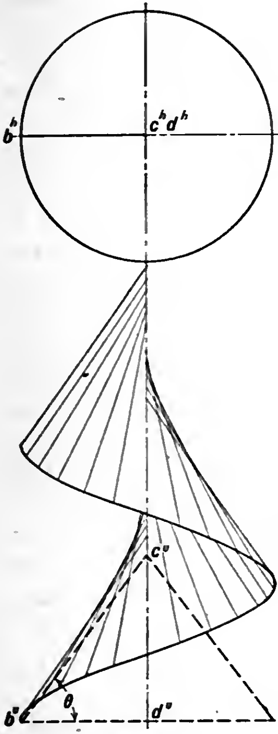



Fig. 190.

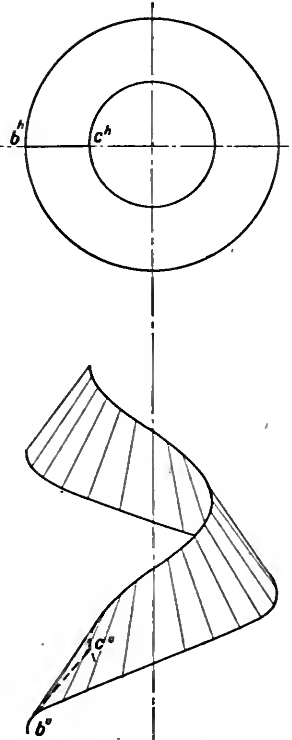



Fig. 191.

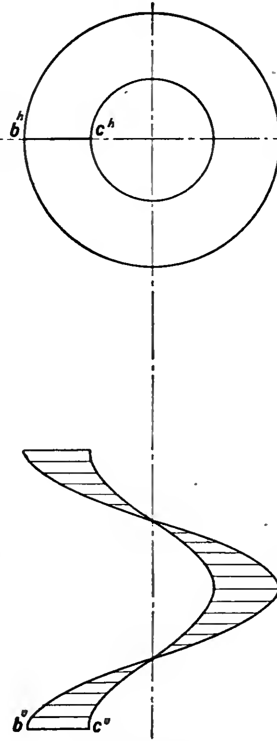



Fig. 192.

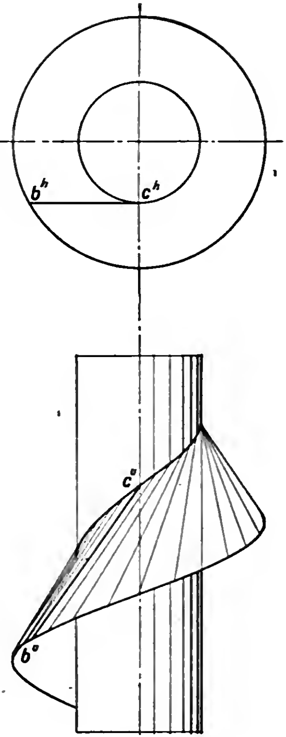



Fig. 193.

In Fig. 194 conceive the generatrix $cd$ as making the constant angle $c^v d^v b^v$ with a horizontal plane, and revolving about a vertical axis through $o$. Point $c$ will describe the circle of the upper base $cgl$, point $d$ will describe the circle of the lower base $dmf$, and the point $e$, the nearest to the axis, will describe the circle $ekn$, which is called the circle of the gorge. All other points of the generatrix will similarly describe circles, and by drawing these parallels of the surface, the meridian line will be determined, and is an hyperbola. Thus point $s$, in the generatrix $cd$, will be in the position $t$ when it lies in the principal meridian plane, and $t^v$ will be a point in the vertical contour, which is an hyperbola.

**156. Through any point of the surface to draw an element.** If one projection of the point be given, draw a parallel of the surface through this point to determine the other projection. Through the horizontal projection of this point draw a tangent to the circle of the gorge, and it will be the horizontal projection of the required element, the extremi-

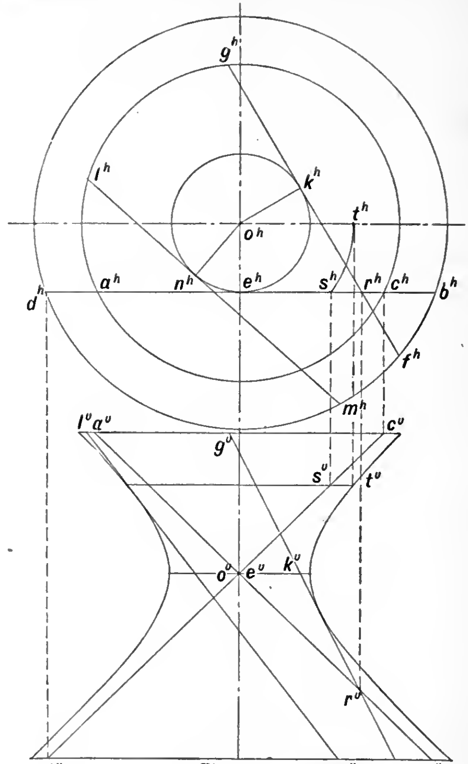



Fig. 194.

ties of which lie in the horizontal projections of the upper and lower bases of the surface. Since either extremity may be regarded as lying in the upper base of the surface, there are two tangents which may be drawn through the given point. They will make equal angles with the horizontal coördinate plane, and intersect at the circle of the gorge.

**157. The Generatrix may be governed by Three Rectilinear Directrices.** Fig. 194. If two elements, *ab* and *cd*, be drawn through point *e* of the circle of the gorge, either may be taken as the generatrix of the surface. One is known as an element of the first generation, and the other as an element of the second generation.

Conceive *ab* as fixed and *cd* as the generatrix. In the revolution about the axis, *cd* will at all times intersect *ab*, if these lines be extended indefinitely. This may be proved as follows: If *cd* be in the position indicated by *gf*, then the horizontal projections of *ab* and *gf* will intersect in $r^h$. This point will be equally distant from the points of tangency $e^h$ and $k^h$, and since *ab* and *gf* make equal angles with *H*, the distances *er* and *kr* must be equal, and hence, *r* must be at the intersection of *ab* and *gf*. If, then, we conceive three elements of the surface, such as *cd*, *gf*, and *lm*, and if we conceive *ab* as the generatrix, it will intersect each of these elements and they may be used as directrices.

Again, if three parallels be the directrices, the generatrix will be fully constrained.

**158. The Generatrix may be governed by Two Curvilinear Directrices and a Cone Director.** Fig. 194. Since the elements of the surface are parallel to the elements of a cone, having the angle $d^r e^r b^r$ for the apex angle, this may be used as a cone director, the generatrix being also governed by two parallels of the surface, such as the bases, or a base and the circle of the gorge.

**159. The tangent plane** to any point of the surface is determined by the elements of the two generations drawn through this point. The plane determined by lines *ab* and *gf* will be tangent to the surface at point *r*. Fig. 194.

**160. Through a right line to pass a plane tangent to any double-curved surface of revolution.**

By the use of the hyperboloid of revolution of one nappe as an auxiliary surface, it is possible to make a general solution of problems requiring the determination of tangent planes to double curved surfaces of revolution, as follows:

PRINCIPLE. If the given right line be revolved about the axis of the double-curved surface of revolution, it will generate an hyperboloid of revolution. A plane tangent to both surfaces and containing the given line, which is an element of one of them, will be the required plane. Since one, and only one meridian plane at a point of tangency will be perpendicular to the tangent plane, and as the surfaces of revolution have a common axis, it follows that one meridian plane will cut a line from the tangent plane which will pass through the points of tangency on both surfaces and be tangent to both meridian curves. This line and the given line will determine the tangent plane.

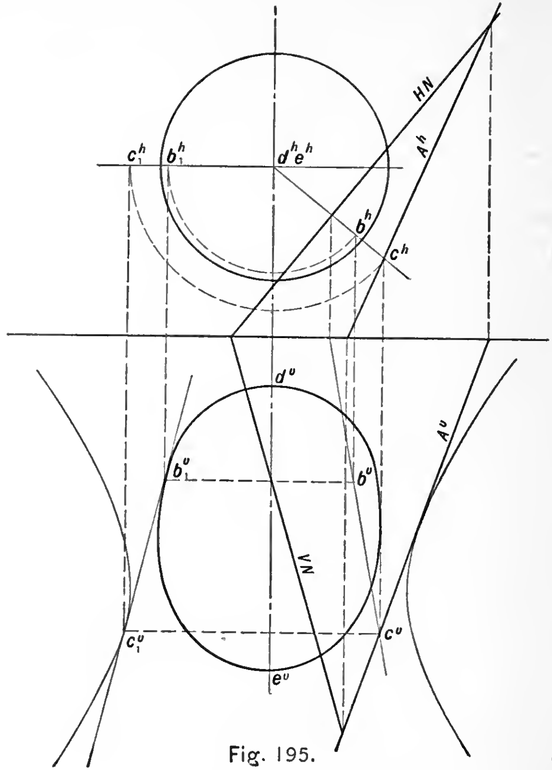



Fig. 195.

METHOD. 1. Draw the principal meridian section of the hyperboloid of revolution which has the given line for its generatrix. 2. Draw a tangent to the principal meridian sections of both surfaces. 3. Revolve this line about the axis of the surfaces until it intersects the given line, observing that its point of tangency with the hyperbola is a point of the given line. 4. Determine the plane of this tangent and the given line.

CONSTRUCTION. Fig. 195. Having described the hyperboloid of revolution with the given line $A$ as its generatrix, draw $c_1^v b_1^v$ tangent to the meridian curves. It will be the vertical projection of the revolved position of a line tangent to both surfaces. In counter-revolution this line will intersect the given line $A$ at $c$, which is the counter-revolved position of the point of tangency $c_1$. This must be so, since line $A$ is an element of the hyperboloid of revolution and must be in contact with the parallel through $c_1$. Point $b_1$ in counter-revolution is at $b$, and $bc$ will be a line of the tangent plane. $N$ will be the plane of $bc$ and the given line $A$, and, therefore, the required tangent plane.

## CHAPTER VIII

### PROBLEMS

**161. Directions for solving the Problems.** The problems are arranged in pairs, allowing an instructor to assign them alternately, inasmuch as it would not be a wise expenditure of time for a student to solve all of them.

The problems are designed to be solved within margin lines measuring $7 \times 10$ inches, one such plate constituting an exercise.

The notation of Art. 6, page 4, is to be used. The student should remember that the correct lettering of every point and line, and the observance of the character of lines, is as much a part of the solution of the problem as is the correct location of point or line.

The following abbreviations will be used in solving the problems:

*V-pr.* signifies Vertical Projection.

*H-pr.* signifies Horizontal Projection.

*P-pr.* signifies Profile Projection.

*V-tr.* signifies Vertical Trace.

*H-tr.* signifies Horizontal Trace.

*P-tr.* signifies Profile Trace.

The coördinates of points will be designated as follows:

1st dimension is the perpendicular distance to $V$.

2nd dimension is the perpendicular distance to $H$.

3rd dimension is the perpendicular distance to $P$.

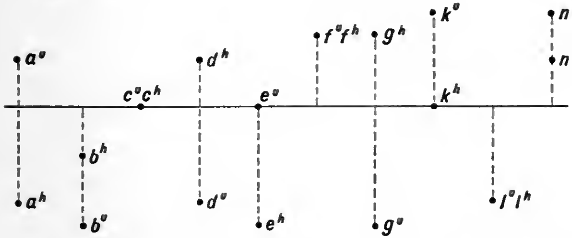
Distances above $H$ are $+$, and below $H$ are $-$.

Distances before $V$ are $+$, and behind $V$ are $-$.

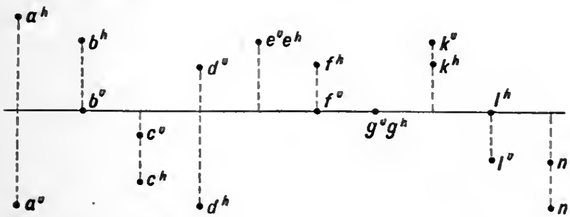
Measurements from $P$ are $+$ and to the left.

**162. Problems.** The space required for each of the first forty-eight problems is $3\frac{1}{2} \times 5$ inches, and the unit of measure is $\frac{1}{8}$ inch.

**1.** Required the distance from *H*, and the *Q*, of each of the following points (Art. 7, page 6).




**2.** Required the distance from *V*, and the *Q*, of each of the following points (Art. 7, page 6).



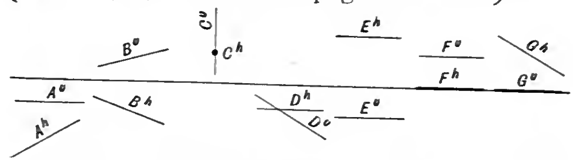

**3.** Construct the *H-pr.* and *V-pr.* of the following points (Art. 7, page 6).

*a*, 6, 2. *b*, 5, −2. *c*, −1, −4. *d*, −4, 3. *e*, 4, 0. *f*, −4, 4. *k*, 0, −2. *l*, 4, −3. *m*, 0, 0. *n*, 4, −4.

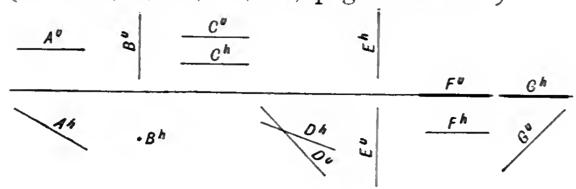
**4.** Construct the *H-pr.* and *V-pr.* of the following points (Art. 7, page 6).

*a*, −6, −4. *b*, −4, 4. *c*, 5, 4. *d*, 9, −4. *e*, −4, 6. *f*, 0, 10. *k*, −5, 0. *l*, 0, 0. *m*, −2, −6. *n*, 8, 8.

**5.** Fully describe the following lines (Arts. 8, 9, 10, 11, 12, pages 7 and 8).




**6.** Fully describe the following lines (Arts. 8, 9, 10, 11, 12, pages 7 and 8).



**7.** Required the *H-pr.* and *V-pr.* of the following lines (Arts. 8, 9, 10, 11, 12, pages 7 and 8).

*A*, inclined to *V*, inclined to *H*, in *3 Q*.

*B*, parallel to *H*, inclined to *V*, in *2 Q*.

*C*, parallel to *P*, inclined to *H* and *V*, in *1 Q*.

*D*, perpendicular to *V*, in *3 Q*.

*E*, parallel to *H*, inclined to *V*, in *4 Q*.

*F*, inclined to *V*, lying in *H*, between *2 Q* and *3 Q*.

**8.** Required the *H-pr.* and *V-pr.* of the following lines (Arts. 8, 9, 10, 11, 12, pages 7 and 8).

*A*, inclined to *V*, inclined to *H*, in *1 Q*.

*B*, inclined to *H*, parallel to *V*, in *3 Q*.

*C*, perpendicular to *V*, in *3 Q*.

*D*, inclined to *V*, parallel to *H*, in *2 Q*.

*E*, parallel to *GL*, in *4 Q*.

*F*, inclined to *H*, lying in *V*, between *3 Q* and *4 Q*.

**9.** Draw the *H-pr.* and *V-pr.* of the following lines (Arts. 8 to 15, pages 7 to 10).

*A* and *B* intersecting in *3 Q*. *A* parallel to *V* and inclined to *H*; *B* inclined to *V* and *H*.

*C* and *D* intersecting in *2 Q*. *C* perpendicular to *H*; *D* parallel to *GL*.

*E* and *F* not intersecting. *E* perpendicular to *V*; *F* inclined to *V* and *H*. Both in *4 Q*.

**10.** Draw the *H-pr.* and *V-pr.* of the following lines (Arts. 8 to 15, pages 7 to 10).

*A* and *B* parallel, and inclined to *V* and *H*, in *3 Q*.

*C* and *D* intersecting in *1 Q*. *C* inclined to *V* and *H*; *D* parallel to *V* only.

*E* and *F* intersecting in *3 Q*. *E* parallel to *H* and inclined to *V*; *F* parallel to *GL*.

**11.** Required the *H-pr.*, *V-pr.*, and *P-pr.* of the following lines (Arts. 21–23, pages 14, 15). State the *Q*'s in which they appear, and the direction of inclination (Art. 17, page 10).

$$ab\begin{cases} a, -6, -4, 8. \\ b, -2, -4, 0. \end{cases} \qquad cd\begin{cases} c, 6, 4, 9. \\ d, 2, -4, 0. \end{cases}$$

$$ef\begin{cases} e, -2, 4, 10. \\ f, -8, 4, 0. \end{cases}$$

**12.** Required the *H-pr.*, *V-pr.*, and *P-pr.* of the following lines (Arts. 21–23, pages 14, 15). State the *Q*'s in which they appear, and the direction of inclination (Art. 17, page 10).

$$ab \begin{cases} a, -2, 6, 7. \\ b, -5, 1, 0. \end{cases} \qquad cd \begin{cases} c, 6, 1, 10. \\ d, 4, 1, 0. \end{cases}$$

$$ef \begin{cases} e, -6, -4, 9. \\ f, -2, 4, 0. \end{cases}$$

**13.** Required the *H-pr.* and *V-pr.* of the following triangles (Art. 21, page 14).

$$abc \begin{cases} a, -8, -3, 11. \\ b, -1, -10, 7. \\ c, 0, -3, 0. \end{cases} \qquad def \begin{cases} d, -2, 2, 8. \\ e, -2, 9, 4. \\ f, -2, 2, 0. \end{cases}$$

**14.** Required the *H-pr.* and *V-pr.* of the following triangles (Art. 21, page 14).

$$abc \begin{cases} a, -6, -6, 8. \\ b, -1, -2, 6. \\ c, 0, -1, 0. \end{cases} \qquad def \begin{cases} d, -1, -1, 11. \\ e, 6, 4, 0. \\ f, 4, -2; 6. \end{cases}$$

**15.** Three points, *a*, *b*, and *c*, lie in *P*, and in *3 Q*. *a* and *b* have their *V-prs.* in the same point, and *b* and *c* have their *H-prs.* in the same point. *a* is 4 units from *V* and *H*; *b* is 10 units from *V*, and *c* is 8 units from *H*. Determine their *H-prs.*, *V-prs.*, and *P-prs.* (Arts. 21–23, pages 14 and 15).

**16.** Three points, *a*, *b*, and *c*, lie in *P*, and in *3 Q*. *a* and *b* have their *H-prs.* in the same point, and *b* and *c* have their *V-prs.* in the same point. *a* is 4 units from *H*, and 8 units from *V*; *c* is 10 units from *H*, and 6 units from *V*. Determine their *H-prs.*, *V-prs.*, and *P-prs.* (Arts. 21–23, pages 14 and 15).

**17.** Draw the *H-pr.*, *V-pr.*, and *P-pr.* of line *ab*. *a*, $-2$, 12, 0. *b*, 9, $-2$, 22. Determine the *prs.* of the following points in *ab*.

*c*, equidistant from *H* and *V*.

$$\left.\begin{array}{l} d\text{, the } H\text{-}tr.\text{ of the line.} \\ e\text{, the } V\text{-}tr.\text{ of the line.} \end{array}\right\} \quad \left\{\begin{array}{l} \text{Art. 16, page 10.} \\ \text{Art. 24, page 16.} \end{array}\right.$$

*f*, the distance from *H* twice that from *V*.

*k*, in *4 Q*, 4 units from *H*.

In what *Q*'s does the line appear?

**18.** Draw the *H-pr.*, *V-pr.*, and *P-pr.* of line *ab*. *a*, 6, $-6$, 20. *b*, $-2$, 11, 0. Solve as for 17.

**19.** Make an oblique projection of 17, representing *V*, *H*, and *P* in their relative positions, and the *prs.* of the line, and points thereon.

**20.** Make an oblique projection of 18, representing *V*, *H*, and *P* in their relative positions, and the *prs.* of the line, and points thereon.

**21.** Draw the *prs.* of line *ab* lying in *P*. *a*, $-2$, 12. *b*, 12, $-4$. Solve as for 17, omitting *c* and *k* (Art. 25, page 18).

**22.** Draw the *prs.* of line *ab* lying in *P*. *a*, 12, 8. *b*, 2, $-8$. Solve as for 17, omitting *c* and *k* (Art. 25, page 18).

**23.** Draw the *H-pr.* and *V-pr.* of lines *A* and *B*, having the following traces. Designate the *Q*'s in which they appear, if produced (Art. 26, page 18). *A*, *H-tr.*, 6 units behind *V*, and 10 units to the right of *V-tr.* *V-tr.*, 8 units below *H*. *B*, *H-tr.*, 5 units before *V*, and 11 units to the right of *V-tr.* *V-tr.*, 10 units below *H*.

**24.** Draw the *H-prs.* and *V-prs.* of lines *C* and *D*, having the following traces. Designate the *Q*'s in which they appear if produced (Art. 26, page 18). *C*, *V-tr.*, 4 units above *H*, and 11 units to the right of *H-tr.* *H-tr.*, 10 units behind *V*. *D*, *H-tr.*, 9 units before *V*, and 12 units to the right of the *V-tr.* *V-tr.*, 6 units above *H*.

**25.** Draw the *H-pr.* and *V-pr.* of lines appearing in the following *Q*'s only (Art. 24, page 16). *A*, 1, 4, 3. *B*, 1, 2, 3. Note. Assume the traces of the lines and proceed by Art. 24, page 18.

**26.** Solve as for 25. *A, 4, 1, 2.* *B, 2, 3, 4.*

**27.** Solve as for 25. *C, 2, 4.* *D, 1, 3.*

**28.** Solve as for 25. *C, 1, 3.* *D, 4.*

**29.** Solve as for 25. *E, 4.* *F, 1, 4.*

**30.** Solve as for 25. *E, 4, 2.* *F, 3, 2.*

**31.** Solve as for 25. *K, 3, 4.* *L, 3.*

**32.** Solve as for 25. *K, 2, 1.* *L, 1.*

**33.** Draw the *H-pr.* and *V-pr.* of line *ab*. *a*, $-1$, $-6$, 14. *b*, $-8$, $-1$, 0. Through the middle point of *ab* draw line *C* parallel to *V*, and making an angle of 30° with *H*. Determine the traces of the plane of these lines (Arts. 27–30, pages 19, 20).

**34.** Draw the *H-pr.* and *V-pr.* of line *ab.* *a*, – 6, – 2, 0. *b*, – 1, 11, 14. Through the middle point of *ab* draw line *C* parallel to *H.* and making an angle of 45° with *V.* Determine the traces of the plane of these lines (Arts. 27–30, pages 19 and 20).

**35.** Given line *ab.* *a*, 3, 3, 10; *b*, 7, 9, 0; and point *c*, 7, 4, 0. Through point *c* draw a line parallel with *ab*, and determine the traces of the plane of these lines (Arts. 27–30, pages 19 and 20).

**36.** Given line *ab.* *a*, – 3, – 8, 9; *b*, 2, – 12, 0; and point *c*, – 6, – 5, 4. Through point *c* draw a line parallel with *ab*, and determine the traces of the plane of these lines (Arts. 27–30, pages 19 and 20).

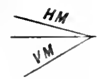
**37.** Given line *ab.* *a*, – 1, – 6, 14; *b*, – 8, – 1, 0; and point *c*, 6, – 1, 7. Determine the traces of the plane of these (Art. 33, page 22).

**38.** Given line *ab.* *a*, – 6, – 2, 0; *b*, – 1, 11, 14; and point *c*, 4, – 4, 8. Determine the traces of the plane of these (Art. 33, page 22).

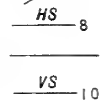
**39.** Determine the traces of the plane in which points *a*, *b*, and *c* lie. *a*, – 8, – 2, 22. *b*, 8, – 2, 0. *c*, 6, – 9, 14 (Art. 34, page 22).

**40.** Determine the traces of the plane in which points *a*, *b*, and *c* lie. *a*, – 8, – 4, 16. *b*, – 3, – 8, 8. *c*, – 6, – 12, 0 (Art. 34, page 22).

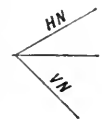
NOTE. In the following problems the traces of the planes are parallel to, or make angles of 15°, or its multiple, with *GL.* This angle may be determined by inspection.




**41.** Draw a triangle on plane *M.* Assume one projection and proceed as in Art. 35, page 22.

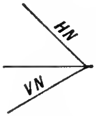



**42.** Draw a triangle on plane *S.* Assume one projection and proceed as in Art. 35, page 22.

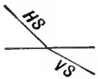



**43.** Determine the *prs.* of a point on plane *N* which is 6 units from *V* and *H* (Arts. 36, 37, page 24). Through this point draw three lines on the plane as follows: *A*, parallel to *H*; *B*, parallel to *V*; *C*, oblique to *V* and *H* (Arts. 27, 28, page 19).

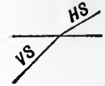
**44.** Determine the *prs.* of a point on plane *N* which is 7 units from *V* and 4 units from *H* (Arts. 36, 37, page 24). Through this point draw three lines on the plane as follows: *A*, parallel to *VN*; *B*, parallel to *HN*; *C*, through a point on *VN* 12 units from *GL* (Arts. 27, 28, page 19).



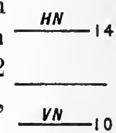

**45.** Determine the *prs.* of a point on plane *S* which is 6 units from *V* and *H* (Arts. 36, 37, page 24). Through this point draw three lines on the plane as follows: *A*, passing through *2 Q* and *3 Q*; *B*, passing through *3 Q* and *4 Q*; *C*, passing through *2 Q*, *3 Q*, and *4 Q* (Arts. 27, 28, page 19).




**46.** Determine the *prs.* of a point on plane *S* which is 7 units from *V* and 4 units from *H* (Arts. 36, 37, page 24). Through this point draw three lines on the plane as follows: *A*, parallel to *V*; *B*, parallel to *H*; *C*, passing through *2 Q*, *3 Q*, and *4 Q* (Arts. 27, 28, page 19).



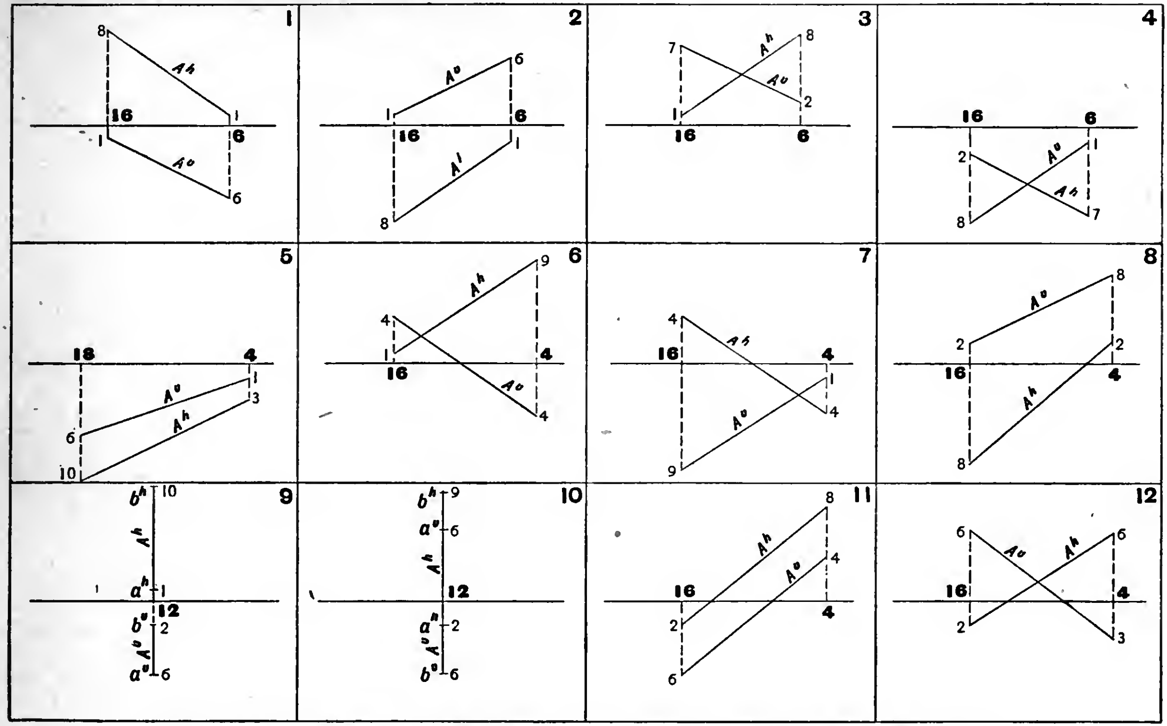

**47.** Determine the *prs.* of the following points on plane *N*, but not lying in a profile plane. *a*, 6 units from *V* in *3 Q*; *b*, 4 units from *H* in *3 Q*; *c*, 2 units from *V* in *4 Q* (Arts. 36, 37, page 24).




**48.** Determine the *prs.* of the following points on plane *R*, but not lying in a profile plane. *a*, 4 units from *H* in *3 Q*; *b*, 2 units from *V* in *1 Q*; *c*, 4 units from *H* in *2 Q* (Arts. 36, 37, page 24).

Unit of measure, $\frac{1}{8}$ inch. Space required for each problem, $2\frac{1}{2} \times 3$ inches. Measurements from GL, in light type, and from right-hand division line, in heavy type.

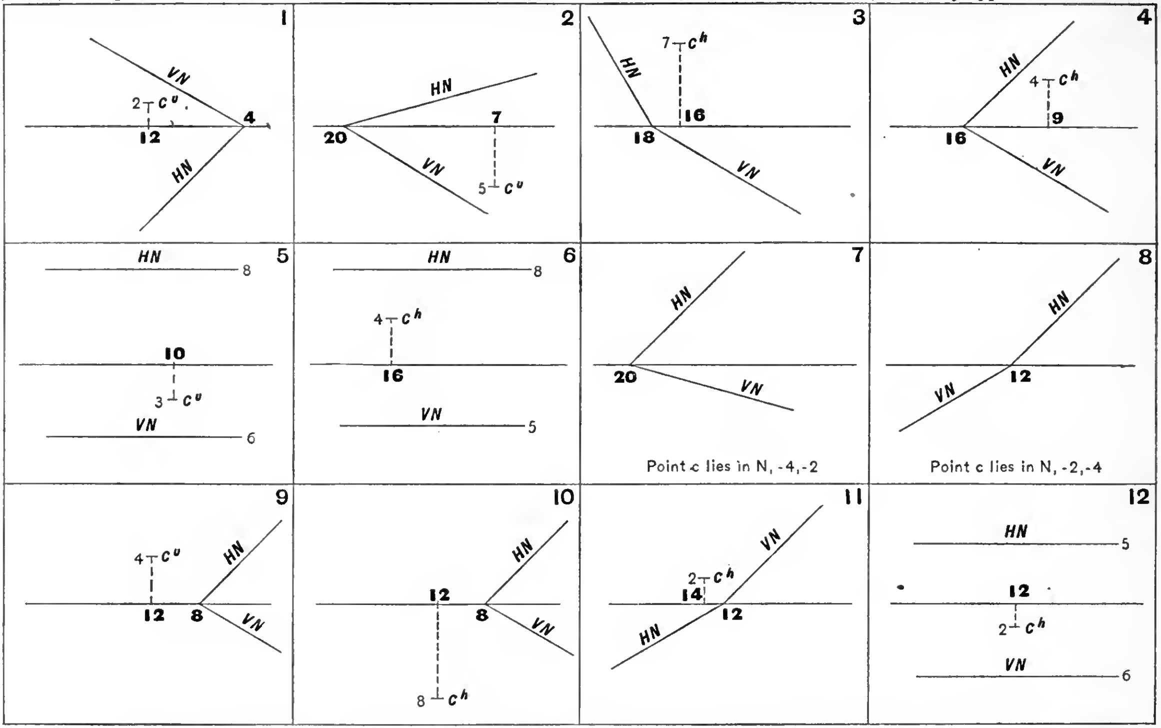
# PLATE 1




Determine the true length of line A. Case 1. (Art. 39, page 27.) Case 2. (Arts. 40, 41, page 28.)

Unit of measure, $\frac{1}{8}$ inch. Space required for each problem, $2\frac{1}{2} \times 3$ inches. Angles between GL and traces of planes, multiples of 15°. Measurements from GL, in light type, and from right-hand division line, in heavy type.

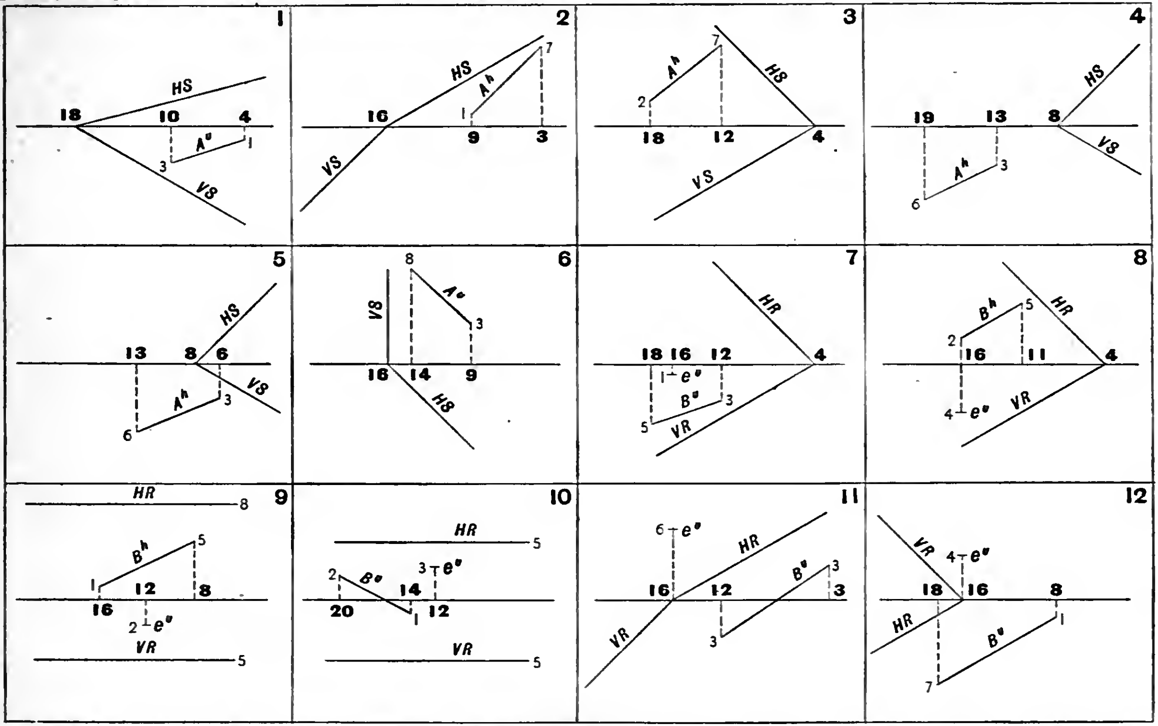
## PLATE 2




Point c lies in plane N. Determine its distance from VN and HN. (Art. 36, page 24; Art. 42, page 30.)

Unit of measure, $\frac{1}{4}$ inch. Space required for each problem, $2\frac{1}{4} \times 3$ inches. Angles between GL and traces of planes, multiples of 15°. Measurements from GL, in light type, and from right-hand division line, in heavy type.

# PLATE 3

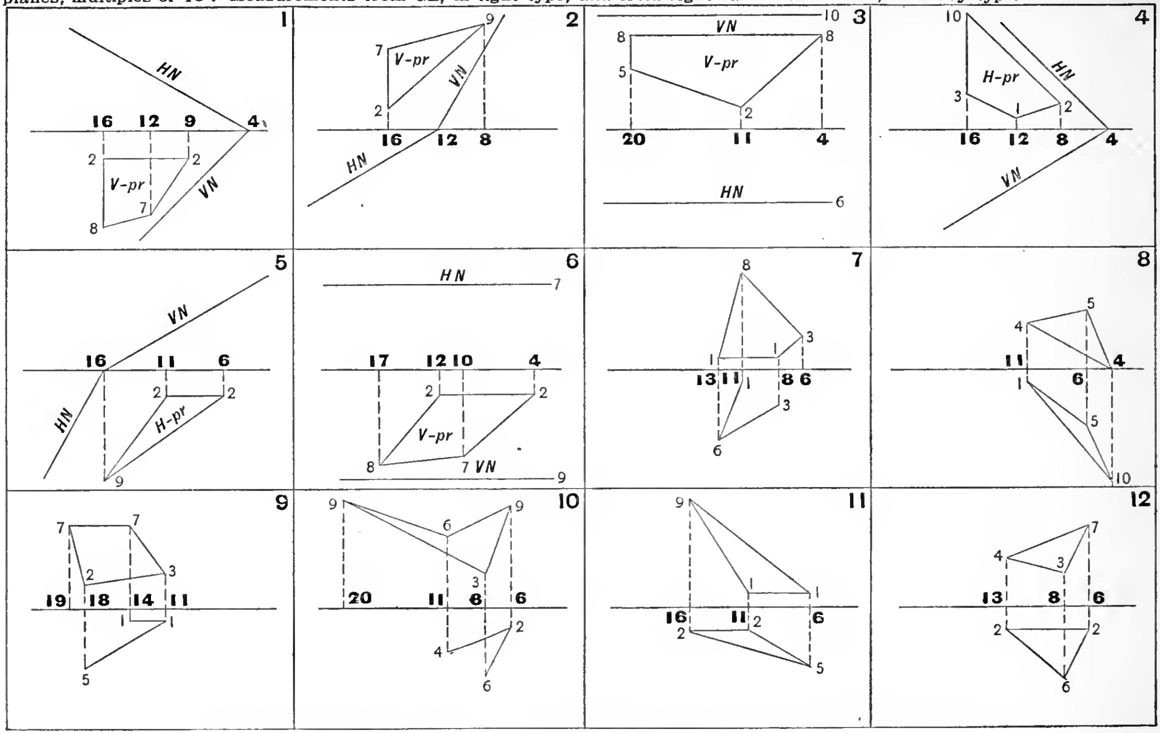



Problems 1-6. Line A lies in plane S. Determine its length by revolving it into V and into H, about VS and HS as axes. (Art. 43, page 31.)

Problems 7-12. Line B and point e lie in plane R. Determine the distance between them. (Arts. 42, 43, pages 30, 31.)

Unit of measure, $\frac{1}{8}$ inch. Space required for each problem, $2\frac{1}{2} \times 3$ inches. Angles between GL and traces of planes, multiples of 15°. Measurements from GL, in light type, and from right-hand division line, in heavy type.

# PLATE 4



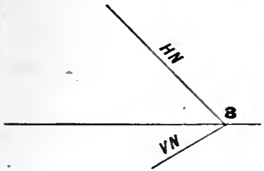

Determine the true size of the polygon by revolving it into V or H. (Art. 43, page 31.)

Note. — In Problems 7-12 the traces of the plane of the polygon should first be determined.

Unit of measure, $\frac{1}{8}$ inch. Space required for each problem, 5 x 7 inches. Angles between GL and traces of planes, multiples of 15°. Measurements from GL, in light type, and from right-hand division line, in heavy type.

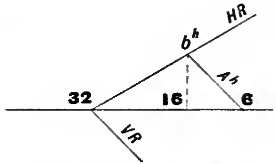
# PLATE 5

**1**



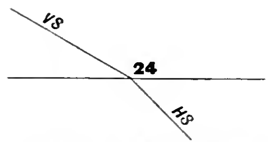

Draw the prs. of an equilateral triangle on plane N. Its center is point a, −6, −4. One side of the triangle is 7 units long and parallel to V.

**2**



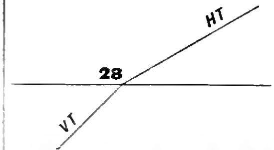

Draw the prs. of a regular hexagon on plane R. Point b is one extremity of a long diameter coinciding with line A. Side of hexagon is 5 units long.

**3**



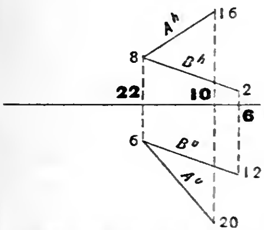

Draw the prs. of a square on plane S. Its center is point c, 8, 5. One side is 8 units long and at an angle of 60° with HS.

**4**



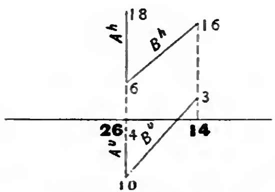

Draw the prs. of a regular octagon on plane T. Its center is point d, −8, −5. Its short diameter is 8 units long and parallel to H.

**5**



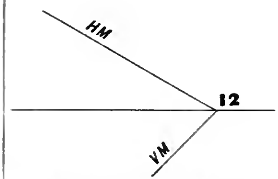

Draw the prs. of a circle tangent to lines A and B. Its diameter is 8 units.

**6**



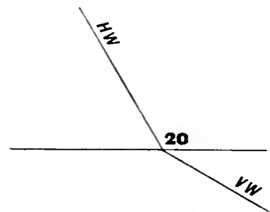

Draw the prs. of a circle tangent to lines A and B. Its diameter is 10 units.

**7**




Draw the prs. of a circle tangent to the traces of plane M. Its diameter is 12 units.

**8**

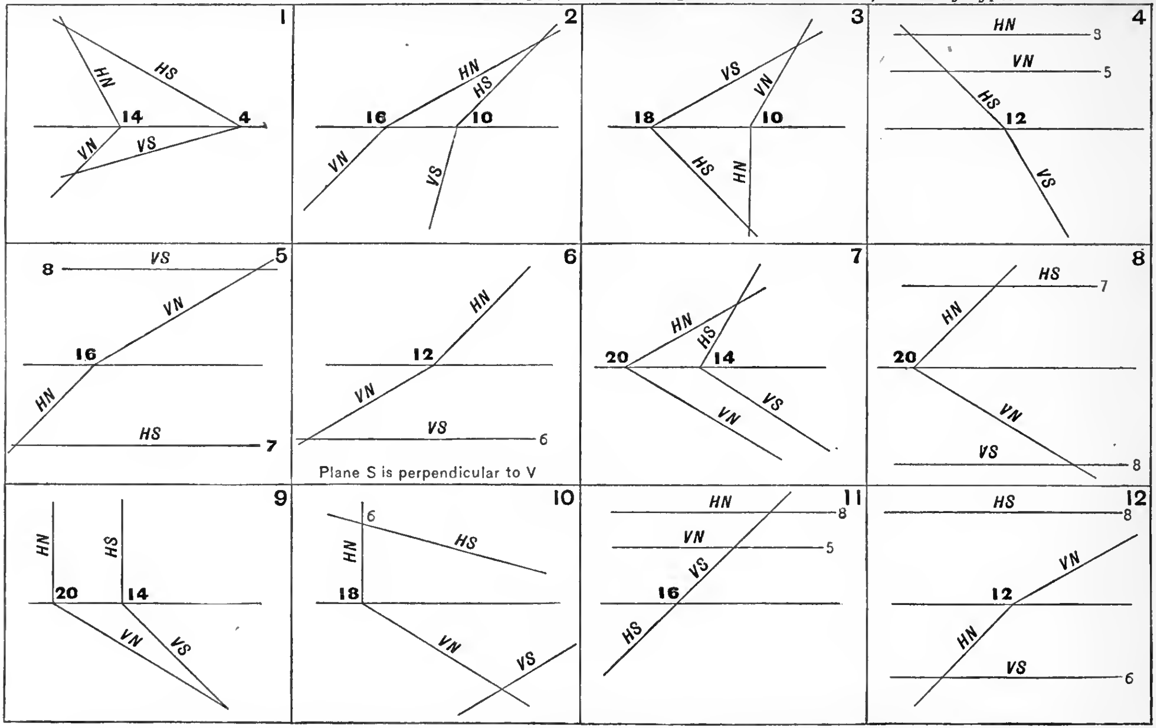



Draw the prs. of a circle tangent to the traces of plane W, and lying in 3Q. Its diameter is 16 units.

Draw the projections of the figures indicated. (Art. 45, page 32.)

Unit of measure, $\frac{1}{8}$ inch. Space required for each problem, $2\frac{1}{2} \times 3$ inches. Angles between GL and traces of planes, multiples of 15°. Measurements from GL, in light type, and from right-hand division line, in heavy type.

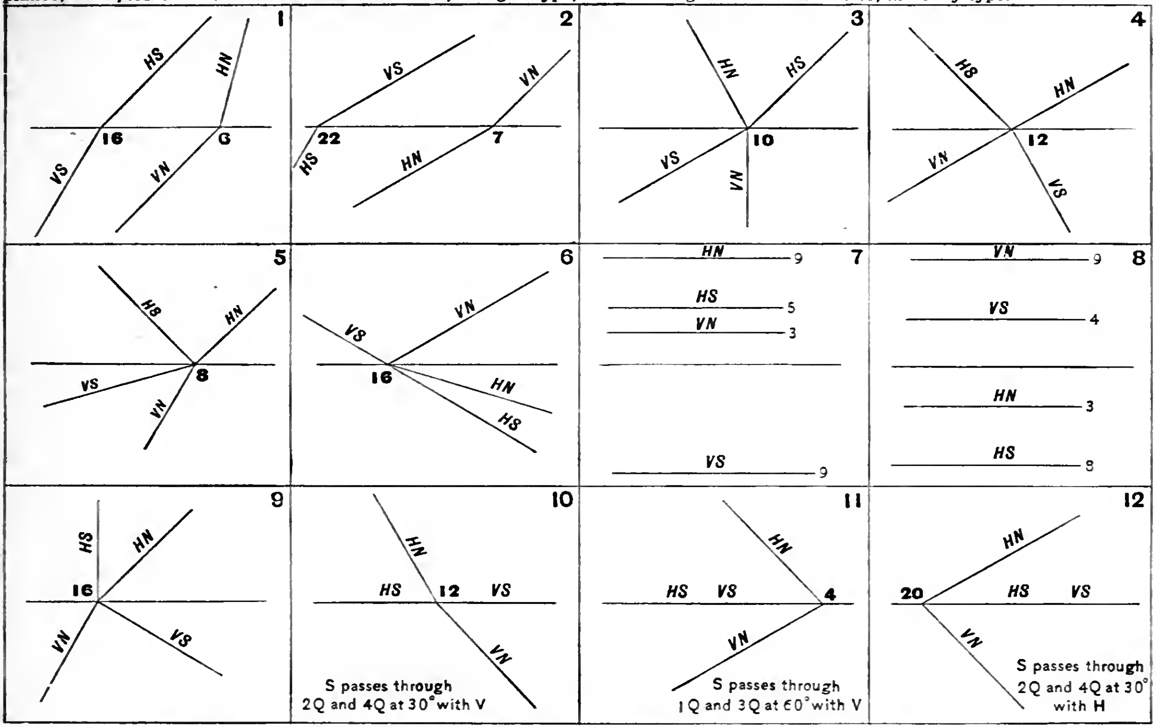
# PLATE 6




Determine the prs. of the line of intersection between planes N and S. (Arts. 49, 50, page 35, Art. 52, page 37.)

Unit of measure, ¼ inch. Space required for each problem, 2½ × 3 inches. Angles between GL and traces of planes, multiples of 15°. Measurements from GL, in light type, and from right-hand division line, in heavy type.

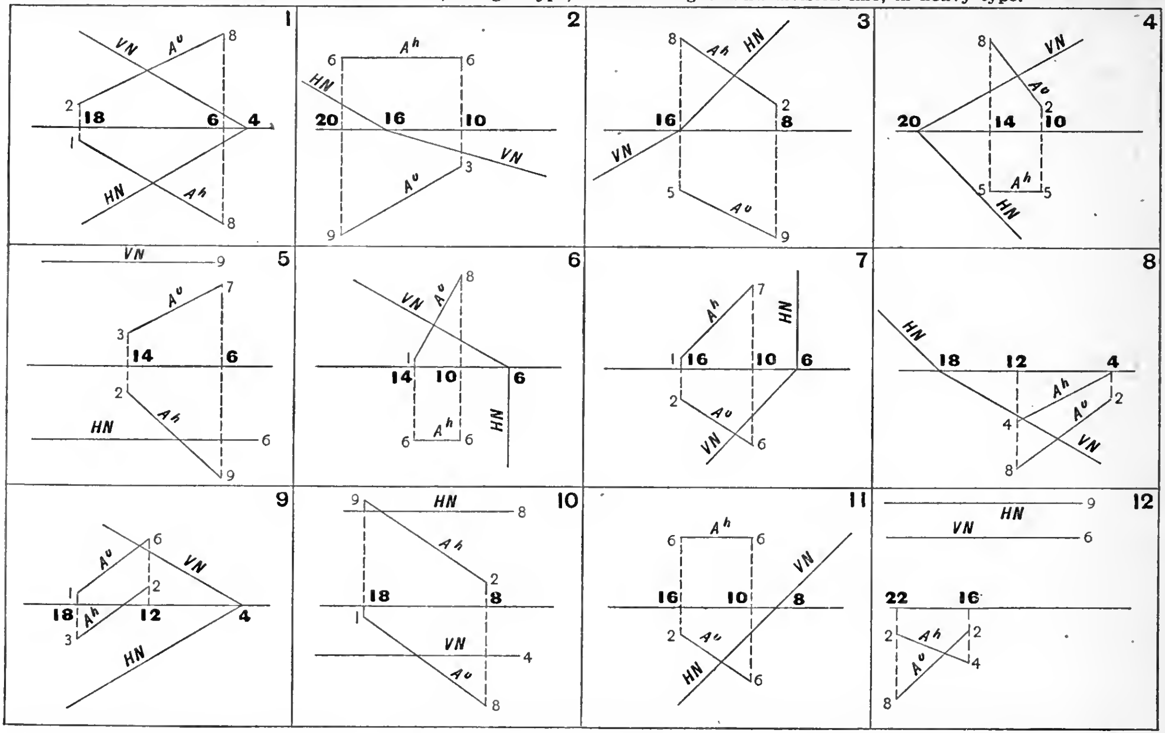
# PLATE 7




Determine the prs. of the line of intersection between planes N and S. (Arts. 52-56, pages 37-40.)
Problems 1-4. Solve by Case 2. (Art. 51, page 36.)
Problems 5-12. Solve by Case 3. (Art. 54, page 38.)

Unit of measure, $\frac{1}{8}$ inch. Space required for each problem, $2\frac{1}{2} \times 3$ inches. Angles between GL and traces of planes, multiples of 15°. Measurements from GL, in light type, and from right-hand division line, in heavy type.

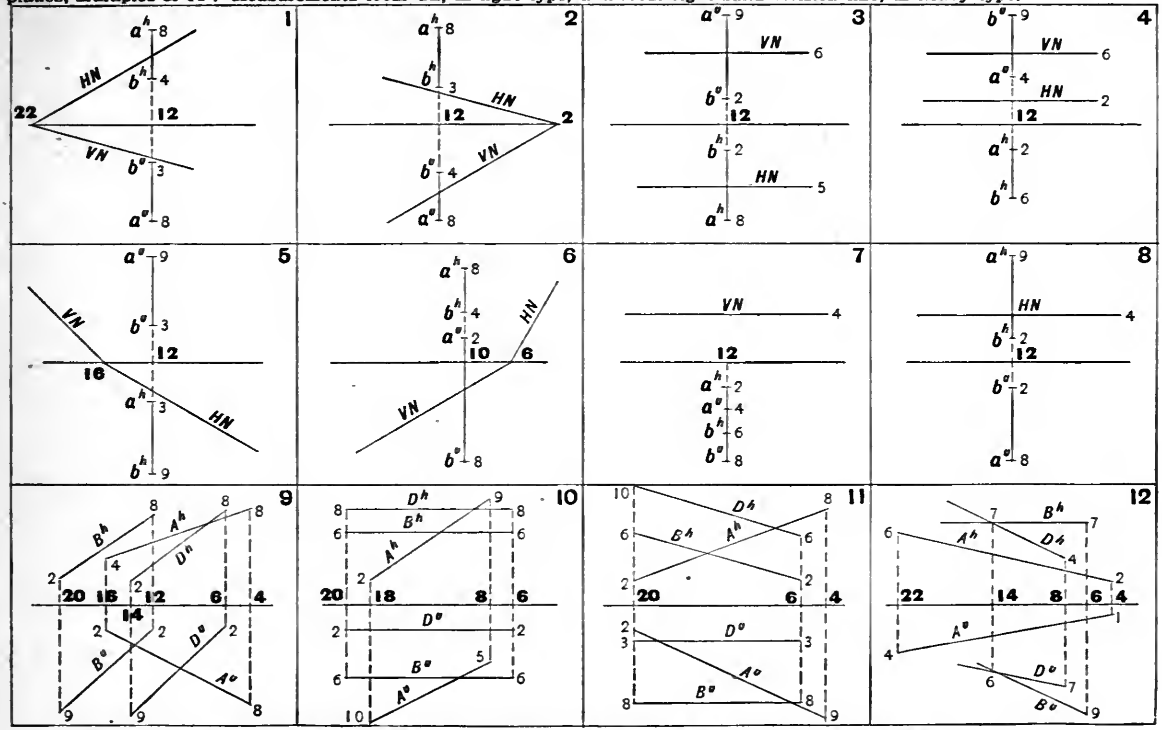
## PLATE 8




Determine the prs. of the point in which line A pierces plane N. Indicate the prs. of the point by $c^v$ and $c^h$. (Arts. 57-59, page 42.)

Unit of measure, ⅛ inch. Space required for each problem, 2½ × 3 inches. Angles between GL and traces of planes, multiples of 15°. Measurements from GL, in light type, and from right-hand division line, in heavy type.

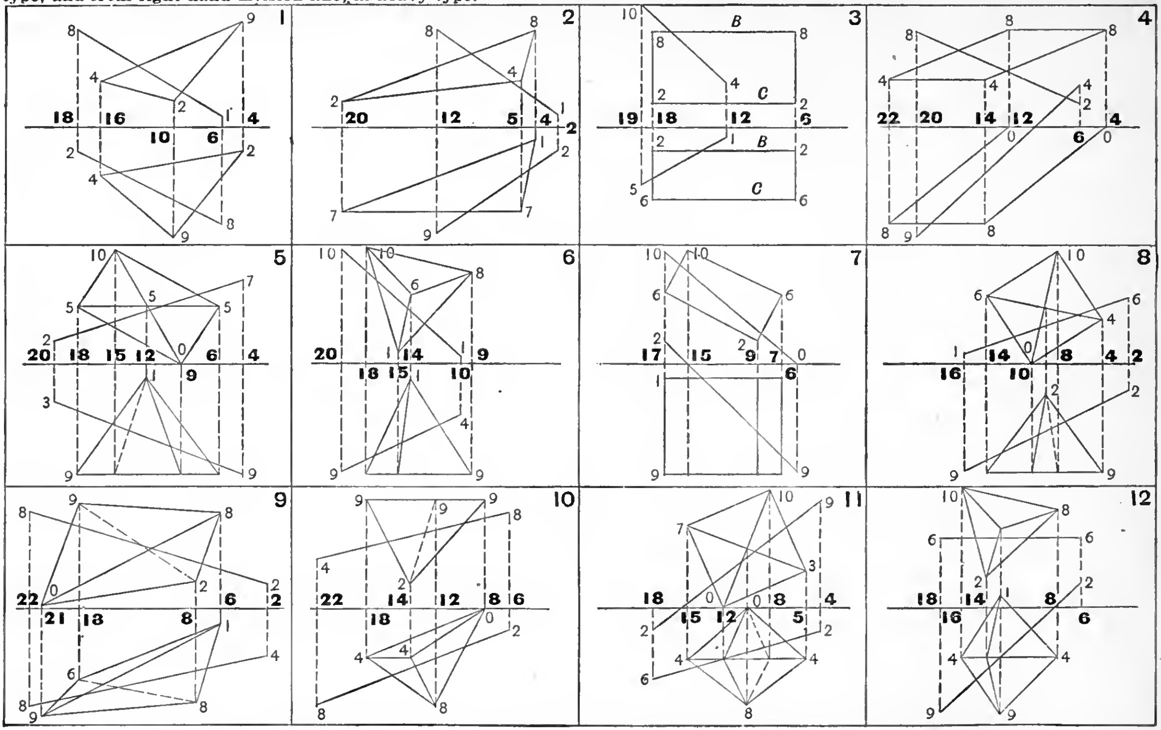
# PLATE 9




Problems 1-8. Determine the prs. of the point in which line ab pierces plane N. (Art. 60, page 42.)

Problems 9-12. Determine the prs. of the point in which line A pierces the plane of lines B and D. (Art. 61, page 44.)

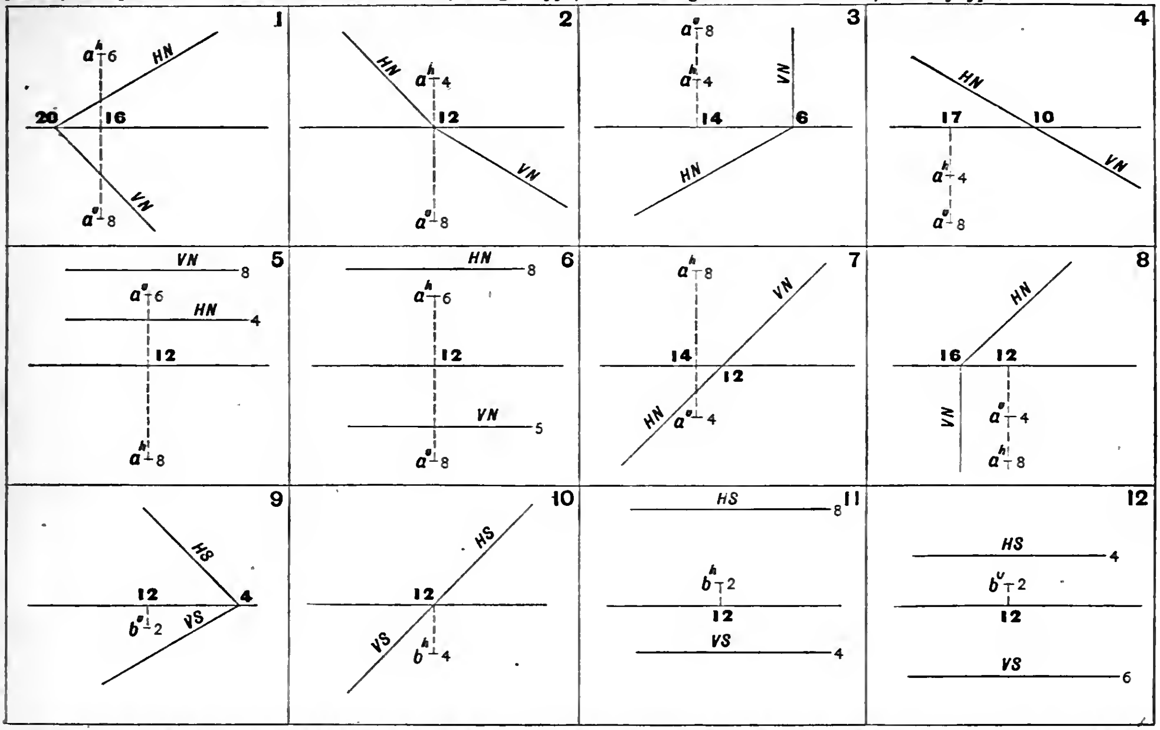
Unit of measure, ⅛ inch. Space required for each problem, 2½ × 3 inches. Measurements from GL, in light type, and from right-hand division line, in heavy type. **PLATE 10**



Problems 1-4. Determine the prs. of the point in which the line pierces the polygon. (Art. 61, page 44.)
Problems 5-15. Determine the prs. of the points in which the line pierces the object. (Art. 61, page 44.)

Unit of measure, ⅛ inch. Space required for each problem, 2½ × 3 inches. Angles between GL and traces of planes, multiples of 15°. Measurements from GL, in light type, and from right-hand division line, in heavy type.

# PLATE 11

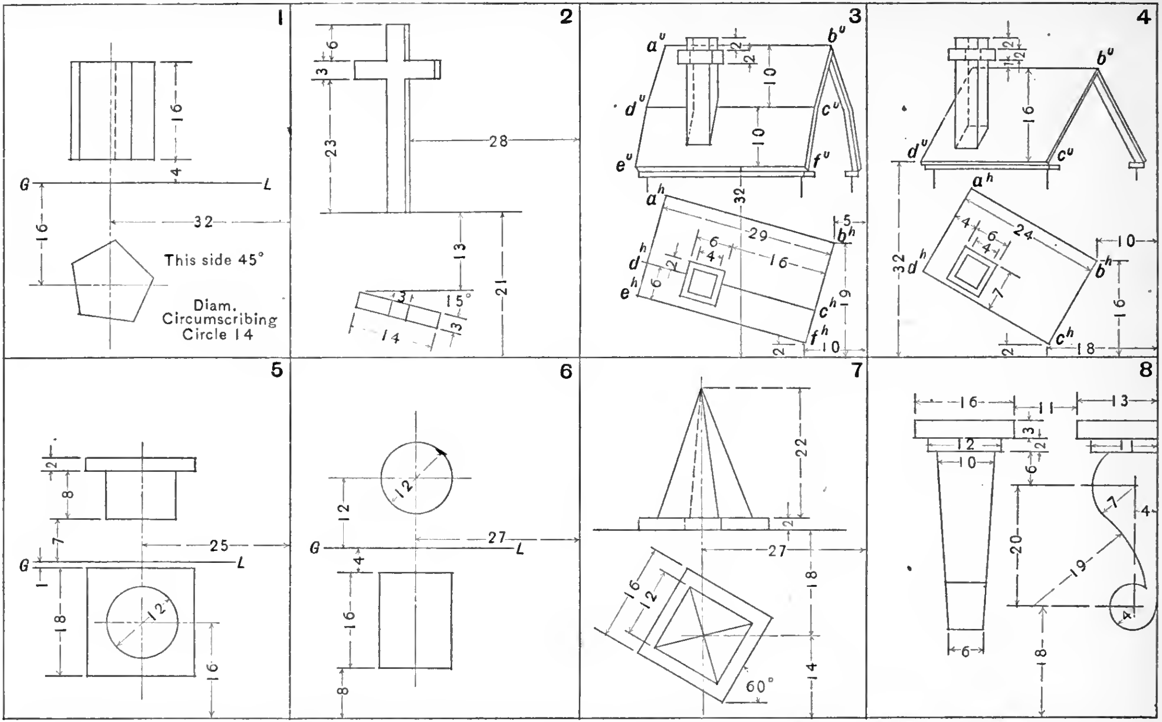



Problems 1-8. Determine the prs. and true length of the line, ab, measuring the shortest distance from point a to plane N. (Arts. 62-65, pages 44, 45.)

Problems 9-12. Determine the prs. of a perpendicular, ab, to plane S at point b on S. Line ab to be 8 units long. (Art. 36, page 24; Arts. 62-65, pages 44, 45.)

# PLATE 12

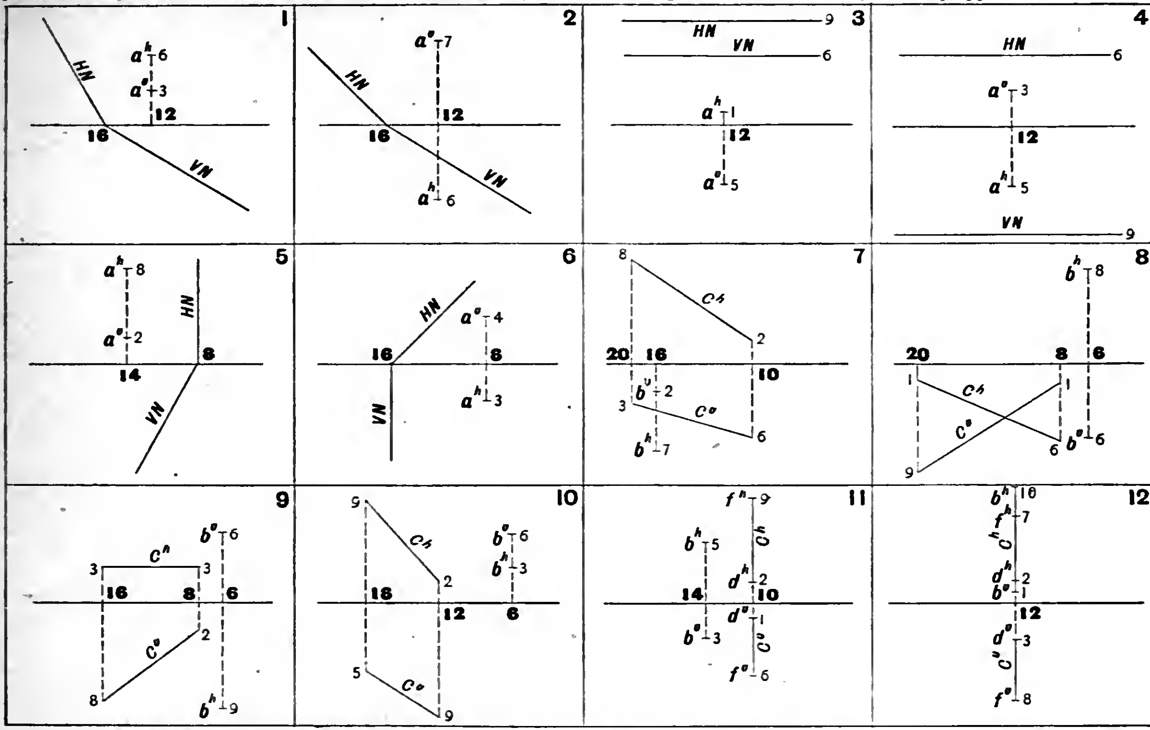
Unit of measure, $\frac{1}{8}$ inch. Space required for each problem, 5 × 7 inches. Measurements from GL, in light type, and from right-hand division line, in heavy type.




Problems 1, 2, 5-7. Determine the prs. of the shadows of the object on itself and on V and H. (Arts. 66-69, pages 46-48.)
Problems 3, 4. Determine the prs. of the shadow of the chimney on itself and on the roof.
Problem 8. Determine the prs. of the shadow of the bracket on itself and on V.

Unit of measure, $\frac{1}{4}$ inch. Space required for each problem, $2\frac{1}{3} \times 3$ inches. Angles between GL and traces of planes, multiples of 15°. Measurements from GL, in light type, and from right-hand division line, in heavy type.

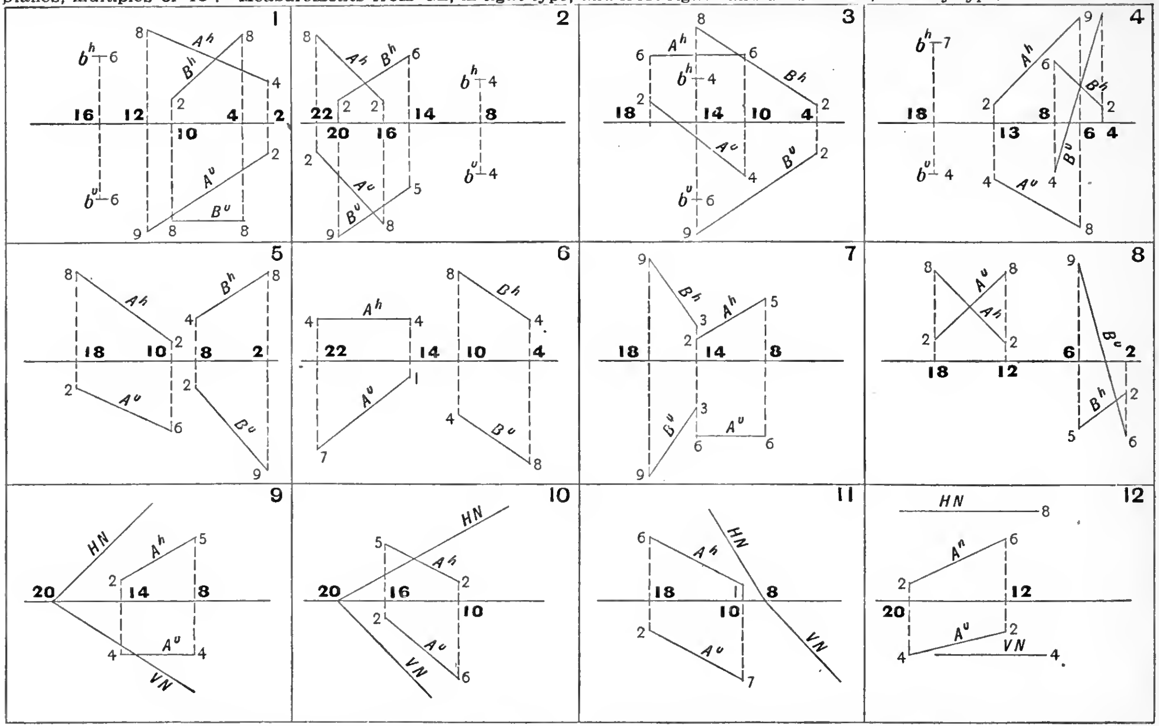
# PLATE 13




Problems 1-6. Determine the traces of the plane containing point a and parallel to plane N. (Art. 71, page 50.)
Problems 7-12. Determine the traces of the plane containing point b and perpendicular to line C. (Art. 72, page 50.)

Unit of measure, $\frac{1}{8}$ inch. Space required for each problem, $2\frac{1}{2} \times 3$ inches. Angles between GL and traces of planes, multiples of 15°. Measurements from GL, in light type, and from right-hand division line, in heavy type.

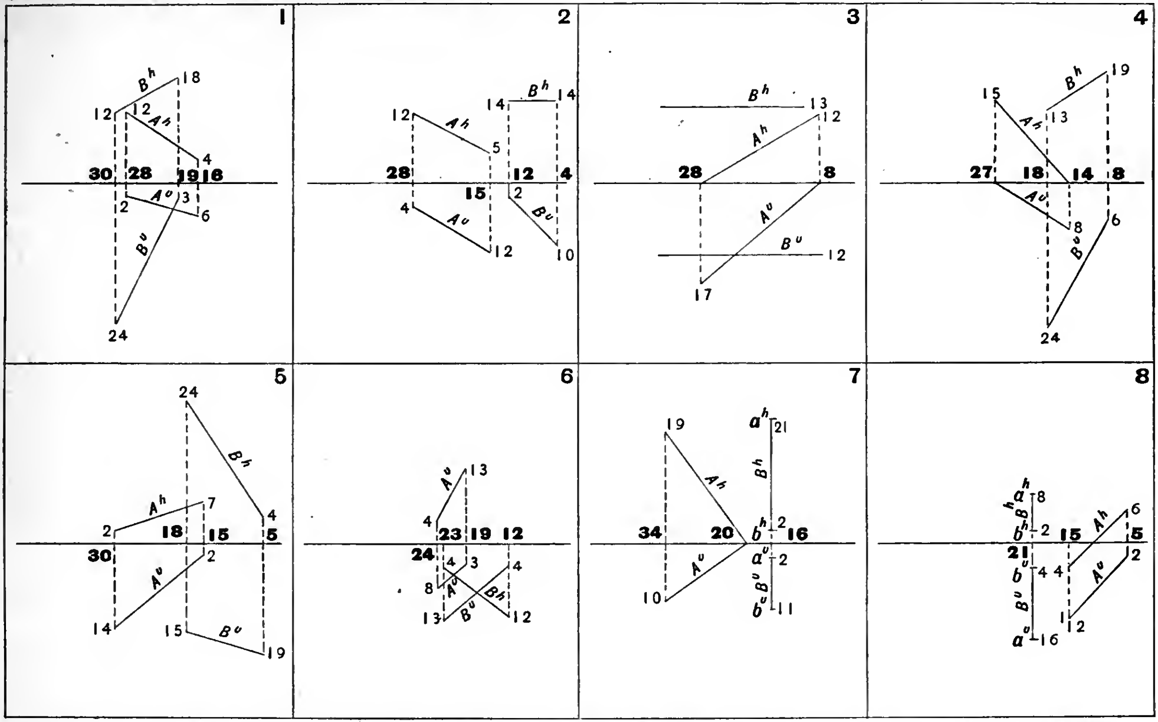
# PLATE 14




Problems 1–4. Through point b pass a plane parallel to lines A and B. (Art. 73, page 51.)
Problems 5–8. Through line A pass a plane parallel to line B. (Art 74, page 52.)
Problems 9–12. Through line A pass a plane perpendicular to plane N. (Art. 75, page 52.)

Unit of measure, ⅛ inch. Space required for each problem, 5 x 7 inches. Measurements from GL, in light type, and from right-hand division line, in heavy type.

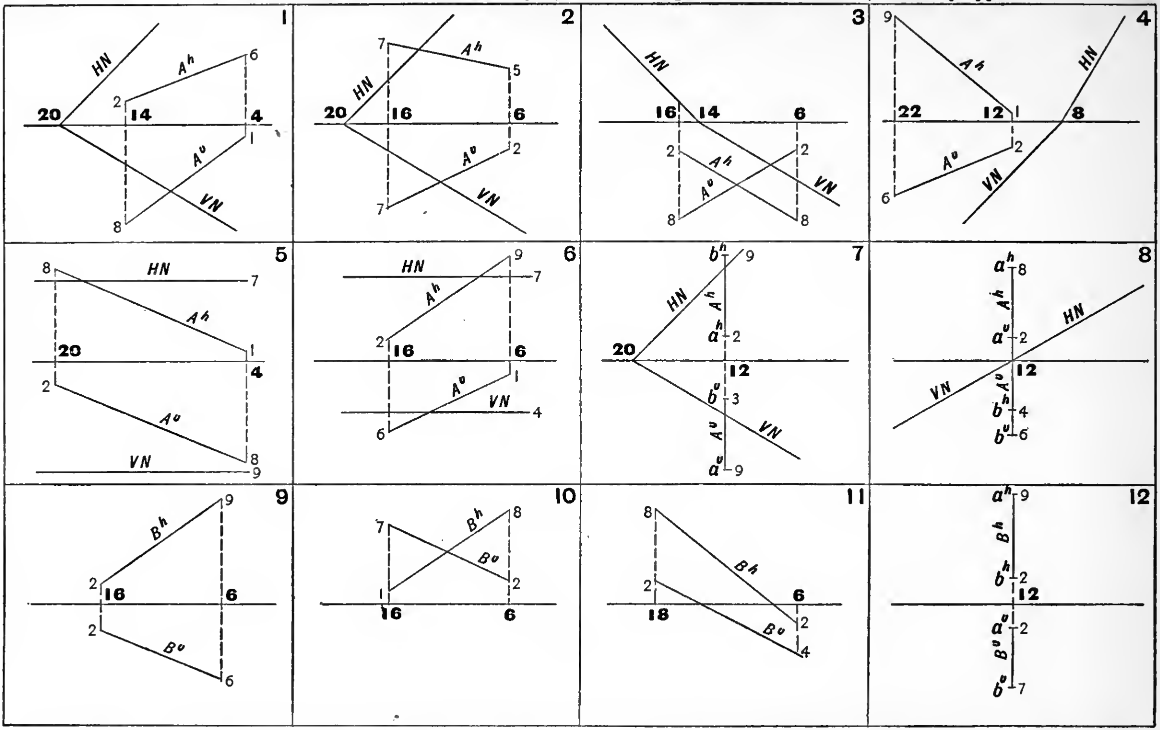
# PLATE 15




Determine the prs. and true length of the line measuring the shortest distance between lines A and B. (Art. 79, page 54.)
Note. — The problem is best solved by passing the auxiliary plane through line A.

Unit of measure, $\frac{1}{8}$ inch. Space required for each problem, $2\frac{1}{2} \times 3$ inches. Angles between GL and traces of planes, multiples of 15°. Measurements from GL, in light type, and from right-hand division line, in heavy type.

# PLATE 16

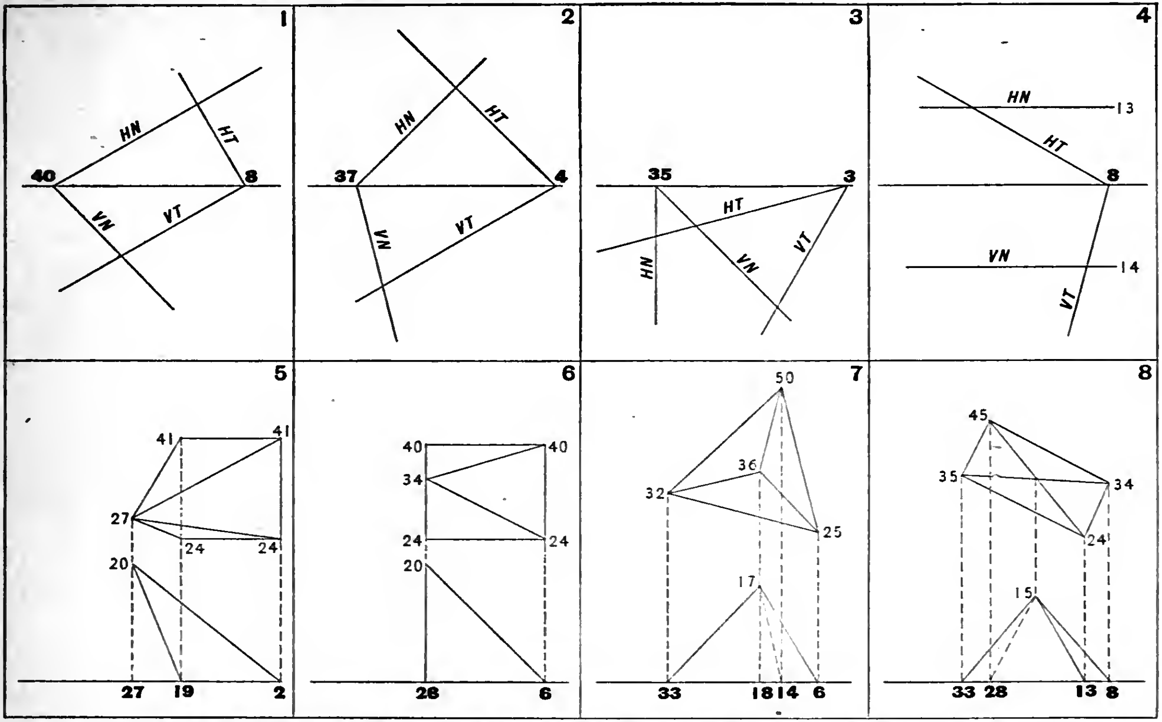



Problems 1-8. Determine the true size of the angle between line A and plane N. (Art. 80, page 54.)

Problems 9-12. Determine the true size of the angle between line B and V and H. (Art. 81, page 55.) Note.—Letter the angle with V as x, and with H as y.

Unit of measure, ¼ inch. Space required for each problem, 5 x 7 inches. Angles between GL and traces of planes, multiples of 15°. Measurements from GL, in light type, and from right-hand division line, in heavy type.

# PLATE 17

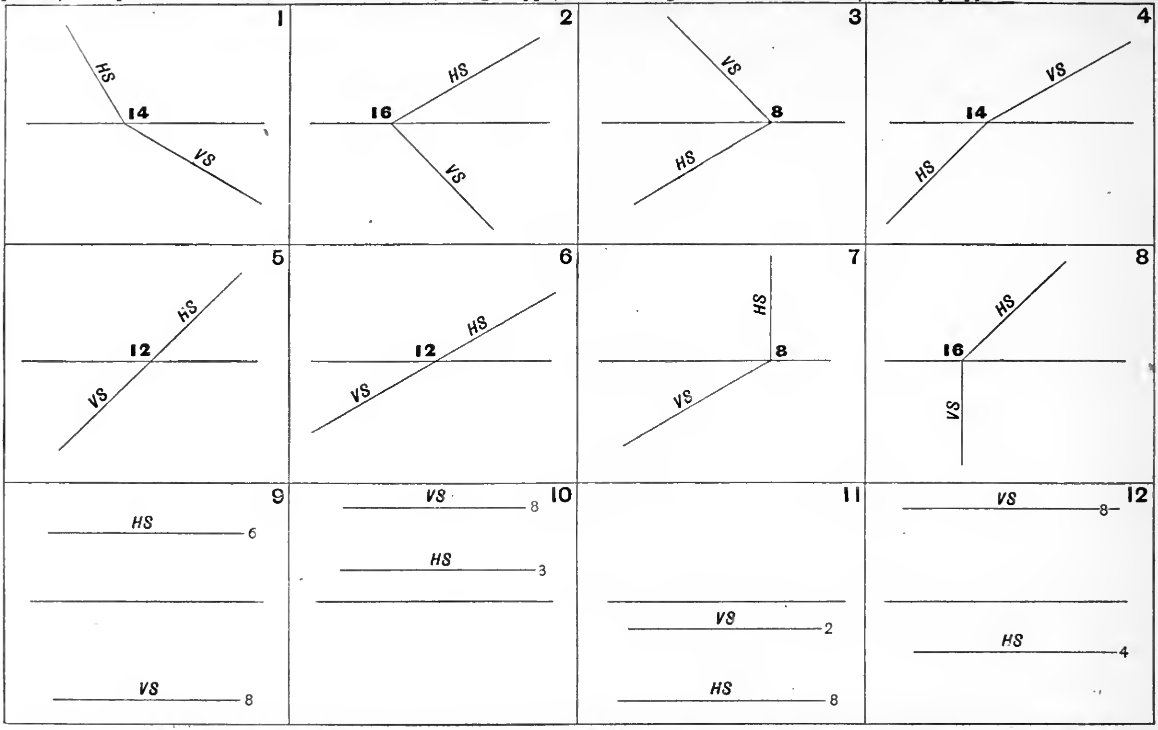



Problems 1-4. Determine the true size of the diedral angle between planes N and T. (Arts. 83-85 pages, 57, 58.)

Problems 5-8. Determine the true sizes of the diedral angles of the objects. (Arts. 83-85, pages 57, 58.)

Unit of measure, $\frac{1}{8}$ inch. Space required for each problem, $2\frac{1}{4} \times 3$ inches. Angles between GL and traces of planes, multiples of 15°. Measurements from GL, in light type, and from right-hand division line, in heavy type.

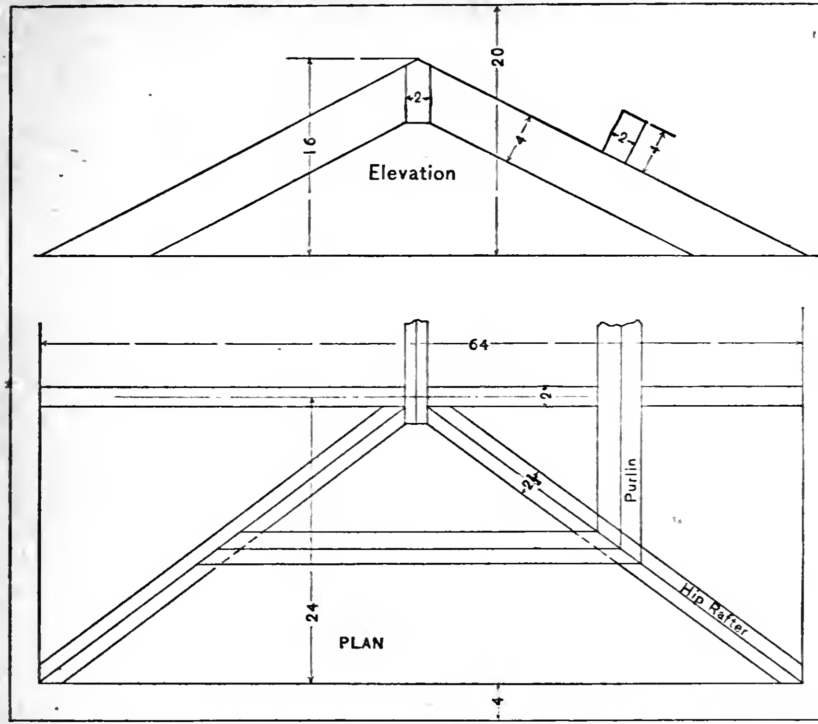
# PLATE 18




Determine the angle between plane S and V and H. (Art. 86, page 58.) Note.—Letter the angle with V as x, and with H as y.

Unit of measure, $\frac{1}{8}$ inch. Space required for the problem, 7 × 10 inches.

# PLATE 19




**HIP RAFTER.**

Determine length — down cut — heel cut — side cut — top bevel.

**JACK RAFTER.**

Determine side cut.

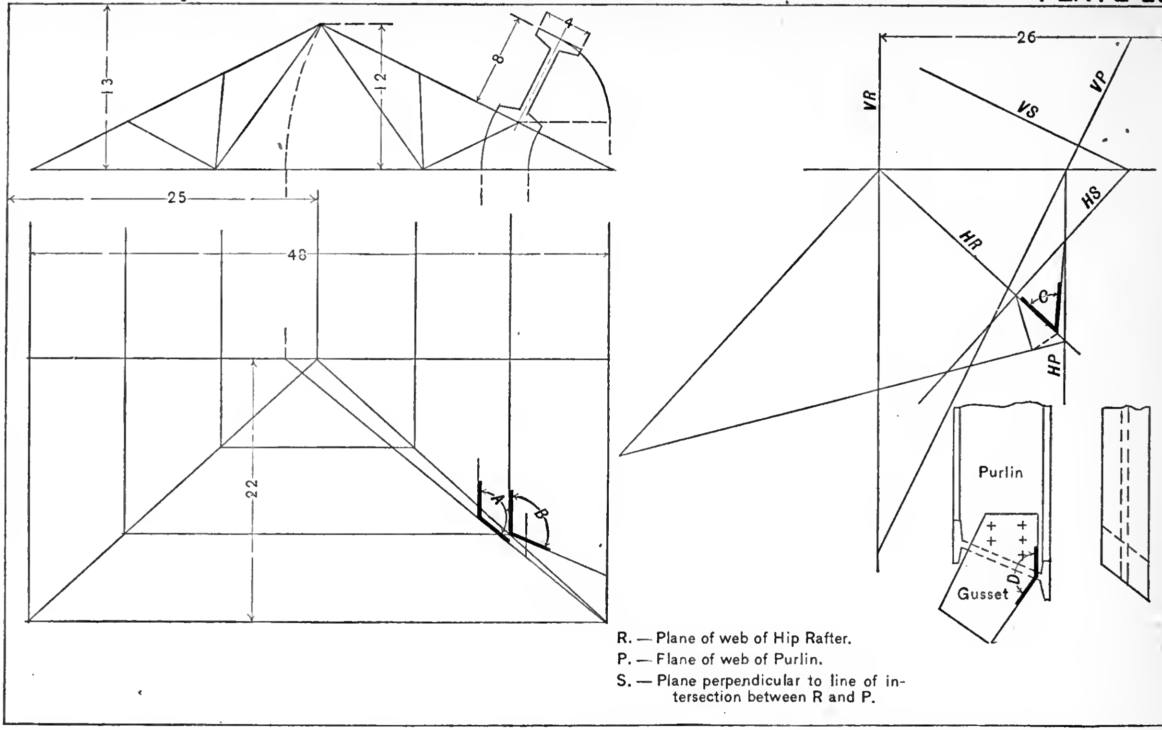
**PURLIN.**

Determine down cut — side cut — angle between face and end.

**Determine the bevels, cuts, and lengths of roof members, as above. (Art. 87, page 60.)**

Unit of measure, $\frac{1}{8}$ inch. Space required for the problem, $7 \times 10$ inches.

# PLATE 20

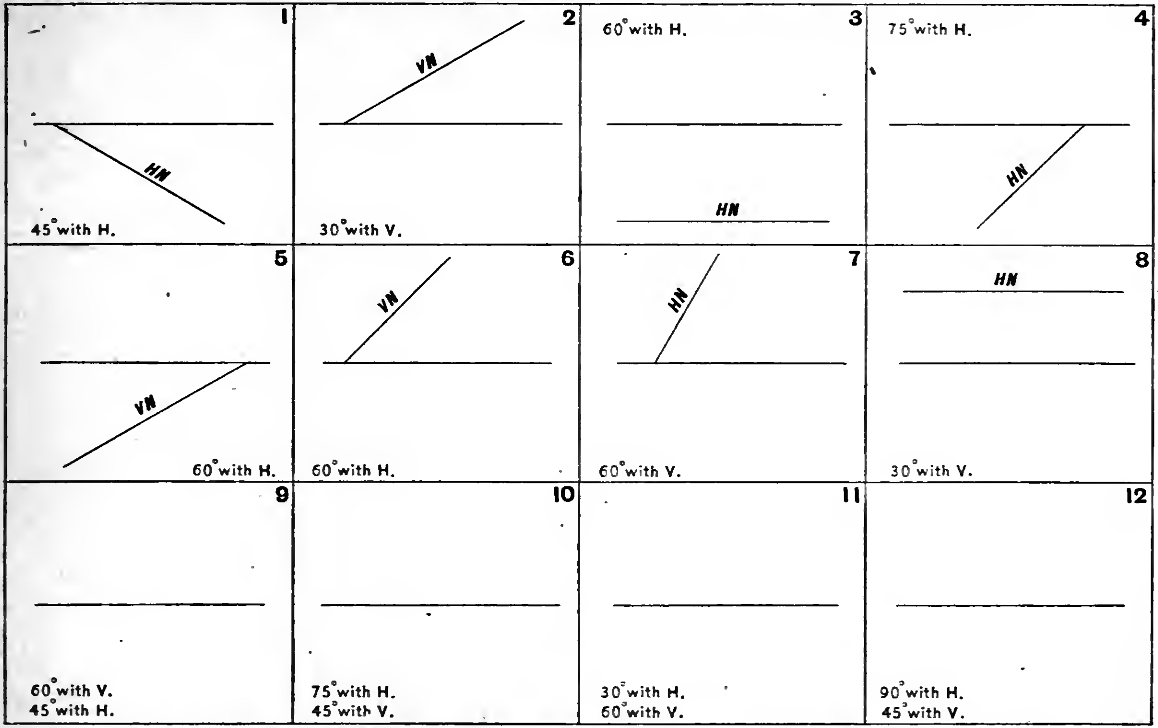



R. — Plane of web of Hip Rafter.
P. — Plane of web of Purlin.
S. — Plane perpendicular to line of intersection between R and P.

Determine angle of cut on top of purlin (A). Bevel on web of purlin (B). Angle between plane of web of hip rafter and purlin, or bend of gusset (C). Angle between top edges of gusset (D). (Art. 87, page 61.)

Unit of measure, $\frac{1}{8}$ inch. Space required for each problem, $2\frac{1}{2} \times 3$ inches. Angles between GL and traces of planes, multiples of 15°. Measurements from GL, in light type, and from right-hand division line, in heavy type.

# PLATE 21

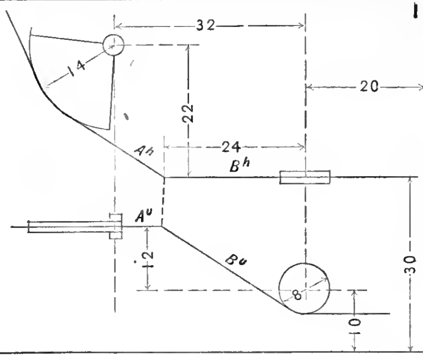



Problems 1-8. Determine one position of the missing trace of plane N. (Arts. 88, 89, page 62.)
Problems 9-12. Determine the traces of plane S making the given angles with V and H. (Art. 90, page 62.)

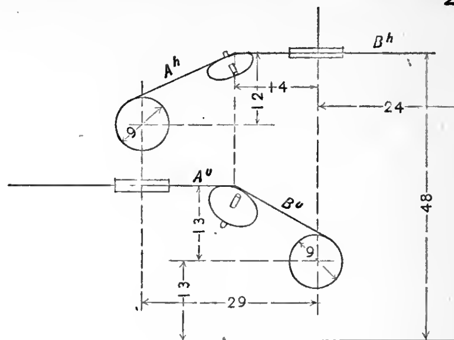
Unit of measure, ⅛ inch. Space required for each problem, 7 × 10 inches.

# PLATE 22

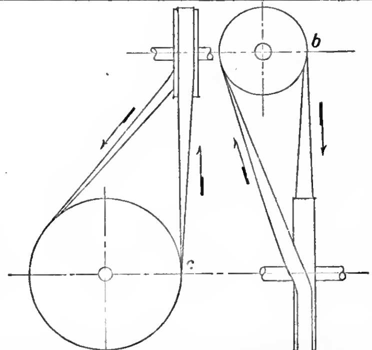
1




2




In belt transmission one condition must always be obtained, namely: the point on a pulley from which the belt is delivered must lie in the mid-plane of the pulley to which it is delivered. If the shafts are at right angles and placed as in the figure, they will run in the direction indicated, point b lying in the mid-plane of the large pulley, and point c lying in the mid plane of the small pulley. But if the direction be reversed a guide pulley would be necessary to compel the belt to fulfill the above conditions. The point at which the direction of the belt is changed by the guide pulley is governed by convenience. Problem 1 is that of a steering gear and requires one guide pulley. Problem 2 illustrates a condition which necessitates two guide pulleys, although but one is to be determined.




Problem 1. Draw the projections of a guide pulley 8 units diameter, 2 units face, and determine the shaft angle with V and H. (Art. 45, page 32.)

Problem 2. Draw the projections of one guide pulley 8 units diameter, 2 units face, and determine the shaft angle with V and H. (Art. 45, page 32.)

Unit of measure, $\frac{1}{8}$ inch. Space required for each problem, $7 \times 10$ inches.



**1**

Locate GL 38 units from lower margin line. Locate point b on GL 16 units from right-hand margin line.

Draw the prs. of a regular hexagonal prism in IQ, resting on a plane, the vertical trace of which makes an angle of 15° with GL, and the horizontal trace, an angle nf 45° with GL. The center of the base of the prism is a point of the plane, 17, 4. The sides of the hexagon are 6 units and two of the sides are perpendicular to the horizontal trace of the plane Altitude of prism is 12 units. The traces of the plane intersect GL in point b.

Determine the shadow of the prism on the plane.

**2**

Locate GL 38 units above lower margin line. Locate point b on GL 20 units from right-hand margin line.

Draw the prs. of a regular pentagonal pyramid in IQ, resting on a plane, the vertical trace of which makes an angle of 15° with GL, and the horizontal trace, an angle of 45° with GL The pyramid rests on its apex at a point 17, 4. Axis of pyramid is perpendicular to the plane and is 12 units long. Circumscribing circle of base of pyramid is 12 units in diameter. One side of pentagon to be parallel to H. The traces of the plane intersect GL in point b.

Determine the shadow of the pyramid on the plane.

**3**

Locate GL 38 units from lower margin line. Locate point b on GL 16 units from right-hand margin line.

Draw the prs. of a regular hexagonal pyramid in IQ, resting on a plane which makes an angle of 20° with H and 75° with V. The pyramid rests on its apex at a point in the plane, 18, 4. Axis of pyramid is perpendicular to the plane, and is 12 units long. The short diameter of the base is parallel to H; the long diameter is 12 units in length. The traces of the plane intersect GL in point b.

Determine the shadow of the pyramid on the plane.

**4**

Locate GL 38 units from lower margin line. Locate point b on GL 60 units from right-hand margin line.

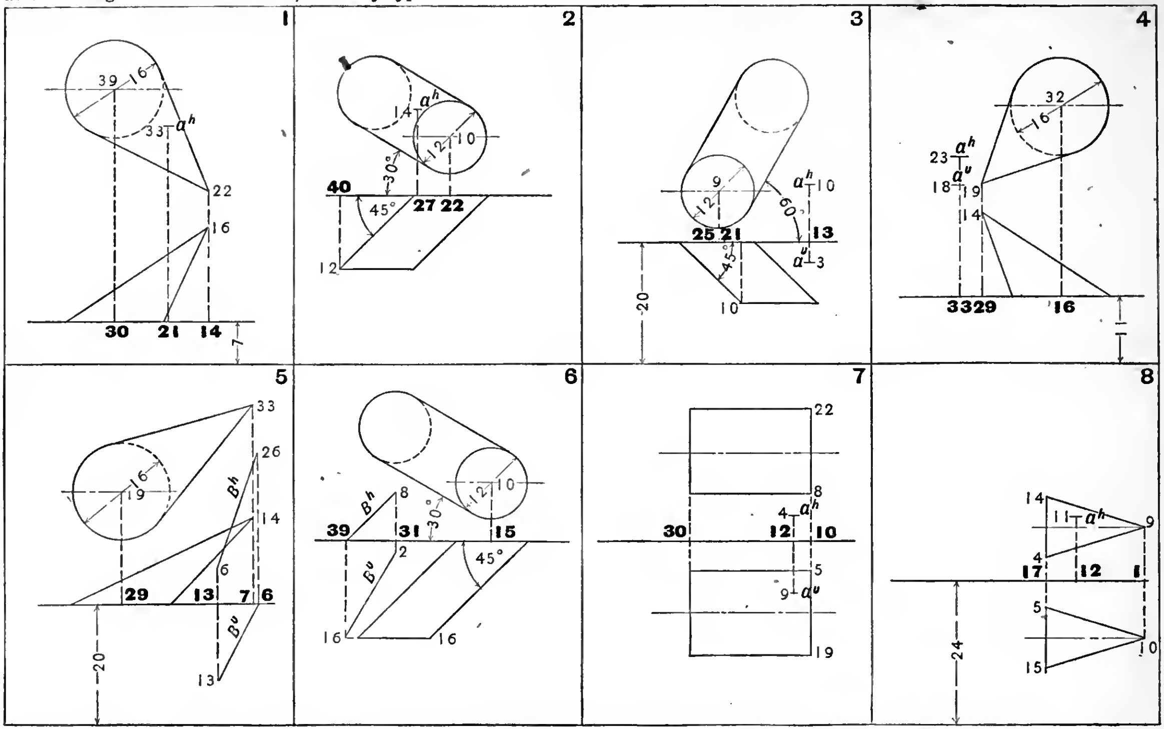
Draw the prs. of a cube in IQ, resting on a plane which makes an angle of 30° with H and 65° with V. The center of the base of the cube is 10 units from the horizontal trace of the plane and 8 units from V. One diagonal of the base is parallel to H. Edge of cube is 8 units. The traces of the plane intersect GL in point b.

Determine the shadow of the cube on the plane.

**Draw the prs. of a solid resting on an oblique plane, and determine the prs. of its shadow on the plane.**

Unit of measure, $\frac{1}{8}$ inch. Space required for each problem, $5 \times 7$ inches. Measurements from GL, in light type, and from right-hand division line, in heavy type.

# PLATE 24



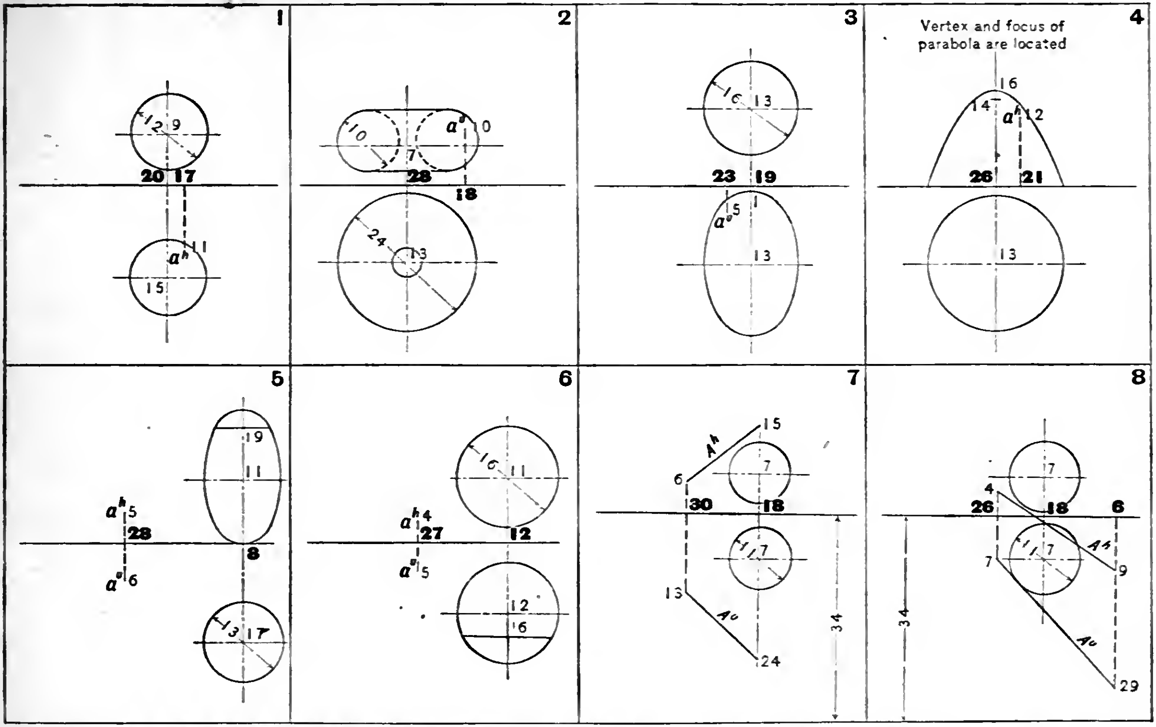

Problems 1, 2, 8. Draw the traces of a plane which is tangent at point a of the surface. (Arts. 104, 105, pages 72-74.)

Problems 3, 4, 7. Draw traces of planes which are tangent to the surface and contain point a. (Art. 106, page 74; Art. 108, page 75.)

Problems 5, 6. Draw traces of planes which are tangent to the surface, and parallel to line B. (Art. 107, page 75; Art. 109, page 76.)

Unit of measure, ⅛ inch. Space required for each problem, 5 × 7 inches. Measurements from GL, in light type, and from right-hand division line, in heavy type.

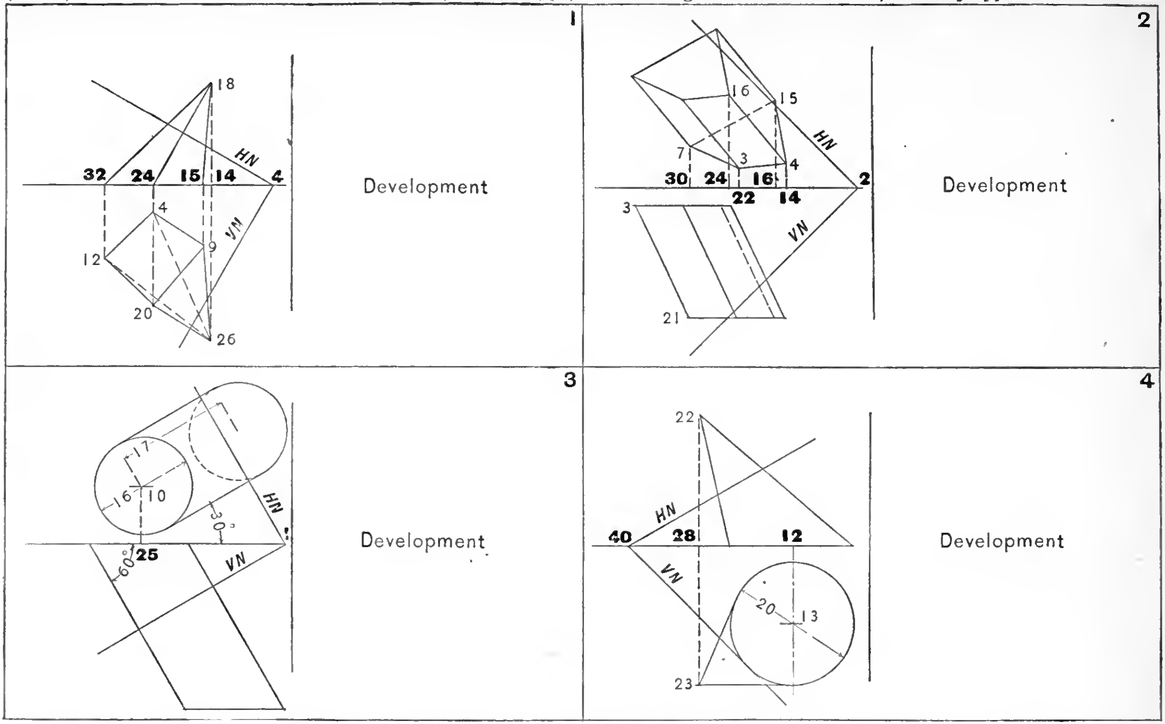
# PLATE 25




**Problems 1-4.** Draw the traces of a plane which is tangent at point a of the surface. (Art. 111, page 78.)
**Problems 5, 6.** Draw traces of planes containing point a and tangent to the surface at given parallel. (Art. 112, page 79.)
**Problems 7, 8.** Draw the traces of planes tangent to the sphere and containing line A. (Art. 114, page 80.)
**Note.** Problems 5-8. Determine the points of tangency.

Unit of measure, $\frac{1}{8}$ inch. Space required for each problem, 7 × 10 inches. Angles between GL and traces of planes, multiples of 15°. Measurements from GL, in light type, and from right-hand division line, in heavy type

# PLATE 26

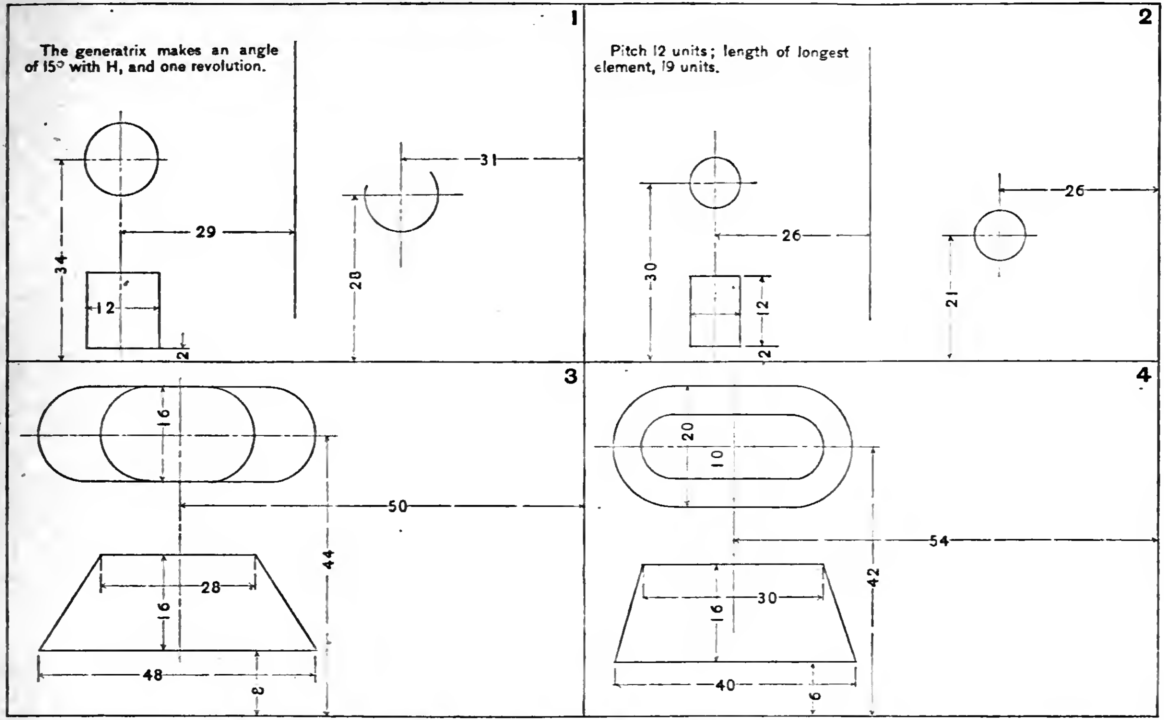



Determine the intersection of the plane with the solid, and develop the surface.

Problem 1. (Arts. 120, 121, pages 83, 84.) Problem 2. (Art. 127, page 92.) Problem 3. (Arts. 124, 125, page 88.) Problem 4. (Arts. 122, 123, page 86.)

Unit of measure, $\frac{1}{8}$ inch. Space required for each problem, 7 x 10 inches.

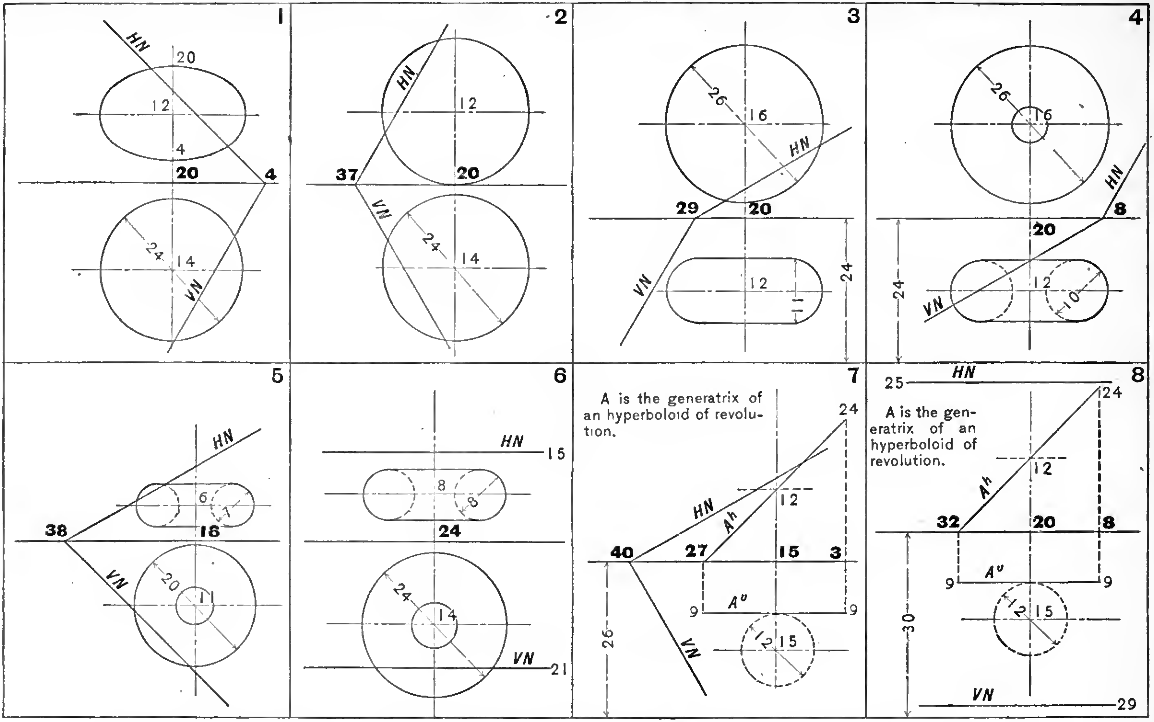
## PLATE 27




**Problems 1, 2. Draw and develop an helical convolute. (Arts. 129-131, pages 93-95.)**
**Problems 3, 4. Delvelop one quarter of the lamp shade. Problem 3. (Art. 126, page 91.) Problem 4. (Art. 123, page 86.)**

Unit of measure, $\frac{1}{8}$ inch. Space required for each problem, 5 × 7 inches. Angles between GL and traces of planes, multiples of 15°. Measurements from GL, in light type, and from right-hand division line, in heavy type.

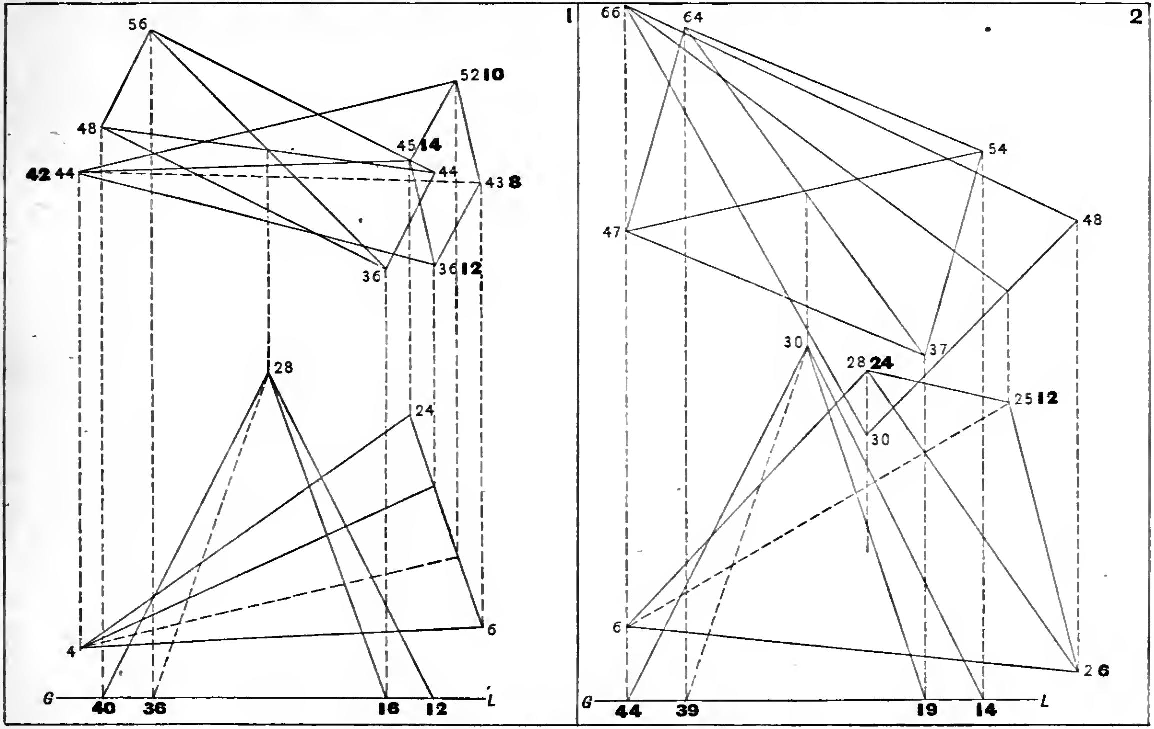
## PLATE 28




Determine the intersection of the plane N with the double-curved surface of revolution. (Art. 132, page 96.)

Unit of measure, $\frac{1}{4}$ inch. Space required for each problem, 7 × 10 inches. Measurements from GL, in light type, and from right-hand division line, in heavy type.

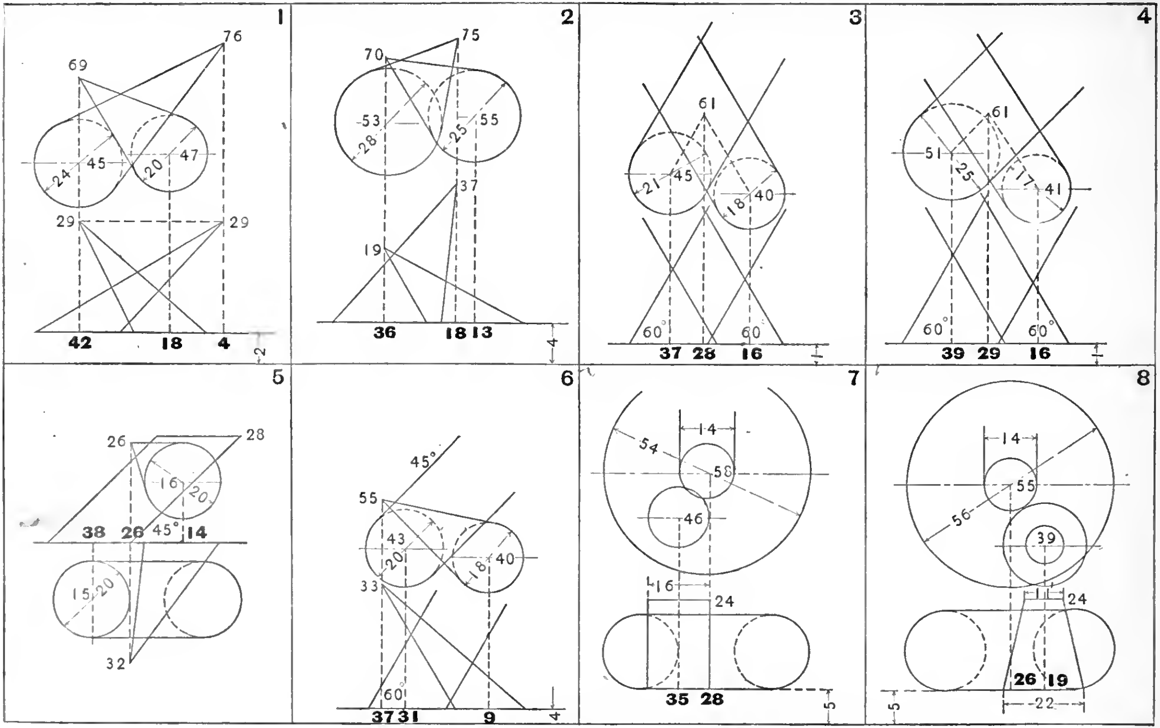
**PLATE 29**




Determine the intersection between the solids.

Unit of measure, $\frac{1}{8}$ inch. Space required for each problem, $7 \times 10$ inches. Measurements from GL, in light type, and from right-hand division line, in heavy type.

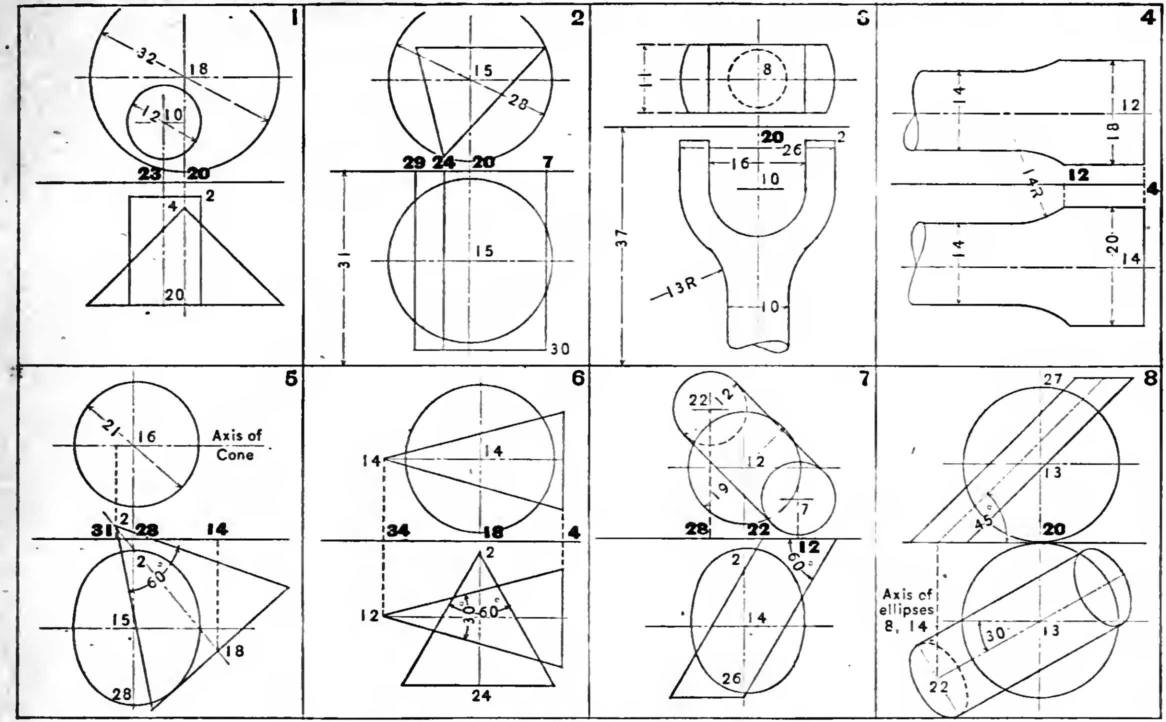
**PLATE 30**




Determine the intersection between the solids.

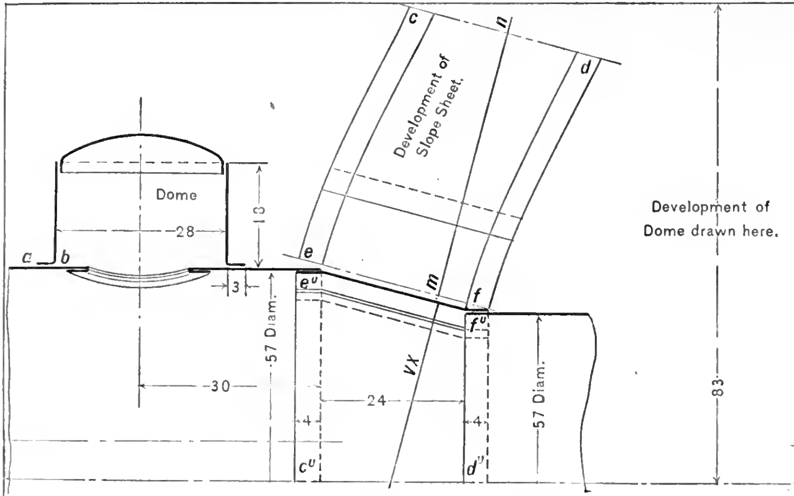
Unit of measure, ¼ inch. Space required for each problem, 5 × 7 inches. Measurements from GL, in light type, and from right-hand division line, in heavy type.

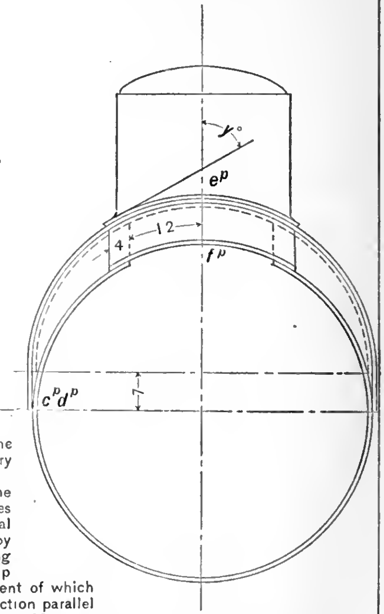
## PLATE 31




Determine the intersection between the solids.

The unit of measurement is 1/16 inch. Space required 7 × 10 inches. **PLATE 32**



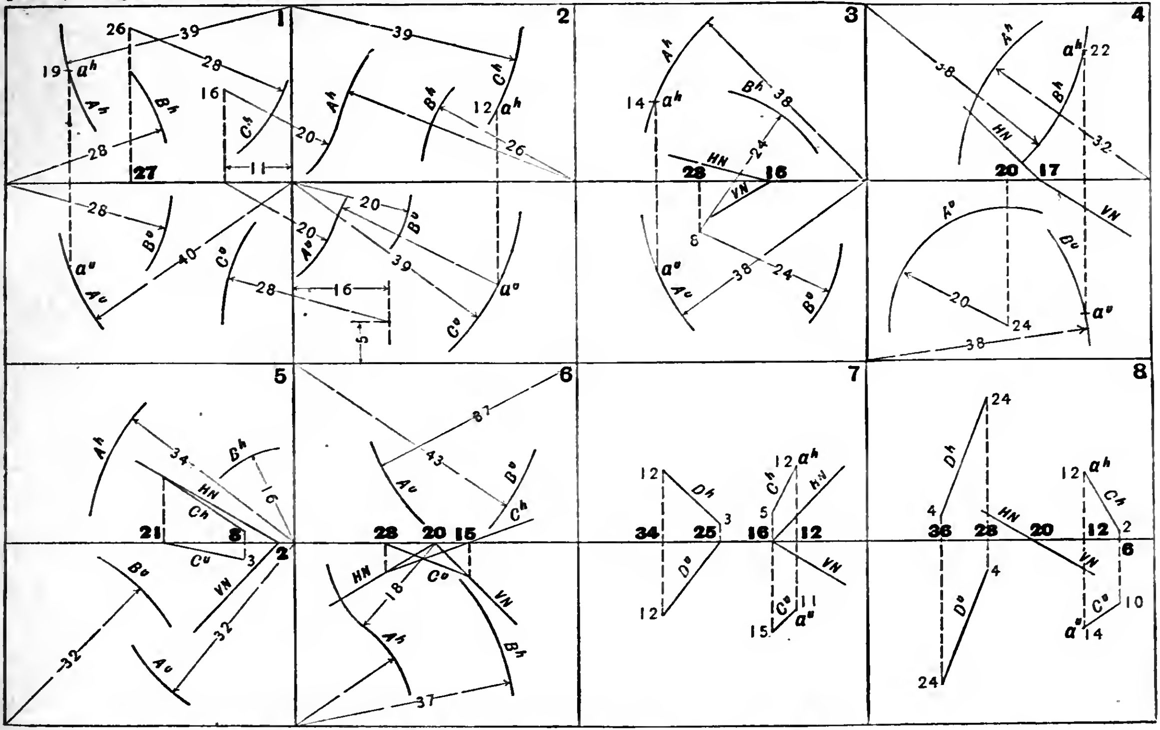

A sectional and end view of a portion of a boiler is shown. It is required to develop one half of the dome, and that part of the slope sheet lying between the elements ef and cd, adding the amount necessary for flanging or lap

Make two full views of the boiler, indicating the thickness of plate by a single line. Determine the projection of the two curves of the flange ab, observing that the radial elements of the flange make angles varying from 0° to y°. The elements of the slope sheet will be parallel to each other and to the vertical plane, thus appearing in their true lengths in vertical projection. The development may be obtained by either of the following methods, the first being that used in practice. First method: the elements being parallel to V, the development may be obtained as in Art. 126, page 91, the amount indicated for the lap being added to the sheet. Second method: obtain a right section VX by Art 118, page 82, the development of which will be a right line, mn. To obtain the distance between the elements revolve the plane of the right section parallel to P, and measure the required distances in the profile projection, or end view. Develop as in Art. 125, page 88.

**Draw and develop the dome and connection sheet, or slope sheet, of a locomotive boiler.**

Unit of measure, ¼ inch. Space required for each problem, 5 × 7 inches. Angles between GL and traces of planes, multiples of 15°. Measurements from GL, in light type, and from right-hand division line, in heavy type.

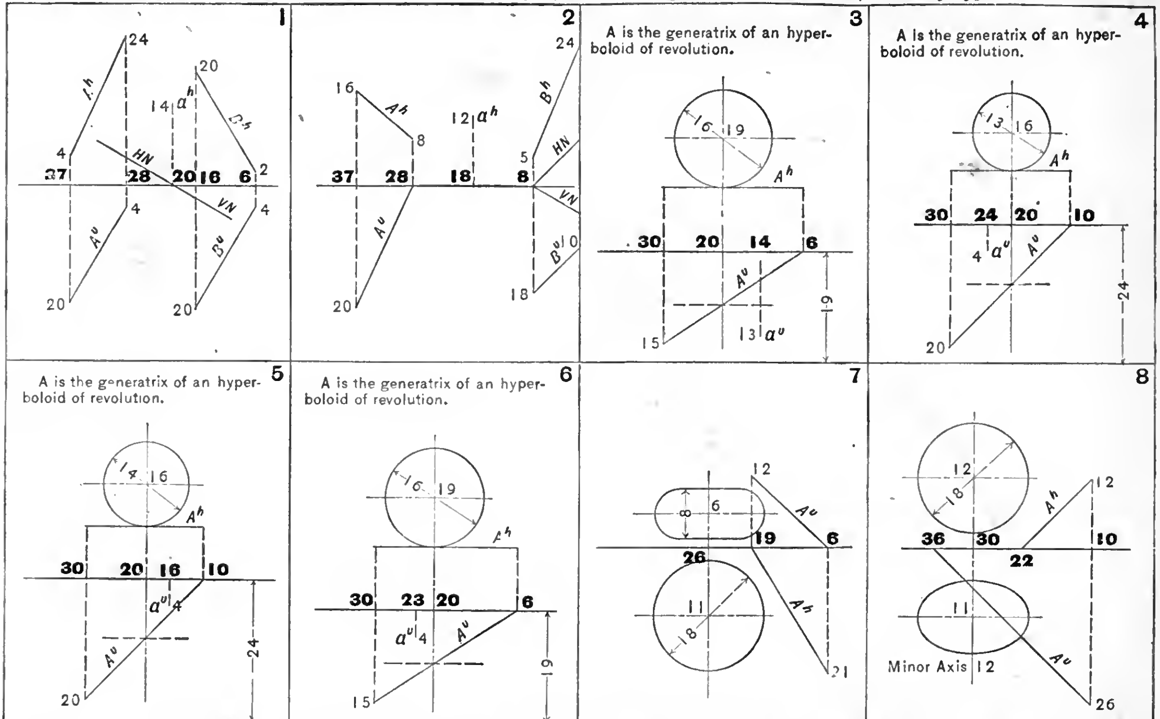
# PLATE 33




Problems 1, 2. Draw an element of the warped surface through point a. (Art. 145, page 110.)
Problems 3, 4. Draw an element of the warped surface through point a. (Art. 146, page 111.)
Problems 5, 6. Draw an element of the warped surface parallel to line C of the plane director, N. (Art. 147, page 112.)
Problems 7, 8. Draw an element of the hyperbolic paraboloid through point a, of a directrix. (Art. 150, page 116.)

Unit of measure, $\frac{1}{8}$ inch. Space required for each problem, 5×7 inches. Angles between GL and traces of planes, multiples of 15°. Measurements from GL, in light type, and from right-hand divison line, in heavy type.

# PLATE 34




Problems 1, 2. Draw an element of the hyperbolic paraboloid through point a. (Art. 151, page 116.)
Problems 3, 4. Draw an element through point a. (Art. 156, page 120.)
Problems 5, 6. Draw a plane tangent to the surface at point a. (Art. 159, page 121.)
Problems 7, 8. Through line A pass a plane tangent to the surface. (Art. 160, page 122.)